

FUNCTIONS DEFINED BY IMPROPER INTEGRALS



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Book: Functions Defined by Improper Integrals
(Trench)

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This text was compiled on 08/26/2026

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1: CHAPTERS

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1.1: Introduction to Improper Functions

In Section 7.2 (pp. 462–484) we considered functions of the form

$$F(y) = \int_a^b f(x, y) dx, \quad c \leq y \leq d. \quad (1.1.1)$$

We saw that if f is continuous on $[a, b] \times [c, d]$, then F is continuous on $[c, d]$ (Exercise 7.2.3, p. 481) and that we can reverse the order of integration in

$$\int_c^d F(y) dy = \int_c^d \left(\int_a^b f(x, y) dx \right) dy \quad (1.1.2)$$

to evaluate it as

$$\int_c^d F(y) dy = \int_a^b \left(\int_c^d f(x, y) dy \right) dx \quad (1.1.3)$$

(Corollary 7.2.3, p. 466).

Here is another important property of F .

[theorem:1] If f and f_y are continuous on $[a, b] \times [c, d]$, then

$$F(y) = \int_a^b f(x, y) dx, \quad c \leq y \leq d, \quad (1.1.4)$$

is continuously differentiable on $[c, d]$ and $F'(y)$ can be obtained by differentiating [eq:1] under the integral sign with respect to y ; that is,

$$F'(y) = \int_a^b f_y(x, y) dx, \quad c \leq y \leq d. \quad (1.1.5)$$

Here $F'(a)$ and $f_y(x, a)$ are derivatives from the right and $F'(b)$ and $f_y(x, b)$ are derivatives from the left.

If y and $y + \Delta y$ are in $[c, d]$ and $\Delta y \neq 0$, then

$$\frac{F(y + \Delta y) - F(y)}{\Delta y} = \int_a^b \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} dx. \quad (1.1.6)$$

From the mean value theorem (Theorem 2.3.11, p. 83), if $x^{\text{int}}[a, b]$ and $y, y + \Delta y^{\text{int}}[c, d]$, there is a $y(x)$ between y and $y + \Delta y$ such that

$$f(x, y + \Delta y) - f(x, y) = f_y(x, y) \Delta y = f_y(x, y(x)) \Delta y + (f_y(x, y(x)) - f_y(x, y)) \Delta y. \quad (1.1.7)$$

From this and [eq:3],

$$\left| \frac{F(y + \Delta y) - F(y)}{\Delta y} - \int_a^b f_y(x, y) dx \right| \leq \int_a^b |f_y(x, y(x)) - f_y(x, y)| dx. \quad (1.1.8)$$

Now suppose $\epsilon > 0$. Since f_y is uniformly continuous on the compact set $[a, b] \times [c, d]$ (Corollary 5.2.14, p. 314) and $y(x)$ is between y and $y + \Delta y$, there is a $\delta > 0$ such that if $|\Delta| < \delta$ then

$$|f_y(x, y) - f_y(x, y(x))| < \epsilon, \quad (x, y)^{\text{int}}[a, b] \times [c, d]. \quad (1.1.9)$$

This and [eq:4] imply that

$$\left| \frac{F(y + \Delta y) - F(y)}{\Delta y} - \int_a^b f_y(x, y) dx \right| < \epsilon(b - a) \quad (1.1.10)$$

if y and $y + \Delta y$ are in $[c, d]$ and $0 < |\Delta y| < \delta$. This implies [eq:2]. Since the integral in [eq:2] is continuous on $[c, d]$ (Exercise 7.2.3, p. 481, with f replaced by f_y), F' is continuous on $[c, d]$.

[example:1] Since

$$f(x, y) = \cos xy \quad \text{and} \quad f_y(x, y) = -x \sin xy \quad (1.1.11)$$

are continuous for all (x, y) , Theorem [theorem:1] implies that if

$$F(y) = \int_0^\pi \cos xy \, dx, \quad -\infty < y < \infty$$

then

$$F'(y) = -\int_0^\pi x \sin xy \, dx, \quad -\infty < y < \infty$$

(In applying Theorem [theorem:1] for a specific value of y , we take $R = [0, \pi] \times [-\rho, \rho]$, where $\rho > |y|$.) This provides a convenient way to evaluate the integral in [eq:6]: integrating the right side of [eq:5] with respect to x yields

$$F(y) = \frac{\sin xy}{y} \Big|_{x=0}^\pi = \frac{\sin \pi y}{y}, \quad y \neq 0. \quad (1.1.12)$$

Differentiating this and using [eq:6] yields

$$\int_0^\pi x \sin xy \, dx = \frac{\sin \pi y}{y^2} - \frac{\pi \cos \pi y}{y}, \quad y \neq 0. \quad (1.1.13)$$

To verify this, use integration by parts.

We will study the continuity, differentiability, and integrability of

$$F(y) = \int_a^b f(x, y) \, dx, \quad y \in S, \quad (1.1.14)$$

where S is an interval or a union of intervals, and F is a convergent improper integral for each $y \in S$. If the domain of f is $[a, b] \times S$ where f is pointwise convergent on S or simply *convergent on S* , and write

$$\int_a^b f(x, y) \, dx = \lim_{r \rightarrow b^-} \int_a^r f(x, y) \, dx \quad (1.1.15)$$

if, for each $y \in S$ and every $\epsilon > 0$, there is an $r = r_0(y)$ (which also depends on ϵ) such that

$$\left| F(y) - \int_a^r f(x, y) \, dx \right| = \left| \int_r^b f(x, y) \, dx \right| < \epsilon, \quad \text{for } r \geq r_0(y) \quad (1.1.16)$$

If the domain of f is $(a, b] \times S$ where f is

$$\int_a^b f(x, y) \, dx = \lim_{r \rightarrow a^+} \int_r^b f(x, y) \, dx \quad (1.1.16)$$

and [eq:8] by

$$\left| F(y) - \int_r^b f(x, y) \, dx \right| = \left| \int_a^r f(x, y) \, dx \right| < \epsilon, \quad \text{for } r \geq r_0(y)$$

In general, pointwise convergence of F for all $y \in S$ does not imply that F is continuous or integrable on $[c, d]$, and the additional assumptions that f_y is continuous and $\int_a^b f_y(x, y) \, dx$ converges do not imply [eq:2].

[example:2] The function

$$f(x, y) = ye^{-|y|x} \quad (1.1.17)$$

is continuous on $[0, \infty) \times (-\infty, \infty)$ and

$$F(y) = \int_0^{\infty} f(x, y) dx = \int_0^{\infty} y e^{-|y|x} dx \quad (1.1.18)$$

converges for all y , with

$$F(y) = \begin{cases} -1 & y < 0, \\ 0 & y = 0, \\ 1 & y > 0; \end{cases} \quad (1.1.19)$$

therefore, F is discontinuous at $y = 0$.

[example:3] The function

$$f(x, y) = y^3 e^{-y^2 x} \quad (1.1.20)$$

is continuous on $[0, \infty) \times (-\infty, \infty)$. Let

$$F(y) = \int_0^{\infty} f(x, y) dx = \int_0^{\infty} y^3 e^{-y^2 x} dx = y, \quad -\infty < y < \infty$$

Then

$$F'(y) = 1, \quad -\infty < y < \infty$$

However,

$$\int_0^{\infty} \frac{\partial}{\partial y} (y^3 e^{-y^2 x}) dx = \int_0^{\infty} (3y^2 - 2y^4 x) e^{-y^2 x} dx = \begin{cases} 1, & y \neq 0, \\ 0, & y = 0, \end{cases} \quad (1.1.21)$$

so

$$F'(y) \neq \int_0^{\infty} \frac{\partial f(x, y)}{\partial y} dx \quad \text{if } y = 0. \quad (1.1.22)$$

1.2: Preparation

We begin with two useful convergence criteria for improper integrals that do not involve a parameter. Consistent with the definition on p. 152, we say that f is locally integrable on an interval I if it is integrable on every finite closed subinterval of I .

[theorem:2] Suppose g is locally integrable on $[a, b)$ and denote

$$G(r) = \int_a^r g(x) dx, \quad a \leq r$$

Then the improper integral $\int_a^b g(x) dx$ converges if and only if, for each $\epsilon > 0$, there is an $r_0^{\text{int}}[a, b)$ such that

$$|G(r) - G(r_1)| < \epsilon, \quad r_0 \leq r, r_1$$

For necessity, suppose $\int_a^b g(x) dx = L$. By definition, this means that for each $\epsilon > 0$ there is an $r_0^{\text{int}}[a, b)$ such that

$$|G(r) - L| < \frac{\epsilon}{2} \quad \text{and} \quad |G(r_1) - L| < \frac{\epsilon}{2}, \quad r_0 \leq r, r_1$$

Therefore

$$|G(r) - G(r_1)| = |(G(r) - L) - (G(r_1) - L)| \leq |G(r) - L| + |G(r_1) - L| < \epsilon, \quad r_0 \leq r, r_1$$

For sufficiency, [eq:9] implies that

$$|G(r)| = |G(r_1) + (G(r) - G(r_1))| \leq |G(r_1)| + |G(r) - G(r_1)| \leq |G(r_1)| + \epsilon, \quad (1.2.1)$$

$$r_0 \leq r \leq r_1$$

$$\overline{G}(r) = \sup\{G(r_1)\}, \quad \underline{G}(r) = \inf\{G(r_1)\}, \quad r_0 \leq r \leq r_1$$

are well defined on $[a, b)$, and

$$\lim_{r \rightarrow b^-} \overline{G}(r) = \overline{L} \quad \text{and} \quad \lim_{r \rightarrow b^-} \underline{G}(r) = \underline{L} \quad (1.2.2)$$

both exist and are finite (Theorem 2.1.11, p. 47). From [eq:9],

$$\begin{aligned} |G(r) - G(r_1)| &= |(G(r) - G(r_0)) - (G(r_1) - G(r_0))| \\ &\leq |G(r) - G(r_0)| + |G(r_1) - G(r_0)| < 2\epsilon, \end{aligned} \quad (1.2.3)$$

so

$$\overline{G}(r) - \underline{G}(r) \leq 2\epsilon, \quad r_0 \leq r \leq r_1$$

Since ϵ is an arbitrary positive number, this implies that

$$\lim_{r \rightarrow b^-} (\overline{G}(r) - \underline{G}(r)) = 0, \quad (1.2.4)$$

so $\overline{L} = \underline{L}$. Let $L = \overline{L} = \underline{L}$. Since

$$\underline{G}(r) \leq G(r) \leq \overline{G}(r), \quad (1.2.5)$$

it follows that $\lim_{r \rightarrow b^-} G(r) = L$.

We leave the proof of the following theorem to you (Exercise [exer:2]).

[theorem:3] Suppose g is locally integrable on $(a, b]$ and denote

$$G(r) = \int_r^b g(x) dx, \quad a \leq r$$

Then the improper integral $\int_a^b g(x) \, dx$ converges if and only if, for each $\epsilon > 0$, there is an $r_0^{\text{int}}(a, b]$ such that

$$|G(r) - G(r_0)| < \epsilon, \quad \forall r \in [r_0, b]$$

To see why we associate Theorems [theorem:2] and [theorem:3] with Cauchy, compare them with Theorem 4.3.5 (p. 204)

1.3: Uniform convergence of improper integrals

Henceforth we deal with functions $f = f(x, y)$ with domains $I \times S$, where S is an interval or a union of intervals and I is of one of the following forms:

- $[a, b)$ with $-\infty < a$
- $(a, b]$ with $-\infty < a$
- (a, b) with $-\infty < a < b < \infty$.

In all cases it is to be understood that f is locally integrable with respect to x on I . When we say that the improper integral $\int_a^b f(x, y) dx$ has a stated property “on S ” we mean that it has the property for every $y \in S$.

[definition:1] If the improper integral

$$\int_a^b f(x, y) dx = \lim_{r \rightarrow b^-} \int_a^r f(x, y) dx \quad (1.3.1)$$

converges on S , it is said to converge uniformly (or be uniformly convergent) on S if, for each $\epsilon > 0$, there is an $r_0^{\text{int}}[a, b)$ such that

$$\left| \int_a^b f(x, y) dx - \int_a^r f(x, y) dx \right| < \epsilon, \quad y \in S, \quad r_0 \leq r$$

or, equivalently,

$$\left| \int_r^b f(x, y) dx \right| < \epsilon, \quad y \in S, \quad r_0 \leq r$$

The crucial difference between pointwise and uniform convergence is that $r_0(y)$ in [eq:8] may depend upon the particular value of y , while the r_0 in [eq:11] does not: one choice must work for all $y \in S$. Thus, uniform convergence implies pointwise convergence, but pointwise convergence does not imply uniform convergence.

(Cauchy Criterion for Uniform Convergence I) [theorem:4] The improper integral in [eq:10] converges uniformly on S if and only if, for each $\epsilon > 0$, there is an $r_0^{\text{int}}[a, b)$ such that

$$\left| \int_r^{r_1} f(x, y) dx \right| < \epsilon, \quad y \in S, \quad r_0 \leq r, r_1$$

Suppose $\int_a^b f(x, y) dx$ converges uniformly on S and $\epsilon > 0$. From Definition [definition:1], there is an $r_0^{\text{int}}[a, b)$ such that

$$\left| \int_r^b f(x, y) dx \right| < \frac{\epsilon}{2} \quad \text{and} \quad \left| \int_{r_1}^b f(x, y) dx \right| < \frac{\epsilon}{2}, \quad y \in S, \quad r_0 \leq r, r_1$$

Since

$$\int_r^{r_1} f(x, y) dx = \int_r^b f(x, y) dx - \int_{r_1}^b f(x, y) dx, \quad (1.3.2)$$

[eq:13] and the triangle inequality imply [eq:12].

For the converse, denote

$$F(y) = \int_a^r f(x, y) dx. \quad (1.3.3)$$

Since [eq:12] implies that

$$|F(r, y) - F(r_1, y)| < \epsilon, \quad y \in S, \quad r_0 \leq r, r_1$$

Theorem [theorem:2] with $G(r) = F(r, y)$ (y fixed but arbitrary in S) implies that $\int_a^b f(x, y) dx$ converges pointwise for $y \in S$. Therefore, if $\epsilon > 0$ then, for each $y \in S$, there is an $r_0(y)^{\text{int}}[a, b)$ such that

$$\left| \int_r^b f(x, y) dx \right| < \epsilon, \quad y \in S, \quad r_0(y) \leq r < b. \quad (1.3.4)$$

For each $y \in S$, choose $r_1(y) \geq \max[r_0(y), r_0]$. (Recall [eq:14]). Then

$$\int_r^b f(x, y) dx = \int_r^{r_1(y)} f(x, y) dx + \int_{r_1(y)}^b f(x, y) dx, \quad (1.3.5)$$

so [eq:12], [eq:15], and the triangle inequality imply that

$$\left| \int_r^b f(x, y) dx \right| < 2\epsilon, \quad y \in S, \quad r \geq r_0$$

In practice, we don't explicitly exhibit r_0 for each given ϵ . It suffices to obtain estimates that clearly imply its existence.

[example:4] For the improper integral of Example [example:2],

$$\left| \int_r^\infty f(x, y) dx \right| = \int_r^\infty |y| e^{-|y|x} = e^{-r|y|}, \quad y \neq 0. \quad (1.3.6)$$

If $|y| \geq \rho$, then

$$\left| \int_r^\infty f(x, y) dx \right| \leq e^{-r\rho}, \quad (1.3.7)$$

so $\int_0^\infty f(x, y) dx$ converges uniformly on $(-\infty, \rho] \cup [\rho, \infty)$ if $\rho > 0$; however, it does not converge uniformly on any neighborhood of $y = 0$, since, for any $r > 0$, $e^{-r|y|} > \frac{1}{2}$ if $|y|$ is sufficiently small.

[definition:2] If the improper integral

$$\int_a^b f(x, y) dx = \lim_{r \rightarrow a+} \int_r^b f(x, y) dx \quad (1.3.8)$$

converges on S , it is said to converge uniformly (or be uniformly convergent) on S if, for each $\epsilon > 0$, there is an $r_0^{\text{int}}(a, b]$ such that

$$\left| \int_a^b f(x, y) dx - \int_{r_0}^b f(x, y) dx \right| < \epsilon, \quad y \in S, \quad r_0 \geq a$$

or, equivalently,

$$\left| \int_a^{r_0} f(x, y) dx \right| < \epsilon, \quad y \in S, \quad r_0 \geq a$$

We leave proof of the following theorem to you (Exercise [exer:3]).

(Cauchy Criterion for Uniform Convergence II) [theorem:5] The improper integral

$$\int_a^b f(x, y) dx = \lim_{r \rightarrow a+} \int_r^b f(x, y) dx \quad (1.3.9)$$

converges uniformly on S if and only if, for each $\epsilon > 0$, there is an $r_0^{\text{int}}(a, b]$ such that

$$\left| \int_{r_1}^{r_2} f(x, y) dx \right| < \epsilon, \quad y \in S, \quad r_1, r_2 \geq a$$

We need one more definition, as follows.

[definition:3] Let $f = f(x, y)$ be defined on $(a, b) \times S$, where $-\infty \leq a$

We leave it to you (Exercise [exer:4]) to show that this definition is independent of c ; that is, if $\int_a^c f(x, y) dx$ and $\int_c^b f(x, y) dx$ both converge uniformly on S for some $c^{\text{int}}(a, b)$, then they both converge uniformly on S for every $c^{\text{int}}(a, b)$.

We also leave it to you (Exercise [exer:5]) to show that if f is bounded on $[a, b] \times [c, d]$ and $\int_a^b f(x, y) dx$ exists as a proper integral for each $y \in [c, d]$, then it converges uniformly on $[c, d]$ according to all three Definitions [definition:1]–[definition:3].

[example:5] Consider the improper integral

$$F(y) = \int_0^{\infty} x^{-1/2} e^{-xy} dx, \quad (1.3.10)$$

which diverges if $y \leq 0$ (verify). Definition [definition:3] applies if $y > 0$, so we consider the improper integrals

$$F_1(y) = \int_0^1 x^{-1/2} e^{-xy} dx \quad \text{and} \quad F_2(y) = \int_1^{\infty} x^{-1/2} e^{-xy} dx \quad (1.3.11)$$

separately. Moreover, we could just as well define

$$F_1(y) = \int_0^c x^{-1/2} e^{-xy} dx \quad \text{and} \quad F_2(y) = \int_c^{\infty} x^{-1/2} e^{-xy} dx, \quad (1.3.12)$$

where c is any positive number.

Definition [definition:2] applies to F_1 . If $\forall \epsilon$

$$\left| \int_r^{r_1} x^{-1/2} e^{-xy} dx \right| < \int_{r_1}^r x^{-1/2} dx < 2r^{1/2}, \quad (1.3.13)$$

so $F_1(y)$ converges for uniformly on $[0, \infty)$.

Definition [definition:1] applies to F_2 . Since

$$\left| \int_r^{r_1} x^{-1/2} e^{-xy} dx \right| < r^{-1/2} \int_r^{\infty} e^{-xy} dx = \frac{e^{-ry}}{yr^{1/2}}, \quad (1.3.14)$$

$F_2(y)$ converges uniformly on $[\rho, \infty)$ if $\rho > 0$. It does not converge uniformly on $(0, \rho)$, since the change of variable $u = xy$ yields

$$\int_r^{r_1} x^{-1/2} e^{-xy} dx = y^{-1/2} \int_{ry}^{r_1 y} u^{-1/2} e^{-u} du, \quad (1.3.15)$$

which, for any fixed $r > 0$, can be made arbitrarily large by taking y sufficiently small and $r = 1/y$. Therefore we conclude that $F(y)$ converges uniformly on $[\rho, \infty)$ if $\rho > 0$.

Note that that the constant c in [eq:16] plays no role in this argument.

[example:6] Suppose we take

$$\int_0^{\infty} \frac{\sin u}{u} du = \frac{\pi}{2} \quad (1.3.16)$$

as given (Exercise [exer:31](b)). Substituting $u = xy$ with $y > 0$ yields

$$\int_0^{\infty} \frac{\sin xy}{x} dx = \frac{\pi}{2}, \quad y > 0. \quad (1.3.17)$$

What about uniform convergence? Since $(\sin xy)/x$ is continuous at $x = 0$, Definition [definition:1] and Theorem [theorem:4] apply here. If $\forall \epsilon > 0$, then

$$\int_r^{r_1} \frac{\sin xy}{x} dx = -\frac{1}{y} \left(\frac{\cos xy}{x} \Big|_r^{r_1} + \int_r^{r_1} \frac{\cos xy}{x^2} dx \right), \quad \text{so} \quad \left| \int_r^{r_1} \frac{\sin xy}{x} dx \right| < \frac{3}{ry}. \quad (1.3.18)$$

Therefore [eq:18] converges uniformly on $[\rho, \infty)$ if $\rho > 0$. On the other hand, from [eq:17], there is a $\delta > 0$ such that

$$\int_{u_0}^{\infty} \frac{\sin u}{u} du > \frac{\pi}{4}, \quad 0 \leq u_0 < \delta. \quad (1.3.19)$$

This and [eq:18] imply that

$$\int_r^\infty \frac{\sin xy}{x} dx = \int_{yr}^\infty \frac{\sin u}{u} du > \frac{\pi}{4} \quad (1.3.20)$$

for any $r > 0$ if $\backslash(0 \ 0\backslash)$.

1.4: Absolutely Uniformly Convergent Improper Integrals

(Absolute Uniform Convergence I) [definition:4] The improper integral

$$\int_a^b f(x, y) dx = \lim_{r \rightarrow b^-} \int_a^r f(x, y) dx \quad (1.4.1)$$

is said to converge absolutely uniformly on S if the improper integral

$$\int_a^b |f(x, y)| dx = \lim_{r \rightarrow b^-} \int_a^r |f(x, y)| dx \quad (1.4.2)$$

converges uniformly on S ; that is, if, for each $\epsilon > 0$, there is an $r_0^{\text{int}}[a, b)$ such that

$$\left| \int_a^b f(x, y) dx - \int_a^r f(x, y) dx \right| < \epsilon, \quad \forall y \in S, \quad \forall r \geq r_0$$

To see that this definition makes sense, recall that if f is locally integrable on $[a, b)$ for all y in S , then so is $|f|$ (Theorem 3.4.9, p. 161). Theorem [theorem:4] with f replaced by $|f|$ implies that $\int_a^b f(x, y) dx$ converges absolutely uniformly on S if and only if, for each $\epsilon > 0$, there is an $r_0^{\text{int}}[a, b)$ such that

$$\left| \int_r^{r_1} f(x, y) dx \right| < \epsilon, \quad \forall y \in S, \quad \forall r \geq r_0$$

Since

$$\left| \int_r^{r_1} f(x, y) dx \right| \leq \int_r^{r_1} |f(x, y)| dx, \quad (1.4.3)$$

Theorem [theorem:4] implies that if $\int_a^b f(x, y) dx$ converges absolutely uniformly on S then it converges uniformly on S .

[theorem:6] (Weierstrass's Test for Absolute Uniform Convergence I) Suppose $M = M(x)$ is nonnegative on $[a, b)$, $\int_a^b M(x) dx < \infty$, and

$$|f(x, y)| \leq M(x), \quad \forall y \in S, \quad \forall x \in [a, b)$$

Then $\int_a^b f(x, y) dx$ converges absolutely uniformly on S .

Denote $\int_a^b M(x) dx = L < \infty$. By definition, for each $\epsilon > 0$ there is an $r_0^{\text{int}}[a, b)$ such that

$$L - \int_a^r M(x) dx \leq \epsilon, \quad \forall r \geq r_0$$

Therefore, if $r_0 < r \leq r_1$, then

$$0 \leq \int_r^{r_1} M(x) dx = \left(\int_a^{r_1} M(x) dx - L \right) - \left(\int_a^r M(x) dx - L \right) < \epsilon \quad (1.4.4)$$

This and [eq:19] imply that

$$\left| \int_r^{r_1} f(x, y) dx \right| \leq \int_r^{r_1} M(x) dx < \epsilon, \quad \forall y \in S, \quad \forall r \geq r_0$$

Now Theorem [theorem:4] implies the stated conclusion.

[example:7] Suppose $g = g(x, y)$ is locally integrable on $[0, \infty)$ for all $y \in S$ and, for some $a_0 \geq 0$, there are constants K and p_0 such that

$$|g(x, y)| \leq K e^{-p_0 x}, \quad y \in S, \quad x \geq a_0. \quad (1.4.5)$$

If $p > p_0$ and $r \geq a_0$, then

$$\begin{aligned} \int_r^\infty e^{-px} |g(x, y)| dx &= \int_r^\infty e^{-(p-p_0)x} e^{-p_0x} |g(x, y)| dx \\ &\leq K \int_r^\infty e^{-(p-p_0)x} dx = \frac{K e^{-(p-p_0)r}}{p-p_0}, \end{aligned} \quad (1.4.6)$$

so $\int_0^\infty e^{-px} g(x, y) dx$ converges absolutely on S . For example, since

$$|x^\alpha \sin xy|$$

for x sufficiently large if $p_0 > 0$, Theorem [theorem:4] implies that $\int_0^\infty e^{-px} x^\alpha \sin xy dx$ and $\int_0^\infty e^{-px} x^\alpha \cos xy dx$ converge absolutely uniformly on $(-\infty, \infty)$ if $p > 0$ and $\alpha \geq 0$. As a matter of fact, $\int_0^\infty e^{-px} x^\alpha \sin xy dx$ converges absolutely on $(-\infty, \infty)$ if $p > 0$ and $\alpha > -1$. (Why?)

(Absolute Uniform Convergence II) [definition:5] The improper integral

$$\int_a^b f(x, y) dx = \lim_{r \rightarrow a+} \int_r^b f(x, y) dx \quad (1.4.7)$$

is said to converge absolutely uniformly on S if the improper integral

$$\int_a^b |f(x, y)| dx = \lim_{r \rightarrow a+} \int_r^b |f(x, y)| dx \quad (1.4.8)$$

converges uniformly on S ; that is, if, for each $\epsilon > 0$, there is an $r_0^{\text{int}}(a, b]$ such that

$$\left| \int_a^b f(x, y) dx - \int_{r_0}^b f(x, y) dx \right| < \epsilon, \quad y \in S, \quad \text{quad } a$$

We leave it to you (Exercise [exer:7]) to prove the following theorem.

[theorem:7] (Weierstrass's Test for Absolute Uniform Convergence II) Suppose $M = M(x)$ is nonnegative on $(a, b]$, $\int_a^b M(x) dx < \infty$, and

$$|f(x, y)| \leq M(x), \quad y^{\text{int}} S, \quad x^{\text{int}}(a, b]. \quad (1.4.9)$$

Then $\int_a^b f(x, y) dx$ converges absolutely uniformly on S .

[example:8] If $g = g(x, y)$ is locally integrable on $(0, 1]$ for all $y^{\text{int}} S$ and

$$|g(x, y)| \leq A x^{-\beta}, \quad \text{quad } 0$$

for each $y^{\text{int}} S$, then

$$\int_0^1 x^\alpha g(x, y) dx \quad (1.4.10)$$

converges absolutely uniformly on S if $\alpha > \beta - 1$. To see this, note that if

$$\int_{r_1}^r x^\alpha |g(x, y)| dx \leq A \int_{r_1}^r x^{\alpha-\beta} dx = \frac{A x^{\alpha-\beta+1}}{\alpha-\beta+1} \Big|_{r_1}^r < \frac{A r^{\alpha-\beta+1}}{\alpha-\beta+1}. \quad (1.4.11)$$

Applying this with $\beta = 0$ shows that

$$F(y) = \int_0^1 x^\alpha \cos xy dx \quad (1.4.12)$$

converges absolutely uniformly on $(-\infty, \infty)$ if $\alpha > -1$ and

$$G(y) = \int_0^1 x^\alpha \sin xy dx \quad (1.4.13)$$

converges absolutely uniformly on $(-\infty, \infty)$ if $\alpha > -2$.

By recalling Theorem 4.4.15 (p. 246), you can see why we associate Theorems [theorem:6] and [theorem:7] with Weierstrass.

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1.5: Dirichlet's Tests

Weierstrass's test is useful and important, but it has a basic shortcoming: it applies only to absolutely uniformly convergent improper integrals. The next theorem applies in some cases where $\int_a^b f(x, y) dx$ converges uniformly on S , but $\int_a^b |f(x, y)| dx$ does not.

[theorem:8] (**Dirichlet's Test for Uniform Convergence I**) If g , g_x , and h are continuous on $[a, b] \times S$, then

$$\int_a^b g(x, y)h(x, y) dx \quad (1.5.1)$$

converges uniformly on S if the following conditions are satisfied:

$$\lim_{x \rightarrow b^-} \left\{ \sup_{y \in S} |g(x, y)| \right\} = 0; \quad (1.5.2)$$

There is a constant M such that

$$\left| \int_a^x h(u, y) du \right| < M, \quad \forall x \in [a, b], \quad \forall y \in S$$

$\int_a^b |g_x(x, y)| dx$ converges uniformly on S .

If

$$H(x, y) = \int_a^x h(u, y) du, \quad (1.5.3)$$

then integration by parts yields

$$\begin{aligned} \int_r^{r_1} g(x, y)h(x, y) dx &= \int_r^{r_1} g(x, y)H_x(x, y) dx \\ &= g(r_1, y)H(r_1, y) - g(r, y)H(r, y) \\ &\quad - \int_r^{r_1} g_x(x, y)H(x, y) dx. \end{aligned} \quad (1.5.4)$$

Since assumption **(b)** and [eq:20] imply that $|H(x, y)| \leq M$, $(x, y) \in (a, b) \times S$, Eqn. [eq:21] implies that

$$\left| \int_r^{r_1} g(x, y)h(x, y) dx \right| < M \left(2 \sup_{x \geq r} |g(x, y)| + \int_r^{r_1} |g_x(x, y)| dx \right) \quad (1.5.5)$$

on $[r, r_1] \times S$.

Now suppose $\epsilon > 0$. From assumption **(a)**, there is an $r_0^{\text{int}}[a, b]$ such that $|g(x, y)| < \epsilon$ on S if $x \in (r_0, b]$ and Theorem [theorem:6], there is an $s_0^{\text{int}}[a, b]$ such that

$$\left| \int_{s_0}^{r_1} |g_x(x, y)| dx \right| < \epsilon, \quad \forall y \in S, \quad \forall s_0 \in [a, b]$$

Therefore [eq:22] implies that

$$\left| \int_r^{r_1} g(x, y)h(x, y) dx \right| < 3M\epsilon, \quad \forall y \in S, \quad \forall r \in [r_0, b], \quad \forall s_0 \in [a, b]$$

Now Theorem [theorem:4] implies the stated conclusion.

The statement of this theorem is complicated, but applying it isn't; just look for a factorization $f = gh$, where h has a bounded antiderivative on $[a, b]$ and g is "small" near b . Then integrate by parts and hope that something nice happens. A similar comment applies to Theorem 9, which follows.

Example 1.5.1

Add text here.

Solution

Add text here.

Let

$$I(y) = \int_0^{\infty} \frac{\cos xy}{x+y} dx, \quad y > 0. \quad (1.5.6)$$

The obvious inequality

$$\left| \frac{\cos xy}{x+y} \right| \leq \frac{1}{x+y} \quad (1.5.7)$$

is useless here, since

$$\int_0^{\infty} \frac{dx}{x+y} = \infty. \quad (1.5.8)$$

However, integration by parts yields

$$\begin{aligned} \int_r^{r_1} \frac{\cos xy}{x+y} dx &= \left. \frac{\sin xy}{y(x+y)} \right|_r^{r_1} + \int_r^{r_1} \frac{\sin xy}{y(x+y)^2} dx \\ &= \frac{\sin r_1 y}{y(r_1+y)} - \frac{\sin ry}{y(r+y)} + \int_r^{r_1} \frac{\sin xy}{y(x+y)^2} dx. \end{aligned} \quad (1.5.9)$$

Therefore, if $0 < r$

$$\left| \int_r^{r_1} \frac{\cos xy}{x+y} dx \right| < \frac{1}{y} \left(\frac{2}{r+y} + \int_r^{\infty} \frac{1}{(x+y)^2} \right) \leq \frac{3}{y(r+y)^2} \leq \frac{3}{\rho(r+\rho)} \quad (1.5.10)$$

if $y \geq \rho > 0$. Now Theorem [theorem:4] implies that $I(y)$ converges uniformly on $[\rho, \infty)$ if $\rho > 0$.

We leave the proof of the following theorem to you (Exercise [exer:10]).

[theorem:9] (**Dirichlet's Test for Uniform Convergence II**) If g , g_x , and h are continuous on $(a, b] \times S$, then

$$\int_a^b g(x, y) h(x, y) dx \quad (1.5.11)$$

converges uniformly on S if the following conditions are satisfied:

$$\lim_{x \rightarrow a+} \left\{ \sup_{y \in S} |g(x, y)| \right\} = 0;$$

There is a constant M such that

$$\sup_{y \in S} \left| \int_x^b h(u, y) du \right| \leq M, \quad a < x \leq b; \quad (1.5.12)$$

$\int_a^b |g_x(x, y)| dx$ converges uniformly on S .

By recalling Theorems 3.4.10 (p. 163), 4.3.20 (p. 217), and 4.4.16 (p. 248), you can see why we associate Theorems [theorem:8] and [theorem:9] with Dirichlet.

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1.6: Consequences of Uniform Convergence

[theorem:10] If $f = f(x, y)$ is continuous on either $[a, b] \times [c, d]$ or $(a, b) \times [c, d]$ and

$$F(y) = \int_a^b f(x, y) dx \quad (1.6.1)$$

converges uniformly on $[c, d]$, then F is continuous on $[c, d]$. Moreover,

$$\int_c^d \left(\int_a^b f(x, y) dx \right) dy = \int_a^b \left(\int_c^d f(x, y) dy \right) dx. \quad (1.6.2)$$

We will assume that f is continuous on $(a, b) \times [c, d]$. You can consider the other case (Exercise [exer:14]).

We will first show that F in [eq:23] is continuous on $[c, d]$. Since F converges uniformly on $[c, d]$, Definition [definition:1] (specifically, [eq:11]) implies that if $\epsilon > 0$, there is an $r^{\text{int}}[a, b]$ such that

$$\left| \int_r^b f(x, y) dx \right| < \epsilon, \quad c \leq y \leq d. \quad (1.6.3)$$

Therefore, if $c \leq y, y_0 \leq d$, then

$$\begin{aligned} |F(y) - F(y_0)| &= \left| \int_a^b f(x, y) dx - \int_a^b f(x, y_0) dx \right| \\ &\leq \left| \int_a^r [f(x, y) - f(x, y_0)] dx \right| + \left| \int_r^b f(x, y) dx \right| \\ &\quad + \left| \int_r^b f(x, y_0) dx \right|, \end{aligned} \quad (1.6.4)$$

so

$$|F(y) - F(y_0)| \leq \int_a^r |f(x, y) - f(x, y_0)| dx + 2\epsilon. \quad (1.6.5)$$

Since f is uniformly continuous on the compact set $[a, r] \times [c, d]$ (Corollary 5.2.14, p. 314), there is a $\delta > 0$ such that

$$|f(x, y) - f(x, y_0)| < \epsilon \quad (1.6.6)$$

if (x, y) and (x, y_0) are in $[a, r] \times [c, d]$ and $|y - y_0| < \delta$. This and [eq:25] imply that

$$|F(y) - F(y_0)| < (r - a)\epsilon + 2\epsilon < (b - a + 2)\epsilon \quad (1.6.7)$$

if y and y_0 are in $[c, d]$ and $|y - y_0| < \delta$. Therefore F is continuous on $[c, d]$, so the integral on left side of [eq:24] exists. Denote

$$I = \int_c^d \left(\int_a^b f(x, y) dx \right) dy. \quad (1.6.8)$$

We will show that the improper integral on the right side of [eq:24] converges to I . To this end, denote

$$I(r) = \int_a^r \left(\int_c^d f(x, y) dy \right) dx. \quad (1.6.9)$$

Since we can reverse the order of integration of the continuous function f over the rectangle $[a, r] \times [c, d]$ (Corollary 7.2.2, p. 466),

$$I(r) = \int_c^d \left(\int_a^r f(x, y) dx \right) dy. \quad (1.6.10)$$

From this and [eq:26],

$$I - I(r) = \int_c^d \left(\int_r^b f(x, y) dx \right) dy. \quad (1.6.11)$$

Now suppose $\epsilon > 0$. Since $\int_a^b f(x, y) dx$ converges uniformly on $[c, d]$, there is an $r_0^{\text{int}}(a, b)$ such that

$$\left| \int_r^b f(x, y) dx \right| < \epsilon, \quad \text{if } r \geq r_0$$

so $|I - I(r)| < (d - c)\epsilon$ if $r \geq r_0$

$$\lim_{r \rightarrow b^-} \int_a^r \left(\int_c^d f(x, y) dy \right) dx = \int_c^d \left(\int_a^b f(x, y) dx \right) dy, \quad (1.6.12)$$

which completes the proof of [eq:24].

[example:10] It is straightforward to verify that

$$\int_0^\infty e^{-xy} dx = \frac{1}{y}, \quad y > 0, \quad (1.6.13)$$

and the convergence is uniform on $[\rho, \infty)$ if $\rho > 0$. Therefore Theorem [theorem:10] implies that if

$$\begin{aligned} \int_{y_1}^{y_2} \frac{dy}{y} &= \int_{y_1}^{y_2} \left(\int_0^\infty e^{-xy} dx \right) dy = \int_0^\infty \left(\int_{y_1}^{y_2} e^{-xy} dy \right) dx \\ &= \int_0^\infty \frac{e^{-xy_1} - e^{-xy_2}}{x} dx. \end{aligned} \quad (1.6.14)$$

Since

$$\int_{y_1}^{y_2} \frac{dy}{y} = \log \frac{y_2}{y_1}, \quad y_2 \geq y_1 > 0, \quad (1.6.15)$$

it follows that

$$\int_0^\infty \frac{e^{-xy_1} - e^{-xy_2}}{x} dx = \log \frac{y_2}{y_1}, \quad y_2 \geq y_1 > 0. \quad (1.6.16)$$

[example:11] From Example [example:6],

$$\int_0^\infty \frac{\sin xy}{x} dx = \frac{\pi}{2}, \quad y > 0, \quad (1.6.17)$$

and the convergence is uniform on $[\rho, \infty)$ if $\rho > 0$. Therefore, Theorem [theorem:10] implies that if

$$\begin{aligned} \frac{\pi}{2}(y_2 - y_1) &= \int_{y_1}^{y_2} \left(\int_0^\infty \frac{\sin xy}{x} dx \right) dy = \int_0^\infty \left(\int_{y_1}^{y_2} \frac{\sin xy}{x} dy \right) dx \\ &= \int_0^\infty \frac{\cos xy_1 - \cos xy_2}{x^2} dx. \end{aligned} \quad (1.6.18)$$

The last integral converges uniformly on $(-\infty, \infty)$ (Exercise 10(h)), and is therefore continuous with respect to y_1 on $(-\infty, \infty)$, by Theorem [theorem:10]; in particular, we can let $y_1 \rightarrow 0^+$ in [eq:27] and replace y_2 by y to obtain

$$\int_0^\infty \frac{1 - \cos xy}{x^2} dx = \frac{\pi y}{2}, \quad y \geq 0. \quad (1.6.19)$$

The next theorem is analogous to Theorem 4.4.20 (p. 252).

[theorem:11] Let f and f_y be continuous on either $[a, b] \times [c, d]$ or $(a, b) \times [c, d]$. Suppose that the improper integral

$$F(y) = \int_a^b f(x, y) dx \quad (1.6.20)$$

converges for some $y_0^{\text{int}}[c, d]$ and

$$G(y) = \int_a^b f_y(x, y) dx \quad (1.6.21)$$

converges uniformly on $[c, d]$. Then F converges uniformly on $[c, d]$ and is given explicitly by

$$F(y) = F(y_0) + \int_{y_0}^y G(t) dt, \quad c \leq y \leq d. \quad (1.6.22)$$

Moreover, F is continuously differentiable on $[c, d]$; specifically,

$$F'(y) = G(y), \quad c \leq y \leq d, \quad (1.6.23)$$

where $F'(c)$ and $f_y(x, c)$ are derivatives from the right, and $F'(d)$ and $f_y(x, d)$ are derivatives from the left.

We will assume that f and f_y are continuous on $[a, b) \times [c, d]$. You can consider the other case (Exercise [exer:15]).

Let

$$F_r(y) = \int_a^r f(x, y) dx, \quad \text{for all } r$$

Since f and f_y are continuous on $[a, r] \times [c, d]$, Theorem [theorem:1] implies that

$$F'_r(y) = \int_a^r f_y(x, y) dx, \quad c \leq y \leq d. \quad (1.6.24)$$

Then

$$\begin{aligned} F_r(y) &= F_r(y_0) + \int_{y_0}^y \left(\int_a^r f_y(x, t) dx \right) dt \\ &= F(y_0) + \int_{y_0}^y G(t) dt \\ &\quad + (F_r(y_0) - F(y_0)) - \int_{y_0}^y \left(\int_r^b f_y(x, t) dx \right) dt, \quad c \leq y \leq d. \end{aligned} \quad (1.6.25)$$

Therefore,

$$\begin{aligned} \left| F_r(y) - F(y_0) - \int_{y_0}^y G(t) dt \right| &\leq |F_r(y_0) - F(y_0)| \\ &\quad + \left| \int_{y_0}^y \int_r^b f_y(x, t) dx dt \right|. \end{aligned} \quad (1.6.26)$$

Now suppose $\epsilon > 0$. Since we have assumed that $\lim_{r \rightarrow b^-} F_r(y_0) = F(y_0)$ exists, there is an r_0 in (a, b) such that

$$|F_r(y_0) - F(y_0)| < \epsilon, \quad \text{for } r \geq r_0$$

Since we have assumed that $G(y)$ converges for $y^{\text{int}}[c, d]$, there is an $r_1^{\text{int}}[a, b)$ such that

$$\left| \int_a^b f_y(x, t) dx \right| < \epsilon, \quad \text{for } t \in [c, d], \quad \text{for } r \geq r_1$$

Therefore, [eq:29] yields

$$\left| F_r(y) - F(y_0) - \int_{y_0}^y G(t) dt \right| < \epsilon(1 + |y - y_0|) \leq \epsilon(1 + d - c) \quad (1.6.27)$$

if $\max(r_0, r_1) \leq r$

$$F(y) = F(y_0) + \int_{y_0}^y G(t) dt, \quad c \leq y \leq d. \quad (1.6.28)$$

Since G is continuous on $[c, d]$ by Theorem [theorem:10], [eq:28] follows from differentiating this (Theorem 3.3.11, p. 141).

[example:12] Let

$$I(y) = \int_0^\infty e^{-yx^2} dx, \quad y > 0. \quad (1.6.29)$$

Since

$$\int_0^r e^{-yx^2} dx = \frac{1}{\sqrt{y}} \int_0^{r\sqrt{y}} e^{-t^2} dt, \quad (1.6.30)$$

it follows that

$$I(y) = \frac{1}{\sqrt{y}} \int_0^\infty e^{-t^2} dt, \quad (1.6.31)$$

and the convergence is uniform on $[\rho, \infty)$ if $\rho > 0$ (Exercise [exer:8](i)). To evaluate the last integral, denote $J(\rho) = \int_0^\rho e^{-t^2} dt$; then

$$J^2(\rho) = \left(\int_0^\rho e^{-u^2} du \right) \left(\int_0^\rho e^{-v^2} dv \right) = \int_0^\rho \int_0^\rho e^{-(u^2+v^2)} du dv. \quad (1.6.32)$$

Transforming to polar coordinates $r = r \cos \theta$, $v = r \sin \theta$ yields

$$J^2(\rho) = \int_0^{\pi/2} \int_0^\rho r e^{-r^2} dr d\theta = \frac{\pi(1 - e^{-\rho^2})}{4}, \quad \text{so} \quad J(\rho) = \frac{\sqrt{\pi(1 - e^{-\rho^2})}}{2}. \quad (1.6.33)$$

Therefore

$$\int_0^\infty e^{-t^2} dt = \lim_{\rho \rightarrow \infty} J(\rho) = \frac{\sqrt{\pi}}{2} \quad \text{and} \quad \int_0^\infty e^{-yx^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{y}}, \quad y > 0. \quad (1.6.34)$$

Differentiating this n times with respect to y yields

$$\int_0^\infty x^{2n} e^{-yx^2} dx = \frac{1 \cdot 3 \cdots (2n-1) \sqrt{\pi}}{2^n y^{n+1/2}} \quad y > 0, \quad n = 1, 2, 3, \dots, \quad (1.6.35)$$

where Theorem [theorem:11] justifies the differentiation for every n , since all these integrals converge uniformly on $[\rho, \infty)$ if $\rho > 0$ (Exercise [exer:8](i)).

Some advice for applying this theorem: Be sure to check first that $F(y_0) = \int_a^b f(x, y_0) dx$ converges for at least one value of y . If so, differentiate $\int_a^b f(x, y) dx$ formally to obtain $\int_a^b f_y(x, y) dx$. Then $F'(y) = \int_a^b f_y(x, y) dx$ if y is in some interval on which this improper integral converges uniformly.

1.7: Applications to Laplace transforms

The *Laplace transform* of a function f locally integrable on $[0, \infty)$ is

$$F(s) = \int_0^{\infty} e^{-sx} f(x) dx \quad (1.7.1)$$

for all s such that integral converges. Laplace transforms are widely applied in mathematics, particularly in solving differential equations.

We leave it to you to prove the following theorem (Exercise [exer:26]).

[theorem:12] Suppose f is locally integrable on $[0, \infty)$ and $|f(x)| \leq Me^{s_0 x}$ for sufficiently large x . Then the Laplace transform of F converges uniformly on $[s_1, \infty)$ if $s_1 > s_0$.

[theorem:13] If f is continuous on $[0, \infty)$ and $H(x) = \int_0^{\infty} e^{-s_0 u} f(u) du$ is bounded on $[0, \infty)$, then the Laplace transform of f converges uniformly on $[s_1, \infty)$ if $s_1 > s_0$.

If $0 \leq r \leq r_1$,

$$\int_r^{r_1} e^{-sx} f(x) dx = \int_r^{r_1} e^{-(s-s_0)x} e^{-s_0 x} f(x) dt = \int_r^{r_1} e^{-(s-s_0)t} H'(x) dt. \quad (1.7.2)$$

Integration by parts yields

$$\int_r^{r_1} e^{-sx} f(x) dt = e^{-(s-s_0)x} H(x) \Big|_r^{r_1} + (s-s_0) \int_r^{r_1} e^{-(s-s_0)x} H(x) dx. \quad (1.7.3)$$

Therefore, if $|H(x)| \leq M$, then

$$\begin{aligned} \left| \int_r^{r_1} e^{-sx} f(x) dx \right| &\leq M \left| e^{-(s-s_0)r_1} + e^{-(s-s_0)r} + (s-s_0) \int_r^{r_1} e^{-(s-s_0)x} dx \right| \\ &\leq 3Me^{-(s-s_0)r} \leq 3Me^{-(s_1-s_0)r}, \quad s \geq s_1. \end{aligned} \quad (1.7.4)$$

Now Theorem [theorem:4] implies that $F(s)$ converges uniformly on $[s_1, \infty)$.

The following theorem draws a considerably stonger conclusion from the same assumptions.

[theorem:14] If f is continuous on $[0, \infty)$ and

$$H(x) = \int_0^x e^{-s_0 u} f(u) du \quad (1.7.5)$$

is bounded on $[0, \infty)$, then the Laplace transform of f is infinitely differentiable on (s_0, ∞) , with

$$F^{(n)}(s) = (-1)^n \int_0^{\infty} e^{-sx} x^n f(x) dx; \quad (1.7.6)$$

that is, the n -th derivative of the Laplace transform of $f(x)$ is the Laplace transform of $(-1)^n x^n f(x)$.

First we will show that the integrals

$$I_n(s) = \int_0^{\infty} e^{-sx} x^n f(x) dx, \quad n = 0, 1, 2, \dots \quad (1.7.7)$$

all converge uniformly on $[s_1, \infty)$ if $s_1 > s_0$. If $\setminus \emptyset$

$$\int_r^{r_1} e^{-sx} x^n f(x) dx = \int_r^{r_1} e^{-(s-s_0)x} e^{-s_0 x} x^n f(x) dx = \int_r^{r_1} e^{-(s-s_0)x} x^n H'(x) dx. \quad (1.7.8)$$

Integrating by parts yields

$$\int_r^{r_1} e^{-sx} x^n f(x) dx = r_1^n e^{-(s-s_0)r_1} H(r) - r^n e^{-(s-s_0)r} H(r) - \int_r^{r_1} H(x) (e^{-(s-s_0)x} x^n)' dx, \quad (1.7.9)$$

where ' indicates differentiation with respect to x . Therefore, if $|H(x)| \leq M \leq \infty$ on $[0, \infty)$, then

$$\left| \int_r^{r_1} e^{-sx} x^n f(x) dx \right| \leq M \left(e^{-(s-s_0)r} r^n + e^{-(s-s_0)r_1} r_1^n + \int_r^{r_1} |(e^{-(s-s_0)x} x^n)'| dx \right). \quad (1.7.10)$$

Therefore, since $e^{-(s-s_0)r} r^n$ decreases monotonically on (n, ∞) if $s > s_0$ (check!),

$$\left| \int_r^{r_1} e^{-sx} x^n f(x) dx \right| \leq M e^{-(s-s_0)r} r^n, \quad n$$

so Theorem [theorem:4] implies that $I_n(s)$ converges uniformly $[s_1, \infty)$ if $s_1 > s_0$. Now Theorem [theorem:11] implies that $F_{n+1} = -F'_n$, and an easy induction proof yields [eq:30] (Exercise [exer:25]).

[example:13] Here we apply Theorem [theorem:12] with $f(x) = \cos ax$ ($a \neq 0$) and $s_0 = 0$. Since

$$\int_0^x \cos au du = \frac{\sin ax}{a} \quad (1.7.11)$$

is bounded on $(0, \infty)$, Theorem [theorem:12] implies that

$$F(s) = \int_0^\infty e^{-sx} \cos ax dx \quad (1.7.12)$$

converges and

$$F^{(n)}(s) = (-1)^n \int_0^\infty e^{-sx} x^n \cos ax dx, \quad s > 0. \quad (1.7.13)$$

(Note that this is also true if $a = 0$.) Elementary integration yields

$$F(s) = \frac{s}{s^2 + a^2}. \quad (1.7.14)$$

Hence, from [eq:31],

$$\int_0^\infty e^{-sx} x^n \cos ax = (-1)^n \frac{d^n}{ds^n} \frac{s}{s^2 + a^2}, \quad n = 0, 1, \dots \quad (1.7.15)$$

1.8: Improper Functions (Exercises)

[exer:1] Suppose g and h are differentiable on $[a, b]$, with

$$a \leq g(y) \leq b \quad \text{and} \quad a \leq h(y) \leq b, \quad c \leq y \leq d. \quad (1.8.1)$$

Let f and f_y be continuous on $[a, b] \times [c, d]$. Derive *Liebniz's rule*:

$$\begin{aligned} \frac{d}{dy} \int_{g(y)}^{h(y)} f(x, y) dx = & f(h(y), y)h'(y) - f(g(y), y)g'(y) \\ & + \int_{g(y)}^{h(y)} f_y(x, y) dx. \end{aligned} \quad (1.8.2)$$

(Hint: Define $H(y, u, v) = \int_u^v f(x, y) dx$ and use the chain rule.)

[exer:2] Adapt the proof of Theorem [theorem:2] to prove Theorem [theorem:3].

[exer:3] Adapt the proof of Theorem [theorem:4] to prove Theorem [theorem:5].

[exer:4] Show that Definition [definition:3] is independent of c ; that is, if $\int_a^c f(x, y) dx$ and $\int_c^b f(x, y) dx$ both converge uniformly on S for some $c^{\text{int}}(a, b)$, then they both converge uniformly on S and every $c^{\text{int}}(a, b)$.

[exer:5]

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Show that if f is bounded on $[a, b] \times [c, d]$ and $\int_a^b f(x, y) dx$ exists as a proper integral for each $y^{\text{int}}[c, d]$, then it converges uniformly on $[c, d]$ according to all of Definition [definition:1]–[definition:3].

Give an example to show that the boundedness of f is essential in **(a)**.

[exer:6] Working directly from Definition [definition:1], discuss uniform convergence of the following integrals:

(a) $\int_0^\infty \frac{dx}{1+y^2x^2}$	(b) $\int_0^\infty e^{-xy}x^2 dx$
(c) $\int_0^\infty x^{2n}e^{-yx^2} dx$	(d) $\int_0^\infty \sin xy^2 dx$
(e) $\int_0^\infty (3y^2 - 2xy)e^{-y^2x} dx$	(f) $\int_0^\infty (2xy - y^2x^2)e^{-xy} dx$

[exer:7] Adapt the proof of Theorem [theorem:6] to prove Theorem [theorem:7].

[exer:8] Use Weierstrass's test to show that the integral converges uniformly on S :

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$$\int_0^\infty e^{-xy} \sin x dx, S = [\rho, \infty), \rho > 0$$

$$\int_0^\infty \frac{\sin x}{x^y} dx, S = [c, d], \setminus 1$$

$$\int_1^{\infty} e^{-px} \frac{\sin xy}{x} dx, p > 0, S = (-\infty, \infty)$$

$$\int_0^1 \frac{e^{xy}}{(1-x)^y} dx, S = (-\infty, b), b < 1$$

$$\int_{-\infty}^{\infty} \frac{\cos xy}{1+x^2y^2} dx, S = (-\infty, -\rho] \cup [\rho, \infty), \rho > 0.$$

$$\int_1^{\infty} e^{-x/y} dx, S = [\rho, \infty), \rho > 0$$

$$\int_{-\infty}^{\infty} e^{xy} e^{-x^2} dx, S = [-\rho, \rho], \rho > 0$$

$$\int_0^{\infty} \frac{\cos xy - \cos ax}{x^2} dx, S = (-\infty, \infty)$$

$$\int_0^{\infty} x^{2n} e^{-yx^2} dx, S = [\rho, \infty), \rho > 0, n = 0, 1, 2, \dots$$

[exer:9]

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Show that

$$\Gamma(y) = \int_0^{\infty} x^{y-1} e^{-x} dx \quad (1.8.3)$$

converges if $y > 0$, and uniformly on $[c, d]$ if $\setminus \emptyset$

Use integration by parts to show that

$$\Gamma(y) = \frac{\Gamma(y+1)}{y}, \quad y \geq 0, \quad (1.8.4)$$

and then show by induction that

$$\Gamma(y) = \frac{\Gamma(y+n)}{y(y+1) \cdots (y+n-1)}, \quad y > 0, \quad n = 1, 2, 3, \dots \quad (1.8.5)$$

How can this be used to define $\Gamma(y)$ in a natural way for all $y \neq 0, -1, -2, \dots$? (This function is called the *gamma function*.)

Show that $\Gamma(n+1) = n!$ if n is a positive integer.

Show that

$$\int_0^{\infty} e^{-st} t^{\alpha} dt = s^{-\alpha-1} \Gamma(\alpha+1), \quad \alpha > -1, \quad s > 0. \quad (1.8.6)$$

[exer:10] Show that Theorem [theorem:8] remains valid with assumption **(c)** replaced by the assumption that $|g_x(x, y)|$ is monotonic with respect to x for all $y \in S$.

[exer:11] Adapt the proof of Theorem [theorem:8] to prove Theorem [theorem:9].

[exer:12] Use Dirichlet's test to show that the following integrals converge uniformly on $S = [\rho, \infty)$ if $\rho > 0$:

(a) $\int_1^{\infty} \frac{\sin xy}{x^y} dx$	(b) $\int_2^{\infty} \frac{\sin xy}{\log x} dx$

$$(c) \int_0^{\infty} \frac{\cos xy}{x+y^2} dx$$

$$(d) \int_1^{\infty} \frac{\sin xy}{1+xy} dx$$

[exer:13] Suppose g, g_x and h are continuous on $[a, b] \times S$, and denote $H(x, y) = \int_a^x h(u, y) du$, \(\forall x\)

$$\lim_{x \rightarrow b^-} \left\{ \sup_{y \in S} |g(x, y)H(x, y)| \right\} = 0 \quad \text{and} \quad \int_a^b g_x(x, y)H(x, y) dx \quad (1.8.7)$$

converges uniformly on S . Show that $\int_a^b g(x, y)h(x, y) dx$ converges uniformly on S .

[exer:14] Prove Theorem [theorem:10] for the case where $f = f(x, y)$ is continuous on $(a, b] \times [c, d]$.

[exer:15] Prove Theorem [theorem:11] for the case where $f = f(x, y)$ is continuous on $(a, b] \times [c, d]$.

[exer:16] Show that

$$C(y) = \int_{-\infty}^{\infty} f(x) \cos xy dx \quad \text{and} \quad S(y) = \int_{-\infty}^{\infty} f(x) \sin xy dx \quad (1.8.8)$$

are continuous on $(-\infty, \infty)$ if

$$\int_{-\infty}^{\infty} |f(x)| dx < \infty. \quad (1.8.9)$$

[exer:17] Suppose f is continuously differentiable on $[a, \infty)$, $\lim_{x \rightarrow \infty} f(x) = 0$, and

$$\int_a^{\infty} |f'(x)| dx < \infty. \quad (1.8.10)$$

Show that the functions

$$C(y) = \int_a^{\infty} f(x) \cos xy dx \quad \text{and} \quad S(y) = \int_a^{\infty} f(x) \sin xy dx \quad (1.8.11)$$

are continuous for all $y \neq 0$. Give an example showing that they need not be continuous at $y = 0$.

[exer:18] Evaluate $F(y)$ and use Theorem [theorem:11] to evaluate I :

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$$F(y) = \int_0^{\infty} \frac{dx}{1+y^2x^2}, y \neq 0; I = \int_0^{\infty} \frac{\tan^{-1} ax - \tan^{-1} bx}{x} dx, a, b > 0$$

$$F(y) = \int_0^{\infty} x^y dx, y > -1; I = \int_0^{\infty} \frac{x^a - x^b}{\log x} dx, a, b > -1$$

$$F(y) = \int_0^{\infty} e^{-xy} \cos x dx, y > 0$$

$$I = \int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{x} \cos x dx, a, b > 0$$

$$F(y) = \int_0^{\infty} e^{-xy} \sin x dx, y > 0$$

$$I = \int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{x} \sin x dx, a, b > 0$$

$$F(y) = \int_0^{\infty} e^{-x} \sin xy dx; I = \int_0^{\infty} e^{-x} \frac{1 - \cos ax}{x} dx$$

$$F(y) = \int_0^\infty e^{-x} \cos xy \, dx; \quad I = \int_0^\infty e^{-x} \frac{\sin ax}{x} \, dx$$

[exer:19] Use Theorem [theorem:11] to evaluate:

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$$\int_0^1 (\log x)^n x^y \, dx, \quad y > -1, n = 0, 1, 2, \dots$$

$$\int_0^\infty \frac{dx}{(x^2 + y)^{n+1}}, \quad y > 0, n = 0, 1, 2, \dots$$

$$\int_0^\infty x^{2n+1} e^{-yx^2} \, dx, \quad y > 0, n = 0, 1, 2, \dots$$

$$\int_0^\infty xy^x \, dx, \quad y > 0$$

[exer:20]

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Use Theorem [theorem:11] and integration by parts to show that

$$F(y) = \int_0^\infty e^{-x^2} \cos 2xy \, dx \quad (1.8.12)$$

satisfies

$$F' + 2yF = 0. \quad (1.8.13)$$

Use part (a) to show that

$$F(y) = \frac{\sqrt{\pi}}{2} e^{-y^2}. \quad (1.8.14)$$

[exer:21] Show that

$$\int_0^\infty e^{-x^2} \sin 2xy \, dx = e^{-y^2} \int_0^y e^{u^2} \, du. \quad (1.8.15)$$

(Hint: See Exercise [exer:20].)

[exer:22] State a condition implying that

$$C(y) = \int_a^\infty f(x) \cos xy \, dx \quad \text{and} \quad S(y) = \int_a^\infty f(x) \sin xy \, dx \quad (1.8.16)$$

are n times differentiable on for all $y \neq 0$. (Your condition should imply the hypotheses of Exercise [exer:16].)

[exer:23] Suppose f is continuously differentiable on $[a, \infty)$,

$$\int_a^\infty |(x^k f(x))'| \, dx < \infty, \quad 0 \leq k \leq n, \quad (1.8.17)$$

and $\lim_{x \rightarrow \infty} x^n f(x) = 0$. Show that if

$$C(y) = \int_a^\infty f(x) \cos xy \, dx \quad \text{and} \quad S(y) = \int_a^\infty f(x) \sin xy \, dx, \quad (1.8.18)$$

then

$$C^{(k)}(y) = \int_a^\infty x^k f(x) \cos xy \, dx \quad \text{and} \quad S^{(k)}(y) = \int_a^\infty x^k f(x) \sin xy \, dx, \quad (1.8.19)$$

$$0 \leq k \leq n.$$

[exer:24] Differentiating

$$F(y) = \int_1^\infty \cos \frac{y}{x} \, dx \quad (1.8.20)$$

under the integral sign yields

$$- \int_1^\infty \frac{1}{x} \sin \frac{y}{x} \, dx, \quad (1.8.21)$$

which converges uniformly on any finite interval. (Why?) Does this imply that F is differentiable for all y ?

[exer:25] Show that Theorem [theorem:11] and induction imply Eq. [eq:30].

[exer:26] Prove Theorem [theorem:12].

[exer:27] Show that if $F(s) = \int_0^\infty e^{-sx} f(x) \, dx$ converges for $s = s_0$, then it converges uniformly on $[s_0, \infty)$. (What's the difference between this and Theorem [theorem:13]?)

[exer:28] Prove: If f is continuous on $[0, \infty)$ and $\int_0^\infty e^{-s_0 x} f(x) \, dx$ converges, then

$$\lim_{s \rightarrow s_0^+} \int_0^\infty e^{-sx} f(x) \, dx = \int_0^\infty e^{-s_0 x} f(x) \, dx. \quad (1.8.22)$$

(Hint: See the proof of Theorem 4.5.12, p. 273.)

[exer:29] Under the assumptions of Exercise [exer:28], show that

$$\lim_{s \rightarrow s_0^+} \int_r^\infty e^{-sx} f(x) \, dx = \int_r^\infty e^{-s_0 x} f(x) \, dx, \quad r > 0. \quad (1.8.23)$$

[exer:30] Suppose f is continuous on $[0, \infty)$ and

$$F(s) = \int_0^\infty e^{-sx} f(x) \, dx \quad (1.8.24)$$

converges for $s = s_0$. Show that $\lim_{s \rightarrow \infty} F(s) = 0$. (Hint: Integrate by parts.)

[exer:31]

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Starting from the result of Exercise [exer:18](d), let $b \rightarrow \infty$ and invoke Exercise [exer:30] to evaluate

$$\int_0^\infty e^{-ax} \frac{\sin x}{x} \, dx, \quad a > 0. \quad (1.8.25)$$

Use **(a)** and Exercise [exer:28] to show that

$$\int_0^\infty \frac{\sin x}{x} \, dx = \frac{\pi}{2}. \quad (1.8.26)$$

[exer:32]

(14pt 8pt22pt0pt)

Suppose f is continuously differentiable on $[0, \infty)$ and

$$|f(x)| \leq M e^{s_0 x}, \quad 0 \leq x \leq \infty. \quad (1.8.27)$$

Show that

$$G(s) = \int_0^\infty e^{-sx} f'(x) dx \quad (1.8.28)$$

converges uniformly on $[s_1, \infty)$ if $s_1 > s_0$. (Hint: Integrate by parts.)

Show from part **(a)** that

$$G(s) = \int_0^\infty e^{-sx} x e^{x^2} \sin e^{x^2} dx \quad (1.8.29)$$

converges uniformly on $[\rho, \infty)$ if $\rho > 0$. (Notice that this does not follow from Theorem [theorem:6] or [theorem:8].)

[exer:33] Suppose f is continuous on $[0, \infty)$,

$$\lim_{x \rightarrow 0^+} \frac{f(x)}{x} \quad (1.8.30)$$

exists, and

$$F(s) = \int_0^\infty e^{-sx} f(x) dx \quad (1.8.31)$$

converges for $s = s_0$. Show that

$$\int_{s_0}^\infty F(u) du = \int_0^\infty e^{-s_0 x} \frac{f(x)}{x} dx. \quad (1.8.32)$$

Answers to selected exercises

If $f(x, y) = 1/y$ for $y \neq 0$ and $f(x, 0) = 1$, then $\int_a^b f(x, y) dx$ does not converge uniformly on $[0, d]$ for any $d > 0$.

, **(d)**, and **(e)** converge uniformly on $(-\infty, \rho] \cup [\rho, \infty)$ if $\rho > 0$; **(b)**, **(c)**, and **(f)** converge uniformly on $[\rho, \infty)$ if $\rho > 0$.

Let $C(y) = \int_1^\infty \frac{\cos xy}{x} dx$ and $S(y) = \int_1^\infty \frac{\sin xy}{x} dx$. Then $C(0) = \infty$ and $S(0) = 0$, while $S(y) = \pi/2$ if $y \neq 0$.

$$F(y) = \frac{\pi}{2|y|}; I = \frac{\pi}{2} \log \frac{a}{b} \quad F(y) = \frac{1}{y+1}; I = \log \frac{a+1}{b+1}$$

$$F(y) = \frac{y}{y^2+1}; I = \frac{1}{2} \frac{b^2+1}{a^2+1}$$

$$\textbf{(d)} \quad F(y) = \frac{1}{y^2+1}; I = \tan^{-1} b - \tan^{-1} a$$

$$\textbf{(e)} \quad F(y) = \frac{y}{y^2+1}; I = \frac{1}{2} \log(1+a^2)$$

$$\textbf{(f)} \quad F(y) = \frac{1}{y^2+1}; I = \tan^{-1} a$$

$$(-1)^n n! (y+1)^{-n-1} \pi 2^{-2n-1} \binom{2n}{n} y^{-n-1/2}$$

$$\textbf{(c)} \quad \frac{n!}{2y^{n+1}} (\log y)^{-2} \quad \textbf{(d)} \quad \frac{1}{(\log x)^2}$$

$$\int_{-\infty}^\infty |x^n f(x)| dx < \infty$$

No; the integral defining F diverges for all y .

$$\frac{\pi}{2} - \tan^{-1} a$$

Beginning of manual

1. If $H(y, u, v) = \int_u^v f(x, y) dx$ then

$$H_u(y, u, v) = -f(u, y), \quad H_v(y, u, v) = f(v, y), \quad (1.8.33)$$

and, by Theorem 1, $H_y(u, v, y) = \int_u^v f_y(x, y) dx$. If

$$F(y) = H(y, g(y), h(y)) = \int_{g(y)}^{h(y)} f(x, y) dx, \quad (1.8.34)$$

then

$$\begin{aligned} F'(y) &= H_v(y, g(y), h(y))h'(y) + H_u(y, g(y), h(y))g'(y) + H_y(y, g(y), h(y)) \\ &= f(h(y), y)h'(y) - f(g(y), y)g'(y) + \int_{g(y)}^{h(y)} f_y(x, y) dx. \end{aligned} \quad (1.8.35)$$

2. Theorem 3 (Cauchy Criterion for Convergence of an Improper Integral II) Suppose g is integrable on every finite closed subinterval of $(a, b]$ and denote

$$G(r) = \int_r^b g(x) dx, \quad a < r \leq b. \quad (1.8.36)$$

Then the improper integral $\int_a^b g(x) dx$ converges if and only if, for each $\epsilon > 0$, there is an $r_0^{\text{int}}(a, b]$ such that

$$|G(r) - G(r_1)| \leq \epsilon, \quad a < r, r_1 \leq r_0.$$

For necessity, suppose $\int_a^b g(x) dx = L$. By definition, this means that for each $\epsilon > 0$ there is an $r_0^{\text{int}}(a, b]$ such that

$$|G(r) - L| < \frac{\epsilon}{2} \quad \text{and} \quad |G(r_1) - L| < \frac{\epsilon}{2}, \quad a < r, r_1 \leq r_0. \quad (1.8.37)$$

Therefore,

$$\begin{aligned} |G(r) - G(r_1)| &= |(G(r) - L) - (G(r_1) - L)| \\ &\leq |G(r) - L| + |G(r_1) - L| \leq \epsilon, \quad a < r, r_1 \leq r_0. \end{aligned} \quad (1.8.38)$$

For sufficiency, (A) implies that

$$|G(r)| = |G(r_1) + (G(r) - G(r_1))| \leq |G(r_1)| + |G(r) - G(r_1)| \leq |G(r_1)| + \epsilon, \quad (1.8.39)$$

$a < r_1 \leq r_0$. Since G is also bounded on the compact set $[r_0, b]$ (Theorem 5.2.11, p. 313), G is bounded on $(a, b]$. Therefore the monotonic functions

$$\overline{G}(r) = \sup_{t \in (r, b]} G(t), \quad \underline{G}(r) = \inf_{t \in (r, b]} G(t),$$

are well defined on $(a, b]$, and

$$\lim_{r \rightarrow a+} \overline{G}(r) = \overline{L} \quad \text{and} \quad \lim_{r \rightarrow a+} \underline{G}(r) = \underline{L} \quad (1.8.40)$$

both exist and are finite (Theorem 2.1.11, p. 47). From (A),

$$\begin{aligned} |G(r) - G(r_1)| &= |(G(r) - G(r_0)) - (G(r_1) - G(r_0))| \\ &\leq |G(r) - G(r_0)| + |G(r_1) - G(r_0)| \leq 2\epsilon, \end{aligned} \quad (1.8.41)$$

so $\overline{G}(r) - \underline{G}(r) \leq 2\epsilon$. Since ϵ is an arbitrary positive number, this implies that

$$\lim_{r \rightarrow a+} (\overline{G}(r) - \underline{G}(r)) = 0, \quad (1.8.42)$$

so $\overline{L} = \underline{L}$. Let $L = \overline{L} = \underline{L}$. Since

$$\underline{G}(r) \leq G(r) \leq \overline{G}(r), \quad (1.8.43)$$

it follows that $\lim_{r \rightarrow a+} G(r) = L$.

3. Theorem 5 (Cauchy Criterion for Uniform Convergence II) The improper integral

$$\int_a^b f(x, y) dx = \lim_{r \rightarrow a+} \int_r^b f(x, y) dx \quad (1.8.44)$$

converges uniformly on S if and only if, for each $\epsilon > 0$, there is an $r_0^{\text{int}}(a, b]$ such that

$$\left| \int_a^r f(x, y) dx \right| < \epsilon, \quad \text{for all } y \in S, \quad \text{for all } a < r \leq r_0.$$

Suppose $\int_a^b f(x, y) dx$ converges uniformly on S and $\epsilon > 0$. From Definition 2, there is an $r_0^{\text{int}}(a, b]$ such that

$$\left| \int_a^r f(x, y) dx \right| < \frac{\epsilon}{2} \quad \text{and} \quad \left| \int_a^{r_1} f(x, y) dx \right| < \frac{\epsilon}{2}, \quad y \in S, \quad a < r, r_1 \leq r_0. \quad (B)$$

Since

$$\int_{r_1}^r f(x, y) dx = \int_a^r f(x, y) dx - \int_a^{r_1} f(x, y) dx \quad (1.8.45)$$

(B) and the triangle inequality imply (A).

For the converse, denote

$$F(y) = \int_a^b f(x, y) dx. \quad (1.8.46)$$

Since (A) implies that

$$|F(r, y) - F(r_1, y)| \leq \epsilon, \quad y \in S, \quad a < r, r_1 \leq r_0, \quad (1.8.47)$$

Theorem 2 with $G(r) = F(r, y)$ (y fixed but arbitrary in S) implies that $\int_a^b f(x, y) dx$ converges pointwise for $y \in S$. Therefore, if $\epsilon > 0$ then, for each $y \in S$, there is an $r_0(y)^{\text{int}}(a, b]$ such that

$$\left| \int_a^r f(x, y) dx \right| < \epsilon, \quad \text{for all } y \in S, \quad \text{for all } a < r \leq r_0(y).$$

For each $y \in S$, choose $r_1(y) \leq \min[r_0(y), r_0]$. Then

$$\int_a^r f(x, y) dx = \int_a^{r_1(y)} f(x, y) dx + \int_{r_1(y)}^r f(x, y) dx, \quad (1.8.48)$$

so (A), (C), and the triangle inequality imply that

$$\left| \int_a^r f(x, y) dx \right| < 2\epsilon, \quad \text{for all } y \in S, \quad \text{for all } a < r \leq r_1(y).$$

4. From Definition 3, $\int_a^b f(x, y) dx$ converges uniformly on S if and only if $\int_a^c f(x, y) dx$ and $\int_c^b f(x, y) dx$ both converge uniformly on S , where $c \in (a, b)$. From Theorems 4 and Theorem 5, this is true if and only if, for any $\epsilon > 0$ there are points r_0 and s_0 in (a, b) such that

$$\left| \int_a^r f(x, y) dx \right| < \epsilon, \quad \text{for all } y \in S, \quad \text{for all } a < r \leq r_0$$

and

$$\left| \int_{r_1}^{r_2} f(x, y) dx \right| \leq \epsilon, \quad y \in S, \quad a < s, s_1$$

These conditions are independent of c .

5. (a) If $|f(x, y)| \leq M$ on $[a, b] \times [c, d]$ then

$$\left| \int_{r_1}^{r_2} f(x, y) dx \right| \leq M|r_2 - r_1| \quad (1.8.49)$$

so the Cauchy convergence theorems imply the conclusion.

Define $f = f(x, y)$ on $[0, 1] \times [0, 1]$ by

$$f(x, y) = \begin{cases} \frac{1}{y} & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}$$

Then

$$\int_{r_1}^{r_2} f(x, y) dx = \begin{cases} \frac{r_2 - r_1}{y} & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}$$

Therefore f does not satisfy the requirements of Cauchy's convergence theorems.

6. In all parts $I(y)$ denotes the given integral.

$I(0) = \infty$. If $y \neq 0$ let $u = xy$; then $I(y) = \frac{1}{y} \int_0^\infty \frac{du}{1+u^2}$. If $\rho > 0$ and $\epsilon > 0$, choose r so that $\int_r^\infty \frac{du}{1+u^2} < \rho\epsilon$. Then $\frac{1}{|y|} \int_r^\infty \frac{du}{1+u^2} < \epsilon$ if $|y| \geq \rho$, so $I(y)$ converges uniformly on $(-\infty, \rho] \cup [\rho, \infty)$ if $\rho > 0$.

$I(y) = \infty$ if $y \leq 0$. If $y > 0$ let $u = xy$; then $I(y) = \frac{1}{y^3} \int_0^\infty e^{-u} u^2 du$. If $\rho > 0$ and $\epsilon > 0$, choose r so that $\int_r^\infty e^{-u} u^2 du < \rho^3 \epsilon$. Then $\frac{1}{y^3} \int_r^\infty e^{-u} u^2 du < \epsilon$ if $y \geq \rho$, so $I(y)$ converges uniformly on $[\rho, \infty)$ if $\rho > 0$.

$I(y) = \infty$ if $y \leq 0$. If $y > 0$ let $u = xy^{1/2}$; then $I(y) = y^{-n-1/2} \int_0^\infty u^{2n} e^{-u} du$. If $\rho > 0$ and $\epsilon > 0$, we can choose r so that $\int_r^\infty u^{2n} e^{-u} du < \epsilon \rho^{n+1/2}$. Then $y^{-n-1/2} \int_r^\infty u^{2n} e^{-u} du < \epsilon$ if $y \geq \rho$, so $I(y)$ converges uniformly on $S = [\rho, \infty)$ if $\rho > 0$.

Since $I(-y) = -I(y)$, it suffices to assume that $y > 0$. If $u = yx^2$ then $I(y) = \frac{1}{2\sqrt{y}} \int_0^\infty \frac{\sin u du}{\sqrt{u}}$. From Example 3.4.14 (p. 162), this integral converges conditionally. If $\rho > 0$ and $\epsilon > 0$, we can choose r so that $\left| \int_r^\infty \frac{\sin u du}{\sqrt{u}} \right| < 2\epsilon\sqrt{\rho}$, so $I(y)$ converges uniformly on $(-\infty, -\rho] \cup [\rho, \infty)$ if $\rho > 0$.

If $u = y^2 x$ then $I(y) = 3 \int_0^\infty e^{-u} du - \frac{2}{y^3} \int_0^\infty u e^{-u} du$. If $\rho > 0$, we can choose r so that $3 \int_r^\infty e^{-u} du < \frac{\epsilon}{2}$ and $\int_r^\rho u e^{-u} du < \frac{\rho^3 \epsilon}{2}$. Then

$$\left| 3 \int_r^\infty e^{-u} du - \frac{3}{y^3} \int_r^\infty u e^{-u} du \right| < \epsilon \quad \text{if } |y| \geq \rho, \quad (1.8.50)$$

so $I(y)$ converges uniformly on $(-\infty, \rho] \cup [\rho, \infty)$ if $\rho > 0$.

$I(y) = -\infty$ if $y \leq 0$. If $y > 0$, let $u = xy$; then $I(y) = \frac{1}{y} \int_0^\infty (2u - u^2) e^{-u} du$. If $\rho > 0$ and $\epsilon > 0$, we can choose r so that $\int_r^\infty |2u - u^2| e^{-u} du < \epsilon \rho$, so $I(y)$ converges uniformly on $S = [\rho, \infty)$ if $\rho > 0$.

7. Theorem 7 (Weierstrass's Test for Absolute Uniform Convergence II) Suppose $f = f(x, y)$ is locally integrable $(a, b]$ and, for some $b_0^{\text{int}}(a, b]$,

$$|f(x, y)| \leq M(x), \quad y^{\text{int}} S, \quad x^{\text{int}}(a, b_0], \quad (\text{A})$$

where

$$\int_a^{b_0} M(x) dx < \infty. \quad (1.8.51)$$

Then $\int_a^b f(x, y) dx$ converges absolutely uniformly on S .

Denote $\int_a^{b_0} M(x) dx = L < \infty$. By definition, for each $\epsilon > 0$ there is an $r_0^{\text{int}}(a, b_0]$ such that

$$|L - \int_{r_1}^{b_0} M(x) dx| \leq \epsilon, \quad \text{for } r_1 \in [r_0, b_0]$$

Therefore, if

$$0 \leq \int_{r_1}^r M(x) dx = \left(\int_{r_1}^{b_0} M(x) dx - L \right) - \left(\int_r^{b_0} M(x) dx - L \right) < \epsilon. \quad (1.8.52)$$

This and (A) imply that

$$\left| \int_{r_1}^r f(x, y) dx \right| \leq \int_{r_1}^r M(x) dx < \epsilon, \quad \text{for } y \in S, \quad r_1 \in [r_0, b_0]$$

Now Theorem 5 implies the stated conclusion.

$$|e^{-xy} \sin x| \leq e^{-\rho x} \quad \text{if } y \geq \rho \quad \text{and} \quad \int_0^\infty e^{-\rho x} dx < \infty.$$

$$\int_0^\infty \frac{\sin x dx}{x^y} = I_1(y) + I_2(y), \quad \text{where}$$

$$I_1(y) = \int_0^1 \frac{\sin x dx}{x^y} \quad \text{and} \quad I_2(y) = \int_1^\infty \frac{\sin x dx}{x^y} \quad (1.8.53)$$

are both improper integrals. Since

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots$$

If $0 \leq 1$ and $y \leq d \leq 2$, then

$$\left| \frac{\sin x}{x^y} \right| \leq x^{1-y} \leq x^{1-d} \quad \text{so} \quad \int_0^1 x^{-1+y} dx < \int_0^1 x^{-1+d} dx = \frac{1}{2-d} \quad (1.8.54)$$

so $I_1(y)$ converges absolutely uniformly on S . Since $c > 1$,

$$\left| \frac{\sin x}{x^y} \right| \leq x^{-c} \quad \text{and} \quad \int_1^\infty x^{-c} dx = \frac{1}{c-1} \quad \text{if} \quad (1.8.55)$$

$I_2(y)$ converges absolutely uniformly on S .

$$\text{If } x \geq 1 \text{ then } e^{-px} \left| \frac{\sin xy}{x} \right| \leq e^{-px} \text{ for all } y \text{ and } \int_1^\infty e^{-px} dx < \infty, \text{ since } p > 0.$$

$$\frac{e^{xy}}{(1-x)^y} \leq \frac{e^b}{(1-x)^b}, \quad \text{if } 0 \leq x < 1 \text{ and } y \leq b, \text{ and } \int_0^1 (1-x)^{-b} dx < \infty \text{ if } b < 1.$$

$$\text{If } |y| \geq \rho > 0 \text{ then } \left| \frac{\cos xy}{1+x^2y^2} \right| \leq \frac{1}{1+\rho^2x^2} \text{ for all } x, \text{ and } \int_0^\infty \frac{dx}{1+\rho^2x^2} < \infty.$$

$$\text{If } y \geq \rho > 0 \text{ then } e^{-x/y} \leq e^{-x/\rho} \text{ and } \int_0^\infty e^{-x/\rho} dx < \infty.$$

If $|y| \leq \rho$ then $e^{xy}e^{-x^2} \leq e^{x\rho}e^{-x^2}$ and $\int_{-\infty}^{\infty} e^{x\rho}e^{-x^2} dx < \infty$.

If $|x| \geq 1$ then $|\cos xy - \cos ax| \leq 2$ and $\int_1^{\infty} \frac{2 dx}{x^2} < \infty$.

If $y \geq \rho > 0$ then $e^{-yx^2} \leq e^{-\rho x^2}$ and $\int_0^{\infty} x^{2n}e^{-\rho x^2} dx < \infty$.

9. (a) If $\forall (\delta, \infty)$, $\int_0^1 x^{y-1}e^{-x} dx$ converges uniformly on $[c, \infty)$ if $c > 0$, by Theorem 7. If $x > 1$ then $|x^{y-1}e^{-x}| \leq x^{d-1}e^{-x}$ if $y \leq d$. Therefore, since $\int_1^{\infty} x^{d-1}e^{-x} dx < \infty$, $\int_1^{\infty} x^{y-1}e^{-x} dx$ converges uniformly on $(-\infty, d]$ for every d , by Theorem 6. Hence, $\int_0^{\infty} x^{y-1}e^{-x} dx$ converges uniformly on $[c, d]$ if $c > 0$.

If $y > 0$ then

$$\Gamma(y) = \int_0^{\infty} x^{y-1}e^{-x} dx = \left. \frac{x^y e^{-x}}{y} \right|_0^{\infty} + \frac{1}{y} \int_0^{\infty} x^y e^{-x} dx = \frac{\Gamma(y+1)}{y}. \quad (\text{A})$$

Therefore

$$\Gamma(y) = \frac{\Gamma(y+n)}{y(y+1)\cdots(y+n-1)} \quad (\text{B})$$

is true when $n = 1$. Now suppose it is true for given positive integer n , and replace y by $y + n$ in (A):

$$\Gamma(y+n) = \frac{\Gamma(y+n+1)}{y+n}. \quad (1.8.56)$$

Substituting this into (B) yields

$$\Gamma(y) = \frac{\Gamma(y+n+1)}{y(y+1)\cdots(y+n)}, \quad (1.8.57)$$

which completes the induction.

If $\forall -n$ (a):

$$\Gamma(y+n) = \int_0^{\infty} x^{y+n-1}e^{-x} dx. \quad (1.8.58)$$

Then we can define $\Gamma(y)$ by (B).

The assertion is true if $n = 1$, since

$$\Gamma(2) = \int_0^{\infty} xe^{-x} dx = -xe^{-x} \Big|_0^{\infty} + \int_0^{\infty} e^{-x} dx = 1. \quad (1.8.59)$$

If $\Gamma(n+1) = n!$ for some $n \geq 1$, then

$$\begin{aligned} \Gamma(n+2) &= \int_0^{\infty} x^{n+1}e^{-x} dx = -e^{-x}x^{n+1} \Big|_0^{\infty} + (n+1) \int_0^{\infty} x^n e^{-x} dx \\ &= (n+1)\Gamma(n+1) = (n+1)n! = (n+1)!, \end{aligned} \quad (1.8.60)$$

which completes the induction proof.

The change of variable $x = st$ yields

$$\int_0^{\infty} e^{-st}t^{\alpha} dt = \frac{1}{s^{\alpha+1}} \int_0^{\infty} x^{\alpha}e^{-x} dx = \frac{1}{s^{\alpha+1}}\Gamma(\alpha+1), \quad (1.8.61)$$

from the definition of the Gamma function.

Since $|g_x(x, y)|$ is monotonic with respect to x ,

$$\left| \int_{r_1}^r g_x(x, y) dx \right| = |g(r, y) - g(r_1, y)|, \quad a \leq r$$

From Assumption **(a)** of Theorem 8, if $\epsilon > 0$ there is an $r_0^{\text{int}}[a, b]$ such that

$$|g(s, y)| \leq \epsilon, \quad y \in S, \quad r_0 \leq s$$

Therefore, (A) implies that

$$\left| \int_{r_1}^r g_x(x, y) dx \right| \leq 2\epsilon, \quad y \in S, \quad r_0 \leq r \leq r_1$$

Now Theorem 4 implies that $\int_a^b |g(x, y)| dx$ converges uniformly on S , which is assumption **(c)** of Theorem 8.

) If g , g_x , and h are continuous on $(a, b] \times S$ then

$$\int_a^b g(x, y) h(x, y) dx \quad (1.8.62)$$

converges uniformly on S if the following conditions are satisfied:

(I)

$$\lim_{x \rightarrow a+} \left\{ \sup_{y \in S} |g(x, y)| \right\} = 0;$$

There is a constant M such that

$$\sup_{y \in S} \left| \int_x^b h(u, y) du \right| \leq M, \quad a < x \leq b; \quad (1.8.63)$$

$\int_a^b |g_x(x, y)| dx$ converges uniformly on S .

If

$$H(x, y) = \int_x^b h(u, y) du \quad (A)$$

then integration by parts yields

$$\begin{aligned} \int_{r_1}^r g(x, y) h(x, y) dx &= - \int_{r_1}^r g(x, y) H_x(x, y) dx \\ &= -g(r, y) H(r, y) + g(r_1, y) H(r_1, y) \\ &\quad + \int_{r_1}^r g_x(x, y) H(x, y) dx. \end{aligned} \quad (1.8.64)$$

Therefore, since assumption **(b)** and (A) imply that $|H(x, y)| \leq M$, $(x, y) \in (a, b] \times S$,

$$\left| \int_{r_1}^r g(x, y) h(x, y) dx \right| \leq M \left(2 \sup_{a \leq x \leq r_1} |g(x, y)| + \int_{r_1}^r |g_x(x, y)| dx \right)$$

on $[r_1, r] \times S$. Now suppose $\epsilon > 0$. From assumption **(a)**, there is an $r_0^{\text{int}}[a, b]$ such that $|g(x, y)| < \epsilon$ on S if $a < x \leq r_0 \leq b$. From assumption **(c)** and Theorem 5, there is an $s_0^{\text{int}}(a, b]$ such that

$$\left| \int_{r_1}^r g_x(x, y) dx \right| < \epsilon, \quad y \in S, \quad r_0 \leq r$$

Therefore (B) implies that

$$\left| \int_{r_1}^r g(x, y) h(x, y) dx \right| < 3M\epsilon, \quad y \in S, \quad r_0 \leq r$$

Now Theorem 5 implies the stated conclusion.

Denote $F(y) = \int_1^\infty \frac{\sin xy}{x^y} dx$ and, with $1 \leq r < r_1$,

$$\begin{aligned} F(r, r_1, y) &= \int_r^{r_1} \frac{\sin xy}{x^y} dx = -\frac{\cos xy}{y x^{y-1}} \Big|_r^{r_1} - \int_r^{r_1} \frac{\cos xy}{x^{y+1}} dx \\ &= \frac{\cos ry}{y r^y} - \frac{\cos r_1 y}{y r_1^y} - \int_r^{r_1} \frac{\cos xy}{x^{y+1}} dx. \end{aligned} \quad (1.8.65)$$

Therefore

$$|F(r, r_1, y)| \leq \frac{2}{y r^y} + \int_r^{r_1} x^{-y-1} dx < \frac{3}{y r^y}, \quad r, y > 0. \quad (1.8.66)$$

Now Theorem 4 implies that $F(y)$ converges uniformly on $[\rho, \infty)$ if $\rho > 0$.

(b) Denote $F(y) = \int_2^\infty \frac{\sin xy}{\log x} dx$ and, with $2 \leq r < r_1$,

$$F(r, r_1, y) = \int_r^{r_1} \frac{\sin xy}{\log x} dx = -\frac{\cos xy}{y \log x} \Big|_r^{r_1} - \frac{1}{y} \int_r^{r_1} \frac{\cos xy}{x(\log x)^2} dx. \quad (1.8.67)$$

Therefore

$$|F(r, r_1, y)| \leq \frac{1}{y} \left| \frac{2}{\log r} + \int_r^{r_1} \frac{dx}{x(\log x)^2} \right| \leq \frac{3}{y \log r}. \quad (1.8.68)$$

Now Theorem 4 implies that $F(y)$ converges uniformly on $[\rho, \infty)$ if $\rho > 0$.

(c) Denote $F(y) = \int_0^\infty \frac{\cos xy}{x + y^2} dx$, and, with $\forall \emptyset$

$$F(r, r_1, y) = \int_r^{r_1} \frac{\cos xy}{x + y^2} dx = \frac{1}{y} \left(\frac{\sin xy}{x + y^2} \Big|_r^{r_1} + \int_r^{r_1} \frac{\sin xy}{(x + y^2)^2} dx \right), \quad (1.8.69)$$

so

$$|F(r, r_1, y)| \leq \frac{3}{y(r + y^2)}. \quad (1.8.70)$$

Now Theorem 4 implies that $F(y)$ converges uniformly on $[\rho, \infty)$ if $\rho > 0$.

(d) Denote $F(y) = \int_0^\infty \frac{\sin xy}{1 + xy} dx$, and, with $\forall \emptyset$

$$F(r, r_1, y) = \int_r^{r_1} \frac{\sin xy}{1 + xy} dx = -\frac{\cos xy}{y(1 + xy)} \Big|_r^{r_1} - \int_r^{r_1} \frac{\cos xy}{y^2(1 + xy)^2} dx, \quad (1.8.71)$$

so

$$|F(r, r_1, y)| \leq \frac{3}{y(1 + ry)}. \quad (1.8.72)$$

Now Theorem 4 implies that $F(y)$ converges uniformly on $[\rho, \infty)$ if $\rho > 0$.

13. Integration by parts yields

$$\begin{aligned} \int_r^{r_1} g(x, y) h(x, y) dx &= \int_r^{r_1} g(x, y) H_x(x, y) dx \\ &= g(r_1, y) H(r_1, y) - g(r, y) H(r, y) \\ &\quad - \int_r^{r_1} g_x(x, y) H(x, y) dx, \end{aligned} \quad (1.8.73)$$

so

$$\left| \int_r^{r_1} g(x, y) h(x, y) dx \right| \leq 2 \sup_{x \geq r} \left\{ \sup_{y \in S} |g(x, y) H(x, y)| \right\} + \left| \int_r^{r_1} g_x(x, y) H(x, y) dx \right|. \quad (1.8.74)$$

Now suppose $\epsilon \geq 0$. From our first assumption, there is an $s_0^{\text{int}}[a, b)$ such that

$$\sup_{x \in [r, r_1]} \left| \int_a^b g(x, y) h(x, y) dy \right| < \epsilon, \quad \text{for } s_0 \leq r$$

Since $\int_a^b g_x(x, y) H(x, y) dx$ converges uniformly on S , Theorem 4 implies that there is an $r_0^{\text{int}}[a, b)$ such that

$$\left| \int_a^b g_x(x, y) H(x, y) dx \right| \leq \epsilon, \quad \text{for } y \in S, \quad \text{for } r_0 \leq r$$

Therefore,

$$\left| \int_a^b g(x, y) h(x, y) dy \right| \leq 2\epsilon, \quad \text{for } y \in S, \quad \text{for } \max(r_0, s_0) \leq r$$

Now Theorem 4 implies that $\int_a^b g_x(x, y) h(x, y) dx$ converges uniformly on S .

14. Theorem 10 If $f = f(x, y)$ is continuous on $(a, b) \times [c, d]$ and

$$F(y) = \int_a^b f(x, y) dx \quad (A)$$

converges uniformly on $[c, d]$, then F is continuous on $[c, d]$. Moreover,

$$\int_c^d \left(\int_a^b f(x, y) dx \right) dy = \int_a^b \left(\int_c^d f(x, y) dy \right) dx. \quad (B)$$

We will first show that F in (A) is continuous on $[c, d]$. Since F converges uniformly on $[c, d]$, Definition 1 implies that if $\epsilon > 0$, there is an $r^{\text{int}}(a, b]$ such that

$$\left| \int_a^r f(x, y) dx \right| \leq \epsilon, \quad c \leq y \leq d. \quad (1.8.75)$$

Therefore, if y and y_0 are in $[c, d]$, then

$$\begin{aligned} |F(y) - F(y_0)| &= \left| \int_a^b f(x, y) dx - \int_a^b f(x, y_0) dx \right| \\ &\leq \left| \int_r^b [f(x, y) - f(x, y_0)] dx \right| + \left| \int_a^r f(x, y) dx \right| \\ &\quad + \left| \int_a^r f(x, y_0) dx \right| \end{aligned} \quad (1.8.76)$$

so

$$|F(y) - F(y_0)| \leq \int_r^b |f(x, y) - f(x, y_0)| dx + 2\epsilon. \quad (C)$$

Since f is uniformly continuous on the compact set $[r, b] \times [c, d]$ (Corollary 5.2.14, p. 314), there is a $\delta > 0$ such that

$$|f(x, y) - f(x, y_0)| < \epsilon \quad (1.8.77)$$

if (x, y) and (x, y_0) are in $[r, b] \times [c, d]$ and $|y - y_0| < \delta$. This and (C) imply that

$$|F(y) - F(y_0)| < (b - r)\epsilon + 2\epsilon < (b - a + 2)\epsilon \quad (1.8.78)$$

if y and y_0 are in $[c, d]$ and $|y - y_0| < \delta$. Therefore F is continuous on $[c, d]$, so the integral on left side of (B) exists. Denote

$$I = \int_c^d \left(\int_a^b f(x, y) dx \right) dy. \quad (D)$$

We will show that the improper integral on the right side of (B) converges to I . To this end, denote

$$I(r) = \int_r^b \left(\int_c^d f(x, y) dy \right) dx. \quad (1.8.79)$$

Since we can reverse the order of integration of the continuous function f over the rectangle $[r, b] \times [c, d]$ (Corollary 7.2.2, p. 466),

$$I(r) = \int_c^d \left(\int_r^b f(x, y) dx \right) dy. \quad (1.8.80)$$

From this and (D),

$$I - I(r) = \int_c^d \left(\int_a^r f(x, y) dx \right) dy. \quad (1.8.81)$$

Now suppose $\epsilon > 0$. Since $\int_a^b f(x, y) dx$ converges uniformly on $[c, d]$, there is an $r_0^{\text{int}}(a, b)$ such that

$$\left| \int_a^r f(x, y) dx \right| < \epsilon, \quad \text{for } r \geq r_0^{\text{int}}(a, b)$$

$$\text{so } |I - I(r)| < (d - c)\epsilon, \quad \text{for } r \geq r_0^{\text{int}}(a, b)$$

$$\lim_{r \rightarrow a+} \int_r^b \left(\int_c^d f(x, y) dy \right) dx = \int_c^d \left(\int_a^b f(x, y) dx \right) dy, \quad (1.8.82)$$

which completes the proof of (B).

15. Theorem 11 Let f and f_y be continuous on $(a, b) \times [c, d]$, and suppose that

$$F(y) = \int_a^b f(x, y) dx \quad (1.8.83)$$

converges for some $y_0^{\text{int}}[c, d]$ and

$$G(y) = \int_a^b f_y(x, y) dx \quad (1.8.84)$$

converges uniformly on $[c, d]$. Then F converges uniformly on $[c, d]$ and is given explicitly by

$$F(y) = F(y_0) + \int_{y_0}^y G(t) dt, \quad c \leq y \leq d. \quad (1.8.85)$$

Moreover, F is continuously differentiable on $[c, d]$ and

$$F'(y) = G(y), \quad c \leq y \leq d, \quad (\text{A})$$

where $F'(c)$ and $f_y(x, c)$ are derivatives from the right, and $F'(d)$ and $f_y(x, d)$ are derivatives from the left.

Let

$$F_r(y) = \int_r^b f(x, y) dx, \quad \text{for } a \leq r \leq b$$

Since f and f_y are continuous on $[r, b] \times [c, d]$, Theorem 1 implies that

$$F'_r(y) = \int_r^b f_y(x, y) dx, \quad c \leq y \leq d. \quad (1.8.86)$$

Therefore

$$\begin{aligned}
 F_r(y) &= F_r(y_0) + \int_{y_0}^y \left(\int_r^b f_y(x, t) dx \right) dt \\
 &= F(y_0) + \int_{y_0}^y G(t) dt \\
 &\quad + (F_r(y_0) - F(y_0)) - \int_{y_0}^y \left(\int_a^r f_y(x, t) dx \right) dt, \quad c \leq y \leq d.
 \end{aligned} \tag{1.8.87}$$

Therefore,

$$\left| F_r(y) - F(y_0) - \int_{y_0}^y G(t) dt \right| \leq |F_r(y_0) - F(y_0)| + \left| \int_{y_0}^y \int_a^r f_y(x, t) dx dt \right|. \tag{B}$$

Now suppose $\epsilon > 0$. Since we have assumed that $\lim_{r \rightarrow a+} F_r(y_0) = F(y_0)$ exists, there is an r_0 in (a, b) such that

$$|F_{r_0}(y_0) - F(y_0)| < \epsilon, \quad a$$

Since we have assumed that $G(y)$ converges for $y^{\text{int}}[c, d]$, there is an $r_1^{\text{int}}(a, b)$ such that

$$\left| \int_a^{r_1} f_y(x, t) dx dt \right| < \epsilon, \quad t \in [c, d], \quad a$$

Therefore, (B) yields

$$\left| F_r(y) - F(y_0) - \int_{y_0}^y G(t) dt \right| < \epsilon(1 + |y - y_0|) \leq \epsilon(1 + d - c) \tag{1.8.88}$$

if a

$$F(y) = F(y_0) + \int_{y_0}^y G(t) dt, \quad c \leq y \leq d. \tag{1.8.89}$$

Since G is continuous on $[c, d]$ by Theorem 10, (A) follows from differentiating this (Theorem 3.3.11, p. 141).

16. Since

$$|f(x) \cos xy| \leq |f(x)|, \quad |f(x) \sin xy| \leq |f(x)|, \quad \text{and} \quad \int_{-\infty}^{\infty} |f(x)| dx < \infty, \tag{1.8.90}$$

Theorems 6 and 7 imply that $\int_{-\infty}^{\infty} f(x) \cos xy dx$ and $\int_{-\infty}^{\infty} f(x) \sin xy dx$ converge uniformly on $(-\infty, \infty)$, so Theorem 10 implies that $C(y)$ and $S(y)$ are continuous on $(-\infty, \infty)$.

If $y \neq 0$, integrating by parts yields

$$\begin{aligned}
 C(y) &= f(x) \frac{\sin xy}{y} \Big|_a^{\infty} - \frac{1}{y} \int_a^{\infty} f'(x) \sin xy dx \\
 &= -f(a) \frac{\sin ay}{y} - \frac{1}{y} \int_a^{\infty} f'(x) \sin xy dx
 \end{aligned} \tag{1.8.91}$$

and

$$\begin{aligned}
 S(y) &= -f(x) \frac{\cos xy}{y} \Big|_a^{\infty} + \frac{1}{y} \int_a^{\infty} f'(x) \cos xy dx \\
 &= f(a) \frac{\cos ay}{y} + \frac{1}{y} \int_a^{\infty} f'(x) \cos xy dx,
 \end{aligned} \tag{1.8.92}$$

since $\lim_{x \rightarrow \infty} f(x) = 0$. From Exercise [exer:17] with f replaced by f' , $\int_1^{\infty} f'(x) \cos xy dx$ and $\int_1^{\infty} f'(x) \sin xy dx$ are continuous on $(-\infty, \infty)$. Therefore $C(y)$ and $S(y)$ are continuous on $(-\infty, 0) \cup (0, \infty)$.

To see that C and S are not necessarily continuous at $y = 0$, let $a = 1$ and $f(x) = 1/x$, so

$$\lim_{x \rightarrow \infty} f(x) = 0 \quad \text{and} \quad \int_1^{\infty} |f'(x)| = \int_1^{\infty} \frac{dx}{x^2} = 1. \quad (1.8.93)$$

Then

$$C(y) = \lim_{r \rightarrow \infty} \int_1^r \frac{\cos xy}{x} dx \quad \text{and} \quad S(y) = \lim_{r \rightarrow \infty} \int_1^r \frac{\sin xy}{x} dx, \quad y \neq 0. \quad (1.8.94)$$

If $y > 0$ make the change of variable $u = xy$ to see that

$$C(y) = \lim_{r \rightarrow \infty} \int_y^{ry} \frac{\cos u}{u} du = \int_y^{\infty} \frac{\cos u}{u} du \quad (1.8.95)$$

and

$$S(y) = \lim_{r \rightarrow \infty} \int_y^{ry} \frac{\sin u}{u} du. S(y) = \int_y^{\infty} \frac{\sin u}{u} du. \quad (1.8.96)$$

Therefore $\lim_{y \rightarrow 0^+} C(y) = \infty$, so C is not continuous at $y = 0$. Since $S(0) = 0$ and $\lim_{y \rightarrow 0^+} S(y) = \int_0^{\infty} \frac{\sin u}{u} du \neq 0$, S is not continuous at $y = 0$.

The integral diverges if $y = 0$. If $y \neq 0$ substitute $u = |y|x$ to obtain

$$F(y) = \int_0^{\infty} \frac{dx}{1+x^2y^2} = \frac{1}{|y|} \int_0^{\infty} \frac{du}{1+u^2} = \frac{1}{|y|} \tan^{-1} u \Big|_0^{\infty} = \frac{\pi}{2|y|}, \quad (A)$$

so $F(y)$ converges for all $y \neq 0$. To test for uniform convergence, suppose $|y| > 0$ and

$$\int_r^{r_1} \frac{dx}{1+x^2y^2} = \frac{1}{|y|} \int_{r|y|}^{r_1|y|} \frac{du}{1+u^2} < \frac{1}{\rho} \int_{r\rho}^{\infty} \frac{du}{1+u^2} \quad (1.8.97)$$

if $|y| \geq \rho$. If $\epsilon > 0$ there is an $\alpha > 0$ such that $\frac{1}{\rho} \int_{\alpha}^{\infty} \frac{du}{1+u^2} < \epsilon$. Therefore $\int_r^{r_1} \frac{dx}{1+x^2y^2} < \epsilon$ if $(\alpha/\rho, \infty)$.

To evaluate

$$I = \int_0^{\infty} \frac{\tan^{-1} ax - \tan^{-1} bx}{x} dx, \quad (1.8.98)$$

we note that

$$\frac{\tan^{-1} ax - \tan^{-1} bx}{x} = \int_b^a \frac{dy}{1+x^2y^2}. \quad (1.8.99)$$

Therefore

$$I = \int_0^{\infty} dx \int_b^a \frac{dy}{1+x^2y^2} = \int_b^a dy \int_0^{\infty} \frac{dx}{1+x^2y^2} = \frac{\pi}{2} \int_b^a \frac{dy}{y} = \frac{\pi}{2} \log \frac{a}{b}, \quad (1.8.100)$$

where the second equality is valid because of the uniform convergence of $F(y)$ on the closed interval with endpoints a and b , and the third equality follows from (A).

$F(y)$ is a proper integral if $y \geq 0$ and it diverges if $y \leq -1$. If -1

$$F(y) = \int_0^1 x^y dx = \frac{x^{y+1}}{y+1} \Big|_0^1 = \frac{1}{y+1} \quad (A)$$

is convergent. Since

$$\int_0^r x^y dx = \frac{x^{y+1}}{y+1} \Big|_0^r = \frac{r^{y+1}}{y+1}. \quad (1.8.101)$$

and

$$\left| \frac{\partial}{\partial y} \left(\frac{r^{y+1}}{y+1} \right) - \frac{r^{y+1}}{y+1} \right| \leq \log r - \frac{1}{y+1} < 0 \quad \text{if } 0 < y < 1,$$

it follows that

$$\left| \int_0^1 x^y dx \right| \leq \frac{r^{\rho+1}}{\rho+1} \quad \text{if } \rho > 0$$

Therefore, Theorem 5 implies that $F(y)$ converges uniformly on $[\rho, \infty)$ if $\rho > -1$.

Now Theorem 11 implies that

$$I = \int_0^1 \frac{x^a - x^b}{\log x} dx = \int_0^1 dx \int_b^a x^y dy = \int_b^a dy \int_0^1 x^y dx = \int_b^a \frac{dy}{y+1} = \log \frac{a+1}{b+1}. \quad (1.8.102)$$

$$F(y) = \int_0^\infty e^{-yx} \cos x dx = \frac{y}{y^2 + 1}. \text{ Since}$$

$$\left| \int_r^\infty e^{-yx} \cos x dx \right| \leq \int_r^\infty e^{-xy} dx = \frac{e^{-yr}}{y}, \quad (1.8.103)$$

Theorem 4 (or Theorem 6) implies that $F(y)$ converges uniformly on $[\rho, \infty)$ if $\rho > 0$. Therefore, Theorem implies that if $a, b > 0$ then

$$\begin{aligned} I &= \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} \cos x dx = \int_0^\infty \cos x dx \int_a^b e^{-yx} dy \\ &= \int_a^b dy \int_0^\infty e^{-yx} \cos x dx = \int_a^b \frac{y}{y^2 + 1} dy = \frac{1}{2} \log \frac{b^2 + 1}{a^2 + 1}. \end{aligned} \quad (1.8.104)$$

$$F(y) = \int_0^\infty e^{-yx} \sin x dx = \frac{1}{y^2 + 1}. \text{ Since}$$

$$\left| \int_r^\infty e^{-yx} \sin x dx \right| \leq \int_r^\infty e^{-xy} dx = \frac{e^{-yr}}{y}, \quad (1.8.105)$$

Theorem 4 (or Theorem 6) implies that $F(y)$ converges uniformly on every $[\rho, \infty)$ if $\rho > 0$. Therefore, if $a, b > 0$ then Theorem 11 implies that

$$\begin{aligned} I &= \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} \sin x dx = \int_0^\infty \sin x dx \int_a^b e^{-yx} dy \\ &= \int_a^b dy \int_0^\infty e^{-yx} \sin x dx = \int_a^b \frac{1}{y^2 + 1} dy = \tan^{-1} b - \tan^{-1} a. \end{aligned} \quad (1.8.106)$$

$$(e) F(y) = \int_0^\infty e^{-x} \sin xy dx = \frac{y}{y^2 + 1}. \text{ Since}$$

$$\left| \int_r^\infty e^{-x} \sin xy dx \right| \leq \int_r^\infty e^{-x} dx = e^{-r}, \quad (1.8.107)$$

Theorem 4 (or Theorem 6) implies that $F(y)$ converges uniformly on $[\rho, \infty)$ if $\rho > 0$. Therefore Theorem 11 implies that

$$\begin{aligned} I &= \int_0^\infty e^{-x} \frac{1 - \cos ax}{x} dx = \int_0^\infty e^{-x} dx \int_0^a \sin xy dy \\ &= \int_0^a dy \int_0^\infty e^{-x} \sin xy dx = \int_0^a \frac{y}{y^2 + 1} dy = \frac{1}{2} \log(1 + a^2). \end{aligned} \quad (1.8.108)$$

$$F(y) = \int_0^\infty e^{-x} \cos xy dx = \frac{1}{y^2 + 1}. \text{ Since}$$

$$\left| \int_r^\infty e^{-x} \cos xy dx \right| \leq \int_r^\infty e^{-x} dx = e^{-r}, \quad (1.8.109)$$

Theorem 4 (or Theorem 6) implies that $F(y)$ converges uniformly on $[\rho, \infty)$ if $\rho > 0$. Therefore Theorem 11 implies that

$$\begin{aligned} I &= \int_0^\infty e^{-x} \sin ax \, dx = \int_0^\infty e^{-x} \, dx \int_0^a \cos xy \, dy \\ &= \int_0^a dy \int_0^\infty e^{-x} \cos xy \, dx = \int_0^a \frac{1}{y^2 + 1} \, dy = \tan^{-1} a. \end{aligned} \quad (1.8.110)$$

19. (a) We start with

$$F(y) = \int_0^1 x^y \, dx = \frac{1}{y+1} \quad y > -1. \quad (A)$$

Formally differentiating this yields

$$F^{(n)}(y) = \int_0^1 (\log x)^n x^y \, dx = \frac{(-1)^n n!}{(y+1)^{n+1}}, \quad y > -1. \quad (B)$$

To justify this we will show by induction that the improper integrals

$$I_n(y) = \int_0^1 (\log x)^n x^y \, dx, \quad n = 0, 1, 2, \dots \quad (1.8.111)$$

converge uniformly on $[\rho, \infty)$ if $\rho > -1$. We begin with $n = 0$:

$$\int_0^r x^y \, dx = \frac{x^{y+1}}{y+1} \Big|_{r_1}^r = \frac{r^{y+1}}{y+1} \leq \frac{r^{y+1}}{\rho+1}, \quad -1 < \rho \leq y. \quad (1.8.112)$$

so $I_0(y) = F(y)$ converges uniformly on $[\rho, \infty)$ if $\rho > -1$. Now suppose that $I_n(y)$ converges uniformly on $[\rho, \infty)$. Integrating by parts yields

$$\begin{aligned} \int_{r_1}^r (\log x)^{n+1} x^y \, dx &= \frac{r^{y+1} (\log r)^{n+1}}{y+1} - \frac{n+1}{y+1} \int_{r_1}^r (\log x)^n x^y \, dx \\ &\quad - \frac{n+1}{y+1} \int_{r_1}^r (\log x)^n x^y \, dx, \quad \text{quad } -1 \end{aligned}$$

Letting $r_1 \rightarrow 0$ yields

$$\int_0^r (\log x)^{n+1} x^y \, dx = \frac{r^{y+1} (\log r)^{n+1}}{y+1} - \frac{n+1}{y+1} \int_0^r (\log x)^n x^y \, dx. \quad (C)$$

Since the integral on the right converges, it follows that the integral on the left converges; in fact

$$\int_0^1 (\log x)^{n+1} x^y \, dx = -\frac{n+1}{y+1} \int_0^1 (\log x)^n x^y \, dx. \quad (1.8.113)$$

We must still show that the integral on the left converges uniformly on $[\rho, \infty)$ if $\rho > -1$. To this end, note from (C) that

$$\left| \int_0^r (\log x)^{n+1} x^y \, dx \right| \leq \left| \frac{r^{y+1} (\log r)^{n+1}}{\rho+1} \right| + \frac{n+1}{\rho+1} \left| \int_0^r (\log x)^n x^y \, dx \right| \quad (D)$$

if $y \geq \rho$. Now suppose $\epsilon > 0$. Since $\lim_{r \rightarrow 0^+} r^{\rho+1} (\log r)^{n+1} = 0$, there is an $r_1^{\text{int}}(0, 1)$ such that

$$\left| \frac{r^{\rho+1} (\log r)^{n+1}}{\rho+1} \right| \leq \frac{\epsilon}{2} \quad \text{if } r \in (0, r_1^{\text{int}})$$

Since $\int_0^r (\log x)^n x^y \, dx$ is uniformly convergent (by our induction assumption), there is $r_2^{\text{int}}(0, 1)$ such that

$$\frac{n+1}{\rho+1} \left| \int_0^r (\log x)^n x^y \, dx \right| \leq \frac{\epsilon}{2}, \quad y \geq \rho, \quad (1.8.114)$$

Now (D) implies that

$$\left| \int_0^r (\log x)^{n+1} x^y \, dx \right| \leq \epsilon, \quad \text{if } y \in [\rho, \infty), \quad \text{quad } r \in (0, r_1^{\text{int}})$$

This, Theorem 11, and an easy induction argument imply (B).

(b) Substituting $x = u\sqrt{y}$ yields

$$F(y) = \int_0^\infty \frac{dx}{x^2 + y} = \frac{1}{\sqrt{y}} \int_0^\infty \frac{du}{u^2 + 1} = \frac{\pi}{2\sqrt{y}}, \quad y > 0. \quad (\text{A})$$

Formally differentiating this yields

$$\begin{aligned} \int_0^\infty \frac{dx}{(x^2 + y)^{n+1}} &= \frac{\pi}{2n+1} 1 \cdot 3 \cdots (2n-1) y^{-n-1/2} = \frac{\pi}{2^{2n+1}} \frac{(2n)!}{n!} y^{-n-1/2} \\ &= \frac{\pi}{2^{2n+1}} \binom{2n}{n} y^{-n-1/2}, \quad y > 0. \end{aligned} \quad (1.8.115)$$

Theorem 11 implies that the formal differentiation is legitimate, since, if $y \geq 0$ and $r > 0$, then

$$\int_r^\infty \frac{dx}{(x^2 + y)^{n+1}} \leq \int_r^\infty x^{-2n-2} dx = \frac{r^{-2n-1}}{(2n-1)}; \quad (1.8.116)$$

hence, the improper integrals $\int_0^\infty \frac{dx}{(x^2 + y)^{n+1}}$, $n = 0, 1, 2, \dots$ converge uniformly on $[0, \infty)$.

Denote $I_n(y) = \int_0^\infty x^{2n+1} e^{-yx^2} dx$. Then

$$I_0(y) = \int_0^\infty x e^{-yx^2} dx = \frac{1}{2} \int_0^\infty 2x e^{-yx^2} dx = -\frac{1}{2y} e^{-yx^2} \Big|_0^\infty = \frac{1}{2y}. \quad (1.8.117)$$

Since

$$\int_r^\infty x^{2n+1} e^{-yx^2} dx \leq \int_r^\infty x^{2n+1} e^{-\rho x^2} dx \quad \text{if } 0 < \rho \leq y, \quad (1.8.118)$$

if $n \geq 0$, we can differentiate I_n formally with respect to $y^{\text{int}}(0, \infty)$ to obtain

$$I_n(y) = (-1)^n I_0^{(n)} = \frac{n!}{2y^{n+1}}. \quad (1.8.119)$$

(d) Denote

$$\begin{aligned} I(y) &= \int_0^\infty y^x dx = \int_0^\infty e^{x \log y} dx = \frac{1}{\log y} \int_0^\infty (\log y)^x dx \\ &= \frac{1}{\log y} \int_0^\infty \frac{y^x}{\log y} dx = -\frac{1}{\log y} \int_0^\infty \frac{y^x}{\log y} dx = 0 \end{aligned}$$

Formally differentiating this yields $I'(y) = \int_0^\infty xy^{x-1} dx$. There are two improper integrals here: $J_1(y) = \int_0^1 xy^{x-1} dx$ and $J_2(y) = \int_1^\infty xy^{x-1} dx$. If $r < 1$ then

$$\int_0^r xy^{x-1} dx = \frac{1}{y} \int_0^r xy^x dx \leq \frac{1}{y} \int_0^r x dx = \frac{r^2}{2y} \leq \frac{r^2}{2\rho_1}, \quad 0 < \rho_1 \leq y \leq 1. \quad (1.8.120)$$

Therefore $J_1(y)$ converges uniformly on $[\rho_1, 1]$. If $x > r > 1$ and $\rho_2 < 1$ then

$$\int_r^\infty xy^{x-1} dx < \int_r^\infty x\rho_2^{x-1} dx = \frac{1}{\rho_2} \int_r^\infty x\rho_2^x dx, \quad (1.8.121)$$

Since

$$\lim_{x \rightarrow \infty} \frac{1}{\rho_2} \int_r^\infty x\rho_2^x dx = 0 \quad (1.8.122)$$

Theorem 7 implies that $J_2(y)$ converges uniformly on $[0, \rho_2]$. Therefore, if $0 < \rho_1 < \rho_2 < 1$ then $\int_0^\infty xy^{x-1} dx$ converges uniformly on

$[\rho_1, \rho_2]$. Now Theorem 11 implies that

$$\int_0^\infty xy^x dx = \frac{1}{y(\log y)^2}.$$

However, since $I(y) = -\frac{1}{\log y}$, we know that $I'(y) = \frac{1}{y(\log y)^2}$. This and (A) imply that $\int_0^\infty xy^x dx = \frac{1}{(\log y)^2}$.

20. Here $F(y) = \int_0^\infty e^{-x^2} \cos 2xy dx$, so Theorem 11 implies that

$$F'(y) = -2 \int_0^\infty x e^{-x^2} \sin 2xy dx, \quad (A)$$

since the integral on the right converges uniformly on $(-\infty, \infty)$, by Theorem 6.

Integration by parts yields

$$\begin{aligned} F(y) &= \frac{1}{2y} \int_0^\infty e^{-x^2} (2y \cos 2xy) dx \\ &= \frac{1}{2y} \left(e^{-x^2} \sin 2xy \Big|_0^\infty + 2 \int_0^\infty x e^{-x^2} \sin 2xy dx \right) \\ &= \frac{1}{y} \int_0^\infty x e^{-x^2} \sin 2xy dx = -\frac{1}{2y} F'(y). \end{aligned} \quad (1.8.123)$$

From this and (A), $F'(y) + 2yF(y) = 0$, so $\frac{F'(y)}{F(y)} = -2y$, $\log F(y) = -y^2 + \log F(0)$, and $F(y) = F(0)e^{-y^2}$. Since $F(0) = \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$ (Example 12), $F(y) = \frac{\sqrt{\pi}}{2} e^{-y^2}$.

Here $F(y) = \int_0^\infty e^{-x^2} \sin 2xy dx$, so Theorem 11 implies that

$$F'(y) = 2 \int_0^\infty x e^{-x^2} \cos 2xy dx, \quad (A)$$

since the integral on the right converges uniformly on $(-\infty, \infty)$. Integrating this by parts yields

$$\begin{aligned} F'(y) &= -e^{-x^2} \cos 2xy \Big|_0^\infty - 2y \int_0^\infty e^{-x^2} \sin 2xy dx \\ &= 1 - 2yF(y), \end{aligned} \quad (1.8.124)$$

so $F'(y) + 2yF(y) = 1$, $e^{y^2} F'(y) + 2e^{y^2} yF(y) = e^{y^2}$, and $(e^{y^2} F(y))' = e^{y^2}$. Therefore, since $F(0) = 0$, $F(y) = \int_0^y e^{u^2} du$.

22. Theorems 6 and 11 imply that S and C are n times continuously differentiable on $(-\infty, \infty)$ if $\int_{-\infty}^\infty |x^n f(x)| dx < \infty$.

23. We will show first that

$$C_k(y) = \int_a^\infty x^k f(x) \cos xy dx \quad \text{and} \quad S_k(y) = \int_a^\infty x^k f(x) \sin xy dx, \quad 0 \leq k \leq n, \quad (1.8.125)$$

converge uniformly on $U_\rho = (-\infty, -\rho] \cup [\rho, \infty)$ if $\rho > 0$. Note that if $\lim_{x \rightarrow \infty} x^n f(x) = 0$, then $\lim_{x \rightarrow \infty} x^k f(x) = 0$, $k = 0, 1, 2, \dots, n$. If $0 \leq k \leq n$, then

$$\int_r^{r_1} x^k f(x) \cos xy dx = \frac{1}{y} \left[x^k f(x) \sin xy \Big|_r^{r_1} - \int_r^{r_1} (x^k f(x))' \sin xy dx \right]. \quad (A)$$

Henceforth k is fixed. Our assumptions imply that if $\rho > 0$ and $\epsilon > 0$ then there is an $r_0^{\text{int}}[a, \infty)$ such that

$$\int_{r_0}^\infty |(x^k f(x))'| dx < \rho\epsilon \quad \text{and} \quad |x^k f(x)| < \rho\epsilon, \quad x \geq r_0. \quad (1.8.126)$$

Therefore (A) implies that

$$\left| \int_{r_1}^r x^k f(x) \cos xy \, dx \right| < 3\epsilon, \quad r_1 \leq r$$

Now Theorem 4 implies that $C_0, C_1, \dots, C_k$ converge uniformly on $(-\infty, -\rho] \cup [\rho, \infty)$. Since every $y \neq 0$ is in such an interval, Theorem 11 now implies that if $y \neq 0$ then

$$C^{(k)}(y) = \int_a^\infty x^k f(x) \sin xy \, dx, \quad 0 \leq k \leq n. \quad (1.8.127)$$

A similar argument applies to $S, S', \dots, S^{(n)}$.

Let $I(y; r, r_1) = \int_r^{r_1} \frac{1}{x} \sin \frac{y}{x} \, dx$. Assume for the moment that $y \geq 0$. Substituting $u = y/x$ yields

$$I(y; r, r_1) = \int_{y/r_1}^{y/r} \left(\frac{u}{y} \right) \sin u \left(-\frac{y}{u^2} \right) du = \int_{y/r_1}^{y/r} \frac{\sin u}{u} du. \quad (1.8.128)$$

Therefore, since $\left| \frac{\sin u}{u} \right| \leq 1$ for all u , $|I(y; r, r_1)| \leq y/r, 1 \leq r \leq r_1$. In fact, since $I(-y; r, r_1) = -I(y; r, r_1)$, we can write $|I(y; r, r_1)| \leq |y|/r, 1 \leq r \leq r_1$. Therefore, Theorem 4 implies that $\int_1^\infty \frac{1}{x} \sin \frac{y}{x} \, dx$ converges uniformly on every finite interval.

Now denote $F_r(y) = \int_1^r \cos \frac{y}{x} \, dx$. substituting $u = y/x$ yields $F_r(y) = y \int_{y/r}^y \frac{\cos u}{u^2} du$, so $\lim_{r \rightarrow \infty} F_r(y) = \infty$ for all $y \geq 0$. Since $F_r(-y) = F_r(y)$, it follows that $\lim_{r \rightarrow \infty} F_r(y) = \infty$ for all y , so the answer to the question is "no."

Let P_n be the induction assumption

$$F^{(n)}(s) = (-1)^n \int_0^\infty e^{-sx} x^n f(x) \, dx, \quad s > s_0, \quad (1.8.129)$$

which is true by the definition of F for $n = 0$. If P_n is true, then Theorems 11 and 13 imply that

$$\begin{aligned} F^{(n+1)}(s) &= (-1)^n \frac{d}{ds} \int_0^\infty e^{-sx} x^n f(x) \, dx = (-1)^n \int_0^\infty \frac{d}{ds} (e^{-sx} x^n f(x)) \, dx \\ &= (-1)^{n+1} \int_0^\infty e^{-sx} x^{n+1} f(x) \, dx, \end{aligned} \quad (1.8.130)$$

so P_n implies P_{n+1} , which completes the induction proof.

Let $G(x) = \int_0^x e^{-s_0 t} f(t) \, dt$. If $s > s_0$ then

$$F(s) = \int_0^\infty e^{-sx} f(x) \, dx = \int_0^\infty e^{-(s-s_0)x} G'(x) \, dx = (s-s_0) \int_0^\infty e^{-(s-s_0)x} G(x) \, dx \quad (A)$$

(integration by parts). Since $(s-s_0) \int_0^\infty e^{-(s-s_0)x} \, dx = 1$, (A) implies that

$$F(s) - F(s_0) = (s-s_0) \int_0^\infty e^{-(s-s_0)x} (G(x) - F(s_0)) \, dx. \quad (B)$$

Now suppose $\epsilon > 0$. Since $F(s_0) = \int_0^\infty e^{-s_0 t} f(t) \, dt = \lim_{t \rightarrow \infty} G(t)$, there is an r such that $|G(x) - F(s_0)| < \epsilon$ if $x \geq r$; hence, from (B), then

$$\begin{aligned} |F(s) - F(s_0)| &\leq (s-s_0) \int_0^r e^{-(s-s_0)x} |G(x) - F(s_0)| \, dx + \epsilon (s-s_0) \int_r^\infty e^{-(s-s_0)x} \, dx \\ &< (s-s_0) \int_0^r e^{-(s-s_0)x} |G(x) - F(s_0)| \, dx + \epsilon. \end{aligned} \quad (1.8.131)$$

Since r is fixed, we can let $s \rightarrow s_0^+$ to conclude that $\limsup_{s \rightarrow s_0^+} |F(s) - F(s_0)| \leq \epsilon$, which implies that $\lim_{s \rightarrow s_0^+} F(s) = F(s_0)$.

If $s \geq s_1 > s_0$ then

$$|e^{-sx}f(x)| = |e^{-(s-s_0)x}e^{s_0x}f(x)| \leq Me^{-(s-s_0)x} \leq Me^{-(s_1-s_0)x}. \quad (1.8.132)$$

Since

$$\int_0^\infty Me^{-(s_1-s_0)x} dx = \frac{M}{s_1-s_0} < \infty, \quad (1.8.133)$$

Theorem 6 implies the stated conclusion.

In Theorem 13 we assumed only that $\int_0^\infty e^{-s_0u}f(u) du$ is bounded; here we are assuming that $\int_0^\infty e^{-s_0u}f(u) du$ is convergent.

Let

$$G(x) = \int_x^\infty e^{-s_0t}f(t) dt \quad \text{and} \quad H(x) = \sup \{|G(t)| : t \geq x\}. \quad (1.8.134)$$

Then

$$|G(x)| \leq H(x) \quad \text{and} \quad \lim_{x \rightarrow \infty} G(x) = \lim_{x \rightarrow \infty} H(x) = 0, \quad (\text{A})$$

since $\int_0^\infty e^{-s_0x}f(x) dx$ converges. Since f is continuous on $[0, \infty)$, $G'(x) = -e^{-s_0x}f(x)$. Integration by parts yields

$$\begin{aligned} \int_r^\infty e^{-sx}f(x) dx &= \int_r^\infty e^{-(s-s_0)x}(e^{-s_0x}f(x)) dx = - \int_0^\infty e^{-(s-s_0)x}G'(x) dx \\ &= -e^{-(s-s_0)x}G(x) \Big|_r^\infty + (s-s_0) \int_r^\infty e^{-(s-s_0)x}G(x) dx \\ &= e^{-(s-s_0)r}G(r) + (s-s_0) \int_r^\infty e^{-(s-s_0)x}G(x) dx, \quad s \geq s_0. \end{aligned} \quad (1.8.135)$$

Therefore

$$\begin{aligned} \left| \int_r^\infty e^{-sx}f(x) dx \right| &\leq |G(r)|e^{-(s-s_0)r} + H(r)(s-s_0) \int_r^\infty e^{-(s-s_0)x} dx \\ &= (G(r) + H(r))e^{-(s-s_0)r} \leq 2H(r)e^{-(s-s_0)r}, \quad s \geq s_0, \end{aligned} \quad (1.8.136)$$

so (A) implies that $F(s)$ converges uniformly on $[s_0, \infty)$.

From Theorem 13, $F(s) = \int_0^\infty e^{-sx}f(x) dx$ converges for all $s > s_0$. Denote $G(x) = \int_0^x e^{-s_0t}f(t) dt$, $x \geq 0$. Then

$$\begin{aligned} F(s) &= \int_0^\infty e^{-(s-s_0)x}e^{-s_0x}f(x) dx = \int_0^\infty e^{-(s-s_0)x}G'(x) dx \\ &= (s-s_0) \int_0^\infty e^{-(s-s_0)x}G(x) dx \end{aligned} \quad (1.8.137)$$

(integration by parts). Since $(s-s_0) \int_0^\infty e^{-(s-s_0)x} dx = 1$,

$$F(s) - F(s_0) = \int_0^\infty e^{-(s-s_0)x}(G(x) - F(s_0)) dx \quad (1.8.138)$$

If $\epsilon > 0$ there is an R such that $|G(x) - F(s_0)| < \epsilon$ if $x \geq R$. Therefore, if $s > s_0$ then

$$\begin{aligned} |F(s) - F(s_0)| &< (s-s_0) \int_0^R e^{-(s-s_0)x}|G(x) - F(s_0)| dx + \epsilon \\ &< (s-s_0) \int_0^R |G(x) - F(s_0)| dx + \epsilon. \end{aligned} \quad (1.8.139)$$

Hence $\limsup_{s \rightarrow s_0+} |F(s) - F(s_0)| \leq \epsilon$. Since ϵ is arbitrary, this implies that

$$\lim_{s \rightarrow s_0+} |F(s) - F(s_0)| = 0.$$

The assumptions of Exercise 28 imply that $\int_r^\infty e^{-s_0 x} f(x) dx$ converges for every $r > 0$. Since

$$\int_r^\infty e^{-s_0 x} f(x) dx = \int_0^\infty e^{-s(r+x)} f(x+r) dx = e^{-sr} \int_0^\infty e^{-sx} f(x+r) dx, \quad (1.8.140)$$

we can apply the result of Exercise 30 with $f(x)$ replaced by $f(x+r)$, to conclude that

$$\begin{aligned} \lim_{s \rightarrow s_0+} \int_r^\infty e^{-sx} f(x) dx &= e^{-s_0 r} \int_0^\infty e^{-s_0 x} f(x+r) dx \\ &= \int_0^\infty e^{-s_0(x+r)} f(x+r) dx \\ &= \int_r^\infty e^{-s_0 x} f(x) dx. \end{aligned} \quad (1.8.141)$$

If $G(x) = \int_0^x e^{-s_0 t} f(t) dt$, then $|G(x)| \leq M$ on $[0, \infty)$ for some M . If $\epsilon > 0$, there is an $r > 0$ such that

$$\int_0^r e^{-s_0 x} |f(x)| dx < \epsilon. \quad (A)$$

If $s > s_0$, then

$$\begin{aligned} \int_r^\infty e^{-sx} f(x) dx &= \int_r^\infty e^{-(s-s_0)x} G'(x) dx \\ &= e^{-(s-s_0)x} G(x) \Big|_r^\infty + (s-s_0) \int_r^\infty G(x) e^{-(s-s_0)x} dx \\ &= -e^{-(s-s_0)r} G(r) + (s-s_0) \int_r^\infty G(x) e^{-(s-s_0)x} dx. \end{aligned} \quad (1.8.142)$$

Therefore, since $|G(x)| \leq M$,

$$\begin{aligned} \left| \int_r^\infty e^{-sx} f(x) dx \right| &\leq M e^{-(s-s_0)r} + M(s-s_0) \int_r^\infty e^{-(s-s_0)x} dx \\ &= M \left(e^{-(s-s_0)r} - e^{-(s-s_0)x} \Big|_r^\infty \right) = 2M e^{-(s-s_0)r}. \end{aligned} \quad (1.8.143)$$

This and (A) imply that

$$\left| \int_0^\infty e^{-sx} f(x) dx \right| \leq \epsilon + 2M e^{-(s-s_0)r}. \quad (1.8.144)$$

Therefore,

$$\limsup_{s \rightarrow \infty} \left| \int_0^\infty e^{-sx} f(x) dx \right| \leq \epsilon. \quad (1.8.145)$$

Since ϵ is arbitrary, this implies that

$$\limsup_{s \rightarrow \infty} \int_0^\infty e^{-sx} f(x) dx = 0, \quad (1.8.146)$$

From Exercise 18(d), $\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} \sin x dx = \tan^{-1} b - \tan^{-1} a$. From Exercise 30, letting $b \rightarrow \infty$ yields

$$\int_0^\infty e^{-ax} \frac{\sin x}{x} dx = \frac{\pi}{2} - \tan^{-1} a, \quad \text{so (b)} \quad \int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}. \quad (1.8.147)$$

Integrating by parts yields

$$\int_0^r e^{-sx} f'(x) dx = e^{-sr} f(r) - f(0) + \int_0^r s e^{-sx} f(x) dx. \quad (\text{A})$$

Suppose $s \geq s_1 > s_0$. Since $|f(x)| \leq M e^{s_0 x}$, $e^{-sr} |f(r)| \leq M e^{-(s_1-s_0)r}$. Therefore $e^{-sr} f(r) \rightarrow 0$ converges uniformly to zero on $[s_1, \infty)$. Since

$$\begin{aligned} \left| \int_r^\infty s e^{-sx} f(x) dx \right| &\leq M |s| \int_r^\infty e^{-(s-s_0)x} dx \leq \frac{M |s| e^{-(s_1-s_0)r}}{s-s_0} \\ &\leq \frac{M(s-s_0+|s_0|)e^{-(s_1-s_0)r}}{s-s_0} \leq M \left(1 + \frac{|s_0|}{s_1-s_0} \right) e^{-(s_1-s_0)r}, \end{aligned} \quad (1.8.148)$$

it follows that $\int_r^\infty s e^{-sx} f(x) dx$ converges to zero uniformly on $[s_1, \infty)$. Since this implies that $\int_0^r s e^{-sx} f(x) dx$ converges uniformly on $[s_1, \infty)$, (A) implies that $G(s)$ converges uniformly on $[s_1, \infty)$.

(b) In this case let $f'(x) = x e^{x^2} \sin e^{x^2}$, so $f(x) = -\frac{1}{2} \cos e^{x^2}$. Since $|\cos e^{x^2}| \leq 1$ for all x , the hypotheses stated in (a) hold with $s_0 = 0$. Therefore $G(s)$ converges uniformly on $[\rho, \infty)$ if $\rho > 0$.

33. We will first show that $\int_0^\infty e^{-s_0 x} \frac{f(x)}{x} dx$ converges. Denote $G(x) = \int_0^x e^{-s_0 t} f(t) dt$. Since $F(s_0)$ is convergent; say $|G(x)| \leq M$, $0 \leq x < \infty$. If

$$\int_r^{r_1} e^{-s_0 x} \frac{f(x)}{x} dx = \int_r^{r_1} \frac{G'(x)}{x} dx = \frac{G(r)}{r} - \frac{G(r_1)}{r_1} - \int_r^{r_1} \frac{G(x)}{x^2} dx. \quad (1.8.149)$$

Therefore

$$\left| \int_0^\infty e^{-s_0 x} \frac{f(x)}{x} dx \right| \leq \frac{3M}{\rho}, \quad \text{quad } \rho$$

so Theorem 2 implies that $H(s) = \int_0^\infty e^{-st} \frac{f(x)}{x} dx$ converge when $s = s_0$. Therefore Exercise 27 implies that it converges uniformly on $[s_0, \infty)$, Therefore Theorem 10 implies that

$$\begin{aligned} \int_{s_0}^s F(u) du &= \int_{s_0}^s \left(\int_0^\infty e^{-ux} f(x) dx \right) du = \int_0^\infty \left(\int_{s_0}^s e^{-ux} du \right) f(x) dx \\ &= \int_0^\infty (e^{-s_0 x} - e^{-sx}) \frac{f(x)}{x} dx \end{aligned} \quad (1.8.150)$$

From Exercise 30 (with $f(x)$ replaced by $f(x)/x$), $\lim_{s \rightarrow \infty} \int_0^\infty e^{-sx} \frac{f(x)}{x} dx = 0$, which implies the stated conclusion.

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