

CONCEPTUAL MATHEMATICS IN SOCIETY



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Preface

About this Open Educational Resource (OER)

This book and its ancillary materials were created by Ricela Feliciano-Semidei at Northern Illinois University for MATH 103: Elements of Mathematical Reasoning and was made possible through an Illinois Secretary of State/Illinois State Library (ISL) OER Grant awarded to Northern Illinois University Libraries' project: *Huskies Unbound: Developing OER for Undergraduate Success in High-Impact Courses*. Funded by a one-time \$3 million allocation to the Illinois Secretary of State's Budget in FY24, and facilitated by the Consortium of Academic and Research Libraries in Illinois (CARLI), the ISL OER Grant Program was created to support the development of textbooks and ancillaries as open educational resources to reduce students' costs, encourage student success, and allow faculty to adapt materials to meet local needs.

I wish to express my gratitude to Illinois Governor JB Pritzker, Illinois Secretary of State Alexander Giannoulas, and Greg McCormick, Director of the Illinois State Library, for their commitment to Open Education.

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The Peer Reviewer process was completed by the following reviewer(s):

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References

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Introduction - About the Book

The idea of creating this textbook is to provide a resource for a course that Lance Vobornik and I developed in 2022 when Dr. Jeff Thunder asked us to be creative in ways we can better support students' college mathematics learning experience. The result of this second edition is a product of collaboration between instructors who have taught this class and provided feedback. For example, Peter Sassman and Alexandra Hill provided insightful approaches to emphasize conceptual understanding, Mike Mutersbaugh and William O'Conner met weekly with me to discuss the course and provided meaningful feedback of approaches we use in this second edition. Other insights from Jeanne Padilla, Roy Quintero, Scott Rexford, and Allison O'Conner have also made this book possible.

The textbook's approach is to develop strong preparation for the precalculus and calculus sequence through college-level skills to complete mathematical work and to develop conceptual understanding, rather than focusing solely on memorization and repetition of processes. I like to think about the book in three parts: Logic and Critical Thinking, Number Sense, and Finances and Statistics.

Logic and Critical Thinking (An introduction to Chapters 1 and 2)

Doing mathematics requires more than just the traditional mathematical skills (e.g., adding fractions, multiplying natural numbers, evaluating variables, solving equations). In the first part of this textbook (Chapters 1 and 2), it is expected that students will learn the foundation of mathematical logic, thinking, and problem-solving. Students will learn different types of thinking, including creative, analytical, and critical thinking. In addition, we expect students to learn different strategies for solving problems and effective ways of explaining their work.

This part has the main objective of explicitly providing students with tools for solving problems and communicating their work to others. Specifically, these tools will help students explore different types of thinking and strategies for solving problems. In addition, the last section of this part provides students with a checklist on how to effectively communicate their work to their instructor when writing their homework. These communication techniques to explain their work will not only help them write clear homework and mathematical solutions, but also develop communication skills to solve problems with other students and future coworkers. Students can look back at these chapters when they are working on their homework or studying for exams, even if they are preparing for future math classes' exams.

As a supplement to this part, we encourage you to check the [Huskie Academic Success Center: Student Success Tips and Tools](#). Especially, the “study smarter” section with the following resources: growth mindset, studying for math and physics, studying for STEM, and using office hours effectively.

Number Sense (An introduction to Chapters 3 and 4)

In college, there are two major concepts that every student is expected to master: fractions and algebra. In this part of our textbook, we provide students with tools to develop a deeper conceptual understanding of a fraction and the rationale behind the processes involving fraction operations. Similarly, we reflect on the foundations of algebra representations.

Regarding fractions, we expect that students develop a deeper understanding and their own answers to the following questions:

1. Is a fraction one number or two numbers? What can I represent with fractions?
2. When adding or subtracting fractions: Why do we need to have the same denominator? Why are we supposed to add or subtract the numerators?
3. When multiplying fractions: Why do we need to multiply the numerator by the numerator and the denominator by the denominator? Why does this process work?
4. When dividing fractions: Why do we need to “invert and multiply”? Why does this process work?

The authors of these chapters will facilitate tools for understanding these questions by presenting the concepts in several ways. It is expected that by understanding these questions, students can master fraction concepts and processes and avoid common fraction mistakes.

Regarding algebra, we expect to develop the concept of variables through understanding patterns. The authors of this chapter facilitate learning algebra by generalizing patterns in figures and using tools to explore unknown values. In addition, we finalize the algebra ideas by connecting to linear and exponential growth. It is expected that at the end of this chapter, students can recognize linear and exponential behaviors in graphs and equations. This knowledge is foundational in finance and calculus courses.

Finance and Statistics (An introduction to Chapters 5 and 6)

In the final part of this textbook (Chapters 5 and 6), students connect their developing mathematical skills to real-life contexts involving personal finance and foundational statistics. These chapters are designed to help students understand how mathematics informs everyday decisions—whether managing a savings account, navigating credit card interest, or interpreting data in school, work, and daily life.

In the finance chapter, we provide students with mathematical models rooted in algebra to analyze common financial situations. Students will explore how savings grow over time, and how interest rates affect loans and credit cards. Rather than memorizing formulas, students will learn why these financial models work and how algebraic relationships help them make informed decisions. Our goal is to empower students to understand the long-term consequences of financial choices and to apply mathematical reasoning to their personal financial planning.

In the statistics chapter, students are introduced to the fundamental concepts of data analysis. They will learn measures of central tendency, including the mean, median, and mode, and explore how each measure helps describe a data set. In addition, students will examine key ideas such as variability, distribution shapes, and the importance of context when interpreting data. Through real-world examples, this chapter emphasizes how statistics allow us to make sense of information, identify meaningful patterns, and avoid common misinterpretations.

1: Ways of Thinking

Doing mathematics requires more than just the traditional mathematical skills (e.g., adding fractions, multiplying natural numbers, evaluating variables, solving equations). In this chapter, you will learn about thinking, a foundation of mathematical logic and problem-solving. You will learn different types of thinking, including creative, analytical, and critical thinking.

Learning Objectives

By the end of this chapter, you will be able to:

- Identify and describe at least three different ways of thinking used in real-life problem situations.
- Apply these thinking approaches to solve a problem in more than one way.

Specifically, these tools will support you by exploring different types of thinking and different strategies for solving problems. You can look back at this chapter when you are working on your homework or studying for exams.

As a supplement to this part, we encourage you to check the [Huskie Academic Success Center: Student Success Tips and Tools](#), especially the video on growth mindset under Study Smarter.

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1.1: What Thinking Means

Student Survey

How do you feel about the ways you think? Take this quick survey to figure it out, ranking questions on a scale of 1–4, 1 meaning “least like me” and 4 meaning “most like me.” These questions will help you determine how the chapter concepts relate to you right now. As you are introduced to new concepts and practices, it can be informative to reflect on how your understanding changes over time. We’ll revisit these questions at the end of the chapter to see whether your feelings have changed.

1. I understand how to approach problem-solving.
2. I have creative potential.
3. I often think about how I’m learning
4. I know how to find and evaluate valid information.

Whether we admit it or not or even consider it or not, we cannot stop thinking. We think during intense work situations, while we’re playing games, when we eat, as we watch a movie, even during meditation that purports to empty the mind of all thought. Skilled and practiced yogis may be able to get into a state that resembles non-thinking, but most of us keep thinking all the time. Perhaps as you read these lines, you doubt their accuracy suggesting that you don’t really think when you’re just relaxing with friends. But you do. You may think about the other people in the group and what you do or do not know about them. You may wonder what you’ll eat for your next meal. Your mind may flit to question whether you locked the door on the way out. Or you may debate internally whether you’ll finish on time the assignment due for one of your classes. Now, you may not act on any of those random thoughts during this relaxing time, but you *are* nonetheless thinking. As you begin this exploration of thinking, consider all the ways we turn to technology to assist with our thinking and how thinking impacts and defines various careers.

When you consider the word *thinking*, does your mind drift toward:

- a. School
- b. Work
- c. Relationships
- d. Free time

Reflect on your answer, and write one or two sentences on why you associate this idea with thinking.

We’ll look more closely at several distinct types of thinking, including creative, analytical, and critical thinking, all of which come into play for problem-solving.

What is Thinking?

Thinking is one of those hard-to-pinpoint aspects of life we typically don’t analyze much—like breathing or walking or sleeping. We constantly think, and becoming more attuned to how we think and what we do when we encounter new ideas is an excellent habit to pursue.

You may have read quotes or inspirational slogans that claim *you are what you think*. What does that mean? Can you *think* yourself into a good mood? Can you *think* you have a million dollars in your pocket? Does that mean you are the next music sensation if you often sing at parties? Not necessarily, but consider Jose, for instance. He isn’t a rock and roll star, but Jose spends a lot of his leisure time thinking about music, analyzing performances, memorizing his favorite musicians’ characteristics, buying fan clothing, and even designing a creative means to explain his excitement for music to his friends through a homemade video. Jose certainly could allow his fascination to seep into other aspects of his life. Do you have a hobby or interest you spend a lot of time thinking about?

Many people go to great lengths to attend a concert by a favorite music star. They think creatively about how to save enough money for tickets; they think analytically about scheduling their other obligations to have time off work to attend or how to make up work in their college classes. This much planning involves a great deal of thinking, and not all about music. In the example about Jose, thinking directs the actions of the person doing the thinking. So, in fact, what we think *does* influence who we are and how we act. We have many resources available to be more effective thinkers, and learning about these resources gives us options.

Thinking and Technology

Apps and search engines literally bring thinking to our fingertips. Consider how often you visit Google. The use of this familiar site has become so commonplace as to render the proper name of the company into a verb—to *google* a topic. Basic calculators or word-processing software programs are other simple examples of technology we often use without recognizing them as thinking aids.

More recently, artificial intelligence (AI) tools—such as ChatGPT, Copilot, or Gemini—have expanded what it means to “think with technology.” These programs can generate ideas, summarize information, or simulate problem-solving conversations. When used thoughtfully, AI can support curiosity, creativity, and deeper understanding. However, relying on AI without critical engagement can limit independent thinking. It’s important to use these tools as partners in reasoning rather than substitutes for it.

While apps, software programs, thinking games, and thought exercises may help you stretch your brain, don’t let their simplicity fool you into thinking that cultivating an inquisitive, thoughtful mind is easy or automatic. Thinking is as complex as it is necessary for our success in life. Some tools you may find useful are applications that provide challenges using mind puzzles are Peak and Elevate. These training apps offer brain training that varies from quick matching memory games to more sophisticated thought-processing speed scenarios. You can even use a straightforward tool like a flashcard app, such as Chegg Prep, to create your own thinking games -- using word associations or pictures to help you connect topics and build your skills.

A familiar element of the thinking apps is a progression tracker to help thinkers improve their focus and memory as well as to learn and practice math and verbal skills over time. Researchers are still investigating the correlation between thought-invoking game playing and the decrease in mental agility, memory, and cognitive vitality. Early studies have produced numerous findings, including a long-term investigation of the onset of Alzheimer’s disease conducted with nuns; you can look up the *Nun Study* to learn more.

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1.2: Creative Thinking

Discussion:

1. How can you go about generating original ideas?
2. What is the best way to approach working with unconventional ideas?

Has anyone ever told you that you have a flair for the creative? If so, celebrate! That's a good personality trait to nurture. Creativity is needed in all occupations and during all stages of life. Learning to be more in tune with your own version of creativity can help you think more clearly, resolve problems, and appreciate setbacks. You're creative if you repurpose old furniture into a new function. You're also creative if you invent a new cookie recipe for a friend who has a nut allergy. And you're using creativity if you can explain complex biological concepts to your classmates in your lab class. Creativity pops up everywhere. When creative thinking comes into play, you'll be looking for both original and unconventional ideas, and learning to recognize those ideas improves your thinking skills all around.

Would you learn more about the French Revolution by eating foods popular in that era? What if you were to stop using your phone for all nonemergency communication to understand how news flowed in the early 20th century? These examples present creative ways to approach learning the experiences of a specific time in history. When actors want to learn about a character they'll be playing, they often engage in method acting to immerse themselves in the role. They may maintain a different accent or wear only clothes their character would wear even when they are not at rehearsals, all so they can feel what it was like for their character. Think of ways you may be able to apply method acting to your learning experiences.

Discussion:

1. In what ways could thinking creatively help you be a better student?
2. In what ways could thinking creatively help you be a better mathematics student?
3. Some people say creativity is the realm of children. Can you think of how a child's curiosity and willingness to explore may help you understand a college discipline that is unfamiliar to you now?

Creativity doesn't always present itself in the guise of a chart-topping musical hit or other artistic expression. We need creative solutions throughout the workplace—whether board room, emergency room, or classroom. It was no fluke that the 2001 revised Bloom's cognitive taxonomy, originally developed in 1948, placed a new word at the apex—*creating*. That is the highest level of thinking skills. As noted in previous chapters, we do all need to use and develop the lower thinking skills of remembering, applying, and analyzing, but true intelligence and successful thinking move beyond these levels to invention.

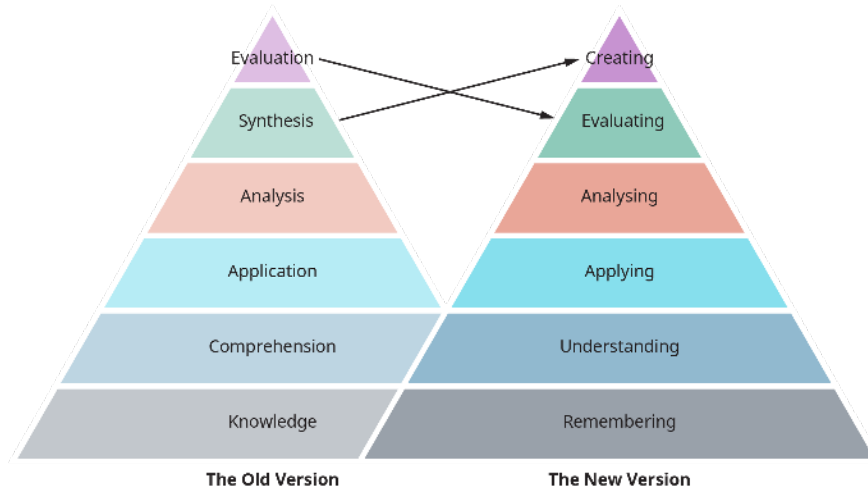


Figure 1.2.5: Bloom's Taxonomy is an important learning theory used by psychologists, cognitive scientists, and educators to demonstrate levels of thinking. Many assessments and lessons you've seen during your schooling have likely been arranged with Bloom's in mind. Researchers recently revised it to place creativity—invention—as the highest level.

Regurgitating the minute details of *Goldilocks* or *Beowulf* demonstrates far less comprehension than fashioning an original ending that turns the tables or developing a board game from the story. Author Gregory Maguire used the base plot of L. Frank Baum's 1900 book, *The Wonderful Wizard of Oz*, and the 1939 film *The Wizard of Oz* to create the smash-hit 2003 Broadway musical *Wicked*, which tells the story from the perspective of the Wicked Witch of the West, making her a sympathetic character. This creative approach calls for far more critical and creative thinking than memorizing facts.

Continuing to support creativity in whatever form it takes will be how we cure cancer, establish peace, and manipulate the time-space continuum. Don't shortchange your own creativity.

Definition: Creative Thinking

Creative thinking is the ability to generate original ideas or work with unconventional ideas, involving the use of one's imagination and curiosity to solve problems in innovative ways.

Generating Original Ideas

Nineteenth-century American writer and humorist Mark Twain may have been partially correct when he said:

"There is no such thing as a new idea. It is impossible. We simply take a lot of old ideas and put them into a sort of mental kaleidoscope. We give them a turn, and they make new and curious combinations. We keep on turning and making new combinations indefinitely, but they are the same old pieces of colored glass that have been in use through all the ages" (*Mark Twain's Own Autobiography* by Mark Twain).



Figure 1.2.6: You may feel like you cannot come up with new ideas, but even the process of combining and recombining familiar concepts and approaches is a creative act. A kaleidoscope creates a nearly infinite number of new images by repositioning the same pieces of glass.

It is certainly a pretty metaphor of idea generation, but even if old ideas are reworked to create new solutions to existing problems or we embellish a current thought to include new ways of living or working, that renewal is the epitome of the creative process.

It's common to think of creativity as something used mostly by traditional artists—people who paint, draw, or sculpt. Indeed, artists are creative, but think of other fields in which people think just a little differently to approach situations in their discipline. The famous heart surgeon Dr. Denton Cooley didn't have an exact model when he first implanted an artificial heart. Chemist Stephanie Kwoleck discovered life-saving Kevlar when she continued work on a substance that would usually be thrown away. Early US astronauts owed their ability to orbit and return to Earth based on creative uses of mathematics by people like Karen Johnson. Inventor and actress Hedy Lamarr used diagrams of fish and birds to help aviation pioneer Howard Hughes produce faster airplanes. Indeed, biomimicry, an approach to innovation that seeks sustainable solutions to human challenges by emulating nature's time-tested patterns and strategies, is now a huge field of study. This list could go on and on.



Figure 1.2.7: Denton Cooley (Credit: Texas Children's Hospital/Public Domain), Stephanie Kwolek (Credit: Chemical Heritage Foundation/Attribution 3.0), Karen Johnson (Credit: NASA/Public Domain), and Hedy Lamarr (Credit: MGM/Public Domain). These individuals employed extensive creativity in the fields of science and math, leading to significant discoveries and accomplishments.

Discussion:

Work with two or three classmates to determine a product or service you could develop. Think of a situation in your life where a new way of doing something or a not-yet-invented process or device would make your life easier, more convenient, or more purposeful. And this is not limited to the creation of something big. Just looking at something you see all the time with a different lens/perspective is also creative, and we can all do that. What adaptation would you need to make? Let your imagination go wild—driverless cars, wireless communication . . . oh wait, already here. Keep thinking! Each member of your group should write a paragraph that describes the product/service, what you would need to create it, and how it will be received by others. Read each other's paragraphs and discuss the merits of the ideas.

You may actually be very good at coming up with original creative ideas. Some people naturally seem to think more creatively than others, but we all have the capacity to create and devise. Do you enjoy rearranging furniture or organizing your closet? If you already think "I could make that so much better!" as you walk through shops or events, you're on the right track. Do you tinker with wood, paper, yarn, or dirt? Are you a doodler? One way to enhance your creativity is to track your ideas. You can keep a running list on your phone, jot down ideas on index cards you can later sort into categories, or keep ideas flowing in a paper journal. Some creative people design storyboards to visualize goals or projects using pictures from magazines or online for creative inspiration. Play around with ways to keep up with ideas you may be able to incorporate in various aspects of your life.

Since the 1980s, Roger von Oech, the president of Creative Think, a California consulting firm, has been encouraging employees in corporations, educational institutions, and government agencies to think more creatively. His pithy stories, examples, scenarios, and challenges present either a barrier to creative thinking that needs to be overcome or an example of how to harness seemingly unproductive ideas. Sometimes creative ideas do not initially seem viable or productive compared to a known process or product, but talking out ideas with others and considering new approaches without fear of ridicule or censure can help individuals and groups think beyond the status quo. Von Oech's discussion starters recommend that thinkers "Avoid Arrogance, Fight for It, Get Rid of Excuses, and Listen to That Hunch." You may be interested in looking these prompts up on the [Creative Think website](https://www.creative-thinking.com/). If you do, you may find some of von Oech's ideas a little out of the ordinary, but great ideas sometimes are, and thinking about them in a different way may be the spark you need to come up with your own version of an idea that will prove effective for you. Stay open to different approaches even if you aren't immediately comfortable with the ideas.

Another creative thinking group you may be interested in investigating is koozai.com, a digital marketing consulting firm based in the United Kingdom with clients worldwide. You may not be in need of help with digital marketing, but the koozai.com website is worth a look to see how creativity can highlight excellent customer service, detail award-winning services, and inject a sense of fun and vitality into a service that may not seem very exciting on the surface, namely helping companies optimize their web presence for increased exposure and profits. The team is a creative mix of engineers, designers, and analysts who use data-based evidence to find the right fit for their clients in a relaxed and productive environment. The actual nuts-and-bolts work involved in web

marketing involves a great deal of tedious coding and specialized web design often performed by software engineers working alone, but you don't get a sense of bored, isolated office workers when you peruse the koozai.com site.

Working with Unconventional Ideas

Working with unconventional ideas can produce anxiety because the ideas are unfamiliar, and the results of implementing these ideas could be unpredictable. People may not immediately accept your nontraditional ideas. Some may never accept them. If your original creation were to require individuals to give up their current cell phones, you can imagine the resistance. Even if the new idea is an improvement in communication, some people would hesitate.

To work in this possibly uncomfortable realm, you have to remain open-minded, focus on your organizational skills, and learn to communicate your ideas well. If a coworker at a café where you work suggests serving breakfast in addition to the already-served lunch and dinner, keeping an open mind means thinking through the benefits of this new plan (e.g., potential new customers, increased profits) instead of merely focusing on the possible drawbacks (e.g., possible scheduling problems, added start-up costs, loss of lunch business). Implementing this plan would mean a new structure for buying, workers' schedules and pay, and advertising, so you would have to organize all of these component areas. And finally, you would need to communicate your ideas on how to make this new plan work not only to the staff who will work the new shift, but also to the public who frequent your café and the others you want to encourage to try your new hours.

"Because we've always done it that way" is not a valid reason to avoid trying a new approach. It may very well be that the old process is a very good way to do things, but it also may just be that the old, comfortable routine is not as effective and efficient as a new process could be.

Can you think of any routine task you do now that you've never questioned, such as doing laundry, studying for exams, spending downtime, or preparing food? Consider how you came to learn this routine. Are you following a pattern your parents set for you as you grew up? Do you ask friends how they perform these tasks and follow their example? How well do these routines work for you? Think of at least one different way you could approach one of these tasks. Would it be a good idea to change the way you do it? How would that benefit you? If not, why is the best approach to keep doing this thing the way you have always done it? Reflect on your thinking behind this routine. How could creative thinking help you identify and assess all of your options?

Another element of working with unconventional ideas is to pay attention to how you organize your thoughts. Organizing includes establishing a clear goal to accomplish, outlining the steps toward that goal, and monitoring progress with specific deadlines. You may be able to add flexibility to this plan since creativity deals in the unknown and that may take longer than you initially expected, but an organized map of your thinking and where you hope to take it can move creative projects forward.

For example, what if you were asked to build a shed for a project or as part of your job? You would need a plan of some sort. It wouldn't be prudent to run to the hardware store and just buy various supplies you see on the spur of the moment. Rather, you would organize your thoughts about this project and determine some specific goals about the size of the shed, its ultimate location and use, the type of materials that would best serve your purposes, and how long the project will take so you can budget time and money toward the accomplishment of the goal. Do you need a building permit in your area for this sort of home improvement project? Will you or others need to sacrifice something (yard space, time, money, a special view) for you to build this shed? Do you have time to complete all the steps? Do you have the skills to put the shed together, or can you learn how to do it? How much are you willing to spend on this? Without an organized plan, you may end up with a good idea, some random supplies, and an incomplete building project that wastes both time and money and does not meet your initial expectations.



Figure 1.2.8: Thinking through a plan isn't just for school. Household activities and projects require forethought and strategic thinking. (Credit: TWP, Inc./Flickr/Attribution 2.0 Generic (CC BY 2.0))

In addition to the need to remain open-minded and organized, creative thinking calls for a dissemination plan. Unconventional ideas typically don't get off the ground without the creator of the ideas communicating those thoughts to others. Do you set yourself up to be in the company of other creative thinkers? It's not a bad idea. Creativity is somewhat contagious. You may not think you have a creative way to approach a situation, but if you were to bounce ideas off like-minded friends and also friends who would offer a completely different way of looking at something, you may discover that indeed you do have some good ideas ready to come to fruition. This creative brainstorming doesn't just happen, though. You need to set aside specific times to work with others to flesh out ideas and think through obstacles. And then you'll need some more time alone for the ideas to gel. Sometimes the creative answers to problems come to you at odd moments once you have laid the groundwork—be ready to capture the ideas in some form of note when your light bulb turns on.

Creative thinking isn't just helpful in solving problems. You may want to enhance an otherwise good plan to make it fantastic and memorable, which is when you can bring in creative thinking. If you want to surprise your best friend with a special birthday celebration but are low on funds, you could think of creative ways to make this event one to remember without spending a lot of money. You could take in a free museum night or window shop at the mall or make a photo collage from pictures on your phone that bring back great memories.

Table 7.1 Creative processes should include a plan that considers the goals of the project and provides opportunities for brainstorming and feedback. The steps in this table may not work for everyone, but you can use them to think about what is needed in a process of your own. See the student resources for a blank version you can adapt.

Creative Process Applied to a Sample Campus Activity	
Creative Process Step	Description and Notes
Problem to Solve or Item/Work to Create	Create a new logo for our Commuter Student Association
Requirements and Needs	- Will be used on social media, merch, print - Must incorporate school colors but be readable in grayscale - Must be understandable at large and small sizes (computer/phone)
Parameters and Limitations	- Cannot look like other logos on campus - Cannot use photos—illustration only - Timeline: 7 weeks (in time for next year's college catalog) - Budget: \$450
Inspiration and Ideas	- Look at Commuter Association logos from other colleges. - Look at city and state transit logos. - Go to library to look at our school's old yearbooks.

Creative Process Applied to a Sample Campus Activity

Resources/Knowledge	• Graphic design • Copyright info (consult student govt) • Market research
Dissemination and Brainstorming	• Create a survey for all our commuters. • Launch a contest for ideas and submissions? • Share drafts with advisor for approval. • Talk to graphic design club?
Implementation Plan	• Samples needed in 3 weeks. From there: • 1 week for survey feedback • 1 week for improvement • 1 week for additional feedback on final candidates • 1 week for finalization and approval
Reflection and Revision	• Ask all new club members in fall semester for feedback. • Consider improving logo during spring semester next year.

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1.3: Analytical Thinking

Discussion:

1. How can you best establish component parts in thinking?
2. How can you use analysis to improve efficiency?

Thinking helps in many situations, as we've discussed throughout this chapter. When we work out a problem or situation systematically, breaking the whole into its component parts for separate analysis to come to a solution or a variety of possible solutions, we call that *analytical thinking*. Characteristics of analytical thinking include setting up the parts, using information literacy, and verifying the validity of any sources you reference. While the phrase *analytical thinking* may sound daunting, we actually do this sort of thinking in our everyday lives when we brainstorm, budget, detect patterns, plan, compare, work puzzles, and make decisions based on multiple sources of information. Think of all the thinking that goes into the logistics of a dinner-and-movie date—where to eat, what to watch, who to invite, what to wear, popcorn or candy—when choices and decisions are rapid-fire, but we do it relatively successfully all the time.

Definition: Analytical Thinking

Analytical thinking is the ability to break down complex problems into smaller parts in order to understand them better, allowing one to examine details closely and use logic to develop solutions.

Employers specifically look for candidates with analytical skills because they need to know employees can use clear and logical thinking to resolve conflicts that cause work to slow down or put the company in jeopardy of not complying with state or national requirements. If everything always went smoothly on the shop floor or in the office, we wouldn't need frontline managers, but everything doesn't always go according to plan or company policy. Your ability to think analytically could be the difference between getting a good job and being passed over by others who prove they are stronger thinkers. A mechanic who takes each car apart piece by piece to see what might be wrong instead of investigating the entire car, gathering customer information, assessing the symptoms, and focusing on a narrow set of possible problems is not an effective member of the team. Some career fields even have set, formulaic analyses that professionals in those fields need to know how to conduct and understand, such as a cost analysis, a statistical analysis, or a return on investment (ROI) analysis.

Discussion:

1. Think about a way analytical thinking would be used in a mathematics class.

Establishing Component Parts

Component parts are the separate elements of a situation or problem. They might include the people involved, the locations of the people, the weather, market fluctuations, or any number of other characteristics of the situation you're examining. If you don't identify all parts of a problem, you run the risk of ignoring a critical element when you offer the solution. For example, if you have a scheduling problem at home and seem to never see your loved ones, the first step in thinking through this problem analytically would be to decide what is contributing to this unfavorable result. To begin, you might examine the family members' individual work, school, and personal schedules, and then create a group calendar to determine if pockets of time exist that are not taken by outside commitments. Perhaps rather than reading your homework assignments at the college library, you could plan for one day a week to read with other members of your family who are doing quiet work. You might also need to determine how time is spent to better understand the family's use of time, perhaps using categories such as work/school, recreation, exercise, sleep, and meals. Once you sort the categories for all the family members, you may see blocks of time spent that would lend themselves to combining with other categories—if you and your significant other both exercise three times a week for an hour each time but at separate locations, one possible solution could be to work out together. You could alternate locations if both people have favorite places to run, or you could compromise and decide on one location for both of you—one week at the park, one week at the campus

rec center. After establishing the component parts and thinking analytically, you have provided at least one viable solution, even if it's not the ultimate one.

What if you look at the situation and decide you have too many component parts? Consider, for instance, how Amazon delivers packages every day. That's a lot of items going to and from seemingly countless locations within a relatively short time—sometimes within just one day. An organization such as Amazon must use a great deal of thinking and organizing to deliver goods and services.



Figure 1.3.9: Warehouse designers must think through complex problems and allow for a range of package sizes and shapes—even ones they haven't yet seen—to work within their systems. (Credit: Scott Lewis/Flickr/Attribution 2.0 Generic (CC-BY 2.0))

One way to maintain clear thinking with so many parts is hyper organization. Proper labeling (for shipping, Amazon uses the foundation of our mailing system, unique ZIP codes that each address must contain to be delivered) and a strong sense of categorization (fulfillment warehouses, customer return warehouses, grocery item warehouses, etc.) are necessary for Amazon to do business. If you were faced with a major research paper your freshman composition professor expects to be polished by the end of the semester, where do you start? What are the component parts of a high-quality research paper? What tasks do you need to finish and how quickly to accomplish the overall goals? A partial list might include generating ideas, selecting a topic, researching, reviewing the available literature, outlining, drafting, and reviewing. What if you encounter setbacks in any of the steps? Do you have a contingency plan? In the construction industry, engineers called this *float*, and they deliberately build in extra time and money in case problems arise on the project. This allows them to avoid getting off schedule, for instance if a severe storm makes access to the worksite impossible.



Figure 1.3.10: Construction planners and engineers allow for a range of contingencies and conditions they cannot control, such as weather, supply problems, safety adjustments, and so on. (Credit: Metropolitan Transportation Authority of New York/Flickr/Attribution Generic 2.0 (CC-BY 2.0))

Forging a Revolution

While most problems require a variety of thinking types, analytical thinking is arguably required in solving every problem. There was a time when manufacturing was completed by a few people who moved around a workspace to complete their projects. As companies grew, this became more and more inefficient, leading to the need for automation. Henry Ford, the early 20th-century American auto inventor, used analytical thinking to revolutionize the way companies increase production by inventing the assembly line. He perceived the problem in his own factory. When the demand for cars increased but his workers continued their work at the same pace, he analyzed their process to create something more efficient in the assembly line. This invention allowed one person to perform the same role over and over before sending the car chassis to another person who also performed the same role over and over as the evolving car moved down a sort of conveyor-belt system. The workers on Ford's assembly lines still had to think and make sure that the task for which they were responsible was properly constructed, free of defects, and ready to move to the next station; they just did this thinking about their one area of expertise. Instead of various skilled workers wasting time and energy moving themselves and their tools around the factory from one incomplete car to the next, possibly getting in the way of each other's work, the cars came to the workers. Ford vastly improved production rates and decreased manufacturing time by thinking about this new way of doing things.

In the 1960s, companies did not have a fast, reliable, and cost-effective way to deliver urgent documents or packages to each other. The standard mail system was slow but inexpensive, and the only alternative was a private courier, which, while faster, was prohibitively expensive. That's when Frederick W. Smith came up with the idea of a national, overnight delivery service as a part of an assignment in his undergraduate economics class at Yale University. As the story goes, Smith received only an average grade because evidently his professor wasn't all that impressed with the concept, but after analyzing the problems with the current system, thinking through his original ideas more fully, and refining his business plan, Smith launched FedEx, the largest, now global, overnight delivery service in the world.¹ This isn't a parable about ignoring your professors, but a testimony to thinking through ideas others may not initially support or even understand; thinking can create change and always has. As with Ford's assembly line and Smith's overnight delivery service, any service we now use and any problem we may still face provides thinkers with opportunities to generate solutions and viable options for improvement. Your thinking may result in a new personal service, a cure for cancer, or a revolutionary way to deliver water to developing countries.

Footnotes

- 1 "Online Extra: Fred Smith on the Birth of FedEx." Bloomberg Business Week. 2004. Retrieved 1/28/20.
<https://www.bloomberg.com/news/artic...birth-of-fedex>

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1.4: Critical Thinking

Discussion

1. How can determining the situation help you think critically?
2. How do you present informed, unbiased thinking?
3. What is the difference between factual arguments and opinions?

Critical thinking has become a buzz phrase in education and corporate environments in recent years. The definitions vary slightly, but most people agree that thinking critically includes some form of judgement that thinkers generate after careful analysis of the perspectives, opinions, or experimental results present for a particular problem or situation. Before you wonder if you're even capable of critical thinking, consider that you think critically every day. When you grab an unwashed T-shirt off the top of the pile on the floor of your bedroom to wear into class but then suddenly remember that you might see the person of your dreams on that route, you may change into something a bit less disheveled. That's thinking critically—you used data (the memory that your potential soul mate walks the same route you use on that day on campus) to change a sartorial decision (dirty shirt for clean shirt), and you will validate your thinking if and when you do have a successful encounter with said soul mate.

Likewise, when you decide to make your lunch rather than grabbing a bag of chips, you're thinking critically. You have to plan ahead, buy the food, possibly prepare it, arrange to carry the lunch with you, and you could have various reasons for doing that—making healthier eating choices, saving money for an upcoming trip, or wanting more quiet time to unwind instead of waiting in a crowded lunch line. You are constantly weighing options, consulting data, gathering opinions, making choices, and then evaluating those decisions, which is a general definition of critical thinking.

Definition: Critical Thinking

Critical thinking is the ability to evaluate information and arguments carefully in order to make reasoned judgments, allowing one to question assumptions, consider multiple perspectives, and make well-informed decisions. It includes, for example, weighing options, consulting data, gathering opinions, making choices, and then evaluating decisions.

Consider the following situations and how each one demands your attention. Which do you find most demanding of critical thinking? Why?

1. Participating in competitive athletic events
2. Watching competitive athletic events
3. Reading a novel for pleasure
4. Reading a textbook passage in science

Critical thinking forces you to determine the actual situation under question and to determine your thoughts and actions regarding that situation.

Determining the Problem

One component to keep in mind to guide your critical thinking is to determine the situation. What problem are you solving? When problems become complex and multifaceted, it is easy to be distracted by the simple parts that may not need as much thinking to resolve but also may not contribute as much to the ultimate problem resolution. What aspect of the situation truly needs your attention and your critical thinking?

Discussion:

Imagine you're planning a fantasy vacation as a group assignment where each person is allowed only \$200. The group doles out specific preliminary tasks to each member to decide where to go, what sort of trip to take, and how to keep costs low, all in the name of a fun fantasy vacation. In this scenario, whose plan demonstrates the most effective critical thinking?

1. DeRhonda creates an elaborate invitation for a dinner party she'll coordinate at an exclusive mountain cabin.
2. Patrick researches cruises, cabin rentals, and staycation options, considering costs for various trip lengths.

3. Rodrigo puts down a deposit for a private dining room for 25 at an expensive local restaurant for a date six weeks from the end of the semester.

Write out what each person's thinking reflects about their expectations for this trip and why their actions may or may not help the group at this stage of the planning.

Critical thinking differs according to the subject you're thinking about, and as such, it can be difficult to pin down any sort of formula to make sure you are doing a good job of thinking critically in all situations. While you may need to adapt this list of critical thinking components, you can get started if you do the following:

- Question everything
- Conduct legitimate research
- Limit your assumptions
- Recognize your own biases
- Gather and weigh all options

Additionally, you must recognize that changes will occur and may alter your conclusions now and in the future. You may eventually have to revisit an issue you effectively resolved previously and adapt to changing conditions. Knowing when to do that is another example of critical thinking. Informed flexibility, or knowing that parts of the plan may need to change and how those changes can work into the overall goal, is also an element of thinking critically.

For example, early in the 20th century, many people considered cigarette smoking a relaxing social pastime that didn't have many negative consequences. Some people may still consider smoking a way to relax; however, years of medical research have proven with mounting evidence that smoking causes cancer and exacerbates numerous other medical conditions. Researchers asked questions about the impact of smoking on people's overall health, conducted regulated experiments, tracked smokers' reactions, and concluded that smoking did impact health. Over time, attitudes, evidence, and opinions change, and as a critical thinker, you must continue to research, synthesize newly discovered evidence, and adapt to that new information.



Figure 1.4.11: Information, attitudes, laws, and acceptance of smoking changed dramatically over time. More recently, vaping and related practices have rekindled debates and launched new research into safety. (Credit: Satish Krishnamurthy/Flickr/Attribution 2.0 Generic (CC-BY 2.0))

Defending Against Bias

Once you have all your information gathered and you have checked your sources for currency and validity, you need to direct your attention to how you're going to present your now well-informed analysis. Be careful on this step to recognize your own possible biases. Facts are verifiable; opinions are beliefs without supporting evidence. Stating an opinion is just that. You could say, "Blue is the best color," and that's your opinion. If you were to conduct research and find evidence to support this claim, you could say, "Researchers at Oxford University recognize that the use of blue paint in mental hospitals reduces heart rates by 25% and contributes to fewer angry outbursts from patients." This would be an informed analysis with credible evidence to support the claim.

Not everyone will accept your analysis, which can be frustrating. Most people resist change and have firm beliefs on both important issues and less significant preferences. With all the competing information surfacing online, on the news, and in general conversation, you can understand how confusing it can be to make any decisions. Look at all the reliable, valid sources that claim different approaches to be the *best* diet for healthy living: ketogenic, low-carb, vegan, vegetarian, high fat, raw foods, paleo, Mediterranean, and so on. All you can do in this sort of situation is conduct your own serious research, check your sources, and write clearly and concisely to provide your analysis of the information for consideration. You cannot force others to accept your stance, but you can show your evidence in support of your thinking, being as persuasive as possible without lapsing into your own personal biases. Then the rest is up to the person reading or viewing your analysis.

Factual Arguments vs. Opinions

Thinking and constructing analyses based on your thinking will bring you in contact with a great deal of information. Some of that information will be factual, and some will not be. You need to be able to distinguish between facts and opinions so you know how to support your arguments. Begin with basic definitions:

- **Fact:** a statement that is true and backed up with evidence; facts can be verified through observation or research
- **Opinion:** a statement someone holds to be true without supporting evidence; opinions express beliefs, assumptions, perceptions, or judgments

Of course, the tricky part is that most people do not label statements as fact or opinion, so you need to be aware and recognize the difference as you go about honing your critical thinking skills.

You probably have heard the old saying “Everyone is entitled to their own opinions,” which may be true, but conversely not everyone is entitled to their own facts. Facts are true for everyone, not just those who want to believe in them. For example, *mice are animals* is a fact; *mice make the best pets* is an opinion.

Discussion:

Determine if the following statements are facts or opinions based on just the information provided here, referring to the basic definitions above. Some people consider scientific findings to be opinions even when they are convincingly backed by reputable evidence and experimentation. However, remember the definition of *fact*—verifiable by research or observation. Think about what other research you may have to conduct to make an informed decision.

- Oregon is a state in the United States. (How would this be proven?)
- Beef is made from cattle. (See current legislation concerning vegetarian “burgers.”)
- Increased street lighting decreases criminal behavior. (What information would you need to validate this claim?)
- In 1952, Elizabeth became Queen of England. (What documents could validate this?)
- Oatmeal tastes plain. (What factors might play into this claim?)
- Kindergarten decreases student dropout rates. (Think of different interest groups that may take sides on this issue.)
- Carbohydrates promote weight gain. (Can you determine if this is a valid statement?)
- Cell phones cause brain tumors. (What research considers this claim?)
- Immigration is good for the US economy. (What research would help you make an informed decision on this topic?)

Many people become very attached to their opinions, even stating them as facts despite the lack of verifiable evidence. Think about political campaigns, sporting rivalries, musical preferences, and religious or philosophical beliefs. When you are reading, writing, and thinking critically, you must be on the lookout for sophisticated opinions others may present as factual information. While it’s possible to be polite when questioning another person’s opinions when engaging in intellectual debate, thinking critically requires that you do conduct this questioning.

For instance, someone may say or write that a particular political party should move its offices to different cities every year—that’s an opinion regardless of whether you side with one party or the other. If, on the other hand, the same person said that one political party is headquartered in a specific city, that is a fact you can verify. You could find sources that can validate or discredit the statement. Even if the city the person lists as the party headquarters is incorrect, the statement itself is still a fact—just an erroneous one. If you use biased and opinionated information or even incorrect facts as your evidence to support your factual arguments, then you have not validated your sources or checked your facts well enough. At this point, you would need to keep researching.

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1.5: Practice Problems

Critical Thinking Problems

(Focus: evaluating reasoning, testing assumptions, and justifying conclusions.)

1. Someone says, “If a number is divisible by 4 and 6, it must also be divisible by 24.” Evaluate this statement and justify your answer.
2. A student writes: “To prove that all odd numbers are prime, note that 3, 5, and 7 are prime. The pattern continues.” Critically evaluate the reasoning.
3. Which grows faster as n increases: n^2 or 2^n ? Explain your reasoning and what “grows faster” means in this context.
4. A student says: “Since $0.999 \dots \neq 1$, that means there are numbers between them.” Critique this argument and provide a correct explanation.
5. A student claims that since $2^3 = 8$ and $3^2 = 9$, then $2^3 < 3^2$ means that $a^b < b^a$ for all positive integers $a < b$. Is this always true? Explain.

Creative Thinking Problems

(Focus: flexible thinking, multiple strategies, and nonstandard connections or representations.)

1. Create a 3-digit number using the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 without repetition, such that it is divisible by both 3 and 11. How many such numbers exist?
2. Find all positive integers a and b such that $a - b = 27$. Explain your reasoning.
3. If you reverse the digits of a two-digit number and subtract the smaller number from the larger, the result is always a multiple of 9. Explain why this happens.
4. Without using algebraic symbols, show (using patterns or diagrams) that the sum of the first n odd numbers equals n^2 .

Analytical Thinking Problems

(Focus: breaking problems into smaller parts, identifying patterns, and applying logic step by step.)

1. Prove that the product of two consecutive integers is always even.
2. A number has a remainder of 2 when divided by 5, and a remainder of 3 when divided by 7. What is the smallest positive number with this property?
3. Show that for any integer, for example n , that number cubed subtracted by the same number, in other words $n^3 - n$, is always divisible by 6.

Problems in this section were generated using artificial intelligence and adapted by the author for inclusion in this book.

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2: Problem Solving

In the previous chapter, we highlighted that doing mathematics requires more than just the traditional mathematical skills (e.g., adding fractions, multiplying natural numbers, evaluating variables, solving equations). In this chapter, you will learn the foundation of mathematical logic, thinking, and problem-solving. You will learn different strategies for solving problems and effective ways of explaining your work.

Learning Objectives

By the end of this chapter, you will be able to:

- Select and apply appropriate tools and strategies to effectively solve a variety of problems.
- Construct and present clear, logical explanations of your solutions and reasoning to others.

Specifically, these tools will support you by exploring different types of thinking and different strategies for solving problems. In addition, there is a checklist you can use to effectively communicate your work to your instructor when writing your homework. These communication techniques to explain your work will help you write clear homework and mathematical solutions, and they will help you develop communication skills to solve problems with other students and future coworkers. You can look back at this chapter when you are working on your homework or studying for exams.

As a supplement to this part, we encourage you to check the [Huskie Academic Success Center: Student Success Tips and Tools](#). Especially, the “study smarter” section with the following resources: growth mindset, studying for math and physics, studying for STEM, and using office hours effectively.

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2.1: Introduction to Problem Solving

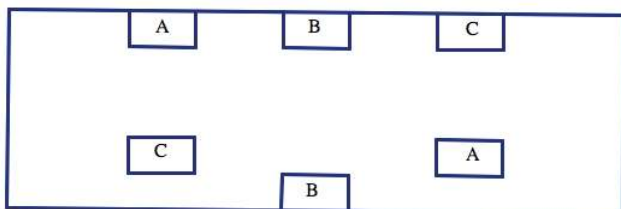
The eight standards of Mathematical Practices (<http://www.corestandards.org/Math/Practice>) describe what mathematics is really about, and why mathematics is so valuable for students to master. While all eight standards are valuable, we will focus on the first standard of Mathematical Practice:

Make sense of problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Let's start with solving a problem!

✓ Example 2.1.1

The ABC Problem. Draw curves connecting A to A, B to B, and C to C. Your curves cannot cross or even touch each other, they cannot cross through any of the lettered boxes, and they cannot go outside the large box or even touch its sides.



Before exploring the given solution, explore the problem on your own for a while, talk through your ideas with a partner (even if you have not solved it).

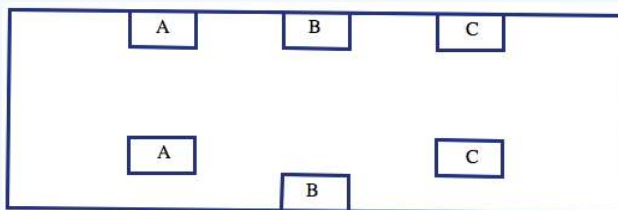
1. What did you try?
2. What makes this problem difficult?
3. Can you change the problem slightly so that it would be easier to solve?

Solution

This solution uses what we call **Wishful Thinking**.

Do you wish something in the problem was different? Would it then be easier to solve the problem?

For example, what if the ABC problem had a picture like this:

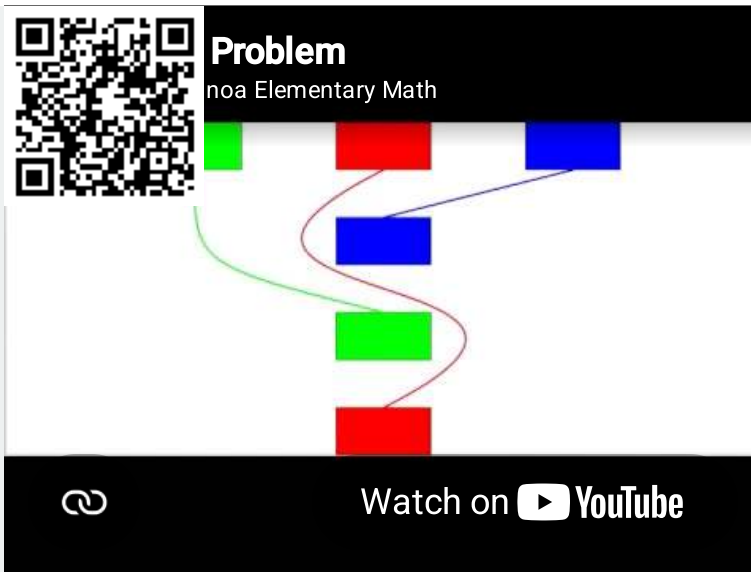


Can you solve this case and use it to help you solve the original case? Think about moving the boxes around once the lines are already drawn.

Here is one possible solution.

📌 The ABC Problem - Wishful Thinking Solution

This video shows a possible solution to the ABC problem by using Wishful Thinking.



 Definition: Problem-Solving Strategy Wishful Thinking

Wishful Thinking is a problem-solving strategy where you explore various scenarios by asking "what if" questions.

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2.2: Problem Solving Strategies

Think back to the first problem in this chapter, the ABC Problem. What did you do to solve it? Even if you did not figure it out completely by yourself, you probably worked toward a solution and figured out some things that *did not* work.

Unlike exercises, there is never a simple recipe for solving a problem. You can get better and better at solving problems, both by building up your background knowledge and by simply practicing. As you solve more problems (and learn how other people solved them), you learn strategies and techniques that can be useful. But no single strategy works every time.



Figure 2.2.1: George Pólya was a great champion in the field of *teaching* effective problem-solving skills. (Image of Pólya by Thane Plambeck from Palo Alto, California (Flickr) [CC BY 2.0 (<http://creativecommons.org/licenses/by/2.0>)], via Wikimedia Commons)

In 1945, Pólya published the short book *How to Solve It*, which gave a four-step method for solving mathematical problems:

Definition: Problem-Solving Strategy Four-Step Method

1. First, you have to understand the problem.
2. After understanding, make a plan.
3. Carry out the plan.
4. Look back on your work. How could it be better?

This is all well and good, but how do you actually do these steps? Steps 1 and 2 are particularly mysterious! How do you “make a plan?” That is where you need some tools in your toolbox and some experience to draw upon.

Much has been written since 1945 to explain these steps in more detail, but the truth is that they are more art than science. This is where math becomes a creative endeavor (and where it becomes so much fun). We will articulate some useful problem-solving strategies, but no such list will ever be complete. This is really just a start to help you on your way. The best way to become a skilled problem solver is to learn the background material well and then solve a lot of problems!

We have already seen one problem-solving strategy, which we call Wishful Thinking. Do not be afraid to change the problem! Ask yourself “what if” questions:

- What if the picture were different?
- What if the numbers were simpler?
- What if I just made up some numbers?

You need to be sure to go back to the original problem at the end, but wishful thinking can be a powerful strategy for getting started.

This brings us to the most important problem-solving strategy of all: Try Something.

Definition: Problem-Solving Strategy Try Something

Try Something is a problem-solving strategy that involves exploring a problem by making an initial attempt, such as gathering information, noticing patterns, and moving your thinking forward.

If you are really trying to solve a problem, the whole point is that you do not know what to do right out of the starting gate. You need to just try something! Put pencil to paper (or stylus to screen or chalk to board or whatever!) and try something. This is often

an important step in understanding the problem; just mess around with it a bit to understand the situation and figure out what is going on.

And equally important: If what you tried first does not work, try something else! Play around with the problem until you have a feel for what is going on.

? Exercise 2.2.1

The Payback Problem. Last week, Alex borrowed money from several of his friends. He finally got paid at work, so he brought cash to school to pay back his debts. First, he saw Brianna, and he gave her $\frac{1}{4}$ of the money he had brought to school. Then Alex saw Chris and gave him $\frac{1}{3}$ of what he had left after paying Brianna. Finally, Alex saw David and gave him $\frac{1}{2}$ of what he had remaining. Who got the most money from Alex?

Answer

All got the same amount.

Discussion:

After you have worked on the problem on your own for a while, talk through your ideas with a partner (even if you have not solved it). What did you try? What did you figure out about the problem? This problem lends itself to two particular strategies. Did you try either of these as you worked on the problem? If not, read about the strategy and then try it out before revealing the solution.

Definition: Problem-Solving Strategy Draw a Picture

Draw a Picture is a problem-solving strategy to support thinking visually about the problem by representing the situation with a picture.

Some problems are obviously about a geometric situation, and it is clear you want to draw a picture and mark down all of the given information before you try to solve it. However, even for a problem that is not geometric, like this one, thinking visually can still be helpful. Can you represent something in the situation with a picture?

Draw a square to represent all of Alex's money. Then shade $\frac{1}{4}$ of the square—that's what he gave away to Brianna. How can the picture help you finish the problem?

After you have worked on the problem yourself using this strategy (or if you are completely stuck), you can watch someone else's solution.

Solving The Payback Problem Using the Draw a Picture Strategy

After you have worked on the problem yourself using the Draw a Picture strategy, or if you are completely stuck, you can watch someone else's solution.

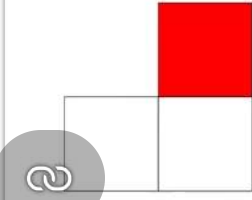


Draw a picture

Manoa Elementary Math

The amount of money Brianna received.

$\frac{1}{3}$ of the remaining money went to Chris



Watch on  YouTube

Definition: Problem-Solving Strategy Make Up Numbers

Make Up Numbers is a problem-solving strategy where you assign specific, convenient numbers to variables or quantities in a problem with the purpose of reasoning about ideas.

Part of what makes the Payback Problem difficult is that it is about money, but there are no numbers given. That means the numbers must not be important. So just make them up!

You can work forward: Assume Alex had some specific amount of money when he showed up at school, say \$100. Then figure out how much he gives to each person. Or you can work backward: suppose he has some specific amount left at the end, like \$10. Since he gave Chris half of what he had left, that means he had \$20 before running into Chris. Now, work backward and figure out how much each person got.

Solving The Payback Problem Using the Make Up Numbers Strategy

Watch the solution only after you tried the Make Up Numbers strategy for yourself.

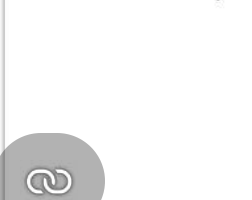


Make up numbers

Manoa Elementary Math

$\frac{1}{4} = 30$

$\frac{1}{4}$ of the money went to Brianna



Watch on  YouTube

If you use the Make Up Numbers strategy, it is really important to remember what the original problem was asking! You do not want to answer something like “Everyone got \$10.” That is not true in the original problem; that is an artifact of the numbers you made up. So after you work everything out, be sure to re-read the problem and answer what was asked!

Exercise 2.2.1

The Squares on a Chess Board Problem. How many squares, of any possible size, are on a 8×8 chess board? (The answer is not 64. The number is a lot bigger!)

Answer

204 squares.

Remember Pólya's first step is to understand the problem. If you are not sure what is being asked, or why the answer is not just 64, be sure to ask someone.

Discussion:

After you have worked on the problem on your own for a while, talk through your ideas with a partner. What did you try? What did you figure out about the problem, even if you have not solved it completely?

It is clear that you want to draw a picture for this problem, but even with the picture it can be hard to know if you have found the correct answer. The numbers get big, and it can be hard to keep track of your work. Your goal at the end is to be absolutely positive that you found the right answer. You should never ask the teacher: *Is this right?* Instead, you should declare: *Here's my answer, and here is why I know it is correct!*

Definition: Problem-Solving Strategy Try a Simpler Problem

Try a Simpler Problem is a problem-solving strategy in which you create an easier version of the original problem and you solve it to discover patterns or methods that can be applied to solve the original problem.

Pólya suggested this strategy: "If you can't solve a problem, then there is an easier problem you can solve: find it." He also said: "If you cannot solve the proposed problem, try to solve first some related problem. Could you imagine a more accessible related problem?"

In the **Squares on a Chess Board Problem** an 8×8 chess board is pretty big. Can you solve the problem for smaller boards? Like 1×1 ? 2×2 ? 3×3 ?

Of course the ultimate goal is to solve the original problem. But working with smaller boards might give you some insight and help you devise your plan to solve it.

Definition: Problem Solving Strategy Work Systematically

Work Systematically is a problem-solving strategy that involves organizing your thinking and following a clear, logical order to explore all possibilities without skipping or repeating steps. By using a consistent plan—such as listing, ordering, or testing options one at a time—you can ensure that no solution is overlooked and that your reasoning is easy to follow.

If you are working on simpler problems, it is useful to keep track of what you have figured out and what changes as the problem gets more complicated.

For example, in this problem you might keep track of how many 1×1 squares are on each board, how many 2×2 squares on are each board, how many 3×3 squares are on each board, and so on. You could keep track of the information in a table like the following table:

size of board	# of 1×1 squares	# of 2×2 squares	# of 3×3 squares	# of 4×4 squares	...
1 by 1	1	0	0	0	
2 by 2	4	1	0	0	
3 by 3	9	4	1	0	
...					

Definition: Problem-Solving Strategy Use Manipulatives

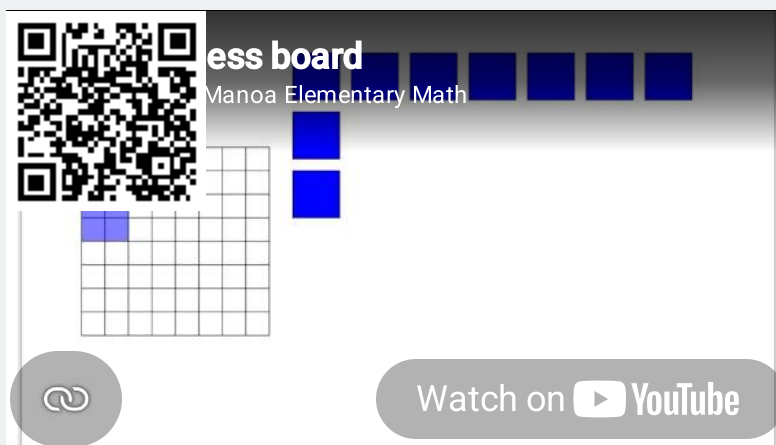
Use Manipulatives is a problem-solving strategy in which you use physical or visual objects—such as blocks, counters, tiles, or drawings—to represent and explore a problem. Manipulatives help you see relationships, test ideas, and better understand abstract concepts by making them more concrete and hands-on.

Sometimes even drawing a picture may not be enough to help you investigate a problem. Having actual materials that you move around can sometimes help a lot!

For example, in the **Squares on a Chess Board Problem** it can be difficult to keep track of which squares you have already counted. You might want to cut out 1×1 squares, 2×2 squares, 3×3 squares, and so on. You can actually move the smaller squares across the chess board in a systematic way, making sure that you count everything once and do not count anything twice.

Solving The Squares on a Chess Board Problem Combining two strategies

This video show a solution to the Squares on a Chess Board Problem combining the Work Systematically and the Use of Manipulatives strategies.



Definition: Look For and Explain Patterns

Look For and Explain Patterns is a problem-solving strategy where you search for things that repeat or follow a rule in a problem. When you notice a pattern, you can use it to make predictions, find shortcuts, or explain why something works. Explaining the pattern helps you understand the problem more deeply and show your reasoning.

If you add odd numbers and notice that adding each number gives sums of 1, 4, 9, 16, ..., you can **look for and explain the pattern** that each result is a square number $1^2, 2^2, 3^2, 4^2, \dots$. The Look For and Explain Patterns helps you predict what will come next.

Sometimes the numbers in a problem are so big, there is no way you will actually count everything up by hand. For example, if the problem in this section were about a 100×100 chess board, you would not want to go through counting all the squares by hand. It would be much more appealing to find a pattern in the smaller boards and then extend that pattern to solve the problem for a 100×100 chess board just with a calculation.

Discussion:

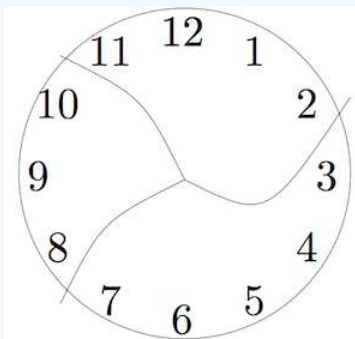
If you have not done so already, extend the table above all the way to an 8×8 chess board, filling in all the rows and columns. Use your table to find the total number of squares in an 8×8 chess board. Then:

- Describe all of the patterns you see in the table.
- Can you explain and justify any of the patterns you see? How can you be sure they will continue?
- What calculation would you do to find the total number of squares on a 100×100 chess board?

We will come back to this question soon. So if you are not sure right now how to explain and justify the patterns you found, that is OK.

? Exercise 2.2.1

The Broken Clock Problem. This clock has been broken into three pieces. If you add the numbers in each piece, the sums are consecutive numbers. We define consecutive numbers as whole numbers that appear one after the other, such as 1, 2, 3, 4 or 13, 14, 15.



Can you break another clock into a different number of pieces so that the sums are consecutive numbers? Assume that each piece has at least two numbers and that no number is damaged (e.g., 12 isn't split into two digits 1 and 2.)

Answer

Multiple answers are possible.

Remember that your first step is to understand the problem. Work out what is going on here. What are the sums of the numbers on each piece? Are they consecutive?

Discussion:

After you have worked on the problem on your own for a while, talk through your ideas with a partner (even if you have not solved it). What did you try? What progress have you made?

Definition: Problem-Solving Strategy Find the Math, Remove the Context

Find the Math, Remove the Context is a problem-solving strategy where you take a word problem or real-life situation and focus on the mathematical ideas behind it. By removing the story or extra details, you rewrite the problem using only numbers, symbols, or equations. This helps you see the structure of the problem more clearly and decide what math operations or relationships are needed to solve it.

The Common Core State Standards describe this process as developing the ability to contextualize and decontextualize at any point during problem solving. This means you can move back and forth between the real-world context of a problem and the underlying mathematics. Sometimes it helps to strip away the context so you can focus on the numbers, operations, or relationships, while other times returning to the context helps you make sense of your solution and check if it is reasonable. Being able to do this lets you see the math more clearly and apply it flexibly in different situations.

Sometimes the problem has a lot of details in it that are unimportant, or at least unimportant for getting started. The goal is to find the underlying math problem, then come back to the original question and see if you can solve it using the math.

In this case, worrying about the clock and exactly how the pieces break is less important than worrying about finding consecutive numbers that sum to the correct total. Ask yourself:

1. What is the sum of all the numbers on the clock's face?
2. Can I find two consecutive numbers that give the correct sum? Or four consecutive numbers? Or some other amount?
3. How do I know when I am done? When should I stop looking?

Of course, solving the question about consecutive numbers is not the same as solving the original problem. You have to go back and see if the clock can actually break apart so that each piece gives you one of those consecutive numbers. Maybe you can solve the math problem, but it does not translate into solving the clock problem.

Definition: Problem-Solving Strategy Check Your Assumptions

Check Your Assumptions is a problem-solving strategy where you stop and think about what you are assuming to be true in a problem. Sometimes we make guesses or take things for granted without realizing it, and these assumptions can affect your solution. By checking them, you make sure you are not missing important information or making mistakes.

When solving problems, it is easy to limit your thinking by adding extra assumptions that are not in the problem. Be sure you ask yourself: Am I constraining my thinking too much?

In the Broken Clock Problem, because the first solution has the clock broken radially, all three pieces meet at the center, many people assume that is how the clock must break. But the problem does not require the clock to break radially. It might break into pieces like this:

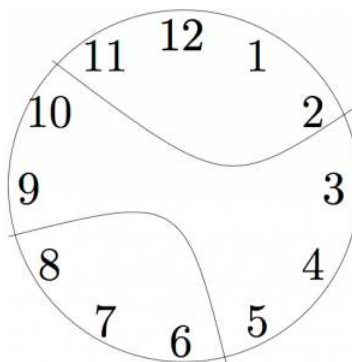


Figure 2.2.1: The Broken Clock Problem breaking the numbers in different ways.

Were you assuming the clock would break in a specific way? Try to solve the problem now, if you have not already.

In this section, we introduced several problem-solving strategies. While learning many strategies at once can feel overwhelming, the goal is not to memorize each one but to view them as tools in your problem-solving toolbox. These strategies often work together rather than in isolation—for example, you might combine the **Use Manipulatives** strategy with **Try a Simpler Problem** or **Work Systematically**. As you apply one strategy, you may naturally find yourself drawing on others to deepen your understanding and strengthen your solution process.

Definitions in this section were generated by artificial intelligence and were adapted appropriately by the author.

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2.3: Explaining Your Work

At its heart, mathematics is a social endeavor. Even if you work on problems all by yourself, you have not really solved the problem until you have explained your work to someone else, and they sign off on it. Professional mathematicians write journal articles, books, grant proposals, training material for their coworkers, and reports for their supervisors. Teachers explain mathematical ideas to their students both in writing and orally. Explaining your work is really an essential part of the problem-solving process, and probably should have been Pólya's step 5.

Writing in mathematics is different from writing poetry or a speech. It's more like technical writing, where the goal is not beauty or persuasion, but clarity. If your reader does not understand, you have not done a good job. Here are some tips for good mathematical writing.

1. **Do Not Turn in Scratch Work:** When you are solving problems and not exercises, you are going to have a lot of false starts. You are going to try a lot of things that do not work. You are going to make a lot of mistakes. You are going to use scratch paper. At some point (hopefully!) you will scribble down an idea that actually solves the problem. Hooray! That paper is not what you want to turn in or share with the world. Take that idea, and write it up carefully, neatly, and clearly. (The rest of these tips apply to that write-up.)
2. **(Re)state the Problem:** Do not assume your reader knows what problem you are solving. (Even if it is the teacher who assigned the problem!) If the problem has a very long description, you can summarize it. You do not have to rewrite it word-for-word or give all of the details, but make sure the question is clear.
3. **Clearly Give the Answer:** It is not a bad idea to state the answer right up front, then show the work to justify your answer. That way, the reader knows what you are trying to justify as they read. It makes their job much easier, and making the reader's job easier should be one of your primary goals! In any case, the answer should be clearly stated somewhere in the write-up, and it should be easy to find.
4. **Be Correct:** Of course, everyone makes mistakes as they are working on a problem. But we are talking about after you have solved the problem, when you are writing up your solution to share with someone else. The best writing in the world cannot save a wrong approach and a wrong answer. Check your work carefully. Ask someone else to read your solution with a critical eye.
5. **Justify Your Answer:** You cannot simply give an answer and expect your reader to "take your word for it." You have to explain how you know your answer is correct. This means "showing your work," explaining your reasoning, and justifying what you say. You need to answer the question, "How do you know your answer is right?"
6. **Be Concise:** There is no bonus prize for writing a lot in math class. Think clearly and write clearly. If you find yourself going on and on, stop, think about what you really want to say, and start over.
7. **Use Variables and Equations:** An equation can be much easier to read and understand than a long paragraph of text describing a calculation. Mathematical writing often has a lot fewer words (and a lot more equations) than other kinds of writing.
8. **Define your Variables:** If you use variables in the solution of your problem, always say what a variable stands for before you use it. If you use an equation, say where it comes from and why it applies to this situation. Do not make your reader guess!
9. **Use Pictures:** If pictures helped you solve the problem, include nice versions of those pictures in your final solution. Even if you did not draw a picture to solve the problem, it still might help your reader understand the solution. And that is your goal!
10. **Use Correct Spelling and Grammar:** Proofread your work. A good test is to read your work aloud (this includes reading the equations and calculations aloud). There should be complete, natural-sounding sentences. Be especially careful with pronouns. Avoid using "it" and "they" for mathematical objects; use the names of the objects (or variables) instead.
11. **Format Clearly:** Do not write one long paragraph. Separate your thoughts. Put complicated equations on a single displayed line rather than in the middle of a paragraph. Do not write too small. Do not make your reader struggle to read and understand your work.
12. **Acknowledge Collaborators:** If you worked with someone else on solving the problem, give them credit!

Consider the following problem:

? Exercise 2.3.1

The Largest Number Problem. Find the largest eight-digit number made up of the digits 1, 1, 2, 2, 3, 3, 4, and 4 such that the 1s are separated by one digit, the 2s are separated by two digits, the 3s by three digits, and the 4s by four digits.

Answer

41,312,432. Please see the discussion below.

Discussion:

Below are several sample solutions submitted by students. Using the criteria provided above, rate each solution on a scale from 1 to 5, and explain the reasons for your ratings.

- (Solution 1). 41,312,432
 - This is the largest eight-digit because the numbers 1, 2, 3, 4 are all separated by the given amount of spaces.
- (Solution 2). 41,312,432
 - With the active rules, I tried putting the highest numbers as far left as possible. Through trying different combinations, I figured out that no two consecutive numbers can be touching in the first two digits. So I instead tried starting with the 4, then 1, then 3, since I'm going for the highest number possible.
 - My answer: 41,312,432
 - You have to have the 4 in the highest place and work down from there. However, unable to follow the rules, the 2 and the 1 in the 10,000 place and 100,000 place must switch.
- (Solution 3). 41,312,432
 - First, I started with the number 4 because that is the largest digit and will make the largest number. Then I had to place the next 4 five spaces away because I knew there had to be four digits separating the two 4s. Next, I placed 1 in the second digit spot because 2 or 3 would interfere with the rule of how many digits could separate them, which allowed me to also place where the next 1 should be. I then placed the 3 because the opening spaces showed me that I could fit three digits in between the two 3s. Lastly, I placed the 2s, which worked out because there are two digits separating them.
- (Solution 4).
 - 1×1
 - 2xx2
 - 3xxx3
 - 4xxxx4
 - Answer: 41,312,432
- (Solution 5).
 - 4 3 2 4 3 2
 - 4 2 2 4
 - 4 1 3 1 4 3
 - *4 1 3 1 2 4 3 2
 - 4 needs to be the first number to make it the biggest. Then check going down from the next largest to the smallest. Ex:
 - 4 3 _____ ×
 - 4 2 _____ ×
 - 4 1 _____ ✓
- (Solution 6). 41,312,432
 - I put 4 at the 10,000,000 place because the largest number should be placed at the highest value. Numbers 2 and 3 could not be placed in the 1,000,000 place because I wasn't able to separate the digits properly. So I ended up placing the number 1 there. In the 100,000 place, I put the number 3 because it was the second-highest number.
- (Solution 7). 41,312,432
 - Since the problem asks you for the largest 8-digit number, I knew 4 had to be the first number since it's the greatest of the set. To solve the rest of the problem, I used the guess-and-test method. I tried many different combinations. First, using the number 3 as the second digit in the sequence, but came to no answer. Then the number 2, but no combination I found correctly finished the sequence. I then finished with the number 1 in the second digit in the sequence and was able to successfully fill out the entire number.
- (Solution 8).
 - 4 _ _ _ _ 4 _ _

- 4 has to be the first digit for the number to be the largest possible. That means the other 4 has to be the sixth digit in the number, because the 4s have to be separated by four digits.
- 4 _ 3 _ _ 4 3 _
- 3 must be the third digit, in order for the number to be the largest possible. 3 cannot be the second digit because the other 3 would have to be the sixth digit in the number, but 4 is already there.
- 4 1 3 1 _ 4 3 _
- 1s must be separated by one digit, so the 1s can only be the second and fourth digit in the number.
- 4 1 3 1 2 4 3 2
- This leaves the 2s to be the fifth and eighth digits.

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2.4: Arithmetic Skills and Operations

Learning Objectives

By the end of this section, you will be able to simplify expressions using the order of operations

Simplify Expressions Using the Order of Operations

To **simplify an expression** means to do all the math possible. For example, to simplify $4 \cdot 2 + 1$ we would first multiply $4 \cdot 2$ to get 8, and then add the 1 to get 9. A good habit to develop is to work down the page, writing each step of the process below the previous step. The example just described would look like this:

$$\begin{aligned} 4 \cdot 2 + 1 \\ = 8 + 1 \\ = 9 \end{aligned} \quad (2.4.1)$$

By not using an equal sign when you simplify an expression, you may avoid confusing expressions with equations.

SIMPLIFY AN EXPRESSION

To **simplify an expression**, do all operations in the expression.

We've introduced most of the symbols and notation used in algebra, but now we need to clarify the **order of operations**. Otherwise, expressions may have different meanings, and they may result in different values.

For example, consider the expression $4 + 3 \cdot 7$. Some students simplify this and get 49 by adding $4 + 3$ and then multiplying that result by 7. Others get 25 by multiplying $3 \cdot 7$ first and then adding 4.

The same expression should give the same result. So mathematicians established some guidelines that are called the order of operations. Following these rules, everyone will get 25 as a result of $4 + 3 \cdot 7$.

USE THE ORDER OF OPERATIONS.

1. Parentheses and other grouping symbols
 - Simplify all expressions inside the parentheses or other grouping symbols, working on the innermost parentheses first.
2. Exponents
 - Simplify all expressions with exponents.
3. Multiplication and division
 - Perform all multiplication and division in order from left to right. These operations have equal priority.
4. Addition and subtraction
 - Perform all addition and subtraction in order from left to right. These operations have equal priority.

Students often ask, "How will I remember the order?" Here is a way to help you remember: Take the first letter of each key word and substitute the silly phrase "Please Excuse My Dear Aunt Sally."

Parentheses	Please
Exponents	Excuse
Multiplication Division	My Dear
Addition Subtraction	Aunt Sally

(2.4.2)

It's good that "My Dear" goes together, as this reminds us that **m**ultiplication and **d**ivision have equal priority. We do not always do multiplication before division or always do division before multiplication. We do them in order from left to right.

Similarly, "Aunt Sally" goes together and so reminds us that **a**ddition and **s**ubtraction also have equal priority, and we do them in order from left to right.

? Example 2.4.10

Simplify: $18 \div 6 + 4(5 - 2)$.

15

Answer

	$18 \div 6 + 4(5 - 2)$
Parentheses? Yes, subtract first.	$18 \div 6 + 4(3)$
Exponents? No.	
Multiplication or division? Yes.	
Divide first because we multiply and divide left to right.	$3 + 4(3)$
Any other multiplication or division? Yes.	
Multiply.	$3 + 12$
Any other multiplication or division? No.	
Any addition or subtraction? Yes.	
Add.	15

? Example 2.4.11

Simplify: $30 \div 5 + 10(3 - 2)$.

Answer

16

? Example 2.4.12

Simplify: $70 \div 10 + 4(6 - 2)$.

Answer

23

When there are multiple grouping symbols, we simplify the innermost parentheses first and work outward.

? Example 2.4.13

Simplify: $5 + 2^3 + 3[6 - 3(4 - 2)]$.

13

Answer

	$5 + 2^3 + 3[6 - 3(4 - 2)]$
Are there any parentheses (or other grouping symbols)? Yes.	$5 + 2^3 + 3[6 - 3(4 - 2)]$
Focus on the parentheses that are inside the brackets. Subtract.	$5 + 2^3 + 3[6 - 3(2)]$
Continue inside the brackets and multiply.	$5 + 2^3 + 3[6 - 6]$
Continue inside the brackets and subtract.	$5 + 2^3 + 3[0]$

The expression inside the brackets requires no further simplification.

Are there any exponents? Yes. Simplify exponents.

$$5 + 8 + 3[0]$$

Is there any multiplication or division? Yes.

Multiply.

$$5 + 8 + 0$$

Is there any addition or subtraction? Yes.

Add.

$$13 + 0$$

Add.

$$13$$

? Example 2.4.14

Simplify: $9 + 5^3 - [4(9 + 3)]$.

Answer

86

? Example 2.4.15

Simplify: $7^2 - 2[4(5 + 1)]$.

Answer

1

Evaluate an Expression

In the last few examples, we simplified expressions using the order of operations. Now we'll evaluate some expressions—again following the order of operations. To **evaluate an expression** means to find the value of the expression when the variable is replaced by a given number.

📌 EVALUATE AN EXPRESSION

To **evaluate an expression** means to find the value of the expression when the variable is replaced by a given number.

To evaluate an expression, substitute that number for the variable in the expression and then simplify the expression.

Evaluate when $x = 4$: a. x^2 b. 3^x c. $2x^2 + 3x + 8$.

Answer

a.

	x^2
Replace x with 4 .	4 ²
Use definition of exponent.	$4 \cdot 4$
Simplify.	16

b.

	3^x
--	-------

Replace x with 4.	3^4
Use definition of exponent.	$3 \cdot 3 \cdot 3 \cdot 3$
Simplify.	81

c.

	$2x^2 + 3x + 8$
Replace x with 4.	$2(4)^2 + 3(4) + 8$
Follow the order of operations.	$2(16) + 3(4) + 8$
	$32 + 12 + 8$
	52

Evaluate when $x = 3$, a. x^2 b. 4^x c. $3x^2 + 4x + 1$.

Answer

- a. 9
- b. 64
- c. 40

? Example 2.4.18

Evaluate when $x = 6$, a. x^3 b. 2^x c. $6x^2 - 4x - 7$.

Answer

- a. 216
- b. 64
- c. 185

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2.5: Practice Problems

Problems for Practicing Strategies and Explaining your Work

1. Choose a number, for example, 12. List all its factors. Now do the same for 8, 9, and 16. What do you notice about how many factors these numbers have? Can you predict which kinds of numbers have an odd number of factors?
2. You build a staircase with 1 block on the first step, 2 on the second, and 3 on the third, and so on. How many blocks are needed to make a staircase with 5 steps? With 10 steps? Can you find a rule that works for any number of steps?
3. Pick any number. Add up its digits. If the sum is more than one digit, add those digits again. Keep going until you get a single digit. What do you notice when you do this for several numbers? What happens if you start with multiples of 9?
4. Can the product of two odd numbers ever be even? Can the sum of two odd numbers ever be odd? Explain your reasoning with examples or patterns you notice.
5. Your calculator's "+" key is broken. How can you add $8 + 6$ using only multiplication, subtraction, or division? Can you find more than one way?
6. Ten students sit in a row. They each have a seat number from 1 to 10. They decide to swap seats so that no one sits in their original seat. In how many different ways could that happen for smaller numbers, like 3 or 4 students? What patterns do you notice?
7. Write the numbers 1 through 9 in a 3 by 3 grid so that every row, column, and diagonal adds to the same total. What patterns do you see? Can you find more than one way to arrange the numbers?
8. Think of a two-digit number. Reverse the digits and subtract the smaller number from the larger number. Try it with a few examples. What do you notice about the results? Why do you think that happens?
9. You and a friend share a pile of 24 candies. You take 3, and your friend takes 5. Then you double what's left and repeat the sharing pattern. Will the candies ever run out? What patterns do you see after each round?
10. There are 100 lockers in a hallway, all closed. Student 1 opens every locker. Student 2 closes every second locker. Student 3 changes (opens if closed, closes if open) every third locker. This continues until Student 100 changes every 100th locker. Which lockers are open at the end? What pattern do you notice?

Problems in this section were generated using artificial intelligence to encourage the use of multiple strategies and algebraic thinking.

Order of Operation Problems

Simplify Expressions Using the Order of Operations

In the following exercises, simplify each expression.

1. $2^3 - 12 \div (9 - 5)$

Answer

5

2. $3^2 - 18 \div (11 - 5)$

3. $2 + 8(6 + 1)$

Answer

58

4. $4 + 6(3 + 6)$

5. $20 \div 4 + 6(5 - 1)$

Answer

29

6. $33 \div 3 + 4(7 - 2)$

7. $3(1 + 9 \cdot 6) - 4^2$

Answer

149

8. $5(2 + 8 \cdot 4) - 7^2$

9. $2[1 + 3(10 - 2)]$

Answer

50

9. $5[2 + 4(3 - 2)]$

10. $8 + 2[7 - 2(5 - 3)] - 3^2$

Answer

5

11. $10 + 3[6 - 2(4 - 2)] - 2^4$

Evaluate an Expression

In the following exercises, evaluate the following expressions.

12. When $x = 2$,

a. x^6

b. 4^x

c. $2x^2 + 3x - 7$

Answer

a. 64

b. 16

c. 7

13. When $x = 3$,

a. x^5

b. $5x$

c. $3x^2 - 4x - 8$

14. When $x = 4$ and $y = 1$

$x^2 + 3xy - 7y^2$

Answer

21

15. When $x = 3$ and $y = 2$

$$6x^2 + 3xy - 9y^2$$

16. When $x = 10$ and $y = 7$

$$(x - y)^2$$

Answer

$$9$$

17. When $a = 3$ and $b = 8$

$$a^2 + b^2$$

Simplify Expressions by Combining Like Terms

In the following exercises, simplify the following expressions by combining like terms.

18. $7x + 2 + 3x + 4$

Answer

$$10x + 6$$

19. $8y + 5 + 2y - 4$

20. $10a + 7 + 5a - 2 + 7a - 4$

Answer

$$22a + 1$$

21. $7c + 4 + 6c - 3 + 9c - 1$

22. $3x^2 + 12x + 11 + 14x^2 + 8x + 5$

Answer

$$17x^2 + 20x + 16$$

23. $5b^2 + 9b + 10 + 2b^2 + 3b - 4$

24. Eric has rock and country songs on his playlist. The number of rock songs is 14 more than twice the number of country songs. Let c represent the number of country songs. Write an expression for the number of rock songs.

Answer

$$2c - 14$$

25. The number of women in a Statistics class is 8 more than twice the number of men. Let m represent the number of men. Write an expression for the number of women.

26. Greg has nickels and pennies in his pocket. The number of pennies is seven less than three times the number of nickels. Let n represent the number of nickels. Write an expression for the number of pennies.

Answer

$$3n - 7$$

27. Jeannette has \$5 and \$10 bills in her wallet. The number of fives is three more than six times the number of tens. Let t represent the number of tens. Write an expression for the number of fives.

Writing Exercises

28. Why is it important to use the order of operations to simplify an expression?

29. Explain how you identify the like terms in the expression $8a^2 + 4a + 9 - a^2 - 1$.

Answer

Answers will vary.

30. Explain the difference between the phrases “4 times the sum of x and y ” and “the sum of 4 times x and y ”.

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CHAPTER OVERVIEW

3: Fractions

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3.1: What is a Fraction?

One of the things that makes fractions such a difficult concept to learn is that you have to think about them in a lot of different ways, depending on the problem at hand. For now, we are going to think of a fraction as the answer to a division problem.

The Pies per Child Model

Suppose 6 pies are to be shared equally among 3 children. This yields 2 pies per kid. We write

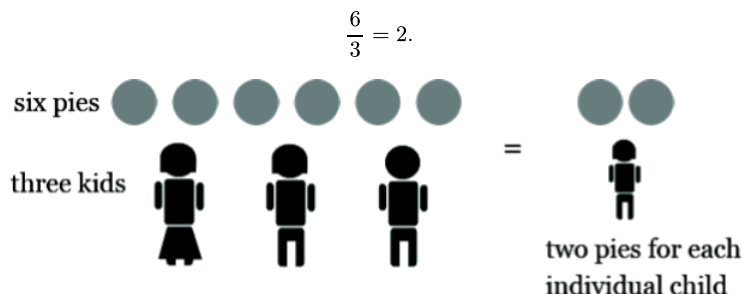


Figure 3.1.1: The pies per child model for six pies and three kids.

The fraction $\frac{6}{3}$ is equivalent to the division problem $6 \div 3 = 2$. It represents the number of pies one whole child receives when three kids share six pies equally.

In the same way:

- Sharing 10 pies among 2 kids yields $\frac{10}{2} = 5$ pies per kid.
- Sharing 8 pies among 2 kids yields $\frac{8}{2} = 4$ pies per kid.
- Sharing 5 pies among 5 kids yields $\frac{5}{5} = 1$ pie per kid.
- Sharing 1 pie among 2 kids yields $\frac{1}{2}$, which we call “one half.”

Notice that sharing 1 pie among 2 kids introduces the concept of fractions less than one. If one pie is shared *equally* between two kids, then each child receives a portion of the pie, which we choose to call “half.”



Figure 3.1.1: The pies per child view for one pie and two kids.

Thus, you can associate the number $\frac{1}{2}$ to the picture of a circle split in two parts with one part shaded. When you split a circle into two equal parts, each part has a size of $\frac{1}{2}$.

In the same way, if you split a circle (a whole) into three equal parts, each part is a third. By shading one of those parts, you represent one third, that is, $\frac{1}{3}$. (And this is indeed the amount of pie an individual child would receive if one pie is shared among three.)

If you split a circle (a whole) into five equal parts, each part is a fifth. By shading one of those parts, you represent one fifth, that is, $\frac{1}{5}$. This is the amount of pie an individual receives if one pie is shared among five kids. Notice that three fifths is three parts of size one fifth and is written as $\frac{3}{5}$, the amount of pie an individual receives if three pies are shared among five children.

Discussion:

Carefully explain why this is true: If five kids share three pies equally, each child receives an amount that looks like this:

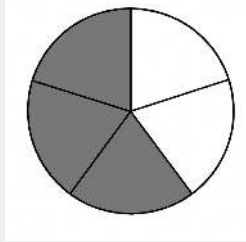


Figure \(\backslash\)
PageIndex

{1}): Amount of pie each child would receive. (This image was AI generated)

Your explanation will probably require both words and pictures.

Definition: Pies per Child Model

In our model, a fraction $\frac{a}{b}$ represents the amount of pie an individual child receives when a pies are shared equally by b kids.

$$\begin{array}{lcl} \text{\#pies} & \rightarrow & a \\ \text{\#kids} & \rightarrow & \frac{a}{b} = \text{amount per individual child} \end{array}$$

Discussion:

- What is $\frac{2}{2}$? What is $\frac{7}{7}$? What is $\frac{100}{100}$? How can you use the Pies per Child Model to make sense of $\frac{a}{a}$ for any positive whole number a ?
- What is $\frac{2}{1}$? What is $\frac{7}{1}$? What is $\frac{1876}{1}$? How can you use the Pies per Child Model to make sense of $\frac{b}{1}$ for any positive whole number b ?
- Write the answer to this division problem: “I have no pies to share among thirteen kids.” How can you generalize this division problem to make a general statement about fractions?

Definition: Fraction $\frac{a}{b}$

For a fraction $\frac{a}{b}$, the top number a (which, for us, is the number of pies) is called the **numerator** of the fraction, and the bottom number b (the number of kids), is called the **denominator** of the fraction.

Most people insist that the numerator and denominator each be whole numbers, but they do not have to be. To understand why the numerator and denominator need not be whole numbers, we must first be a little gruesome. Instead of dividing pies, let’s divide kids! Here is one kid:



Figure 3.1.1: A figure of a kid.

Using the Pie per Child Model, the fraction $\frac{1}{\left(\frac{1}{2}\right)}$ would represent one pie to each “group” of half a child. So how much would a whole child receive? Well, we would have a picture like this:

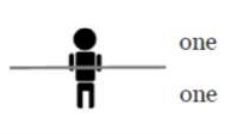


Figure 3.1.1: A figure of a kid "split" in half.

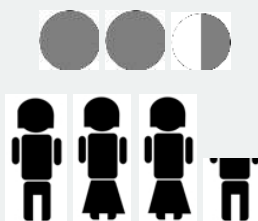
Notice that in this context, the whole child gets two pies, which can be justified by the following process:

$$\frac{1}{\left(\frac{1}{2}\right)} = 2.$$

Discussion:

Draw pictures for these problems if it helps!

1. What does $\frac{1}{\left(\frac{1}{3}\right)}$ represent? Justify your answer using the Pies per Child Model.
2. What is $\frac{1}{\left(\frac{1}{6}\right)}$? Justify your answer.
3. Explain why the fraction $\frac{5}{\left(\frac{1}{2}\right)}$ represents the number 10. (How much pie is given to half a kid? To a whole kid?)
4. What is $\frac{4}{\left(\frac{1}{3}\right)}$? Justify your answer.
5. **Challenge:** Two-and-a-half pies are to be shared equally among four-and-a-half children. How much pie does an individual (whole) child receive? Justify your answer.



Fractions Less than One, Greater than One, and Mixed Numbers

A fraction with a numerator smaller than its denominator is called a fraction less than one. In school math jargon, it is sometimes called a *proper fraction*. For example, $\frac{45}{58}$ is "proper."

A fraction with a numerator larger than its denominator is called a fraction greater than one. In school math jargon, it is sometimes called an *improper fraction*. For example, $\frac{7}{3}$ is "improper." Despite their name and these prejudices, improper fractions are useful nonetheless!

You can write an improper fraction as a combination of a whole number and a proper fraction, which is often called a "mixed number."

With a mixed number, you have a good sense of the overall size of the number: "a little more than five," or "a bit less than 17." But it is often easier to do calculations with improper fractions instead of using the mixed numbers. Let's explore some examples.

✓ Example: $\frac{7}{3}$

If seven pies are shared among three kids, then each kid will certainly receive two whole pies, leaving one pie to share among the three children.

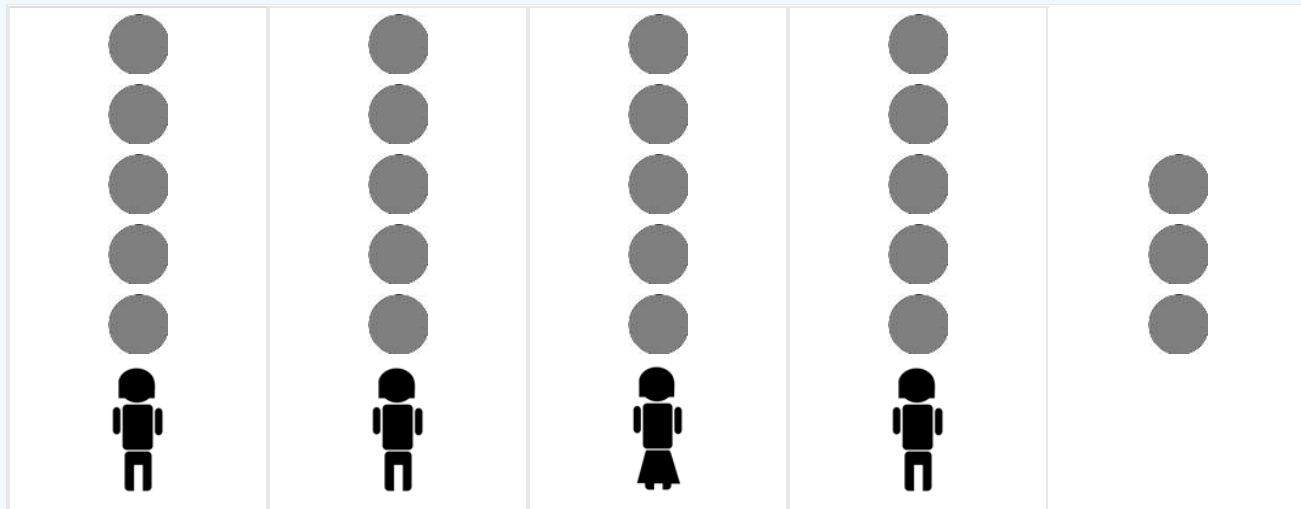




Thus, $\frac{7}{3}$ equals 2 plus $\frac{1}{3}$. People write: $\frac{7}{3} = 2\frac{1}{3}$ and call the result a *mixed number*. One can also write: $2 + \frac{1}{3}$ which is what $2\frac{1}{3}$ really means. But most people choose to omit the plus sign.

✓ Example: $\frac{23}{4}$

If 4 kids share 23 pies, we can give them each 5 whole pies. That uses 20 pies, and there are 3 pies left over.



Those three pies are still to be shared equally by the four kids. We have:

$$\frac{23}{4} = 5\frac{3}{4}.$$

✓ Example: $2\frac{1}{5}$

For fun, let's write the number 2 as a fraction with denominator 5:

$$2 = \frac{10}{5}.$$

So:

$$2\frac{1}{5} = 2 + \frac{1}{5} = \frac{10}{5} + \frac{1}{5} = \frac{11}{5}.$$

We have written the mixed number $2\frac{1}{5}$ as the improper fraction $\frac{11}{5}$.

Discussion:

- Write each of the following as a mixed number. Explain how you got your answer. $\frac{17}{3}$, $\frac{8}{5}$, $\frac{100}{3}$, $\frac{200}{199}$.
- Convert each of these mixed numbers into “improper” fractions. Explain how you got your answer. $3\frac{1}{4}$, $5\frac{1}{6}$, $1\frac{3}{11}$, $200\frac{1}{200}$.

Students are often asked to memorize the names “proper fractions,” “improper fractions,” and “mixed number” so that they can follow directions on tests and problem sets.

But, to a mathematician, these names are not at all important! There is no “correct” way to express an answer (assuming, that the answer is mathematically the right number). We often wish to express our answer in a simpler form, but sometimes the context will tell you what form is “simple” and what form is more complicated.

As you work on problems in this chapter, decide for yourself which type of fraction would be best to work with as you do your task.

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3.2: The Key Fraction Rule

In the previous section, we defined $\frac{a}{b}$ as the amount of pie an individual kid receives when a pies are shared equally by b kids. In this view, the fraction $\frac{a}{b}$ is the answer to a division problem.

Discussion:

What happens if we double the number of pies and double the number of kids?

Notice that the amount of pie per child is still the same:

$$\frac{2a}{2b} = \frac{a}{b}.$$

For example, in the figure below, $\frac{6}{3}$ and $\frac{12}{6}$ both give two pies for each child.

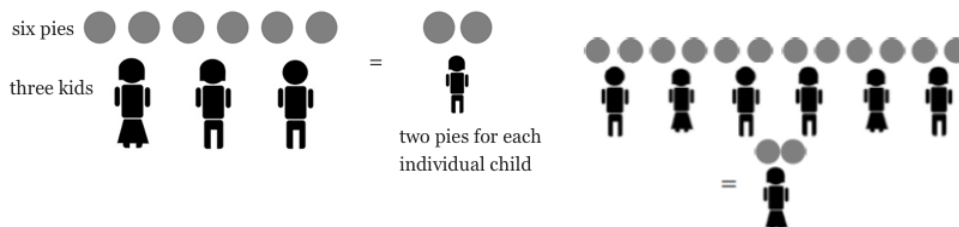


Figure 3.2.1: Representation of $\frac{6}{3}$ and $\frac{12}{6}$.

Tripling the number of pies and the number of kids also does not change the final amount of pies per child, nor does quadrupling each number, or one trillion-billion-tupling the numbers!

$$\frac{6}{3} = \frac{12}{6} = \frac{18}{9} = \dots = \text{two pies per child.}$$

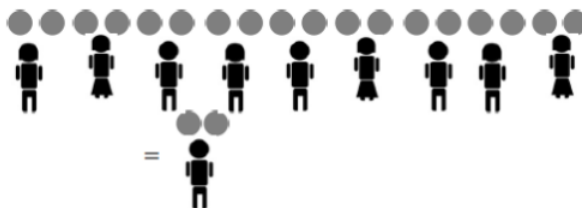


Figure 3.2.1: Representation of $\frac{18}{9}$

We call the generalization of this result the Key Fraction Rule.

Definition: Key Fraction Rule and Equivalent Fractions

$$\frac{xa}{xb} = \frac{a}{b},$$

(at least for positive whole numbers x).

We say that $\frac{xa}{xb}$ and $\frac{a}{b}$ are **equivalent fractions**.

For example,

$$\frac{3}{5} \text{ (sharing three pies among five kids)}$$

yields the same result as

$$\frac{3 \cdot 2}{5 \cdot 2} = \frac{6}{10} \text{ (sharing six pies among ten kids)}$$

and as

$$\frac{3 \cdot 100}{5 \cdot 100} = \frac{300}{500} \text{ (sharing 300 pies among 500 kids).}$$

Discussion:

Determine equivalent fractions for $\frac{1}{2}$, for $\frac{10}{3}$, and for 1.

$$\frac{20}{32} \text{ (sharing 20 pies among 32 kids)}$$

is the same problem as:

$$\frac{5 \cdot 4}{8 \cdot 4} = \frac{5}{8} \text{ (sharing five pies among eight kids).}$$

We have made the numerator and denominator as small as possible. We call this process **reducing** a fraction to its lowest terms, or **simplifying** a fraction.

✓ Example: Reducing a Fraction to Its Lowest Terms

Consider the fraction $\frac{280}{350}$. Notice that there is a common factor of 10 in both the numerator and the denominator:

$$\frac{280}{350} = \frac{28 \cdot 10}{35 \cdot 10} = \frac{28}{35}.$$

We can go further as 28 and 35 are both multiples of 7:

$$\frac{28}{35} = \frac{4 \cdot 7}{5 \cdot 7} = \frac{4}{5}.$$

Thus, sharing 280 pies among 350 children gives the same result as sharing 4 pies among 5 kids.

$$\frac{280}{350} = \frac{4}{5}.$$

Since 4 and 5 share no common factors, this is as far as we can go with this example to keep both the numerator and denominator whole numbers.

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3.3: Adding and Subtracting Fractions

Consider the fractions: $\frac{2}{7}$ and $\frac{3}{7}$. What might it mean to add them? It might seem reasonable to say:

$\frac{2}{7}$ represents 2 pies shared by 7 kids.

$\frac{3}{7}$ represents 3 pies shared by 7 kids.

So maybe $\frac{2}{7} + \frac{3}{7}$ represents 5 pies among 14 kids, giving the answer $\frac{5}{14}$. It is very tempting to say that “adding fractions” means “adding pies and adding kids.”

The trouble is that a fraction is not a pie, and a fraction is not a kid. So adding pies and adding children is not actually adding fractions. A fraction is something different. It is related to pies and kids, but something more subtle. A fraction is an **amount of pie per child**.

We cannot add pies or children. Instead, we must add the amounts each kid receives.

✓ Example 3.3.1

Solve $\frac{2}{7} + \frac{3}{7}$ by using the **amount of pie per child** view.

Solution

Let us take it slowly. Consider the fraction $\frac{2}{7}$. The figure below shows the amount an individual child receives when two pies are given to seven kids:



Consider the fraction $\frac{3}{7}$. The figure below shows the amount an individual child receives when three pies are given to seven children:



The sum $\frac{2}{7} + \frac{3}{7}$ corresponds to the sum:



The answer is then $\frac{5}{7}$. Which means one child gets $\frac{5}{7}$ of a pie, which is equivalent to the amount of pie that one child gets when five pies are shared by seven children.

Discussion:

From the previous example, $\frac{5}{7}$ means “the amount of pie that one child gets when five pies are shared by seven children.” Carefully explain **why** this result makes sense. Your explanation should use both words and representations.

Another way to understand the addition of fractions is to use statements like “two sevenths plus three sevenths gives five sevenths.” Adding fractions with the same denominator is the same as adding a number of pieces of the same size. For example, if you add 4, 3, and 8 pieces all of size $\frac{1}{10}$, then you have: 4 tenths + 3 tenths + 8 tenths = 15 tenths.

$$\frac{4}{10} + \frac{3}{10} + \frac{8}{10} = \frac{15}{10}.$$

82 sixty-fifths + 91 sixty-fifths = 173 sixty-fifths:

$$\frac{82}{65} + \frac{91}{65} = \frac{173}{65}.$$

We can also use the **Pies per Child Model** to explain why adding fractions with like denominators works the way it does.

✓ Example 3.3.1

$$\frac{2}{7} + \frac{3}{7}$$

Solution

To represent $\frac{2}{7} + \frac{3}{7}$ using the Pies per Child Model:

$$\begin{array}{r} \text{amount of pie each kid gets when 7 kids share 2 pies} \\ + \text{amount of pie each kid gets when 7 kids share 3 pies} \\ \hline ? \end{array}$$

Since in both cases we have 7 kids sharing the pies, we can imagine that it is the same 7 kids in both cases. First, they share 2 pies. Then they share 3 more pies. The total each child gets by the time all the pie-sharing is done is the same as if the 7 kids had just shared 5 pies to begin with. That is:

$$\begin{array}{r} \text{amount of pie each kids gets when 7 kids share 2 pies} \\ + \text{amount of pie each kids gets when 7 kids share 3 pies} \\ \hline \text{amount of pie each kids gets when 7 kids share 5 pies.} \end{array}$$

$$\frac{2}{7} + \frac{3}{7} = \frac{5}{7}.$$

Now, let's think about the general case. Our claim is that

$$\frac{a}{d} + \frac{b}{d} = \frac{a+b}{d}.$$

Translating into our model, we have d kids. First, they share a pies between them, and $\frac{a}{d}$ represents the amount each child gets. Then they share b more pies, so the additional amount of pie each child gets is $\frac{b}{d}$. The total each kid gets is $\frac{a}{d} + \frac{b}{d}$.

But it does not really matter that the kids first share a pies and then share b pies. The amount each child gets is the same as if they had started with all of the pies — all $a+b$ of them — and shared them equally. That amount of pie is represented by $\frac{a+b}{d}$.

Discussion:

- How can you *subtract* fractions with the same denominator? For example, what is $\frac{400}{903} - \frac{170}{903}$?
- Use the Pies per Child Model to *carefully explain why* $\frac{a}{d} - \frac{b}{d} = \frac{a-b}{d}$.
- Explain why the fact that the denominators are the same is *essential* to this addition and subtraction method. Where is that fact used in the explanations?

Fractions with Different Denominators

This approach to adding fractions suddenly becomes tricky when the denominators are not the same. For example, what is $\frac{2}{5} + \frac{1}{3}$?

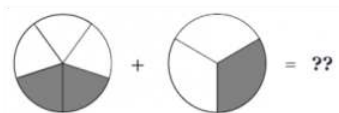


Figure 3.3.1: Representation of $\frac{2}{5} + \frac{1}{3} = ?$

Consider the following discussion to explore the fraction using the concept of pies per kids.

Discussion:

Suppose Félix is part of a team of five kids that shares two pies. Then later, Félix is part of a team of three kids that shares one pie. How much pie does Félix receive in total?

Discuss the meaning of this problem and possible solutions to it. Do you see that this is the same problem as computing $\frac{2}{5} + \frac{1}{3}$? What might be the best approach to answering the problem?

One way to think about answering this addition question is to write $\frac{2}{5}$ in a series of alternative forms using our key fraction rule. For example, multiply the numerator and denominator each by 2, and then each by 3, and then each by 4, and so on, and to do the same for $\frac{1}{3}$:

$$\begin{aligned}\frac{2}{5} + \frac{1}{3} &= \frac{4}{10} + \frac{2}{6} \\ &= \frac{6}{15} + \frac{3}{9} \\ &= \frac{8}{20} + \frac{4}{12} \\ &= \frac{10}{25} + \frac{5}{15} \\ &= \vdots \quad \vdots\end{aligned}$$

We see that the problem $\frac{2}{5} + \frac{1}{3}$ is actually the same as $\frac{6}{15} + \frac{5}{15}$. So we can find the answer using the same-denominator method:

$$\frac{2}{5} + \frac{1}{3} = \frac{6}{15} + \frac{5}{15} = \frac{11}{15}.$$

✓ Example 3.3.1

$$\frac{3}{8} + \frac{3}{10}$$

Solution

The fractions $\frac{3}{8}$ and $\frac{3}{10}$ have different denominators. To provide some context, imagine Félix is part of a group of 8 kids who share 3 pies. Later, Félix is part of a group of 10 kids who share 3 different pies. How much total pie did Félix get?

$$\begin{aligned}\frac{3}{8} + \frac{3}{10} &= \frac{6}{16} + \frac{6}{20} \\ &= \frac{9}{24} + \frac{9}{30} \\ &= \frac{12}{32} + \frac{12}{40} \\ &= \frac{15}{40} + \frac{15}{50} \\ &= \vdots \quad \vdots\end{aligned}$$

$$\frac{3}{8} + \frac{3}{10} = \frac{15}{40} + \frac{12}{40} = \frac{17}{40}.$$

So Félix got $\frac{17}{40}$ of a pie.

Of course, you do not need to list all of the equivalent forms of each fraction in order to find a common denominator. If you can see a denominator right away, go for it!

Discussion:

Cassie suggests the following method for the example above:

When the denominators are the same, we just add the numerators. So when the numerators are the same, shouldn't we just add the denominators? Like this:

$$\frac{3}{8} + \frac{3}{10} = \frac{3}{18}.$$

What do you think of Cassie's suggestion? Does it make sense?

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3.4: The Whole as a Unit- Fraction Comparison and Arithmetic Sequences

So far, we have been thinking about a fraction as the answer to a division problem. For example, $\frac{2}{3}$ is the result of sharing two pies among three children.

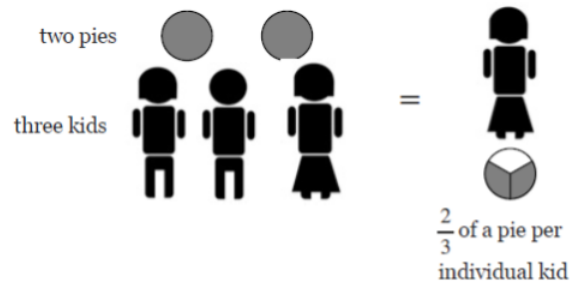


Figure 3.4.1: Representation of two pies for three kids.

Of course, pies do not have to be round. We can have square pies, or triangular pies or squiggly pies or any shape you please.

This **Pies per Child Model** has served us perfectly well in thinking about the meaning of fractions, equivalent fractions, and even adding and subtracting fractions.

However, there is no way to use this model to make sense of multiplying fractions! What would this mean?

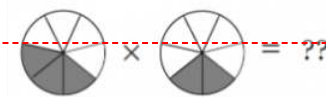
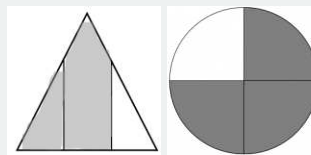


Figure 3.4.1: Incorrect model for multiplication.

So what *are* fractions, if we are asked to multiply them? We are forced to switch models and think about fractions in a new way.

Discussion:

Developing a conceptual understanding of fractions is fundamental for solving problems at any level. For example, a bad representation can lead to inaccurate answers, such as justifying that $\frac{2}{3} > \frac{3}{4}$ with the following representations.



Discuss what issues you see these representations have, and how this can help you think about the definition of a fraction.

Similarly, someone can fail to understand that when two rectangles of the same area are split in three equal parts, then each part is a third. This must ensure the pieces have equal area, but not necessarily the same shape. For example, the following rectangles are both representing $\frac{1}{3}$.





While it is easier to conclude that both figures represent $\frac{1}{3}$, it could be confusing to conclude that both figures represent the same area. Why do you think someone might struggle with this concept?

The following views of fractions will provide conceptual approaches to reflect on these questions.

The Whole as the Unit

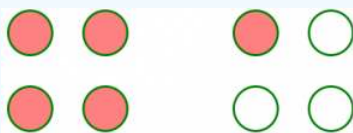
In thinking about fractions, it is important to remember that there are always units attached to a fraction, even if the units are hidden. If you see the number $\frac{1}{2}$ in a problem, you may ask yourself, “Half of what?” The answer to that question is your unit, the amount that equals 1.

So far, our **units** have been consistent: **the whole** (or unit) was a whole pie, and fractions were represented by pies cut into equal-sized pieces. But this is just a model, and we can take anything, cut it into equal-sized pieces, and talk about fractions of that whole.

One thing that can make fraction problems so difficult is that the fractions in the problem may be given in different units, which are known as different parts of different wholes.

✓ Example 3.4.1

"Which number is shown by the shaded region?"



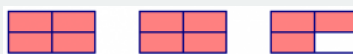
Solution

Because there is no clarity on what the whole is, the answers will vary. Some correct answers are:

- 5, if we consider each circle to be a whole (1).
- $2\frac{1}{2}$, if we consider two circles to be a whole (1).
- $\frac{5}{8}$, if we consider eight circles to be a whole (1).
- $1\frac{1}{4}$, if we consider four circles to be a whole (1).

Discussion:

Consider the following figure:



- If the shaded region represents $3\frac{2}{3}$, what is the whole?
- Find three other numbers that could be represented by the shaded region, and explain what the whole is for each answer.

Fraction Comparison and Number Line Representation

If we think of fractions as portions of a segment, we can talk about their location on a number line. To locate a fraction on a number line, the unit is the distance between 0 and 1.



Figure 3.4.1: The whole or unit when working with a fraction is the distance between 0 and 1.

This measurement model makes it much easier to tackle questions about the relative size of fractions based on where they are on the number line. We can mark off different fractions as parts of the unit segment. Just as with whole numbers, fractions that appear farther to the right are larger.



Figure 3.4.1: Representation of $\frac{3}{5}$ and $\frac{5}{8}$ in a number line

Notice that $\frac{3}{5}$ and $\frac{5}{8}$ are very close, but $\frac{5}{8}$ is bigger. To compare fractions, we can use intuitive methods such as benchmarks, relying on intuition to determine how a fraction compares to a number you know well.

Benchmark Method:

A fraction is greater than 1 if its numerator is greater than its denominator. This is a consequence of the denominator representing how many pieces are in one whole (one unit), and the numerator representing how many pieces are in your portion. So if the numerator is bigger, that means you have more than the number of pieces needed to make one whole.

Another common benchmark is $\frac{1}{2}$. A fraction is greater than $\frac{1}{2}$ if the numerator is more than half the denominator. It is equivalent to check that a fraction is greater than $\frac{1}{2}$ if twice the numerator is bigger than the denominator.

Discussion:

Organize these fractions from smallest to largest using *benchmarks*: 0 to $\frac{1}{2}$, $\frac{1}{2}$ to 1, and greater than 1. Justify your choices.

$$\frac{25}{23}, \frac{4}{7}, \frac{17}{35}, \frac{2}{9}, \frac{14}{15}.$$

You probably used your creative thinking to organize these numbers by considering the relative sizes of fractions. The next discussion allows you to expand your problem-solving skills by exploring different strategies before we officially introduce them. Use your creative thinking to solve the problem in the discussion.

Discussion:

Arrange each group of fractions in *ascending order*. Keep track of your thinking and your methods.

- $\frac{7}{17}, \frac{4}{17}, \frac{12}{17}.$
- $\frac{5}{6}, \frac{10}{11}, \frac{2}{3}.$

Methods Using Denominators or Numerators Comparison:

If two fractions have the **same denominator**, look at the numerators—in this case, the larger numerator appears in the larger fraction. For example, $\frac{5}{7} < \frac{6}{7}$. Why? Well, the pieces are the same size since the denominators are the same. If you have more pieces of the same size, you have a bigger number.

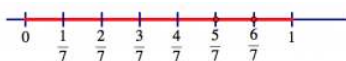


Figure 3.4.1: Number line representation: $\frac{5}{7} < \frac{6}{7}$ using the same denominator for the comparison.

If the fractions have the **same numerators**, pay attention to the denominator—the larger denominator actually makes the smaller fraction. For example, $\frac{3}{4} > \frac{3}{5}$. Notice that $\frac{3}{4}$ and $\frac{3}{5}$ both have the same number of pieces, but the $\frac{3}{4}$ are bigger pieces.

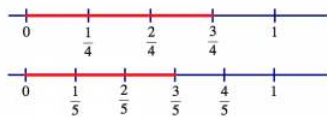


Figure 3.4.1: Number line representation: $\frac{3}{4} > \frac{3}{5}$ using the same numerator for the comparison.

If the **numerator is the denominator**–1, you can easily compare them. The fractions will be in the same order as the denominators. Think of each fraction as a pie with one piece missing. The greater the denominator, the smaller the missing piece, so the greater the amount remaining. For example, $\frac{6}{7} < \frac{10}{11}$, since $\frac{6}{7} = 1 - \frac{1}{7}$ and $\frac{10}{11} = 1 - \frac{1}{11}$.

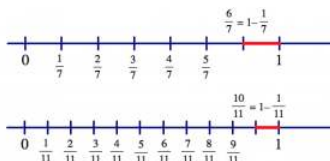


Figure 3.4.1: Number line representation $\frac{6}{7} < \frac{10}{11}$ using the numerator is the denominator –1 for the comparison.

You can extend the test above to fractions whose **numerators are the denominator minus a constant**. The fractions will again be in the same order as the denominators, for exactly the same reason. For example, $\frac{3}{7} < \frac{7}{11}$, because both are four “pieces” less than one whole, and the $\frac{1}{11}$ pieces are smaller than the $\frac{1}{7}$ pieces.

Equivalent fractions:

Using the denominator or numerator comparisons is not always possible, so you may need to find equivalent fractions of the given fractions before you use these comparisons. To find equivalent fractions, you can use the Key Fraction Rule.

Definition: Key Fraction Rule

$$\frac{xa}{xb} = \frac{a}{b},$$

(at least for positive whole numbers x).

We say that $\frac{xa}{xb}$ and $\frac{a}{b}$ are equivalent fractions.

(at least for positive whole numbers).

We say that and are **equivalent fractions**.

Arithmetic Sequences

Consider the following pattern:

5, 8, 11, 14, 17, 20, 23, 26, ...

Discussion:

Work with a classmate to answer the following questions:

- Can you predict the next five numbers of the pattern above?
- Can you predict the 100th number?
- Describe the pattern in words.

The pattern above is an **arithmetic sequence**, because you can get the next number by adding the same amount each time.

Definition: Arithmetic Sequences

An **arithmetic sequence** is a sequence of numbers where the difference between consecutive terms is a constant.

✓ Example 3.4.1

Consider the following sequences:

A. 1, 2, 3, 4, 5, ...

B. 2, 4, 6, 8, 10, ...

C. $\frac{1}{3}, 1, \frac{5}{3}, \frac{7}{3}, 3, \dots$

Verify that these are arithmetic sequences. How do you know?

Solution

To verify that these are arithmetic sequences, we need to find the difference between the consecutive terms.

Pattern A:

$$\begin{array}{ccccccc} \underbrace{1,} & \underbrace{2,} & \underbrace{3,} & \underbrace{4,} & 5, & \dots \\ +1 & +1 & +1 & +1 & & \end{array}$$

Pattern B:

$$\begin{array}{ccccccc} \underbrace{2,} & \underbrace{4,} & \underbrace{6,} & \underbrace{8,} & 10, & \dots \\ +2 & +2 & +2 & +2 & & \end{array}$$

Pattern C:

$$\begin{array}{ccccccc} \underbrace{\frac{1}{3},} & \underbrace{1,} & \underbrace{\frac{5}{3},} & \underbrace{\frac{7}{3},} & 3, & \dots \\ +\frac{2}{3} & +\frac{2}{3} & +\frac{2}{3} & +\frac{2}{3} & & \end{array}$$

Discussion:

Make up two fraction sequences of your own, one that is an arithmetic sequence and one that is *not* an arithmetic sequence.

Exchange your sequences with a partner, but do not tell your partner which is which.

When you get your partner's sequences, decide which is an arithmetic sequence and which is not. Check if you and your partner agree.

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3.5: Multiplying Fractions

Area Model

We can represent multiplication of numbers by using area models. For example, the product 23×37 is the area (number of 1×1 squares) of a 23-by-37 rectangle:

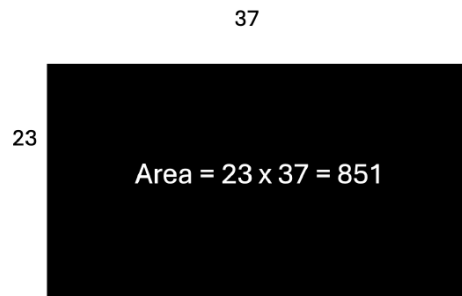


Figure 3.5.1: The product of 23×37 represented in an area model.

So the product of two fractions, say, $\frac{4}{7} \times \frac{2}{3}$ should also correspond to an area problem. The figures below illustrate the process of connecting the multiplication $\frac{4}{7} \times \frac{2}{3}$ with an area model.

Figure 3.5.1: Step 1: Draw a segment of some length; we call this a 1 unit.

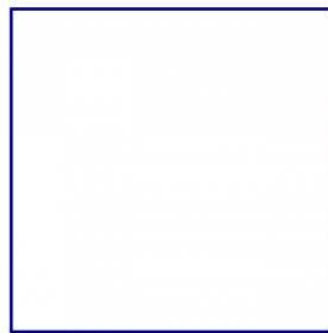


Figure 3.5.1: Step 2: Draw a square with one unit on each side.

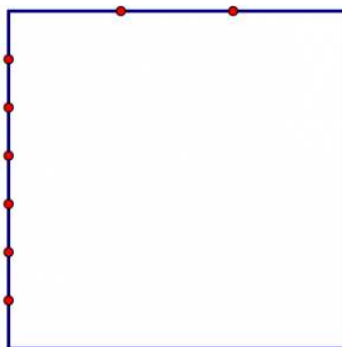


Figure 3.5.1: Step 3: Mark the square. The segment on the top represents a segment split into 3 parts, each of size $\frac{1}{3}$ and the segment on the side represents 7 pieces of size $\frac{1}{7}$.

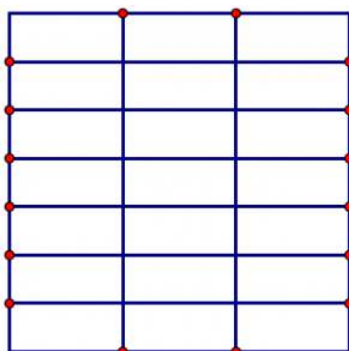


Figure 3.5.1: Step 4: Draw rectangles with one side that measures $\frac{1}{3}$ and another side that measures $\frac{1}{7}$.

The area of the square is $1 \times 1 = 1$ square unit. The marks in the square divide the whole square into small, equal-sized rectangles. Resulting in 21 smaller rectangles, each representing the multiplication $\frac{1}{7} \times \frac{1}{3}$.

We can now mark off four sevenths on one side and two thirds on the other side. This supports the appropriate shading for the rectangles to solve the multiplication $\frac{4}{7} \times \frac{2}{3}$

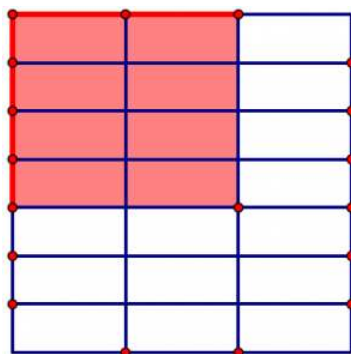


Figure 3.5.1: $\frac{4}{7} \times \frac{2}{3}$

The result of the multiplication should be the area of the rectangle with $\frac{4}{7}$ on one side and $\frac{2}{3}$ on the other. What is that area?

Remember, the whole square was one unit. That one-unit square is divided into 21 equal-sized pieces, and the shaded area contains eight of those rectangles. Since the shaded area is the answer to our multiplication problem, we conclude that

$$\frac{4}{7} \times \frac{2}{3} = \frac{8}{21}.$$

Discuss:

1. The area problem $\frac{4}{7} \times \frac{2}{3}$ yielded a diagram with a *total* of 21 small rectangles. Explain why 21 appears as the total number of equal-sized rectangles.
2. The area problem $\frac{4}{7} \times \frac{2}{3}$ yielded a diagram with 8 small *shaded* rectangles. Explain why 8 appears as the number of shaded rectangles.
3. How can you extend the area model for fractions greater than 1? Try to draw a picture for: $\frac{3}{4} \cdot \frac{3}{2}$.

Explaining the Multiplication Rule



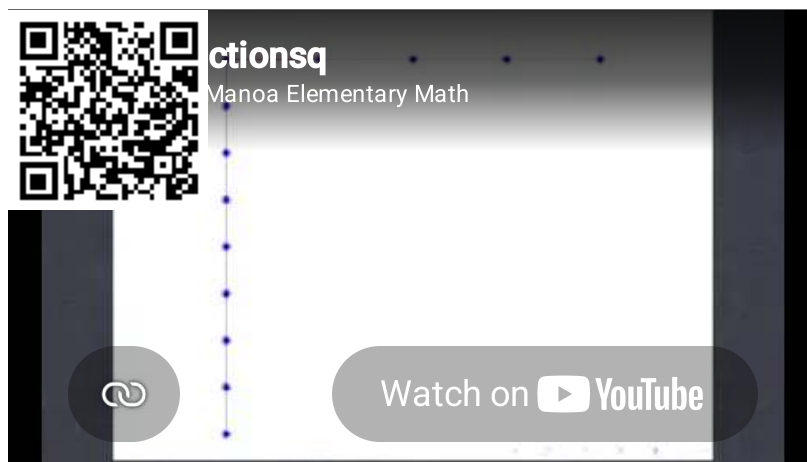
Definition: Multiplication of Fractions

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}.$$

Of course, you may then choose to simplify the final answer, but the answer is always **equivalent** to this one. Why? The area model can help us explain what is going on.

First, let us clearly write out how the area model says to multiply $\frac{a}{b} \cdot \frac{c}{d}$. We want to build a rectangle where one side has length $\frac{a}{b}$ and the other side has length $\frac{c}{d}$. We start with a square, one unit on each side.

- Divide the top segment into b equal-sized pieces. Shade a of those pieces. This will be the side of the rectangle with length $\frac{a}{b}$.
- Divide the left segment into d equal-sized pieces. Shade c of those pieces. This will be the side of the rectangle with length $\frac{c}{d}$.
- Divide the whole rectangle according to the tick marks on the sides, making equal-sized rectangles.
- Shade the rectangle bounded by the shaded segments.



If the answer is $\frac{a \cdot c}{b \cdot d}$, that means there are $b \cdot d$ total equal-sized pieces in the square, and $a \cdot c$ of them are shaded. We can see from the model why this is the case:

- The top segment was divided into b equal-sized pieces. So there are b columns in the rectangle.
- The side segment was divided into d equal-sized pieces. So there are d rows in the rectangle.
- A rectangle with b columns and d rows has $b \cdot d$ pieces, which is the area model for whole-number multiplication!

Discussion:

Consider the Definition of the Multiplication of Fraction:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}.$$

Write a clear explanation for why $a \cdot c$ of the small rectangles will be shaded.

Multiplying Fractions by Whole Numbers

To multiply $2 \cdot \frac{3}{7}$, you may think of “2” as $\frac{2}{1}$, the **whole number with a denominator of one**, and compute this way

$$2 \cdot \frac{3}{7} = \frac{2}{1} \cdot \frac{3}{7} = \frac{2 \cdot 3}{1 \cdot 7} = \frac{6}{7}.$$

We can also think in terms of our original **Pies per Child Model** to answer questions like this. In this case, we know that $\frac{3}{7}$ means the amount of pie each child gets when 7 children evenly share 3 pies.

If we compute $2 \cdot \frac{3}{7}$, that means we double the amount of pie each kid gets. We can do this by doubling the number of pies. So the answer is the same as $\frac{6}{7}$: the amount of pie each child gets when 7 children evenly share 6 pies.

Finally, we can think in terms of the **whole as a unit**. The fraction $\frac{3}{7}$ means that I have 7 equal pieces of a whole, and I take 3 of them. So $2 \cdot \frac{3}{7}$ means do that twice. If I take 3 pieces and then 3 pieces again, I get a total of 6 pieces. There are still 7 equal pieces in the whole, so the answer is $\frac{6}{7}$.

Discussion:

1. Use all three methods to explain how to find the product: $3 \cdot \frac{2}{5}$
2. Compare these different ways of thinking about fraction multiplication. Are any of them more natural to you? Does one make more sense than the others? Do the particular numbers in the problem affect your answer? Compare your thoughts with what others think in your class. Do they agree?

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3.6: Dividing Fractions

You are probably familiar with the computational rule on how to divide fractions: to divide by a fraction, multiply by its reciprocal, which is also known as the **invert and multiply** rule. In this section, we provide context to understand why this rule works. Does it really make sense? Can you explain why it makes sense?

Definition: Invert an Multiply Rule

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$$

We are going to build up to the **invert and multiply rule**, but along the way, we'll find some more meaningful ways to understand division of fractions. So please play along: *pretend that you don't already know the invert and multiply rule*, and solve the problems in this chapter with other methods.

Equal-Sized Groups Approach

The division $18 \div 3$ means: How many groups of 3 can I find in 18?

We start with 18 dots, and we make groups of 3 dots. We ask: How many groups can we make?

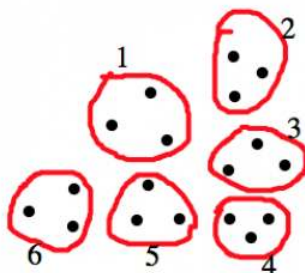


Figure 3.6.1: 18 dots, split into groups of 3 dots. Since there are 6 groups, we have $18 \div 3 = 6$.

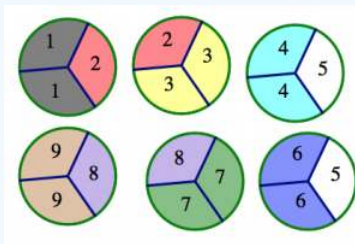
This same idea applies when we divide fractions. Consider the following example. For example, $6 \div \frac{2}{3}$ means: How many groups of $\frac{2}{3}$ can I find in 6?

✓ Example 3.6.1

How many groups of size $\frac{2}{3}$ can I find in 6?

Solution

Let's draw a picture of 6 pies, and see how many groups of $\frac{2}{3}$ we can find:



We found nine equal groups of size $\frac{2}{3}$, so we conclude that

$$6 \div \frac{2}{3} = 9.$$

Unfortunately, it's not always quite so straightforward to find the equal groups. For example, $\frac{3}{4} \div \frac{1}{3}$ asks the question: How many groups of $\frac{1}{3}$ can I find in $\frac{3}{4}$?

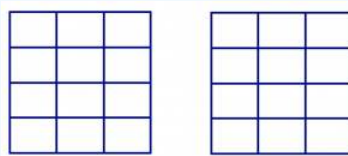
Here's a method that lets you compute exactly. We'll use rectangular pies and divide them into rows and columns based on the denominators of the numbers we're dividing.

✓ Example 3.6.1

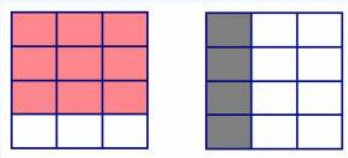
✓ Represent the division: $\frac{3}{4} \div \frac{1}{3}$

Solution

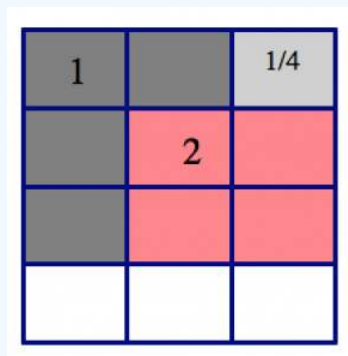
Start by drawing two identical rectangles, each with 4 rows (from the denominator of $\frac{3}{4}$ and 3 columns (from the denominator of $\frac{1}{3}$).



Shade $\frac{3}{4}$ of the first rectangle (this is exactly three rows), and shade $\frac{1}{3}$ of the second rectangle (so that's one column).



Now ask: how many copies of $\frac{1}{3}$ can I find in $\frac{3}{4}$? Well, $\frac{1}{3}$ is equal to four of the smaller squares. So we find groups equal to that:



In the picture of $\frac{3}{4}$, we can find:

- two groups of four squares (two groups of $\frac{1}{3}$), and
- one square left over, which is $\frac{1}{4}$ of the group we're looking for.

We conclude:

$$\frac{3}{4} \div \frac{1}{3} = 2\frac{1}{4}.$$

Discussion:

Determine the quotient by using the equal-sized groups approach: $\frac{3}{4} \div \frac{1}{2}$

Common Denominator Method

Discussion:

Solve each of the following fraction division problems using the equal-sized groups approach:

- $\frac{6}{4} \div \frac{3}{4}$
- $\frac{6}{6} \div \frac{3}{6}$
- $\frac{10}{10} \div \frac{3}{10}$
- $\frac{8}{9} \div \frac{4}{9}$
- $\frac{6}{6} \div \frac{2}{6}$
- $\frac{33}{33} \div \frac{2}{33}$
- $\frac{4}{5} \div \frac{4}{5}$
- $\frac{2}{5} \div \frac{2}{5}$
- $\frac{2}{10} \div \frac{2}{10}$

What do you notice?

When dividing fractions of the same denominator, the result is the ratio of both numerators. We leverage this as a tool for creating our first method for dividing two fractions.

Definition: Common Denominator Method

If two fractions have the same denominator, then when you divide them, you can just divide the numerators. In symbols,

$$\frac{a}{d} \div \frac{b}{d} = \frac{a}{b}.$$

Discussion:

Use the common denominator method to find these quotients:

- $\frac{1}{3} \div \frac{2}{3}$

What if the fractions do not have a common denominator? Is the method useless, or can you find a way to make it work? Use the following problem to help you think about these questions.

- $\frac{3}{5} \div \frac{3}{4}$

Missing Factor Method

We know we can always turn a division problem into a missing-factor multiplication problem. This is a tool for computing fraction division.

Discussion:

Rewrite the problems below as a missing factor multiplication question. Then find the quotient using what you know about multiplying fractions.

- $\frac{9}{10} \div \frac{3}{5}$
- $\frac{7}{8} \div \frac{1}{4}$

Unfortunately, the missing factor method doesn't always work out so nicely. For example,

$$\frac{3}{4} \div \frac{1}{3} = -$$

can be rewritten as

$$\frac{1}{3} \cdot \frac{3}{4} = \frac{3}{4}.$$

You want to ask:

- For the numerator: $1 \cdot \underline{\quad} = 3$. We can fill in the blank with a 3.
- For the denominator: $3 \cdot \underline{\quad} = 4$. We can fill in the blank with $\frac{4}{3}$. (Why does that work?)

So we have: $\frac{1}{3} \cdot \frac{3}{\frac{4}{3}} = \frac{3}{4}$.

In the context of the Pie per Child Model, $\frac{3}{\frac{4}{3}}$ means that each $\frac{4}{3}$ of a kid gets 3 pies. So how much does an individual kid get? You could draw a picture to help you figure it out. But we can also use the key fraction rule to help us out.

$$\frac{3}{\frac{4}{3}} = \frac{3 \cdot 3}{3 \cdot \frac{4}{3}} = \frac{9}{4}.$$

The Simplification Method

To understand the invert and multiply rule, let's consider the following problem: $7\frac{2}{3}$ pies are shared equally by $5\frac{1}{4}$ children. How much pie does each child get?

Technically, we could just write down the answer as $\frac{7\frac{2}{3}}{5\frac{1}{4}}$ and be done! The answer is equivalent to this fraction, so why not?

Is there a way to make this look friendlier? Well, if we change those mixed numbers to improper fractions, it helps a little:

$$\frac{7\frac{2}{3}}{5\frac{1}{4}} = \frac{\frac{23}{3}}{\frac{21}{4}}$$

That's a bit better, but it's still not clear how much pie each kid gets. Let's use the key fraction rule to make the fraction even friendlier. Let's multiply the numerator and denominator each by 3. Remember, multiplying by 3 means we're multiplying the fraction by $\frac{3}{3}$, which is just a special form of 1, so we don't change its value.

$$\frac{3 \cdot \frac{23}{3}}{3 \cdot \frac{21}{4}} = \frac{23}{\frac{63}{4}}.$$

Now multiply numerator and denominator each by 4, which will not change the value as we would be multiplying by $\frac{4}{4}$.

$$\frac{4 \cdot 23}{4 \cdot \frac{63}{4}} = \frac{92}{63}.$$

We now see that the answer is $\frac{92}{63}$. That means that sharing $7\frac{2}{3}$ pies among $5\frac{1}{4}$ children is the same as sharing 92 pies among 63 children. Notice that, in both situations, the individual child gets exactly the same amount of pie.

Thinking about this process leads to an equivalent way of justifying the invert and multiply rule. Without a context, we can consider the following problem.

✓ Example 3.6.1

Use the simplification method illustrated above to solve the following: $\frac{3}{5} \div \frac{2}{3}$

Solution

First, rewrite the problem in the following way:

$$\frac{\frac{3}{5}}{\frac{2}{3}}.$$

Multiplying the numerator and denominator each by 5 (why did we choose 5?) gives

$$\frac{\frac{3}{5}}{\frac{2}{3}} = \frac{5 \cdot \frac{3}{5}}{5 \cdot \frac{2}{3}} = \frac{3}{\frac{10}{3}}.$$

Now multiply the numerator and denominator each by 3 (why did we choose 3?):

$$\frac{3 \cdot 3}{3 \cdot \frac{10}{3}} = \frac{9}{10}.$$

Perhaps without realizing it, we have rediscovered the invert multiply method for dividing fractions.

Invert and Multiply

We can think about the invert and multiply method as a shortcut of the simplification method--this is because the process justifies the following result:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc} \quad (3.6.1)$$

✓ Example 3.6.1

Consider $\frac{5}{9} \div \frac{8}{11}$:

Solution

$$\frac{5}{9} \div \frac{8}{11} = \frac{\frac{5}{9}}{\frac{8}{11}}.$$

Let's multiply numerator and denominator each by 9 and by 11 at the same time. (Why not?)

$$\frac{\frac{5}{9}}{\frac{8}{11}} = \frac{(\frac{5}{9}) \cdot 9 \cdot 11}{(\frac{8}{11}) \cdot 9 \cdot 11} = \frac{5 \cdot 11}{8 \cdot 9}.$$

(Do you see what happened here?)

So we have

$$\frac{\frac{5}{9}}{\frac{8}{11}} = \frac{5 \cdot 11}{8 \cdot 9} = \frac{55}{72}.$$

Discussion:

Consider the problem $\frac{5}{12} \div \frac{7}{11}$. Janine wrote:

$$\frac{\frac{5}{12}}{\frac{7}{11}} = \frac{\frac{5}{12} \cdot 12 \cdot 11}{\frac{7}{11} \cdot 12 \cdot 11} = \frac{5 \cdot 11}{7 \cdot 12} = \frac{5}{12} \cdot \frac{11}{7}.$$

She stopped before completing her final step and exclaimed: "Dividing one fraction by another is the same as multiplying the first fraction with the second fraction upside down!"

Check each step of Janine's work here and ensure it is correct. Then answer these questions:

- Do you understand what Janine is saying? Explain it very clearly.
- Work out $\frac{3}{4} \div \frac{13}{4}$ using the simplification method. Is the answer the same as $\frac{3}{7} \cdot \frac{13}{4}$?
- Work out $\frac{2}{5} \div \frac{10}{3}$ using the simplification method. Is the answer the same as $\frac{2}{5} \cdot \frac{10}{3}$?
- Work out $\frac{a}{b} \div \frac{d}{c}$ using the simplification method. Is the answer the same as $\frac{a}{b} \cdot \frac{d}{c}$?
- Is Janine right? Is dividing two fractions always the same as multiplying the two fractions with the second one turned upside down? What do you think? Do not just think about examples. This is a question of whether something is *always true*.

Summary

We now have several methods for solving problems that require dividing fractions:

Dividing fractions:

- Equal-sized groups approach: Draw a picture using the rectangle method, and use that to solve the division problem.
- Common Denominator Method: Find a common denominator and divide the numerators.
- Missing Factor Method: Rewrite the division as a missing factor multiplication problem, and solve that problem.
- Simplification Method: Simplify an ugly fraction.
- Invert and Multiply Method: Invert the second fraction (the dividend) and then multiply.

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3.7: Dividing Fractions - Word Problems

We have discussed dividing fractions: how to make sense of the operation, how to picture what's going on, and how to do the computations. But all of this kind of begs the question: When would you ever want to divide fractions, anyway? How does that even come up?

To deepen our understanding of the meaning of dividing a fraction, we will focus on the operations and when they are used. This will also help us strengthen our problem-solving skills.

Discussion:

- Use one of our methods to compute $1\frac{3}{4} \div \frac{1}{2}$.
- Come up with a situation where you would want to compute $1\frac{3}{4} \div \frac{1}{2}$. That is, write a word problem that would require you to do this computation to solve it.

When to Multiply, When to Divide?

A common answer to

Come up with a situation where you would want to compute $1\frac{3}{4} \div \frac{1}{2}$.

Is something like this:

My recipe calls for $1\frac{3}{4} \div \frac{1}{2}$ cups of flour, but I only want to make half a recipe. How much flour should I use?

But that problem doesn't ask you to divide fractions. It asks you to cut your recipe in half, which means dividing by 2 or multiplying by $\frac{1}{2}$.

Why is it so hard to come up with division problems that use fractions? Maybe it's because fractions are already the answer to a division problem, so you're dividing and then dividing some more. Maybe it's because they just make it look so complicated. In any case, it's worth spending some time thinking about division problems that involve fractions and how to recognize and solve them.

One handy trick: Write a problem that involves division of whole numbers, and then see if you can change the numbers to fractions in a sensible way.

Here are some division problems involving whole numbers:

- I have 10 feet of ribbon. How many 2-inch pieces can I cut from it?
- I have a fancy old clock that rings once every 15 minutes. How many times will it ring over the course of 2 hours (120 minutes)?
- My fish tank needs 6 gallons of water, and my bucket holds 3 gallons. How many times will I need to fill my bucket in order to fill the tank?
- A recipe calls for 6 cups of flour, and my largest scoop measures exactly 2 cups. How many times should I use it?
- I ran 12 miles and went around the the same route 3 times. How long was the route?

Here are some very similar problems, rewritten to use fractions instead:

- I have $1\frac{3}{4}$ feet of ribbon. How many 6-inch (that's $\frac{1}{2}$ a foot) pieces can I cut from it?
- My watch alarm goes off every half hour, and I don't know how to shut it off. How many times will it go off during the $1\frac{3}{4}$ hour movie?
- My fish tank needs $1\frac{3}{4}$ gallons of water, and my bucket holds $\frac{1}{2}$ gallon. How many times will I need to fill my bucket in order to fill the tank?
- I want to measure $1\frac{3}{4}$ cups of flour for a recipe, but I only have a $\frac{1}{2}$ cup measuring cup. How many times should I fill it?
- I ran $1\frac{3}{4}$ miles before I twisted my ankle. I only finished half the race. How long was the race course?

We can think of the division problem $20 \div 4$ as asking the question: 20 is 4 groups of what size?

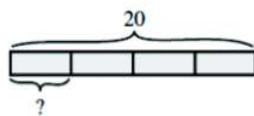


Figure 3.7.1: Representation of 20 split in 4 groups. Then each group is of size ____?

So for $1\frac{3}{4} \div \frac{1}{2}$, we ask: $1\frac{3}{4}$ is half a group of what size?

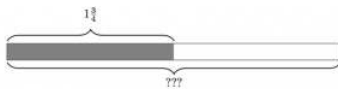


Figure 3.7.1: Representation of the problem: $1\frac{3}{4}$ is half a group of what size?

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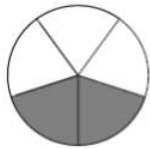
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3.8: Practice Problems

Problems for Section 3.1

Work on the following exercises on your own or with a partner.

1. Draw a picture associated with the fraction $\frac{1}{6}$.
2. Draw a picture associated with the fraction $\frac{3}{7}$. Is your picture really the amount of pie an individual would receive if three pies are shared among seven kids? Be very clear on this!
3. Let's work backwards! Here's the answer to a division problem:



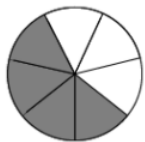
This represents the amount of pie an individual kid receives if some number of pies is shared among some number of children. How many pies? How many children? How can you justify your answers?

4. Here's another answer to a division problem:



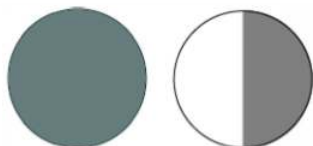
How many pies? How many children? How can you justify your answers?

5. Here is another answer to a division problem:



How many pies? How many children? How can you justify your answers?

6. Leigh says that " $\frac{3}{5}$ is three times as big as $\frac{1}{5}$." Is this right? Explain your answer.
7. Draw a picture for the answer to the division problem $\frac{4}{8}$. Describe what you notice about the answer.
8. Draw a picture for the answer to the division problem $\frac{2}{10}$. Describe what you notice about the answer.
9. What does the division problem $\frac{1}{1}$ represent? How much pie does an individual child receive?
10. What does the division problem $\frac{5}{1}$ represent? How much pie does an individual child receive?
11. What does the division problem $\frac{5}{5}$ represent? How much pie does an individual child receive?
12. Here is the answer to another division problem. This is the amount of pie an individual child receives:



How many pies were in the division problem? How many kids were in the division problem? Justify your answers.

13. Here is the answer to another division problem. This is the amount of pie an individual child receives:



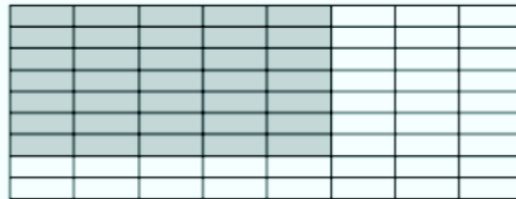
How many pies were in the division problem? How many kids were in the division problem? Justify your answers.

14. Many teachers have young students divide differently shaped pies into fractions. For example, a hexagonal pie is good for illustrating the fractions: $\frac{1}{6}$, $\frac{2}{6}$, $\frac{3}{6}$, $\frac{4}{6}$, $\frac{5}{6}$, and $\frac{6}{6}$.



- Why is this shape used? What does $\frac{1}{6}$ of a pie look like?
- What does $\frac{6}{6}$ of a pie look like?
- What shape pie would be good for illustrating the fractions $\frac{1}{8}$ up to $\frac{8}{8}$?

15. Some rectangular pies are distributed to some number of kids. This picture represents the amount of pie an individual child receives. The large rectangle represents one whole pie.



How many pies? How many kids? Carefully justify your answers!

Problems for Section 3.2

1. Mix and Match: On the left, there are some fractions that have not been simplified. On the right are the simplified answers, in random order. Which simplified answer goes with which fraction? Notice that there are fewer answers than questions.

1. $\frac{10}{20}$	a. $\frac{2}{3}$
2. $\frac{75}{24000}$	b. $\frac{16}{12}$
3. $\frac{36000}{24}$	c. $\frac{1}{7}$
4. $\frac{14}{18}$	d. $\frac{1}{40}$
5. $\frac{32}{1}$	e. $\frac{1}{2}$
6. $\frac{1}{40}$	

2. Use the Pies per Child Model to explain **why** the key fraction rule holds. That is, explain why each individual child gets the same amount of pie in these two situations:

- if you have a pies and b kids, or
- if you have xa pies and xb kids.

Problems for Section 3.3

Try these exercises on your own. For each addition exercise, also write down a Pies per Child interpretation of the problem. You might also want to draw a picture.

1. What is $\frac{1}{2} + \frac{1}{3}$?
2. What is $\frac{2}{5} + \frac{37}{10}$?
3. What is $\frac{1}{2} + \frac{3}{10}$?
4. What is $\frac{2}{3} + \frac{5}{7}$?
5. What is $\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$?
6. What is $\frac{3}{10} + \frac{4}{25} + \frac{7}{20} + \frac{3}{5} + \frac{49}{50}$?

Now try these subtraction exercises.

1. What is $\frac{7}{10} - \frac{3}{10}$?
2. What is $\frac{7}{10} - \frac{3}{20}$?
3. What is $\frac{1}{3} - \frac{1}{5}$?
4. What is $\frac{2}{35} - \frac{2}{7} + \frac{2}{5}$?
5. What is $\frac{1}{2} - \frac{1}{4} - \frac{1}{8} - \frac{1}{16}$?

Problems for Section 3.4

1. Consider the following figure:



- If the shaded region represents $4\frac{3}{4}$, what is the whole?
 - If the shaded region represents $2\frac{3}{8}$, what is the whole?
 - If the shaded region represents $\frac{19}{20}$, what is the whole?
2. Arrange each group of fractions in *ascending order*. Keep track of your thinking and your methods.
 - $\frac{3}{7}$, $\frac{3}{4}$, $\frac{3}{8}$.
 - $\frac{5}{6}$, $\frac{7}{8}$, $\frac{3}{4}$.
 - $\frac{8}{13}$, $\frac{12}{17}$, $\frac{1}{6}$.
 3. Verify that the following are arithmetic sequences. Justify your answers.
 - 2, 9, 16, 23, 30, 37, 44, 51, ...
 - $\frac{1}{5}$, $\frac{3}{5}$, 1, $\frac{7}{5}$, $\frac{9}{5}$, $\frac{11}{5}$, $\frac{13}{5}$, 3, ...
 - 1, 2, 4, 8, 16, 32, 64, 128, ...
 - 1, 3, 6, 10, 15, 21, 28, 36, ...
 - $\frac{2}{5}$, $\frac{7}{10}$, 1, $\frac{13}{10}$, $\frac{8}{5}$, $\frac{19}{10}$, $\frac{11}{5}$, $\frac{5}{2}$, ...
 - $\frac{3}{5}$, $\frac{6}{5}$, $\frac{12}{5}$, $\frac{24}{5}$, $\frac{48}{5}$, $\frac{96}{5}$, ...
 4. Make up your own arithmetic sequence using whole numbers. Exchange sequences with a partner, and check if your partner's sequence is really an arithmetic sequence.
 5. Consider:

$$\frac{1}{4}, \quad \text{---}, \quad \text{---}, \quad \frac{1}{3}$$

- Find two fractions between $\frac{1}{4}$ and $\frac{1}{3}$.
 - Are the resulting four fractions in an arithmetic sequence? Justify your answer.
6. Find two fractions between $\frac{1}{6}$ and $\frac{1}{5}$ so the resulting four numbers are in an arithmetic sequence.

$$\frac{1}{6}, \quad \text{---}, \quad \text{---}, \quad \frac{1}{5}$$

7. Find two fractions between $\frac{1}{6}$ and $\frac{1}{5}$ so the resulting four numbers are in an arithmetic sequence.

$$\frac{1}{6}, \quad \frac{1}{3}, \quad \frac{1}{2}, \quad \frac{1}{5}$$

Problems for Section 3.5

1. Use an area model to compute each of the following products. Connect your answer to your representation.

- $\frac{3}{4} \times \frac{5}{6}$
- $\frac{1}{3} \times \frac{5}{4}$
- $\frac{5}{8} \times \frac{3}{7}$

2. Use an area model to compute each of the following products. Connect your answer to your representation.

- $\frac{2}{5} \cdot \frac{4}{3}$
- $\frac{10}{5} \cdot \frac{4}{7}$
- $\frac{5}{2} \cdot \frac{4}{4}$

3. Compute the following products, simplifying each of the answers as much as possible. You do not need to draw pictures, but you may certainly choose to do so if it helps.

- $\frac{5}{11} \times \frac{7}{12}$
- $\frac{4}{7} \times \frac{4}{8}$
- $\frac{1}{1} \times \frac{1}{3}$
- $\frac{2}{2} \times \frac{3}{3}$
- $\frac{1}{1} \times \frac{1}{5}$
- $\frac{1}{5} \times \frac{1}{1}$

4. Explain how to find the products below by using three different methods (converting the whole number to a fraction with a denominator of one, Pies per Child Model, units and unitizing):

- $4 \cdot \frac{3}{8}$
- $6 \cdot \frac{1}{5}$

5. Compute the following products. Consider using the fraction rule, $\frac{xa}{xb} = \frac{a}{b}$, to help you calculate.

- $\frac{3}{4} \times \frac{1}{3}$
- $\frac{4}{5} \times \frac{3}{7}$
- $\frac{88}{88} \times \frac{541}{788}$
- $\frac{77876}{311} \times \frac{311}{77876}$
- $\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5} \times \frac{5}{6} \times \frac{6}{7} \times \frac{7}{8} \times \frac{8}{9} \times \frac{9}{10}$

6. What do you think about Roy's reasoning? Does it make sense? How would Roy explain the general rule for positive whole numbers x :

$$\frac{xa}{xb} = \frac{a}{b}?$$

Roy says that the fraction rule

$$\frac{xa}{xb} = \frac{a}{b}$$

is "obvious" if you think in terms of multiplying fractions. He reasons as follows:

We know multiplying anything by 1 does not change a number:

$$\begin{aligned} 1 \cdot 4 &= 4 \\ 1 \cdot 2014 &= 2014 \\ 1 \cdot \frac{5}{7} &= \frac{5}{7} \end{aligned}$$

So, in general,

$$1 \cdot \frac{a}{b} = \frac{a}{b}.$$

Now, $\frac{2}{2} = 1$, so that means that

$$\frac{2}{2} \cdot \frac{a}{b} = 1 \cdot \frac{a}{b} = \frac{a}{b},$$

which means

$$\frac{2a}{2b} = \frac{a}{b}.$$

By the same reasoning, $\frac{3}{3} = 1$, so that means that

$$\frac{3}{3} \cdot \frac{a}{b} = 1 \cdot \frac{a}{b} = \frac{a}{b},$$

which means

$$\frac{3a}{3b} = \frac{a}{b}.$$

Problems for Section 3.6

1. Determine the following quotients using the finding equal-sized groups approach.

- $\frac{1}{2} \div \frac{1}{2}$
- $\frac{2}{4} \div \frac{1}{3}$
- $\frac{3}{3} \div \frac{3}{3}$
- $\frac{3}{3} \div \frac{4}{1}$
- $\frac{2}{2} \div \frac{2}{1}$
- $\frac{3}{3} \div \frac{2}{2}$

2. Use the common denominator method to find these quotients. If the fractions don't have common denominators, find their common denominator first.

- $\frac{5}{3} \div \frac{3}{8}$
- $\frac{8}{15} \div \frac{8}{33}$
- $\frac{1}{33} \div \frac{1}{8}$
- $\frac{4}{2} \div \frac{7}{1}$
- $\frac{3}{3} \div \frac{2}{1}$
- $\frac{5}{8} \div \frac{1}{4}$

3. Rewrite the problems below as a missing factor multiplication question. Then find the quotient using what you know about multiplying fractions.

- $\frac{6}{7} \div \frac{3}{7}$
- $\frac{10}{25} \div \frac{2}{5}$
- $\frac{9}{12} \div \frac{3}{6}$

4. Write each of the following in a simpler form using the Simplification Method.

- $\frac{2}{3} \div \frac{1}{3}$
- $2\frac{1}{5}$
- $2\frac{1}{4}$
- $\frac{5}{7} \div \frac{3}{5}$

o $\frac{\frac{3}{7}}{\frac{4}{5}}$

5. Jessica calculated $2\frac{1}{5} \div 2\frac{1}{4}$ in the following way: $\frac{2\frac{1}{5}}{2\frac{1}{4}} = \frac{\frac{1}{5}}{\frac{1}{4}} = \frac{\frac{1}{5} \cdot 4}{\frac{1}{4} \cdot 4} = \frac{\frac{4}{5}}{1} = \frac{4}{5}$. Is this solution correct, or is Jessica

misunderstanding something? Carefully explain what is going on with this solution.

6. Isaac calculated $\frac{3}{7} \div \frac{4}{5}$ this way: $\frac{\frac{3}{7}}{\frac{4}{5}} = \frac{\frac{3}{7} \cdot 7}{\frac{4}{5} \cdot 5} = \frac{3}{4}$. Is this solution correct, or is Isaac misunderstanding something? Carefully explain what is going on with this solution.

7. Compute each of the following, using the invert and multiply method.

- o $\frac{1}{2} \div \frac{1}{3}$
- o $\frac{4}{5} \div \frac{3}{7}$
- o $\frac{2}{3} \div \frac{1}{5}$
- o $\frac{45}{59} \div \frac{902}{902}$
- o $\frac{10}{13} \div \frac{2}{13}$

Problems for Section 3.7

- Write five different division word problems that use whole numbers. Then change the problems so that they are fraction division problems instead. You might need to rewrite the problem a bit so that it makes sense.
- Solve the problems you created above. Provide your answer in the context of the problem.

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CHAPTER OVERVIEW

4: Algebra

- 4.1: Introduction
- 4.2: Algebraic Skills
- 4.3: Borders on a Square
- 4.4: Appropriate Use of the Equal Sign
- 4.5: Growing Patterns
- 4.6: Solving Equations Using the Balance Scale Representations
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- 4.8: Common Mistakes in Algebra
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- 4.10: Practice Problems

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4.1: Introduction

Algebra skills are essential because:

- Mathematics, and especially algebra, is the language of science and modern technology. Thinking algebraically helps you to make sense of the world, to understand and interact with technology more productively, and to succeed in other fields.
- Algebra is a tool for solving problems. This may not be your experience so far, but it is true. If you are able to “algebratize” a problem, that often helps lead you to a solution.
- Algebra helps you to think abstractly. It is a tool for thinking about operations like addition, subtraction, multiplication, and division separate from doing calculations on particular numbers. Algebra helps you to *understand and explain why* the operations work the way they do, to *describe* their properties clearly, and to *manipulate* expressions to see the bigger picture.

As a motivation for this chapter, let's start with a problem: Use four 4's to make 0.

For example, one can think of $44 - 44$, because $44 - 44 = 0$. Are there other ways of using four 4's to make 0? Use your creative thinking and problem-solving strategies to solve this problem.

✓ Example 4.1.1

Use four 4's to make all of the numbers between 0 and 20.

Solution

Answers will vary. Examples of answers:

- I can use four 4's to make 0: $(4 \times 4) - (4 \times 4) = 0$
- I can use four 4's to make the number 10: $(4 \times 4) - 4 - \sqrt{4} = 10$

Discussion:

- What does “algebra” mean to you?
- What does the example above have to do with “algebra”?
- What do you imagine when you think about using algebra to solve problems?
- Have you ever used algebra to solve problems outside of the school context?

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4.2: Algebraic Skills

Identify and Combining Like Terms

Algebraic expressions are made up of terms. A **term** is a constant, or the product of a constant and one or more variables.

Definition: Term

A **term** is a constant or the product of a constant and one or more variables.

Examples of terms are 7, y , $5x^2$, $9a$, and b^5 .

The constant that multiplies the variable is called the **coefficient**.

Definition: Coefficient

The **coefficient** of a term is the constant that multiplies the variable in a term.

Think of the coefficient as the number in front of the variable. The coefficient of the term $3x$ is 3. When we write x , the coefficient is 1, since $x = 1 \cdot x$.

Some terms share common traits. When two terms are constants or have the same variable and exponent, we say they are **like terms**.

Discussion:

Look at the following 6 terms. Which ones seem to have traits in common?

$$5x \quad 7 \quad n^2 \quad 4 \quad 3x \quad 9n^2 \quad (4.2.1)$$

We say,

7 and 4 are like terms.

$5x$ and $3x$ are like terms.

n^2 and $9n^2$ are like terms.

Definition: Like Terms

Terms that are either constants or have the same variables raised to the same powers are called **like terms**.

If there are like terms in an expression, you can simplify the expression by combining the like terms. We add the coefficients and keep the same variable.

$$\begin{array}{ll} \text{Simplify.} & 4x + 7x + x \\ \text{Add the coefficients.} & 12x \end{array} \quad (4.2.2)$$

✓ Example 4.2.1

Simplify: $2x^2 + 3x + 7 + x^2 + 4x + 5$.

Solution

Step 1. Identify the like terms.

$$2x^2 + 3x + 7 + x^2 + 4x + 5$$

$$\underline{2x^2} + \underline{3x} + \underline{7} + \underline{x^2} + \underline{4x} + \underline{5}$$

$$2x^2 + 3x + 7 + x^2 + 4x + 5$$

Step 2. Rearrange the expression so the like terms are together.

$$\underline{2x^2} + \underline{x^2} + \underline{3x} + \underline{4x} + \underline{7} + \underline{5}$$

$$2x^2 + x^2 + 3x + 4x + 7 + 5$$

Step 3. Combine like terms.

$$3x^2 + 7x + 12$$

To combine like terms, you may follow the following steps:

1. Identify the like terms.
2. Rearrange the expression so the like terms are together.
3. Add or subtract the coefficients and keep the same variable for each group of like terms.

Talking Mathematics: Words and Algebraic Expressions

When you have the mathematical expression $a + b$, you can read and interpret the sum in different ways. For example, a plus b , the sum of a and b , a increased by b , b more than a , the total of a and b , or b added to a . The following table provides support on how to read different expressions, including addition, subtraction, multiplication, and division.

Operation	In Words	Algebraic Expression
Addition	a plus b the sum of a and b a increased by b b more than a the total of a and b b added to a	$a + b$
Subtraction	a minus b the difference of a and b a decreased by b b less than a b subtracted from a	$a - b$
Multiplication	a times b the product of a and b twice a	$a \cdot b$, ab , $a(b)$, $(a)(b)$ $2a$
Division	a divided by b the quotient of a and b the ratio of a and b b divided into a	$a \div b$, a/b , $\frac{a}{b}$, $b \overline{)a}$

Evaluate an Expression

In the last few examples, we simplified expressions using the order of operations. Now we'll evaluate some expressions—again following the order of operations. To **evaluate an expression** means to find the value of the expression when the variable is replaced by a given number.

EVALUATE AN EXPRESSION

To **evaluate an expression** means to find the value of the expression when the variable is replaced by a given number.

To evaluate an expression, substitute that number for the variable in the expression and then simplify the expression.

Evaluate when $x = 4$: a. x^2 b. 3^x c. $2x^2 + 3x + 8$.

Answer

a.

	x^2
Replace x with 4.	4^2
Use definition of exponent.	$4 \cdot 4$
Simplify.	16

b.

	3^x
Replace x with 4.	3^4
Use definition of exponent.	$3 \cdot 3 \cdot 3 \cdot 3$
Simplify.	81

c.

	$2x^2 + 3x + 8$
Replace x with 4.	$2(4)^2 + 3(4) + 8$
Follow the order of operations.	$2(16) + 3(4) + 8$
	$32 + 12 + 8$
	52

Evaluate when $x = 3$, a. x^2 b. 4^x c. $3x^2 + 4x + 1$.

Answer

- a. 9
- b. 64
- c. 40

? Example 4.2.18

Evaluate when $x = 6$, a. x^3 b. 2^x c. $6x^2 - 4x - 7$.

Answer

- a. 216
- b. 64
- c. 185

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4.3: Borders on a Square

To encourage algebraic thinking, we will go back to representations. When we discussed fractions, we used area models to represent multiplication of fractions. We will leverage the idea of area models to encourage creative thinking through the algebratization of problems. In this section, we will focus on the borders of a square. This is a classical application to introduce algebra through problem solving.

Before we start with the first problem, let's think about what a square is. A square is a four-sided polygon with all sides of equal length and right angles. When representing a square with an area model, say, for example, a square that measures 5 ft on each side, it would be represented by the following:

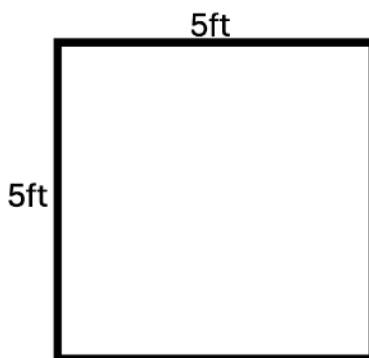


Figure 4.3.1: An area model of a square with sides of 5ft.

Consider you want to put tiles of size $1\text{ft} \times 1\text{ft}$ on this square, then, how many tiles do we need? Notice that the answer is just the area of this square, which is $5\text{ft} \times 5\text{ft}$, which is 25ft^2 . The $1\text{ft} \times 1\text{ft}$ tiles in this problem represent unit squares.

Discussion:

How would you represent the tiling of the given square that measures 5 ft on each side to put tiles of size $1\text{ft} \times 1\text{ft}$.

This problem provides a good foundation for solving the problems in this section. For example, consider a large square made up of 100 smaller unit squares. The unit squares along the border of the large square are colored red. Without counting one by one, can you figure out how many red squares there are in the picture?

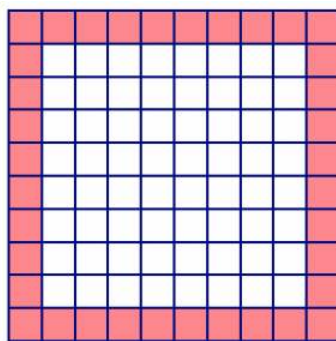


Figure 4.3.1: An area model of a square with sides of 10, with its border in color red.

Clearly describe how you figured out the number of red squares, and how you know your answer is correct. We will present different approaches below to the same problem; your solution may differ from those provided.

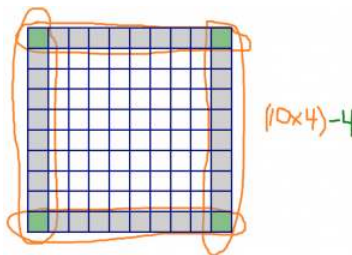
Solution 1:

Justin calculated the number of squares as $(10 \times 4) - 4$. He justified his answer this way:

Since the dimensions of the big square are 10×10 , there are 10 squares along each of the four sides. So that gives me 40 red squares. But then each corner is part of two different

sides. I've counted each of the corners twice. So I need to make up for that by subtracting 4 at the end.

Justin showed this picture to justify his work:



Discussion:

1. Consider Justin's solution above:

- What do you think about Justin's solution? Are you convinced? Could he have explained it more clearly?
- Was Justin's solution different from your solution or the same?
- Notice the color coding in Justin's picture. What do the colors represent? Why did he use the colors the way he did?

2. There are lots of different ways to calculate the number of colored squares along the border of a 10×10 square. Below are the calculations several other students did. For each calculation, write a justification and draw a picture to show why it calculates the number of squares correctly. Think about using color in your picture to make your work clearer.

- Valerie calculated $10 + 10 + 8 + 8$.
- Kayla calculated 4×9 .
- Linda calculated $(10 \times 10) - (8 \times 8)$.
- Mark calculated $(4 \times 8) + 4$.
- Allan calculated $10 + 9 + 9 + 8$.

Now that we have worked with several approaches for this problem, one of our goals in algebra is generalization. For example, if we have a large $n \times n$ square with the border squares colored red, can we determine the number of red squares resulting? Here the number n could be any number. More specifically, we can just focus on whole numbers. To answer this question, we can take several approaches; you are welcome to try to solve this problem. A way we can start thinking about the answer to this problem is by trying cases and comparing the cases to determine the possibility of determining a pattern that can support a generalization.

So let's consider a similar problem, like a 6×6 square with the unit squares along the border colored red. Could we adapt two of the techniques above to calculate the number of red unit squares?

✓ Example 4.3.1

Consider a 6×6 square with the unit squares along the border colored red. How many red squares do we have? Use different techniques. For each technique you use, write an explanation and include a picture. Think about how to use colors or other methods to make your picture and explanation clearer

Solution

Solution 1: $6 + 6 + 4 + 4$

Solution 2: 4×5

Solution 3: $(6 \times 6) - (4 \times 4)$

Solution 4: $(4 \times 4) + 4$

Solution 5: $6 + 5 + 5 + 4$

Notice that we just connected the solutions above with the solutions in the discussion provided by Valerie, Kayla, Linda, Mark, and Allan. Do you see the patterns? Let's unwrap these patterns. Discuss the following with a partner.

Discussion:

Describe some general rules:

- If you have a large $n \times n$ square with the border squares colored red, how many red squares will there be? Connect your answer with each of the five solutions we have discussed. Justify your answer with words and a picture.
- If you have k red squares, is there a quick test you can do to decide if you can use all of those squares to make the border of a large square? Can you tell how big the square will be?

After discussing with a partner, you probably noticed that for Solution 1, when we had the 10×10 square, we used $10 + 10 + 8 + 8$. Considering that $n = 10$, we have that 8 can also be written as: $8 = n - 2$. So the pattern will be $n + n + (n - 2) + (n - 2)$. When using the 6×6 square, we used $6 + 6 + 4 + 4$, considering that $n = 6$, we have that 4 can also be written as: $4 = n - 2$. This leads to $n + n + (n - 2) + (n - 2)$, which is the same result. We now have a good guess that $n + n + (n - 2) + (n - 2) = 4n - 4$ is the number of red squares for any $n \times n$ big square. Did you have the same result for the other four solutions?

Discussion:

Is this information enough for saying that any $n \times n$ big square with the border colored red will have $n + n + (n - 2) + (n - 2) = 4n - 4$ red squares? Discuss how you can justify the result so that it is considered a valid generalization.

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4.4: Appropriate Use of the Equal Sign

The notion of equality is fundamental in mathematics, and especially in algebra and algebraic thinking. The equal symbol “=” expresses a *relationship*. It is *not* an operation and it should not be read left-to-right. It also does not mean “... and the answer is ...”.

For your work to be clear and easily understood by others, it is essential that you use the equal symbol appropriately. We will strengthen your understanding of the equal sign by considering **when** a statement is true, and by using a **pan-balance** model for the equal sign.

Always, Sometimes, or Never

We will explore three ways of thinking about statements. There are three possibilities for a statement: always true, sometimes true, or never true.

For example, the statements $10 = 10$ and $11 = 13 - 2$ are always true for these whole numbers. However, the statements $1 = 2$ and $20 = 4 + 17$ are never true. Statements that are sometimes true can be trickier to think about. For example, $11 = 13 - x$ could be true only if $x = 2$, but other values like $x = 3$ will make it false. So this means that $11 = 13 - x$ is sometimes true.

Discussion:

- Examine the following equations. Decide: Is the statement always true, sometimes true, or never true? Justify your answers.
 - $5 + 3 = 8$
 - $\frac{2}{3} + \frac{1}{2} = \frac{3}{5}$
 - $5 + 3 = y$
 - $y = y$
- Consider the equation $18 - 7 = \underline{\hspace{1cm}}$.
 - Fill in the blank with something that makes the equation *always true*.
 - Fill in the blank with something that makes the equation *always false*.
 - Fill in the blank with something that makes the equation *sometimes true and sometimes false*.

When using the equal sign to solve problems, it is common to misuse it. While you could get a correct answer, the process can still be incorrect. The following discussion discusses an incorrect process to solve $2.30 + 3.20 - 1.50$.

Discussion:

What issues do you perceive when using the following process: $2.30 + 3.20 = 5.50 - 1.50 = 4$? How can we solve $2.30 + 3.20 - 1.50$ using a correct process?

The answer is 4, and that is correct. However, the solution is unclear and incorrect. It is true that

$$2.30 + 3.20 = 5.50.$$

But it is *not* true that

$$2.30 + 3.20 = 5.50 - 1.40.$$

Notice that each equal sign used in a process means that everything is equal to the original expression, so this process would claim that $2.30 + 3.20 = 4$, which is also not true. The incorrect use of the equal symbol makes a calculation difficult to understand.

Discussion:

- Can you write a good *definition* of the symbol “=”? What does it mean and what does it represent?
- Give some examples. When should the symbol “=” be used, and when should it *not* be used?

- Do these two equations express the same relationships or different relationships? Explain your answer: $x^2 - 1 = (x + 1)(x - 1)$. $(x + 1)(x - 1) = x^2 - 1$.

Pan-Balance Scale Model

This picture shows a (very simplistic) two-pan balance scale. Such a scale allows you to *compare* the weight of two objects. Place one object in each pan. If one side is lower than the other, then that side holds heavier objects. If the two sides are balanced, then the objects on each side weigh the same.

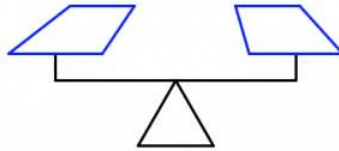


Figure 4.4.1: Pan-Balance Scale Representation

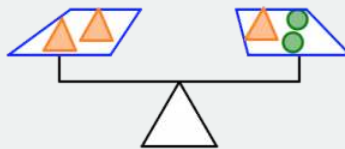
In the following discussion and examples, we will explore the idea of keeping the two sides balanced and connect it to algebraic thinking.

Discussion:

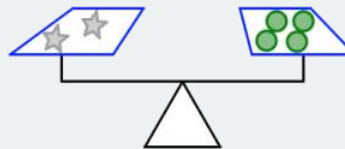
In the pictures below:

- The orange triangles all weigh the same.
- The green circles all weigh the same.
- The purple squares all weigh the same.
- The silver stars all weigh the same.
- The scale is balanced.

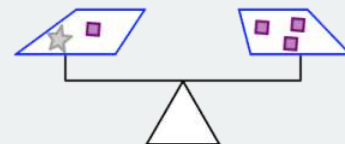
1. In the picture below, what do you know about the weights of the triangles and the circles? How do you know it?



2. In the picture below, what do you know about the weights of the circles and the stars? How do you know it?



3. In the picture below, what do you know about the weights of the stars and the squares? How do you know it?



Notice that the picture starts by giving you five assumptions, the first four confirm that all figures that look the same will weigh the same, and the last one supposes that the scale is balanced. From the rest of the section and for the assigned problems, please make these assumptions.

In the first representation above, you could remove one triangle on each side and still keep the balance. Which will result in one triangle on one side and two circles on the other. This means that one triangle weighs the same as two circles.

In the second representation, you could group the figures. If you have two stars on the left and four circles on the right, an approach to work on this is to distribute the weight equally for each of the figures. In other words, we could assign two circles to each star. So each star has the weight of two circles.

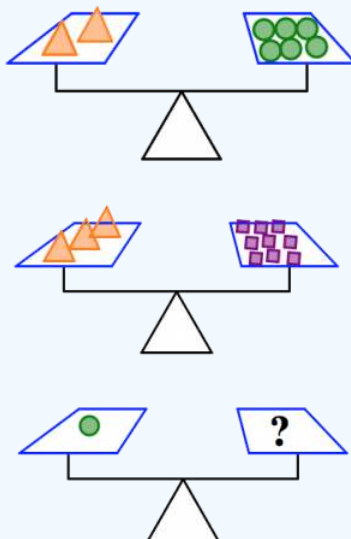
In the third representation, you could remove one square on each side, which will result in one star on the left and two squares on the right. Therefore, one star weighs the same as two squares.

This foundational discussion will drive the problems and examples for this section.

✓ Example 4.4.1

In the pictures below:

- The orange triangles all weigh the same.
- The green circles all weigh the same.
- The purple squares all weigh the same.
- The scale is balanced.



How many purple squares will balance with one circle? Justify your answer.

Solution

The solutions will vary. Here is one solution:

From the second representation, one triangle is the same as three squares. Then, in the first representation, we could replace six squares with two triangles, and then we have that six squares are the same as six circles. Therefore, one circle is the same as one purple square.

Discussion:

What do these type of problems have to do with the “=” symbol?

Notice that while we provided a solution in words, we could also represent this by using equations. Below, we provide a different solution and we also represent it in a different way. We use equations and the equal sign for solving the question from the previous example: How many purple squares will balance with one circle?

Using the first representation we have:

$$2 \text{ triangles} = 6 \text{ circles} \quad (4.4.1)$$

This is equivalent to saying:

$$1 \text{ triangle} = 3 \text{ circles} \quad (4.4.2)$$

Now, using the second representation, we have the following equation:

$$3 \text{ triangles} = 9 \text{ squares} \quad (4.4.3)$$

Which is equivalent to saying:

$$1 \text{ triangle} = 3 \text{ squares} \quad (4.4.4)$$

Now by using the two results: $1 \text{ triangle} = 3 \text{ circles}$ and that $1 \text{ triangle} = 3 \text{ squares}$, notice that both include 1 triangle. So, we have that

$$3 \text{ circles} = 1 \text{ triangle} = 3 \text{ squares} \quad (4.4.5)$$

Which means that:

$$3 \text{ circles} = 3 \text{ squares} \quad (4.4.6)$$

And we can conclude that:

$$1 \text{ circle} = 1 \text{ square} \quad (4.4.7)$$

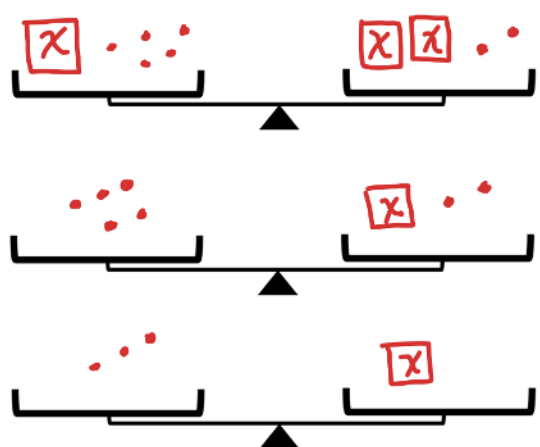
To purposefully connect algebraic processes with balance scales, you could think about figures to represent a specific variable. Let's start by thinking about a scale that is balanced with one square and five circles on the left side and two squares and two circles on the right side. If you think of squares as a representation of x and circles as units or just whole numbers, then you will have the equation:

$$x + 5 = 2x + 2 \quad (4.4.8)$$

✓ Example 4.4.1

Use a pan balance representation to solve the equation $x + 5 = 2x + 2$. Include a pan-balance representation to the left and annotate your representation with the corresponding equation to the right of the representation.

Solution



$$\begin{aligned} x + 5 &= 2x + 2 \\ \hline 5 &= x + 2 \\ \hline 3 &= x \end{aligned}$$

Then we say that $x=3$.

We will expand on this idea in Section 4.6.

Although this representation has limitations, especially with negative numbers, the key idea of keeping an equation balanced is key to understanding algebraic reasoning and the meaning of the equal sign.

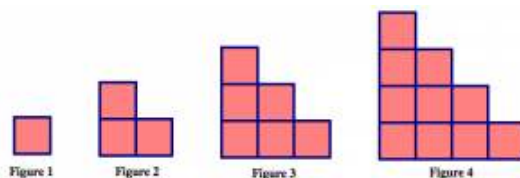
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4.5: Growing Patterns

We have seen many ways of developing algebraic thinking through different approaches, including the borders of a square problem, the use of the equal sign, and the balancing approach. In this section, we will continue exploring algebraic thinking by examining patterns in figures, as we did in the border-of-a-square applications. However, our focus in this section is on generalization through the consideration of cases and the logical explanation of this generalization.

Consider the following pattern made from square tiles.

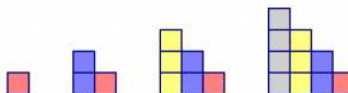


Discussion:

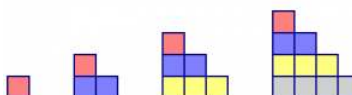
1. Describe how you see this pattern growing. Be as specific as you can. Draw pictures and write an explanation to make your answer clear.
2. Say as much as you can about this growing pattern. Can you draw pictures to extend the pattern?
3. What mathematical questions can you ask about this pattern? Can you answer any of them?

You can think about the pattern of these figures by using different approaches. Consider the following rationales.

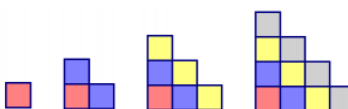
Solution 1: Think about columns. Each time the pattern adds a new column, it creates columns of size 1, 2, 3, 4, and so on.



Solution 2: Think about rows. Each time the pattern adds a new row, it creates rows of size 1, 2, 3, 4, and so on.



Solution 3: Think about diagonals. Each time the pattern adds a new diagonal, it creates diagonals of size 1, 2, 3, 4, and so on.



Discussion:

- How many tiles would you need to build the 5th figure in the pattern?
- How many tiles would you need to build the 10th figure in the pattern?
- How can you compute the number of tiles in any figure in the pattern? Consider, for example, the figure: n , where n is any whole number.

Notice that the 5th figure will have an extra set of 5 tiles, so it will be a total of $1 + 2 + 3 + 4 + 5$ tiles. We start to see a pattern. By drawing more figures, you will notice that the total of tiles in the 10th figure is: $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$.

Consider the n^{th} figure, where n can be any whole number. Then we would expect that the number of tiles is the sum of all whole numbers from 1 to n . In other words, it will be: $1 + 2 + \dots + (n - 1) + n$. We can try this formula and find that for $n = 1$, we will have 1 tile, for $n = 2$, we will have $1 + 2$ tiles, for $n = 3$, we will have $1 + 2 + 3$ tiles, and so on. Using n supports the generalization of the pattern. This will allow us to calculate the number of tiles for the 100th figure, which will be $1 + 2 + 3 + 4 + 5 + \dots + 99 + 100$.

Discussion:

Consider the formula $1 + 2 + \cdots + (n - 1) + n$. Is there another way of representing this in a simpler way? Use your creative thinking to solve this problem.

Notice that $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$, so the 100th figure, will have $1 + 2 + 3 + 4 + 5 + \cdots + 99 + 100 = \frac{100(100+1)}{2} = \frac{100(101)}{2} = 5050$. How do we get the formula: $\frac{n(n+1)}{2}$. Think of pairing the numbers from the ends.

We write the sum:

$$1 + 2 + 3 + \cdots + (n - 2) + (n - 1) + n. \quad (4.5.1)$$

Now pair the first and last terms:

$$1 + n, \quad (4.5.2)$$

the second and the second-to-last:

$$2 + (n - 1), \quad (4.5.3)$$

the third and the third-to-last:

$$3 + (n - 2), \quad (4.5.4)$$

and so on. Each pair adds up to the same value:

$$1 + n = 2 + (n - 1) = 3 + (n - 2) = n + 1. \quad (4.5.5)$$

There are $\frac{n}{2}$ such pairs (or the idea still works if n is odd).

Therefore, the total sum is

$$\text{Sum} = \frac{n}{2}(n + 1) = \frac{n(n + 1)}{2}. \quad (4.5.6)$$

Discussion:

Explore the following approach and connect to the previous generalization.

Hy saw the pattern in a different way. Here's what Hy drew:



Hy's picture.

1. Describe in words how Hy saw the pattern grow.
2. How would Hy calculate the number of tiles needed to build the 10th figure in the pattern?
3. How would Hy calculate the number of tiles needed to build the 100th figure in the pattern?
4. How would Hy calculate the number of tiles needed to build any figure in the pattern?

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4.6: Solving Equations Using the Balance Scale Representations

In Section 4.4, you examined the concept of the equal sign. Balance scales can effectively illustrate this concept by showing that the quantities on both sides of the equal sign are the same.

Using balance scales to understand how to solve linear equations provides a visual and concrete way to grasp the idea of equality and variable manipulation. A balance scale represents an equation where both sides must be equal. Solving the equation means keeping this balance while isolating the variable.

Consider the equation $x + 3 = 5$. This can be visualized with balance scales as follows:



In this balance scale, x is represented by a box. Discuss with a partner: How would you use a balance scale to solve this equation?

The concept of using a balance scale to solve linear equations involves keeping x (the box) isolated on one side of the scale. This means you need to maintain the balance while manipulating the equation.

To connect the visual representation (balance scale) with the algebraic notation (equation with symbols), follow this structure: Draw the balance scales on the left-hand side and write the equation on the right-hand side. This approach helps track each step, ensuring that any operation performed on one side of the equation is also applied to the other side.

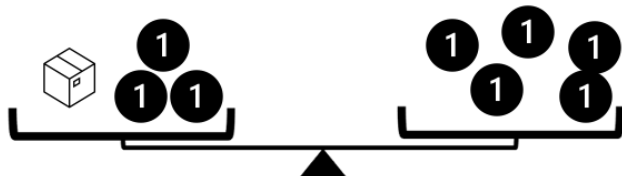
✓ Example 4.6.1

$$x + 3 = 5$$

(4.6.1)

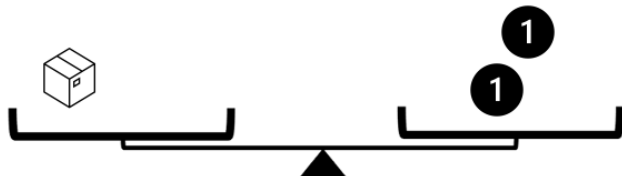
Solution

Step 1. Represent the equation as it is.



$$x + 3 = 5$$

Step 2. Take three 1's out of the scale. To keep the balance, we will do the same on both sides.



$$x + 3 = 5$$

$$x + 3 - 3 = 5 - 3$$

$$x = 2$$

Once you manage to isolate the box, the value of x will be determined by the remaining quantity on the other side. In this example, we removed three 1's from each side of the balance, which is equivalent to subtracting 3 from both sides of the equation. After isolating the box, we were left with two 1's on the other side of the balance. Therefore, the value of x is 2.

What happens if we have more than one box?

✓ Example 4.6.1

$$2x + 1 = 3$$

(4.6.2)

Solution

Step 1. Represent the equation as it is.



$$2x + 1 = 3$$

Step 2. Take one 1 out of the scale. To keep the balance, we will do the same in both sides.



$$2x + 1 = 3$$

$$2x + 1 - 1 = 3 - 1$$

$$2x = 2$$

Now that we have two boxes isolated on one side and two 1's on the other side, what's the next step? We need to determine how to distribute the 1's equally among the boxes. The number of 1's each box receives will be the value of x .

Step 3. Determine the value for each box.



$$x = 1$$

A

Now, consider, for example, the equation $2x + 4 = 3x + 1$. Notice that now we have boxes in both sides.

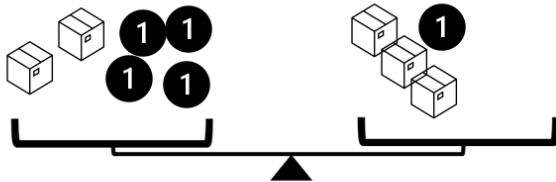
✓ Example 4.6.1

$$2x + 4 = 3x + 1$$

(4.6.3)

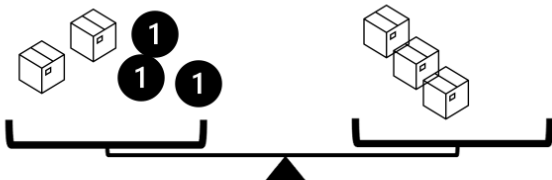
Solution

Step 1. Represent the equation as it is.



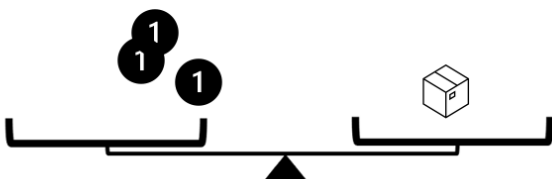
$$2x+4=3x+1$$

Step 2. Take one 1 out of each side of the scale.



$$\begin{aligned} 2x+4 &= 3x+1 \\ 2x+4-1 &= 3x+1-1 \\ 2x+3 &= 3x \end{aligned}$$

Step 3. Take two boxes out of each side of the scale.



$$\begin{aligned} 2x+3 &= 3x \\ 2x+3-2x &= 3x-2x \\ 3 &= x \end{aligned}$$

In this example, we also removed boxes from the scale.

Using balance scales can be tricky and has some limitations. For instance, representing negative numbers with balance scales can be challenging. Consider the difficulties in solving an equation like $2x + 1 = 3x + 4$ using this tool.

In this course, we will use balance scales only for problems that are easy to visualize with this method.

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4.7: Solving Linear Equations with Fractions and Decimals

The goal of this section is to provide traditional practice exercises that will help you assess your current understanding, sharpen your skills, engage in discussions with your classmates, and reflect on your learning process in solving algebraic equations. We will specifically focus on linear equations and include those linear equations with fractions and decimals.

The idea for solving linear equations is to isolate the x to one side of the equation and leave the corresponding value in the other side. In previous sections we discussed an approach for solving linear equations using a balanced scale. For some problems, this can be tricky, for example, involving negatives like: $-4x = 24$. While a balance scale may not make sense for solving all linear equations, the core idea of performing the same operation in both sides will still work.

✓ Example 4.7.1

Solve the equation $-4x = 24$

Solution

Notice we only have x to the left-hand side of the equation. However this is multiplied by -4 . To isolate the x we need to make the coefficient of x to be equal to one. This can be done by multiplying by $-\frac{1}{4}$, then, you need to make sure to keep the balance. So we will multiply by $-\frac{1}{4}$ in both sides of the equation. This is,

$$-4x = 24$$

$$-\frac{1}{4} \cdot -4x = -\frac{1}{4} \cdot 24$$

$$-\frac{1(-4)}{4} \cdot x = -\frac{1(24)}{4}$$

$$1 \cdot x = -\frac{1(24)}{4}$$

$$x = -6$$

Equivalently to this process, you could also divide by -4 in both sides of the equation

$$-4x = 24$$

$$\frac{-4}{-4} \cdot x = \frac{24}{-4}$$

$$1 \cdot x = -6$$

$$x = -6$$

If the x is multiplied by a number, the general idea is to multiply by the reciprocate or to divide by the coefficient. This approach will make the coefficient of x to be equal to 1, therefore you have $1 \cdot x$, which is just 1. In the case of a number that is added or subtracted to the x , then we will need this addition or subtraction to be equal to 0.

✓ Example 4.7.1

Solve the equation $x + 27 = 24$

Solution

Notice that we have a 27 added to the x . To isolate the x , we need to subtract 27. To keep the balance, we need to do the same to both sides.

$$x + 27 = 24$$

$$x + 27 - 27 = 24 - 27$$

$$x + 0 = -3$$

$$x = -3$$

If the equations have both, a coefficient for the variable and a number added or subtracted, there is a specific order to follow.

Discussion:

How can we solve the following equations? What order should we follow when solving linear equations?

$$12p + 9 - 11p = 2 - 4p + 7$$

$$2(x + 3) = -46$$

To solve linear equations, here is an approach you can follow:

1. Simplify each side of the equation.
2. Combine all terms in each side.
3. Rearrange the terms so that all linear terms are in one side of the equation and all constant terms are in the other side of the equation.
4. Multiply by the reciprocal of the coefficient or divide by the coefficient.

These steps can also be done even if the numbers involved in the linear equation are decimals or fractions.

When working with decimals, you can add, subtract, multiply, and divide decimals, as you know. Similarly, you can use the same approaches to work with equations involving fractions. However, some students prefer to use equivalent forms of the linear equations to simplify the process. You should be comfortable with both processes.

✓ Example 4.7.1

Solve the equation $\frac{1}{3}x + \frac{1}{5} = -1$

Solution

$$\frac{1}{3}x + \frac{1}{5} = -1$$

There is nothing to simplify or combine in each side of the equation. Let's start by rearranging the terms so that all linear terms are on one side of the equation and all constant terms are on the other side of the equation.

$$\frac{1}{3}x = -1 - \frac{1}{5}$$

We will treat fractions as always. Notice that $-1 - \frac{1}{5}$ would require us to add or subtract equally sized parts, in this case fifths.

This is, $-\frac{5}{5} - \frac{1}{5}$, so we have -5 fifths -1 fifths is the same as -6 fifths.

$$\frac{1}{3}x = -\frac{6}{5}$$

Now, we have a linear term in the left-hand side, and a constant to the right-hand side. So we can multiply by the reciprocal of the coefficient of the variable.

$$\frac{3}{1} \cdot \frac{1}{3}x = \frac{3}{1} \cdot -\frac{6}{5}$$

Recall that the process of multiplying fractions is as follows:

$$\frac{3 \cdot 1}{1 \cdot 3}x = -\frac{3 \cdot 6}{1 \cdot 5}$$

$$1 \cdot x = -\frac{18}{5}$$

$$x = -\frac{18}{5}$$

Alternatively, you could use a different (but equivalent approach), where you multiply the linear equation by 15. As this is a common multiple of all fractions, all numbers of the equation will now be integers.

$$15 \cdot \left(\frac{1}{3}x + \frac{1}{5}\right) = 15 \cdot -1$$

$$15 \cdot \left(\frac{1}{3}x\right) + 15 \cdot \left(\frac{1}{5}\right) = 15 \cdot -1$$

$$\frac{15}{3}x + \frac{15}{5} = -15$$

$$5x + 3 = -15$$

$$5x + 3 - 3 = -15 - 3$$

$$5x + 0 = -15 - 3$$

$$5x = -18$$

$$\frac{1}{5} \cdot 5x = \frac{1}{5} \cdot -18$$

$$1 \cdot x = \frac{-18}{5}$$

$$x = \frac{-18}{5}$$

Solving equations with decimals can also use similar approaches to the ones in the previous example.

✓ Example 4.7.1

Solve the equation $2x - 0.7 = 1.03x - 1.05$

Solution

$$2x - 0.7 = 1.03x - 1.05$$

There is nothing to simplify or combine in each side of the equation. Let's start by rearranging the terms so that all linear terms are on one side of the equation and all constant terms are on the other side of the equation.

$$2x - 1.03x = 0.7 - 1.05$$

$$0.97x = -0.35$$

$$\frac{0.97}{0.97}x = \frac{-0.35}{0.97}$$

$$x \approx -0.36082474$$

Equivalently, you can multiply by 100 and work with integers.

$$2x - 0.7 = 1.03x - 1.05$$

$$100 \cdot (2x - 0.7) = 100 \cdot (1.03x - 1.05)$$

$$100 \cdot (2x) - 100 \cdot (0.7) = 100 \cdot (1.03x) - 100 \cdot (1.05)$$

$$200x - 70 = 103x - 105$$

$$200x - 103x = -105 + 70$$

$$97x = -35$$

$$x = \frac{-35}{97}$$

$$x \approx -0.36082474$$

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4.8: Common Mistakes in Algebra

In previous sections, we focused on building a conceptual understanding of algebra, helping you understand why certain algebraic steps are allowed. Developing this conceptual foundation is a crucial step in learning algebra. However, in this section, we will complement that understanding by raising awareness of common mistakes students make in algebra.

As a learner of algebra, understanding the common mistakes students make is crucial. Recognizing mistakes made by others can serve as a valuable learning tool. By studying these errors, you can learn vicariously, avoiding similar issues in your own work, and gaining a broader perspective on how to approach algebraic problems.

Knowing common errors helps you recognize potential pitfalls in your own work. This awareness can prevent you from repeating the same mistakes, leading to a more accurate and efficient learning process. The skill of identifying and correcting common mistakes in algebra improves your problem-solving abilities and helps build your confidence in tackling algebraic problems. Understanding typical errors allows you to develop strategies to avoid them. This proactive approach leads to more effective problem-solving techniques and enhances your overall mathematical thinking.

Analyzing why certain mistakes occur can deepen your understanding of algebraic concepts. When you explore the reasons behind common errors, you gain insights into the underlying principles, reinforcing your grasp of the material.

While there are many errors in algebra, here is a list of some of them.

1. **Incorrect Sign Handling:** Moving terms to the other side of the equation is equivalent to subtracting the same term from both sides. While this strategy can make the process faster, it can lead to mistakes if you forget to change the sign. Instead of thinking of it as just "moving" terms, it's important to conceptualize the process: you are actually subtracting from both sides, which naturally requires using the opposite sign.
2. **Mistakes in Distribution:** Misapplying the distributive property, such as not distributing a factor to all terms inside parentheses or forgetting to distribute negative signs when multiplying.
3. **Combining Unlike Terms:** Attempting to combine terms that are not like terms (e.g., combining a constant with a variable term).
4. **Incorrectly Isolating the Variable:** Adding or subtracting the wrong term when trying to isolate the variable. Forgetting to perform an operation on both sides of the equation.
5. **Dividing Incorrectly:** Failing to divide all terms by the same coefficient when isolating the variable. Incorrectly applying division, especially when fractions are involved.
6. **Order of Operations Errors:** Misapplying the order of operations, leading to incorrect simplification. Solving in the wrong order, such as subtracting before dividing.
7. **Misinterpreting the Equal Sign:** Treating the equal sign as an operator rather than a symbol indicating balance between two sides. Failing to maintain balance by not applying the same operation to both sides.
8. **Dropping or Adding Terms:** Forgetting to include all terms in each step, leading to incomplete or incorrect equations. Accidentally adding extra terms that don't belong in the equation.
9. **Misunderstanding Zero:** Confusing the role of zero, especially when it acts as a placeholder or when dividing by zero.
10. **Errors in Fraction Operations:** Incorrectly adding, subtracting, multiplying, or dividing fractions, particularly when clearing fractions from an equation. Failing to find a common denominator when required.

This is not an exhaustive list of common algebra mistakes, but it serves as a reflective guide to help you be aware of and avoid these errors. In the exercises below, we will practice addressing common mistakes students make.

✓ Example 4.8.1

The following example shows a common mistake students make when doing algebra. Find the mistake, explain it, and then solve the problem correctly.^[1]

A student solved: $3x + 4 = 6$ by showing the following steps:

1. $3x + 4 = 6$ (4.8.1)

2. $\frac{3x}{3} + 4 = \frac{6}{3}$ (4.8.2)

3. $x + 4 = 2$ (4.8.3)

4. $x + 4 - 4 = 2 - 4$ (4.8.4)

5. $x = -2$ (4.8.5)

Solution

The error of this process is in the second step, because the student divided by 3 some of the terms, but not all. All other processes look fine. Here is a correct way of solving the equation:

1. $3x + 4 = 6$ (4.8.6)

2. $3x = 6 - 4$ (4.8.7)

3. $3x = 2$ (4.8.8)

4. $x = \frac{2}{3}$ (4.8.9)

Notice that we can always verify the solution by evaluating x in the original equation:

$$3x + 4 = 3 \times \frac{2}{3} + 4 = 2 + 4 = 6 \quad (4.8.10)$$

Therefore, the solution is now correct.

Discussion:

The following work shows a common mistake students make when doing algebra. Discuss what the mistake is, explain it, and then solve the problem correctly.[\[1\]](#)

A student solved: $2x - 6 = 2(x - 3)$ by showing the following steps:

1. $2x - 6 = 2(x - 3)$ (4.8.11)

2. $2x - 6 = 2x - 6$ (4.8.12)

3. $2x - 6 + 6 = 2x - 6 + 6$ (4.8.13)

4. $2x = 2x$ (4.8.14)

5. $\frac{2x}{2} = \frac{2x}{2}$ (4.8.15)

6. $x = x$ (4.8.16)

7. $x - x = x - x$ (4.8.17)

8. $x = 0$ (4.8.18)

[\[1\]](#) These examples and discussions were created by William O'Conner as resources for MATH 103 at Northern Illinois University.

4.9: Applications of Additive Growth and Multiplicative Growth- Patterns and Finances

Additive growth and *multiplicative growth* are two fundamental concepts used to describe how quantities increase over time. Understanding these can be particularly useful in recognizing patterns and analyzing financial situations like investments, interest rates, or savings.

Understanding additive growth

Additive growth occurs when a fixed amount is added to a quantity at each step or time interval. This type of growth is linear, meaning it increases by the same amount repeatedly.

Definition: Additive Growth

If you start with an initial or fixed amount I and add a constant m at each step, the amount y after x steps is given by: $y = I + m \times x$, where m is the fixed amount added each time.

We could see this in patterns.

✓ Example 4.9.1

Suppose you have a sequence where you add 3 to the previous number: 2, 5, 8, 11, etc.

The following table shows this sequence:

x	1	2	3	4	5	6
y	2	5	8	11	14	17

Explain why this is an additive growth and provide an equation to represent this relationship.

Solution

This is an example of additive growth where each number is increasing by 3. To represent this growth with an equation, we start with $I = 2$, then adding 3 each time makes $m = 3$. To know the amount y after x steps, we will have:

$$y = 2 + 3x \quad (4.9.1)$$

Discussion:

What would be the graphical representation of the sequence from the previous example?

In a financial context, if you save \$100 every month, your total savings after 12 months would grow by \$100 each month. After 12 months, you'd have saved \$1,200. In this example, the initial amount is 0, on the first month you have \$100, on the second month \$200, and so on.

This is also an additive growth. If x is the number of months and y is the total savings in dollars, the following table represents this context.

x	1	2	3	4	5	6
y	\$100	\$200	\$300	\$400	\$500	\$600

Notice that every time x is increased by 1 month, there is an increment of \$100. When the growth is linear, the growth will be constant everywhere in the graph. That is, the *rate of change* is constant.

The rate of change in a linear context is the ratio of the change in the dependent variable (usually y) to the change in the independent variable (usually x). This is often referred to as the slope of the line.

Understanding multiplicative growth

Multiplicative growth occurs when a quantity increases by a fixed percentage or factor at each step or time interval. This type of growth is exponential, meaning the increase becomes larger with each step as it is based on multiplying by a constant factor rather than adding a fixed amount.

Definition: Multiplicative Growth

If you start with an initial amount I and multiply by a constant r at each step, the amount y after x steps is given by: $y = I \times r^x$, where r is the growth factor, representing the percentage increase at each step.

We could see this in patterns.

✓ Example 4.9.1

Suppose you have a sequence where you multiply the previous number by 2: 3, 6, 12, 24, etc. The following table shows this sequence:

x	1	2	3	4	5	6
y	3	6	12	24	48	96

Explain why this is a multiplicative growth and provide an equation to represent this relationship.

Solution

This is an example of multiplicative growth where each number is doubled (multiplied by 2). To represent this growth with an equation, we start with $I = 3$ and a growth factor of $r = 2$. To know the amount y after x steps, we will have:

$$y = 3 \times 2^x \quad (4.9.2)$$

Discuss the graphical representation of this sequence. Unlike linear growth, where the graph is a straight line, the graph of multiplicative growth is a curve that becomes steeper as x increases. The rate of change is not constant but increases with each step.

In the financial context, if you invest \$1,000 in an account with a 5% annual interest rate compounded yearly, your total investment grows multiplicatively. After each year, the interest is calculated on the new total, so the amount grows exponentially.

The following table represents this context, assuming x is the number of years and y is the total value of the investment:

x	1	2	3	4	5	6
y	\$1,050	\$1,102.50	\$1,157.63	\$1,215.51	\$1,276.28	\$1,340.09

Notice that every time x increases by 1 year, the amount increases by a larger dollar amount than before. This is because the growth is based on a percentage of the previous total, so the increase gets larger as the total grows.

In an exponential context, the rate of change itself changes. Unlike in linear growth, where the rate of change is constant, the rate in exponential growth increases as the quantity increases, leading to a rapidly rising curve in the graph. This means the growth accelerates over time, making exponential growth a powerful concept in finance and other fields. While the amount of change grows at an increasing rate, the growth rate is determined by a constant percentage, which is why the function is considered exponential.

Comparing additive and multiplicative growth

It's important to note that a nonlinear relationship does not necessarily imply exponential growth, nor does an increasing rate of change. These misunderstandings are common but incorrect. A deeper conceptual understanding of additive and multiplicative growth is essential to distinguish between these two types of growth accurately. Additive growth occurs when the same quantity is added to an initial amount at each step or time interval. This results in a linear increase, where the change in the quantity remains constant over time. In contrast, multiplicative growth involves increasing a quantity by a constant percentage or factor at each step. This results in exponential growth, where the amount of increase becomes larger with each interval.

The graphical representations of additive and multiplicative growth highlight their distinct characteristics. In additive growth, the graph is a straight line, reflecting a constant rate of change. The slope of this line represents the fixed amount added per step. For example, saving \$100 each month creates a straight-line graph with a consistent upward slope, indicating that each additional month contributes the same \$100. Multiplicative growth, however, is represented by a curve on the graph that becomes steeper over time. This curvature illustrates the accelerating nature of exponential growth. The graph starts off less steep but rises more sharply as x increases, reflecting the growing rate of change. For example, an investment with compound interest will show a curve that rises rapidly, illustrating the effects of exponential growth.

The rate of change in additive growth is constant, meaning the amount of increase per step remains the same. This results in a linear pattern where each interval contributes equally to the total change. The rate of change in additive growth is straightforward and easy to calculate, represented by the slope of the line. In contrast, the rate of change in multiplicative growth varies. The growth rate increases exponentially as the quantity grows, leading to a rapidly accelerating curve. The amount of increase is proportional to the current quantity, resulting in more substantial changes as time progresses. This exponential nature makes multiplicative growth more complex to calculate and predict, as it involves exponential functions or logarithms.

Additive growth is commonly observed in scenarios where quantities are added at a constant rate. For instance, regularly depositing a fixed amount into a savings account leads to linear growth, making it a suitable model for budgeting and straightforward financial planning. Similarly, distance traveled at a constant speed represents additive growth, where the total distance increases by a fixed amount over time. Multiplicative growth is crucial in contexts involving compound effects or percentage increases. In finance, compound interest models multiplicative growth, where interest is calculated on an increasing total, leading to rapid growth. This model is also applicable to population studies, where populations grow by a constant percentage each period, resulting in exponential increases.

Note: The narrative of the text in this chapter was generated using the support of artificial intelligence.

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4.10: Practice Problems

Problems for Section 4.2

In the following exercises, simplify each expression.

- $2^3 - 12 \div (9 - 5)$
- $3^2 - 18 \div (11 - 5)$
- $2 + 8(6 + 1)$
- $4 + 6(3 + 6)$
- $20 \div 4 + 6(5 - 1)$
- $33 \div 3 + 4(7 - 2)$
- $3(1 + 9 \cdot 6) - 4^2$
- $2[1 + 3(10 - 2)]$
- $5[2 + 4(3 - 2)]$
- $8 + 2[7 - 2(5 - 3)] - 3^2$
- $(3x + 5) + (2x + 7)$
- $4x^2 + x^2 + 6x + 3x$
- $5x^2 + 3x + 9 + 2x^2 + 7x + 4$
- $(6x + 8) - (2x - 5) + (x + 3)$
- $3x^3 + 4x^2 + 7x + 9 + x^3 + 5x^2 + 2x + 1$
- $(8x^2 - 3x + 6) + (4x^2 + 9x - 2) + (x - 1)$
- $10x^3 + 6x^2 - x + 4 - (3x^3 - 2x^2 + 5x - 7)$
- $(7x^3 + 5x^2 - 4x + 1) + (2x^3 - x^2 + 9x - 6) + (x^2 - 3x + 4)$
- $12x^4 - 3x^3 + 8x^2 - x + 5 + 4x^4 + x^3 - 6x^2 + 9x - 2$
- $(5x^4 - 7x^3 + 12x^2 - 8x + 3) - (2x^4 + x^3 - 5x^2 + 6x - 9) + (x^4 - 3x^2 + 4)$
- Why is it important to use the order of operations to simplify an expression?

Problems for Section 4.3

- The 25 case: Suppose that you have a large 25×25 square with the unit squares along the border colored red. Adapt two of the techniques above to calculate the number of red unit squares. For each technique you used, write an explanation and include a picture. Think about how to use colors or other methods to make your picture and explanation clearer.
- Exploration: Suppose that you have 64 red squares. Can you use all of those squares to make the border of a larger square in a picture like the one above? If yes, what are the dimensions of the larger square? If not, why not?
 - What if you have 30 red squares? Same questions.
 - What if you have 256 red squares? Same questions.
- Generalization: If you have k red squares, is there a quick test you can do to decide if you can use all of those squares to make the border of a large square? Can you tell how big the square will be?

Problems for Section 4.4

- Examine the following equations. Decide: Is the statement always true, sometimes true, or never true? Justify your answers.
 - $\frac{a}{5} = \frac{5}{a}$
 - $n + 3 = m$
 - $3x = 2x + x$
 - $5k = 5k + 1$
- Consider the equation $20 + 3 = \underline{\hspace{1cm}}$.
 - Fill in the blank with something that makes the equation *always true*.
 - Fill in the blank with something that makes the equation *always false*.
 - Fill in the blank with something that makes the equation *sometimes true and sometimes false*.
- Consider the equation $5 = 40 - \underline{\hspace{1cm}}$.

- Is it possible to fill in the blank with something that makes the equation *always true*? If yes, provide the answer. If not, explain why not.
 - Is it possible to fill in the blank with something that makes the equation *always false*? If yes, provide the answer. If not, explain why not.
 - Is it possible to fill in the blank with something that makes the equation *sometimes true and sometimes false*? If yes, provide the answer. If not, explain why not.
4. Consider the equation $x + 3 = \underline{\hspace{1cm}}$.
- Is it possible to fill in the blank with something that makes the equation *always true*? If yes, provide the answer. If not, explain why not.
 - Is it possible to fill in the blank with something that makes the equation *always false*? If yes, provide the answer. If not, explain why not.
 - Is it possible to fill in the blank with something that makes the equation *sometimes true and sometimes false*? If yes, provide the answer. If not, explain why not.
5. Consider the given information in the first set of figures to determine the number of squares that will balance each scale in the second set of figures.

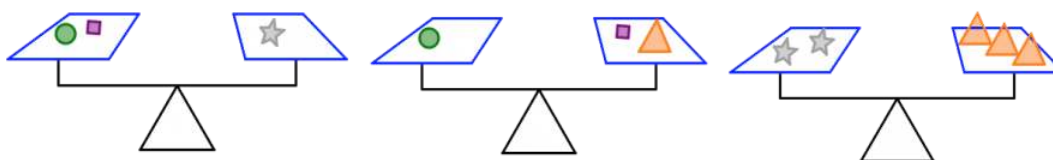


Figure 4.10.1: Given information: Assume all figures that look the same also weigh the same, and the scale is balanced.

How many purple squares will balance the scale in each case? Justify your answers.

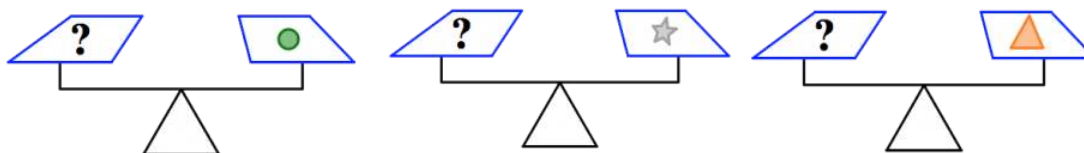


Figure 4.10.1: Three balance scales: (a) a balance with one green circle on the right; (b) a balance with one silver star on the right; and (c) a balance with one orange triangle on the right.

6. In the pictures below, what will balance the last scale? Can you find more than one answer? Assume all figures that look the same also weigh the same, and the scale is balanced.

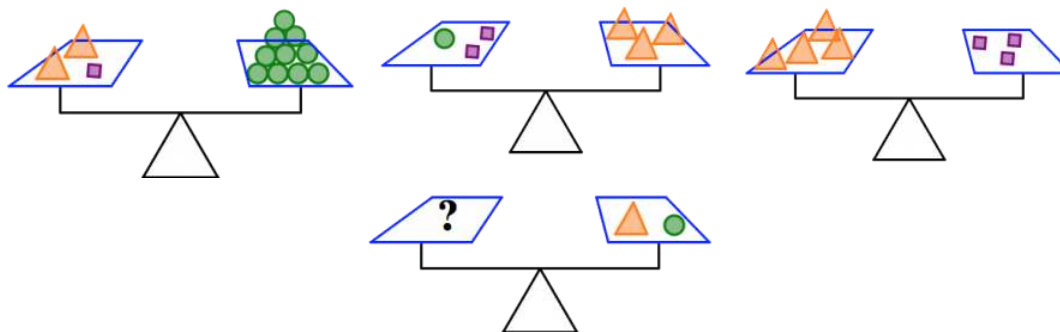


Figure 4.10.1: Balanced-scale representation

7. Use the figure below to answer the questions. Assume all figures that look the same also weigh the same, and the scale is balanced.
- Which shape weighs the most: the square, the triangle, or the circle? Which shape weighs the least? Justify your answers.
 - Which of the two scales is holding the most total weight? How do you know you're right?

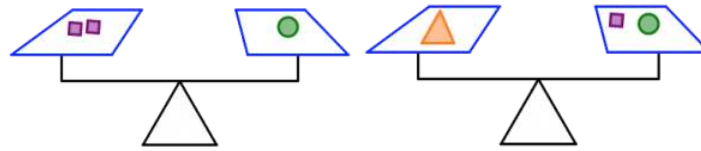


Figure 4.10.1: Balanced-scale representation

7. Use a pan-balance representation, like the one provided in the figure below, to solve the equation $5x + 2 = 7$. Include a pan-balance representation to the left and annotate your representation with the corresponding equation to the right of the representation. Reflect on the importance of the equal sign when solving equations.



Figure 4.10.1: Pan-balance connected to equation representation. You may need to add rows, or you might not use all the rows provided.

Problems for Section 4.5

1. The next patterns (A, B, C, D, E), do the following:
 - a. Describe in words and pictures how you see the pattern growing.
 - b. Calculate the number of tiles you would need to build the 10th figure in the pattern. Justify your answer based on how the pattern grows.
 - c. Calculate the number of tiles you would need to build the 100th figure in the pattern.
 - d. Describe how you can figure out the number of tiles in any figure in the pattern. Be sure to justify your answer based on how the pattern grows.
 - e. Could you make one of the figures in the pattern using exactly 25 tiles? If yes, which figure? If no, why not? Justify your answer.
 - f. Could you make one of the figures in the pattern using exactly 100 tiles? If yes, which figure? If no, why not? Justify your answer.

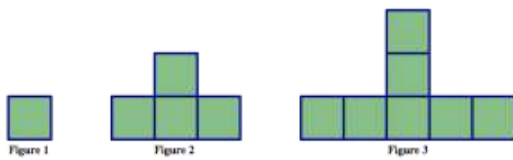


Figure 4.10.1: Pattern A

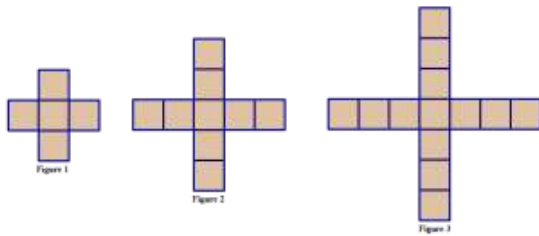


Figure 4.10.1: Pattern B

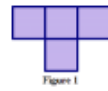


Figure 1



Figure 2



Figure 3

Figure 4.10.1: Pattern C



Figure 1

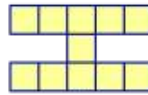


Figure 2

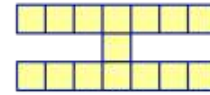


Figure 3

Figure 4.10.1: Pattern D

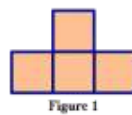


Figure 1

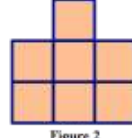


Figure 2

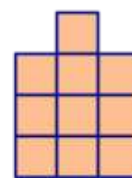


Figure 3

Figure 4.10.1:

Pattern E

Problems for Section 4.6

1. Use a pan-balance representation, like the one provided in the figure below, to solve each equation. Include a pan-balance representation to the left and annotate your representation with the corresponding equation to the right of the representation.

- $x+3=10$
- $x+7=14$
- $5x+2=7$
- $3x+5=20$
- $2x=10$
- $4x=12$
- $5x+1=3x+7$
- $x+5=3x+1$
- $7x+7=6x+10$
- $5x+8=3x+12$

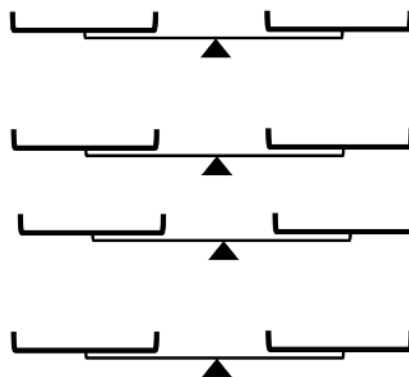


Figure 4.10.1: Pan-balance connected to equation representation. You may need to add rows, or you might not use all the rows provided.

Problems for Section 4.7

For this section, please use the following links:

Practice Problems for linear equations: <https://www.myopenmath.com/assess2/?cid=220955&aid=15606762#/>

Practice Problems for linear equations with fractions and decimals: <https://www.myopenmath.com/assess2/?cid=220955&aid=15607015#/>

Use the following information in MyOpenMath for more practice.

Course ID: 220955

Course name: Elements of Mathematical Reasoning

Enrollment key: 103-niu

Problems for Section 4.8

1. The following work shows common mistakes students make when doing algebra. Find what the mistake is, explain it, and then solve the problem correctly.[\[1\]](#)

Problem 1: A student solved: $x + 6 = \frac{4x}{3}$ by showing the following steps:

1.
$$x + 6 = \frac{4x}{3} \quad (4.10.1)$$

2.
$$x + 6 \times 3 = \frac{4x}{3} \times 3 \quad (4.10.2)$$

3.
$$x + 18 = 4x \quad (4.10.3)$$

4.
$$x + 18 - x = 4x - x \quad (4.10.4)$$

5.
$$18 = 3x \quad (4.10.5)$$

6.
$$\frac{18}{3} = \frac{3x}{3} \quad (4.10.6)$$

7.
$$x = 6 \quad (4.10.7)$$

Problem 2: A student solved: $8x + 7 = 2x$ by showing the following steps:

1.
$$8x + 7 = 2x \quad (4.10.8)$$

2.
$$\frac{8x + 7}{2} = \frac{2x}{2} \quad (4.10.9)$$

3.
$$4x + 7 = x \quad (4.10.10)$$

4.
$$4x + 7 - 7 = x - 7 \quad (4.10.11)$$

5.
$$4x = x - 7 \quad (4.10.12)$$

6.
$$4x - x = x - 7 - x \quad (4.10.13)$$

7.
$$3x = -7 \quad (4.10.14)$$

8.
$$\frac{3x}{3} = \frac{-7}{3} \quad (4.10.15)$$

9.
$$x = -\frac{7}{3} \quad (4.10.16)$$

Problems for Section 4.9

1. What is additive growth, and how does it differ from multiplicative growth?
2. If you have a savings account that increases by \$50 each month, how does this additive growth affect the total amount over time?
3. Imagine you receive \$10 more each week as your allowance. How does the total amount you receive change each week compared to if it were a fixed amount?
4. How would the total amount in your savings account change if it increased by a fixed percentage each month instead of a fixed amount?

5. What is multiplicative growth, and how does it differ from additive growth?
6. If a population of bacteria doubles every hour, how does the population size change over time compared to additive growth?
7. Consider a situation where your investment grows by 5% each year. How does this multiplicative growth affect the total value of your investment compared to a fixed annual increase?
8. How does the total amount of money in an account grow if it compounds at a fixed interest rate versus increasing by a fixed amount?
9. Examine the following table. Determine whether the growth represented is linear, multiplicative, or neither. Provide a detailed explanation for your answer.

x	0	1	2	3	4
y	1	2	5	10	17

10. Draw a graph to represent both additive and multiplicative growth over time. How do the shapes of these graphs differ?

Selected Solutions for Section 4.2

1. 5
2. 6
3. 58
4. 58
5. 29
6. 31
7. 149
8. 50
9. 30
10. 5
11. $5x + 12$
12. $5x^2 + 9x$
13. $7x^2 + 10x + 13$
14. $5x + 16$
15. $4x^3 + 9x^2 + 9x + 10$
16. $12x^2 + 7x + 3$
17. $7x^3 + 8x^2 - 6x + 11$
18. $10x^3 + 5x^2 + 2x - 1$
19. $16x^4 - 2x^3 + 2x^2 + 8x + 3$
20. $4x^4 - 8x^3 + 10x^2 - 14x + 16$

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CHAPTER OVERVIEW

5: Understanding Financial Literacy

- 5.1: Prelude to Understanding Financial literacy
- 5.2: Personal Financial Planning
- 5.3: Savings, Expenses, and Budgeting
- 5.4: Banking and Emergency Funds
- 5.5: Credit Cards and Other Debt
- 5.6: Education Debt- Paying for College
- 5.7: Practice Problems

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5.1: Prelude to Understanding Financial literacy



Figure 5.1.1: Financial success depends on getting a good start and avoiding setbacks and wrong turns. It's a lifelong process, more like a marathon than a sprint. (Credit: Bengt Nyman / Flickr / Attribution 2.0 Generic (CC-BY 2.0))

Now that we have explored algebraic concepts, this chapter will focus on how algebra can be a helpful tool for understanding real life. We will specialize in building financial literacy throughout arithmetic, and model it building on the concepts of **additive** and **multiplicative** growth.

How financially literate are you? This survey will help you determine how the chapter concepts relate to you right now. As we are introduced to new concepts and practices, it can be informative to reflect on how your understanding changes over time. We'll revisit these questions at the end of the chapter to see whether your feelings have changed. Take this quick survey to figure it out, ranking the statements on a scale of 1–4, 1 meaning “least like me” and 4 meaning “most like me.”

1. I actively and regularly plan and/or monitor my finances.
2. I understand the benefits and risks of credit.
3. I have a plan to repay my student loans.
4. I regularly take steps to protect my identity and assets.

“A big part of the college experience for many students is the art of the student loan process. This has been both a painful and challenging experience for me over the course of the first semester. The biggest struggle for me has been simply understanding what everything means and what I’m supposed to do. Another challenge has been determining how exactly I’m going to pay these loans back while also saving for rent, utilities, additional expenses, and a study abroad fund with a part-time job that I don’t even have yet.”

—Hanna Moyster

Mathematical tools can support your financial planning. In this chapter, you will learn to reach your personal life goals by implementing financial planning and strategies to protect yourself, manage your money today, and put yourself in a better position for tomorrow. How you act today impacts your tomorrow.

What Would You Do?

Everything was working out for Elan. They got into the college they wanted to, and some friends were planning to attend as well. They felt like an adult, and were looking forward to new freedoms and opportunities. Elan’s parents let them get a credit card after high school graduation. Elan shared an apartment with their friends just off campus, and was able to get where they needed to go because they had a car. Elan had also saved over \$1,000 from gifts and a summer job. They needed a new laptop.

Elan planned to stay within set limits. They went to the store and found a very knowledgeable salesperson, Jermain, who said he knew exactly what Elan needed. Jermain pointed out that the laptop in Elan’s budget would do schoolwork just fine, but it was not as powerful as the best top-of-the line unit with advanced gaming features. Plus, the better computer came with new headphones! Jermain suggested that Elan could later sell the computer to incoming students. (Most freshmen bought used computers if they did

not have one when they came to school.) The high-powered computer was \$2,000, though, and Elan didn't have that much money. Maybe they should use the credit card? Maybe their new part-time job would pay for it. But Jermain arranged for a small down payment and monthly payments of only \$100. That did not seem too bad to Elan. The future looked bright!

At least, that's what Elan thought. They soon realized that working more hours meant fewer hours to study. Meanwhile, Elan's rent and gas usage went up, and, as a young car owner, their insurance was through the roof. Only three months into the first semester, Elan missed a payment on the laptop and accrued a late fee. They put the next laptop payment on the credit card. Soon, Elan was alternating payments between the credit card, laptop, and car, building up interest and late charges. Now Elan was having trouble paying their rent and started getting calls from creditors. Everything had seemed so promising. Elan didn't know where they had gone wrong.

Elan comes to you and shares the situation. They ask, "What could I have done differently?"

This chapter offers you insight into your finances so that you can make good decisions and avoid costly mistakes. We all face chances to spend money and try to get what we want. Many think only about now and not next month, next year, or ten years from now, but our behavior now has consequences later. Not everyone can own all the latest technology, drive their dream car, continually invest for their retirement, or live in the perfect home at this moment. But by understanding the different components of earning money, banking, credit, and budgeting, you can begin working toward your personal and financial goals. We'll also discuss a related topic, safeguarding your accounts and personal information, which is critical to protecting everything you've worked for. By the end of this chapter, you will have good insights for Elan . . . and you!

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5.2: Personal Financial Planning

Discussion:

- What simple steps do I take to create a financial plan?
- How do I use financial planning in everyday life?
- How is the financial planning process implemented for every purchase?
- How is financial planning related to additive and multiplicative growth?

Honestly, practicing money management isn't that hard to figure out. In many ways it's similar to playing a video game. The first time you play a game, you may feel awkward or have the lowest score. Playing for a while can make you OK at the game. But if you learn the rules of the game, figure out how to best use each tool in the game, read strategy guides from experts, and practice, you can get really good at it.

Money management is the same. It's not enough to "figure it out as you go." If you want to get good at managing your money, you must treat money like you treat your favorite game. You have to come at it with a well-researched plan. Research has shown that people with stronger finances are healthier¹ and happier,² have better marriages,³ and even have better cognitive functioning.⁴

Student Survey

When asking students about their financial priorities, concerns, and plans, students expressed the following:

What is your top immediate financial priority?

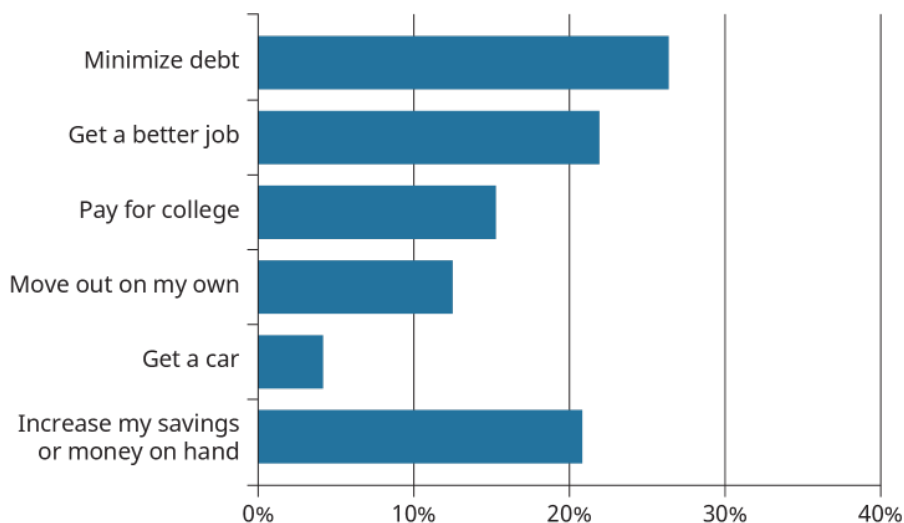


Figure 5.2.2

Which aspect of your finances concerns you the most?

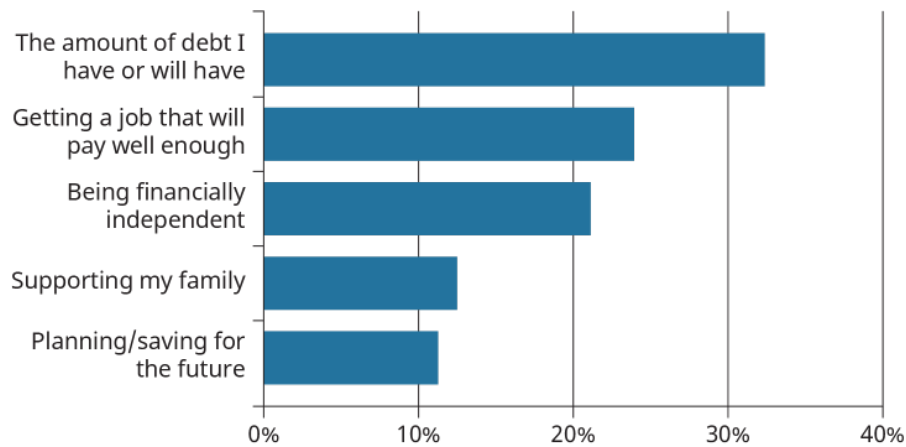


Figure 5.2.3

When considering how to pay for college, which of the following do you know *least* about?

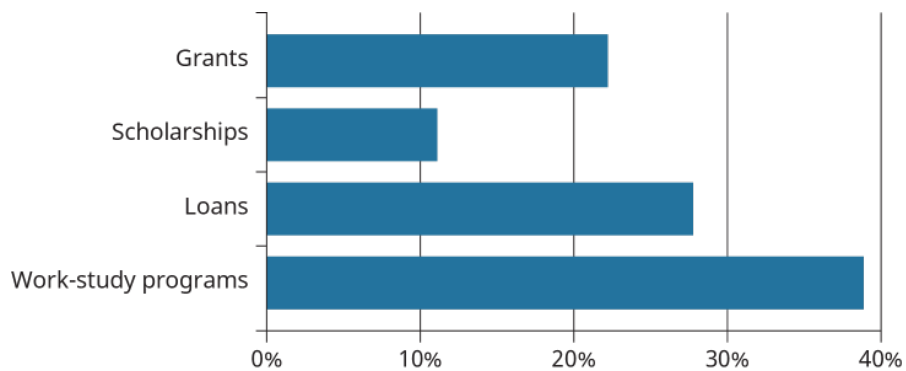


Figure 5.2.4

You can also take the anonymous [What Students Say surveys](#) to add your voice to this textbook. Your responses will be included in updates.

Financial Planning Process

Personal goals and behaviors have a financial component or consequence. To make the most of your financial resources, you need to do some financial planning. The financial planning process consists of five distinct steps: goal setting, evaluating, planning, implementing, and monitoring.

Financial Planning in Five Steps

1. Develop Personal Goals
 - What do I want my life to look like?
2. Identify and Evaluate Alternatives for Achieving Goals for My Situation
 - What do my savings, debt, income, and expenses look like?
 - Explore mathematical models and predict my financial situation.
 - What creative ways are available to get the life I want?
3. Write My Financial Plan
 - What small steps can I take to start working toward my goals?
4. Implement the Plan
 - Begin taking those steps, even if I can only do a few small things each week.
5. Monitor and Adjust the Plan
 - Make sure I don't get distracted by life. Keep taking those small steps each week. Make adjustments when needed.



Figure 5.2.5: Steps of financial planning.

How to Use Financial Planning in Everyday Life

The financial planning process isn't only about creating one big financial plan. You can also use it to get a better deal when you buy a car or computer or rent an apartment. In fact, anytime you are thinking about spending a lot of money, you can use the financial planning process to pay less and get more.

To explore financial planning in depth, we'll use the example of buying a car.

1. Develop Goals

First, what do you really need? If you're looking for a car, you probably need transportation. Before you decide to buy a car, consider alternatives to buying a car. Could you take a bus, walk, or bike instead? Often one goal can impact another goal. Cars are typically not good financial investments. We have cars for convenience and necessity, to earn an income and to enjoy life. Financially, they are an expense. They lose value, or depreciate, rather than increase in value, like savings. So buying a car may slow your savings or retirement plan goals. Cars continually use up cash for gas, repairs, taxes, parking, and so on. Keep this in mind throughout the planning process.

2. Identify and Evaluate Alternatives for Achieving Goals in Your Current Situation.

For this example, let's assume that you have determined the best alternative is to buy a car. Do you need a new car? Will your current car last with some upkeep? Consider a used car over a new one. On average, a new car will lose one-fifth of its value during its first year.⁵ Buying a one-year-old car is like getting a practically new car for a 20 percent discount. So in many cases, the best deal may be to buy a five- or six-year-old car. Perhaps someone in your family has a car they will sell you at a discount.

Sites such as the Kelley Blue Book website (KBB.com) and Edmunds.com can show you depreciation tables for the cars you are considering. You can build strong financial understanding by using mathematical models suitable for a specific vehicle make and model. For example, say that the depreciation value for a car you are interested to buy in 2026 is as follows:

Years after buying the car	Depreciation each year
1	16%
2	12%

3	11%
4	9%
5	7%

Discussion:

1. Browse online for two car models you are curious about and check their prices when buying them new. Use a table of depreciation value that is appropriate for your cars, if you do not find one, you can use the previous table. Create new tables and graphs to explore your cars' values in 1, 2, 3, 4, 5 years. For each table and graph, assume the x axis is the years after buying the car and the y axis is the actual value of the car each year.
2. How would you describe the relationship between the year after buying the car and the car value each year? Is this an additive or a multiplicative growth?
3. The graph you created should not be a linear graph; what would need to be different in order for the graph to be linear?

Definition: Linear Relationship

A **linear relationship** describes a connection between two variables where a change in one results in a constant *additive* change in the other. When plotted on a graph, this relationship always forms a straight line. In other words, the dependent variable increases or decreases by a constant amount for equal changes in the independent variable.

Do you know how much it will cost in total to own the car? It will help to check out the total cost of ownership tools (also on KBB.com and Edmunds.com) to estimate how much each car will cost you in maintenance, repairs, gas, and insurance. A cheap car that gets poor gas mileage and breaks down all the time will actually cost you more in the long run.

We can use mathematical models to help us understand the cost of having a car, not just buying a car. Specifically, we can think about a linear relationship as two variables where a change in one variable results in a constant change in the other variable. In this case, we will model the cost of the car, y , in dollars and how this value relates to x years of having the car. To create the model, discuss the following questions.

Discussion:

1. What costs are a one-time expense? Create a list of these costs, estimate the fixed cost and label them as "fixed cost."
2. What costs are recurring costs? Create a list of these costs, estimate the cost per year, and label the list as "variable cost."
3. Consider the fixed cost as one value. For example, in your value account for all expenses that will be a one-time thing together. This means that you will no longer think about down payment, car plate, etc. Instead, you will use "fixed cost" for all of these expenses together. Fill in the table for the cumulative fixed cost value.

Years after having my car	Cumulative fixed cost value
1 year	
2 years	
3 years	
...	

4. Consider the variable cost as one value. For example, in your value account for all variables that will be recurrent together. This means that you will no longer think about gas per year, insurance per year, car loan, etc. Instead, you will use the "variable cost" per year of all of these expenses together. Fill in the table for the cumulative variable cost value.

Years after having my car	Cumulative variable cost value
1 year	
2 years	

3 years

...

5. Use the previous information to create a graph and an equation to model the cost of the car y in dollars and how this value relates to x years of having the car.

In the previous chapter, we discussed that to represent an additive growth equation, we start with I , the initial amount, then adding m each time makes $y = I + mx$. We can also think about the initial amount, I , as the fixed cost because this does not change with x and the m as the variable cost.



Figure 5.2.6: Weighing all the factors is critical when deciding on any purchase, especially a major one like a car. (Credit: Greg Gjerdingen / Flickr / Attribution 2.0 Generic (CC-BY 2.0))

The information you generated in the discussion builds a strong base for creating a budget in your financial plan.

3. Write Down Your Financial Plan

Table 10.1 Examples of financial plans for a car and a computer.

Goal	Item	Details	Budget	Timeline
Transportation/Car	2014 Toyota Camry	Black, A/C, power windows, less than 60,000 miles	Car \$12,000 (max) Down payment \$3,000 Insurance \$100/mo Sales tax \$900 + Licensing \$145 Cash needed \$4,145	Have \$3600 in savings for this. Save \$50/week. Purchase in approximately 11 weeks.
Computer	Used or refurbished laptop	Dell w/ Windows, minimum 13", 128G hard drive, HD Graphics	\$300 Use free Windows update from school. Use free Wi-Fi at school.	Sell current laptop for \$100. Buy refurbished from Dell site for \$289. \$189 on credit card. Pay off when statement comes.

4. Implement Your Plan

Once you've narrowed down which car you are looking for, do more online research with resources such as Kelley Blue Book to see what is for sale in your area. You can also begin contacting dealerships and asking them if they have the car you are looking for with the features you want. Ask the dealerships with the car you want to give you their best offer, then compare their price to your

researched price. You may have to spend more time looking at other dealerships to compare offers, but one goal of online research is to save time and avoid driving from place to place if possible.

When you do go to buy the car, bring a copy of your written plan into the dealership and stick to it. If a dealership tries to switch you to a more expensive option, just say no, or you can leave to go to another dealership. Remember Elan in our opening scenario? He went shopping alone and caved to the pressure and persuasion of the salesperson. If you feel it is helpful, take a responsible friend or family member with you for support.

5. Monitor and Adjust the Plan to Changing Circumstances and New Life Goals

Life changes, and things wear out. Keep up the recommended maintenance on the car (or any other purchase). Keep saving money for your emergency fund, then for your next car. The worst time to buy a car is when your current car breaks down, because you are easier to take advantage of when you are desperate. When your car starts giving you trouble or your life circumstances start to change, you will be ready to shop smart again.

A good practice is to keep making car payments once the car loan is paid off. If you are paying \$300 per month for a car loan, when the loan is paid off, put \$300 per month into a savings account for a new car instead. Do it long enough and you can buy your next car using your own money!

Use the Financial Planning Process for Everything

The same process can be used to make every major purchase in your life. When you rent an apartment, begin with the same assessment of your current financial situation, what you need in an apartment, and what goals it will affect or fulfill. Then look for an apartment using a written plan to avoid being sold on a more expensive place than you want.

You can even use the process of assessing and planning for small things such as buying textbooks or weekly groceries. While saving a few bucks each week may seem like a small deal, you will gain practice using the financial planning process, so it will become automatic for when you make the big decisions in life. Stick to your plan.

Footnotes

1. <https://www.sciencedirect.com/scienc...77953613002839>
2. <https://academic.oup.com/geronj/arti...8/5/626/578092>
3. onlinelibrary.wiley.com/doi/...9.2012.00715.x and onlinelibrary.wiley.com/doi/1...715.x/abstract
4. science.sciencemag.org/conte...41/6149/976%20
5. Krome, Charles. "Car Depreciation." 2018, Carfax. <https://www.carfax.com/blog/car-depreciation>

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5.3: Savings, Expenses, and Budgeting

Discussion:

- How is the flow of money best measured?
- How do I keep things balanced?

“Do not save what is left after spending; instead spend what is left after saving.”

—Warren Buffett⁶

What is the best way to get to the Mississippi River from here? Do you know? To answer the question, even with a map app, you would need to know where you are starting from and exactly where on the river you want to arrive before you can map the best route. Our financial lives need maps, too. You need to know where you are now and where you want to end up in order to map a course to meet the goal.

You map your financial path using a spending and savings plan, or budget, which tracks your income, savings, and spending. You check on your progress using a balance sheet that lists your *assets*, or what you own, and your *liabilities*, or what you owe. A balance sheet is like a snapshot, a moment in time, that we use to check our progress.

Budgets

The term *budget* is unpleasant to some people because it just looks like work. But who will care more about your money than you? We all want to know if we have enough money to pay our bills, travel, get an education, buy a car, etc. Technically, a budget is a specific financial plan for a specified time. Budgets have three elements: income, saving and investing, and expenses.

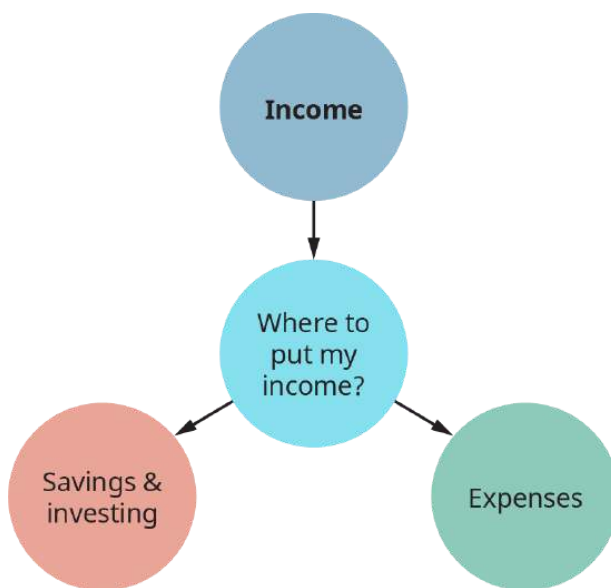


Figure 5.3.7: A budget is a specific financial plan for a finite amount of time. For example, you can set a budget for your family for a year.

Income

Income most often comes from our jobs in the form of a paper or electronic paycheck. When listing your income for your monthly budget, you should use your *net pay*, also called your disposable income. It is the only money you can use to pay bills. If you currently have a job, look at the pay stub or statement. You will find *gross pay*, then some money deducted for a variety of taxes, leaving a smaller amount—your net pay. Sometimes you have the opportunity to have some other, optional deductions taken from your paycheck before you get your net pay. Examples of optional deductions include 401(k) or health insurance payments. You can change these amounts, but you should still use your net pay when considering your budget.

Some individuals receive disability income, social security income, investment income, alimony, child support, and other forms of payment on a regular basis. All of these payments go under income. During school, you may receive support from family that could

be considered income. You may also receive scholarships, grants, or student loan money. Some of these sources may have tax implications. In general, scholarships and grants used for qualified educational expenses are not taxable, while funds used for other purposes may be. Student loans are not considered taxable income because they must be repaid. If taxes apply, they are typically the responsibility of the student, even if the student is claimed as a dependent.

Saving and Investing

The first bill you should pay is to yourself. You owe yourself today and tomorrow. That means you should set aside a certain amount of money for savings and investments, before paying bills and making discretionary, or optional, purchases. Savings can be for an emergency fund or for short-term goals such as education, a wedding, travel, or a car. Investing, such as putting your money into stocks, bonds, or real estate, offers higher returns at a higher risk than money saved in a bank. Investments include retirement accounts that can be automatically funded with money deducted from your paycheck. Automatic payroll deductions are an effective way to save money before you can get your hands on it. Setting saving as a priority assures that you will work to make the payment to yourself as hard as you work to make your car or housing payment. The money you “pay” toward saving or investing will earn you back your money, plus some money earned on your money. Compare this to the cost of buying an item on credit and paying your money plus interest to a creditor. Paying yourself first is a habit that pays off!

Pay yourself first! Put something in savings from every paycheck or gift.

Let's suppose you set aside your savings in a piggy bank. We will call this the piggy bank approach, where you do not earn any money for your actual savings. Instead you are planning on not using a credit card, which will save you a lot of money in the end. In the next sections, we will discuss how savings work with mathematical models when you use a savings account in a bank.

Discussion:

Imagine you want to go to a Chicago Bulls vs. Los Angeles Lakers game. When you check ticket prices, they start at \$200, and you would like to spend around \$800. You know you could pay with a credit card and take on debt, or you could wait until next season and start saving now. If you save \$25 each week, will you be able to afford the game? Notice that you may think there is not enough information for solving the problem, so you will need to make some assumptions. For example, are you going alone? Will you be okay with a \$250 ticket?

Support your argument by creating a table and a graph. Then answer the following questions:

1. Are your planned savings represented by additive growth? How do you know?
2. Which equation represents your planned savings? Make sure you label your variables appropriately.

Expenses

Expenses are categorized in two ways. One method separates them into fixed expenses and variable expenses. Rent, insurance costs, and utilities (power, water) are fixed: they cost about the same every month and are predictable based on your arrangement with the provider. Variable expenses, on the other hand, change based on your priorities and available funds; they include groceries, restaurants, cell phone plans, gas, clothing, and so on. You have a good degree of control over your variable expenses. You can begin organizing your expenses by categorizing each one as either fixed or variable.

Notice that classifying your expenses as fixed or variable can help you see the additive growth and create linear models that can help you organize your expenses.

Discussion:

1. Create a list of fixed expenses and a list of estimated variable expenses per month.
2. Notice that the fixed and variable expenses create an additive growth relationship. Model the expenses using a linear graph, a table, and a linear equation. While creating your model, think about the following questions:
 - a. What would make sense to use as x and y variables?

b. What is the interpretation of the model you created?

A second way to categorize expenses is to identify them as either needs or wants. Your needs come first: food, basic clothing, safe housing, medical care, and water. Your wants come afterward, if you can afford them while sticking to a savings plan. Wants may include meals at a restaurant, designer clothes, video games, other forms of entertainment, or a new car. After you identify an item as a need or want, you must exercise self-control to avoid caving to your desire for too many wants.

Discussion:

List the last ten purchases you made, and place each of them in the category you think is correct.

Table 10.2

Item	Need Expense \$	Want Expense \$
Totals		

How do your total “need” expenses compare to your total “want” expenses? Should either of them change?

Budgets are done in a chart or spreadsheet format and often look like the ones below. Pay attention to how the first budget differs from the second.

Table 10.3 This budget balances because all money is accounted for.

Income (use net monthly pay)	
Paycheck	\$2200
Other	\$300
Total Income	\$2,500
Saving and Investing	
Savings Account	\$120
Investments	\$240
Amount Left for Expenses	\$2,140
Expenses (Monthly)	
Housing	\$750
Car Payment/Insurance	\$450
Groceries	\$400
Restaurants/Food Delivery	\$100
Internet	\$60

Phone	\$60
Medical Insurance and Copays	\$120
Gas	\$200
Total Expenses	\$2,140
Balance (Amount left for expenses minus total expenses)	\$0

Table 10.4 Note that restaurants, phone, and gas are more expensive in this budget, so the total expenses are more than the amount left for them.

Income (use net monthly pay)	
Paycheck	\$2,200
Other	\$300
Total Income	\$2,500
Saving and Investing	
Savings Account	\$120
Investments	\$240
Amount Left for Expenses	\$2,140
Expenses (Monthly)	
Housing	\$750
Car Payment/Insurance	\$450
Groceries	\$400
Restaurants/Food Delivery	\$225
Internet	\$60
Phone	\$75
Medical Insurance and Copays	\$120
Gas	\$250
Total Expenses	\$2,330
Balance (Amount left for expenses minus total expenses)	-\$190

Balancing Your Budget

Would you take all your cash outside and throw it up in the air on a windy day? Probably not. We want to hold on to every cent and decide where we want it to go. Our budget allows us to find a place for each dollar. We should not regularly have money left over. If we do, we should consider increasing our saving and investing. We also should not have a negative balance, meaning we don't have enough to pay our bills. If we are short on money, we can look at all three categories of our budget: income, savings, and expenses.

We could increase our income by taking a second job or working overtime, although this is rarely advisable alongside college coursework. The time commitment quickly becomes overwhelming. Another option is to cut savings, or there's always the possibility of reducing expenses. Any of these options in combination can work.

Another, even less desirable option is to take on debt to make up the shortfall. This is usually only a short-term solution that makes future months and cash shortages worse as we pay off the debt. When we budget for each successive month, we can look at what we actually spent the month before and make adjustments.

Tracking the Big Picture

When you think about becoming more financially secure, you're usually considering your *net worth*, or the total measure of your wealth. Earnings, savings, and investments build up your assets—that is, the valuable things you own. Borrowed money, or debt, increases your liabilities, or what you owe. If you subtract what you owe from what you own, the result is your net worth. Your goal is to own more than you owe.

When people first get out of college and have student debt, they often owe more than they own. But over time and with good financial strategies, they can reverse that situation. You can track information about your assets, liabilities, and net worth on a balance sheet or part of a personal financial statement. This information will be required to get a home loan or other types of loans. For your net worth to grow in a positive direction, you must increase your assets and decrease your liabilities over time.

$$\text{Assets (Owned)} - \text{Liabilities (Owed)} = \text{Net Worth}$$

Discussion:

1. Identify areas in your life where you are losing money by paying fees on your checking account or interest on your loans. If you are not losing money, imagine a scenario of something that could happen to you or someone in the future.
2. What actions could you take to stop giving away money and instead set yourself up to start earning money? You may use your imaginary scenario.

Table 10.5

Good Practices that Build Wealth	Bad Practices that Dig a Debt Hole
Tracking all spending and saving	Living paycheck to paycheck with no plan
Knowing the difference between needs and wants	Spending money on wants instead of saving
Resisting impulse buying and emotional spending	Using credit to buy more than you need and increasing what you owe

You can write down your budget on paper or using a computer spreadsheet program such as Excel, or you can find popular budgeting apps that work for you.⁷ Some apps link to your accounts and offer other services such as tracking credit cards and your credit score. The key is to find an app that does what you need and use it.

Here are some examples:

- [Mint](#)
- [Mvelopes](#)
- [Wally](#)
- [Goodbudget](#)

Footnotes

6. Buffett, Warren. The Essays of Warren Buffett: Lessons for Corporate America. 1991. Cardozo Law Review.

7. <http://www.techtimes.com/articles/80...t-and-more.htm>

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5.4: Banking and Emergency Funds

Discussion:

- How do I plan appropriately for a financial emergency?
- What do I need to keep in mind when it comes to the banking system?

Emergency Funds

“Time and unexpected events [affect us] all.”

—King Solomon

Plan on the unplanned happening to you. It happens to all of us: a car repair, a broken computer, an unplanned visit to the doctor, a friend or relative in desperate need, etc. How will you pay for it? A recent study found that over 60 percent of households could not pay cash for a \$400 unexpected expense.⁸ Could you?

What Is an Emergency Fund?

An emergency fund is a cash reserve that’s specifically set aside for unplanned expenses or financial emergencies⁹. Some common examples include car repairs, home repairs, medical bills, and a loss of income. In general, emergency savings can be used for large or small unplanned bills or payments that are not part of your routine monthly expenses and spending.

Why Do I Need an Emergency Fund?

Without savings, even a minor financial shock could set you back, and if it turns into debt, it can potentially have a lasting impact.

Research suggests that individuals who struggle to recover from a financial shock have less savings to help protect against a future emergency. They may rely on credit cards or loans, which can lead to debt that’s generally harder to pay off. They may also pull from other savings, such as retirement funds, to cover these costs.

How Much Money Should I Keep in My Emergency Fund?

There is no magic or “official” amount to keep in an emergency fund, but you can look at your own life to get an idea to start with. How much could you put into a bank account to have for emergencies? Some students and their parents will not have a problem paying for most emergencies, but many students are on their own. What can you save up over time? A common recommendation for graduates with full-time jobs is perhaps three to six months’ worth of expenses. This may not be practical for you. A large sampling of students in financial literacy classes recommends approximately \$1,000.

One thousand dollars can cover a lot of small to medium unexpected expenses, such as last-minute textbooks, computer repair or replacement, car repair, or a prescription or doctor’s visit. The emergency fund is best kept separate from other money for living expenses to protect it as emergency money. While you could keep cash, an emergency fund is often best kept in a bank, in order to avoid theft or loss and still have easy access by debit card or automated teller machine (ATM). Pizza is not an emergency!



Figure 5.4.8: Emergency funds can cover the cost of a broken phone. (Credit: Simon Clancy / Flickr / Attribution 2.0 Generic (CC-BY 2.0))

How Do I Create an Emergency Fund?

Emergency funds can be created quickly if you have the money, or over time if you need to save a little from each paycheck, loan, or gift. You can use a financial planning tool. Follow these steps:

- Set an emergency fund goal.
- Identify an amount to keep on hand.
- Determine how to fund it, monthly or all at once.
- Decide where you will keep your fund (e.g., a savings account), and set specific dates to deposit money in it.
- Start now!

Discussion:

1. How do you feel when you do not have enough money for something?
2. Will you stop spending on some wants while you build up your emergency fund?

Safety and Success: Bank on It!

The banking system in the United States is one of the safest and most regulated banking systems in the world. A host of federal and state agencies regulate financial institutions to keep them from accidentally or purposefully losing customer money.

Banks, Credit Unions, and Online Banking

In the United States, financial institutions (FIs) are divided into multiple types of companies. The banking system is generally divided into banks and credit unions, which have similar offerings and are both regulated and insured by the federal government.

Choosing a Bank or Credit Union

When choosing a bank or credit union, it is important to understand what you are looking for and what benefits each company provides. Generally, large national banks offer the most advanced technology and a large network of branches. There are also smaller community banks that serve specific groups of people and may offer products to meet the specific needs of the community. For example, a community bank that serves Latino customers might make it easier to send money to family in South American countries, while a bank that focuses on small businesses will promote products specifically needed by business owners.

Credit unions differ from banks in that they don't have a profit motive. Instead, they are not-for-profit organizations that are owned by the people who bank with them. Each member of a credit union gets one vote for the board of directors, which runs the credit union. This means that whether you have \$5 in your account or \$5 million, you get the same vote. Credit unions tend to offer better rates and lower fees, on average, than banks.

There is no single best answer for what bank or credit union you should choose. The most vital question to ask and answer about a financial institution is whether it meets both your current and your future needs. Use figure 10.3 to compare different options and

determine the best one for you.

Many banks and credit unions do not publish the interest rates paid on deposit accounts or charged on loans. While some colleges have their own bank or credit union right on campus, you should consider visiting at least one other bank or one credit union to compare. You may also explore at least one online bank, which will publish interest rates on their website. Consider interest rates, access to ATMs, online transfers, automatic paycheck deposits, branch locations if you will use one, and other services that are important to you. Since you will select a bank or credit union that is insured, do not feel pressure to use any specific institution.

Table 10.6

Comparison Chart for Choosing a Bank/Credit Union			
Item	Local Bank	Credit Union	Online Only Bank
	Yes/No/Amount	Yes/No/Amount	Yes/No/Amount
Checking Accounts			
• Monthly Fees			
• Methods to Avoid Fees			
Savings Accounts			
• Interest Percentage			
• Monthly Fees			
• Methods to Avoid Fees			
Loans			
• Auto Loans			
• Home Loans			
• Credit Cards			
Nearest Branch Location*			
• Near Home?			
• Near School or Work?			
• Convenient ATMs?			
Online Banking Services*			
• Transfer Funds between Accounts			
• Open New Accounts			
• Pay Bills			
• Fees			
• Budgeting/Daily Transaction Access			
Additional Services or Fees			
* Branch locations are less important if you use online banking for most banking activity.			

Banking Products and Services

Banks and credit unions offer a similar set of financial products or services, called account types. The difference between the account types lies primarily in how easy it is to put money into or take money out of an account. Regulations set maximum numbers of transactions (deposits or withdrawals) for each type of account at a bank or credit union.

How you use these accounts is less about the rules and more about how long you plan to keep the money in the account. The main reason to use a bank is to keep your money safe and available. Banks may offer other services that benefit you, such as certificates of deposit (which allow you to earn higher interest over a longer time), retirement accounts, and car and home loans.

Checking

Checking accounts allow you to deposit money and take money out anytime you want. There are no government limits on the number of transactions, although a bank or credit union might begin to charge you if you make too many transactions. Checking accounts often don't pay any interest or pay an extremely low rate of interest. They are used to keep money safe and pay bills conveniently.

Checking accounts are ideal for depositing paychecks, cashing paper checks, buying everyday items, and paying your bills. The money you have in your checking account should be money you plan to spend by the end of the month. Any money you don't plan to spend within a month should be transferred from your checking account to a savings account. Your savings account should be the *first* bill that you pay each month. You can still add extra at the end of the month!

Savings Accounts

Savings accounts allow you a specific number of transactions each month or each quarter. If you go over the maximum number of transactions, the bank won't let you take any more money out or put any more money into the account until the next month. Savings accounts pay a small amount of interest on your money, but usually not enough to keep up with inflation or overcome banking fees (see the section on fees below).

This pattern actually causes your savings to go backwards. If you earn 2% on a savings account but inflation is 3% per year, you are losing 1% of purchasing power each year. For this reason, money in a savings account should be money you plan to spend within the next 12–48 months. The only exception to this is money you have saved for an emergency, called an emergency fund. Since you never know when an emergency (such as losing your job) is going to happen, you want the money to be available to you in a savings account.



Figure 5.4.9: Banks and credit unions can be accessed in many forms, both physical and online. (Credit David Hilowitz / Flickr / Attribution 2.0 Generic (CC-BY 2.0))

Debit Cards

When you get a checking account, you'll also get a debit card, or check card. This card allows you to access the money in your checking account (and savings account at an ATM) using a plastic card similar to a credit card. But it is not a credit card.

A debit card only uses money available in your account. Paying with a debit card is like paying with a paper check, but more immediate and convenient. You will have the option of selecting overdraft protection, which means the bank or credit union will allow you to buy stuff even if you don't have enough money in your account; they'll just charge you a fee, perhaps \$25, for each

event. This can be compared to a high-interest loan. Depending on how many things you buy in a week, overdraft protection could add many fees to your statement and use up your cash so it will not be available for your planned expenses. Consider opting out of overdraft protection and carefully keeping track of your account balance. This way you can only spend the money that you have.

Be aware that by using your debit card at an ATM associated with a different bank, you can incur fees—sometimes from both banks!

Debit cards offer a lot of security benefits over carrying around cash, including the ability to cancel a lost or stolen debit card. While the legal protections on debit cards are not as great as the legal protections on credit cards, you can't go into debt using a debit card. This inability to go into significant debt is a major advantage for those who struggle with debt.

Banking Fees

Banks and credit unions charge fees to operate. Many charge fees for a checking or savings account, overdrafts, and other services. You should seek to avoid fees for which you receive no extra services or when you can get similar services elsewhere for free. Two areas that are most subject to fees are services and “triggered” events. Triggered events are primarily caused by actions such as overdrawing your account (an overdraft). Overdraft fees are avoidable. The best way to avoid an overdraft fee is to continually monitor your bank balance and only spend money that you have.

Standard bank fees can often be avoided by taking one or more measures as specified by the bank, such as maintaining a minimum balance or using direct deposit. Avoid getting paid on a payroll or prepaid card unless you know all related costs or have a reason to want to be paid in that manner. Payroll cards often lead to ATM and banking fees, so federal law requires employers to offer you an alternative.¹⁰ Ask at your financial institution for assistance in setting up an account or accounts that are best for you.

Online and Mobile Banking

There are other important banking tools you should consider. Online and mobile banking are among the most important activities in banking. You should list all the things you might want to do regularly with your bank accounts and make sure you can do them through the bank's website and app. This might include making payments on loans, transferring money between your checking and savings accounts, paying bills through automated bill pay, and creating new savings accounts.

Learn the rules of your account, and keep track of how you use it. This can help you keep costs down and develop a positive banking relationship.

Earning Interest and Compound Interest

Interest refers to money paid for the privilege of borrowing money. When banks use our money for their investments, they pay us interest. (Remember, our bank accounts are insured, so you can't lose your money even when the banks use it.) When you take out a loan using the bank's money, you pay the bank interest.

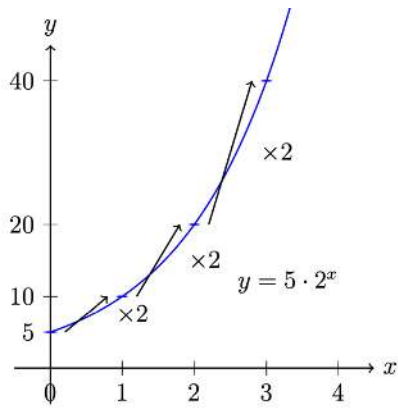
Compound interest means that you earn interest on the money that you deposit, called the principal, first. After that, you earn money on your money *plus* all the interest that has been paid to your account. Your earnings are reinvested. Interest on interest! You are being paid on the basis of other people's money—the interest that they paid you. Over time, this compound interest results in more and more money in your account. The same principle holds true for investing. Banks sometimes have investment services that pay higher interest but include risk to your money, which you typically access after completing your degree or certificate and obtaining full-time work.

The concept of multiplicative growth can help you understand how this works. Multiplicative growth is mathematically known as an exponential relationship.

Definition: Exponential Relationship

An **exponential relationship** describes a connection between two variables in which a change in one variable results in a constant *multiplicative* change in the other. When plotted on a graph, this relationship produces a curve that increases or decreases at an accelerating rate. In other words, the dependent variable changes by a constant *factor* for equal additive changes in the independent variable.

For example, the graph of $y = 5 \times 2^x$ shows that when x increases by 1, the value of y is multiplied by 2. Specifically, the y values follow the pattern: 5, 10, 20, 40, ... This constant multiplicative change is the defining feature of an exponential relationship.



Consider the example in the table below, and notice that the amount of interest paid each year is larger than the year before. That is compound interest. The only money deposited by the account owner was the first \$2,000.

Year	Starting Deposit or Beginning Amount from End of Last Year	Annual Interest Rate	Interest Paid over Period (1 year) (Interest x Beginning Amount)	Total at End of Year / Beginning Amount for Next Year
1	\$2,000.00	6%	\$120	\$2,120.00
2	\$2,120.00	6%	\$127.20	\$2,247.20
3	\$2,247.20	6%	\$134.83	\$2,382.03
4	\$2,382.03	6%	\$142.92	\$2,524.95

In this example, at the end of four years, the account owner has put in \$2,000 and has added \$524.95 of other people's money in earned interest!

Discussion:

1. Why is this an exponential relationship? Expand the table to represent years 5, 6, 7, 8, 9, 10.
2. Write an equation to represent, y , the total amount at the end of each year x .
3. Graph the relationship.
4. What happens to the amount of compound interest when you save for a longer time? When should you start saving?

You should always compare interest rates across multiple banks and understand how each one calculates interest. While some banks provide online calculators, it is good practice to verify the results using mathematical formulas and ask the bank for clarification when needed.

Footnotes

8. Board of Governors of the Federal Reserve System. "Report on the Economic Wellbeing of US Households, 2018" <https://www.federalreserve.gov/publications/and-expenditures.htm>
9. Consumer Financial Protection Bureau. "An Essential Guide to Building an Emergency Fund." https://www.consumerfinance.gov/star...ent=FY20_Jan_P
10. Dratch, Dana. "It pays to know these five things about payroll cards." Credit Cards.com. <https://www.creditcards.com/credit-c...loyer-1271.php>

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5.5: Credit Cards and Other Debt

Discussion:

1. What do I need to know about student loans?
2. How dangerous is debt?
3. What should I think about when getting and using a credit card?

Yes, taking on too much debt can (and does) have disastrous effects on people's personal finances, but if used appropriately, debt can be a tool to help you build wealth. Debt is like fire. You can use it to keep yourself warm, cook food, and ward off animals—but if you don't know how to control it, it'll burn your house down.

The Danger of Debt

When you take out a loan, you take on an obligation to pay the money back, with interest, through a monthly payment. You will take this debt with you when you apply for auto loans or home loans, when you enter into a marriage, and so on. Effectively, you have committed your future income to the loan. While this can be a good idea with student loans, take on too many loans and your future self will be poor, no matter how much money you make. Worse, you'll be transferring more and more of your money to the bank through interest payments.

Compounding Interest

While compounding works to make you money when you are earning interest on savings or investments, it works against you when you are paying the interest on loans. To avoid compounding interest on loans, make sure your payments are at least enough to cover the interest charged each month. The good news is that the interest you are charged will be listed each month on the loan account statements you are sent by the bank or credit union, and fully amortized loans will always cover the interest costs plus enough principal to pay off what you owe by the end of the loan term.

The two most common loans on which people get stuck paying compounding interest are credit cards and student loans. Paying the minimum payment each month on a credit card will just barely cover the interest charged that month, while anything you buy with the credit card will begin to accrue interest on the day you make the purchase. Since credit cards charge interest daily, you'll begin paying interest on the interest immediately, starting the compound interest snowball working against you. When you get a credit card, always pay the credit card balance down to \$0 each month to avoid the compound interest trap.

Student loans are another way you can be caught in the compound interest trap. When you have an unsubsidized student loan or put your loans into deferment, the interest continues to rack up on the loans. Again, you'll be charged interest on the interest, not just on the original loan amount, forcing you to pay compound interest on the loan.

Let's see the example of your deposit in a savings account, but now with a different perspective, you owe the money, this is not money you are getting for free, but money you didn't use that will be added to your debt.

✓ Example 5.5.1

Imagine you have a debt of \$2,000 and the bank charge an annual interest rate of 6%. How much money will you owe in 5 years?

Solution

Year	Starting Debt or Beginning Amount from End of Last Year	Annual Interest Rate	Interest Added to the Debt over Period (1 year) (Interest x Beginning Amount)	Total of Debt at End of Year / Beginning Amount for Next Year
1	\$2,000.00	6%	\$120	\$2,120.00
2	\$2,120.00	6%	\$127.20	\$2,247.20
3	\$2,247.20	6%	\$134.83	\$2,382.03

Year	Starting Debt or Beginning Amount from End of Last Year	Annual Interest Rate	Interest Added to the Debt over Period (1 year) (Interest x Beginning Amount)	Total of Debt at End of Year / Beginning Amount for Next Year
4	\$2,382.03	6%	\$142.92	\$2,524.95
5	\$2,524.95	6%	\$151.50	\$2676.45

In this example, at the end of five years, the original debt of \$2,000 will have an extra amount of \$676.45.

Discussion:

1. What happens to the amount of the debt if you wait 10 years to start paying? When should you start paying your debt?

Sacrificing Your Future Fun

When you graduate college, you are most likely to graduate with student loan debt and credit card debt.¹¹ Many students use credit cards and student loans to allow them to pay for fun today, such as trips, clothing, and expensive meals.

Getting into debt while in college forces you to sacrifice your future fun. Say you take out \$100,000 in student loans instead of the \$50,000 you need, doubling your monthly payment. You are not just making an extra \$338 payment; you are also sacrificing anything else you can do with that money. You sacrifice that extra \$338 a month, every month, for the next twenty-five years. You can't use it to go to the movies, pay down other debt, save for a home, take a vacation, or throw a party. When you sign those papers, you sacrifice all those opportunities every month for decades. As a result, when you take out a loan, you should make sure it's a good loan.

How Much Good Debt to Take On

A drink of water is refreshing on a hot day and is required to stay alive. Too much water, however, and you will drown.

During college and for the first few years after graduation, most students should only have two loans: student loans and possibly a car loan. We've already discussed your student loans, which should be equal to or less than your first year's expected salary after graduation.

When you get a car, you should keep your car payment to between 10% and 20% of your monthly take-home pay. This means if your paycheck is \$200 per week, your car payment should be no more than \$80–\$160 each month.

In total, you want your debt payments (plus rent if you are renting) to be no more than 44% of your take-home pay. If you are planning to build wealth, however, you want to cap it at 30% of take-home pay.

Signs You Have Too Much Debt

You can consider yourself in too much debt if you have any of the following situations:

- You cannot make your minimum credit card payments.
- Your money is gone before your next paycheck.
- Bill collectors are contacting you.
- You are unable to get a loan.
- Your paycheck is being garnished by creditors.
- You are considering a debt consolidation loan with extra fees added.
- Your items are repossessed.
- You do not know your debt or financial situation.

Getting and Using a Credit Card

One of the most controversial aspects of personal finance is the use of credit cards. While credit cards can be an incredibly useful tool, their high interest rates, combined with how easily credit cards can bury you in debt, make them extremely dangerous if not managed correctly.

Reflect on Elan from the chapter introduction and how he felt. How would you (or did you) feel to hold a new credit card with a \$2,000 spending limit?

Benefits of a Credit Card

There are three main benefits of getting a credit card. The first is that credit cards offer a secure and convenient method of making purchases, similar to using a debit card. When you carry cash, you have the potential of having the money lost or stolen. A credit card or debit card, on the other hand, can be canceled and replaced at no cost to you.

Additionally, credit cards offer greater consumer protections than debit cards do. These consumer protections are written into law, and with credit cards you have a maximum liability of \$50. With a debit card, you are responsible for transfers made up until the point you report the card stolen. In order to have the same protections as with credit cards, you need to report the card lost or stolen within 48 hours. The longer you wait to report the loss of the card, or the longer it takes you to realize you lost your card, the more money you may be responsible for, up to an unlimited amount.¹²

The final benefit is that a credit card will allow you to build your credit score, which is helpful in many aspects of life. While most people associate a credit score with getting better rates on loans, credit scores are also important to getting a job, lowering car insurance rates, and finding an apartment.¹³

What Is a Good Credit Score?

Most credit scores have a 300–850 score range. The higher the score, the lower the risk to lenders. A “good” credit score is considered to be in the 670–739 score range.

Table 10.8

Credit Score Ranges	Rating	Description
< 580	Poor	This credit score is well below the average score of US consumers and demonstrates to lenders that the borrower may be a risk.
580–669	Fair	This credit score is below the average score of US consumers, though many lenders will approve loans with this score.
670–739	Good	This credit score is near or slightly above the average of US consumers, and most lenders consider this a good score.
740–799	Very Good	This credit score is above the average of US consumers and demonstrates to lenders that the borrower is very dependable.
800+	Exceptional	This credit score is well above the average score of US consumers and clearly demonstrates to lenders that the borrower is an exceptionally low risk.

Components of a Credit Score and How to Improve Your Credit

Credit scores contain a total of five components. These components are credit payment history (35%), credit utilization (30%), length of credit history (15%), new credit (10%), and credit mix (10%). The main action you can take to improve your credit score is to stop charging and pay all bills on time. Even if you cannot pay the full amount of the credit card balance, which is the best practice, pay the minimum on time. Paying more is better for your debt load but does not improve your score. Carrying a balance on a credit card does not improve your score. Your score will go down if you pay bills late and owe more than 30% of your credit available. Your credit score is a reflection of your willingness and ability to do what you say you will do—pay your debts on time.

How to Use a Credit Card

All the benefits of credit cards are destroyed if you carry credit card debt. Credit cards should be used as a method of paying for things you can afford, meaning you should only use a credit card if the money is already sitting in your bank account and is

budgeted for the item you are buying. If you use credit cards as a loan, you are losing the game.

Every month, you should pay your credit card off in full, meaning you will be bringing the loan amount down to \$0. If your statement says you charged \$432.56 that month, make sure you can pay off all \$432.56. If you do this, you won't pay any interest on the credit card.

But what happens if you don't pay it off in full? If you are even one cent short on the payment, meaning you pay \$432.55 instead, you must pay daily interest on the entire amount from the date you made the purchases. Your credit card company, of course, will be perfectly happy for you to make smaller payments—that's how they make money. It is not uncommon for people to pay twice as much as the amount purchased and take years to pay off a credit card when they only pay the minimum payment each month.

What to Look for in Your First Credit Card

1. Find a Low-Rate Credit Card

Even though you plan to never pay interest, mistakes will happen, and you don't want to be paying high interest while you fix a misstep. Start by narrowing the hundreds of card options to the few with the lowest APR (annual percentage rate).

2. Avoid Cards with Annual Fees or Minimum Usage Requirements

Your first credit card should ideally be one you can keep forever, but that's expensive to do if they charge you an annual fee or have other requirements just for having the card. There are many options that won't require you to spend a minimum amount each month and won't charge you an annual fee.

3. Keep the Credit Limit Equal to Two Weeks' Take-Home Pay

Even though you want to pay your credit card off in full, most people will max out their credit cards once or twice while they are building their good financial habits. If this happens to you, having a small credit limit makes that mistake a small mistake instead of a \$5,000 mistake.

4. Avoid Rewards Cards

Everyone loves to talk about rewards cards, but credit card companies wouldn't offer rewards if they didn't earn them a profit. Rewards systems with credit cards are designed by experts to get you to spend more money and pay more interest than you otherwise would. Until you build a strong habit of paying off your card in full each month, don't step into that trap.

Footnotes

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12. Federal Trade Commission. "Lost Or Stolen Credit, ATM, and Debit Cards." 2012.

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5.6: Education Debt- Paying for College

Discussion:

- What choices should you consider when taking on student debt?
- How do you match debt to postgraduate income?
- What types of financial aid are available?
- How do you apply for financial aid?
- What are the best repayment strategies?

“An investment in knowledge always pays the best interest.”

—Benjamin Franklin, *The Way to Wealth: Ben Franklin on Money and Success*

As you progress through your college experience, the cost of college can add up rapidly. Worse, your anxiety about the cost of college may rise faster as you hear about the rising costs of college and horror stories regarding the “student loan crisis.” It is important to remember that *you* are in control of your choices and the cost of your college experience, and you do not have to be a sad statistic.

Education Choices

Education is vital to living. Education starts at the beginning of our life, and as we grow, we learn language, sharing, and to look both ways before crossing the street. We also generally pursue a secular or public education that often ends at high school graduation. After that, we have many choices, including getting a job and stopping our education, working at a trade or business started by our parents and bypassing additional schooling, earning a certificate from a community college or four-year college or university, earning a two-year or associate degree from one of the same schools, and completing a bachelor’s or advanced degree at a college or university. We can choose to attend a public or private school. We can live at home or on a campus.

Each of these choices impacts our debt, happiness, and earning power. The average income goes up with an increase in education, but that is not an absolute rule. The New York Federal Reserve Bank reported in 2017 that approximately 34% of college graduates worked in a job that did not require a college degree,¹⁴ and in 2013, CNN Money reported on a study from Georgetown University’s Center on Education and the Workforce showing that nearly 30% of Americans with two-year degrees are now earning more than graduates with bachelor’s degrees.¹⁵ Of course, many well-paying occupations do require a bachelor’s or master’s degree. You have started on a path that may be perfect for you, but you might also choose to make adjustments.

College success from a financial perspective means that you must:

- Know the total cost of the education
- Consider job market trends
- Work hard at school during the education
- Pursue ways to reduce costs

Most importantly: Buy only the amount of education that returns more than you invest.

According to *US News & World Report*, the average cost of college (including university) tuition and fees varies widely. In-state colleges average \$9,716 while out-of-state students pay \$21,629 for the same state college. Private colleges average \$35,676. The local community college averages approximately \$3,726. On-campus housing and meals, if available, can add approximately \$10,000 per year.¹⁶ See the table below, and create your own chart after you research.

Sample College Costs			
Type of School	Annual Tuition without Housing	Tuition If Living on Campus	Total Cost at Planned Completion
Community College (2 yr.)	\$3,726	Live at Home	\$7,452
Public University, In State (4 yr.)	\$9,716	Live at Home	\$38,864

Sample College Costs			
Public University, In State (4 yr.)		\$19,716	\$78,864
Public University, Out of State (4 yr.)	\$21,629	\$31,629	\$126,516
Private College (4 yr.)	\$35,676	\$45,676	\$182,704

You may need to adjust your college plan as circumstances change for you and in the job market. You can modify plans based on funding opportunities available to you (see next sections) and your location. You may prefer a strictly community college education, or you may complete two years at a community college and then transfer to a university to complete a bachelor's degree. Living at home for the first two years or all of your college education will save a lot of money if your circumstances allow. Be creative!

Key to Success: Matching Student Debt to Postgraduation Income

Students and parents often ask, "How much debt should I have?" The problem is that the correct answer depends on your personal situation. A big-firm attorney in a major city might make \$120,000 in their first year as a lawyer. Having \$100,000 or even \$200,000 in student debt in this situation may be reasonable. But a high school teacher making \$40,000 in their first year would never be able to pay off the debt.

The amount of debt you take on should be tied to the income you expect.

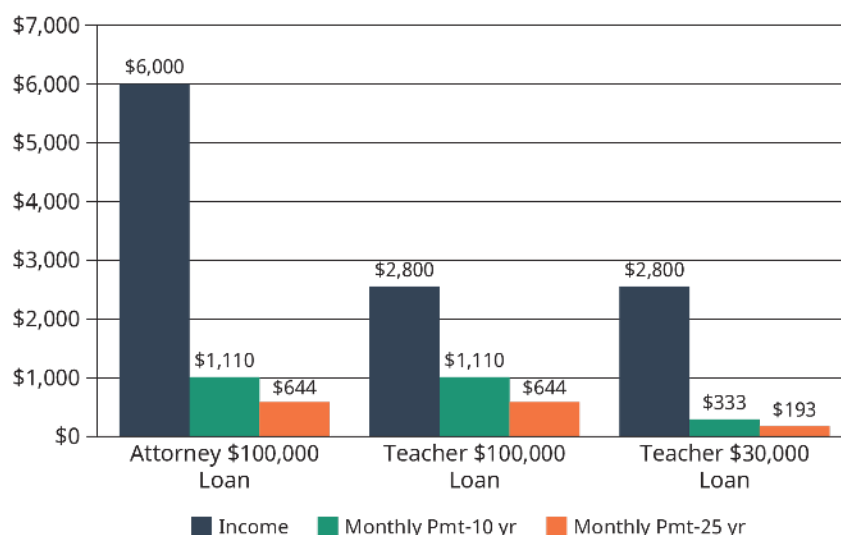


Figure 5.6.10: Each field of employment brings with it an average income and assumed debt. This graph shows the impact of an attorney's income versus debt, and then compares a teacher who took a \$100,000 loan with one who took a \$30,000 loan. Note the teacher's income is the same in both cases. (Credit: Based on information from National Association of Colleges and Employers and US Bureau of Labor Statistics.)

Research Your Starting Salary

Begin by researching your expected starting salary when you graduate. Most students expect to make significantly more than they will actually make.¹⁷ As a result, your salary expectations are likely much higher than reality. Ask professors at your college what is typical for a recent graduate in your field, or do informational interviews with human resource managers at local companies. Explore the US Bureau of Labor Statistics' [Occupational Outlook Handbook](#). [PayScale](#) also has a handy tool for getting general information based on your personal experience and location. Search websites and talk to employees of companies that interest you for future employment to identify real starting salaries.

Undergraduate Degree: 1 x Annual Salary

For students working toward a bachelor's or associate degree, both forms of undergraduate degrees, you should try to keep your student loans equal to or less than your expected first year's salary. So if, based on research, you expect to make \$40,000 in your

first year out of college, then \$33,000 in student loans would be a reasonable amount for you to pay out of a monthly budget with some sacrifice.

Advanced Degrees: 1–2 x Annual Salary

Once you've graduated with your bachelor's degree, you may want to get an advanced degree such as a master's degree, a law degree, a medical degree, or a doctorate. While these degrees can greatly increase your income, you still need to match your student debt to your expected income. Advanced degrees can often double your expected annual salary, meaning your total debt for all your degrees should be equal to or less than twice your expected first job income. A lower number for the debt portion of your education would be more manageable.

Your goal should be to pay for college using multiple methods so your student loan debt can be as small as possible, rather than just making low monthly payments on a large loan that will lead to a higher overall cost.

Types of Financial Aid: How to Pay for College

The true cost of college may be more than you expected, but you can make an effort to make the cost less than many might think. While the price tag for a school might say \$40,000, the net cost of college may be significantly less. The net price for a college is the true cost a family will pay when grants, scholarships, and education tax benefits are factored in. The net cost for the average family at a public in-state school is only \$3,980. And for a private school, free financial aid money reduces the cost to the average family from \$32,410 per year to just \$14,890.

If you haven't visited your college's financial aid office recently, it's probably worth it to talk with them. You must seek out opportunities, complete paperwork, and learn and meet criteria, but it can save you thousands of dollars.

Type of College	Average Published Yearly Tuition and Fees
Public Two-Year College (in-district students)	\$3,440
Public Four-Year College (in-state students)	\$9,410
Public Two-Year College (out-of-state students)	\$23,890
Private Four-Year College	\$32,410

Grants and Scholarships

Grants and scholarships are free money you can use to pay for college. Unlike loans, you never have to pay back a grant or a scholarship. All you have to do is go to school. And you don't have to be a straight-A student to get grants and scholarships. There is so much free money, in fact, that billions of dollars go unclaimed every year.¹⁸

While some grants and scholarships are based on a student's academic record, many are given to average students based on their major, ethnic background, gender, religion, or other factors. There are likely dozens or hundreds of scholarships and grants available to you personally if you look for them.

Federal Grants

Federal Pell Grants are awarded to students based on financial need, although there is no income or wealth limit on the grant program. The Pell Grant can give you more than \$6,000 per year in free money toward tuition, fees, and living expenses.¹⁹ If you qualify for a Pell Grant based on your financial need, you will automatically get the money.

Federal Supplemental Educational Opportunity Grants (FSEOGs) is additional free money available to students with financial need. Through the FSEOG program, you can receive up to an additional \$4,000 in free money. These grants are distributed through your school's financial aid department on a first-come, first-served basis, so pay close attention to deadlines.

Teacher Education Assistance for College and Higher Education (TEACH) Grants are designed to help students who plan to go into the teaching profession. You can receive up to \$4,000 per year through the TEACH Grant. To be eligible for a TEACH Grant, you must take specific classes and majors and must hold a qualifying teaching job for at least four years after graduation. If you do not fulfill these obligations, your TEACH Grant will be converted to a loan, which you will have to pay back with both interest and back interest.

There are numerous other grants available through individual states, employers, colleges, and private organizations.

State Grants

Most states also have grant programs for their residents, often based on financial need. Eleven states have even implemented free college tuition programs for residents who plan to continue to live in the state. Even some medical schools are beginning to be tuition free. Check your school's financial aid office and your state's department of education for details.

College/University Grants and Scholarships

Most colleges and universities have their own scholarships and grants. These are distributed through a wide variety of sources, including the school's financial aid office, the school's endowment fund, individual departments, and clubs on campus.

Private Organization Grants and Scholarships

A wide variety of grants and scholarships are awarded by foundations, civic groups, companies, religious groups, professional organizations, and charities. Most are small awards under \$4,000, but multiple awards can add up to large amounts of money each year. Your financial aid office can help you find these opportunities.

Employer Grants and Scholarships

Many employers also offer free money to help employees go to school. A common work benefit is a tuition reimbursement program, where employers will pay students extra money to cover the cost of tuition once they've earned a passing grade in a college class. And some companies are going even further, offering to pay 100% of college costs for employees. Check to see whether your employer offers any kind of educational support.



Figure 5.6.11: Employers in certain fields, such as healthcare, may offer their own grants and scholarships. (Credit: Ano Lobb / Flickr / Attribution 2.0 Generic (CC-BY 2.0))

Additional Federal Support

The federal government offers a handful of additional options for college students to find financial support.

Education Tax Credits

The IRS gives out free money to students and their parents through two tax credits, although you will have to choose between them. The American opportunity tax credit (AOTC) will refund up to \$2,500 of qualifying education expenses per eligible student, while the lifetime learning credit (LLC) refunds up to \$2,000 per year regardless of the number of qualifying students.

While the AOTC may be a better tax credit to choose for some, it can only be claimed for four years for each student, and it has other limitations. The LLC has fewer limitations, and there is no limit on the number of years you can claim it. Lifetime learners and nontraditional students may consider the LLC a better choice. Calculate the benefits for your situation.

The IRS warns taxpayers to be careful when claiming the credits. There are potential penalties for incorrectly claiming the credits, and you or your family should consult a tax professional or financial adviser when claiming these credits.

Federal Work-Study Program

The Federal Work-Study Program provides part-time jobs through colleges and universities to students who are enrolled in the school. The program offers students the opportunity to work in their field, for their school, or for a nonprofit or civic organization.

to help pay for the cost of college. If your school participates in the program, it will be offered through your school's financial aid office.

Student Loans

Federal student loans are offered through the US Department of Education and are designed to give easy and inexpensive access to loans for school. You don't have to make payments on the loans while you are in school, and the interest on the loans is tax deductible for most people. Direct Loans, also called Federal Stafford Loans, have a competitive fixed interest rate and don't require a credit check or cosigner.

Direct Subsidized Loans

Direct Subsidized Loans are federal student loans on which the government pays the interest while you are in school. Direct Subsidized Loans are made based on financial need as calculated from the information you provide in your application. Qualifying students can get up to \$3,500 in subsidized loans in their first year, \$4,500 in their second year, and \$5,500 in later years of their college education.

Direct Unsubsidized Loans

Direct Unsubsidized Loans are federal loans on which you are charged interest while you are in school. If you don't make interest payments while in school, the interest will be added to the loan amount each year and will result in a larger student loan balance when you graduate. The amount you can borrow each year depends on numerous factors, with a maximum of \$12,500 annually for undergraduates and \$20,500 annually for professional or graduate students.

There are also aggregate loan limits that apply to put a maximum cap on the total amount you can borrow for student loans.

Direct PLUS Loans

Direct PLUS Loans are additional loans a parent, grandparent, or graduate student can take out to help pay for additional costs of college. PLUS loans require a credit check and have higher interest rates, but the interest is still tax deductible. The maximum PLUS loan you can receive is the remaining cost of attending the school.

Parents and other family members should be careful when taking out PLUS loans on behalf of a child. Whoever is on the loan is responsible for the loan forever, and the loan generally cannot be forgiven in bankruptcy. The government can also take Social Security benefits should the loan not be repaid.

Private Loans

Private loans are also available for students who need them from banks, credit unions, private investors, and even predatory lenders. But with all of the other resources available for paying for college, a private loan is generally unnecessary and unwise. Private loans will require a credit check and potentially a cosigner, they will likely have higher interest rates, and the interest is not tax deductible. As a general rule, you should be wary of private student loans or avoid them altogether.

Repayment Strategies

Payments on student loans will begin shortly after you graduate. While many websites, financial "gurus," and talking heads in the media will encourage you to pay off your student loans as quickly as possible, you should give careful consideration to your repayment options and how they may impact your financial plans. Quickly paying off your student loans or refinancing your student loans into a private loan may be the worst option available to you.

Payment Plans

The federal government has eight separate loan repayment programs, each with their own way of calculating the payment you owe. Five of the programs tie loan payments to your income, which can make it easier to afford your student loans when you are just starting off in your career. The programs are described briefly below, but you should seek the help of a licensed fiduciary financial adviser familiar with student loans when making decisions related to student loan payment plans.

The standard repayment plan sets a consistent monthly payment to pay off your loan within 10 years (or up to 30 years for consolidated loans). You can also choose a graduated repayment plan, which will begin with lower payments and then increase the payment every two years. The graduated plan is also designed to pay off your student loans in 10 years (or up to 30 years for consolidated loans). A third option is the extended repayment plan, which provides a fixed or graduated payment for up to 25 years. However, none of these programs are ideal for individuals planning to seek loan forgiveness options, which are discussed below.

Beyond the “normal” repayment options, the government offers five income-based repayment options: (1) the Pay As You Earn (PAYE) repayment plan, (2) the Revised Pay As You Earn (REPAYE) repayment plan, (3) the Income-Based Repayment (IBR) plan, (4) the Income-Contingent Repayment (ICR) plan, and (5) the Income-Sensitive Repayment (ISR) plan. Each program has its method of calculating payments, along with specific requirements for eligibility and rules for staying eligible in the program. Many income-based repayment plans are also eligible for loan forgiveness after a set period of time, assuming you follow all the rules and remain eligible.

Loan Forgiveness Programs

Many income-based repayment options also have a loan forgiveness feature built into the repayment plan. If you make 100% of your payments on time and follow all the other plan rules, any remaining loan balance at the end of the plan repayment term (typically 20 to 30 years) will be forgiven. This means you will not have to pay the remainder on your student loans.

This loan forgiveness, however, comes with a catch: taxes. Any forgiven balance will be counted and taxed as income during that year. So if you have a \$100,000 loan forgiven, you could be looking at an additional \$20,000 tax bill that year (assuming you were in the 20% marginal tax rate).

Another option is the Public Service Loan Forgiveness (PSLF) program for students who go on to work for a nonprofit or government organization. If eligible, you can have your loans forgiven after working for 10 years in a qualifying public service job and making 120 on-time payments on your loans. A major advantage of PSLF is that the loan forgiveness may not be taxed as income in the year the loan is forgiven.

Consider Professional Advice

The complexity of the payment and forgiveness programs makes it difficult for nonexperts to choose the best strategy to minimize costs. Additionally, the strict rules and potential tax implications create a minefield of potential financial problems. In 2017, the first year graduates were eligible for the PSLF program, 99% of applicants were denied due to misunderstanding the programs or having broken one of the many requirements for eligibility.²⁰

Your Rights as a Loan Recipient

As a recipient of a federal student loan, you have the same rights and protections as you would for any other loan. This includes the right to know the terms and conditions for any loan before signing the paperwork. You also have the right to know information on your credit report and to dispute any loan or information on your credit file.

If you end up in collections, you also have several rights, even though you have missed loan payments. Debt collectors can only call you between 8 a.m. and 9 p.m. They also cannot harass you, threaten you, or call you at work once you’ve told them to stop. The United States doesn’t have debtors’ prisons, so anyone threatening you with arrest or jail time is automatically breaking the law.

Federal student loans also come with many other rights, including the right to put your loan in deferment or forbearance (pushing pause on making payments) under qualifying circumstances. Deferment or forbearance can be granted if you lose your job, go back to school, or have an economic hardship. If you have a life event that makes it difficult to make your payments, immediately contact the student loan servicing company on your loan statements to see if you can pause your student loan payments.

The Consumer Financial Protection Bureau (CFPB) has created a series of [sample letters](#) you can use to respond to a debt collector. You can also [file a complaint](#) with the CFPB if you believe your rights have been violated.

Applying for Financial Aid, FAFSA, and Everything Else

Take this first step—you will need to do it. The federal government offers a standard form called the Free Application for Federal Student Aid (FAFSA), which qualifies you for federal financial aid and also opens the door for nearly all other financial aid. Most grants and scholarships require you to fill out the FAFSA, and they base their decisions on the information in the application.

The FAFSA only requests financial aid for the specific year you file your application. This means you will need to file a FAFSA for each year you are in college. Since your financial needs will change over time, you may qualify for financial aid even if you did not qualify before.

You can apply for the FAFSA through your college’s financial aid office or at studentaid.gov if you don’t have access to a financial aid office. Once you file a FAFSA, any college can gain access to the information (with your approval), so you can shop around for financial aid offers from colleges.

Maintaining Financial Aid

To maintain your financial aid throughout your college, you need to make sure you meet the eligibility requirements for each year you are in school, not just the year of your initial application. The basic requirements include being a US citizen or eligible noncitizen, having a valid Social Security number, and registering for selective service if required. Undocumented residents may receive financial aid as well and should check with their school's financial aid office.

You also must make satisfactory academic progress, including meeting a minimum grade-point average, taking and completing a minimum number of classes, and making progress toward graduation or a certificate. Your school will have a policy for satisfactory academic progress, which you can get from the financial aid office.

What to Do with Extra Financial Aid Money

One expensive mistake that students make with financial aid money is spending the money on noneducation expenses. Students often use financial aid, including student loans, to purchase clothing, take vacations, or dine out at restaurants. Nearly 3% of students spend student loan money on alcohol and drugs.²¹ While this seems like fun now, these noneducation expenses are major contributors to student loan debt, which will make it harder for you to afford a home, take vacations, or save for your retirement after you graduate.

When you have extra student loan money, consider saving it for future education expenses. Just like you will need an emergency fund all your adult life, you will want an emergency fund for college when expensive books or travel abroad programs present unexpected costs. If you make it through your college years with extra money in your savings, you can use the money to help pay down debt.

Discussion:

1. How much student loan debt do you currently have, and how much do you think you'll have by the end of college?
2. How could this debt impact your future?
3. How can mathematical models such as linear and exponential relationships help you understand your finances?

Footnotes

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5.7: Practice Problems

1. Identifying Linear and Exponential Relationships

Question:

Where in your personal financial organization do **linear relationships** show? Where do **exponential relationships** show? You can consider the following and other examples: monthly savings with a fixed deposit, rent or subscription payments, credit card balances, savings accounts with interest. Explain why each example is linear or exponential.

2. Understanding Additive Growth

Question:

Explain what the following statement means in your own words:

“**Additive growth represents linear relationships.**”

Your explanation should describe: What happens to the amount over equal time intervals? Why is the rate of change constant? Provide an example.

3. Understanding Multiplicative Growth

Question:

Explain what the following statement means in your own words:

“**Multiplicative growth represents exponential relationships.**”

Your explanation should describe how the amount changes over time and why the growth depends on the current value. Provide an example.

4. Modeling Monthly Expenses (Linear Model)

Question:

Create a list of your **fixed expenses** and **variable expenses** for one month. Then:

1. Create a **table** showing total monthly expenses over several months.
 2. Draw a **graph** representing your expenses.
 3. Write a **linear equation** that models your total monthly expenses.
 4. Identify the **independent variable** and the **dependent variable**, and explain why.
-

5. Buying a Car: Planning with a Mathematical Model

Question:

Imagine you plan to buy a car. Create a model of expenses to consider when buying a car (not only the initial value).

Tasks:

1. Identify whether your model is **linear or exponential**, and explain why.
 2. Create a **mathematical model** to represent the total cost over time.
 3. Describe a **financial plan** that would allow you to afford the car responsibly.
-

6. Planning Your Dream Vacation

Question:

Imagine you are planning your dream vacation. Create a model of expenses. Consider: total cost, time until the trip, and how you will pay for it.

Tasks:

1. Create a **savings plan** using a mathematical model.
 2. Determine whether the relationship is **linear or exponential**.
 3. Explain how changing your monthly savings affects the model.
-

7. Buying a House: Long-Term Financial Planning

Question:

Imagine you plan to buy a house. Create a model of expenses to consider when buying a house (not only the initial value). Consider, for example, purchase price, down payment, mortgage interest rate, length of the loan

Tasks:

1. Explain which parts of the situation are **linear** and which are **exponential**.
 2. Create a model representing the total amount paid over time.
 3. Describe important financial considerations when planning to buy a home.
-

8. Linear Savings Problem

Question:

If you save **\$50 per month** in a piggy bank, how much money will you have after **3 years**?

Tasks:

1. Write a **linear equation** to represent the situation.
 2. Identify the slope and interpret its meaning.
 3. Find the total amount saved.
-

9. Exponential Savings with Compound Interest

Question:

You win **\$5,000** from a lottery ticket and deposit it into a savings account that compounds **annually at 5%**.

Tasks:

1. Write an **exponential equation** to model the account balance.
 2. Determine how much money you will have after **10 years**.
 3. Explain why this situation represents **multiplicative growth**.
-

Reflection

Question:

Why is it important to understand the difference between **linear** and **exponential** growth when making financial decisions?

Note: Problems in this section were generated using artificial intelligence and adapted by the author for inclusion in this book.

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CHAPTER OVERVIEW

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6.1: Introduction

Like most people, you probably feel that it is important to "take control of your life." But what does this mean? Partly it means being able to properly evaluate the data and claims that bombard you every day. If you cannot distinguish good from faulty reasoning, then you are vulnerable to manipulation and to decisions that are not in your best interest. Statistics provides tools that you need in order to react intelligently to information you hear or read. In this sense, statistics is one of the most important things that you can study.

To be more specific, here are some claims that we have heard on several occasions. (We are *not* saying that each one of these claims is true!)

- 4 out of 5 dentists recommend Dentyne.
- Almost 85% of lung cancers in men and 45% in women are tobacco-related.
- Condoms are effective 94% of the time.
- Native Americans are significantly more likely to be hit crossing the streets than are people of other ethnicities.
- People tend to be more persuasive when they look others directly in the eye and speak loudly and quickly.
- Women make 75 cents to every dollar a man makes when they work the same job.
- A surprising new study shows that eating egg whites can increase one's life span.
- People predict that it is very unlikely there will ever be another baseball player with a batting average over 400.
- There is an 80% chance that in a room full of 30 people that at least two people will share the same birthday.
- 79.48% of all statistics are made up on the spot.

All of these claims are statistical in character. We suspect that some of them sound familiar; if not, we bet that you have heard other claims like them. Notice how diverse the examples are; they come from psychology, health, law, sports, business, etc. Indeed, data and data interpretation show up in discourse from virtually every facet of contemporary life.

Statistics are often presented in an effort to add credibility to an argument or advice. You can see this by paying attention to television advertisements. Many of the numbers thrown about in this way do not represent careful statistical analysis. They can be misleading, and push you into decisions that you might find cause to regret. For these reasons, learning about statistics is a long step toward taking control of your life. (It is not, of course, the only step needed for this purpose.) This chapter will help you learn statistical essentials. It will help you become an intelligent consumer of statistical claims.

You can take the first step right away. To be an intelligent consumer of statistics, your first reflex must be to question the statistics that you encounter. The British Prime Minister Benjamin Disraeli famously said, "There are three kinds of lies -- lies, damned lies, and statistics." This quote reminds us why it is so important to understand statistics. So let us invite you to reform your statistical habits from now on. No longer will you blindly accept numbers or findings. Instead, you will begin to think about the numbers, their sources, and most importantly, the procedures used to generate them.

We have put the emphasis on defending ourselves against fraudulent claims wrapped up as statistics. Just as important as detecting the deceptive use of statistics is the appreciation of the proper use of statistics. You must also learn to recognize statistical evidence that supports a stated conclusion. When a research team is testing a new treatment for a disease, statistics allows them to conclude based on a relatively small trial that there is good evidence their drug is effective. Statistics allowed prosecutors in the 1950's and 60's to demonstrate racial bias existed in jury panels. Statistics are all around you, sometimes used well, sometimes not. We must learn how to distinguish the two cases.

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6.2: Categorizing data

Once we have gathered data, we might wish to classify it. Roughly speaking, data can be classified as **categorical data** or **quantitative data**.

Definition: Categorical and Quantitative Data

- **Categorical (qualitative) data** are pieces of information that allow us to classify the objects under investigation into various categories.
- **Quantitative data** are responses that are numerical in nature and with which we can perform meaningful arithmetic calculations.

✓ Example 6.2.1

We might conduct a survey to determine the name of the favorite movie that each person in a math class saw in a movie theater.

When we conduct such a survey, the responses would look like: *Finding Nemo*, *The Hulk*, or *Terminator 3: Rise of the Machines*. What type of data would be collected by conducting this survey?

Solution

This is categorical data. We might count the number of people who give each answer, but the answers themselves do not have any numerical values; we cannot perform computations with an answer like "*Finding Nemo*."

✓ Example 6.2.1

A survey could ask the number of movies you have seen in a movie theater in the past 12 months (0, 1, 2, 3, 4...). What type of data would be collected by conducting this survey?

Solution

This would be quantitative data.

Other examples of quantitative data would be the running time of the movie you saw most recently (104 minutes, 137 minutes, 104 minutes...) or the amount of money you paid for a movie ticket the last time you went to a movie theater (\$5.50, \$7.75, \$9...).

Discussion:

Classify each measurement as categorical or quantitative.

- a. Eye color of a group of people
- b. Daily high temperature of a city over several weeks
- c. Annual income

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6.3: Measures of Central Tendency

Let's begin by trying to find the most "typical" value of a data set.

Note that we just used the word "typical" although in many cases you might think of using the word "average." We need to be careful with the word "average" as it means different things to different people in different contexts. One of the most common uses of the word "average" is what mathematicians and statisticians call the **arithmetic mean**, or just plain old **mean** for short. "Arithmetic mean" sounds rather fancy, but you have likely calculated a mean many times without realizing it; the mean is what most people think of when they use the word "average."

Definition: Mean

The **mean** of a set of data is the sum of the data values divided by the number of values.

The interpretation of the mean is based on the definition. Let's see some examples to explore how to compute and interpret the mean.

✓ Example 6.3.1

Marci's exam scores for her last math class were: 79, 86, 82, 94. Marci's final course grade is calculated based on the mean of these four exams. What would be Marci's final course grade? What is the interpretation of the mean?

Solution

$$\frac{79 + 86 + 82 + 94}{4} = 85.25.$$

Typically we round means to one more decimal place than the original data had. In this case, we would round 85.25 to 85.3. In a traditional US system, Marci would expect a B in the class.

Interpretation: If we equally distribute all the score points into the four exams, each exam will account for approximately 85.25 points.

✓ Example 6.3.1

The number of touchdown (TD) passes thrown by each of the 31 teams in the National Football League in the 2000 season are shown below.

37 33 33 32 29

28 28 23 22 22

22 21 21 21 20

20 19 19 18 18

18 18 16 15 14

14 14 12 12 9

6

Calculate and interpret the mean.

Solution

Adding these values, we get 634 total TDs. Dividing by 31, the number of data values, we get $\frac{634}{31} = 20.4516$. It would be appropriate to round this to 20.5.

It would be most correct for us to report that "The mean number of touchdown passes thrown in the NFL in the 2000 season was 20.5 passes," but it is not uncommon to see the more casual word "average" used in place of "mean."

Interpretation: If we equally distribute the number of touchdowns among the 31 teams, each team will get approximately 20.5 points.

Discussion:

The price of a jar of peanut butter at 5 stores was: \$3.29, \$3.59, \$3.79, \$3.75, and \$3.99.

1. Find the mean price.
2. What is the unit?
3. Write an interpretation for your value.

✓ Example 6.3.1

The one hundred families in a particular neighborhood are asked their annual household income, to the nearest \$5,000. The results are summarized in a frequency table below.

Income (thousands of dollars)	Frequency
15	6
20	8
25	11
30	17
35	19
40	20
45	12
50	7

Calculate and interpret the mean.

Solution

Calculating the mean by hand could get tricky if we try to type in all 100 values:

$$\frac{15 + \dots + 15 + 20 + \dots + 20 + 25 + \dots + 25 + \dots}{100}$$

We could calculate this more easily by noticing that adding 15 to itself six times is the same as $15 \cdot 6 = 90$. Using this simplification, we get

$$\frac{15 \cdot 6 + 20 \cdot 8 + 25 \cdot 11 + 30 \cdot 17 + 35 \cdot 19 + 40 \cdot 20 + 45 \cdot 12 + 50 \cdot 7}{100} = \frac{3390}{100} = 33.9$$

The mean household income of our sample is 33.9 thousand dollars (\$33,900).

Interpretation: If the total amount of money earned by all households is equally split in each of them, each household would get \$33,900.

Notice all these values are between \$15,000 and \$50,000. Having a mean of \$33,900 is reasonable. Sometimes, the value of the mean can be biased. This can happen when we have atypical values. In this case, the median can support our analysis.

✓ Example 6.3.1

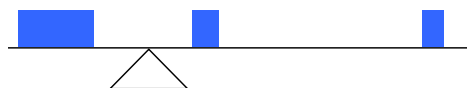
Extending off the last example, suppose a new family moves into the neighborhood, which has a household income of \$5 million (equivalent to \$5000 thousand). What is the new value of the mean?

Solution

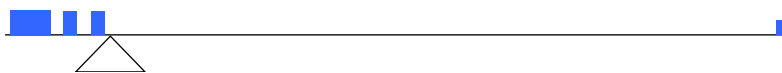
$$\frac{15 \cdot 6 + 20 \cdot 8 + 25 \cdot 11 + 30 \cdot 17 + 35 \cdot 19 + 40 \cdot 20 + 45 \cdot 12 + 50 \cdot 7 + 5000 \cdot 1}{101} = \frac{8390}{101} = 83.069$$

While 83.1 thousand dollars (\$83,069) is the correct mean household income, it no longer represents a “typical” value.

Imagine the data values on a see-saw or balance scale. The mean is the value that keeps the data in balance, like in the picture below.



If we graph our household data, the \$5 million data value is so far out to the right that the mean has to adjust up to keep things in balance



For this reason, when working with data that have **outliers** – values far outside the primary grouping – it is common to use a different measure of center, the **median**.

Definition: Median

The **median** represents the positional center of the data. In other words, about half of the data is above the median and half is below the median. To find the median, begin by listing the data in order from smallest to largest, or largest to smallest.

If the number of data values, N , is odd, then the median is the middle data value. This value can be found by rounding $\frac{N}{2}$ up to the next whole number.

If the number of data values is even, there is no one middle value, so we find the mean of the two middle values (values $\frac{N}{2}$ and $\frac{N}{2} + 1$).

✓ Example 6.3.1

The number of touchdown (TD) passes thrown by each of the 31 teams in the National Football League in the 2000 season are shown below.

37 33 33 32 29

28 28 23 22 22

22 21 21 21 20

20 19 19 18 18

18 18 16 15 14

14 14 12 12 9

6

Calculate and interpret the median.

Solution

We would start by listing the data in order. Luckily, it was already in decreasing order, so we can work with it without needing to reorder it first.

Since there are 31 data values, an odd number, the median will be the middle number, the 16th data value ($\frac{31}{2} = 15.5$), round up to 16, leaving 15 values below and 15 above). The 16th data value is 20, so the median number of touchdown passes in the 2000 season was 20 passes. Notice that for this data, the median is fairly close to the mean we calculated earlier, 20.5.

Interpretation: In the 2000 season, about half of the teams had more than 20 touchdown passes and about half had fewer than 20 passes.

✓ Example 6.3.1

Find the median of these quiz scores: 5 10 8 6 4 8 2 5 7 7.

Solution

We start by listing the data in order: 2 4 5 5 6 7 7 8 8 10.

Since there are 10 data values, an even number, there is no one middle number. So we find the mean of the two middle numbers, 6 and 7, and get $\frac{6+7}{2} = 6.5$.

The median quiz score was 6.5.

Interpretation: About half of the quiz scores are below 6.5 and half are above 6.5.

Discussion

The price of a jar of peanut butter at 5 stores was: \$3.29, \$3.59, \$3.79, \$3.75, and \$3.99. Calculate and interpret the median price. If you were to select the median or the mean to represent this value, which one would you select?

✓ Example 6.3.1

Calculate and interpret the median of the original household problem:

Income (thousands of dollars)	Frequency
15	6
20	8
25	11
30	17
35	19
40	20
45	12
50	7

Solution

Here we have 100 data values. If we didn't already know that, we could find it by adding the frequencies. Since 100 is an even number, we need to find the mean of the middle two data values -- the 50th and 51st data values. To find these values, we start counting up from the bottom:

There are 6 data values of \$15, so	Values 1 to 6 are \$15 thousand
The next 8 data values are \$20, so	Values 7 to $(6+8) = 14$ are \$20 thousand
The next 11 data values are \$25, so	Values 15 to $(14+11) = 25$ are \$25 thousand
The next 17 data values are \$30, so	Values 26 to $(25+17) = 42$ are \$30 thousand
The next 19 data values are \$35, so	Values 43 to $(42+19) = 61$ are \$35 thousand

From this we can tell that values 50 and 51 will be \$35 thousand, and the mean of these two values is \$35 thousand. The median income in this neighborhood is \$35 thousand.

Interpretation: About half of the neighborhood's households have an income of \$35,000 or less and about half have more than or equal to \$35,000.

Discussion:

If we add in the new neighbor with a \$5 million household income, then there will be 101 data values, and the 51st value will be the median. What would be the median? If you were to select the median or the mean to represent the household income of this neighborhood, would you choose the mean or the median? Explain your rationale.

As we discovered in the last example, the 51st value is \$35 thousand. Notice that the new neighbor did not affect the median in this case. The median is not swayed as much by outliers as the mean is.

In addition to the mean and the median, there is one other common measurement of the "typical" value of a data set: the **mode**.

 Definition: Mode

The **mode** is the element of the data set that occurs most frequently.

The mode is fairly useless with data like weights or heights where there are a large number of possible values. The mode is most commonly used for categorical data, for which median and mean cannot be computed.

✓ Example 6.3.1

Calculate and interpret the mode of the original household problem:

Income (thousands of dollars)	Frequency
15	6
20	8
25	11
30	17
35	19
40	20
45	12
50	7

Solution

The mode is \$40,000.

Interpretation: The most frequent household income is \$40,000.

It is possible for a data set to have more than one mode if several categories have the same frequency, or no modes if each every category occurs only once.

Discussion:

Reviewers were asked to rate a product on a scale of 1 to 5. Find

- The mean rating
- The median rating
- The mode rating

Rating	Frequency
1	4
2	8
3	7
4	3
5	1

For the previous discussion, notice that the mean is $\frac{1 \cdot 4 + 2 \cdot 8 + 3 \cdot 7 + 4 \cdot 3 + 5 \cdot 1}{23} \approx 2.5$. There are 23 data values, so the median will be the 12th data value. Ratings of 1 are the first 4 values, while a rating of 2 are the next 8 values, so the 12th value will be a rating of 2. The median is 2. The mode is the most frequent rating, which is 2.

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6.4: Measures of Variation

Consider these three sets of quiz scores:

- Section A: 5 5 5 5 5 5 5 5
- Section B: 0 0 0 0 10 10 10 10
- Section C: 4 4 4 5 5 5 6 6

All three of these sets of data have a mean of 5 and median of 5, yet the sets of scores are clearly quite different. In section A, everyone had the same score; in section B half the class got no points and the other half got a perfect score, assuming this was a 10-point quiz. Section C was not as consistent as section A, but not as widely varied as section B.

In addition to the mean and median, which are measures of the "typical" or "middle" value, we also need a measure of how "spread out" or varied each data set is.

There are several ways to measure this "spread" of the data. The first is the simplest and is called the **range**.

Definition: Range

The **range** is the difference between the maximum value and the minimum value of the data set. This is,

$$\text{range} = \text{maximum value} - \text{minimum value}$$

Example 6.4.1

Using the quiz scores from above, calculate the range.

Solution

For section A, the range is 0 since both maximum and minimum are 5 and $5 - 5 = 0$

For section B, the range is 10 since $10 - 0 = 10$

For section C, the range is 2 since $6 - 4 = 2$

In the last example, the range seems to be revealing how spread out the data is. However, suppose we add a fourth section, Section D, with scores 0 5 5 5 5 5 5 10.

This section also has a mean and median of 5. The range is 10, yet this data set is quite different than Section B. To better illuminate the differences, we'll have to turn to more sophisticated measures of variation.

Definition: Standard Deviation

The **standard deviation** is a measure of variation based on measuring how far each data value deviates, or is different, from the mean. A few important characteristics:

- Standard deviation is positive or zero. Standard deviation will be zero if all the data values are equal, and will get larger as the data spreads out.
- Standard deviation has the same units as the original data.
- Standard deviation, like the mean, can be highly influenced by outliers.

Using the data from section D, we could compute for each data value the difference between the data value and the mean:

data value	deviation: data value - mean
0	$0 - 5 = -5$
5	$5 - 5 = 0$
5	$5 - 5 = 0$
5	$5 - 5 = 0$
5	$5 - 5 = 0$
5	$5 - 5 = 0$
5	$5 - 5 = 0$
5	$5 - 5 = 0$
5	$5 - 5 = 0$
10	$10 - 5 = 5$

We would like to get an idea of the "average" deviation from the mean, but if we find the average of the values in the second column the negative and positive values cancel each other out (this will always happen), so to prevent this we square every value in the second column:

We then add the squared deviations up to get $25 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 25 = 50$. Ordinarily we would then divide by the number of scores, n , (in this case, 10) to find the mean of the deviations. But we only do this if the data set represents a population; if the data set represents a sample (as it almost always does), we instead divide by $n - 1$ (in this case, $10 - 1 = 9$).[1]

So in our example, we would have $\frac{50}{10} = 5$ if section D represents a population and $\frac{50}{9} =$ about 5.56 if section D represents a sample. These values (5 and 5.56) are called, respectively, the **population variance** and the **sample variance** for section D.

Variance can be a useful statistical concept, but note that the units of variance in this instance would be points-squared since we squared all of the deviations. What are points-squared? Good question. We would rather deal with the units we started with (points in this case), so to convert back we take the square root and get:

$$\text{population standard deviation} = \sqrt{\frac{50}{10}} = \sqrt{5} \approx 2.2$$

or

$$\text{sample standard deviation} = \sqrt{\frac{50}{9}} \approx 2.4$$

If we are unsure whether the data set is a sample or a population, we will usually assume it is a sample, and we will round answers to one more decimal place than the original data, as we have done above.



Definition: Computing Standard Deviation

1. Find the deviation of each data from the mean. In other words, subtract the mean from the data value.
2. Square each deviation.
3. Add the squared deviations.
4. Divide by n , the number of data values, if the data represents a whole population; divide by $n-1$ if the data is from a sample.
5. Compute the square root of the result.



Example 6.4.1

Compute the standard deviation for Section B.

Solution

Computing the standard deviation for Section B above, we first calculate that the mean is 5. Using a table can help keep track of your computations for the standard deviation:

data value	deviation: data value - mean	deviation squared
0	$0 - 5 = -5$	$(-5)^2 = 25$
0	$0 - 5 = -5$	$(-5)^2 = 25$
0	$0 - 5 = -5$	$(-5)^2 = 25$
0	$0 - 5 = -5$	$(-5)^2 = 25$
0	$0 - 5 = -5$	$(-5)^2 = 25$
10	$10 - 5 = 5$	$(5)^2 = 25$
10	$10 - 5 = 5$	$(5)^2 = 25$
10	$10 - 5 = 5$	$(5)^2 = 25$
10	$10 - 5 = 5$	$(5)^2 = 25$
10	$10 - 5 = 5$	$(5)^2 = 25$

Assuming this data represents a population, we will add the squared deviations, divide by 10, the number of data values, and compute the square root:

$$\sqrt{\frac{25 + 25 + 25 + 25 + 25 + 25 + 25 + 25 + 25 + 25}{10}} = \sqrt{\frac{250}{10}} = 5$$

Notice that the standard deviation of this data set is much larger than that of section D since the data in this set is more spread out.

For comparison, the standard deviations of all four sections are:

Section A: 5 5 5 5 5 5 5 5 5 5	Standard deviation: 0
Section B: 0 0 0 0 0 10 10 10 10 10	Standard deviation: 5
Section C: 4 4 4 5 5 5 5 6 6 6	Standard deviation: 0.8
Section D: 0 5 5 5 5 5 5 5 5 10	Standard deviation: 2.2

Discussion:

The price of a jar of peanut butter at 5 stores was: \$3.29, \$3.59, \$3.79, \$3.75, and \$3.99. Find the standard deviation of the prices.

Note: Earlier we found the mean of the data was \$3.682.

To compute the standard deviation of the price of a jar of peanut butter at 5 stores, we have the following:

data value	deviation: data value - mean	deviation squared
3.29	$3.29 - 3.682 = -0.391$	0.153664
3.59	$3.59 - 3.682 = -0.092$	0.008464
3.79	$3.79 - 3.682 = 0.108$	0.011664
3.75	$3.75 - 3.682 = 0.068$	0.004624
3.99	$3.99 - 3.682 = 0.308$	0.094864

This data is from a sample, so we will add the squared deviations, divide by 4, the number of data values minus 1, and compute the square root:

$$\sqrt{\frac{0.153664 + 0.008464 + 0.011664 + 0.004624 + 0.094864}{4}} \approx \$0.261$$

While standard deviation is a measure of variation related to the mean, **quartiles** are a variation and position measure, which is more related to the median.

Definition: Quartiles

Quartiles are values that divide the data in quarters.

- The first quartile (Q_1) is the value so that 25% of the data values are below it
- The third quartile (Q_3) is the value so that 75% of the data values are below it.
- The second quartile is the same as the median, since the median is the value so that 50% of the data values are below it.

This divides the data into quarters; 25% of the data is between the minimum and Q_1 , 25% is between Q_1 and the median, 25% is between the median and Q_3 , and 25% is between Q_3 and the maximum value.

While quartiles are not a 1 number summary of variation like standard deviation, the quartiles are used with the median, minimum, and maximum values to form a **5 number summary** of the data.

Definition: Five Number Summary

The five number summary takes this form:

Minimum, Q_1 , Median, Q_3 , Maximum

To find the first quartile, we need to find the data value so that 25% of the data is below it. If n is the number of data values, we compute a locator by finding 25% of n . If this locator is a decimal value, we round up, and find the data value in that position. If the locator is a whole number, we find the mean of the data value in that position and the next data value. This is identical to the process we used to find the median, except we use 25% of the data values rather than half the data values as the locator.

Definition: Computing Quartiles

You are already familiar with the computation of the median, which is the second quartile. The process to compute Q_1 and Q_3 is similar.

1. Begin by ordering the data from smallest to largest.
2. Compute the locator: $L = 0.25n$, where n is the size of the data set.
 1. If L is a decimal value: Round up to $L + 1$. Use the data value in the $L + 1^{\text{th}}$ position
 2. If L is a whole number: Find the mean of the data values in the L^{th} and $L + 1^{\text{th}}$ positions.

Note: To compute Q_3 use the same procedure as for Q_1 , but with locator: $L = 0.75n$

Examples should help make this clearer.

Example 6.4.1

Suppose we have measured 9 students and their heights (in inches), sorted from smallest to largest are:

59 60 62 64 66 67 69 70 72

Calculate and interpret the first and third quartiles.

Solution

To find the first quartile we first compute the locator: 25% of 9 is $L = 0.25(9) = 2.25$. Since this value is not a whole number, we round up to 3. The first quartile will be the third data value: 62 inches.

Interpretation of Q_1 : About 25% of the students measure less or equal than 62 inches and about 75% measure more or equal to 62 inches.

To find the third quartile, we again compute the locator: 75% of 9 is $0.75(9) = 6.75$. Since this value is not a whole number, we round up to 7. The third quartile will be the seventh data value: 69 inches.

Interpretation of Q_3 : About 75% of the students measure less or equal than 69 inches and about 25% measure more or equal to 69 inches.

Example 6.4.1

Suppose we had measured 8 students and their heights (in inches), sorted from smallest to largest are:

59 60 62 64 66 67 69 70

Find and interpret the first and third quartiles.

Solution

To find the first quartile we first compute the locator: 25% of 8 is $L = 0.25(8) = 2$. Since this value is a whole number, we will find the mean of the 2nd and 3rd data values: $(60+62)/2 = 61$, so the first quartile is 61 inches.

Interpretation of Q_1 : About 25% of the students measure less or equal than 61 inches and about 75% measure more or equal to 61 inches.

The third quartile is computed similarly, using 75% instead of 25%. $L = 0.75(8) = 6$. This is a whole number, so we will find the mean of the 6th and 7th data values: $\frac{67+69}{2} = 68$, so Q_3 is 68.

Interpretation of Q_3 : About 75% of the students measure less or equal than 68 inches and about 25% measure more or equal to 68 inches.

Note that the median could be computed the same way, using 50%.

The 5 number summary combines the first and third quartile with the minimum, median, and maximum values.

Discussion:

Calculate the five point summary for the 9 students sample and the 8 students sample above.

Notice that for the 9 students sample, the median is 66, the minimum is 59, and the maximum is 72. The 5 number summary is: 59, 62, 66, 69, 72. For the 8 students sample, the median is 65, the minimum is 59, and the maximum is 70, so the 5 number summary would be: 59, 61, 65, 68, 70.

✓ Example 6.4.1

Consider the three sets of quiz scores we discussed before.

- Section A: 5 5 5 5 5 5 5 5 5
- Section B: 0 0 0 0 0 10 10 10 10
- Section C: 4 4 4 5 5 5 5 6 6

Compute the five-number summaries for each section.

Solution

Returning to our quiz score data. In each case, the first quartile locator is $0.25(10) = 2.5$, so the first quartile will be the 3rd data value, and the third quartile will be the 8th data value. Creating the 5 number summaries:

Section and data	5-number summary
Section A: 5 5 5 5 5 5 5 5 5	5, 5, 5, 5, 5
Section B: 0 0 0 0 0 10 10 10 10	0, 0, 5, 10, 10
Section C: 4 4 4 5 5 5 5 6 6	4, 4, 5, 6, 6
Section D: 0 5 5 5 5 5 5 5 10	0, 5, 5, 5, 10

Of course, with a relatively small data set, finding a 5 number summary is a bit silly, since the summary contains almost as many values as the original data.

Discussion:

The total cost of textbooks for the term was collected from 36 students. Find the 5 number summary of this data.

\$140 \$160 \$160 \$165 \$180 \$220 \$235 \$240 \$250 \$260 \$280 \$285
 \$285 \$285 \$290 \$300 \$300 \$305 \$310 \$310 \$315 \$315 \$320 \$320
 \$330 \$340 \$345 \$350 \$355 \$360 \$360 \$380 \$395 \$420 \$460 \$460

Because the data is already in order, we don't need to sort it first. The minimum value is \$140 and the maximum is \$460.

There are 36 data values so $n = 36$. $n/2 = 18$, which is a whole number, so the median is the mean of the 18th and 19th data values, \$305 and \$310. The median is \$307.50

To find the first quartile, we calculate the locator, $L = 0.25(36) = 9$. since this is a whole number, we know Q_1 is the mean of the 9th and 10th data values, \$250 and \$260. $Q_1 = \$255$

To find the third quartile, we calculate the locator, $L = 0.75(36) = 27$. since this is a whole number, we know Q_3 is the mean of the 27th and 28th data values, \$345 and \$350.

$Q_3 = \$347.50$

Thus, the 5 number summary of this data is: \$140, \$255, \$307.50, \$347.50, \$460.

✓ Example 6.4.1

Returning to the household income data from earlier, create the 5 number summary.

Income (thousands of dollars)	Frequency
15	6
20	8
25	11
30	17
35	19
40	20
45	12
50	7

Solution

By adding the frequencies, we can see there are 100 data values represented in the table. In Example 20, we found the median was \$35 thousand. We can see in the table that the minimum income is \$15 thousand, and the maximum is \$50 thousand.

To find Q_1 , we calculate the locator: $L = 0.25(100) = 25$. This is a whole number, so Q_1 will be the mean of the 25th and 26th data values.

Counting up in the data as we did before,

There are 6 data values of \$15, so Values 1 to 6 are \$15 thousand
 The next 8 data values are \$20, so Values 7 to $(6+8)=14$ are \$20 thousand
 The next 11 data values are \$25, so Values 15 to $(14+11)=25$ are \$25 thousand
 The next 17 data values are \$30, so Values 26 to $(25+17)=42$ are \$30 thousand

The 25th data value is \$25 thousand, and the 26th data value is \$30 thousand, so Q_1 will be the mean of these: $(25 + 30)/2 = \$27.5$ thousand.

To find Q_3 , we calculate the locator: $L = 0.75(100) = 75$. This is a whole number, so Q_3 will be the mean of the 75th and 76th data values. Continuing our counting from earlier,

The next 19 data values are \$35, so Values 43 to $(42+19)=61$ are \$35 thousand
 The next 20 data values are \$40, so Values 61 to $(61+20)=81$ are \$40 thousand

Both the 75th and 76th data values lie in this group, so Q_3 will be \$40 thousand.

Putting these values together into a five-number summary, we get: 15, 27.5, 35, 40, 50

Note that the 5 number summary divides the data into four intervals, each of which will contain about 25% of the data. In the previous example, that means about 25% of households have income between \$40 thousand and \$50 thousand.

For visualizing data, there is a graphical representation of a 5-number summary called a **box plot**, or box and whisker graph.

Definition: Box Plot

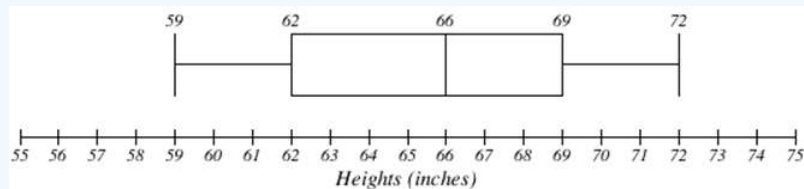
A **box plot** is a graphical representation of a five-number summary.

To create a box plot, a number line is first drawn. A box is drawn from the first quartile to the third quartile, and a line is drawn through the box at the median. “Whiskers” are extended out to the minimum and maximum values.

✓ Example 6.4.1

Create a box plot based on the 9 students height data with 5 number summary: 59, 62, 66, 69, 72.

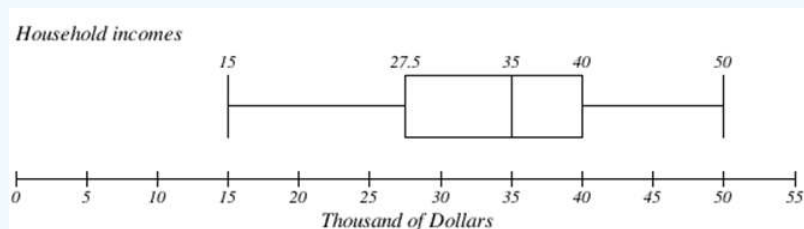
Solution



✓ Example 6.4.1

Create a box plot based on the household income data with 5 number summary: 15, 27.5, 35, 40, 50

Solution



Discussion:

Create a box plot based on the textbook price data from the total cost of textbooks in the previous discussion.

\$140 \$160 \$160 \$165 \$180 \$220 \$235 \$240 \$250 \$260 \$280 \$285

\$285 \$285 \$290 \$300 \$300 \$305 \$310 \$310 \$315 \$315 \$320 \$320

\$330 \$340 \$345 \$350 \$355 \$360 \$360 \$380 \$395 \$420 \$460 \$460

To create this box plot, we need to first find the five point summary. We worked on this already, the five point summary is: \$140, \$255, \$307.50, \$347.50, \$460. To start the graph, we need to start with a number line, which should include the numbers 140 and 460. This means, that we could start with 100 and end with 500. A box is drawn from $(Q_1=255)$ to $(Q_3=347.50)$, and a line is drawn through the box at the median, 307.50. “Whiskers” are extended out to 140, which is the minimum value, and to 460, which is the maximum value.

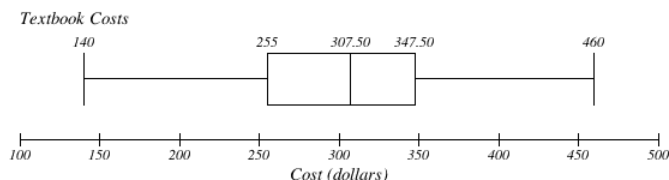
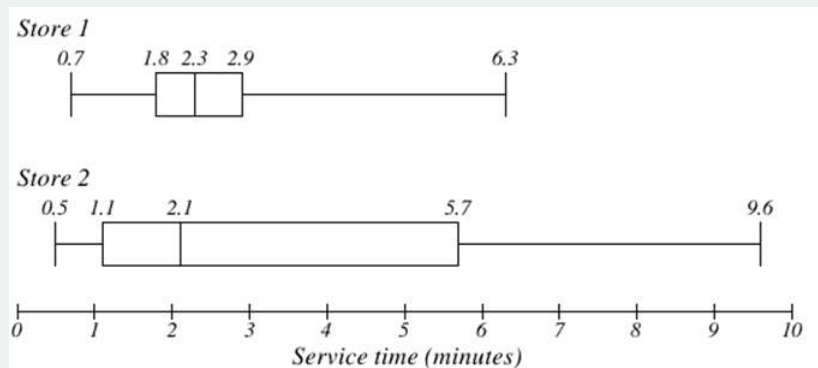


Figure 6.4.1: Box plot for the textbook cost data.

Box plots are particularly useful for comparing data from two populations.

Discussion:

The box plot of service times for two fast-food restaurants is shown below. Which store should you go to in a hurry? Support your argument by using the concepts learned in this chapter.

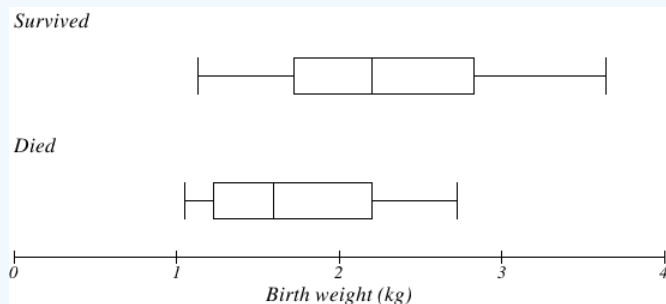


While store 2 had a slightly shorter median service time (2.1 minutes vs. 2.3 minutes), store 2 is less consistent, with a wider spread of the data. At store 1, 75% of customers were served within 2.9 minutes, while at store 2, 75% of customers were served within 5.7 minutes.

Which store should you go to in a hurry? That depends upon your opinions about luck – 25% of customers at store 2 had to wait between 5.7 and 9.6 minutes.

✓ Example 6.4.1

Compare the two box plots below that are based on the birth weights of infants with severe idiopathic respiratory distress syndrome (SIRDS)[2]. The box plot is separated to show the birth weights of infants who survived and those who did not.



Solution

Comparing the two groups, the box plot reveals that the birth weights of the infants who died appear to be, overall, smaller than the weights of infants who survived. In fact, we can see that the median birth weight of infants who survived is the same as the third quartile of the infants who died.

Similarly, we can see that the first quartile of the survivors is larger than the median weight of those who died, meaning that over 75% of the survivors had a birth weight larger than the median birth weight of those who died.

Looking at the maximum value for those who died and the third quartile of the survivors, we can see that over 25% of the survivors had birth weights higher than the heaviest infant who died.

The box plot gives us a quick, albeit informal, way to determine that birth weight is quite likely linked to survival of infants with SIRDS.

[1] The reason we do this is highly technical, but we can see how it might be useful by considering the case of a small sample from a population that contains an outlier, which would increase the average deviation: the outlier very likely won't be included in the sample, so the mean deviation of the sample would underestimate the mean deviation of the population; thus we divide by a slightly smaller number to get a slightly bigger average deviation.

[2] van Vliet, P.K. and Gupta, J.M. (1973) Sodium bicarbonate in idiopathic respiratory distress syndrome. *Arch. Disease in Childhood*, **48**, 249–255. As quoted on <http://openlearn.open.ac.uk/mod/ouco...§ion=1.1.3>

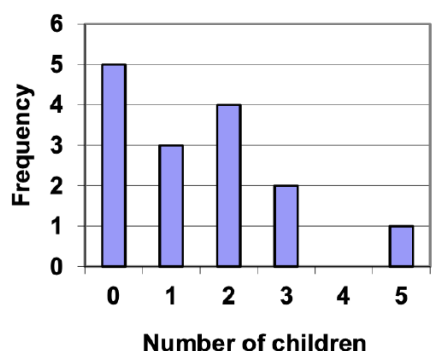
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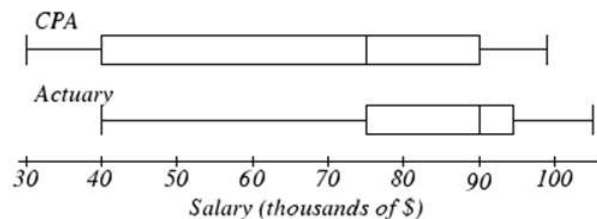
6.5: Practice Problems

Practice Problems for the Chapter

- A group of diners were asked how much they would pay for a meal. Their responses were: \$7.50, \$8.25, \$9.00, \$8.00, \$7.25, \$7.50, \$8.00, \$7.00.
 - Find the mean
 - Find the median
 - Write the 5 number summary for this data
- You recorded the time in seconds it took for 8 participants to solve a puzzle. The times were: 15.2, 18.8, 19.3, 19.7, 20.2, 21.8, 22.1, 29.4.
 - Find the mean
 - Find the median
 - Write the 5 number summary for this data
 - Find the standard deviation of this data
- A group of adults were asked how many children they have in their families. The graph below shows the number of adults who indicated each number of children.



- Compute the mean number of children for the group surveyed
 - Compute the median number of children for the group surveyed
 - Write the 5 number summary for this data.
 - Create box plot.
- The box plot below shows salaries for Actuaries and CPAs. Kendra makes the median salary for an Actuary. Kelsey makes the first quartile salary for a CPA. Who makes more money? How much more?



- Referring to the box plot above, what percentage of actuaries makes more than the median salary of a CPA?
- Studies are often done by pharmaceutical companies to determine the effectiveness of a treatment program. Suppose that a new AIDS antibody drug is currently under study. It is given to patients once the AIDS symptoms have revealed themselves. Of interest is the average length of time in months patients live once starting the treatment. Two researchers each follow a different set of 40 AIDS patients from the start of treatment until their deaths. The following data (in months) are collected
 - Researcher 1: 3; 4; 11; 15; 16; 17; 22; 44; 37; 16; 14; 24; 25; 15; 26; 27; 33; 29; 35; 44; 13; 21; 22; 10; 12; 8; 40; 32; 26; 27; 31; 34; 29; 17; 8; 24; 18; 47; 33; 34
 - Researcher 2: 3; 14; 11; 5; 16; 17; 28; 41; 31; 18; 14; 14; 26; 25; 21; 22; 31; 2; 35; 44; 23; 21; 21; 16; 12; 18; 41; 22; 16; 25; 33; 34; 29; 13; 18; 24; 23; 42; 33; 29

Create and compare box plots of the data. Draw the two box plots using the same number line, as shown in the comparison examples, so that you can build a strong argument for the comparison of these box plots.

7. Make up three data sets with 5 numbers each that have:
 - a. the same mean but different standard deviations.
 - b. the same mean but different medians.
 - c. the same median but different means.
8. In your class, design a poll on a topic of interest to you and give it to the class.
 - a. Summarize the data, computing the mean and five number summary.
 - b. Create a box plot of the data.
 - c. Write several sentences about the topic, using your computed statistics as evidence in your writing.
9. In the example from the section “*Measures of Variation*,” we examined a comparison of two boxplots displaying the birth weights of infants diagnosed with severe idiopathic respiratory distress syndrome (SIRDS). The boxplots distinguish between infants who survived and those who did not. However, note that these data were collected in the 1970s. Recall that statistics provides tools to help us respond thoughtfully and critically to information we encounter. This context offers an opportunity to apply those tools. How might a similar boxplot from the current decade differ? Locate a recent dataset online, construct an updated boxplot, and interpret your results. Then, compare your findings with the boxplots presented in the textbook example.

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