CCSF CALCULUS II – INTEGRAL CALCULUS



Kendra Lockman
City College of San Francisco



CCSF Calculus II – Integral Calculus .
Lockman Spring 2024

This text is disseminated via the Open Education Resource (OER) LibreTexts Project (https://LibreTexts.org) and like the thousands of other texts available within this powerful platform, it is freely available for reading, printing, and "consuming."

The LibreTexts mission is to bring together students, faculty, and scholars in a collaborative effort to provide an accessible, and comprehensive platform that empowers our community to develop, curate, adapt, and adopt openly licensed resources and technologies; through these efforts we can reduce the financial burden born from traditional educational resource costs, ensuring education is more accessible for students and communities worldwide.

Most, but not all, pages in the library have licenses that may allow individuals to make changes, save, and print this book. Carefully consult the applicable license(s) before pursuing such effects. Instructors can adopt existing LibreTexts texts or Remix them to quickly build course-specific resources to meet the needs of their students. Unlike traditional textbooks, LibreTexts' web based origins allow powerful integration of advanced features and new technologies to support learning.



LibreTexts is the adaptable, user-friendly non-profit open education resource platform that educators trust for creating, customizing, and sharing accessible, interactive textbooks, adaptive homework, and ancillary materials. We collaborate with individuals and organizations to champion open education initiatives, support institutional publishing programs, drive curriculum development projects, and more.

The LibreTexts libraries are Powered by NICE CXone Expert and was supported by the Department of Education Open Textbook Pilot Project, the California Education Learning Lab, the UC Davis Office of the Provost, the UC Davis Library, the California State University Affordable Learning Solutions Program, and Merlot. This material is based upon work supported by the National Science Foundation under Grant No. 1246120, 1525057, and 1413739.

Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation nor the US Department of Education.

Have questions or comments? For information about adoptions or adaptions contact info@LibreTexts.org or visit our main website at https://LibreTexts.org.

This text was compiled on 04/03/2026

TABLE OF CONTENTS

Licensing

1: Fast Review of Important Calc I Stuff

- 1.1: Trigonometry
  - 1.1E: Exercises
- 1.2: Derivatives Review
  - 1.2E: Exercises
- 1.3: Integrals Review
  - 1.3.1: Exercises
- 1.4: Integration by Substitution
  - 1.4E: Exercises

2: Applications of Integration

- 2.1: Areas Between Curves
  - 2.1E: Exercises
- 2.2: Volume by Cross-Sectional Area
  - 2.2E: Exercises
- 2.3: Volumes of Revolution - Cylindrical Shells
  - 2.3E: Exercises
- 2.4: Arc Length of a Curve and Surface Area
  - 2.4E: Exercises
- 2.5: Work
  - 2.5E: Exercises
- 2.6: Hydrostatic Force and Pressure
  - 2.6E: Exercises
- 2.7: Moments and Centers of Mass
  - 2.7E: Exercises
- 2.8: The Mean Value Theorem for Integrals
  - 2.8E: Exercises
- 2.9: Chapter 2 Review Exercises

3: Techniques of Integration

- 3.1: Integration by Parts
  - 3.1E: Exercises
- 3.2: Trigonometric Integrals
  - 3.2E: Exercises
- 3.3: Trigonometric Substitution
  - 3.3E: Exercises
- 3.4: Partial Fractions

- 3.4E: Exercises
- 3.5: Numerical Integration
  - 3.5E: Exercises
- 3.6: Indeterminate Forms and L'Hospital's Rule
  - 3.6E: Exercises
- 3.7: Improper Integrals
  - 3.7E: Exercises
- 3.8: Chapter 3 Review Exercises

4: Sequences and Series

- 4.1: Sequences
  - 4.1E: Exercises
- 4.2: Infinite Series
  - 4.2E: Exercises
- 4.3: The Divergence and Integral Tests
  - 4.3E: Exercises
- 4.4: The Comparison Tests
  - 4.4E: Exercises
- 4.5: The Alternating Series Test
  - 4.5E: Exercises
- 4.6: The Ratio and Root Tests
  - 4.6E: Exercises
- 4.7: Chapter 4 Review Exercises

5: Power Series

- 5.1: Power Series and Functions
  - 5.1E: Exercises
- 5.2: Properties of Power Series
  - 5.2E: Exercises
- 5.3: Taylor and Maclaurin Series
  - 5.3E: Exercises
- 5.4: Working with Taylor Series
  - 5.4E: Exercises
- 5.5: Chapter 5 Review Exercises

6: Parametric Equations and Polar Coordinates

- 6.1: Parametric Equations
  - 6.1E: Exercises
- 6.2: Calculus of Parametric Curves
  - 6.2E: Exercises
- 6.3: Polar Coordinates
  - 6.3E: Exercises
- 6.4: Area and Arc Length in Polar Coordinates

- 6.4E: Exercises
- 6.5: Conic Sections
  - 6.5E: Exercises
- 6.6: Chapter 6 Review Exercises

7: Introduction to Differential Equations

- 7.1: Basics of Differential Equations
  - 7.1E: Exercises for Section 8.1
- 7.2: Direction Fields and Numerical Methods
  - 7.2E: Exercises for Section 8.2
- 7.3: Separable Equations
  - 7.3E: Exercises for Section 8.3
- 7.4: The Logistic Equation
  - 7.4E: Exercises for Section 8.4
- 7.5: First-order Linear Equations
  - 7.5E: Exercises for Section 8.5
- 7.6: Chapter 7 Review Exercises

8: Appendices

- 8.1: Other Strategies for Integration
  - 8.1E: Exercises
- 8.2: Volumes of Revolution - Cylindrical Shells
  - 8.2E: Exercises for Section 6.3
- 8.A: Table of Derivatives
- 8.B: Table of Integrals

Index

Glossary

Detailed Licensing

Licensing

*A detailed breakdown of this resource's licensing can be found in **Back Matter/Detailed Licensing**.*

CHAPTER OVERVIEW

1: Fast Review of Important Calc I Stuff

> This page is a draft and is under active development.

1.1: Trigonometry
1.1E: Exercises
1.2: Derivatives Review
1.2E: Exercises
1.3: Integrals Review
1.3.1: Exercises
1.4: Integration by Substitution
1.4E: Exercises

This page titled 1: Fast Review of Important Calc I Stuff is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Gilbert Strang & Edwin “Jed” Herman (OpenStax) via source content that was edited to the style and standards of the LibreTexts platform.

1.1: Trigonometry

Learning Objectives

- Convert angle measures between degrees and radians.
- Recognize the triangular and circular definitions of the basic trigonometric functions.
- Write the basic trigonometric identities.
- Identify the graphs and periods of the trigonometric functions.
- Describe the shift of a sine or cosine graph from the equation of the function.

Trigonometric functions are used to model many phenomena, including sound waves, vibrations of strings, alternating electrical current, and the motion of pendulums. In fact, almost any repetitive, or cyclical, motion can be modeled by some combination of trigonometric functions. In this section, we define the six basic trigonometric functions and look at some of the main identities involving these functions.

Radian Measure

To use trigonometric functions, we first must understand how to measure the angles. Although we can use both radians and degrees, **radians** are a more natural measurement because they are related directly to the **unit circle**, a circle with radius 1. The radian measure of an angle is defined as follows. Given an angle θ, let s be the length of the corresponding arc on the unit circle (Figure 1.1.1). We say the angle corresponding to the arc of length 1 has radian measure 1.

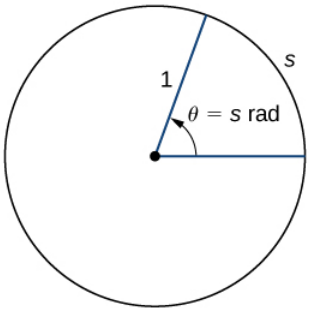


Figure 1.1.1: The radian measure of an angle θ is the arc length s of the associated arc on the unit circle.

Since an angle of $360°$ corresponds to the circumference of a circle, or an arc of length 2π, we conclude that an angle with a degree measure of $360°$ has a radian measure of 2π. Similarly, we see that $180°$ is equivalent to π radians. Table 1.1.1 shows the relationship between common degree and radian values.

Table 1.1.1: Common Angles Expressed in Degrees and Radians

Degrees	Radians	Degrees	Radians
0	0	120	$2\pi/3$
30	$\pi/6$	135	$3\pi/4$
45	$\pi/4$	150	$5\pi/6$
60	$\pi/3$	180	π
90	$\pi/2$		

Example 1.1.1: Converting between Radians and Degrees

a. Express $225°$ using radians.
b. Express $5\pi/3$ rad using degrees.

Solution

Use the fact that 180° is equivalent to π radians as a conversion factor (Table 1.1.1):

$$1 = \frac{\pi \text{ rad}}{180^{\circ}} = \frac{180^{\circ}}{\pi \text{ rad}}.$$

a. $225^{\circ} = 225^{\circ} \cdot \left(\frac{\pi}{180^{\circ}}\right) = \left(\frac{5\pi}{4}\right)$ rad

b. $\frac{5\pi}{3}$ rad $= \frac{5\pi}{3} \cdot \frac{180^{\circ}}{\pi} = 300^{\circ}$

? Exercise 1.1.1

a. Express 210° using radians.

b. Express $11\pi/6$ rad using degrees.

Hint

π radians is equal to 180°

Answer

a. $7\pi/6$

b. 330°

The Six Basic Trigonometric Functions

Trigonometric functions allow us to use angle measures, in radians or degrees, to find the coordinates of a point on any circle - not only on a unit circle - or to find an angle given a point on a circle. They also define the relationship between the sides and angles of a triangle.

To define the trigonometric functions, first consider the unit circle centered at the origin and a point $P = (x, y)$ on the unit circle. Let θ be an angle with an initial side that lies along the positive x-axis and with a terminal side that is the line segment OP. An angle in this position is said to be in standard position (Figure 1.1.2). We can then define the values of the six trigonometric functions for θ in terms of the coordinates x and y.

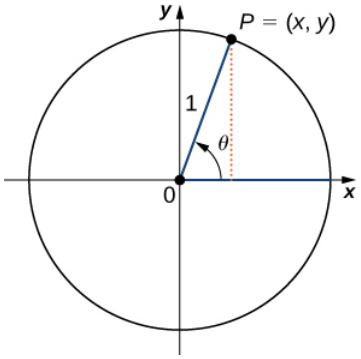


Figure 1.1.2: The angle θ is in standard position. The values of the trigonometric functions for θ are defined in terms of the coordinates x and y.

Definition: Trigonometric functions

Let $P = (x, y)$ be a point on the unit circle centered at the origin O. Let θ be an angle with an initial side along the positive x-axis and a terminal side given by the line segment OP. The trigonometric functions are then defined as

$\sin\theta = y$	$\csc\theta = \frac{1}{y}$
$\cos\theta = x$	$\sec\theta = \frac{1}{x}$
$\tan\theta = \frac{y}{x}$	$\cot\theta = \frac{x}{y}$

If $x = 0, \sec\theta$ and $\tan\theta$ are undefined. If $y = 0$, then $\cot\theta$ and $\csc\theta$ are undefined.

We can see that for a point $P = (x, y)$ on a circle of radius r with a corresponding angle θ, the coordinates x and y satisfy

$$\cos\theta = \frac{x}{r} \tag{1.1.1}$$
$$x = r\cos\theta \tag{1.1.2}$$

and

$$\sin\theta = \frac{y}{r} \tag{1.1.3}$$
$$y = r\sin\theta. \tag{1.1.4}$$

The values of the other trigonometric functions can be expressed in terms of x, y, and r (Figure 1.1.3).

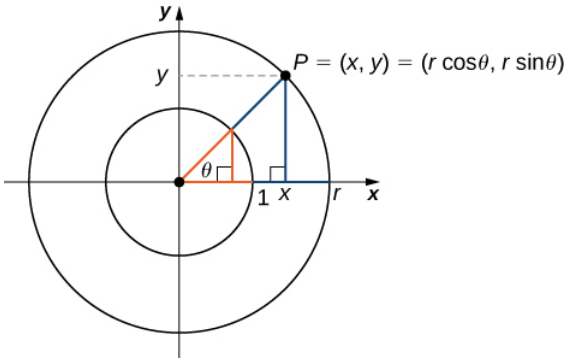


Figure 1.1.3: For a point $P = (x, y)$ on a circle of radius r, the coordinates x and y satisfy $x = r\cos\theta$ and $y = r\sin\theta$.

Table 1.1.2 shows the values of sine and cosine at the major angles in the first quadrant. From this table, we can determine the values of sine and cosine at the corresponding angles in the other quadrants. The values of the other trigonometric functions are calculated easily from the values of $\sin\theta$ and $\cos\theta$.

Table 1.1.2: Values of $\sin\theta$ and $\cos\theta$ at Major Angles θ in the First Quadrant

θ	$\sin\theta$	$\cos\theta$
0	0	1
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1	0

Example 1.1.2: Evaluating Trigonometric Functions

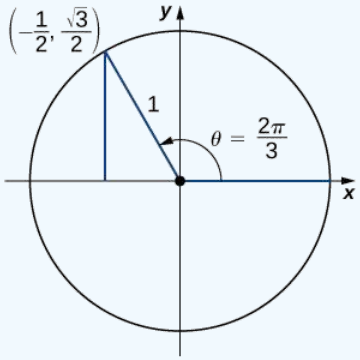
Evaluate each of the following expressions.

a. $\sin\left(\frac{2\pi}{3}\right)$
b. $\cos\left(-\frac{5\pi}{6}\right)$
c. $\tan\left(\frac{15\pi}{4}\right)$

Solution

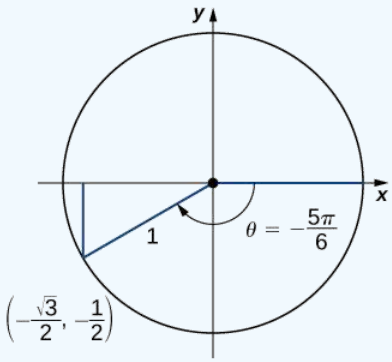
a) On the unit circle, the angle $\theta = \frac{2\pi}{3}$ corresponds to the point $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Therefore,

$$\sin\left(\frac{2\pi}{3}\right) = y = \left(\frac{\sqrt{3}}{2}\right).$$



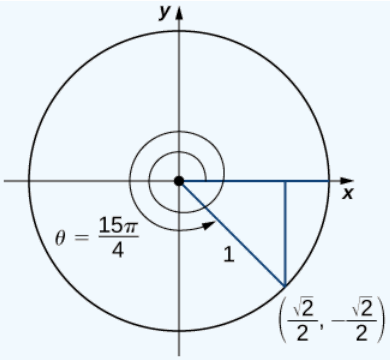
b) An angle $\theta = -\frac{5\pi}{6}$ corresponds to a revolution in the negative direction, as shown. Therefore,

$$\cos\left(-\frac{5\pi}{6}\right) = x = -\frac{\sqrt{3}}{2}.$$



c) An angle $\theta = \frac{15\pi}{4} = 2\pi + \frac{7\pi}{4}$. Therefore, this angle corresponds to more than one revolution, as shown. Knowing the fact that an angle of $\frac{7\pi}{4}$ corresponds to the point $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$, we can conclude that

$$\tan\left(\frac{15\pi}{4}\right) = \frac{y}{x} = -1.$$



? Exercise 1.1.2

Evaluate $\cos(3\pi/4)$ and $\sin(-\pi/6)$.

Hint

Look at angles on the unit circle.

Answer

$$\cos(3\pi/4) = -\sqrt{2}/2$$

$$\sin(-\pi/6) = -1/2$$

As mentioned earlier, the ratios of the side lengths of a right triangle can be expressed in terms of the trigonometric functions evaluated at either of the acute angles of the triangle. Let θ be one of the acute angles. Let A be the length of the adjacent leg, O be the length of the opposite leg, and H be the length of the hypotenuse. By inscribing the triangle into a circle of radius H, as shown in Figure 1.1.4, we see that A, H, and O satisfy the following relationships with θ:

$\sin\theta = \frac{O}{H}$	$\csc\theta = \frac{H}{O}$
$\cos\theta = \frac{A}{H}$	$\sec\theta = \frac{H}{A}$
$\tan\theta = \frac{O}{A}$	$\cot\theta = \frac{A}{O}$

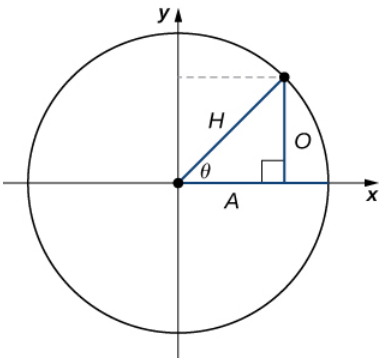


Figure 1.1.4: By inscribing a right triangle in a circle, we can express the ratios of the side lengths in terms of the trigonometric functions evaluated at θ.

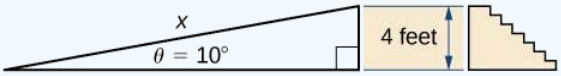
✓ Example 1.1.3: Constructing a Wooden Ramp

A wooden ramp is to be built with one end on the ground and the other end at the top of a short staircase. If the top of the staircase is 4 ft from the ground and the angle between the ground and the ramp is to be $10°$, how long does the ramp need to

be?

Solution

Let x denote the length of the ramp. In the following image, we see that x needs to satisfy the equation $\sin(10°) = 4/x$. Solving this equation for x, we see that $x = 4\sin(10°) \approx 23.035$ ft.



Exercise 1.1.3

A house painter wants to lean a **20**-ft ladder against a house. If the angle between the base of the ladder and the ground is to be **60°**, how far from the house should she place the base of the ladder?

Hint

Draw a right triangle with hypotenuse 20.

Answer

10 ft

Trigonometric Identities

A **trigonometric identity** is an equation involving trigonometric functions that is true for all angles θ for which the functions are defined. We can use the identities to help us solve or simplify equations. The main trigonometric identities are listed below.

Theorem 1.1.1: Trigonometric Identities

Reciprocal Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\csc\theta = \frac{1}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1 \tag{1.1.5}$$

$$1 + \tan^2\theta = \sec^2\theta \tag{1.1.6}$$

$$1 + \cot^2\theta = \csc^2\theta \tag{1.1.7}$$

Addition and Subtraction Identities

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

Double-Angle Identities

$$\sin(2\theta) = 2\sin\theta\cos\theta \tag{1.1.8}$$

$$\cos(2\theta) = 2\cos^2\theta - 1 \tag{1.1.9}$$
$$= 1 - 2\sin^2\theta \tag{1.1.10}$$
$$= \cos^2\theta - \sin^2\theta \tag{1.1.11}$$

Example 1.1.4: Solving Trigonometric Equations

For each of the following equations, use a trigonometric identity to find all solutions.

a. $1 + \cos(2\theta) = \cos\theta$
b. $\sin(2\theta) = \tan\theta$

Solution

a) Using the Double-Angle Identity for $\cos(2\theta)$, we see that θ is a solution of

$$1 + \cos(2\theta) = \cos\theta$$

if and only if

$$1 + 2\cos^2\theta - 1 = \cos\theta,$$

which is true if and only if

$$2\cos^2\theta - \cos\theta = 0.$$

To solve this equation, it is important to note that we need to factor the left-hand side and not divide both sides of the equation by $\cos\theta$. The problem with dividing by $\cos\theta$ is that it is possible that $\cos\theta$ is zero. In fact, if we did divide both sides of the equation by $\cos\theta$, we would miss some of the solutions of the original equation. Factoring the left-hand side of the equation, we see that θ is a solution of this equation if and only if

$$\cos\theta(2\cos\theta - 1) = 0.$$

Since $\cos\theta = 0$ when

$$\theta = \frac{\pi}{2}, \frac{\pi}{2} \pm \pi, \frac{\pi}{2} \pm 2\pi, \ldots,$$

and $\cos\theta = 1/2$ when

$$\theta = \frac{\pi}{3}, \frac{\pi}{3} \pm 2\pi, \ldots \text{ or } \theta = -\frac{\pi}{3}, -\frac{\pi}{3} \pm 2\pi, \ldots,$$

we conclude that the set of solutions to this equation is

$$\theta = \frac{\pi}{2} + n\pi, \ \theta = \frac{\pi}{3} + 2n\pi$$

and

$$\theta = -\frac{\pi}{3} + 2n\pi, \ n = 0, \pm 1, \pm 2, \ldots.$$

b) Using the Double-Angle Identity for $\sin(2\theta)$ and the reciprocal identity for $\tan(\theta)$, the equation can be written as

$$2\sin\theta\cos\theta = \frac{\sin\theta}{\cos\theta}.$$

To solve this equation, we multiply both sides by $\cos\theta$ to eliminate the denominator, and say that if θ satisfies this equation, then θ satisfies the equation

$$2\sin\theta\cos^2\theta - \sin\theta = 0.$$

However, we need to be a little careful here. Even if θ satisfies this new equation, it may not satisfy the original equation because, to satisfy the original equation, we would need to be able to divide both sides of the equation by $\cos\theta$. However, if $\cos\theta = 0$, we cannot divide both sides of the equation by $\cos\theta$. Therefore, it is possible that we may arrive at extraneous

solutions. So, at the end, it is important to check for extraneous solutions. Returning to the equation, it is important that we factor $\sin\theta$ out of both terms on the left-hand side instead of dividing both sides of the equation by $\sin\theta$. Factoring the left-hand side of the equation, we can rewrite this equation as

$$\sin\theta(2\cos^2\theta - 1) = 0.$$

Therefore, the solutions are given by the angles θ such that $\sin\theta = 0$ or $\cos^2\theta = 1/2$. The solutions of the first equation are $\theta = 0, \pm\pi, \pm 2\pi, \ldots$. The solutions of the second equation are $\theta = \pi/4, (\pi/4) \pm (\pi/2), (\pi/4) \pm \pi, \ldots$. After checking for extraneous solutions, the set of solutions to the equation is

$$\theta = n\pi$$

and

$$\theta = \frac{\pi}{4} + \frac{n\pi}{2}$$

with $n = 0, \pm 1, \pm 2, \ldots$.

? Exercise 1.1.4

Find all solutions to the equation $\cos(2\theta) = \sin\theta$.

Hint

Use the Double-Angle Identity for cosine (Equation 1.1.8).

Answer

$\theta = \frac{3\pi}{2} + 2n\pi, \frac{\pi}{6} + 2n\pi, \frac{5\pi}{6} + 2n\pi$

for $n = 0, \pm 1, \pm 2, \ldots$.

✓ Example 1.1.5: Proving a Trigonometric Identity

Prove the trigonometric identity $1 + \tan^2\theta = \sec^2\theta$.

Solution

We start with the Pythagorean Identity (Equation 1.1.5)

$$\sin^2\theta + \cos^2\theta = 1.$$

Dividing both sides of this equation by $\cos^2\theta$, we obtain

$$\frac{\sin^2\theta}{\cos^2\theta} + 1 = \frac{1}{\cos^2\theta}.$$

Since $\sin\theta/\cos\theta = \tan\theta$ and $1/\cos\theta = \sec\theta$, we conclude that

$$\tan^2\theta + 1 = \sec^2\theta.$$

? Exercise 1.1.5

Prove the trigonometric identity $1 + \cot^2\theta = \csc^2\theta$.

Answer

Divide both sides of the identity $\sin^2\theta + \cos^2\theta = 1$ by $\sin^2\theta$.

Graphs and Periods of the Trigonometric Functions

We have seen that as we travel around the unit circle, the values of the trigonometric functions repeat. We can see this pattern in the graphs of the functions. Let $P = (x, y)$ be a point on the unit circle and let θ be the corresponding angle . Since the angle θ and $\theta + 2\pi$ correspond to the same point P, the values of the trigonometric functions at θ and at $\theta + 2\pi$ are the same. Consequently, the trigonometric functions are **periodic functions**. The period of a function f is defined to be the smallest positive value p such that $f(x + p) = f(x)$ for all values x in the domain of f. The sine, cosine, secant, and cosecant functions have a period of 2π. Since the tangent and cotangent functions repeat on an interval of length π, their period is π (Figure 1.1.5).

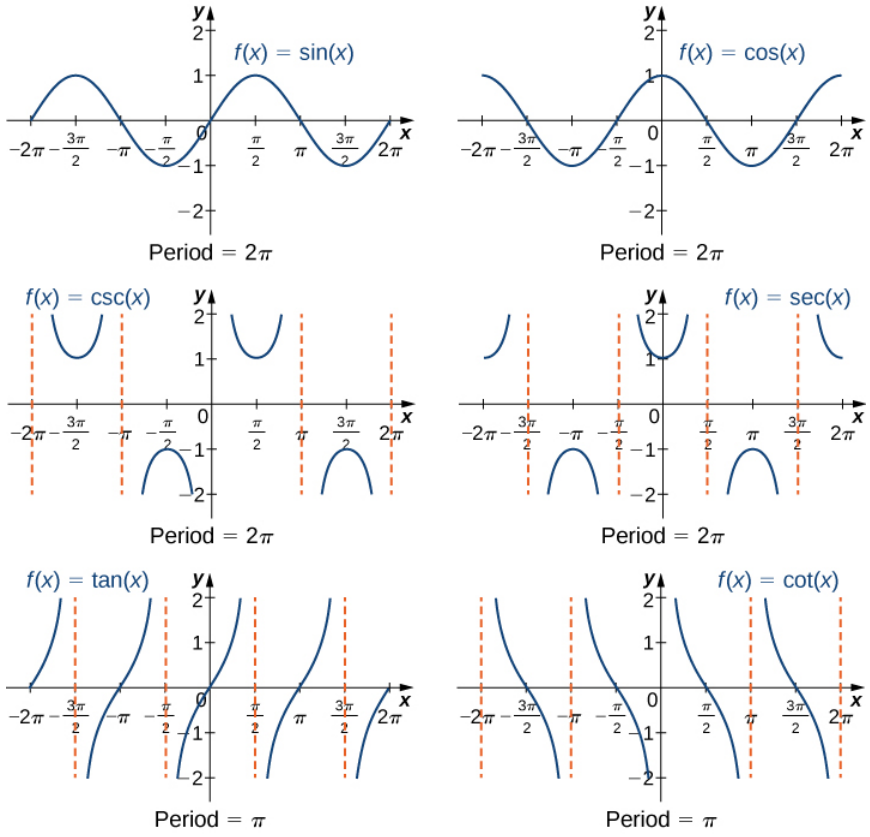


Figure 1.1.5: The six trigonometric functions are periodic.

Just as with algebraic functions, we can apply transformations to trigonometric functions. In particular, consider the following function:

$$f(x) = A\sin(B(x - \alpha)) + C.$$

In Figure 1.1.6, the constant α causes a horizontal or phase shift. The factor B changes the period. This transformed sine function will have a period $2\pi/|B|$. The factor A results in a vertical stretch by a factor of $|A|$. We say $|A|$ is the "amplitude of f." The constant C causes a vertical shift.

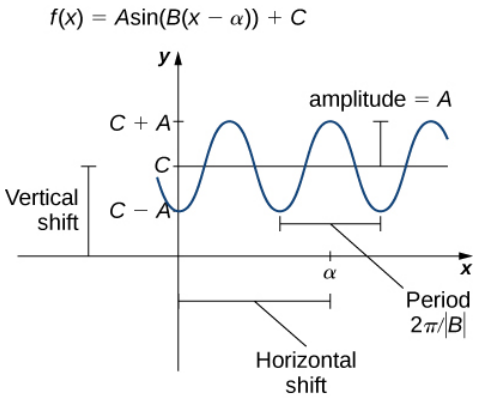


Figure 1.1.6: A graph of a general sine function.

Notice in Figure 1.1.6 that the graph of $y = \cos x$ is the graph of $y = \sin x$ shifted to the left $\pi/2$ units. Therefore, we can write

$$\cos x = \sin(x + \pi/2).$$

Similarly, we can view the graph of $y = \sin x$ as the graph of $y = \cos x$ shifted right $\pi/2$ units, and state that $\sin x = \cos(x - \pi/2)$.

A shifted sine curve arises naturally when graphing the number of hours of daylight in a given location as a function of the day of the year. For example, suppose a city reports that June 21 is the longest day of the year with 15.7 hours and December 21 is the shortest day of the year with 8.3 hours. It can be shown that the function

$$h(t) = 3.7 \sin\left(\frac{2\pi}{365}(x - 80.5)\right) + 12$$

is a model for the number of hours of daylight h as a function of day of the year t (Figure 1.1.7).

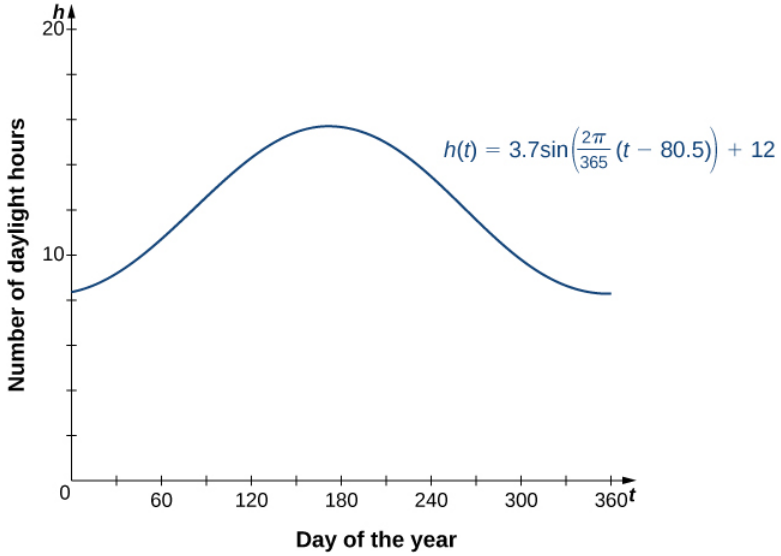


Figure 1.1.7: The hours of daylight as a function of day of the year can be modeled by a shifted sine curve.

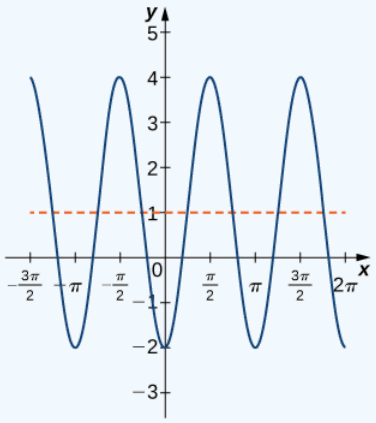
Example 1.1.6: Sketching the Graph of a Transformed Sine Curve

Sketch a graph of $f(x) = 3\sin(2(x - \frac{\pi}{4})) + 1$.

Solution

This graph is a phase shift of $y = \sin(x)$ to the right by $\pi/4$ units, followed by a horizontal compression by a factor of 2, a vertical stretch by a factor of 3, and then a vertical shift by 1 unit. The period of f is π.

$f(x) = 3\sin\left(2\left(x - \frac{\pi}{4}\right)\right) + 1$



? Exercise 1.1.6

Describe the relationship between the graph of $f(x) = 3\sin(4x) - 5$ and the graph of $y = \sin(x)$.

Hint

The graph of f can be sketched using the graph of $y = \sin(x)$ and a sequence of three transformations.

Answer

To graph $f(x) = 3\sin(4x) - 5$, the graph of $y = \sin(x)$ needs to be compressed horizontally by a factor of 4, then stretched vertically by a factor of 3, then shifted down 5 units. The function f will have a period of $\pi/2$ and an amplitude of 3.

Inverse Trigonometric Functions

The six basic trigonometric functions are periodic, and therefore they are not one-to-one. However, if we restrict the domain of a trigonometric function to an interval where it is one-to-one, we can define its inverse. Consider the sine function. The sine function is one-to-one on an infinite number of intervals, but the standard convention is to restrict the domain to the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. By doing so, we define the inverse sine function on the domain $[-1, 1]$ such that for any x in the interval $[-1, 1]$, the inverse sine function tells us which angle θ in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ satisfies $\sin\theta = x$. Similarly, we can restrict the domains of the other trigonometric functions to define **inverse trigonometric functions**, which are functions that tell us which angle in a certain interval has a specified trigonometric value.

Definition: Inverse Trigonometric Functions

The inverse sine function, denoted $\sin^{-1}$ or **arcsin**, and the inverse cosine function, denoted $\cos^{-1}$ or **arccos**, are defined on the domain $D = \{y \mid -1 \le y \le 1\}$ as follows:

$$\begin{aligned} \sin^{-1}(y) = \theta \quad &\text{if and only if} \quad \sin(\theta) = y \quad \text{where} \quad \theta \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}] \\ \cos^{-1}(y) = \theta \quad &\text{if and only if} \quad \cos(\theta) = y \quad \text{where} \quad \theta \in [0, \pi]. \end{aligned}$$

The inverse tangent function, denoted $\tan^{-1}$ or **arctan**, and inverse cotangent function, denoted $\cot^{-1}$ or **arccot**, are defined on the domain $D = \{y \mid -\infty < y < \infty\}$ as follows:

$$\begin{aligned} \tan^{-1}(y) = \theta \quad &\text{if and only if} \quad \tan(\theta) = y \quad \text{where} \quad \theta \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2}) \\ \cot^{-1}(y) = \theta \quad &\text{if and only if} \quad \cot(\theta) = y \quad \text{where} \quad \theta \in (0, \pi). \end{aligned}$$

The inverse cosecant function, denoted $\csc^{-1}$ or **arccsc**, and inverse secant function, denoted $\sec^{-1}$ or **arcsec**, are defined on the domain $D = \{y \mid |y| \ge 1\}$ as follows:

$$\csc^{-1}(y) = \theta \quad \text{if and only if} \quad \csc(\theta) = y \quad \text{where} \quad \theta \in \left(0, \tfrac{\pi}{2}\right] \cup \left(\pi, \tfrac{3\pi}{2}\right]$$
$$\sec^{-1}(y) = \theta \quad \text{if and only if} \quad \sec(\theta) = y \quad \text{where} \quad \theta \in \left[0, \tfrac{\pi}{2}\right) \cup \left[\pi, \tfrac{3\pi}{2}\right).$$

To graph the inverse trigonometric functions, we use the graphs of the trigonometric functions restricted to the domains defined earlier and reflect the graphs about the line $y = x$ (Figures 1.1.8 – 1.1.10).

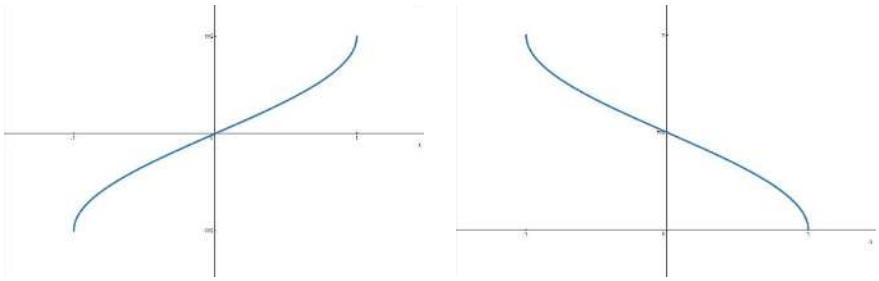


Figure 1.1.8: The graphs of the arcsine (left) and arccosine (right).

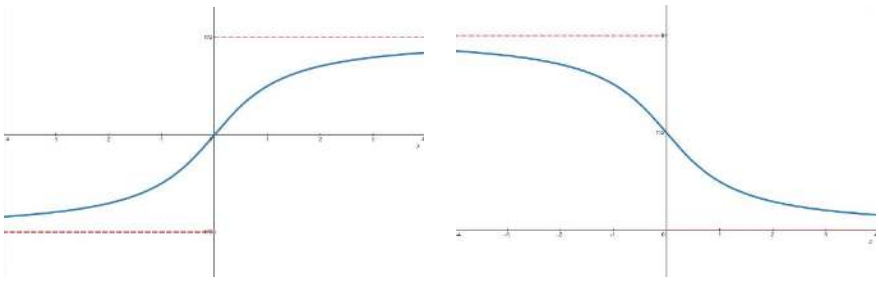


Figure 1.1.9: The graphs of the arctangent (left) and arccotangent (right).

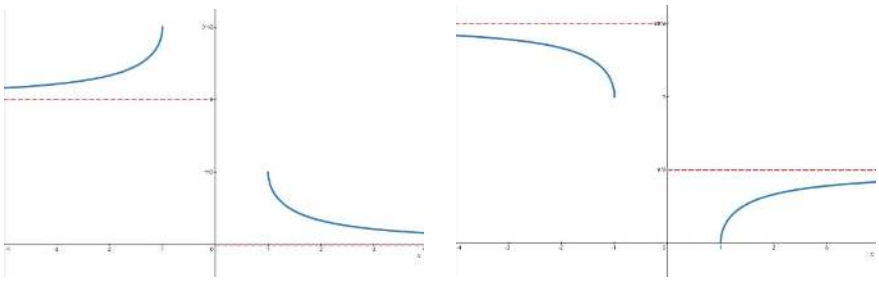


Figure 1.1.10: The graphs of the arccosecant (left) and arcsecant (right).

Note

All the inverse trigonometric functions return angles in quadrant I (where all trigonometric functions are positive) and one other quadrant where the corresponding trigonometric function is negative. The returned quadrants where the corresponding trigonometric functions are negative are fairly standard and agreed upon (quadrant IV for arctangent and arcsine, and quadrant II for arccosine and arccotangent); **however**, the quadrants chosen for the ranges of the arcsecant and arccosecant are *not* universally agreed upon.

In this text, we choose arcsecant and arccosecant to return angles in quadrants I and **III** for a special reason - it makes our work in calculus slightly easier. I mention this because you might have seen a slightly different choice for the ranges of these functions in another textbook. The difference is insignificant other than the ease our choice makes for our work in calculus.

When evaluating an inverse trigonometric function, the output is an angle. For example, to evaluate $\cos^{-1}\left(\frac{1}{2}\right)$, we need to find an angle θ such that $\cos\theta = \frac{1}{2}$. Clearly, many angles have this property. However, given the definition of $\cos^{-1}$, we need the angle θ that not only solves this equation, but also lies in the interval $[0, \pi]$. We conclude that $\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$.

We now consider a composition of a trigonometric function and its inverse. For example, consider the two expressions $\sin\left(\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$ and $\sin^{-1}(\sin(\pi))$.

For the first one, we simplify as follows:

$$\sin\left(\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$

For the second one, we have

$$\sin^{-1}(\sin(\pi)) = \sin^{-1}(0) = 0.$$

The inverse function is supposed to "undo" the original function, so why isn't $\sin^{-1}(\sin(\pi)) = \pi$? Recalling our definition of inverse functions, a function f and its inverse f^{-1} satisfy the conditions $f(f^{-1}(y)) = y$ for all y in the domain of f^{-1} and $f^{-1}(f(x)) = x$ for all x in the domain of f, so what happened here? The issue is that the inverse sine function, $\sin^{-1}$, is the inverse of the restricted sine function defined on the domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Therefore, for x in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, it is true that $\sin^{-1}(\sin x) = x$. However, for values of x outside this interval, the equation does not hold, even though $\sin^{-1}(\sin x)$ is defined for all real numbers x.

What about $\sin(\sin^{-1} y)$? Does that have a similar issue? The answer is no. Since the domain of $\sin^{-1}$ is the interval $[-1, 1]$, we conclude that $\sin\left(\sin^{-1} y\right) = y$ if $-1 \le y \le 1$ and the expression is not defined for other values of y. To summarize,

$$\sin(\sin^{-1} y) = y \text{ if } -1 \le y \le 1$$

and

$$\sin^{-1}(\sin x) = x \text{ if } -\tfrac{\pi}{2} \le x \le \tfrac{\pi}{2}.$$

Similarly, for the cosine function,

$$\cos(\cos^{-1} y) = y \text{ if } -1 \le y \le 1$$

and

$$\cos^{-1}(\cos x) = x \text{ if } 0 \le x \le \pi.$$

Similar properties hold for the other trigonometric functions and their inverses.

Example 1.1.7: Evaluating Expressions Involving Inverse Trigonometric Functions

Evaluate each of the following expressions.

a. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
b. $\tan\left(\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right)$
c. $\cos^{-1}\left(\cos\left(\frac{5\pi}{4}\right)\right)$
d. $\sin^{-1}\left(\cos\left(\frac{2\pi}{3}\right)\right)$

Solution

a. Evaluating $\sin^{-1}(-\sqrt{3}/2)$ is equivalent to finding the angle θ such that $\sin\theta = -\sqrt{3}/2$ and $-\pi/2 \le \theta \le \pi/2$. The angle $\theta = -\pi/3$ satisfies these two conditions. Therefore, $\sin^{-1}(-\sqrt{3}/2) = -\pi/3$.
b. First we use the fact that $\tan^{-1}(-1/\sqrt{3}) = -\pi/6$. Then $\tan(-\pi/6) = -1/\sqrt{3}$. Therefore, $\tan(\tan^{-1}(-1/\sqrt{3})) = -1/\sqrt{3}$.
c. To evaluate $\cos^{-1}(\cos(5\pi/4))$,first use the fact that $\cos(5\pi/4) = -\sqrt{2}/2$. Then we need to find the angle θ such that $\cos(\theta) = -\sqrt{2}/2$ and $0 \le \theta \le \pi$. Since $3\pi/4$ satisfies both these conditions, we have $\cos^{-1}(\cos(5\pi/4)) = \cos^{-1}(-\sqrt{2}/2)) = 3\pi/4$.
d. Since $\cos(2\pi/3) = -1/2$, we need to evaluate $\sin^{-1}(-1/2)$. That is, we need to find the angle θ such that $\sin(\theta) = -1/2$ and $-\pi/2 \le \theta \le \pi/2$. Since $-\pi/6$ satisfies both these conditions, we can conclude that $\sin^{-1}(\cos(2\pi/3)) = \sin^{-1}(-1/2) = -\pi/6$.

The Maximum Value of a Function

In many areas of science, engineering, and mathematics, it is useful to know the maximum value a function can obtain, even if we don't know its exact value at a given instant. For instance, if we have a function describing the strength of a roof beam, we would want to know the maximum weight the beam can support without breaking. If we have a function that describes the

speed of a train, we would want to know its maximum speed before it jumps off the rails. Safe design often depends on knowing maximum values.

This project describes a simple example of a function with a maximum value that depends on two equation coefficients. We will see that maximum values can depend on several factors other than the independent variable x.

1. Consider the graph in Figure 1.1.9 of the function $y = \sin x + \cos x$. Describe its overall shape. Is it periodic? How do you know?

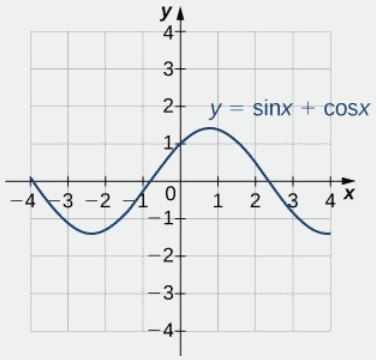


Figure 1.1.9: The graph of $y = \sin x + \cos x$.

Using a graphing calculator or other graphing device, estimate the x- and y-values of the maximum point for the graph (the first such point where $x > 0$). It may be helpful to express the x-value as a multiple of π.

2. Now consider other graphs of the form $y = A\sin x + B\cos x$ for various values of A and B. Sketch the graph when $A = 2$ and $B = 1$, and find the x- and y-values for the maximum point. (Remember to express the x-value as a multiple of π, if possible.) Has it moved?

3. Repeat for $A = 1,\ B = 2$. Is there any relationship to what you found in part (2)?

4. Complete the following table, adding a few choices of your own for A and B:

A	B	x	y	A	B	x	y
0	1			3	4		
1	0			4	3		
1	1			$\sqrt{3}$	1		
1	2			1	$\sqrt{3}$		
2	1			12	5		
2	2			5	12		

5. Try to figure out the formula for the y-values.

6. The formula for the x-values is a little harder. The most helpful points from the table are $(1,1)$, $(1,\sqrt{3})$, $(\sqrt{3},1)$. (Hint: Consider inverse trigonometric functions.)

7. If you found formulas for parts (5) and (6), show that they work together. That is, substitute the x-value formula you found into $y = A\sin x + B\cos x$ and simplify it to arrive at the y-value formula you found.

Key Concepts

- Radian measure is defined such that the angle associated with the arc of length 1 on the unit circle has radian measure 1. An angle with a degree measure of $180°$ has a radian measure of π rad.
- For acute angles θ,the values of the trigonometric functions are defined as ratios of two sides of a right triangle in which one of the acute angles is θ.

- For a general angle θ, let (x, y) be a point on a circle of radius r corresponding to this angle θ. The trigonometric functions can be written as ratios involving x, y, and r.
- The trigonometric functions are periodic. The sine, cosine, secant, and cosecant functions have period 2π. The tangent and cotangent functions have period π.

Key Equations

- **Generalized sine function**

$$f(x) = A\sin(B(x - \alpha)) + C$$

Glossary

periodic function

a function is periodic if it has a repeating pattern as the values of x move from left to right

radians

for a circular arc of length s on a circle of radius 1, the radian measure of the associated angle θ is s

trigonometric functions

functions of an angle defined as ratios of the lengths of the sides of a right triangle

trigonometric identity

an equation involving trigonometric functions that is true for all angles θ for which the functions in the equation are defined

1.1: Trigonometry is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by LibreTexts.

- **1.3: Trigonometric Functions** by Edwin “Jed” Herman, Gilbert Strang is licensed CC BY-NC-SA 4.0. Original source: https://openstax.org/details/books/calculus-volume-1.
- **1.4: Inverse Functions** by Edwin “Jed” Herman, Gilbert Strang is licensed CC BY-NC-SA 4.0. Original source: https://openstax.org/details/books/calculus-volume-1.

1.1E: Exercises

In exercises 1 - 5, convert each angle in degrees to radians. Write the answer as a multiple of π.

1) 240°

Answer

$\frac{4\pi}{3}$ rad

2) 15°

3) 60°

Answer

$\frac{\pi}{3}$ rad

4) -225°

5) 330°

Answer

$\frac{11\pi}{6}$ rad

In exercises 6 - 10, convert each angle in radians to degrees.

6) $\frac{\pi}{2}$ rad

7) $\frac{7\pi}{6}$ rad

Answer

210°

8) $\frac{11\pi}{2}$ rad

9) -3π rad

Answer

-540°

10) $\frac{5\pi}{12}$ rad

In exercises 11 - 16, evaluate the functional values.

11) $\cos\frac{4\pi}{3}$

Answer

$\cos\frac{4\pi}{3} = -0.5$

12) $\tan\frac{19\pi}{4}$

13) $\sin\left(-\frac{3\pi}{4}\right)$

Answer

$\sin\left(-\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

14) $\sec\left(-\frac{\pi}{6}\right)$

15) $\sin\left(-\frac{\pi}{12}\right)$

Answer

$$\sin\left(-\frac{\pi}{12}\right)=\frac{\sqrt{3}-1}{2\sqrt{2}}$$

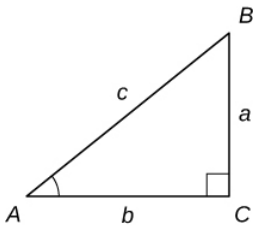
16) $\cos\left(-\frac{5\pi}{12}\right)$

In exercises 17 - 22, consider triangle ABC, a right triangle with a right angle at C.

a. Find the missing side of the triangle.

b. Find the six trigonometric function values for the angle at A.

Where necessary, round to one decimal place.



= 225^{\circ} \cdot

17) $a=4,\ c=7$

Answer

a. $b=5.7$

b. $\sin A=\frac{4}{7},\ \cos A=\frac{5.7}{7},\ \tan A=\frac{4}{5.7},\ \csc A=\frac{7}{4},\ \sec A=\frac{7}{5.7},\ \cot A=\frac{5.7}{4}$

18) $a=21,\ c=29$

19) $a=85.3,\ b=125.5$

Answer

a. $c=151.7$

b. $\sin A=0.5623,\ \cos A=0.8273,\ \tan A=0.6797,\ \csc A=1.778,\ \sec A=1.209,\ \cot A=1.471$

20) $b=40,\ c=41$

21) $a=84,\ b=13$

Answer

a. $c=85$

b. $\sin A=\frac{84}{85},\ \cos A=\frac{13}{85},\ \tan A=\frac{84}{13},\ \csc A=\frac{85}{84},\ \sec A=\frac{85}{13},\ \cot A=\frac{13}{84}$

22) $b=28,\ c=35$

In exercises 23 - 26, P is a point on the unit circle.

a. Find the (exact) missing coordinate value of each point and

b. find the values of the six trigonometric functions for the angle θ with a terminal side that passes through point P.

Rationalize denominators.

23) $P\left(\frac{7}{25},y\right),\quad y>0$

Answer

a. $y=\frac{24}{25}$

b. $\sin\theta=\frac{24}{25},\ \cos\theta=\frac{7}{25},\ \tan\theta=\frac{24}{7},\ \csc\theta=\frac{25}{24},\ \sec\theta=\frac{25}{7},\ \cot\theta=\frac{7}{24}$

24) $P\left(-\frac{15}{17},y\right),\quad y>0$

25) $P\left(x, \frac{\sqrt{7}}{3}\right), \quad x > 0$

Answer

a. $x = -\frac{\sqrt{2}}{3}$

b. $\sin\theta = \frac{\sqrt{7}}{3}, \ \cos\theta = -\frac{\sqrt{2}}{3}, \ \tan\theta = -\frac{\sqrt{14}}{2}, \ \csc\theta = \frac{3\sqrt{7}}{7}, \ \sec\theta = -\frac{3\sqrt{2}}{2}, \ \cot\theta = -\frac{\sqrt{14}}{7}$

26) $P\left(x, -\frac{\sqrt{15}}{4}\right), \quad y > 0$

In exercises 27 - 34, simplify each expression by writing it in terms of sines and cosines, then simplify. The final answer does not have to be in terms of sine and cosine only.

27) $\tan^2 x + \sin x \csc x$

Answer

$\sec^2 x$

28) $\sec x \sin x \cot x$

29) $\frac{\tan^2 x}{\sec^2 x}$

Answer

$\sin^2 x$

30) $\sec x - \cos x$

31) $(1 + \tan\theta)^2 - 2\tan\theta$

Answer

$\sec^2\theta$

32) $(\sin x)(\csc x - \sin x)$

33) $\frac{\cos t}{\sin t} + \frac{\sin t}{1 + \cos t}$

Answer

$\frac{1}{\sin t} = \csc t$

34) $\frac{1 + \tan^2\alpha}{1 + \cot^2\alpha}$

In exercises 35 - 42, verify that each equation is an identity.

35) $\frac{\tan\theta\cot\theta}{\csc\theta} = \sin\theta$

36) $\frac{\sec^2\theta}{\tan\theta} = \sec\theta\csc\theta$

37) $\frac{\sin t}{\csc t} + \frac{\cos t}{\sec t} = 1$

38) $\frac{\sin x}{\cos x + 1} + \frac{\cos x - 1}{\sin x} = 0$

39) $\cot\gamma + \tan\gamma = \sec\gamma\csc\gamma$

40) $\sin^2\beta + \tan^2\beta + \cos^2\beta = \sec^2\beta$

41) $\frac{1}{1 - \sin\alpha} + \frac{1}{1 + \sin\alpha} = 2\sec^2\alpha$

42) $\frac{\tan\theta - \cot\theta}{\sin\theta\cos\theta} = \sec^2\theta - \csc^2\theta$

In exercises 43 - 50, solve the trigonometric equations on the interval $0 \le \theta < 2\pi$.

43) $2\sin\theta - 1 = 0$

Answer

$\left\{\frac{\pi}{6}, \frac{5\pi}{6}\right\}$

44) $1 + \cos\theta = \frac{1}{2}$

45) $2\tan^2\theta = 2$

Answer

$\left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}$

46) $4\sin^2\theta - 2 = 0$

47) $\sqrt{3}\cot\theta + 1 = 0$

Answer

$\left\{\frac{2\pi}{3}, \frac{5\pi}{3}\right\}$

48) $3\sec\theta - 2\sqrt{3} = 0$

49) $2\cos\theta\sin\theta = \sin\theta$

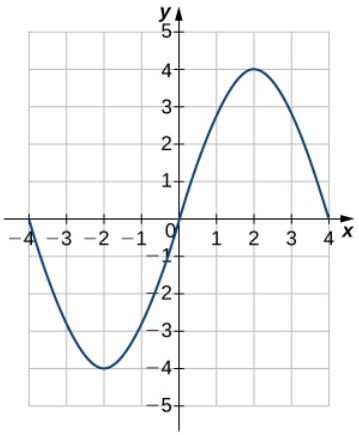
Answer

$\left\{0, \pi, \frac{\pi}{3}, \frac{5\pi}{3}\right\}$

50) $\csc^2\theta + 2\csc\theta + 1 = 0$

In exercises 51 - 54, each graph is of the form $y = A\sin Bx$ or $y = A\cos Bx$, where $B > 0$. Write the equation of the graph.

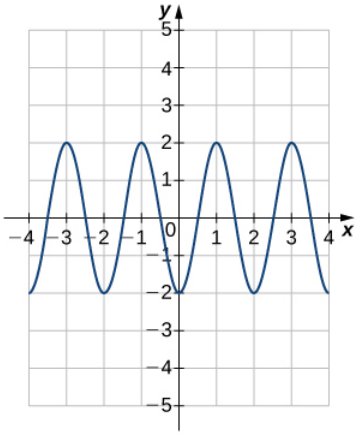
51)



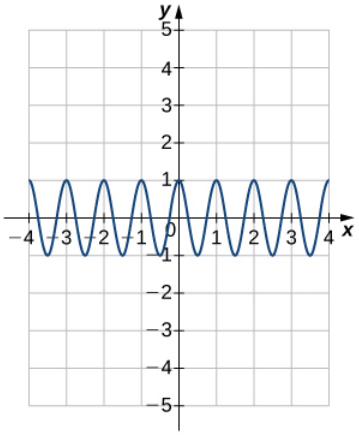
Answer

$y = 4\sin\left(\frac{\pi}{4}x\right)$

52)



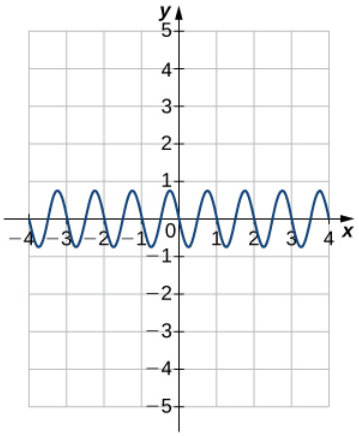
53)



Answer

$y = \cos 2\pi x$

54)



In exercises 55 - 60, find

a. the amplitude,

b. the period, and

c. the phase shift with direction for each function.

55) $y = \sin\left(x - \frac{\pi}{4}\right)$

Answer

a. 1
b. 2π
c. $\frac{\pi}{4}$ units to the right

56) $y = 3\cos(2x + 3)$

57) $y = -\frac{1}{2}\sin\left(\frac{1}{4}x\right)$

Answer

a. $\frac{1}{2}$
b. 8π
c. No phase shift

58) $y = 2\cos\left(x - \frac{\pi}{3}\right)$

59) $y = -3\sin(\pi x + 2)$

Answer

a. 3
b. 2
c. $\frac{2}{\pi}$ units to the left

60) $y = 4\cos\left(2x - \frac{\pi}{2}\right)$

61) [Technology Required] The diameter of a wheel rolling on the ground is 40 in. If the wheel rotates through an angle of $120°$, how many inches does it move? Approximate to the nearest whole inch.

Answer

Approximately 42 in.

62) [Technology Required] Find the length of the arc intercepted by central angle θ in a circle of radius r. Round to the nearest hundredth.

a. $r = 12.8$ cm, $\theta = \frac{5\pi}{6}$ rad b. $r = 4.378$ cm, $\theta = \frac{7\pi}{6}$ rad c. $r = 0.964$ cm, $\theta = 50°$ d. $r = 8.55$ cm, $\theta = 325°$

63) [Technology Required] As a point P moves around a circle, the measure of the angle changes. The measure of how fast the angle is changing is called angular speed, ω, and is given by $\omega = \theta/t$, where θ is in radians and t is time. Find the angular speed for the given data. Round to the nearest thousandth.

a. $\theta = \frac{7\pi}{4}$ rad, $t = 10$ sec b. $\theta = \frac{3\pi}{5}$ rad, $t = 8$ sec c. $\theta = \frac{2\pi}{9}$ rad, $t = 1$ min d. $\theta = 23.76$ rad, $t = 14$ min

Answer

a. 0.550 rad/sec
b. 0.236 rad/sec
c. 0.698 rad/min
d. 1.697 rad/min

64) [Technology Required] A total of $250,000\ \text{m}^2$ of land is needed to build a nuclear power plant. Suppose it is decided that the area on which the power plant is to be built should be circular.

a)Find the radius of the circular land area.

b)If the land area is to form a $45°$ sector of a circle instead of a whole circle, find the length of the curved side.

65) [Technology Required] The area of an isosceles triangle with equal sides of length x is $\frac{1}{2}x^2 \sin\theta$, where θ is the angle formed by the two sides. Find the area of an isosceles triangle with equal sides of length 8 in. and angle $\theta = \frac{5\pi}{12}$ rad.

Answer

$\approx 30.9 \text{ in}^2$

66) [Technology Required] A particle travels in a circular path at a constant angular speed ω. The angular speed is modeled by the function $\omega = 9\,|\cos(\pi t - \pi/12)|$. Determine the angular speed at $t = 9$ sec.

67) [Technology Required] An alternating current for outlets in a home has voltage given by the function $V(t) = 150\cos 368t$,

where V is the voltage in volts at time t in seconds.

a) Find the period of the function and interpret its meaning.

b) Determine the number of periods that occur when 1 sec has passed.

Answer

a. $\frac{\pi}{184}$; the voltage repeats every $\frac{\pi}{184}$ sec
b. Approximately 59 periods

68) [Technology Required] The number of hours of daylight in a northeast city is modeled by the function

$$N(t) = 12 + 3\sin\left[\frac{2\pi}{365}(t - 79)\right],$$

where t is the number of days after January 1.

a) Find the amplitude and period.

b) Determine the number of hours of daylight on the longest day of the year.

c) Determine the number of hours of daylight on the shortest day of the year.

d) Determine the number of hours of daylight 90 days after January 1.

e) Sketch the graph of the function for one period starting on January 1.

69) [Technology Required] Suppose that $T = 50 + 10\sin\left[\frac{\pi}{12}(t - 8)\right]$ is a mathematical model of the temperature (in degrees Fahrenheit) at t hours after midnight on a certain day of the week.

a) Determine the amplitude and period.

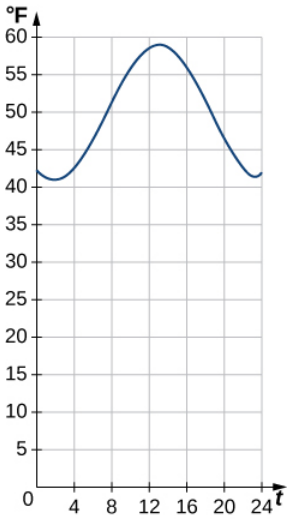
b) Find the temperature 7 hours after midnight.

c) At what time does $T = 60°$?

d) Sketch the graph of T over $0 \le t \le 24$.

Answer

a. Amplitude = 10; Period=24
b. $47.4°$ F
c. 14 hours later, or 2 p.m.
d.



70) [Technology Required] The function $H(t) = 8\sin\left(\frac{\pi}{6}t\right)$ models the height H (in feet) of the tide t hours after midnight. Assume that $t = 0$ is midnight.

a) Find the amplitude and period.

b) Graph the function over one period.

c) What is the height of the tide at 4:30 a.m.?

In exercises 71 - 79, evaluate the functions. Give the exact value.

71) $\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$

Answer

$\frac{\pi}{6}$

72) $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

73) $\cot^{-1}(1)$

Answer

$\frac{\pi}{4}$

74) $\sin^{-1}(-1)$

75) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

Answer

$\frac{\pi}{6}$

76) $\cos\left(\tan^{-1}(\sqrt{3})\right)$

77) $\sin\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$

Answer

$\frac{\sqrt{2}}{2}$

78) $\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$

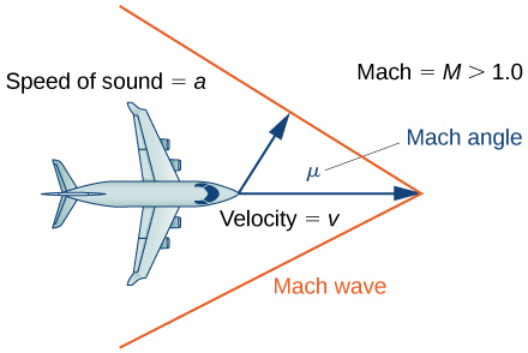
79) $\tan^{-1}\left(\tan\left(-\frac{\pi}{6}\right)\right)$

Answer

$-\frac{\pi}{6}$

80) [Technology Required] An airplane's Mach number M is the ratio of its speed to the speed of sound. When a plane is flying at a constant altitude, then its Mach angle is given by $\mu = 2\sin^{-1}\left(\frac{1}{M}\right)$.

Find the Mach angle (to the nearest degree) for the following Mach numbers.



a. $\mu = 1.4$

b. $\mu = 2.8$

c. $\mu = 4.3$

Answer

a. $\sim 92°$ b. $\sim 42°$ c. $\sim 27°$

81) [Technology Required] Using $\mu = 2\sin^{-1}\left(\frac{1}{M}\right)$, find the Mach number M for the following angles.

a. $\mu = \frac{\pi}{6}$

b. $\mu = \frac{2\pi}{7}$

c. $\mu = \frac{3\pi}{8}$

82) [Technology Required] The temperature (in degrees Celsius) of a city in the northern United States can be modeled by the function

$$T(x) = 5 + 18\sin\left[\frac{\pi}{6}(x - 4.6)\right],$$

where x is time in months and $x = 1.00$ corresponds to January 1. Determine the month and day when the temperature is $21°C$.

Answer

$x \approx 6.69, 8.51$; so, the temperature occurs on June 21 and August 15

83) [Technology Required] The depth (in feet) of water at a dock changes with the rise and fall of tides. It is modeled by the function $D(t) = 5\sin\left(\frac{\pi}{6}t - \frac{7\pi}{6}\right) + 8$, where t is the number of hours after midnight. Determine the first time after midnight when the depth is 11.75 ft.

84) [Technology Required] An object moving in simple harmonic motion is modeled by the function $s(t) = -6\cos\left(\frac{\pi t}{2}\right)$, where s is measured in inches and t is measured in seconds. Determine the first time when the distance moved is 4.5 in.

Answer

~ 1.5 sec

85) [Technology Required] A local art gallery has a portrait 3 ft in height that is hung 2.5 ft above the eye level of an average person. The viewing angle θ can be modeled by the function $\theta = \tan^{-1}\frac{5.5}{x} - \tan^{-1}\frac{2.5}{x}$, where x is the distance (in feet) from the portrait. Find the viewing angle when a person is 4 ft from the portrait.

86) [Technology Required] Use a calculator to evaluate $\tan^{-1}(\tan(2.1))$ and $\cos^{-1}(\cos(2.1))$. Explain the results of each.

Answer

$\tan^{-1}(\tan(2.1)) \approx -1.0416$; the expression does not equal 2.1 since $2.1 > 1.57 = \frac{\pi}{2}$—in other words, it is not in the restricted domain of $\tan x$. $\cos^{-1}(\cos(2.1)) = 2.1$, since 2.1 is in the restricted domain of $\cos x$.

87) [Technology Required] Use a calculator to evaluate $\sin(\sin^{-1}(-2))$ and $\tan(\tan^{-1}(-2))$. Explain the results of each.

Contributors

Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

1.1E: Exercises is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Gilbert Strang & Edwin “Jed” Herman.

1.2: Derivatives Review

Defining the Derivative of a Function at a Point

Recall that the derivative of a function f at a point a is denoted as $f'(a)$ and can be thought of in one of two ways:

- $f'(a)$ is the slope of the tangent line of the graph of f at the point $(a, f(a))$
- $f'(a)$ is a number that describes how the output values of f are changing for x-values near a

Using what we know about slopes of lines, we gather that

- $f'(a) > 0 \Leftrightarrow f$ is increasing at a
- $f'(a) < 0 \Leftrightarrow f$ is decreasing at a
- The steeper the graph of f is at $x = a$, the larger $|f'(a)|$ is



Figure 1.2.1: A function graph with a moveable tangent line, showing the slope

By the way, a great way to think of what a *tangent line* is, is as follows:

> The **tangent line** to the graph of f at the point a is the *line that the graph looks like if you zoom way in to the point* $(a, f(a))$.

We define the derivative of a function f at a by taking the slope of a secant line through $(a, f(a))$ and another nearby point on the graph, and then taking the limit as that nearby point approaches $(a, f(a))$. Notationally, this looks like one of two definitions.

Definition: Derivative of f at a

The derivative of f at a point a in its domain is defined by

$$1. \quad f'(a) = \lim_{X \to a} \frac{f(X) - f(a)}{X - a}$$

or

$$2. \quad f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided these limits exist.

Note that these two definitions are equivalent. The idea behind each of these definitions is demonstrated in the following interactive figures.



Figure 1.2.2: Showing the calculation of $f'(x)$ using version 1 of the definition of derivative



Figure 1.2.3: Showing the calculation of $f'(x)$ using version 2 of the definition of derivative

Defining the Derivative As a Function

We define the derivative function f' by taking the derivative of f at every point in its domain. The derivative function is defined essentially by the second version of the derivative of the function at a point, and letting that point vary throughout the domain of f.

Definition: The Derivative Function of f

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Recall that back in Calculus I, we used this definition of the derivative as a limit every time we calculated the derivative of a new function. See if you can do it yourself with the following exercise, before looking at the solution.

Exercise 1.2.1

Use the definition of the derivative of a function to calculate $g'(x)$ for $g(x) = x^2 - 3x + 1$.

Solution

We use the definition of derivative for the given function.

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\ &= \lim_{h\to 0} \frac{\left[(x+h)^2-3(x+h)+1\right]-\left[x^2-3x+1\right]}{h} \\ &= \lim_{h\to 0} \frac{x^2+2hx+h^2-3x-3h+1-x^2+3x-1}{h} \\ &= \lim_{h\to 0} \frac{2hx+h^2-3h}{h} \\ &= \lim_{h\to 0} (2x+h-3) \\ &= 2x-3 \end{aligned}$$

Note that we can then use the derivative function to calculate the derivative at any given point, as in $g'(4) = 2(4) - 3$, so $g'(4) = 5$.

Derivative Notations

There are several different ways to write derivatives. Below are the most common ways that we will be using throughout this class.

For the derivative as a function, the following statements are equivalent:

- If $g(x) = x^2 - 3x + 1$, then $g'(x) = 2x - 3$
- $\frac{d}{dx}\left(x^2 - 3x + 1\right) = 2x - 3$
- $\left(x^2 - 3x + 1\right)' = 2x - 3$

For the derivative at a point, the following statements are equivalent:

- If $g(x) = x^2 - 3x + 1$, then $g'(4) = 5$
- $\frac{d}{dx}\Big|_{x=4}\left(x^2 - 3x + 1\right) = 5$

Derivatives of Basic Functions

In Precalculus math we learn 5 ways that you can combine two functions: using the four arithmetic operations of addition, subtraction, multiplication, or division; or through function composition. We can also modify a single function by multiplying it by a constant k. In Calculus I we learn that if we know the derivatives of two given functions, then there are some basic rules that tell us how to take derivatives of any combinations of the functions.

Rules of Differentiation

If f and g are differentiable functions, then

- $(k \cdot f)' = k \cdot f'$ (where k is a constant)
- $(f + g)' = f' + g'$
- $(f - g)' = f' - g'$
- $(fg)' = f'g + g'f$ (the **Product Rule**)
- $\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$ (the **Quotient Rule**)
- $(f(g(x)))' = f'(g(x)) \cdot g'(x)$ (the **Chain Rule**)

Using these rules, we're able to calculate the derivatives of any combination of functions if we know the derivatives of the basic functions that are combined. Throughout Calculus I, we used the definition of the derivative of a function to calculate derivatives of all of our basic functions. Part of our work in Calculus II will be related to integration and antidifferentiation, which requires fluent knowledge of all of these basic derivatives.

Table 1.2.1: Differentiation Formulas

$\frac{d}{dx}(k)=0$	$\frac{d}{dx}(x^n)=nx^{n-1}$ (the Power Rule)
$\frac{d}{dx}(e^x)=e^x$	$\frac{d}{dx}(b^x)=b^x\ln(b)$
$\frac{d}{dx}(\ln\lvert x\rvert)=\frac{1}{x}$	$\frac{d}{dx}(\log_a x)=\frac{1}{x\ln a},\quad x>0$
$\frac{d}{dx}(\sin x)=\cos x$	$\frac{d}{dx}(\cos x)=-\sin x$
$\frac{d}{dx}(\tan x)=\sec^2 x$	$\frac{d}{dx}(\cot x)=-\csc^2 x$
$\frac{d}{dx}(\sec x)=\sec x\tan x$	$\frac{d}{dx}(\csc x)=-\csc x\cot x$
$\frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}(\cos^{-1}x)=-\frac{1}{\sqrt{1-x^2}}$
$\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}$	$\frac{d}{dx}(\cot^{-1}x)=-\frac{1}{1+x^2}$
$\frac{d}{dx}(\sec^{-1}x)=\frac{1}{x\sqrt{x^2-1}}$	$\frac{d}{dx}(\csc^{-1}x)=-\frac{1}{x\sqrt{x^2-1}}$
$\frac{d}{dx}(\sinh x)=\cosh x$	$\frac{d}{dx}(\cosh x)=\sinh x$

? Exercise 1.2.2

Calculate the following derivatives.

1. $\frac{d}{dx}\left(\frac{3}{\sqrt[4]{x}}\right)$
2. $\frac{d}{dx}\left(\frac{3x^2-4x}{\cos x}\right)$
3. $\frac{d}{dx}(\sin(e^x+5x+3))$
4. $\frac{d}{dx}(\sin x(e^x+5x+3))$

Solution

1. We can write the function as a power function and then use the Power Rule.

$$\begin{aligned}\frac{d}{dx}\left(\frac{3}{\sqrt[4]{x}}\right)&=\frac{d}{dx}\left(3x^{-1/4}\right)\\&=3\left(-\frac{1}{4}x^{-\frac{1}{4}-1}\right)\\&=-\frac{3}{4}x^{-5/4}\end{aligned}$$

2. For this we use the Quotient Rule.

$$\begin{aligned}\frac{d}{dx}\left(\frac{3x^2-4x}{\cos x}\right)&=\frac{\cos x\cdot\frac{d}{dx}(3x^2-4x)-(3x^2-4x)\cdot\frac{d}{dx}(\cos x)}{(\cos x)^2}\\&=\frac{\cos x(6x-4)-(3x^2-4x)(-\sin x)}{\cos^2 x}\\&=\frac{(6x-4)\cos x+(3x^2-4x)\sin x}{\cos^2 x}\end{aligned}$$

3. For this we use the Chain Rule, since we are differentiating a composition of functions.

$$\begin{aligned}\frac{d}{dx}\left(\sin\left(e^x+5x+3\right)\right) &= \sin\left(e^x+5x+3\right)\cdot\frac{d}{dx}\left(e^x+5x+3\right)\\ &= \sin\left(e^x+5x+3\right)\cdot\left(e^x+5\right)\\ &= \left(e^x+5\right)\sin\left(e^x+5x+3\right)\end{aligned}$$

4. This one is a product of functions, so we'll use the Product Rule.

$$\begin{aligned}\frac{d}{dx}\left(\sin x\left(e^x+5x+3\right)\right) &= \sin x\cdot\frac{d}{dx}\left(e^x+5x+3\right)+\left(e^x+5x+3\right)\cdot\frac{d}{dx}\left(\sin x\right)\\ &= \sin x\cdot\left(e^x+5\right)+\left(e^x+5x+3\right)\cdot\left(\cos x\right)\\ &= \left(e^x+5\right)\sin x+\left(e^x+5x+3\right)\cos x\end{aligned}$$

Exercise 1.2.3

Write an equation for the line tangent to the graph of $h(x)=\frac{1}{(3x-5)^2}$ at $x=2$. Verify your answer by using a graphing utility to graph the function along with the tangent line.

Solution

Because we are finding an equation of a line, we need a point. The x-coordinate of the point is 2. To find the y-coordinate, substitute 2 into $h(x)$. Since $h(2)=\frac{1}{(3(2)-5)^2}=1$, the point is $(2,1)$.

For the slope, we need $h'(2)$. To find $h'(x)$, first we rewrite $h(x)=(3x-5)^{-2}$ and apply the Power Rule and the Chain Rule to obtain

$$h'(x)=-2(3x-5)^{-3}(3)=-6(3x-5)^{-3}.$$

By substituting, we have $h'(2)=-6(3(2)-5)^{-3}=-6$.

Therefore, with slope -6 and point $(2,1)$, the line has equation $y=1-6(x-2)$. The following graph confirms that this equation is tangent to the graph of h at $x=2$.



Figure 1.2.4: A graph of $h(x) = \frac{1}{(3x-5)^2}$ and $y = 1 - 6(x - 2)$

This page titled 1.2: Derivatives Review is shared under a not declared license and was authored, remixed, and/or curated by .

1.2E: Exercises

This page is a draft and is under active development.

Differentiation Review Exercises

True or False? Justify the answer with a proof or a counterexample.

1) Every function has a derivative.

Answer

False

2) A continuous function has a continuous derivative.

3) A continuous function has a derivative.

Answer

False

4) If a function is differentiable, it is continuous.

In exercises 5 and 6, use the limit definition of the derivative to exactly evaluate the derivative.

5) $f(x) = \sqrt{x+4}$

Answer

$$f'(x) = \frac{1}{2\sqrt{x+4}}$$

6) $f(x) = \frac{3}{x}$

In exercises 7 - 15, find the derivatives of the given functions.

7) $f(x) = 3x^3 - \frac{4}{x^2}$

Answer

$$f'(x) = 9x^2 + \tfrac{8}{x^3}$$

9) $f(x) = (4 - x^2)^3$

10) $f(x) = e^{\sin x}$

Answer

$$f'(x) = e^{\sin x}\cos x$$

11) $f(x) = \ln(x+2)$

12) $f(x) = x^2\cos x + x\tan x$

Answer

$$f'(x) = x\sec^2 x + 2x\cos x + \tan x - x^2\sin x$$

13) $f(x) = \sqrt{3x^2+2}$

14) $f(x) = \frac{x}{4}\sin^{-1}(x)$

Answer

$$f'(x) = \frac{1}{4}\left(\frac{x}{\sqrt{1-x^2}} + \sin^{-1} x\right)$$

15) $x^2y = (y+2) + xy\sin x$

In exercises 16 - 18, find the indicated derivatives of various orders.

16) First derivative of $y = x(\ln x)\cos x$

Answer

$$\frac{dy}{dx} = \cos x \cdot (\ln x + 1) - x(\ln x)\sin x$$

17) Third derivative of $y = (3x+2)^2$

18) Second derivative of $y = 4^x + x^2 \sin x$

Answer

$$\frac{d^2y}{dx^2} = 4^x(\ln 4)^2 + 2\sin x + 4x\cos x - x^2\sin x$$

In exercises 19 and 20, find the equation of the tangent line to the following equations at the specified point.

19) $y = \cos^{-1}(x) + x$ at $x = 0$

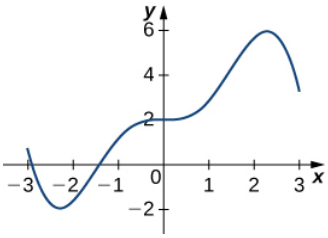
20) $y = x + e^x - \dfrac{1}{x}$ at $x = 1$

Answer

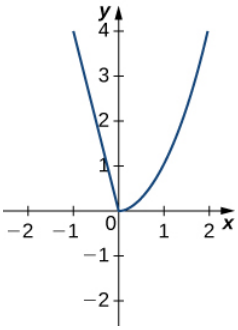
$$y = (2+e)(x-1) + e$$

In exercises 21 and 22, draw the derivative of the functions with the given graphs.

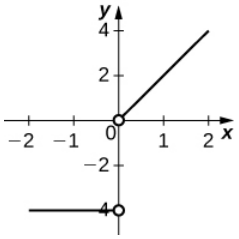
21)



22)



Answer



Questions 22 and 23 concern the water level in Ocean City, New Jersey, in January, which can be approximated by $w(t) = 1.9 + 2.9\cos(\frac{\pi}{6}t)$, where t is measured in hours after midnight, and the height is measured in feet.

22) Find and graph the derivative. What is the physical meaning?

23) Find $w'(3)$. What is the physical meaning of this value?

Answer

$w'(3) = -\frac{2.9\pi}{6}$. At 3 a.m. the tide is decreasing at a rate of 1.514 ft/hr.

Questions 24 and 25 consider the wind speeds of Hurricane Katrina, which affected New Orleans, Louisiana, in August 2005. The data are displayed in a table.

Hours after Midnight, August 26	Wind Speed (mph)
1	45
5	75
11	100
29	115
49	145
58	175
73	155
81	125
85	95
107	35

Wind Speeds of Hurricane KatrinaSource: news.nationalgeographic.com/n..._timeline.html.

24) Using the table, estimate the derivative of the wind speed at hour 39. What is the physical meaning?

25) Estimate the derivative of the wind speed at hour 83. What is the physical meaning?

Answer

−7.5. The wind speed is decreasing at a rate of 7.5 mph/hr

This page titled 1.2E: Exercises is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Gilbert Strang & Edwin “Jed” Herman (OpenStax) via source content that was edited to the style and standards of the LibreTexts platform.

- **3.R: Chapter 3 Review Exercises** by Edwin “Jed” Herman, Gilbert Strang is licensed CC BY-NC-SA 4.0. Original source: https://openstax.org/details/books/calculus-volume-1.

1.3: Integrals Review

This page is a draft and is under active development.

Learning Objectives

- Use Riemann sums to approximate area.
- State the definition of the definite integral.
- Describe the relationship between the definite integral and net signed area.
- State the Fundamental Theorem of Calculus, Parts 1 and 2.
- Explain the relationship between differentiation and integration.
- Know the general antiderivative of several functions, based on derivatives of known basic functions.
- Use properties of integrals (definite and indefinite) to integrate linear combinations of the functions with known antiderivatives

Approximating Area

The topic of function integration can begin with a question of how to calculate areas of curved regions. Let $f(x)$ be a continuous, nonnegative function defined on the closed interval $[a, b]$. We want to calculate the area A bounded by $f(x)$ above, the x-axis below, the line $x = a$ on the left, and the line $x = b$ on the right (Figure 1.3.1).

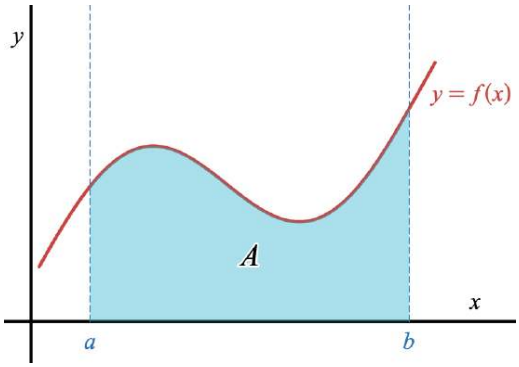


Figure 1.3.1: An area (shaded region) bounded by the curve $f(x)$ at the top, the x-axis at the bottom, the line $x = a$ to the left, and the line $x = b$ at right.

We begin by approximating the curved area with a collection of rectangular areas. We divide the interval $[a, b]$ into n subintervals of equal width, $\Delta x = \frac{b-a}{n}$, where the sub-intervals have endpoints points $a = x_0, x_1, x_2, \ldots, x_n = b$. With this setup, we have

$$x_i = x_0 + i\Delta x \quad \text{for} \quad i = 1, 2, 3, \ldots, n.$$

Then, within each sub-interval (x_{i-1}, x_i), we choose a representative point x_i^*, so that the ith rectangle has width Δx and height $f(x_i^*)$ and area $f(x_i^*)\Delta x$. Then the entire area is approximated with the sum

$$A \approx \sum_{i=1}^{n} f(x_i^*)\Delta x$$

.

Area Approximation

Given a function f that is continuous and non-negative on an interval (a, b) and a positive integer n, we approximate the area under the curve $y = f(x)$ on the interval $[a, b]$ as follows. Divide the interval $[a, b]$ into n equal-length subintervals, each with width $\Delta x = \frac{b-a}{n}$, and with endpoints $a = x_0, x_1, x_2, \ldots, x_{n-1}, x_n$. Within each subinterval $[x_{i-1}, x_i]$ (for $i = 1, 2, 3, \ldots, n$), choose a

representative number x_i^* and construct a rectangle with width Δx and height equal to $f(x_i^*)$, which is the function value at the representative input value. Then the area of this rectangle is $f(x_i^*)\Delta x$. Adding the areas of all these rectangles, we get an approximate value for A (Figure 1.3.2). We use the notation A_n to denote that this is an approximation of A using n subintervals.

$$A \approx A_n = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x = \sum_{i=1}^{n} f(x_i^*)\Delta x$$



Figure 1.3.2: In the rectangular approximation of the area under a curve, the height of each rectangle is determined by the function value at some representative point x_i^* of each subinterval.

Note that if we choose x_i^* to be the left endpoint of each sub-interval, $x_i^* = x_{i-1} = a + (i-1)\Delta x$, then we get the **left-endpoint approximation**,

$$A \approx L_n = \sum_{i=0}^{n-1} f(x_i)\Delta x$$

If we choose x_i^* to be the right endpoint of each sub-interval, $x_i^* = x_i = a + i\Delta x$, then we get the **right-endpoint approximation**,

$$A \approx R_n = \sum_{i=1}^{n} f(x_i)\Delta x$$

If we choose x_i^* to be the midpoint of each sub-interval, $\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i)$, then we get the **midpoint approximation**,

$$A \approx M_n = \sum_{i=1}^{n} f(\bar{x}_i)\Delta x$$

Example 1.3.1: Approximating the Area Under a Curve

Use left-endpoint, right-endpoint, and midpoint approximations to approximate the area under the curve of $f(x) = 4 - x^2$ on the interval $[0, 2]$, for $n = 4$.

Solution

First, divide the interval $[0, 2]$ into 4 equal subintervals, so $\Delta x = \frac{(2-0)}{4} = 0.5$. This is the width of each rectangle.

Using a left-endpoint approximation, the heights are $f(0) = 4$, $f(0.5) = 3.75$, $f(1) = 3$, and $f(1.5) = 1.75$. Then,

$$\begin{aligned} L_4 &= f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x \\ &= 4(0.5) + 3.75(0.5) + 3(0.5) + 1.75(0.5) \\ &= 6.25 \text{ units}^2 \end{aligned}$$

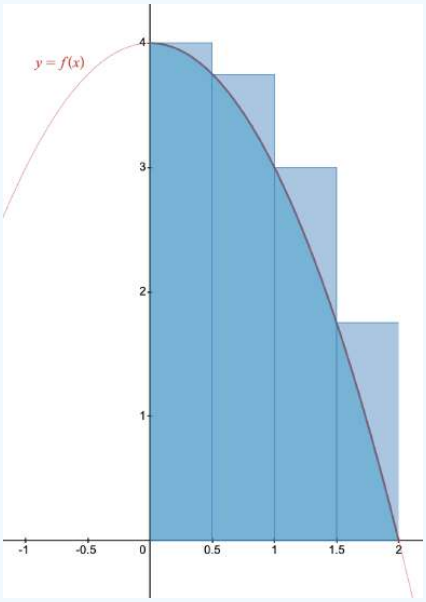


Figure 1.3.3: The graph shows the left-endpoint approximation of the area under $f(x) = 4 - x^2$ from 0 to 2.

The right-endpoint approximation is shown in Figure 1.3.4. The intervals are the same, $\Delta x = 0.5$, but now use the right endpoint to calculate the height of the rectangles. We have

$$\begin{aligned} R_4 &= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x \\ &= 3.75(0.5) + 3(0.5) + 1.75(0.5) + 0(0.5) \\ &= 4.25 \text{ units}^2 \end{aligned}$$

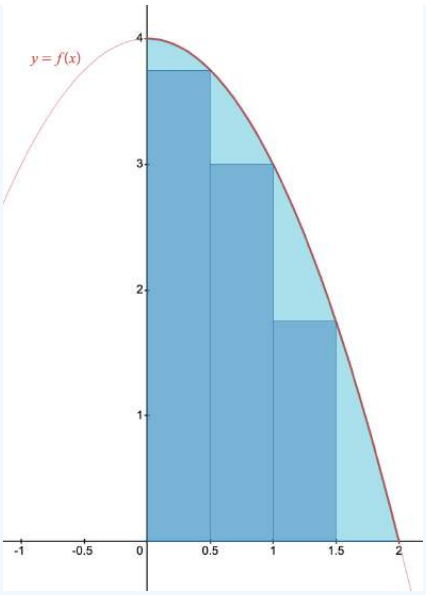


Figure 1.3.4: The graph shows the right-endpoint approximation of the area under $f(x) = 4 - x^2$ from 0 to 2.

Using a midpoint approximation, the heights are $f(0.25) = 3.9375$, $f(0.75) = 3.4375$, $f(1.25) = 2.4375$, and $f(1.75) = 0.9375$. Then,

$$\begin{aligned} M_4 &= f(\bar{x}_1)\Delta x + f(\bar{x}_2)\Delta x + f(\bar{x}_3)\Delta x + f(\bar{x}_4)\Delta x \\ &= 3.9375(0.5) + 3.4375(0.5) + 2.4375(0.5) + 0.9375(0.5) \\ &= 5.375 \text{ units}^2 \end{aligned}$$

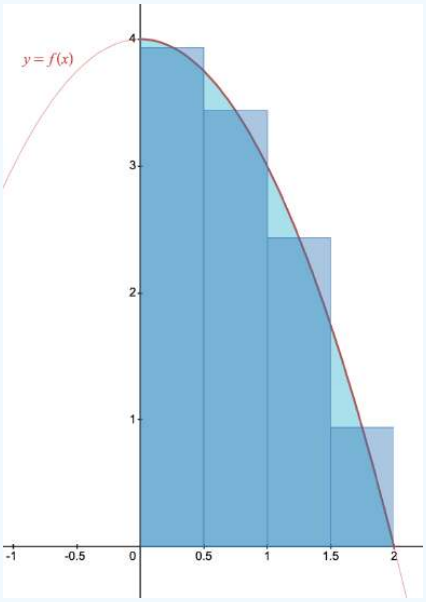


Figure 1.3.5: The graph shows the midpoint approximation of the area under $f(x) = 4 - x^2$ from 0 to 2.

The left-endpoint approximation is $L_4 = 6.25 \text{ units}^2$; the right-endpoint approximation is $R_4 = 4.25 \text{ units}^2$; and the midpoint approximation is $M_4 = 5.375 \text{ units}^2$.

Forming Riemann Sums

Regardless of whether we use left endpoints, right endpoints, midpoints, or random points of sub-intervals, we have always ended up with sums of the form $A_n = \sum_{i=1}^{n} f(x_i^*)\ \Delta x$. A sum of this form is called a **Riemann sum**, named for the 19th-century mathematician Bernhard Riemann, who developed the idea.

Definition: Riemann sum

Let $f(x)$ be defined on a closed interval $[a, b]$ and let P be any partition of $[a, b]$. Let Δx_i be the width of each subinterval $[x_{i-1}, x_i]$ and for each i, let x_i^* be any point in $[x_{i-1},\ x_i]$. A **Riemann sum** is defined for $f(x)$ as

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x_i.$$

At this point, we'll choose a regular partition P, as we have in our examples above. This forces all Δx_i to be equal to $\Delta x = \frac{b-a}{n}$ for any natural number of intervals n.

All of these sums will provide a better approximation of the curved area as n increases, and they all converge to the same number A as $n \to \infty$. This allows us to define the area under a curve formally.

Definition: Area Under the Curve

Let $f(x)$ be a continuous, nonnegative function on an interval $[a, b]$, and let $\sum_{i=1}^{n} f(x_i^*)\ \Delta x$ be a Riemann sum for $f(x)$ with a regular partition P. Then, the **area under the curve** $y = f(x)$ on $[a, b]$ is given by

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x.$$

It is worth noting that we are putting a lot of faith in this limit converging, and converging to the same number regardless of the choices of the x_i^* in each sub-interval. Proof of this convergence is beyond the scope of this course, but rest assured that it has been rigorously proven.

The Definite Integral

In the preceding section, we defined the area under a curve in terms of Riemann sums:$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$.. For the sake of calculating the area under the curve, we required $f(x)$ to be continuous and nonnegative. However, all of the calculations that we did could have been done with any function f that is defined at every point in the interval (a, b). This gives us the **definite integral** of f on the interval (a, b).

Definition: Definite Integral

If $f(x)$ is a function defined on an interval $[a, b]$, the **definite integral** of f from a to b is given by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x,$$

provided the limit exists. If this limit exists, the function $f(x)$ is said to be **integrable** on $[a, b]$, or is an **integrable function**.

In the notation $\int_a^b f(x)\ dx$, we call the function $f(x)$ the **integrand**, and the dx indicates that $f(x)$ is a function with respect to x, called the **variable of integration**. Note that, like the index in a sum, the variable of integration is a **dummy variable** and has no impact on the computation of the integral. We could use any variable we like as the variable of integration:

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(u)\,du$$

Previously, we discussed the fact that if $f(x)$ is continuous on $[a,b]$, then the limit $\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$ exists and is unique. This leads to the following theorem, which we state without proof.

Theorem: Continuous Functions Are Integrable

If $f(x)$ is continuous on $[a,b]$, then f is integrable on $[a,b]$.

Functions that are not continuous on $[a,b]$ may still be integrable, depending on the nature of the discontinuities. For example, functions with a finite number of jump discontinuities or removable discontinuities on a closed interval are integrable.

Area and the Definite Integral

When we defined the definite integral, we lifted the requirement that $f(x)$ be nonnegative. But how do we interpret "the area under the curve" when $f(x)$ is negative?

Let us return to the Riemann sum. Consider, for example, the function $f(x) = 2 - 2x^2$ (shown in Figure 1.3.2) on the interval $[0,2]$. Use $n = 8$ and choose x_i^* as the midpoint of each interval. Construct a rectangle on each subinterval of height $f(x_i^*)$ and width Δx. When $f(x_i^*)$ is positive, the product $f(x_i^*)\Delta x$ represents the area of the rectangle, as before. When $f(x_i^*)$ is negative, however, the product $f(x_i^*)\Delta x$ represents the negative of the area of the rectangle. The Riemann sum then becomes

$$\sum_{i=1}^{8} f(x_i^*)\Delta x = (\text{Area of rectangles above the } x\text{-axis}) - (\text{Area of rectangles below the } x\text{-axis})$$

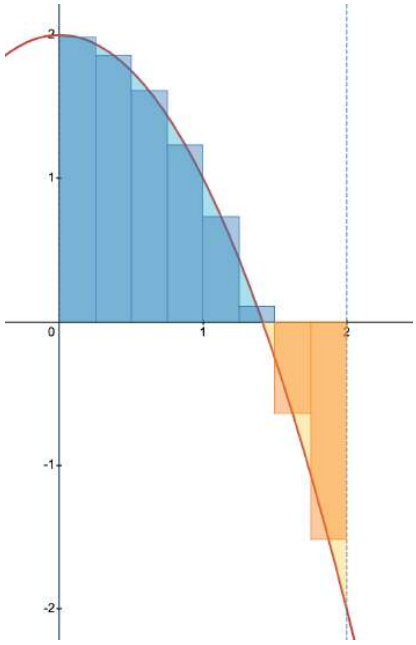


Figure 1.3.6: For a function that is partly negative, the Riemann sum is the area of the rectangles above the x-axis less the area of the rectangles below the x-axis.

Taking the limit as $n \to \infty$, the Riemann sum approaches the area between the curve above the x-axis and the x-axis, less the area between the curve below the x-axis and the x-axis, as shown in Figure 1.3.3. Then,

$$\int_0^2 f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x = A_1 - A_2.$$

The quantity $A_1 - A_2$ is called the **net signed area.**

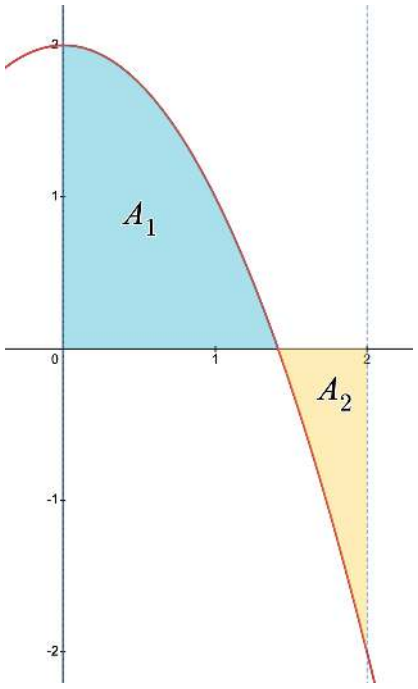


Figure **1.3.7**: In the limit, the definite integral equals area A_1 less area A_2, or the net signed area.

Properties of the Definite Integral

These properties, along with the rules of integration that we examine later in this course, help us manipulate expressions to evaluate definite integrals.

Theorem: Properties of the Definite Integral

1. If the bounds of integration are the same, the integral is just a vertical line and contains no area. That is,
$$\int_a^a f(x)\,dx = 0.$$
2. If the bounds of integration are reversed, the integral gets negated. That is,
$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$
3. The integral of a sum is the sum of the integrals. That is,
$$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$
4. The integral of a difference is the difference of the integrals. That is,
$$\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx.$$
5. The integral of a constant multiple of a function is equal to the constant multiplied by the integral of the function. That is, for a constant c,
$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx.$$
6. An integral can be split into pieces along the interval. That is,
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Although this formula normally applies when c is between a and b, the formula holds for all values of a, b, and c, provided $f(x)$ is integrable on the largest interval.

Example 1.3.2: Using the Properties of the Definite Integral

Use the properties of the definite integral to express the definite integral of $f(x) = -3x^3 + 2x + 2$ over the interval $[-2, 1]$ as the sum of three definite integrals.

Solution

Using integral notation, we have $\int_{-2}^{1} (-3x^3 + 2x + 2)\, dx$. We apply properties 3. and 5. to get

$$\begin{aligned} \int_{-2}^{1}(-3x^3 + 2x + 2)\, dx &= \int_{-2}^{1} -3x^3\, dx + \int_{-2}^{1} 2x\, dx + \int_{-2}^{1} 2\, dx \\ &= -3\int_{-2}^{1} x^3\, dx + 2\int_{-2}^{1} x\, dx + \int_{-2}^{1} 2\, dx. \end{aligned}$$

Checkpoint 1.3.1

Use the properties of the definite integral to express the definite integral of $f(x) = 6x^3 - 4x^2 + 2x - 3$ over the interval $[1, 3]$ as the sum of four definite integrals.

Answer

$$6\int_1^3 x^3\, dx - 4\int_1^3 x^2\, dx + 2\int_1^3 x\, dx - \int_1^3 3\, dx$$

Example 1.3.3: Using the Properties of the Definite Integral

If it is known that $\int_0^8 f(x)\, dx = 10$ and $\int_0^5 f(x)\, dx = 5$, find the value of $\int_5^8 f(x)\, dx$.

Solution

By property 6,

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

Thus,

$$\begin{aligned} \int_0^8 f(x)\, dx &= \int_0^5 f(x)\, dx + \int_5^8 f(x)\, dx \\ 10 &= 5 + \int_5^8 f(x)\, dx \\ 5 &= \int_5^8 f(x)\, dx. \end{aligned}$$

Checkpoint 1.3.2

If it is known that $\int_1^5 f(x)\, dx = -3$ and $\int_2^5 f(x)\, dx = 4$, find the value of $\int_1^2 f(x)\, dx$.

Answer

-7

The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus comes in two parts, and it tells us that integration and differentiation are somewhat inverse process of each other. You can find the proofs of these theorems in Section 5.4 of the Calculus I textbook.

Theorem: The Fundamental Theorem of Calculus, Part 1 (FTC1)

If $f(x)$ is continuous over an interval $[a, b]$, and the function $F(x)$ is defined by

$$F(x) = \int_a^x f(t)\,dt,$$

then $F'(x) = f(x)$ over $[a, b]$.

$F(x)$ is called an **integral function** (also known as an **area function**) because it represents the definite integral of a function, $f(t)$, over an interval determined by the argument of F. That is, $F(x)$ is the area under the curve $f(t)$ from $t = a$ to some, yet to be determined value, $t = x$.

The **Fundamental Theorem of Calculus, Part 2**, is perhaps the most important theorem in Calculus. This is the theorem that we use to evaluate most definite integrals. It seems too simple that the area of an entire curved region can be calculated by just evaluating an antiderivative at the first and last endpoints of an interval.

Theorem: The Fundamental Theorem of Calculus, Part 2

If $f(x)$ is continuous over the interval $[a, b]$ and $F(x)$ is any antiderivative of $f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a). \tag{1.3.1}$$

We often see the notation $F(x)\Big|_a^b$ to denote the expression $F(b) - F(a)$. We use this vertical bar and associated limits a and b to indicate that we should evaluate the function $F(x)$ at the upper limit (in this case, b), and subtract the value of the function $F(x)$ evaluated at the lower limit (in this case, a).

The Fundamental Theorem of Calculus, Part 2 (also known as the **Evaluation Theorem**) states that if we can find an antiderivative for the integrand, then we can evaluate the definite integral by evaluating the antiderivative at the endpoints of the interval and subtracting.

Example 1.3.4: Evaluating an Integral with the Fundamental Theorem of Calculus

Use Equation 1.3.1 to evaluate

$$\int_{-2}^{2} (t^2 - 4)\,dt.$$

Solution

An antiderivative for $f(t) = t^2 - 4$ is \(F(t)=\dfrac{t^3}{3}-4t \). Therefore we have\[\begin{array}{rcl}

\int^2_{−2}(t^2−4)dt & = & \left(\dfrac{t^3}{3}−4t \right)\Big|^2_{−2} \[6pt]
& = & \left[\dfrac{(2)^3}{3}−4(2)\right]−\left[\dfrac{(−2)^3}{3}−4(−2)\right] \[6pt]
& = & \left[\dfrac{8}{3}−8\right] − \left[−\dfrac{8}{3}+8 \right] \[6pt]
& = & \dfrac{8}{3}−8+\dfrac{8}{3}−8 \[6pt]
& = & \dfrac{16}{3}−16 \[6pt]
& = & −\dfrac{32}{3}. \[6pt]
\end{array}\nonumber \]

Checkpoint 1.3.3

Evaluate $\int_1^2 x^{-4}\,dx$.

Answer

$\frac{7}{24}$

Antiderivatives and Indefinite Integrals

Recall the definition of antiderivative.

Definition: Antiderivative

A function F is an **antiderivative** of the function f if

$$F'(x) = f(x)$$

for all x in the domain of f.

And recall that antiderivatives come in *families* of functions, all of which are constant differences of one another.

Theorem: General Form of an Antiderivative

Let F be an antiderivative of f over an interval I. Then,

I. for each constant C, the function $F(x) + C$ is also an antiderivative of f over I;
II. if G is an antiderivative of f over I, there is a constant C for which $G(x) = F(x) + C$ over I.

In other words, the most general form of the antiderivative of f over I is $F(x) + C$.

Indefinite Integrals

Since the Fundamental Theorem of Calculus (both parts) gives us a sort of inverse relationship between differentiation and integration, it makes sense to use the integral symbol $\int$ to denote antidifferentiation.

Definition: Indefinite Integrals

Given a function f, the **indefinite integral** of f, denoted

$$\int f(x)\,dx,$$

is the most general antiderivative of f. If F is an antiderivative of f, then

$$\int f(x)\,dx = F(x) + C.$$

The expression $f(x)$ is called the **integrand** and the variable x is the **variable of integration**.

Given the terminology introduced in this definition, the act of finding the antiderivatives of a function f is usually referred to as **integrating** f.

For a function f and an antiderivative F, the set of functions $F(x) + C$, where C is any real number, is often referred to as the **family of antiderivatives** of f. For example, since x^2 is an antiderivative of $2x$ and any antiderivative of $2x$ is of the form $x^2 + C$, we write

$$\int 2x\,dx = x^2 + C.$$

The collection of all functions of the form $x^2 + C$, where C is any real number, is known as the family of antiderivatives of $2x$. Figure 1.3.1 shows a graph of this family of antiderivatives.

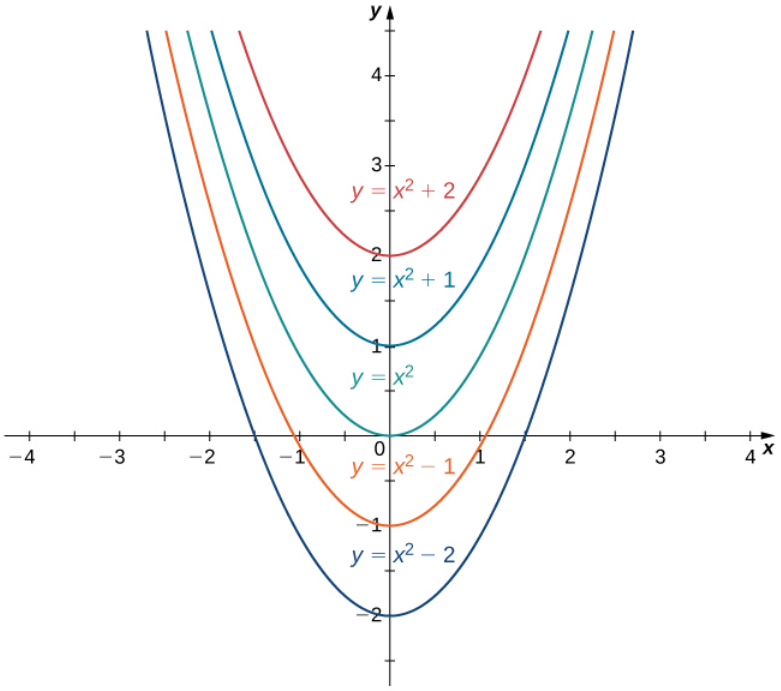


Figure 1.3.1: The family of antiderivatives of $2x$ consists of all functions of the form $x^2 + C$, where C is any real number.

For some functions, evaluating indefinite integrals follows directly from properties of derivatives. For example, for $n \neq -1$,

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,$$

which comes directly from

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = (n+1)\frac{x^n}{n+1} = x^n.$$

This fact is known as the Power Rule for integrals.

Theorem: Power Rule for Antidifferentiaion

For $n \neq -1$,

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C.$$

Power functions are pretty much the only function for which there is a formula for antidifferentiation. Outside of power functions, the primary way that one finds an antiderivative of a function is to recognize it as the derivative of a function that they know. For this reason, it is very helpful to fluently know the derivatives (forward and backward) of all of your basic functions. The following table lists the indefinite integrals for several common functions.

Table 1.3.1: Integration Formulas

Differentiation Formula	Indefinite Integral
$\frac{d}{dx}(k) = 0$	$\int k\,dx = \int kx^0\,dx = kx + C$
$\frac{d}{dx}(x^n) = nx^{n-1}$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ for $n \neq -1$
$\frac{d}{dx}(\ln\lvert x\rvert) = \frac{1}{x}$	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$
$\frac{d}{dx}(e^x) = e^x$	$\int e^x\,dx = e^x + C$
$\frac{d}{dx}(b^x) = b^x \ln(b)$	$\int b^x\,dx = \frac{b^x}{\ln(b)} + C$
$\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x\,dx = \sin x + C$
$\frac{d}{dx}(\cos x) = -\sin x$	$\int \sin x\,dx = -\cos x + C$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$
$\frac{d}{dx}(\csc x) = -\csc x \cot x$	$\int \csc x \cot x\,dx = -\csc x + C$
$\frac{d}{dx}(\sec x) = \sec x \tan x$	$\int \sec x \tan x\,dx = \sec x + C$
$\frac{d}{dx}(\cot x) = -\csc^2 x$	$\int \csc^2 x\,dx = -\cot x + C$
$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$
$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$
$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1} x + C$
$\frac{d}{dx}(\sinh(x)) = \cosh(x)$	$\int \cosh(x)\,dx = \sinh(x) + C$
$\frac{d}{dx}(\cosh(x)) = \sinh(x)$	$\int \sinh(x)\,dx = \cosh(x) + C$

From the definition of indefinite integral of f, we know

$$\int f(x)\,dx = F(x) + C$$

if and only if F is an antiderivative of f.

Therefore, when claiming that

$$\int f(x)\,dx = F(x) + C,$$

it is essential to check whether this statement is correct by verifying that $F'(x) = f(x)$.

Because differentiation respects sums, differences, and constant multiples of functions, antidifferentiation also does. These properties are summaried below.

Theorem: Properties of Antiderivatives

Let F and G be antiderivatives of f and g, respectively, and let k be any real number.

Sums and Differences

$$\int (f(x) \pm g(x))\,dx = F(x) \pm G(x) + C$$

Constant Multiples

$$\int kf(x)\,dx = kF(x) + C$$

From this theorem, we can evaluate any integral involving a sum, difference, or constant multiple of functions with known antiderivatives. Unlike differentiation, there are not rules that allow you to antidifferentiate products or quotients or compositions of functions that you know the antiderivatives of. We will spend some time throughout this course learning techniques of integration, but they nearly all boil down to re-working the integrand to make it look like one of the derivatives we know.

Example 1.3.5: Evaluating Indefinite Integrals

Evaluate each of the following indefinite integrals:

a. $\int \left(5x^3 - 7x^2 + 3x + 4\right)\,dx$

b. $\int \dfrac{x^2 + 4\sqrt[3]{x}}{x}\,dx$

c. $\int \dfrac{4}{1+x^2}\,dx$

d. $\int \tan x \cos x\,dx$

Solutions

a. Using Properties of Indefinite Integrals, we can integrate the four terms in the integrand separately. We obtain

$$\int \left(5x^3 - 7x^2 + 3x + 4\right)\,dx = \int 5x^3\,dx - \int 7x^2\,dx + \int 3x\,dx + \int 4\,dx.$$

From the second part of Properties of Indefinite Integrals, each coefficient can be written in front of the integral sign, which gives

$$\int 5x^3\,dx - \int 7x^2\,dx + \int 3x\,dx + \int 4\,dx = 5\int x^3\,dx - 7\int x^2\,dx + 3\int x\,dx + 4\int 1\,dx.$$

Using the Power Rule for integrals, we conclude that

$$\int \left(5x^3 - 7x^2 + 3x + 4\right)\,dx = \frac{5}{4}x^4 - \frac{7}{3}x^3 + \frac{3}{2}x^2 + 4x + C.$$

b. Rewrite the integrand as

$$\frac{x^2 + 4\sqrt[3]{x}}{x} = \frac{x^2}{x} + \frac{4\sqrt[3]{x}}{x}.$$

Then, to evaluate the integral, integrate each of these terms separately. Using the Power Rule, we have

$$\begin{aligned}\int \left(x + \frac{4}{x^{2/3}}\right)\,dx &= \int x\,dx + 4\int x^{-2/3}\,dx \\ &= \frac{1}{2}x^2 + 4\frac{1}{\left(\frac{-2}{3}\right)+1}x^{(-2/3)+1} + C \\ &= \frac{1}{2}x^2 + 12x^{1/3} + C.\end{aligned}$$

c. Using Properties of Indefinite Integrals, write the integral as

$$4\int \frac{1}{1+x^2}\,dx.$$

Then, use the fact that $\tan^{-1}(x)$ is an antiderivative of $\frac{1}{1+x^2}$ to conclude that

$$\int \frac{4}{1+x^2}\,dx = 4\tan^{-1}(x) + C.$$

d. Rewrite the integrand as

$$\tan x \cos x = \frac{\sin x}{\cos x} \cdot \cos x = \sin x.$$

Therefore,

$$\int \tan x \cos x \, dx = \int \sin x \, dx = -\cos x + C.$$

? Checkpoint 1.3.4

Evaluate $\int \left(4x^3 - 5x^2 + x - 7\right) \, dx$.

Answer

$$\int \left(4x^3 - 5x^2 + x - 7\right) \, dx = \quad x^4 - \frac{5}{3}x^3 + \frac{1}{2}x^2 - 7x + C$$

Key Concepts

- Riemann sums are expressions of the form $\sum_{i=1}^{n} f(x_i^*)\Delta x$ and can be used to estimate the area under the curve $y = f(x)$. Left- and right-endpoint and midpoint approximations are special kinds of Riemann sums where the values of x_i^* are chosen to be the left or right endpoints or midpoints of the subintervals, respectively.
- The definite integral can be used to calculate the net signed area, which is the area above the x-axis less the area below the x-axis. The net signed area can be positive, negative, or zero.
- Functions that are piecewise continuous on a closed interval are integrable.
- The properties of definite integrals can be used to evaluate integrals.
- The Fundamental Theorem of Calculus, Parts 1 and 2, show an inverse relationship between the derivative and the integral.
- The Fundamental Theorem of Calculus Part 2 gives a formula for evaluating a definite integral in terms of an antiderivative of its integrand.
- The area under the curve of many functions can be calculated using geometric formulas rather than through antidifferentiation.
- We use the integral symbol without bounds of integration to denote antiderivative families.
- If F is an antiderivative of f, then every antiderivative of f is of the form $F(x) + C$ for some constant C. That is, $\int f(x) \, dx = F(x) + C$.
- A definite integral (with bounds of integration) results in a *number*, whereas an indefinite integral (without bounds of integration) results in a *family of functions*.

Key Equations

- **Left-Endpoint Approximation**

$$A \approx L_n = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x = \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

- **Right-Endpoint Approximation**

$$A \approx R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x = \sum_{i=1}^{n} f(x_i)\Delta x$$

- **Definite Integral**

$$\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

- **Properties of the Definite Integral**

$$\int_a^a f(x) \, dx = 0$$

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

$$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, for constant c

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

- **Fundamental Theorem of Calculus, Part 1**

  If $f(x)$ is continuous over an interval $[a, b]$, and the function $F(x)$ is defined by

  $$F(x) = \int_a^x f(t)\,dt,$$

  then

  $$F'(x) = f(x).$$

- **Fundamental Theorem of Calculus, Part 2**

  If f is continuous over the interval $[a, b]$ and $F(x)$ is any antiderivative of $f(x)$, then

  $$\int_a^b f(x)\,dx = F(b) - F(a).$$

Glossary

fundamental theorem of calculus
: the theorem, central to the entire development of Calculus, that establishes the relationship between differentiation and integration

fundamental theorem of calculus, part 1
: uses a definite integral to define an antiderivative of a function

fundamental theorem of calculus, part 2
: (also, **evaluation theorem**) we can evaluate a definite integral by evaluating the antiderivative of the integrand at the endpoints of the interval and subtracting

antiderivative
: a function F such that $F'(x) = f(x)$ for all x in the domain of f is an antiderivative of f

indefinite integral
: the most general antiderivative of $f(x)$ is the indefinite integral of f; we use the notation $\int f(x)\,dx$ to denote the indefinite integral of f

initial value problem
: a problem that requires finding a function y that satisfies the differential equation $\frac{dy}{dx} = f(x)$ together with the initial condition $y(x_0) = y_0$

1.3: Integrals Review is shared under a not declared license and was authored, remixed, and/or curated by LibreTexts.

1.3.1: EXERCISES

This page is a draft and is under active development.

INDEFINITE INTEGRALS AND ANTIDERIVATIVES

In exercises 1 - 20, find the antiderivative $F(x)$ of each function $f(x)$.

1) $f(x) = \frac{1}{x^2} + x$

2) $f(x) = e^x - 3x^2 + \sin x$

Answer

$F(x) = e^x - x^3 - \cos x + C$

3) $f(x) = e^x + 3x - x^2$

4) $f(x) = x - 1 + 4\sin(2x)$

Answer

$F(x) = \frac{x^2}{2} - x - 2\cos(2x) + C$

5) $f(x) = 5x^4 + 4x^5$

6) $f(x) = x + 12x^2$

Answer

$F(x) = \frac{1}{2}x^2 + 4x^3 + C$

7) $f(x) = \frac{1}{\sqrt{x}}$

8) $f(x) = (\sqrt{x})^3$

Answer

$F(x) = \frac{2}{5}(\sqrt{x})^5 + C$

9) $f(x) = x^{1/3} + (2x)^{1/3}$

10) $f(x) = \frac{x^{1/3}}{x^{2/3}}$

Answer

$F(x) = \frac{3}{2}x^{2/3} + C$

11) $f(x) = 2\sin(x) + \sin(2x)$

12) $f(x) = \sec^2 x + 1$

Answer

$F(x) = x + \tan x + C$

13) $f(x) = \sin x \cos x$

14) $f(x) = \sin^2(x)\cos(x)$

Answer

$F(x) = \frac{1}{3}\sin^3(x) + C$

15) $f(x) = 0$

16) $f(x) = \frac{1}{2}\csc^2 x + \frac{1}{x^2}$

Answer

$$F(x) = -\tfrac{1}{2}\cot x - \frac{1}{x} + C$$

17) $f(x) = \csc x \cot x + 3x$

18) $f(x) = 4\csc x \cot x - \sec x \tan x$

Answer

$$F(x) = -\sec x - 4\csc x + C$$

19) $f(x) = 8(\sec x)\left(\sec x - 4\tan x\right)$

20) $f(x) = \cosh(x) + \sinh(x)$

Answer

$$F(x) = \cosh(x) + \sinh(x) + C$$

For exercises 21 - 31, evaluate the integral.

21) $\int (-1)\, dx$

22) $\int \sin x\, dx$

Answer

$$\int \sin x\, dx = -\cos x + C$$

23) $\int \left(4x + \sqrt{x}\right) dx$

24) $\int \frac{3x^2 + 2}{x^2}\, dx$

Answer

$$\int \frac{3x^2 + 2}{x^2}\, dx = 3x - \frac{2}{x} + C$$

25) $\int \left(\sec x \tan x + 4x\right) dx$

26) $\int \left(4\sqrt{x} + \sqrt[4]{x}\right) dx$

Answer

$$\int \left(4\sqrt{x} + \sqrt[4]{x}\right) dx = \frac{8}{3}x^{3/2} + \frac{4}{5}x^{5/4} + C$$

27) $\int \left(x^{-1/3} - x^{2/3}\right)\, dx$

28) $\int \frac{14x^3 + 2x + 1}{x^3}\, dx$

Answer

$$\int \frac{14x^3 + 2x + 1}{x^3}\, dx = 14x - \frac{2}{x} - \frac{1}{2x^2} + C$$

29) $\int \left(e^x + e^{-x}\right) dx$

30) $\int \frac{dx}{\sqrt{x^2 + 1}}$

31) $\int -\frac{dx}{x\sqrt{1 - x^2}}$

In exercises 32 - 36, solve the initial value problem.

32) $f'(x) = x^{-3}, \quad f(1) = 1$

Answer

$$f(x) = -\frac{1}{2x^2} + \frac{3}{2}$$

33) $f'(x) = \sqrt{x} + x^2, \quad f(0) = 2$

34) $f'(x) = \cos x + \sec^2(x), \quad f(\frac{\pi}{4}) = 2 + \frac{\sqrt{2}}{2}$

Answer

$f(x) = \sin x + \tan x + 1$

35) $f'(x) = x^3 - 8x^2 + 16x + 1, \quad f(0) = 0$

36) $f'(x) = \dfrac{2}{x^2} - \dfrac{x^2}{2}, \quad f(1) = 0$

Answer

$f(x) = -\frac{1}{6}x^3 - \dfrac{2}{x} + \dfrac{13}{6}$

In exercises 37 - 40, find two possible functions f given the second- or third-order derivatives

37) $f''(x) = x^2 + 2$

38) $f''(x) = e^{-x}$

Answer

Answers may vary; one possible answer is $f(x) = e^{-x}$

39) $f''(x) = 1 + x$

40) $f'''(x) = \cos x$

Answer

Answers may vary; one possible answer is $f(x) = -\sin x$

41) A car is being driven at a rate of 40 mph when the brakes are applied. The car decelerates at a constant rate of $10\ \text{ft/sec}^2$. How long before the car stops?

Answer

5.867 sec

42) In the preceding problem, calculate how far the car travels in the time it takes to stop.

43) You are merging onto the freeway, accelerating at a constant rate of $12\ \text{ft/sec}^2$. How long does it take you to reach merging speed at 60 mph?

Answer

7.333 sec

44) Based on the previous problem, how far does the car travel to reach merging speed?

45) A car company wants to ensure its newest model can stop in 8 sec when traveling at 75 mph. If we assume constant deceleration, find the value of deceleration that accomplishes this.

Answer

$13.75\ \text{ft/sec}^2$

46) A car company wants to ensure its newest model can stop in less than 450 ft when traveling at 60 mph. If we assume constant deceleration, find the value of deceleration that accomplishes this.

In exercises 47 - 51, find the antiderivative of the function, assuming $F(0) = 0$.

47) [T] $f(x) = x^2 + 2$

Answer

$F(x) = \frac{1}{3}x^3 + 2x$

48) [T] $f(x) = 4x - \sqrt{x}$

49) [T] $f(x) = \sin x + 2x$

Answer

$F(x) = x^2 - \cos x + 1$

50) [T] $f(x) = e^x$

51) [T] $f(x) = \dfrac{1}{(x+1)^2}$

Answer

$F(x) = -\dfrac{1}{x+1} + 1$

In exercises 52 - 55, determine whether the statement is true or false. Either prove it is true or find a counterexample if it is false.

52) If $f(x)$ is the antiderivative of $v(x)$, then $2f(x)$ is the antiderivative of $2v(x)$.

Answer

True

53) If $f(x)$ is the antiderivative of $v(x)$, then $f(2x)$ is the antiderivative of $v(2x)$.

54) If $f(x)$ is the antiderivative of $v(x)$, then $f(x) + 1$ is the antiderivative of $v(x) + 1$.

Answer

False

55) If $f(x)$ is the antiderivative of $v(x)$, then $(f(x))^2$ is the antiderivative of $(v(x))^2$.

In exercises 43 - 45, use the fact that a falling body with friction equal to velocity squared obeys the equation $\dfrac{dv}{dt} = g - v^2$.

43) Show that $v(t) = \sqrt{g}\tanh(\sqrt{g}t)$ satisfies this equation.

Answer

Answers may vary

44) Derive the previous expression for $v(t)$ by integrating $\dfrac{dv}{g - v^2} = dt$.

45) [Technology Required] Estimate how far a body has fallen in 12seconds by finding the area underneath the curve of $v(t)$.

Answer

37.30

In exercises 46 - 48, use this scenario: A cable hanging under its own weight has a slope $S = \dfrac{dy}{dx}$ that satisfies $\dfrac{dS}{dx} = c\sqrt{1 + S^2}$. The constant c is the ratio of cable density to tension.

46) Show that $S = \sinh(cx)$ satisfies this equation.

47) Integrate $\dfrac{dy}{dx} = \sinh(cx)$ to find the cable height $y(x)$ if $y(0) = 1/c$.

Answer

$y = \frac{1}{c}\cosh(cx)$

48) Sketch the cable and determine how far down it sags at $x = 0$.

FUNDAMENTAL THEOREM OF CALCULUS

In exercises 1 - 12, use the Fundamental Theorem of Calculus, Part 1, to find each derivative.

1) $\dfrac{d}{dx}\left[\displaystyle\int_1^x e^{-t^2}\,dt\right]$

2) $\dfrac{d}{dx}\left[\displaystyle\int_1^x e^{\cos t}\,dt\right]$

Answer

$$\frac{d}{dx}\left[\int_1^x e^{\cos t}\,dt\right] = e^{\cos t}$$

3) $\frac{d}{dx}\left[\int_3^x \sqrt{9-y^2}\,dy\right]$

4) $\frac{d}{dx}\left[\int_3^x \frac{ds}{\sqrt{16-s^2}}\right]$

Answer

$$\frac{d}{dx}\left[\int_3^x \frac{ds}{\sqrt{16-s^2}}\right] = \frac{1}{\sqrt{16-x^2}}$$

5) $\frac{d}{dx}\left[\int^{2x} x_t\,dt\right]$

6) $\frac{d}{dx}\left[\int_0^{\sqrt{x}} t\,dt\right]$

Answer

$$\frac{d}{dx}\left[\int_0^{\sqrt{x}} t\,dt\right] = \sqrt{x}\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2}$$

7) $\frac{d}{dx}\left[\int_0^{\sin x} \sqrt{1-t^2}\,dt\right]$

8) $\frac{d}{dx}\left[\int_{\cos x}^1 \sqrt{1-t^2}\,dt\right]$

Answer

$$\frac{d}{dx}\left[\int_{\cos x}^1 \sqrt{1-t^2}\,dt\right] = -\sqrt{1-\cos^2 x}\frac{d}{dx}\left(\cos x\right) = |\sin x|\sin x$$

9) $\frac{d}{dx}\left[\int_1^{\sqrt{x}} \frac{t^2}{1+t^4}\,dt\right]$

10) $\frac{d}{dx}\left[\int_1^{x^2} \frac{\sqrt{t}}{1+t}\,dt\right]$

Answer

$$\frac{d}{dx}\left[\int_1^{x^2} \frac{\sqrt{t}}{1+t}\,dt\right] = 2x\frac{|x|}{1+x^2}$$

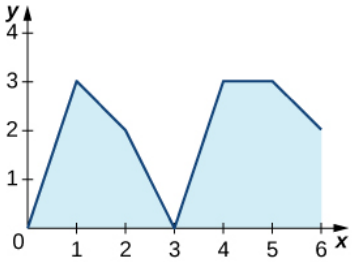
11) $\frac{d}{dx}\left[\int_0^{\ln x} e^t\,dt\right]$

12) $\frac{d}{dx}\left[\int_1^{e^x} \ln u^2\,du\right]$

Answer

$$\frac{d}{dx}\left[\int_1^{e^x} \ln u^2\,du\right] = \ln(e^{2x})\frac{d}{dx}\left(e^x\right) = 2xe^x$$

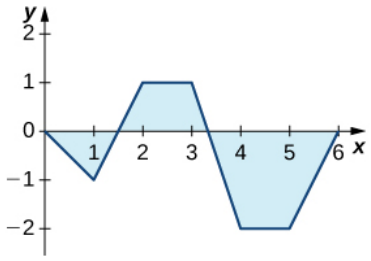
13) The graph of $y = \int_0^x f(t)\,dt$, where f is a piecewise constant function, is shown here.



a. Over which intervals is f positive? Over which intervals is it negative? Over which intervals, if any, is it equal to zero?

b. What are the maximum and minimum values of f?

14) The graph of $y = \int_0^x f(t)\,dt$, where f is a piecewise constant function, is shown here.



a. Over which intervals is f positive? Over which intervals is it negative? Over which intervals, if any, is it equal to zero?

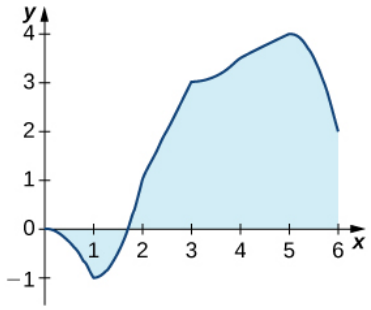
b. What are the maximum and minimum values of f?

Answer

a. f is positive over $[1, 2]$ and $[5, 6]$, negative over $[0, 1]$ and $[3, 4]$, and zero over $[2, 3]$ and $[4, 5]$.

b. The maximum value is 2 and the minimum is -3.

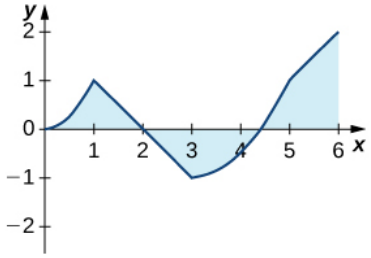
15) The graph of $y = \int_0^x \ell(t)\,dt$, where ℓ is a piecewise linear function, is shown here.



a. Over which intervals is ℓ positive? Over which intervals is it negative? Over which, if any, is it zero?

b. Over which intervals is ℓ increasing? Over which is it decreasing? Over which, if any, is it constant?

16) The graph of $y = \int_0^x \ell(t)\,dt$, where ℓ is a piecewise linear function, is shown here.



a. Over which intervals is ℓ positive? Over which intervals is it negative? Over which, if any, is it zero?

b. Over which intervals is ℓ increasing? Over which is it decreasing? Over which intervals, if any, is it constant?

Answer

a. ℓ is positive over $[0, 1]$ and $[3, 6]$, and negative over $[1, 3]$.

b. It is increasing over $[0, 1]$ and $[3, 5]$, and it is constant over $[1, 3]$ and $[5, 6]$.

In exercises 17 - 36, evaluate each definite integral using the Fundamental Theorem of Calculus, Part 2.

17) $\int_{-1}^{2} (x^2 - 3x)\,dx$

18) $\int_{-2}^{3} (x^2 + 3x - 5)\,dx$

Answer

$$F(x) = \frac{x^3}{3} + \frac{3x^2}{2} - 5x, \quad F(3) - F(-2) = -\frac{35}{6}$$

19) $\int_{-2}^{3} (t+2)(t-3)dt$

20) $\int_{2}^{3} (t^2-9)(4-t^2)dt$

Answer

$$F(x) = -\frac{t^5}{5} + \frac{13t^3}{3} - 36t, \quad F(3) - F(2) = \frac{62}{15}$$

21) $\int_{1}^{2} x^9\,dx$

22) $\int_{0}^{1} x^{99}\,dx$

Answer

$$F(x) = \frac{x^{100}}{100}, \quad F(1) - F(0) = \frac{1}{100}$$

23) $\int_{4}^{8} (4t^{5/2} - 3t^{3/2})dt$

24) $\int_{1/4}^{4} \left(x^2 - \frac{1}{x^2}\right) dx$

Answer

$$F(x) = \frac{x^3}{3} + \frac{1}{x}, \quad F(4) - F\left(\tfrac{1}{4}\right) = \frac{1125}{64}$$

25) $\int_{1}^{2} \frac{2}{x^3}\,dx$

26) $\int_{1}^{4} \frac{1}{2\sqrt{x}}\,dx$

Answer

$$F(x) = \sqrt{x}, \quad F(4) - F(1) = 1$$

27) $\int_{1}^{4} \frac{2-\sqrt{t}}{t^2}\,dt$

28) $\int_{1}^{16} \frac{dt}{t^{1/4}}$

Answer

$$F(t) = \tfrac{4}{3}t^{3/4}, \quad F(16) - F(1) = \tfrac{28}{3}$$

29) $\int_{0}^{2\pi} \cos\theta\,d\theta$

30) $\int_{0}^{\pi/2} \sin\theta\,d\theta$

Answer

$$F(\theta) = -\cos\theta, \quad F\left(\tfrac{\pi}{2}\right) - F(0) = 1$$

31) $\int_{0}^{\pi/4} \sec^2\theta\,d\theta$

32) $\int_{0}^{\pi/4} \sec\theta\tan\theta\,d\theta$

Answer

$$F(\theta) = \sec\theta, \quad F\left(\tfrac{\pi}{4}\right) - F(0) = \sqrt{2} - 1$$

33) $\int_{\pi/3}^{\pi/4} \csc\theta\cot\theta\,d\theta$

34) $\int_{\pi/4}^{\pi/2} \csc^2\theta\,d\theta$

Answer

$F(\theta) = -\cot\theta, \quad F(\frac{\pi}{2}) - F(\frac{\pi}{4}) = 1$

35) $\int_1^2 \left(\frac{1}{t^2} - \frac{1}{t^3}\right) dt$

36) $\int_{-2}^{-1} \left(\frac{1}{t^2} - \frac{1}{t^3}\right) dt$

Answer

$F(t) = -\frac{1}{t} + \frac{1}{2t^2}, \quad F(-1) - F(-2) = \frac{7}{8}$

In exercises 37 - 40, use the evaluation theorem to express the integral as a function $F(x)$.

37) $\int_a^x t^2\,dt$

38) $\int_1^x e^t\,dt$

Answer

$F(x) = e^x - e$

39) $\int_0^x \cos t\,dt$

40) $\int_{-x}^x \sin t\,dt$

Answer

$F(x) = 0$

In exercises 41 - 44, identify the roots of the integrand to remove absolute values, then evaluate using the Fundamental Theorem of Calculus, Part 2.

41) $\int_{-2}^3 |x|\,dx$

42) $\int_{-2}^4 |\,t^2 - 2t - 3\,|\,dt$

Answer

$\int_{-2}^{-1} (t^2 - 2t - 3)\,dt - \int_{-1}^3 (t^2 - 2t - 3)\,dt + \int_3^4 (t^2 - 2t - 3)\,dt = \frac{46}{3}$

43) $\int_0^\pi |\cos t|\,dt$

44) $\int_{-\pi/2}^{\pi/2} |\sin t|\,dt$

Answer

$-\int_{-\pi/2}^0 \sin t\,dt + \int_0^{\pi/2} \sin t\,dt = 2$

1.3.1: Exercises is shared under a not declared license and was authored, remixed, and/or curated by LibreTexts.

- **6.9E: Exercises for Section 6.9** by Edwin "Jed" Herman, Gilbert Strang is licensed CC BY-NC-SA 4.0. Original source: https://openstax.org/details/books/calculus-volume-1.
- **4.10E: Exercises for Section 4.10** is licensed CC BY-NC-SA 4.0.
- **5.3E: Exercises for Section 5.3** is licensed CC BY-NC-SA 4.0.

1.4: Integration by Substitution

Learning Objectives

- Use substitution to evaluate indefinite integrals.
- Use substitution to evaluate definite integrals.

Hawk A.I. Section-Specific Tutor

Please note that, to access the Hawk A.I. Tutor, you will need a (free) OpenAI account.

The Fundamental Theorem of Calculus allows us to evaluate integrals without using Riemann sums. The drawback of this method is that we must be able to find an antiderivative, which can be challenging. This section examines **integration by substitution** - a technique to help us find antiderivatives. Specifically, this method allows us to find antiderivatives when the integrand is the result of a Chain Rule derivative.

At first, the approach to the substitution procedure may appear obscure. However, it is primarily a visual task - that is, the integrand shows you what to do; it is a matter of recognizing the form of the function. So, what are we supposed to see?

We are looking for an integrand of the form $f\,(g(x))\,g'(x)\,dx$. For example, in the integral

$$\int (x^2-3)^3\,2x\,dx. \tag{1.4.1}$$

we have

$$f(x)=x^3$$

and

$$g(x)=x^2-3.$$

Then

$$g'(x)=2x.$$

Therefore,

$$f(g(x))g'(x)=(x^2-3)^3(2x),$$

and we see that our integrand is in the correct form. The method is called substitution, or the **Substitution Method**, because we substitute part of the integrand with the variable u and part of the integrand with du. It is also referred to as **change of variables** because we are changing variables to obtain an easier expression to work with for applying the integration rules.

Theorem: Substitution Method with Indefinite Integrals

Let $u=g(x)$, where $g'(x)$ is continuous over an interval, let $f(x)$ be continuous over the corresponding range of g, and let $F(x)$ be an antiderivative of $f(x)$. Then,

$$\begin{aligned}\int f\,(g(x))\,g'(x)\,dx &= \int f(u)\,du\\ &= F(u)+C\\ &= F(g(x))+C\end{aligned}$$

Proof

Let f, g, u, and F be as specified in the theorem. Then

$$\frac{d}{dx}\left[F(g(x))\right] = F'(g(x))g'(x) = f[g(x)]g'(x).$$

Integrating both sides with respect to x, we see that

$$\int f\left(g(x)\right) g'(x)\, dx = F(g(x)) + C.$$

If we now substitute $u = g(x)$, and $du = g'(x)\, dx$, we get

$$\int f\left(g(x)\right) g'(x)\, dx = \int f(u)\, du = F(u) + C = F(g(x)) + C.$$

Q.E.D.

Returning to the problem we looked at originally, we let $u = x^2 - 3$ and then $du = 2x\, dx$.

Rewrite the integral (Equation 1.4.1) in terms of u:

$$\int (x^2 - 3)^3 (2x\, dx) = \int u^3\, du.$$

Using the Power Rule for integrals, we have

$$\int u^3\, du = \frac{u^4}{4} + C.$$

Substitute the original expression for x back into the solution:

$$\frac{u^4}{4} + C = \frac{(x^2 - 3)^4}{4} + C.$$

At this point, it is important to note that integration is mostly a heuristic method. That is, integration is approached more often by using a "calculated guess" derived from previous experiences. As you move through the middle part of Calculus (also known as Calculus II, or Integral Calculus), you will learn many different integration techniques. You must practice as many problems as possible to get a "natural feel" for what method to use and when to use it.

In the case of Integration by Substitution, you are often (but not always) looking for an integrand involving a function, $g(x)$, and its *approximate* derivative, $g'(x)$. You let $u(x) = g(x)$ so that $du = g'(x)dx$. At that point, you will have rewritten your original integral into a form that (hopefully) looks like

$$\int u\, du.$$

Example 1.4.1: Using Substitution to Find an Antiderivative

Use substitution to find the antiderivative of $\int 6x(3x^2 + 4)^4\, dx$.

Solution

The first step is to choose an expression for u. We choose $u = 3x^2 + 4$ because then $du = 6x\, dx$ and we already have du in the integrand. Write the integral in terms of u:

$$\int 6x(3x^2 + 4)^4\, dx = \int u^4\, du.$$

Remember that du is the derivative of the expression chosen for u, regardless of what is inside the integrand. Now we can evaluate the integral with respect to u:

$$\int u^4\, du = \frac{u^5}{5} + C = \frac{(3x^2 + 4)^5}{5} + C.$$

Analysis

We can check our answer by taking the derivative of the result of integration. We should obtain the integrand. Picking a value for C of 1, we let $y = \frac{1}{5}(3x^2+4)^5+1$. We have

$$y = \frac{1}{5}(3x^2+4)^5+1,$$

so

$$\begin{aligned} y' &= \left(\frac{1}{5}\right)5(3x^2+4)^4 6x \\ &= 6x(3x^2+4)^4. \end{aligned}$$

This is exactly the expression we started with inside the integrand.

? Checkpoint 1.4.1

Use substitution to find the antiderivative of $\int 3x^2(x^3-3)^2\,dx$.

Answer

$$\int 3x^2(x^3-3)^2\,dx = \frac{1}{3}(x^3-3)^3+C$$

Sometimes, we need to adjust the constants in our integral if they don't match up exactly with the expressions we are substituting.

✓ Example 1.4.2: Using Substitution with Alteration

Use substitution to find the antiderivative of

$$\int z\sqrt{z^2-5}\,dz.$$

Solution

Rewrite the integral as $\int z(z^2-5)^{1/2}\,dz$. Let $u = z^2-5$ and $du = 2z\,dz$. We have a problem because $du = 2z\,dz$ and the original expression has only $z\,dz$. We must alter our expression for du, or the integral in u will be twice as large as it should be. If we multiply both sides of the du equation by $\frac{1}{2}$. we can solve this problem. Thus,

$$u = z^2-5$$

$$du = 2z\,dz$$

$$\frac{1}{2}du = \frac{1}{2}(2z)\,dz = z\,dz.$$

Write the integral in terms of u, but pull the $\frac{1}{2}$ outside the integration symbol:

$$\int z(z^2-5)^{1/2}\,dz = \frac{1}{2}\int u^{1/2}\,du.$$

Integrate the expression in u:

$$\begin{aligned}\frac{1}{2}\int u^{1/2}\,du &= \left(\frac{1}{2}\right)\frac{u^{3/2}}{\frac{3}{2}}+C\\ &= \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)u^{3/2}+C\\ &= \frac{1}{3}u^{3/2}+C\\ &= \frac{1}{3}(z^2-5)^{3/2}+C\end{aligned}$$

? Checkpoint 1.4.2

Use substitution to find the antiderivative of $\int x^2(x^3+5)^9\,dx$.

Answer

$$\int x^2(x^3+5)^9\,dx = \frac{(x^3+5)^{10}}{30}+C$$

✔ Example 1.4.3: Using Substitution with Integrals of Trigonometric Functions

Use substitution to evaluate the integral $\int \frac{\sin t}{\cos^3 t}\,dt$.

Solution

We know the derivative of $\cos t$ is $-\sin t$, so we set $u=\cos t$. Then $du=-\sin t\,dt$.

Substituting into the integral, we have

$$\int \frac{\sin t}{\cos^3 t}\,dt = -\int \frac{du}{u^3}.$$

Evaluating the integral, we get

$$-\int \frac{du}{u^3} = -\int u^{-3}\,du = -\left(-\frac{1}{2}\right)u^{-2}+C.$$

Putting the answer back in terms of t, we get

$$\int \frac{\sin t}{\cos^3 t}\,dt = \frac{1}{2u^2}+C = \frac{1}{2\cos^2 t}+C.$$

Example 1.4.3 needs a little more explanation. How did we know to let $u=\cos(t)$ rather than the sine function? To be honest, this is something that becomes more of an instinct after a while; however, there is a science to it - it's the science of *considering ramifications*.

If we let $u=\sin(t)$, then $du=\cos(t)dt$. This might not jump off the page to you and scream, "This is bad," so let's see the ramifications of this choice.

The original integrand was $\frac{\sin(t)}{\cos^3(t)}$. Since this does not have a $\cos(t)$ to "steal" in the numerator, we need to solve $du=\cos(t)dt$ for dt. Doing so yields $dt=\frac{1}{\cos(t)}du$. If we make this substitution, the integrand would become

$$\frac{\sin(t)}{\cos^3(t)}\,dt = \frac{u}{\cos^3(t)}\cdot\frac{1}{\cos(t)}\,du = \frac{u}{\cos^4(t)}\,du.$$

There are two major issues here:

1. the integrand would have a mixture of variables, so we cannot leave it this way, and
2. the integrand would become *more* complex - not less.

The first issue can be overcome by using our Trigonometry knowledge.

$$\begin{aligned}\frac{u}{\cos^4(t)}\,du &= \frac{u}{(\cos^2(t))^2}\,du \\ &= \frac{u}{(1-\sin^2(t))^2}\,du \\ &= \frac{u}{(1-u^2)^2}\,du \qquad \text{(Substitution: Recall } u=\sin(t)\text{)}\end{aligned}$$

However, the second issue (complexity) still exists. Instead of

$$\int \frac{\sin(t)}{\cos^3(t)}\,dt,$$

we are now faced with

$$\int \frac{u}{(1-u^2)^2}\,du.$$

In either case, we do not immediately know of an appropriate antiderivative.

? Checkpoint 1.4.3

Use substitution to evaluate each integral

a. $\int \frac{\cos t}{\sin^2 t}\,dt$
b. $\int \cos^3 t \sin t\,dt$

Answers

a. $\int \frac{\cos t}{\sin^2 t}\,dt = -\frac{1}{\sin t} + C$
b. $\int \cos^3 t \sin t\,dt = -\frac{\cos^4 t}{4} + C$

Sometimes, we need to manipulate an integral in more complicated ways than just multiplying or dividing by a constant. We need to eliminate all the expressions within the integrand that are in terms of the original variable. When done, u should be the only variable in the integrand. This sometimes means solving for the original variable in terms of u. This technique should become clear in the next example.

✓ Example 1.4.4: Finding an Antiderivative Using u-Substitution

Use substitution to find the antiderivative of

$$\int \frac{x}{\sqrt{x-1}}\,dx.$$

Solution

If we let $u = x-1$, then $du = dx$. But this does not account for the x in the numerator of the integrand. We must express x in terms of u. If $u = x-1$, then $x = u+1$. Now we can rewrite the integral in terms of u:

$$\int \frac{x}{\sqrt{x-1}}\,dx = \int \frac{u+1}{\sqrt{u}}\,du = \int \left(\sqrt{u} + \frac{1}{\sqrt{u}}\right)du = \int \left(u^{1/2} + u^{-1/2}\right)du.$$

Then we integrate in the usual way, replace u with the original expression, and factor and simplify the result. Thus,

$$\begin{aligned}\int (u^{1/2}+u^{-1/2})\,du &= \frac{2}{3}u^{3/2}+2u^{1/2}+C\\ &= \frac{2}{3}(x-1)^{3/2}+2(x-1)^{1/2}+C\\ &= (x-1)^{1/2}\left[\frac{2}{3}(x-1)+2\right]+C\\ &= (x-1)^{1/2}\left(\frac{2}{3}x-\frac{2}{3}+\frac{6}{3}\right)\\ &= (x-1)^{1/2}\left(\frac{2}{3}x+\frac{4}{3}\right)\\ &= \frac{2}{3}(x-1)^{1/2}(x+2)+C.\end{aligned}$$

The final example in this subsection illustrates a common tripping point for students - the dreaded substitution with resubstitution.

Example 1.4.5: Using Substitution with a Resubstitution

Evaluate.

$$\int 3x^{17}\sqrt[4]{4x^9+8}\,dx$$

Solution

On first inspection, there doesn't seem to be an expression whose exact derivative (or constant multiple of the derivative) is contained within the integrand. However, practicing some faith, let's try the u-substitution, $u=4x^9+8$. This would imply that $du=36x^8\,dx$. Looking at the integrand, we don't have a $36x^8\,dx$ to "steal," but we *do* have $3x^8\,dx$. Thus, we let $\frac{1}{12}\,du = 3x^8\,dx$. Therefore,

$$\begin{aligned}\int 3x^{17}\sqrt[4]{4x^9+8}\,dx &= \int x^9\sqrt[4]{4x^9+8}\cdot 3x^8\,dx\\ &= \frac{1}{12}\int x^9\sqrt[4]{u}\,du\end{aligned}$$

At this point, I want to be clear that the notation in that final line is not traditionally acceptable. Our original integral was in terms of the variable x (and only that variable). The integral we arrived at in the last line *should* be in terms of only u; however, we have a pesky x^9 in there. Luckily, we know that $u=4x^9+8$. Hence, $\frac{u-8}{4}=x^9$. Substituting (also known as resubstituting) this into the last line of the computation above, we get the following.

$$\begin{aligned}\int 3x^{17}\sqrt[4]{4x^9+8}\,dx &= \frac{1}{12}\int x^9\sqrt[4]{u}\,du\\ &= \frac{1}{12}\int \frac{u-8}{4}u^{1/4}\,du\\ &= \frac{1}{48}\int (u-8)u^{1/4}\,du\\ &= \frac{1}{48}\int u^{5/4}-8u^{1/4}\,du\\ &= \frac{1}{48}\left(\frac{4}{9}u^{9/4}-\frac{32}{5}u^{5/4}\right)+C\\ &= \frac{1}{12}\left(\frac{1}{9}u^{9/4}-\frac{8}{5}u^{5/4}\right)+C\\ &= \frac{1}{12}\left(\frac{1}{9}(4x^9+8)^{9/4}-\frac{8}{5}(4x^9+8)^{5/4}\right)+C\end{aligned}$$

Before moving away from Example 1.4.5, it's worthwhile to mention that we should simplify expressions when possible. Unfortunately, factoring out common expressions from the answer in Example 1.4.5 does not improve the look and feel of the solution.

$$\begin{aligned}\frac{1}{12}\left(\frac{1}{9}(4x^9+8)^{9/4}-\frac{8}{5}(4x^9+8)^{5/4}\right)+C &= \frac{1}{12\cdot 9\cdot 5}(4x^9+8)^{5/4}\left(5(4x^9+8)-9\cdot 8\right)+C\\ &= \frac{1}{540}(4x^9+8)^{5/4}\left(20x^9+40-72\right)+C\\ &= \frac{1}{540}(4x^9+8)^{5/4}\left(20x^9-32\right)+C\\ &= \frac{4}{540}(4x^9+8)^{5/4}\left(5x^9-8\right)+C\\ &= \frac{1}{135}(4x^9+8)^{5/4}\left(5x^9-8\right)+C\end{aligned}$$

Substitution for Definite Integrals

Substitution can be used with definite integrals, too. However, using substitution to evaluate a definite integral requires changing the limits of integration. If we change variables in the integrand, the limits of integration change as well.

Theorem: Substitution with Definite Integrals

Let $u=g(x)$ and let g' be continuous over an interval $[a,b]$, and let f be continuous over the range of $u=g(x)$. Then,

$$\int_a^b f(g(x))g'(x)\,dx=\int_{g(a)}^{g(b)} f(u)\,du.$$

Although we will not formally prove this theorem, we justify it with some calculations. From the Substitution Method for indefinite integrals, if $F(x)$ is an antiderivative of $f(x)$, we have

$$\int f(g(x))g'(x)\,dx=F(g(x))+C.$$

Then

$$\begin{aligned}\int_a^b f[g(x)]g'(x)\,dx &= F(g(x))\Big|_{x=a}^{x=b}\\ &= F(g(b))-F(g(a))\\ &= F(u)\Big|_{u=g(a)}^{u=g(b)}\\ &= \int_{g(a)}^{g(b)} f(u)\,du\end{aligned}$$

and we have the desired result.

Example 1.4.6: Using Substitution to Evaluate a Definite Integral

Use substitution to evaluate

$$\int_0^1 x^2(1+2x^3)^5\,dx.$$

Solution

Let $u=1+2x^3$, so $du=6x^2\,dx$. Since the original function includes one factor of x^2 and $du=6x^2\,dx$, multiply both sides of the du equation by $1/6$. Then,

$$du = 6x^2\,dx$$
$$\Rightarrow \quad \frac{1}{6}du = x^2\,dx.$$

To adjust the limits of integration, note that when $x = 0, u = 1 + 2(0) = 1$, and when $x = 1,\ u = 1 + 2(1) = 3$.

Then

$$\int_0^1 x^2(1+2x^3)^5 dx = \frac{1}{6}\int_1^3 u^5\,du.$$

Evaluating this expression, we get

$$\begin{aligned}\frac{1}{6}\int_1^3 u^5\,du &= \left(\frac{1}{6}\right)\left(\frac{u^6}{6}\right)\Big|_{u=1}^{u=3} \\ &= \frac{1}{36}\left[(3)^6 - (1)^6\right] \\ &= \frac{182}{9}.\end{aligned}$$

Checkpoint 1.4.6

Use substitution to evaluate the definite integral.

a. $\int_{-1}^{0} y(2y^2-3)^5\,dy$

b. $\int_0^1 x^2\cos\left(\frac{\pi}{2}x^3\right)\,dx$

Answers

a. $\int_{-1}^{0} y(2y^2-3)^5\,dy = \frac{91}{3}$

b. $\int_0^1 x^2\cos\left(\frac{\pi}{2}x^3\right)\,dx = \frac{2}{3\pi} \approx 0.2122$

Example 1.4.7: Using Substitution with an Exponential Function

Use substitution to evaluate

$$\int_0^1 xe^{4x^2+3}\,dx.$$

Solution

Let $u = 4x^3 + 3$. Then, $du = 8x\,dx$. To adjust the limits of integration, we note that when $x = 0,\ u = 3$, and when $x = 1,\ u = 7$. So our substitution gives

$$\begin{aligned}\int_0^1 xe^{4x^2+3}\,dx &= \frac{1}{8}\int_3^7 e^u\,du \\ &= \frac{1}{8}e^u\Big|_{u=3}^{u=7} \\ &= \frac{e^7 - e^3}{8} \\ &\approx 134.568\end{aligned}$$

Caution: Change Limits of Integration!

When performing a substitution on a definite integral, it is critical that you *immediately* change the limits of integration as well.

Substitution may be only one of the techniques needed to evaluate a definite integral. All of the properties and rules of integration apply independently, and trigonometric functions may need to be rewritten using a trigonometric identity before we can apply substitution.

Example 1.4.8: Using Substitution to Evaluate a Trigonometric Integral

Use substitution to evaluate

$$\int_0^{\pi/2} \cos^2\theta\, d\theta.$$

Solution

First, use a trigonometric identity to rewrite the integral. The trigonometric identity $\cos^2\theta = \frac{1+\cos 2\theta}{2}$ allows us to rewrite the integral as

$$\int_0^{\pi/2} \cos^2\theta\, d\theta = \int_0^{\pi/2} \frac{1+\cos 2\theta}{2}\, d\theta.$$

Then,

$$\begin{aligned} \int_0^{\pi/2} \left(\frac{1+\cos 2\theta}{2}\right) d\theta &= \int_0^{\pi/2} \left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right) d\theta \\ &= \frac{1}{2}\int_0^{\pi/2} d\theta + \int_0^{\pi/2} \cos 2\theta\, d\theta. \end{aligned}$$

We can evaluate the first integral as it is. Still, we need to make a substitution to evaluate the second integral. Let $u = 2\theta$. Then, $du = 2\, d\theta$, or $\frac{1}{2}\, du = d\theta$. Also, when $\theta = 0$, $u = 0$, and when $\theta = \pi/2$, $u = \pi$. Expressing the second integral in terms of u, we have

$$\begin{aligned} \frac{1}{2}\int_0^{\pi/2} d\theta + \frac{1}{2}\int_0^{\pi/2} \cos 2\theta\, d\theta &= \frac{1}{2}\int_0^{\pi/2} d\theta + \frac{1}{2}\left(\frac{1}{2}\right)\int_0^{\pi} \cos u\, du \\ &= \frac{\theta}{2}\Big|_{\theta=0}^{\theta=\pi/2} + \frac{1}{4}\sin u\Big|_{u=0}^{u=\theta} \\ &= \left(\frac{\pi}{4}-0\right)+(0-0) \\ &= \frac{\pi}{4} \end{aligned}$$

1.4E: EXERCISES

This page is a draft and is under active development.

READING QUESTIONS

1. What is the main goal of the integration by substitution method?
2. If we choose $u = g(x)$ for a substitution, what is du equal to?
3. When is integration by substitution particularly useful? What form of integrand are we often looking for?
4. After performing a u-substitution for an indefinite integral and finding the antiderivative in terms of u, what is the final step?
5. When performing a u-substitution for a definite integral $\int_a^b f(g(x))g'(x)\,dx$, what must be done to the limits of integration a and b?
6. In Example 2, where $u = z^2 - 5$ and $du = 2z\,dz$, but the integral had $z\,dz$, what "alteration" was made?
7. Is it permissible to have both x and u (or dx and du) in the integrand after making a substitution?
8. If, after choosing $u = g(x)$, there are still terms involving x in the integrand that are not part of du, what algebraic step might be necessary?
9. What is meant by "substitution with resubstitution" as illustrated in Example 5?
10. Why is it critical to change the limits of integration immediately when performing a substitution on a definite integral?

HOMEWORK

In exercises 1 - 10, find the antiderivatives for the given functions.

1) $\cosh(2x+1)$

Answer

$\frac{1}{2}\sinh(2x+1)+C$

2) $\tanh(3x+2)$

3) $x\cosh(x^2)$

Answer

$\frac{1}{2}\sinh^2(x^2)+C$

4) $3x^3\tanh(x^4)$

5) $\cosh^2(x)\sinh(x)$

Answer

$\frac{1}{3}\cosh^3(x)+C$

6) $\tanh^2(x)\text{sech}^2(x)$

7) $\dfrac{\sinh(x)}{1+\cosh(x)}$

Answer

$\ln(1+\cosh(x))+C$

8) $\coth(x)$

9) $\cosh(x)+\sinh(x)$

Answer

$\cosh(x)+\sinh(x)+C$

10) $(\cosh(x)+\sinh(x))^n$

In exercises 11 - 17, find the antiderivatives for the functions.

11) $\int \dfrac{dx}{4-x^2}$

12) $\int \frac{dx}{a^2 - x^2}$

Answer

$\frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right) + C$

13) $\int \frac{dx}{\sqrt{x^2+1}}$

14) $\int \frac{x\,dx}{\sqrt{x^2+1}}$

Answer

$\sqrt{x^2+1} + C$

15) $\int -\frac{dx}{x\sqrt{1-x^2}}$

16) $\int \frac{e^x\,dx}{\sqrt{e^{2x}-1}}$

Answer

$\cosh^{-1}(e^x) + C$

17) $\int -\frac{2x}{x^4-1}\,dx$

18) Why is u-substitution referred to as a *change of variable*?

19) If $f = g \circ h$, when reversing the chain rule, $\frac{d}{dx}(g \circ h)(x) = g'(h(x))h'(x)$, should you take $u = g(x)$ or $u = h(x)$?

Answer

$u = h(x)$

In exercises 20 - 24, verify each identity using differentiation. Then, using the indicated u-substitution, identify f such that the integral takes the form $\int f(u)\,du$.

20) $\int x\sqrt{x+1}\,dx = \frac{2}{15}(x+1)^{3/2}(3x-2) + C; \quad u = x+1$

21) $\int \frac{x^2}{\sqrt{x-1}}\,dx = \frac{2}{15}\sqrt{x-1}(3x^2+4x+8) + C, \quad (x>1); \quad u = x-1$

Answer

$f(u) = \frac{(u+1)^2}{\sqrt{u}}$

22) $\int x\sqrt{4x^2+9}\,dx = \frac{1}{12}(4x^2+9)^{3/2} + C; \quad u = 4x^2+9$

23) $\int \frac{x}{\sqrt{4x^2+9}}\,dx = \frac{1}{4}\sqrt{4x^2+9} + C; \quad u = 4x^2+9$

Answer

$du = 8x\,dx; \quad f(u) = \frac{1}{8\sqrt{u}}$

24) $\int \frac{x}{(4x^2+9)^2}\,dx = -\frac{1}{8(4x^2+9)} + C; \quad u = 4x^2+9$

In exercises 25 - 34, find the antiderivative using the indicated substitution.

25) $\int (x+1)^4\,dx; \quad u = x+1$

Answer

$\int (x+1)^4\,dx = \frac{1}{5}(x+1)^5 + C$

26) $\int (x-1)^5\,dx; \quad u = x-1$

27) $\int (2x-3)^{-7}\,dx; \quad u=2x-3$

Answer

$$\int (2x-3)^{-7}\,dx = -\frac{1}{12(2x-3)^6}+C$$

28) $\int (3x-2)^{-11}\,dx; \quad u=3x-2$

29) $\int \frac{x}{\sqrt{x^2+1}}\,dx; \quad u=x^2+1$

Answer

$$\int \frac{x}{\sqrt{x^2+1}}\,dx = \sqrt{x^2+1}+C$$

30) $\int \frac{x}{\sqrt{1-x^2}}\,dx; \quad u=1-x^2$

31) $\int (x-1)(x^2-2x)^3\,dx; \quad u=x^2-2x$

Answer

$$\int (x-1)(x^2-2x)^3\,dx = \frac{1}{8}(x^2-2x)^4+C$$

32) $\int (x^2-2x)(x^3-3x^2)^2\,dx; \quad u=x^3=3x^2$

33) $\int \cos^3\theta\,d\theta; \quad u=\sin\theta$ (Hint: $\cos^2\theta=1-\sin^2\theta$)

Answer

$$\int \cos^3\theta\,d\theta = \sin\theta-\frac{\sin^3\theta}{3}+C$$

34) $\int \sin^3\theta\,d\theta; \quad u=\cos\theta$ (Hint: $\sin^2\theta=1-\cos^2\theta$)

In exercises 35 - 51, use a suitable change of variables to determine the indefinite integral.

35) $\int x(1-x)^{99}\,dx$

Answer

$$\begin{aligned}\int x(1-x)^{99}\,dx &= \frac{(1-x)^{101}}{101}-\frac{(1-x)^{100}}{100}+C\\ &= -\frac{(1-x)^{100}}{10100}\left[100x+1\right]+C\end{aligned}$$

36) $\int t(1-t^2)^{10}\,dt$

37) $\int (11x-7)^{-3}\,dx$

Answer

$$\int (11x-7)^{-3}\,dx = -\frac{1}{22(11x-7)^2}+C$$

38) $\int (7x-11)^4\,dx$

39) $\int \cos^3\theta\sin\theta\,d\theta$

Answer

$$\int \cos^3\theta\sin\theta\,d\theta = -\frac{\cos^4\theta}{4}+C$$

40) $\int \sin^7\theta\cos\theta\,d\theta$

41) $\int \cos^2(\pi t)\sin(\pi t)\,dt$

Answer

$$\int \cos^2(\pi t)\sin(\pi t)\,dt = -\frac{cos^3(\pi t)}{3\pi} + C$$

42) $\int \sin^2 x \cos^3 x\,dx$ (Hint: $\sin^2 x + \cos^2 x = 1$)

43) $\int t\sin(t^2)\cos(t^2)\,dt$

Answer

$$\int t\sin(t^2)\cos(t^2)\,dt = -\frac{1}{4}\cos^2(t^2) + C$$

44) $\int t^2\cos^2(t^3)\sin(t^3)\,dt$

45) $\int \frac{x^2}{(x^3-3)^2}\,dx$

Answer

$$\int \frac{x^2}{(x^3-3)^2}\,dx = -\frac{1}{3(x^3-3)} + C$$

46) $\int \frac{x^3}{\sqrt{1-x^2}}\,dx$

47) $\int \frac{y^5}{(1-y^3)^{3/2}}\,dy$

Answer

$$\int \frac{y^5}{(1-y^3)^{3/2}}\,dy = -\frac{2(y^3-2)}{3\sqrt{1-y^3}} + C$$

48) $\int \cos\theta(1-\cos\theta)^{99}\sin\theta\,d\theta$

49) $\int (1-\cos^3\theta)^{10}\cos^2\theta\sin\theta\,d\theta$

Answer

$$\int (1-\cos^3\theta)^{10}\cos^2\theta\sin\theta\,d\theta = \frac{1}{33}(1-\cos^3\theta)^{11} + C$$

50) $\int (\cos\theta - 1)(\cos^2\theta - 2\cos\theta)^3\sin\theta\,d\theta$

51) $\int (\sin^2\theta - 2\sin\theta)(\sin^3\theta - 3\sin^2\theta)^3\cos\theta\,d\theta$

Answer

$$\int (\sin^2\theta - 2\sin\theta)(\sin^3\theta - 3\sin^2\theta)^3\cos\theta\,d\theta = \frac{1}{12}(\sin^3\theta - 3\sin^2\theta)^4 + C$$

In exercises 52 - 55, use a calculator to estimate the area under the curve using left Riemann sums with 50 terms, then use substitution to solve for the exact answer.

52) [Technology Required] $y = 3(1-x)^2$ over $[0, 2]$

53) [Technology Required] $y = x(1-x^2)^3$ over $[-1, 2]$

Answer

$L_{50} = -8.5779$. The exact area is $\frac{-81}{8}$ units^2.

54) [Technology Required] $y = \sin x(1-\cos x)^2$ over $[0, \pi]$

55) [Technology Required] $y = \frac{x}{(x^2+1)^2}$ over $[-1, 1]$

Answer

$L_{50} = -0.006399$. The exact area is 0.

In exercises 56 - 61, use a change of variables to evaluate the definite integral.

56) $\int_0^1 x\sqrt{1-x^2}\,dx$

57) $\int_0^1 \dfrac{x}{\sqrt{1+x^2}}\,dx$

Answer

$$u=1+x^2,\quad du=2x\,dx,\quad \int_0^1 \frac{x}{\sqrt{1+x^2}}\,dx=\frac{1}{2}\int_1^2 u^{-1/2}du=\sqrt{2}-1$$

58) $\int_0^2 \dfrac{t}{\sqrt{5+t^2}}\,dt$

59) $\int_0^1 \dfrac{t^2}{\sqrt{1+t^3}}\,dt$

Answer

$$u=1+t^3,\quad du=3t^2,\quad \int_0^1 \frac{t^2}{\sqrt{1+t^3}}\,dt=\frac{1}{3}\int_1^2 u^{-1/2}du=\frac{2}{3}(\sqrt{2}-1)$$

60) $\int_0^{\pi/4} \sec^2\theta\tan\theta\,d\theta$

61) $\int_0^{\pi/4} \dfrac{\sin\theta}{\cos^4\theta}\,d\theta$

Answer

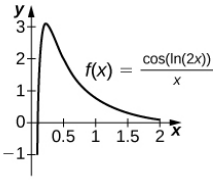
$$u=\cos\theta,\quad du=-\sin\theta\,d\theta,\quad \int_0^{\pi/4}\frac{\sin\theta}{\cos^4\theta}\,d\theta=-\int_1^{\sqrt{2}/2}u^{-4}\,du=\int_{\sqrt{2}/2}^1 u^{-4}\,du=\frac{1}{3}(2\sqrt{2}-1)$$

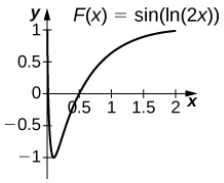
In exercises 62 - 67, evaluate the indefinite integral $\int f(x)\,dx$ with constant $C=0$ using u-substitution. Then, graph the function and the antiderivative over the indicated interval. If possible, estimate a value of C that would need to be added to the antiderivative to make it equal to the definite integral $F(x)=\int_a^x f(t)\,dt$, with a the left endpoint of the given interval.

62) [Technology Required] $\int (2x+1)e^{x^2+x-6}\,dx$ over $[-3,2]$

63) [Technology Required] $\int \dfrac{\cos(\ln(2x))}{x}\,dx$ on $[0,2]$

Answer



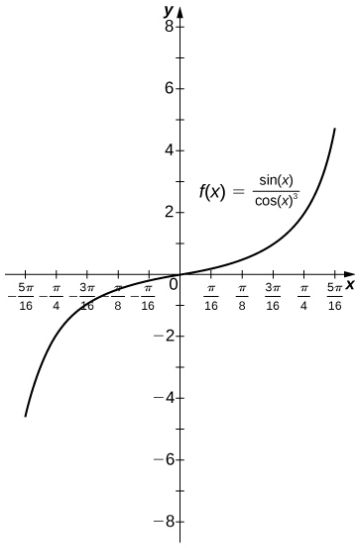


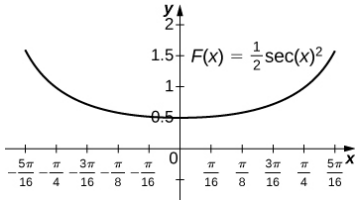
The antiderivative is $y=\sin(\ln(2x))$. Since the antiderivative is not continuous at $x=0$, one cannot find a value of C that would make $y=\sin(\ln(2x))-C$ work as a definite integral.

64) [Technology Required] $\int \dfrac{3x^2+2x+1}{\sqrt{x^3+x^2+x+4}}\,dx$ over $[-1,2]$

65) [Technology Required] $\int \dfrac{\sin x}{\cos^3 x}\,dx$ over $[-\frac{\pi}{3},\frac{\pi}{3}]$

Answer



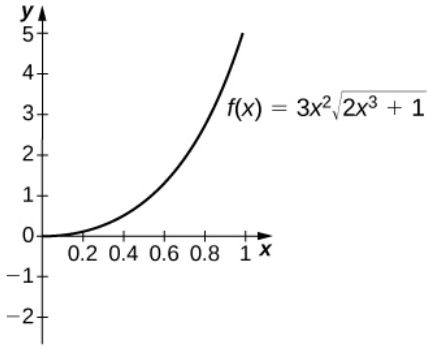


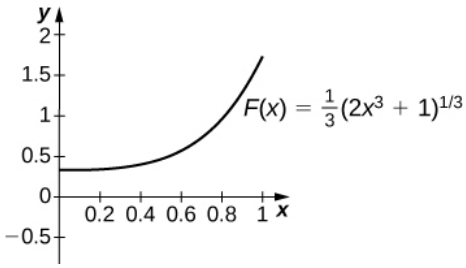
The antiderivative is $y = \frac{1}{2}\sec^2 x$. You should take $C = -2$ so that $F(-\frac{\pi}{3}) = 0$.

66) [Technology Required] $\int (x+2)e^{-x^2-4x+3}\,dx$ over $[-5, 1]$

67) [Technology Required] $\int 3x^2\sqrt{2x^3+1}\,dx$ over $[0, 1]$

Answer





The antiderivative is $y = \frac{1}{3}(2x^3+1)^{3/2}$. One should take $C = -\frac{1}{3}$.

68) If $h(a) = h(b)$ in $\int_a^b g'(h(x))h(x)\,dx$, what can you say about the value of the integral?

69) Is the substitution $u = 1 - x^2$ in the definite integral $\int_0^2 \frac{x}{1-x^2}\,dx$ okay? If not, why not?

Answer

No, because the integrand is discontinuous at $x = 1$.

In exercises 70 - 76, use a change of variables to show that each definite integral is equal to zero.

70) $\int_0^{\pi} \cos^2(2\theta)\sin(2\theta)\,d\theta$

71) $\int_0^{\sqrt{\pi}} t\cos(t^2)\sin(t^2)\,dt$

Answer

$u = \sin(t^2)$; the integral becomes $\frac{1}{2}\int_0^0 u\,du$.

72) $\int_0^1 (1-2t)\,dt$

73) $\int_0^1 \frac{1-2t}{1+(t-\frac{1}{2})^2}\,dt$

Answer

$u = 1 + (t - \frac{1}{2})^2$; the integral becomes $-\int_{5/4}^{5/4} \frac{1}{u}\,du$.

74) $\int_0^{\pi} \sin\left((t-\frac{\pi}{2})^3\right)\cos\left(t-\frac{\pi}{2}\right)\,dt$

75) $\int_0^2 (1-t)\cos(\pi t)\,dt$

Answer

$u = 1 - t$; Since the integrand is odd, the integral becomes

$$\int_1^{-1} u\cos\left(\pi(1-u)\right)du = \int_1^{-1} u[\cos\pi\cos u - \sin\pi\sin u]\,du = -\int_1^{-1} u\cos u\,du = \int_{-1}^{1} u\cos u\,du = 0$$

76) $\int_{\pi/4}^{3\pi/4} \sin^2 t\cos t\,dt$

77) Show that the average value of $f(x)$ over an interval $[a, b]$ is the same as the average value of $f(cx)$ over the interval $[\frac{a}{c}, \frac{b}{c}]$ for $c > 0$.

Answer

Setting $u = cx$ and $du = c\,dx$ gets you $\frac{1}{\frac{b}{c}-\frac{a}{c}}\int_{a/c}^{b/c} f(cx)\,dx = \frac{c}{b-a}\int_{u=a}^{u=b} f(u)\frac{du}{c} = \frac{1}{b-a}\int_a^b f(u)\,du$.

78) Find the area under the graph of $f(t) = \frac{t}{(1+t^2)^a}$ between $t = 0$ and $t = x$ where $a > 0$ and $a \neq 1$ is fixed, and evaluate the limit as $x \to \infty$.

79) Find the area under the graph of $g(t) = \frac{t}{(1-t^2)^a}$ between $t = 0$ and $t = x$, where $0 < x < 1$ and $a > 0$ is fixed. Evaluate the limit as $x \to 1$.

Answer

$\int_0^x g(t)\,dt = \frac{1}{2}\int_{u=1-x^2}^{1} \frac{du}{u^a} = \frac{1}{2(1-a)}u^{1-a}\,|\,1u = \frac{1}{2(1-a)}(1-(1-x^2)^{1-a})$ As $x \to 1$ the limit is $\frac{1}{2(1-a)}$ if $a < 1$, and the limit diverges to $+\infty$ if $a > 1$.

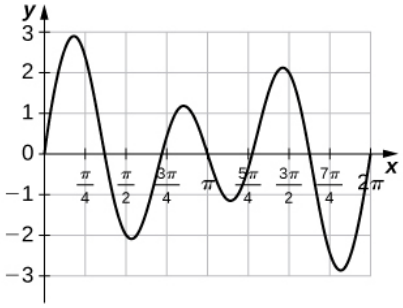
80) The area of a semicircle of radius 1 can be expressed as $\int_{-1}^{1} \sqrt{1-x^2}\,dx$. Use the substitution $x = \cos t$ to express the area of a semicircle as the integral of a trigonometric function. You do not need to compute the integral.

81) The area of the top half of an ellipse with a major axis that is the x-axis from $x = -1$ to a and with a minor axis that is the y-axis from $y = -b$ to $y = b$ can be written as $\int_{-a}^{a} b\sqrt{1 - \frac{x^2}{a^2}}\, dx$. Use the substitution $x = a\cos t$ to express this area in terms of an integral of a trigonometric function. You do not need to compute the integral.

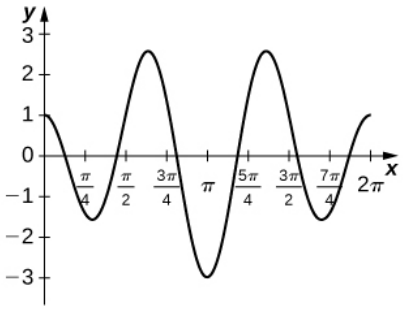
Answer

$$\int_{t=\pi}^{t=0} b\sqrt{1-\cos^2 t} \times (-a\sin t)\, dt = \int_{t=0}^{t=\pi} ab\sin^2 t\, dt$$

82) [Technology Required] The following graph is of a function of the form $f(t) = a\sin(nt) + b\sin(mt)$. Estimate the coefficients a and b and the frequency parameters n and m. Use these estimates to approximate $\int_0^{\pi} f(t)\, dt$.



83) [Technology Required] The following graph is of a function of the form $f(x) = a\cos(nt) + b\cos(mt)$. Estimate the coefficients a and b and the frequency parameters n and m. Use these estimates to approximate $\int_0^{\pi} f(t)\, dt$.



Answer

$$f(t) = 2\cos(3t) - \cos(2t); \quad \int_0^{\pi/2} (2\cos(3t) - \cos(2t))\, dt = -\frac{2}{3}$$

CHAPTER OVERVIEW

2: Applications of Integration

In this chapter, we use definite integrals to calculate the force exerted on the dam when the reservoir is full and we examine how changing water levels affect that force. Hydrostatic force is only one of the many applications of definite integrals we explore in this chapter. From geometric applications such as surface area and volume, to physical applications such as mass and work, to growth and decay models, definite integrals are a powerful tool to help us understand and model the world around us.

2.1: Areas Between Curves

2.1E: Exercises

2.2: Volume by Cross-Sectional Area

2.2E: Exercises

2.3: Volumes of Revolution - Cylindrical Shells

2.3E: Exercises

2.4: Arc Length of a Curve and Surface Area

2.4E: Exercises

2.5: Work

2.5E: Exercises

2.6: Hydrostatic Force and Pressure

2.6E: Exercises

2.7: Moments and Centers of Mass

2.7E: Exercises

2.8: The Mean Value Theorem for Integrals

2.8E: Exercises

2.9: Chapter 2 Review Exercises

This page titled 2: Applications of Integration is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Roy Simpson.

2.1: Areas Between Curves

https://math.libretexts.org/@go/page/168405

Note to the Instructor (click to expand)

This section only covers areas between curves (with a clarification of the difference between a direct integration problem and being requested to find areas). Focus should be on requiring graphing skills (algebraic, trigonometric, exponential, etc.).

To succeed in this section, you'll need to use some skills from previous courses. While you should already know them, this is the first time they've been required. You can review these skills in CRC's Corequisite Codex. If you have a support class, it might cover some, **but not all**, of these topics.

Prerequisite Skills and Support Topics (click to expand)

- **Geometry**
  - The Pythagorean Theorem, Distance Formula, and Midpoint Formula: The concept of distance, particularly the distance formula, is reviewed in the section as foundational for understanding $y_T - y_B$ and $x_R - x_L$.
- **Simplifying Expressions**
  - Multiplying and Distributing Polynomial Expressions: Necessary for simplifying the integrand $f(x) - g(x)$ after substitution.
- **Solving Equations**
  - Solving Linear Equations: Essential for finding intersection points when the bounding curves are lines.
  - Solving Quadratic Equations: Required for finding intersection points when one or both bounding curves are quadratic functions (as seen in Example 2).
  - Solving Polynomial Equations: Generally needed to find points where $f(x) = g(x)$ if the functions are polynomials, which define the limits of integration.
  - Solving Radical Equations: May be needed if functions involve roots and intersections are sought, or when converting $y = f(x)$ to $x = v(y)$.
- **Graphs**
  - The Cartesian Coordinate System and the Quadrants: Fundamental for setting up and visualizing the area problem in a 2D plane.
  - Ordered Pairs and Graphing Relations by Point-Plotting: Essential for sketching the functions to correctly identify the region, the upper/lower or right/left functions, and the limits of integration.
- **An Overview of Functions**
  - The Definition of a Function: Crucial for understanding the relationships $y = f(x)$ and $x = u(y)$ that define the boundaries of the area.
  - Evaluating Functions: Necessary for finding specific points on curves, comparing function values, and for numerical approximation methods.

Hawk A.I. Section-Specific Tutor



To access the Hawk A.I. Tutor, you will need to be logged into your campus Gmail account.

- The Domain and Range of a Function: Important for understanding the intervals over which functions are defined and where intersections might occur.

- **Graphing Functions**
  - Graphs of Functions: The ability to sketch or visualize graphs is paramount for setting up the correct integral for the area between curves.
  - Base Graphs: Recognizing the shapes of common functions (lines, parabolas, square roots, trigonometric functions) helps in quickly sketching the region.
- **Trigonometry**
  - Evaluating Trigonometric Functions at Special Angles: Required for calculations when limits of integration or intersection points involve these specific angles (e.g., $\frac{\pi}{4}$).
  - Base Graphs of the Trigonometric Functions: Important for visualizing regions bounded by trigonometric curves like sine and cosine.
  - Solving Simple Trigonometric Equations: Necessary for finding the intersection points of trigonometric functions, which determine the limits of integration.
  - Trigonometric Identities - The Pythagorean Identities: Often used in simplifying expressions or solving trigonometric equations.
  - Trigonometric Identities - Sum and Difference of Angles Identities: Useful in various trigonometric manipulations.
  - Trigonometric Identities - Double-Angle Identities: Can be needed for solving equations or simplifying integrands.
  - Trigonometric Identities - Half-Angle and Power Reduction Identities: Particularly power reduction formulas can be useful in integration, though not directly for area setup here, they are part of the expected mastery.

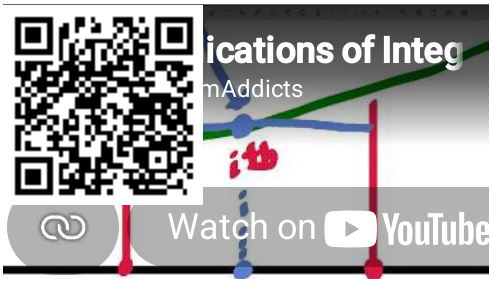
The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Determine the area of a region between two curves by integrating with respect to the independent variable.
- Find the area of a compound region.
- Determine the area of a region between two curves by integrating with respect to the dependent variable.

A Review of Necessary Prerequisite Theory

The following video is an excellent place to start before your first class meeting. It contains a review of the necessary theory from Differential Calculus (a.k.a. Calculus I) to begin your pathway to success in this course.



In Differential Calculus, we developed the definite integral concept to calculate the area between a curve and an axis on a given interval. In this section, we expand that idea to calculate the area of more complex regions. We start by finding the area between two curves that are functions of x, beginning with the simple case in which one function is always greater than the other. We then look at instances where the graphs of the functions cross. Lastly, we consider how to calculate the area between two curves that are functions of y.

An Aside: Measuring Distances

Before we dive into Integral Calculus (a.k.a., Calculus II), let's take a moment to discuss something simple and familiar - distance.[1] Specifically, we need to recall the Distance Formula. The distance between any two points, $P(x_1, y_1)$ and $Q(x_2, y_2)$, in the Cartesian plane is

$$d(P,Q) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}. \tag{2.1.1}$$

If the two points share the same x-coordinate (i.e., they lie on the same vertical line), then Equation 2.1.1 becomes

$$d(P,Q) = \sqrt{(y_1 - y_2)^2} = |y_1 - y_2| = \begin{cases} y_1 - y_2 & \text{if} \quad y_1 \geq y_2 \\ y_2 - y_1 & \text{if} \quad y_1 < y_2 \end{cases}.$$

This can be read as follows:

> *The distance between any two points on a vertical line is* $y_T - y_B$*, where* $y_T = y_{\text{Top}} = \max\{y_1, y_2\}$ *and* $y_B = y_{\text{Bottom}} = \min\{y_1, y_2\}$.

This interpretation will be *critical* as we move forward.

If we, instead, assume that the points P and Q lie on the same horizontal line (thereby having the same y-coordinates), then Equation 2.1.1 becomes

$$d(P,Q) = \sqrt{(x_1 - x_2)^2} = |x_1 - x_2| = \begin{cases} x_1 - x_2 & \text{if} \quad x_1 \geq x_2 \\ x_2 - x_1 & \text{if} \quad x_1 < x_2 \end{cases}.$$

This can be read as follows:

> *The distance between any two points on a horizontal line is* $x_R - x_L$*, where* $x_R = x_{\text{Right}} = \max\{x_1, x_2\}$ *and* $x_L = x_{\text{Left}} = \min\{x_1, x_2\}$.

The two statements, that the distance between two points on a vertical line is $y_{\text{Top}} - y_{\text{Bottom}}$ and the distance between two points on the horizontal line is $x_{\text{Right}} - x_{\text{Left}}$, are going to be used repeatedly in this course. For example, we will often need to determine the distance between a function, say $f(x)$, and a horizontal line, say $y = L$. If we know that $L \geq f(x)$ on the interval $[a, b]$, then the vertical distance between $y = L$ and the function $f(x)$ is always going to be $y_{\text{Top}} - y_{\text{Bottom}} = L - f(x)$, for all $x \in [a, b]$. Likewise, if we know the function $g(y)$ is always to the left of $h(y)$ for $y \in [c, d]$, then the *horizontal* distance between these two functions for any value of y in $[c, d]$ is $x_{\text{Right}} - x_{\text{Left}} = h(y) - g(y)$.

While the previous paragraph is (hopefully) simple to understand, the results are immensely helpful for us. It's now time to return to Calculus!

Area of a Region between Two Curves

Let $f(x)$ and $g(x)$ be continuous functions over an interval $[a, b]$ such that $f(x) \geq g(x)$ on $[a, b]$. We want to find the area between the graphs of the functions, as shown in Figure 2.1.1.

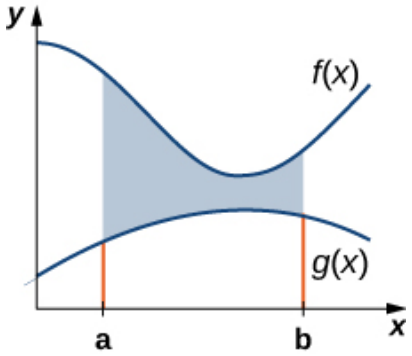
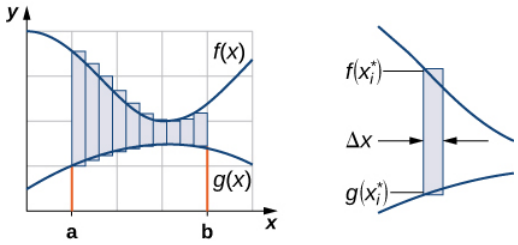


Figure 2.1.1: The area between the graphs of two functions, $f(x)$ and $g(x)$, on the interval $[a, b]$

As we did before, we will partition the interval on the x-axis and approximate the area between the graphs of the functions with rectangles. So, for $i = 0, 1, 2, \ldots, n$, let $P = \{x_i\}$ be a regular partition of $[a, b]$. Then, for $i = 1, 2, \ldots, n$, choose a point $x_i^* \in [x_{i-1}, x_i]$, and on each interval $[x_{i-1}, x_i]$ construct a rectangle that extends vertically from $g(x_i^*)$ to $f(x_i^*)$. Figure 2.1.2(a) shows the rectangles when x_i^* is selected to be the left endpoint of the interval and $n = 10$. Figure 2.1.2(b) shows a representative rectangle in detail.



(a) (b)

Figure 2.1.2: (a) We can approximate the area between the graphs of two functions, $f(x)$ and $g(x)$, with rectangles. (b) The area of a typical rectangle goes from one curve to the other.

The height of each individual rectangle is $y_{T_i} - y_{B_i} = f(x_i^*) - g(x_i^*)$ and the width of each rectangle is Δx. Adding the areas of all the rectangles, we see that the area between the curves is approximated by

$$A \approx \sum_{i=1}^{n} [y_{T_i} - y_{B_i}] \Delta x = \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)] \Delta x.$$

This is a Riemann sum, so we take the limit as $n \to \infty$ and we get

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)] \Delta x = \int_a^b [f(x) - g(x)] dx.$$

These findings are summarized in the following theorem.

Theorem: Area between Two Curves

Let $f(x)$ and $g(x)$ be continuous functions such that $f(x) \geq g(x)$ over an interval $[a, b]$. Let $\mathbf{R}$ denote the region bounded above by the graph of $f(x)$, below by the graph of $g(x)$, and on the left and right by the lines $x = a$ and $x = b$, respectively. Then, the area of $\mathbf{R}$ is given by

$$A = \int_a^b [f(x) - g(x)] dx.$$

We apply this theorem in the following example.

Example 2.1.1

If $\mathbf{R}$ is the region bounded above by the graph of the function $f(x) = x + 4$ and below by the graph of the function $g(x) = 3 - \frac{x}{2}$ over the interval $[1, 4]$, find the area of region $\mathbf{R}$.

Solution

Graphing the requested region, depicted in the following figure, will be a requirement.

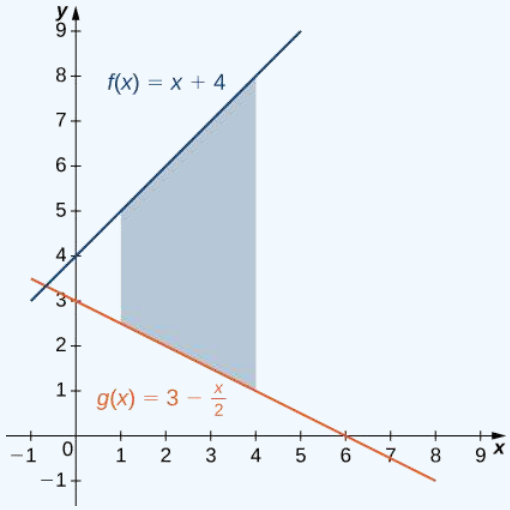


Figure 2.1.3: A region between two curves is shown where one curve is always greater than the other.

We can see that the values of y_T are always "played by" $f(x)$ and the values of y_B are always "played by" $g(x)$. Therefore, we have

$$\begin{aligned} A &= \int_a^b [y_T - y_B]\,dx \\ &= \int_a^b [f(x) - g(x)]\,dx \\ &= \int_1^4 \left[(x+4) - \left(3 - \frac{x}{2}\right)\right] dx \\ &= \int_1^4 \left[\frac{3x}{2} + 1\right] dx \\ &= \left[\frac{3x^2}{4} + x\right]\Big|_1^4 \\ &= \left(16 - \frac{7}{4}\right) \\ &= \frac{57}{4}. \end{aligned}$$

The area of the region is $\frac{57}{4}$ units^2.

? Checkpoint 2.1.1

If $\mathbf{R}$ is the region bounded by the graphs of the functions $f(x) = \frac{x}{2} + 5$ and $g(x) = x + \frac{1}{2}$ over the interval $[1, 5]$, find the area of region $\mathbf{R}$.

Answer

12 units^2

In Example 2.1.1, we defined the interval of interest as part of the problem statement. Quite often, though, we want to define our interval of interest based on where the graphs of the two functions intersect. This is illustrated in the following example.

✔ Example 2.1.2

If $\mathbf{R}$ is the region bounded above by the graph of the function $f(x) = 9 - (\frac{x}{2})^2$ and below by the graph of the function $g(x) = 6 - x$, find the area of region $\mathbf{R}$.

Solution

The region is depicted in the following figure, and we can see that $y_T = f(x)$ and $y_B = g(x)$ over the entire region.

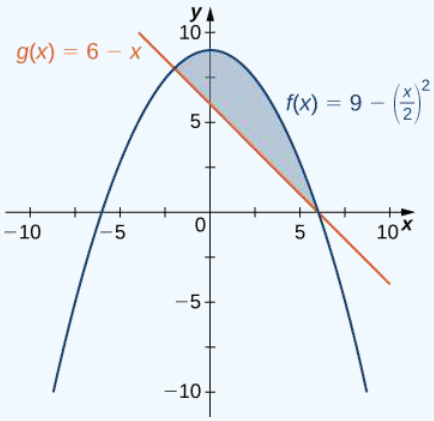


Figure 2.1.4: This graph shows the region below the graph of $f(x)$ and above the graph of $g(x)$.

We first need to compute where the graphs of the functions intersect. Setting $f(x) = g(x)$, we get

$$\begin{aligned} f(x) &= g(x) \\ \Rightarrow \quad 9 - \left(\frac{x}{2}\right)^2 &= 6 - x \\ \Rightarrow \quad 9 - \frac{x^2}{4} &= 6 - x \\ \Rightarrow \quad 36 - x^2 &= 24 - 4x \\ \Rightarrow \quad x^2 - 4x - 12 &= 0 \\ \Rightarrow \quad (x-6)(x+2) &= 0. \end{aligned}$$

The graphs of the functions intersect when $x = 6$ or $x = -2$, so we want to integrate from -2 to 6. Therefore,

$$\begin{aligned} A &= \int_a^b [y_T - y_B]\,dx \\ &= \int_a^b [f(x) - g(x)]\,dx \\ &= \int_{-2}^6 \left[9 - \left(\frac{x}{2}\right)^2 - (6 - x)\right]\,dx \\ &= \int_{-2}^6 \left[3 - \frac{x^2}{4} + x\right]\,dx \\ &= \left[3x - \frac{x^3}{12} + \frac{x^2}{2}\right]\Big|_{-2}^6 \\ &= \frac{64}{3}. \end{aligned}$$

The area of the region is $\frac{64}{3}$ units^2.

? Checkpoint 2.1.2

If $\mathbf{R}$ is the region bounded above by the graph of the function $f(x) = x$ and below by the graph of the function $g(x) = x^4$, find the area of region $\mathbf{R}$.

Answer

$\frac{3}{10}$ units^2

Areas of Compound Regions

What if we want to examine regions bounded by graphs of functions that cross one another? In that case, we modify the process we just developed by using the absolute value function.

Theorem: Finding the Area of a Region between Curves That Cross

Let $f(x)$ and $g(x)$ be continuous functions over an interval $[a, b]$. Let $\mathbf{R}$ denote the region between the graphs of $f(x)$ and $g(x)$, and be bounded on the left and right by the lines $x = a$ and $x = b$, respectively. Then, the area of $\mathbf{R}$ is given by

$$A = \int_a^b |f(x) - g(x)|dx.$$

In practice, applying this theorem requires us to break up the interval $[a, b]$ and evaluate several integrals, depending on which function values are greater over a given part of the interval. We study this process in the following example.

✓ Example 2.1.3

If $\mathbf{R}$ is the region between the graphs of the functions $f(x) = \sin(x)$ and $g(x) = \cos(x)$ over the interval $[0, \pi]$, find the area of region $\mathbf{R}$.

Solution

The region is depicted in the following figure.

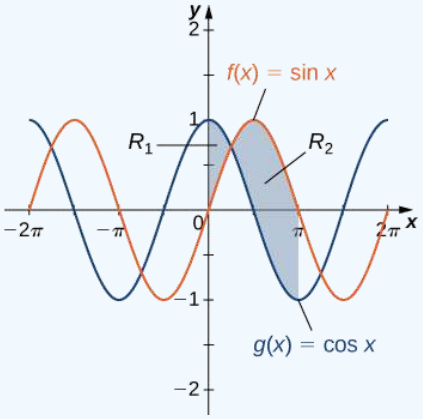


Figure **2.1.5**: The region between two curves can be broken into two sub-regions.

The graphs of the functions intersect at $x = \frac{\pi}{4}$. Since the roles of y_T and y_B switch between regions $\mathbf{R}_1$ and $\mathbf{R}_2$, our notation for y_{top} and y_{bottom} needs to gain a little more complexity.

For $x \in [0, \frac{\pi}{4}]$,

$$|f(x) - g(x)| = y_{T_1} - y_{B_1} = \cos(x) - \sin(x).$$

On the other hand, for $x \in [\frac{\pi}{4}, \pi]$,

$$|f(x) - g(x)| = y_{T_2} - y_{B_2} = \sin(x) - \cos(x).$$

Then

$$\begin{aligned} A &= \int_a^b |f(x) - g(x)|dx \\ &= \int_0^{\pi} |\sin(x) - \cos(x)|dx \\ &= \int_0^{\pi/4} (y_{T_1} - y_{B_1})dx + \int_{\pi/4}^{\pi} (y_{T_2} - y_{B_2})dx \\ &= \int_0^{\pi/4} (\cos(x) - \sin(x))dx + \int_{\pi/4}^{\pi} (\sin(x) - \cos(x))dx \\ &= [\sin(x) + \cos(x)]\Big|_0^{\pi/4} + [-\cos(x) - \sin(x)]\Big|_{\pi/4}^{\pi} \\ &= (\sqrt{2} - 1) + (1 + \sqrt{2}) \\ &= 2\sqrt{2}. \end{aligned}$$

The area of the region is $2\sqrt{2}$ **units**^{2}.

Caution: Mastery of Trigonometry is required in this course

Your success in Calculus II will rely heavily on your mastery of concepts from Trigonometry. You will frequently need to recall, without any prompting, the following identities and theorems:

- Pythagorean Identities,
- Sum and Difference Formulas (for sine and cosine),

- Double-Angle Identities (for sine and cosine),
- Half-Angle Formulas (for sine and cosine) - these are also known as the Power Reduction Formulas, and
- the Even/Odd Identities.

Moreover, it is assumed you can perform each of the following skills without using technology for help:

- evaluate all trigonometric functions at special angles, or at angles having special angles as reference angles, as well as at axial angles,
- accurately graph transformations of trigonometric functions,
- solve trigonometric equations,
- evaluate inverse trigonometric functions,
- understand the domain and range restrictions of all base trigonometric functions, and
- understand the domain and range restrictions of all base inverse trigonometric functions.

If you need a quick review of your Trigonometry, please see Section 1.6 of the Differential Calculus textbook. If you need a deeper review, an entire course in Trigonometry (i.e., Math 373 at CRC) is a great choice.

? Checkpoint 2.1.3

If $\mathbf{R}$ is the region between the graphs of the functions $f(x) = \sin(x)$ and $g(x) = \cos(x)$ over the interval $[\frac{\pi}{2}, 2\pi]$, find the area of region $\mathbf{R}$.

Answer

$2 + 2\sqrt{2}$ units^2

✓ Example 2.1.4

Consider the region depicted in Figure 2.1.6. Find the area of $\mathbf{R}$.

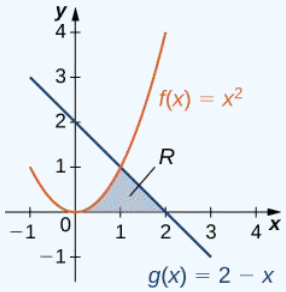


Figure 2.1.6: Two integrals are required to calculate the area of this region.

Solution

As with Example 2.1.3, we need to divide the interval into two pieces. The graphs of the functions intersect at $x = 1$ (set $f(x) = g(x)$ and solve for x), so we evaluate two separate integrals: one over the interval $[0, 1]$ and one over the interval $[1, 2]$.

Over the interval $[0, 1]$, the region is bounded above by $y_{T_1} = f(x) = x^2$ and below by $y_{B_1} = 0$, which is the x-axis, so we have

$$A_1 = \int_0^1 (y_{T_1} - y_{B_1})\,dx = \int_0^1 (x^2 - 0)dx = \int_0^1 x^2 dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}.$$

Over the interval $[1, 2]$, the region is bounded above by $y_{T_2} = g(x) = 2 - x$ and below by $y_{B_2} = 0$, which again is the x-axis, so we have

$$A_2 = \int_1^2 (y_{T_2} - y_{B_2})\,dx = \int_1^2 (2 - x - 0)dx = \int_1^2 (2 - x)dx = \left[2x - \frac{x^2}{2}\right]\Big|_1^2 = \frac{1}{2}.$$

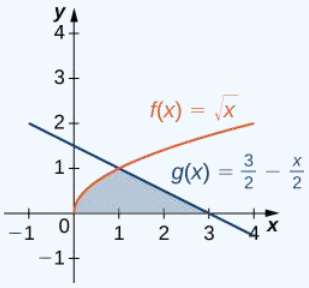
Adding these areas together, we obtain

$$A = A_1 + A_2 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}.$$

The area of the region is $\frac{5}{6}$ units^2.

Checkpoint 2.1.4

Consider the region depicted in the following figure. Find the area of $\mathbf{R}$.



Answer

$\frac{5}{3}$ units^2

Regions Defined with Respect to y

In Example 2.1.4, we had to evaluate two separate integrals to calculate the area of the region. However, there is another approach that requires only one integral. What if we treat the curves as functions of y instead of as functions of x? Review Figure 2.1.6. Note that the left graph, shown in red, is represented by the function $y = f(x) = x^2$. We could just as easily solve this for x and represent the curve by the function of y, $x = v(y) = \sqrt{y}$.[2] Similarly, the right graph is represented by the function $y = g(x) = 2 - x$, but could just as easily be represented by the function $x = u(y) = 2 - y$. When the graphs are represented as functions of y, we see the region is bounded on the left by the graph of one function and on the right by the graph of the other function. Therefore, if we integrate with respect to y, we only need to evaluate one integral. Let's develop a formula for this type of integration.

Let $u(y)$ and $v(y)$ be continuous functions over an interval $[c, d]$ such that $u(y) \geq v(y)$ for all $y \in [c, d]$. We want to find the area between the graphs of the functions, as shown in Figure 2.1.7.

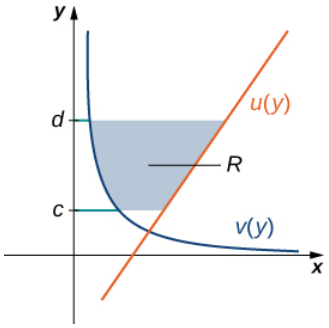
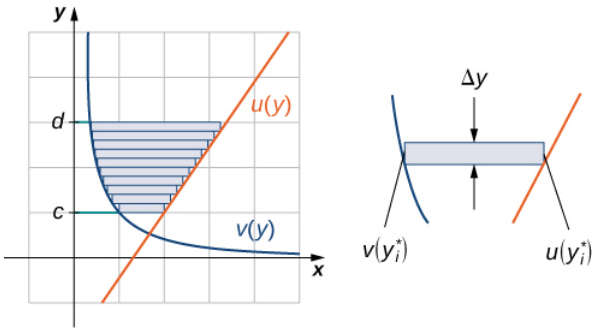


Figure 2.1.7: We can find the area between the graphs of two functions, $u(y)$ and $v(y)$.

This time, we are going to partition the interval on the y-axis and use horizontal rectangles to approximate the area between the functions. So, for $i = 0, 1, 2, \ldots, n$, let $Q = \{y_i\}$ be a regular partition of $[c, d]$. Then, for $i = 1, 2, \ldots, n$, choose a point $y_i^* \in [y_{i-1}, y_i]$, then over each interval $[y_{i-1}, y_i]$ construct a rectangle that extends horizontally from $v(y_i^*)$ to $u(y_i^*)$. Figure 2.1.8(a) shows the rectangles when y_i^* is selected to be the lower endpoint of the interval and $n = 10$. Figure 2.1.8(b) shows a representative rectangle in detail.



(a) (b)

Figure 2.1.8: (a) Approximating the area between the graphs of two functions, $u(y)$ and $v(y)$, with rectangles. (b) The area of a typical rectangle.

The height of each individual rectangle is Δy and the width of each rectangle is $x_{R_i} - x_{L_i} = u(y_i^*) - v(y_i^*)$. Therefore, the area between the curves is approximately

$$A \approx \sum_{i=1}^{n} [x_{R_i} - x_{L_i}]\Delta y = \sum_{i=1}^{n} [u(y_i^*) - v(y_i^*)]\Delta y.$$

This is a Riemann sum, so we take the limit as $n \to \infty$, obtaining

$$\begin{aligned} A &= \lim_{n\to\infty} \sum_{i=1}^{n} [x_{R_i} - x_{L_i}]\Delta y \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} [u(y_i^*) - v(y_i^*)]\Delta y \\ &= \int_c^d [u(y) - v(y)]dy. \end{aligned}$$

These findings are summarized in the following theorem.

Theorem: Finding the Area between Two Curves, Integrating along the y-axis

Let $u(y)$ and $v(y)$ be continuous functions such that $u(y) \geq v(y)$ for all $y \in [c, d]$. Let $\mathbf{R}$ denote the region bounded on the right by the graph of $u(y)$, on the left by the graph of $v(y)$, and above and below by the lines $y = d$ and $y = c$, respectively. Then, the area of $\mathbf{R}$ is given by

$$A = \int_c^d [u(y) - v(y)]dy.$$

Example 2.1.5

Let's revisit Example 2.1.4; only this time, let's integrate with respect to y. Let $\mathbf{R}$ be the region depicted in Figure 2.1.9. Find the area of $\mathbf{R}$ by integrating with respect to y.

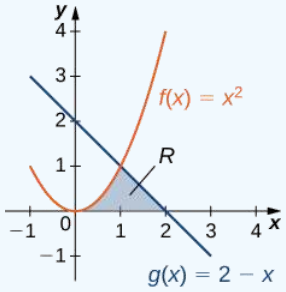


Figure 2.1.9: The area of region $\mathbf{R}$ can be calculated using one integral only when the curves are treated as functions of y.

Solution

We must first express the graphs as functions of y. As we saw at the beginning of this section, the curve on the left can be represented by the function $x_L = v(y) = \sqrt{y}$, and the curve on the right can be represented by the function $x_R = u(y) = 2 - y$.

Now, we have to determine the limits of integration. The region is bounded below by the x-axis, so the lower limit of integration is $y = 0$. The upper limit of integration is determined by the point where the two graphs intersect, which is the point $(1, 1)$, so the upper limit of integration is $y = 1$. Thus, we have $[c, d] = [0, 1]$.

Calculating the area of the region, we get

$$\begin{aligned} A &= \int_c^d [x_R - x_L]dy \\ &= \int_c^d [u(y) - v(y)]dy \\ &= \int_0^1 [(2 - y) - \sqrt{y}]dy \\ &= \left[2y - \frac{y^2}{2} - \frac{2}{3}y^{3/2}\right]\Big|_0^1 \\ &= \frac{5}{6}. \end{aligned}$$

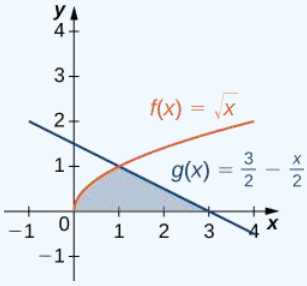
The area of the region is $\frac{5}{6}$ units^2.

Caution: Understand the meaning of $|f(x) - g(x)|$

While the notations, y_T, y_B, x_R, and x_L are incredibly helpful in properly setting up integrals to determine areas between curves (and, later, for rotational volumes), you are not excused from understanding the true definition of $|f(x) - g(x)|$.

Checkpoint 2.1.5

Let's revisit Checkpoint 2.1.4; only this time, let's integrate with respect to y. Let $\mathbf{R}$ be the region depicted in the following figure. Find the area of $\mathbf{R}$ by integrating with respect to y.



Answer

$\frac{5}{3}$ units^2

Approximating Areas Between Curves

We can combine concepts from Calculus I with our new knowledge to approximate the area between two curves.

Example 2.1.6

Approximate the area between $f(x) = \cosh(x)$ and $g(x) = -\frac{x}{2x^2+1}$ on the interval $[-2, 2]$ using vertical slices and right endpoints with $n = 8$.

Solution

First, it is actually possible for us to compute the exact area between these two curves. Consider the Figure 2.1.10 below.

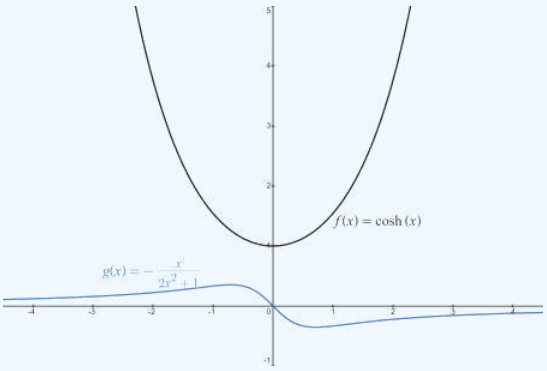


Figure 2.1.10

We can easily see that the area can be computed using the definite integral

$$
\begin{aligned}
A &= \int_{-2}^{2} (y_T - y_B)\,dx \\
&= \int_{-2}^{2} (f(x) - g(x))\,dx \\
&= \int_{-2}^{2} \left(\cosh(x) + \frac{x}{2x^2+1}\right) dx \\
&= \int_{-2}^{2} \cosh(x)dx + \int_{-2}^{2} \frac{x}{2x^2+1}dx \\
&= \int_{-2}^{2} \cosh(x)dx + 0 \qquad \left(\frac{x}{2x^2+1} \text{ is an odd function being integrated over a symmetric interval.}\right) \\
&= 2\int_{0}^{2} \cosh(x)dx \qquad (\cosh(x) \text{ is an even function being integrated over a symmetric interval.}) \\
&= 2\sinh(x)\Big|_0^2 \\
&= 2\sinh(2) - 2\sinh(0) \\
&= 2\sinh(2) - 0 \\
&= 2\sinh(2)
\end{aligned}
$$

However, we are being asked to *approximate* the value of this area, so let's practice our skills from Calculus I.

We are given $n = 8$, which implies $\Delta x = \frac{b-a}{n} = \frac{2-(-2)}{8} = \frac{1}{2}$ and $x_i = a + i\Delta x = -2 + \frac{1}{2}i$. Therefore,

$$
\begin{aligned}
A &\approx R_8 \\
&= \sum_{i=1}^{8} (y_{T_i} - y_{B_i})\,\Delta x \\
&= \sum_{i=1}^{8} (f(x_i) - g(x_i))\,\Delta x \\
&= \sum_{i=1}^{8} \left(f\left(-2+\frac{1}{2}i\right) - g\left(-2+\frac{1}{2}i\right)\right)\frac{1}{2} \\
&= \frac{1}{2}\sum_{i=1}^{8} \left(\cosh\left(-2+\frac{1}{2}i\right) + \frac{-2+\frac{1}{2}i}{2\left(-2+\frac{1}{2}i\right)^2+1}\right) \\
&= \frac{1}{2}\left\{\left[\cosh\left(-\frac{3}{2}\right) + \frac{-\frac{3}{2}}{2\left(-\frac{3}{2}\right)^2+1}\right] + \left[\cosh(-1) + \frac{-1}{2(-1)^2+1}\right] + \cdots + \left[\cosh(2) + \frac{2}{2(2)^2+1}\right]\right\} \\
&\approx 7.5153
\end{aligned}
$$

Advice: Review hyperbolic functions

The hyperbolic functions you learned from Calculus I will creep up at random points throughout Integral Calculus and Differential Equations. If your Calculus I instructor did not cover them, then you can always review the necessary concepts in Section 1.7 of the Differential Calculus textbook.

Footnotes

[1] We will use the language and notation of this mini-review of distances throughout the remainder of Calculus II.

[2] Note that $x = -\sqrt{y}$ is also a valid representation of the function $y = f(x) = x^2$ as a function of y. However, based on the graph, it is clear we are interested in the positive square root.

2.1E: EXERCISES

READING QUESTIONS

1. What is the general formula for finding the area of a region R bounded above by $y = f(x)$, below by $y = g(x)$, and between $x = a$ and $x = b$, assuming $f(x) \geq g(x)$ on $[a, b]$?

   Answer: $A = \int_a^b [f(x) - g(x)]dx$.

2. How does the formula change if the curves $f(x)$ and $g(x)$ intersect one or more times between $x = a$ and $x = b$?

   Answer: The area is given by $A = \int_a^b |f(x) - g(x)|dx$. In practice, this means you must find the points of intersection, split the interval $[a, b]$ into subintervals based on these intersections, and then integrate the difference of $(y_{Top} - y_{Bottom})$ over each subinterval, ensuring the integrand is always positive, then sum the results.

3. Under what circumstances might you choose to integrate with respect to y instead of x to find the area between two curves?

   Answer: You might choose to integrate with respect to y if the curves are more easily expressed as functions of y (i.e., $x = u(y)$ and $x = v(y)$), or if integrating with respect to x would require splitting the region into multiple parts while integrating with respect to y would only require a single integral. This often happens when the region is bounded more simply by a "right" curve and a "left" curve over an interval of y-values.

4. What do the expressions $y_{Top} - y_{Bottom}$ and $x_{Right} - x_{Left}$ represent geometrically when setting up integrals for area?

   Answer: $y_{Top} - y_{Bottom}$ represents the height of a typical vertical rectangular strip used to approximate the area when integrating with respect to x. $x_{Right} - x_{Left}$ represents the width of a typical horizontal rectangular strip used to approximate the area when integrating with respect to y.

5. If a region R is bounded on the right by $x = u(y)$, on the left by $x = v(y)$, and between $y = c$ and $y = d$, with $u(y) \geq v(y)$ on $[c, d]$, what is the integral for its area?

   Answer: $A = \int_c^d [u(y) - v(y)]dy$.

6. What is the typical first algebraic step if you are asked to find the area of a region bounded by two curves, say $y = f(x)$ and $y = g(x)$, but the interval of integration is not given?

   Answer: The typical first step is to find the points of intersection of the two curves by setting $f(x) = g(x)$ and solving for x. These x-values will typically define the limits of integration a and b.

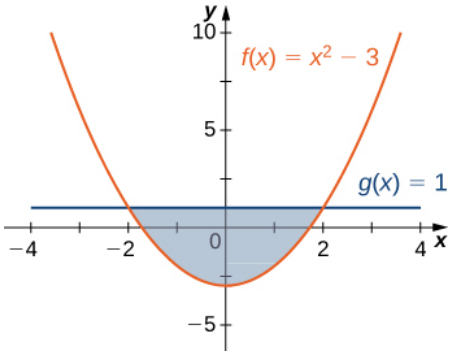
7. The section mentions a "caution" regarding trigonometry. What are students expected to be proficient in regarding trigonometry for success in Calculus II?

   Answer: Students are expected to have mastered various trigonometric identities (Pythagorean, Sum/Difference, Double-Angle, Half-Angle/Power Reduction, Even/Odd), be able to evaluate trig functions at special/axial angles, graph transformations of trig functions, solve trig equations, evaluate inverse trig functions, and understand their domain/range restrictions, all without relying on technology.

HOMEWORK

For exercises 1 - 2, determine the area of the region between the two curves in the given figure by integrating over the x-axis.

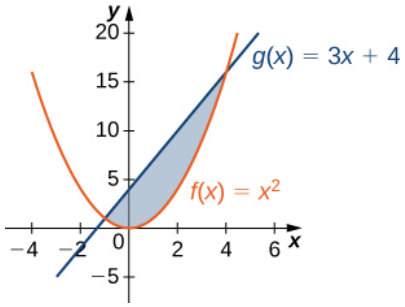
1) $y = x^2 - 3$ and $y = 1$



Answer

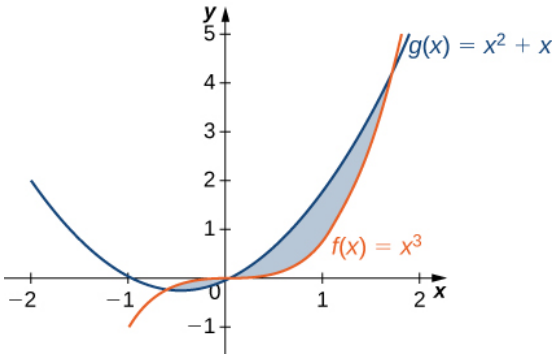
$\frac{32}{3}$ units^2

2) $y = x^2$ and $y = 3x + 4$



For exercises 3 - 4, split the region between the two curves into two smaller regions, then determine the area by integrating over the x-axis. Note that you will have two integrals to solve.

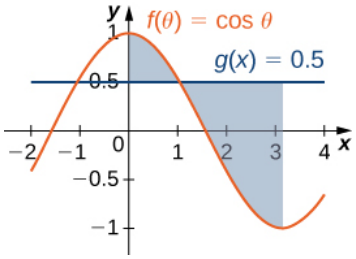
3) $y = x^3$ and $y = x^2 + x$



Answer

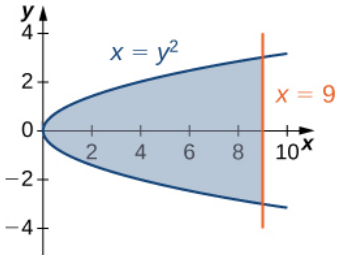
$\frac{13}{12}$ **units**^{2}

4) $y = \cos\theta$ and $y = 0.5$, for $0 \le \theta \le \pi$



For exercises 5-6, determine the area of the region between the two curves by integrating over the y-axis.

5) $x = y^2$ and $x = 9$

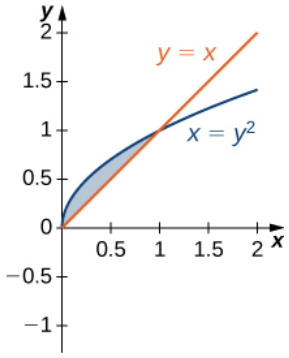


Answer

36 units^2

6) $y = x$ and $x = y^2$

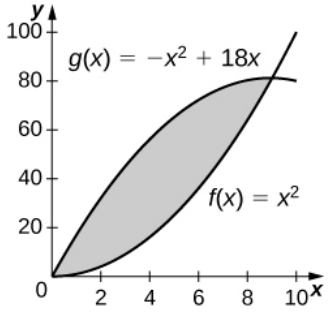
https://math.libretexts.org/@go/page/168406



For exercises 7 - 13, graph the equations and shade the area of the region between the curves. Determine its area by integrating over the x-axis.

7) $y = x^2$ and $y = -x^2 + 18x$

Answer

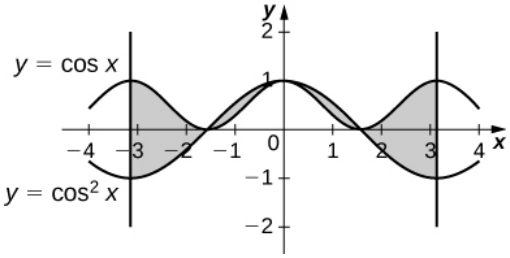


243 square units

8) $y = \frac{1}{x}$, $y = \frac{1}{x^2}$, and $x = 3$

9) $y = \cos x$ and $y = \cos^2 x$ on $x \in [-\pi, \pi]$

Answer

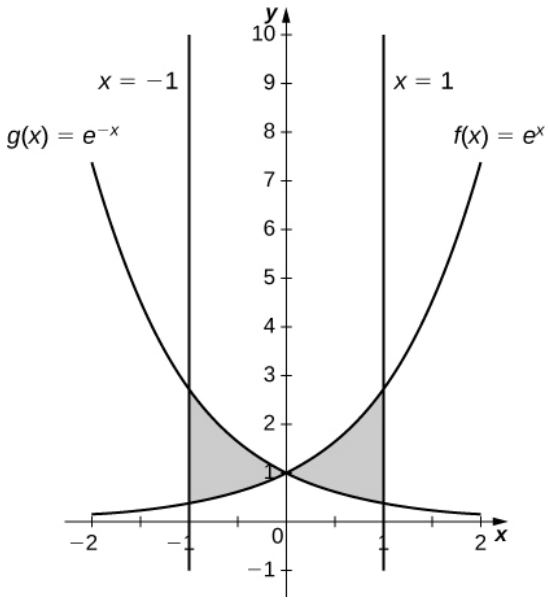


4 square units

10) $y = e^x$, $y = e^{2x-1}$, and $x = 0$

11) $y = e^x$, $y = e^{-x}$, $x = -1$ and $x = 1$

Answer

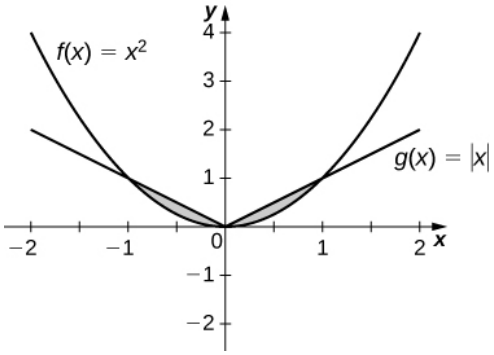


$\frac{2(e-1)^2}{e}$ units^2

12) $y = e$, $y = e^x$, and $y = e^{-x}$

13) $y = |x|$ and $y = x^2$

Answer



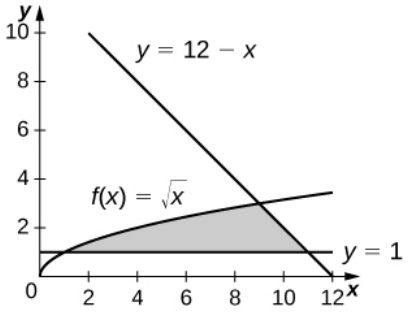
$\frac{1}{3}$ units^2

For exercises 14 - 19, graph the equations and shade the area of the region between the curves. If necessary, break the region into sub-regions to determine its entire area.

14) $y = \sin(\pi x)$, $y = 2x$, and $x > 0$

15) $y = 12 - x$, $y = \sqrt{x}$, and $y = 1$

Answer

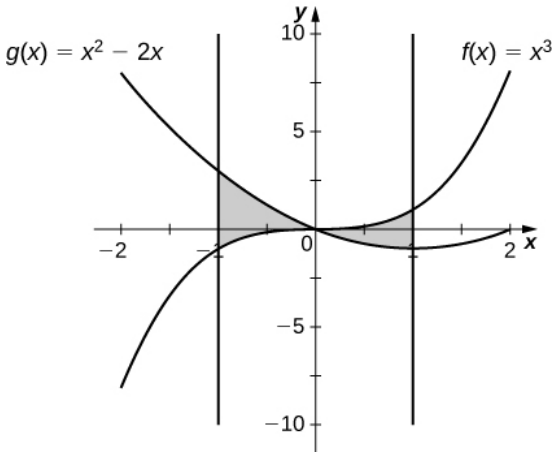


$\frac{34}{3}$ units^2

16) $y = \sin x$ and $y = \cos x$ over $x \in [-\pi, \pi]$

17) $y = x^3$ and $y = x^2 - 2x$ over $x \in [-1, 1]$

Answer

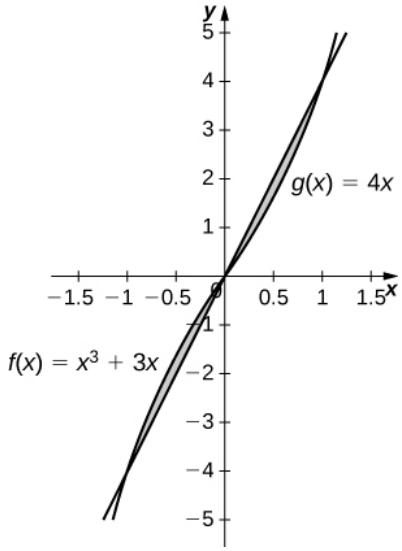


$\frac{5}{2}$ units^2

18) $y = x^2 + 9$ and $y = 10 + 2x$ over $x \in [-1, 3]$

19) $y = x^3 + 3x$ and $y = 4x$

Answer



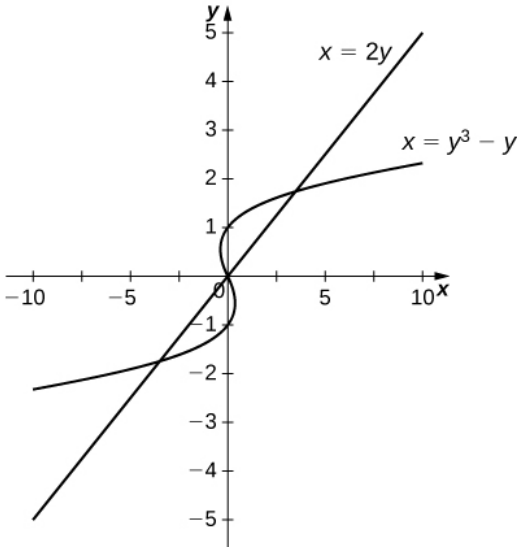
$\frac{1}{2}$ units^2

For exercises 20 -25, graph the equations and shade the area of the region between the curves. Determine its area by integrating over the y-axis.

20) $x = y^3$ and $x = 3y - 2$

21) $x = y$ and $x = y^3 - y$

Answer

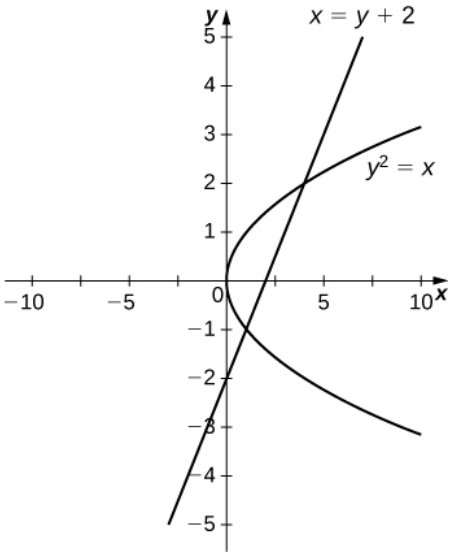


$\frac{9}{2}$ units^2

22) $x = -3 + y^2$ and $x = y - y^2$

23) $y^2 = x$ and $x = y + 2$

Answer

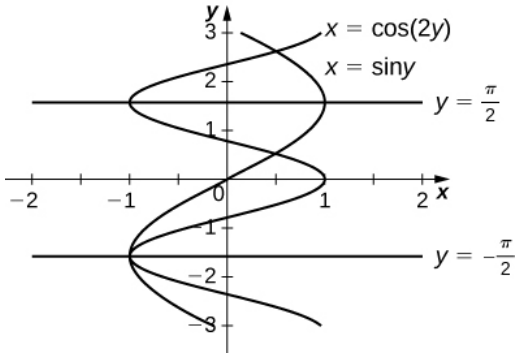


$\frac{9}{2}$ units^2

24) $x = |y|$ and $2x = -y^2 + 2$

25) $x = \sin y, \quad x = \cos(2y), \quad y = \pi/2$, and $y = -\pi/2$

Answer



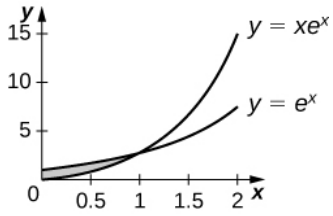
$\frac{3\sqrt{3}}{2}$ units^2

For exercises 26 - 37, graph the equations and shade the area of the region between the curves. Determine its area by integrating over the x-axis or y-axis, whichever seems more convenient.

26) $x = y^4$ and $x = y^5$

27) $y = xe^x$, $y = e^x$, $x = 0$, and $x = 1$.

Answer

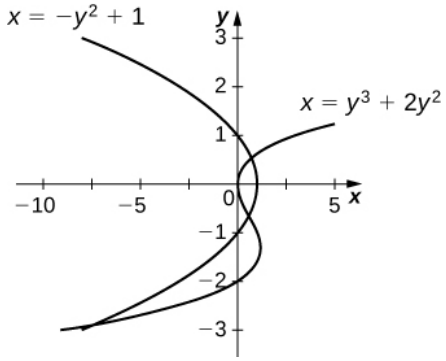


e^{-2} units^2

28) $y = x^6$ and $y = x^4$

29) $x = y^3 + 2y^2 + 1$ and $x = -y^2 + 1$

Answer

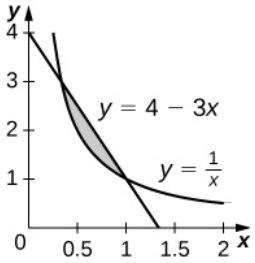


$\frac{27}{4}$ units^2

30) $y = |x|$ and $y = x^2 - 1$

31) $y = 4 - 3x$ and $y = \frac{1}{x}$

Answer

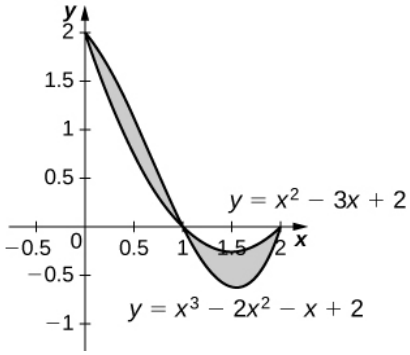


$\left(\frac{4}{3}-\ln(3)\right)$ units^2

32) $y=\sin x, \quad x=-\pi/6, \quad x=\pi/6$, and $y=\cos^3 x$

33) $y=x^2-3x+2$ and $y=x^3-2x^2-x+2$

Answer

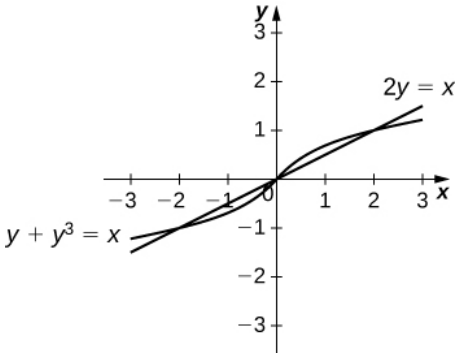


$\frac{1}{2}$ square units

34) $y=2\cos^3(3x), \quad y=-1, \quad x=\frac{\pi}{4}$, and $x=-\frac{\pi}{4}$

35) $y+y^3=x$ and $2y=x$

Answer



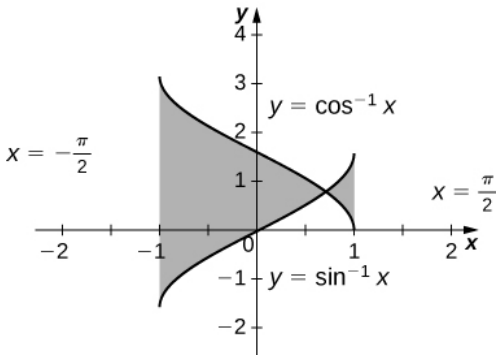
$\frac{1}{2}$ square units

36) $y=\sqrt{1-x^2}$ and $y=x^2-1$

37) $y=\cos^{-1}x, \quad y=\sin^{-1}x, \quad x=-1$, and $x=1$

Answer

https://math.libretexts.org/@go/page/168406



$-2(\sqrt{2}-\pi)$ square units

For exercises 38 - 47, find the exact area of the region bounded by the given equations if possible. If you are unable to determine the intersection points analytically, use a calculator to approximate the intersection points with three decimal places and determine the approximate area of the region.

38) [Technology Required] $x = e^y$ and $y = x - 2$

39) [Technology Required] $y = x^2$ and $y = \sqrt{1-x^2}$

Answer

1.067 square units

40) [Technology Required] $y = 3x^2 + 8x + 9$ and $3y = x + 24$

41) [Technology Required] $x = \sqrt{4-y^2}$ and $y^2 = 1 + x^2$

Answer

0.852 square units

42) [Technology Required] $x^2 = y^3$ and $x = 3y$

43) [Technology Required] $y = \sin^3 x + 2, \quad y = \tan x, \quad x = -1.5,$ and $x = 1.5$

Answer

7.523 square units

44) [Technology Required] $y = \sqrt{1-x^2}$ and $y^2 = x^2$

45) [Technology Required] $y = \sqrt{1-x^2}$ and $y = x^2 + 2x + 1$

Answer

$\frac{3\pi - 4}{12}$ square units

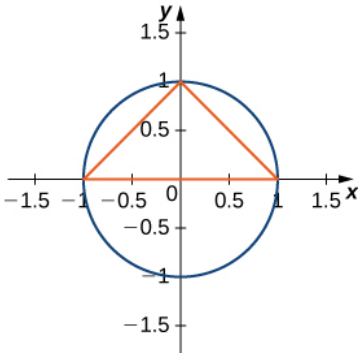
46) [Technology Required] $x = 4 - y^2$ and $x = 1 + 3y + y^2$

47) [Technology Required] $y = \cos x, \quad y = e^x, \quad x = -\pi, \quad$ and $\quad x = 0$

Answer

1.429 square units

48) The largest triangle with a base on the x-axis that fits inside the upper half of the unit circle $y^2 + x^2 = 1$ is given by $y = 1 + x$ and $y = 1 - x$. See the following figure. What is the area inside the semicircle but outside the triangle?



49) A factory selling cell phones has a marginal cost function $C(x) = 0.01x^2 - 3x + 229$, where x represents the number of cell phones, and a marginal revenue function given by $R(x) = 429 - 2x$. Find the area between the graphs of these curves and $x = 0$. What does this area represent?

Answer

$33,333.33 total profit for **200** cell phones sold

50) An amusement park has a marginal cost function $C(x) = 1000e - x + 5$, where x represents the number of tickets sold, and a marginal revenue function given by $R(x) = 60 - 0.1x$. Find the total profit generated when selling **550** tickets. Use a calculator to determine intersection points, if necessary, to two decimal places.

51) The tortoise versus the hare: The speed of the hare is given by the sinusoidal function $H(t) = 1 - \cos((\pi t)/2)$ whereas the speed of the tortoise is $T(t) = (1/2)\tan^{-1}(t/4)$, where t is time measured in hours and the speed is measured in miles per hour. Find the area between the curves from time $t = 0$ to the first time after one hour when the tortoise and hare are traveling at the same speed. What does it represent? Use a calculator to determine the intersection points, if necessary, accurate to three decimal places.

Answer

3.263 mi represents how far ahead the hare is from the tortoise

52) The tortoise versus the hare: The speed of the hare is given by the sinusoidal function $H(t) = (1/2) - (1/2)\cos(2\pi t)$ whereas the speed of the tortoise is $T(t) = \sqrt{t}$, where t is time measured in hours and speed is measured in kilometers per hour. If the race is over in 1 hour, who won the race and by how much? Use a calculator to determine the intersection points, if necessary, accurate to three decimal places.

For exercises 53 - 55, find the area between the curves by integrating with respect to x and then with respect to y. Is one method easier than the other? Do you obtain the same answer?

53) $y = x^2 + 2x + 1$ and $y = -x^2 - 3x + 4$

Answer

$\frac{343}{24}$ square units

54) $y = x^4$ and $x = y^5$

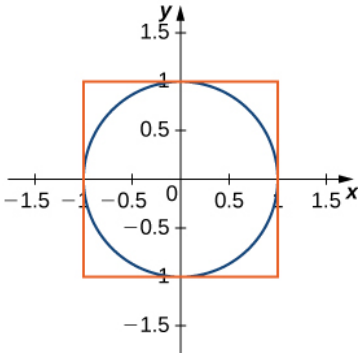
55) $x = y^2 - 2$ and $x = 2y$

Answer

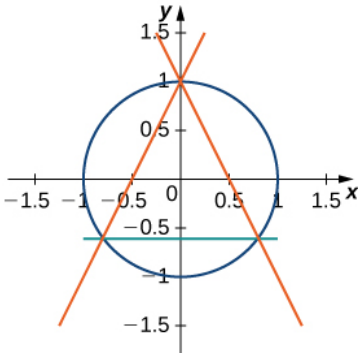
$4\sqrt{3}$ square units

For exercises 56 - 57, solve using calculus, then check your answer with geometry.

56) Determine the equations for the sides of the square that touches the unit circle on all four sides, as seen in the following figure. Find the area between the perimeter of this square and the unit circle. Is there another way to solve this without using calculus?



57) Find the area between the perimeter of the unit circle and the triangle created from $y = 2x + 1$, $y = 1 - 2x$ and $y = -\frac{3}{5}$, as seen in the following figure. Is there a way to solve this without using calculus?



Answer

$\left(\pi - \frac{32}{25}\right)$ square units

In exercises 58 - 61, if you are unable to find intersection points analytically, use a calculator.

58) Find the area of the region enclosed by $x = 1$ and $y = 5$ above $y = \ln x$.

Answer

$(e^5 - 6)$ units^2

59) Find the area under $y = 1/x$ and above the x-axis from $x = 1$ to $x = 4$.

60) Find the area of the hyperbolic quarter-circle enclosed by $x = 2$ and $y = 2$ above $y = 1/x$.

61) Find the area between $\ln x$ and the x-axis from $x = 1$ to $x = 2$.

Answer

$\ln(4) - 1)$ units^2

CONTRIBUTORS

Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

This page titled 2.1E: Exercises is shared under a CC BY-NC license and was authored, remixed, and/or curated by Roy Simpson.

- **6.7E: Exercises for Section 6.7** by Edwin “Jed” Herman, Gilbert Strang is licensed CC BY-NC-SA 4.0. Original source: https://openstax.org/details/books/calculus-volume-1.

2.2: Volume by Cross-Sectional Area

The volume of a general right cylinder, as shown in Figure 2.2.1, is

$$\text{Area of the base} \times \text{height}. \tag{2.2.1}$$

We can use this fact as the building block in finding volumes of a variety of shapes.

Given an arbitrary solid, we can *approximate* its volume by cutting it into n thin slices. When the slices are thin, each slice can be approximated well by a general right cylinder. Thus the volume of each slice is approximately its cross-sectional area $\times$ thickness. (These slices are the differential elements.)

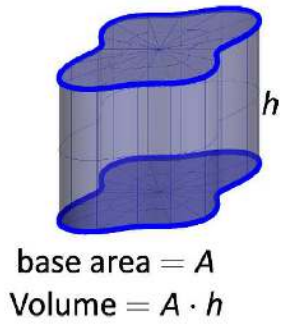


Figure* 2.2.1*: *The volume of a general right cylinder.*

By orienting a solid along the x-axis, we can let $A(x_i)$ represent the cross-sectional area of the ith slice, and let dx_i represent the thickness of this slice (the thickness is a small change in x). The total volume of the solid is approximately:

$$\text{Volume} \approx \sum_{i=1}^{n} \Big[\text{Area} \times \text{thickness}\Big] \tag{2.2.2}$$

$$= \sum_{i=1}^{n} A(x_i)\, dx_i. \tag{2.2.3}$$

Recognize that this is a Riemann Sum. By taking a limit (as the thickness of the slices goes to 0) we can find the volume exactly.

Theorem 2.2.1: Volume by Cross-Sectional Area

The volume V of a solid, oriented along the x-axis with cross-sectional area $A(x)$ from $x = a$ to $x = b$, is

$$V = \int_a^b A(x)\, dx. \tag{2.2.4}$$

Example 2.2.1: Finding the volume of a solid

Find the volume of a pyramid with a square base of side length 10 in and a height of 5 in.

Solution

There are many ways to "orient" the pyramid along the x-axis; Figure 2.2.2 gives one such way, with the pointed top of the pyramid at the origin and the x-axis going through the center of the base.

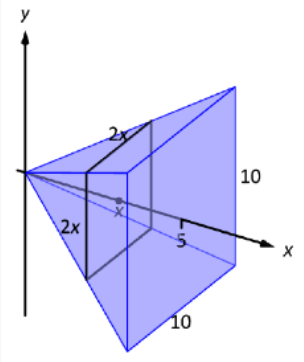


***Figure* 2.2.2:** *Orienting a pyramid along the x-axis in Example* **2.2.1**.

Each cross section of the pyramid is a square; this is a sample differential element. To determine its area $A(x)$, we need to determine the side lengths of the square.

When $x = 5$, the square has side length 10; when $x = 0$, the square has side length 0. Since the edges of the pyramid are lines, it is easy to figure that each cross-sectional square has side length $2x$, giving $A(x) = (2x)^2 = 4x^2$.

If one were to cut a slice out of the pyramid at $x = 3$, as shown in Figure **2.2.3**, one would have a shape with square bottom and top with sloped sides. If the slice were thin, both the bottom and top squares would have sides lengths of about 6, and thus the cross--sectional area of the bottom and top would be about 36in^2. Letting Δx_i represent the thickness of the slice, the volume of this slice would then be about $36\Delta x_i$in^3.

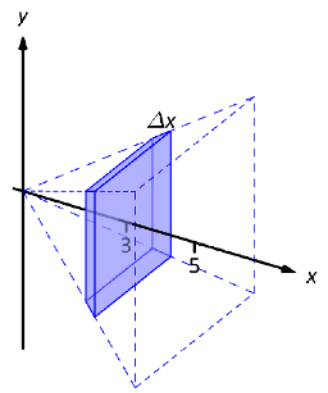


***Figure* 2.2.3:** *Cutting a slice in they pyramid in Example* **2.2.1** *at $x = 3$.*

Cutting the pyramid into n slices divides the total volume into n equally--spaced smaller pieces, each with volume $(2x_i)^2\Delta x$, where x_i is the approximate location of the slice along the x-axis and Δx represents the thickness of each slice. One can approximate total volume of the pyramid by summing up the volumes of these slices:

$$\text{Approximate volume} = \sum_{i=1}^{n} (2x_i)^2\Delta x. \tag{2.2.5}$$

Taking the limit as $n \to \infty$ gives the actual volume of the pyramid; recoginizing this sum as a Riemann Sum allows us to find the exact answer using a definite integral, matching the definite integral given by Theorem **2.2.1**.

We have

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} (2x_i)^2 \Delta x \tag{2.2.6}$$

$$= \int_0^5 4x^2\, dx \tag{2.2.7}$$

$$= \frac{4}{3}x^3\Big|_0^5 \tag{2.2.8}$$

$$= \frac{500}{3}\ \text{in}^3 \approx 166.67\ \text{in}^3. \tag{2.2.9}$$

We can check our work by consulting the general equation for the volume of a pyramid (see the back cover under "Volume of A General Cone"):

$$\frac{1}{3} \times \text{area of base} \times \text{height}. \tag{2.2.10}$$

Certainly, using this formula from geometry is faster than our new method, but the calculus--based method can be applied to much more than just cones.

An important special case of Theorem **2.2.1** is when the solid is a **solid of revolution**, that is, when the solid is formed by rotating a shape around an axis.

Start with a function $y = f(x)$ from $x = a$ to $x = b$. Revolving this curve about a horizontal axis creates a three-dimensional solid whose cross sections are disks (thin circles). Let $R(x)$ represent the radius of the cross-sectional disk at x; the area of this disk is $\pi R(x)^2$. Applying Theorem **2.2.1** gives the Disk Method.

Key Idea 23: The Disk Method

Let a solid be formed by revolving the curve $y = f(x)$ from $x = a$ to $x = b$ around a horizontal axis, and let $R(x)$ be the radius of the cross-sectional disk at x. The volume of the solid is

$$V = \pi \int_a^b R(x)^2\, dx. \tag{2.2.11}$$

Example 2.2.2: Finding volume using the Disk Method

Find the volume of the solid formed by revolving the curve $y = 1/x$, from $x = 1$ to $x = 2$, around the x-axis.

Solution

A sketch can help us understand this problem. In Figure **2.2.4a** the curve $y = 1/x$ is sketched along with the differential element -- a disk -- at x with radius $R(x) = 1/x$. In Figure **2.2.4b** the whole solid is pictured, along with the differential element.

The volume of the differential element shown in part (a) of the figure is approximately $\pi R(x_i)^2 \Delta x$, where $R(x_i)$ is the radius of the disk shown and Δx is the thickness of that slice. The radius $R(x_i)$ is the distance from the x-axis to the curve, hence $R(x_i) = 1/x_i$.

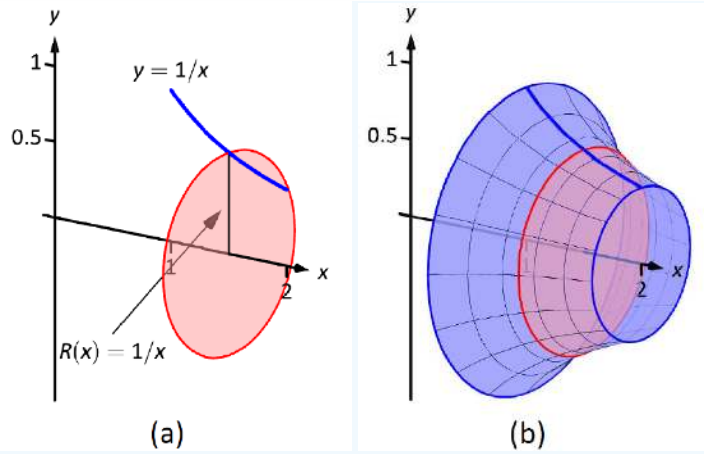


Figure* 2.2.4*: *Sketching a solid in Example* **2.2.2**.

Slicing the solid into n equally--spaced slices, we can approximate the total volume by adding up the approximate volume of each slice:

$$\text{Approximate volume } = \sum_{i=1}^{n} \pi \left(\frac{1}{x_i}\right)^2 \Delta x. \tag{2.2.12}$$

Taking the limit of the above sum as $n \to \infty$ gives the actual volume; recognizing this sum as a Riemann sum allows us to evaluate the limit with a definite integral, which matches the formula given in Key Idea 23:

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} \pi \left(\frac{1}{x_i}\right)^2 \Delta x \tag{2.2.13}$$

$$= \pi \int_1^2 \left(\frac{1}{x}\right)^2 dx \tag{2.2.14}$$

$$= \pi \int_1^2 \frac{1}{x^2}\, dx \tag{2.2.15}$$

$$= \pi \left[-\frac{1}{x}\right]\Big|_1^2 \tag{2.2.16}$$

$$= \pi \left[-\frac{1}{2} - (-1)\right] \tag{2.2.17}$$

$$= \frac{\pi}{2} \text{ units}^3. \tag{2.2.18}$$

While Key Idea 23 is given in terms of functions of x, the principle involved can be applied to functions of y when the axis of rotation is vertical, not horizontal. We demonstrate this in the next example.

Example 2.2.3: Finding volume using the Disk Method

Find the volume of the solid formed by revolving the curve $y = 1/x$, from $x = 1$ to $x = 2$, about the y-axis.

Solution

Since the axis of rotation is vertical, we need to convert the function into a function of y and convert the x-bounds to y-bounds. Since $y = 1/x$ defines the curve, we rewrite it as $x = 1/y$. The bound $x = 1$ corresponds to the y-bound $y = 1$, and the bound $x = 2$ corresponds to the y-bound $y = 1/2$.

Thus we are rotating the curve $x = 1/y$, from $y = 1/2$ to $y = 1$ about the y-axis to form a solid. The curve and sample differential element are sketched in Figure 2.2.5*a*, with a full sketch of the solid in Figure 2.2.5*b*.

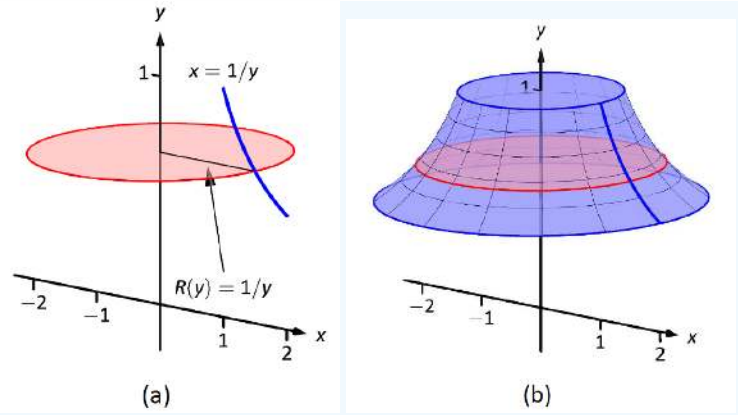


***Figure* 2.2.5:** *Sketching a solid in Example* 2.2.3.

We integrate to find the volume:

$$V = \pi \int_{1/2}^{1} \frac{1}{y^2}\, dy \tag{2.2.19}$$

$$= -\frac{\pi}{y}\Big|_{1/2}^{1} \tag{2.2.20}$$

$$= \pi \text{ units}^3. \tag{2.2.21}$$

We can also compute the volume of solids of revolution that have a hole in the center. The general principle is simple: compute the volume of the solid irrespective of the hole, then subtract the volume of the hole. If the outside radius of the solid is $R(x)$ and the inside radius (defining the hole) is $r(x)$, then the volume is

$$V = \pi \int_a^b R(x)^2\, dx - \pi \int_a^b r(x)^2\, dx = \pi \int_a^b \left(R(x)^2 - r(x)^2\right)\, dx. \tag{2.2.22}$$

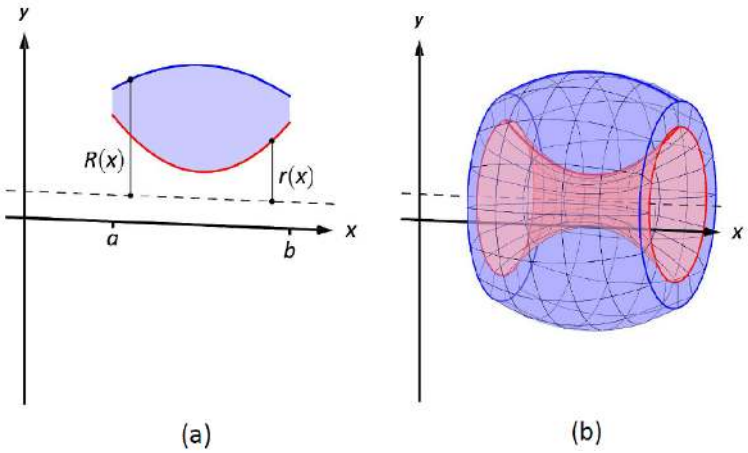


***Figure* 2.2.6:** *Establishing the Washer Method; see also Figure* 2.2.7.

One can generate a solid of revolution with a hole in the middle by revolving a region about an axis. Consider Figure 2.2.6*a*, where a region is sketched along with a dashed, horizontal axis of rotation. By rotating the region about the axis, a solid is formed as sketched in Figure 2.2.6*b*. The outside of the solid has radius $R(x)$, whereas the inside has radius $r(x)$. Each cross section of this solid will be a washer (a disk with a hole in the center) as sketched in Figure 2.2.6*b*. This leads us to the Washer Method.

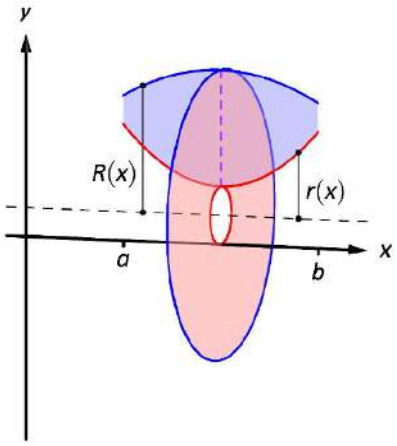


***Figure* 2.2.7:** *Establishing the Washer Method; see also Figure* **2.2.6**.

Key Idea 24: The Washer Method

Let a region bounded by $y = f(x)$, $y = g(x)$, $x = a$ and $x = b$ be rotated about a horizontal axis that does not intersect the region, forming a solid. Each cross section at x will be a washer with outside radius $R(x)$ and inside radius $r(x)$. The volume of the solid is

$$V = \pi \int_a^b \left(R(x)^2 - r(x)^2 \right) dx. \tag{2.2.23}$$

Even though we introduced it first, the Disk Method is just a special case of the Washer Method with an inside radius of $r(x) = 0$.

Example 2.2.4: Finding volume with the Washer Method

Find the volume of the solid formed by rotating the region bounded by $y = x^2 - 2x + 2$ and $y = 2x - 1$ about the x-axis.

Solution

A sketch of the region will help, as given in Figure **2.2.8***a*.

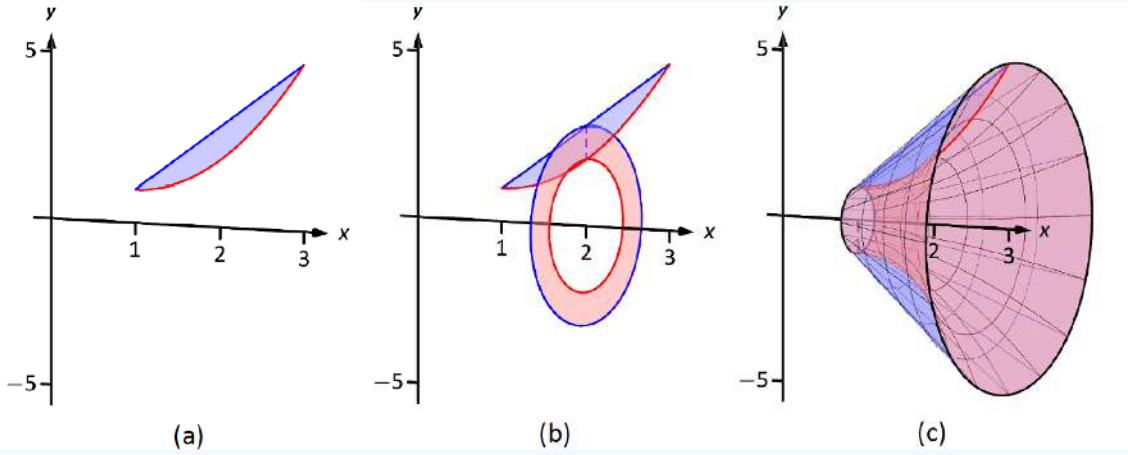


***Figure* 2.2.8:** *Sketching the differential element and solid in Example* **2.2.4**.

Rotating about the x-axis will produce cross sections in the shape of washers, as shown in Figure **2.2.8***b*; the complete solid is shown in part (c). The outside radius of this washer is $R(x) = 2x + 1$; the inside radius is $r(x) = x^2 - 2x + 2$. As the region is bounded from $x = 1$ to $x = 3$, we integrate as follows to compute the volume.

$$V = \pi \int_1^3 \left((2x-1)^2 - (x^2-2x+2)^2 \right) dx \tag{2.2.24}$$
$$= \pi \int_1^3 \left(-x^4 + 4x^3 - 4x^2 + 4x - 3 \right) dx \tag{2.2.25}$$
$$= \pi \left[-\frac{1}{5}x^5 + x^4 - \frac{4}{3}x^3 + 2x^2 - 3x \right]\Big|_1^3 \tag{2.2.26}$$
$$= \frac{104}{15}\pi \approx 21.78 \text{ units}^3. \tag{2.2.27}$$

When rotating about a vertical axis, the outside and inside radius functions must be functions of y.

Example 2.2.5: Finding volume with the Washer Method

Find the volume of the solid formed by rotating the triangular region with vertices at $(1,1)$, $(2,1)$ and $(2,3)$ about the y-axis.

Solution

The triangular region is sketched in Figure 2.2.9; the differential element is sketched in (b) and the full solid is drawn in (c). They help us establish the outside and inside radii. Since the axis of rotation is vertical, each radius is a function of y.

The outside radius $R(y)$ is formed by the line connecting $(2,1)$ and $(2,3)$; it is a constant function, as regardless of the y-value the distance from the line to the axis of rotation is 2. Thus $R(y) = 2$.

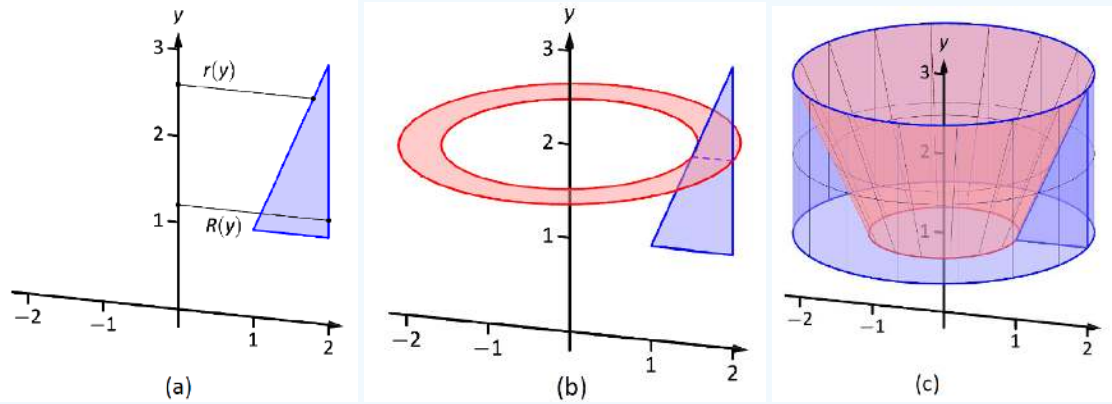


Figure 2.2.9: *Sketching the solid in Example 2.2.5.*

The inside radius is formed by the line connecting $(1,1)$ and $(2,3)$. The equation of this line is $y = 2x - 1$, but we need to refer to it as a function of y. Solving for x gives $r(y) = \frac{1}{2}(y+1)$.

We integrate over the y-bounds of $y = 1$ to $y = 3$. Thus the volume is

$$V = \pi \int_1^3 \left(2^2 - \left(\frac{1}{2}(y+1) \right)^2 \right) dy \tag{2.2.28}$$
$$= \pi \int_1^3 \left(-\frac{1}{4}y^2 - \frac{1}{2}y + \frac{15}{4} \right) dy \tag{2.2.29}$$
$$= \pi \left[-\frac{1}{12}y^3 - \frac{1}{4}y^2 + \frac{15}{4}y \right]\Big|_1^3 \tag{2.2.30}$$
$$= \frac{10}{3}\pi \approx 10.47 \text{ units}^3. \tag{2.2.31}$$

This section introduced a new application of the definite integral. Our default view of the definite integral is that it gives "the area under the curve." However, we can establish definite integrals that represent other quantities; in this section, we computed volume.

The ultimate goal of this section is not to compute volumes of solids. That can be useful, but what is more useful is the understanding of this basic principle of integral calculus, outlined in Key Idea 24: to find the exact value of some quantity,

- we start with an approximation (in this section, slice the solid and approximate the volume of each slice),
- then make the approximation better by refining our original approximation (i.e., use more slices),

- then use limits to establish a definite integral which gives the exact value.

We practice this principle in the next section where we find volumes by slicing solids in a different way.

Contributors and Attributions

- Gregory Hartman (Virginia Military Institute). Contributions were made by Troy Siemers and Dimplekumar Chalishajar of VMI and Brian Heinold of Mount Saint Mary's University. This content is copyrighted by a Creative Commons Attribution - Noncommercial (BY-NC) License. http://www.apexcalculus.com/
- Integrated by Justin Marshall.

This page titled 2.2: Volume by Cross-Sectional Area is shared under a CC BY-NC 3.0 license and was authored, remixed, and/or curated by Gregory Hartman et al. via source content that was edited to the style and standards of the LibreTexts platform.

- **7.2: Volume by Cross-Sectional Area- Disk and Washer Methods** by Gregory Hartman et al. is licensed CC BY-NC 3.0. Original source: http://www.apexcalculus.com/.

2.2E: Exercises

Reading Questions

1. What is the definition of a solid's cross-section?
2. What is the general definite integral formula for the volume of a solid using the slicing method, where $A(x)$ is the cross-sectional area at x?
3. List the three steps in the problem-solving strategy for finding volumes by the slicing method.
4. In Example 1.2.1, what geometric principle is used to find the side length s of the square cross-section in terms of its distance x from the apex of the pyramid?
5. If the cross-sections of a solid are taken perpendicular to the y-axis, what variable must the area function and the integration be in terms of?
6. According to the text, what is the mathematical definition of a cylinder? Does its base have to be a circle?

1) Derive the formula for the volume of a sphere using the slicing method.

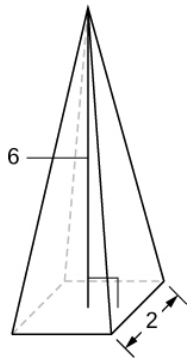
2) Use the slicing method to derive the formula for the volume of a cone.

3) Use the slicing method to derive the formula for the volume of a tetrahedron with side length a.

Volumes by Slicing

For exercises 4 - 8, draw a typical slice and find the volume using the slicing method for the given volume.

4) A pyramid with height 6 units and square base of side 2 units, as pictured here.



Solution:

Here the cross-sections are squares taken perpendicular to the y-axis.
We use the vertical cross-section of the pyramid through its center to obtain an equation relating x and y.
Here this would be the equation, $y = 6 - 6x$. Since we need the dimensions of the square at each y-level, we solve this equation for x to get, $x = 1 - \frac{y}{6}$.
This is half the distance across the square cross-section at the y-level, so the side length of the square cross-section is, $s = 2\left(1 - \frac{y}{6}\right)$.
Thus, we have the area of a cross-section is,

$$A(y) = \left[2\left(1 - \tfrac{y}{6}\right)\right]^2 = 4\left(1 - \tfrac{y}{6}\right)^2.$$

Then, $V = \int_0^6 4\left(1 - \frac{y}{6}\right)^2 \, dy$

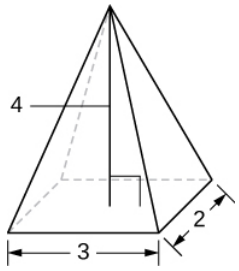
$= -24 \int_1^0 u^2 \, du, \quad \text{where } u = 1 - \frac{y}{6}, \text{ so } du = -\frac{1}{6} \, dy, \quad \Longrightarrow \quad -6 \, du = dy$

$= 24 \int_0^1 u^2 \, du = 24 \frac{u^3}{3} \Big|_0^1$

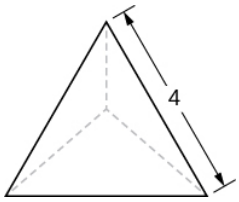
$= 8u^3 \Big|_0^1$

$= 8\left(1^3 - 0^3\right) \quad = \quad 8 \text{ units}^3$

5) A pyramid with height 4 units and a rectangular base with length 2 units and width 3 units, as pictured here.



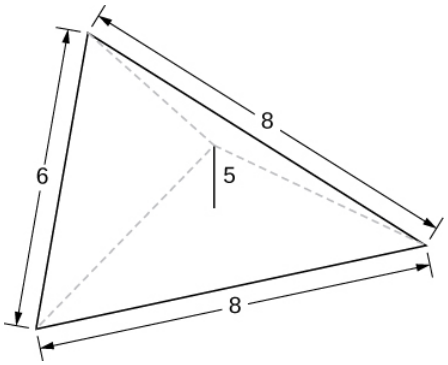
6) A tetrahedron with a base side of 4 units,as seen here.



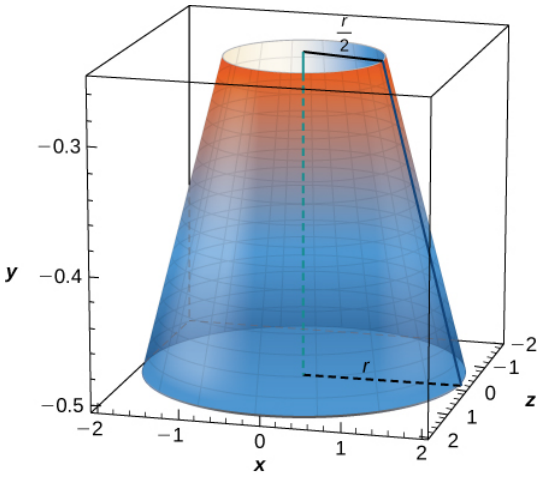
Answer

$V = \frac{32}{3\sqrt{2}} = \frac{16\sqrt{2}}{3}$ units^3

7) A pyramid with height 5 units, and an isosceles triangular base with lengths of 6 units and 8 units, as seen here.



8) A cone of radius r and height h has a smaller cone of radius $r/2$ and height $h/2$ removed from the top, as seen here. The resulting solid is called a frustum.



Answer

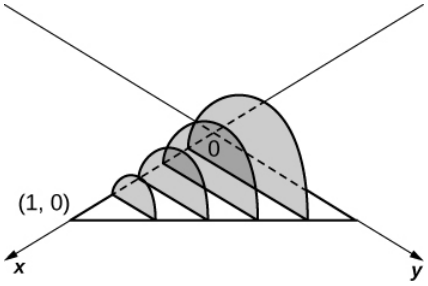
$V = \frac{7\pi}{12} hr^2$ units^3

For exercises 9 - 14, draw an outline of the solid and find the volume using the slicing method.

9) The base is a circle of radius a. The slices perpendicular to the base are squares.

10) The base is a triangle with vertices $(0,0), (1,0)$, and $(0,1)$. Slices perpendicular to the xy-plane are semicircles.

Answer

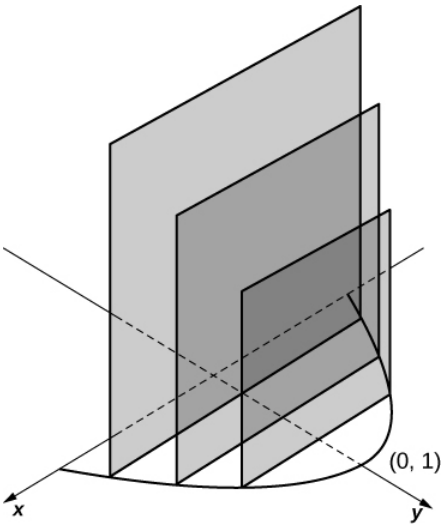


$V = \int_0^1 \frac{\pi(1-x)^2}{8}\,dx \quad = \quad \frac{\pi}{24}$ units^3

11) The base is the region under the parabola $y = 1 - x^2$ in the first quadrant. Slices perpendicular to the xy-plane are squares.

12) The base is the region under the parabola $y = 1 - x^2$ and above the x-**axis**. Slices perpendicular to the y-**axis** are squares.

Answer

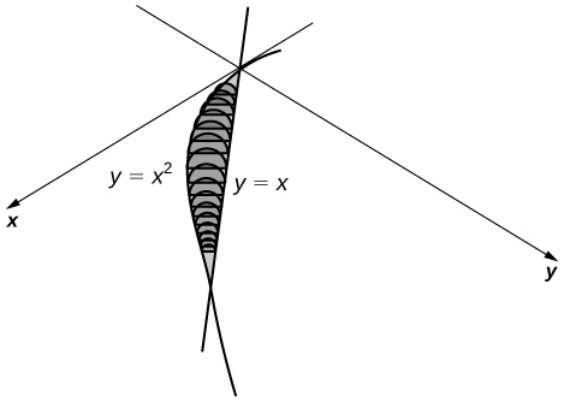


$V = \int_0^1 4(1-y)\,dy \quad = \quad 2 \text{ units}^3$

13) The base is the region enclosed by $y = x^2$ and $y = 9$. Slices perpendicular to the x-axis are right isosceles triangles.

14) The base is the area between $y = x$ and $y = x^2$. Slices perpendicular to the x-axis are semicircles.

Answer



$V = \int_0^1 \frac{\pi}{8}(x-x^2)^2\,dx \quad = \quad \frac{\pi}{240} \text{ units}^3$

For exercises 15 - 16, find the volume of the solid described.

15) The base is the region between $y = x$ and $y = x^2$. Slices perpendicular to the x-axis are semicircles.

16) The base is the region enclosed by the generic ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Slices perpendicular to the x-axis are semicircles.

Answer

$V = \frac{2ab^2\pi}{3} \text{ units}^3$

Contributors

Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

This page titled 2.2E: Exercises is shared under a CC BY-NC 4.0 license and was authored, remixed, and/or curated by Roy Simpson.

2.3: Volumes of Revolution - Cylindrical Shells

Note to the Instructor (click to expand)

This is the typical "cylindrical shells" topic.

To succeed in this section, you'll need to use some skills from previous courses. While you should already know them, this is the first time they've been required. You can review these skills in CRC's Corequisite Codex. If you have a support class, it might cover some, **but not all**, of these topics.

Prerequisite Skills and Support Topics (click to expand)

- **Geometry**
  - Common Volumes: The volume of a cylindrical shell is derived by considering it as a cylinder whose height is the height of the function and whose base is an annulus. The "unrolling" analogy uses the volume of a rectangular prism (a flat plate).

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Calculate the volume of a solid of revolution using the Method of Cylindrical Shells.
- Compare the different methods for calculating the volume of revolution.

Hawk A.I. Section-Specific Tutor



To access the Hawk A.I. Tutor, you will need to be logged into your campus Gmail account.

In this section, we examine the Method of Cylindrical Shells, the final method for finding the volume of a solid of revolution. We can use this method on the same kinds of solids as the Disk Method or the Washer Method; however, with the Disk and Washer Methods, we integrate along the coordinate axis parallel to the axis of revolution. With the Method of Cylindrical Shells, we integrate along the coordinate axis *perpendicular* to the axis of revolution. The ability to choose which variable of integration we want to use can be a significant advantage with more complicated functions. Also, the specific geometry of the solid sometimes makes the method of using cylindrical shells more appealing than using the Washer Method.

The Method of Cylindrical Shells

Again, we are working with a solid of revolution. As before, we define a region $\mathbf{R}$, bounded above by the graph of a function $y = f(x)$, below by the x-axis, and on the left and right by the lines $x = a$ and $x = b$, respectively, as shown in Figure $\mathbf{2.3.1}(a)$. We then revolve this region around the y-axis, as shown in Figure $\mathbf{2.3.1}(b)$. Note that this is different from what we have done before. Previously, regions defined in terms of functions of x were revolved around the x-axis or a line parallel to it.

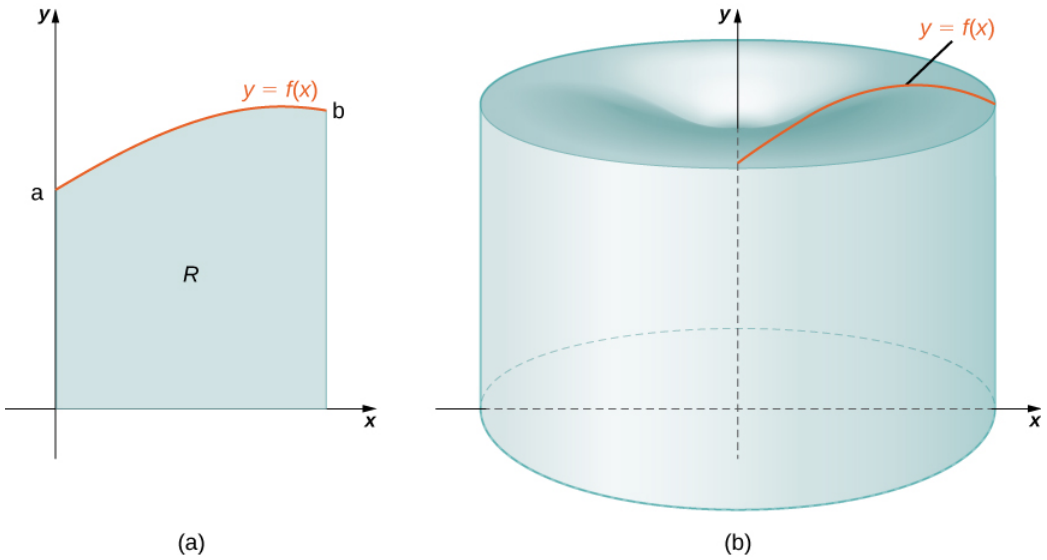


Figure 2.3.1(a): A region bounded by the graph of a function of x.
Figure 2.3.1(b): The solid of revolution formed when the region revolves around the y-axis.

As we have done many times before, partition the interval $[a, b]$ using a regular partition, $P = \{x_0, x_1, \ldots, x_n\}$ and, for $i = 1, 2, \ldots, n$, choose a point $x_i^* \in [x_{i-1}, x_i]$. Then, construct a rectangle over the interval $[x_{i-1}, x_i]$ of height $f(x_i^*)$ and width Δx. A representative rectangle is shown in Figure 2.3.2(a). When that rectangle is revolved around the y-axis, instead of a disk or a washer, we get a **cylindrical shell**, as shown in Figure 2.3.2(b).

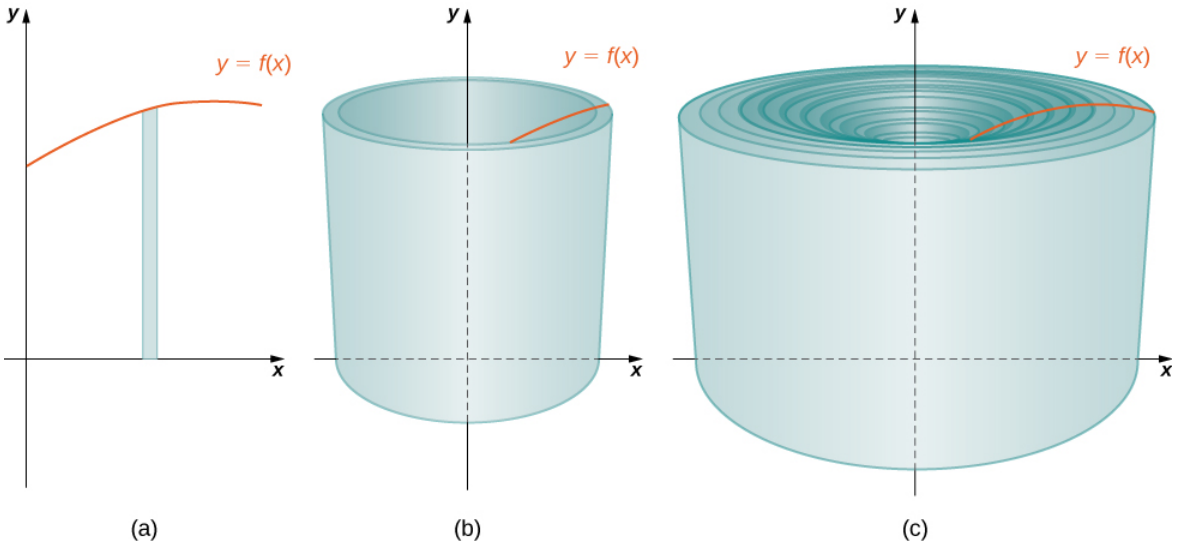


Figure 2.3.2(a): A representative rectangle.
Figure 2.3.2(b): When this rectangle revolves around the y-axis, the result is a cylindrical shell.
Figure 2.3.2(c): When we put all the shells together, we get an approximation of the original solid.

To calculate the volume of this shell, consider Figure 2.3.3.

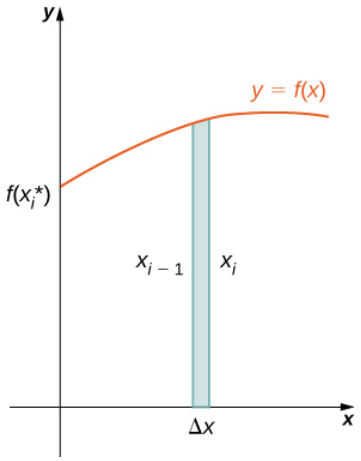


Figure **2.3.3**: Calculating the volume of the shell.

The shell is a cylinder, so its volume is the cross-sectional area multiplied by the height of the cylinder. The cross-sections are **annuli** (ring-shaped regions - essentially, circles with a hole in the center), with outer radius x_i and inner radius x_{i-1}. Thus, the cross-sectional area is $\pi x_i^2 - \pi x_{i-1}^2$. The height of the cylinder is $f(x_i^*)$. Then the volume of the shell is

$$\begin{aligned} V_{\text{shell}} &= f(x_i^*)(\pi x_i^2 - \pi x_{i-1}^2) \\ &= \pi f(x_i^*)(x_i^2 - x_{i-1}^2) \\ &= \pi f(x_i^*)(x_i + x_{i-1})(x_i - x_{i-1}) \\ &= 2\pi f(x_i^*)\left(\frac{x_i + x_{i-1}}{2}\right)(x_i - x_{i-1}). \end{aligned}$$

Note that $x_i - x_{i-1} = \Delta x$, so we have

$$V_{\text{shell}} = 2\pi f(x_i^*)\left(\frac{x_i + x_{i-1}}{2}\right)\Delta x.$$

Furthermore, $\frac{x_i + x_{i-1}}{2}$ is both the midpoint of the interval $[x_{i-1}, x_i]$ and the average radius of the shell. We can approximate this by x_i^*. We then have

$$V_{\text{shell}} \approx 2\pi f(x_i^*)x_i^* \Delta x.$$

Another way to think of this is to think of making a vertical cut in the shell and then opening it up to form a flat plate (Figure **2.3.4**).

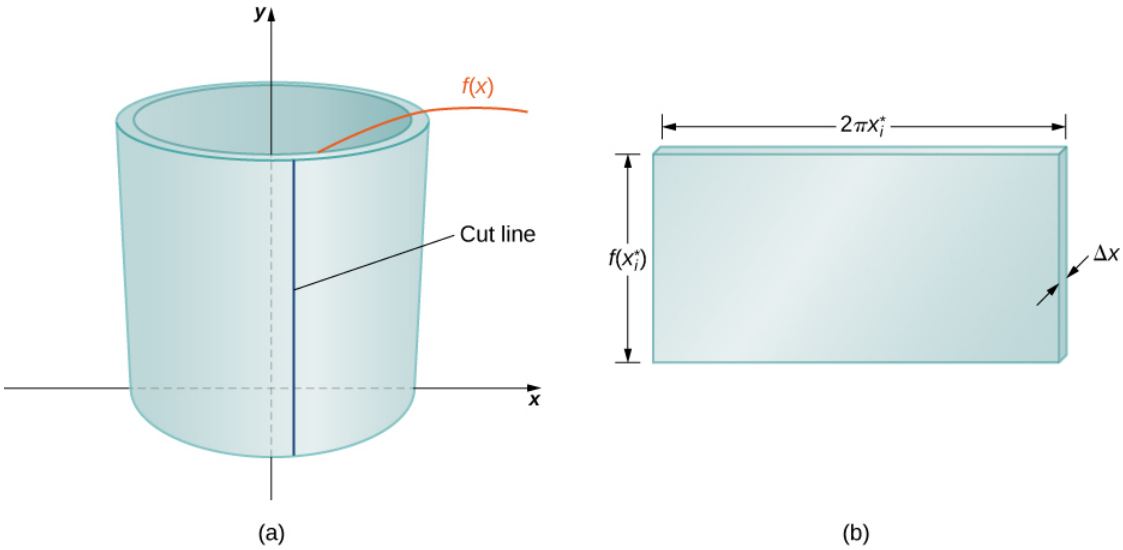


Figure 2.3.4(a): Make a vertical cut in a representative shell.
Figure 2.3.4(b): Open the shell up to form a flat plate.

In reality, the outer radius of the shell is greater than the inner radius. Hence, the back edge of the plate would be slightly longer than the front edge of the plate. However, we can approximate the flattened shell by a flat plate of height $f(x_i^*)$, width $2\pi x_i^*$, and thickness Δx (Figure 2.3.4(b)). Then, the shell's volume is approximately the volume of the flat plate. Multiplying the height, width, and depth of the plate, we get

$$V_{\text{shell}} \approx f(x_i^*)(2\pi x_i^*)\,\Delta x,$$

which is the same formula we had before.

To calculate the volume of the entire solid, we then add the volumes of all the shells and obtain

$$V \approx \sum_{i=1}^{n} (2\pi x_i^* f(x_i^*)\,\Delta x).$$

Here we have another Riemann sum, this time for the function $2\pi x\, f(x)$. Taking the limit as $n \to \infty$ gives us

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} (2\pi x_i^* f(x_i^*)\,\Delta x) = \int_a^b (2\pi x\, f(x))\,dx.$$

This leads to the following rule for the **Method of Cylindrical Shells**.

Theorem: The Method of Cylindrical Shells About the y-axis

Let $f(x)$ be continuous and nonnegative. Define $\mathbf{R}$ as the region bounded above by the graph of $f(x)$, below by the x-axis, on the left by the line $x = a$, and on the right by the line $x = b$. Then the volume of the solid of revolution formed by revolving $\mathbf{R}$ around the y-axis is given by

$$V = \int_a^b (2\pi x\, f(x))\,dx.$$

As stated in the previous section, you want to understand how this works rather than memorizing the formula. We will work hard to showcase the thought process you should follow when using the Method of Cylindrical Shells.

Example 2.3.1

Define $\mathbf{R}$ as the region bounded above by the graph of $f(x) = 1/x$ and below by the x-axis over the interval $[1, 3]$. Find the volume of the solid of revolution formed by revolving $\mathbf{R}$ around the y-axis.

Solution

First, we must graph the region $\mathbf{R}$ and the associated solid of revolution, as shown in Figure 2.3.5. You should also get used to graphing a representative shell (use the CalcPlot3D applet below to visualize this for this example).

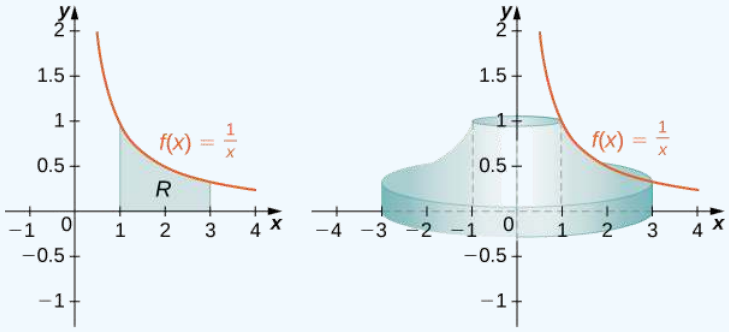


Figure 2.3.5(a): The region $\mathbf{R}$ under the graph of $f(x) = 1/x$ over the interval $[1, 3]$.
Figure 2.3.5(b): The solid of revolution generated by revolving $\mathbf{R}$ about the y-axis.

If we had not been told to use the Method of Cylindrical Shells, we would have a choice to make. If you can visualize horizontal slices being rotated about the y-axis, you would see that the outer radius changes functions at $y = 1/3$. This means that you would need two groups of integrals (one for the bottom set of washers, and one for the washers starting at a height of $y = 1/3$). This is highly inefficient.

On the other hand, choosing to make vertical slices and rotating those about the y-axis shows that the top function of each slice is always $f(x) = 1/x$ and the bottom function is always $y = 0$. Therefore, vertical slicing is a more attractive option.

The volume of the ith slice is given *in words* by

$$V_i = (\text{Circumference of the rotated slice})\,(\text{Height of the rotated slice})\,(\text{Thickness of the rotated slice}).$$

As per our usual approach, we let $r(x_i^*)$ be the radius of rotation. We will also let $h(x_i^*)$ be the height of the slice and naturally select Δx as the thickness. Then our language formula transforms to

$$\begin{array}{rcccc} & & \text{Circumference} & \text{Height} & \text{Thickness} \\ V_i & = & 2\pi r(x_i^*) & h(x_i^*) & \Delta x. \end{array}$$

The radius of rotation is the distance between the slice and the axis of rotation (the y-axis). Since this is a horizontal distance, we measure it as $x_R - x_L$. In this case, $x_R = x_i^*$ and x_L is the y-axis. That is, or $x_L = 0$. The height of the slice is a vertical distance, so this should be $y_T - y_B$. The top of the slice is $f(x_i^*) = \frac{1}{x_i^*}$ and the bottom is $y = 0$. Putting this altogether, we get

$$\begin{array}{rcccc} & & \text{Circumference} & \text{Height} & \text{Thickness} \\ V_i & = & 2\pi r(x_i^*) & h(x_i^*) & \Delta x \\ & = & 2\pi x_i^* & \frac{1}{x_i^*} & \Delta x. \end{array}$$

Hence, the true volume is

$$\begin{aligned} V &= \int_{x=1}^{x=3} \left(2\pi x \left(\frac{1}{x}\right)\right) dx \\ &= \int_{x=1}^{x=3} 2\pi \, dx \\ &= 2\pi x \Big|_{x=1}^{x=3} \\ &= 4\pi \text{ units}^3. \end{aligned}$$

? Checkpoint 2.3.1

Define $\mathbf{R}$ as the region bounded above by the graph of $f(x) = x^2$ and below by the x-axis over the interval $[1, 2]$. Find the volume of the solid of revolution formed by revolving $\mathbf{R}$ around the y-axis.

Answer

$\frac{15\pi}{2}$ units^3

✔ Example 2.3.2

Define $\mathbf{R}$ as the region bounded above by the graph of $f(x) = 2x - x^2$ and below by the x-axis over the interval $[0, 2]$. Find the volume of the solid of revolution formed by revolving $\mathbf{R}$ around the y-axis.

Solution

First graph the region $\mathbf{R}$ and the associated solid of revolution, as shown in Figure 2.3.6.

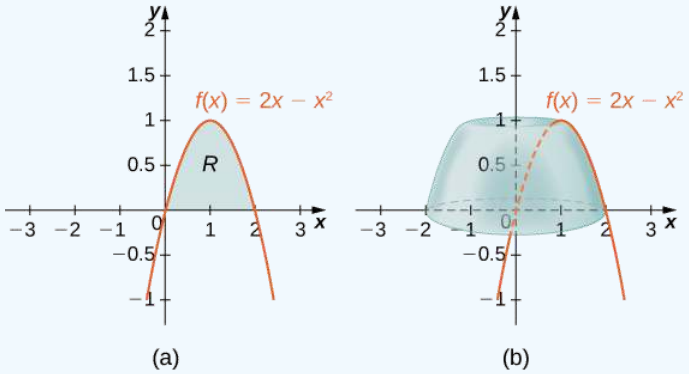


Figure 2.3.6(a): The region $\mathbf{R}$ under the graph of $f(x) = 2x - x^2$ over the interval $[0, 2]$.
Figure 2.3.6(b): The volume of revolution obtained by revolving $\mathbf{R}$ about the y-axis.

If we chose horizontal slices, we would get washers; however, each washer's outer and inner edges would be described by the same function. While we *could* find a way to get the volume of this washer, there are more efficient methods than this one. Instead, let's try slicing vertically.

Figure 2.3.6 shows that vertical slices would result in cylindrical shells. From Example 2.3.1, we know the volume of the ith such shell would be

$$\begin{array}{rcccc} & & \textbf{Circumference} & \textbf{Height} & \textbf{Thickness} \\ V_i & = & 2\pi r(x_i^*) & h(x_i^*) & \Delta x, \end{array}$$

where $r(x_i^*) = x_i^*$ and $h(x_i^*) = 2x_i^* - (x_i^*)^2$. Hence,

$$\begin{array}{rcccc} & & \textbf{Circumference} & \textbf{Height} & \textbf{Thickness} \\ V_i & = & 2\pi r(x_i^*) & h(x_i^*) & \Delta x \\ & = & 2\pi x_i^* & (2x_i^* - (x_i^*)^2) & \Delta x. \end{array}$$

Thus,

$$\begin{aligned} V &= \int_0^2 (2\pi x(2x - x^2))\, dx \\ &= 2\pi \int_0^2 (2x^2 - x^3)\, dx \\ &= 2\pi \left[\frac{2x^3}{3} - \frac{x^4}{4} \right] \Big|_0^2 \\ &= \frac{8\pi}{3} \text{ units}^3 \end{aligned}$$

? Checkpoint 2.3.2

Define $\mathbf{R}$ as the region bounded above by the graph of $f(x) = 3x - x^2$ and below by the x-axis over the interval $[0, 2]$. Find the volume of the solid of revolution formed by revolving $\mathbf{R}$ around the y-axis.

Answer

8π units^3

As with the Disk Method and the Washer Method, we can use the Method of Cylindrical Shells with solids of revolution, revolved around the x-axis, when we want to integrate with respect to y. The analogous rule for this type of solid is given here. To be clear, you should not memorize this formula without truly understanding how to derive it yourself. Committing this formula to memory is worthless if you genuinely understand the derivation of the process.

Theorem: The Method of Cylindrical Shells About the x-axis

Let $g(y)$ be continuous and nonnegative. Define $\mathbf{Q}$ as the region bounded on the right by the graph of $g(y)$, on the left by the y-axis, below by the line $y = c$, and above by the line $y = d$. Then, the volume of the solid of revolution formed by revolving $\mathbf{Q}$ around the x-axis is given by

$$V = \int_c^d (2\pi y\, g(y))\, dy.$$

✓ Example 2.3.3

Define $\mathbf{Q}$ as the region bounded on the right by the graph of $g(y) = 2\sqrt{y}$ and on the left by the y-axis for $y \in [0, 4]$. Find the volume of the solid of revolution formed by revolving $\mathbf{Q}$ around the x-axis.

Solution

First, we need to graph the region $\mathbf{Q}$ and the associated solid of revolution, as shown in Figure 2.3.7.

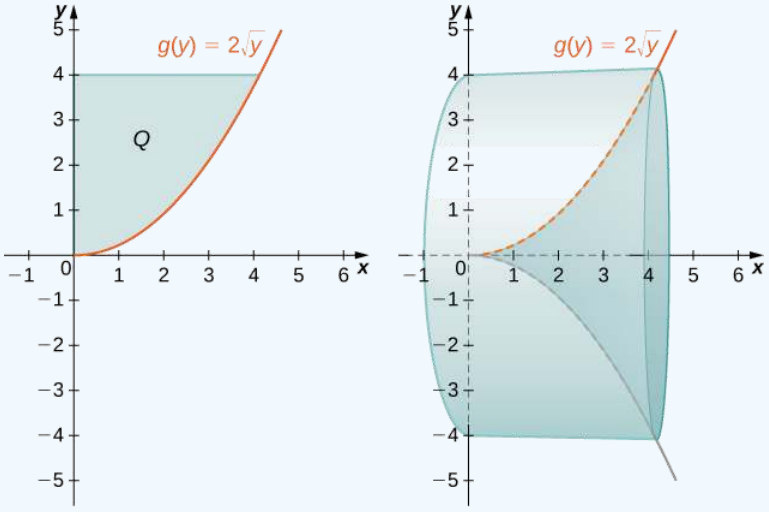


Figure 2.3.7(a): The region $\mathbf{Q}$ to the left of the function $g(y)$ over the interval $[0, 4]$.
Figure 2.3.7(b): The solid of revolution generated by revolving $\mathbf{Q}$ around the x-axis.

This is an excellent example of how we could easily use *either* the Washer Method or the Method of Cylindrical Shells. It's advisable to set up both integrals for practice and to see which one looks easier to evaluate.

VERTICAL SLICES: If we choose vertical slices, we will get washers, which implies the Washer Method. Moreover, each washer will have width Δx. This informs us that all of our work should eventually be in terms of x. Let's state the required information first.

$$r_O(x_i^*) = y_T - y_B = 4 - 0 = 4.$$

Finding $r_I(x_i^*)$ requires us to solve $x = 2\sqrt{y}$ for y. This gives $\frac{x^2}{4} = y$. Therefore,

$$r_I(x_i^*) = y_T - y_B = \frac{x^2}{4} - 0 = \frac{x^2}{4}.$$

The volume of the ith slice is

$$\begin{aligned} V_i &= \pi \left[r_O(x_i^*)\right]^2 \Delta x - \pi \left[r_I(x_i^*)\right]^2 \Delta x \\ &= \pi \left(16 - \frac{x^4}{16}\right) \Delta x \end{aligned}$$

Thus,

$$V = \pi \int_{x=0}^{x=4} 16 - \frac{x^4}{16}\, dx.$$

HORIZONTAL SLICES: If, on the other hand, we decide on horizontal slices, the rotation will result in shells. Hence, we will use the Method of Cylindrical Shells. The thickness of each shell will be Δy. This informs us that all of our work should eventually be only in terms of y.

The radius of rotation is y_i^* and the "height" of each shell is $x_R - x_L = 2\sqrt{y_i^*} - 0 = 2\sqrt{y_i^*}$. Therefore, the volume of the ith slice is

		Circumference	Height	Thickness
V_i	$=$	$2\pi r(y_i^*)$	$h(y_i^*)$	Δy
	$=$	$2\pi y_i^*$	$\left(2\sqrt{y_i^*}\right)$	$\Delta y.$

Then the volume of the solid is given by

$$V = 4\pi \int_{y=0}^{y=4} y^{3/2} dy.$$

You be the judge of which integral looks nicer. While they will both yield the same result, I am choosing the second because... well, it's nicer.

$$\begin{aligned} V &= 4\pi \int_{y=0}^{y=4} y^{3/2}\, dy \\ &= 4\pi \left[\frac{2y^{5/2}}{5} \right] \Big|_0^4 \\ &= \frac{256\pi}{5} \text{ units}^3 \end{aligned}$$

Example **2.3.3** showcases two fundamental concepts. First, you should refrain from getting married to a method. That is, always be willing to try *both* horizontal *and* vertical slices. This takes little time once you get used to things and can make an impossible problem simple. The second important concept is based on notation. If you look back through Example **2.3.3**, you will notice that I wrote the limits of integration as $x = 0$ or $y = 0$, and $x = 4$ or $y = 4$, accordingly. While the limits didn't change in this example, they often are not the same, and writing $x =$ or $y =$ will keep your work "honest" and remind you that, yes, you already changed those limits into the proper variable.

? Checkpoint 2.3.3

Define $\mathbf{Q}$ as the region bounded on the right by the graph of $g(y) = 3/y$ and on the left by the y-axis for $y \in [1, 3]$. Find the volume of the solid of revolution formed by revolving $\mathbf{Q}$ around the x-axis.

Answer

12π units^3

✓ Example 2.3.4

Define $\mathbf{R}$ as the region bounded above by the graph of $f(x) = x$ and below by the x-axis over the interval $[1, 2]$. Find the volume of the solid of revolution formed by revolving $\mathbf{R}$ around the line $x = -1$.

Solution

First, graph the region $\mathbf{R}$ and the associated solid of revolution, as shown in Figure **2.3.8**.

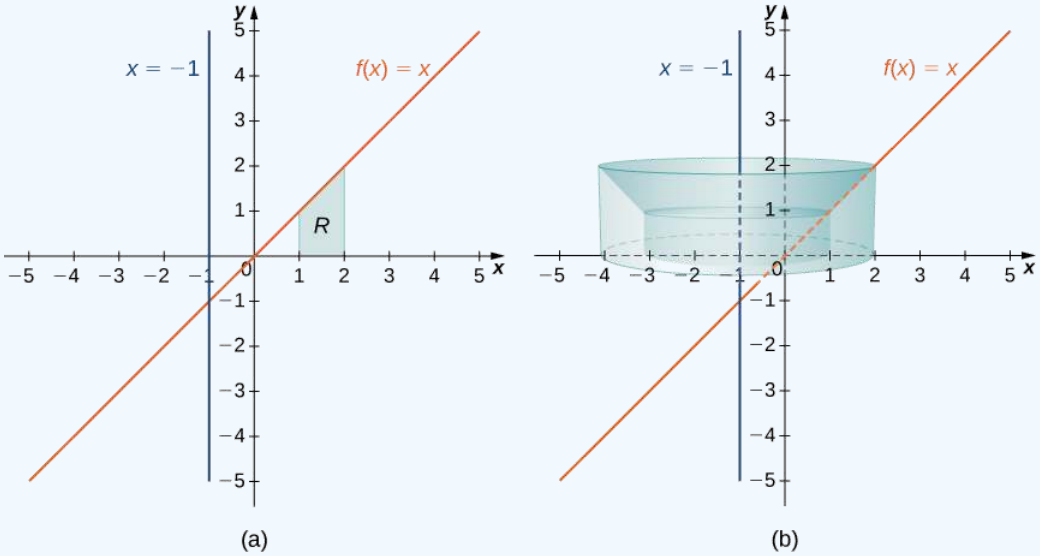


Figure 2.3.8(a): The region $\mathbf{R}$ between the graph of $f(x)$ and the x-axis over the interval $[1, 2]$.
Figure 2.3.8(b): The solid of revolution generated by revolving $\mathbf{R}$ around the line $x = -1$.

HORIZONTAL SLICES: If we chose to slice $\mathbf{R}$ into horizontal slices, we can easily see that we would need two sets of computations - one for the region where the left edge is $x = 1$ and the right edge is $x = 2$, and one for the region where the left edge is $f(x) = x$ and the right edge is $x = 2$. This should motivate us to try vertical slices.

VERTICAL SLICES: Slicing the region $\mathbf{R}$ into vertical slices means that each has a width of Δx, and so all of our work needs to be in terms of x. Moreover, a rotation about the vertical line $x = -1$ means we are creating shells. The radius of rotation to the ith slice is $x_R - x_L = x_i^* - (-1) = x_i^* + 1$. The height of the slice is $h(x_i^*) = y_T - y_B = x_i^* - 0 = x_i^*$. Therefore, the volume of the ith slice is

$$V_i = 2\pi r(x_i^*)h(x_i^*)\Delta x = 2\pi(x_i^* + 1)x_i^*\Delta x.$$

Thus, the volume of the solid is given by

$$\begin{aligned} V &= 2\pi \int_{x=1}^{x=2} x^2 + x\, dx \\ &= 2\pi \left[\frac{x^3}{3} + \frac{x^2}{2}\right]\Big|_1^2 \\ &= \frac{23\pi}{3} \text{ units}^3 \end{aligned}$$

? Checkpoint 2.3.4

Define $\mathbf{R}$ as the region bounded above by the graph of $f(x) = x^2$ and below by the x-axis over the interval $[0, 1]$. Find the volume of the solid of revolution formed by revolving $\mathbf{R}$ around the line $x = -2$.

Answer

$\frac{11\pi}{6}$ units^3

For our final example in this section, let's look at the volume of a solid of revolution for which the graphs of two functions bound the region of revolution.

Example 2.3.5

Define $\mathbf{R}$ as the region bounded above by the graph of the function $f(x) = \sqrt{x}$, below by the graph of the function $g(x) = 1/x$, and on the right by $x = 4$. Find the volume of the solid of revolution generated by revolving $\mathbf{R}$ around the y-axis.

Solution

First, graph the region $\mathbf{R}$ and the associated solid of revolution, as shown in Figure 2.3.9. During this process, you will need to find the point of intersection of these two curves, which is $(1, 1)$.

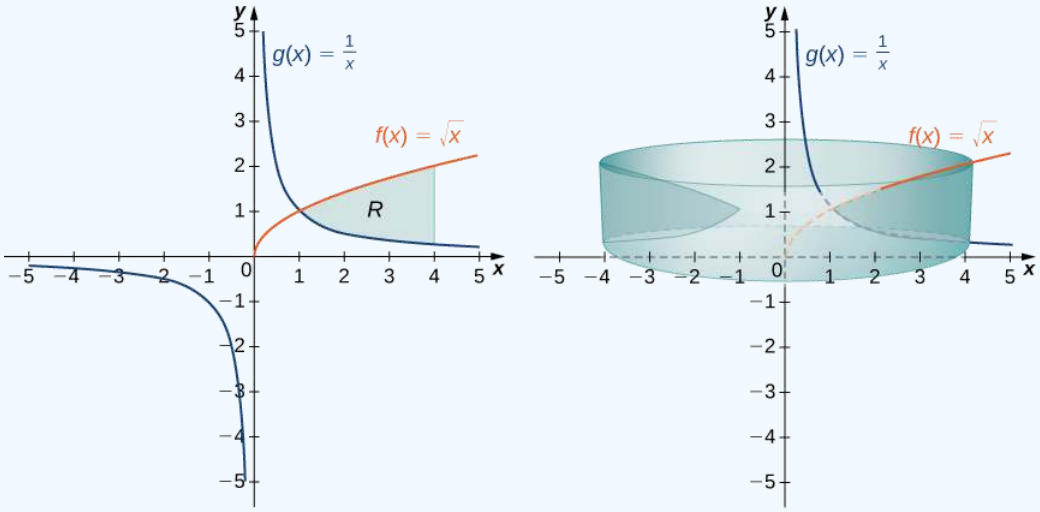


Figure 2.3.9(a): The region $\mathbf{R}$ between the graph of $f(x)$ and the graph of $g(x)$ over the interval $[1, 4]$.
Figure 2.3.9(b): The solid of revolution generated by revolving $\mathbf{R}$ around the y-axis.

HORIZONTAL OR VERTICAL SLICES? Since we are rotating about the y-axis, a quick inspection reveals that horizontal slices yield two separate regions (one below $y = 1$ and one above $y = 1$), so this is likely not the best choice.

Now that we know to slice vertically, we also know that each slice has width Δx. Again, this means all of our eventual work must be in terms of x. Moreover, vertical slices rotated about the y-axis leads to shells. Hence, we are using the Method of Cylindrical Shells. The radius of rotation for the ith shell is $x_R - x_L = x_i^* - 0 = x_i^*$. The height of this slice is $h(x_i^*) = y_T - y_B = \sqrt{x_i^*} - \frac{1}{x_i^*}$. Combining this information, we setup the volume of the ith slice to be

$$V_i = 2\pi r(x_i^*)h(x_i^*)\Delta x = 2\pi x_i^* \left(\sqrt{x_i^*} - \frac{1}{x_i^*}\right)\Delta x.$$

Then the volume of the solid is given by

$$\begin{aligned} V &= \int_{x=1}^{x=4} \left(2\pi x \left(\sqrt{x} - \frac{1}{x}\right)\right) dx \\ &= 2\pi \int_{x=1}^{x=4} (x^{3/2} - 1)\, dx \\ &= 2\pi \left[\frac{2x^{5/2}}{5} - x\right]\Big|_1^4 \\ &= \frac{94\pi}{5} \text{ units}^3. \end{aligned}$$

Checkpoint 2.3.5

Define $\mathbf{R}$ as the region bounded above by the graph of $f(x) = x$ and below by the graph of $g(x) = x^2$ over the interval $[0, 1]$. Find the volume of the solid of revolution formed by revolving $\mathbf{R}$ around the y-axis.

Answer

$\frac{\pi}{6}$ units^3

2.3E: Exercises

Reading Questions

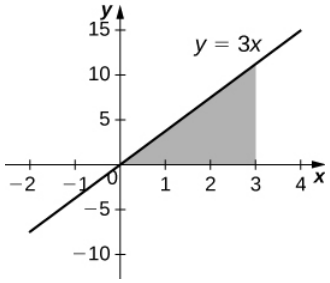
1. What is the primary difference between the Method of Cylindrical Shells and the Disk/Washer methods regarding the variable of integration?
2. When a vertical rectangle is revolved around the y-axis, what is the resulting three-dimensional shape called?
3. The volume of a single cylindrical shell can be approximated by "unrolling" it into what simple geometric shape?
4. What are the three components that are multiplied together to find the approximate volume of a single cylindrical shell?
5. When revolving a region bounded by $y = f(x)$ around the y-axis, how are the shell's radius and height typically expressed in terms of x and y?
6. In Example 1.4.4, the region is revolved around the vertical line $x = -1$. How is the radius of rotation for a slice at position x calculated?

For exercises 1 - 6, find the volume generated when the region between the two curves is rotated around the given axis. Use both the shell method and the Washer Method. Use technology to graph the functions and draw a typical slice by hand.

1) [Technology Required] Over the curve of $y = 3x$, $x = 0$, and $y = 3$ rotated around the y-axis.

2) [Technology Required] Under the curve of $y = 3x$, $x = 0$, and $x = 3$ rotated around the y-axis.

Answer

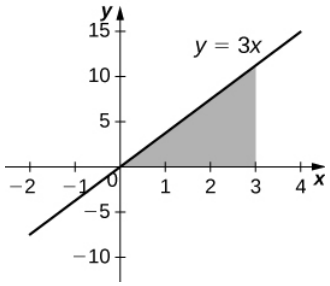


$V = 54\pi$ units^3

3) [Technology Required] Over the curve of $y = 3x$, $x = 0$, and $y = 3$ rotated around the x-axis.

4) [Technology Required] Under the curve of $y = 3x$, $x = 0$, and $x = 3$ rotated around the x-axis.

Answer

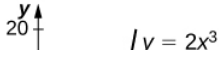


$V = 81\pi$ units^3

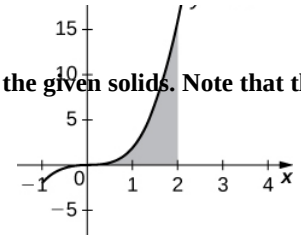
5) [Technology Required] Under the curve of $y = 2x^3$, $x = 0$, and $x = 2$ rotated around the y-axis.

6) [Technology Required] Under the curve of $y = 2x^3$, $x = 0$, and $x = 2$ rotated around the x-axis.

Answer



$V = \frac{512\pi}{7}$ units^3



For exercises 7 - 16, use shells to find the volumes of the given solids. Note that the rotated regions lie between the curve and the x-axis and are rotated around the y-axis.

7) $y = 1 - x^2$, $x = 0$, and $x = 1$

8) $y = 5x^3$, $x = 0$, and $x = 1$

Answer

$V = 2\pi$ units^3

9) $y = \frac{1}{x}$, $x = 1$, and $x = 100$

10) $y = \sqrt{1 - x^2}$, $x = 0$, and $x = 1$

Answer

$V = \frac{2\pi}{3}$ units^3

11) $y = \frac{1}{1 + x^2}$, $x = 0$,and $x = 3$

12) $y = \sin x^2, x = 0$, and $x = \sqrt{\pi}$

Answer

$V = 2\pi$ units^3

13) $y = \frac{1}{\sqrt{1 - x^2}}$, $x = 0$, and $x = \frac{1}{2}$

14) $y = \sqrt{x}$, $x = 0$, and $x = 1$

Answer

$V = \frac{4\pi}{5}$ units^3

15) $y = (1 + x^2)^3$, $x = 0$, and $x = 1$

16) $y = 5x^3 - 2x^4$, $x = 0$, and $x = 2$

Answer

$V = \frac{64\pi}{3}$ units^3

For exercises 17 - 26, use shells to find the volume generated by rotating the regions between the given curve and $y = 0$ around the x-axis.

17) $y = \sqrt{1 - x^2}$, $x = 0$, and $x = 1$

18) $y = x^2$, $x = 0$, and $x = 2$

Answer

$V = \frac{32\pi}{5}$ units^3

19) $y = e^x$, $x = 0$, and $x = 1$

20) $y = \ln(x)$, $x = 1$, and $x = e$

Answer

$V = \pi(e - 2)$ units^3

21) $x = \frac{1}{1 + y^2}$, $y = 1$, and $y = 4$

22) $x = \dfrac{1+y^2}{y}$, $y = 0$, and $y = 2$

Answer

$V = \frac{28\pi}{3}$ units^3

23) $x = \cos y$, $y = 0$, and $y = \pi$

24) $x = y^3 - 4y^2$, $x = -1$, and $x = 2$

Answer

$V = \frac{84\pi}{5}$ units^3

25) $x = ye^y$, $x = -1$, and $x = 2$

26) $x = e^y \cos y$, $x = 0$, and $x = \pi$

Answer

$V = e^\pi \pi^2$ units^3

For exercises 27 - 36, find the volume generated when the region between the curves is rotated around the given axis.

27) $y = 3 - x$, $y = 0$, $x = 0$, and $x = 2$ rotated around the y-axis.

28) $y = x^3$, $y = 0$, $x = 0$, and $y = 8$ rotated around the y-axis.

Answer

$V = \frac{64\pi}{5}$ units^3

29) $y = x^2$, $y = x$, rotated around the y-axis.

30) $y = \sqrt{x}$, $x = 0$, and $x = 1$ rotated around the line $x = 2$.

Answer

$V = \frac{28\pi}{15}$ units^3

31) $y = \dfrac{1}{4-x}$, $x = 1$, and $x = 2$ rotated around the line $x = 4$.

32) $y = \sqrt{x}$ and $y = x^2$ rotated around the y-axis.

Answer

$V = \frac{3\pi}{10}$ units^3

33) $y = \sqrt{x}$ and $y = x^2$ rotated around the line $x = 2$.

34) $x = y^3$, $y = \dfrac{1}{x}$, $x = 1$, and $y = 2$ rotated around the x-axis.

Answer

$\frac{52\pi}{5}$ units^3

35) $x = y^2$ and $y = x$ rotated around the line $y = 2$.

36) [Technology Required] Left of $x = \sin(\pi y)$, right of $y = x$, around the y-axis.

Answer

$V \approx 0.9876$ units^3

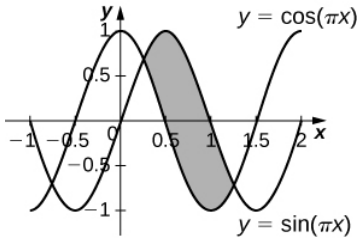
For exercises 37 - 44, use technology to graph the region. Determine which method you think would be easiest to use to calculate the volume generated when the function is rotated around the specified axis. Then, use your chosen method to find

the volume.

37) [Technology Required] $y = x^2$ and $y = 4x$ rotated around the y-axis.

38) [Technology Required] $y = \cos(\pi x), y = \sin(\pi x), x = \frac{1}{4}$, and $x = \frac{5}{4}$ rotated around the y-axis.

Answer

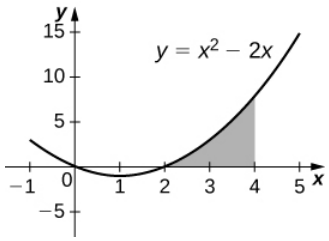


$V = 3\sqrt{2}$ units^3

39) [Technology Required] $y = x^2 - 2x,\ x = 2$, and $x = 4$ rotated around the y-axis.

40) [Technology Required] $y = x^2 - 2x,\ x = 2$, and $x = 4$ rotated around the x-axis.

Answer

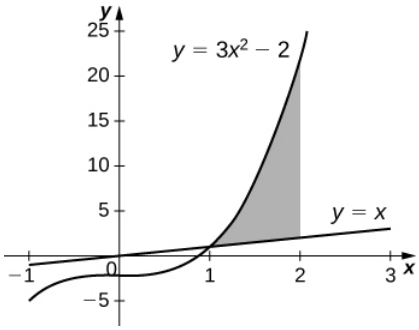


$V = \frac{496\pi}{15}$ units^3

41) [Technology Required] $y = 3x^3 - 2,\ y = x$, and $x = 2$ rotated around the x-axis.

42) [Technology Required] $y = 3x^3 - 2,\ y = x$, and $x = 2$ rotated around the y-axis.

Answer

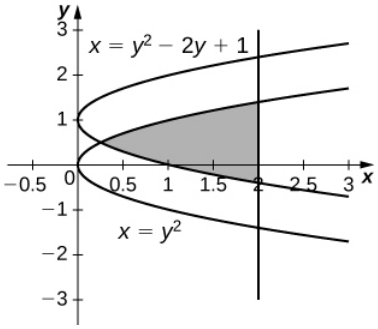


$V = \frac{398\pi}{15}$ units^3

43) [Technology Required] $x = \sin(\pi y^2)$ and $x = \sqrt{2}y$ rotated around the x-axis.

44) [Technology Required] $x = y^2,\ x = y^2 - 2y + 1$, and $x = 2$ rotated around the y-axis.

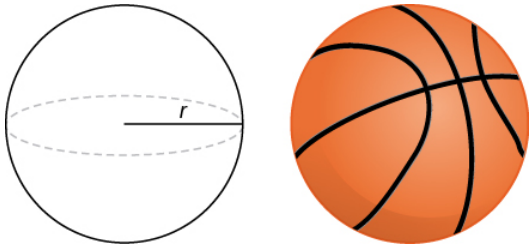
Answer



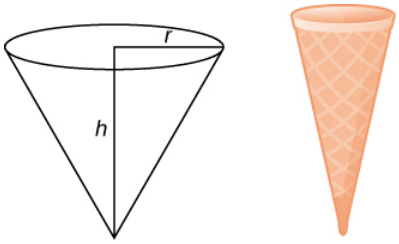
$V = 15.9074$ units^3

For exercises 45 - 51, use the method of shells to approximate the volumes of some common objects, which are pictured in accompanying figures.

45) Use the method of shells to find the volume of a sphere of radius r.



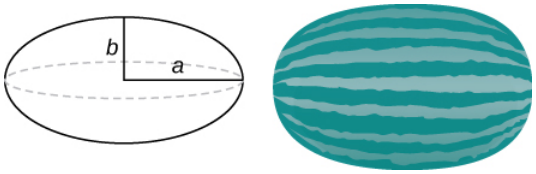
46) Use the method of shells to find the volume of a cone with radius r and height h.



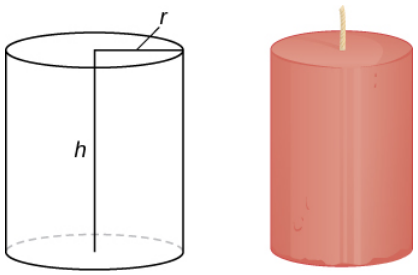
Answer

$V = \frac{1}{3}\pi r^2 h$ units^3

47) Use the method of shells to find the volume of an ellipse $(x^2/a^2) + (y^2/b^2) = 1$ rotated around the x-axis.



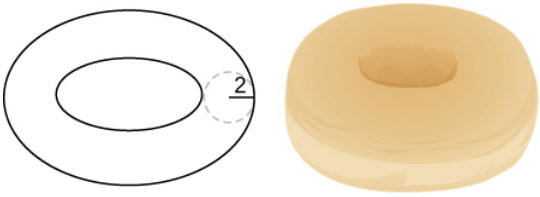
48) Use the method of shells to find the volume of a cylinder with radius r and height h.



Answer

$V = \pi r^2 h$ units^3

49) Use the method of shells to find the volume of the donut created when the circle $x^2 + y^2 = 4$ is rotated around the line $x = 4$.



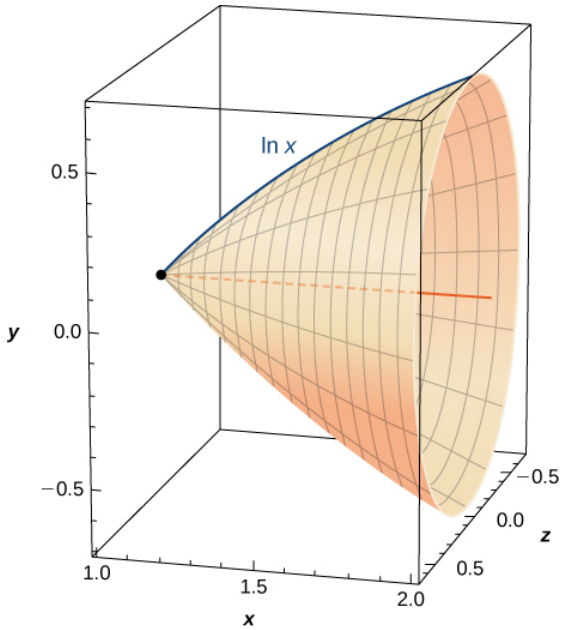
50) Consider the region enclosed by the graphs of $y = f(x)$, $y = 1 + f(x)$, $x = 0$, $y = 0$, and $x = a > 0$. What is the volume of the solid generated when this region is rotated around the y-axis? Assume that the function is defined over the interval $[0, a]$.

Answer

$V = \pi a^2$ units^3

51) Consider the function $y = f(x)$, which decreases from $f(0) = b$ to $f(1) = 0$. Set up the integrals for determining the volume, using both the shell method and the Disk Method, of the solid generated when this region, with $x = 0$ and $y = 0$, is rotated around the y-axis. Prove that both methods approximate the same volume. Which method is easier to apply? (Hint: Since $f(x)$ is one-to-one, there exists an inverse $f^{-1}(y)$.)

52) Find the volume of the shape created when rotating this curve from $x = 1$ to $x = 2$ around the x-axis, as pictured here.



Contributors

Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

This page titled 2.3E: Exercises is shared under a CC BY-NC 4.0 license and was authored, remixed, and/or curated by Roy Simpson.

- **6.3E: Exercises for Section 6.3** by Edwin “Jed” Herman, Gilbert Strang is licensed CC BY-NC-SA 4.0. Original source: https://openstax.org/details/books/calculus-volume-1.
- **6.7E: Exercises for Section 6.7** by Edwin “Jed” Herman, Gilbert Strang is licensed CC BY-NC-SA 4.0. Original source: https://openstax.org/details/books/calculus-volume-1.

2.4: Arc Length of a Curve and Surface Area

Note to the Instructor (click to expand)

The title of this section pretty much sums it up. If you have taught out of the Stewart text, you will know this material occurs *much* earlier in this text. This is because it flows naturally from the previous topics. Any integrals requiring special (read as, "untaught") integration techniques are avoided. If you choose to skip this and teach it with parametrics and polars (as in the Stewart text), be warned - the homework in the integration techniques chapter frequently uses this material.

To succeed in this section, you'll need to use some skills from previous courses. While you should already know them, this is the first time they've been required. You can review these skills in CRC's Corequisite Codex. If you have a support class, it might cover some, **but not all**, of these topics.

Prerequisite Skills and Support Topics (click to expand)

The following is a list of prerequisite skills (all of which can be reviewed in CRC's Corequisite Codex) needed for this section **that have not already been mentioned in any previous section**. If you are enrolled in a course with a Support section, some (but definitely not all) of these topics might be covered or reviewed in the Support section of your course.

- **Simplifying Expressions**
  - Simplifying Radical Expressions: The integrands for both arc length and surface area involve a square root, $\sqrt{1+[f'(x)]^2}$. Simplifying the expression inside the radical is a critical step.
- **An Overview of Functions**
  - The Difference Quotient: The derivation of the arc length formula uses the expression $\frac{\Delta y_i}{\Delta x}$, which is the difference quotient over a small interval, and relates it to the derivative $f'(x_i^*)$ via the Mean Value Theorem.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Determine the length of a curve, $y=f(x)$, between two points.
- Determine the length of a curve, $x=g(y)$, between two points.
- Find the surface area of a solid of revolution.

Hawk A.I. Section-Specific Tutor



To access the Hawk A.I. Tutor, you will need to be logged into your campus Gmail account.

In this section, we use definite integrals to find the arc length of a curve. We can think of arc length as the distance you would travel if you walked along the curve's path. Many real-world applications involve arc length. If a rocket is launched along a

parabolic path, we want to know how far the rocket travels. Or, if a curve on a map represents a road, we might want to know how far we have to drive to reach our destination.

We begin by calculating the arc length of curves defined as functions of x. We examine the same process for curves defined as functions of y.[1] The techniques we use to find arc length can be extended to find the surface area of a surface of revolution. We close the section by examining this concept.

Arc Length of the Curve $y = f(x)$

In previous applications of integration, we required the function $f(x)$ to be integrable, or at most continuous. However, for calculating arc length, we have a more stringent requirement for $f(x)$. Here, we require $f(x)$ to be differentiable, and we require its derivative, $f'(x)$, to be continuous. Functions like this, which have continuous derivatives, are called **continuously differentiable**.[2]

Let $f(x)$ be a continuously differentiable function defined over $[a, b]$. We want to calculate the length, L, of the curve from the point $(a, f(a))$ to the point $(b, f(b))$. We start by using line segments to approximate the length of the curve. For $i = 0, 1, 2, \ldots, n$, let $P = \{x_i\}$ be a regular partition of the interval $[a, b]$. Then, for $i = 1, 2, \ldots, n$, construct a line segment from the point $(x_{i-1}, f(x_{i-1}))$ to the point $(x_i, f(x_i))$. Although it might seem logical to use either horizontal or vertical line segments, we want our line segments to approximate the curve as closely as possible. Figure 2.4.1 depicts this construct for $n = 5$.

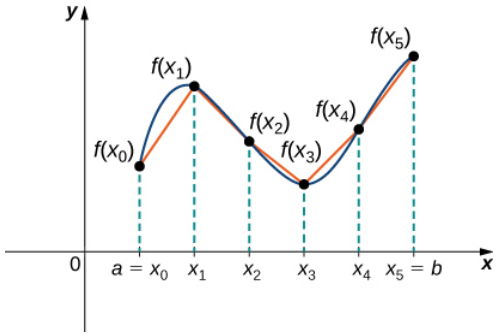


Figure 2.4.1: We can approximate the length of a curve by adding line segments.

To help us find the length of each line segment, we look at the change in vertical distance as well as the change in horizontal distance over each interval. Because we have used a regular partition, the change in horizontal distance over each interval is given by Δx. The change in vertical distance varies from interval to interval, though, so we use

$$\Delta y_i = f(x_i) - f(x_{i-1})$$

to represent the change in vertical distance over the interval $[x_{i-1}, x_i]$, as shown in Figure 2.4.2.[3] We will denote the length of the line segment from the point $(x_{i-1}, f(x_{i-1}))$ to the point $(x_i, f(x_i))$ as l_i.

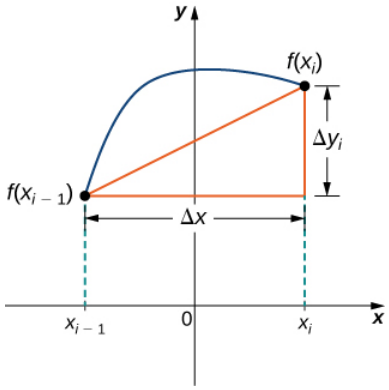


Figure 2.4.2: A representative line segment approximates the curve over the interval $[x_{i-1}, x_i]$.

By the Pythagorean Theorem, the length of the ith line segment is

$$\begin{aligned} l_i &= \sqrt{(\Delta x)^2 + (\Delta y_i)^2} \\ &= \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2}\Delta x. \end{aligned}$$

Moreover, by the Mean Value Theorem for Derivatives, there is a point $x_i^* \in [x_{i-1}, x_i]$ such that $f'(x_i^*) = \frac{\Delta y_i}{\Delta x}$. Therefore, we can rewrite the last formula as

$$l_i = \sqrt{1 + [f'(x_i^*)]^2}\Delta x.$$

Adding up the lengths of all the line segments, we get

$$L \equiv \text{Arc Length} \approx \sum_{i=1}^{n} l_i = \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\Delta x.$$

This is a Riemann sum. Taking the limit as $n \to \infty$, we have

$$\begin{aligned} L &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\Delta x \\ &= \int_{x=a}^{x=b} \sqrt{1 + [f'(x)]^2}\, dx. \end{aligned}$$

Before we formalize all the work up to this point into a theorem, it is useful to create an **arc length function**, $s(x)$, that allows us to find the length of the continuously differentiable curve f from the constant point $(a, f(a))$ to the arbitrary point $(x, f(x))$. Using our knowledge of integral functions, we determine that

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt,$$

is the arc length function.[4] From the Fundamental Theorem of Calculus, Part 1 (FTC1) and our previous work with differentials,

$$ds = s'(x)dx \implies ds = \sqrt{1 + [f'(x)]^2}\, dx.$$

We will discover throughout the remainder of this course that the differential we just found, ds, has four equivalent forms. The choice of forms will depend on the structure of the curve with which we work; however, in all forms, the *meaning* of this differential is the same.

> *ds is the infinitesimal change in arc length given an infinitesimal change in some other value.*

In our current case, ds is the infinitesimal change in arc length given an infinitesimal change in x.

We now have enough mathematical language to streamline our discussion of arc length (and other future topics). We summarize these findings in the following theorem.

Theorem: Arc Length of a Curve (with respect to x)

Let $f(x)$ be a continuously differentiable function over the interval $[a, b]$. Then the arc length, L, of the portion of the graph of $f(x)$ from the point $(a, f(a))$ to the point $(b, f(b))$ is given by

$$L = \int_{x=a}^{x=b} ds, \tag{2.4.1}$$

where[5]

$$ds = \sqrt{1 + [f'(x)]^2}\, dx. \tag{2.4.2}$$

A few things need to be pointed out about this previous theorem. First, the limits of the integral in Equation 2.4.1 are in terms of x - not s. Therefore, evaluating Equation 2.4.1 as written is "mathematically illegal." To evaluate a definite integral, the limits of integration must be in terms of the variable of integration. Luckily, substituting in Equation 2.4.2 allows us to evaluate Equation 2.4.1 "legally."

This brings us to our next point. We did all that work to get ds, but why not just say, "The arc length of f over $[a, b]$ is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx?"$$

As was mentioned before this theorem, ds will eventually have four equivalent (and often interchangeable) forms. Throughout these forms, the basic structure of the arc length integral, $L = \int ds$, will not change - we will only need to worry about changing the limits of the integral and swapping out ds for an appropriate form.

Finally, we are integrating an expression involving $f'(x)$, so we need to be sure $f'(x)$ is integrable. This is why we require $f(x)$ to be continuously differentiable.

The following example shows how to apply the theorem.

Example 2.4.1: Calculating the Arc Length of a Function of x

Let $f(x) = 2x^{3/2}$. Calculate the arc length of the graph of $f(x)$ over the interval $[0, 1]$.

Solution

To get us into a "good habit," let's start by recognizing that f is a function of x.[6]

We have $f'(x) = 3x^{1/2}$, so $[f'(x)]^2 = 9x$. Therefore,

$$ds = \sqrt{1 + [f'(x)]^2}\, dx = \sqrt{1 + 9x}\, dx.$$

Then, the arc length is

$$\begin{aligned} L &= \int_{x=a}^{x=b} ds && \text{(arc length formula)} \\ &= \int_{x=0}^{x=1} \sqrt{1 + 9x}\, dx \\ &= \frac{1}{9}\int_{u=1}^{u=10} \sqrt{u}\, du && \text{(Let } u = 1 + 9x \implies du = 9\, dx\text{, and change the limits of integration accordingly)} \\ &= \frac{1}{9} \cdot \frac{2}{3} u^{3/2}\Big|_1^{10} \\ &= \frac{2}{27}[10\sqrt{10} - 1] \end{aligned}$$

Checkpoint 2.4.1

Let $f(x) = (4/3)x^{3/2}$. Calculate the arc length of the graph of $f(x)$ over the interval $[0, 1]$.

Answer

$$\frac{1}{6}(5\sqrt{5} - 1)$$

Although it is nice to have a formula for calculating arc length, this particular theorem often generates expressions that are difficult to integrate. We will gain some ability to evaluate these integrals in the next chapter. Still, in some cases, we may have to use a computer or calculator to approximate the value of the integral.

Example 2.4.2: Using a Computer or Calculator to Determine the Arc Length of a Function of x

Let $y = x^2$. Calculate the arc length of the graph of y over the interval $[1, 3]$.

Solution

Noting that y is a function of x, we compute $y' = 2x$, so that $(y')^2 = 4x^2$. Therefore,

$$ds = \sqrt{1 + 4x^2}\, dx.$$

Then the arc length is given by

$$L = \int_{x=a}^{x=b} ds = \int_1^3 \sqrt{1 + 4x^2}\, dx.$$

Using the Right Endpoint Method with $n = 10$,[7] we get

$$\int_1^3 \sqrt{1 + 4x^2}\, dx \approx R_{10} = \sum_{n=1}^{10} \sqrt{1 + 4\left(1 + \frac{1}{5}n\right)^2} \cdot \frac{1}{5} \approx 8.65347.$$

Checkpoint 2.4.2

Let $y = \sin(x)$. Calculate the arc length of the graph of y over the interval $[0, \pi]$. Use a computer or calculator to approximate the value of the integral.

Answer

Depending on how many subintervals you choose (and whether you are using the Left Endpoint or the Right Endpoint Method), the value should approach

$$\text{Arc Length} \approx 3.8202.$$

Arc Length of the Curve $x = g(y)$

We have just seen how to approximate the length of a curve with line segments. If we want to find the arc length of the graph of a function of y, we can repeat the same process, except we partition the y-axis instead of the x-axis. Figure 2.4.3 shows a representative line segment.

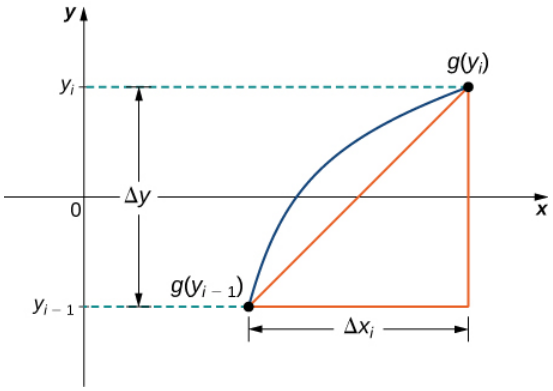


Figure 2.4.3: A representative line segment over the interval $[y_{i-1}, y_i]$.

Then the length of the line segment is

$$l_i = \sqrt{(\Delta y)^2 + (\Delta x_i)^2} = \sqrt{1 + \left(\frac{\Delta x_i}{\Delta y}\right)^2}\, \Delta y.$$

If we now follow the same development we did earlier, we get a formula for the arc length of a function $x = g(y)$.

Theorem: Arc Length of a Curve (with respect to y)

Let $g(y)$ be a continuously differentiable function over the (y-axis) interval $[c,d]$. Then the arc length, L, of the portion of the graph of $g(y)$ from the point $(c,g(c))$ to the point $(d,g(d))$ is given by

$$L=\int_{y=c}^{y=d} ds,$$

where[8]

$$ds=\sqrt{1+[g'(y)]^2}\,dy. \tag{2.4.3}$$

Example 2.4.3: Approximating the Arc Length of a Function of y

Let $x=3y^3$. Approximate the arc length of the curve over the (y-axis) interval $[1,2]$.

Solution

Note that x is definitely a function of y. So we have $x'=9y^2$, so $(x'(y))^2=81y^4$. Therefore,

$$ds=\sqrt{1+81y^4}\,dy.$$

Thus, the arc length is

$$L=\int_{y=c}^{y=d} ds=\int_1^2 \sqrt{1+81y^4}\,dy.$$

Choosing to approximate this value using 20 slices,[9] we get

$$\begin{aligned} L &\approx L_{20} \\ &= \sum_{n=1}^{20}\sqrt{1+81\left(1+\frac{1}{20}(n-1)\right)^4}\cdot\frac{1}{20} \\ &\approx 20.3576 \end{aligned}$$

Checkpoint 2.4.3

Let $g(y)=\frac{1}{y}$. Calculate the arc length of the graph of $g(y)$ over the interval $[1,4]$. Use a computer or calculator to approximate the value of the integral.

Answer

$$\text{Arc Length}\approx 3.15018$$

Arc Lengths - A Wealth of Choices

In the derivation of the arc length formulas, both with respect to x and with respect to y, the base integral for arc length had the form

$$s=\int ds,$$

where s was the *initial* independent variable. We chose to trade out ds for an expression involving dx or dy depending on what we were learning at the moment; however, how would we know what form to use in a general scenario? Is the correct choice to trade out ds for an expression involving dx? Or would it be better to trade ds for its equivalent form in terms of dy? The good news is that

there is a way to determine which option is viable. The bad news is that sometimes *both* options are viable and you have the extra work of determining which is the best to use.

Determining Which Form of ds to Use

Simply put, the form of ds available for use depends solely on whether the given curve is a function of x, y, or both over the given interval. That is if the given curve is a function of x over the given interval, then

$$ds = \sqrt{1 + [f'(x)]^2}\, dx$$

is a viable form. If the given curve is a function of y over the given interval, then

$$ds = \sqrt{1 + [g'(y)]^2}\, dy$$

is a viable form. If the curve is *both* a function of x *and* a function of y over the given interval, then you can use either form of ds; however, it is often the case that one such form will be easier to work with than the other.

Let's see this in an example.

Example 2.4.4

Find the length of the curve $y = \left(\frac{3}{2}x\right)^{2/3} + 1$ on the (x-axis) interval $\left[0, \frac{2}{3}(3)^{3/2}\right]$.

Solution

First, all arc length problems boil down to evaluating

$$L = \int ds.$$

We need to determine the proper limits of integration and which form of ds to use. Since we were not explicitly told which variable to integrate with respect to, we need to do a little investigation.

With Respect to x

It's simple to see that, over the given interval of x-values, y is a function of x. So we know that we can use $ds = \sqrt{1 + [y'(x)]^2}\, dx$. Let's set up the corresponding arc length integral to see how it looks.

$$y' = \frac{2}{3}\left(\frac{3}{2}x\right)^{-1/3} \cdot \frac{3}{2} = \left(\frac{3}{2}x\right)^{-1/3} \implies (y')^2 = \left(\frac{3}{2}x\right)^{-2/3}.$$

Therefore, the arc length could be found by evaluating

$$L = \int_{x=0}^{x=\frac{2}{3}(3)^{3/2}} \sqrt{1 + \left(\frac{3}{2}x\right)^{-2/3}}\, dx.$$

While not impossible to evaluate, the complex structure of this integrand should give you pause - especially when you know there *might* be a way to approach this problem. Let's pause on this approach and see if the curve is a function of y over the given interval. If it's not, then we will come back to this and see if we can evaluate this integral analytically.[10]

With Respect to y

Solving the given equation for x, we arrive at $\left[\frac{2}{3}(y-1)\right]^{3/2} = x$. This already looks uglier than the original equation, but do not let that stop you - Integral Calculus is full of pleasant surprises. It should be somewhat obvious that x is a function of y wherever the function is defined (when $y \geq 1$), so we should now investigate the interval of y-values corresponding to the original interval. When $x = 0$, $y = 1$, and when $x = \frac{2}{3}(3)^{3/2}$, $y = \left(\frac{3}{2} \cdot \frac{2}{3}(3)^{3/2}\right)^{2/3} + 1 = 3 + 1 = 4$.

Finding the derivatives for the corresponding arc length integral, we get

$$x' = \frac{3}{2}\left[\frac{2}{3}(y-1)\right]^{1/2} \cdot \frac{2}{3} = \left[\frac{2}{3}(y-1)\right]^{1/2} \implies (x')^2 = \frac{2}{3}(y-1).$$

Hence, the arc length function is

$$\begin{aligned} L &= \int_{y=1}^{y=4} \sqrt{1+\frac{2}{3}(y-1)}\,dy \\ &= \int_{y=1}^{y=4} \sqrt{\frac{2}{3}y+\frac{1}{3}}\,dy \\ &= \frac{3}{2}\int_{u=1}^{u=3} \sqrt{u}\,du \qquad \left(\text{Let } u=\tfrac{2}{3}y+\tfrac{1}{3} \implies du=\tfrac{2}{3}\,dy\text{, and change the limits of integration correspondingly}\right) \\ &= \frac{3}{2}\cdot\frac{2}{3}u^{3/2}\Big|_{u=1}^{u=3} \\ &= 3^{3/2}-1 \end{aligned}$$

Example 2.4.4 illustrates the need to investigate whether a given curve is a function of *both* x and y over a given interval. As we found, the difference in computational expense can be vastly different between approaches. I leave it to the reader to verify that the first integral (with respect to x) results in the same value as the second.[11]

Applications

One physical application of hyperbolic functions involves hanging cables. Suppose a cable of uniform density is suspended between two supports without any load other than its weight. In that case, the cable forms a **catenary** curve. High-voltage power lines, chains hanging between two posts, and spider web strands all form catenaries. The following figure shows chains hanging from a row of posts.



Figure 2.4.4: Chains between these posts take the shape of a catenary. (credit: modification of work by OKFoundryCompany, Flickr)

Hyperbolic functions can be used to model catenaries. Specifically, functions of the form $y = a\cdot\cosh\left(\frac{x}{a}\right)$ are catenaries. Figure 2.4.5 shows the graph of $y = 2\cosh\left(\frac{x}{2}\right)$.

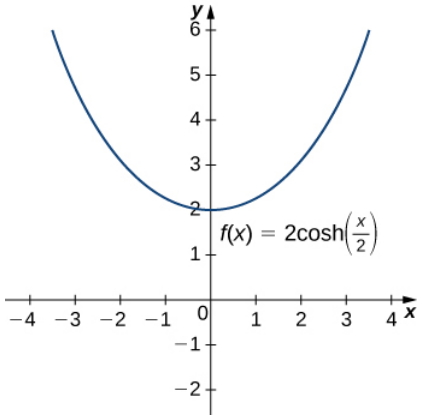


Figure 2.4.5: A hyperbolic cosine function forms the shape of a catenary.

Example 2.4.5: Using a Catenary to Find the Length of a Cable

Assume a hanging cable has the shape $10\cosh\left(\frac{x}{10}\right)$ for $-15 \le x \le 15$, where x is measured in feet. Determine the length of the cable (in feet).

Solution

We have $f(x) = 10\cosh\left(\frac{x}{10}\right)$, so $f'(x) = \sinh\left(\frac{x}{10}\right)$. Then the arc length is

$$\int_a^b \sqrt{1+[f'(x)]^2}\,dx = \int_{-15}^{15} \sqrt{1+\sinh^2\left(\frac{x}{10}\right)}\,dx.$$

Now recall that

$$1+\sinh^2(x) = \cosh^2(x),$$

so we have

$$\begin{aligned}
\text{Arc Length} &= \int_{-15}^{15} \sqrt{1+\sinh^2\left(\frac{x}{10}\right)}\,dx \\
&= \int_{-15}^{15} \cosh\left(\frac{x}{10}\right)\,dx \\
&= 10\sinh\left(\frac{x}{10}\right)\Big|_{-15}^{15} \\
&= 10\left[\sinh\left(\frac{3}{2}\right) - \sinh\left(-\frac{3}{2}\right)\right] \\
&= 20\sinh\left(\frac{3}{2}\right) \\
&\approx 42.586 \text{ ft.}
\end{aligned}$$

Checkpoint 2.4.5

Assume a hanging cable has the shape $15\cosh\left(\frac{x}{15}\right)$ for $-20 \le x \le 20$. Determine the length of the cable (in feet).

Answer

52.95 ft

Area of a Surface of Revolution

The concepts we used to find the arc length of a curve can be extended to find the surface area of a surface of revolution. **Surface area** is the total area of the outer layer of an object. For objects such as cubes or bricks, the object's surface area is the sum of the areas of all of its faces. For curved surfaces, the situation is more complex. Let $f(x)$ be a nonnegative continuously differentiable function over the interval $[a, b]$. We wish to find the surface area of the surface of revolution created by revolving the graph of $y = f(x)$ around the x-axis as shown in the following figure.

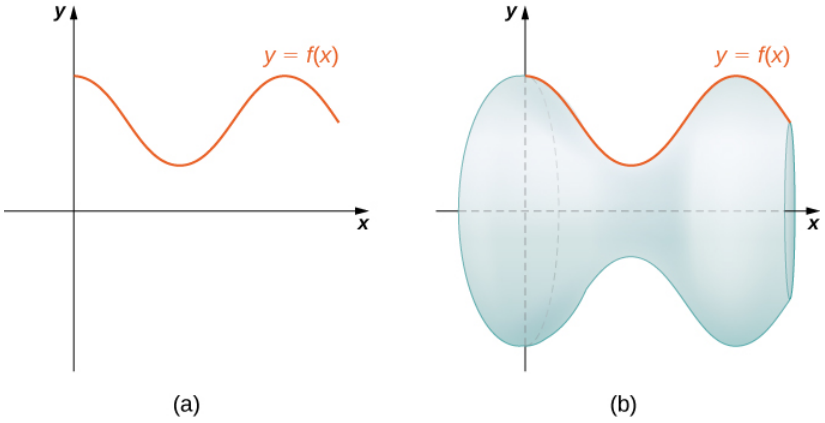


Figure **2.4.6***a*: A curve representing the function $f(x)$.

Figure **2.4.6***b*: The surface of revolution formed by revolving the graph of $f(x)$ around the x-axis.

As we have done many times before, we are going to partition the interval $[a, b]$ and approximate the surface area by calculating the surface area of simpler shapes. As we did earlier in this section, we start by using line segments to approximate the curve. For $i = 0, 1, 2, \ldots, n$, let $P = \{x_i\}$ be a regular partition of $[a, b]$. Then, for $i = 1, 2, \ldots, n$, construct a line segment from the point $(x_{i-1}, f(x_{i-1}))$ to the point $(x_i, f(x_i))$. Now, revolve these line segments around the x-axis to generate an approximation of the surface of revolution as shown in the following figure.

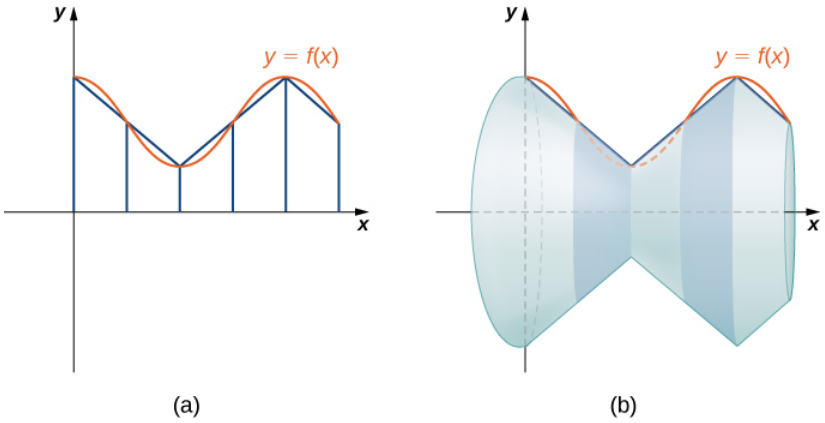


Figure **2.4.7***a*: Approximating $f(x)$ with line segments.

Figure **2.4.7***b*: The surface of revolution formed by revolving the line segments around the x-axis.

Notice that when each line segment is revolved around the axis, it produces a band. These bands are pieces of cones.[12] A piece of a cone like this is called a **frustum** of a cone.

Deriving the Surface Area of a Frustum

In order to derive a formula for the surface area of the *entire* rotational volume, we must first focus our efforts on deriving a formula for the surface area of *one* of the bands of the overall rotated volume. To do so, we take a few steps back to Geometry.

To find the surface area of the band, we need to find the frustum's lateral surface area, S.[13] Let r_1 and r_2 be the radii of the wide end and the narrow end of the frustum, respectively, and let l be the slant height of the frustum as shown in the Figure **2.4.8**.

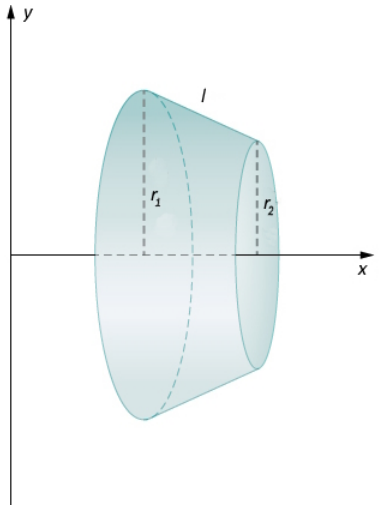


Figure 2.4.8: A frustum of a cone can approximate a small part of the surface area.

From Geometry, we know the lateral surface area of a *cone* is given by

$$\text{Lateral Surface Area} = \pi r s,$$

where r is the radius of the base of the cone and s is the slant height (Figure 2.4.9).

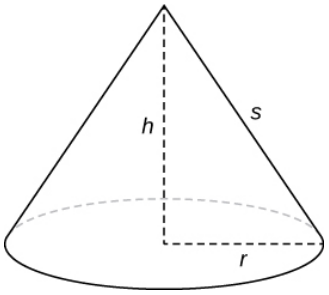


Figure 2.4.9: The lateral surface area of the cone is given by $\pi r s$.

Since a frustum can be thought of as a piece of a cone, the lateral surface area of the frustum is given by the lateral surface area of the whole cone less the lateral surface area of the smaller cone (the pointy tip) that was cut off (Figure 2.4.10).

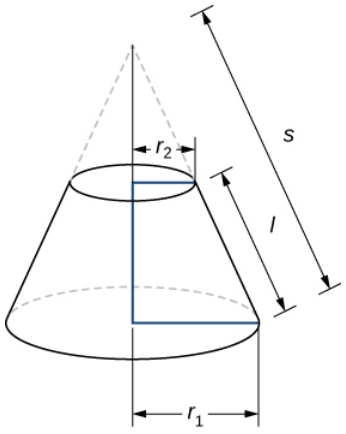


Figure 2.4.10: Calculating the lateral surface area of a frustum of a cone.

The cross-sections of the small cone and the large cone are similar triangles, so we see that

$$\frac{r_2}{r_1} = \frac{s-l}{s}, \tag{2.4.4}$$

which can also be written as

$$\frac{r_2 s}{r_1} = s - l. \tag{2.4.5}$$

Solving Equation 2.4.5 for s, we get

$$\begin{aligned} \frac{r_2 s}{r_1} &= s - l \\ \Rightarrow \quad r_2 s &= r_1(s-l) \\ \Rightarrow \quad r_2 s &= r_1 s - r_1 l \\ \Rightarrow \quad r_1 l &= r_1 s - r_2 s \\ \Rightarrow \quad r_1 l &= (r_1 - r_2)s \\ \Rightarrow \quad \frac{r_1 l}{r_1 - r_2} &= s \end{aligned}$$

We will use this last result,

$$s = \frac{r_1 l}{r_1 - r_2}, \tag{2.4.6}$$

to help compute the surface area of the frustum.

If S is the surface area of the frustum, then

$$\begin{aligned} S &= \text{(Lateral surface area of large cone)} \\ &\quad - \text{(Lateral surface area of small cone)} \\ &= \pi r_1 s - \pi r_2 (s-l) \\ &= \pi r_1 s - \pi r_2 \left(\frac{r_2 s}{r_1}\right) && \text{(from Equation 2.4.5)} \\ &= \pi s \left(r_1 - \frac{r_2^2}{r_1}\right) \\ &= \pi \left(\frac{r_1 l}{r_1 - r_2}\right)\left(\frac{r_1^2 - r_2^2}{r_1}\right) && \text{(from Equation 2.4.6)} \\ &= \pi \left(\frac{\cancel{r_1} l}{\cancel{(r_1 - r_2)}}\right)\left(\frac{\cancel{(r_1 - r_2)}\,(r_1 + r_2)}{\cancel{r_1}}\right) \\ &= \pi(r_1 + r_2)l. \end{aligned}$$

Multiplying the numerator and denominator of this last result by 2, we get

$$S = 2\pi \left(\frac{r_1 + r_2}{2}\right) l = 2\pi \bar{r} l, \tag{2.4.7}$$

where $\bar{r} = \frac{r_1 + r_2}{2}$ is the average value of the radii of the band.

Deriving and Computing the Surface Area of a Rotational Volume

We now turn our attention back to the fact that we are rotating a function about an axis. To calculate the surface area of each of the bands formed by revolving the line segments around the x-axis, we consider the ith subinterval and exchange l_i for l. Moreover, without loss of generality, we can assume r_1 in our previous discussion is $f(x_{i-1})$, and r_2 is $f(x_i)$.[14] The ith band is shown in Figure 2.4.11.

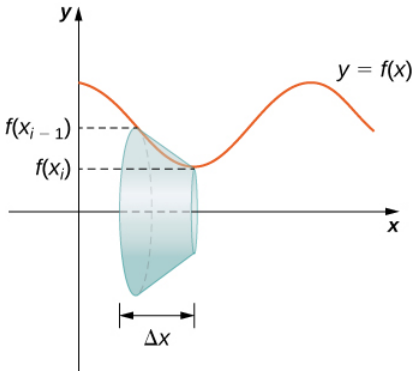


Figure **2.4.11**: A representative band used for determining surface area.

The surface area, S_i, of this i^{th} band is therefore

$$\begin{aligned} S_i &= 2\pi \bar{r} l_i \\ &= 2\pi \left(\frac{f(x_{i-1}) + f(x_i)}{2} \right) \sqrt{1 + (f'(x_i^*))^2} \Delta x \end{aligned}$$

Furthermore, since $f(x)$ is continuous, by the Intermediate Value Theorem, there is a point $x_i^{**} \in [x_{i-1}, x_i]$ such that $f(x_i^{**}) = \frac{f(x_{i-1}) + f(x_i)}{2}$, so we get

$$S_i = 2\pi f(x_i^{**}) \sqrt{1 + (f'(x_i^*))^2} \Delta x.$$

Then the approximate surface area of the whole surface of revolution is given by

$$S \equiv \text{Surface Area} \approx \sum_{i=1}^{n} 2\pi f(x_i^{**}) \sqrt{1 + (f'(x_i^*))^2} \Delta x.$$

This almost looks like a Riemann sum, except we have functions evaluated at two different points, x_i^* and x_i^{**}, over the interval $[x_{i-1}, x_i]$. Although we do not examine the details here, it turns out that because $f(x)$ is continuously differentiable, if we let $n \to \infty$, the limit works the same as a Riemann sum even with the two different evaluation points. This makes sense intuitively. Both x_i^* and x_i^{**} are in the interval $[x_{i-1}, x_i]$, so it makes sense that as $n \to \infty$, both x_i^* and x_i^{**} approach x.[15]

Taking the limit as $n \to \infty$, we get

$$\begin{aligned} S \equiv \text{Surface Area} &= \lim_{n \to \infty} \sum_{i=1}^{n} 2\pi f(x_i^{**}) \sqrt{1 + (f'(x_i^*))^2} \Delta x \\ &= \int_{x=a}^{x=b} 2\pi f(x) \sqrt{1 + (f'(x))^2)}\, dx \end{aligned}$$

As with arc length, we can conduct a similar development for functions of y to get a formula for the surface area of surfaces of revolution about the y-axis. These findings are summarized in the following theorem.

Theorem: Surface Area of a Surface of Revolution

The surface area of the surface of revolution formed by revolving the graph of a nonnegative, continuously differentiable function about an axis is

$$S = \int_a^b 2\pi r ds,$$

where r is the radius of rotation and

$$ds = \sqrt{1 + (f'(x))^2} dx$$

or

$$ds = \sqrt{1 + (g'(y))^2}\,dy.$$

Recall, the differential ds represents an infinitesimal change in arc length. Which form you choose for ds will, again, depend on whether the given curve is a function of x or y (or both). The radius of rotation function, r, is a function that gives the distance between the axis of rotation and the curve.

A great interpretation of the surface area integral

$$\int 2\pi r\,ds$$

is that the surface area of the ith band is the product of its circumference and its width. The circumference of the band is $2\pi r_i$ and its width is Δs.

Example 2.4.6: Calculating the Surface Area of a Surface of Revolution 1.

Let $f(x) = \sqrt{x}$ over the interval $[1, 4]$. Find the surface area of the surface generated by revolving the graph of $f(x)$ around the x-axis.

Solution

As always, begin by sketching the function, the volume, and a representative slice (not shown). The graph of $f(x)$ and the surface of rotation are shown in Figure 2.4.12.

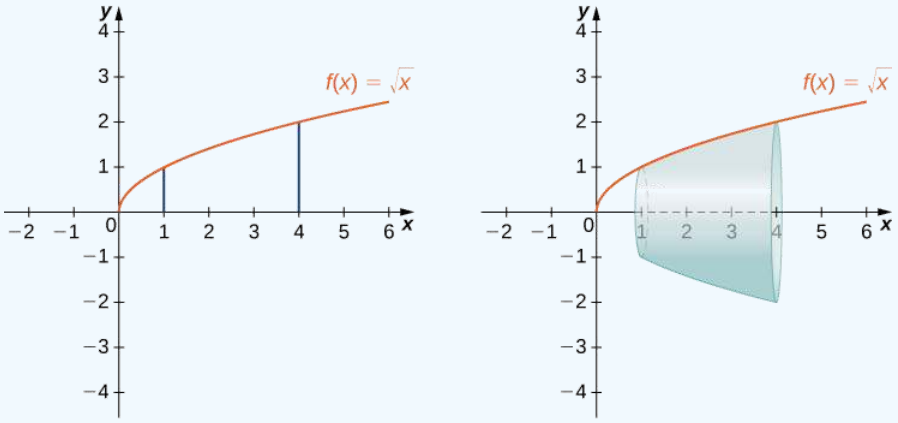


Figure 2.4.12*a*: The graph of $f(x)$.
Figure 2.4.12*b*: The surface of revolution.

To showcase how flexible we can be, we are going to perform several different setups for this problem (we will only compute the value of one of them, as they all result in the same value).

With Respect to x

This is the approach most students would take because we know $y = \sqrt{x}$ is a function of x on the given interval. Therefore, it makes sense to build an integral with respect to x.

All surface area integrals have the common form

$$S = \int 2\pi r\,ds.$$

Since we are rotating the curve about the x-axis, the radius of rotation is $y_T - y_B = \sqrt{x} - 0 = \sqrt{x}$. Furthermore, replacing ds with a function of x, we get

$$\begin{aligned} ds &= \sqrt{1+(y')^2}\,dx \\ &= \sqrt{1+\left(\frac{1}{2\sqrt{x}}\right)^2}\,dx \\ &= \sqrt{1+\tfrac{1}{4x}}\,dx \end{aligned}$$

Therefore,

$$\begin{aligned} S &= \int_{x=a}^{x=b} 2\pi r\,ds \\ &= \int_{x=1}^{x=4} 2\pi\sqrt{x}\sqrt{1+\frac{1}{4x}}\,dx \\ &= 2\pi\int_{x=1}^{x=4} \sqrt{x}\sqrt{\frac{4x+1}{4x}}\,dx \\ &= 2\pi\int_{x=1}^{x=4} \frac{1}{2}\sqrt{4x+1}\,dx \\ &= \pi\int_{x=1}^{x=4} \sqrt{4x+1}\,dx \\ &= \frac{\pi}{4}\int_{u=5}^{u=17} \sqrt{u}\,du \qquad \text{(Let } u=4x+1 \implies du=4\,dx\text{, and change the limits of integration accordingly)} \\ &= \frac{\pi}{4}\cdot\frac{2}{3}u^{3/2}\Big|_{u=5}^{u=17} \\ &= \frac{\pi}{6}\left(17^{3/2}-5^{3/2}\right) \end{aligned}$$

With Respect to y

This is *not* the approach most students would take, but I want to make certain that you understand all of your options. We were given $y=\sqrt{x}$ on the x-interval $[1,4]$. On this interval, the curve is one-to-one. That is, it passes both the Vertical Line Test and the Horizontal Line Test. Hence, we could represent the curve as a function of y on this interval. Specifically, $y^2=x$ on the y-interval $[1,2]$.

All surface area integrals have the common form

$$S=\int 2\pi r\,ds.$$

Since we are rotating the curve about the x-axis, the radius of rotation is $y_T-y_B=y-0=y$. Moreover, since we are choosing to integrate with respect to y,

$$ds=\sqrt{1+(x')^2}\,dy=\sqrt{1+(2y)^2}\,dy=\sqrt{1+4y^2}\,dy.$$

Putting this altogether, we get

$$
\begin{aligned}
S &= \int_{y=1}^{y=2} 2\pi r\, ds \\
&= \int_{y=1}^{y=2} 2\pi y\sqrt{1+4y^2}\, dy \\
&= 2\pi \int_{y=1}^{y=2} y\sqrt{1+4y^2}\, dy \\
&= \frac{\pi}{4}\int_{u=5}^{y=17} \sqrt{u}\, du \qquad \text{(Let } u = 1+4y^2 \implies du = 8y\, dy\text{, and change the limits of integration accordingly)} \\
&= \frac{\pi}{4}\cdot\frac{2}{3}u^{3/2}\Big|_{u=5}^{u=17} \\
&= \frac{\pi}{6}\left(17^{3/2} - 5^{3/2}\right)
\end{aligned}
$$

Example **2.4.6** is another great illustration that knowing we have a choice can simplify our computations. While not the intuitive approach, integrating with respect to y led to a simpler integral.

? Checkpoint 2.4.6

Let $f(x) = \sqrt{1-x}$ over the interval $[0, 1/2]$. Find the surface area of the surface generated by revolving the graph of $f(x)$ around the x-axis.

Answer

$$\frac{\pi}{6}(5\sqrt{5} - 3\sqrt{3})$$

✓ Example 2.4.7: Calculating the Surface Area of a Surface of Revolution 2

Let $y = \sqrt[3]{3x}$. Consider the portion of the curve where $0 \le y \le 2$.

a. Find the surface area of the surface generated by revolving the graph of $f(x)$ around the y-axis.
b. Setup, but do not evaluate, and integral for the surface area of the surface generated by revolving the graph of $f(x)$ about the line $x = 4$.

Solution

a. A quick sketch of the curve, the solid, and a slice (not shown) is the starting point.

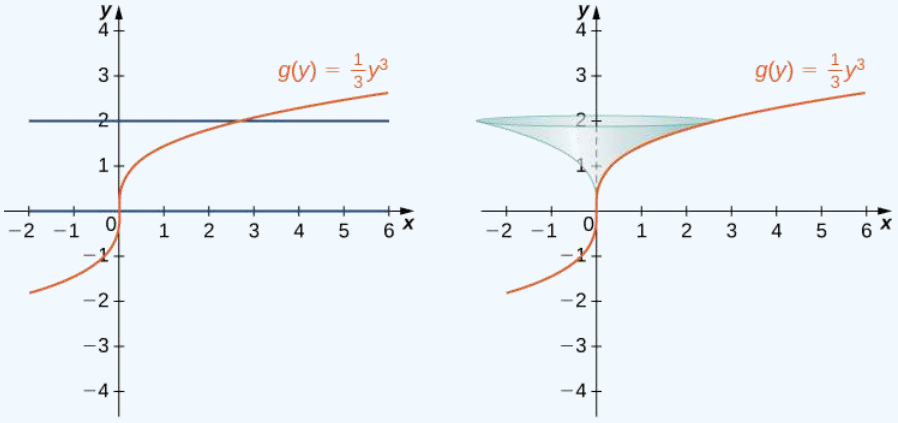


Figure 2.4.13a: The graph of $g(y)$.
Figure 2.4.13b: The surface of revolution.

We know that all surface area integrals share the form

$$S = \int 2\pi r \, ds.$$

If we decided to integrate with respect to x, then the entire integrand must be written in terms of x. The radius of rotation is $x_R - x_L = x - 0 = x$ and our choice for ds would be

$$\begin{aligned} ds &= \sqrt{1 + (y')^2}\, dx \\ &= \sqrt{1 + \left(\frac{1}{3}(3x)^{-2/3} \cdot 3\right)^2}\, dx \\ &= \sqrt{1 + \left((3x)^{-2/3}\right)^2}\, dx \\ &= \sqrt{1 + (3x)^{-4/3}}\, dx \end{aligned}$$

and so the surface area of this solid of revolution would be computed using

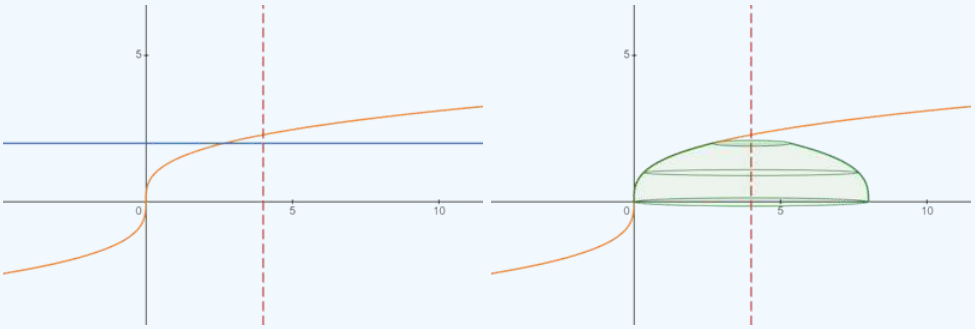
$$\begin{aligned} S &= \int_{x=0}^{x=8/3} 2\pi r \, ds \\ &= 2\pi \int_{x=0}^{x=8/3} x\sqrt{1 + (3x)^{-4/3}}\, dx \end{aligned}$$

While we could evaluate this integral, it might be best to check if an integral in terms of y is cleaner.

Solving for x, we have $x = \frac{1}{3}y^3$, so $x' = y^2$ and $(x')^2 = y^4$. Moreover, the radius of rotation is $x_R - x_L = \frac{1}{3}y^3 - 0 = \frac{1}{3}y^3$. Hence,

$$
\begin{aligned}
S &= \int_{y=0}^{y=2} 2\pi r\, ds \\
&= 2\pi \int_{y=0}^{y=2} \frac{1}{3} y^3 \sqrt{1+y^4}\, dy \\
&= \frac{\pi}{6} \int_{u=1}^{u=17} \sqrt{u}\, du \qquad \text{(Let } u = 1+y^4 \implies du = 4y^3\, dy\text{, and change the limits of integration accordingly)} \\
&= \frac{\pi}{9} \cdot u^{3/2} \Big|_{u=1}^{u=17} \\
&= \frac{\pi}{9} \left(17^{3/2} - 1\right)
\end{aligned}
$$

A sketch of the curve and solid is shown below.



b. Building the integral with respect to y (I will leave building the integral with respect to x for the interested reader), we get

$$
\begin{aligned}
S &= \int_{y=0}^{y=2} 2\pi r\, ds \\
&= 2\pi \int_{y=0}^{y=2} (x_R - x_L)\, ds \\
&= 2\pi \int_{y=0}^{y=2} \left(4 - \frac{1}{3} y^3\right) \sqrt{1+y^4}\, dy
\end{aligned}
$$

Notice that the radius of rotation is the distance from the axis of rotation, $x = 4$, to the curve, $x = \frac{1}{3}y^3$. If you draw a radius line from the function to the axis of rotation, the *right* edge, x_R, is at $x = 4$, and the left edge, x_L, is at the curve, $x = \frac{1}{3}y^3$. This is why the value of r is $4 - \frac{1}{3}y^3$.

Also, note that ds didn't change. When rotating about a line that is *not* an axis, the only factor that changes in the formula for the surface area integral is r (because all you are doing is changing the radius of rotation).

? Checkpoint 2.4.7

Let $g(y) = \sqrt{9-y^2}$ over the interval $y \in [0,2]$. Find the surface area of the surface generated by revolving the graph of $g(y)$ around the y-axis.

Answer

12π

Footnotes

[1] The process is identical, with the roles of x and y reversed.

[2] This property comes up again in later chapters.

[3] Note that some (or all) Δy_i may be negative.

[4] To be clear, we defined the arc length of a given function over a *constant* interval using the dependent variable L, but we defined the arc length *function* over a *varying* interval using the dependent variable s.

[5] There is often a confusion as to why we use s for the arc length function and L for the arc length of a curve. Why not just use s in both cases? In fact, wouldn't it make more sense to use s for the arc length of a curve given that we used s in Trigonometry for the arc length of the circle subtended by an angle θ? The answer is more complex than most students care to hear, but since you are curious enough to investigate...

Recall that we arrived at the differential, ds, by allowing $s(x) = \int_a^x \sqrt{1+[f'(t)]^2}\,dt$. This was all to streamline the eventual arc length formula into the simple statement in Equation 2.4.1. Had we *also* used the dependent variable s for this computed arc length, we would have arrived at

$$s = \int_{x=a}^{x=b} ds.$$

This would be completely nonsensical as we would be stating that the definite integral, *which is a constant*, is equal to s; however, if s is constant, then its differential, ds, would be 0, and so $s \equiv 0$ always.

[6] I get it that this is obvious, but there is an actual reason for recognizing that this is a function of x. We will see why in a little bit.

[7] The choice of using the Right Endpoint Method as opposed to the Left Endpoint Method was completely arbitrary here, as was the decision to use $n = 10$ subintervals. In Differential Calculus, we were explicitly told what method to use and how many subintervals to consider, however, now that we are beyond that course, there will be times when *we* have to play the scientific investigator and make the decision of method and number of subintervals for ourselves.

[8] As was promised, here is another, equivalent form for ds.

[9] Another arbitrary choice.

[10] "Analytically" means "by hand."

[11] I promise, you *can* actually finish that first integral. It just takes some work, but it's worth the practice.

[12] Think of an ice cream cone with the pointy end cut off.

[13] The area of just the slanted outside surface of the frustum, not including the areas of the top or bottom faces.

[14] The phrase, "without loss of generality," is used quite extensively in mathematics. It means that, had the alternative situation occurred (in this case, had r_1 in our previous discussion been $f(x_i)$ and r_2 been $f(x_{i-1})$), the results (with the appropriate adjustments) would be the same.

[15] Those of you who are interested in the details should consult an advanced calculus text.

2.4E: Exercises

Reading Questions

1. What does it mean for a function to be "continuously differentiable," and why is this a requirement for calculating arc length?
2. The formula for arc length is derived by approximating the curve with a series of straight line segments. What theorem is used to find the length of each of these small segments?
3. What is the formula for the arc length differential, ds, for a curve defined by $y = f(x)$?
4. What is a catenary, and what type of function is used to model it?
5. What is the general integral formula for the surface area of a surface of revolution, using the terms "radius of rotation" (r) and "arc length differential" (ds)?
6. When a line segment is revolved around an axis, it creates a band. What is the geometric name for this shape?
7. In the surface area integral, $S = \int 2\pi r ds$, how is the radius of rotation, r, defined?

For exercises 1 - 3, find the length of the functions over the given interval.

1) $y = 5x$ from $x = 0$ to $x = 2$

Answer

$s = 2\sqrt{26}$ units

2) $y = -\frac{1}{2}x + 25$ from $x = 1$ to $x = 4$

3) $x = 4y$ from $y = -1$ to $y = 1$

Answer

$s = 2\sqrt{17}$ units

4) Pick an arbitrary linear function $x = g(y)$ over any interval of your choice (y_1, y_2). Determine the length of the function and then prove the length is correct by using geometry.

5) Find the surface area of the volume generated when the curve $y = \sqrt{x}$ revolves around the x-axis from $(1, 1)$ to $(4, 2)$, as seen here.

Answer

$A = \frac{\pi}{6}(17\sqrt{17} - 5\sqrt{5})$ units^2

6) Find the surface area of the volume generated when the curve $y = x^2$ revolves around the y-axis from $(1, 1)$ to $(3, 9)$.

For exercises 7 - 16, find the lengths of the functions of x over the given interval. If you cannot evaluate the integral exactly, use technology to approximate it.

7) $y = x^{3/2}$ from $(0, 0)$ to $(1, 1)$

Answer

$s = \frac{13\sqrt{13} - 8}{27}$ units

8) $y = x^{2/3}$ from $(1, 1)$ to $(8, 4)$

9) $y = \frac{1}{3}(x^2 + 2)^{3/2}$ from $x = 0$ to $x = 1$

Answer

$s = \frac{4}{3}$ units

10) $y = \frac{1}{3}(x^2 - 2)^{3/2}$ from $x = 2$ to $x = 4$

11) [Technology Required] $y = e^x$ on $x = 0$ to $x = 1$

Answer

$s \approx 2.0035$ units

12) $y = \dfrac{x^3}{3} + \dfrac{1}{4x}$ from $x = 1$ to $x = 3$

13) $y = \dfrac{x^4}{4} + \dfrac{1}{8x^2}$ from $x = 1$ to $x = 2$

Answer

$s = \frac{123}{32}$ units

14) $y = \dfrac{2x^{3/2}}{3} - \dfrac{x^{1/2}}{2}$ from $x = 1$ to $x = 4$

15) $y = \frac{1}{27}(9x^2 + 6)^{3/2}$ from $x = 0$ to $x = 2$

Answer

$s = 10$ units

16) [Technology Required] $y = \sin x$ on $x = 0$ to $x = \pi$

For exercises 17 - 26, find the lengths of the functions of y over the given interval. If you cannot evaluate the integral exactly, use technology to approximate it.

17) $y = \dfrac{5 - 3x}{4}$ from $y = 0$ to $y = 4$

Answer

$s = \frac{20}{3}$ units

18) $x = \frac{1}{2}(e^y + e^{-y})$ from $y = -1$ to $y = 1$

19) $x = 5y^{3/2}$ from $y = 0$ to $y = 1$

Answer

$s = \frac{1}{675}(229\sqrt{229} - 8)$ units

20) [Technology Required] $x = y^2$ from $y = 0$ to $y = 1$

21) $x = \sqrt{y}$ from $y = 0$ to $y = 1$

Answer

$s = \frac{1}{8}(4\sqrt{5} + \ln(9 + 4\sqrt{5}))$ units

22) $x = \frac{2}{3}(y^2 + 1)^{3/2}$ from $y = 1$ to $y = 3$

23) [Technology Required] $x = \tan y$ from $y = 0$ to $y = \frac{3}{4}$

Answer

$s \approx 1.201$ units

24) [Technology Required] $x = \cos^2 y$ from $y = -\frac{\pi}{2}$ to $y = \frac{\pi}{2}$

25) [Technology Required] $x = 4^y$ from $y = 0$ to $y = 2$

Answer

$s \approx 15.2341$ units

26) [Technology Required] $x = \ln(y)$ on $y = \dfrac{1}{e}$ to $y = e$

For exercises 27 - 34, find the surface area of the volume generated when the following curves revolve around the x-axis. If you cannot evaluate the integral exactly, use your calculator to approximate it.

27) $y = \sqrt{x}$ from $x = 2$ to $x = 6$

Answer

$A = \frac{49\pi}{3}$ units^2

28) $y = x^3$ from $x = 0$ to $x = 1$

29) $y = 7x$ from $x = -1$ to $x = 1$

Answer

$A = 70\pi\sqrt{2}$ units^2

30) [Technology Required] $y = \frac{1}{x^2}$ from $x = 1$ to $x = 3$

31) $y = \sqrt{4 - x^2}$ from $x = 0$ to $x = 2$

Answer

$A = 8\pi$ units^2

32) $y = \sqrt{4 - x^2}$ from $x = -1$ to $x = 1$

33) $y = 5x$ from $x = 1$ to $x = 5$

Answer

$A = 120\pi\sqrt{26}$ units^2

34) [Technology Required] $y = \tan x$ from $x = -\frac{\pi}{4}$ to $x = \frac{\pi}{4}$

For exercises 35 - 42, find the surface area of the volume generated when the following curves revolve around the y-axis. If you cannot evaluate the integral exactly, use your calculator to approximate it.

35) $y = x^2$ from $x = 0$ to $x = 2$

Answer

$A = \frac{\pi}{6}(17\sqrt{17} - 1)$ units^2

36) $y = \frac{1}{2}x^2 + \frac{1}{2}$ from $x = 0$ to $x = 1$

37) $y = x + 1$ from $x = 0$ to $x = 3$

Answer

$A = 9\sqrt{2}\pi$ units^2

38) [Technology Required] $y = \frac{1}{x}$ from $x = \frac{1}{2}$ to $x = 1$

39) $y = \sqrt[3]{x}$ from $x = 1$ to $x = 27$

Answer

$A = \frac{10\sqrt{10}\pi}{27}(73\sqrt{73} - 1)$ units^2

40) [Technology Required] $y = 3x^4$ from $x = 0$ to $x = 1$

41) [Technology Required] $y = \frac{1}{\sqrt{x}}$ from $x = 1$ to $x = 3$

Answer

$A \approx 25.645$ units^2

42) [Technology Required] $y = \cos x$ from $x = 0$ to $x = \frac{\pi}{2}$

43) The base of a lamp is constructed by revolving a quarter circle $y = \sqrt{2x - x^2}$ around the y-axis from $x = 1$ to $x = 2$, as seen here. Create an integral for the surface area of this curve and compute it.

Answer

$A = 2\pi$ units^2

44) A light bulb is a sphere with radius $1/2$ in. with the bottom sliced off to fit exactly onto a cylinder of radius $1/4$ in. and length $1/3$ in., as seen here. The sphere is cut off at the bottom to fit exactly onto the cylinder, so the radius of the cut is $1/4$ in. Find the surface area (not including the top or bottom of the cylinder).

45) [Technology Required] A lampshade is constructed by rotating $y = 1/x$ around the x-axis from $y = 1$ to $y = 2$, as seen here. Determine how much material you would need to construct this lampshade—that is, the surface area—accurate to four decimal places.

Answer

10.5017 units^2

46) [Technology Required] An anchor drags behind a boat according to the function $y = 24e^{-x/2} - 24$, where y represents the depth beneath the boat and x is the horizontal distance of the anchor from the back of the boat. If the anchor is 23 ft below the boat, how much rope do you have to pull to reach the anchor? Round your answer to three decimal places.

47) [Technology Required] You are building a bridge that will span 10 ft. You intend to add decorative rope in the shape of $y = 5|\sin((x\pi)/5)|$, where x is the distance in feet from one end of the bridge. Find out how much rope you need to buy, rounded to the nearest foot.

Answer

23 ft

For exercise 48, find the exact arc length for the following problems over the given interval.

48) $y = \ln(\sin x)$ from $x = \frac{\pi}{4}$ to $x = \frac{3\pi}{4}$. (Hint: Recall trigonometric identities.)

49) Draw graphs of $y = x^2, y = x^6$, and $y = x^{10}$. For $y = x^n$, as n increases, formulate a prediction on the arc length from $(0, 0)$ to $(1, 1)$. Now, compute the lengths of these three functions and determine whether your prediction is correct.

Answer

2

50) Compare the lengths of the parabola $x = y^2$ and the line $x = by$ from $(0, 0)$ to (b^2, b) as b increases. What do you notice?

51) Solve for the length of $x = y^2$ from $(0, 0)$ to $(1, 1)$. Show that $x = \dfrac{y^2}{2}$ from $(0, 0)$ to $(2, 2)$ is twice as long. Graph both functions and explain why this is so.

Answer

Answers may vary

52) [Technology Required] Which is longer between $(1, 1)$ and $(2, \frac{1}{2})$: the hyperbola $y = \dfrac{1}{x}$ or the graph of $x + 2y = 3$?

53) Explain why the surface area is infinite when $y = 1/x$ is rotated around the x-axis for $1 \leq x < \infty$, but the volume is finite.

Answer

For more information, look up Gabriel's Horn.

In exercises 54 - 57, solve each problem.

54) [Technology Required] A chain hangs from two posts 2m apart to form a catenary described by the equation $y = 2\cosh(x/2) - 1$. Find the slope of the catenary at the left fence post.

Answer

-0.521095

55) A telephone line is a catenary described by $y = a\cosh(x/a)$. Find the ratio of the area under the catenary to its arc length. Does this confirm your answer for the previous question?

56) [Technology Required] A chain hangs from two posts four meters apart to form a catenary described by the equation $y = 4\cosh(x/4) - 3$. Find the total length of the catenary (arc length).

57) [Technology Required] A high-voltage power line is a catenary described by $y = 10\cosh(x/10)$. Find the ratio of the area under the catenary to its arc length. What do you notice?

Answer

10

58) [Technology Required] Find the arc length of $y = 1/x$ from $x = 1$ to $x = 4$.

59) [Technology Required] Find the surface area of the shape created when rotating the curve in the previous exercise from $x = 1$ to $x = 2$ around the x-axis.

60) [Technology Required] Find the arc length of $\ln x$ from $x = 1$ to $x = 2$.

Contributors

Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

This page titled 2.4E: Exercises is shared under a CC BY-NC 4.0 license and was authored, remixed, and/or curated by Roy Simpson.

- **6.4E: Exercises for Section 6.4** by Edwin “Jed” Herman, Gilbert Strang is licensed CC BY-NC-SA 4.0. Original source: https://openstax.org/details/books/calculus-volume-1.
- **6.9E: Exercises for Section 6.9** by Edwin “Jed” Herman, Gilbert Strang is licensed CC BY-NC-SA 4.0. Original source: https://openstax.org/details/books/calculus-volume-1.
- **6.7E: Exercises for Section 6.7** by Edwin “Jed” Herman, Gilbert Strang is licensed CC BY-NC-SA 4.0. Original source: https://openstax.org/details/books/calculus-volume-1.

2.5: Work

Note to the Instructor (click to expand)

This section covers the physics of mechanical work, including spring-mass problems (as an introduction to force), and the work to lift fluid and non-fluid objects.

To succeed in this section, you'll need to use some skills from previous courses. While you should already know them, this is the first time they've been required. You can review these skills in CRC's Corequisite Codex. If you have a support class, it might cover some, **but not all**, of these topics.

Prerequisite Skills and Support Topics (click to expand)

The following is a list of prerequisite skills (all of which can be reviewed in CRC's Corequisite Codex) needed for this section **that have not already been mentioned in any previous section**. If you are enrolled in a course with a Support section, some (but definitely not all) of these topics might be covered or reviewed in the Support section of your course.

- **Sets and Numbers**
  - Scientific Notation: Some homework problems, particularly those involving gravitational force, use scientific notation for physical constants (e.g., Homework Problem #18).
- **Linear Functions**
  - Linear Models: Hooke's Law, which states that the force required to stretch a spring is proportional to the distance stretched ($F = kx$), is a fundamental linear model used in this section.
- **Trigonometry**
  - Trigonometric Functions - Right Triangle Definition: This is needed to resolve a force vector into its components. In Example 1.6.1, the horizontal component of a force applied at an angle is found using cosine.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Calculate the work done by a variable force acting along a line.
- Calculate the work done in lifting objects.
- Calculate the work done in pumping a liquid from one height to another.

Hawk A.I. Section-Specific Tutor



To access the Hawk A.I. Tutor, you will need to be logged into your campus Gmail account.

In this section, we turn our attention to work. The concept of work is the backbone of many Physics problems, and we will investigate the energy expenditure in several increasingly complex situations.

Work Done by a Constant Force

You have likely read about **Newton's Second Law of Motion**, which states that **force** (which is often intuitively defined as a push or a pull on an object) is the product of an object's mass and its acceleration. That is,

$$F = ma.$$

For our current conversation, we will only consider the acceleration due to gravity, traditionally denoted g. Therefore, we replace a with g to get.[1]

$$F = mg. \tag{2.5.1}$$

When a force moves an object, we say the force does **work** on the object. In other words, work can be thought of as the amount of energy it takes to move an object through a distance. According to Physics, work can be expressed as the product of force and distance. That is,

$$W = Fd. \tag{2.5.2}$$

You would not be wrong to state that we could combine Equations 2.5.1 and 2.5.2 to arrive at

$$W = mgd; \tag{2.5.3}$$

however, the choice to do our mathematics with Equation 2.5.2 or Equation 2.5.3 will depend on which measurement system we are given.

The Metric and U.S. Customary Systems

There are two measurement systems we deal with in science - the **English system** (also known as the **U.S. Customary System**) and the **metric system** (also known as the **International System of Units**, or the **S.I. System**). We work with both systems in this course. For simplicity, the following table gives the unit names in each system (the base unit for time in both systems is the second).

	Distance d	**Mass** m	**Gravitational Constant (at sea level)** g	**Force** $F = mg$	**Work** $W = Fd$
English System	foot (ft)	slug	$\approx 32.2 \left(\frac{\text{ft}}{\text{s}^2}\right)$	pound $\left(\frac{\text{slug}\cdot\text{ft}}{\text{s}^2}\right)$	foot-pound $(\text{ft}\cdot\text{lbs})$
Metric System	meter (m)	kilogram (kg)	$\approx 9.8 \left(\frac{\text{m}}{\text{s}^2}\right)$	newton $\left(\frac{\text{kg m}}{\text{s}^2} \equiv \text{N}\right)$	joule $\left(\frac{\text{kg m}}{\text{s}^2}\cdot\text{m} = \text{N m}\right)$

In the English system, the unit of force is the pound and the unit of distance is the foot. Therefore, according to Equation 2.5.2, work is given in **foot-pounds**. It is incredibly rare in the English system to state the mass of an object - instead, the *force* the Earth is exerting on that object (which we call pounds) is typically given. Therefore, when asked to compute the work done in a scenario involving the English system, you will use Equation 2.5.2.

In the metric system, kilograms and meters are used. The unit of force is called the **newton**.[2] Hence, according Equation 2.5.2, work is given in **newton-meters**. Unlike the English system, the metric system has a special unit name for work - the **joule**. Therefore, you can use the terminology "joule" and "newton-meter" interchangeably. Another difference between the English and metric systems is that, in the metric system, you will often be given the object's mass. Therefore, when computing work in scenarios involving the metric system, you will use Equation 2.5.3.

✔ Example 2.5.1: The Work Required to Push a Lawn Mower

Suppose you are pushing a lawn mower along flat, even ground. The handle makes an angle of $60°$ with the ground and you are pushing with a constant force of 40 lbs over a distance of 80 feet. Compute the amount of work done on the lawn mower over this distance.

Solution

Only the horizontal component of the force adds work to the mower in this situation (the vertical component is wasted by pushing the mower into the ground), so we first need to compute the horizontal force we are applying to the mower. Suppose the force of 40 lbs is along the handle. In that case, the horizontal component of the force is computed using some spectacular Right Triangle Trigonometry:

$$F_h = F\cos(60^\circ) = (40 \text{ lbs})(0.5) = 20 \text{ lbs}.$$

Hence, according to Equation 2.5.2, the work done on the mower over an 80-foot stretch of ground is

$$W = F_h \cdot d = (20 \text{ lbs}) \cdot (80 \text{ feet}) = 160 \text{ ft-lbs}.$$

Notice in Example **2.5.1** that I said "the work done on *the mower*," not "the work *we* did." When moving the mower, we are imparting energy to the mower; however, the energy we are imparting to the mower might not be (and, in this case, is not) the same as the energy we are expending in the process of moving the mower. Part of the energy we are using is pushing downward (into the ground) as we move the mower. This energy is not imparted to the mower and is completely wasted in the entire process. The only way to counteract this waste is to either push the mower from closer to ground level or build a mower tall enough that we are pushing from chest level. The issue with the former is that we would have to crawl or lean over the entire time, and the issue with the latter is that we would have to push with more force because the mower would be larger.

Before diving into an example, we need to discuss the concept of density briefly. The **volumetric density** (referred to as **density**) of a three-dimensional object is its mass per unit volume. In some disciplines, this is also known as the object's **specific mass**. If we let ρ, m, and V be the density, mass, and volume, respectively, then

$$\rho = \frac{m}{V} \implies m = \rho V. \tag{2.5.4}$$

Caution: Tricky Wording

What most people refer to as "density" in the U.S. Customary System is the **weight density** (also referred to as the **specific weight**) of the object. That is, since density is mass per unit volume, the units for density in the U.S. Customary System are technically slugs per cubic foot; however, you would search far and wide before you found anyone stating that the density of water is approximately 1.9 slugs/ft^3. Most people, even in scientific circles, cite the density of water (at $32^\circ F$ as approximately 62.4 lbs/ft^3.

This "loose" definition of density is scientifically inaccurate as the pound is a force and, therefore, the quantity being used is *force* per unit volume - not *mass* per unit volume. To avoid confusion, I use "weight density" when referring to force per unit volume. The Greek letter γ (lowercase "gamma") is traditionally used when referring to the weight density of an object, so I will use that convention in this text as well.

In light of the Cautionary statement above, it's best to (re)summarize our current knowledge in a table.

	Distance d	**Mass** m	**Gravitational Constant (at sea level)** g	**Force** $F = mg$	**Density** $\rho = \frac{m}{d^3}$	**Weight Density** $\gamma = \rho g = \frac{F}{d^3}$
English System	foot (ft)	slug	$\approx 32.2 \left(\frac{\text{ft}}{\text{s}^2}\right)$	pound $\left(\frac{\text{slug-ft}}{\text{s}^2}\right)$	$\left(\frac{\text{slugs}}{\text{ft}^3}\right)$	$\left(\frac{\text{lbs}}{\text{ft}^3}\right)$
Metric System	meter (m)	kilogram (kg)	$\approx 9.8 \left(\frac{\text{m}}{\text{s}^2}\right)$	newton $\left(\frac{\text{kg m}}{\text{s}^2} \equiv \text{N}\right)$	$\left(\frac{\text{kg}}{\text{m}^3}\right)$	$\left(\frac{\text{N}}{m^3}\right)$

The equations in 2.5.4 will become incredibly important as we move forward. It's best to get used to them now.

Example 2.5.2: The Work Required to Lift a Bucket (Part 1)

A 5-liter bucket is filled with water. By itself, the bucket has a mass of 2 kg.[3] How much work is done in lifting the bucket from the bottom of a 60-meter-deep well? (Water has a density of $\rho_{\mathrm{w}} = 1000\,\mathrm{kg/m^3}$, where $1\,\mathrm{L} = 0.001\,\mathrm{m^3}$).

Solution

The total mass of the bucket, including water, is

$$\begin{aligned} m &= m_{\mathrm{b}} + m_{\mathrm{w}} \\ &= m_{\mathrm{b}} + \rho_{\mathrm{w}} \cdot V_{\mathrm{w}} \\ &= 2\,\mathrm{kg} + 1\frac{\mathrm{kg}}{\cancel{\mathrm{L}}} \cdot 5\,\cancel{\mathrm{L}} \\ &= 7\,\mathrm{kg} \end{aligned}$$

Hence, according to Equation 2.5.3, the work done in lifting the bucket is

$$\begin{aligned} W &= m \cdot g \cdot d \\ &\approx (7\,\mathrm{kg})\left(9.8\,\frac{\mathrm{m}}{\mathrm{s^2}}\right)(60\,\mathrm{m}) \\ &= 4116\,\frac{\mathrm{kg\,m}}{\mathrm{s^2}} \cdot \mathrm{m} \\ &= 4116\,\mathrm{N\text{-}m} \\ &= 4116\,\mathrm{J} \end{aligned}$$

The end of Example 2.5.2 showcases a couple important "unit concepts." First, the newton is often denoted N, and the joule is often denoted J. Second, force in the metric system (which is measured in newtons) is $\frac{\mathrm{kg\,m}}{\mathrm{s^2}}$.

Work Done by a Variable Force

When we have a constant force, things are pretty easy. It is rare, however, for a force to be constant. For example, the work done to compress (or elongate) a spring varies depending on how far the spring has already been compressed (or stretched). We look at springs in more detail later in this section.

Suppose we have a variable force, $F(x)$, that moves an object in a positive direction along the x-axis from point a to point b. To calculate the work done, we partition the interval $[a, b]$ and estimate the work done over each subinterval. So, for $i = 0, 1, 2, \ldots, n$, let $P = \{x_i\}$ be a regular partition of the interval $[a, b]$, and for $i = 1, 2, \ldots, n$, choose an arbitrary point $x_i^* \in [x_{i-1}, x_i]$. To calculate the work done to move an object from point x_{i-1} to point x_i, we assume the force is roughly constant over this short interval, and use $F(x_i^*)$ to approximate the force used to move the object through this interval. The work done over the interval $[x_{i-1}, x_i]$, then, is given by

$$W_i \approx F(x_i^*)(x_i - x_{i-1}) = F(x_i^*)\Delta x.$$

Therefore, the work done over the interval $[a, b]$ is approximately

$$W \approx \sum_{i=1}^{n} W_i \approx \sum_{i=1}^{n} F(x_i^*)\Delta x.$$

Taking the limit of this expression as $n \to \infty$ gives us the exact value for work:

$$W = \lim_{n\to\infty}\sum_{i=1}^{n} F(x_i^*)\Delta x$$

$$= \int_a^b F(x)dx.$$

Thus, we can define work as follows.

Definition: Work

If a variable force $F(x)$ moves an object in a positive direction along the x-axis from point a to point b, then the **work** done on the object is

$$W = \int_a^b F(x)\,dx. \tag{2.5.5}$$

Note that if F is constant, the integral evaluates to $F\cdot(b-a) = Fd$, which is Equation 2.5.2.

Example 2.5.3

When a particle is located a distance x meters from the origin, a force of $\cos\left(\frac{\pi x}{3}\right)$ newtons acts on it. Compute the amount of work done on the particle in moving it between the following x-values.

a. From $x = 1$ to $x = 2$.
b. From $x = 1$ to $x = 1.5$.
c. From $x = 1.5$ to $x = 2$.

Solution

a. From Equation 2.5.5, we get

$$W = \int_1^2 \cos\left(\frac{\pi x}{3}\right)dx$$

$$= \frac{3}{\pi}\sin\left(\frac{\pi x}{3}\right)\Big|_{x=1}^{x=2}$$

$$= \frac{3}{\pi}\left(\sin\left(\frac{2\pi}{3}\right) - \sin\left(\frac{\pi}{3}\right)\right)$$

$$= \frac{3}{\pi}\left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right)$$

$$= 0$$

Therefore, no work is done on the particle in moving it from $x = 1$ to $x = 2$. If this seems strange, the next two parts *might* clarify what happened.

b. Repeating the setup from part a, we get

$$\begin{aligned} W &= \int_1^{3/2} \cos\left(\frac{\pi x}{3}\right) dx \\ &= \frac{3}{\pi}\sin\left(\frac{\pi x}{3}\right)\Big|_{x=1}^{x=3/2} \\ &= \frac{3}{\pi}\left(\sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{3}\right)\right) \\ &= \frac{3}{\pi}\left(1 - \frac{\sqrt{3}}{2}\right) \\ &\approx 0.1279 \end{aligned}$$

Thus, work of approximately 0.1279 joules have been imparted to the particle over the interval $[1, 1.5]$.

c. For our final calculation, we get

$$\begin{aligned} W &= \int_{3/2}^{2} \cos\left(\frac{\pi x}{3}\right) dx \\ &= \frac{3}{\pi}\sin\left(\frac{\pi x}{3}\right)\Big|_{x=3/2}^{x=2} \\ &= \frac{3}{\pi}\left(\sin\left(\frac{2\pi}{3}\right) - \sin\left(\frac{\pi}{2}\right)\right) \\ &= \frac{3}{\pi}\left(\frac{\sqrt{3}}{2} - 1\right) \\ &\approx -0.1279 \end{aligned}$$

In this case, work of approximately -0.1279 joules have been extracted from the particle over the interval $[1.5, 2]$. The work done during the first 0.5 meters of movement is undone in the second 0.5 meter. This explains why the total amount of work done over the interval $[1, 2]$ is zero.[4]

Spring-Mass Problems and Hooke's Law

Now let's look at the specific example of the work done to compress or elongate a spring. Consider a block attached to a horizontal spring.[5] The block moves back and forth as the spring stretches and compresses. This type of problem is called a **spring-mass system**, and it is a common topic in Differential Equations. Although, in the real world, we would have to account for the friction between the block and the surface on which it is resting, we ignore this here and assume the block is resting on a frictionless surface. When the spring is at its natural length (at rest), the system is said to be at **equilibrium**. In this state, the spring is neither elongated nor compressed, and in this **equilibrium position** the block does not move until some force is introduced. We orient the system such that $x = 0$ corresponds to the equilibrium position (Figure 2.5.1).

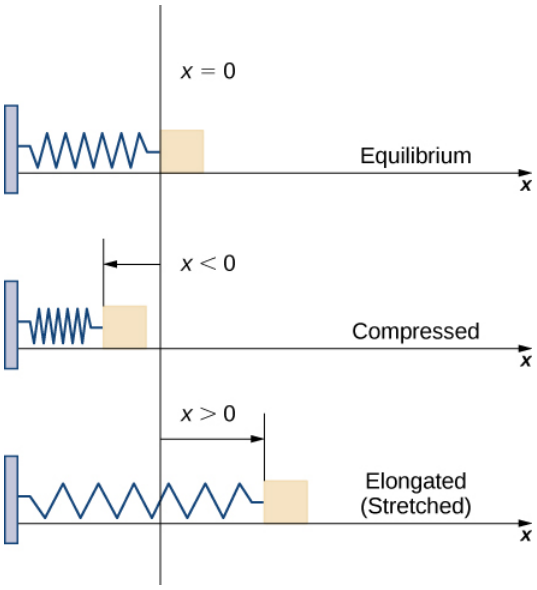


Figure 2.5.1: A block attached to a horizontal spring at equilibrium, compressed, and elongated.

According to Hooke's law, the force required to compress or stretch a spring from an equilibrium position is given by

$$F(x) = kx, \tag{2.5.6}$$

where $k > 0$ depends on the physical characteristics of the spring and, as such, is called the **spring constant**. We can use this information to calculate the work done to compress or elongate a spring, as shown in the following example.

Example 2.5.4: The Work Required to Stretch or Compress a Spring

Suppose it takes a force of 10 N (in the *negative* direction) to compress a spring 0.2 m from the equilibrium position. How much work is done to *stretch* the spring 0.5 m from the equilibrium position?

Solution

First, we find the spring constant, k.

When $x = -0.2$, we know $F(x) = -10$, so

$$\begin{aligned} F(x) &= kx \\ \implies -10 &= k(-0.2) \\ \implies k &= 50 \end{aligned}$$

and $F(x) = 50x$. Then, to calculate work, we integrate the force function, obtaining

$$\begin{aligned} W &= \int_a^b F(x)\,dx \\ &= \int_0^{0.5} 50x\,dx \\ &= 25x^2\Big|_0^{0.5} \\ &= 6.25. \end{aligned}$$

The work done to stretch the spring 0.5 m from equilibrium is 6.25 J.

? Checkpoint 2.5.4

Suppose it takes a force of 8 lb to stretch a spring 6 in. from the equilibrium position. How much work is done to stretch the spring 1 ft from the equilibrium position?

Answer

8 ft-lb

Lifting Distributed Masses

In Example 2.5.2, we lifted a simple object - a bucket filled with water. We can consider this bucket as a point mass. A **point mass** appropriately represents any object whenever its size, shape, and structure are irrelevant in a given context. That is, the shape and size of the bucket never entered our conversation when talking about lifting it - the mathematics doesn't change if we draw the bucket as a car, a dog, or a point. In fact, in the context of work, buckets, cars, dogs, books, baseballs, and dumbbells are all examples of point masses. There is no difference in the work done lifting them if we compress them to a single point with the same mass.

A **distributed mass**, on the other hand, refers to a system where the mass is spread out or distributed over a certain region or volume, instead of being concentrated at a single point. It is often used to describe objects or systems that have a continuous mass distribution. For our purposes, we will consider an object to be a distributed mass if we cannot consider it a point mass. Therefore, chains, cables, and ropes are all common examples of distributed masses in this text.

Computing the work to lift an object that can be considered a point mass is simple - replace the object with a point having the same mass and compute the work necessary to lift it using Equations 2.5.2, 2.5.3, or 2.5.5.

Computing the work to lift an object that cannot be considered a point mass (and is, therefore, a distributed mass) requires a little more thinking. For example, if we had a cable of length l hanging over a cliff of height h that we needed to pull to the top, we would consider that cable to be a distributed mass. As we pull the cable, the weight we are lifting (the force) becomes less and less because there is less cable to lift. How do we account for this phenomenon mathematically?

As with most of our processes in mathematics, we start by drawing a picture to help us conceptualize what is going on.

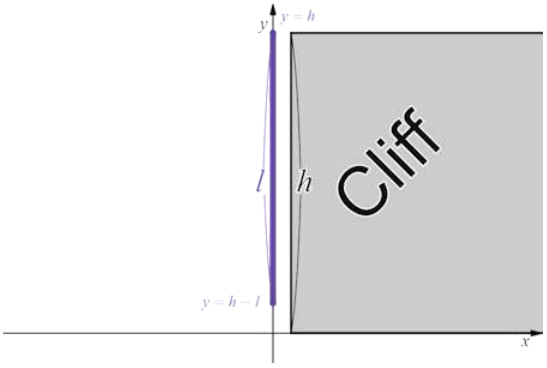


Figure 2.5.2: A cable hanging from the edge of a cliff.

I like to place the bottom of the cable on the x-axis, but if you prefer (or your instructor prefers) to place the x-axis at the top of the cliff, the only difference will be minor perspective changes.

Lifting the entire cable to the top of the cliff is the same as lifting small segments (slices) of the cable to the top, one by one. That is, we will partition the y-interval, $[h-l, h]$ into n equally-spaced subintervals. We need to compute the work required to lift the ith slice to the top of the cliff. From Equation 2.5.2, we know that

$$W_i = F_i d_i,$$

where F_i is the weight of the ith slice and d_i is the distance the ith slice needs to move to get to the top of the cliff. If we let $y_i^* \in [y_{i-1}, y_i]$, then the distance from the ith slice to the top of the cliff is approximately $d_i^* = h - y_i^*$ (see Figure 2.5.3 below).

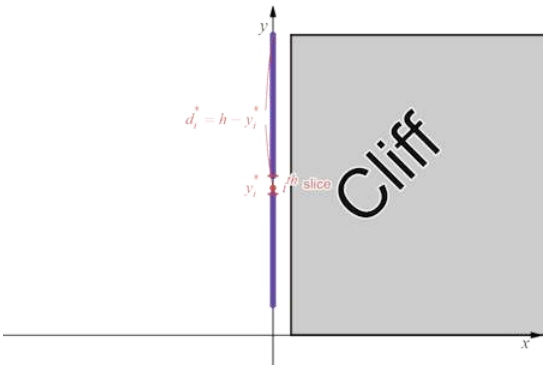


Figure 2.5.3: Lifting the ith slice a distance d_i^*.

That is,

$$W_i \approx F_i d_i^* = F_i (h - y_i^*) .$$

Moreover, if F represents the force (weight) of the entire cable we need to lift, we can compute the unit force of the cable, $\frac{F}{l}$, and multiply that result by the width of the ith slice, Δy, to arrive at

$$F_i = \frac{F}{l} \cdot \Delta y.$$

A little unit analysis shows that our argument is solid: since the unit length is in pounds per foot and Δy is in feet, the result is in pounds - a force!

In the end, the work to lift the ith slice is

$$W_i \approx F_i d_i^* = F_i (h - y_i^*) = \frac{F}{l} \Delta y (h - y_i^*) .$$

As we have done many times up to this point, we would sum all of these little pieces of work, which creates a Riemann sum, take the limit as $n \to \infty$, and arrive at an integral; however, rather than providing a closed-form integral, it's best to derive a proper integral for each situation - this is because these types of problems can vary wildly, and knowing the derivation will provide you with a *much* better understanding of the physics in the background.

Example 2.5.5: The Work Required to Lift a Bucket (Part 2)

A chain is attached to a 5-liter bucket which is filled with molten gold. The bucket has a mass of 80 kg (it has to be made of strong stuff to hold molten metals). The chain has a mass of 3.1 kg per meter. How much work is done in lifting the bucket from the bottom of a 60-meter-deep well? (Molten gold has a density of $\rho_g = 17,310\,\text{kg/m}^3$, and $1,000\,\text{L} = 1\,\text{m}^3$).

Solution

It is wise to treat this problem as lifting two separate masses - the point mass (the bucket of molten gold) and the distributed mass (the chain).

Work to Lift the Point Mass

The work to lift the bucket is

$$\begin{aligned} W_{\mathrm{b}} &= F_{\mathrm{b}} d_{\mathrm{b}} \\ &= \left(m_{\mathrm{b}} + m_{\mathrm{g}}\right) g d_{\mathrm{b}} \\ &\approx \left(80\,\mathrm{kg} + \rho_{\mathrm{g}} V_{\mathrm{g}}\right)\left(9.8\,\mathrm{m/s^2}\right)\left(60\,\mathrm{m}\right) \\ &\approx \left(80\,\mathrm{kg} + 17,310\,\tfrac{\mathrm{kg}}{\mathrm{m^3}} \cdot 5\,\mathrm{L}\right)\left(9.8\,\tfrac{\mathrm{m}}{\mathrm{s^2}}\right)\left(60\,\mathrm{m}\right) \\ &= \left(80\,\mathrm{kg} + 17,310\,\tfrac{\mathrm{kg}}{\cancel{\mathrm{m^3}}} \cdot 5\,\tfrac{1}{1000}\,\cancel{\mathrm{m^3}}\right)\left(9.8\,\tfrac{\mathrm{m}}{\mathrm{s^2}}\right)\left(60\,\mathrm{m}\right) \\ &= \left(80\,\mathrm{kg} + 86.55\,\mathrm{kg}\right)\left(9.8\,\tfrac{\mathrm{m}}{\mathrm{s^2}}\right)\left(60\,\mathrm{m}\right) \\ &= \left(166.55\right)\left(9.8\right)\left(60\right)\tfrac{\mathrm{kg\,m}}{\mathrm{s^2}} \cdot \mathrm{m} \\ &\approx 97,931\,\mathrm{J} \end{aligned}$$

Work to Lift the Distributed Mass

The work to lift the ith slice of the chain is

$$W_i = F_i d_i = m_i g d_i \approx \left(9.8\frac{\mathrm{m}}{\mathrm{s^2}}\right) m_i d_i.$$

Assuming the bottom of the well is the x-axis and $y_i^* \in [y_{i-1}, y_i]$, then $d_i \approx d_i^* = 60\,\mathrm{m} - y_i^*\,\mathrm{m} = (60 - y_i^*)\,\mathrm{m}$. Therefore,

$$W_i \approx \left(9.8\frac{\mathrm{m}}{\mathrm{s^2}}\right) m_i \left(60 - y_i^*\right)\,\mathrm{m}.$$

To compute the mass of the ith slice, we need the mass per unit length of the chain and then we need to multiply that by the width of the ith slice. Luckily, we were told the mass per unit length of the chain was 3.1 kg per meter. Thus, $m_i = \left(3.1\,\tfrac{\mathrm{kg}}{\cancel{\mathrm{m}}}\right)\Delta y\,\cancel{\mathrm{m}} = 3.1\Delta y\,\mathrm{kg}$. Therefore,

$$W_i \approx \left(9.8\frac{\mathrm{m}}{\mathrm{s^2}}\right)\left(3.1\Delta y\,\mathrm{kg}\right)\left(60 - y_i^*\right)\,\mathrm{m} = (9.8)(3.1)(60 - y_i^*)\Delta y\,\mathrm{J}.$$

Forming a Riemann sum, taking the limit, and transposing it into an integral, we get the work to lift the chain (by itself) to be

$$\begin{aligned} W_{\mathrm{c}} &= \int_{y=0}^{y=60} (9.8)(3.1)(60 - y)\,dy \\ &= 30.38\left(60y - \tfrac{y^2}{2}\right)\Big|_0^{60} \\ &= 54,684\,\mathrm{J} \end{aligned}$$

Total Work

We now need to add these two computations.

$$W = W_{\mathrm{b}} + W_{\mathrm{c}} \approx 152,615\,\mathrm{J}.$$

The limits of the integral in Example $2.5.5$ can be read as, "We are moving slices starting at the height of $y = 0$ and ending at the height of $y = 60$." It is important to understand this. If the well were 60 meters deep, but we were only lifting the chain 10 meters, then we would consider the length of the chain from $y = 0$ to $y = 50$ as a point mass being lifted only 10 meters. We would then lift the upper 10 meters using the integral

$$\int_{y=50}^{y=60} (9.8)(3.1)(60-y)\,dy,$$

because it is only that section that loses mass as we lift the chain.

Lifting Point Masses that Gain or Lose Mass

Suppose an object is a point mass that starts with a mass of m_{initial} and ends with a mass m_{final}, where $m_{\text{initial}} \neq m_{\text{final}}$ and where the gain or loss of mass is linear (i.e., the mass gains or loses the same amount of mass per unit length). For purposes of this discussion, we will assume that the object has lost mass. Still, you only need to adjust a single part of your thought process to accommodate an object gaining mass.

To see how we approach this type of problem, it's best to consider an example.

Example 2.5.6: The Work Required to Lift a Bucket (Part 3)

A chain is attached to a 5-liter *leaky* bucket which is filled with molten gold. The leak is such that the molten gold drains from the bucket at a constant rate, and by the time the bucket reaches the top of the well, the bucket is 80% empty. The bucket has a mass of 80 kg. The chain has a mass of 3.1 kg per meter. How much work is done in lifting the bucket from the bottom of a 60-meter-deep well? (Molten gold has a density of $\rho_g = 17,310\ \text{kg/m}^3$, and $1,000\,\text{L} = 1\,\text{m}^3$).

Solution

Again, we treat this as two separate problems - the leaky point mass (the leaky bucket of molten gold) and the distributed mass (the chain).

Work to Lift the Leaky Point Mass

From our previous work, we know that the bucket and the molten gold have a combined mass of 166.55 kg; however, we are now told that the bucket loses 80% of mass due to molten gold by the time it reaches the top of the well. Since it drained at a constant rate, we can safely say that the bucket loses $0.80\left(\frac{1731}{20}\right) = 69.24$ kg of mass over the 60-meter lift. That is, it loses 1.154 kg per meter.

Consider partitioning the interval $[0, 60]$ into n equally-spaced subintervals. The distance to move the mass from y_{i-1} to y_i is Δy. Therefore, the work needed to lift this mass in this small interval is

$$W_i = F_i d_i = m_i g d_i = m_i \left(9.8\,\frac{\text{m}}{\text{s}^2}\right)\Delta y.$$

If $y_i^* \in [y_{i-1}, y_i]$, then we can consider the mass of the object in the small interval $[y_{i-1}, y_i]$ to approximately be

$$m_i^* \approx m_{\text{initial}} - m_{\text{lost}} = 166.55\,\text{kg} - 1.154\,\frac{\text{kg}}{\cancel{\text{m}}}\,y_i^*\,\cancel{\text{m}} = (166.55 - 1.154y_i^*)\,\text{kg}.$$

Thus, the work to lift the leaky bucket is

$$\begin{aligned} W_{\text{b}} &= \int_{y=0}^{y=60} (166.55 - 1.154y)(9.8)\,dy \\ &= 9.8\left(166.55y - 0.577y^2\right)\Big|_{y=0}^{y=60} \\ &= 77,574.84\,\text{J} \end{aligned}$$

Work to Lift the Distributed Mass

Nothing changes about this part, so we can use the 54,684 J computed previously.

Total Work

The total work done is

$$W = W_b + W_c = 132,258.84\,\text{J}.$$

A quick logic check helps here. Notice this is *less* work than before. That should make sense as we are lifting a bucket with a hole!

Lifting Fluids

Consider the work done to pump water (or some other liquid) out of a tank. Pumping problems are more complicated than spring problems because many of the calculations depend on the shape and size of the tank. In addition, instead of being concerned about the work done to move a single mass, we are looking at the work done to move a volume of water. It takes more work to move the water from the bottom of the tank (which has to be moved farther) than moving the water from the top of the tank.

We examine the process in the context of a cylindrical tank, then look at a couple of examples using tanks of different shapes.

Assume a cylindrical tank of radius 4 m and height 10 m is filled to a depth of 8 m. How much work does it take to pump all the water over the top edge of the tank?

The first thing we need to do is define a frame of reference. We let x represent the vertical distance below the top of the tank. That is, we orient the x-axis vertically, with the origin at the top of the tank and the downward direction being positive (Figure 2.5.4).[6]

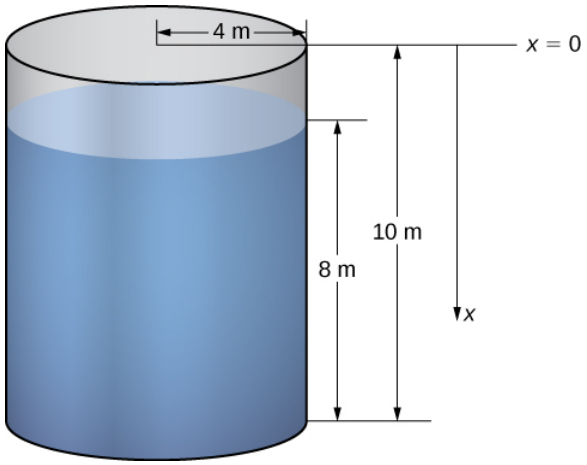


Figure 2.5.4: How much work is needed to empty a tank partially filled with water?

Using this coordinate system, the water extends from $x = 2$ to $x = 10$.[7] Therefore, we partition the interval $[2, 10]$ and look at the work required to lift each individual "layer" of water. So, for $i = 0, 1, 2, \ldots, n$, let $P = x_i$ be a regular partition of the interval $[2, 10]$, and for $i = 1, 2, \ldots, n$, choose an arbitrary point $x_i^* \in [x_{i-1}, x_i]$. Figure 2.5.5 shows a representative layer.[8]

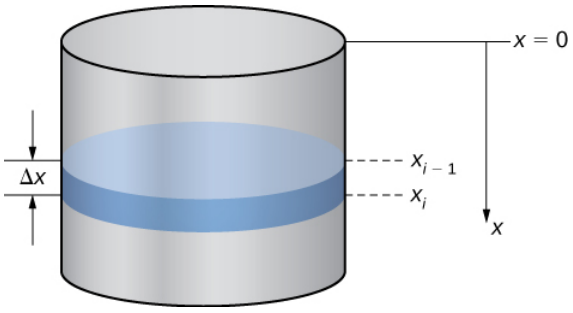


Figure 2.5.5: A representative layer of water.

In pumping problems, the force required to lift the water to the top of the tank is the force required to overcome gravity, so it is equal to the weight of the water. In the metric system, the density of water is $\rho_w = 1000\,\text{kg/m}^3$, which means the *weight density* is

$$\gamma_w = \rho_w g \approx 1000\,\frac{\text{kg}}{\text{m}^3} \cdot 9.8\,\frac{\text{m}}{\text{s}^2} = 9800\,\frac{\text{kg m/s}^2}{\text{m}^3} = 9800\,\frac{\text{N}}{\text{m}^3}. \tag{2.5.7}$$

In the U.S. Customary System, the *weight density* of water is $\gamma_w \approx 62.4\,\text{lb/ft}^3$.

Calculating the volume of each layer gives us the weight. In this case, we have

$$V = \pi(4)^2 \Delta x = 16\pi\Delta x.$$

The force needed to lift each layer is

$$F = 9800 \cdot 16\pi\Delta x = 156,800\pi\Delta x.$$

Note that this step becomes a little more difficult if we have a noncylindrical tank (that will be our next example).

We also need to know the distance the water must be lifted. Based on our choice of coordinate systems, we can use x_i^* as an approximation of the distance the layer must be lifted.[9] Then the work to lift the ith layer of water W_i is approximately[10]

$$W_i \approx 156,800\pi x_i^* \Delta x.$$

Adding the work for each layer, we see the approximate work to empty the tank is given by

$$\begin{aligned} W &\approx \sum_{i=1}^{n} W_i \\ &\approx \sum_{i=1}^{n} 156,800\pi x_i^* \Delta x. \end{aligned}$$

This is a Riemann sum, so taking the limit as $n \to \infty$, we get

$$\begin{aligned} W &= \lim_{n\to\infty} \sum_{i=1}^{n} 156,800\pi x_i^* \Delta x \\ &= 156,800\pi \int_2^{10} x\, dx \\ &= 156,800\pi \left(\frac{x^2}{2}\right)\Big|_2^{10} \\ &= 7,526,400\pi \\ &\approx 23,644,883. \end{aligned}$$

The work required to empty the tank is approximately 23,650,000 J.[11]

For pumping problems, the calculations vary depending on the shape of the tank or container. The following problem-solving strategy lays out a step-by-step process for solving pumping problems.

Problem-Solving Strategy: Solving Pumping Problems

1. Sketch a picture of the tank and select an appropriate frame of reference.
2. Calculate the volume of a representative layer of water.
3. Multiply the volume by the weight-density of water to get the force.
4. Calculate the distance the layer of water must be lifted.
5. Multiply the force and distance to estimate the work needed to lift the layer of water.
6. Sum the work required to lift all the layers. This expression is an estimate of the work needed to pump out the desired amount of water, and it is in the form of a Riemann sum.
7. Take the limit as $n \to \infty$ and evaluate the resulting integral to get the exact work required to pump out the desired amount of water.

We now apply this problem-solving strategy in an example with a noncylindrical tank.

Example 2.5.7: A Pumping Problem with a Noncylindrical Tank

Assume a tank in the shape of an inverted cone, with height 12 ft and base radius 4 ft. The tank is full to start with, and water is pumped over the upper edge of the tank until the height of the water remaining in the tank is 4 ft. How much work is required to pump out that amount of water?

Solution

The tank is depicted in Figure 2.5.6. As we did in the example with the cylindrical tank, we orient the x-axis vertically, with the origin at the top of the tank and the downward direction being positive.

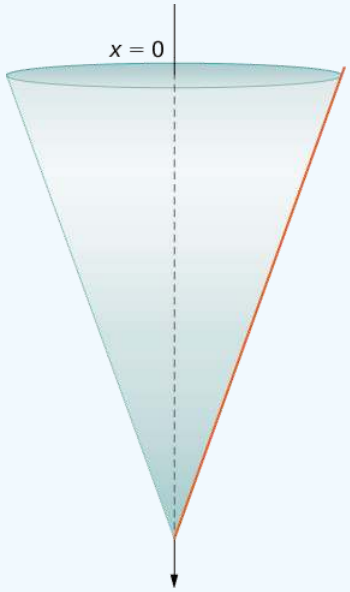


Figure 2.5.6: A water tank shaped like an inverted cone.

The tank starts out full and ends with 4 ft of water left, so, based on our chosen frame of reference, we need to partition the interval $[0, 8]$. Then, for $i = 0, 1, 2, \ldots, n$, let $P = x_i$ be a regular partition of the interval $[0, 8]$, and for $i = 1, 2, \ldots, n$, choose an arbitrary point $x_i^* \in [x_{i-1}, x_i]$. We can approximate the volume of a layer by using a disk, then use similar triangles to find the radius of the disk (Figure 2.5.7).

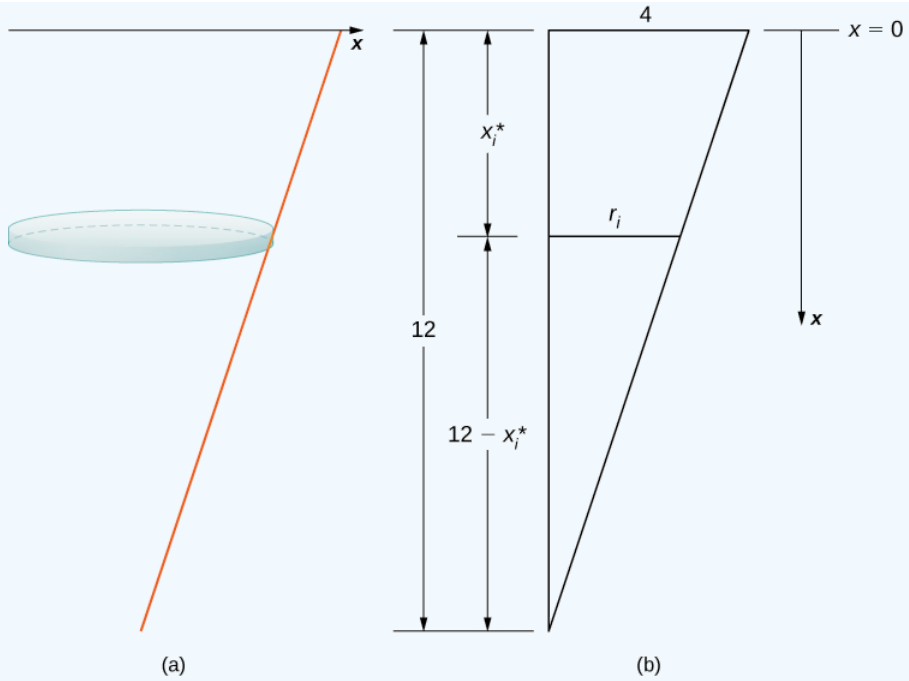


Figure **2.5.7**: Using similar triangles to express the radius of a disk of water.

From properties of similar triangles, we have

$$\begin{aligned} \frac{r_i}{12-x_i^*} &= \frac{4}{12} = \frac{1}{3} \\ \Longrightarrow \quad 3r_i &= 12 - x_i^* \\ \Longrightarrow \quad r_i &= \frac{12-x_i^*}{3} \\ \Longrightarrow \quad r_i &= 4 - \frac{x_i^*}{3}. \end{aligned}$$

Then the volume of the disk is

$$V_i = \pi \left(4 - \frac{x_i^*}{3}\right)^2 \Delta x.$$

The *weight density* of water is $62.4\ \text{lb/ft}^3$, so the force needed to lift each layer is approximately

$$F_i \approx 62.4\pi \left(4 - \frac{x_i^*}{3}\right)^2 \Delta x$$

Based on the diagram, the distance the water must be lifted is approximately x_i^* feet, so the approximate work needed to lift the layer is

$$W_i \approx 62.4\pi x_i^* \left(4 - \frac{x_i^*}{3}\right)^2 \Delta x.$$

Summing the work required to lift all the layers, we get an approximate value of the total work:

$$W = \sum_{i=1}^{n} W_i \approx \sum_{i=1}^{n} 62.4\pi x_i^* \left(4 - \frac{x_i^*}{3}\right)^2 \Delta x.$$

Taking the limit as $n \to \infty$, we obtain

$$
\begin{aligned}
W &= \lim_{n\to\infty}\sum_{i=1}^{n} 62.4\pi x_i^*(4-\tfrac{x_i^*}{3})^2\Delta x \\
&= \int_0^8 62.4\pi x\left(4-\frac{x}{3}\right)^2 dx \\
&= 62.4\pi\int_0^8 x\left(16-\frac{8x}{3}+\frac{x^2}{9}\right) dx \\
&= 62.4\pi\int_0^8 \left(16x-\frac{8x^2}{3}+\frac{x^3}{9}\right) dx \\
&= 62.4\pi\left[8x^2-\frac{8x^3}{9}+\frac{x^4}{36}\right]\Big|_0^8 \\
&= 10,649.6\pi \\
&\approx 33,456.7.
\end{aligned}
$$

It takes approximately $33,450$ ft-lb of work to empty the tank to the desired level.[12]

? Checkpoint 2.5.7

A tank is shaped like an inverted cone, with height 10 ft and a base radius of 6 ft. The tank is filled to a depth of 8 ft to start with, and water is pumped over the upper edge of the tank until 3 ft of water remains in the tank. How much work is required to pump out that amount of water?

Answer

Approximately $43,255.2$ ft-lb

Footnotes

[1] In more complex systems, you could have several factors contributing different accelerations. For example, in a car, you experience acceleration due to gravity (otherwise, your car would begin to rise in the air) and acceleration due to the movement of the car.

[2] For a great diversion into why some argue the unit of force in the metric system is the Newton, and not the newton, I encourage the reader to take a look at this excerpt by the Australian physicist, Don Koks.

[3] We do not say, "The bucket has a *weight* of 2 kg," because kilograms is a measure of mass - weight is a measure of force. Mass is unchangeable (unless something gets added to or removed from the object); however, weight *can* change depending on the mass of the planet and your position on that planet. That is, according to Equation 2.5.1, the 2-kg bucket has a *weight* of

$$F = mg \approx (2\,\text{kg})\left(9.8\,\frac{\text{kg m}}{\text{s}^2}\right) \approx 19.6\,\text{N}$$

at sea level on the Earth. However, on the surface of Jupiter, the bucket will still have a mass of 2 kg, but its weight would be

$$F = mg \approx (2\,\text{kg})\left(24.5\,\frac{\text{kg m}}{\text{s}^2}\right) = 49\,\text{N}.$$

[4] As confounding as this might seem, from a physics perspective it is correct. The "tripping" point for most students is a perspective issue. For example, if a book is lifted straight up 3 feet and then lowered the same 3 feet, it makes sense to all of us that *we* expended energy in the lifting and lowering process; however, we need to change our perspective when it comes to work problems and physics. Instead of asking how much work (energy) have *we* have spent, we need to focus on how much work (energy) the

object (the book) has gained or lost over that process. This concept is related to the conservation of energy. When you lift the object, you provide potential energy to it, which is converted from the energy you expend. When you lower the object back down, the potential energy is converted back into the same amount of energy you initially provided. No additional work is done on the object in the overall process.

[5] The choice of having the spring oriented horizontally is not essential. We could have written this entire derivation assuming the spring was attached to the ceiling and oriented vertically. In that case, the block would have been connected to the bottom of the spring. It is important to clarify that, in the vertical orientation, the equilibrium position is the point where the **restorative force** of the spring is counteracted by the weight of the mass. That is, the equilibrium position would be where the spring stretches to once the mass is attached.

[6] Your instructor might take a different approach for the frame of reference for fluid-work problems. For example, it is perfectly acceptable to place the *bottom* of the tank on the x-axis (centered at the origin) and still consider the upward direction as the positive y-axis - this is how I normally teach it.

[7] Or, from $y = 0$ to $y = 8$, if your instructor is using the standard coordinate system.

[8] Again, if your instructor is using the standard coordinate system, then the intervals are $[0, 8]$.

[9] If using the standard coordinate system, the layer we are considering is at a "height" (from the bottom of the tank) of y_i^* m. To move this layer to the top of the tank, we must move it $(10 - y_i^*)$ m.

[10] Again, if using the standard coordinate system, the work to lift this i^{th} layer is $W_i = F_i \times d_i \approx 156,800\pi \times (10 - y_i^*)$.

[11] You would have reached this same value using the standard coordinate system and computing

$$W = \int_{y=0}^{y=8} 156,800\pi(10 - y)\, dy.$$

Conceptually, you would read this as, "I am lifting plates of fluid, starting at a height of 0 meters (the bottom), and the final plate is at a height of 8 meters. Each plate weighs $156,800$ newtons and is being lifted 10 minus its height in meters."

[12] Done using the standard coordinate system, we would place the *bottom* of the tank on the x-axis, centered at the origin. We would need to lift the slices of water starting with the slice at $y = 4$ feet and ending with the slice at $y = 12$ feet (leaving the bottom 4 feet). The volume of the ith slice is

$$V_i = \text{surface area} \times \Delta y = \pi r_i^2 \Delta y.$$

Visually, we can see that $r_i = x_i^*$, but we need a relationship between x_i^* and y_i^* to continue. Luckily, we can find the equation of the edge of the tank. It's a line going through the origin having slope $\frac{12}{4} = 3$. Hence, $y_i^* = 3x_i^*$. That is, $x_i^* = \frac{y_i^*}{3}$. Therefore,

$$V_i = \pi \left(\frac{y_i^*}{3}\right)^2 \Delta y = \frac{\pi}{9}(y_i^*)^2 \Delta y.$$

Hence, the force needed to lift this layer is

$$F_i \approx 62.4 \cdot \frac{\pi}{9}(y_i^*)^2 \Delta y.$$

We need to lift this layer $(12 - y_i^*)$ feet, so the work done moving this specific layer is

$$W_i \approx 62.4 \cdot \frac{\pi}{9}(y_i^*)^2 (12 - y_i^*) \Delta y.$$

Thus,

$$W = 62.4 \cdot \frac{\pi}{9} \int_{y=4}^{y=12} y^2(12 - y)\, dy$$

This will become approximately 33,456.7 ft-lb.

2.5E: Exercises

Reading Questions

1. What is the formula for the work done by a constant force F over a distance d?
2. How is work calculated when the force, $F(x)$, is not constant over the interval from $x = a$ to $x = b$?
3. What is Hooke's Law, and what does the spring constant k represent?
4. What is the key difference between a "point mass" and a "distributed mass" in the context of work problems?
5. When setting up an integral to find the work required to pump a liquid out of a tank, what three quantities must be determined for a representative "slice" of the liquid?
6. Why is the force required to wind up a hanging cable considered a variable force?
7. What is the difference between mass density and weight density, and which one is used to calculate the force on a slice of water in a pumping problem?

Basic Work Problems

For exercises 1 - 6, find the work done.

1) Find the work done when a constant force $F = 12$ lb moves a chair from $x = 0.9$ to $x = 1.1$ ft.

2) How much work is done when a person lifts a 50 lb box of comics onto a truck that is 3 ft off the ground?

Answer

$W = 150$ ft-lb

3) What is the work done lifting a 20 kg child from the floor to a height of 2 m? (Note that 1 kg equates to 9.8 N)

4) Find the work done when you push a box along the floor 2 m, when you apply a constant force of $F = 100$ N.

Answer

$W = 200$ J

5) Compute the work done for a force $F = \frac{12}{x^2}$ N from $x = 1$ to $x = 2$ m.

6) What is the work done moving a particle from $x = 0$ to $x = 1$ m if the force acting on it is $F = 3x^2$ N?

Answer

$W = 1$ J

Spring Work Problems

7) A 12-in. spring is stretched to 15 in. by a force of 75 lb. What is the spring constant?

8) A spring has a natural length of 10 cm. It takes 2 J to stretch the spring to 15 cm. How much work would it take to stretch the spring from 15 cm to 20 cm?

Answer

$W = 6$ J

9) A 1-m spring requires 10 J to stretch the spring to 1.1 m. How much work would it take to stretch the spring from 1 m to 1.2 m?

10) A spring requires 5 J to stretch the spring from 8 cm to 12 cm, and an additional 4 J to stretch the spring from 12 cm to 14 cm. What is the natural length of the spring?

Answer

The natural length is5 cm.

11) A shock absorber is compressed 1 in. by a weight of 1 ton. What is the spring constant?

12) A force of $F = (20x - x^3)$ N stretches a nonlinear spring by x meters. What work is required to stretch the spring from $x = 0$ to $x = 2$ m?

Answer

$W = 36$ J

Cable and Chain Work Problems

13) Find the work done by winding up a hanging cable of length 100 ft and *weight density* 5 lb/ft.

14) For the cable in the preceding exercise, how much work is done to lift the cable 50 ft?

Answer

$W = 18,750$ ft-lb

15) For the cable in the preceding exercise, how much additional work is done by hanging a 200 lb weight at the end of the cable?

Pyramid & Satellite/Rocket Work Problems

16) [Technology Required] A pyramid of height 500 ft has a square base 800 ft by 800 ft. Find the area A at height h. If the rock used to build the pyramid weighs approximately $w = 100\ \text{lb/ft}^3$, how much work did it take to lift all the rock?

Answer

$W = \frac{32}{3} \times 10^9$ ft-lb

17) [Technology Required] For the pyramid in the preceding exercise, assume there were 1000 workers each working 10 hours a day, 5 days a week, 50 weeks a year. If each of the workers, on average, lifted ten 100-lb rocks 2 ft/hr, how long did it take to build the pyramid?

18) [Technology Required] The force of gravity on a mass m is $F = -((GMm)/x^2)$ newtons. For a rocket of mass $m = 1000$ kg, compute the work to lift the rocket from $x = 6400$ to $x = 6500$ km. (Note: $G = 6 \times 10^{-17}\ \text{N m}^2/\text{kg}^2$ and $M = 6 \times 10^{24}$ kg.)

Answer

$W = 8.65 \times 10^5$ J

19) [Technology Required] For the rocket in the preceding exercise, find the work to lift the rocket from $x = 6400$ to $x = \infty$.

Pumping Work Problems

20) [Technology Required] Find the work required to pump all the water out of a cylinder that has a circular base of radius 5ft and height 200 ft. Use the fact that the *weight density* of water is 62 lb/ft^3.

21) [Technology Required] Find the work required to pump all the water out of the cylinder in the preceding exercise if the cylinder is only half full.

Answer

$W = 23.25\pi$ million ft-lb

22) [Technology Required] How much work is required to pump out a swimming pool if the area of the base is $800\ \text{ft}^2$, the water is 4 ft deep, and the top is 1 ft above the water level? Assume that the *weight density* of water is 62 lb/ft^3.

23) A cylinder of depth H and cross-sectional area A stands full of water at density ρ. Compute the work to pump all the water to the top.

Answer

$W = \frac{A\rho H^2}{2}$

24) For the cylinder in the preceding exercise, compute the work to pump all the water to the top if the cylinder is only half full.

25) A cone-shaped tank has a cross-sectional area that increases with its depth: $A = \frac{\pi r^2 h^2}{H^3}$. Show that the work to empty it is half the work for a cylinder with the same height and base.

Answer

Answers may vary.

Contributors

Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

This page titled 2.5E: Exercises is shared under a CC BY-NC 4.0 license and was authored, remixed, and/or curated by Roy Simpson.

- **6.5E: Exercises for Section 6.5** by Edwin “Jed” Herman, Gilbert Strang is licensed CC BY-NC-SA 4.0. Original source: https://openstax.org/details/books/calculus-volume-1.

2.6: Hydrostatic Force and Pressure

Note to the Instructor (click to expand)

This topic flows naturally from Work. It's a short section, but can be entertaining and strangely useful.

To succeed in this section, you'll need to use some skills from previous courses. While you should already know them, this is the first time they've been required. You can review these skills in CRC's Corequisite Codex. If you have a support class, it might cover some, **but not all**, of these topics.

Prerequisite Skills and Support Topics (click to expand)

- **Linear Functions**
  - Equations of Lines: In problems involving non-rectangular submerged surfaces, such as the trapezoidal dam in Example 1.7.2, it is necessary to find a linear function for the width of the plate, $w(x)$. This often involves finding the equation of the line that forms the slanted edge of the object.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Find the hydrostatic force against a submerged vertical plate.

Hawk A.I. Section-Specific Tutor

COSUMNES RIVER COLLEGE MATHEMATICS

To access the Hawk A.I. Tutor, you will need to be logged into your campus Gmail account.

Hydrostatic Force and Pressure

Let's look at the force and pressure exerted on an object submerged in a liquid. **Pressure**, P, is force per unit area. That is,

$$P = \frac{F}{A}. \tag{2.6.1}$$

Therefore, in the English system, we have pounds per square foot (or, perhaps more commonly, pounds per square inch, denoted psi). In the metric system, we have newtons per square meter, also called **pascals**.

We begin with the simple case of a plate of area A submerged horizontally in water at a depth s (Figure 2.6.1). The force exerted on the plate is simply the weight of the water above it, which gives

$$F = \rho A s, \tag{2.6.2}$$

where ρ is the volumetric density of water. To find the **hydrostatic pressure** - that is, the pressure exerted by water on a submerged object - we substitute F from Equation 2.6.2 into Equation 2.6.1 so that

$$P = \frac{F}{A} = \frac{\rho A s}{A} = \rho s.$$

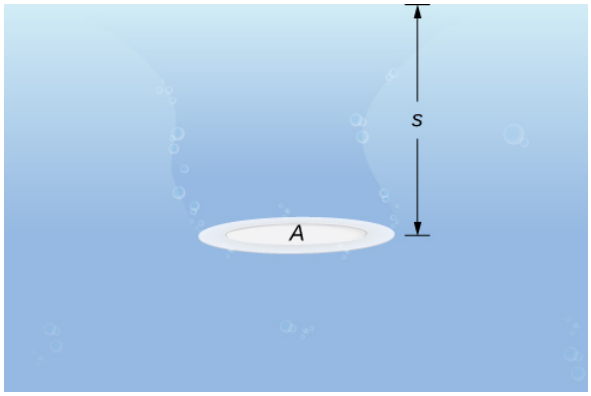


Figure 2.6.1: A plate submerged horizontally in water.

By Pascal's principle, the pressure at a given depth is the same in all directions, so it does not matter if the plate is submerged horizontally or vertically. So, as long as we know the depth, we know the pressure. We can apply Pascal's principle to find the force exerted on surfaces, such as dams, that are oriented vertically. We *cannot*, however, apply the Equation 2.6.2 directly, because the depth varies from point to point on a vertically-oriented surface. So, as we have done many times before, we form a partition, a Riemann sum, and, ultimately, a definite integral to calculate the force.

Suppose a thin plate is submerged in water. We choose our frame of reference such that the x-axis is oriented vertically, with the downward direction being positive, and point $x = 0$ corresponding to a logical reference point.[1] Let $s(x)$ denote the depth at point x. Note we often let $x = 0$ correspond to the surface of the water. In this case, depth at any point is given by $s(x) = x$. However, in some cases we may want to select a different reference point for $x = 0$, so we proceed with the development in the more general case. Finally, let $w(x)$ denote the width of the plate at the point x.

Assume the top edge of the plate is at point $x = a$ and the bottom edge of the plate is at point $x = b$. Then, for $i = 0, 1, 2, \ldots, n$, let $P = \{x_i\}$ be a regular partition of the interval $[a, b]$, and for $i = 1, 2, \ldots, n$, choose an arbitrary point $x_i^* \in [x_{i-1}, x_i]$. The partition divides the plate into several thin, rectangular strips (Figure 2.6.2).

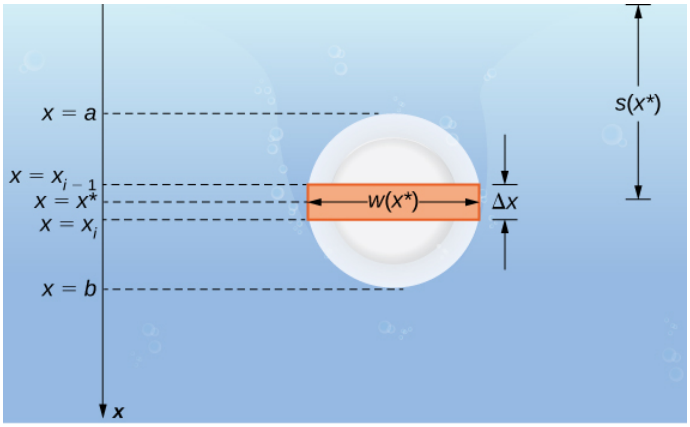


Figure 2.6.2: A thin plate submerged vertically in water.

Let's now estimate the force on a representative strip. If the strip is thin enough, we can treat it as if it is at a constant depth, $s(x_i^*)$. We then have

$$F_i = \rho A s = \rho[w(x_i^*)\Delta x]s(x_i^*).$$

Adding the forces, we get an estimate for the force on the plate:

$$F \approx \sum_{i=1}^{n} F_i = \sum_{i=1}^{n} \rho[w(x_i^*)\Delta x]s(x_i^*).$$

This is a Riemann sum, so taking the limit gives us the exact force. We obtain

$$F = \lim_{n\to\infty} \sum_{i=1}^{n} \rho[w(x_i^*)\Delta x]s(x_i^*) = \int_a^b \rho w(x)s(x)\,dx. \tag{2.6.3}$$

Evaluating this integral gives us the force on the plate. We summarize this in the following problem-solving strategy.

Problem-Solving Strategy: Finding Hydrostatic Force

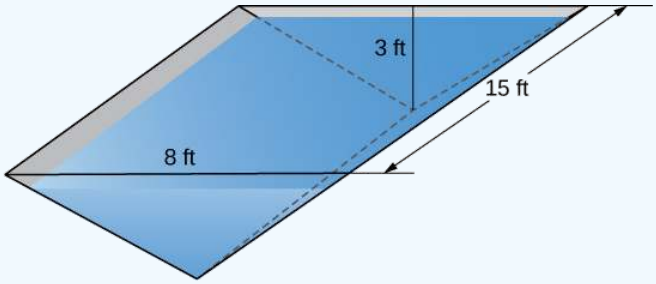
1. Sketch a picture and select an appropriate frame of reference. (Note that if we select a frame of reference other than the one used earlier, we may have to adjust Equation 2.6.3 accordingly.)
2. Determine the depth and width functions, $s(x)$ and $w(x)$.
3. Determine the *weight density* of whatever liquid with which you are working. The *weight density* of water is $62.4\,\text{lb/ft}^3$, or $9800\,\text{N/m}^3$.
4. Use the equation to calculate the total force.

Example 2.6.1: Finding Hydrostatic Force

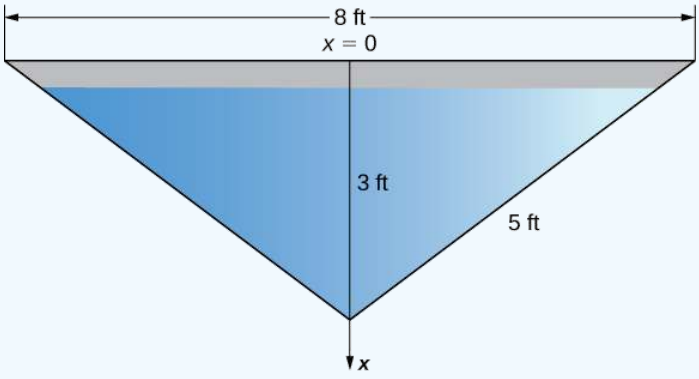
A water trough 15 ft long has ends shaped like inverted isosceles triangles, with a base of 8 ft and a height of 3 ft. Find the force on one end of the trough if the trough is full of water.

Solution

Figure 2.6.3 shows the trough and a more detailed view of one end.



(a)



(b)

Figure 2.6.3*a*: A water trough with a triangular cross-section.
Figure 2.6.3*b*: Dimensions of one end of the water trough.

Select a frame of reference with the x-axis oriented vertically and the downward direction being positive. Select the top of the trough as the point corresponding to $x = 0$. The depth function, then, is $s(x) = x$. Using similar triangles, we see that $w(x) = 8 - \frac{8}{3}x$. Now, the *weight density* of water is approximately $62.4\,\text{lb/ft}^3$, so applying Equation 2.6.3, we obtain

$$
\begin{aligned}
F &= \int_a^b \rho w(x) s(x)\, dx \\
&= \int_0^3 62.4 \left(8 - \frac{8}{3}x\right) x\, dx \\
&= 62.4 \int_0^3 \left(8x - \frac{8}{3}x^2\right) dx \\
&= 62.4 \left[4x^2 - \frac{8}{9}x^3\right] \Big|_0^3 \\
&\approx 748.8.
\end{aligned}
$$

The water exerts a force of approximately 748.8 lb on the end of the trough.

Checkpoint 2.6.1

A water trough 12 m long has ends shaped like inverted isosceles triangles, with a base of 6 m and height of 4 m. Find the force on one end of the trough if the trough is full of water.

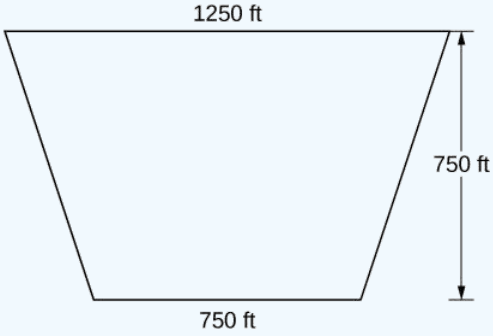
Solution

156,800 N

Example 2.6.2: Finding Hydrostatic Force

We now turn our attention to the Hoover Dam. The actual dam is arched rather than flat, but we are going to make some simplifying assumptions to help us with the calculations. Assume the face of the Hoover Dam is shaped like an isosceles trapezoid with a lower base of 750 ft, upper base of 1250 ft, and a height of 750 ft (see the following figure).





When the reservoir is full, Lake Mead's maximum depth is about 530 ft, and the lake's surface is about 10 ft below the top of the dam (see the following figure).

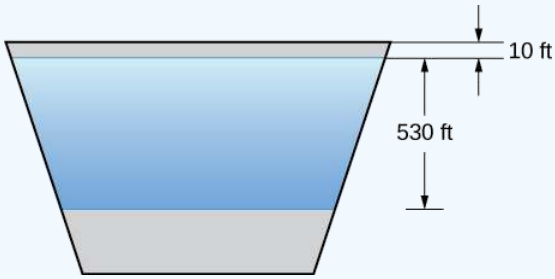


Figure 2.6.4: A simplified model of the Hoover Dam with assumed dimensions.

a. Find the force on the face of the dam when the reservoir is full.
b. The southwest United States has been experiencing a drought, and the surface of Lake Mead is about 125 ft below where it would be if the reservoir were full. What is the force on the face of the dam under these circumstances?

Solution

a. We begin by establishing a frame of reference. As usual, we choose to orient the x-axis vertically, with the downward direction being positive. This time, however, we will let $x = 0$ represent the top of the dam, rather than the water's surface. When the reservoir is full, the surface of the water is 10 ft below the top of the dam, so $s(x) = x - 10$ (see the following figure).

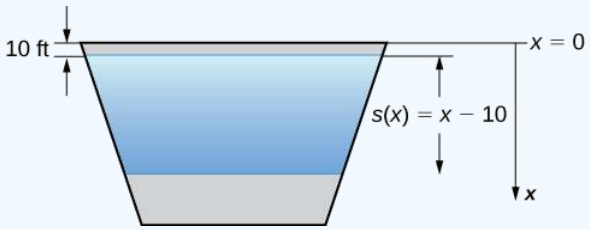


Figure 2.6.5: We first choose a frame of reference.

To find the width function, we again turn to similar triangles, as shown in the figure below.

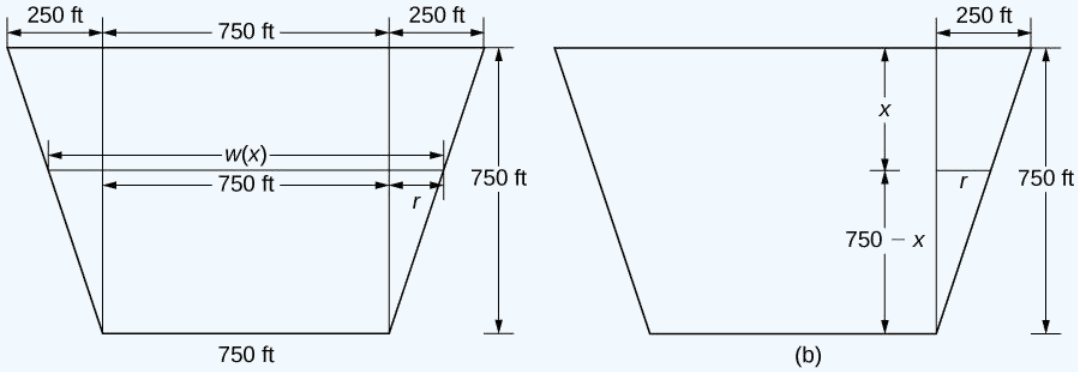


Figure 2.6.6: We use similar triangles to determine a function for the width of the dam. (a) Assumed dimensions of the dam; (b) highlighting the similar triangles.

The figure shows that $w(x) = 750 + 2r$. Using properties of similar triangles, we get $r = 250 - \frac{1}{3}x$. Thus,

$$w(x) = 1250 - \frac{2}{3}x$$

Using a *weight density* of $62.4\,\text{lb/ft}^3$ and applying Equation 2.6.3, we get

$$\begin{aligned} F &= \int_a^b \rho\, w(x) s(x)\, dx \\ &= \int_{10}^{540} 62.4 \left(1250 - \frac{2}{3}x\right)(x-10)\, dx \\ &= 62.4 \int_{10}^{540} -\frac{2}{3}(x^2 - 1885x + 18750)\, dx \\ &= -62.4 \left(\frac{2}{3}\right)\left(\frac{x^3}{3} - \frac{1885x^2}{2} + 18750x\right)\Big|_{10}^{540} \\ &\approx 8,832,245,000 \text{ lb} \\ &= 4,416,122.5 \text{ tons.} \end{aligned}$$

Note the change from pounds to tons (2000lb = 1 ton).

b. This changes our depth function, $s(x)$, and our limits of integration. We have $s(x) = x - 135$. The lower limit of integration is 135. The upper limit remains 540. Evaluating the integral, we get

$$\begin{aligned} F &= \int_a^b \rho\, w(x) s(x)\, dx \\ &= \int_{135}^{540} 62.4 \left(1250 - \frac{2}{3}x\right)(x-135)\, dx \\ &= -62.4(\frac{2}{3}) \int_{135}^{540} (x-1875)(x-135)\, dx \\ &= -62.4 \left(\frac{2}{3}\right) \int_{135}^{540} (x^2 - 2010x + 253125)\, dx \\ &= -62.4 \left(\frac{2}{3}\right)\left(\frac{x^3}{3} - 1005x^2 + 253125x\right)\Big|_{135}^{540} \\ &\approx 5,015,230,000 \text{ lb} \\ &= 2,507,615 \text{ tons.} \end{aligned}$$

? Checkpoint 2.6.2

When the reservoir is at its average level, the surface of the water is about 50 ft below where it would be if the reservoir were full. What is the force on the face of the dam under these circumstances?

Solution

Approximately 7,164,520,000 lb or 3,582,260 tons

Footnotes

[1] As before, you could treat this entire subsection using the standard coordinate system. The reader is trusted to be able to derive the relevant mechanics.

2.6E: Exercises

Reading Questions

1. What is the definition of pressure, and what are its common units in the English and metric systems?
2. What is the formula for the force exerted on a plate of area A submerged horizontally at a depth s?
3. According to Pascal's principle, if you know the pressure at a specific depth, do you need to worry about the orientation (horizontal or vertical) of a surface at that depth to determine the pressure on it?
4. In the integral for hydrostatic force, $F = \int_a^b \rho w(x) s(x) dx$, what do the functions $w(x)$ and $s(x)$ represent?
5. What is the approximate weight density of water in lb/ft^3?
6. When setting up a hydrostatic force problem, why is it important to first sketch a picture and establish a frame of reference?

Hydrostatic Force and Pressure

1) [Technology Required] A rectangular dam is 40 ft high and 60 ft wide. Compute the total force F on the dam when

a. the surface of the water is at the top of the dam and

b. the surface of the water is halfway down the dam.

Answer

a. $3,000,000$ lb,
b. $749,000$ lb

Contributors

Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

This page titled 2.6E: Exercises is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Roy Simpson.

- **6.5E: Exercises for Section 6.5** by Edwin “Jed” Herman, Gilbert Strang is licensed CC BY-NC-SA 4.0. Original source: https://openstax.org/details/books/calculus-volume-1.

2.7: Moments and Centers of Mass

Note to the Instructor (click to expand)

This is the typical moments and centers of mass topic. There is nothing out of the ordinary here from other texts with the exception of its placement. Since it is an application of intergration, I have moved it from its traditional place (later in the course) to the first chapter. Exercises from this material will occur throughout many sections from this point forward.

To succeed in this section, you'll need to use some skills from previous courses. While you should already know them, this is the first time they've been required. You can review these skills in CRC's Corequisite Codex. If you have a support class, it might cover some, **but not all**, of these topics.

Prerequisite Skills and Support Topics (click to expand)

- **Graphing Functions**
  - Symmetry: The Symmetry Principle is a key tool in this section. Recognizing if a region is symmetric about the x-axis, y-axis, or origin can significantly simplify the process of finding the centroid.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Determine the mass of a one-dimensional object from its linear density function.
- Determine the mass of a two-dimensional circular object from its radial density function.
- Find the center of mass of objects distributed along a line.
- Locate the center of mass of a thin plate.
- Use symmetry to help locate the centroid of a thin plate.
- Apply the theorem of Pappus for volume.

Hawk A.I. Section-Specific Tutor

EST. 1970 COSUMNES RIVER COLLEGE MATHEMATICS

To access the Hawk A.I. Tutor, you will need to be logged into your campus Gmail account.

In this section, we consider centers of mass (also called centroids, under certain conditions) and moments. The basic idea of the center of mass is a balancing point. Many of us have seen performers spin plates on the ends of sticks. The performers try to keep several spinning without allowing them to drop. If we look at a single plate (without spinning it), there is a sweet spot where it balances perfectly on the stick. If we put the stick anywhere other than that sweet spot, the plate does not balance and falls to the ground. (That is why performers spin the plates; the spin helps keep the plates from falling even if the stick is not exactly in the right place.) Mathematically, that sweet spot is called the center of mass of the plate.

This section examines these concepts in a one-dimensional context. Then, we expand our development to consider centers of mass of two-dimensional regions and symmetry. Last, we use centroids to find the volume of certain solids by applying the Theorem of Pappus.

Mass and Linear Density

We can use integration to develop a formula for calculating mass based on a **linear density** function. First, we consider a thin rod or wire. Orient the rod so it aligns with the x-axis, with the left end of the rod at $x = a$ and the right end of the rod at $x = b$ (Figure

2.7.1). Note that although we depict the rod with some thickness in the figures, for mathematical purposes, we assume the rod is thin enough to be treated as a one-dimensional object.

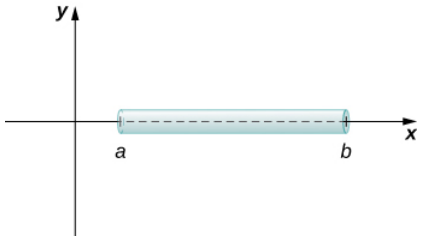


Figure 2.7.1: We can calculate the mass of a thin rod oriented along the x-axis by integrating its linear density function.

If the rod has constant linear density ρ, given in terms of mass per unit length (i.e., $\rho = m/l$), then the mass of the rod is just the product of the linear density and the length of the rod. That is,

$$m = \rho(b-a).$$

If the linear density of the rod is not constant, however, the problem becomes a little more challenging. When the linear density of the rod varies from point to point, we use a **linear density function**, $\rho(x)$, to denote the density of the rod at any point, x. Let $\rho(x)$ be an integrable linear density function. Now, for $i = 0, 1, 2, \ldots, n$ let $P = \{x_i\}$ be a regular partition of the interval $[a, b]$, and for $i = 1, 2, \ldots, n$ choose an arbitrary point $x_i^* \in [x_{i-1}, x_i]$. Figure 2.7.2 shows a representative segment of the rod.

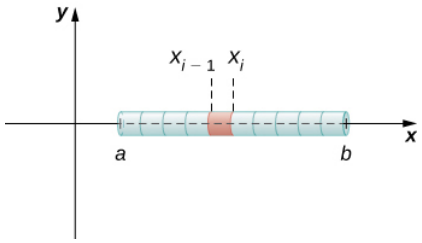


Figure 2.7.2: A representative segment of the rod.

The mass, m_i, of the segment of the rod from x_{i-1} to x_i is approximated by

$$\begin{aligned} m_i &\approx \rho(x_i^*)(x_i - x_{i-1}) \\ &= \rho(x_i^*)\Delta x. \end{aligned}$$

Adding the masses of all the segments gives us an approximation for the mass of the entire rod:

$$\begin{aligned} m &= \sum_{i=1}^{n} m_i \\ &\approx \sum_{i=1}^{n} \rho(x_i^*)\Delta x. \end{aligned}$$

This is a Riemann sum. Taking the limit as $n \to \infty$, we get an expression for the exact mass of the rod:

$$\begin{aligned} m &= \lim_{n\to\infty} \sum_{i=1}^{n} \rho(x_i^*)\Delta x \\ &= \int_a^b \rho(x)\,dx. \end{aligned}$$

We state this result in the following theorem.

Theorem: Mass–Linear Density and Weight-Linear Weight Density Formulas of a One-Dimensional Object

Given a thin rod oriented along the x-axis over the interval $[a, b]$, let $\rho(x)$ denote a linear density function giving the density of the rod at a point x in the interval. Then the mass of the rod is given by

$$m = \int_a^b \rho(x)\,dx. \tag{2.7.1}$$

If given the linear *weight* density function, $\gamma(x)$, then the *weight* of the rod is given by

$$F = \int_a^b \gamma(x)\,dx. \tag{2.7.2}$$

Before we move on, we must understand how the units work with Equations 2.7.1 and 2.7.2.

In Equation 2.7.1, we are given the linear density function $\rho(x)$. The unit types for this density function are

$$\frac{\text{mass}}{\text{length}},$$

where the mass unit could be in kilograms, grams, slugs, or any other unit of mass we are interested in, and the length unit is in meters, centimeters, feet, inches, or whatever unit of length in which we are interested. The unit type for dx is length (which we would make sure matches the units of the length in the density function). Hence, the unit type for the integral is

$$\frac{\text{mass}}{\cancel{\text{length}}^{1}} \cdot \cancel{\text{length}}^{1} = \text{mass},$$

which is exactly what we are trying to compute.

Likewise, in Equation 2.7.2, we are given the linear *weight* density function $\gamma(x)$. The unit types for this *weight* density function are

$$\frac{\text{force}}{\text{length}},$$

where the force unit could be either in pounds or newtons (and only those two choices), and the length unit is in meters, centimeters, feet, inches, or whatever unit of length in which we are interested. The unit type for dx is length (which we would make sure matches the units of the length in the *weight* density function). Hence, the unit type for the integral is

$$\frac{\text{force}}{\cancel{\text{length}}^{1}} \cdot \cancel{\text{length}}^{1} = \text{force},$$

which is, again, exactly what we are trying to compute.

The previous discussion using unit analysis will be especially helpful in physics and engineering (and, luckily, the homework). We apply this theorem in the next example.

Example 2.7.1: Calculating Mass from Linear Density

Consider a thin rod oriented on the x-axis over the interval $[\frac{\pi}{2}, \pi]$. If the linear density of the rod is given by $\rho(x) = \sin x$, what is the mass of the rod?

Solution

Applying Equation 2.7.1 directly, we have

$$\begin{aligned} m &= \int_a^b \rho(x)\,dx \\ &= \int_{\pi/2}^{\pi} \sin x\,dx \\ &= -\cos x\Big|_{\pi/2}^{\pi} \\ &= 1. \end{aligned}$$

? Checkpoint 2.7.1

Consider a thin rod oriented on the x-axis over the interval $[1,3]$. If the linear density of the rod is given by $\rho(x) = 2x^2 + 3$, what is the mass of the rod?

Answer

$70/3$

Mass and Radial Density

We now extend this concept to find the mass of a two-dimensional disk of radius r. As with the rod we looked at in the one-dimensional case, here we assume the disk is thin enough that, for mathematical purposes, we can treat it as a two-dimensional object. We assume the density is given in terms of mass per unit area (i.e., $\rho = m/A$), which is why this is often called the **area density**. For this discussion, we will further assume the area density varies only along the disk's radius. In this situation, we could call the area density by a different name - the **radial density**. This is because it is a function of how far away a point is from the disk's center.

We orient the disk in the xy-plane, with the center at the origin. Then, the density of the disk can be treated as a function of x, denoted $\rho(x)$. We assume $\rho(x)$ is integrable. Because density is a function of x, we partition the interval from $[0,r]$ along the x-axis. For $i = 0, 1, 2, \ldots, n$, let $P = \{x_i\}$ be a regular partition of the interval $[0,r]$, and for $i = 1, 2, \ldots, n$, choose an arbitrary point $x_i^* \in [x_{i-1}, x_i]$. Use the partition to break the disk into thin (two-dimensional) washers. A disk and a representative washer are depicted in the following figure.

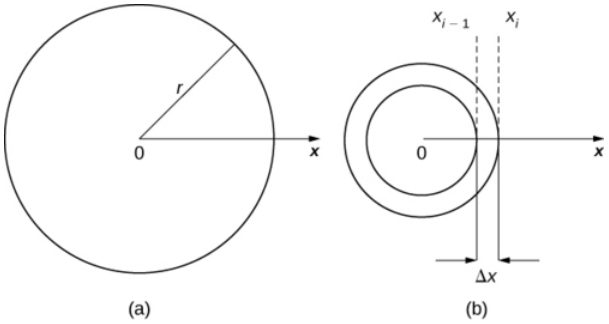


Figure **2.7.3***a*: A thin disk in the xy-plane.
Figure **2.7.3***b*: A representative washer.

We now approximate the density and area of the washer to calculate an approximate mass, m_i. Note that the area of the washer is given by

$$\begin{aligned} A_i &= \pi(x_i)^2 - \pi(x_{i-1})^2 \\ &= \pi[x_i^2 - x_{i-1}^2] \\ &= \pi(x_i + x_{i-1})(x_i - x_{i-1}) \\ &= \pi(x_i + x_{i-1})\Delta x. \end{aligned}$$

You may recall that we had an expression similar to this when we were computing volumes by the Method of Cylindrical Shells. As we did there, we use $x_i^* \approx (x_i + x_{i-1})/2$ to approximate the average radius of the washer. We obtain

$$A_i = \pi(x_i + x_{i-1})\Delta x \approx 2\pi x_i^* \Delta x.$$

Using $\rho(x_i^*)$ to approximate the (radial) density of the washer, we approximate the mass of the washer by

$$m_i \approx 2\pi x_i^* \rho(x_i^*)\Delta x.$$

Adding up the masses of the washers, we see the mass m of the entire disk is approximated by

$$m = \sum_{i=1}^{n} m_i \approx \sum_{i=1}^{n} 2\pi x_i^* \rho(x_i^*)\Delta x.$$

We again recognize this as a Riemann sum and take the limit as $n \to \infty$. This gives us

$$\begin{aligned} m &= \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi x_i^* \rho(x_i^*)\Delta x \\ &= \int_0^r 2\pi x \rho(x)\, dx. \end{aligned}$$

We summarize these findings in the following theorem.

Theorem: Mass–Radial Density and Weight-Radial Weight Density Formulas of a Circular Object

Let $\rho(x)$ be an integrable function representing the radial density of a disk of radius r. Then the disk's mass is given by

$$m = \int_0^r 2\pi x \rho(x)\, dx. \tag{2.7.3}$$

If given the radial *weight* density function, $\gamma(x)$, the *weight* of the disk is given by

$$F = \int_0^r 2\pi x \gamma(x)\, dx. \tag{2.7.4}$$

As we did with linear density, a unit analysis discussion for radial density is helpful.

In Equation 2.7.3, we are given the radial density function $\rho(x)$. The unit types for this radial density function are

$$\frac{\text{mass}}{\text{length}^2},$$

where, just like before, the mass unit could be in kilograms, grams, slugs, or any other unit of mass we are interested in, and the length unit is in meters, centimeters, feet, inches, or whatever unit of length in which we are interested. The unit type for dx is length, and the unit type for x is also length. Hence, the unit type for the integral is

$$\text{length} \cdot \frac{\text{mass}}{\text{length}^2} \cdot \text{length} = \frac{\text{mass}}{\cancel{\text{length}^2}^{\,1}} \cdot \cancel{\text{length}^2}^{\,1} = \text{mass},$$

which is exactly what we are trying to compute.

Likewise, in Equation 2.7.4, we are given the radial *weight* density function $\gamma(x)$. The unit types for this radial *weight* density function are

$$\frac{\text{force}}{\text{length}^2},$$

where, yet again, the force unit could be either in pounds or newtons. The length unit is in meters, centimeters, feet, inches, or whatever unit of length we are interested in. The unit type for dx is length, and the unit type for x is also length. Hence, the unit type for the integral is

$$\text{length} \cdot \frac{\text{force}}{\text{length}^2} \cdot \text{length} = \frac{\text{force}}{\cancel{\text{length}^2}^{\,1}} \cdot \cancel{\text{length}^2}^{\,1} = \text{force},$$

which is, again, exactly what we are trying to compute.

Example 2.7.2: Calculating Mass from Radial Density

Let $\rho(x) = \sqrt{x}$ represent the radial density of a disk. Calculate the mass of a disk of radius 4.

Solution

Applying Equation 2.7.3, we find

$$\begin{aligned} m &= \int_0^r 2\pi x \rho(x)\, dx \\ &= \int_0^4 2\pi x \sqrt{x}\, dx \\ &= 2\pi \int_0^4 x^{3/2}\, dx \\ &= \frac{4\pi}{5} x^{5/2} \Big|_0^4 \\ &= \frac{128\pi}{5}. \end{aligned}$$

Checkpoint 2.7.2

Let $\rho(x) = 3x + 2$ represent the radial density of a disk. Calculate the mass of a disk of radius 2.

Answer

24π

Center of Mass and Moments

We now step away from standard shapes like rods and circles and instead try to focus on irregular-shaped regions. Specifically, we want to discuss a given two-dimensional object's center of mass (the balancing point). To do so, however, we first need a bit of theory.

Let's begin by looking at the center of mass in a one-dimensional context. Consider a long, thin wire or rod of negligible mass resting on a **fulcrum**, as shown in Figure 2.7.4*a*. Now suppose we place objects having masses m_1 and m_2 at distances d_1 and d_2 from the fulcrum, respectively, as shown in Figure 2.7.4*b*.

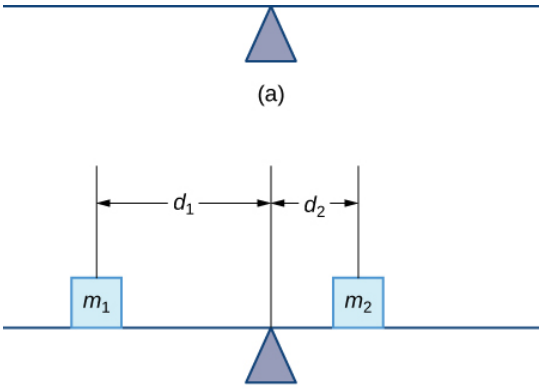


Figure 2.7.4*a*: A thin rod rests on a fulcrum.
Figure 2.7.4*b*: Masses are placed on the rod.

The most common real-life example of a system like this is a playground seesaw, or teeter-totter, with children of different weights sitting at different distances from the center. On a seesaw, if one child sits at each end, the heavier child sinks, and the lighter child is lifted into the air. Suppose the heavier child slides in toward the center, though the seesaw balances. Applying this concept to the masses on the rod, we note that the masses balance each other if and only if

$$m_1 d_1 = m_2 d_2.$$

In the seesaw example, we balanced the system by moving the masses (children) with respect to the fulcrum. However, we are interested in systems where the masses cannot move. Instead, we balance the system by moving the fulcrum. Suppose we have two **point masses**,[1] m_1 and m_2, located on a number line at points x_1 and x_2, respectively (Figure 2.7.5). The center of mass, $\bar{x}$, is where the fulcrum should be placed to balance the system.

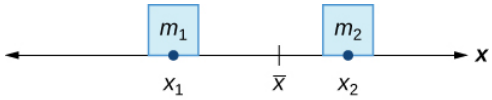


Figure 2.7.5: The center of mass $\bar{x}$ is the balance point of the system.

Thus, we have

$$\begin{aligned} & m_1|x_1 - \bar{x}| &&= m_2|x_2 - \bar{x}| \\ \Longrightarrow \quad & m_1(\bar{x} - x_1) &&= m_2(x_2 - \bar{x}) \\ \Longrightarrow \quad & m_1\bar{x} - m_1x_1 &&= m_2x_2 - m_2\bar{x} \\ \Longrightarrow \quad & \bar{x}(m_1 + m_2) &&= m_1x_1 + m_2x_2 \end{aligned}$$

or

$$\bar{x} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} \tag{2.7.5}$$

The expression in the numerator of Equation 2.7.5, $m_1x_1 + m_2x_2$, is called the **first moment of the system** with respect to the origin. If the context is clear, we often drop the word first and refer to this expression as the **moment of the system**. The expression in the denominator, $m_1 + m_2$, is the system's total mass. Thus, the **center of mass of the system** is the point at which the system's total mass could be concentrated without changing the moment.

This idea is more expansive than two point masses. In general, if n masses, $m_1, m_2, \ldots, m_n$, are placed on a number line at points $x_1, x_2, \ldots, x_n$, respectively, then the center of mass of the system is given by

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}.$$

Theorem: Center of Mass of Objects on a Line

Let $m_1, m_2, \ldots, m_n$ be point masses placed on a number line at points $x_1, x_2, \ldots, x_n$, respectively, and let $m = \sum_{i=1}^{n} m_i$ denote the total mass of the system. Then, the moment of the system with respect to the origin is given by

$$M = \sum_{i=1}^{n} m_i x_i \tag{2.7.6}$$

and the center of mass of the system is given by

$$\bar{x} = \frac{M}{m}. \tag{2.7.7}$$

We apply this theorem in the following example.

Example 2.7.3: Finding the Center of Mass of Objects along a Line

Suppose four point masses are placed on a number line as follows:

- $m_1 = 30\,\text{kg}$, placed at $x_1 = -2\,\text{m}$
- $m_2 = 5\,\text{kg}$, placed at $x_2 = 3\,\text{m}$
- $m_3 = 10\,\text{kg}$, placed at $x_3 = 6\,\text{m}$
- $m_4 = 15\,\text{kg}$, placed at $x_4 = -3\,\text{m}$.

Find the moment of the system with respect to the origin and find the center of mass of the system.

Solution

First, we need to calculate the moment of the system (Equation 2.7.6):

$$\begin{aligned} M &= \sum_{i=1}^{4} m_i x_i \\ &= -60 + 15 + 60 - 45 \\ &= -30. \end{aligned}$$

Now, to find the center of mass, we need the total mass of the system:

$$\begin{aligned} m &= \sum_{i=1}^{4} m_i \\ &= 30 + 5 + 10 + 15 \\ &= 60\,\text{kg} \end{aligned}$$

Then we have (from Equation 2.7.7)

$$\bar{x} = \frac{M}{m} = -\frac{30}{60} = -\frac{1}{2}.$$

The center of mass is located 1/2 m to the left of the origin.

Checkpoint 2.7.3

Suppose four point masses are placed on a number line as follows:

- $m_1 = 12\,\text{kg}$ placed at $x_1 = -4\,\text{m}$
- $m_2 = 12\,\text{kg}$ placed at $x_2 = 4\,\text{m}$

- $m_3 = 30\,\text{kg}$ placed at $x_3 = 2\,\text{m}$
- $m_4 = 6\,\text{kg}$, placed at $x_4 = -6\,\text{m}$.

Find the moment of the system with respect to the origin and find the center of mass of the system.

Answer

$M = 24, \bar{x} = \frac{2}{5}\,\text{m}$

We can generalize this concept to find the center of mass of a system of point masses in a plane.

Let m_1 be a point mass located at point (x_1, y_1) in the plane. Then the **moment M_x of the mass with respect to the x-axis** is given by $M_x = m_1 y_1$. Similarly, the **moment M_y with respect to the y-axis** is given by $M_y = m_1 x_1$.

Notice that the x-coordinate of the point is used to calculate the moment with respect to the y-axis, and vice versa. The reason is that the x-coordinate gives the distance from the point mass to the y-axis, and the y-coordinate gives the distance to the x-axis (see Figure 2.7.6).

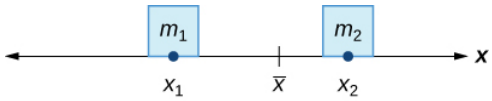


Figure 2.7.6: Point mass m_1 is located at point (x_1, y_1) in the plane.

If we have several point masses in the xy-plane, we can use the moments with respect to the x- and y-axes to calculate the x- and y-coordinates of the center of mass of the system.

Theorem: Center of Mass of Objects in a Plane

Let $m_1, m_2, \ldots, m_n$ be point masses located in the xy-plane at points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, respectively, and let $m = \sum_{i=1}^{n} m_i$ denote the total mass of the system. Then the moments M_x and M_y of the system with respect to the x- and y-axes, respectively, are given by

$$M_x = \sum_{i=1}^{n} m_i y_i \tag{2.7.8}$$

and

$$M_y = \sum_{i=1}^{n} m_i x_i. \tag{2.7.9}$$

Also, the coordinates of the center of mass $(\bar{x}, \bar{y})$ of the system are

$$\bar{x} = \frac{M_y}{m} \tag{2.7.10}$$

and

$$\bar{y} = \frac{M_x}{m}. \tag{2.7.11}$$

The next example demonstrates how the center of mass formulas (Equations 2.7.8 - 2.7.11) may be applied.

Example 2.7.4: Finding the Center of Mass of Objects in a Plane

Suppose three point masses are placed in the xy-plane as follows (assume coordinates are given in meters):

- $m_1 = 2\,\text{kg}$ placed at $(-1, 3)$,
- $m_2 = 6\,\text{kg}$ placed at $(1, 1)$,
- $m_3 = 4\,\text{kg}$ placed at $(2, -2)$.

Find the center of mass of the system.

Solution

First we calculate the total mass of the system:

$$m = \sum_{i=1}^{3} m_i = 2 + 6 + 4 = 12\,\text{kg}.$$

Next we find the moments with respect to the x- and y-axes:

$$M_y = \sum_{i=1}^{3} m_i x_i = -2 + 6 + 8 = 12,$$

$$M_x = \sum_{i=1}^{3} m_i y_i = 6 + 6 - 8 = 4.$$

Then we have

$$\bar{x} = \frac{M_y}{m} = \frac{12}{12} = 1$$

and

$$\bar{y} = \frac{M_x}{m} = \frac{4}{12} = \frac{1}{3}.$$

The center of mass of the system is $\left(1, \frac{1}{3}\right)$, in meters.

? Checkpoint 2.7.4

Suppose three point masses are placed on a number line as follows (assume coordinates are given in meters):

- $m_1 = 5\,\text{kg}$ placed at $(-2, -3)$,
- $m_2 = 3\,\text{kg}$ placed at $(2, 3)$,
- $m_3 = 2\,\text{kg}$ placed at $(-3, -2)$.

Find the center of mass of the system.

Answer

$(-1, -1)$ m

Center of Mass of Thin Plates

So far, we have looked at systems of point masses on a line and in a plane. Now, instead of having the mass of a system concentrated at discrete points, we want to look at systems in which the system's mass is distributed continuously across a thin sheet of material. For our purposes, we assume the sheet is thin enough that it can be treated as if it is two-dimensional. Such a sheet is called a **lamina**. Next, we develop techniques to find the center of mass of a lamina. In this section, we also assume the area density of the lamina is constant.

Laminas are often represented by a two-dimensional region in a plane. The geometric center of such a region is called its **centroid**. Since we have assumed the area density of the lamina is constant, the center of mass of the lamina depends only on the shape of the corresponding region in the plane; it does *not* depend on the area density. In this case, the center of mass of the lamina corresponds to the centroid of the delineated region in the plane. As with systems of point masses, we need to find the total mass of the lamina, as well as the moments of the lamina with respect to the x- and y-axes.

We first consider a lamina in a rectangle shape. Recall that the center of mass of a lamina is the point where the lamina balances. For a rectangle, that point is the rectangle's horizontal and vertical center. Based on this understanding, it is clear that the center of

mass of a rectangular lamina is the point where the diagonals intersect, resulting from the **symmetry principle**, and it is stated here without proof.

Theorem: Symmetry Principle

If a region $\mathbf{R}$ is symmetric about a line l, then the centroid of $\mathbf{R}$ lies on l.

Let's turn to more general laminas. Suppose we have a lamina bounded above by the graph of a continuous function $f(x)$, below by the x-axis, and on the left and right by the lines $x = a$ and $x = b$, respectively, as shown in Figure 2.7.7.

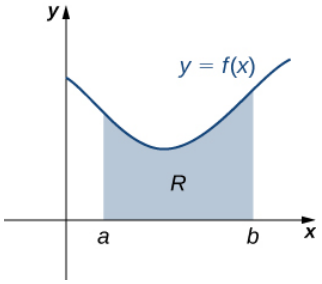


Figure 2.7.7: A region in the plane representing a lamina.

As with systems of point masses, to find the center of mass of the lamina, we need to find the total mass of the lamina, as well as the moments of the lamina with respect to the x- and y-axes. As we have often done, we approximate these quantities by partitioning the interval $[a, b]$ and constructing rectangles.

For $i = 0, 1, 2, \ldots, n$, let $P = \{x_i\}$ be a regular partition of $[a, b]$. Recall that we can choose any point within the interval $[x_{i-1}, x_i]$ as our x_i^*. In this case, we want x_i^* to be the x-coordinate of the centroid of our rectangles. Thus, for $i = 1, 2, \ldots, n$, we select $x_i^* \in [x_{i-1}, x_i]$ such that x_i^* is the midpoint of the interval. That is, $x_i^* = \frac{x_{i-1}+x_i}{2}$. Now, for $i = 1, 2, \ldots, n$, construct a rectangle of height $y_T - y_B = f(x_i^*) - 0 = f(x_i^*)$ on $[x_{i-1}, x_i]$. The center of mass of this rectangle is $\left(x_i^*, \frac{f(x_i^*)}{2}\right)$, as shown in the following figure.

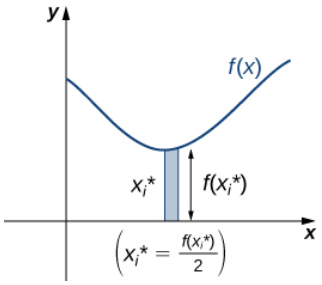


Figure 2.7.8: A representative rectangle of the lamina.

Next, we need to find the total mass of the rectangle. Let ρ represent the area density of the lamina (note that ρ is a constant). Therefore, ρ is expressed in mass per unit area. Thus, to find the total mass of the rectangle, we multiply the area of the rectangle by ρ. Then, the mass of the rectangle is given by $\rho f(x_i^*)\Delta x$.

To get the approximate mass of the lamina, we add the masses of all the rectangles to get

$$m \approx \sum_{i=1}^{n} \rho f(x_i^*)\Delta x. \tag{2.7.12}$$

Equation 2.7.12 is a Riemann sum. Taking the limit as $n \to \infty$ gives the exact mass of the lamina:

$$\begin{aligned} m &= \lim_{n\to\infty} \sum_{i=1}^{n} \rho f(x_i^*)\Delta x \\ &= \rho \int_a^b f(x)\, dx. \end{aligned}$$

Next, we calculate the moment of the lamina with respect to the x-axis. Returning to the representative rectangle, recall its center of mass is $\left(x_i^*, \frac{f(x_i^*)}{2}\right)$. Recall also that treating the rectangle as if it is a point mass located at the center of mass does not change the moment. Thus, the moment of the rectangle with respect to the x-axis is given by the mass of the rectangle, $\rho f(x_i^*)\Delta x$, multiplied by the distance from the center of mass to the x-axis: $\frac{f(x_i^*)}{2}$. Therefore, the moment with respect to the x-axis of the rectangle is $\rho\left(\frac{[f(x_i^*)]^2}{2}\right)\Delta x$. Adding the moments of the rectangles and taking the limit of the resulting Riemann sum, we see that the moment of the lamina with respect to the x-axis is

$$\begin{aligned} M_x &= \lim_{n\to\infty}\sum_{i=1}^{n}\rho\frac{[f(x_i^*)]^2}{2}\Delta x \\ &= \rho\int_a^b \frac{[f(x)]^2}{2}\,dx. \end{aligned}$$

We derive the moment with respect to the y-axis similarly, noting that the distance from the center of mass of the rectangle to the y-axis is x_i^*. Then the moment of the lamina with respect to the y-axis is given by

$$\begin{aligned} M_y &= \lim_{n\to\infty}\sum_{i=1}^{n}\rho x_i^* f(x_i^*)\Delta x \\ &= \rho\int_a^b xf(x)\,dx. \end{aligned}$$

We find the coordinates of the center of mass by dividing the moments by the total mass to give $\bar{x} = \frac{M_y}{m}$ and $\bar{y} = \frac{M_x}{m}$. If we look closely at the expressions for M_x, M_y, and m, we notice that the constant ρ cancels out when $\bar{x}$ and $\bar{y}$ are calculated.

We summarize these findings in the following theorem.

Theorem: Center of Mass of a Thin Plate in the xy-Plane

Let $\mathbf{R}$ denote a region bounded above by the graph of a continuous function $f(x)$, below by the x-axis, and on the left and right by the lines $x = a$ and $x = b$, respectively. Let ρ denote the area density of the associated lamina. Then we can make the following statements:

i. The mass of the lamina is

$$m = \rho\int_a^b f(x)\,dx. \tag{2.7.13}$$

ii. The moments M_x and M_y of the lamina with respect to the x- and y-axes, respectively, are

$$M_x = \rho\int_a^b \frac{[f(x)]^2}{2}dx \tag{2.7.14}$$

and

$$M_y = \rho\int_a^b xf(x)dx. \tag{2.7.15}$$

iii. The coordinates of the center of mass $(\bar{x}, \bar{y})$ are

$$\bar{x} = \frac{M_y}{m} \tag{2.7.16}$$

and

$$\bar{y} = \frac{M_x}{m}. \tag{2.7.17}$$

In the next example, we use this theorem to find the center of mass of a lamina.

Example 2.7.5: Finding the Center of Mass of a Lamina

Let $\mathbf{R}$ be the region bounded above by the graph of the function $f(x) = \sqrt{x}$ and below by the x-axis over the interval $[0, 4]$. Find the centroid of the region.

Solution

The region is depicted in the following figure.

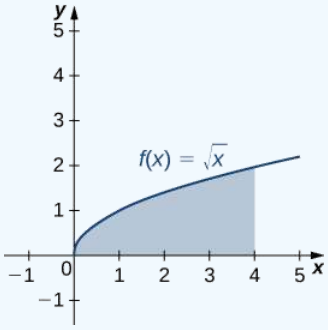


Figure 2.7.9: Finding the center of mass of a lamina.

Since we are only asked for the region's centroid rather than the mass or moments of the associated lamina, we know the area density constant ρ cancels out of the calculations eventually. Therefore, for the sake of convenience, let's assume $\rho = 1$.

First, we need to calculate the total mass (Equation 2.7.13):

$$\begin{aligned} m &= \rho \int_a^b f(x)\,dx \\ &= \int_0^4 \sqrt{x}\,dx \\ &= \frac{2}{3}x^{3/2}\Big|_0^4 \\ &= \frac{2}{3}[8-0] \\ &= \frac{16}{3}. \end{aligned}$$

Next, we compute the moments (Equation 2.7.16):

$$\begin{aligned} M_x &= \rho \int_a^b \frac{[f(x)]^2}{2}\,dx \\ &= \int_0^4 \frac{x}{2}\,dx \\ &= \frac{1}{4}x^2\Big|_0^4 \\ &= 4 \end{aligned}$$

and (Equation 2.7.15):

$$\begin{aligned} M_y &= \rho \int_a^b x f(x)\, dx \\ &= \int_0^4 x\sqrt{x}\, dx \\ &= \int_0^4 x^{3/2}\, dx \\ &= \frac{2}{5}x^{5/2}\Big|_0^4 \\ &= \frac{2}{5}[32-0] \\ &= \frac{64}{5}. \end{aligned}$$

Thus, we have (Equation 2.7.16):

$$\begin{aligned} \bar{x} &= \frac{M_y}{m} \\ &= \frac{64/5}{16/3} \\ &= \frac{64}{5}\cdot\frac{3}{16} \\ &= \frac{12}{5} \end{aligned}$$

and (Equation 2.7.17):

$$\begin{aligned} \bar{y} &= \frac{M_x}{m} \\ &= \frac{4}{16/3} \\ &= 4\cdot\frac{3}{16} \\ &= \frac{3}{4}. \end{aligned}$$

The centroid of the region is $\left(\frac{12}{5}, \frac{3}{4}\right)$.

? Checkpoint 2.7.5

Let $\mathbf{R}$ be the region bounded above by the graph of the function $f(x) = x^2$ and below by the x-axis over the interval $[0, 2]$. Find the centroid of the region.

Answer

The centroid of the region is $\left(\frac{3}{2}, \frac{6}{5}\right)$.

We can also adapt this approach to find centroids of more complex regions. Suppose our region is bounded above by the graph of a continuous function $f(x)$, as before, but now, instead of having the lower bound for the region be the x-axis, suppose the region is bounded below by the graph of a second continuous function, $g(x)$, as shown in Figure 2.7.10.

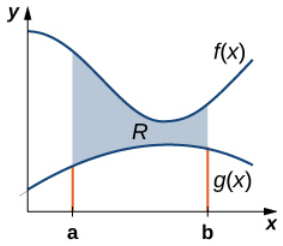


Figure 2.7.10: A region between two functions.

Again, we partition the interval $[a, b]$ and construct rectangles. A representative rectangle is shown in Figure 2.7.11.

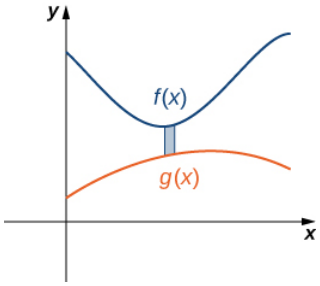


Figure 2.7.11: A representative rectangle of the region between two functions.

Note that the centroid of this rectangle is $\left(x_i^*, \frac{f(x_i^*)+g(x_i^*)}{2}\right)$. We won't go through all the details of the Riemann sum development, but let's look at some of the key steps. In the development of the formulas for the mass of the lamina and the moment with respect to the y-axis, the height of each rectangle is given by $y_T - y_B = f(x_i^*) - g(x_i^*)$, which leads to the expression $f(x) - g(x)$ in the integrands.

In the development of the formula for the moment with respect to the x-axis, the moment of each rectangle is found by multiplying the area density of the rectangle, $\rho[f(x_i^*) - g(x_i^*)]\Delta x$, by the distance of the centroid from the x-axis, $\frac{f(x_i^*)+g(x_i^*)}{2}$, which gives $\rho\frac{1}{2}\left[f(x_i^*)\right]^2 - [g(x_i^*)]^2] \Delta x$. Summarizing these findings, we arrive at the following theorem.

Theorem: Center of Mass of a Lamina Bounded by Two Functions

Let $\mathbf{R}$ denote a region bounded above by the graph of a continuous function $f(x)$, below by the graph of the continuous function $g(x)$, and on the left and right by the lines $x = a$ and $x = b$, respectively. Let ρ denote the area density of the associated lamina. Then we can make the following statements:

i. The mass of the lamina is

$$m = \rho \int_a^b [f(x) - g(x)]\, dx.$$

ii. The moments M_x and M_y of the lamina with respect to the x- and y-axes, respectively, are

$$M_x = \rho \int_a^b \frac{1}{2}\left([f(x)]^2 - [g(x)]^2\right)\, dx$$

and

$$M_y = \rho \int_a^b x[f(x) - g(x)]\, dx.$$

iii. The coordinates of the center of mass $(\bar{x}, \bar{y})$ are

$$\bar{x} = \frac{M_y}{m}$$

and

$$\bar{y} = \frac{M_x}{m}.$$

We illustrate this theorem in the following example.

Example 2.7.6: Finding the Centroid of a Region Bounded by Two Functions

Let $\mathbf{R}$ be the region bounded above by the graph of the function $f(x) = 1 - x^2$ and below by the graph of the function $g(x) = x - 1$. Find the centroid of the region.

Solution

The region is depicted in the following figure.

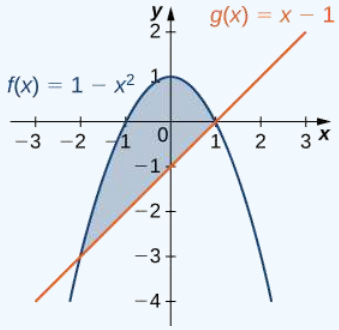


Figure 2.7.12: Finding the centroid of a region between two curves.

The graphs of the functions intersect at $(-2, -3)$ and $(1, 0)$, so we integrate from -2 to 1. Once again, for the sake of convenience, assume $\rho = 1$.

First, we need to calculate the total mass:

$$\begin{aligned} m &= \rho \int_a^b [f(x) - g(x)]\, dx \\ &= \int_{-2}^{1} [1 - x^2 - (x - 1)]\, dx \\ &= \int_{-2}^{1} (2 - x^2 - x)\, dx \\ &= \left[2x - \frac{1}{3}x^3 - \frac{1}{2}x^2\right]\Big|_{-2}^{1} \\ &= \left[2 - \frac{1}{3} - \frac{1}{2}\right] - \left[-4 + \frac{8}{3} - 2\right] \\ &= \frac{9}{2}. \end{aligned}$$

Next, we compute the moments:

$$\begin{aligned} M_x &= \rho \int_a^b \frac{1}{2}\left([f(x)]^2 - [g(x)]^2\right)\,dx \\ &= \frac{1}{2}\int_{-2}^{1}((1-x^2)^2 - (x-1)^2)\,dx \\ &= \frac{1}{2}\int_{-2}^{1}(x^4 - 3x^2 + 2x)\,dx \\ &= \frac{1}{2}\left[\frac{x^5}{5} - x^3 + x^2\right]\Big|_{-2}^{1} \\ &= -\frac{27}{10} \end{aligned}$$

and

$$\begin{aligned} M_y &= \rho \int_a^b x[f(x) - g(x)]\,dx \\ &= \int_{-2}^{1} x[(1-x^2) - (x-1)]\,dx \\ &= \int_{-2}^{1} x[2 - x^2 - x]\,dx \\ &= \int_{-2}^{1}(2x - x^4 - x^2)\,dx \\ &= \left[x^2 - \frac{x^5}{5} - \frac{x^3}{3}\right]\Big|_{-2}^{1} \\ &= -\frac{9}{4}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \bar{x} &= \frac{M_y}{m} \\ &= -\frac{9}{4}\cdot\frac{2}{9} \\ &= -\frac{1}{2} \end{aligned}$$

and

$$\begin{aligned} \bar{y} &= \frac{M_x}{m} \\ &= -\frac{27}{10}\cdot\frac{2}{9} \\ &= -\frac{3}{5}. \end{aligned}$$

The centroid of the region is $\left(-\frac{1}{2}, -\frac{3}{5}\right)$.

Checkpoint 2.7.6

Let $\mathbf{R}$ be the region bounded above by the graph of the function $f(x) = 6 - x^2$ and below by the graph of the function $g(x) = 3 - 2x$. Find the centroid of the region.

Answer

The centroid of the region is $\left(1, \frac{13}{5}\right)$.

The Symmetry Principle

We stated the symmetry principle earlier when we were looking at the centroid of a rectangle. The symmetry principle can greatly help when finding centroids of symmetric regions. Consider the following example.

Example 2.7.7: Finding the Centroid of a Symmetric Region

Let $\mathbf{R}$ be the region bounded above by the graph of the function $f(x) = 4 - x^2$ and below by the x-axis. Find the centroid of the region.

Solution

The region is depicted in the following figure

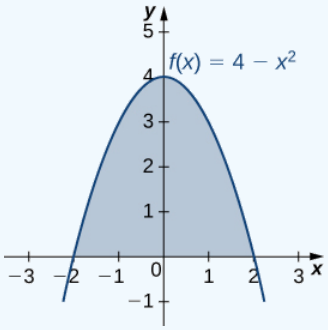


Figure 2.7.13: We can use the symmetry principle to help find the centroid of a symmetric region.

The region is symmetric with respect to the y-axis. Therefore, the x-coordinate of the centroid is zero. We need only calculate $\bar{y}$. Once again, for the sake of convenience, assume $\rho = 1$.

First, we calculate the total mass:

$$\begin{aligned} m &= \rho \int_a^b f(x)\,dx \\ &= \int_{-2}^{2} (4 - x^2)\,dx \\ &= \left[4x - \frac{x^3}{3}\right]\Big|_{-2}^{2} \\ &= \frac{32}{3}. \end{aligned}$$

Next, we calculate the moments. We only need M_x:

$$
\begin{aligned}
M_x &= \rho \int_a^b \frac{[f(x)]^2}{2}\, dx \\
&= \frac{1}{2}\int_{-2}^{2} \left[4 - x^2\right]^2 dx \\
&= \frac{1}{2}\int_{-2}^{2} (16 - 8x^2 + x^4)\, dx \\
&= \frac{1}{2}\left[\frac{x^5}{5} - \frac{8x^3}{3} + 16x\right]\Big|_{-2}^{2} \\
&= \frac{256}{15}
\end{aligned}
$$

Then we have

$$\bar{y} = \frac{M_x}{m} = \frac{256}{15} \cdot \frac{3}{32} = \frac{8}{5}.$$

The centroid of the region is $\left(0, \frac{8}{5}\right)$.

? Checkpoint 2.7.7

Let $\mathbf{R}$ be the region bounded above by the graph of the function $f(x) = 1 - x^2$ and below by x-axis. Find the centroid of the region.

Answer

The centroid of the region is $\left(0, \frac{2}{5}\right)$.

The Grand Canyon Skywalk

The Grand Canyon Skywalk opened on March 28, 2007. This engineering marvel is a horseshoe-shaped observation platform suspended 4000 ft above the Colorado River on the Grand Canyon's West Rim. Its crystal-clear glass floor allows stunning views of the canyon below (see the following figure).



Figure 2.7.14: The Grand Canyon Skywalk offers magnificent canyon views. (credit: 10da_ralta, Wikimedia Commons)

The Skywalk is a cantilever design, meaning that the observation platform extends over the canyon's rim with no visible means of support below it. Despite the lack of visible support posts or struts, cantilever structures are engineered to be very stable, and the Skywalk is no exception. The observation platform is attached firmly to support posts that extend 46 ft down into bedrock. The structure was built to withstand 100-mph winds and an 8.0-magnitude earthquake within 50 mi and is capable of supporting more than 70,000,000 lb.

One factor affecting the Skywalk's stability is the structure's center of gravity. We will calculate the center of gravity of the Skywalk and examine how the center of gravity changes when tourists walk out onto the observation platform.

The observation platform is U-shaped. The legs of the U are 10 ft wide and begin on land, under the visitors' center, 48 ft from the canyon's edge. The platform extends 70 ft over the edge of the canyon.

To calculate the center of mass of the structure, we treat it as a lamina and use a two-dimensional region in the xy-plane to represent the platform. We begin by dividing the area into three subregions so we can consider each subregion separately. The first region, denoted $\mathbf{R}_1$, consists of the curved part of the U. We model $\mathbf{R}_1$ as a semicircular annulus, with an inner radius of 25 ft and outer radius 35 ft, centered at the origin (Figure 2.7.12).

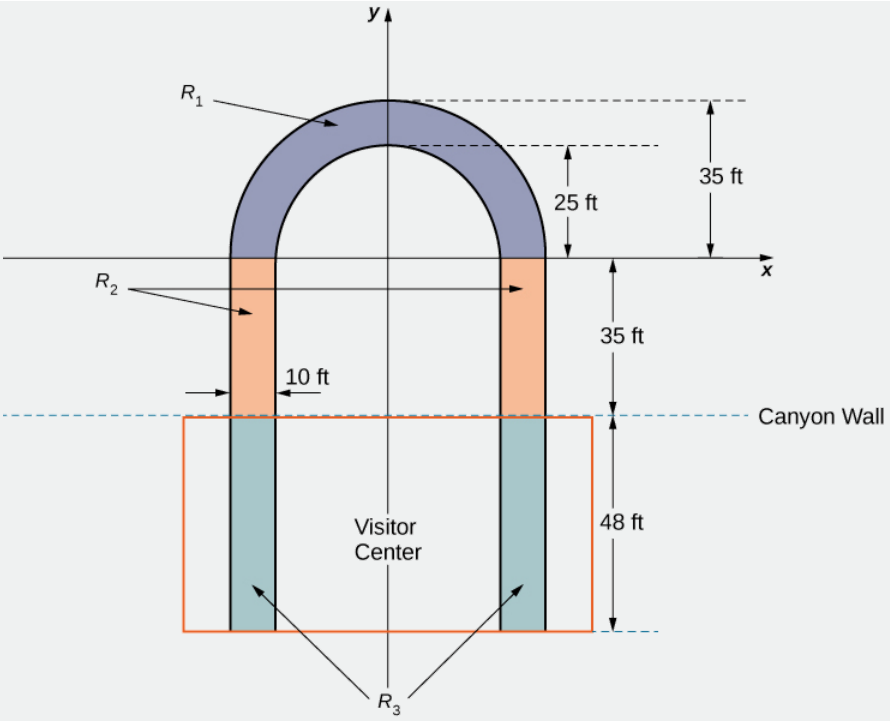


Figure **2.7.12**: We model the Skywalk with three sub-regions.

The legs of the platform, extending 35 ft between $\mathbf{R_1}$ and the canyon wall, comprise the second sub-region, $\mathbf{R_2}$. Last, the ends of the legs, which extend 48 ft under the visitor center, comprise the third sub-region, $\mathbf{R_3}$. Assume the area density of the lamina is constant and assume the total weight of the platform is 1,200,000 lb (not including the weight of the visitor center; we will consider that later). Use $g = 32\ \text{ft/sec}^2$.

1. Compute the area of each of the three sub-regions. Note that the areas of regions $\mathbf{R_2}$ and $\mathbf{R_3}$ should include the areas of the legs only, not the open space between them. Round answers to the nearest square foot.
2. Determine the mass associated with each of the three sub-regions.
3. Calculate the center of mass of each of the three sub-regions.
4. Now, treat each of the three sub-regions as a point mass located at the center of mass of the corresponding sub-region. Using this representation, calculate the center of mass of the entire platform.
5. Assume the visitor center weighs 2,200,000 lb, with a center of mass corresponding to the center of mass of $\mathbf{R_3}$. Treating the visitor center as a point mass, recalculate the system's center of mass. How does the center of mass change?
6. Although the Skywalk was built to limit the number of people on the observation platform to 120, the platform can support up to 800 people weighing 200 lb each. If all 800 people were allowed on the platform, and all of them went to the farthest end of the platform, how would the system's center of gravity be affected? (Include the visitor center in the calculations and represent the people by a point mass located at the farthest edge of the platform, 70 ft from the canyon wall.)

Theorem of Pappus

This section ends with a discussion of the **Theorem of Pappus for volume**, which allows us to find the volume of particular kinds of solids by using the centroid.[2]

Theorem of Pappus for Volume

Let $\mathbf{R}$ be a region in the plane and let l be a line in the plane that does not intersect $\mathbf{R}$. Then the volume of the solid of revolution formed by revolving $\mathbf{R}$ around l is equal to the area of $\mathbf{R}$ multiplied by the distance, d, traveled by the centroid of $\mathbf{R}$.

Proof

We can prove the case when the region is bounded above by the graph of a function $f(x)$ and below by the graph of a function $g(x)$ over an interval $[a, b]$, and for which the axis of revolution is the y-axis. In this case, the area of the region is $A = \int_a^b [f(x) - g(x)]\,dx$. Since the axis of rotation is the y-axis, the distance traveled by the centroid of the region depends only on the x-coordinate of the centroid, $\bar{x}$, which is

$$x = \frac{M_y}{m},$$

where

$$m = \rho \int_a^b [f(x) - g(x)]\,dx$$

and

$$M_y = \rho \int_a^b x[f(x) - g(x)]\,dx.$$

Then,

$$d = 2\pi \frac{\rho \int_a^b x[f(x) - g(x)]\,dx}{\rho \int_a^b [f(x) - g(x)]\,dx}$$

and thus

$$d \cdot A = 2\pi \int_a^b x[f(x) - g(x)]\,dx.$$

However, using the Method of Cylindrical Shells, we have

$$V = 2\pi \int_a^b x[f(x) - g(x)]\,dx.$$

So,

$$V = d \cdot A$$

and the proof is complete.

Example 2.7.8: Using the Theorem of Pappus for Volume

Let $\mathbf{R}$ be a circle of radius 2 centered at $(4, 0)$. Use the Theorem of Pappus for Volume to find the volume of the torus generated by revolving $\mathbf{R}$ around the y-axis.

Solution

The region and torus are depicted in the following figure.

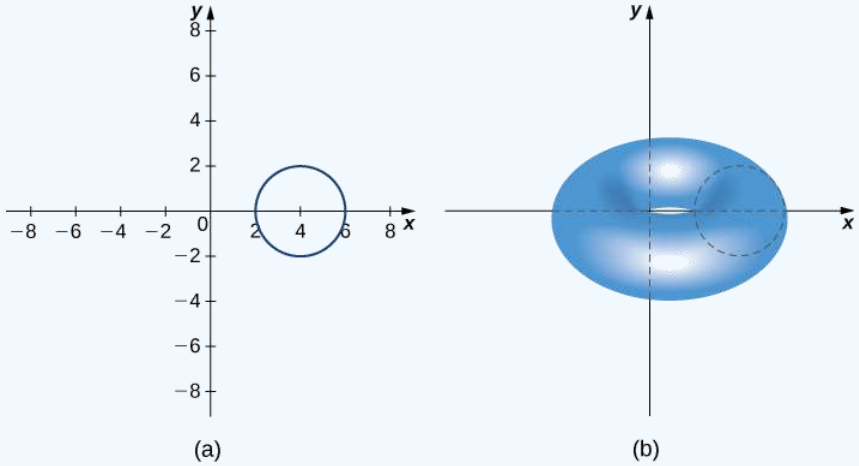


Figure 2.7.15: Determining a torus's volume using the Pappus theorem. (a) A circular region $\mathbf{R}$ in the plane; (b) the torus generated by revolving $\mathbf{R}$ about the y-axis.

The region $\mathbf{R}$ is a circle of radius 2, so the area of $\mathbf{R}$ is $A = 4\pi \text{ units}^2$. By the symmetry principle, the centroid of $\mathbf{R}$ is the circle's center. The centroid travels around the y-axis in a circular path of radius 4, so the centroid travels $d = 8\pi$ units. Then, the volume of the torus is $A \cdot d = 32\pi^2 \text{ units}^3$.

? Checkpoint 2.7.8

Let $\mathbf{R}$ be a circle of radius 1 centered at $(3, 0)$. Use the Theorem of Pappus for Volume to find the volume of the torus generated by revolving $\mathbf{R}$ around the y-axis.

Answer

$6\pi^2 \text{ units}^3$.

Footnotes

[1] A point mass refers to an idealized object that is considered to have mass but negligible size and shape.

[2] There is also a **Theorem of Pappus for Surface Area**, but it is much less useful than the theorem for volume.

2.7E: Exercises

Reading Questions

1. What is the physical interpretation of the center of mass?
2. What is the difference between a center of mass and a centroid, and under what condition are they the same?
3. For a system of point masses in a plane, why does the calculation for the moment about the x-axis (M_x) use the y-coordinates of the masses?
4. What is the formula for the mass of a thin rod on the interval $[a, b]$ with a linear density function $\rho(x)$?
5. How can the Symmetry Principle be used to simplify the process of finding the centroid of a region?
6. For a lamina bounded by $y = f(x)$ and the x-axis, what are the integral expressions for the moments M_x and M_y?
7. State the Theorem of Pappus for Volume. What three quantities does it relate?

In exercises 1 - 4, find the weight of the one-dimensional object.

1) A wire that is 2 ft long (starting at $x = 0$) and has a *weight density* function of $\gamma(x) = x^2 + 2x$ lb/ft

2) A car antenna that is 3 ft long (starting at $x = 0$) and has a *weight density* function of $\gamma(x) = 3x + 2$ lb/ft

Answer

$\frac{39}{2}$ lbs.

3) A metal rod that is 8 in. long (starting at $x = 0$) and has a *weight density* function of $\gamma(x) = e^{1/2x}$ lb/in.

4) A pencil that is 4 in. long (starting at $x = 2$) and has a *weight density* function of $\gamma(x) = \frac{5}{x}$ oz/in.

Answer

$\ln(243)$ oz.

5) Using the Left Endpoint Method with $n = 6$, approximate the weight of a one-dimensional ruler that is 12 in. long (starting at $x = 5$) and has a *weight density* function of $\gamma(x) = \ln(x) + (1/2)x^2$ oz/in.

In exercises 6 - 7, find the weight of the two-dimensional object that is centered at the origin.

6) An oversized hockey puck of radius 2 in. with radial *weight* density function $\gamma(x) = x^3 - 2x + 5$ lb/in^2.

Answer

$\frac{332\pi}{15}$ lbs.

7) A disk of radius 5 cm with radial *weight* density function $\gamma(x) = \sqrt{3x}$ g/cm^2

Answer

$20\pi\sqrt{15}$ g.

8) Using the Right Endpoint Method with $n = 6$, approximate the weight of a two-dimensional frisbee (centered at the origin) of radius 6 in. with radial *weight* density function $\gamma(x) = e^{-x}$ oz/in^2.

In exercises 9 - 14, calculate the center of mass for the collection of masses given.

9) $m_1 = 2$ at $x_1 = 1$ and $m_2 = 4$ at $x_2 = 2$

10) $m_1 = 1$ at $x_1 = -1$ and $m_2 = 3$ at $x_2 = 2$

Answer

$x = \frac{5}{4}$

11) $m = 3$ at $x = 0, 1, 2, 6$

12) Unit masses at $(x, y) = (1, 0), (0, 1), (1, 1)$

Answer

$\left(\frac{2}{3}, \frac{2}{3}\right)$

13) $m_1 = 1$ at $(1, 0)$ and $m_2 = 4$ at $(0, 1)$

14) $m_1 = 1$ at $(1, 0)$ and $m_2 = 3$ at $(2, 2)$

Answer

$\left(\frac{7}{4}, \frac{3}{2}\right)$

In exercises 15 - 24, compute the center of mass $\bar{x}$.

15) $\rho = 1$ for $x \in (-1, 3)$

16) $\rho = x^2$ for $x \in (0, L)$

Answer

$\frac{3L}{4}$

17) $\rho = 1$ for $x \in (0, 1)$ and $\rho = 2$ for $x \in (1, 2)$

18) $\rho = \sin x$ for $x \in (0, \pi)$

Answer

$\frac{\pi}{2}$

19) $\rho = \cos x$ for $x \in \left(0, \frac{\pi}{2}\right)$

20) $\rho = e^x$ for $x \in (0, 2)$

Answer

$\frac{e^2 + 1}{e^2 - 1}$

21) $\rho = x^3 + xe^{-x}$ for $x \in (0, 1)$

22) $\rho = x \sin x$ for $x \in (0, \pi)$

Answer

$\frac{\pi^2 - 4}{\pi}$

23) $\rho = \sqrt{x}$ for $x \in (1, 4)$

24) $\rho = \ln x$ for $x \in (1, e)$

Answer

$\frac{1}{4}(1 + e^2)$

In exercises 25 - 27, compute the center of mass $(\bar{x}, \bar{y})$. Use symmetry to help locate the center of mass whenever possible.

25) $\rho = 7$ in the square $0 \le x \le 1,\ 0 \le y \le 1$

26) $\rho = 3$ in the triangle with vertices $(0, 0)$, $(a, 0)$, and $(0, b)$

Answer

$\left(\frac{a}{3}, \frac{b}{3}\right)$

27) $\rho = 2$ for the region bounded by $y = \cos(x)$, $y = -\cos(x)$, $x = -\frac{\pi}{2}$, and $x = \frac{\pi}{2}$

In exercises 28 - 34, use a calculator to draw the region, then compute the center of mass $(\bar{x}, \bar{y})$. Use symmetry to help locate the center of mass whenever possible.

28) [Technology Required] The region bounded by $y = \cos(2x)$, $x = -\frac{\pi}{4}$, and $x = \frac{\pi}{4}$

Answer

$\left(0, \frac{\pi}{8}\right)$

29) [Technology Required] The region between $y = 2x^2$, $y = 0$, $x = 0$, and $x = 1$

30) [Technology Required] The region between $y = \frac{5}{4}x^2$ and $y = 5$

Answer

$(0, 3)$

31) [Technology Required] Region between $y = \sqrt{x}$, $y = \ln x$, $x = 1$, and $x = 4$

32) [Technology Required] The region bounded by $y = 0$ and $\frac{x^2}{4} + \frac{y^2}{9} = 1$

Answer

$\left(0, \frac{4}{\pi}\right)$

33) [Technology Required] The region bounded by $y = 0$, $x = 0$, and $\frac{x^2}{4} + \frac{y^2}{9} = 1$

34) [Technology Required] The region bounded by $y = x^2$ and $y = x^4$ in the first quadrant

Answer

$\left(\frac{5}{8}, \frac{1}{3}\right)$

In exercises 35 - 39, use the Theorem of Pappus to determine the volume of the shape.

35) Rotating $y = mx$ around the x-axis between $x = 0$ and $x = 1$

36) Rotating $y = mx$ around the y-axis between $x = 0$ and $x = 1$

Answer

$V = \frac{m\pi}{3}$ units^3

37) A general cone created by rotating a triangle with vertices $(0, 0)$, $(a, 0)$, and $(0, b)$ around the y-axis. Does your answer agree with the volume of a cone?

38) A general cylinder created by rotating a rectangle with vertices $(0, 0)$, $(a, 0)$, $(0, b)$, and (a, b) around the y-axis. Does your answer agree with the volume of a cylinder?

Answer

$V = \pi a^2 b$ units^3

39) A sphere created by rotating a semicircle with radius a around the y-axis. Does your answer agree with the volume of a sphere?

In exercises 40 - 44, use a calculator to draw the region enclosed by the curve. Find the area M and the centroid $(\bar{x}, \bar{y})$ for the given shapes. Use symmetry to help locate the center of mass whenever possible.

40) [Technology Required] Quarter-circle: $y = \sqrt{1 - x^2}$, $y = 0$, and $x = 0$

Answer

$\left(\frac{4}{3\pi}, \frac{4}{3\pi}\right)$

41) [Technology Required] Triangle: $y = x$, $y = 2 - x$, and $y = 0$

42) [Technology Required] Lens: $y = x^2$ and $y = x$

Answer

$\left(\frac{1}{2}, \frac{2}{5}\right)$

43) [Technology Required] Ring: $y^2 + x^2 = 1$ and $y^2 + x^2 = 4$

44) [Technology Required] Half-ring: $y^2 + x^2 = 1$, $y^2 + x^2 = 4$, and $y = 0$

Answer

$\left(0, \frac{28}{9\pi}\right)$

45) Find the generalized center of mass in the sliver between $y = x^a$ and $y = x^b$ with $a > b$. Then, use the Pappus theorem to find the volume of the solid generated when revolving around the y-axis.

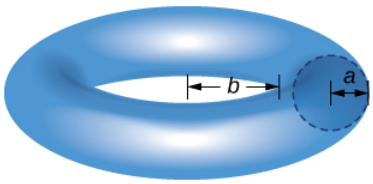
46) Find the generalized center of mass between $y = a^2 - x^2$, $x = 0$, and $y = 0$. Then, use the Pappus theorem to find the volume of the solid generated when revolving around the y-axis.

Answer

Center of mass: $\left(\frac{a}{6}, \frac{4a^2}{5}\right)$,
Volume: $\frac{2\pi a^4}{9}$ units^3

47) Find the generalized center of mass between $y = b\sin(ax)$, $x = 0$, and $x = \frac{\pi}{a}$. Then, use the Theorem of Pappus to find the volume of the solid generated when revolving around the y-axis.

48) Use the Theorem of Pappus to find the volume of a torus (pictured here). Assume that a disk of radius a is positioned with the left end of the circle at $x = b$, $b > 0$, and is rotated around the y-axis.



Answer

Volume: $V = 2\pi^2 a^2(b + a)$

49) Find the center of mass $(\bar{x}, \bar{y})$ for a thin wire along the semicircle $y = \sqrt{1 - x^2}$ with unit mass. (Hint: Use the Theorem of Pappus.)

Contributors

Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

This page titled 2.7E: Exercises is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Roy Simpson.

- **6.5E: Exercises for Section 6.5** by Edwin “Jed” Herman, Gilbert Strang is licensed CC BY-NC-SA 4.0. Original source: https://openstax.org/details/books/calculus-volume-1.
- **6.6E: Exercises for Section 6.6** by Edwin “Jed” Herman, Gilbert Strang is licensed CC BY-NC-SA 4.0. Original source: https://openstax.org/details/books/calculus-volume-1.

2.8: The Mean Value Theorem for Integrals

Note to the Instructor (click to expand)

This is an *incredibly* short section - perhaps a half lecture.

To succeed in this section, you'll need to use some skills from previous courses. While you should already know them, this is the first time they've been required. You can review these skills in CRC's Corequisite Codex. If you have a support class, it might cover some, **but not all**, of these topics.

Prerequisite Skills and Support Topics (click to expand)

- **Solving Equations**
  - Solving Trigonometric Equations Using Identities: Homework problem #9 requires solving a trigonometric equation to find the times when the number of daylight hours equals the average value.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Describe the meaning of the Mean Value Theorem for Integrals.

Hawk A.I. Section-Specific Tutor



To access the Hawk A.I. Tutor, you will need to be logged into your campus Gmail account.

This short section is dedicated to an examination of another essential theorem, the Mean Value Theorem for Integrals.

The Mean Value Theorem for Integrals

Recall, for a discrete list of values, $\{a_1, a_2, \ldots, a_n\}$, we compute the average using the formula

$$a_{\text{avg}} = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

We now have the "mathematical technology" to create the analog for a continuous function, $f(x)$, over a closed interval $[a, b]$.

We can *approximate* the average value of f as follows

$$f_{\text{avg}} \approx \frac{\sum_{i=1}^{n} f(x_i)}{n},$$

where x_i is defined using our traditional definitions from Riemann sums. However, we now perform a little manipulation to force the numerator to become a Riemann sum.

$$f_{\text{avg}} \approx \frac{\Delta x \sum_{i=1}^{n} f(x_i)}{n\Delta x} = \frac{\sum_{i=1}^{n} f(x_i)\Delta x}{n\Delta x} = \frac{\sum_{i=1}^{n} f(x_i)\Delta x}{b-a},$$

where we used the fact that $\Delta x = \frac{b-a}{n}$ in our last step.

Taking the limit as $n \to \infty$, we obtain the exact value of the average for f over the closed interval $[a, b]$, simply called the **average value** of f,

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)dx.$$

This definition leads us to a new theorem to add to our "toolkit." The **Mean Value Theorem for Integrals** states that a continuous function on a closed interval takes on its average value at the same point in that interval. The theorem guarantees that if $f(x)$ is continuous, a point c exists in an interval $[a, b]$ such that the value of the function at c is equal to the average value of $f(x)$ over $[a, b]$.

We state this theorem mathematically with the help of the formula for the average value of a function that we presented in Calculus I.

Theorem: The Mean Value Theorem for Integrals

If $f(x)$ is continuous over an interval $[a, b]$, then there is at least one point $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

This formula can also be stated as

$$\int_a^b f(x)\, dx = f(c)(b-a).$$

Proof

Since $f(x)$ is continuous on $[a, b]$, by the Extreme Value Theorem, it assumes minimum and maximum values - m and M, respectively - on $[a, b]$. Then, for all x in $[a, b]$, we have $m \leq f(x) \leq M$. Therefore, by the Comparison Theorem, we have

$$m(b-a) \leq \int_a^b f(x)\, dx \leq M(b-a).$$

Dividing by $b-a$ gives us

$$m \leq \frac{1}{b-a} \int_a^b f(x)\, dx \leq M.$$

Since $\frac{1}{b-a} \int_a^b f(x)\, dx$ is a number between m and M, and since $f(x)$ is continuous and assumes the values m and M over $[a, b]$, by the Intermediate Value Theorem, there is a number c over $[a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\, dx,$$

and the proof is complete.

Example 2.8.1: Finding the Average Value of a Function

Find the average value of the function $f(x) = 8 - 2x$ over the interval $[0, 4]$ and find c such that $f(c)$ equals the average value of the function over $[0, 4]$.

Solution

The formula states the mean value of $f(x)$ is given by

$$\frac{1}{4-0} \int_0^4 (8 - 2x)\, dx.$$

We can see in Figure 2.8.1 that the function represents a straight line and forms a right triangle bounded by the x- and y-axes. The area of the triangle is $A = \frac{1}{2}(base)(height)$. We have

$$A = \frac{1}{2}(4)(8) = 16.$$

The average value is found by multiplying the area by $1/(4-0)$. Thus, the average value of the function is

$$\frac{1}{4}(16) = 4$$

Set the average value equal to $f(c)$ and solve for c.

$$\begin{aligned} 8-2c &= 4 \\ c &= 2 \end{aligned}$$

At $c=2, f(2)=4$.

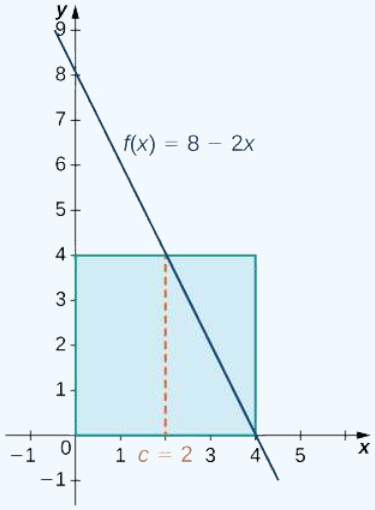


Figure 2.8.1: By the Mean Value Theorem, the continuous function $f(x)$ takes on its average value at c at least once over a closed interval.

Checkpoint 2.8.1

Find the average value of the function $f(x)=\frac{x}{2}$ over the interval $[0,6]$ and find c such that $f(c)$ equals the average value of the function over $[0,6]$.

Answer

The average value is 1.5 and $c=3$.

Example 2.8.2: Finding the Point Where a Function Takes on Its Average Value

Given $\int_0^3 x^2\,dx=9$, find c such that $f(c)$ equals the average value of $f(x)=x^2$ over $[0,3]$.

Solution

We are looking for the value of c such that

$$f(c)=\frac{1}{3-0}\int_0^3 x^2\,dx=\frac{1}{3}(9)=3.$$

Replacing $f(c)$ with c^2, we have

$$\begin{aligned} c^2 &= 3 \\ c &= \pm\sqrt{3}. \end{aligned}$$

Since $-\sqrt{3}$ is outside the interval, take only the positive value. Thus, $c=\sqrt{3}$ (Figure 2.8.2).

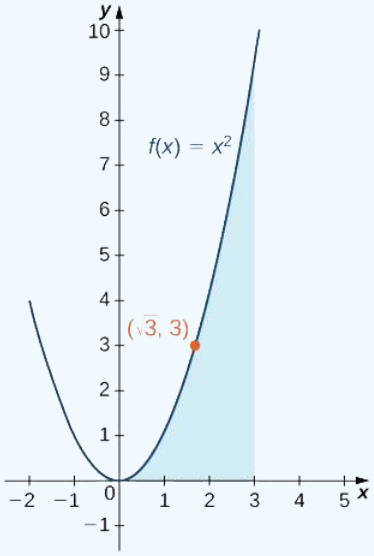


Figure 2.8.2: Over the interval $[0, 3]$, the function $f(x) = x^2$ takes on its average value at $c = \sqrt{3}$.

? Checkpoint 2.8.2

Given $\int_0^3 (2x^2 - 1)\,dx = 15$, find c such that $f(c)$ equals the average value of $f(x) = 2x^2 - 1$ over $[0, 3]$.

Answer

$c = \sqrt{3}$

2.8E: EXERCISES

READING QUESTIONS

1. What is the formula for the average value of a continuous function $f(x)$ over an interval $[a, b]$?
2. What does the Mean Value Theorem for Integrals guarantee about a continuous function on a closed interval?
3. What are the two equivalent ways to write the conclusion of the Mean Value Theorem for Integrals?
4. In Example 1.9.2, why is the value $c = -\sqrt{3}$ discarded as a valid solution?
5. How is the average value of a continuous function derived from the concept of averaging a discrete list of numbers?

1) Consider two athletes running at variable speeds $v_1(t)$ and $v_2(t)$. The runners start and finish a race at exactly the same time. Explain why the two runners must be going the same speed at some point.

2) Two mountain climbers start their climb at base camp, taking two different routes, one steeper than the other, and arrive at the peak at exactly the same time. Is it necessarily true that, at some point, both climbers increased in altitude at the same rate?

Answer

Yes. It is implied by the Mean Value Theorem for Integrals.

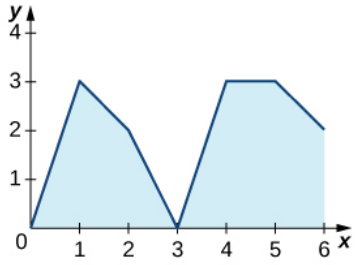
3) To get on a certain toll road a driver has to take a card that lists the mile entrance point. The card also has a timestamp. When going to pay the toll at the exit, the driver is surprised to receive a speeding ticket along with the toll. Explain how this can happen.

4) Set $F(x) = \int_1^x (1 - t)\, dt$. Find $F'(2)$ and the average value of F' over $[1, 2]$.

Answer

$F'(2) = -1$; average value of F' over $[1, 2]$ is $-1/2$.

5) The graph of $y = \int_0^x f(t)\, dt$, where f is a piecewise constant function, is shown here.

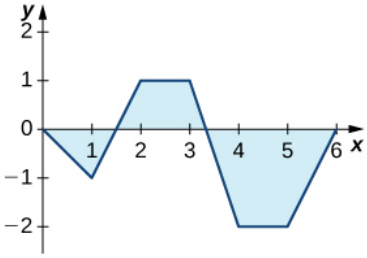


a. Over which intervals is f positive? Over which intervals is it negative? Over which intervals, if any, is it equal to zero?

b. What are the maximum and minimum values of f?

c. What is the average value of f?

6) The graph of $y = \int_0^x f(t)\, dt$, where f is a piecewise constant function, is shown here.



a. Over which intervals is f positive? Over which intervals is it negative? Over which intervals, if any, is it equal to zero?

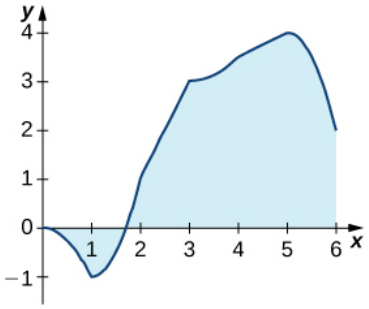
b. What are the maximum and minimum values of f?

c. What is the average value of f?

Answer

a. f is positive over $[1,2]$ and $[5,6]$, negative over $[0,1]$ and $[3,4]$, and zero over $[2,3]$ and $[4,5]$.
b. The maximum value is 2 and the minimum is -3.
c. The average value is 0.

7) The graph of $y = \int_0^x \ell(t)\,dt$, where ℓ is a piecewise linear function, is shown here.

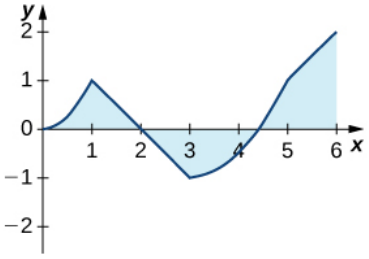


a. Over which intervals is ℓ positive? Over which intervals is it negative? Over which, if any, is it zero?

b. Over which intervals is ℓ increasing? Over which is it decreasing? Over which, if any, is it constant?

c. What is the average value of ℓ?

8) The graph of $y = \int_0^x \ell(t)\,dt$, where ℓ is a piecewise linear function, is shown here.



a. Over which intervals is ℓ positive? Over which intervals is it negative? Over which, if any, is it zero?

b. Over which intervals is ℓ increasing? Over which is it decreasing? Over which intervals, if any, is it constant?

c. What is the average value of ℓ?

Answer

a. ℓ is positive over $[0,1]$ and $[3,6]$, and negative over $[1,3]$.
b. It is increasing over $[0,1]$ and $[3,5]$, and it is constant over $[1,3]$ and $[5,6]$.
c. Its average value is $\frac{1}{3}$.

9) Suppose that the number of hours of daylight on a given day in Seattle is modeled by the function $-3.75\cos(\frac{\pi t}{6}) + 12.25$, with t given in months and $t = 0$ corresponding to the winter solstice.

a. What is the average number of daylight hours in a year?

b. At which times t_1 and t_2, where $0 \le t_1 < t_2 < 12$, do the number of daylight hours equal the average number?

c. Write an integral that expresses the total number of daylight hours in Seattle between t_1 and t_2

d. Compute the mean hours of daylight in Seattle between t_1 and t_2, where $0 \le t_1 < t_2 < 12$, and then between t_2 and t_1, and show that the average of the two is equal to the average day length.

10) Suppose the rate of gasoline consumption in the United States can be modeled by a sinusoidal function of the form $(11.21 - \cos(\frac{\pi t}{6})) \times 10^9$ gal/mo.

a. What is the average monthly consumption, and for which values of t is the rate at time t equal to the average rate?

b. What is the number of gallons of gasoline consumed in the United States in a year?

c. Write an integral that expresses the average monthly U.S. gas consumption during the part of the year between the beginning of April $(t = 3)$ and the end of September $(t = 9)$.

Answer

a. The average is 11.21×10^9 since $\cos(\frac{\pi t}{6})$ has period 12 and integral 0 over any period. Consumption is equal to the average when $\cos(\frac{\pi t}{6}) = 0$, when $t = 3$, and when $t = 9$.

b. Total consumption is the average rate times duration: $11.21 \times 12 \times 10^9 = 1.35 \times 10^{11}$

c. $10^9(11.21 - \frac{1}{6}\int_3^9 \cos(\frac{\pi t}{6})\,dt) = 10^9(11.21 + 2\pi) = 11.84x10^9$

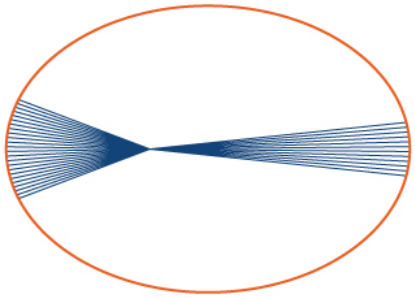
11) Explain why, if f is continuous over $[a, b]$, there is at least one point $c \in [a, b]$ such that $f(c) = \frac{1}{b-a}\int_a^b f(t)\,dt$.

12) Explain why, if f is continuous over $[a, b]$ and is not equal to a constant, there is at least one point $M \in [a, b]$ such that $f(M) = \frac{1}{b-a}\int_a^b f(t)\,dt$ and at least one point $m \in [a, b]$ such that $f(m) < \frac{1}{b-a}\int_a^b f(t)\,dt$.

Answer

If f is not constant, then its average is strictly smaller than the maximum and larger than the minimum, which are attained over $[a, b]$ by the Extreme Value Theorem.

13) Kepler's first law states that the planets move in elliptical orbits with the Sun at one focus. The closest point of a planetary orbit to the Sun is called the **perihelion** (for Earth, it currently occurs around January 3) and the farthest point is called the **aphelion** (for Earth, it currently occurs around July 4). Kepler's second law states that planets sweep out equal areas of their elliptical orbits in equal times. Thus, the two arcs indicated in the following figure are swept out in equal times. At what time of year is Earth moving fastest in its orbit? When is it moving slowest? Kepler's first law states that the planets move in elliptical orbits with the Sun at one focus. The closest point of a planetary orbit to the Sun is called the perihelion (for Earth, it currently occurs around January 3) and the farthest point is called the aphelion (for Earth, it currently occurs around July 4). Kepler's second law states that planets sweep out equal areas of their elliptical orbits in equal times. Thus, the two arcs indicated in the following figure are swept out in equal times. At what time of year is Earth moving fastest in its orbit? When is it moving slowest?



14) A point on an ellipse with major axis length $2a$ and minor axis length $2b$ has the coordinates $(a\cos\theta, b\sin\theta), \quad 0 \le \theta \le 2\pi$.

a. Show that the distance from this point to the focus at $(-c, 0)$ is $d(\theta) = a + c\cos\theta$, where $c = \sqrt{a^2 - b^2}$.

Use these coordinates to show that the average distance $\bar{d}$ from a point on the ellipse to the focus at $(-c, 0)$, with respect to angle θ, is a.

Answer

a. $d^2\theta = (a\cos\theta + c)^2 + b^2\sin^2\theta = a^2 + c^2\cos^2\theta + 2ac\cos\theta = (a + c\cos\theta)^2$;

b. $\bar{d} = \frac{1}{2\pi}\int_0^{2\pi}(a + 2c\cos\theta)\,d\theta = a$

15) As implied earlier, according to Kepler's laws, Earth's orbit is an ellipse with the Sun at one focus. The perihelion for Earth's orbit around the Sun is 147,098,290 km and the aphelion is 152,098,232 km.

a. By placing the major axis along the x-axis, find the average distance from Earth to the Sun.

b. The classic definition of an astronomical unit (AU) is the distance from Earth to the Sun, and its value was computed as the average of the perihelion and aphelion distances. Is this definition justified?

16) The force of gravitational attraction between the Sun and a planet is $F(\theta) = \frac{GmM}{r^2(\theta)}$, where m is the mass of the planet, M is the mass of the Sun, G is a universal constant, and $r(\theta)$ is the distance between the Sun and the planet when the planet is at an angle θ with the major axis of its orbit. Assuming that M, m, and the ellipse parameters a and b (half-lengths of the major and minor axes) are given, set up—but do not evaluate—an integral that expresses in terms of G, m, M, a, b the average gravitational force between the Sun and the planet.

Answer

Mean gravitational force $= \dfrac{GmM}{2}\displaystyle\int_0^{2\pi} \dfrac{1}{(a+2\sqrt{a^2-b^2}\cos\theta)^2}\,d\theta.$

The displacement from rest of a mass attached to a spring satisfies the simple harmonic motion equation $x(t) = A\cos(\omega t - \phi)$, where ϕ is a phase constant, ω is the angular frequency, and A is the amplitude. Find the average velocity, the average speed (magnitude of velocity), the average displacement, and the average distance from rest (magnitude of displacement) of the mass.

This page titled 2.8E: Exercises is shared under a not declared license and was authored, remixed, and/or curated by Roy Simpson.

- **5.3E: Exercises for Section 5.3** is licensed CC BY-NC-SA 4.0.

2.9: CHAPTER 2 REVIEW EXERCISES

***True or False?* Justify your answer with a proof or a counterexample.**

1) The amount of work to pump the water out of a half-full cylinder is half the amount of work to pump the water out of the full cylinder.

Answer

False

2) If the force is constant, the amount of work to move an object from $x = a$ to $x = b$ is $F(b - a)$.

3) The disk method can be used in any situation in which the washer method is successful at finding the volume of a solid of revolution.

Answer

False

4) If the half-life of $seaborgium - 266$ is 360 ms, then $k = \dfrac{\ln 2}{360}$.

For exercises 5 - 8, use the requested method to determine the volume of the solid.

5) The volume that has a base of the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$ and cross-sections of an equilateral triangle perpendicular to the y-axis. Use the method of slicing.

Answer

$V = 32\sqrt{3}\,\text{units}^3$

6) $y = x^2 - x$, from $x = 1$ to $x = 4$, rotated around the y-axis using the washer method

7) $x = y^2$ and $x = 3y$ rotated around the y-axis using the washer method

Answer

$V = \frac{162\pi}{5}\,\text{units}^3$

8) $x = 2y^2 - y^3,\ x = 0$,and $y = 0$ rotated around the x-axis using cylindrical shells

For exercises 9 - 14, find

a. the area of the region,

b.the volume of the solid when rotated around the x-axis, and

c. the volume of the solid when rotated around the y-axis. Use whichever method seems most appropriate to you.

9) $y = x^3, x = 0, y = 0$, and $x = 2$

Answer

a. $A = 4$ units^2
b. $V = \frac{128\pi}{7}$ units^3
c. $V = \frac{64\pi}{5}$ units^3

10) $y = x^2 - x$ and $x = 0$

11) [T] $y = \ln(x) + 2$ and $y = x$

Answer

a. $A \approx 1.949$ units^2
b. $V \approx 21.952$ units^3
c. $V = \approx 17.099$ units^3

12) $y = x^2$ and $y = \sqrt{x}$

13) $y = 5 + x, y = x^2, x = 0$, and $x = 1$

Answer

a. $A = \frac{31}{6}$ units^2
b. $V = \frac{452\pi}{15}$ units^3
c. $V = \frac{31\pi}{6}$ units^3

14) Below $x^2 + y^2 = 1$ and above $y = 1 - x$

15) Find the mass of $\rho = e^{-x}$ on a disk centered at the origin with radius 4.

Answer

$m \approx 245.282$

16) Find the center of mass for $\rho = \tan^2 x$ on $x \in (-\frac{\pi}{4}, \frac{\pi}{4})$.

17) Find the mass and the center of mass of $\rho = 1$ on the region bounded by $y = x^5$ and $y = \sqrt{x}$.

Answer

Mass: $\frac{1}{2}$,
Center of mass: $(\frac{18}{35}, \frac{9}{11})$

For exercises 18 - 19, find the requested arc lengths.

18) The length of x for $y = \cosh(x)$ from $x = 0$ to $x = 2$.

19) The length of y for $x = 3 - \sqrt{y}$ from $y = 0$ to $y = 4$

Answer

$s = [\sqrt{17} + \frac{1}{8}\ln(33 + 8\sqrt{17})]$ units

For exercises 20 - 21, find the surface area and volume when the given curves are revolved around the specified axis.

20) The shape created by revolving the region between $y = 4 + x$, $y = 3 - x$, $x = 0$, and $x = 2$ rotated around the y-axis.

21) The loudspeaker created by revolving $y = \dfrac{1}{x}$ from $x = 1$ to $x = 4$ around the x-axis.

Answer

Volume: $V = \frac{3\pi}{4}$ units^3
Surface area: $A = \pi\left(\sqrt{2} - \sinh^{-1}(1) + \sinh^{-1}(16) - \frac{\sqrt{257}}{16}\right)$ units^2

For exercise 22, consider the Karun-3 dam in Iran. Its shape can be approximated as an isosceles triangle with height 205 m and width 388 m. Assume the current depth of the water is 180 m. The density of water is 1000 kg/m^3.

22) Find the total force on the wall of the dam.

23) You are a crime scene investigator attempting to determine the time of death of a victim. It is noon and 45 °F outside and the temperature of the body is 78 °F. You know the cooling constant is $k = 0.00824$ °F/min. When did the victim die, assuming that a human's temperature is 98 °F?

Answer

11:02 a.m.

For the following exercise, consider the stock market crash in 1929 in the United States. The table lists the Dow Jones industrial average per year leading up to the crash.

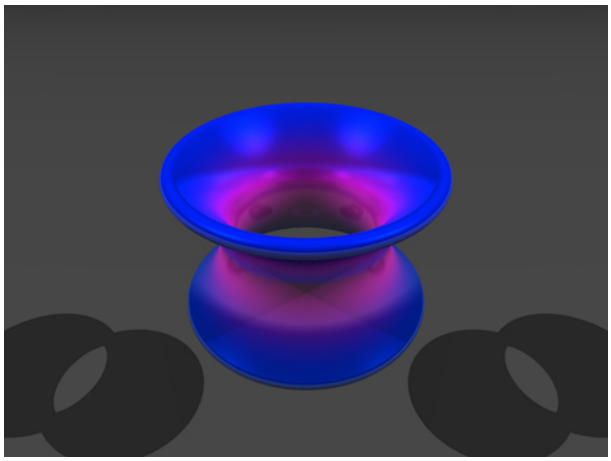
Year after 1920	Value ($)
1	63.90
3	100
5	110
7	160
9	381.17

Source: http:/stockcharts.com/freecharts/hi...a19201940.html

24) [T] The best-fit exponential curve to these data is given by $y = 40.71 + 1.224^x$. Why do you think the gains of the market were unsustainable? Use first and second derivatives to help justify your answer. What would this model predict the Dow Jones industrial average to be in **2014** ?

For exercises 25 - 26, consider the catenoid, the only solid of revolution that has a minimal surface, or zero mean curvature. A catenoid in nature can be found when stretching soap between two rings.

25) Find the volume of the catenoid $y = \cosh(x)$ from $x = -1$ to $x = 1$ that is created by rotating this curve around the x-**axis,** as shown here.



Answer

$V = \pi\left(1 + \sinh(1)\cosh(1)\right)$ units^3

26) Find surface area of the catenoid $y = \cosh(x)$ from $x = -1$ to $x = 1$ that is created by rotating this curve around the x-**axis.**

This page titled 2.9: Chapter 2 Review Exercises is shared under a not declared license and was authored, remixed, and/or curated by Roy Simpson.

- **6.R: Chapter 6 Review Exercises** is licensed CC BY-NC-SA 4.0.

CHAPTER OVERVIEW

3: Techniques of Integration

We saw in the previous chapter how important integration can be for all kinds of different topics—from calculations of volumes to flow rates, and from using a velocity function to determine a position to locating centers of mass. It is no surprise, then, that techniques for finding antiderivatives (or indefinite integrals) are important to know for everyone who uses them. We have already discussed some basic integration formulas and the method of integration by substitution. In this chapter, we study some additional techniques, including some ways of approximating definite integrals when normal techniques do not work.

3.1: Integration by Parts

3.1E: Exercises

3.2: Trigonometric Integrals

3.2E: Exercises

3.3: Trigonometric Substitution

3.3E: Exercises

3.4: Partial Fractions

3.4E: Exercises

3.5: Numerical Integration

3.5E: Exercises

3.6: Indeterminate Forms and L'Hospital's Rule

3.6E: Exercises

3.7: Improper Integrals

3.7E: Exercises

3.8: Chapter 3 Review Exercises

This page titled 3: Techniques of Integration is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Roy Simpson.

3.1: Integration by Parts

Note to the Instructor (click to expand)

This is the standard Integration by Parts section with the inclusion of the integral of $\sec^3(\theta)$ and power reduction formulas. It's fun and no-frills.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Recognize when to use Integration by Parts.
- Use the Integration by Parts formula to solve integration problems.
- Use the Integration by Parts formula for definite integrals.

Hawk A.I. Section-Specific Tutor



To access the Hawk A.I. Tutor, you will need to be logged into your campus Gmail account.

By now, we have a fairly thorough procedure for evaluating many basic integrals. However, although we can integrate $\int x\sin(x^2)\,dx$ by using the substitution, $u = x^2$, something as simple looking as $\int x\sin x\;dx$ defies us. Many students want to know whether there is a Product Rule for integration. There is not, but a technique based on the Product Rule for differentiation allows us to exchange one integral for another. We call this technique Integration by Parts.

The Integration by Parts Formula

If, $h(x) = f(x)g(x)$, then by using the Product Rule, we obtain

$$h'(x) = f'(x)g(x) + g'(x)f(x). \tag{3.1.1}$$

Although at first it may seem counterproductive, let's now integrate both sides of Equation 3.1.1:

$$\int h'(x)\;dx = \int (g(x)f'(x) + f(x)g'(x))\;dx.$$

This gives us

$$h(x) = f(x)g(x) = \int g(x)f'(x)\,dx + \int f(x)g'(x)\;dx.$$

Now we solve for $\int f(x)g'(x)\;dx$:

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\;dx.$$

By making the substitutions $u = f(x)$ and $v = g(x)$, which in turn make $du = f'(x)\,dx$ and $dv = g'(x)\,dx$, we have the more compact form

$$\int u\,dv = uv - \int v\,du.$$

Theorem: Integration by Parts

Let $u = f(x)$ and $v = g(x)$ be functions with continuous derivatives. Then, the **Integration by Parts** formula (also known as IbP) for the integral involving these two functions is:

$$\int u\,dv = uv - \int v\,du. \tag{3.1.2}$$

The advantage of using the Integration by Parts formula is that we can exchange one integral for another, possibly more accessible integral. The following example illustrates its use.

Example 3.1.1: Using Integration by Parts

Use Integration by Parts with $u = x$ and $dv = \sin x\ dx$ to evaluate

$$\int x \sin x\ dx.$$

Solution

By choosing $u = x$, we have $du = 1\ dx$. Since $dv = \sin x\ dx$, we get

$$v = \int \sin x\ dx = -\cos x.$$

It is handy to keep track of these values in a table, as follows:

$u = x$	$dv = \sin x\, dx$
$du = 1\, dx$	$v = \int \sin x\, dx = -\cos x$

Applying the Integration by Parts formula (Equation 3.1.2) results in

$$\begin{aligned} \int x \sin x\ dx &= (x)(-\cos x) - \int (-\cos x)(1\ dx) && \text{(substitution)} \\ &= -x\cos x + \int \cos x\ dx && \text{(simplifying)} \end{aligned}$$

Then use

$$\int \cos x\ dx = \sin x + C$$

to obtain

$$\int x \sin x\ dx = -x\cos x + \sin x + C.$$

Analysis

At this point, a few items need clarification. First, what would have happened if we had chosen $u = \sin x$ and $dv = x,\ dx$? If we had done so, then we would have $du = \cos x\, dx$ and $v = \frac{1}{2}x^2$. Thus, after applying Integration by Parts (Equation 3.1.2), we have

$$\int x \sin x\ dx = \frac{1}{2}x^2 \sin x - \int \frac{1}{2}x^2 \cos x\ dx.$$

Unfortunately, we are in no better position than before with the new integral. It is important to remember that when we apply Integration by Parts, we may need to try several choices for u and dv before finding a choice that works.

Second, you may wonder why, when we find $v = \int \sin x\ dx = -\cos x$, we do not use $v = -\cos x + K$. To see that it makes no difference, we can rework the problem using $v = -\cos x + K$:

$$\begin{aligned} \int x \sin x\ dx &= (x)(-\cos x + K) - \int (-\cos x + K)(1\ dx) \\ &= -x\cos x + Kx + \int \cos x\ dx - \int K\ dx \\ &= -x\cos x + Kx + \sin x - Kx + C \\ &= -x\cos x + \sin x + C. \end{aligned}$$

As you can see, it makes no difference in the final solution.

Last, we can check to make sure that our antiderivative is correct by differentiating $-x\cos x+\sin x+C$:

$$\begin{aligned}\frac{d}{dx}(-x\cos x+\sin x+C) &= \cancel{(-1)\cos x}+(-x)(-\sin x)+\cancel{\cos x}\\ &= x\sin x\end{aligned}$$

Therefore, the antiderivative checks out.

? Checkpoint 3.1.1

Evaluate $\int xe^{2x}\,dx$ using the Integration by Parts formula (Equation 3.1.2) with $u=x$ and $dv=e^{2x}\,dx$.

Answer

$$\int xe^{2x}\,dx=\frac{1}{2}xe^{2x}-\frac{1}{4}e^{2x}+C$$

Choosing u and dv

The natural question to ask at this point is: How do we know what to choose for u and dv?

Most authors will immediately give you a trick to picking the "right" choice for u; however, the trick they give is not flawless. It often fails. Therefore, it is best to develop an approach to Integration by Parts from the other direction - by asking, "What is a good choice for dv?

Our choice for dv will immediately inform us as to what u needs to be. Since we are going to have to integrate dv, you want to choose it to be something you actually *can* integrate. This is so important that it bears repeating:

> ***Integration by Parts boils down to selecting a factor, preferably the most complex, of the integrand that you can integrate either by direct integration or by the Substitution Method.***

That's it! The rest of this subsection discusses *why* we make certain choices for dv and u. Still, in the end, it's all about asking yourself, "What's the most complex factor of this integrand I can integrate?"

Now, think of all of the "traditional" functions you have encountered up to this point in mathematics, and ask yourself which ones you can easily integrate.

Basic Function Type	Easily Integrable?
Algebraic (e.g., x^3, $\sqrt{x}$, and $\frac{1}{x}$)	Yes
Exponential (e.g., e^x and 2^x)	Yes
Logarithmic (e.g., $\ln(x)$ and $\log_3(x)$)	No
Trigonometric (e.g., $\sin(x)$ and $\sec^2(x)$, but not $\sec(x)$)	Yes (in some cases)
Inverse Trigonometric (e.g., $\tan^{-1}(x)$ and $\csc^{-1}(x)$)	No

Since we do not have integration formulas that allow us to integrate simple **L**ogarithmic functions and **I**nverse trigonometric functions,[1] it makes sense that they should not be chosen for dv. On the other hand, **E**xponential and **T**rigonometric functions are, in general, easy to integrate and make good choices for dv. Finally, we can always integrate basic **A**lgebraic functions. However,

their antiderivatives require slightly more work than the straightforward antiderivatives of the exponential and trigonometric functions. Therefore, if given the option, we would rather integrate exponential or trigonometric functions than algebraic functions.

Given that discussion, we have the following tactic.

Tactic for Choosing dv

Given an integral where you wish to use Integration by Parts, the order of preference for choosing dv is as follows:

1. **E**xponential functions
2. **T**rigonometric functions
3. **A**lgebraic functions
4. **I**nverse trigonometric functions
5. **L**ogarithmic functions

As was mentioned previously, most authors approach this conversation by stating what your choice of u should be rather than dv. To accommodate students with instructors who take this same approach, we could reverse the direction of our tactic for choosing dv and, instead, create the commonly-taught tactic for choosing u. If logarithms are the worst choice for dv, then they will be our first choice for u, and so on. This gives us the following tactic for choosing u.

Tactic for Choosing u

Given an integral where you wish to use Integration by Parts, the order of preference for choosing u is as follows:

1. **L**ogarithmic functions
2. **I**nverse trigonometric functions
3. **A**lgebraic functions
4. **T**rigonometric functions
5. **E**xponential functions

This last tactic gives a common mnemonic, **LIATE**, to take some of the guesswork out of our choices for u. The type of function in the integral that appears first in the list should be our first choice of u.[2] When we have chosen u, dv is selected to be the remaining part of the integrand.

Caution: My Approach

I stick with finding dv first instead of finding u so that you get a more natural and understandable approach to Integration by Parts. All of the examples will reflect this.

Thus, if an integral contains the product of a logarithmic function and an algebraic function, we would choose dv to be the algebraic function (because we don't know how to integrate a logarithm as of yet). Therefore, we would choose u to be the logarithmic function. Following the typical textbook tactic, our choice for u is correct because L comes before A in LIATE.

The integral in Example **3.1.1** has a trigonometric function, $\sin x$, and an algebraic function, x. Both of these are easily integrable, but we would rather integrate $\sin x$ over x because antiderivatives of trigonometric functions are simple (as opposed to algebraic functions, which require raising powers and dividing by new powers). Thus, we would choose $dv = \sin x\,dx$ and $u = x$. Again, this coincides with our decision to use the common textbook tactic because A comes before T in LIATE.

There is a second, more important reason as to why we are selecting to differentiate the algebraic function and integrate the trigonometric function in Example **3.1.1**. When we tried to go the opposite direction (choosing to integrate the algebraic function and differentiate the trigonometric function), we arrived at an integral that got *worse* as we ended with an algebraic function of *higher* power and we *still* had a trigonometric function. A hidden goal when dealing with products of polynomials and other functions is to remove the polynomial. This can only be done if we take derivatives of the polynomial. While this might seem confusing right now, things will get more apparent as you use Integration by Parts.

Finally, here is the rule of thumb I tell my students:

Rule of Thumb: Choosing dv and u

When using Integration by Parts:

- Choose dv to be the *largest* factor of the integrand that you can integrate, either directly or by using the Substitution Method - u will be the rest of the integrand. You will often need to rewrite the integrand to see this largest factor and, remember, $\int f(x)\,dx = \int f(x)\cdot 1\,dx$. This is especially useful if you cannot integrate any *obvious* factor within the integrand.
- If you can integrate *all* factors within the integrand, then switch gears and instead choose u to be the factor whose derivative (or eventual, higher-order derivative) changes forms or becomes a constant.
- If you have to use Integration by Parts more than once while evaluating an integral, be sure to stay with the same "function type" choice for all Integrations by Parts.

The best way to understand this Rule of Thumb is through examples.

Example 3.1.2: Using Integration by Parts

Evaluate

$$\int \frac{\ln x}{x^3}\,dx.$$

Solution

☒ **Direct Integration:** Since the integrand is not the derivative of an established, well-known function, we cannot directly integrate.

☒ **Simple u-Substitution:** I leave it for the reader to try as they might to find a proper u-substitution - none will work.

☒ **Radical Substitution:** Since a simple u-substitution will not work, a radical substitution is out of the question.

☒ **Trigonometric Substitution:** As written, there isn't any obvious trigonometric substitution.

? **By Parts:** Since the integrand is the product of two different function types, it is a good candidate for Integration by Parts.[3]

It is helpful, at times, to begin by rewriting the integral:

$$\int \frac{\ln x}{x^3}\,dx = \int x^{-3}\ln x\,dx.$$

Consider the factors of the integrand - x^{-3} and $\ln(x)$. Which one can we easily integrate? The obvious answer here is x^{-3}. Hence, we choose $dv = x^{-3}\,dx$ (don't forget the dx) and we let u be the rest of the integrand. That is, $u = \ln(x)$.

For the LIATE Folks

We choose $u = \ln x$ and $dv = x^{-3}\,dx$, since L comes before A in LIATE.

Next, since $u = \ln x$, we have $du = \frac{1}{x}\,dx$. Also, $v = \int x^{-3}\,dx = -\frac{1}{2}x^{-2}$. Summarizing,

$u = \ln(x)$	$dv = x^{-3}\,dx$
$du = \dfrac{1}{x}\,dx$	$v = -\dfrac{1}{2}x^{-2}$

Substituting into the Integration by Parts formula (Equation 3.1.2) gives

$$\begin{aligned}\int \frac{\ln x}{x^3}\,dx &= \int x^{-3}\ln x\,dx \\ &= (\ln x)\left(-\frac{1}{2}x^{-2}\right) - \int \left(-\frac{1}{2}x^{-2}\right)\left(\frac{1}{x}\,dx\right) \\ &= -\frac{1}{2}x^{-2}\ln x + \int \frac{1}{2}x^{-3}\,dx \\ &= -\frac{1}{2}x^{-2}\ln x - \frac{1}{4}x^{-2} + C \\ &= -\frac{1}{2x^2}\ln x - \frac{1}{4x^2} + C\end{aligned}$$

? Checkpoint 3.1.2

Evaluate

$$\int x\ln x\,dx.$$

Answer

$$\int x\ln x\;dx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C$$

More Complex Uses of Integration by Parts

In some cases, as in the next two examples, it may be necessary to apply Integration by Parts more than once.

✓ Example 3.1.3: Applying Integration by Parts More Than Once

Evaluate

$$\int x^2 e^{3x}\,dx.$$

Solution

☒ **Direct Integration:** Since the integrand is not the derivative of an established, well-known function, we cannot directly integrate.

☒ **Simple u-Substitution:** I leave it for the reader to try as they might to find a proper u-substitution - none will work.

☒ **Radical Substitution:** Since a simple u-substitution will not work, a radical substitution is out of the question.

☒ **Trigonometric Substitution:** As written, there isn't any obvious trigonometric substitution.

? **By Parts:** Since the integrand is the product of two different function types, it is a good candidate for Integration by Parts.

Since we can easily integrate both x^2 and e^{3x}, we switch gears and ask ourselves which one has a derivative (or eventual, higher-order derivative) that will change forms. Thus, we choose $u = x^2$ because a couple of derivatives will result in the function completely devolving into a constant function. Therefore, $dv = e^{3x}\,dx$.

$u = x^2$	$dv = e^{3x}\,dx$
$du = 2x\,dx$	$v = \frac{1}{3}e^{3x}$

For the LIATE Folks

Choose $u = x^2$ and $dv = e^{3x}\,dx$ because A comes before E in LIATE.

Substituting into Equation 3.1.2 produces

$$\int x^2 e^{3x}\,dx = \frac{1}{3}x^2e^{3x} - \int \frac{2}{3}xe^{3x}\,dx = \frac{1}{3}x^2e^{3x} - \frac{2}{3}\int xe^{3x}\,dx. \tag{3.1.3}$$

We still cannot integrate $\int xe^{3x}\,dx$ directly, but the integral now has a lower power on x. We can evaluate this new integral by using Integration by Parts again. Since we have already started the Integration by Parts process on this integral, we stick with the same "function type" choices for u and dv. We chose u to be the algebraic function on our first Integration by Parts, so our choice of u on this new Integration by Parts must also be an algebraic function.

For the LIATE Folks

Choose $u = x$ and $dv = e^{3x}\,dx$ because A comes before E in LIATE.

$u = x$	$dv = e^{3x}\,dx$
$du = dx$	$v = \frac{1}{3}e^{3x}$

Substituting back into Equation 3.1.3 yields

$$\int x^2 e^{3x}\,dx = \frac{1}{3}x^2e^{3x} - \frac{2}{3}\left(\frac{1}{3}xe^{3x} - \int \frac{1}{3}e^{3x}\,dx\right).$$

After evaluating the last integral and simplifying, we obtain

$$\int x^2 e^{3x}\,dx = \frac{1}{3}x^2e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + C.$$

Example **3.1.3** showcases a vital difference between LIATE and having a more natural understanding of Integration by Parts using the Rule of Thumb. When using LIATE, we chose $u = x^2$ by rote memorization without understanding the consequences of the choice. The Rule of Thumb, on the other hand, delivered an extra bit of "hidden" knowledge - since we chose $u = x^2$ because it changes form after two derivatives, we could have immediately recognized that this integral would require *two* integrations (in this case, two applications of Integration by Parts).

Example 3.1.4: Applying Integration by Parts When LIATE Does not Quite Work

Evaluate

$$\int t^3 e^{t^2}\,dt.$$

Solution

☒ **Direct Integration:** Since the integrand is not the derivative of an established, well-known function, we cannot directly integrate.

? **Simple u-Substitution:** Letting $u = t^2$ *might* work.

☒ **Radical Substitution:** There is not a radical in the integrand.

☒ **Trigonometric Substitution:** As written, there isn't any obvious trigonometric substitution.

? **By Parts:** Since the integrand is the product of two different function types, it is a good candidate for Integration by Parts.

This is an excellent example of having many approaches to a problem that are all correct - welcome to the wonderful world of real integration!

Approach #1 (Ineffective): Using LIATE

For the LIATE-lovers out there, this one's for you. If we use a strict interpretation of the mnemonic LIATE to make our choice of u, we end up with $u = t^3$ and $dv = e^{t^2}\,dt$.

$u = t^3$	$dv = e^{t^2}\,dt$
$du = 3t^2\,dt$	$v = \ldots$ (we're stuck)

Unfortunately, we cannot continue with this approach because we cannot integrate e^{t^2}.

Approach #2: Starting with a Substitution

Using the Substitution Method, we could let $w = t^2$ so that $dw = 2t\,dt$. This means $\frac{1}{2}dw = t\,dt$. Hence,

$$\begin{aligned}\int t^3 e^{t^2}\,dt &= \int t^2 e^{t^2} t\,dt \\ &= \frac{1}{2}\int we^w\,dw \qquad \left(\text{substituting } w = t^2 \implies \frac{1}{2}dw = t\,dt\right)\end{aligned}$$

We can integrate *both* w and e^w, so we focus on choosing u to be the factor whose derivative (or eventual derivative) changes form. This is w. Thus, we choose $u = w$ and $dv = e^w\,dw$.

$u = w$	$dv = e^w\,dw$
$du = dw$	$v = e^w$

Using Integration by Parts, we get

$$\begin{aligned}\int t^3 e^{t^2}\,dt &= \int t^2 e^{t^2} t\,dt \\ &= \frac{1}{2}\int we^w\,dw && \left(\text{substituting } w = t^2 \implies \frac{1}{2}dw = t\,dt\right) \\ &= \frac{1}{2}\left(we^w - \int e^w\,dw\right) && \text{(IbP)} \\ &= \frac{1}{2}\left(we^w - e^w + C_1\right) \\ &= \frac{1}{2}\left(t^2e^{t^2} - e^{t^2} + C_1\right) && \text{(resubstituting } w = t^2\text{)} \\ &= \frac{1}{2}t^2e^{t^2} - \frac{1}{2}e^{t^2} + C && \left(\frac{1}{2}C_1 \text{ is a constant, so we call it } C\right)\end{aligned}$$

Approach #3 (Most Efficient): Starting with IBP by Choosing dv to be the Largest Integrable Factor

If we rewrite the given integral as

$$\int t^2\left(te^{t^2}\right)\,dt,$$

we can see that both t^2 and te^{t^2} are integrable. Therefore, we choose dv to be the "larger" (read as "more complex") of these two factors. Hence, we choose $dv = te^{t^2}\,dt$ and $u = t^2$.[4]

$u = t^2$	$dv = te^{t^2}\,dt$
$du = 2t\,dt$	$v = \frac{1}{2}e^{t^2}$ (Substitution Method)

Therefore, we get

$$\begin{aligned}\int t^3 e^{t^2}\,dt &= \int t^2\left(te^{t^2}\right)\,dt \\ &= \frac{1}{2}t^2e^{t^2} - \int te^{t^2}\,dt. \qquad \text{(IbP)} \\ &= \frac{1}{2}t^2e^{t^2} - \frac{1}{2}e^{t^2} + C\end{aligned}$$

There are a couple of things I want to mention about Example **3.1.4**. First, notice that LIATE failed us here, but a natural understanding of our goal - to find the largest factor of the integrand that we *can* integrate - still led us to a solution. This is why I do not teach LIATE in my classes (again, I only include it here because you will hear of it from students in other classes, and those students might have the illusion that it always works).

The second thing to mention is that Approach #3 in this example demonstrates the level of comfort with integration for which you want to strive. There is nothing wrong with Approach #2, but you *definitely* want to build the skill set to perform complex integrations like this. However, The only way you will build that skill is through lots of practice!

Caution: Practice Makes Better

Integration by Parts (and most other integration techniques) is almost a completely **heuristic** method. That is, you will only learn how to identify when to use a technique properly and what pitfalls to avoid by practicing.

Example 3.1.5: Self-Relating Integration by Parts

Evaluate

$$\int \sin(\ln x)\,dx.$$

Solution

☒ **Direct Integration:** Since the integrand is not the derivative of an established, well-known function, we cannot directly integrate.

☒ **Simple u-Substitution:** I leave it for the reader to try as they might to find a proper u-substitution - none will work.

☒ **Radical Substitution:** Since a simple u-substitution will not work, a radical substitution is out of the question.

☒ **Trigonometric Substitution:** As written, there isn't any obvious trigonometric substitution.

? **By Parts:** Since the integrand is *not* the product of two different function types, it doesn't *seem* to be a good candidate for Integration by Parts.[5]

Again, LIATE fails us here, but the Rule of Thumb is still helpful. We begin by rewriting the integral as

$$\int \sin(\ln x)\cdot 1\,dx.$$

We can integrate 1, but not $\sin(\ln(x))$, so we let $dv = 1\,dx$.

$u = \sin(\ln(x))$	$dv = 1\,dx$
$du = \dfrac{\cos(\ln(x))}{x}\,dx$	$v = x$

Therefore, we have

$$\int \sin\left(\ln(x)\right)\,dx = x\sin(\ln(x)) - \int \cos(\ln(x))\,dx.$$

Unfortunately, this process leaves us with a new integral that is very similar to the original. However, let's see what happens when we apply Integration by Parts again.

Since we chose u to be the composition of the trigonometric function and the logarithmic function on our first pass of the Integration by Parts method, we choose the same *style* of function for u on our second pass.

$u = \cos(\ln(x))$	$dv = 1\,dx$
$du = -\dfrac{\sin(\ln(x))}{x}\,dx$	$v = x$

Substituting, we have

$$\int \sin(\ln(x))\,dx = x\sin(\ln x) - \left(x\cos(\ln(x)) - \int -\sin(\ln(x))\,dx\right).$$

After simplifying, we obtain

$$\int \sin(\ln(x))\,dx = x\sin(\ln(x)) - x\cos(\ln(x)) - \int \sin(\ln(x))\,dx.$$

The last integral is now the same as the original. It may seem that we have gone in circles, but now we can evaluate the integral. To see how to do this more clearly, substitute $I = \int \sin(\ln(x))\,dx$. Thus, the equation becomes

$$I = x\sin(\ln(x)) - x\cos(\ln(x)) - I.$$

First, add I to both sides of the equation to obtain

$$2I = x\sin(\ln(x)) - x\cos(\ln(x)).$$

Next, divide by 2:

$$I = \frac{1}{2}x\sin(\ln(x)) - \frac{1}{2}x\cos(\ln(x)).$$

Substituting $I = \int \sin(\ln(x))\,dx$ again, we have

$$\int \sin(\ln(x))\,dx = \frac{1}{2}x\sin(\ln(x)) - \frac{1}{2}x\cos(\ln(x)).$$

From this we see that $\frac{1}{2}x\sin(\ln(x)) - \frac{1}{2}x\cos(\ln(x))$ is an antiderivative of $\sin(\ln(x))$. For the most general antiderivative, add C:

$$\int \sin(\ln(x))\,dx = \frac{1}{2}x\sin(\ln(x)) - \frac{1}{2}x\cos(\ln(x)) + C.$$

Analysis

If this method feels a little strange at first, we can check the answer by differentiation:

$$\frac{d}{dx}\left(\frac{1}{2}x\sin(\ln x)-\frac{1}{2}x\cos(\ln x)\right) = \frac{1}{2}(\sin(\ln x))+\cos(\ln x)\cdot\frac{1}{x}\cdot\frac{1}{2}x-\left(\frac{1}{2}\cos(\ln x)-\sin(\ln x)\cdot\frac{1}{x}\cdot\frac{1}{2}x\right)$$
$$= \sin(\ln x).$$

Example 3.1.5 is an integral I call a "Self-Relating Integration by Parts." This is because, when we use Integration by Parts, we obtain the original integral as part of our evaluated integral. This style of integration comes up frequently enough that you should practice it until it is comfortable.

? Checkpoint 3.1.5

Evaluate

$$\int x^2\sin x\,dx.$$

Answer

$$\int x^2\sin x\,dx = -x^2\cos x+2x\sin x+2\cos x+C$$

In Section 2.1, we introduced the following Problem-Solving Strategy for dealing with integrals involving products of powers of the tangent and secant functions. In this case, we complete the strategy by filling out the fourth tactic - one that requires Integration by Parts.

Problem-Solving Strategy: Integrating $\int \tan^k(x)\sec^j(x)\,dx$

To integrate $\int \tan^k(x)\sec^j(x)\,dx$, use the following strategies:

1. If j is even and $j\geq 2$, rewrite $\sec^j(x)=\sec^{j-2}(x)\sec^2(x)$ and use $\sec^2(x)=\tan^2(x)+1$ to rewrite $\sec^{j-2}(x)$ in terms of $\tan(x)$. Let $u=\tan(x)$ and $du=\sec^2(x)\,dx$.
2. If k is odd and $j\geq 1$, rewrite $\tan^k(x)\sec^j(x)=\tan^{k-1}(x)\sec^{j-1}(x)\sec(x)\tan(x)$ and use $\tan^2(x)=\sec^2(x)-1$ to rewrite $\tan^{k-1}(x)$ in terms of $\sec(x)$. Let $u=\sec(x)$ and $du=\sec(x)\tan(x)\,dx$. (Note: If j is even and k is odd, then either strategy 1 or strategy 2 may be used.)
3. If k is odd where $k\geq 3$ and $j=0$, rewrite $\tan^k(x)=\tan^{k-2}(x)\tan^2(x)=\tan^{k-2}(x)(\sec^2(x)-1)=\tan^{k-2}(x)\sec^2(x)-\tan^{k-2}(x)$. It may be necessary to repeat this process on the $\tan^{k-2}(x)$ term.
4. If k is even and j is odd, then use $\tan^2(x)=\sec^2(x)-1$ to express $\tan^k(x)$ in terms of $\sec(x)$. Use Integration by Parts to integrate odd powers of $\sec(x)$.

✓ Example 3.1.6: Integrating $\sec^3(x)$

Evaluate $\int \sec^3(x)\,dx$.

Solution

☒ **Direct Integration:** Since the integrand is not the derivative of an established, well-known function, we cannot directly integrate.

☒ **Simple u-Substitution:** Any choice of substitution will lead nowhere. For example, letting $u=\tan(x)$ will definitely "steal" two powers off the secant (via $du=\sec^2(x)\,dx$); however, the secant would be left with an *odd* power - we don't want that. Letting $u=\sec(x)$ would steal a power off tangent (via $du=\sec(x)\tan(x)\,dx$), leaving $(\tan(x))^{-1}$; however, we would need that power on tangent to be even, and that's not going to happen. I leave it for the reader to try as they might to find a proper u-substitution - none will work.

☒ **Radical Substitution:** The integrand doesn't involve a radical.

☒ **Trigonometric Substitution:** As written, there isn't any obvious trigonometric substitution.

? **By Parts:** Since all other techniques have failed, we *might* be able to use Integration by Parts; however, as written, we have no clue where to start.

This integral is dangerously popular in some texts!

Since we are stuck from the get-go, let's use the Mathematical Mantra and see if there is anything from Trigonometry that might help. We would like to rewrite the integrand as the product of two functions (which is not technically always needed for Integration by Parts, but it is helpful here):

$$\int \sec^3(x)\,dx = \int \sec(x)\sec^2(x)\,dx.$$

We definitely can integrate $sec^2(x)$, so we choose $dv = \sec^2(x)\,dx$ and $u = \sec(x)$.

$u = \sec(x)$	$dv = \sec^2(x)\,dx$
$du = \sec(x)\tan(x)\,dx$	$v = \tan(x)$

Thus,

$$\begin{aligned}
\int \sec^3(x)\,dx &= \sec(x)\tan(x) - \int \tan(x)\sec(x)\tan(x)\,dx \\
&= \sec(x)\tan(x) - \int \tan^2(x)\sec(x)\,dx \\
&= \sec(x)\tan(x) - \int (\sec^2(x) - 1)\sec(x)\,dx \\
&= \sec(x)\tan(x) + \int \sec(x)\,dx - \int \sec^3(x)\,dx \\
&= \sec(x)\tan(x) + \ln|\sec(x) + \tan(x)| - \int \sec^3(x)\,dx.
\end{aligned}$$

We now have

$$\int \sec^3(x)\,dx = \sec(x)\tan(x) + \ln|\sec(x) + \tan(x)| - \int \sec^3(x)\,dx.$$

Since the integral $\int \sec^3(x)\,dx$ has reappeared on the right-hand side, we can solve for $\int \sec^3(x)\,dx$ by adding it to both sides. In doing so, we obtain

$$2\int \sec^3(x)\,dx = \sec(x)\tan(x) + \ln|\sec(x) + \tan(x)|.$$

Dividing by 2, we arrive at

$$\int \sec^3(x)\,dx = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln|\sec(x) + \tan(x)| + C.$$

Interesting Pattern!

While I love the derivation of this integral (and you could be expected to derive it on an exam), I cannot help but mention the cool pattern it generates. The integral of the cube of the secant is one-half the sum of the secant's derivative and the secant's integral.

Integration by Parts for Definite Integrals

Now that we have used Integration by Parts successfully to evaluate indefinite integrals, we turn our attention to definite integrals. The integration technique is the same; we only add a step to evaluate the integral at the upper and lower limits of integration.

Theorem: Integration by Parts for Definite Integrals

Let $u = f(x)$ and $v = g(x)$ be functions with continuous derivatives on $[a, b]$. Then

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

Example 3.1.7: Finding the Area of a Region

Find the area of the region bounded above by the graph of $y = \tan^{-1} x$ and below by the x-axis over the interval $[0, 1]$.

Solution

This region is shown in Figure 3.1.1. To find the area, we must evaluate

$$\int_0^1 \tan^{-1} x\ dx.$$

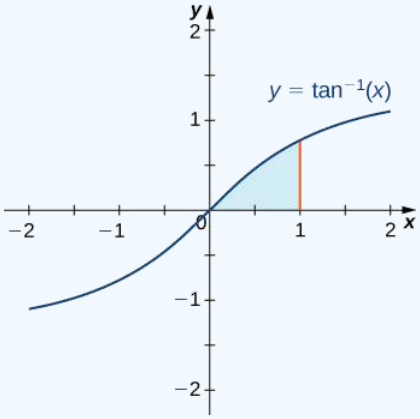


Figure 3.1.1

☒ **Direct Integration:** Since the integrand is not the derivative of an established, well-known function, we cannot directly integrate.

☒ **Simple u-Substitution:** I leave it for the reader to try as they might to find a proper u-substitution - none will work.

☒ **Radical Substitution:** Since a simple u-substitution will not work, a radical substitution is out of the question.

☒ **Trigonometric Substitution:** As written, there isn't any obvious trigonometric substitution.

? **By Parts:** Since all other integration techniques have been exhausted, Integration by Parts *might* be feasible.

Since we can integrate 1, this will be our choice for dv.

$u = \tan^{-1}(x)$	$dv = 1\,dx$
$du = \dfrac{1}{1+x^2}\,dx$	$v = x$

After applying the Integration by Parts formula (Equation 3.1.2) we obtain

$$\text{Area} = x\tan^{-1} x\Big|_0^1 - \int_0^1 \frac{x}{x^2+1}\,dx.$$

Use u-substitution to obtain

$$\int_0^1 \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln\left(x^2+1\right)\Big|_0^1.$$

Thus,

$$\text{Area} = x\tan^{-1}x\Big|_0^1 - \frac{1}{2}\ln\left(x^2+1\right)\Big|_0^1 = \left(\frac{\pi}{4} - \frac{1}{2}\ln 2\right)\ \text{units}^2.$$

At this point, it might not be a bad idea to do a "reality check" on the reasonableness of our solution. Since $\frac{\pi}{4} - \frac{1}{2}\ln 2 \approx 0.4388\ \text{units}^2$, and from Figure 3.1.1 we expect our area to be slightly less than $0.5\ \text{units}^2$, this solution appears to be reasonable.

Example 3.1.8: Finding a Volume of Revolution

Find the volume of the solid obtained by revolving the region bounded by the graph of $f(x) = e^{-x}$, the x-axis, the y-axis, and the line $x = 1$ about the y-axis.

Solution

The best solution to this problem is to use the Method of Cylindrical Shells. Begin by sketching the region to be revolved, along with a typical rectangle (Figure 3.1.2).

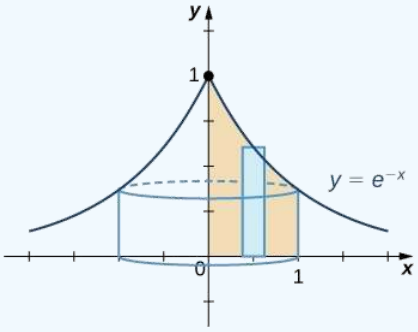


Figure 3.1.2: We can use the shell method to find a volume of revolution.

To find the volume using shells, we must evaluate

$$2\pi\int_0^1 xe^{-x}\,dx. \tag{3.1.4}$$

☒ **Direct Integration:** Since the integrand is not the derivative of an established, well-known function, we cannot directly integrate.

☒ **Simple u-Substitution:** I leave it for the reader to try as they might to find a proper u-substitution - none will work.

☒ **Radical Substitution:** Since a simple u-substitution will not work, a radical substitution is out of the question.

☒ **Trigonometric Substitution:** As written, there isn't any obvious trigonometric substitution.

? **By Parts:** The fact that the integrand is the product of two different types of functions is promising and invokes thoughts of using Integration by Parts.

Both x and e^{-x} are easily integrable, so we instead resort to choosing for u the function whose form eventually changes through differentiation - this is x. Thus, we let $u = x$ and $dv = e^{-x}\,dx$.

$u = x$	$dv = e^{-x}\,dx$
$du = dx$	$v = -e^{-x}$

Using the Shell Method formula and substituting these values in, we obtain

$$\begin{aligned}
\text{Volume} &= 2\pi \int_0^1 xe^{-x}\,dx \\
&= 2\pi \left(-xe^{-x}\Big|_0^1 + \int_0^1 e^{-x}\,dx\right) \\
&= -2\pi \left(xe^{-x}\Big|_0^1 - \int_0^1 e^{-x}\,dx\right) \\
&= -2\pi \left(e^{-1} - 0 + e^{-x}\Big|_0^1\right) \\
&= -2\pi \left(e^{-1} + e^{-1} - 1\right) \\
&= -2\pi \left(2e^{-1} - 1\right) \\
&= 2\pi - \frac{4\pi}{e} \text{ units}^3.
\end{aligned}$$

Analysis

Again, it is a good idea to check the reasonableness of our solution. We observe that the solid has a volume slightly less than that of a cylinder of radius 1 and height of $1/e$ added to the volume of a cone of base radius 1 and height of $1 - \frac{1}{e}$. Consequently, the solid should have a volume a bit less than

$$\pi(1)^2 \frac{1}{e} + \left(\frac{\pi}{3}\right)(1)^2 \left(1 - \frac{1}{e}\right) = \frac{2\pi}{3e} + \frac{\pi}{3} \approx 1.8177 \text{ units}^3.$$

Since $2\pi - \frac{4\pi}{e} \approx 1.6603$, we see that our calculated volume is reasonable.

? Checkpoint 3.1.8

Evaluate

$$\int_0^{\pi/2} x \cos x\,dx.$$

Answer

$$\int_0^{\pi/2} x \cos x\,dx = \frac{\pi}{2} - 1$$

✓ Example 3.1.9: Finding the Area of a Region

Find the area of the region between the graph of $f(x) = \sqrt{x^2 - 9}$ and the x-axis over the interval $[3, 5]$.

Solution

First, sketch a rough graph of the region described in the problem, as shown in the following figure.

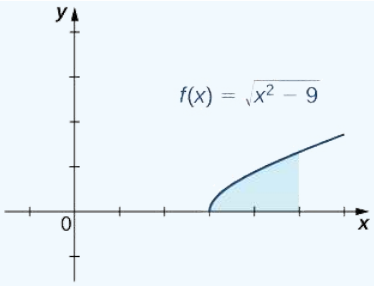


Figure 3.1.7: Calculating the area of the shaded region requires evaluating an integral with a trigonometric substitution.

We can see that the area is $A = \int_3^5 \sqrt{x^2-9}\,dx$. To evaluate this definite integral, substitute $x = 3\sec\theta$ and $dx = 3\sec\theta\tan\theta\,d\theta$. We must also change the limits of integration. If $x = 3$, then $3 = 3\sec\theta$ and hence $\theta = 0$. If $x = 5$, then $\theta = \sec^{-1}(\frac{5}{3})$. After making these substitutions and simplifying, we have

$$\begin{aligned}
\text{Area} &= \int_3^5 \sqrt{x^2-9}\,dx \\
&= \int_0^{\sec^{-1}(5/3)} 9\tan^2\theta\sec\theta\,d\theta \\
&= \int_0^{\sec^{-1}(5/3)} 9(\sec^2\theta-1)\sec\theta\,d\theta && \text{(Pythagorean Identity)} \\
&= \int_0^{\sec^{-1}(5/3)} 9(\sec^3\theta-\sec\theta)\,d\theta \\
&= \left(\frac{9}{2}\ln|\sec\theta+\tan\theta| + \frac{9}{2}\sec\theta\tan\theta - 9\ln|\sec\theta+\tan\theta|\right)\Big|_0^{\sec^{-1}(5/3)} \\
&= \left(\frac{9}{2}\sec\theta\tan\theta - \frac{9}{2}\ln|\sec\theta+\tan\theta|\right)\Big|_0^{\sec^{-1}(5/3)} \\
&= \frac{9}{2}\cdot\frac{5}{3}\cdot\frac{4}{3} - \frac{9}{2}\ln\left|\frac{5}{3}+\frac{4}{3}\right| - \left(\frac{9}{2}\cdot 1\cdot 0 - \frac{9}{2}\ln|1+0|\right) \\
&= 10 - \frac{9}{2}\ln 3
\end{aligned}$$

? Checkpoint 3.1.9: Finding an Arc Length

Find the length of the curve $y = x^2$ over the interval $[0, \frac{1}{2}]$.

Answer

$\frac{1}{4}\left(\sqrt{2}+\ln(\sqrt{2}+1)\right)$

Reduction Formulas

Evaluating $\int \sec^n(x)\,dx$ for values of n where n is odd requires Integration by Parts. In addition, we must also know the value of $\int \sec^{n-2}(x)\,dx$ to evaluate $\int \sec^n(x)\,dx$. The evaluation of $\int \tan^n(x)\,dx$ also requires being able to integrate $\int \tan^{n-2}(x)\,dx$. We can derive and apply the following **Power Reduction Identities for Integration** to simplify the process. These rules allow us to replace the integral of a power of $\sec(x)$ or $\tan(x)$ with the integral of a lower power of $\sec(x)$ or $\tan(x)$.

Theorem: Power Reduction Identities for $\int \sec^n(x)\,dx$ and $\int \tan^n(x)\,dx$

$$\int \sec^n(x)\,dx = \frac{1}{n-1}\sec^{n-2}(x)\tan(x) + \frac{n-2}{n-1}\int \sec^{n-2}(x)\,dx$$

and

$$\int \tan^n(x)\,dx = \frac{1}{n-1}\tan^{n-1}(x) - \int \tan^{n-2}(x)\,dx.$$

Note

The Power Reduction Identity for Integrating Powers of Secant may be verified by applying Integration by Parts. The Power Reduction Identity for Integrating Powers of Tangent may be verified by following the strategy outlined for integrating odd powers of $\tan(x)$.

✔ Example 3.1.10: Revisiting $\int \sec^3(x)\,dx$

Apply a reduction formula to evaluate $\int \sec^3(x)\,dx$.

Solution

By applying the first reduction formula, we obtain

$$\begin{aligned}\int \sec^3(x)\,dx &= \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\int \sec(x)\,dx \\ &= \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln|\sec(x)+\tan(x)| + C.\end{aligned}$$

✔ Example 3.1.11: Using a Power Reduction Identity for Integration

Evaluate $\int \tan^4(x)\,dx$.

Solution

Applying the Power Reduction Identity for $\int \tan^4(x)\,dx$ we have

$$\begin{aligned}\int \tan^4(x)\,dx &= \frac{1}{3}\tan^3(x) - \int \tan^2(x)\,dx \\ &= \frac{1}{3}\tan^3(x) - (\tan(x) - \int \tan^0(x)\,dx) \\ &= \frac{1}{3}\tan^3(x) - \tan(x) + \int 1\,dx \\ &= \frac{1}{3}\tan^3(x) - \tan(x) + x + C\end{aligned}$$

? Checkpoint 3.1.11

Apply the Power Reduction Identity to $\int \sec^5(x)\,dx$.

Answer

$$\int \sec^5(x)\,dx = \frac{1}{4}\sec^3(x)\tan(x) + \frac{3}{4}\int \sec^3(x)\,dx$$

Footnotes

[1] The capital, bold letters are purposeful here, we will see why momentarily.

[2] Note that we put **LI** at the beginning of the mnemonic; however, we could just as easily have started with **IL**, since these two types of functions won't appear together in an Integration by Parts problem and they are both terrible choices for dv.

[3] I use a ? instead of ✓ because having a product of two different types of functions within the integrand is *not* a guarantee that Integration by Parts will work.

[4] Another way we could have made our choice for u and dv here is to note that, since both t^2 and te^{t^2} are integrable, we switch gears and choose u to be the factor that will *eventually* change forms given enough derivatives. This will be $u = t^2$ because, after two derivatives, it will become a constant function.

[5] It turns out that Integration by Parts does not require the integrand to be a product of two different function types. It can be used to integrate compositions of functions and it is sometimes used to integrate *single* functions!

This page titled 3.1: Integration by Parts is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Roy Simpson.

3.1E: EXERCISES

READING QUESTIONS

1. The formula for Integration by Parts is derived from which rule in differential calculus?
2. What is the formula for Integration by Parts for indefinite integrals?
3. When choosing which part of the integrand will be u and which will be dv, what is the most important characteristic of the expression chosen for dv?
4. What does the mnemonic LIATE stand for, and how is it used as a guideline for choosing u?
5. What is a "self-relating integral," and how do you solve it once the original integral reappears in your work?
6. If you need to apply Integration by Parts more than once to solve an integral, why is it important to be consistent with your choices of function types for u?
7. What is the purpose of a reduction formula?

In using the technique of Integration by Parts, you must carefully choose which expression is u. For each of the following problems, use the guidelines in this section to choose u. Do <u>not</u> evaluate the integrals.

1) $\int x^3 e^{2x}\,dx$

Answer

$u = x^3$

2) $\int x^3 \ln(x)\,dx$

3) $\int y^3 \cos y\,dy$

Answer

$u = y^3$

4) $\int x^2 \arctan x\,dx$

5) $\int e^{3x} \sin(2x)\,dx$

Answer

$u = \sin(2x)$

In exercises 6 - 40, find the integral by using the simplest method. Not all problems require Integration by Parts.

6) $\int v \sin v\,dv$

7) $\int \ln x\,dx$ (Hint: $\int \ln x\,dx$ is equivalent to $\int 1 \cdot \ln(x)\,dx$.)

Answer

$$\int \ln x\,dx \quad = \quad -x + x \ln x + C$$

8) $\int x \cos x\,dx$

9) $\int \tan^{-1} x\,dx$

Answer

$$\int \tan^{-1} x\,dx \quad = \quad x \tan^{-1} x - \tfrac{1}{2}\ln(1+x^2) + C$$

10) $\int x^2 e^x\,dx$

11) $\int x \sin(2x)\,dx$

Answer

$$\int x\sin(2x)\,dx \quad = \quad -\tfrac{1}{2}x\cos(2x)+\tfrac{1}{4}\sin(2x)+C$$

12) $\int xe^{4x}\,dx$

13) $\int xe^{-x}\,dx$

Answer

$$\int xe^{-x}\,dx \quad = \quad e^{-x}(-1-x)+C$$

14) $\int x\cos 3x\,dx$

15) $\int x^2\cos x\,dx$

Answer

$$\int x^2\cos x\,dx \quad = \quad 2x\cos x+(-2+x^2)\sin x+C$$

16) $\int x\ln x\,dx$

17) $\int \sqrt{x^2+9}\,dx$

Answer

$$\int \sqrt{x^2+9}\,dx \quad = \quad 9\left[\frac{x\sqrt{x^2+9}}{18}+\tfrac{1}{2}\ln\left|\frac{\sqrt{x^2+9}}{3}+\frac{x}{3}\right|\right]+C$$

18) $\int \ln(2x+1)\,dx$

Answer

$$\int \ln(2x+1)\,dx \quad = \quad \tfrac{1}{2}(1+2x)(-1+\ln(1+2x))+C$$

19) $\int x^2e^{4x}\,dx$

20) $\int e^x\sin x\,dx$

Answer

$$\int e^x\sin x\,dx \quad = \quad \tfrac{1}{2}e^x(-\cos x+\sin x)+C$$

21) $\int e^x\cos x\,dx$

22) $\int xe^{-x^2}\,dx$

Answer

$$\int xe^{-x^2}\,dx \quad = \quad -\frac{e^{-x^2}}{2}+C$$

23) $\int x^2e^{-x}\,dx$

24) $\int \sin(\ln(2x))\,dx$

Answer

$$\int \sin(\ln(2x))\,dx \quad = \quad -\tfrac{1}{2}x\cos[\ln(2x)]+\tfrac{1}{2}x\sin[\ln(2x)]+C$$

25) $\int \cos(\ln x)\,dx$

26) $\int (\ln x)^2\,dx$

Answer

$$\int (\ln x)^2\,dx \quad = \quad 2x - 2x\ln x + x(\ln x)^2 + C$$

27) $\int \ln(x^2)\,dx$

28) $\int \dfrac{x^2}{\sqrt{1+x^2}}\,dx$

29) $\int x^2 \ln x\,dx$

Answer

$$\int x^2 \ln x\,dx \quad = \quad -\frac{x^3}{9} + \tfrac{1}{3}x^3 \ln x + C$$

30) $\int \sin^{-1} x\,dx$

31) $\int \cos^{-1}(2x)\,dx$

Answer

$$\int \cos^{-1}(2x)\,dx \quad = \quad -\tfrac{1}{2}\sqrt{1-4x^2} + x\cos^{-1}(2x) + C$$

32) $\int x \arctan x\,dx$

33) $\int x^2 \sin x\,dx$

Answer

$$\int x^2 \sin x\,dx \quad = \quad -(-2+x^2)\cos x + 2x\sin x + C$$

34) $\int x^3 \cos x\,dx$

35) $\int x^3 \sin x\,dx$

Answer

$$\int x^3 \sin x\,dx \quad = \quad -x(-6+x^2)\cos x + 3(-2+x^2)\sin x + C$$

36) $\int x^3 e^x\,dx$

37) $\int x \sec^{-1} x\,dx$

Answer

$$\int x \sec^{-1} x\,dx \quad = \quad \tfrac{1}{2}x\left(-\sqrt{1-\frac{1}{x^2}} + x \cdot \sec^{-1} x\right) + C$$

38) $\int x \sec^2 x\,dx$

39) $\int x \cosh x\,dx$

Answer

$$\int x \cosh x\,dx \quad = \quad -\cosh x + x\sinh x + C$$

40) $\int \tan^2 x \sec x\,dx$

Answer

$$\int \tan^2 x \sec x\,dx \quad = \quad \tfrac{1}{2}\sec x \tan x - \tfrac{1}{2}\ln(\sec x + \tan x) + C$$

In exercises 41 - 50, compute the definite integrals. Use a graphing utility to confirm your answers.

41) $\int_{1/e}^{1} \ln x \, dx$

42) $\int_{0}^{1} xe^{-2x} \, dx$ (Express the answer in exact form.)

Answer

$$\int_{0}^{1} xe^{-2x} \, dx \quad = \quad \frac{1}{4} - \frac{3}{4e^2}$$

43) $\int_{0}^{1} e^{\sqrt{x}} \, dx \quad (\text{let } u = \sqrt{x})$

44) $\int_{1}^{e} \ln(x^2) \, dx$

Answer

$$\int_{1}^{e} \ln(x^2) \, dx \quad = \quad 2$$

45) $\int_{0}^{\pi} x \cos x \, dx$

46) $\int_{-\pi}^{\pi} x \sin x \, dx$ (Express the answer in exact form.)

Answer

$$\int_{-\pi}^{\pi} x \sin x \, dx \quad = \quad 2\pi$$

47) $\int_{0}^{3} \ln(x^2 + 1) \, dx$ (Express the answer in exact form.)

48) $\int_{0}^{\pi/2} x^2 \sin x \, dx$ (Express the answer in exact form.)

Answer

$$\int_{0}^{\pi/2} x^2 \sin x \, dx \quad = \quad -2 + \pi$$

49) $\int_{0}^{1} x5^x \, dx$ (Express the answer using five significant digits.)

50) Evaluate $\int \cos x \ln(\sin x) \, dx$

Answer

$$\int \cos x \ln(\sin x) \, dx \quad = \quad -\sin(x) + \ln[\sin(x)] \sin x + C$$

In exercises 51 - 53, derive the following formulas using the technique of Integration by Parts. Assume that n is a positive integer. These formulas are called *reduction formulas* because the exponent in the x term has been reduced by one in each case. The second integral is simpler than the original integral.

51) $\int x^n e^x \, dx = x^n e^x - n \int x^{n-1} e^x \, dx$

52) $\int x^n \cos x \, dx = x^n \sin x - n \int x^{n-1} \sin x \, dx$

Answer

Answers vary

53) $\int x^n \sin x \, dx =$_____

54) Integrate $\int 2x\sqrt{2x-3} \, dx$ using two methods:

a. Using Integration by Parts, letting $dv = \sqrt{2x-3} \, dx$

b. Substitution Method, letting $u = 2x - 3$

Answer

a. $\int 2x\sqrt{2x-3}\,dx = \frac{2}{5}(1+x)(-3+2x)^{3/2}+C$

b. $\int 2x\sqrt{2x-3}\,dx = \frac{2}{5}(1+x)(-3+2x)^{3/2}+C$

In exercises 55 - 60, state whether you would use Integration by Parts to evaluate the integral. If so, identify u and dv. If not, describe the technique used to perform the integration without actually doing the problem.

55) $\int x\ln x\,dx$

56) $\int \frac{\ln^2 x}{x}\,dx$

Answer

Do not use Integration by Parts. Choose u to be $\ln x$, and the integral is of the form $\int u^2\,du$.

57) $\int xe^x\,dx$

58) $\int xe^{x^2-3}\,dx$

Answer

Do not use Integration by Parts. Let $u = x^2-3$, and the integral can be put into the form $\int e^u\,du$.

59) $\int x^2\sin x\,dx$

60) $\int x^2\sin(3x^3+2)\,dx$

Answer

Do not use Integration by Parts. Choose u to be $u = 3x^3+2$ and the integral can be put into the form $\int \sin(u)\,du$.

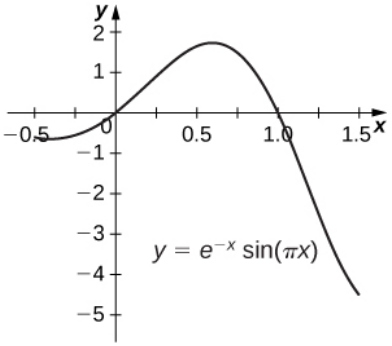
In exercises 61 - 62, sketch the region bounded above by the curve, the x-axis, and $x = 1$, and find the area of the region. Provide the exact form or round answers to the number of places indicated.

61) $y = 2xe^{-x}$ (Approximate answer to four decimal places.)

62) $y = e^{-x}\sin(\pi x)$ (Approximate answer to five decimal places.)

Answer

The area under graph is $0.39535\ \text{units}^2$.



In exercises 63 - 64, find the volume generated by rotating the region bounded by the given curves about the specified line. Express the answers in exact form or approximate to the number of decimal places indicated.

63) $y = \sin x$, $y = 0$, $x = 2\pi$, $x = 3\pi$; about the y-axis (Express the answer in exact form.)

64) $y = e^{-x}$, $y = 0$, $x = -1$, $x = 0$; about $x = 1$ (Express the answer in exact form.)

Answer

$V = 2\pi e\ \text{units}^3$

65) A particle moving along a straight line has a velocity of $v(t) = t^2e^{-t}$ after t sec. How far does it travel in the first 2 sec? (Assume the units are in feet and express the answer in exact form.)

66) Find the area under the graph of $y = \sec^3 x$ from $x = 0$ to $x = 1$. (Round the answer to two significant digits.)

Answer

$A = 2.05 \text{ units}^2$

67) Find the area between $y = (x - 2)e^x$ and the x-axis from $x = 2$ to $x = 5$. (Express the answer in exact form.)

68) Find the area of the region enclosed by the curve $y = x \cos x$ and the x-axis for $\frac{11\pi}{2} \le x \le \frac{13\pi}{2}$. (Express the answer in exact form.)

Answer

$A = 12\pi \text{ units}^2$

69) Find the volume of the solid generated by revolving the region bounded by the curve $y = \ln x$, the x-axis, and the vertical line $x = e^2$ about the x-axis. (Express the answer in exact form.)

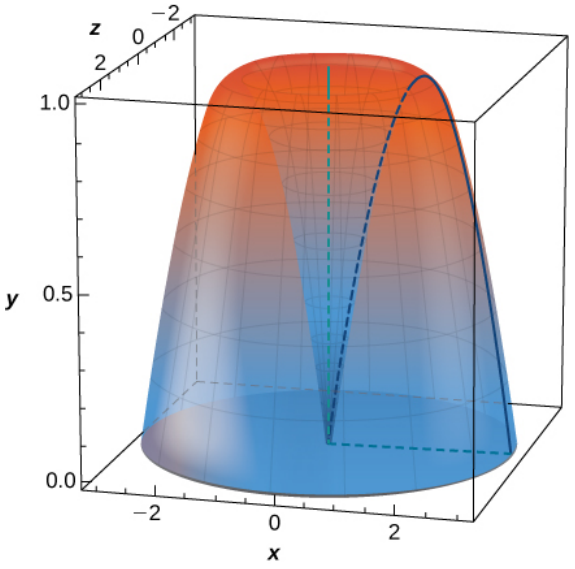
70) Find the volume of the solid generated by revolving the region bounded by the curve $y = 4 \cos x$ and the x-axis, $\frac{\pi}{2} \le x \le \frac{3\pi}{2}$, about the x-axis. (Express the answer in exact form.)

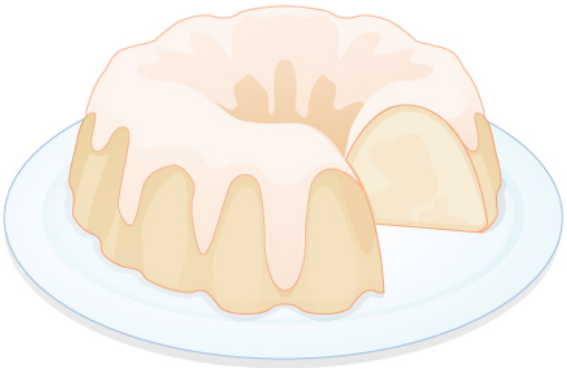
Answer

$V = 8\pi^2 \text{ units}^3$

71) Find the volume of the solid generated by revolving the region in the first quadrant bounded by $y = e^x$ and the x-axis, from $x = 0$ to $x = \ln(7)$, about the y-axis. (Express the answer in exact form.)

72) What is the volume of the Bundt cake that comes from rotating $y = \sin x$ around the y-axis from $x = 0$ to $x = \pi$?





Answer

$V = 2\pi^2$ units^3

73) Find the weight of a ruler that is 12 in. long (starting at $x = 5$) and has a weight density function of $\gamma(x) = \ln(x) + (1/2)x^2$ oz/in. You should assume the ruler is a one-dimensional object.

In exercises 74 - 75, find the weight of the two-dimensional object that is centered at the origin.

74) A plate of radius 10 in. with radial *weight* density function $\gamma(x) = 1 + \cos(\pi x)$ lb/in^2

Answer

100π lbs.

75) A jar lid of radius 3 in. with radial *weight* density function $\gamma(x) = \ln(x + 1)$ oz/in^2

For each pair of integrals in exercises 76 - 77, determine which one is more difficult to evaluate. Explain your reasoning.

76) $\int \sin^{456} x \cos x \, dx$ or $\int \sin^2 x \cos^2 x \, dx$

77) $\int \tan^{350} x \sec^2 x \, dx$ or $\int \tan^{350} x \sec x \, dx$

Answer

The second integral is more difficult because the first integral is simply a u-substitution type.

3.2: Trigonometric Integrals

Note to the Instructor (click to expand)

This section is unlike most other introductory section to integration techniques. Instead of starting with Integration by Parts, we showcase that *most* of the new integration techniques are just glorified substitution methods. Specifically, in this section we introduce radical substitution and substitutions involving powers of the trigonometric functions. We also use power reduction formulas to help us evaluate otherwise impossible integrals.

To succeed in this section, you'll need to use some skills from previous courses. While you should already know them, this is the first time they've been required. You can review these skills in CRC's Corequisite Codex. If you have a support class, it might cover some, **but not all**, of these topics.

Prerequisite Skills and Support Topics (click to expand)

- **Simplifying Expressions**
  - Laws of Exponents: This skill is essential for manipulating powers of trigonometric functions, such as rewriting $\sin^5(x)$ as $(\sin^2(x))^2\sin(x)$.
  - Dividing Polynomials - Long Division: This algebraic technique is sometimes required after a radical substitution to simplify a rational function, as shown in Example 2.1.1.
- **Polynomial Functions**
  - The Binomial Theorem: The text mentions that this theorem can be used to quickly expand expressions like $(u-1)^5$, which can arise after a substitution.
- **Trigonometry**
  - Trigonometric Identities - Sum-to-Product and Product-to-Sum Identities: The Product-to-Sum identities are used to evaluate integrals of products like $\sin(ax)\cos(bx)$, as seen in Example 2.1.8.
  - The Inverse Trigonometric Functions: The integral of $1/(a^2+x^2)$ is derived, resulting in an inverse tangent function.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Solve integration problems involving radicals.
- Solve integration problems involving products and powers of $\sin x$ and $\cos x$.
- Solve integration problems involving products and powers of $\tan x$ and $\sec x$.
- Use reduction formulas to solve trigonometric integrals.

Hawk A.I. Section-Specific Tutor

COSUMNES RIVER COLLEGE MATHEMATICS

To access the Hawk A.I. Tutor, you will need to be logged into your campus Gmail account.

We begin our exploration of "new" integration techniques by revisiting the only technique we have so far (besides direct integration) - the Substitution Method. As we delve into the first couple sections of this chapter, we expand our use of the Substitution Method to more creative substitutions. Let's start by reviewing one of the most useful creative substitutions up to this point - a technique dubbed **radical substitution**.

Radical Substitutions

In truth, a **radical substitution** is just a simple u-substitution we make when our integrand involves a radical. As an example, consider the indefinite integral

$$\int \frac{x}{\sqrt{x-3}}\,dx.$$

Most students fresh from Calculus I will perform the substitution $u = x - 3$ so that $du = dx$ and $u + 3 = x$. This gives us

$$\begin{aligned}\int \frac{x}{\sqrt{x-3}}\,dx &= \int \frac{u+3}{\sqrt{u}}\,du \\ &= \int u^{1/2} + 3u^{-1/2}\,du \\ &= \frac{2}{3}u^{3/2} + 6u^{1/2} + C \\ &= \frac{2}{3}(x-3)^{3/2} + 6(x-3)^{1/2} + C.\end{aligned}$$

However, we could have arrived at the same result using the more creative (and cleaner) substitution $u = \sqrt{x-3}$. We avoid differentiating the radical by first squaring both sides of this equation to get $u^2 = x - 3$. Employing implicit differentiation from Calculus I, we arrive at $2u\,du = dx$. Noting that we still have an x in the numerator of our integrand, we solve $u^2 = x - 3$ for x to get $u^2 + 3 = x$. Hence, our integral becomes

$$\begin{aligned}\int \frac{x}{\sqrt{x-3}}\,dx &= 2\int \frac{u^2+3}{u}\cdot u\,du \\ &= 2\int u^2 + 3\,du\end{aligned}$$

Using a radical substitution often leaves us with an integrand that is much nicer to work with. Continuing the process, we get

$$\begin{aligned}&= 2\left(\frac{1}{3}u^3 + 3u\right) + C \\ &= \frac{2}{3}u^3 + 6u + C \\ &= \frac{2}{3}(\sqrt{x-3})^3 + 6\sqrt{x-3} + C \\ &= \frac{2}{3}(x-3)^{3/2} + 6(x-3)^{1/2} + C.\end{aligned}$$

While this process might *look* a little longer, we benefit by not having such a complicated integrand.

Truth be told, this technique often leads to a rational function requiring a special integration technique that we will learn later in this chapter; however, we avoid such integrals for now and practice the method when the result only requires simple integration. While there is no "hard and fast" rule to how or when to apply this method, it is a good technique to try if an integrand contains an expression of the form $\sqrt[n]{g(x)}$.

✔ Example 3.2.1: Using a Radical Substitution

Evaluate

$$\int \frac{\sqrt{x}}{1+\sqrt[4]{x}}\,dx$$

Solution

I will approach this as if I were a student. I begin by thinking of all the methods we have so far to help us evaluate integrals (of which, we only have three at this moment):

- **Direct Integration:** Since the integrand is not the derivative of an established, well-known function, we cannot directly integrate.
- **Simple u-Substitution:** If we let $u = 1 + \sqrt[4]{x}$, then $du = \frac{1}{4}x^{-3/4}\,dx$. While this might seem messy, it is an option.
- **Radical Substitution:** If we let $u = \sqrt[4]{x}$ so that $u^4 = x$, we would get $4u^3\,du = dx$. Moreover, the $\sqrt{x}$ in the numerator of the integrand can be replaced using $u^2 = \sqrt{x}$.

Of the two viable options, the radical substitution looks the most attractive. Substituting all of the expressions into the original integral, we get

$$\begin{aligned}\int \frac{\sqrt{x}}{1+\sqrt[4]{x}}\,dx &= \int \frac{u^2}{1+u}\cdot 4u^3\,du \\ &= 4\int \frac{u^5}{1+u}\,du\end{aligned}$$

At this point, most people feel stuck. It is important to remember the Mathematical Mantra - Arithmetic before Algebra, Algebra before Trigonometry, and Trigonometry before Calculus. Calculus should be the last thing we do. Something we know from Algebra is long division!

$$\begin{aligned}\int \frac{\sqrt{x}}{1+\sqrt[4]{x}}\,dx &= 4\int \frac{u^5}{1+u}\,du \\ &= 4\int u^4 - u^3 + u^2 - u + 1 - \frac{1}{1+u}\,du \\ &= 4\left(\frac{1}{5}u^5 - \frac{1}{4}u^4 + \frac{1}{3}u^3 - \frac{1}{2}u^2 + u - \ln|1+u|\right) + C \\ &= \frac{4}{5}x^{5/4} - x + \frac{4}{3}x^{3/4} - 2x^{1/2} + 4x^{1/4} - 4\ln|1+\sqrt[4]{x}| + C\end{aligned}$$

Let's spend a moment talking about two important points concerning Example 3.2.1. First, as is often the case with integration, we had more than one option for evaluating this integral - simple u-substitution or a radical substitution; however, there is yet another way to evaluate this integral. If $u = \sqrt[4]{x}$ is an attractive option, then why not make it slightly "bigger" by including any neighboring constants?

Indeed, had we let

$$u = 1 + \sqrt[4]{x}, \tag{3.2.1}$$

then our work simplifies quite nicely. Before finding the differential, du, let's get rid of radicals by isolating $\sqrt[4]{x}$ in Equation 3.2.1 and raising both sides of to the fourth power. Therefore,

$$u - 1 = \sqrt[4]{x} \implies (u-1)^4 = x. \tag{3.2.2}$$

Then

$$4(u-1)^3\,du = dx.$$

Also, looking at the original integrand, we have $\sqrt{x}$ in the numerator, so taking the square root of both sides of the result in Equation 3.2.2, we get

$$(u-1)^2 = \sqrt{x}.$$

Therefore,

$$\begin{aligned}
\int \frac{\sqrt{x}}{1+\sqrt[4]{x}}\,dx &= \int \frac{(u-1)^2}{u}\cdot 4(u-1)^3\,du \\
&= 4\int \frac{(u-1)^5}{u}\,du \\
&= 4\int \frac{u^5-5u^4+10u^3-10u^2+5u-1}{u}\,du \\
&= 4\int u^4-5u^3+10u^2-10u+5-\frac{1}{u}\,du \\
&= 4\left(\frac{1}{5}u^5-\frac{5}{4}u^4+\frac{10}{3}u^3-5u^2+5u-\ln|u|\right)+C \\
&= \frac{4}{5}u^5-5u^4+\frac{40}{3}u^3-20u^2+20u-4\ln|u|+C \\
&= \frac{4}{5}\left(1+\sqrt[4]{x}\right)^5-5\left(1+\sqrt[4]{x}\right)^4+\frac{40}{3}\left(1+\sqrt[4]{x}\right)^3-20\left(1+\sqrt[4]{x}\right)^2+20\left(1+\sqrt[4]{x}\right) \\
&\quad -4\ln\left(1+\sqrt[4]{x}\right)+C
\end{aligned}$$

Notice that the answer in this last computation does not *look* like the answer we arrived at in Example **3.2.1**. I assure you they are equivalent, but you should check by taking their derivatives and seeing if they match.

The second point to mention has to do with our method of solution using $u=1+\sqrt[4]{x}$. The binomial expansion on the third line of the last set of computations is achieved quickly using either the Binomial Theorem (from Precalculus) or Pascal's Triangle (also from Precalculus). Knowing these concepts will be critical as we progress in this course.

There is not much else (for now) to radical substitutions - it's a great method when available. We now venture into integrands involving trigonometric functions.

Integrands of the form $\frac{1}{a^2+x^2}$

Another integral commonly encountered throughout Calculus is of the type

$$\int \frac{1}{a^2+x^2}\,dx.$$

If this integral looks familiar, it is because you dealt with its more elementary form

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C$$

in Calculus I. Let's see how to derive the antiderivative of this new form.

$$\begin{aligned}
\int \frac{1}{a^2+x^2}\,dx &= \frac{1}{a^2}\int \frac{1}{1+\left(\frac{x}{a}\right)^2}\,dx && \text{(Factoring out } a^2 \text{ from the denominator)} \\
&= \frac{a}{a^2}\int \frac{1}{1+u^2}\,du && \left(u=\tfrac{x}{a} \implies du=\tfrac{1}{a}\,dx \implies a\,du=dx\right) \\
&= \frac{1}{a}\arctan u + C \\
&= \frac{1}{a}\arctan\left(\frac{x}{a}\right)+C
\end{aligned}$$

We have proved the following theorem.

Theorem

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)+C$$

Strive to Derive

As with most (but not all) of Mathematics, it's best to be able to derive a theorem as easy as this one. Your memory might fail you, but your ability to logically derive a result will likely not.

Example 3.2.2

Evaluate

$$\int \frac{e^x}{3+e^{2x}}\,dx$$

Solution

Letting $u = e^x$, we find $du = e^x\,dx$. Therefore,

$$\int \frac{e^x}{3+e^{2x}}\,dx = \int \frac{1}{3+u^2}\,du = \int \frac{1}{(\sqrt{3})^2+u^2}\,du.$$

Notice that I rewrote 3 as $(\sqrt{3})^2$. This is because, in our theorem, we are interested in a - not a^2.
Using our newly-derived theorem, this yields

$$\int \frac{e^x}{3+e^{2x}}\,dx = \frac{1}{\sqrt{3}}\arctan\left(\frac{u}{\sqrt{3}}\right)+C = \frac{1}{\sqrt{3}}\arctan\left(\frac{e^x}{\sqrt{3}}\right)+C.$$

Integrating Products and Powers of $\sin(x)$ and $\cos(x)$

For the remainder of this section, we look at integrating various products of trigonometric functions. As a collection, these integrals are called **trigonometric integrals**. They are an important part of the integration technique called trigonometric substitution, which is featured in Section 2.2. This technique allows us to convert algebraic expressions that we may not be able to integrate into expressions involving trigonometric functions that we might be able to integrate using the techniques described in this section. In addition, these types of integrals frequently appear when we study polar, cylindrical, and spherical coordinate systems (the first two will be studied later in this course, and the final one is left for Calculus III). Let's begin our study with products of $\sin x$ and $\cos x$.

Most authors spend an inordinate amount of time having students memorize cases; however, all of these cases can be summarized into the following statement:

Rule of Thumb

When dealing with integrands involving products of trigonometric functions, we strive to make a u-substitution to force even powers on the factor u is not equal to.

This Rule of Thumb is a bit oversimplified, and there are plenty of exceptions; however, as you gain experience working with this special substitution method, you will begin to understand why this Rule of Thumb is important and when it can be used. For now, let's see it in action.

Example 3.2.3: Integrating $\cos^j(x)\sin(x)$

Evaluate $\int \cos^3(x)\sin(x)\,dx$.

Solution

Since we are at the beginning of learning new techniques, let's list out our options and discuss which ones are viable.

☒ **Direct Integration:** Since the integrand is not the derivative of an established, well-known function, we cannot directly integrate.

☑ **Simple u-Substitution:** We have two options:

1. If we let $u = \sin x$, then $du = \cos x\, dx$.
2. If we let $u = \cos x$, then $du = -\sin x\, dx$.

☒ **Radical Substitution:** Since the integrand does not have a radical, a radical substitution is out of the question.

Our only viable option is the simplue u-substitution; however, which one do we use?

Option 1.

If $u = \sin x$, the differential, du, would steal away a single power of the cosine function leaving the integrand with $\cos^2 x$. The Pythagorean Identity tells us that $\cos^2 x = 1 - \sin^2 x = 1 - u^2$. Hence

$$\begin{aligned}\int \cos^3 x \sin x\, dx &= \int (1-u^2)u\, du \\ &= \int u - u^3\, du \\ &= \frac{1}{2}u^2 - \frac{1}{4}u^4 + C \\ &= \frac{1}{2}\sin^2 x - \frac{1}{4}\sin^4 x + C\end{aligned}$$

Option 2.

If $u = \cos(x)$, then $du = -\sin x\, dx$. Since the integrand doesn't have a $-\sin x\, dx$ to steal, we rewrite this as $-du = \sin x\, dx$. Thus,

$$\int \cos^3(x)\sin(x)\, dx = -\int u^3\, du = -\frac{1}{4}u^4 + C = -\frac{1}{4}\cos^4(x) + C.$$

Let's dissect Example **3.2.3** to get a better intuition of what is happening.

The first question that springs to students' minds when looking at the two different substitutions we used is, "Why are the answers different?" The short answer is, they aren't! Trigonometric functions (especially powers of trigonometric functions) are incredibly malleable - they can be rewritten in terms of other trigonometric functions somewhat easily. In fact,

$$\begin{aligned}-\frac{1}{4}\cos^4 x + C &= -\frac{1}{4}\left(\cos^2 x\right)^2 + C \\ &= -\frac{1}{4}\left(1 - \sin^2 x\right)^2 + C \\ &= -\frac{1}{4}\left(1 - 2\sin^2 x + \sin^4 x\right) + C \\ &= -\frac{1}{4} + \frac{1}{2}\sin^2 x - \frac{1}{4}\sin^4 x + C\end{aligned}$$

However, $-\frac{1}{4} + C$ is just a constant, so we could easily call that C_1. Hence,

$$-\frac{1}{4}\cos^4 x + C = \frac{1}{2}\sin^2 x - \frac{1}{4}\sin^4 x + C_1.$$

Note: Get Used to Equivalent Answers That Look Different

Since the constant of integration can "absorb" constants, be prepared to see answers from other students (and in the answer keys to homework) that *look* different from your answers. Before assuming you are wrong, take a few moments to

1. algebraically manipulate your answer to see if it is equivalent to what you are comparing it to (but, possibly, off by a constant), or
2. use graphing technology to see if the graph of your answer is equivalent to what you are comparing it to (but, again, possibly off by a constant), or
3. take the derivative of your answer and the answer you are comparing with to see if their derivatives are the same as the integrand.

The second thing to notice about our solution to **3.2.2** is that in both cases, we made u-substitutions that forced an even power on the factor that u was *not* equal to. In the first case, our substitution of $u = \sin x$ left us with $\cos^2 x$. In the second case, our substitution of $u = \cos x$ left us with $\sin^0 x$. Note that **2** and **0** are both even numbers - this aligns nicely with the Rule of Thumb!

? Checkpoint 3.2.3

Evaluate $\int \sin^4(x)\cos(x)\,dx$.

Answer

$$\int \sin^4(x)\cos(x)\,dx = \frac{1}{5}\sin^5 x + C$$

✓ Example 3.2.4: A Preliminary Example: Integrating $\cos^j(x)\sin^k(x)$ where k is Odd

Evaluate $\int \cos^2(x)\sin^3(x)\,dx$.

Solution

☒ **Direct Integration**

☑ **Simple u-Substitution**

$$u = \cos x \implies du = -\sin x\,dx \quad \text{or} \quad u = \sin x \implies du = \cos x\,dx$$

☒ Radical Substitution

Again, a simple substitution is the only option; however, this time our hand is forced. If we choose $u = \sin x \implies du = \cos x\,dx$, the ultimate power left on the cosine (which is what u is *not* equal to) is odd and we would not be able to rewrite $\cos x$ in terms of $u = \sin x$. Therefore, our only choice is to make the substitution $u = \cos x \implies -du = \sin x\,dx$. This would leave our integrand with $\sin^2(x) = 1 - \cos^2(x) = 1 - u^2$. Thus,

$$\begin{aligned}
\int \cos^2(x)\sin^3(x)\,dx &= \int \cos^2(x)\sin^2 x \sin(x)\,dx \\
&= \int \cos^2(x)(1-\cos^2(x))\sin(x)\,dx \\
&= -\int u^2(1-u^2)\,du \\
&= \int (u^4 - u^2)\,du \\
&= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C \\
&= \frac{1}{5}\cos^5(x) - \frac{1}{3}\cos^3(x) + C.
\end{aligned}$$

? Checkpoint 3.2.4

Evaluate $\int \cos^3(x)\sin^2(x)\,dx$.

Answer

$$\int \cos^3(x)\sin^2(x)\,dx = \frac{1}{3}\sin^3(x) - \frac{1}{5}\sin^5(x) + C$$

In the next example, we see the strategy that must be applied when there are only even powers of $\sin(x)$ and $\cos(x)$. For integrals of this type, the identities

$$\sin^2(x) = \frac{1-\cos(2x)}{2} = \frac{1}{2} - \frac{1}{2}\cos(2x)$$

and

$$\cos^2(x) = \frac{1+\cos(2x)}{2} = \frac{1}{2} + \frac{1}{2}\cos(2x)$$

are invaluable. These identities are sometimes known as the **Power Reduction Identities** and they may be derived from the Double-Angle Identity $\cos(2x) = \cos^2(x) - \sin^2(x)$ and the Pythagorean Identity $\cos^2(x) + \sin^2(x) = 1$.

✔ Example 3.2.5: Integrating an Even Power of $\sin(x)$

Evaluate $\int \sin^2(x)\,dx$.

Solution

☒ **Direct Integration**

☒ **Simple u-Substitution**

☒ **Radical Substitution**

When you are facing an integral and none of your methods work, you should immediately think of the Mathematical Mantra. There isn't any Arithmetic or Algebra that will clean up the integrand, but we have a couple of options from Trigonometry. The first is the Pythagorean Identity, $\sin^2 x = 1 - \cos^2 x$; however, this will leave us with a similar quandary - how do we integrate the square of cosine?

Thus, we change identities to the Power Reduction Formula, $\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$. Using this yields

$$\int \sin^2(x)\,dx = \int \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right) dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C.$$

? Checkpoint 3.2.5

Evaluate $\int \cos^2(x)\,dx$.

Answer

$$\int \cos^2(x)\,dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C$$

Again, the general process for integrating products of powers of $\sin(x)$ and $\cos(x)$ boils down to making a u-substitution (for either $\sin(x)$ or $\cos(x)$) so that the differential, du, "steals" the odd power off of the other trigonometric function. Using the Pythagorean Identity, you then rewrite the entire integrand in terms of u. You must use the Power Reduction Identities if there are no odd powers. Rather than writing out a "Problem-Solving Strategy" (which is too bulky and algorithmic), we showcase the process through examples.

✔ Example 3.2.6: Integrating $\cos^j(x)\sin^k(x)$ where k is Odd

Evaluate $\int \cos^8(x)\sin^5(x)\,dx$.

Solution

☒ **Direct Integration**

☑ **Simple u-Substitution**

☒ **Radical Substitution**

We would prefer to have all powers even (this allows us to use identities from Trigonometry). Therefore, we let $u = \cos(x)$ so that $du = -\sin(x)\,dx$ "steals" the odd power of the sine function away. Thus,

$$\begin{aligned}
\int \cos^8(x)\sin^5(x)\,dx &= \int \cos^8(x)\sin^4(x)\sin(x)\,dx \\
&= \int \cos^8(x)(\sin^2(x))^2\sin(x)\,dx && \text{(Laws of Exponents)} \\
&= \int \cos^8(x)(1-\cos^2(x))^2\sin(x)\,dx && \text{(Pythagorean Identity)} \\
&= \int u^8(1-u^2)^2(-du) && \text{(Let } u=\cos(x) \text{ and } du=-\sin(x)\,dx\text{)} \\
&= \int (-u^8+2u^{10}-u^{12})\,du \\
&= -\frac{1}{9}u^9+\frac{2}{11}u^{11}-\frac{1}{13}u^{13}+C \\
&= -\frac{1}{9}\cos^9(x)+\frac{2}{11}\cos^{11}(x)-\frac{1}{13}\cos^{13}(x)+C && \text{(resubstitute)}
\end{aligned}$$

? Checkpoint 3.2.6

Evaluate $\int \cos^3(x)\,dx$.

Answer

$$\int \cos^3(x)\,dx = \sin(x)-\frac{1}{3}\sin^3(x)+C$$

✓ Example 3.2.7: Integrating $\cos^j(x)\sin^k(x)$ where k and j are Even

Evaluate $\int \sin^4(x)\,dx$.

Solution

☒ **Direct Integration**

☒ **Simple u-Substitution:** Since the power on both sine and cosine are even (4 and 0, respectively), any u-substitution will result in "stealing" a power off of sine (or cosine), resulting in an expression not having an odd power. This will leave us stuck.

☒ **Radical Substitution**

Using the Mathematical Mantra, we see we must resort to using the Power Reduction Identities.

$$\begin{aligned}
\int \sin^4(x)\,dx &= \int \left(\sin^2(x)\right)^2\,dx && \text{(Laws of Exponents)}\\
&= \int \left(\frac{1}{2}-\frac{1}{2}\cos(2x)\right)^2\,dx && \text{(Power Reduction Identities)}\\
&= \int \left(\frac{1}{4}-\frac{1}{2}\cos(2x)+\frac{1}{4}\cos^2(2x)\right)\,dx && \text{(distributing)}\\
&= \int \left(\frac{1}{4}-\frac{1}{2}\cos(2x)+\frac{1}{4}\left(\frac{1}{2}+\frac{1}{2}\cos(4x)\right)\right)\,dx && \text{(Power Reduction Identities)}\\
&= \int \left(\frac{3}{8}-\frac{1}{2}\cos(2x)+\frac{1}{8}\cos(4x)\right)\,dx\\
&= \frac{3}{8}x-\frac{1}{4}\sin(2x)+\frac{1}{32}\sin(4x)+C
\end{aligned}$$

? Checkpoint 3.2.7

Evaluate $\int \cos^2(3x)\,dx$.

Answer

$$\int \cos^2(3x)\,dx = \frac{1}{2}x+\frac{1}{12}\sin(6x)+C$$

In some areas of physics, such as quantum mechanics, signal processing, and the computation of Fourier series, it is often necessary to integrate products that include $\sin(ax), \sin(bx), \cos(ax)$, and $\cos(bx)$. These integrals are evaluated by applying the **Product-to-Sum Identities** from Trigonometry. These identities are listed below because no student should be made to memorize them.

Theorem: Product-to-Sum Identities

$$\begin{aligned}
\sin(ax)\sin(bx) &= \frac{1}{2}\cos((a-b)x)-\frac{1}{2}\cos((a+b)x)\\
\sin(ax)\cos(bx) &= \frac{1}{2}\sin((a-b)x)+\frac{1}{2}\sin((a+b)x)\\
\cos(ax)\cos(bx) &= \frac{1}{2}\cos((a-b)x)+\frac{1}{2}\cos((a+b)x)
\end{aligned}$$

✓ Example 3.2.8: Evaluating $\int \sin(ax)\cos(bx)\,dx$

Evaluate $\int \sin(5x)\cos(3x)\,dx$.

Solution

☒ **Direct Integration**

☒ **Simple u-Substitution**

☒ **Radical Substitution**

Again, using the Mathematical Mantra, we begin by applying the identity $\sin(5x)\cos(3x) = \frac{1}{2}\sin(2x)+\frac{1}{2}\sin(8x)$. Thus,

$$\int \sin(5x)\cos(3x)\,dx = \int \frac{1}{2}\sin(2x)+\frac{1}{2}\sin(8x)\,dx = -\frac{1}{4}\cos(2x)-\frac{1}{16}\cos(8x)+C.$$

? Checkpoint 3.2.8

Evaluate $\int \cos(6x)\cos(5x)\,dx$.

Answer

$$\int \cos(6x)\cos(5x)\,dx = \frac{1}{2}\sin x + \frac{1}{22}\sin(11x) + C$$

Integrating Products and Powers of $\tan(x)$ and $\sec(x)$

Before discussing the integration of products and powers of $\tan(x)$ and $\sec(x)$, we need to introduce the antiderivative of the secant.

Theorem: $\int \sec(x)\,dx$ and $\int \csc(x)\,dx$

$$\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + C$$

and

$$\int \csc(x)\,dx = -\ln|\csc(x) + \cot(x)| + C$$

Proof

$$\begin{aligned}
\int \sec(x)\,dx &= \int \frac{\sec(x)(\sec(x)+\tan(x))}{\sec(x)+\tan(x)}\,dx && \text{(multiplying numerator and denominator by } \sec(x)+\tan(x)) \\
&= \int \frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)}\,dx \\
&= \int \frac{1}{u}\,du && (u = \sec(x)+\tan(x) \implies du = (\sec(x)\tan(x)+\sec^2(x))\,dx) \\
&= \ln|u| + C \\
&= \ln|\sec(x)+\tan(x)| + C && \text{(resubstitute)}
\end{aligned}$$

The proof for the integral of the cosecant is generally left as a homework exercise.

For completeness, let's list the integrals involving $\tan(x)$ and $\sec(x)$ we have learned so far:

$$\begin{aligned}
\int \sec^2(x)\,dx &= \tan(x) + C \\
\int \sec(x)\tan(x)\,dx &= \sec(x) + C \\
\int \tan(x)\,dx &= \ln|\sec(x)| + C \\
\int \sec(x)\,dx &= \ln|\sec(x)+\tan(x)| + C
\end{aligned}$$

For *most* integrals of products and powers of $\tan(x)$ and $\sec(x)$, we rewrite the expression we wish to integrate as the sum or difference of integrals of the form $\int \tan^j(x)\sec^2(x)\,dx$ or $\int \sec^j(x)\tan(x)\,dx$. In the former case, we can see that allowing $u = \tan(x)$ would cause du to "steal" the $\sec^2(x)\,dx$. In the latter case, allowing $u = \sec(x)$ causes du to "steal" a single power of $\sec(x)$ along with the $\tan(x)\,dx$. In both cases, the integral simplifies nicely.

Let's see this in action.

✔ Example 3.2.9: Evaluating $\int \sec^j(x)\tan(x)\,dx$

Evaluate $\int \sec^5(x)\tan(x)\,dx$.

Solution

☒ **Direct Integration**

☑ **Simple u-Substitution:** Choosing to let $u=\tan(x)$ would result in $du=\sec^2(x)\,dx$. This would "steal" two powers off the secant function and result in an odd power remaining on secant. This is *not* desirable; however, if we let $u=\sec(x)$ so that $du=\sec(x)\tan(x)\,dx$, the differential would "steal" a single power from the tangent, leaving an even power on the tangent (a power of 0), which is what we want.

☒ **Radical Substitution**

$$\begin{aligned}\int \sec^5(x)\tan(x)\,dx &= \int \sec^4(x)\sec(x)\tan(x)\,dx \\ &= \int u^4\,du && (u=\sec(x),\text{ then } du=\sec(x)\tan(x)\,dx) \\ &= \frac{1}{5}u^5+C \\ &= \frac{1}{5}\sec^5(x)+C && (\text{resubstitute})\end{aligned}$$

? Checkpoint 3.2.9

Evaluate $\int \tan^5(x)\sec^2(x)\,dx$.

Answer

$$\int \tan^5(x)\sec^2(x)\,dx=\frac{1}{6}\tan^6(x)+C$$

When working with products and powers of the sine and cosine, the Rule of Thumb was to make a u-substitution so that the "non-u" function ended with an even power. We then use the Pythagorean Identity to rewrite the integrand entirely in terms of u. The only exception to this strategy was if *both* sine and cosine have even powers. In that case, we start by using the Power Reduction Identities. When dealing with products and powers of the secant and the tangent, things are not so clean. As such, writing out a "Problem-Solving Strategy" is helpful, but still not as good as experience.

Problem-Solving Strategy: Integrating $\int \tan^k(x)\sec^j(x)\,dx$

To integrate $\int \tan^k(x)\sec^j(x)\,dx$, use the following strategies:

1. If j is even and $j\geq 2$, rewrite $\sec^j(x)=\sec^{j-2}(x)\sec^2(x)$ and use $\sec^2(x)=\tan^2(x)+1$ to rewrite $\sec^{j-2}(x)$ in terms of $\tan(x)$. Let $u=\tan(x)$ and $du=\sec^2(x)\,dx$.
2. If k is odd and $j\geq 1$, rewrite $\tan^k(x)\sec^j(x)=\tan^{k-1}(x)\sec^{j-1}(x)\sec(x)\tan(x)$ and use $\tan^2(x)=\sec^2(x)-1$ to rewrite $\tan^{k-1}(x)$ in terms of $\sec(x)$. Let $u=\sec(x)$ and $du=\sec(x)\tan(x)\,dx$. (Note: If j is even and k is odd, then either strategy 1 or strategy 2 may be used.)
3. If k is odd where $k\geq 3$ and $j=0$, rewrite $\tan^k(x)=\tan^{k-2}(x)\tan^2(x)=\tan^{k-2}(x)(\sec^2(x)-1)=\tan^{k-2}(x)\sec^2(x)-\tan^{k-2}(x)$. It may be necessary to repeat this process on the $\tan^{k-2}(x)$ term.
4. The case where k is even and j is odd requires a method we will introduce later in this chapter. For now, we focus only on integrands of the form 1, 2, or 3.

Caution: Strive to Understand - Not to Memorize

The preceding "Problem-Solving Strategy" is complex, but there is no way around it. Integration techniques are mostly heuristic methods. As such, it's best to learn them by working through many problems - experience is a *much* better teacher than memorizing the preceding strategy.

Example 3.2.10: Integrating $\tan^k(x)\sec^j(x)$ when j is Even

Evaluate $\int \tan^6(x)\sec^4(x)\,dx$.

Solution

☒ **Direct Integration**

☑ **Simple u-Substitution:** Both powers on tangent and secant are already even, so whatever u-substitution (if any) that we perform, we would like to keep those even powers there (this allows us to use our trigonometric identities, if needed). To keep the powers even after a u-substitution, we set $u=\tan(x)$ so that $du=\sec^2(x)\,dx$. This differential "steals" two powers off the secant function (keeping the power even in the end).

☒ **Radical Substitution**

$$\begin{aligned}
\int \tan^6(x)\sec^4(x)\,dx &= \int \tan^6(x)(\tan^2(x)+1)\sec^2(x)\,dx && \text{(Pythagorean Identity)}\\
&= \int u^6(u^2+1)\,du && (u=\tan(x) \text{ and } du=\sec^2(x)dx)\\
&= \int (u^8+u^6)\,du\\
&= \frac{1}{9}u^9+\frac{1}{7}u^7+C\\
&= \frac{1}{9}\tan^9(x)+\frac{1}{7}\tan^7(x)+C && \text{(resubstitute)}
\end{aligned}$$

Example 3.2.11: Integrating $\tan^k(x)\sec^j(x)$ when k is Odd

Evaluate $\int \tan^5(x)\sec^3(x)\,dx$.

Solution

☒ **Direct Integration**

☑ **Simple u-Substitution:** If we let $u=\sec(x)$, then $du=\sec(x)\tan(x)\,dx$ will "steal" enough power so that the tangent (the non-u function) is left with an even power.

☒ **Radical Substitution**

$$\begin{aligned}
\int \tan^5(x)\sec^3(x)\,dx &= \int \tan^4(x)\sec^2(x)\sec(x)\tan(x)\,dx \\
&= \int (\tan^2(x))^2\sec^2(x)\sec(x)\tan(x)\,dx && \text{(Laws of Exponents)} \\
&= \int (\sec^2(x)-1)^2\sec^2(x)\sec(x)\tan(x)\,dx && \text{(Pythagorean Identity)} \\
&= \int (u^2-1)^2u^2\,du && (u=\sec(x) \text{ and } du=\sec(x)\tan(x)\,dx) \\
&= \int (u^6-2u^4+u^2)\,du \\
&= \frac{1}{7}u^7-\frac{2}{5}u^5+\frac{1}{3}u^3+C \\
&= \frac{1}{7}\sec^7(x)-\frac{2}{5}\sec^5(x)+\frac{1}{3}\sec^3(x)+C && \text{(resubstitute)}
\end{aligned}$$

✔ Example 3.2.12: Integrating $\tan^k(x)$ where k is Odd and $k \geq 3$

Evaluate $\int \tan^3(x)\,dx$.

Solution

☒ **Direct Integration**

☒ **Simple u-Substitution:** We could probably stare at this for a while and think, "What kind of substitution would lead to an even power on that tangent?" The answer is, in the form this integral is written, no substitution will help!

☒ **Radical Substitution**

Since there is no "obvious" integration technique, we go back to the Mathematical Mantra to get creative. If we recognize that

$$\tan^3(x) = \tan(x)\tan^2(x) = \tan(x)\left(\sec^2(x)-1\right) = \tan(x)\sec^2(x)-\tan(x),$$

we can make some progress.

$$\begin{aligned}
\int \tan^3(x)\,dx &= \int (\tan(x)\sec^2(x)-\tan(x))\,dx \\
&= \int \tan(x)\sec^2(x)\,dx - \int \tan(x)\,dx \\
&= \frac{1}{2}\tan^2(x) - \ln|\sec(x)| + C.
\end{aligned}$$

For the first integral, we used the substitution $u=\tan(x)$. For the second integral, we used the formula.

? Checkpoint 3.2.12

Evaluate $\int \tan^3(x)\sec^7(x)\,dx$.

Answer

$$\int \tan^3(x)\sec^7(x)\,dx = \frac{1}{9}\sec^9(x)-\frac{1}{7}\sec^7(x)+C$$

3.2E: Exercises

Reading Questions

1. When an integrand contains a radical, such as $\sqrt[n]{g(x)}$, what is a "radical substitution" you could try?
2. What is the general antiderivative of the function $f(x) = \frac{1}{a^2+x^2}$?
3. When integrating $\int \sin^m(x)\cos^n(x)dx$, what is the general strategy if the power of sine (m) is odd?
4. What trigonometric identities are required to integrate $\int \sin^m(x)\cos^n(x)dx$ when both m and n are even?
5. When integrating $\int \tan^k(x)\sec^j(x)dx$, if the power of secant (j) is even, what is the recommended choice for u and du?
6. When integrating $\int \tan^k(x)\sec^j(x)dx$, if the power of tangent (k) is odd and there is at least one secant factor, what is the recommended choice for u and du?
7. What type of trigonometric identities are used to evaluate integrals like $\int \sin(5x)\cos(3x)dx$?

Fill in the blank to make a true statement.

1) $\sin^2 x + \_\_\_\_\_ = 1$

Answer

$\cos^2 x$

2) $\sec^2 x - 1 = \_\_\_\_\_$

Answer

$\tan^2 x$

Use an identity to reduce the power of the trigonometric function to a trigonometric function raised to the first power.

3) $\sin^2 x = \_\_\_\_\_$

Answer

$$\frac{1-\cos(2x)}{2}$$

4) $\cos^2 x = \_\_\_\_\_$

Answer

$$\frac{1+\cos(2x)}{2}$$

Evaluate each of the following integrals by u-substitution.

5) $\int \sin^3 x \cos x \, dx$

Answer

$$\int \sin^3 x \cos x \, dx = \frac{\sin^4 x}{4} + C$$

6) $\int \sqrt{\cos x} \sin x \, dx$

7) $\int \tan^5(2x)\sec^2(2x)\, dx$

Answer

$$\int \tan^5(2x)\sec^2(2x)\, dx = \frac{1}{12}\tan^6(2x) + C$$

8) $\int \sin^7(2x)\cos(2x)\, dx$

9) $\int \tan(\frac{x}{2})\sec^2(\frac{x}{2})\,dx$

Answer

$$\int \tan(\frac{x}{2})\sec^2(\frac{x}{2})\,dx \quad = \quad \tan^2(\frac{x}{2}) + C$$

10) $\int \tan^2 x \sec^2 x\,dx$

Compute the following integrals using the guidelines for integrating powers of trigonometric functions. Use a CAS to check the solutions. (*Note*: Some of the problems may be done using techniques of integration learned previously.)

11) $\int \sin^3 x\,dx$

Answer

$$\int \sin^3 x\,dx \quad = \quad -\frac{3\cos x}{4} + \tfrac{1}{12}\cos(3x) + C = -\cos x + \frac{\cos^3 x}{3} + C$$

12) $\int \cos^3 x\,dx$

13) $\int \sin x \cos x\,dx$

Answer

$$\int \sin x \cos x\,dx \quad = \quad -\tfrac{1}{2}\cos^2 x + C$$

14) $\int \cos^5 x\,dx$

15) $\int \sin^5 x \cos^2 x\,dx$

Answer

$$\int \sin^5 x \cos^2 x\,dx \quad = \quad -\frac{5\cos x}{64} - \tfrac{1}{192}\cos(3x) + \tfrac{3}{320}\cos(5x) - \tfrac{1}{448}\cos(7x) + C$$

16) $\int \sin^3 x \cos^3 x\,dx$

17) $\int \sqrt{\sin x}\cos x\,dx$

Answer

$$\int \sqrt{\sin x}\cos x\,dx \quad = \quad \tfrac{2}{3}(\sin x)^{3/2} + C$$

18) $\int \sqrt{\sin x}\cos^3 x\,dx$

19) $\int \sec x \tan x\,dx$

Answer

$$\int \sec x \tan x\,dx \quad = \quad \sec x + C$$

20) $\int \tan(5x)\,dx$

21) $\int \tan x \sec^3 x\,dx$

22) $\int \sec^4 x\,dx$

Answer

$$\int \sec^4 x\,dx \quad = \quad \frac{2\tan x}{3} + \tfrac{1}{3}\sec^2 x \tan x = \tan x + \frac{\tan^3 x}{3} + C$$

23) $\int \cot x\,dx$

24) $\int \csc x\,dx$

Answer

$$\int \csc x\,dx \quad = \quad -\ln|\cot x + \csc x| + C$$

25) $\int \frac{\tan^3 x}{\sqrt{\sec x}}\,dx$

For exercises 26 - 27, find a general formula for the integrals.

26) $\int \sin^2 ax \cos ax\,dx$

Answer

$$\int \sin^2 ax \cos ax\,dx \quad = \quad \frac{\sin^3(ax)}{3a} + C$$

27) $\int \sin ax \cos ax\,dx.$

Use the Double-Angle Identities to evaluate the integrals in exercises 28 - 33.

28) $\int_0^{\pi} \sin^2 x\,dx$

Answer

$$\int_0^{\pi} \sin^2 x\,dx \quad = \quad \frac{\pi}{2}$$

29) $\int_0^{\pi} \sin^4 x\,dx$

30) $\int \cos^2 3x\,dx$

Answer

$$\int \cos^2 3x\,dx \quad = \quad \frac{x}{2} + \tfrac{1}{12}\sin(6x) + C$$

31) $\int \sin^2 x \cos^2 x\,dx$

32) $\int \sin^2 x\,dx + \int \cos^2 x\,dx$

Answer

$$\int \sin^2 x\,dx + \int \cos^2 x\,dx \quad = \quad x + C$$

33) $\int \sin^2 x \cos^2(2x)\,dx$

For exercises 34 - 42, evaluate the definite integrals. Express answers in exact form whenever possible.

34) $\int_0^{2\pi} \cos x \sin 2x\,dx$

Answer

$$\int_0^{2\pi} \cos x \sin 2x\,dx \quad = \quad 0$$

35) $\int_0^{\pi} \sin 3x \sin 5x \, dx$

36) $\int_0^{\pi} \cos(99x) \sin(101x) \, dx$

Answer

$$\int_0^{\pi} \cos(99x) \sin(101x) \, dx = 0$$

37) $\int_{-\pi}^{\pi} \cos^2(3x) \, dx$

38) $\int_0^{2\pi} \sin x \sin(2x) \sin(3x) \, dx$

Answer

$$\int_0^{2\pi} \sin x \sin(2x) \sin(3x) \, dx = 0$$

39) $\int_0^{4\pi} \cos(x/2) \sin(x/2) \, dx$

40) $\int_{\pi/6}^{\pi/3} \frac{\cos^3 x}{\sqrt{\sin x}} \, dx$ (Round this answer to three decimal places.)

Answer

$$\int_{\pi/6}^{\pi/3} \frac{\cos^3 x}{\sqrt{\sin x}} \, dx \approx 0.239$$

41) $\int_{-\pi/3}^{\pi/3} \sqrt{\sec^2 x - 1} \, dx$

42) $\int_0^{\pi/2} \sqrt{1 - \cos(2x)} \, dx$

Answer

$$\int_0^{\pi/2} \sqrt{1 - \cos(2x)} \, dx = \sqrt{2}$$

43) Find the area of the region bounded by the graphs of the equations $y = \sin x$, $y = \sin^3 x$, $x = 0$, and $x = \frac{\pi}{2}$.

44) Find the area of the region bounded by the graphs of the equations $y = \cos^2 x$, $y = \sin^2 x$, $x = -\frac{\pi}{4}$, and $x = \frac{\pi}{4}$.

Answer

$A = 1 \text{ unit}^2$

45) A particle moves in a straight line with the velocity function $v(t) = \sin(\omega t) \cos^2(\omega t)$. Find its position function $x = f(t)$ if $f(0) = 0$.

46) Find the average value of the function $f(x) = \sin^2 x \cos^3 x$ over the interval $[-\pi, \pi]$.

Answer

0

For exercises 47 - 48, solve the differential equations.

47) $\frac{dy}{dx} = \sin^2 x$. The curve passes through point $(0, 0)$.

48) $\frac{dy}{d\theta} = \sin^4(\pi\theta)$

Answer

$$f(x) = \frac{3\theta}{8} - \frac{1}{4\pi}\sin(2\pi\theta) + \frac{1}{32\pi}\sin(4\pi\theta) + C$$

49) Find the length of the curve $y = \ln(\csc x)$, for $\frac{\pi}{4} \le x \le \frac{\pi}{2}$.

50) Find the length of the curve $y = \ln(\sin x)$, for $\frac{\pi}{3} \le x \le \frac{\pi}{2}$.

Answer

$s = \ln(\sqrt{3})$

51) Find the volume generated by revolving the curve $y = \cos(3x)$ about the x-axis, for $0 \le x \le \frac{\pi}{36}$.

For exercises 52 - 53, use this information: The inner product of two functions f and g over $[a, b]$ is defined by $f(x) \cdot g(x) = \langle f, g \rangle = \int_a^b f \cdot g \, dx$. Two distinct functions f and g are said to be orthogonal if $\langle f, g \rangle = 0$.

52) Show that $\sin(2x)$, $\cos(3x)$ are orthogonal over the interval $[-\pi, \pi]$.

Answer

$$\int_{-\pi}^{\pi} \sin(2x)\cos(3x)\,dx = 0$$

53) Evaluate $\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,dx$.

54) Integrate $y' = \sqrt{\tan x}\sec^4 x$.

Answer

$$y = \int \sqrt{\tan x}\sec^4 x\,dx \quad = \quad \tfrac{2}{3}(\tan x)^{3/2} + \tfrac{2}{7}(\tan x)^{7/2} + C = \tfrac{2}{21}(\tan x)^{3/2}\left[7 + 3\tan^2 x\right] + C$$

3.3: Trigonometric Substitution

Note to the Instructor (click to expand)

This section is a continuation of expanding the Substitution Method by introducing trig subs. We painstakingly avoid the integral of $\sec^3(\theta)$ (despite it normally appearing in this material) because we need Integration by Parts to deal with that. We hold off on that gem for a future section.

To succeed in this section, you'll need to use some skills from previous courses. While you should already know them, this is the first time they've been required. You can review these skills in CRC's Corequisite Codex. If you have a support class, it might cover some, **but not all**, of these topics.

Prerequisite Skills and Support Topics (click to expand)

- **Quadratic Functions**
  - Vertex Form and Completing the Square: This algebraic technique is required to transform an integral with a general quadratic expression into a form containing a sum or difference of squares, making it suitable for trigonometric substitution. This is demonstrated in Example 2.2.8.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Solve integration problems involving the square root of a sum or difference of two squares.

Hawk A.I. Section-Specific Tutor

EST. 1970 COSUMNES RIVER COLLEGE MATHEMATICS

To access the Hawk A.I. Tutor, you will need to be logged into your campus Gmail account.

In this section, we explore integrals containing expressions of the form $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, and $\sqrt{x^2-a^2}$, where the values of a are positive. We have already encountered and evaluated integrals containing some expressions of this type, but many remain inaccessible. The **trigonometric substitution** technique is convenient when evaluating these integrals. This technique, which is a specific use of the Substitution Method, rewrites these integrals as trigonometric integrals.

Before diving into the instruction, it's best to recall the following identities from Trigonometry (all based on the Pythagorean Identity):

- $\cos^2\theta = 1-\sin^2\theta$,
- $\sec^2\theta = 1+\tan^2\theta$, and
- $\tan^2\theta = \sec^2\theta - 1$.

Note that the *forms* of these expressions match the radicands in

- $\sqrt{a^2-x^2}$ (where $x = a\sin\theta$),
- $\sqrt{a^2+x^2}$ (where $x = a\tan\theta$), and
- $\sqrt{x^2-a^2}$ (where $x = a\sec\theta$).

This observation will be compelling. It would be best to convince yourself that these forms are incredibly similar.

The final bit of "recall" we will need before developing the method of trigonometric substitution are the ranges of the inverse trigonometric functions. Specifically, we will need to remember, from Section 1.6 in the Differential Calculus textbook, the following range restrictions:

$$\sin^{-1}(y)=\theta \quad \text{if and only if} \quad \sin(\theta)=y \quad \text{where} \quad \theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$$

$$\tan^{-1}(y)=\theta \quad \text{if and only if} \quad \tan(\theta)=y \quad \text{where} \quad \theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$$

$$\sec^{-1}(y)=\theta \quad \text{if and only if} \quad \sec(\theta)=y \quad \text{where} \quad \theta\in\left[0,\frac{\pi}{2}\right)\cup\left[\pi,\frac{3\pi}{2}\right).$$

You should already be comfortable with the restrictions on θ for both the arcsine and arctangent from your Trigonometry or Precalculus course; however, the following snippet from Section 1.6 in the Differential Calculus textbook should begin to justify our choice of range for the arcsecant.

> All the inverse trigonometric functions return angles in quadrant I (where all trigonometric functions are positive) and one other quadrant where the corresponding trigonometric function is negative. The returned quadrants where the corresponding trigonometric functions are negative are fairly standard and agreed upon (quadrant IV for arctangent and arcsine, and quadrant II for arccosine and arccotangent); **however**, the quadrants chosen for the ranges of the arcsecant and arccosecant are *not* universally agreed upon.
>
> In this text, we choose arcsecant and arccosecant to return angles in quadrants I and **III** for a special reason - it makes our work in calculus slightly easier. I mention this because you might have seen a slightly different choice for the ranges of these functions in another textbook. The difference is insignificant other than the ease our choice makes for our work in calculus.

Integrals Involving $\sqrt{a^2-x^2}$

Before developing a general strategy for integrals containing $\sqrt{a^2-x^2}$, consider the integral $\int \sqrt{9-x^2}\,dx$. This integral cannot be evaluated using any of the techniques we have discussed so far. However, if we make the substitution $x = 3\sin\theta$, we have $dx = 3\cos\theta\,d\theta$. After substituting into the integral, we have

$$\int \sqrt{9-x^2}\,dx = \int \sqrt{9-(3\sin\theta)^2}\cdot 3\cos\theta\,d\theta.$$

After simplifying, we have

$$\int \sqrt{9-x^2}\,dx = \int 9\sqrt{1-\sin^2\theta}\cdot\cos\theta\,d\theta.$$

Letting $1-\sin^2\theta = \cos^2\theta$, we now have

$$\int \sqrt{9-x^2}\,dx = \int 9\sqrt{\cos^2\theta}\cos\theta\,d\theta.$$

Assuming that $\cos\theta \geq 0$,[1] we have

$$\int \sqrt{9-x^2}\,dx = \int 9\cos^2\theta\,d\theta.$$

At this point, we can evaluate the integral using the techniques developed for integrating powers and products of trigonometric functions. Before completing this example, let's examine the general theory behind this idea.

To evaluate integrals involving $\sqrt{a^2-x^2}$, we make the substitution $x = a\sin\theta$ and $dx = a\cos\theta\,d\theta$. To see that this makes sense, consider the following argument: The domain of $\sqrt{a^2-x^2}$ is $[-a,a]$. Thus,

$$-a \leq x \leq a. \tag{3.3.1}$$

Consequently,

$$-1 \leq \frac{x}{a} \leq 1.$$

Since the range of $\sin\theta$ over $[-\frac{\pi}{2},\frac{\pi}{2}]$ is $[-1,1]$, there is a unique angle θ satisfying $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ so that $\sin\theta = \frac{x}{a}$, or equivalently, so that $x = a\sin\theta$. If we substitute $x = a\sin\theta$ into $\sqrt{a^2-x^2}$, we get

$$\begin{aligned}
\sqrt{a^2-x^2} &= \sqrt{a^2-(a\sin\theta)^2} && \left(x = a\sin\theta \text{ where } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\right)\\
&= \sqrt{a^2-a^2\sin^2\theta}\\
&= \sqrt{a^2(1-\sin^2\theta)} && \text{(factoring)}\\
&= \sqrt{a^2\cos^2\theta} && \text{(Pythagorean Identity)}\\
&= |a\cos\theta| && \left(\sqrt{a^2\cos^2\theta} = |a\cos\theta|\right)
\end{aligned}$$

Since $\cos\theta \geq 0$ on $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ and $a > 0$, $|a\cos\theta| = a\cos\theta$.

Therefore, by making the substitution $x = a\sin\theta$, we can convert an integral involving a radical into an integral involving trigonometric functions, where we have the added benefit of not worrying about absolute values. From this point, we need to evaluate the integral. Once this is done, however, we must convert the solution to an expression involving x. To do this, we begin by drawing a reference triangle for θ (see Figure 3.3.1).

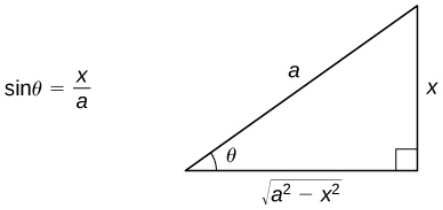


Figure 3.3.1: A reference triangle can help express the trigonometric functions evaluated at θ in terms of x.

Since $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, this reference triangle can only be drawn in quadrants I or IV. In either case,

$$\begin{aligned}\sin\theta &= \frac{x}{a}\\ \cos\theta &= \frac{\sqrt{a^2-x^2}}{a}\\ \tan\theta &= \frac{x}{\sqrt{a^2-x^2}}.\end{aligned}$$

Hence, no matter what trigonometric function is in the result of our evaluated integral, we can rewrite it as a function of x. Should θ appear by itself, we use $\theta = \sin^{-1}\left(\frac{x}{a}\right)$.[2]

The essential part of this discussion is summarized in the following problem-solving strategy.

Problem-Solving Strategy: Integrating Expressions Involving $\sqrt{a^2-x^2}$

1. It is an excellent idea to make sure the integral cannot be evaluated easily in another way. For example, although this method can be applied to integrals of the form
$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx, \quad \int \frac{x}{\sqrt{a^2-x^2}}\,dx, \quad \text{and} \quad \int x\sqrt{a^2-x^2}\,dx,$$
they can each be integrated directly either by formula or by a simple u-substitution.
2. Make the substitution $x = a\sin\theta$ and $dx = a\cos\theta\,d\theta$. Note: This substitution yields $\sqrt{a^2-x^2} = a\cos\theta$.
3. Simplify the expression.
4. Evaluate the integral using techniques from the Section 2.1.
5. Use a reference triangle to rewrite the result in terms of x. You may also need to use some trigonometric identities and the relationship $\theta = \sin^{-1}\left(\frac{x}{a}\right)$.

The following example demonstrates the application of this problem-solving strategy.

Example 3.3.1: Integrating an Expression Involving $\sqrt{a^2-x^2}$

Evaluate

$$\int \sqrt{9-x^2}\,dx.$$

Solution

☒ **Direct Integration**

☒ **Simple u-Substitution:** Letting $u = 9-x^2$ results in $du = -2x\,dx$. I leave it to the reader to verify that this results in an uglier (and more difficult) integrand.

☒ **Radical Substitution:** Letting $u = \sqrt{9-x^2}$ means $u^2 = 9-x^2$. Therefore, $2u\,du = -2x\,dx \implies u\,du = -x\,dx$. Again, I leave it to the reader to verify that this results in an uglier (and more difficult) integrand.

☑ **Trigonometric Substitution:** We can try the trigonometric substitution $x = 3\sin\theta$ so that $dx = 3\cos\theta\, d\theta$. Since $\sin\theta = \frac{x}{3}$, we can construct the reference triangle shown in Figure **3.3.2**.

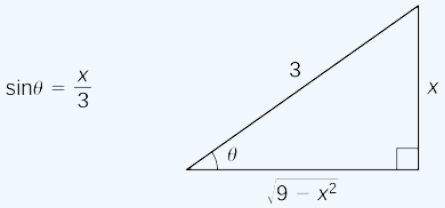


Figure **3.3.2**

Thus,

$$\begin{aligned}
\int \sqrt{9-x^2}\,dx &= \int \sqrt{9-(3\sin\theta)^2}\cdot 3\cos\theta\,d\theta && (x = 3\sin\theta \implies dx = 3\cos\theta\,d\theta)\\
&= \int \sqrt{9(1-\sin^2\theta)}\cdot 3\cos\theta\,d\theta\\
&= \int \sqrt{9\cos^2\theta}\cdot 3\cos\theta\,d\theta && \text{(Pythagorean Identity)}\\
&= \int 3|\cos\theta|3\cos\theta\,d\theta && \left(\sqrt{\blacksquare^2} = |\blacksquare|\right)\\
&= \int 9\cos^2\theta\,d\theta && \left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \implies \cos\theta \ge 0 \implies |\cos\theta| = \cos\theta\right)\\
&= \int 9\left(\frac{1}{2}+\frac{1}{2}\cos(2\theta)\right)d\theta && \text{(Power Reduction Identity)}\\
&= \frac{9}{2}\theta + \frac{9}{4}\sin(2\theta) + C\\
&= \frac{9}{2}\theta + \frac{9}{4}(2\sin\theta\cos\theta) + C && \text{(Double-Angle Identity)}\\
&= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{9}{2}\cdot\frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3} + C && \left(\text{resubstitute } \sin^{-1}\left(\frac{x}{3}\right) = \theta \text{ and use the reference triangle}\right)\\
&= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{x\sqrt{9-x^2}}{2} + C
\end{aligned}$$

Example **3.3.1** showcases something that I tell my students repeatedly - mastery of Trigonometry is a requirement for your success in Calculus. If you need to brush up on lost Trigonometry skills, please see Section 1.6 of the Differential Calculus textbook for a single-section review of all basic material required from Trigonometry for survival in Calculus. More topics are required from your Trigonometry and Precalculus course than are shown in that section; however, those missing topics (e.g., polar coordinates and parametric equations) will be reviewed later.

✔ Example **3.3.2**: Integrating an Expression Involving $\sqrt{a^2-x^2}$

Evaluate

$$\int \frac{\sqrt{4-\frac{25}{9}x^2}}{x}\,dx.$$

Solution

☒ **Direct Integration**

☒ **Simple u-Substitution:** Letting $u = 4 - \frac{25}{9}x^2$ results in $du = -\frac{50}{9}x\,dx$. I leave it to the reader to verify that this results in $-\frac{1}{2}\int \frac{\sqrt{u}}{4-u}\,du$, which is no better than where we started.

☒ **Radical Substitution:** For the same reasons that the simple u-substitution didn't work, a radical substitution is not viable either.

☑ **Trigonometric Substitution:** We want $\frac{25}{9}x^2 = 4\sin^2\theta$, so we make the substitution $\frac{5}{3}x = 2\sin\theta$, which leads to $\frac{5}{3}\,dx = 2\cos\theta\,d\theta$. Solving this for dx, we get $dx = \frac{6}{5}\cos\theta\,d\theta$. We also need to rewrite the denominator in the integrand in terms of θ. Collecting

everything in a nice group, we have the following:

- $\frac{25}{9}x^2 = 4\sin^2\theta$
- $x = \frac{6}{5}\sin\theta$
- $dx = \frac{6}{5}\cos\theta\, d\theta$

As in Example **3.3.1**, we draw a reference triangle to help our conversions, and we get

$$\begin{aligned}
\int \frac{\sqrt{4-\frac{25}{9}x^2}}{x}\,dx &= \int \frac{\sqrt{4-4\sin^2\theta}}{\frac{6}{5}\sin\theta}\cdot\frac{6}{5}\cos\theta\,d\theta \\
&= \int \frac{2\sqrt{\cos^2\theta}\cdot\cos\theta}{\sin\theta}\,d\theta && \text{(Pythagorean Identity)} \\
&= \int \frac{2|\cos\theta|\cdot\cos\theta}{\sin\theta}\,d\theta && \left(\sqrt{\blacksquare^2} = |\blacksquare|\right) \\
&= \int \frac{2\cos^2\theta}{\sin\theta}\,d\theta \\
&= \int \frac{2(1-\sin^2\theta)}{\sin\theta}\,d\theta && \text{(Pythagorean Identity)} \\
&= \int (2\csc\theta - 2\sin\theta)\,d\theta \\
&= 2\ln|\csc\theta-\cot\theta| + 2\cos\theta + C \\
&= 2\ln\left|\frac{6}{5x} - \frac{3\sqrt{4-\frac{25}{9}x^2}}{5x}\right| + \sqrt{4-\frac{25}{9}x^2} + C && \text{(using the reference triangle)} \\
&= 2\ln\left|\frac{6-\sqrt{36-25x^2}}{5x}\right| + \frac{1}{3}\sqrt{36-25x^2} + C
\end{aligned}$$

In the next example, we see that we sometimes have a choice of methods.

✔ Example 3.3.3: Integrating an Expression Involving $\sqrt{a^2-x^2}$ Two Ways

Evaluate $\int x^3\sqrt{1-x^2}dx$ two ways: first by using the substitution $u = 1-x^2$ and then by using a trigonometric substitution.

Solution

Method 1

Let $u = 1-x^2$. Hence, $x^2 = 1-u$. Thus, $du = -2x\,dx$. In this case, the integral becomes

$$\begin{aligned}
\int x^3\sqrt{1-x^2}\,dx &= -\frac{1}{2}\int x^2\sqrt{1-x^2}(-2x\,dx) \\
&= -\frac{1}{2}\int (1-u)\sqrt{u}\,du \\
&= -\frac{1}{2}\int (u^{1/2}-u^{3/2})\,du \\
&= -\frac{1}{2}\left(\frac{2}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right) + C \\
&= -\frac{1}{3}(1-x^2)^{3/2} + \frac{1}{5}(1-x^2)^{5/2} + C
\end{aligned}$$

Method 2

Let $x = \sin\theta$. In this case, $dx = \cos\theta\,d\theta$. Using this substitution, we have

$$\begin{aligned}
\int x^3\sqrt{1-x^2}\,dx &= \int \sin^3\theta\cos^2\theta\,d\theta \\
&= \int (1-\cos^2\theta)\cos^2\theta\sin\theta\,d\theta \\
&= \int (u^4-u^2)\,du && (u=\cos\theta \implies du=-\sin\theta\,d\theta) \\
&= \frac{1}{5}u^5-\frac{1}{3}u^3+C \\
&= \frac{1}{5}\cos^5\theta-\frac{1}{3}\cos^3\theta+C \\
&= \frac{1}{5}(1-x^2)^{5/2}-\frac{1}{3}(1-x^2)^{3/2}+C && \text{(using the reference triangle)}
\end{aligned}$$

? Checkpoint 3.3.3

Rewrite the integral $\int \frac{x^3}{\sqrt{25-x^2}}\,dx$ using the appropriate trigonometric substitution (do not evaluate the integral).

Answer

$$\int 125\sin^3\theta\,d\theta$$

Integrating Expressions Involving $\sqrt{a^2+x^2}$

For integrals containing $\sqrt{a^2+x^2}$, let's first consider the domain of this expression. Since $\sqrt{a^2+x^2}$ is defined for all real values of x, we restrict our choice to those trigonometric functions that have a range of all real numbers. Thus, we are restricted to selecting either $x=a\tan\theta$ or $x=a\cot\theta$. Either of these substitutions would work, but the standard substitution is $x=a\tan\theta$ or, equivalently, $\tan\theta=\frac{x}{a}$. With this substitution, we make the assumption that $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$, so that we also have $\theta=\tan^{-1}\left(\frac{x}{a}\right)$. As before, let's derive the concept.

Since the range of $\tan\theta$ over $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ is $(-\infty,\infty)$, there is a unique angle θ satisfying $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$ so that $\tan\theta=\frac{x}{a}$, or equivalently, so that $x=a\tan\theta$. If we substitute $x=a\tan\theta$ into $\sqrt{a^2+x^2}$, we get

$$\begin{aligned}
\sqrt{a^2+x^2} &= \sqrt{a^2+(a\tan\theta)^2} && \left(x=a\tan\theta \text{ where } -\frac{\pi}{2}<\theta<\frac{\pi}{2}\right) \\
&= \sqrt{a^2+a^2\tan^2\theta} \\
&= \sqrt{a^2(1+\tan^2\theta)} && \text{(factoring out } a^2\text{)} \\
&= \sqrt{a^2\sec^2\theta} && \text{(Pythagorean Identity)} \\
&= |a\sec\theta| && \left(\sqrt{\blacksquare^2}=|\blacksquare|\right)
\end{aligned}$$

Since $\sec\theta\geq 0$ on $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$ and $a>0$, $|a\sec\theta|=a\sec\theta$.

We again arrive at the beautiful result that making the substitution $x=a\tan\theta$ allows us to convert an integral involving a radical into an integral involving trigonometric functions, where we have the added benefit of not worrying about absolute values. To convert our solution back to an expression involving x, we draw a reference triangle for θ (see Figure 3.3.3).

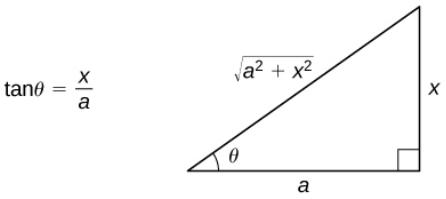


Figure 3.3.3: A reference triangle can be constructed to express the trigonometric functions evaluated at θ in terms of x.

Since $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$, this reference triangle can only be drawn in quadrants I or IV. In either case,

$$\sin\theta = \frac{x}{\sqrt{a^2+x^2}}$$
$$\cos\theta = \frac{a}{\sqrt{a^2+x^2}}$$
$$\tan\theta = \frac{x}{a}.$$

Hence, no matter what trigonometric function is in the result of our evaluated integral, we can rewrite it as a function of x. Should θ appear by itself, we use $\theta = \tan^{-1}\left(\frac{x}{a}\right)$. The procedure for using this substitution is outlined in the following problem-solving strategy.

Problem-Solving Strategy: Integrating Expressions Involving $\sqrt{a^2+x^2}$

1. Check whether the integral can be evaluated easily using another method. In some cases, it is more convenient to use an alternative method.[3]
2. Substitute $x = a\tan\theta$ and $dx = a\sec^2\theta\, d\theta$. This substitution yields
$$\sqrt{a^2+x^2} = \sqrt{a^2+(a\tan\theta)^2} = \sqrt{a^2(1+\tan^2\theta)} = \sqrt{a^2sec^2\theta} = |a\sec\theta| = a\sec\theta.$$
Since $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and $\sec\theta > 0$ over this interval, $|a\sec\theta| = a\sec\theta$.
3. Simplify the expression.
4. Evaluate the integral using techniques from the section on trigonometric integrals.
5. Use the reference triangle from Figure **3.3.3** to rewrite the result in terms of x. You may also need to use some trigonometric identities and the relationship $\theta = \tan^{-1}\left(\frac{x}{a}\right)$.[4,5]

Example 3.3.4: Integrating an Expression Involving $\sqrt{a^2+x^2}$

Evaluate $\int \frac{dx}{\sqrt{1+x^2}}$ and check the solution by differentiating.

Solution

☒ **Direct Integration***

☒ **Simple u-Substitution:** Letting $u = 1+x^2$ results in $du = 2x\,dx$. I leave it to the reader to verify that this results in an uglier (and more difficult) integrand.

☒ **Radical Substitution:** At this this point, you might have guessed that if the simple substitution doesn't work out nicely, nor will the related radical substitution.

☑ **Trigonometric Substitution:** Begin with the substitution $x = \tan\theta$ and $dx = \sec^2\theta\, d\theta$. Since $\tan\theta = x$, draw the reference triangle in Figure **3.3.4**.

$\tan\theta = x = \frac{x}{1}$

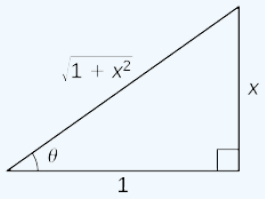


Figure **3.3.4**

Thus,

$$\int \frac{dx}{\sqrt{1+x^2}} = \int \frac{\sec^2\theta}{\sec\theta}\,d\theta \qquad \left(x = \tan\theta \implies dx = \sec^2\theta\, d\theta \implies \sqrt{1+x^2} = \sec\theta\right)$$
$$= \int \sec\theta\, d\theta$$
$$= \ln|\sec\theta + \tan\theta| + C$$
$$= \ln|\sqrt{1+x^2} + x| + C \quad \text{(using the reference triangle)}$$

To check the solution, differentiate:

$$\frac{d}{dx}\left(\ln|\sqrt{1+x^2}+x|\right)=\frac{1}{\sqrt{1+x^2}+x}\cdot\left(\frac{x}{\sqrt{1+x^2}}+1\right)=\frac{1}{\sqrt{1+x^2}+x}\cdot\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}=\frac{1}{\sqrt{1+x^2}}.$$

Since $\sqrt{1+x^2}+x>0$ for all values of x, we could rewrite $\ln|\sqrt{1+x^2}+x|+C=\ln(\sqrt{1+x^2}+x)+C$, if desired.

* In Differential Calculus, you learned about the hyperbolic functions, their inverses, and the derivatives of both the base hyperbolic functions and their inverses. At the time, it was very likely that you were *not required* to memorize the derivatives of the inverse hyperbolic functions; however, if you happened to have stored those derivatives in your brain, then Example **3.3.4** could be much easier. Let's take a look at what I mean.

✔ Example 3.3.5: Evaluating $\int \frac{dx}{\sqrt{1+x^2}}$ Using a Different Substitution

Use the substitution $x=\sinh t$ to evaluate $\displaystyle\int\frac{dx}{\sqrt{1+x^2}}$.

Solution

Because $\sinh t$ has a range of all real numbers, and $1+\sinh^2 t=\cosh^2 t$, we may use the substitution $x=\sinh t$ to evaluate this integral. In this case, $dx=\cosh t\,dt$, Consequently,

$$\begin{aligned}
\int\frac{dx}{\sqrt{1+x^2}} &= \int\frac{\cosh t}{\sqrt{1+\sinh^2 t}}\,dt && (x=\sinh t \implies dx=\cosh t\,dt)\\
&= \int\frac{\cosh t}{\sqrt{\cosh^2 t}}\,dt && (1+\sinh^2 t=\cosh^2 t)\\
&= \int\frac{\cosh t}{|\cosh t|}\,dt\\
&= \int\frac{\cosh t}{\cosh t}\,dt && (\cosh t>0 \text{ for all } t \implies |\cosh t|=\cosh t)\\
&= \int dt\\
&= t+C\\
&= \sinh^{-1}x+C
\end{aligned}$$

Example **3.3.5** deserves a couple comments. First, I did not use θ as the argument for **sinh** and **cosh** because we traditionally reserve θ for angles. The argument of the hyperbolic functions are *not* angles, so it's a bit safer to use a variable that does not invoke thoughts of angles. If you were unaware of this fact, please see Section 1.7 of the Differential Calculus textbook for a discussion that the arguments of the hyperbolic functions are neither angles nor arc lengths.

The second comment is a continuation of my comments before Example **3.3.5**. Had we recalled that

$$\frac{d}{dt}\sinh^{-1}t=\frac{1}{\sqrt{1+t^2}},$$

we would have easily finished Example **3.3.5** in a single step; *however*, the answers to Examples **3.3.4** and **3.3.5** don't appear to match. To rectify this, we would need to recall that

$$\sinh^{-1}x=\ln(x+\sqrt{x^2+1}).$$

? Checkpoint 3.3.5

Rewrite $\displaystyle\int x^3\sqrt{x^2+4}\,dx$ by using a substitution involving $\tan\theta$.

Answer

$$\int 32\tan^3\theta\sec^3\theta\,d\theta$$

Integrating Expressions Involving $\sqrt{x^2-a^2}$

The domain of the expression $\sqrt{x^2-a^2}$ is $(-\infty,-a]\cup[a,\infty)$. Thus, either $x\le -a$ or $x\ge a$. Hence, $\frac{x}{a}\le -1$ or $\frac{x}{a}\ge 1$. Since these intervals correspond to the range of $\sec\theta$ on the set $[0,\frac{\pi}{2})\cup[\pi,\frac{3\pi}{2})$, it makes sense to use the substitution $\sec\theta=\frac{x}{a}$ or, equivalently, $x=a\sec\theta$, where $0\le\theta<\frac{\pi}{2}$ or $\pi\le\theta<\frac{3\pi}{2}$. The corresponding substitution for dx is $dx=a\sec\theta\tan\theta\,d\theta$. Again, we derive the process below.[6]

Since the range of $\sec\theta$ over $[0,\frac{\pi}{2})\cup[\pi,\frac{3\pi}{2})$ is $(-\infty,-1]\cup[1,\infty)$, there is a unique angle θ satisfying $0\le\theta<\frac{\pi}{2}$ or $\pi\le\theta<\frac{3\pi}{2}$ so that $\sec\theta=\frac{x}{a}$, or equivalently, so that $x=a\sec\theta$. If we substitute $x=a\sec\theta$ into $\sqrt{x^2-a^2}$, we get

$$\begin{aligned}
\sqrt{x^2-a^2} &= \sqrt{(a\sec\theta)^2-a^2} && \left(x=a\sec\theta \text{ where } 0\le\theta<\frac{\pi}{2} \text{ or } \pi\le\theta<\frac{3\pi}{2}\right)\\
&= \sqrt{a^2\sec^2\theta-a^2}\\
&= \sqrt{a^2(\sec^2\theta-1)} && (\text{factoring out } a^2)\\
&= \sqrt{a^2\tan^2\theta} && (\text{Pythagorean Identity})\\
&= |a\tan\theta| && \left(\sqrt{\blacksquare^2}=|\blacksquare|\right)
\end{aligned}$$

Since $\tan\theta\ge 0$ on $0\le\theta<\frac{\pi}{2}$ and $\pi\le\theta<\frac{3\pi}{2}$, and because $a>0$, $|a\tan\theta|=a\tan\theta$.

Once more, by making the substitution $x=a\sec\theta$, we can convert an integral involving a radical into an integral involving trigonometric functions, where we have the added benefit of not worrying about absolute values. To convert our solution back to an expression involving x, we draw a reference triangle for θ (see Figure 3.3.5 below).

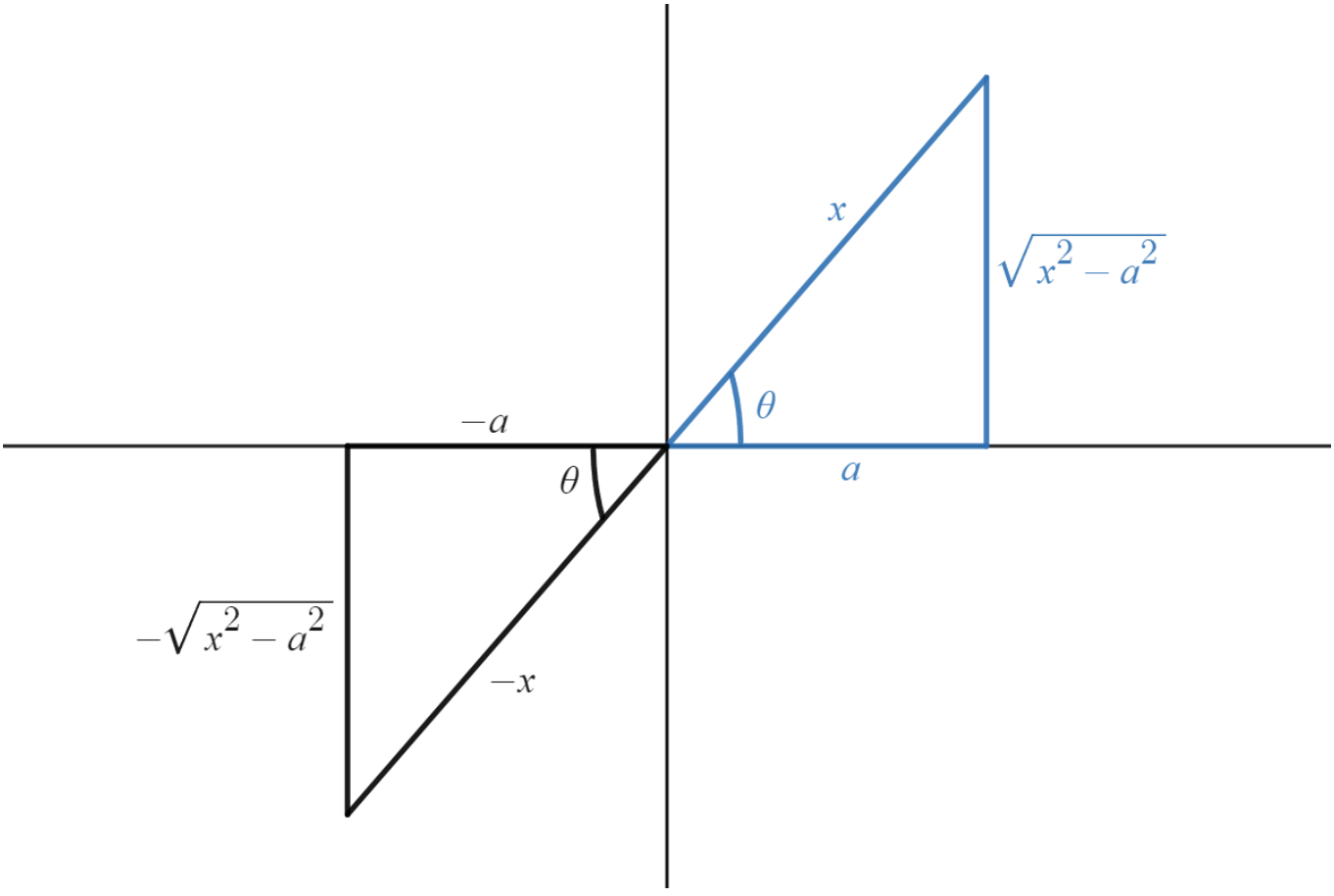


Figure 3.3.5: Use the appropriate reference triangle to express the trigonometric functions evaluated at θ in terms of x.

Since $\theta\in[0,\frac{\pi}{2})\cup[\pi,\frac{3\pi}{2})$, this reference triangle can only be drawn in quadrants I or III.[7] In either case,

$$\sin\theta = \frac{\sqrt{x^2-a^2}}{x}$$
$$\cos\theta = \frac{a}{x}$$
$$\tan\theta = \frac{\sqrt{x^2-a^2}}{a}.$$

Hence, no matter what trigonometric function is in the result of our evaluated integral, we can rewrite it as a function of x. Should θ appear by itself, we use $\theta = \sec^{-1}\left(\frac{x}{a}\right)$. The procedure for using this substitution is outlined in the following problem-solving strategy.

The procedure for using this substitution is outlined in the following problem-solving strategy.

Problem-Solving Strategy: Integrals Involving $\sqrt{x^2-a^2}$

1. Check whether the integral cannot be evaluated using another method. If so, we may wish to consider applying an alternative technique.
2. Substitute $x = a\sec\theta$ and $dx = a\sec\theta\tan\theta\, d\theta$. This substitution yields
$$\sqrt{x^2-a^2} = \sqrt{(a\sec\theta)^2 - a^2} = \sqrt{a^2(\sec^2\theta - 1)} = \sqrt{a^2\tan^2\theta} = |a\tan\theta| = a\tan\theta.$$
3. Simplify the expression.
4. Evaluate the integral using techniques from the section on trigonometric integrals.
5. Use the reference triangles from Figure **3.3.5** to rewrite the result in terms of x.
6. You may also need to use some trigonometric identities and the relationship $\theta = \sec^{-1}\left(\frac{x}{a}\right)$.

Example 3.3.7: Finding the Area of a Region

Find the area of the region between the graph of $f(x) = \frac{3\sqrt{x^2-25}}{x^4}$ and the x-axis over the interval $[5, 10]$.

Solution

First, we use graphing technology to obtain a graph of the region described in the problem, as shown in the following figure.

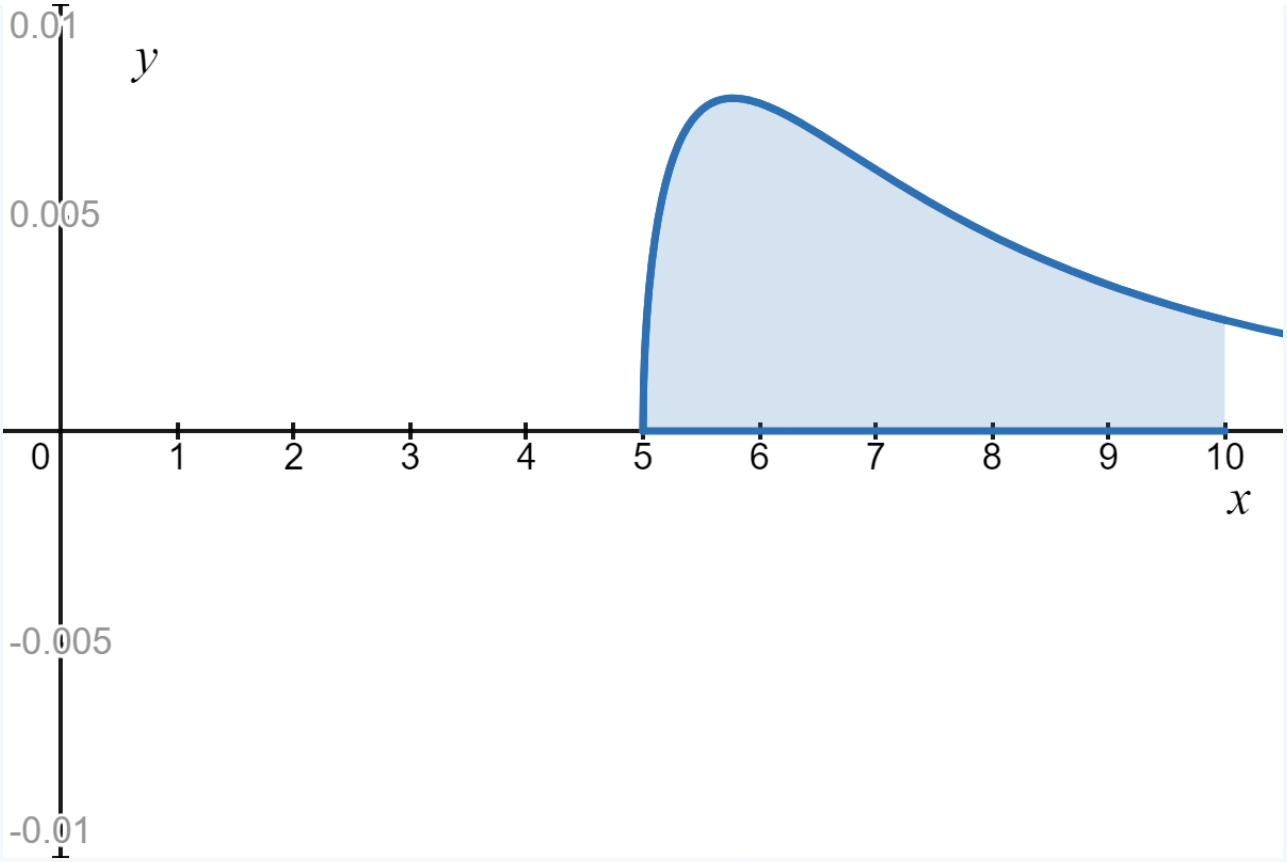


Figure 3.3.7

We can see that the area is $A = \int_5^{10} \frac{\sqrt{x^2-25}}{x^4}\,dx$.

☒ **Direct Integration**

☒ **Simple u-Substitution:** Letting $u = x^2 - 25$ results in $du = 2x\,dx$. I leave it to the reader to verify that this results in an uglier (and more difficult) integrand.

☒ **Radical Substitution**

☑ **Trigonometric Substitution:** We can substitute $x = 5\sec\theta$ so that $dx = 5\sec\theta\tan\theta\,d\theta$. We must also change the limits of integration. If $x = 5$, then $5 = 5\sec\theta$ and hence $\theta = 0$. If $x = 10$, then $10 = 5\sec\theta$ and so $\sec\theta = 2 \implies \theta = \frac{\pi}{3}$. After making these substitutions and simplifying, we have

$$
\begin{aligned}
\text{Area} &= \int_5^{10} \frac{\sqrt{x^2-25}}{x^4}\,dx \\
&= \int_0^{\pi/3} \frac{\sqrt{25\sec^2\theta - 25}}{625\sec^4\theta} \cdot 5\sec\theta\tan\theta\,d\theta \\
&= \frac{5}{625}\int_0^{\pi/3} \frac{5\sqrt{\sec^2\theta - 1}\cdot\tan\theta}{\sec^3\theta}\,d\theta \\
&= \frac{25}{625}\int_0^{\pi/3} \frac{\sqrt{\tan^2\theta}\cdot\tan\theta}{\sec^3\theta}\,d\theta && \text{(Pythagorean Identity)} \\
&= \frac{1}{25}\int_0^{\pi/3} \frac{\tan\theta\cdot\tan\theta}{\sec^3\theta}\,d\theta \\
&= \frac{1}{25}\int_0^{\pi/3} \frac{\tan^2\theta}{\sec^3\theta}\,d\theta \\
&= \frac{1}{25}\int_0^{\pi/3} \frac{\sin^2\theta/\cos^2\theta}{1/\cos^3\theta}\,d\theta && \text{(Ratio Identities)} \\
&= \frac{1}{25}\int_0^{\pi/3} \sin^2\theta\cos\theta\,d\theta \\
&= \frac{1}{25}\int_0^{\sqrt{3}/2} u^2\,du && (u = \sin\theta \implies du = \cos\theta\,d\theta) \\
&= \frac{1}{75}u^3\Big|_0^{\sqrt{3}/2} \\
&= \frac{3^{3/2}}{600} \\
&= \frac{3\cdot 3^{1/2}}{600} \\
&= \frac{3^{1/2}}{200}
\end{aligned}
$$

? Checkpoint 3.3.7

Evaluate

$$\int \frac{dx}{\sqrt{x^2-4}}.$$

Assume that $x > 2$.

Answer

$$\ln\left|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right| + C$$

Rewriting Integrals to Use Trigonometric Substitution

There are many situations in which a given integral seems too challenging to be evaluated using known techniques. In these (and, in fact, in all) cases, you should remember the Mathematical Mantra:

> *Perform mathematics in the order you learned mathematics. That is, perform*
>
> *Arithmetic before Algebra,*
>
> *Algebra before Trigonometry,*
>
> *Trigonometry before Precalculus, and*
>
> *Precalculus before Differential Calculus.*

> *Always be willing to ask yourself, "Is there something I learned previously that can simplify my work here?"*

The following example illustrates how using the Mathematical Mantra can reduce a seemingly impossible problem into a more manageable one.

Example 3.3.8: Completing the Square to Assist in Evaluating an Integral

Evaluate the integral.

$$\int \frac{x^2}{(3+4x-4x^2)^{3/2}}\,dx$$

Solution

☒ **Direct Integration:** Since the integrand is not the derivative of an established, well-known function, we cannot directly integrate.

☒ **Simple u-Substitution:**[8] If we let $u = 3+4x-4x^2$, then $du = (4-8x)\,dx$. At this point, we are experienced enough to recognize that this substitution will lead nowhere.

☒ **Radical Substitution:** Since a simple u-substitution will not work, a radical substitution is out of the question.

☒ **Trigonometric Substitution:** As written, this does not look to have a sum or difference of squares, so it doesn't *look* to be a candidate for a trigonometric substitution.

Now that we have exhausted all our integration techniques learned up to this point, we should realize that there must be some manipulation of the integrand using our previously learned mathematics to help us out. There are, in fact, *two* previously learned topics in Mathematics that can help here - both from Algebra; however, I will only mention completing the square as the other topic will lead to an integral we do not yet have a way to evaluate.

Back in Algebra, we learned how to complete the square for quadratic functions. This idea is beneficial with the given integral.

$$3+4x-4x^2 = -4(x^2-x)+3 = -4\left(x^2-x+\left(-\frac{1}{2}\right)^2\right)+3+4\left(-\frac{1}{2}\right)^2 = -4\left(x-\frac{1}{2}\right)^2+4 = 4-4\left(x-\frac{1}{2}\right)^2.$$

Therefore,

$$\int \frac{x^2}{(3+4x-4x^2)^{3/2}}\,dx = \int \frac{x^2}{\left(4-4\left(x-\frac{1}{2}\right)^2\right)^{3/2}}\,dx = \frac{1}{8}\int \frac{x^2}{\left(1-\left(x-\frac{1}{2}\right)^2\right)^{3/2}}\,dx.$$

The denominator of the integrand includes a difference of squares. This should encourage us to try a trigonometric substitution. The form is a^2-x^2, which reminds us of the trigonometric identity $1-\sin^2\theta = \cos^2\theta$. Hence, we want $\left(x-\frac{1}{2}\right)^2 = a^2\sin^2\theta = \sin^2\theta$, which gives $x-\frac{1}{2} = \sin\theta$. Moreover, this gives $dx = \cos\theta\,d\theta$. Finally, the numerator of the integrand has x^2, so it is necessary to determine x as a function of θ. Since $x-\frac{1}{2} = \sin\theta$, $x = \sin\theta+\frac{1}{2}$, and so $x^2 = \left(\sin\theta+\frac{1}{2}\right)^2$. Summarizing, we have

- $\left(x-\frac{1}{2}\right)^2 = \sin^2\theta$,
- $dx = \cos\theta\,d\theta$, and
- $x^2 = \left(\sin\theta+\frac{1}{2}\right)^2$.

Moreover, drawing a reference triangle at this point is a good idea. Figure 3.3.8 shows our reference triangle.

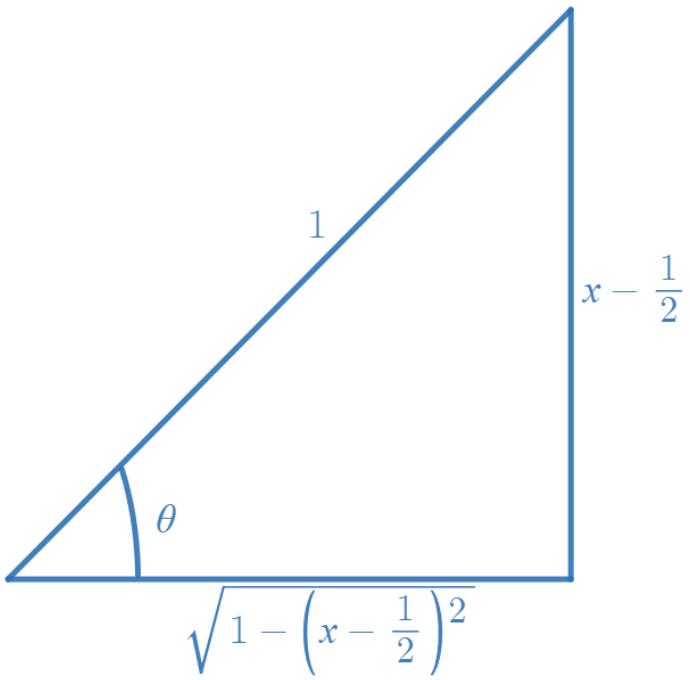


Figure **3.3.8**: The reference triangle.

Thus,

$$
\begin{aligned}
\int \frac{x^2}{(3+4x-4x^2)^{3/2}}\,dx &= \frac{1}{8}\int \frac{x^2}{\left(1-\left(x-\frac{1}{2}\right)^2\right)^{3/2}}\,dx \\
&= \frac{1}{8}\int \frac{\left(\sin\theta+\frac{1}{2}\right)^2}{(1-\sin^2\theta)^{3/2}}\cos\theta\,d\theta && \text{(completing the square)} \\
&= \frac{1}{8}\int \frac{\left(\sin\theta+\frac{1}{2}\right)^2}{(\cos^2\theta)^{3/2}}\cos\theta\,d\theta && \text{(Trigonometric substitutions)} \\
&= \frac{1}{8}\int \frac{\left(\sin\theta+\frac{1}{2}\right)^2}{\cos^3\theta}\cos\theta\,d\theta && \text{(Pythagorean Identity)} \\
&= \frac{1}{8}\int \frac{\left(\sin\theta+\frac{1}{2}\right)^2}{\cos^2\theta}\,d\theta \\
&= \frac{1}{8}\int \frac{\sin^2\theta+\sin\theta+\frac{1}{4}}{\cos^2\theta}\,d\theta \\
&= \frac{1}{8}\int \tan^2\theta+\sec\theta\tan\theta+\frac{1}{4}\sec^2\theta\,d\theta \\
&= \frac{1}{8}\int \sec^2\theta-1+\sec\theta\tan\theta+\frac{1}{4}\sec^2\theta\,d\theta && \text{(Pythagorean Identity)} \\
&= \frac{1}{8}\int -1+\sec\theta\tan\theta+\frac{5}{4}\sec^2\theta\,d\theta \\
&= -\frac{1}{8}\theta+\frac{1}{8}\sec\theta+\frac{5}{32}\tan\theta+C \\
&= -\frac{1}{8}\sin^{-1}\left(x-\frac{1}{2}\right)+\frac{1}{8\sqrt{1-\left(x-\frac{1}{2}\right)^2}}+\frac{5\left(x-\frac{1}{2}\right)}{32\sqrt{1-\left(x-\frac{1}{2}\right)^2}}+C && \text{(using the reference triangle)} \\
&= -\frac{1}{8}\sin^{-1}\left(x-\frac{1}{2}\right)+\frac{4+5\left(x-\frac{1}{2}\right)}{32\sqrt{1-\left(x-\frac{1}{2}\right)^2}}+C \\
&= -\frac{1}{8}\sin^{-1}\left(x-\frac{1}{2}\right)+\frac{5x+\frac{3}{2}}{32\sqrt{1-\left(x-\frac{1}{2}\right)^2}}+C \\
&= -\frac{1}{8}\sin^{-1}\left(x-\frac{1}{2}\right)+\frac{10x+3}{32\sqrt{4-4\left(x-\frac{1}{2}\right)^2}}+C \\
&= -\frac{1}{8}\sin^{-1}\left(x-\frac{1}{2}\right)+\frac{10x+3}{32\sqrt{3+4x-4x^2}}+C
\end{aligned}
$$

Footnotes

[1] We will justify this assumption momentarily.

[2] After integrating the trigonometric integral as a function of θ, you sometimes have a term of the form $k\theta$. If your original substitution was $x = a\sin\theta$, then it is a fact that $\theta = \arcsin\left(\frac{x}{a}\right)$. It is permissible to leave your final answer in this form *if and only if* $\frac{x}{a}$ is not a special trigonometric ratio. That is if evaluating a definite integral where you end up with $\arcsin\left(-\frac{1}{2}\right)$, you *must* use your prerequisite Trigonometry to simplify this to $-\frac{\pi}{6}$.

[3] The idea of constantly checking to see if something from the past will work first (i.e., a simple u-substitution) should not be new to you - especially if you took Differential Calculus using this text (well, the Calculus I version, at least). In that text, I continually referenced the Mathematical Mantra. The Mathematical Mantra boils down to doing mathematics in the order you learned mathematics - Arithmetic before Algebra, Algebra before Trigonometry, Trigonometry before Precalculus, Precalculus before Differential Calculus, and, in this case, Differential Calculus before Integral Calculus. In Differential Calculus, you learned about the Substitution Method. If you can evaluate an integral using the Substitution Method, then there is no need to get any fancier than that - don't jump to Trigonometric Substitutions or any other, more obscure technique if a good 'ol u-substitution does the job.

[4] As was stated previously, if given a definite integral, then you are expected to be able to evaluate the arctangent at special ratios.

[5] The reference triangle is based on the assumption that $x > 0$; however, the trigonometric ratios produced from the reference triangle are the same as the ratios for which $x \leq 0$.

[6] Of the three processes we have developed, it is important to pay close attention to this one in particular. The reason behind the choice of restriction for the range from Section 1.6 in the Differential Calculus textbook of the arcsecant should become apparent.

[7] Again, this is the distinct difference between the previous two derivations and this one. It is *very* important to note that we *could* have used the non-elegant range restriction of $\theta \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$ for the arcsecant; however, doing so would result in a more complex derivation of this trigonometric substitution due to the sign of the tangent in quadrant II being negative - this would *not* allow for $|a \tan \theta| = a \tan \theta$.

[8] We have learned enough integration techniques at this point to realize that the Substitution Method is an "umbrella method" encapsulating at least two different "types" of substitution. The first was a simple u-substitution from our Differential Calculus course. We were introduced to the second in this section, trigonometric substitutions.

3.3E: Exercises

Reading Questions

1. What are the three forms of radical expressions that indicate that a trigonometric substitution might be useful?
2. For an integral involving the expression $\sqrt{a^2-x^2}$, what is the standard trigonometric substitution for x, and what does the radical simplify to after this substitution?
3. For an integral involving the expression $\sqrt{a^2+x^2}$, what is the standard trigonometric substitution for x?
4. What is the purpose of drawing a reference triangle as part of the trigonometric substitution process?
5. What algebraic technique must be applied to an expression like $\sqrt{3+4x-4x^2}$ before a trigonometric substitution can be performed?
6. After making the substitution and evaluating the new integral in terms of θ, what must be done to complete the problem?

Simplify the expressions in exercises 1 - 5 by writing each one using a single trigonometric function.

1) $4-4\sin^2\theta$

2) $9\sec^2\theta-9$

Answer

$$9\sec^2\theta-9 \quad = \quad 9\tan^2\theta$$

3) $a^2+a^2\tan^2\theta$

4) $a^2+a^2\sinh^2\theta$

Answer

$$a^2+a^2\sinh^2\theta \quad = \quad a^2\cosh^2\theta$$

5) $16\cosh^2\theta-16$

Use the technique of completing the square to express each trinomial in exercises 6 - 8 as the square of a binomial.

6) $4x^2-4x+1$

Answer

$4(x-\frac{1}{2})^2$

7) $2x^2-8x+3$

8) $-x^2-2x+4$

Answer

$-(x+1)^2+5$

In exercises 9 - 26, integrate using the method of trigonometric substitution. Express the final answer in terms of the original variable.

9) $\int \frac{dx}{\sqrt{4-x^2}}$

10) $\int \frac{dx}{\sqrt{x^2-a^2}}$

Answer

$$\int \frac{dx}{\sqrt{x^2-a^2}} \quad = \quad \ln|x+\sqrt{-a^2+x^2}|+C$$

11) $\int \sqrt{4-x^2}\,dx$

12) $\int \frac{dx}{\sqrt{1+9x^2}}$

Answer

$$\int \frac{dx}{\sqrt{1+9x^2}} = \frac{1}{3}\ln\left|\sqrt{9x^2+1}+3x\right|+C$$

13) $\int \frac{x^2\,dx}{\sqrt{1-x^2}}$

14) $\int \frac{dx}{x^2\sqrt{1-x^2}}$

Answer

$$\int \frac{dx}{x^2\sqrt{1-x^2}} = -\frac{\sqrt{1-x^2}}{x}+C$$

15) $\int \frac{dx}{(1+x^2)^2}$

16) $\int \frac{\sqrt{x^2-25}}{x}\,dx$

17) $\int \frac{\theta^3}{\sqrt{9-\theta^2}}\,d\theta$

Answer

$$\int \frac{\theta^3 d\theta}{\sqrt{9-\theta^2}}\,d\theta = -\frac{1}{3}\sqrt{9-\theta^2}(18+\theta^2)+C$$

18) $\int \frac{dx}{\sqrt{x^6-x^2}}$

19) $\int \sqrt{x^6-x^8}\,dx$

Answer

$$\int \sqrt{x^6-x^8}\,dx = \frac{(-1+x^2)(2+3x^2)\sqrt{x^6-x^8}}{15x^3}+C$$

20) $\int \frac{dx}{(1+x^2)^{3/2}}$

21) $\int \frac{dx}{(x^2-9)^{3/2}}$

Answer

$$\int \frac{dx}{(x^2-9)^{3/2}} = -\frac{x}{9\sqrt{x^2-9}}+C$$

22) $\int \frac{\sqrt{1+x^2}}{x}\,dx$

23) $\int \frac{x^2}{\sqrt{x^2-1}}\,dx$

Answer

$$\int \frac{x^2}{\sqrt{x^2-1}}\,dx = \frac{1}{2}\left(\ln\left|x+\sqrt{x^2-1}\right|+x\sqrt{x^2-1}\right)+C$$

24) $\int \frac{x^2}{x^2+4}\,dx$

25) $\int \frac{dx}{x^2\sqrt{x^2+1}}$

Answer

$$\int \frac{dx}{x^2\sqrt{x^2+1}} = -\frac{\sqrt{1+x^2}}{x} + C$$

26) $\int_{-1}^{1} (1-x^2)^{3/2}\,dx$

Answer

$$\int_{-1}^{1} (1-x^2)^{3/2}\,dx = \tfrac{1}{8}\left(x(5-2x^2)\sqrt{1-x^2} + 3\arcsin x\right) + C$$

In exercises 27 - 32, use the substitutions $x = \sinh\theta$, $\cosh\theta$, or $\tanh\theta$. Express the final answers in terms of the variable x.

27) $\int \frac{dx}{\sqrt{x^2-1}}$

28) $\int \frac{dx}{x\sqrt{1-x^2}}$

Answer

$$\int \frac{dx}{x\sqrt{1-x^2}} = \ln x - \ln|\,1+\sqrt{1-x^2}\,| + C$$

29) $\int \sqrt{x^2-1}\,dx$

30) $\int \frac{\sqrt{x^2-1}}{x^2}\,dx$

Answer

$$\int \frac{\sqrt{x^2-1}}{x^2}\,dx = -\frac{\sqrt{-1+x^2}}{x} + \ln\left|x+\sqrt{-1+x^2}\right| + C$$

31) $\int \frac{dx}{1-x^2}$

32) $\int \frac{\sqrt{1+x^2}}{x^2}\,dx$

Answer

$$\int \frac{\sqrt{1+x^2}}{x^2}\,dx = -\frac{\sqrt{1+x^2}}{x} + \operatorname{arcsinh} x + C$$

Use the technique of completing the square to evaluate the integrals in exercises 33 - 37.

33) $\int \frac{1}{x^2-6x}\,dx$

34) $\int \frac{1}{x^2+2x+1}\,dx$

Answer

$$\int \frac{1}{x^2+2x+1}\,dx = -\frac{1}{1+x} + C$$

35) $\int \frac{1}{\sqrt{-x^2+2x+8}}\,dx$

36) $\int \frac{1}{\sqrt{-x^2+10x}}\,dx$

Answer

$$\int \frac{1}{\sqrt{-x^2+10x}}\,dx = \arcsin\left(\frac{x-5}{5}\right) + C$$

37) $\int \frac{1}{\sqrt{x^2+4x-12}}\,dx$

38) Evaluate the integral without using calculus: $\int_{-3}^{3} \sqrt{9-x^2}\,dx$.

Answer

$\int_{-3}^{3} \sqrt{9-x^2}\,dx = \frac{9\pi}{2}$; area of a semicircle with radius 3

39) Find the area enclosed by the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$.

40) Evaluate the integral $\int \frac{dx}{\sqrt{1-x^2}}$ using two different substitutions. First, let $x=\cos\theta$ and evaluate using trigonometric substitution. Second, let $x=\sin\theta$ and use trigonometric substitution. Are the answers the same?

Answer

$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin(x)+C$ is the common answer.

41) Evaluate the integral $\int \frac{dx}{x\sqrt{x^2-1}}$ using the substitution $x=\sec\theta$. Next, evaluate the same integral using the substitution $x=\csc\theta$. Show that the results are equivalent.

42) Evaluate the integral $\int \frac{x}{x^2+1}\,dx$ using the form $\int \frac{1}{u}\,du$. Next, evaluate the same integral using $x=\tan\theta$. Are the results the same?

Answer

$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln(1+x^2)+C$ is the result using either method.

43) State the method of integration you would use to evaluate the integral $\int x\sqrt{x^2+1}\,dx$. Why did you choose this method?

44) State the method of integration you would use to evaluate the integral $\int x^2\sqrt{x^2-1}\,dx$. Why did you choose this method?

Answer

Use trigonometric substitution. Let $x=\sec(\theta)$.

45) Evaluate $\int_{-1}^{1} \frac{x}{x^2+1}\,dx$

46) Find the length of the arc of the curve over the specified interval: $y=\ln x, \quad [1,5]$. Round the answer to three decimal places.

Answer

$s=4.367$ units

47) Find the surface area of the solid generated by revolving the region bounded by the graphs of $y=x^2$, $y=0$, $x=0$, and $x=\sqrt{2}$ about the x-axis. (Round the answer to three decimal places).

48) The region bounded by the graph of $f(x)=\frac{1}{1+x^2}$ and the x-axis between $x=0$ and $x=1$ is revolved about the x-axis. Find the volume of the solid that is generated.

Answer

$V=\left(\frac{\pi^2}{8}+\frac{\pi}{4}\right)$ units^3

In exercises 49 - 50, solve the initial-value problem for y as a function of x.

49) $(x^2+36)\frac{dy}{dx}=1, \quad y(6)=0$

50) $(64-x^2)\frac{dy}{dx}=1, \quad y(0)=3$

Answer

$$y = \frac{1}{16}\ln\left|\frac{x+8}{x-8}\right| + 3$$

51) Find the area bounded by $y = \dfrac{2}{\sqrt{64-4x^2}}$, $x = 0$, $y = 0$, and $x = 2$.

52) An oil storage tank can be described as the volume generated by revolving the area bounded by $y = \dfrac{16}{\sqrt{64+x^2}}$, $x = 0$, $y = 0$, $x = 2$ about the x-axis. Find the volume of the tank (in cubic meters).

Answer

$V = 24.6$ m^3

53) During each cycle, the velocity v (in feet per second) of a robotic welding device is given by $v = 2t - \dfrac{14}{4+t^2}$, where t is time in seconds. Find the expression for the displacement s (in feet) as a function of t if $s = 0$ when $t = 0$.

54) Find the length of the curve $y = \sqrt{16-x^2}$ between $x = 0$ and $x = 2$.

Answer

$s = \frac{2\pi}{3}$ units

This page titled 3.3E: Exercises is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Roy Simpson.

- **7.3E: Exercises for Section 7.3** by Edwin “Jed” Herman, Gilbert Strang is licensed CC BY-NC-SA 4.0. Original source: https://openstax.org/details/books/calculus-volume-1.

3.4: Partial Fractions

Note to the Instructor (click to expand)

This is the nuts and bolts of PFD.

To succeed in this section, you'll need to use some skills from previous courses. While you should already know them, this is the first time they've been required. You can review these skills in CRC's Corequisite Codex. If you have a support class, it might cover some, **but not all**, of these topics.

Prerequisite Skills and Support Topics (click to expand)

- **Factoring Techniques**
  - Greatest Common Factor: Factoring the denominator is a critical step, and it often begins with factoring out a greatest common factor, as seen in Example 2.4.2 where $x^3 - x^2 - 2x$ is factored.
  - Factoring Trinomials: After factoring out a GCF, the remaining polynomial often needs to be factored, such as factoring $x^2 - x - 2$ in Example 2.4.2.
  - Factoring Sums and Differences of Cubes: This technique is needed to factor denominators like $x^3 - 8$ in Example 2.4.7.
  - Factoring Expressions Using the Rational Roots Theorem: The homework mentions that this may be needed to factor higher-degree polynomials in the denominator.
- **Systems**
  - Solving 2x2 Systems of Linear Equations: The method of equating coefficients for finding the constants in the decomposition often results in a system of linear equations that must be solved.
  - Solving 3x3 Systems of Linear Equations: For decompositions with three or more unknown constants, solving a larger system of equations is necessary.
- **Quadratic Functions**
  - The Discriminant: The text mentions using the discriminant ($b^2 - 4ac$) to determine if a quadratic factor is irreducible.
- **Polynomial Functions**
  - The Basics of Polynomial Functions: Understanding the degree of a polynomial is fundamental to the first step of the process: comparing the degree of the numerator to the degree of the denominator.

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Integrate a rational function using the method of partial fractions.
- Recognize simple linear factors in a rational function.
- Recognize repeated linear factors in a rational function.
- Recognize quadratic factors in a rational function.

Hawk A.I. Section-Specific Tutor

COSUMNES RIVER COLLEGE MATHEMATICS

To access the Hawk A.I. Tutor, you will need to be logged into your campus Gmail account.

We have seen some techniques that allow us to integrate specific rational functions. For example, we know that

$$\int \frac{du}{u} = \ln|u| + C$$

and

$$\int \frac{du}{u^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C.$$

However, we do not yet have a technique that allows us to tackle arbitrary quotients of this type. Thus, it is not immediately obvious how to go about evaluating

$$\int \frac{3x}{x^2 - x - 2}\, dx.$$

However, we know from material previously developed that

$$\int \left(\frac{1}{x+1} + \frac{2}{x-2}\right) dx = \ln|x+1| + 2\ln|x-2| + C.$$

By getting a common denominator, we see that

$$\frac{1}{x+1} + \frac{2}{x-2} = \frac{3x}{x^2 - x - 2}.$$

Consequently,

$$\int \frac{3x}{x^2 - x - 2}\, dx = \int \left(\frac{1}{x+1} + \frac{2}{x-2}\right) dx.$$

In this section, we examine the method of **Partial Fraction Decomposition**, which allows us to decompose rational functions into sums of simpler, more easily integrated rational functions. Using this method, we can rewrite an expression such as:

$$\frac{3x}{x^2 - x - 2}$$

as an expression such as

$$\frac{1}{x+1} + \frac{2}{x-2}.$$

The key to the partial fraction decomposition method is anticipating the form that the decomposition of a rational function will take. As we shall see, this form is both predictable and highly dependent on the factorization of the denominator of the rational function. It is also imperative to keep in mind that partial fraction decomposition can be applied to a rational function $\frac{P(x)}{Q(x)}$ only if $\deg(P(x)) < \deg(Q(x))$. In the case when $\deg(P(x)) \geq \deg(Q(x))$, we must first perform long division to rewrite the quotient $\frac{P(x)}{Q(x)}$ in the form $A(x) + \frac{R(x)}{Q(x)}$, where $\deg(R(x)) < \deg(Q(x))$.[1] We then do a partial fraction decomposition on $\frac{R(x)}{Q(x)}$. The following example, although not requiring partial fraction decomposition, illustrates our approach to integrals of rational functions of the form $\int \frac{P(x)}{Q(x)}\, dx$, where $\deg(P(x)) \geq \deg(Q(x))$.

✔ Example 3.4.1: Evaluating $\int \frac{P(x)}{Q(x)}\, dx$, where $deg(P(x)) \geq deg(Q(x))$

Evaluate

$$\int \frac{x^2 + 3x + 5}{x+1}\, dx.$$

Solution

Since $\deg(x^2 + 3x + 5) \geq \deg(x+1)$, we perform long division to obtain

$$\frac{x^2+3x+5}{x+1} = x+2+\frac{3}{x+1}.$$

Thus,

$$\int \frac{x^2+3x+5}{x+1}\,dx = \int \left(x+2+\frac{3}{x+1}\right)\,dx = \frac{1}{2}x^2+2x+3\ln|x+1|+C.$$

? Checkpoint 3.4.1

Evaluate

$$\int \frac{x-3}{x+2}\,dx.$$

Answer

$$x-5\ln|x+2|+C$$

To integrate $\int \frac{P(x)}{Q(x)}\,dx$, where $\deg(P(x)) < \deg(Q(x))$, we must begin by factoring $Q(x)$.

Nonrepeated Linear Factors

If $Q(x)$ can be factored as $(a_1x+b_1)(a_2x+b_2)\ldots(a_nx+b_n)$, where each linear factor is distinct, then it is possible to find constants A_1, A_2, ..., A_n satisfying the following.[2]

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1}+\frac{A_2}{a_2x+b_2}+\cdots+\frac{A_n}{a_nx+b_n}. \tag{3.4.1}$$

In this next example, we see how to use partial fractions to integrate a rational function of this type.

✓ Example 3.4.2: Partial Fractions with Nonrepeated Linear Factors

Evaluate

$$\int \frac{3x+2}{x^3-x^2-2x}\,dx.$$

Solution

Since $\deg(3x+2) < \deg(x^3-x^2-2x)$, we begin by factoring the denominator of $\frac{3x+2}{x^3-x^2-2x}$.

We can see that $x^3-x^2-2x = x(x-2)(x+1)$. Thus, there are constants A, B, and C satisfying Equation 3.4.1 such that

$$\frac{3x+2}{x(x-2)(x+1)} = \frac{A}{x}+\frac{B}{x-2}+\frac{C}{x+1}.$$

We must now find these constants. To do so, we begin by multiplying both sides of this equation by the least common denominator: $x(x-2)(x+1)$. Thus,

$$3x+2 = A(x-2)(x+1)+Bx(x+1)+Cx(x-2). \tag{3.4.2}$$

There are two different strategies for finding the coefficients A, B, and C. We refer to these as the **method of equating coefficients** and the **method of strategic substitution**.

Strategy One: Method of Equating Coefficients

Rewrite Equation 3.4.2 in the form

$$3x+2 = (A+B+C)x^2+(-A+B-2C)x+(-2A).$$

Equating coefficients produces the system of equations

$$\begin{aligned} A+B+C &= 0 \\ -A+B-2C &= 3 \\ -2A &= 2. \end{aligned}$$

To solve this system, we first observe that $-2A=2 \implies A=-1$. Substituting this value into the first two equations gives us the system

$$\begin{aligned} B+C &= 1 \\ B-2C &= 2. \end{aligned}$$

Multiplying the second equation by -1 and adding the resulting equation to the first produces $-3C=1$, which in turn implies that $C=-\frac{1}{3}$. Substituting this value into the equation $B+C=1$ yields $B=\frac{4}{3}$. Thus, solving these equations yields $A=-1$, $B=\frac{4}{3}$, and $C=-\frac{1}{3}$.

It is important to note that the system produced by this method is consistent if and only if we have set up the decomposition correctly. If the system is inconsistent, there is an error in our decomposition.

Strategy Two: Method of Strategic Substitution

The method of strategic substitution is also based on the assumption that we have set up the decomposition correctly. If the decomposition is set up correctly, then there must be values of A, B, and C that satisfy Equation 3.4.2 for all values of x. This equation must be valid for any value of x we care to substitute into it. Therefore, by choosing values of x carefully and substituting them into the equation, we may find A, B, and C easily.

For example, if we substitute $x=0$, the equation reduces to $2=A(-2)(1)$. Solving for A yields $A=-1$. Next, by substituting $x=2$, the equation reduces to $8=B(2)(3)$, or equivalently $B=\frac{4}{3}$. Last, we substitute $x=-1$ into the equation and obtain $-1=C(-1)(-3)$. Solving, we have $C=-\frac{1}{3}$.

It is important to remember that if we attempt to use this method with a decomposition that has not been set up correctly, we can still find values for the constants, but these constants are meaningless. If we opt to use the method of strategic substitution, then it is a good idea to check the result by recombining the terms algebraically.

Now that we have the values of A, B, and C, we rewrite the original integral:

$$\int \frac{3x+2}{x^3-x^2-2x}\,dx = \int \left(-\frac{1}{x}+\frac{4}{3}\cdot\frac{1}{x-2}-\frac{1}{3}\cdot\frac{1}{x+1}\right)dx.$$

Evaluating the integral gives us

$$\int \frac{3x+2}{x^3-x^2-2x}\,dx = -\ln|x|+\frac{4}{3}\ln|x-2|-\frac{1}{3}\ln|x+1|+C.$$

In the next example, we integrate a rational function in which the degree of the numerator is not less than the degree of the denominator.

Example 3.4.3: Dividing before Applying Partial Fractions

Evaluate

$$\int \frac{x^2+3x+1}{x^2-4}\,dx.$$

Solution

Since $\deg(x^2+3x+1) \geq \deg(x^2-4)$, we must perform long division of polynomials. This results in

$$\frac{x^2+3x+1}{x^2-4} = 1+\frac{3x+5}{x^2-4}.$$

Next, we perform partial fraction decomposition on $\frac{3x+5}{x^2-4} = \frac{3x+5}{(x+2)(x-2)}$. We have

$$\frac{3x+5}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}.$$

Thus,

$$3x+5 = A(x+2) + B(x-2).$$

Solving for A and B using either method, we obtain $A = \frac{11}{4}$ and $B = \frac{1}{4}$.

Rewriting the original integral, we have

$$\int \frac{x^2+3x+1}{x^2-4}\,dx = \int \left(1 + \frac{11}{4}\cdot\frac{1}{x-2} + \frac{1}{4}\cdot\frac{1}{x+2}\right)dx.$$

Evaluating the integral produces

$$\int \frac{x^2+3x+1}{x^2-4}\,dx = x + \frac{11}{4}\ln|x-2| + \frac{1}{4}\ln|x+2| + C.$$

As we see in the following example, it may be possible to apply the technique of partial fraction decomposition to a nonrational function. To do so, we must first convert the nonrational function to a rational function through a substitution.

Example 3.4.4: Applying Partial Fractions after a Substitution

Evaluate

$$\int \frac{\cos x}{\sin^2 x - \sin x}\,dx.$$

Solution

Let's begin by letting $u = \sin x$. Consequently, $du = \cos x\,dx$. After making these substitutions, we have

$$\int \frac{\cos x}{\sin^2 x - \sin x}\,dx = \int \frac{du}{u^2-u} = \int \frac{du}{u(u-1)}.$$

Applying partial fraction decomposition to $\frac{1}{u(u-1)}$ gives $\frac{1}{u(u-1)} = -\frac{1}{u} + \frac{1}{u-1}$.

Thus,

$$\int \frac{\cos x}{\sin^2 x - \sin x}\,dx = -\ln|u| + \ln|u-1| + C = -\ln|\sin x| + \ln|\sin x - 1| + C.$$

Checkpoint 3.4.4

Evaluate

$$\int \frac{x+1}{(x+3)(x-2)}\,dx.$$

Answer

$$\frac{2}{5}\ln|x+3| + \frac{3}{5}\ln|x-2| + C$$

Repeated Linear Factors

For some applications, we need to integrate rational expressions that have denominators with repeated linear factors - that is, rational functions with at least one factor of the form $(ax+b)^n$, where n is a positive integer greater than or equal to 2. If the denominator contains the repeated linear factor $(ax+b)^n$, then the decomposition must contain

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}. \tag{3.4.3}$$

As we see in our next example, the basic technique for solving the coefficients is the same. Still, more algebra is required to determine the numerators of the partial fractions.

Example 3.4.5: Partial Fractions with Repeated Linear Factors

Evaluate

$$\int \frac{x-2}{(2x-1)^2(x-1)}\,dx.$$

Solution

We have $\deg(x-2) < \deg((2x-1)^2(x-1))$, so we can proceed with the decomposition. Since $(2x-1)^2$ is a repeated linear factor, include

$$\frac{A}{2x-1}+\frac{B}{(2x-1)^2}$$

in the decomposition in Equation 3.4.3. Thus,

$$\frac{x-2}{(2x-1)^2(x-1)}=\frac{A}{2x-1}+\frac{B}{(2x-1)^2}+\frac{C}{x-1}.$$

After multiplying both sides by the least common denominator, we have

$$x-2=A(2x-1)(x-1)+B(x-1)+C(2x-1)^2. \tag{3.4.4}$$

We then use the method of equating coefficients to find the values of A, B, and C.

$$x-2=(2A+4C)x^2+(-3A+B-4C)x+(A-B+C).$$

Equating coefficients yields $2A+4C=0$, $-3A+B-4C=1$, and $A-B+C=-2$. Solving this system yields $A=2$, $B=3$, and $C=-1$.

Alternatively, we can use the method of strategic substitution. In this case, substituting $x=1$ and $x=\frac{1}{2}$ into Equation 3.4.4 easily produces the values $B=3$ and $C=-1$. At this point, it may seem that we have run out of good choices for x; however, since we already have values for B and C, we can substitute these values and choose any value for x not previously used. The value $x=0$ is a good option. In this case, we obtain the equation $-2=A(-1)(-1)+3(-1)+(-1)(-1)^2$ or, equivalently, $A=2$.

Now that we have the values for A, B, and C, we rewrite the original integral and evaluate it:

$$\begin{aligned}\int \frac{x-2}{(2x-1)^2(x-1)}\,dx &= \int\left(\frac{2}{2x-1}+\frac{3}{(2x-1)^2}-\frac{1}{x-1}\right)dx\\ &= \ln|2x-1|-\frac{3}{2(2x-1)}-\ln|x-1|+C.\end{aligned}$$

Checkpoint 3.4.5

Set up the partial fraction decomposition for

$$\int \frac{x+2}{(x+3)^3(x-4)^2}\,dx.$$

(Do not solve for the coefficients or complete the integration.)

Answer

$$\frac{x+2}{(x+3)^3(x-4)^2}=\frac{A}{x+3}+\frac{B}{(x+3)^2}+\frac{C}{(x+3)^3}+\frac{D}{(x-4)}+\frac{E}{(x-4)^2}$$

The General Method

Now that we are beginning to understand how the partial fraction decomposition technique works let's outline the basic method in the following problem-solving strategy.

Problem-Solving Strategy: Partial Fraction Decomposition

To decompose the rational function $\frac{P(x)}{Q(x)}$, use the following steps:

1. Make sure that $\deg(P(x)) < \deg(Q(x))$. If not, perform a long division of polynomials.
2. Factor $Q(x)$ into the product of linear and irreducible quadratic factors.[3]
3. Assuming that $\deg(P(x)) < \deg(Q(x))$, the factors of $Q(x)$ determine the form of the decomposition of $\frac{P(x)}{Q(x)}$.
   a. If $Q(x)$ can be factored as $(a_1x+b_1)(a_2x+b_2)\ldots(a_nx+b_n)$, where each linear factor is distinct, then it is possible to find constants $A_1, A_2, \ldots, A_n$ satisfying
   $$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_n}{a_nx+b_n}.$$
   b. If $Q(x)$ contains the repeated linear factor $(ax+b)^n$, then the decomposition must contain
   $$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}.$$
   c. For each irreducible quadratic factor ax^2+bx+c that $Q(x)$ contains, the decomposition must include
   $$\frac{Ax+B}{ax^2+bx+c}.$$
   d. For each repeated irreducible quadratic factor $(ax^2+bx+c)^n$, the decomposition must include
   $$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_nx+B_n}{(ax^2+bx+c)^n}.$$
   e. After determining the appropriate decomposition, solve for the constants.
   f. Last, rewrite the integral in its decomposed form and evaluate it using previously developed techniques or integration formulas.

Irreducible Quadratic Factors

Now, let's consider integrating a rational expression in which the denominator contains an irreducible quadratic factor. Recall that the quadratic ax^2+bx+c is irreducible if $ax^2+bx+c=0$ has no real zeros - that is, if b^2-4ac (the discriminant) is less than 0.

Example 3.4.6: Rational Expressions with an Irreducible Quadratic Factor

Evaluate
$$\int \frac{2x-3}{x^3+x}\,dx.$$

Solution

Since $\deg(2x-3) < \deg(x^3+x)$, factor the denominator and proceed with partial fraction decomposition. In this case, $x^3+x = x(x^2+1)$ contains the irreducible quadratic factor x^2+1. Therefore, we include $\frac{Ax+B}{x^2+1}$ as part of the decomposition, along with $\frac{C}{x}$ for the linear term x. Thus, the decomposition has the form
$$\frac{2x-3}{x(x^2+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x}.$$

After multiplying both sides by the least common denominator, we obtain the equation
$$2x-3 = (Ax+B)x + C(x^2+1).$$

Solving for A, B, and C, we get $A=3$, $B=2$, and $C=-3$.

Thus,

$$\frac{2x-3}{x^3+x}=\frac{3x+2}{x^2+1}-\frac{3}{x}.$$

Substituting back into the integral, we obtain

$$\begin{aligned}\int\frac{2x-3}{x^3+x}\,dx &= \int\left(\frac{3x+2}{x^2+1}-\frac{3}{x}\right)dx\\ &= 3\int\frac{x}{x^2+1}\,dx+2\int\frac{1}{x^2+1}\,dx-3\int\frac{1}{x}\,dx\\ &= \frac{3}{2}\ln\left|x^2+1\right|+2\tan^{-1}x-3\ln|x|+C.\end{aligned}$$

Note: We may rewrite $\ln\left|x^2+1\right|=\ln(x^2+1)$, if we wish to do so, since $x^2+1>0$; however, this is not required.

Example 3.4.7: Partial Fractions with an Irreducible Quadratic Factor

Evaluate

$$\int\frac{dx}{x^3-8}.$$

Solution

We can start by factoring $x^3-8=(x-2)(x^2+2x+4)$. We see that the quadratic factor x^2+2x+4 is irreducible since $2^2-4(1)(4)=-12<0$. Using the decomposition described in the problem-solving strategy, we get

$$\frac{1}{(x-2)(x^2+2x+4)}=\frac{A}{x-2}+\frac{Bx+C}{x^2+2x+4}.$$

Multiplying both sides by the LCD, we arrive at

$$1=A(x^2+2x+4)+(Bx+C)(x-2).$$

Applying either method to find the constants, we get $A=\frac{1}{12}$, $B=-\frac{1}{12}$, and $C=-\frac{1}{3}$.

Rewriting $\int\frac{dx}{x^3-8}$, we have

$$\int\frac{dx}{x^3-8}=\frac{1}{12}\int\frac{1}{x-2}\,dx-\frac{1}{12}\int\frac{x+4}{x^2+2x+4}\,dx.$$

We can see that

$$\int\frac{1}{x-2}\,dx=\ln|x-2|+C,$$

but

$$\int\frac{x+4}{x^2+2x+4}\,dx$$

requires a bit more effort. Let's begin by completing the square on x^2+2x+4 to obtain

$$x^2+2x+4=(x+1)^2+3.$$

By letting $u=x+1$ and consequently $du=dx$, we see that

$$\begin{aligned}\int \frac{x+4}{x^2+2x+4}\,dx &= \int \frac{x+4}{(x+1)^2+3}\,dx \\ &= \int \frac{u+3}{u^2+3}\,du \\ &= \int \frac{u}{u^2+3}\,du + \int \frac{3}{u^2+3}\,du \\ &= \frac{1}{2}\ln\left|u^2+3\right| + \frac{3}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right) + C \\ &= \frac{1}{2}\ln\left|x^2+2x+4\right| + \sqrt{3}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + C\end{aligned}$$

Substituting back into the original integral and simplifying gives

$$\int \frac{dx}{x^3-8} = \frac{1}{12}\ln|x-2| - \frac{1}{24}\ln|x^2+2x+4| - \frac{\sqrt{3}}{12}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + C.$$

Here again, we can drop the absolute value if we wish to do so, since $x^2+2x+4>0$ for all x.

Example 3.4.8: Finding a Volume

Find the volume of the solid of revolution obtained by revolving the region enclosed by the graph of $f(x)=\frac{x^2}{(x^2+1)^2}$ and the x-axis over the interval $[0,1]$ about the y-axis.

Solution

Let's begin by sketching the region to be revolved (see Figure 3.4.1). The sketch shows that the shell method is a good choice for solving this problem.

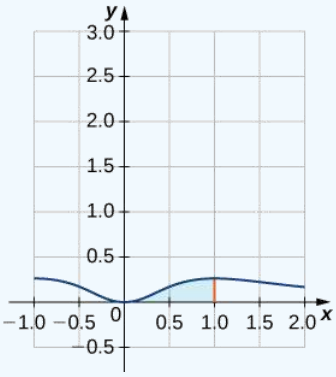


Figure 3.4.1: We can use the shell method to find the volume of revolution obtained by revolving the region shown about the y-axis.

The volume is given by

$$V = 2\pi\int_0^1 x\cdot\frac{x^2}{(x^2+1)^2}\,dx = 2\pi\int_0^1 \frac{x^3}{(x^2+1)^2}\,dx.$$

Since $\deg((x^2+1)^2)=4>3=\deg(x^3)$, we can proceed with partial fraction decomposition. Note that $(x^2+1)^2$ is a repeated irreducible quadratic. Using the decomposition described in the problem-solving strategy, we get

$$\frac{x^3}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}.$$

Multiplying both sides by the LCD gives

$$x^3 = (Ax + B)(x^2 + 1) + Cx + D.$$

Solving, we obtain $A = 1$, $B = 0$, $C = -1$, and $D = 0$. Substituting back into the integral, we have

$$V = 2\pi \int_0^1 \frac{x^3}{(x^2+1)^2}\,dx = 2\pi \int_0^1 \left(\frac{x}{x^2+1} - \frac{x}{(x^2+1)^2}\right) dx = 2\pi \left(\frac{1}{2}\ln(x^2+1) + \frac{1}{2}\cdot\frac{1}{x^2+1}\right)\Big|_0^1 = \pi\left(\ln 2 - \tfrac{1}{2}\right).$$

? Checkpoint 3.4.8

Set up the partial fraction decomposition for

$$\int \frac{x^2+3x+1}{(x+2)(x-3)^2(x^2+4)^2}\,dx.$$

Answer

$$\frac{x^2+3x+1}{(x+2)(x-3)^2(x^2+4)^2} = \frac{A}{x+2} + \frac{B}{x-3} + \frac{C}{(x-3)^2} + \frac{Dx+E}{x^2+4} + \frac{Fx+G}{(x^2+4)^2}$$

Footnotes

[1] Visit this website for a review of long division of polynomials.

[2] The proof that such constants exist is beyond the scope of this course.

[3] An irreducible quadratic is a quadratic that has no real zeros.

3.4E: Exercises

Reading Questions

1. What condition must be met regarding the degrees of the numerator and denominator of a rational function before you can apply the method of partial fraction decomposition? What must you do if this condition is not met?
2. If the denominator of a rational function contains the nonrepeated linear factor $(x-3)$, what is the corresponding term in its partial fraction decomposition?
3. If the denominator contains the repeated linear factor $(x+5)^3$, what three terms must be included in the decomposition corresponding to this factor?
4. If the denominator contains the irreducible quadratic factor (x^2+x+1), what is the form of the corresponding term in the decomposition?
5. What are the two algebraic strategies mentioned in the text for finding the unknown constants in a partial fraction decomposition?
6. What algebraic technique is often required to integrate a term with an irreducible quadratic factor in the denominator?

Use partial fraction decomposition (or a simpler technique) to express the rational function as a sum or difference of two or more simpler rational expressions.

1) $\dfrac{1}{(x-3)(x-2)}$

2) $\dfrac{x^2+1}{x(x+1)(x+2)}$

Answer

$$\frac{x^2+1}{x(x+1)(x+2)} = -\frac{2}{x+1}+\frac{5}{2(x+2)}+\frac{1}{2x}$$

3) $\dfrac{1}{x^3-x}$

4) $\dfrac{3x+1}{x^2}$

Answer

$$\frac{3x+1}{x^2} = \frac{1}{x^2}+\frac{3}{x}$$

5) $\dfrac{3x^2}{x^2+1}$ (Hint: Use long division first.)

6) $\dfrac{2x^4}{x^2-2x}$

Answer

$$\frac{2x^4}{x^2-2x} = 2x^2+4x+8+\frac{16}{x-2}$$

7) $\dfrac{1}{(x-1)(x^2+1)}$

8) $\dfrac{1}{x^2(x-1)}$

Answer

$$\frac{1}{x^2(x-1)} = -\frac{1}{x^2}-\frac{1}{x}+\frac{1}{x-1}$$

9) $\dfrac{x}{x^2-4}$

10) $\dfrac{1}{x(x-1)(x-2)(x-3)}$

Answer

Answer

$$\frac{1}{x(x-1)(x-2)(x-3)} = -\frac{1}{2(x-2)} + \frac{1}{2(x-1)} - \frac{1}{6x} + \frac{1}{6(x-3)}$$

11) $\frac{1}{x^4-1} = \frac{1}{(x+1)(x-1)(x^2+1)}$

12) $\frac{3x^2}{x^3-1} = \frac{3x^2}{(x-1)(x^2+x+1)}$

Answer

$$\frac{3x^2}{x^3-1} = \frac{1}{x-1} + \frac{2x+1}{x^2+x+1}$$

13) $\frac{2x}{(x+2)^2}$

14) $\frac{3x^4+x^3+20x^2+3x+31}{(x+1)(x^2+4)^2}$

Answer

$$\frac{3x^4+x^3+20x^2+3x+31}{(x+1)(x^2+4)^2} = \frac{2}{x+1} + \frac{x}{x^2+4} - \frac{1}{(x^2+4)^2}$$

In exercises 15 - 25, use the method of partial fractions to evaluate each of the following integrals.

15) $\int \frac{dx}{(x-3)(x-2)}$

16) $\int \frac{3x}{x^2+2x-8}\,dx$

Answer

$$\int \frac{3x}{x^2+2x-8}\,dx = 2\ln|x+4| + \ln|x-2| + C = \ln\left|(x+4)^2(x-2)\right| + C$$

17) $\int \frac{dx}{x^3-x}$

18) $\int \frac{x}{x^2-4}\,dx$

Answer

Note that you don't need Partial Fractions here. We use a simple u-substitution.

$$\int \frac{x}{x^2-4}\,dx = \tfrac{1}{2}\ln|4-x^2| + C$$

19) $\int \frac{dx}{x(x-1)(x-2)(x-3)}$

20) $\int \frac{2x^2+4x+22}{x^2+2x+10}\,dx$

Answer

Note that since the degree of the numerator is equal to the degree of the denominator, we need to start with long division. Then note that we will need to use completing the square to continue since we cannot factor the trinomial in the denominator.

$$\int \frac{2x^2+4x+22}{x^2+2x+10}\,dx = 2\left(x + \tfrac{1}{3}\arctan\left(\frac{1+x}{3}\right)\right) + C$$

21) $\int \frac{dx}{x^2-5x+6}$

22) $\int \frac{2-x}{x^2+x}\,dx$

Answer

$$\int \frac{2-x}{x^2+x}\,dx = 2\ln|x| - 3\ln|1+x| + C = \ln\left|\frac{x^2}{(1+x)^3}\right| + C$$

23) $\int \frac{2}{x^2-x-6}\,dx$

24) $\int \frac{dx}{x^3-2x^2-4x+8}$

Answer

$$\int \frac{dx}{x^3-2x^2-4x+8} \quad = \quad \tfrac{1}{16}\left(-\frac{4}{-2+x}-\ln|-2+x|+\ln|2+x|\right)+C = \tfrac{1}{16}\left(-\frac{4}{-2+x}+\ln\left|\frac{x+2}{x-2}\right|\right)+C$$

25) $\int \frac{dx}{x^4-10x^2+9}$

In exercises 26 - 29, evaluate the integrals with irreducible quadratic factors in the denominators.

26) $\int \frac{2}{(x-4)(x^2+2x+6)}\,dx$

Answer

$$\int \frac{2}{(x-4)(x^2+2x+6)}\,dx \quad = \quad \tfrac{1}{30}(-2\sqrt{5}\arctan\left[\frac{1+x}{\sqrt{5}}\right]+2\ln|-4+x|-\ln|6+2x+x^2|)+C$$

27) $\int \frac{x^2}{x^3-x^2+4x-4}\,dx$

28) $\int \frac{x^3+6x^2+3x+6}{x^3+2x^2}\,dx$

Answer

Note that we need to use long division first, since the degree of the numerator is greater than the degree of the denominator.

$$\int \frac{x^3+6x^2+3x+6}{x^3+2x^2}\,dx \quad = \quad -\frac{3}{x}+4\ln|x+2|+x+C$$

29) $\int \frac{x}{(x-1)(x^2+2x+2)^2}\,dx$

In exercises 30 - 32, use the method of partial fractions to evaluate the integrals.

30) $\int \frac{3x+4}{(x^2+4)(3-x)}\,dx$

Answer

$$\int \frac{3x+4}{(x^2+4)(3-x)}\,dx \quad = \quad -\ln|3-x|+\tfrac{1}{2}\ln|x^2+4|+C$$

31) $\int \frac{2}{(x+2)^2(2-x)}\,dx$

32) $\int \frac{3x+4}{x^3-2x-4}\,dx$ (Hint: Use the rational root theorem.)

Answer

$$\int \frac{3x+4}{x^3-2x-4}\,dx \quad = \quad \ln|x-2|-\tfrac{1}{2}\ln|x^2+2x+2|+C$$

In exercises 33 - 46, use substitution to convert the integrals to integrals of rational functions. Then use partial fractions to evaluate the integrals.

33) $\int_0^1 \frac{e^x}{36-e^{2x}}\,dx$ (Give the exact answer and the decimal equivalent. Round to five decimal places.)

34) $\int \frac{e^x\,dx}{e^{2x}-e^x}\,dx$

Answer

$$\int \frac{e^x\,dx}{e^{2x}-e^x}\,dx \quad = \quad -x+\ln|1-e^x|+C$$

35) $\int \frac{\sin x\,dx}{1-\cos^2 x}$

36) $\int \frac{\sin x}{\cos^2 x+\cos x-6}\,dx$

Answer

$$\int \frac{\sin x}{\cos^2 x+\cos x-6}\,dx \quad = \quad \tfrac{1}{5}\ln\left|\frac{\cos x+3}{\cos x-2}\right|+C$$

37) $\int \frac{1-\sqrt{x}}{1+\sqrt{x}}\,dx$

38) $\int \frac{dt}{(e^t-e^{-t})^2}$

Answer

$$\int \frac{dt}{(e^t-e^{-t})^2} \quad = \quad \frac{1}{2-2e^{2t}}+C$$

39) $\int \frac{1+e^x}{1-e^x}\,dx$

40) $\int \frac{dx}{1+\sqrt{x+1}}$

Answer

$$\int \frac{dx}{1+\sqrt{x+1}} \quad = \quad 2\sqrt{1+x}-2\ln|1+\sqrt{1+x}|+C$$

41) $\int \frac{dx}{\sqrt{x}+\sqrt[4]{x}}$

42) $\int \frac{\cos x}{\sin x(1-\sin x)}\,dx$

Answer

$$\int \frac{\cos x}{\sin x(1-\sin x)}\,dx \quad = \quad \ln\left|\frac{\sin x}{1-\sin x}\right|+C$$

43) $\int \frac{e^x}{(e^{2x}-4)^2}\,dx$

44) $\int_1^2 \frac{1}{x^2\sqrt{4-x^2}}\,dx$

Answer

$$\int_1^2 \frac{1}{x^2\sqrt{4-x^2}}\,dx \quad = \quad \frac{\sqrt{3}}{4}$$

45) $\int \frac{1}{2+e^{-x}}\,dx$

46) $\int \frac{1}{1+e^x}\,dx$

Answer

$$\int \frac{1}{1+e^x}\,dx \quad = \quad x-\ln(1+e^x)+C$$

In exercises 47 - 48, use the given substitution to convert the integral to an integral of a rational function, then evaluate.

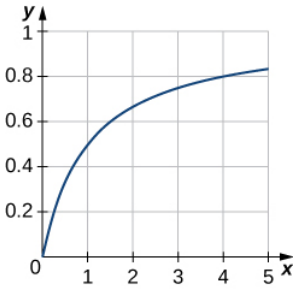
47) $\int \frac{1}{t-\sqrt[3]{t}}\,dt; \quad t=x^3$

48) $\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx; \quad x=u^6$

Answer

$$\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx \quad = \quad 6x^{1/6}-3x^{1/3}+2\sqrt{x}-6\ln(1+x^{1/6})+C$$

49) Graph the curve $y=\dfrac{x}{1+x}$ over the interval $[0,5]$. Then, find the area of the region bounded by the curve, the x-axis, and the line $x=4$.



50) Find the volume of the solid generated when the region bounded by $y=\dfrac{1}{\sqrt{x(3-x)}}$, $y=0$, $x=1$, and $x=2$ is revolved about the x-axis.

Answer

$V=\frac{4}{3}\pi\text{arctanh}\,\left[\frac{1}{3}\right]=\frac{1}{3}\pi\ln 4\,\text{units}^3$

51) The velocity of a particle moving along a line is a function of time given by $v(t)=\dfrac{88t^2}{t^2+1}$. Find the distance that the particle has traveled after $t=5$ sec.

In exercises 52 - 54, solve the initial-value problem for x as a function of t.

52) $(t^2-7t+12)\dfrac{dx}{dt}=1, \quad t>4,\ x(5)=0$

Answer

$x=-\ln|t-3|+\ln|t-4|+\ln 2=\ln\left|\dfrac{2(t-4)}{t-3}\right|$

53) $(t+5)\dfrac{dx}{dt}=x^2+1, \quad t>-5,\ x(1)=\tan 1$

54) $(2t^3-2t^2+t-1)\dfrac{dx}{dt}=3, \quad x(2)=0$

Answer

$x=\ln|t-1|-\sqrt{2}\arctan(\sqrt{2}t)-\frac{1}{2}\ln(t^2+\frac{1}{2})+\sqrt{2}\arctan(2\sqrt{2})+\frac{1}{2}\ln 4.5$

55) Find the x-coordinate of the centroid of the area bounded by $y(x^2-9)=1$, $y=0$, $x=4$, and $x=5$. (Round the answer to two decimal places.)

56) Find the volume generated by revolving the area bounded by $y=\dfrac{1}{x^3+7x^2+6x}$, $x=1$, $x=7$, and $y=0$ about the y-axis.

Answer

$V=\frac{2}{5}\pi\ln\frac{28}{13}\,\text{units}^3$

57) Find the area bounded by $y=\dfrac{x-12}{x^2-8x-20}$, $y=0$, $x=2$, and $x=4$. (Round the answer to the nearest hundredth.)

58) Evaluate the integral $\displaystyle\int\frac{dx}{x^3+1}$.

Answer

$$\int\frac{dx}{x^3+1} \quad = \quad \frac{\arctan\left[\frac{-1+2x}{\sqrt{3}}\right]}{\sqrt{3}}+\frac{1}{3}\ln|1+x|-\frac{1}{6}\ln|\,1-x+x^2\,|+C$$

For problems 59 - 62, use the substitutions $\tan\left(\frac{x}{2}\right)=t$, $dx=\frac{2}{1+t^2}\,dt$, $\sin x=\frac{2t}{1+t^2}$, and $\cos x=\frac{1-t^2}{1+t^2}$.

59) $\int \frac{dx}{3-5\sin x}$

60) Find the area under the curve $y=\frac{1}{1+\sin x}$ between $x=0$ and $x=\pi$. (Assume the dimensions are in inches.)

Answer

2.0 in.^{2}

61) Given $\tan\left(\frac{x}{2}\right)=t$, derive the formulas $dx=\frac{2}{1+t^2}dt$, $\sin x=\frac{2t}{1+t^2}$, and $\cos x=\frac{1-t^2}{1+t^2}$.

62) Evaluate $\int \frac{\sqrt[3]{x-8}}{x}\,dx$.

Answer

$$\int \frac{\sqrt[3]{x-8}}{x}\,dx = 3(-8+x)^{1/3}-2\sqrt{3}\arctan\left[\frac{-1+(-8+x)^{1/3}}{\sqrt{3}}\right]-2\ln\left[2+(-8+x)^{1/3}\right]$$
$$+\ln\left[4-2(-8+x)^{1/3}+(-8+x)^{2/3}\right]+C$$

This page titled 3.4E: Exercises is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Roy Simpson.

- **7.4E: Exercises for Section 7.4** by Edwin “Jed” Herman, Gilbert Strang is licensed CC BY-NC-SA 4.0. Original source: https://openstax.org/details/books/calculus-volume-1.

3.5: Numerical Integration

Note to the Instructor (click to expand)

We finally get to the final frontier of integration techniques. All of this is core content for Calculus II.

To succeed in this section, you'll need to use some skills from previous courses. While you should already know them, this is the first time they've been required. You can review these skills in CRC's Corequisite Codex. If you have a support class, it might cover some, **but not all**, of these topics.

Prerequisite Skills and Support Topics (click to expand)

- **Graphing Functions**
  - Local and Absolute Extrema: Finding the error bounds for the Midpoint, Trapezoidal, and Simpson's Rules requires finding the absolute maximum value of a function's second or fourth derivative over a closed interval

---

The following is a list of learning objectives for this section.

Learning Objectives (click to expand)

- Approximate the value of a definite integral by using the Midpoint and Trapezoidal Rules.
- Determine the absolute and relative error in using a numerical integration technique.
- Estimate the absolute and relative error using an error-bound formula.
- Recognize when the midpoint and Trapezoidal Rules over- or underestimate the true value of an integral.
- Use Simpson's Rule to approximate the value of a definite integral to a given accuracy.

Hawk A.I. Section-Specific Tutor



To access the Hawk A.I. Tutor, you will need to be logged into your campus Gmail account.

The antiderivatives of many functions either cannot be expressed or cannot be expressed easily in **closed form** (that is, in terms of known functions). Consequently, rather than directly evaluating definite integrals of these functions, we resort to various **numerical integration** techniques to approximate their values. In this section, we explore several of these techniques. In addition, we examine the process of estimating the error when using these techniques.

The Midpoint Rule

Earlier in this text, we defined the definite integral of a function over an interval as the limit of Riemann sums. In general, any Riemann sum of a function $f(x)$ over an interval $[a, b]$ may be viewed as an estimate of $\int_a^b f(x)\,dx$. Recall that a Riemann sum of a function $f(x)$ over an interval $[a, b]$ is obtained by selecting a partition

$$P = \{x_0, x_1, x_2, \ldots, x_n\},$$

where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, and a set

$$S = \{x_1^*, x_2^*, \ldots, x_n^*\},$$

where $x_{i-1} \le x_i^* \le x_i$ for all $i = 1, 2, \ldots, n$.

The Riemann sum corresponding to the partition P and the set S is given by

$$\sum_{i=1}^{n} f(x_i^*)\Delta x_i,$$

where $\Delta x_i = x_i - x_{i-1}$ is the length of the ith subinterval.

In Differential Calculus, we learned the Left Endpoint and Right Endpoint approximation methods for estimating the value of a definite integral. The **Midpoint Rule** for estimating the value of a definite integral uses a Riemann sum with subintervals of equal width and the midpoints, m_i, of each subinterval in place of x_i^*. Formally, we state a theorem regarding the convergence of the Midpoint Rule as follows.

Theorem: Midpoint Rule

Let $f(x)$ be continuous on $[a, b]$, n be a positive integer, and $\Delta x = \frac{b-a}{n}$. If $[a, b]$ is divided into n subintervals, each of length Δx, and $m_i = \frac{x_{i-1}+x_i}{2}$ is the midpoint of the ith subinterval, set

$$M_n = \sum_{i=1}^{n} f(m_i)\Delta x = \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x.$$

Then

$$\lim_{n\to\infty} M_n = \int_a^b f(x)\,dx.$$

As we can see in Figure 3.5.1, if $f(x) \geq 0$ over $[a, b]$, then $\sum_{i=1}^{n} f(m_i)\Delta x$ corresponds to the sum of the areas of rectangles approximating the area between the graph of $f(x)$ and the x-axis over $[a, b]$. The graph shows the rectangles corresponding to M_4 for a nonnegative function over a closed interval $[a, b]$.

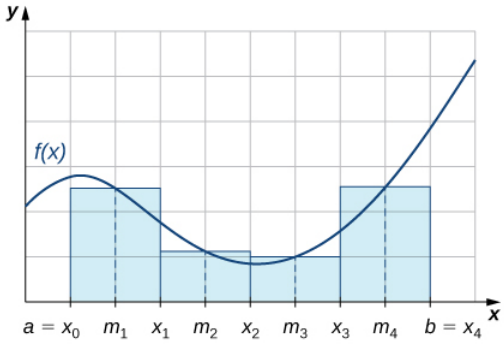


Figure 3.5.1: The Midpoint Rule approximates the area between the graph of $f(x)$ and the x-axis by summing the areas of rectangles with midpoints that are points on $f(x)$.

If we are lucky enough to know the exact value of a definite integral, then we can define the absolute error of an estimation as follows:

Definition: Absolute Error

The **absolute error**, E_*, of a numerical approximation, A, to the true value of the quantity, T, is defined to be

$$E_* = |T - A|.$$

The notation "$*$" is reserved for the method being used. For example, the absolute error if using the Midpoint Rule with $n = 10$ to approximate the value of a definite integral is

$$E_M = |T - M_{10}|.$$

Example 3.5.1: Using the Midpoint Rule with M_4

Use the Midpoint Rule to estimate

$$\int_0^1 x^2\,dx$$

using four subintervals. Compare the result with the actual value of this integral.

Solution

Each subinterval has length $\Delta x = \frac{1-0}{4} = \frac{1}{4}$. Therefore, the subintervals consist of

$$\left[0, \frac{1}{4}\right], \left[\frac{1}{4}, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{3}{4}\right], \text{ and } \left[\frac{3}{4}, 1\right].$$

The midpoints of these subintervals are $\left\{\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}\right\}$. Thus,

$$\begin{aligned} M_4 &= \frac{1}{4}\cdot f\left(\frac{1}{8}\right) + \frac{1}{4}\cdot f\left(\frac{3}{8}\right) + \frac{1}{4}\cdot f\left(\frac{5}{8}\right) + \frac{1}{4}\cdot f\left(\frac{7}{8}\right) \\ &= \frac{1}{4}\cdot\frac{1}{64} + \frac{1}{4}\cdot\frac{9}{64} + \frac{1}{4}\cdot\frac{25}{64} + \frac{1}{4}\cdot\frac{49}{64} \\ &= \frac{21}{64} \\ &= 0.328125. \end{aligned}$$

Since the true value of this definite integral is

$$T = \int_0^1 x^2\,dx = \frac{1}{3},$$

the absolute error in this approximation is

$$E_M = |T - M_4| = \left|\frac{1}{3} - \frac{21}{64}\right| = \frac{1}{192} \approx 0.0052,$$

and we see that the Midpoint Rule produces an estimate that is somewhat close to the actual value of the definite integral.

Example 3.5.2: Using the Midpoint Rule with M_6

Use M_6 to estimate the length of the curve

$$y = \frac{1}{2}x^2,$$

on $[1, 4]$.

Solution

The length of $y = \frac{1}{2}x^2$ on $[1, 4]$ is

$$s = \int_1^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx.$$

Since $\frac{dy}{dx} = x$, this integral becomes $\int_1^4 \sqrt{1 + x^2}\,dx$.

If $[1, 4]$ is divided into six subintervals, then each subinterval has length $\Delta x = \frac{4-1}{6} = \frac{1}{2}$. The midpoints of the subintervals are $\left\{\frac{5}{4}, \frac{7}{4}, \frac{9}{4}, \frac{11}{4}, \frac{13}{4}, \frac{15}{4}\right\}$. If we set $f(x) = \sqrt{1 + x^2}$, we get

$$\begin{aligned} M_6 &= \frac{1}{2}\cdot f\left(\frac{5}{4}\right)+\frac{1}{2}\cdot f\left(\frac{7}{4}\right)+\frac{1}{2}\cdot f\left(\frac{9}{4}\right)+\frac{1}{2}\cdot f\left(\frac{11}{4}\right)+\frac{1}{2}\cdot f\left(\frac{13}{4}\right)+\frac{1}{2}\cdot f\left(\frac{15}{4}\right) \\ &\approx \frac{1}{2}(1.6008+2.0156+2.4622+2.9262+3.4004+3.8810) \\ &= 8.1431 \text{ units.} \end{aligned}$$

? Checkpoint 3.5.2

Use M_2 to estimate

$$\int_1^2 \frac{1}{x}\,dx.$$

Answer

$\frac{24}{35} \approx 0.685714$

The Trapezoidal Rule

We can also approximate the value of a definite integral by using trapezoids rather than rectangles (see Figure 3.5.2).

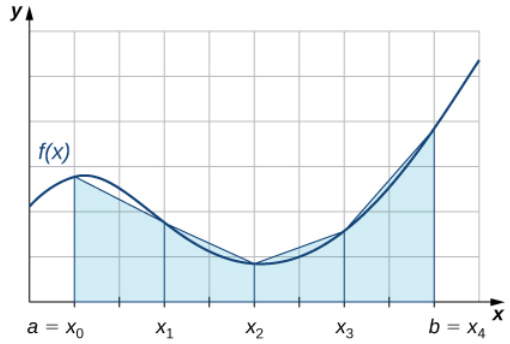


Figure 3.5.2: Trapezoids may be used to approximate the area under a curve, hence approximating the definite integral.

The **Trapezoidal Rule** for estimating definite integrals uses trapezoids rather than rectangles to approximate the area under a curve. Consider the trapezoids in Figure 3.5.2 to gain insight into the rule's final form. We assume that the length of each subinterval is given by Δx. First, recall that the area of a trapezoid with a height of h and bases of length b_1 and b_2 is given by

$$\text{Area} = \frac{1}{2}h(b_1+b_2).$$

We see that the first trapezoid has a "height" Δx and parallel "bases" of length $f(x_0)$ and $f(x_1)$. Thus, the area of the first trapezoid in Figure 3.5.2 is

$$\frac{1}{2}\Delta x\left(f(x_0)+f(x_1)\right).$$

The areas of the remaining three trapezoids are

$$\frac{1}{2}\Delta x\left(\left(f(x_1)+f(x_2)\right)\right),\ \frac{1}{2}\Delta x\left(f(x_2)+f(x_3)\right),\ \text{and}\ \frac{1}{2}\Delta x\left(f(x_3)+f(x_4)\right).$$

Consequently,

$$\begin{aligned} &\int_a^b f(x)\,dx \\ &\approx \frac{1}{2}\Delta x\left(f(x_0)+f(x_1)\right)+\frac{1}{2}\Delta x\left(f(x_1)+f(x_2)\right)+\frac{1}{2}\Delta x\left(f(x_2)+f(x_3)\right)+\frac{1}{2}\Delta x\left(f(x_3)+f(x_4)\right). \end{aligned}$$

After taking out a common factor of $\frac{1}{2}\Delta x$ and combining like terms, we have

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\left[f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+f(x_4)\right].$$

Generalizing, we formally state the following rule.

Theorem: The Trapezoidal Rule

Let $f(x)$ be continuous over $[a,b]$, n be a positive integer, and $\Delta x = \frac{b-a}{n}$. If $[a,b]$ is divided into n subintervals, each of length Δx, with endpoints at $P = \{x_0, x_1, x_2, \ldots, x_n\}$, set

$$T_n = \frac{\Delta x}{2}\left[f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)\right].$$

Then,

$$\lim_{n\to\infty} T_n = \int_a^b f(x)\,dx.$$

Before continuing, let's make a few observations about the Trapezoidal Rule. First of all, it is useful to note that

$$T_n = \frac{1}{2}(L_n + R_n), \text{ where } L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x \text{ and } R_n = \sum_{i=1}^{n} f(x_i)\Delta x.$$

That is, the Trapezoidal Rule is the average of the Left Endpoint Approximation, L_n, and the Right Endpoint Approximation, R_n. In addition, a careful examination of Figure **3.5.3** (see below) leads us to make the following observations about using the Trapezoidal Rules and Midpoint Rules to estimate the definite integral of a nonnegative function. The Trapezoidal Rule tends to overestimate the value of a definite integral systematically over intervals where the function is concave up and to underestimate the value systematically over intervals where the function is concave down. On the other hand, the Midpoint Rule averages out these errors somewhat by partially overestimating and underestimating the value of the definite integral over these same types of intervals. This leads us to hypothesize that, in general, the Midpoint Rule tends to be more accurate than the Trapezoidal Rule.

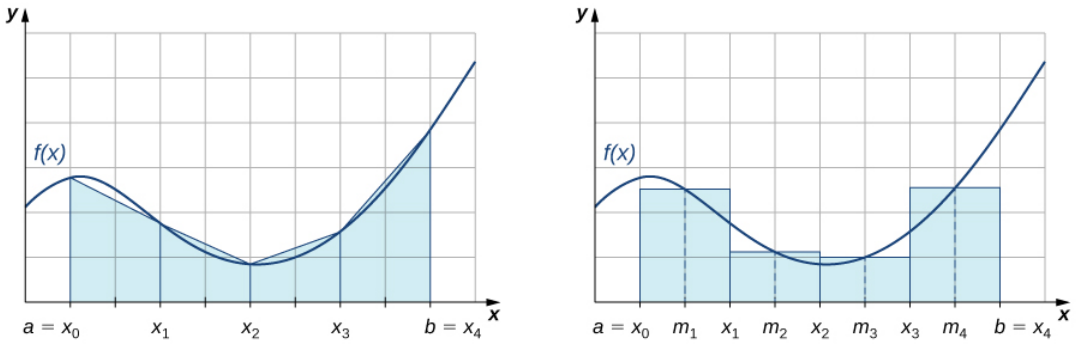


Figure **3.5.3**: The Trapezoidal Rule tends to be less accurate than the Midpoint Rule.

Example 3.5.3: Using the Trapezoidal Rule

Use the Trapezoidal Rule to estimate

$$\int_0^1 x^2\,dx$$

using four subintervals.

Solution

The endpoints of the subintervals consist of elements of the set $P = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ and $\Delta x = \frac{1-0}{4} = \frac{1}{4}$. Thus,

$$\begin{aligned}\int_0^1 x^2\,dx &\approx \frac{1}{2}\cdot\frac{1}{4}\left[f(0)+2f\left(\frac{1}{4}\right)+2f\left(\frac{1}{2}\right)+2f\left(\frac{3}{4}\right)+f(1)\right]\\ &= \frac{1}{8}\left(0+2\cdot\frac{1}{16}+2\cdot\frac{1}{4}+2\cdot\frac{9}{16}+1\right)\\ &= \frac{11}{32}\\ &= 0.34375\end{aligned}$$

Checkpoint 3.5.3

Use T_2 to estimate

$$\int_1^2 \frac{1}{x}\,dx.$$

Answer

$\frac{17}{24} \approx 0.708333$

Absolute and Relative Error

An important aspect to consider when using numerical integration techniques is calculating their errors from the true value of their associated definite integrals. As mentioned, a method's absolute error is given by

$$E_* = |T - A|\,.$$

Another important measure is the **relative error** of an approximation.

Definition: Relative Error

The **relative error** of an approximation is the error as a percentage of the actual value and is given by

$$\left|\frac{T - A_*}{T}\right| \cdot 100\% = \frac{E_*}{|T|} \cdot 100\%.$$

Example 3.5.4: Calculating Error in the Midpoint Rule

Calculate the relative error in the estimate of $\int_0^1 x^2\,dx$ using the Midpoint Rule, found in Example 3.5.1.

Solution

We computed the absolute error to be $E_M = \frac{1}{192} \approx 0.0052$. Therefore, the relative error is

$$\left|\frac{T - M_4}{T}\right| = \frac{E_M}{|T|} = \frac{1/192}{1/3} = \frac{1}{64} \approx 0.015625 \approx 1.6\%.$$

Example 3.5.5: Calculating Error in the Trapezoidal Rule

Calculate the absolute and relative error in the estimate of $\int_0^1 x^2\,dx$ using the Trapezoidal Rule, found in Example 3.5.3.

Solution

The true value is $\int_0^1 x^2\,dx = \frac{1}{3}$, and our estimate from the example is $T_4 = \frac{11}{32}$. Thus, the absolute error is given by

$$E_T = |T - T_4| = \left|\frac{1}{3} - \frac{11}{32}\right| = \frac{1}{96} \approx 0.0104.$$

The relative error is given by

$$\left|\frac{T - T_4}{T}\right| = \frac{E_T}{|T|} = \frac{1/96}{1/3} = 0.03125 \approx 3.1\%.$$

? Checkpoint 3.5.5

In an earlier checkpoint, we estimated $\int_1^2 \frac{1}{x}\,dx$ to be $\frac{24}{35}$ using M_2. The actual value of this integral is $\ln 2$. Using $\frac{24}{35} \approx 0.6857$ and $\ln 2 \approx 0.6931$, calculate the absolute error and the relative error.

Answer

The absolute error is ≈ 0.0074, and the relative error is $\approx 1.1\%$.

Error Bounds on the Midpoint and Trapezoidal Rules

In the two previous examples, we could compare our estimate of an integral with the actual value of the integral; however, we do not typically have this luxury. In general, if we are approximating an integral, we are doing so because we cannot easily compute the exact value of the integral. Therefore, determining an upper bound for the error in an approximation of an integral is often helpful. The following theorem provides error bounds for the Midpoint and Trapezoidal Rules. The theorem is stated without proof.[1]

Theorem: Error Bounds for the Midpoint and Trapezoidal Rules

Let $f(x)$ be a continuous function over $[a, b]$, where $f''(x)$ exists over this same interval interval. If M is the maximum value of $|f''(x)|$ over $[a, b]$, then the upper bounds for the error in using M_n and T_n to estimate $\int_a^b f(x)\,dx$ are

$$E_M \le \frac{M(b-a)^3}{24n^2} \tag{3.5.1}$$

and

$$E_T \le \frac{M(b-a)^3}{12n^2}. \tag{3.5.2}$$

We can use these bounds to determine the value of n necessary to guarantee that the error in an estimate is less than a specified value.

✓ Example 3.5.6: Determining the Number of Intervals to Use

What value of n should be used to guarantee that an estimate of

$$\int_0^1 e^{x^2}\,dx$$

is accurate to within 0.01 if we use the Midpoint Rule?

Solution

We begin by determining the value of M, the maximum value of $|f''(x)|$ over $[0, 1]$ for $f(x) = e^{x^2}$.

Since $f'(x) = 2xe^{x^2}$, we have

$$f''(x) = 2e^{x^2} + 4x^2e^{x^2}.$$

Thus,

$$|f''(x)| = 2e^{x^2}(1+2x^2) \le 2 \cdot e \cdot 3 = 6e.$$

From the error-bound Equation 3.5.1, we have

$$E_M \le \frac{M(b-a)^3}{24n^2} \le \frac{6e(1-0)^3}{24n^2} = \frac{6e}{24n^2}.$$

Now we solve the following inequality for n:

$$\frac{6e}{24n^2} \le 0.01.$$

Thus, $n \ge \sqrt{\frac{600e}{24}} \approx 8.24$. Since n must be an *integer* satisfying this inequality, a choice of $n = 9$ would guarantee that

$$E_M = |T - M_n| = \left| \int_0^1 e^{x^2}\,dx - M_n \right| < 0.01.$$

Analysis

We might have been tempted to round 8.24 down and choose $n = 8$, but this would be incorrect because we must have an integer greater than or equal to 8.24. We must remember that the error estimates provide an upper bound only for the error. The actual estimate may be a much better approximation than the error bound indicates.

? Checkpoint 3.5.6

Use Equation 3.5.1 to find an upper bound for the error in using M_4 to estimate $\int_0^1 x^2\,dx$.

Answer

$\frac{1}{192}$

Simpson's Rule

With the Midpoint Rule, we estimated areas of regions under curves by using rectangles. In a sense, we approximated the curve with piecewise *constant* functions. With the Trapezoidal Rule, we approximated the curve using piecewise *linear* functions. What if we were to approximate a curve using piecewise quadratic functions instead? With **Simpson's Rule,** we do just this. Consider the function in Figure 3.5.4.

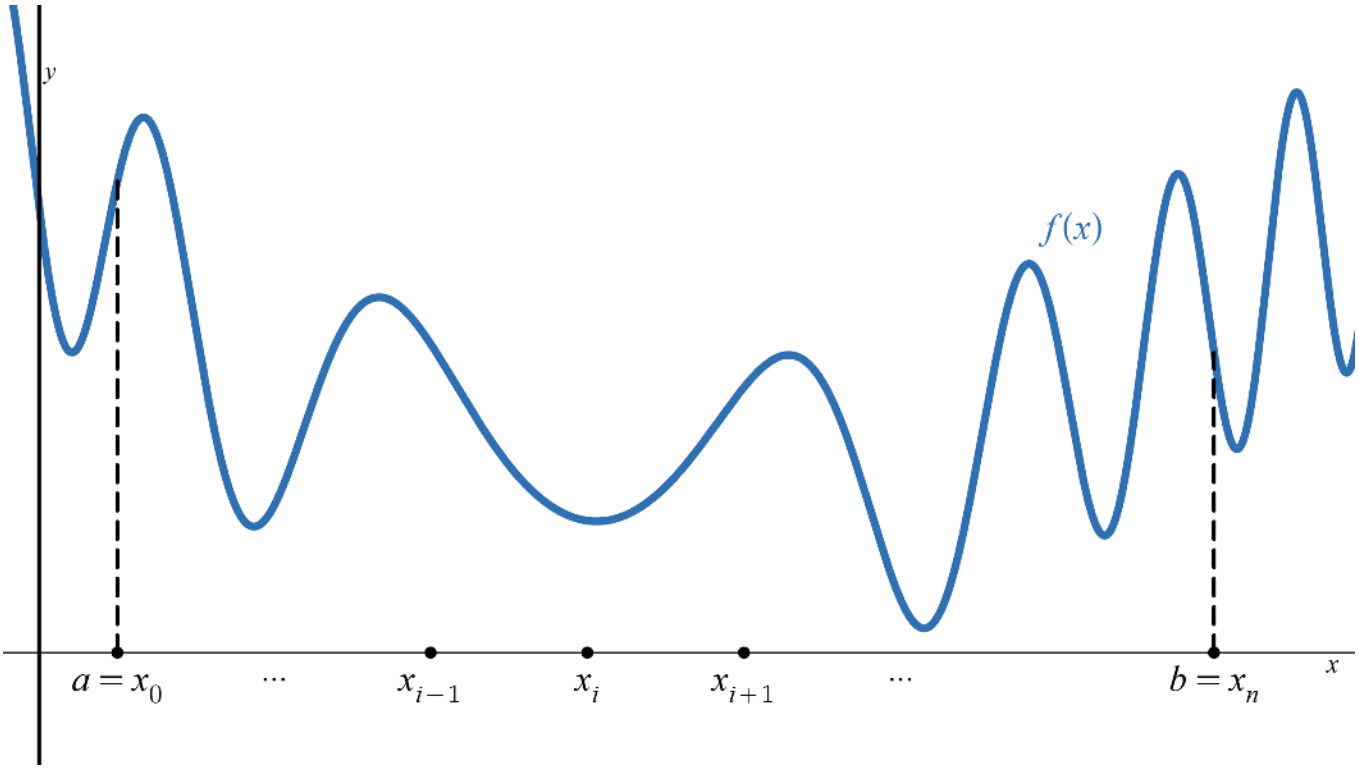


Figure 3.5.4

We partition the interval into an *even* number of subintervals of equal width (the necessity of the even number of subintervals will be revealed shortly). If we wish to string together quadratic functions to help us approximate the area under this curve, we face a new challenge - the Left Endpoint, Right Endpoint, Midpoint, and Trapezoidal Rules each required "tops" to our slices that were linear functions (horizontal lines in all cases but the Trapezoidal Rule). Since we only need two points to build a line, we only needed our subintervals to include neighboring points in the partition - namely, $(x_{i-1}, f(x_{i-1}))$ and $(x_i, f(x_i))$. This is why the ith subinterval was $[x_{i-1}, x_i]$ in each of these methods; however, quadratic functions graph as parabolas, and from our Algebra, we know we need *three* points to find the equation of a parabola. Hence, we need an "expanded" interval containing *three* neighboring points to uniquely find the approximating parabola, as see in Figure 3.5.5.

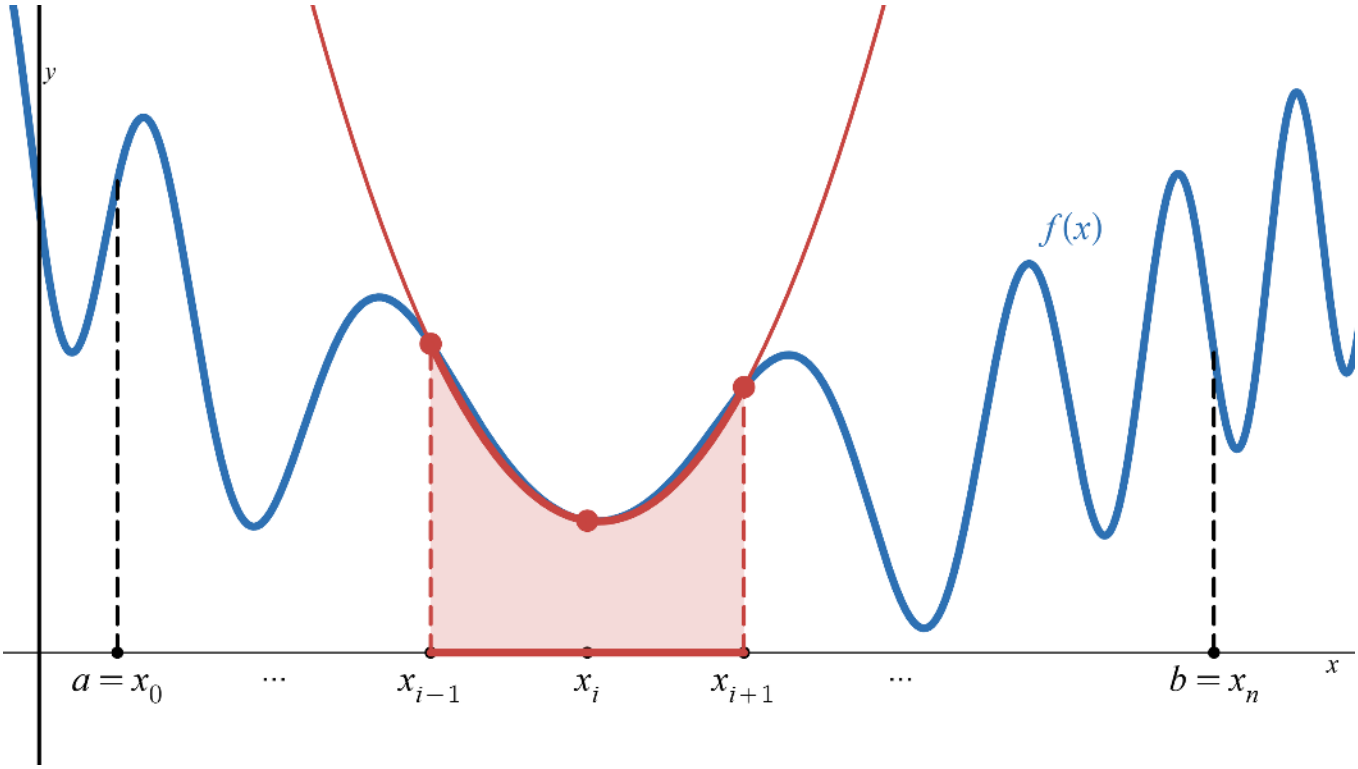


Figure 3.5.5

Our goal is to derive a formula for the area of this i^{th} slice over the interval $[x_{i-1}, x_{i+1}]$. Before continuing, let's make sure we all agree that the red points in Figure 3.5.5 are $(x_{i-1}, f(x_{i-1}))$, $(x_i, f(x_i))$, and $(x_{i+1}, f(x_{i+1}))$. For notational simplicity, let's replace those function values with a cleaner notation - (x_{i-1}, y_{i-1}), (x_i, y_i), and (x_{i+1}, y_{i+1}).

There are several ways to approach this, however, we will use the simple concept that shifting a function left or right does not affect the area between the function and the x-axis (another way to say this is that the area under the curve on a given interval is **invariant** under horizontal translations). Therefore, we shift the function so $x_i = 0$ and temporarily let $h = \Delta x$. Figure 3.5.6 reflects this change.

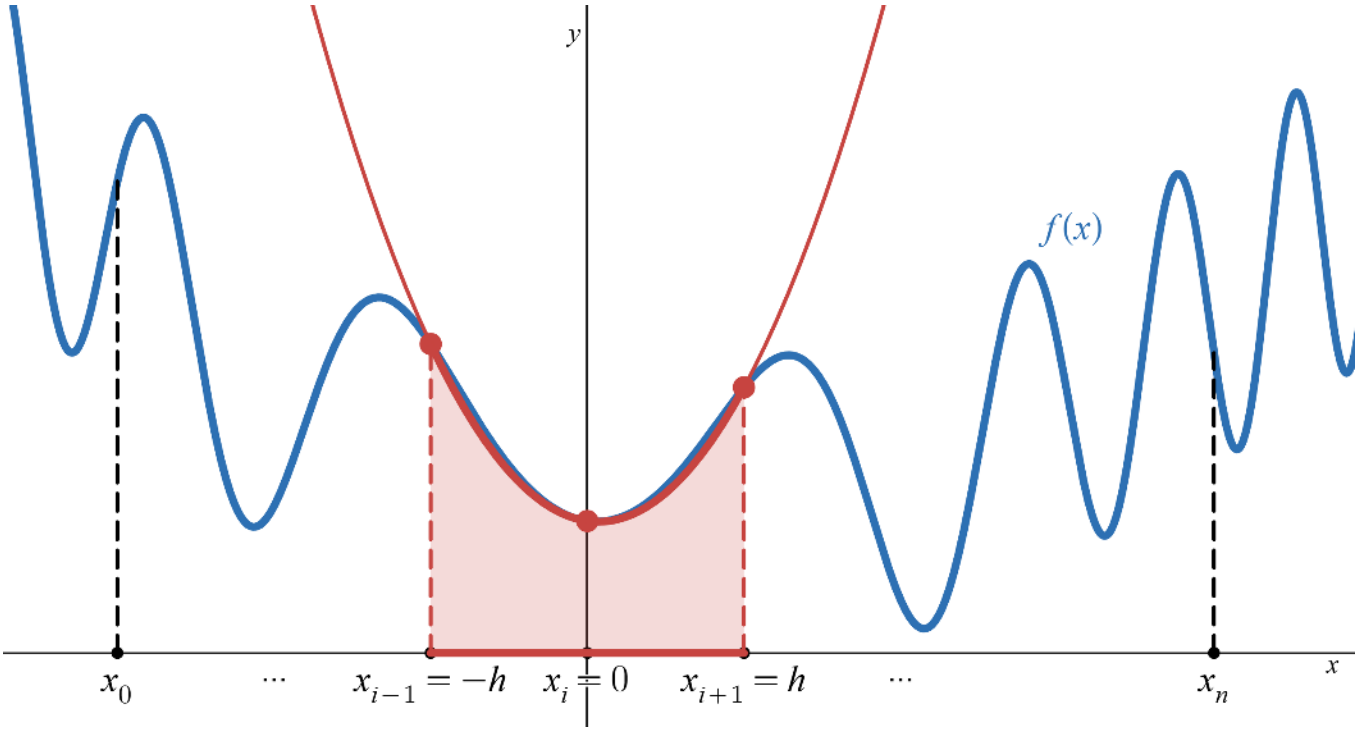


Figure 3.5.6

As mentioned, computationally, this shift has no effect; however, it simplifies our derivation quite a bit. The most important part we note is that the y-values of those points are still y_{i-1}, y_i, and y_{i+1}. Thus, the three red points in Figure 3.5.6 are $(-h, y_{i-1})$, $(0, y_i)$, and (h, y_{i+1}). We know the equation of a parabola is

$$y = Ax^2 + Bx + C.$$

Substituting in the three points, we get the equations

$$\begin{aligned} y_{i-1} &= Ah^2 - Bh + C \\ y_i &= C \\ y_{i+1} &= Ah^2 + Bh + C \end{aligned} \tag{3.5.3}$$

We now turn our focus to computing the approximated area under f on the interval $[x_{i-1}, x_{i+1}]$.

$$\text{Area of the } i^{\text{th}} \text{ slice} \approx \int_{x_{i-1}}^{x_{i+1}} Ax^2 + Bx + C\,dx = \int_{-h}^{h} Ax^2 + Bx + C\,dx$$
$$= \left(\frac{A}{3}x^3 + \frac{B}{2}x^2 + Cx\right)\Big|_{-h}^{h} = \frac{2A}{3}h^3 + 2Ch = \frac{h}{3}\left(2Ah^2 + 6C\right).$$

A little ingenuity reveals that the expression $2Ah^2 + 6C$ is a combination of the equations in 3.5.3. Specifically, if you took one of the first equation, four of the second, and one of the third, you would get

$$\begin{aligned} & y_{i-1} &= Ah^2 - Bh + C \\ & 4y_i &= 4C \\ + \quad & y_{i+1} &= Ah^2 + Bh + C \\ \hline & y_{i-1} + 4y_i + y_{i+1} &= 2Ah^2 + 6C \end{aligned}$$

Does this look familiar? I hope so because we just spent a little while deriving that right side! This leads to a clean approximation to the area of the ith slice:

$$\text{Area of the } i^{\text{th}} \text{ slice} \approx \frac{h}{3}\left(2Ah^2 + 6C\right) = \frac{h}{3}\left(y_{i-1} + 4y_i + y_{i+1}\right) = \frac{h}{3}\left(f(x_{i-1}) + 4f(x_i) + f(x_{i+1})\right).$$

Now that we have an explicit formula for the area of the ith slice, we can use a Riemann sum to approximate the area under the curve from x_0 to x_n. Before we setup the summation, however, it is important to visualize that the edges of the ith slice occur at $x = x_{i-1}$ and x_{i+1}. Hence, our summation will go from $i = 1$ to $i = n - 1$ (the $(n-1)^{\text{st}}$ slice having x_n as its rightmost edge). Moreover, the center of the first slice is at x_1, the second at x_3, the third at x_5, and so on. The rightmost edge of the entire interval must be x_n, where n is even (thus, the requirement for an even number of slices). Thus,

$$\begin{aligned} A &\approx \left[\frac{h}{3}\left(f(x_0)+4f(x_1)+f(x_2)\right)+\frac{h}{3}\left(f(x_2)+4f(x_3)+f(x_4)\right)\right. \\ &\quad +\frac{h}{3}\left(f(x_4)+4f(x_5)+f(x_6)\right)+\cdots \\ &\quad \left.+\frac{h}{3}\left(f(x_{n-4})+4f(x_{n-3})+f(x_{n-2})\right)+\frac{h}{3}\left(f(x_{n-2})+4f(x_{n-1})+f(x_n)\right)\right] \\ &= \frac{h}{3}\left[\left(f(x_0)+4f(x_1)+f(x_2)\right)+\left(f(x_2)+4f(x_3)+f(x_4)\right)+\left(f(x_4)+4f(x_5)+f(x_6)\right)+\cdots\right. \\ &\quad \left.+\left(f(x_{n-4})+4f(x_{n-3})+f(x_{n-2})\right)+\left(f(x_{n-2})+4f(x_{n-1})+f(x_n)\right)\right] \\ &= \frac{\Delta x}{3} \\ &\quad \left[\left(f(x_0)+4f(x_1)+f(x_2)\right)+\left(f(x_2)+4f(x_3)+f(x_4)\right)+\left(f(x_4)+4f(x_5)+f(x_6)\right)+\cdots\right. \\ &\quad \left.+\left(f(x_{n-4})+4f(x_{n-3})+f(x_{n-2})\right)+\left(f(x_{n-2})+4f(x_{n-1})+f(x_n)\right)\right] \\ &= \frac{\Delta x}{3}\left(f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\right) \end{aligned}$$

Other than the outer edges of the main interval, the doubling of the function values at even indexed partition values occurs because the edges of *two* parabolas meet there and their function values get double-counted. The general rule may be stated as follows.

Theorem: Simpson's Rule

Assume that $f(x)$ is continuous over $[a, b]$. Let n be a positive even integer and $\Delta x = \frac{b-a}{n}$. Let $[a, b]$ be divided into n subintervals, each of length Δx, with endpoints at $P = \{x_0, x_1, x_2, \ldots, x_n\}$. Set

$$S_n = \frac{\Delta x}{3}\left[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\right].$$

Then,

$$\lim_{n\to\infty} S_n = \int_a^b f(x)\,dx.$$

Just as the Trapezoidal Rule is the average of the Left Endpoint and Right Endpoint approximations for estimating definite integrals, Simpson's Rule may be obtained using a weighted average from the Midpoint and Trapezoidal Rules. It can be shown that $S_{2n} = \left(\frac{2}{3}\right) M_n + \left(\frac{1}{3}\right) T_n$.

It is also possible to put a bound on the error when using Simpson's Rule to approximate a definite integral. The bound in the error is given by the following rule:

Theorem: Error Bound for Simpson's Rule

Let $f(x)$ be a continuous function over $[a, b]$ having a fourth derivative, $f^{(4)}(x)$, over this interval. If M is the maximum value of $\left|f^{(4)}(x)\right|$ over $[a, b]$, then the upper bound for the error in using S_n to estimate $\int_a^b f(x)\,dx$ is given by

$$E_S \leq \frac{M(b-a)^5}{180n^4}.$$

Example 3.5.7: Applying Simpson's Rule 1

Use S_2 to approximate

$$\int_0^1 x^3\,dx.$$

Estimate a bound for the error in S_2.

Solution

Since $[0,1]$ is divided into two intervals, each subinterval has length $\Delta x = \frac{1-0}{2} = \frac{1}{2}$. The endpoints of these subintervals are $\left\{0, \frac{1}{2}, 1\right\}$. If we set $f(x) = x^3$, then

$$S_2 = \frac{1/2}{3}\left(f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right) = \frac{1}{6}\left(0 + 4\cdot\frac{1}{8} + 1\right) = \frac{1}{4}.$$

Since $f^{(4)}(x) = 0$ and consequently $M = 0$, we see that

$$E_S \leq \frac{0(1)^5}{180\cdot 2^4} = 0.$$

This bound indicates that the value obtained through Simpson's Rule is exact. A quick check will verify that, in fact, $\int_0^1 x^3\,dx = \frac{1}{4}$.

Example 3.5.8: Applying Simpson's Rule 2

Use S_6 to estimate the length of the curve

$$y = \frac{1}{2}x^2$$

over $[1,4]$

Solution

The length of $y = \frac{1}{2}x^2$ over $[1,4]$ is $\int_1^4 \sqrt{1+x^2}\,dx$. If we divide $[1,4]$ into six subintervals, then each subinterval has length $\Delta x = \frac{4-1}{6} = \frac{1}{2}$, and the endpoints of the subintervals are $\left\{1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4\right\}$. Setting $f(x) = \sqrt{1+x^2}$,

$$S_6 = \frac{1/2}{3}\left(f(1) + 4f\left(\frac{3}{2}\right) + 2f(2) + 4f\left(\frac{5}{2}\right) + 2f(3) + 4f\left(\frac{7}{2}\right) + f(4)\right).$$

After substituting, we have

$$S_6 = \frac{1}{6}\left(1.4142 + 4\cdot 1.80278 + 2\cdot 2.23607 + 4\cdot 2.69258 + 2\cdot 3.16228 + 4\cdot 3.64005 + 4.12311\right) \approx 8.14594 \text{ units}.$$

Checkpoint 3.5.8

Use S_2 to estimate

$$\int_1^2 \frac{1}{x}\,dx.$$

Answer

$\frac{25}{36} \approx 0.694444$

Footnotes

[1] The exploration of numerical methods, their error bounds, and the trade-off between the accuracy of the method and its computational expense is the realm of a subfield of Mathematics called Numerical Analysis. If you are intrigued by how such methods (and their errors) are developed, I recommend taking a series of upper-division courses in Numerical Analysis.

This page titled 3.5: Numerical Integration is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Roy Simpson.

3.5E: Exercises

Reading Questions

1. What is the primary reason for using numerical integration methods like the Midpoint, Trapezoidal, or Simpson's Rule?
2. What geometric shapes are used to approximate the area under a curve in the Trapezoidal Rule and Simpson's Rule, respectively?
3. What is the difference between absolute error and relative error in a numerical approximation?
4. The error bound formula for the Midpoint Rule depends on finding the maximum value of which derivative of the function being integrated?
5. What specific requirement does Simpson's Rule have for the number of subintervals, n?
6. The Trapezoidal Rule approximation, T_n, can be found by taking the average of which two other Riemann sum approximations?
7. For a function that is concave up over an interval, will the Trapezoidal Rule produce an overestimate or an underestimate of the definite integral?

In exercises 1 - 5, approximate the following integrals using either the Midpoint Rule, Trapezoidal Rule, or Simpson's Rule as indicated. (Round answers to three decimal places.)

1) $\int_1^2 \frac{dx}{x}$; Trapezoidal Rule; $n = 5$

Answer

0.696

2) $\int_0^3 \sqrt{4+x^3}\ dx$; Trapezoidal Rule; $n = 6$

3) $\int_0^3 \sqrt{4+x^3}\ dx$; Simpson's Rule; $n = 6$

Answer

9.279

4) $\int_0^{12} x^2\ dx$; Midpoint Rule; $n = 6$

5) $\int_0^1 \sin^2(\pi x)\ dx$; Midpoint Rule; $n = 3$

Answer

0.500

6) Use the Midpoint Rule with eight subdivisions to estimate $\int_2^4 x^2\ dx$.

7) Use the Trapezoidal Rule with four subdivisions to estimate $\int_2^4 x^2\ dx$.

Answer

$T_4 = 18.75$

8) Find the exact value of $\int_2^4 x^2\ dx$. Find the error of approximation between the exact value and the value calculated using the Trapezoidal Rule with four subdivisions. Draw a graph to illustrate.

Approximate the integral to four decimal places using the indicated rule.

9) $\int_0^1 \sin^2(\pi x)\ dx$; Trapezoidal Rule; $n = 6$

Answer

0.5000

10) $\int_0^3 \frac{1}{1+x^3}\,dx$; Trapezoidal Rule; $n=6$

11) $\int_0^3 \frac{1}{1+x^3}\,dx$; Simpson's Rule; $n=6$

Answer

1.1614

12) $\int_0^{0.8} e^{-x^2}\,dx$; Trapezoidal Rule; $n=4$

13) $\int_0^{0.8} e^{-x^2}\,dx$; Simpson's Rule; $n=4$

Answer

0.6577

14) $\int_0^{0.4} \sin(x^2)\,dx$; Trapezoidal Rule; $n=4$

15) $\int_0^{0.4} \sin(x^2)\,dx$; Simpson's Rule; $n=4$

Answer

0.0213

16) $\int_{0.1}^{0.5} \frac{\cos x}{x}\,dx$; Trapezoidal Rule; $n=4$

17) $\int_{0.1}^{0.5} \frac{\cos x}{x}\,dx$; Simpson's Rule; $n=4$

Answer

1.5629

18) Evaluate $\int_0^1 \frac{dx}{1+x^2}$ exactly and show that the result is $\pi/4$. Then, find the approximate value of the integral using the Trapezoidal Rule with $n=4$ subdivisions. Use the result to approximate the value of π.

19) Approximate $\int_2^4 \frac{1}{\ln x}\,dx$ using the Midpoint Rule with four subdivisions to four decimal places.

Answer

1.9133

20) Approximate $\int_2^4 \frac{1}{\ln x}\,dx$ using the Trapezoidal Rule with eight subdivisions to four decimal places.

21) Use the Trapezoidal Rule with four subdivisions to estimate $\int_0^{0.8} x^3\,dx$ to four decimal places.

Answer

$T(4)=0.1088$

22) Use the Trapezoidal Rule with four subdivisions to estimate $\int_0^{0.8} x^3\,dx$. Compare this value with the exact value and find the error estimate.

23) Using Simpson's Rule with four subdivisions, find $\int_0^{\pi/2} \cos(x)\,dx$.

Answer

$\int_0^{\pi/2} \cos(x)\, dx \approx \quad 1.0$

24) Show that the exact value of $\int_0^1 xe^{-x}\, dx = 1 - \frac{2}{e}$. Find the absolute error if you approximate the integral using the Midpoint Rule with 16 subdivisions.

25) Given $\int_0^1 xe^{-x}\, dx = 1 - \frac{2}{e}$, use the Trapezoidal Rule with 16 subdivisions to approximate the integral and find the absolute error.

Answer

Approximate error is **0.000325.**

26) Find an upper bound for the error in estimating $\int_0^3 (5x + 4)\, dx$ using the Trapezoidal Rule with six steps.

27) Find an upper bound for the error in estimating $\int_4^5 \frac{1}{(x-1)^2}\, dx$ using the Trapezoidal Rule with seven subdivisions.

Answer

$\frac{1}{7938}$

28) Find an upper bound for the error in estimating $\int_0^3 (6x^2 - 1)\, dx$ using Simpson's Rule with $n = 10$ steps.

29) Find an upper bound for the error in estimating $\int_2^5 \frac{1}{x-1}\, dx$ using Simpson's Rule with $n = 10$ steps.

Answer

$\frac{81}{25{,}000}$

30) Find an upper bound for the error in estimating $\int_0^{\pi} 2x\cos(x)\, dx$ using Simpson's Rule with four steps.

31) Estimate the minimum number of subintervals needed to approximate the integral $\int_1^4 (5x^2 + 8)\, dx$ with an error magnitude of less than 0.0001 using the Trapezoidal Rule.

Answer

475

32) Determine a value of n such that the Trapezoidal Rule will approximate $\int_0^1 \sqrt{1 + x^2}\, dx$ with an error of no more than 0.01.

33) Estimate the minimum number of subintervals needed to approximate the integral $\int_2^3 (2x^3 + 4x)\, dx$ with an error of magnitude less than 0.0001 using the Trapezoidal Rule.

Answer

174

34) Estimate the minimum number of subintervals needed to approximate the integral $\int_3^4 \frac{1}{(x-1)^2}\, dx$ with an error magnitude of less than 0.0001 using the Trapezoidal Rule.

35) Use Simpson's Rule with four subdivisions to approximate the area under the probability density function $y = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ from $x = 0$ to $x = 0.4$.

Answer

0.1544

36) Use Simpson's Rule with $n = 14$ to approximate (to three decimal places) the area of the region bounded by the graphs of $y = 0, x = 0$, and $x = \pi/2$.

37) The length of one arch of the curve $y = 3\sin(2x)$ is given by $L = \int_0^{\pi/2} \sqrt{1 + 36\cos^2(2x)}\ dx$. Estimate L using the Trapezoidal Rule with $n = 6$.

Answer

6.2807

38) The length of the ellipse $x = a\cos(t), y = b\sin(t), 0 \le t \le 2\pi$ is given by $L = 4a\int_0^{\pi/2} \sqrt{1 - e^2\cos^2(t)}dt$, where e is the eccentricity of the ellipse. Use Simpson's Rule with $n = 6$ subdivisions to estimate the length of the ellipse when $a = 2$ and $e = 1/3$.

39) Estimate the area of the surface generated by revolving the curve $y = \cos(2x), 0 \le x \le \frac{\pi}{4}$ about the x-axis. Use the Trapezoidal Rule with six subdivisions.

Answer

4.606

40) Estimate the area of the surface generated by revolving the curve $y = 2x^2, 0 \le x \le 3$ about the x-axis. Use Simpson's Rule with $n = 6$.

41) The growth rate of a certain tree (in feet) is given by $y = \dfrac{2}{t+1} + e^{-t^2/2}$, where t is time in years. Estimate the growth of the tree through the end of the second year by using Simpson's Rule, using two subintervals. (Round the answer to the nearest hundredth.)

Answer

3.41 ft

42) [Technology Required] Use a calculator to approximate $\int_0^1 \sin(\pi x)\ dx$ using the Midpoint Rule with 25 subdivisions. Compute the relative error of approximation.

43) [Technology Required] Given $\int_1^5 (3x^2 - 2x)\ dx = 100$, approximate the value of this integral using the Midpoint Rule with 16 subdivisions and determine the absolute error.

Answer

$T_{16} = 100.125$; absolute error = 0.125

44) Given that we know the Fundamental Theorem of Calculus, why would we want to develop numerical methods for definite integrals?

45) The table represents the coordinates (x, y) that give the boundary of a lot. The units of measurement are meters. Use the Trapezoidal Rule to estimate the number of square meters of land that is in this lot.

x	y	x	y
0	125	600	95
100	125	700	88
200	120	800	75
300	112	900	35
400	90	1000	0
500	90		

Answer

about 89,250 m^2

46) Choose the correct answer. When Simpson's Rule is used to approximate the definite integral, it is necessary that the number of partitions be____

a. an even number

b. odd number

c. either an even or an odd number

d. a multiple of 4

47) The “Simpson” sum is based on the area under a ____.

Answer

parabola

48) The error formula for Simpson's Rule depends on___.

a. $f(x)$

b. $f'(x)$

c. $f^{(4)}(x)$

d. the number of steps

This page titled 3.5E: Exercises is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Roy Simpson.

- **7.6E: Exercises for Section 7.6** by Edwin “Jed” Herman, Gilbert Strang is licensed CC BY-NC-SA 4.0. Original source: https://openstax.org/details/books/calculus-volume-1.

3.6: Indeterminate Forms and L'Hospital's Rule

Learning Objectives

- Recognize when to apply l'Hôpital's Rule.
- Identify indeterminate forms produced by quotients, products, subtractions, and powers, and apply l'Hôpital's Rule in each case.



Hawk A.I. Section-Specific Tutor

Please note that, to access the Hawk A.I. Tutor, you will need a (free) OpenAI account.

This section examines a powerful tool for evaluating limits. This tool, **l'Hôpital's Rule**, uses derivatives to calculate limits. With this rule, we will be able to evaluate many limits we have not yet been able to determine. Instead of relying on numerical evidence to conjecture that a limit exists, we can show definitively that a limit exists and determine its exact value.

Applying l'Hôpital's Rule

l'Hôpital's Rule can be used to evaluate limits involving the quotient of two functions. Consider

$$\lim_{x\to a}\frac{f(x)}{g(x)}.$$

If $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2 \neq 0$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{L_1}{L_2}.$$

However, what happens if $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$? We call this one of the **indeterminate forms,** of type $\frac{0}{0}$. This is considered an indeterminate form because we cannot *determine* the exact behavior of $\frac{f(x)}{g(x)}$ as $x \to a$ without further analysis. We have seen examples of this earlier in the text. For example, consider

$$\lim_{x\to 2}\frac{x^2-4}{x-2}$$

and

$$\lim_{x\to 0}\frac{\sin x}{x}.$$

For the first of these examples, we can evaluate the limit by factoring the numerator and writing

$$\lim_{x\to 2}\frac{x^2-4}{x-2} = \lim_{x\to 2}\frac{(x+2)(x-2)}{x-2} = \lim_{x\to 2}(x+2) = 2+2 = 4.$$

For $\lim_{x\to 0}\frac{\sin x}{x}$, we were able to show, using a geometric argument, that

$$\lim_{x\to 0}\frac{\sin x}{x} = 1.$$

Here, we use a different technique for evaluating limits such as these. Not only does this technique provide an easier way to evaluate these limits, but it also (and more importantly) provides us with a way to evaluate many other limits we could not previously calculate.

The idea behind l'Hôpital's Rule can be explained using local linear approximations. Consider two differentiable functions f and g such that $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$ and such that $g'(a) \neq 0$. For x near a,we can write

$$f(x) \approx f(a) + f'(a)(x-a)$$

and

$$g(x) \approx g(a) + g'(a)(x - a).$$

Therefore,

$$\frac{f(x)}{g(x)} \approx \frac{f(a) + f'(a)(x - a)}{g(a) + g'(a)(x - a)}.$$

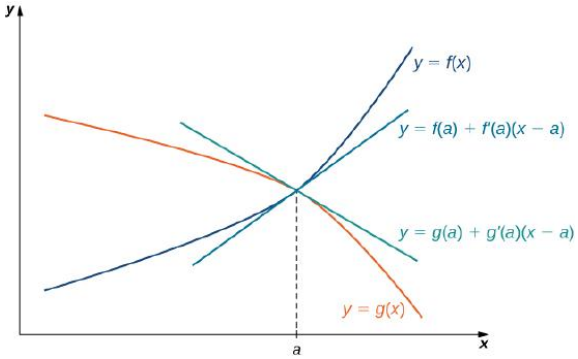


Figure **3.6.1**: *If* $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$, *then the ratio* $f(x)/g(x)$ *is approximately equal to the ratio of their linear approximations near* a.

Since f is differentiable at a, then f is continuous at a, and therefore $f(a) = \lim_{x \to a} f(x) = 0$. Similarly, $g(a) = \lim_{x \to a} g(x) = 0$. If we also assume that f' and g' are continuous at $x = a$, then $f'(a) = \lim_{x \to a} f'(x)$ and $g'(a) = \lim_{x \to a} g'(x)$. Using these ideas, we conclude that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)(x - a)}{g'(x)(x - a)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Note that the assumption that f' and g' are continuous at a and $g'(a) \neq 0$ can be loosened. We state l'Hôpital's Rule formally for the indeterminate form $\frac{0}{0}$.

Caution: Abusive Notation

Note that the notation $\frac{0}{0}$ does not mean we are actually dividing zero by zero. Rather, we are using the notation $\frac{0}{0}$ to *represent* a quotient of limits, each of which is zero.

Theorem: l'Hôpital's Rule (0/0 Case)

Suppose f and g are differentiable functions over an open interval containing a, except possibly at a. If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming the limit on the right exists or is ∞ or $-\infty$. This result also holds if we consider one-sided limits, or if $a = \infty$ or $a = -\infty$.

Proof

We provide a proof of this theorem in the special case when f, g, f', and g' are all continuous over an open interval containing a. In that case, since $\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$ and f and g are continuous at a, it follows that $f(a) = 0 = g(a)$. Therefore,

$$\begin{aligned}
\lim_{x\to a}\frac{f(x)}{g(x)} &= \lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)} && (f(a)=0=g(a))\\
&= \lim_{x\to a}\frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} && \left(\text{multiplying numerator and denominator by } \tfrac{1}{x-a}\right)\\
&= \frac{\lim\limits_{x\to a}\frac{f(x)-f(a)}{x-a}}{\lim\limits_{x\to a}\frac{g(x)-g(a)}{x-a}} && \text{(Limit Laws)}\\
&= \frac{f'(a)}{g'(a)} && \text{(definition of the derivative)}\\
&= \frac{\lim\limits_{x\to a}f'(x)}{\lim\limits_{x\to a}g'(x)} && \text{(continuity of } f' \text{ and } g')\\
&= \lim_{x\to a}\frac{f'(x)}{g'(x)} && \text{(The limit of a quotient)}
\end{aligned}$$

Note that l'Hôpital's Rule states we can calculate the limit of a quotient $\frac{f}{g}$ by considering the limit of the quotient of the derivatives $\frac{f'}{g'}$. It is essential to realize that we are not calculating the derivative of the quotient $\frac{f}{g}$.

Q.E.D.

Example 3.6.1

Evaluate each of the following limits by applying l'Hôpital's Rule.

a. $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x}$
b. $\lim\limits_{x\to 1}\dfrac{\sin(\pi x)}{\ln x}$
c. $\lim\limits_{x\to\infty}\dfrac{e^{1/x}-1}{1/x}$
d. $\lim\limits_{x\to 0}\dfrac{\sin x - x}{x^2}$

Solutions

a. Since the numerator $1-\cos x\to 0$ and the denominator $x\to 0$, we can apply l'Hôpital's Rule to evaluate this limit. We have

$$\begin{aligned}
\lim_{x\to 0}\frac{1-\cos x}{x} &\overset{\text{l'H}}{=} \lim_{x\to 0}\frac{\frac{d}{dx}(1-\cos x)}{\frac{d}{dx}(x)}\\
&= \lim_{x\to 0}\frac{\sin x}{1}\\
&= \frac{\lim\limits_{x\to 0}\sin x}{\lim\limits_{x\to 0}1}\\
&= \frac{0}{1}=0.
\end{aligned}$$

b. As $x\to 1$, the numerator $\sin(\pi x)\to 0$ and the denominator $\ln(x)\to 0$. Therefore, we can apply l'Hôpital's Rule. We obtain

$$\lim_{x\to 1}\frac{\sin(\pi x)}{\ln x} \overset{\text{l'H}}{=} \lim_{x\to 1}\frac{\pi\cos(\pi x)}{1/x}$$
$$= \lim_{x\to 1}(\pi x)\cos(\pi x)$$
$$= (\pi\cdot 1)(-1)$$
$$= -\pi.$$

c. As $x\to\infty$, the numerator $e^{1/x}-1\to 0$ and the denominator $\frac{1}{x}\to 0$. Therefore, we can apply l'Hôpital's Rule. We obtain

$$\lim_{x\to\infty}\frac{e^{1/x}-1}{\frac{1}{x}} \overset{\text{l'H}}{=} \lim_{x\to\infty}\frac{e^{1/x}\left(\frac{-1}{x^2}\right)}{\left(\frac{-1}{x^2}\right)}$$
$$= \lim_{x\to\infty} e^{1/x}$$
$$= e^0$$
$$= 1.$$

d. As $x\to 0$, both the numerator and denominator approach zero. Therefore, we can apply l'Hôpital's Rule. We obtain

$$\lim_{x\to 0}\frac{\sin x - x}{x^2} \overset{\text{l'H}}{=} \lim_{x\to 0}\frac{\cos x - 1}{2x}.$$

Since the numerator and denominator of this new quotient both approach zero as $x\to 0$, we apply l'Hôpital's Rule again. In doing so, we see that

$$\lim_{x\to 0}\frac{\cos x - 1}{2x} \overset{\text{l'H}}{=} \lim_{x\to 0}\frac{-\sin x}{2} = 0.$$

Therefore, we conclude that

$$\lim_{x\to 0}\frac{\sin x - x}{x^2} = 0.$$

? Checkpoint 3.6.1

Evaluate

$$\lim_{x\to 0}\frac{x}{\tan x}.$$

Answer

1

We can also use l'Hôpital's Rule to evaluate limits of quotients $\frac{f(x)}{g(x)}$ in which $f(x)\to\pm\infty$ and $g(x)\to\pm\infty$. Limits of this form are classified as indeterminate forms of type $\frac{\infty}{\infty}$.

Caution: More Notational Abuse

Again, note that we are not actually dividing ∞ by ∞. Since ∞ is not a real number, that is impossible; rather, ∞/∞ is used to *represent* a quotient of limits, each of which is ∞ or $-\infty$.

Theorem: l'Hôpital's Rule (∞/∞ Case)

Suppose f and g are differentiable functions over an open interval containing a, except possibly at a. Suppose $\lim_{x\to a} f(x)=\infty$ (or $-\infty$) and $\lim_{x\to a} g(x)=\infty$ (or $-\infty$). Then,

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

assuming the limit on the right exists or is ∞ or $-\infty$. This result also holds if the limit is infinite, if $a = \infty$ or $-\infty$, or if the limit is one-sided.

Example 3.6.2

Evaluate each of the following limits by applying l'Hôpital's Rule.

a. $\lim\limits_{x\to\infty} \dfrac{3x+5}{2x+1}$

b. $\lim\limits_{x\to 0^+} \dfrac{\ln x}{\cot x}$

Solutions

a. Since $3x+5$ and $2x+1$ are first-degree polynomials with positive leading coefficients, $\lim\limits_{x\to\infty}(3x+5)=\infty$ and $\lim\limits_{x\to\infty}(2x+1)=\infty$. Therefore, we apply l'Hôpital's Rule and obtain

$$\begin{aligned}\lim_{x\to\infty}\frac{3x+5}{2x+1} &\overset{\text{l'H}}{=} \lim_{x\to\infty}\frac{3}{2}\\ &= \frac{3}{2}\end{aligned}$$

Note that this limit can also be calculated without invoking l'Hôpital's Rule. Earlier in the textbook, we showed how to evaluate such a limit by dividing the numerator and denominator by the highest power of x in the denominator. In doing so, we saw that

$$\lim_{x\to\infty}\frac{3x+5}{2x+1}=\lim_{x\to\infty}\frac{3+5/x}{2+1/x}=\frac{3}{2}.$$

l'Hôpital's Rule provides us with an alternative means of evaluating this type of limit.

b. Here, $\lim\limits_{x\to 0^+}\ln x=-\infty$ and $\lim\limits_{x\to 0^+}\cot x=\infty$. Therefore, we can apply l'Hôpital's Rule and obtain

$$\begin{aligned}\lim_{x\to 0^+}\frac{\ln(x)}{\cot(x)} &\overset{\text{l'H}}{=} \lim_{x\to 0^+}\frac{1/x}{-\csc^2(x)}\\ &= \lim_{x\to 0^+}\frac{1}{-x\csc^2(x)}\end{aligned}$$

Now as $x\to 0^+, \csc^2 x\to\infty$. Therefore, the first term in the denominator is approaching zero, and the second term is getting really large. In such a case, anything can happen with the product. Therefore, we cannot make any conclusion yet. To evaluate the limit, we use the definition of $\csc x$ to write

$$\lim_{x\to 0^+}\frac{1}{-x\csc^2 x}=\lim_{x\to 0^+}\frac{\sin^2 x}{-x}.$$

Now $\lim\limits_{x\to 0^+}\sin^2 x=0$ and $\lim\limits_{x\to 0^+}-x=0$, so we apply l'Hôpital's Rule again. We find

$$\begin{aligned}\lim_{x\to 0^+}\frac{\sin^2(x)}{-x} &\overset{\text{l'H}}{=} \lim_{x\to 0^+}\frac{2\sin(x)\cos(x)}{-1}\\ &= \frac{0}{-1}\\ &= 0.\end{aligned}$$

We conclude that

$$\lim_{x\to 0^+}\frac{\ln x}{\cot x}=0.$$

Checkpoint 3.6.2

Evaluate

$$\lim_{x\to\infty}\frac{\ln x}{5x}.$$

Answer

0

As mentioned, l'Hôpital's Rule is a handy tool for evaluating limits. It is important to remember, however, that to apply l'Hôpital's Rule to a quotient $\frac{f(x)}{g(x)}$, it is essential that the limit of $\frac{f(x)}{g(x)}$ be of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Consider the following example.

Example 3.6.3

Consider $\lim_{x\to 1}\frac{x^2+5}{3x+4}$. Show that the limit cannot be evaluated by applying l'Hôpital's Rule.

Solution

Because the limits of the numerator and denominator are not zero or infinite, we cannot apply l'Hôpital's Rule. If we try to do so, we get

$$\frac{d}{dx}(x^2+5)=2x$$

and

$$\frac{d}{dx}(3x+4)=3.$$

At which point we would conclude erroneously that

$$\lim_{x\to 1}\frac{x^2+5}{3x+4}=\lim_{x\to 1}\frac{2x}{3}=\frac{2}{3}.$$

However, since $\lim_{x\to 1}(x^2+5)=6$ and $\lim_{x\to 1}(3x+4)=7$, we actually have

$$\lim_{x\to 1}\frac{x^2+5}{3x+4}=\frac{6}{7}.$$

We can conclude that

$$\lim_{x\to 1}\frac{x^2+5}{3x+4}\neq\lim_{x\to 1}\frac{\frac{d}{dx}(x^2+5)}{\frac{d}{dx}(3x+4)}.$$

Checkpoint 3.6.3

Explain why we cannot apply l'Hôpital's Rule to evaluate $\lim_{x\to 0^+}\frac{\cos x}{x}$. Evaluate $\lim_{x\to 0^+}\frac{\cos x}{x}$ by other means.

Answer

$\lim_{x\to 0^+}\cos x=1$. Therefore, we cannot apply l'Hôpital's Rule. The limit of the quotient is ∞.

Other Indeterminate Forms

l'Hôpital's Rule is very useful for evaluating limits involving the indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$. However, we can also use l'Hôpital's Rule to help evaluate limits involving other indeterminate forms that arise when evaluating limits. The expressions $0\cdot\infty$, $\infty-\infty$, 1^∞, ∞^0, and 0^0 are all considered indeterminate forms. These expressions are not real numbers. Rather, they represent

forms that arise when trying to evaluate certain limits. Next, we realize why these are indeterminate forms and then understand how to use l'Hôpital's Rule in these cases. The key idea is that we must rewrite the indeterminate forms in such a way that we arrive at the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Indeterminate Form of Type $0 \cdot \infty$

Suppose we want to evaluate $\lim\limits_{x \to a} (f(x) \cdot g(x))$, where $f(x) \to 0$ and $g(x) \to \infty$ (or $-\infty$) as $x \to a$. Since one term in the product is approaching zero, but the other is becoming arbitrarily large (in magnitude), anything can happen to the product. We use the notation $0 \cdot \infty$ to denote the form in this situation. The expression $0 \cdot \infty$ is considered indeterminate because we cannot *determine* without further analysis the exact behavior of the product $f(x)g(x)$ as $x \to \infty$. For example, let n be a positive integer and consider

$$f(x) = \frac{1}{(x^n + 1)}$$

and

$$g(x) = 3x^2.$$

As $x \to \infty, f(x) \to 0$ and $g(x) \to \infty$. However, the limit as $x \to \infty$ of $f(x)g(x) = \frac{3x^2}{(x^n+1)}$ varies, depending on n. If $n = 2$, then $\lim\limits_{x \to \infty} f(x)g(x) = 3$. If $n = 1$, then $\lim\limits_{x \to \infty} f(x)g(x) = \infty$. If $n = 3$, then $\lim\limits_{x \to \infty} f(x)g(x) = 0$. Here, we consider another limit involving the indeterminate form $0 \cdot \infty$ and show how to rewrite the function as a quotient to use l'Hôpital's Rule.

Example 3.6.4

Evaluate $\lim\limits_{x \to 0^+} x \ln(x)$.

Solution

We first note that, as $x \to 0^+$, $\ln(x) \to -\infty$. Therefore, this limit is indeterminate of the form $0 \cdot \infty$.

Rewrite the function $x \ln(x)$ as a quotient to apply l'Hôpital's Rule. If we write

$$x \ln(x) = \frac{\ln(x)}{1/x},$$

we see that $\ln(x) \to -\infty$ as $x \to 0^+$ and $\frac{1}{x} \to \infty$ as $x \to 0^+$. Therefore, we can apply l'Hôpital's Rule and obtain

$$\begin{aligned} \lim_{x \to 0^+} \frac{\ln(x)}{1/x} &\overset{\text{l'H}}{=} \lim_{x \to 0^+} \frac{\frac{d}{dx}(\ln(x))}{\frac{d}{dx}(1/x)} \\ &= \lim_{x \to 0^+} \frac{1/x}{-1/x^2} \\ &= \lim_{x \to 0^+} (-x) \\ &= 0. \end{aligned}$$

We conclude that

$$\lim_{x \to 0^+} x \ln(x) = 0.$$

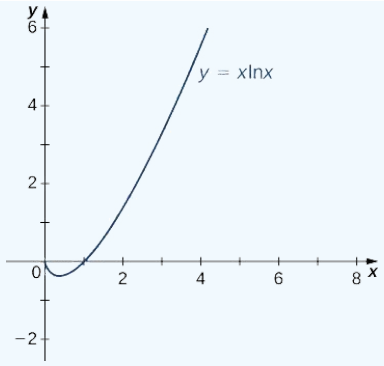


Figure **3.6.2***: Finding the limit at* $x = 0$ *of the function* $f(x) = x \ln x$.

Checkpoint 3.6.4

Evaluate

$$\lim_{x \to 0} x \cot x.$$

Answer

1

Indeterminate Form of Type $\infty - \infty$

Another type of indeterminate form is $\infty - \infty$. Consider the following example. Let n be a positive integer and let $f(x) = 3x^n$ and $g(x) = 3x^2 + 5$. As $x \to \infty$, $f(x) \to \infty$ and $g(x) \to \infty$. We are interested in $\lim_{x \to \infty} (f(x) - g(x))$. Depending on whether $f(x)$ grows faster, $g(x)$ grows faster, or they grow at the same rate, as we see next, anything can happen in this limit. Since $f(x) \to \infty$ and $g(x) \to \infty$, we write $\infty - \infty$ to denote the form of this limit.

Caution: Yet More Notational Abuse

As with our other indeterminate forms, $\infty - \infty$ has no meaning on its own, and we must do more analysis to determine the value of the limit.

For example, suppose the exponent n in the function $f(x) = 3x^n$ is $n = 3$, then

$$\lim_{x \to \infty} (f(x) - g(x)) = \lim_{x \to \infty} (3x^3 - 3x^2 - 5) = \infty.$$

On the other hand, if $n = 2$, then

$$\lim_{x \to \infty} (f(x) - g(x)) = \lim_{x \to \infty} (3x^2 - 3x^2 - 5) = -5.$$

However, if $n = 1$, then

$$\lim_{x \to \infty} (f(x) - g(x)) = \lim_{x \to \infty} (3x - 3x^2 - 5) = -\infty.$$

Therefore, the limit cannot be determined by considering only $\infty - \infty$. Next, we see how to rewrite an expression involving the indeterminate form $\infty - \infty$ as a fraction to apply l'Hôpital's Rule.

Example 3.6.5

Evaluate

$$\lim_{x \to 0^+} \left(\frac{1}{x^2} - \frac{1}{\tan x} \right).$$

Solution

As $x \to 0^+$, $\frac{1}{x^2} \to \infty$ and $\frac{1}{\tan(x)} \to \infty$. Therefore, this limit is of the form $\infty - \infty$.

By combining the fractions, we can write the function as a quotient. Since the least common denominator is $x^2 \tan x$, we have

$$\frac{1}{x^2} - \frac{1}{\tan x} = \frac{(\tan x) - x^2}{x^2 \tan x}.$$

As $x \to 0^+$, the numerator $\tan x - x^2 \to 0$ and the denominator $x^2 \tan x \to 0$. Therefore, we can apply l'Hôpital's Rule. Taking the derivatives of the numerator and the denominator, we have

$$\lim_{x \to 0^+} \frac{(\tan x) - x^2}{x^2 \tan x} \overset{\text{l'H}}{=} \lim_{x \to 0^+} \frac{(\sec^2 x) - 2x}{x^2 \sec^2 x + 2x \tan x}.$$

As $x \to 0^+$, $(\sec^2 x) - 2x \to 1$ and $x^2 \sec^2 x + 2x \tan x \to 0$. Since the denominator is positive as x approaches zero from the right, we conclude that

$$\lim_{x \to 0^+} \frac{(\sec^2 x) - 2x}{x^2 \sec^2 x + 2x \tan x} = \infty.$$

Therefore,

$$\lim_{x \to 0^+} \left(\frac{1}{x^2} - \frac{1}{\tan x} \right) = \infty.$$

? Checkpoint 3.6.5

Evaluate $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x} \right)$.

Answer

0

Another indeterminate form that arises when evaluating limits involves exponents. The expressions 0^0, ∞^0, and 1^∞ are all indeterminate forms. These expressions are meaningless because we cannot evaluate them as we would evaluate an expression involving real numbers. Rather, these expressions represent forms that arise when finding limits. Now, we examine how l'Hôpital's Rule can be used to evaluate limits involving these indeterminate forms.

Since l'Hôpital's Rule applies to quotients, we use the natural logarithm function and its properties to reduce a problem evaluating a limit involving exponents to a related problem involving a limit of a quotient. For example, suppose we want to evaluate $\lim_{x \to a} f(x)^{g(x)}$ and we arrive at the indeterminate form ∞^0. We proceed as follows. Let

$$y = f(x)^{g(x)}.$$

Then,

$$\ln y = \ln(f(x)^{g(x)}) = g(x) \ln(f(x)).$$

Therefore,

$$\lim_{x \to a} [\ln(y)] = \lim_{x \to a} [g(x) \ln(f(x))].$$

Since $\lim_{x \to a} f(x) = \infty$, we know that $\lim_{x \to a} \ln(f(x)) = \infty$. Therefore, $\lim_{x \to a} g(x) \ln(f(x))$ is of the indeterminate form $0 \cdot \infty$, and we can use the techniques discussed earlier to rewrite the expression $g(x) \ln(f(x))$ in a form so that we can apply l'Hôpital's Rule.

The indeterminate forms 0^0 and 1^∞ can be handled similarly.

Example 3.6.6

Evaluate

$$\lim_{x\to\infty} x^{1/x}.$$

Solution

As $x \to \infty$, $\frac{1}{x} \to 0$. Therefore, this limit is indeterminate of the form ∞^0.

Let $y = x^{1/x}$. Then,

$$\ln(x^{1/x}) = \frac{1}{x}\ln x = \frac{\ln x}{x}.$$

We need to evaluate $\lim_{x\to\infty} \frac{\ln x}{x}$, which is now of the form $\frac{\infty}{\infty}$. Applying l'Hôpital's Rule, we obtain

$$\begin{aligned} \lim_{x\to\infty} \ln(y) &= \lim_{x\to\infty} \frac{\ln(x)}{x} \\ &\overset{\text{l'H}}{=} \lim_{x\to\infty} \frac{1/x}{1} \\ &= 0. \end{aligned}$$

Therefore, $\lim_{x\to\infty} \ln y = 0$. Since the natural logarithm function is continuous, we conclude that

$$\ln\left(\lim_{x\to\infty} y\right) = 0,$$

which leads to

$$\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} y = e^{\ln(\lim_{x\to\infty} y)} = e^0 = 1.$$

Hence,

$$\lim_{x\to\infty} x^{1/x} = 1.$$

Checkpoint 3.6.6

Evaluate

$$\lim_{x\to\infty} x^{1/\ln(x)}.$$

Answer

e

Example 3.6.7

Evaluate

$$\lim_{x\to 0^+} x^{\sin(x)}.$$

Solution

This limit is definitely indeterminate of the form 0^0.

Let

$$y = x^{\sin(x)}.$$

Therefore,

$$\ln(y) = \ln\left(x^{\sin(x)}\right) = \sin(x)\ln(x).$$

We now evaluate $\lim_{x\to 0^+} \sin(x)\ln(x)$. Since $\lim_{x\to 0^+} \sin(x) = 0$ and $\lim_{x\to 0^+} \ln(x) = -\infty$, we have the indeterminate form $0\cdot\infty$. To apply l'Hôpital's Rule, we need to rewrite $\sin(x)\ln(x)$ as a fraction. We could write

$$\sin(x)\ln(x) = \frac{\sin(x)}{1/\ln(x)}$$

or

$$\sin(x)\ln(x) = \frac{\ln(x)}{1/\sin(x)} = \frac{\ln(x)}{\csc(x)}.$$

Let's consider the first option. In this case, applying l'Hôpital's Rule, we would obtain

$$\begin{aligned}\lim_{x\to 0^+} \sin(x)\ln(x) &= \lim_{x\to 0^+} \frac{\sin(x)}{1/\ln(x)} \\ &= \lim_{x\to 0^+} \frac{\cos(x)}{-1/(x(\ln(x))^2)} \\ &= \lim_{x\to 0^+} \left(-x(\ln(x))^2\cos(x)\right).\end{aligned}$$

Unfortunately, we not only have another expression involving the indeterminate form $0\cdot\infty$, but the new limit is even more complicated to evaluate than the one with which we started. Instead, we try the second option. By writing

$$\sin(x)\ln(x) = \frac{\ln(x)}{1/\sin(x)} = \frac{\ln(x)}{\csc(x)},$$

and applying l'Hôpital's Rule, we obtain

$$\begin{aligned}\lim_{x\to 0^+} \sin(x)\ln(x) &= \lim_{x\to 0^+} \frac{\ln(x)}{\csc(x)} \\ &\overset{\text{l'H}}{=} \lim_{x\to 0^+} \frac{1/x}{-\csc(x)\cot(x)} \\ &= \lim_{x\to 0^+} \frac{-1}{x\csc(x)\cot(x)}.\end{aligned}$$

Using the fact that $\csc(x) = \frac{1}{\sin(x)}$ and $\cot(x) = \frac{\cos(x)}{\sin(x)}$, we can rewrite the expression on the right-hand side as

$$\lim_{x\to 0^+} \frac{-\sin^2(x)}{x\cos(x)} = \lim_{x\to 0^+} \left[\frac{\sin(x)}{x}\cdot(-\tan(x))\right] = \left(\lim_{x\to 0^+} \frac{\sin(x)}{x}\right)\cdot\left(\lim_{x\to 0^+}(-\tan(x))\right) = 1\cdot 0 = 0.$$

We conclude that $\lim_{x\to 0^+} \ln(y) = 0$. Therefore, $\ln\left(\lim_{x\to 0^+} y\right) = 0$ and we have

$$\lim_{x\to 0^+} y = \lim_{x\to 0^+} x^{\sin(x)} = e^0 = 1.$$

Hence,

$$\lim_{x\to 0^+} x^{\sin(x)} = 1.$$

? Checkpoint 3.6.7

Evaluate $\lim_{x\to 0^+} x^x$.

Answer

1

Growth Rates of Functions

We have shown how to use a function's first and second derivatives to describe the shape of a graph. To graph a function f defined on an unbounded domain, we also need to know the behavior of f as $x \to \pm\infty$.

Suppose the functions f and g both approach infinity as $x \to \infty$. Although the values of both functions become arbitrarily large as the values of x become sufficiently large, sometimes one function grows more quickly than the other.

We say g grows more rapidly than f as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = \infty \quad \text{or, equivalently,} \quad \lim_{x\to\infty} \frac{f(x)}{g(x)} = 0.$$

On the other hand, if there exists a constant $M \neq 0$ such that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = M,$$

we say f and g grow at the same rate as $x \to \infty$.

Next, we see how to use l'Hôpital's Rule to compare the growth rates of power, exponential, and logarithmic functions.

Example 3.6.8

For each of the following pairs of functions, use l'Hôpital's Rule to evaluate

$$\lim_{x\to\infty} \frac{f(x)}{g(x)}.$$

a. $f(x) = x^2$ and $g(x) = e^x$
b. $f(x) = \ln(x)$ and $g(x) = x^2$

Solutions

a. Since $\lim_{x\to\infty} x^2 = \infty$ and $\lim_{x\to\infty} e^x = \infty$, we can use l'Hôpital's Rule to evaluate $\lim_{x\to\infty} \left[\frac{x^2}{e^x}\right]$. We obtain

$$\lim_{x\to\infty} \frac{x^2}{e^x} \overset{\text{l'H}}{=} \lim_{x\to\infty} \frac{2x}{e^x}.$$

Since $\lim_{x\to\infty} 2x = \infty$ and $\lim_{x\to\infty} e^x = \infty$, we can apply l'Hôpital's Rule again. Since

$$\lim_{x\to\infty} \frac{2x}{e^x} \overset{\text{l'H}}{=} \lim_{x\to\infty} \frac{2}{e^x} = 0,$$

we conclude that

$$\lim_{x\to\infty} \frac{x^2}{e^x} = 0.$$

Therefore, e^x grows more rapidly than x^2 as $x \to \infty$ (See Figure 3.6.3 and Table 3.6.1)

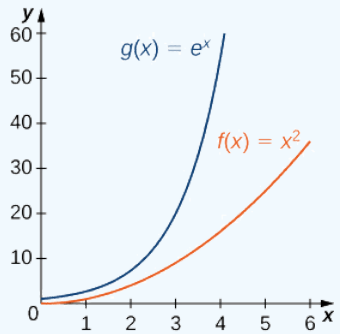


Figure 3.6.3: An exponential function grows at a faster rate than a power function.

Table 3.6.1: Growth rates of a power function and an exponential function.

x	5	10	15	20
x^2	25	100	225	400
e^x	148	22,026	3,269,017	485,165,195

b. Since $\lim_{x\to\infty} \ln x = \infty$ and $\lim_{x\to\infty} x^2 = \infty$, we can use l'Hôpital's Rule to evaluate $\lim_{x\to\infty} \frac{\ln x}{x^2}$. We obtain

$$\lim_{x\to\infty} \frac{\ln x}{x^2} \overset{\text{l'H}}{=} \lim_{x\to\infty} \frac{1/x}{2x} = \lim_{x\to\infty} \frac{1}{2x^2} = 0.$$

Thus, x^2 grows more rapidly than $\ln x$ as $x \to \infty$ (see Figure 3.6.4 and Table 3.6.2).

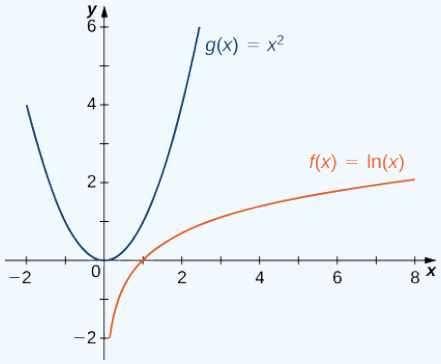


Figure 3.6.4: A power function grows at a faster rate than a logarithmic function.

Table 3.6.2: Growth rates of a power function and a logarithmic function

x	10	100	1000	10,000
$\ln(x)$	2.303	4.605	6.908	9.210
x^2	100	10,000	1,000,000	100,000,000

? Checkpoint 3.6.8

Compare the growth rates of x^{100} and 2^x.

Answer

The function 2^x grows faster than x^{100}.

Using the same ideas as in Example 3.6.8a, it is not difficult to show that e^x grows more rapidly than x^p for any $p > 0$. In Figure 3.6.5 and Table 3.6.3, we compare e^x with x^3 and x^4 as $x \to \infty$.

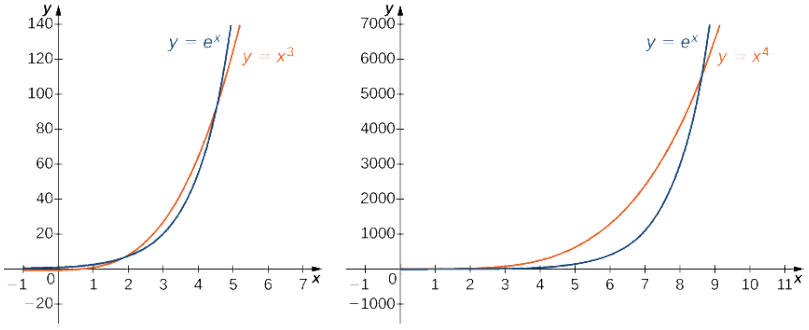


Figure 3.6.5: The exponential function e^x grows faster than x^p for any $p > 0$. (a) A comparison of e^x with x^3. (b) A comparison of e^x

with x^4.

Table 3.6.3: An exponential function grows at a faster rate than any power function

x	5	10	15	20
x^3	125	1000	3375	8000
x^4	625	10,000	50,625	160,000
e^x	148	22,026	3,269,017	485,165,195

Similarly, it is not difficult to show that x^p grows more rapidly than $\ln x$ for any $p > 0$. In Figure 3.6.6 and Table 3.6.4, we compare $\ln x$ with $\sqrt[3]{x}$ and $\sqrt{x}$.

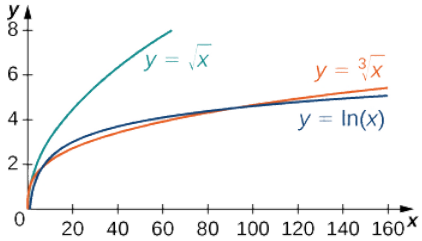


Figure 3.6.6*: The function* $y = \ln(x)$ *grows more slowly than* x^p *for any* $p > 0$ *as* $x \to \infty$.

Table 3.6.4: A logarithmic function grows at a slower rate than any root function

x	10	100	1000	10,000
$\ln(x)$	2.303	4.605	6.908	9.210
$\sqrt[3]{x}$	2.154	4.642	10	21.544
$\sqrt{x}$	3.162	10	31.623	100

3.6E: EXERCISES

This page is a draft and is under active development.

READING QUESTIONS

1. What are the two primary indeterminate forms for which L'Hôpital's Rule can be directly applied?
2. State L'Hôpital's Rule for the $\frac{0}{0}$ case. What conditions must be met by the functions f and g?
3. If $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$ is of the form $\frac{0}{0}$, what limit do you evaluate according to L'Hôpital's Rule?
4. Does L'Hôpital's Rule apply if $\lim\limits_{x\to a} f(x)=0$ but $\lim_{x\to a} g(x)=5$? Why or why not?
5. How can the indeterminate form $0\cdot\infty$ be rewritten so that L'Hôpital's Rule might be applicable?
6. How can the indeterminate form $\infty-\infty$ be rewritten so that L'Hôpital's Rule might be applicable?
7. What technique is used to handle indeterminate forms involving exponents, such as 0^0, ∞^0, or 1^∞, before L'Hôpital's Rule can be applied?
8. If $\lim\limits_{x\to a}\ln(y)=L$, what is $\lim\limits_{x\to a} y$?
9. When comparing the growth rates of two functions $f(x)$ and $g(x)$ that both approach ∞ as $x\to\infty$, what does it mean if $\lim\limits_{x\to\infty}\dfrac{f(x)}{g(x)}=0$? Which function grows more rapidly?
10. Based on the examples, which type of function generally grows faster as $x\to\infty$: an exponential function (e^x), a power function ($x^p, p>0$), or a logarithmic function ($\ln x$)?

HOMEWORK

In exercises 1 - 6, evaluate the limit.

1) Evaluate the limit $\lim\limits_{x\to\infty}\dfrac{e^x}{x}$.

2) Evaluate the limit $\lim\limits_{x\to\infty}\dfrac{e^x}{x^k}$.

Answer

$$\lim_{x\to\infty}\frac{e^x}{x^k} \quad = \quad \infty$$

3) Evaluate the limit $\lim\limits_{x\to\infty}\dfrac{\ln x}{x^k}$.

4) Evaluate the limit $\lim\limits_{x\to a}\dfrac{x-a}{x^2-a^2}$.

Answer

$$\lim_{x\to a}\frac{x-a}{x^2-a^2} \quad = \quad \frac{1}{2a}$$

5. Evaluate the limit $\lim\limits_{x\to a}\dfrac{x-a}{x^3-a^3}$.

6. Evaluate the limit $\lim\limits_{x\to a}\dfrac{x-a}{x^n-a^n}$.

Answer

$$\lim_{x\to a}\frac{x-a}{x^n-a^n} \quad = \quad \frac{1}{na^{n-1}}$$

In exercises 7 - 11, determine whether you can apply l'Hôpital's Rule directly. Explain why or why not. Then, indicate if there is some way you can alter the limit so you can apply l'Hôpital's Rule.

7) $\lim\limits_{x\to 0^+} x^2\ln x$

8) $\lim\limits_{x\to\infty} x^{1/x}$

Answer

Cannot apply directly; use logarithms

9) $\lim\limits_{x\to 0} x^{2/x}$

10) $\lim_{x\to 0} \frac{x^2}{1/x}$

Answer

Cannot apply directly; rewrite as $\lim_{x\to 0} x^3$

11) $\lim_{x\to\infty} \frac{e^x}{x}$

In exercises 12 - 40, evaluate the limits with either l'Hôpital's Rule or previously learned methods.

12) $\lim_{x\to 3} \frac{x^2-9}{x-3}$

Answer

$$\lim_{x\to 3} \frac{x^2-9}{x-3} = 6$$

13) $\lim_{x\to 3} \frac{x^2-9}{x+3}$

14) $\lim_{x\to 0} \frac{(1+x)^{-2}-1}{x}$

Answer

$$\lim_{x\to 0} \frac{(1+x)^{-2}-1}{x} = -2$$

15) $\lim_{x\to \pi/2} \frac{\cos x}{\frac{\pi}{2}-x}$

16) $\lim_{x\to \pi} \frac{x-\pi}{\sin x}$

Answer

$$\lim_{x\to \pi} \frac{x-\pi}{\sin x} = -1$$

17) $\lim_{x\to 1} \frac{x-1}{\sin x}$

18) $\lim_{x\to 0} \frac{(1+x)^n-1}{x}$

Answer

$$\lim_{x\to 0} \frac{(1+x)^n-1}{x} = n$$

19) $\lim_{x\to 0} \frac{(1+x)^n-1-nx}{x^2}$

20) $\lim_{x\to 0} \frac{\sin x-\tan x}{x^3}$

Answer

$$\lim_{x\to 0} \frac{\sin x-\tan x}{x^3} = -\frac{1}{2}$$

21) $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}$

22) $\lim_{x\to 0} \frac{e^x-x-1}{x^2}$

Answer

$$\lim_{x\to 0} \frac{e^x-x-1}{x^2} = \frac{1}{2}$$

23) $\lim_{x\to 0} \frac{\tan x}{\sqrt{x}}$

24) $\lim_{x\to 1} \frac{x-1}{\ln x}$

Answer

$$\lim_{x\to 1} \frac{x-1}{\ln x} = 1$$

25) $\lim_{x\to 0}(x+1)^{1/x}$

26) $\lim_{x\to 1}\dfrac{\sqrt{x}-\sqrt[3]{x}}{x-1}$

Answer

$$\lim_{x\to 1}\frac{\sqrt{x}-\sqrt[3]{x}}{x-1} = \frac{1}{6}$$

27) $\lim_{x\to 0^+} x^{2x}$

28) $\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right)$

Answer

$$\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right) = 1$$

29) $\lim_{x\to 0}\dfrac{\sin x - x}{x^2}$

30) $\lim_{x\to 0^+} x\ln\left(x^4\right)$

Answer

$$\lim_{x\to 0^+} x\ln\left(x^4\right) = 0$$

31) $\lim_{x\to\infty}(x-e^x)$

32) $\lim_{x\to\infty} x^2e^{-x}$

Answer

$$\lim_{x\to\infty} x^2e^{-x} = 0$$

33) $\lim_{x\to 0}\dfrac{3^x-2^x}{x}$

34) $\lim_{x\to 0}\dfrac{1+1/x}{1-1/x}$

Answer

$$\lim_{x\to 0}\frac{1+1/x}{1-1/x} = -1$$

35) $\lim_{x\to\pi/4}(1-\tan x)\cot x$

36) $\lim_{x\to\infty} xe^{1/x}$

Answer

$$\lim_{x\to\infty} xe^{1/x} = \infty$$

37) $\lim_{x\to 0} x^{1/\cos x}$

38) $\lim_{x\to 0^+} x^{1/x}$

Answer

$$\lim_{x\to 0^+} x^{1/x} = 0$$

39) $\lim_{x\to 0}\left(1-\dfrac{1}{x}\right)^x$

40) $\lim_{x\to\infty}\left(1-\dfrac{1}{x}\right)^x$

Answer

$$\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^x = \frac{1}{e}$$

For exercises 41 - 50, use a calculator to graph the function and estimate the value of the limit, then use l'Hôpital's Rule to find the limit directly.

41) [Technology Required] $\lim\limits_{x\to 0}\dfrac{e^x-1}{x}$

42) [Technology Required] $\lim\limits_{x\to 0} x\sin\left(\frac{1}{x}\right)$

Answer

$$\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) \;=\; 0$$

43) [Technology Required] $\lim\limits_{x\to 1}\dfrac{x-1}{1-\cos(\pi x)}$

44) [Technology Required] $\lim\limits_{x\to 1}\dfrac{e^{x-1}-1}{x-1}$

Answer

$$\lim_{x\to 1}\frac{e^{x-1}-1}{x-1} \;=\; 1$$

45) [Technology Required] $\lim\limits_{x\to 1}\dfrac{(x-1)^2}{\ln x}$

46) [Technology Required] $\lim\limits_{x\to \pi}\dfrac{1+\cos x}{\sin x}$

Answer

$$\lim_{x\to \pi}\frac{1+\cos x}{\sin x} \;=\; 0$$

47) [Technology Required] $\lim\limits_{x\to 0}\left(\csc x-\dfrac{1}{x}\right)$

48) [Technology Required] $\lim\limits_{x\to 0^+}\tan\left(x^x\right)$

Answer

$$\lim_{x\to 0^+}\tan\left(x^x\right) \;=\; \tan 1$$

49) [Technology Required] $\lim\limits_{x\to 0^+}\dfrac{\ln x}{\sin x}$

50) [Technology Required] $\lim\limits_{x\to 0}\dfrac{e^x-e^{-x}}{x}$

Answer

$$\lim_{x\to 0}\frac{e^x-e^{-x}}{x} \;=\; 2$$

3.7: Improper Integrals

Learning Objectives

- Evaluate an integral over an infinite interval.
- Evaluate an integral over a closed interval with an infinite discontinuity within the interval.
- Use the comparison theorem to determine whether a definite integral is convergent.

Before we dive too deeply into the focus of this chapter (sequences and series), we must investigate the concept of integrals with infinite upper and/or lower limits - this will significantly help us with the material for the rest of this chapter.

Is the area between the graph of $f(x) = \frac{1}{x}$ and the x-axis over the interval $[1, \infty)$ finite or infinite? If this same region is revolved about the x-axis, is the volume finite or infinite? Surprisingly, the area of the region described is infinite, but the volume of the solid obtained by revolving this region about the x-axis is finite.

In this section, we define integrals over an infinite interval and integrals of functions containing a discontinuity on the interval. Integrals of these types are called improper integrals. We examine several techniques for evaluating improper integrals, all involving taking limits.

Integrating over an Infinite Interval

How should we go about defining an integral of the type $\int_a^\infty f(x)\,dx$? We can integrate $\int_a^t f(x)\,dx$ for any value of t, so it is reasonable to look at the behavior of this integral as we substitute larger values of t. Figure 3.7.1 shows that $\int_a^t f(x)\,dx$ may be interpreted as area for various values of t. In other words, we may define an **improper integral** as a limit, taken as one of the limits of integration increases or decreases without bound.

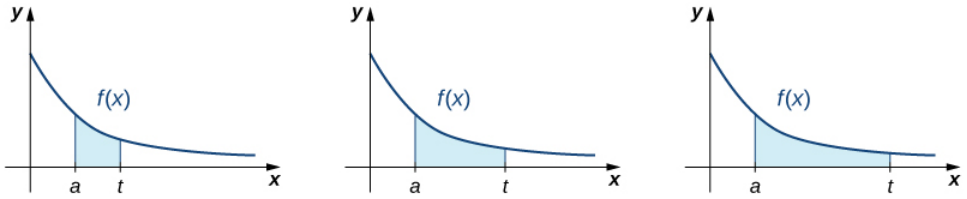


Figure 3.7.1: To integrate a function over an infinite interval, we consider the limit of the integral as the upper limit increases without bound.

Definition: Improper Integral

1. Let $f(x)$ be continuous over an interval of the form $[a, \infty)$. Then $\int_a^\infty f(x)\,dx$ is called an **improper integral**, and

$$\int_a^\infty f(x)\,dx \equiv \lim_{t\to\infty} \int_a^t f(x)\,dx, \tag{3.7.1}$$

   provided this limit exists.[1,2]

2. Let $f(x)$ be continuous over an interval of the form $(-\infty, b]$. Then $\int_{-\infty}^b f(x)\,dx$ is also called an improper integral, and

$$\int_{-\infty}^b f(x)\,dx \equiv \lim_{t\to-\infty} \int_t^b f(x)\,dx, \tag{3.7.2}$$

   provided this limit exists.

In each case, if the limit exists, then the improper integral is said to **converge**. If the limit does not exist, then the improper integral is said to **diverge**.

3. Let $f(x)$ be continuous over $(-\infty, \infty)$. Then $\int_{-\infty}^\infty f(x)\,dx$ is also called an improper integral, and

$$\int_{-\infty}^{\infty} f(x)\,dx \equiv \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx \tag{3.7.3}$$

for any value of a, provided that $\int_{-\infty}^{a} f(x)\,dx$ *and* $\int_{a}^{\infty} f(x)\,dx$ both converge.

If *either* of these two integrals diverge, then $\int_{-\infty}^{\infty} f(x)\,dx$ diverges.

In our first example, we return to the question we posed at the start of this section: Is the area between the graph of $f(x) = \frac{1}{x}$ and the x-axis over the interval $[1, \infty)$ finite or infinite?

Example 3.7.1: Finding an Area

Determine whether the area between the graph of $f(x) = \frac{1}{x}$ and the x-axis over the interval $[1, \infty)$ is finite or infinite.

Solution

We first do a quick sketch of the region in question, as shown in Figure 3.7.2.

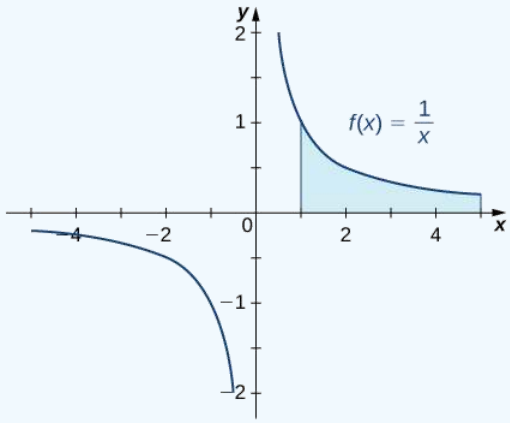


Figure 3.7.2: We can find the area between the curve $f(x) = 1/x$ and the x-axis on an infinite interval.

We can see that the area of this region is given by

$$A = \int_{1}^{\infty} \frac{1}{x}\,dx.$$

which can be evaluated using Equation 3.7.1:

$$\begin{aligned} A &= \int_{1}^{\infty} \frac{1}{x}\,dx \\ &= \lim_{t\to\infty} \int_{1}^{t} \frac{1}{x}\,dx \\ &= \lim_{t\to\infty} \ln|x|\Big|_{1}^{t} \\ &= \lim_{t\to\infty} (\ln|t| - \ln 1) \\ &= \infty. \end{aligned}$$

Since the limit tends to ∞, it technically does not exist (as a finite number).[2] Thus, the improper integral is divergent. In fact, it diverges to infinity, and so the area of the region is infinite.

Caution: Anything Involving ∞ in Calculus Requires Limits!!!

A *very* lousy habit that some students get into is to evaluate an improper integral without using limit notation. To be clear:

To evaluate an improper integral, you *must* use limits!

That is, in Example 3.7.1, it would be incorrect to write

$$A = \int_1^{\infty} \frac{1}{x}\,dx = \ln|x|\Big|_1^{\infty} = \ln|\infty| - \ln|1| = \infty - 0 = \infty.$$

This makes no sense mathematically since ∞ is not a number and, as such, you cannot directly evaluate *any* function "at ∞." Hence, $\ln|\infty|$ has no meaning. On the other hand, $\lim_{t\to\infty} \ln|t| = \infty$ makes complete sense.

Example 3.7.2: Finding a Volume

Find the volume of the solid obtained by revolving the region bounded by the graph of $f(x) = \frac{1}{x}$ and the x-axis over the interval $[1, \infty)$ about the x-axis.

Solution

The solid is shown in Figure 3.7.3. Using the Disk Method, we see that the volume V is

$$V = \pi \int_1^{\infty} \frac{1}{x^2}\,dx.$$

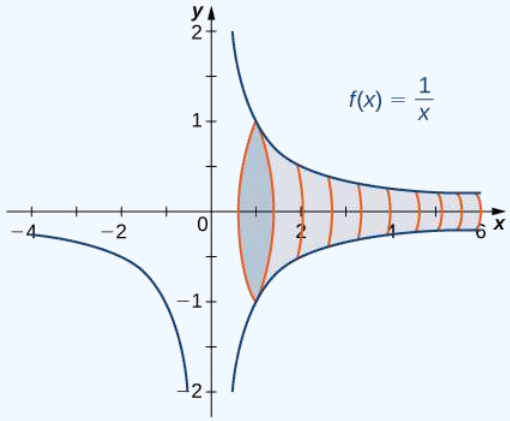


Figure 3.7.3: The solid of revolution can be generated by rotating an infinite area about the x-axis.

Then we have

$$\begin{aligned} V &= \pi \int_1^{\infty} \frac{1}{x^2}\,dx \\ &= \pi \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx \\ &= \pi \lim_{t\to\infty} -\frac{1}{x}\Big|_1^t \\ &= \pi \lim_{t\to\infty} \left(-\frac{1}{t}+1\right) \\ &= \pi \lim_{t\to\infty} \left(-\frac{1}{\not t^{\,0}}+1\right) \\ &= \pi \end{aligned}$$

Since this limit exists, the improper integral converges. In fact, it converges to π. Therefore, the volume of the solid of revolution is π.

In conclusion, although the area of the region between the x-axis and the graph of $f(x) = \frac{1}{x}$ over the interval $[1, \infty)$ is infinite, the volume of the solid generated by revolving this region about the x-axis is finite. The solid generated is known as **Gabriel's Horn**.[3]

Example 3.7.3: Traffic Accidents in a City

Suppose that at a busy intersection, traffic accidents occur at an average rate of one every three months. After residents complained, changes were made to the traffic lights at the intersection. It has been eight months since the changes were made, and no accidents have occurred. Were the changes effective, or is the 8-month interval without an accident a result of chance?



Figure 3.7.4: Modification of work by David McKelvey, Flickr.

Probability theory tells us that if the *average* time between events is k, then the probability that the *actual* time between events is between a and b is given by

$$P(a \le X \le b) = \int_a^b f(x)\,dx,$$

where

$$f(x) = \begin{cases} 0, & \text{if } x < 0 \\ ke^{-kx}, & \text{if } x \ge 0 \end{cases}.$$

Thus, if accidents are occurring at a rate of one every 3 months, then the probability that *actual* time between accidents is between a and b is given by

$$P(a \le X \le b) = \int_a^b f(x)\,dx,$$

where

$$f(x) = \begin{cases} 0, & \text{if } x < 0 \\ \\ 3e^{-3x}, & \text{if } x \geq 0 \end{cases}.$$

To answer the question, we must ask ourselves, "What is the probability that the first accident in the intersection occurs after 8 months?" Hence, we need to compute $P(X \geq 8) = \int_8^{\infty} 3e^{-3x}\,dx$ and decide whether it is likely that 8 months could have passed without an accident if there had been no improvement in the traffic situation.

Solution

We need to calculate the probability as an improper integral:

$$\begin{aligned} P(X \geq 8) &= \int_8^{\infty} 3e^{-3x}\,dx \\ &= \lim_{t\to\infty} \int_8^t 3e^{-3x}\,dx \\ &= \lim_{t\to\infty} -e^{-3x}\Big|_8^t \\ &= \lim_{t\to\infty} (-e^{-3t} + e^{-24}) \\ &\approx 3.8 \times 10^{-11}. \end{aligned}$$

The value 3.8×10^{-11} represents the probability of no accidents in 8 months under the initial conditions. Since this value is very small, concluding the changes were effective is reasonable.

Example 3.7.4: Evaluating an Improper Integral over an Infinite Interval

Evaluate

$$\int_{-\infty}^{0} \frac{1}{x^2+4}\,dx.$$

State whether the improper integral converges or diverges.

Solution

Begin by rewriting $\int_{-\infty}^{0} \frac{1}{x^2+4}\,dx$ as a limit using Equation 3.7.2 from the definition. Thus,

$$\begin{aligned} \int_{-\infty}^{0} \frac{1}{x^2+4}\,dx &= \lim_{t\to-\infty} \int_t^0 \frac{1}{x^2+4}\,dx \\ &= \lim_{t\to-\infty} \frac{1}{2}\tan^{-1}\frac{x}{2}\Big|_t^0 \\ &= \lim_{t\to-\infty} \left(\frac{1}{2}\tan^{-1}0 - \frac{1}{2}\tan^{-1}\frac{t}{2}\right) \\ &= \frac{\pi}{4} \end{aligned}$$

Therefore, the improper integral converges to $\frac{\pi}{4}$.

Example 3.7.5: Evaluating an Improper Integral on $(-\infty, \infty)$

Evaluate

$$\int_{-\infty}^{\infty} xe^x\,dx.$$

State whether the improper integral converges or diverges.

Solution

Start by splitting up the integral:

$$\int_{-\infty}^{\infty} xe^x\,dx = \int_{-\infty}^{0} xe^x\,dx + \int_{0}^{\infty} xe^x\,dx.$$

If either $\int_{-\infty}^{0} xe^x\,dx$ or $\int_{0}^{\infty} xe^x\,dx$ diverges, then $\int_{-\infty}^{\infty} xe^x\,dx$ diverges.

For the first integral,

$$\begin{aligned}\int_{-\infty}^{0} xe^x\,dx &= \lim_{t\to-\infty}\int_{t}^{0} xe^x\,dx \\ &= \lim_{t\to-\infty}(xe^x - e^x)\Big|_{t}^{0} \\ &= \lim_{t\to-\infty}(-1 - te^t + e^t) \\ &= -1.\end{aligned}$$

Note: $\lim_{t\to-\infty} te^t$ is indeterminate of the form $0\cdot\infty$. Thus,

$$\lim_{t\to-\infty} te^t = \lim_{t\to-\infty}\frac{t}{e^{-t}} \overset{\text{l'H}}{=} \lim_{t\to-\infty}\frac{-1}{e^{-t}} = \lim_{t\to-\infty} -e^t = 0$$

by l'Hospital's Rule. Hence, the first improper integral converges.

For the second integral,

$$\begin{aligned}\int_{0}^{\infty} xe^x\,dx &= \lim_{t\to\infty}\int_{0}^{t} xe^x\,dx \\ &= \lim_{t\to\infty}(xe^x - e^x)\Big|_{0}^{t} \\ &= \lim_{t\to\infty}(te^t - e^t + 1) \\ &= \lim_{t\to\infty}((t-1)e^t + 1) \\ &= \infty.\end{aligned}$$

Thus, $\int_{0}^{\infty} xe^x\,dx$ diverges. Since this integral diverges, $\int_{-\infty}^{\infty} xe^x\,dx$ diverges as well.

Checkpoint 3.7.5

Evaluate

$$\int_{-3}^{\infty} e^{-x}\,dx.$$

State whether the improper integral converges or diverges.

Answer

It converges to e^3.

Integrating a Discontinuous Integrand

Now, let's examine integrals of functions containing an infinite discontinuity in the interval over which the integration occurs. Consider an integral of the form $\int_a^b f(x)\,dx$, where $f(x)$ is continuous over $[a,b)$ and discontinuous at b. Since the function $f(x)$ is continuous over $[a,t]$ for all values of t satisfying $a \le t < b$, the integral $\int_a^t f(x)\,dx$ is defined for all such values of t. Thus, it makes sense to consider the values of $\int_a^t f(x)\,dx$ as t approaches b for $a \le t < b$. That is, we define $\int_a^b f(x)\,dx \equiv \lim_{t\to b^-} \int_a^t f(x)\,dx$, provided this limit exists. Figure 3.7.5 illustrates $\int_a^t f(x)\,dx$ as areas of regions for values of t approaching b.

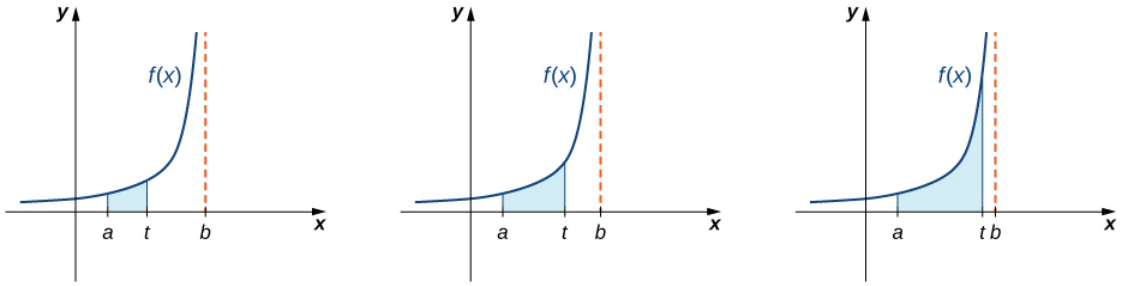


Figure 3.7.5: As t approaches b from the left, the value of the area from a to t approaches the area from a to b.

We use a similar approach to define $\int_a^b f(x)\,dx$, where $f(x)$ is continuous over $(a,b]$ and discontinuous at a. We now proceed with a formal definition.

Definition: Converging and Diverging Improper Integral

1. Let $f(x)$ be continuous over $[a,b)$. Then,

$$\int_a^b f(x)\,dx \equiv \lim_{t\to b^-} \int_a^t f(x)\,dx, \tag{3.7.4}$$

provided this limit exists.

2. Let $f(x)$ be continuous over $(a,b]$. Then,

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx, \tag{3.7.5}$$

provided this limit exists.

3. If $f(x)$ is continuous over $[a,b]$ except at a point c in (a,b), then

$$\int_a^b f(x)\,dx \equiv \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, \tag{3.7.6}$$

provided both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ converge. If either of these integrals diverges, then $\int_a^b f(x)\,dx$ diverges.

In each case, if the limit exists, then the improper integral is said to converge. If the limit does not exist, then the improper integral is said to diverge.

The following examples demonstrate the application of this definition.

Example 3.7.6: Integrating a Discontinuous Integrand

Evaluate

$$\int_0^4 \frac{1}{\sqrt{4-x}}\,dx,$$

if possible. State whether the integral converges or diverges.

Solution

The function $f(x) = \frac{1}{\sqrt{4-x}}$ is continuous over $[0,4)$ and discontinuous at 4. Using Equation 3.7.4 from the definition, rewrite $\int_0^4 \frac{1}{\sqrt{4-x}}\,dx$ as a limit:

$$\begin{aligned}\int_0^4 \frac{1}{\sqrt{4-x}}\,dx &= \lim_{t\to 4^-}\int_0^t \frac{1}{\sqrt{4-x}}\,dx \\ &= \lim_{t\to 4^-}\left(-2\sqrt{4-x}\right)\Big|_0^t \\ &= \lim_{t\to 4^-}\left(-2\sqrt{4-t}+4\right) \\ &= 4.\end{aligned}$$

The improper integral converges (to 4).

Example 3.7.7: Integrating a Discontinuous Integrand

Evaluate

$$\int_0^2 x\ln x\,dx.$$

State whether the integral converges or diverges.

Solution

Since $f(x) = x\ln x$ is continuous over $(0,2]$ and is discontinuous at zero, we can rewrite the integral in limit form using Equation 3.7.5:

$$\begin{aligned}\int_0^2 x\ln x\,dx &= \lim_{t\to 0^+}\int_t^2 x\ln x\,dx \\ &= \lim_{t\to 0^+}\left(\frac{1}{2}x^2\ln x - \frac{1}{4}x^2\right)\Big|_t^2 \\ &= \lim_{t\to 0^+}\left(2\ln 2 - 1 - \frac{1}{2}t^2\ln t + \frac{1}{4}t^2\right) \\ &= 2\ln 2 - 1\end{aligned}$$

Note that $\lim_{t\to 0^+} t^2\ln t$ is indeterminate of the form $0\cdot\infty$. Therefore,

$$\lim_{t\to 0^+} t^2 \ln t = \lim_{t\to 0^+} \frac{\ln t}{t^{-2}} \qquad \left(\text{indeterminate of the form } \frac{\infty}{\infty}\right)$$

$$\overset{\text{l'H}}{=} \lim_{t\to 0^+} \frac{t^{-1}}{-2t^{-3}}$$

$$= \lim_{t\to 0^+} -\frac{t^2}{2}$$

$$= 0$$

Thus, the improper integral converges to $2\ln 2 - 1$.

Example 3.7.8: Integrating a Discontinuous Integrand

Evaluate

$$\int_{-1}^{1} \frac{1}{x^3}\, dx.$$

State whether the improper integral converges or diverges.

Solution

Since $f(x) = \frac{1}{x^3}$ is discontinuous at zero, using Equation 3.7.6, we can write

$$\int_{-1}^{1} \frac{1}{x^3}\, dx = \int_{-1}^{0} \frac{1}{x^3}\, dx + \int_{0}^{1} \frac{1}{x^3}\, dx.$$

If either of the two integrals diverges, the original integral diverges. Begin with $\int_{-1}^{0} \frac{1}{x^3}\, dx$:

$$\int_{-1}^{0} \frac{1}{x^3}\, dx = \lim_{t\to 0^-} \int_{-1}^{t} \frac{1}{x^3}\, dx$$

$$= \lim_{t\to 0^-} \left(-\frac{1}{2x^2}\right)\Big|_{-1}^{t}$$

$$= \lim_{t\to 0^-} \left(-\frac{1}{2t^2} + \frac{1}{2}\right)$$

$$= \infty.$$

Therefore, $\int_{-1}^{0} \frac{1}{x^3}\, dx$ diverges.

Since $\int_{-1}^{0} \frac{1}{x^3}\, dx$ diverges, $\int_{-1}^{1} \frac{1}{x^3}\, dx$ also diverges.

Checkpoint 3.7.8

Evaluate

$$\int_{0}^{2} \frac{1}{x}\, dx.$$

State whether the integral converges or diverges.

Answer

∞, It diverges.

The Comparison Test for Improper Integrals

It is not always easy, or even possible, to evaluate an improper integral directly; however, by comparing it with another carefully chosen integral, it may be possible to determine its convergence or divergence. To see this, consider two continuous functions $f(x)$ and $g(x)$ satisfying $0 \le f(x) \le g(x)$ for $x \ge a$ (Figure 3.7.6). In this case, we may view integrals of these functions over intervals of the form $[a, t]$ as areas, so we have the relationship

$$0 \le \int_a^t f(x)\,dx \le \int_a^t g(x)\,dx$$

for $t \ge a$.

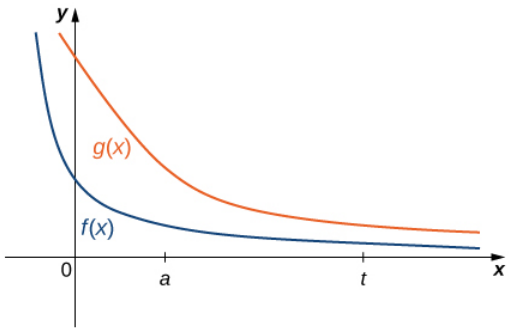


Figure 3.7.6: If $0 \le f(x) \le g(x)$ for $x \ge a$, then for $t \ge a$, $\int_a^t f(x)\,dx \le \int_a^t g(x)\,dx$.

Thus, if

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx = \infty,$$

then

$$\int_a^\infty g(x)\,dx = \lim_{t\to\infty} \int_a^t g(x)\,dx = \infty$$

as well. That is, if the area of the region between the graph of $f(x)$ and the x-axis over $[a, \infty)$ is infinite, then the area of the region between the graph of $g(x)$ and the x-axis over $[a, \infty)$ is infinite too.

On the other hand, if

$$\int_a^\infty g(x)\,dx = \lim_{t\to\infty} \int_a^t g(x)\,dx = L$$

for some real number L, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

must converge to some value *less than or equal to* L, since $\int_a^t f(x)\,dx$ increases as t increases and $\int_a^t f(x)\,dx \le L$ for all $t \ge a$.

If the area of the region between the graph of $g(x)$ and the x-axis over $[a, \infty)$ is finite, then the area of the region between the graph of $f(x)$ and the x-axis over $[a, \infty)$ is also finite.

These conclusions are summarized in the following theorem.

Theorem: Comparison Test for Improper Integrals

Let $f(x)$ and $g(x)$ be continuous over $[a, \infty)$. Assume that $0 \le f(x) \le g(x)$ for $x \ge a$.

i. If

$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx = \infty,$$

then

$$\int_a^{\infty} g(x)\,dx = \lim_{t\to\infty}\int_a^t g(x)\,dx = \infty.$$

That is, if the area between $f(x)$ and the x-axis is divergent (in this case, infinite), then so is the area between $g(x)$ and the x-axis.

ii. If

$$\int_a^{\infty} g(x)\,dx = \lim_{t\to\infty}\int_a^t g(x)\,dx = L,$$

where L is a real number, then

$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx = M$$

for some real number $M \le L$. That is, if the area between $g(x)$ and the x-axis is convergent (i.e., finite), then so is the area between $f(x)$ and the x-axis.

Example 3.7.9: Applying the Comparison Theorem

Use the Comparison Test for Improper Integrals to show that

$$\int_1^{\infty} \frac{1}{xe^x}\,dx$$

converges.

Solution

We can see that $0 \le \frac{1}{xe^x} \le \frac{1}{e^x} = e^{-x}$, so if $\int_1^{\infty} e^{-x}\,dx$ converges, then so does $\int_1^{\infty} \frac{1}{xe^x}\,dx$.

To evaluate $\int_1^{\infty} e^{-x}\,dx$, first rewrite it as a limit:

$$\begin{aligned}\int_1^{\infty} e^{-x}\,dx &= \lim_{t\to\infty}\int_1^t e^{-x}\,dx \\ &= \lim_{t\to\infty}\left(-e^{-x}\right)\Big|_1^t \\ &= \lim_{t\to\infty}\left(-e^{-t}+e^{-1}\right) \\ &= e^{-1}.\end{aligned}$$

Since $\int_1^{\infty} e^{-x}\,dx$ converges, so does $\int_1^{\infty} \frac{1}{xe^x}\,dx$. Note that we do not know the value the improper integral converges to, but we know it is less than or equal to e^{-1}.

Finding a function to compare an integrand to is an art form. I always advise my students to look for a *similar* function that they can integrate. Be sure to set up the bounding behavior. That is, once you find a function to compare your integrand with, be sure to develop an inequality relating the two functions.

The Comparison Test for Improper Integrals gives us the following very useful theorem.

Theorem: p-Integral Test

The improper integral

$$\int_a^\infty \frac{1}{x^p}\,dx$$

where $a > 0$, converges for $p > 1$ and diverges otherwise.

Proof

Assuming for now that $p \neq 1$,

$$\begin{aligned}\int_a^\infty \frac{1}{x^p}\,dx &= \lim_{t\to\infty}\int_a^t \frac{1}{x^p}\,dx \\ &= \lim_{t\to\infty}\left(\frac{x^{-p+1}}{-p+1}\right)\Big|_a^t \\ &= \frac{1}{1-p}\lim_{t\to\infty}\left(t^{1-p}-a^{1-p}\right)\end{aligned}$$

If $p > 1$, then $0 > 1-p$ and so $\lim_{t\to\infty} t^{1-p} = 0$. Hence, the improper integral converges (specifically, it converges to $\frac{a^{1-p}}{p-1}$).

On the other hand, if $p < 1$, then $0 < 1-p$ and so $\lim_{t\to\infty} t^{1-p} = \infty$. Hence, the improper integral diverges.

Finally, if $p = 1$, then

$$\begin{aligned}\int_a^\infty \frac{1}{x^p}\,dx &= \lim_{t\to\infty}\int_a^t \frac{1}{x}\,dx \\ &= \lim_{t\to\infty}\left(\ln(x)\right)\Big|_a^t \\ &= \lim_{t\to\infty}\left(\ln(t)-\ln(a)\right)\end{aligned}$$

However, $\lim_{t\to\infty}\ln(t) = \infty$. Hence, the improper integral diverges.

Therefore,

$$\int_a^\infty \frac{1}{x^p}\,dx$$

where $a > 0$, converges for $p > 1$ and diverges otherwise.

Example 3.7.10: Applying the Comparison Theorem

Determine whether the improper integral is convergent or divergent.

$$\int_3^\infty \frac{\sin^2(x)}{x^2+x+1}\,dx$$

Solution

For $x \geq 3$,

$$x^2 \leq x^2 + x + 1.$$

Therefore,

$$\frac{1}{x^2} \geq \frac{1}{x^2+x+1}.$$

Moreover, since $1 \geq \sin^2(x) \geq 0$,

$$\frac{1}{x^2} \geq \frac{1}{x^2+x+1} \geq \frac{\sin^2(x)}{x^2+x+1} \geq 0.$$

Thus,

$$0 \leq \frac{\sin^2(x)}{x^2+x+1} \leq \frac{1}{x^2}.$$

Since $\int_3^\infty \frac{1}{x^2}\,dx$ converges by the p-Integral Test, the Comparison Test for Improper Integrals implies

$$\int_3^\infty \frac{\sin^2(x)}{x^2+x+1}\,dx$$

converges as well.

? Checkpoint 3.7.10

Show that

$$\int_e^\infty \frac{\ln x}{x}\,dx$$

diverges.

Answer

Since $\int_e^\infty \frac{1}{x}\,dx = \infty$, $\int_e^\infty \frac{\ln x}{x}\,dx$ diverges.

Laplace Transforms

In the last chapter, we have looked at several ways to use integration for solving real-world problems. For this next project, we will explore a more advanced application of integration: **integral transforms**. Specifically, we describe the Laplace transform and some of its properties.

The **Laplace transform** is used in engineering and physics to simplify the computations needed to solve some problems. It takes functions expressed in terms of time and *transforms* them to functions expressed in terms of frequency. In many cases, the computations needed to solve problems in the frequency domain are much simpler than those required in the time domain.

The Laplace transform is defined in terms of an integral as

$$\mathscr{L}(f(t)) = F(s) = \int_0^\infty e^{-st} f(t)\,dt.$$

Note that the input to a Laplace transform is a function of time, $f(t)$, and the output is a function of frequency, $F(s)$. Although many real-world examples require the use of complex numbers (involving the imaginary number $i = \sqrt{-1}$), in this project, we limit ourselves to functions of real numbers.

Let's start with a simple example. We calculate the Laplace transform of $f(t) = t$. We have

$$\mathscr{L}(t) = \int_0^\infty te^{-st}\,dt.$$

This is an improper integral, so we express it in terms of a limit, which gives

$$\mathscr{L}(t) = \int_0^\infty te^{-st}\,dt = \lim_{z\to\infty} \int_0^z te^{-st}\,dt.$$

We use Integration by Parts to evaluate the integral. Note that we are integrating with respect to t, so we treat the variable s as a constant. We have

$$u = t \qquad dv = e^{-st}\,dt$$

$$du = dt \quad v = -\frac{1}{s}e^{-st}$$

Then we obtain

$$\begin{aligned}
\lim_{z\to\infty}\int_0^z te^{-st}\,dt &= \lim_{z\to\infty}\left[-\frac{t}{s}e^{-st}\Big|_0^z + \frac{1}{s}\int_0^z e^{-st}\,dt\right] \\
&= \lim_{z\to\infty}\left[\left(-\frac{z}{s}e^{-sz} + \frac{0}{s}e^{-0s}\right) + \frac{1}{s}\int_0^z e^{-st}\,dt\right] \\
&= \lim_{z\to\infty}\left[\left(-\frac{z}{s}e^{-sz} + 0\right) - \frac{1}{s}\left(\frac{e^{-st}}{s}\right)\Big|_0^z\right] \\
&= \lim_{z\to\infty}\left[-\frac{z}{s}e^{-sz} - \frac{1}{s^2}\left(e^{-sz} - 1\right)\right] \\
&= \lim_{z\to\infty}\left[-\frac{z}{se^{sz}}\right] - \lim_{z\to\infty}\frac{1}{s^2e^{sz}} + \lim_{z\to\infty}\frac{1}{s^2} \\
&= 0 - 0 + \frac{1}{s^2} \\
&= \frac{1}{s^2}.
\end{aligned}$$

Note, $\lim\limits_{z\to\infty}\left[-\frac{z}{se^{sz}}\right]$ is indeterminate of the form $-\frac{\infty}{\infty}$ and requires l'Hospital's Rule to evaluate. Using l'Hospital's Rule,

$$\lim_{z\to\infty}\left[-\frac{z}{se^{sz}}\right] = \lim_{z\to\infty}\left[-\frac{1}{s^2e^{sz}}\right] = 0.$$

Laplace transforms are often used to solve differential equations. While we saw some differential equations as homework problems in Differential Calculus, they are not covered in detail until later in this book; but, for now, let's look at the relationship between the Laplace transform of a function and the Laplace transform of its derivative.

Let's start with the definition of the Laplace transform. We have

$$\mathscr{L}\big(f(t)\big) = \int_0^\infty e^{-st}f(t)\,dt = \lim_{z\to\infty}\int_0^z e^{-st}f(t)\,dt.$$

Use Integration by Parts to evaluate $\lim\limits_{z\to\infty}\int_0^z e^{-st}f(t)\,dt$. (Let $u = f(t)$ and $dv = e^{-st}dt$.)

After integrating by parts and evaluating the limit, you should see that

$$\mathscr{L}\big(f(t)\big) = \frac{f(0)}{s} + \frac{1}{s}\left[\mathscr{L}\big(f'(t)\big)\right].$$

Then,

$$\mathscr{L}\big(f'(t)\big) = s\mathscr{L}\big(f(t)\big) - f(0).$$

Thus, differentiation in the time domain simplifies to multiplication by s in the frequency domain.

The final thing we look at in this project is how the Laplace transforms of $f(t)$ and its antiderivative are related. Let $g(t) = \int_0^t f(u)\,du$.

Then,

$$\mathscr{L}\big(g(t)\big) = \int_0^{\infty} e^{-st} g(t)\, dt = \lim_{z\to\infty} \int_0^{z} e^{-st} g(t)\, dt.$$

Use Integration by Parts to evaluate $\lim_{z\to\infty} \int_0^{z} e^{-st} g(t)\, dt$. (Let $u = g(t)$ and $dv = e^{-st} dt$. Note, by the way that we have defined $g(t)$, $du = f(t)\, dt$.)

As you might expect, you should see that

$$\mathscr{L}\big(g(t)\big) = \frac{1}{s} \cdot \mathscr{L}\big(f(t)\big).$$

Integration in the time domain simplifies to division by s in the frequency domain.

Now it's your turn.

1. Calculate the Laplace transform of $f(t) = 1$.
2. Calculate the Laplace transform of $f(t) = e^{-3t}$.
3. Calculate the Laplace transform of $f(t) = t^2$. (Note: you will have to integrate by parts twice.)

Footnotes

[1] Recall that the notation "$\equiv$" means "is equivalent to," and is often used in Mathematics to clearly state that an expression is *defined* to be some other expression. In this case, $\int_a^{\infty} f(x)\, dx$ is *defined* to be $\lim_{t\to\infty} \int_a^{t} f(x)\, dx$.

[2] In your Differential Calculus class, your instructor should have been very clear that the words "provided this limit exists" means "provided this limit exists as a *finite number*." That is, if $\lim_{t\to a} g(t) = \infty$, we would say that this limit does not exist (as a finite number). We still use the notation $\lim_{t\to a} g(t) = \infty$ to describe the divergent behavior of the limit, but, in truth, the limit does not exist. This distinction will be critical when dealing with improper integrals.

[3] Gabriel's horn (also called **Torricelli's Trumpet**) is a geometric figure which has infinite surface area, but finite volume. The name refers to the tradition identifying the Archangel Gabriel as the angel who blows the horn to announce Judgment Day, associating the divine, or infinite, with the finite. Italian physicist and mathematician Evangelista Torricelli first studied the properties of this figure in the 17th century.

Key Concepts

- Integrals of functions over infinite intervals are defined in terms of limits.
- Integrals of functions over an interval for which the function has a discontinuity at an endpoint may be defined in terms of limits.
- The convergence or divergence of an improper integral may be determined by comparing it with the value of an improper integral for which the convergence or divergence is known.

Key Equations

- **Improper integrals**

$$\int_a^{+\infty} f(x)\, dx = \lim_{t\to+\infty} \int_a^{t} f(x)\, dx$$

$$\int_{-\infty}^{b} f(x)\, dx = \lim_{t\to-\infty} \int_t^{b} f(x)\, dx$$

$$\int_{-\infty}^{+\infty} f(x)\, dx = \int_{-\infty}^{0} f(x)\, dx + \int_0^{+\infty} f(x)\, dx$$

Glossary

improper integral

an integral over an infinite interval or an integral of a function containing an infinite discontinuity on the interval; an improper integral is defined as a limit. The improper integral converges if this limit is a finite real number; otherwise, the improper integral diverges

Review Topics (from previous courses)

Algebra

- function

Differential Calculus

- continuous function
- definite integrals, properties of
- discontinuities, their types, and discontinuous functions
- indeterminate forms and l'Hospital's Rule
- limits (precise definition of infinite limits at infinity and finite limits at infinity)
- limit of integration

3.7E: Exercises

In exercises 1 - 8, evaluate the following integrals. If the integral is not convergent, answer "It diverges."

1) $\int_2^4 \frac{dx}{(x-3)^2}$

Answer

It diverges.

2) $\int_0^\infty \frac{1}{4+x^2}\,dx$

3) $\int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx$

Answer

Converges to $\frac{\pi}{2}$

4) $\int_1^\infty \frac{1}{x\ln x}\,dx$

5) $\int_1^\infty xe^{-x}\,dx$

Answer

Converges to $\frac{2}{e}$

6) $\int_{-\infty}^\infty \frac{x}{x^2+1}\,dx$

7) Without integrating, determine whether the integral $\int_1^\infty \frac{1}{\sqrt{x^3+1}}\,dx$ converges or diverges by comparing the function $f(x) = \frac{1}{\sqrt{x^3+1}}$ with $g(x) = \frac{1}{\sqrt{x^3}}$.

Answer

It converges.

8) Without integrating, determine whether the integral $\int_1^\infty \frac{1}{\sqrt{x+1}}\,dx$ converges or diverges.

In exercises 9 - 25, determine whether the improper integrals converge or diverge. If possible, determine the value of the integrals that converge.

9) $\int_0^\infty e^{-x}\cos x\,dx$

Answer

Converges to $\frac{1}{2}$.

10) $\int_1^\infty \frac{\ln x}{x}\,dx$

11) $\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx$

Answer

Converges to -4.

12) $\int_0^1 \ln x\,dx$

13) $\int_{-\infty}^\infty \frac{1}{x^2+1}\,dx$

Answer

Converges to π.

14) $\int_1^5 \frac{dx}{\sqrt{x-1}}$

15) $\int_{-2}^2 \frac{dx}{(1+x)^2}$

Answer

It diverges.

16) $\int_0^\infty e^{-x}\,dx$

17) $\int_0^\infty \sin x\,dx$

Answer

It diverges.

18) $\int_{-\infty}^\infty \frac{e^x}{1+e^{2x}}\,dx$

19) $\int_0^1 \frac{dx}{\sqrt[3]{x}}$

Answer

Converges to 1.5.

20) $\int_0^2 \frac{dx}{x^3}$

21) $\int_{-1}^2 \frac{dx}{x^3}$

Answer

It diverges.

22) $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

23) $\int_0^3 \frac{1}{x-1}\,dx$

Answer

It diverges.

24) $\int_1^\infty \frac{5}{x^3}\,dx$

25) $\int_3^5 \frac{5}{(x-4)^2}\,dx$

Answer

It diverges.

In exercises 26 and 27, determine the convergence of each of the following integrals by comparison with the given integral. If the integral converges, find the number to which it converges.

26) $\int_1^\infty \frac{dx}{x^2+4x}$; compare with $\int_1^\infty \frac{dx}{x^2}$.

27) $\int_1^\infty \frac{dx}{\sqrt{x}+1}$; compare with $\int_1^\infty \frac{dx}{2\sqrt{x}}$.

Answer

Both integrals diverge.

In exercises 28 - 38, evaluate the integrals. If the integral diverges, answer "It diverges."

28) $\int_1^\infty \frac{dx}{x^e}$

29) $\int_0^1 \frac{dx}{x^\pi}$

Answer

It diverges.

30) $\int_0^1 \frac{dx}{\sqrt{1-x}}$

31) $\int_0^1 \frac{dx}{1-x}$

Answer

It diverges.

32) $\int_{-\infty}^0 \frac{dx}{x^2+1}$

33) $\int_{-1}^1 \frac{dx}{\sqrt{1-x^2}}$

Answer

Converges to π.

34) $\int_0^1 \frac{\ln x}{x}\,dx$

35) $\int_0^e \ln(x)\,dx$

Answer

Converges to **0**.

36) $\int_0^\infty xe^{-x}\,dx$

37) $\int_{-\infty}^\infty \frac{x}{(x^2+1)^2}\,dx$

Answer

Converges to **0**.

38) $\int_0^\infty e^{-x}\,dx$

In exercises 39 - 44, evaluate the improper integrals. Each of these integrals has an infinite discontinuity either at an endpoint or at an interior point of the interval.

39) $\int_0^9 \frac{dx}{\sqrt{9-x}}$

Answer

Converges to **6**.

40) $\int_{-27}^1 \frac{dx}{x^{2/3}}$

41) $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$

Answer

Converges to $\frac{\pi}{2}$.

42) $\int_6^{24} \frac{dt}{t\sqrt{t^2-36}}$

43) $\int_0^4 x\ln(4x)\,dx$

Answer

Converges to $8\ln(16)-4$.

44) $\int_0^3 \frac{x}{\sqrt{9-x^2}}\,dx$

45) Evaluate $\int_{.5}^t \frac{dx}{\sqrt{1-x^2}}$. (Be careful!) (Express your answer using three decimal places.)

Answer

Converges to about 1.047.

46) Evaluate $\int_1^4 \frac{dx}{\sqrt{x^2-1}}$. (Express the answer in exact form.)

47) Evaluate $\int_2^\infty \frac{dx}{(x^2-1)^{3/2}}$.

Answer

Converges to $-1+\frac{2}{\sqrt{3}}$.

48) Find the area of the region in the first quadrant between the curve $y=e^{-6x}$ and the x-axis.

49) Find the area of the region bounded by the curve $y=\frac{7}{x^2}$, the x-axis, and on the left by $x=1$.

Answer

$A=7.0$ units.^{2}

50) Find the area under the curve $y=\frac{1}{(x+1)^{3/2}}$, bounded on the left by $x=3$.

51) Find the area under $y=\frac{5}{1+x^2}$ in the first quadrant.

Answer

$A=\frac{5\pi}{2}$ units.^{2}

52) Find the volume of the solid generated by revolving about the x-axis the region under the curve $y=\frac{3}{x}$ from $x=1$ to $x=\infty$.

53) Find the volume of the solid generated by revolving about the y-axis the region under the curve $y=6e^{-2x}$ in the first quadrant.

Answer

$V=3\pi$ units^3

54) Find the volume of the solid generated by revolving about the x-axis the area under the curve $y=3e^{-x}$ in the first quadrant.

The **Laplace transform** of a continuous function over the interval $[0,\infty)$ is defined by $F(s)=\int_0^\infty e^{-sx}f(x)\,dx$ (see the Student Project). This definition is used to solve some important initial-value problems in differential equations, as discussed later. The

domain of F is the set of all real numbers s such that the improper integral converges. Find the Laplace transform F of each of the following functions and give the domain of F.

55) $f(x) = 1$

Answer

$\frac{1}{s}, \quad s > 0$

56) $f(x) = x$

57) $f(x) = \cos(2x)$

Answer

$\frac{s}{s^2 + 4}, \quad s > 0$

58) $f(x) = e^{ax}$

59) Use the formula for arc length to show that the circumference of the circle $x^2 + y^2 = 1$ is 2π.

Answer

Answers will vary.

A function is a **probability density function** if it satisfies the following definition: $\int_{-\infty}^{\infty} f(t)\,dt = 1$. The probability that a random variable x lies between a and b is given by $P(a \le x \le b) = \int_a^b f(t)\,dt$.

60) Show that $f(x) = \begin{cases} 0, & \text{if } x < 0 \\ 7e^{-7x}, & \text{if } x \ge 0 \end{cases}$ is a probability density function.

61) Find the probability that x is between 0 and 0.3. (Use the function defined in the preceding problem.) Use four-place decimal accuracy.

Answer

0.8775

This page titled 3.7E: Exercises is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Roy Simpson.

- **7.7E: Exercises for Section 7.7** by Edwin “Jed” Herman, Gilbert Strang is licensed CC BY-NC-SA 4.0. Original source: https://openstax.org/details/books/calculus-volume-1.

3.8: Chapter 3 Review Exercises

In exercises 1 - 4, determine whether the statement is true or false. Justify your answer with a proof or a counterexample.

1) $\int e^x \sin(x)\,dx$ cannot be integrated by parts.

2) $\int \frac{1}{x^4+1}\,dx$ cannot be integrated using partial fractions.

Answer

False

3) In numerical integration, increasing the number of points decreases the error.

4) Integration by parts can always yield the integral.

Answer

False

In exercises 5 - 10, evaluate the integral using the specified method.

5) $\int x^2 \sin(4x)\,dx$, using integration by parts

6) $\int \frac{1}{x^2\sqrt{x^2+16}}\,dx$, using trigonometric substitution

Answer

$$\int \frac{1}{x^2\sqrt{x^2+16}}\,dx = -\frac{\sqrt{x^2+16}}{16x} + C$$

7) $\int \sqrt{x}\ln x\,dx$, using integration by parts

8) $\int \frac{3x}{x^3+2x^2-5x-6}\,dx$, using partial fractions

Answer

$$\int \frac{3x}{x^3+2x^2-5x-6}\,dx = \frac{1}{10}\left(4\ln|2-x| + 5\ln|x+1| - 9\ln|x+3|\right) + C$$

9) $\int \frac{x^5}{(4x^2+4)^{5/2}}\,dx$, using trigonometric substitution

10) $\int \frac{\sqrt{4-\sin^2(x)}}{\sin^2(x)}\cos(x)\,dx$, using a table of integrals or a CAS

Answer

$$\int \frac{\sqrt{4-\sin^2(x)}}{\sin^2(x)}\cos(x)\,dx = -\frac{\sqrt{4-\sin^2(x)}}{\sin(x)} - \frac{x}{2} + C$$

In exercises 11 - 15, integrate using whatever method you choose.

11) $\int \sin^2 x \cos^2 x\,dx$

12) $\int x^3\sqrt{x^2+2}\,dx$

Answer

$$\int x^3\sqrt{x^2+2}\,dx = \frac{1}{15}(x^2+2)^{3/2}(3x^2-4) + C$$

13) $\int \frac{3x^2+1}{x^4-2x^3-x^2+2x}\,dx$

14) $\int \frac{1}{x^4+4}\,dx$

Answer

$$\int \frac{1}{x^4+4}\,dx = \frac{1}{16}\ln\left(\frac{x^2+2x+2}{x^2-2x+2}\right) - \frac{1}{8}\tan^{-1}(1-x) + \frac{1}{8}\tan^{-1}(x+1) + C$$

15) $\int \frac{\sqrt{3+16x^4}}{x^4}\,dx$

In exercises 16 - 18, approximate the integrals using the midpoint rule, trapezoidal rule, and Simpson's rule using four subintervals, rounding to three decimals.

16) [T] $\int_1^2 \sqrt{x^5+2}\,dx$

Answer

$M_4 = 3.312,$
$T_4 = 3.354,$
$S_4 = 3.326$

17) [T] $\int_0^{\sqrt{\pi}} e^{-\sin(x^2)}\,dx$

18) [T] $\int_1^4 \frac{\ln(1/x)}{x}\,dx$

Answer

$M_4 = -0.982,$
$T_4 = -0.917,$
$S_4 = -0.952$

In exercises 19 - 20, evaluate the integrals, if possible.

19) $\int_1^\infty \frac{1}{x^n}\,dx$, for what values of n does this integral converge or diverge?

20) $\int_1^\infty \frac{e^{-x}}{x}\,dx$

Answer

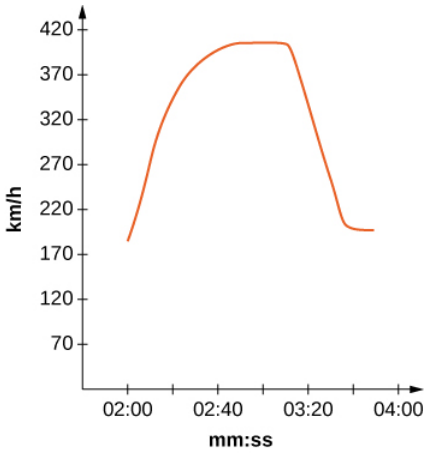
approximately 0.2194

In exercises 21 - 22, consider the gamma function given by $\Gamma(a) = \int_0^\infty e^{-y}y^{a-1}\,dy$.

21) Show that $\Gamma(a) = (a-1)\Gamma(a-1)$.

22) Extend to show that $\Gamma(a) = (a-1)!$, assuming a is a positive integer.

The fastest car in the world, the Bugati Veyron, can reach a top speed of 408 km/h. The graph represents its velocity.



23) [T] Use the graph to estimate the velocity every 20 sec and fit to a graph of the form $v(t) = ae^{bx}\sin(cx) + d$. (Hint: Consider the time units.)

24) [T] Using your function from the previous problem, find exactly how far the Bugati Veyron traveled in the 1 min 40 sec included in the graph.

Answer

Answers may vary. Ex: 9.405 km

3.8: Chapter 3 Review Exercises is shared under a not declared license and was authored, remixed, and/or curated by LibreTexts.

- **7.R: Chapter 7 Review Exercises** by Edwin “Jed” Herman, Gilbert Strang is licensed CC BY-NC-SA 4.0. Original source: https://openstax.org/details/books/calculus-volume-1.

CHAPTER OVERVIEW

4: Sequences and Series

The topic of infinite series may seem unrelated to differential and integral calculus. In fact, an infinite series whose terms involve powers of a variable is a powerful tool that we can use to express functions as “infinite polynomials.” We can use infinite series to evaluate complicated functions, approximate definite integrals, and create new functions. In addition, infinite series are used to solve differential equations that model physical behavior, from tiny electronic circuits to Earth-orbiting satellites.

4.1: Sequences

4.1E: Exercises

4.2: Infinite Series

4.2E: Exercises

4.3: The Divergence and Integral Tests

4.3E: Exercises

4.4: The Comparison Tests

4.4E: Exercises

4.5: The Alternating Series Test

4.5E: Exercises

4.6: The Ratio and Root Tests

4.6E: Exercises

4.7: Chapter 4 Review Exercises

This page titled 4: Sequences and Series is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Roy Simpson.

4.1: Sequences

Learning Objectives

- Find the formula for the general term of a sequence.
- Calculate the limit of a sequence if it exists.
- Determine the convergence or divergence of a given sequence.

In this section, we reintroduce the concept of sequences that you learned back in Algebra. The calculus of sequences allows us to define what it means for a sequence to converge or diverge. We show how to find limits of sequences that converge, often by using the properties of limits for functions discussed in Calculus I. We close this section with the Monotone Convergence Theorem, a tool we can use to prove that certain types of sequences converge.

Terminology of Sequences

We need new terms and definitions for this topic. First, an **infinite sequence** is an ordered list of numbers of the form

$$a_1, a_2, a_3, \ldots, a_n, \ldots.$$

Each of the numbers in the sequence is called a **term**. The symbol n is called the **index variable** for the sequence. We use the notation

$$\{a_n\}_{n=1}^{\infty},$$

or simply $\{a_n\}$, to denote this sequence.[1] Because, in general, a particular number a_n exists for each positive integer n, we can also define a sequence as a function whose domain is the set of positive integers.

Let's consider the infinite, ordered list

$$2, 4, 8, 16, 32, \ldots.$$

This is a sequence in which the first, second, and third terms are given by $a_1 = 2$, $a_2 = 4$, and $a_3 = 8$. You can probably see that the terms in this sequence have the following pattern:

$$a_1 = 2^1,\ a_2 = 2^2,\ a_3 = 2^3,\ a_4 = 2^4 \text{ and } a_5 = 2^5.$$

Assuming this pattern continues, we can write the nth term in the sequence by the explicit formula $a_n = 2^n$. Using this notation, we can write this sequence as

$$\{2^n\}_{n=1}^{\infty},$$

or

$$\{2^n\},$$

or

$$a_n = 2^n. \tag{4.1.1}$$

The formula in Equation 4.1.1 is called an **explicit formula** (or **explicit relation**) for the sequence.

Alternatively, we can describe this sequence differently. Since each term is twice the previous term, this sequence can be defined **recursively** by expressing the nth term, a_n, in terms of the previous term, a_{n-1}. In particular, we can define this sequence as the sequence $\{a_n\}$ where $a_1 = 2$ and, for all $n \geq 2$, each term a_n is defined by the **recurrence relation**

$$a_n = 2a_{n-1}.$$

In a recurrence relation, one term (or more) of the sequence is given explicitly, and subsequent terms are defined in terms of earlier terms in the sequence.

Note that the index does not have to start at $n = 1$ but could start with other integers. For example, a sequence given by the explicit formula $a_n = f(n)$ could start at $n = 0$, in which case the sequence would be

$$a_0, a_1, a_2, \ldots.$$

Similarly, for a sequence defined by a recurrence relation, the term a_0 may be given explicitly, and the terms a_n for $n \geq 1$ may be defined in terms of a_{n-1}.

The graph of a sequence $\{a_n\}$ consists of all points (n, a_n) for all positive integers n.[2] Figure 4.1.1 shows the graph of 2^n.

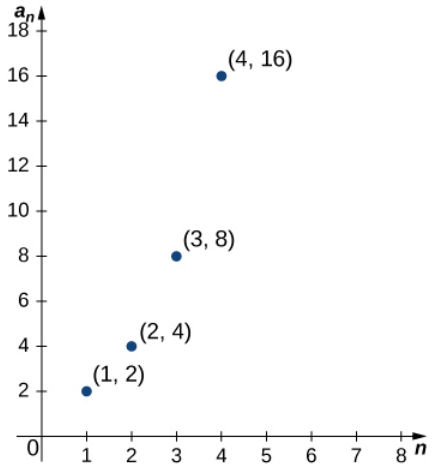


Figure 4.1.1: The plotted points are a graph of the sequence $\{2^n\}$.

Two types of sequences occur often and are given special names: **arithmetic sequences** and **geometric sequences**. The difference between every pair of consecutive terms is the same in an arithmetic sequence. For example, consider the sequence

$$3, 7, 11, 15, 19, \ldots.$$

You can see that the difference between every consecutive pair of terms is 4. Assuming that this pattern continues, this sequence is an arithmetic sequence. It can be described by using the recurrence relation

$$\begin{cases} a_1 = 3 \\ a_n = a_{n-1} + 4, \text{ for } n \geq 2 \end{cases}.$$

Note that

$$\begin{aligned} a_2 &= 3 + 4 \\ a_3 &= 3 + 4 + 4 = 3 + 2 \cdot 4 \\ a_4 &= 3 + 4 + 4 + 4 = 3 + 3 \cdot 4 \end{aligned}$$

Thus, the sequence can also be described using the explicit formula

$$a_n = 3 + 4(n-1) = 4n - 1.$$

In general, an arithmetic sequence is any sequence that can be written in the form $a_n = cn + b$.

In a geometric sequence, the ratio of every pair of consecutive terms is the same. For example, consider the sequence

$$2, -\frac{2}{3}, \frac{2}{9}, -\frac{2}{27}, \frac{2}{81}, \ldots.$$

We see that the ratio of any term to the preceding term is $-\frac{1}{3}$. Assuming this pattern continues, this sequence is geometric. It can be defined recursively as

$$\begin{cases} a_1 = 2 \\ a_n = -\frac{1}{3} \cdot a_{n-1}, \text{ for } n \geq 2 \end{cases}$$

Alternatively, since

$$a_2 = -\frac{1}{3}\cdot(2)$$

$$a_3 = \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)(2) = \left(-\frac{1}{3}\right)^2\cdot(2)$$

$$a_4 = \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)(2) = \left(-\frac{1}{3}\right)^3\cdot(2),$$

we see that the sequence can be described by using the explicit formula[3]

$$a_n = 2\left(-\frac{1}{3}\right)^{n-1}.$$

The sequence $\{2^n\}$ that we discussed earlier is a geometric sequence where the ratio of any term to the previous term is 2. Generally, a geometric sequence is any sequence that can be written in the form $a_n = cr^n$.

✔ Example 4.1.1: Finding Explicit Formulas

For each of the following sequences, find an explicit formula for the nth term of the sequence.

a. $-\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, -\frac{5}{6}, \ldots$
b. $\frac{3}{4}, \frac{9}{7}, \frac{27}{10}, \frac{81}{13}, \frac{243}{16}, \ldots$.

Solutions

a. First, note that the sequence alternates from negative to positive. The odd terms in the sequence are negative, and the even terms are positive. Therefore, the nth term includes a factor of $(-1)^n$. Next, consider the sequence of numerators $1, 2, 3, \ldots$ and the sequence of denominators $2, 3, 4, \ldots$. We can see that both of these sequences are arithmetic sequences. The nth term in the sequence of numerators is n, and the nth term in the sequence of denominators is $n+1$. Therefore, the sequence can be described by the explicit formula

$$a_n = \frac{(-1)^n n}{n+1}.$$

b. The sequence of numerators $3, 9, 27, 81, 243, \ldots$ is a geometric sequence. The numerator of the nth term is 3^n. The sequence of denominators $4, 7, 10, 13, 16, \ldots$ is an arithmetic sequence. The denominator of the nth term is $4 + 3(n-1) = 3n+1$. Therefore, we can describe the sequence by the explicit formula

$$a_n = \frac{3^n}{3n+1}.$$

? Checkpoint 4.1.1

Find an explicit formula for the nth term of the sequence $\left\{\frac{1}{5}, -\frac{1}{7}, \frac{1}{9}, -\frac{1}{11}, \ldots\right\}$.

Answer

$$a_n = \frac{(-1)^{n+1}}{3+2n}$$

✔ Example 4.1.2: Defined by Recurrence Relations

For each of the following recursively defined sequences, find an explicit formula for the sequence.

a. $a_1 = 2, a_n = -3a_{n-1}$ for $n \geq 2$
b. $a_1 = \left(\frac{1}{2}\right), a_n = a_{n-1} + \left(\frac{1}{2}\right)^n$ for $n \geq 2$

Solutions

a. Writing out the first few terms, we have

$$\begin{aligned} a_1 &= 2 \\ a_2 &= -3a_1 = -3(2) \\ a_3 &= -3a_2 = (-3)^2(2) \\ a_4 &= -3a_3 = (-3)^3(2). \end{aligned}$$

In general,

$$a_n = 2(-3)^{n-1}.$$

b. Write out the first few terms:

$$a_1 = \frac{1}{2}$$

$$a_2 = a_1 + \left(\frac{1}{2}\right)^2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$a_3 = a_2 + \left(\frac{1}{2}\right)^3 = \frac{3}{4} + \frac{1}{8} = \frac{7}{8}$$

$$a_4 = a_3 + \left(\frac{1}{2}\right)^4 = \frac{7}{8} + \frac{1}{16} = \frac{15}{16}.$$

From this pattern, we derive the explicit formula

$$a_n = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}.$$

? Checkpoint 4.1.2

Find an explicit formula for the sequence defined recursively such that $a_1 = -4$ and $a_n = a_{n-1} + 6$.

Answer

$a_n = 6n - 10$

Limit of a Sequence

A fundamental question regarding infinite sequences is the behavior of the terms as n gets larger. Since a sequence is a function defined on the positive integers, it makes sense to discuss the limit of the terms as $n \to \infty$. For example, consider the following four sequences and their different behaviors as $n \to \infty$ (Figure 4.1.2):

a. $\{1 + 3n\} = \{4, 7, 10, 13, ...\}$. The terms $1 + 3n$ become arbitrarily large as $n \to \infty$. In this case, we say that $1 + 3n \to \infty$ as $n \to \infty$.
b. $\{1 - (\frac{1}{2})^n\} = \{\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16} ...\}$. The terms $1 - (\frac{1}{2})^n \to 1$ as $n \to \infty$.
c. $\{(-1)^n\} = \{-1, 1, -1, 1, ...\}$. The terms alternate but do not approach one value as $n \to \infty$.
d. $\left\{\frac{(-1)^n}{n}\right\} = \{-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, ...\}$. The terms alternate for this sequence as well, but $\frac{(-1)^n}{n} \to 0$ as $n \to \infty$.

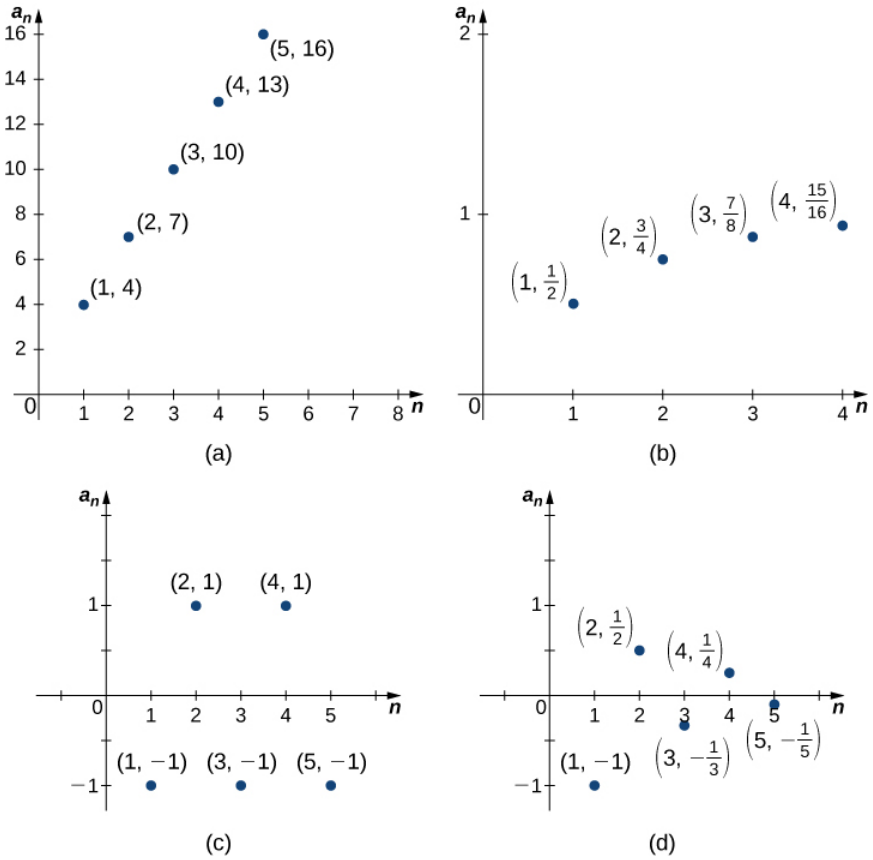


Figure 4.1.2(a): The terms in the sequence become arbitrarily large as $n \to \infty$.

Figure 4.1.2(b): The terms in the sequence approach 1 as $n \to \infty$.

Figure 4.1.2(c): The terms in the sequence alternate between 1 and -1 as $n \to \infty$.

Figure 4.1.2(d): The terms in the sequence alternate between positive and negative values but approach 0 as $n \to \infty$.

From these examples, we see several possibilities for the behavior of the terms of a sequence as $n \to \infty$. The terms approach a finite number in two sequences as $n \to \infty$. In the other two sequences, the terms do not. If the terms of a sequence approach a finite number L as $n \to \infty$, we say that the sequence is a **convergent sequence** and the real number L is the limit of the sequence. We can give an informal definition here.

Definition: The Informal Definition of the Limit of a Sequence

Given a sequence $\{a_n\}$, if the terms a_n become arbitrarily close to a finite number L as n becomes sufficiently large, we say $\{a_n\}$ is a **convergent sequence** and L is the **limit of the sequence**. In this case, we write

$$\lim_{n\to\infty} a_n = L.$$

If a sequence $\{a_n\}$ is not convergent, we say it is a **divergent sequence.**

From Figure 4.1.2(b), we see that the terms in the sequence $\left\{1-\left(\frac{1}{2}\right)^n\right\}$ are becoming arbitrarily close to 1 as n becomes very large. We conclude that $\left\{1-\left(\frac{1}{2}\right)^n\right\}$ is a convergent sequence and its limit is 1. In contrast, from Figure 4.1.2(a), we see that the terms in the sequence $1+3n$ are not approaching a finite number as n becomes larger. We say that $\{1+3n\}$ is a divergent sequence.

In the informal definition for the limit of a sequence, we used the terms "arbitrarily close" and "sufficiently large." Although these phrases help illustrate the meaning of a converging sequence, they could be more specific. To be more precise, we now present the more formal definition of limit for a sequence and show these ideas graphically in Figure 4.1.2.

Definition: The Precise Definition of the Limit of a Sequence

A sequence $\{a_n\}$ converges to a real number L if for all $\epsilon > 0$, there exists an integer N such that for all $n \geq N$, $|a_n - L| < \epsilon$. The number L is the limit of the sequence, and we write

$$\lim_{n\to\infty} a_n = L \text{ or } a_n \to L.$$

In this case, we say the sequence $\{a_n\}$ is a convergent sequence. If a sequence does not converge, it is divergent, and we say the limit does not exist.[4]

Figure **4.1.3** illustrates the precise definition of the limit of a convergent sequence.

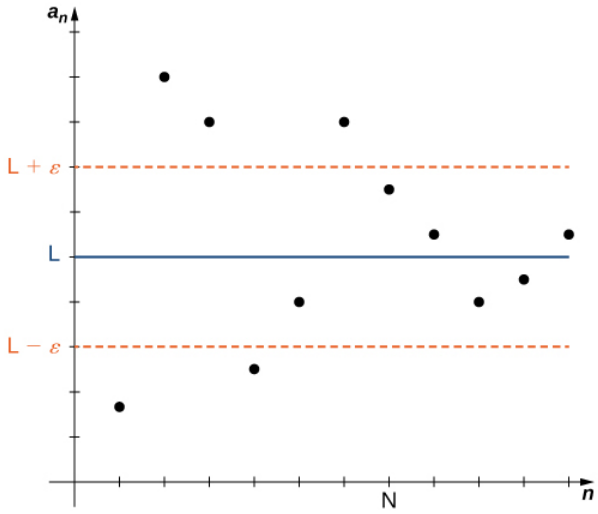


Figure **4.1.3**: As n increases, the terms a_n become closer to L. For values of $n \geq N$, the distance between each point (n, a_n) and the line $y = L$ is less than ϵ.

We remark that the convergence or divergence of a sequence $\{a_n\}$ depends only on what happens to the terms a_n as $n \to \infty$. Therefore, if a finite number of terms $b_1, b_2, \ldots, b_N$ are placed before a_1 to create a new sequence

$$b_1,\ b_2,\ \ldots,\ b_N,\ a_1,\ a_2,\ \ldots,$$

this new sequence will converge if $\{a_n\}$ converges and diverge if $\{a_n\}$ diverges. Furthermore, if the sequence $\{a_n\}$ converges to L, this new sequence will also converge to L.

As defined above, if a sequence does not converge, it is considered a divergent sequence. For example, the sequences $\{1 + 3n\}$ and $\{(-1)^n\}$ shown in Figure **4.1.2** diverge. However, different sequences can diverge in different ways. The sequence $\{(-1)^n\}$ diverges because the terms alternate between 1 and -1, but do not approach one value as $n \to \infty$. On the other hand, the sequence $\{1 + 3n\}$ diverges because the terms $1 + 3n \to \infty$ as $n \to \infty$. We say the sequence $\{1 + 3n\}$ **diverges to infinity** and write $\lim_{n\to\infty}(1 + 3n) = \infty$. It is important to recognize that this notation does not imply the limit of the sequence $\{1 + 3n\}$ exists. The sequence is, in fact, divergent. Writing that the limit is infinite is intended only to provide more information about why the sequence is divergent. A sequence can also diverge to negative infinity. For example, the sequence $\{-5n + 2\}$ diverges to negative infinity because $-5n + 2 \to -\infty$ as $n \to \infty$. We write this as $\lim_{n\to\infty}(-5n + 2) = -\infty$.

Because a sequence is a function whose domain is the set of positive integers, we can use properties of limits of functions to determine whether a sequence converges. For example, consider a sequence $\{a_n\}$ and a related function f defined on all positive real numbers such that $f(n) = a_n$ for all integers $n \geq 1$. Since the domain of the sequence is a subset of the domain of f, if $\lim_{x\to\infty} f(x)$ exists, then the sequence converges and has the same limit. For example, consider the sequence $\left\{\frac{1}{n}\right\}$ and the related function $f(x) = \frac{1}{x}$. Since the function f defined on all real numbers $x > 0$ satisfies $f(x) = \frac{1}{x} \to 0$ as $x \to \infty$, the sequence $\left\{\frac{1}{n}\right\}$ must satisfy $\frac{1}{n} \to 0$ as $n \to \infty$. We summarize this result as a theorem.

Theorem

Consider a sequence $\{a_n\}$ such that $a_n = f(n)$ for all $n \geq 1$. If there exists a real number L such that

$$\lim_{x\to\infty} f(x) = L,$$

then $\{a_n\}$ converges and

$$\lim_{n\to\infty} a_n = L.$$

Caution: The Converse of this Theorem is NOT True (in general)

In general, the converse of this theorem is *not* true. For example, consider the sequence $\{\sin(\pi n)\}$. For all integer values of n, the value of this sequence is 0. Hence, $\lim_{n\to\infty} \sin(\pi n) = 0$; however, from our work in Calculus I, we know that $\lim_{x\to\infty} \sin(\pi x)$ does not exist.

It's best to think of this theorem as follows:

> *If the infinite limit of the related function exists (and is finite), then the sequence converges (to the same value).*

We can use this theorem to evaluate $\lim_{n\to\infty} r^n$ for $0 \leq r \leq 1$. For example, consider the sequence $\{(1/2)^n\}$ and the related exponential function $f(x) = (1/2)^x$. Since $\lim_{x\to\infty} (1/2)^x = 0$, we conclude that the sequence $\{(1/2)^n\}$ converges and its limit is 0. Similarly, for any real number r such that $0 \leq r < 1$, $\lim_{x\to\infty} r^x = 0$, and therefore the sequence $\{r^n\}$ converges (to 0). On the other hand, if $r = 1$, then $\lim_{x\to\infty} r^x = 1$, and therefore the limit of the sequence $\{1^n\}$ is 1. If $r > 1$, $\lim_{x\to\infty} r^x = \infty$, and therefore we cannot apply this theorem. However, in this case, just as the function r^x grows without bound as $n \to \infty$, the terms r^n in the sequence become arbitrarily large as $n \to \infty$, and we conclude that the sequence $\{r^n\}$ diverges to infinity if $r > 1$.

We summarize these results regarding the geometric sequence r^n:

$$\begin{array}{lcll} r^n & \to & 0 & \text{if } 0 < r < 1 \\ r^n & \to & 1 & \text{if } r = 1 \\ r^n & \to & \infty & \text{if } r > 1 \end{array}$$

Later in this section we consider the case when $r < 0$.

We now consider slightly more complicated sequences. For example, consider the sequence $\{(2/3)^n + (1/4)^n\}$. The terms in this sequence are more complicated than other sequences we have discussed, but luckily the limit of this sequence is determined by the limits of the two sequences $\{(2/3)^n\}$ and $\{(1/4)^n\}$. As we describe in the following algebraic Limit Laws, since $\{(2/3)^n\}$ and $\{1/4)^n\}$ both converge to 0, the sequence $\{(2/3)^n + (1/4)^n\}$ converges to $0 + 0 = 0$. Just as we were able to evaluate a limit involving an algebraic combination of functions f and g by looking at the limits of f and g in Calculus I, we can evaluate the limit of a sequence whose terms are algebraic combinations of a_n and b_n by evaluating the limits of $\{a_n\}$ and $\{b_n\}$.

Theorem: Algebraic Limit Laws

Given sequences $\{a_n\}$ and $\{b_n\}$ and any real number c, if there exist constants A and B such that $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, then

i. $\lim_{n\to\infty} c = c$
ii. $\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n = cA$
iii. $\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n = A \pm B$
iv. $\lim_{n\to\infty} (a_n \cdot b_n) = \left(\lim_{n\to\infty} a_n\right) \cdot \left(\lim_{n\to\infty} b_n\right) = A \cdot B$
v. $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{A}{B}$, provided $B \neq 0$ and each $b_n \neq 0$.

Proof of part iii.

Let $\epsilon > 0$. Since $\lim_{n\to\infty} a_n = A$, there exists a constant positive integer N_1 such that $|a_n - A| < \frac{\epsilon}{2}$ for all $n \geq N_1$. Since $\lim_{n\to\infty} b_n = B$, there exists a constant N_2 such that $|b_n - B| < \epsilon/2$ for all $n \geq N_2$. Let $N = \max\{N_1, N_2\}$. Therefore, for all $n \geq N$, $|(a_n + b_n) - (A + B)| \leq |a_n - A| + |b_n - B| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

The Algebraic Limit Laws allow us to evaluate limits for many sequences. For example, consider the sequence $a_n = \frac{1}{n^2}$. As shown earlier, $\lim_{n\to\infty} \frac{1}{n} = 0$. Similarly, for any positive integer k, we can conclude that

$$\lim_{n\to\infty} \frac{1}{n^k} = 0.$$

In the next example, we make use of this fact along with the Algebraic Limit Laws to evaluate limits for other sequences.

Example 4.1.3: Determining Convergence and Finding Limits

For each of the following sequences, determine whether or not the sequence converges. If it converges, find its limit.

a. $\left\{5 - \frac{3}{n^2}\right\}$
b. $\left\{\frac{3n^4 - 7n^2 + 5}{6 - 4n^4}\right\}$
c. $\left\{\frac{2^n}{n^2}\right\}$
d. $\left\{\left(1 + \frac{4}{n}\right)^n\right\}$

Solutions

a. We know that $\lim_{n\to\infty} \frac{1}{n} = 0$. Using this fact, we conclude that

$$\lim_{n\to\infty} \frac{1}{n^2} = \lim_{n\to\infty} \frac{1}{n} \cdot \lim_{n\to\infty} \frac{1}{n} = 0.$$

Therefore,

$$\lim_{n\to\infty} \left(5 - \frac{3}{n^2}\right) = \lim_{n\to\infty} 5 - 3 \lim_{n\to\infty} \frac{1}{n^2} = 5 - 3.0 = 5.$$

The sequence converges and its limit is 5.

b. By factoring n^4 out of the numerator and denominator and using the Algebraic Limit Laws above, we have

$$\begin{aligned}
\lim_{n\to\infty}\frac{3n^4-7n^2+5}{6-4n^4} &= \lim_{n\to\infty}\frac{3-\frac{7}{n^2}+\frac{5}{n^4}}{\frac{6}{n^4}-4}\\
&= \frac{\lim_{n\to\infty}(3-\frac{7}{n^2}+\frac{5}{n^4})}{\lim_{n\to\infty}(\frac{6}{n^4}-4)}\\
&= \frac{\lim_{n\to\infty}(3)-\lim_{n\to\infty}\frac{7}{n^2}+\lim_{n\to\infty}\frac{5}{n^4}}{\lim_{n\to\infty}\frac{6}{n^4}-\lim_{n\to\infty}(4)}\\
&= \frac{\lim_{n\to\infty}(3)-7\cdot\lim_{n\to\infty}\frac{1}{n^2}+5\cdot\lim_{n\to\infty}\frac{1}{n^4}}{6\cdot\lim_{n\to\infty}\frac{1}{n^4}-\lim_{n\to\infty}(4)}\\
&= \frac{3-7\cdot 0+5\cdot 0}{6\cdot 0-4}\\
&= -\frac{3}{4}.
\end{aligned}$$

The sequence converges and its limit is $-3/4$.

c. Consider the related function $f(x)=2^x/x^2$ defined on all real numbers $x>0$. Since $2^x\to\infty$ and $x^2\to\infty$ as $x\to\infty$, apply l'Hospital's Rule and write

$$\begin{aligned}
\lim_{x\to\infty}\frac{2^x}{x^2} &\overset{\text{L.R.}}{=} \lim_{x\to\infty}\frac{2^x\ln 2}{2x}\\
&\overset{\text{L.R.}}{=} \lim_{x\to\infty}\frac{2^x(\ln 2)^2}{2}\\
&= \infty.
\end{aligned}$$

We conclude that the sequence diverges.

d. Consider the function $f(x)=\left(1+\frac{4}{x}\right)^x$ defined on all real numbers $x>0$. We start by letting

$$y=\left(1+\frac{4}{x}\right)^x.$$

Hence,

$$\begin{aligned}
\ln(y) &= \ln\left[\left(1+\frac{4}{x}\right)^x\right]\\
&= x\ln\left(1+\frac{4}{x}\right) \qquad \text{(Properties of Logarithms)}
\end{aligned}$$

Taking the limit of both sides, we get

$$\begin{aligned}\lim_{x\to\infty}\ln(y) &= \lim_{x\to\infty} x\ln\left(1+\frac{4}{x}\right)\\ &= \lim_{x\to\infty}\frac{\ln\left(1+\frac{4}{x}\right)}{\frac{1}{x}} \qquad \text{(rewriting as the indeterminate form } \tfrac{0}{0}\text{)}\\ &\overset{\text{L.R.}}{=} \lim_{x\to\infty}\frac{4}{1+4/x}\\ &= 4\end{aligned}$$

Hence,

$$\begin{aligned}\lim_{x\to\infty}\left(1+\frac{4}{x}\right)^x &= \lim_{x\to\infty} y\\ &= \lim_{x\to\infty} e^{\ln(y)}\\ &= e^{\lim_{x\to\infty}\ln(y)}\\ &= e^4\end{aligned}$$

Hence, since $\lim_{x\to\infty}\left(1+\frac{4}{x}\right)^x = e^4$, we can conclude that the sequence $\left\{\left(1+\frac{4}{n}\right)^n\right\}$ converges to e^4.

? Checkpoint 4.1.3

Consider the sequence $\{(5n^2+1)/e^n\}$. Determine whether or not the sequence converges. If it converges, find its limit.

Answer

The sequence converges, and its limit is 0

Recall that if f is a continuous function at a value L, then $f(x)\to f(L)$ as $x\to L$. This idea applies to sequences as well. Suppose a sequence $a_n\to L$, and a function f is continuous at L. Then $f(a_n)\to f(L)$. This property often enables us to find limits for complicated sequences. For example, consider the sequence $\sqrt{5-\frac{3}{n^2}}$. From Example 4.1.3a. we know the sequence $5-\frac{3}{n^2}\to 5$. Since $\sqrt{x}$ is a continuous function at $x=5$,

$$\lim_{n\to\infty}\sqrt{5-\frac{3}{n^2}} = \sqrt{\lim_{n\to\infty}(5-\frac{3}{n^2})} = \sqrt{5}.$$

Theorem

Consider a sequence $\{a_n\}$ and suppose there exists a real number L such that the sequence $\{a_n\}$ converges to L. Suppose f is a continuous function at L. Then there exists an integer N such that f is defined at all values an for $n\geq N$, and the sequence $\{f(a_n)\}$ converges to $f(L)$ (see Figure 4.1.4).

Proof

Let $\epsilon>0$. Since f is continuous at L, there exists $\delta>0$ such that $|f(x)-f(L)|<\epsilon$ if $|x-L|<\delta$. Since the sequence $\{a_n\}$ converges to L, there exists N such that $|a_n-L|<\delta$ for all $n\geq N$. Therefore, for all $n\geq N$, $|a_n-L|<\delta$, which implies $|f(a_n)-f(L)|<\epsilon$. We conclude that the sequence $\{f(a_n)\}$ converges to $f(L)$.

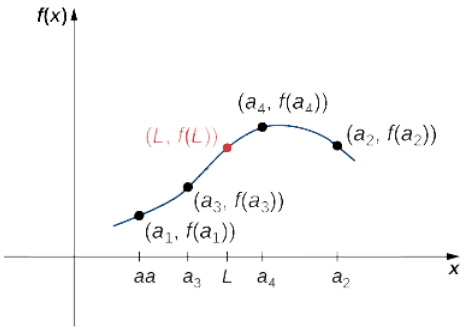


Figure 4.1.4: Because f is a continuous function as the inputs $a_1, a_2, a_3, \ldots$ approach L, the outputs $f(a_1), f(a_2), f(a_3), \ldots$ approach $f(L)$.

Example 4.1.4: Limits Involving Continuous Functions Defined on Convergent Sequences

Determine whether the sequence $\{\cos(3/n^2)\}$ converges. If it converges, find its limit.

Solution

Since the sequence $\{3/n^2\}$ converges to 0 and $\cos x$ is continuous at $x = 0$, we can conclude that the sequence $\{\cos(3/n^2)\}$ converges and

$$\lim_{n\to\infty} \cos\left(\frac{3}{n^2}\right) = \cos 0 = 1.$$

Checkpoint 4.1.4

Determine if the sequence $\left\{\sqrt{\frac{2n+1}{3n+5}}\right\}$ converges. If it converges, find its limit.

Answer

The sequence converges, and its limit is $\sqrt{2/3}$.

Another theorem involving limits of sequences is an extension of the Squeeze Theorem for limits discussed in Calculus I.

Theorem: Squeeze Theorem for Sequences

Consider sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$. Suppose there exists an integer N such that

$$a_n \le b_n \le c_n \text{ for all } n \ge N.$$

If there exists a real number L such that

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n,$$

then $\{b_n\}$ converges and $\lim_{n\to\infty} b_n = L$ (see Figure 4.1.5).

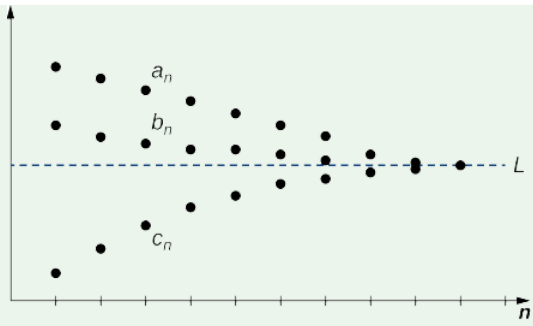


Figure 4.1.5: Each term b_n satisfies $a_n \le b_n \le c_n$ and the sequences $\{a_n\}$ and $\{c_n\}$ converge to the same limit, so the sequence $\{b_n\}$ must converge to the same limit as well.

Proof

Let $\epsilon > 0$. Since the sequence $\{a_n\}$ converges to L, there exists an integer N_1 such that $|a_n - L| < \epsilon$ for all $n \ge N_1$. Similarly, since $\{c_n\}$ converges to L, there exists an integer N_2 such that $|c_n - L| < \epsilon$ for all $n \ge N_2$. By assumption, there exists an integer N such that $a_n \le b_n \le c_n$ for all $n \ge N$. Let $M = \max\{N_1, N_2, N\}$. We must show that $|b_n - L| < \epsilon$ for all $n \ge M$.

For all $n \ge M$,

$$-\epsilon < -|a_n - L| \le a_n - L \le b_n - L \le c_n - L \le |c_n - L| < \epsilon.$$

Therefore, $-\epsilon < b_n - L < \epsilon$, and we conclude that $|b_n - L| < \epsilon$ for all $n \ge M$, and we conclude that the sequence b_n converges to L.

Example 4.1.5: Using the Squeeze Theorem

Use the Squeeze Theorem to find the limit of each of the following sequences.

a. $\left\{\frac{\cos n}{n^2}\right\}$
b. $\left\{\left(-\frac{1}{2}\right)^n\right\}$

Solutions

a. Since $-1 \le \cos n \le 1$ for all integers n, we have

$$-\frac{1}{n^2} \le \frac{\cos n}{n^2} \le \frac{1}{n^2}.$$

Since $-1/n^2 \to 0$ and $1/n^2 \to 0$ as $n \to \infty$, we conclude that $\cos n/n^2 \to 0$ by the Squeeze Theorem.

b. Since

$$-\frac{1}{2^n} \le \left(-\frac{1}{2}\right)^n \le \frac{1}{2^n}$$

for all positive integers n, $-1/2^n \to 0$ and $1/2^n \to 0$ as $n \to \infty$, we can conclude that $(-1/2)^n \to 0$ by the Squeeze Theorem.

Checkpoint 4.1.5

Find $\lim\limits_{n\to\infty} \dfrac{2n - \sin n}{n}$.

Answer

2

Using the idea from Example 4.1.5b we conclude that $r^n \to 0$ for any real number r such that $-1 < r < 0$. If $r < -1$, the sequence r^n diverges because the terms oscillate and become arbitrarily large in magnitude. If $r = -1$, the sequence $r^n = (-1)^n$ diverges, as discussed earlier. Here is a summary of the properties for geometric sequences.

$$\begin{aligned} r^n &\to 0 \quad \text{if} \quad |r| < 1 \\ r^n &\to 1 \quad \text{if} \quad r = 1 \\ r^n &\to \infty \quad \text{if} \quad r > 1 \\ \{r^n\} &\text{ diverges} \quad \text{if} \quad r \leq -1 \end{aligned}$$

Bounded Sequences

We now focus on one of the most important theorems involving sequences: the Monotone Convergence Theorem. Before stating the theorem, we need to introduce some terminology and motivation. We begin by defining what it means for a sequence to be bounded.

Definition: Bound Sequences

A sequence $\{a_n\}$ is **bounded above** if there exists a real number M such that

$$a_n \leq M$$

for all $n \in \mathbb{N}$.

A sequence $\{a_n\}$ is **bounded below** if there exists a real number m such that

$$m \leq a_n$$

for all $n \in \mathbb{N}$.

A sequence $\{a_n\}$ is a **bounded sequence** if it is bounded above and bounded below.

If a sequence is not bounded, it is an **unbounded sequence.**

For example, the sequence $\{1/n\}$ is bounded above because $1/n \leq 1$ for all $n \in \mathbb{N}$. It is also bounded below because $1/n \geq 0$ for all $n \in \mathbb{N}$. Therefore, $\{1/n\}$ is a bounded sequence. On the other hand, consider the sequence $\{2^n\}$. Because $2^n \geq 2$ for all $n \in \mathbb{N}$, the sequence is bounded below. However, the sequence is not bounded above. Therefore, $\{2^n\}$ is an unbounded sequence.

We now discuss the relationship between boundedness and convergence. Suppose a sequence $\{a_n\}$ is unbounded. Then, it is not bounded above or below. In any case, there are terms a_n that are arbitrarily large in magnitude as n gets larger. As a result, the sequence $\{a_n\}$ cannot converge. Therefore, being bounded is a *necessary* condition for a sequence to converge.[5]

Theorem

If a sequence $\{a_n\}$ converges, then it is bounded.

Note that a sequence being bounded is not a sufficient condition for a sequence to converge. For example, the sequence $\{(-1)^n\}$ is bounded, but the sequence diverges because the sequence oscillates between 1 and -1 and never approaches a finite number. We now discuss a sufficient (but not necessary) condition for a bounded sequence to converge.

Consider a bounded sequence $\{a_n\}$. Suppose the sequence $\{a_n\}$ is increasing. That is, $a_1 \leq a_2 \leq a_3 \ldots$. Since the sequence is increasing, the terms are not oscillating. Therefore, there are two possibilities - the sequence could diverge to infinity or converge. However, since the sequence is bounded above, it cannot diverge to infinity. We conclude that $\{a_n\}$ converges. For example, consider the sequence

$$\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right\}.$$

Since this sequence is increasing and bounded above, it converges. Next, consider the sequence

$$\left\{2, 0, 3, 0, 4, 0, 1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \ldots\right\}.$$

Even though the sequence is not increasing for all values of $n \in \mathbb{N}$, we see that $-1/2 < -1/3 < -1/4 < \cdots$. Therefore, starting with the eighth term, $a_8 = -1/2$, the sequence is increasing. In this case, we say the sequence is *eventually* increasing. Since the sequence is bounded above, it converges. It is also true that if a sequence is decreasing (or eventually decreasing) and bounded below, it also converges.

Definition: Increasing, Decreasing, and Monotone Sequences

A sequence $\{a_n\}$ is said to be **increasing** for all $n \geq n_0$ if

$$a_n \leq a_{n+1} \text{ for all } n \geq n_0.$$

A sequence $\{a_n\}$ is said to be **decreasing** for all $n \geq n_0$ if

$$a_n \geq a_{n+1} \text{ for all } n \geq n_0.$$

A sequence $\{a_n\}$ is called a **monotone sequence** for all $n \geq n_0$ if it is increasing for all $n \geq n_0$ or decreasing for all $n \geq n_0$.

We now have the necessary definitions to state the Monotone Convergence Theorem, which gives a sufficient condition for sequence convergence.

Theorem: Monotone Convergence Theorem

If $\{a_n\}$ is a bounded sequence and there exists a $n_0 \in \mathbb{N}$ such that $\{a_n\}$ is monotone for all $n \geq n_0$, then $\{a_n\}$ converges.

The proof of this theorem is beyond the scope of this text. Instead, we provide a graph to show intuitively why this theorem makes sense (Figure 4.1.6).

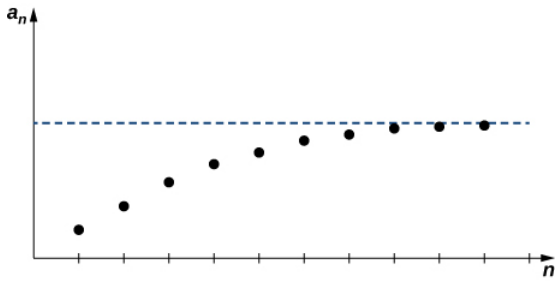


Figure 4.1.6: Since the sequence $\{a_n\}$ is increasing and bounded above, it must converge.

In the following example, we show how the Monotone Convergence Theorem can be used to prove the convergence of a sequence.

Example 4.1.6: Using the Monotone Convergence Theorem

For each of the following sequences, use the Monotone Convergence Theorem to show the sequence converges and find its limit.

a. $\left\{\frac{4^n}{n!}\right\}$
b. $\{a_n\}$ defined recursively such that

$$a_1 = 2 \text{ and } a_{n+1} = \frac{a_n}{2} + \frac{1}{2a_n} \text{ for all } n \geq 2.$$

Solutions

a. Writing out the first few terms, we see that

$$\left\{\frac{4^n}{n!}\right\} = \left\{4, 8, \frac{32}{3}, \frac{32}{3}, \frac{128}{15}, \ldots\right\}.$$

At first, the terms increase. However, after the third term, the terms decrease. In fact, the terms decrease for all $n \geq 3$. We can show this as follows.

$$a_{n+1} = \frac{4^{n+1}}{(n+1)!} = \frac{4}{n+1} \cdot \frac{4^n}{n!} = \frac{4}{n+1} \cdot a_n \leq a_n \text{ if } n \geq 3.$$

Therefore, the sequence is decreasing for all $n \geq 3$. Further, the sequence is bounded below by 0 because $4n/n! \geq 0$ for all $n \in \mathbb{N}$. Therefore, by the Monotone Convergence Theorem, the sequence converges.

To find the limit, we use the fact that the sequence converges and let $L = \lim_{n\to\infty} a_n$. Now note this important observation. Consider $\lim_{n\to\infty} a_{n+1}$. Since

$$\{a_{n+1}\} = \{a_2, a_3, a_4, ...\},$$

the only difference between the sequences $\{a_{n+1}\}$ and $\{a_n\}$ is that $\{a_{n+1}\}$ omits the first term. Since a finite number of terms does not affect the convergence of a sequence,

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = L.$$

Combining this fact with the equation

$$a_{n+1} = \frac{4}{n+1} a_n$$

and taking the limit of both sides of the equation

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{4}{n+1} a_n,$$

we can conclude that

$$L = 0.$$

b. Writing out the first several terms,

$$\left\{2, \frac{5}{4}, \frac{41}{40}, \frac{3281}{3280}, ...\right\}.$$

we can conjecture that the sequence is decreasing and bounded below by 1. To show that the sequence is bounded below by 1, we can show that

$$\frac{a_n}{2} + \frac{1}{2a_n} \geq 1.$$

To show this, first rewrite

$$\frac{a_n}{2} + \frac{1}{2a_n} = \frac{a_n^2 + 1}{2a_n}.$$

Since $a_1 > 0$ and a_2 is defined as a sum of positive terms, $a_2 > 0$. Similarly, all terms $a_n > 0$. Therefore,

$$\frac{a^2 n + 1}{2a_n} \geq 1$$

if and only if

$$a_n^2 + 1 \geq 2a_n.$$

Rewriting the inequality $a_n^2 + 1 \geq 2a_n$ as $a_n^2 - 2a_n + 1 \geq 0$ and using the fact that

$$a_n^2 - 2a_n + 1 = (a_n - 1)^2 \geq 0$$

because the square of any real number is nonnegative, we can conclude that

$$\frac{a^n}{2} + \frac{1}{2a_n} \geq 1.$$

To show that the sequence is decreasing, we must show that $a_{n+1} \leq a_n$ for all $n \geq 1$. Since $1 \leq a_n^2$, it follows that

$$a_n^2 + 1 \leq 2a_n^2.$$

Dividing both sides by $2a_n$, we obtain

$$\frac{a_n}{2} + \frac{1}{2a_n} \leq a_n.$$

Using the definition of a_{n+1}, we conclude that

$$a_{n+1} = \frac{a_n}{2} + \frac{1}{2a_n} \leq a_n.$$

Since $\{a_n\}$ is bounded below and decreasing, by the Monotone Convergence Theorem, it converges.

To find the limit, let $L = \lim_{n\to\infty} a_n$. Then using the recurrence relation and the fact that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$, we have

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \left(\frac{a_n}{2} + \frac{1}{2a_n}\right),$$

and therefore

$$L = \frac{L}{2} + \frac{1}{2L}.$$

Multiplying both sides of this equation by $2L$, we arrive at the equation

$$2L^2 = L^2 + 1.$$

Solving this equation for L, we conclude that $L^2 = 1$, which implies $L = \pm 1$. Since all the terms are positive, the limit $L = 1$.

? Checkpoint 4.1.6

Consider the sequence $\{a_n\}$ defined recursively such that $a_1 = 1$, $a_n = a_{n-1}/2$. Use the Monotone Convergence Theorem to show that this sequence converges and find its limit.

Answer

0.

Definition: Fibonacci Numbers

The **Fibonacci numbers** are defined recursively by the sequence $\{F_n\}$ where $F_0 = 0$, $F_1 = 1$, and for $n \geq 2$,

$$F_n = F_{n-1} + F_{n-2}.$$

Here, we look at the properties of the Fibonacci numbers.

1. Write out the first twenty Fibonacci numbers.

2. Find a closed formula for the Fibonacci sequence by using the following steps.

   a. Consider the recursively defined sequence x_n where $x_0 = c$ and $x_{n+1} = ax_n$. Show that this sequence can be described by the closed formula $x_n = ca^n$ for all $n \geq 0$.

   b. Using the result from part a. as motivation, look for a solution of the equation

   $$F_n = F_{n-1} + F_{n-2}$$

   of the form $F_n = c\lambda^n$. Determine what two values for λ will allow F_n to satisfy this equation.

   c. Consider the two solutions from part b.: λ_1 and λ_2. Let $F_n = c_1\lambda_1^n + c_2\lambda_2^n$. Use the initial conditions F_0 and F_1 to determine the values for the constants c_1 and c_2 and write the closed formula F_n.

3. Use the answer in 2 c. to show that

$$\lim_{n\to\infty}\frac{F_{n+1}}{F_n}=\frac{1+\sqrt{5}}{2}.$$

The number $\phi=(1+\sqrt{5})/2$ is known as the **golden ratio** (Figure 4.1.7 and Figure 4.1.8).



Figure 4.1.7: The seeds in a sunflower exhibit spiral patterns curving to the left and the right. The number of spirals in each direction is always a Fibonacci number. (credit: modification of work by Esdras Calderan, Wikimedia Commons)

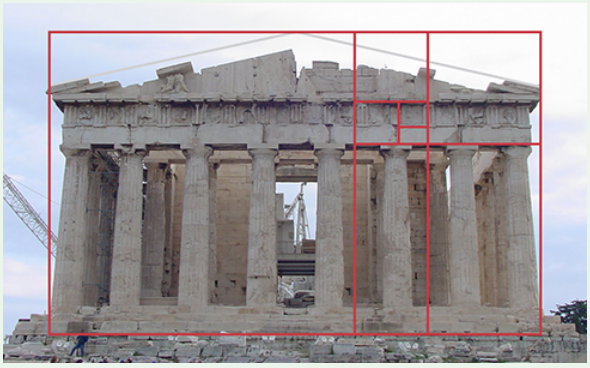


Figure 4.1.8: The proportion of the golden ratio appears in many famous examples of art and architecture. The ancient Greek temple, the Parthenon, was designed with these proportions, and the ratio appears again in many smaller details. (credit: modification of work by TravelingOtter, Flickr).

Footnotes

[1] A similar notation is used for sets, but a sequence is an ordered list, whereas a set is not ordered.

[2] Remember, the domain of a sequence, in general, is the set of natural numbers (also called the positive integers). As such, the graph of a sequence will not be continuous.

[3] Since pattern recognition is an important skill when working with sequences (and it's sister-subject, series), it is often best to *not* simplify your arithmetic when writing out computations. This allows you to see the "hidden" patterns that make the sequence simple to write succinctly.

[4] This definition should be hauntingly familiar - you worked with the precise definition of a limit in Calculus I.

[5] When it comes to conditions for something to be true, the words "necessary" and "sufficient" are important in mathematics, and understanding the difference between the two will be helpful.

If a condition is *necessary* for a result to be true, it means that the necessary condition must exist for the result to even be possible. It does not mean, however, that the necessary condition immediately implies the result is true. For example, a necessary condition for you to pass this class is to attend all of the exams; however, you need more than just attending the exams to guarantee you will pass the course.

A *sufficient* condition, on the other hand, means that, as long as the sufficient condition is met, the result will be true. For example, scoring 80% on all of the exams and assignments in this class is a sufficient condition to pass the class. Note, however, that you can pass this class in other ways. For example, earning 96% on all of the exams and assignments will allow you to pass as well.

The takeaway:

- A sufficient condition guarantees the truth of another condition but is not necessary for that other condition to happen.
- A necessary condition is required for something else to happen. Still, it does not guarantee that the something else happens.

Much confusion can be had with necessary and sufficient conditions. Still, that investigation is left for a course in logic.

Key Concepts

- To determine the convergence of a sequence given by an explicit formula $a_n = f(n)$, we use the properties of limits for functions.
- If $\{a_n\}$ and $\{b_n\}$ are convergent sequences that converge to A and B, respectively, and c is any real number, then the sequence $\{ca_n\}$converges to $c \cdot A$, the sequences $\{a_n \pm b_n\}$ converge to $A \pm B$, the sequence $\{a_n \cdot b_n\}$ converges to $A \cdot B$, and the sequence $\{a_n/b_n\}$ converges to A/B, provided $B \neq 0$.
- If a sequence is bounded and monotone, it converges, but not all convergent sequences are monotone.
- If a sequence is unbounded, it diverges, but not all divergent sequences are unbounded.
- The geometric sequence $\{r^n\}$ converges if and only if $|r| < 1$ or $r = 1$.

Glossary

arithmetic sequence

a sequence in which the difference between every pair of consecutive terms is the same is called an arithmetic sequence

bounded above

a sequence $\{a_n\}$ is bounded above if there exists a constant M such that $a_n \leq M$ for all positive integers n

bounded below

a sequence $\{a_n\}$ is bounded below if there exists a constant M such that $M \leq a_n$ for all positive integers n

bounded sequence

a sequence $\{a_n\}$ is bounded if there exists a constant M such that $|a_n| \leq M$ for all positive integers n

convergent sequence

a convergent sequence is a sequence $\{a_n\}$ for which there exists a real number L such that a_n is arbitrarily close to L as long as n is sufficiently large

divergent sequence

a sequence that is not convergent is divergent

explicit formula

a sequence may be defined by an explicit formula such that $a_n = f(n)$

geometric sequence

a sequence $\{a_n\}$ in which the ratio a_{n+1}/a_n is the same for all positive integers n is called a geometric sequence

index variable

the subscript used to define the terms in a sequence is called the index

limit of a sequence

the real number L to which a sequence converges is called the limit of the sequence

monotone sequence

an increasing or decreasing sequence

recurrence relation

a recurrence relation is a relationship in which a term a_n in a sequence is defined in terms of earlier terms in the sequence

sequence

an ordered list of numbers of the form $a_1, a_2, a_3, \ldots$ is a sequence

term

the number a_n in the sequence $\{a_n\}$ is called the nth term of the sequence

unbounded sequence

a sequence that is not bounded is called unbounded

4.1E: Exercises

In exercises 1 - 4, find the first six terms of each sequence, starting with $n = 1$.

1) $a_n = 1 + (-1)^n$ for $n \geq 1$

Answer

$a_n = 0$ if n is odd and $a_n = 2$ if n is even

2) $a_n = n^2 - 1$ for $n \geq 1$

3) $a_1 = 1$ and $a_n = a_{n-1} + n$ for $n \geq 2$

Answer

$a_n = 1, 3, 6, 10, 15, 21, ...$

4) $a_1 = 1, a_2 = 1$ and $a_n + 2 = a_n + a_{n+1}$ for $n \geq 1$

5) Find an explicit formula for a_n where $a_1 = 1$ and $a_n = a_{n-1} + n$ for $n \geq 2$.

Answer

$$a_n = \frac{n(n+1)}{2}$$

6) Find a formula a_n for the nth term of the arithmetic sequence whose first term is $a_1 = 1$ such that $a_{n-1} - a_n = 17$ for $n \geq 1$.

7) Find a formula a_n for the nth term of the arithmetic sequence whose first term is $a_1 = -3$ such that $a_{n-1} - a_n = 4$ for $n \geq 1$.

Answer

$a_n = 4n - 7$

8) Find a formula a_n for the nth term of the geometric sequence whose first term is $a_1 = 1$ such that $\frac{a_{n+1}}{a_n} = 10$ for $n \geq 1$.

9) Find a formula a_n for the nth term of the geometric sequence whose first term is $a_1 = 3$ such that $\frac{a_{n+1}}{a_n} = 1/10$ for $n \geq 1$.

Answer

$a_n = 3.10^{1-n} = 30.10^{-n}$

10) Find an explicit formula for the nth term of the sequence whose first several terms are $0, 3, 8, 15, 24, 35, 48, 63, 80, 99, ...$ (Hint: First add one to each term.)

11) Find an explicit formula for the nth term of the sequence satisfying $a_1 = 0$ and $a_n = 2a_{n-1} + 1$ for $n \geq 2$.

Answer

$a_n = 2^n - 1$

In exercises 12 and 13, find a formula for the general term a_n of each of the following sequences.

12) $1, 0, -1, 0, 1, 0, -1, 0, ...$ (Hint: Find where $\sin x$ takes these values)

13) $1, -1/3, 1/5, -1/7, ...$

Answer

$$a_n = \frac{(-1)^{n-1}}{2n-1}$$

In exercises 14-18, find a function $f(n)$ that identifies the nth term a_n of the following recursively defined sequences, as $a_n = f(n)$.

14) $a_1 = 1$ and $a_{n+1} = -a_n$ for $n \geq 1$

15) $a_1 = 2$ and $a_{n+1} = 2a_n$ for $n \geq 1$

Answer

$f(n) = 2^n$

16) $a_1 = 1$ and $a_{n+1} = (n+1)a_n$ for $n \geq 1$

17) $a_1 = 2$ and $a_{n+1} = (n+1)a_n/2$ for $n \geq 1$

Answer

$f(n) = \dfrac{n!}{2^{n-2}}$

18) $a_1 = 1$ and $a_{n+1} = a_n/2^n$ for $n \geq 1$

In exercises 19 - 22, plot the first N terms of the given sequence. State whether the graphical evidence suggests that the sequence converges or diverges.

19) [Technology Required] $a_1 = 1, a_2 = 2$, and for $n \geq 2, a_n = \frac{1}{2}(a_{n-1} + a_{n-2})$; $N = 30$

20) [Technology Required] $a_1 = 1, a_2 = 2, a_3 = 3$ and for $n \geq 4, a_n = \frac{1}{3}(a_{n-1} + a_{n-2} + a_{n-3}), N = 30$

21) [Technology Required] $a_1 = 1, a_2 = 2$, and for $n \geq 3, a_n = \sqrt{a_{n-1}a_{n-2}}; N = 30$

22) [Technology Required] $a_1 = 1, a_2 = 2, a_3 = 3$, and for $n \geq 4, a_n = \sqrt{a_{n-1}a_{n-2}a_{n-3}}; N = 30$

In exercises 23 - 16, suppose that $\lim_{n\to\infty} a_n = 1$, $\lim_{n\to\infty} b_n = -1$, and $0 < -b_n < a_n$ for all n.

Using this information, evaluate each of the following limits, state that the limit does not exist, or state that there is not enough information to determine whether the limit exists.

23) $\lim_{n\to\infty} 3a_n - 4b_n$

Answer

$\lim_{n\to\infty} 3a_n - 4b_n \quad = \quad 7$

24) $\lim_{n\to\infty} \dfrac{1}{2}b_n - \dfrac{1}{2}a_n$

25) $\lim_{n\to\infty} \dfrac{a_n + b_n}{a_n - b_n}$

Answer

$\lim_{n\to\infty} \dfrac{a_n + b_n}{a_n - b_n} \quad = \quad 0$

26) $\lim_{n\to\infty} \dfrac{a_n - b_n}{a_n + b_n}$

In exercises 27 - 30, find the limit of each of the following sequences, using L'Hôpital's rule when appropriate.

27) $\dfrac{n^2}{2^n}$

Answer

$\lim_{n\to\infty} \dfrac{n^2}{2^n} \quad = \quad 0$

28) $\dfrac{(n-1)^2}{(n+1)^2}$

29) $\dfrac{\sqrt{n}}{\sqrt{n+1}}$

Answer

$$\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} = 1$$

30) $n^{1/n}$ (Hint: $n^{1/n} = e^{\frac{1}{n}\ln n}$)

In exercises 31 - 37, state whether each sequence is bounded and whether it is eventually monotone, increasing, or decreasing.

31) $n/2^n, n \geq 2$

Answer

bounded, decreasing for $n \geq 1$

32) $\ln\left(1+\frac{1}{n}\right)$

33) $\sin n$

Answer

bounded, not monotone

34) $\cos(n^2)$

35) $n^{1/n}, \quad n \geq 3$

Answer

bounded, decreasing

36) $n^{-1/n}, \quad n \geq 3$

37) $\tan n$

Answer

not monotone, not bounded

In exercises 38 - 39, determine whether the given sequence has a limit. If it does, find the limit.

38) $a_1 = \sqrt{2}, a_2 = \sqrt{2\sqrt{2}}. a_3 = \sqrt{2\sqrt{2\sqrt{2}}}$ etc.

39) $a_1 = 3, a_n = \sqrt{2a_{n-1}}, n = 2, 3, \ldots.$

Answer

a_n is decreasing and bounded below by 2. The limit a must satisfy $a = \sqrt{2a}$ so $a = 2$, independent of the initial value.

Use the Squeeze Theorem to find the limit of each sequence in exercises 40 - 43.

40) $n\sin(1/n)$

41) $\frac{\cos(1/n)-1}{1/n}$

Answer

0

42) $a_n = \frac{n!}{n^n}$

43) $a_n = \sin n \sin(1/n)$

Answer

0 since $|\sin x| \leq |x|$ and $|\sin x| \leq 1$ so $-\frac{1}{n} \leq a_n \leq \frac{1}{n}$).

For the sequences in exercises 44 and 45, plot the first 25 terms of the sequence and state whether the graphical evidence suggests that the sequence converges or diverges.

44) [Technology Required] $a_n = \sin n$

45) [Technology Required] $a_n = \cos n$

In exercises 46 - 52, determine the limit of the sequence or show that the sequence diverges. If it converges, find its limit.

46) $a_n = \tan^{-1}(n^2)$

47) $a_n = (2n)^{1/n} - n^{1/n}$

Answer

$n^{1/n} \to 1$ and $2^{1/n} \to 1$, so $a_n \to 0$

48) $a_n = \dfrac{\ln(n^2)}{\ln(2n)}$

49) $a_n = \left(1 - \frac{2}{n}\right)^n$

Answer

Since $(1 + 1/n)^n \to e$, one has $(1 - 2/n)^n \approx (1 + k)^{-2k} \to e^{-2}$ as $k \to \infty$.

50) $a_n = \ln\left(\dfrac{n+2}{n^2-3}\right)$

51) $a_n = \dfrac{2^n + 3^n}{4^n}$

Answer

$2^n + 3^n \le 2 \cdot 3^n$ and $3^n/4^n \to 0$ as $n \to \infty$, so $a_n \to 0$ as $n \to \infty$.

52) $a_n = \dfrac{(1000)^n}{n!}$

53) $a_n = \dfrac{(n!)^2}{(2n)!}$

Answer

$\dfrac{a_{n+1}}{a_n} = n!/(n+1)(n+2)\cdots(2n) = \dfrac{1 \cdot 2 \cdot 3 \cdots n}{(n+1)(n+2)\cdots(2n)} < 1/2^n$. In particular, $a_{n+1}/a_n \le 1/2$, so $a_n \to 0$ as $n \to \infty$.

Newton's method seeks to approximate a solution $f(x) = 0$ that starts with an initial approximation x_0 and successively defines a sequence $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$. For the given choice of f and x_0, write out the formula for x_{n+1}. If the sequence appears to converge, give an exact formula for the solution x, then identify the limit x accurate to four decimal places and the smallest n such that x_n agrees with x up to four decimal places.

54) [Technology Required] $f(x) = x^2 - 2, \quad x_0 = 1$

55) [Technology Required] $f(x) = (x-1)^2 - 2, \quad x_0 = 2$

Answer

$x_{n+1} = x_n - ((x_n - 1)^2 - 2)/2(x_n - 1);\ x = 1 + \sqrt{2},\ x \approx 2.4142,\ n = 5$

56) [Technology Required] $f(x) = e^x - 2, \quad x_0 = 1$

57) [Technology Required] $f(x) = \ln x - 1, \quad x_0 = 2$

Answer

$x_{n+1} = x_n - x_n(\ln(x_n) - 1);\ x = e,\ x \approx 2.7183,\ n = 5$

58) [Technology Required] Suppose you start with one liter of vinegar and repeatedly remove 0.1 L, replace with water, mix, and repeat.

a. Find a formula for the concentration after n steps.

b. After how many steps does the mixture contain less than 10% vinegar?

59) [Technology Required] A lake initially contains 2000 fish. Suppose that in the absence of predators or other causes of removal, the fish population increases by 6% each month. However, factoring in all causes, 150 fish are lost each month.

a. Explain why the fish population after n months is modeled by $P_n = 1.06P_{n-1} - 150$ with $P_0 = 2000$.

b. How many fish will be in the pond after one year?

Answer

a. Without losses, the population would obey $P_n = 1.06P_{n-1}$. The subtraction of 150 accounts for fish losses.
b. After 12 months, we have $P_{12} \approx 1494$.

60) [Technology Required] A bank account earns 5% interest compounded monthly. Suppose that $\$1000$ is initially deposited into the account, but that $\$10$ is withdrawn each month.

a. Show that the amount in the account after n months is $A_n = (1 + .05/12)A_{n-1} - 10;\ A_0 = 1000$.

b. How much money will be in the account after 1 year?

c. Is the amount increasing or decreasing?

d. Suppose that instead of $\$10$, a fixed amount d dollars is withdrawn each month. Find a value of d such that the amount in the account after each month remains $\$1000$.

e. What happens if d is greater than this amount?

61) [Technology Required] A student takes out a college loan of $\$10,000$ at an annual percentage rate of 6%, compounded monthly.

a. If the student makes payments of $\$100$ per month, how much does the student owe after 12 months?

b. After how many months will the loan be paid off?

Answer

a. The student owes $\$9383$ after 12 months.
b. The loan will be paid in full after 139 months or eleven and a half years.

62) [Technology Required] Consider a series combining geometric growth and arithmetic decrease. Let $a_1 = 1$. Fix $a > 1$ and $0 < b < a$. Set $a_{n+1} = a.a_n - b$. Find a formula for a_{n+1} in terms of a_n, a, and b and a relationship between a and b such that a_n converges.

63) [Technology Required] The binary representation $x = 0.b_1b_2b_3...$ of a number x between 0 and 1 can be defined as follows. Let $b_1 = 0$ if $x < 1/2$ and $b_1 = 1$ if $1/2 \le x < 1$. Let $x_1 = 2x - b_1$. Let $b_2 = 0$ if $x_1 < 1/2$ and $b_2 = 1$ if $1/2 \le x < 1$. Let $x_2 = 2x_1 - b_2$ and in general, $x_n = 2x_{n-1} - b_n$ and $b_n - 1 = 0$ if $x_n < 1/2$ and $b_{n-1} = 1$ if $1/2 \le x_n < 1$. Find the binary expansion of $1/3$.

Answer

$b_1 = 0, x_1 = 2/3, b_2 = 1, x_2 = 4/3 - 1 = 1/3$, so the pattern repeats, and $1/3 = 0.010101\ldots$.

64) [Technology Required] To find an approximation for π, set $a_0 = \sqrt{2+1}, a_1 = \sqrt{2+a_0}$, and, in general, $a_{n+1} = \sqrt{2+a_n}$. Finally, set $p_n = 3.2^n\sqrt{2-a_n}$. Find the first ten terms of p_n and compare the values to π.

For the following two exercises, assume that you have access to a computer program or Internet source that can generate a list of zeros and ones of any desired length. Pseudo-random number generators (PRNGs) play an important role in simulating random noise in physical systems by creating sequences of zeros and ones that appear like the result of flipping a coin repeatedly. One of the simplest types of PRNGs recursively defines a random-looking sequence of N integers $a_1, a_2, \ldots, a_N$ by fixing two special integers (K and M and letting a_{n+1} be the remainder after dividing $K.a_n$ into M, then creates a bit sequence of zeros and ones

whose n^{th} term b_n is equal to one if a_n is odd and equal to zero if a_n is even. If the bits b_n are pseudo-random, then the behavior of their average $(b_1 + b_2 + \cdots + b_N)/N$ should be similar to behavior of averages of truly randomly generated bits.

65) [Technology Required] Starting with $K = 16,807$ and $M = 2,147,483,647$, using ten different starting values of a_1, compute sequences of bits b_n up to $n = 1000$, and compare their averages to ten such sequences generated by a random bit generator.

Answer

For the starting values $a_1 = 1, a_2 = 2, \ldots, a_1 = 10$, the corresponding bit averages calculated by the method indicated are $0.5220, 0.5000, 0.4960, 0.4870, 0.4860, 0.4680, 0.5130, 0.5210, 0.5040$, and 0.4840. Here is an example of ten corresponding averages of strings of 1000 bits generated by a random number generator:
$0.4880, 0.4870, 0.5150, 0.5490, 0.5130, 0.5180, 0.4860, 0.5030, 0.5050, 0.4980$. There is no real pattern in either type of average. The random-number-generated averages range between 0.4860 and 0.5490, a range of 0.0630, whereas the calculated PRNG bit averages range between 0.4680 and 0.5220, a range of 0.0540.

66) [Technology Required] Find the first 1000 digits of π using either a computer program or Internet resource. Create a bit sequence b_n by letting $b_n = 1$ if the nth digit of π is odd and $b_n = 0$ if the nth digit of π is even. Compute the average value of b_n and the average value of $d_n = |b_{n+1} - b_n|, n = 1, \ldots, 999$. Does the sequence b_n appear random? Do the differences between successive elements of b_n appear random?

This page titled 4.1E: Exercises is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Roy Simpson.

4.2: Infinite Series

Learning Objectives

- Explain the meaning of the sum of an infinite series.
- Calculate the sum of a geometric series.
- Evaluate a telescoping series.

We have seen that a sequence is an ordered set of terms. Adding these terms together gives you a **series**. In this section, we define an infinite series and show how series are related to sequences. We also define what it means for a series to converge or diverge. We introduce one of the most important series types: the geometric series. In the next chapter, we will use geometric series to write certain functions as polynomials with infinite terms. This process is important because it allows us to evaluate, differentiate, and integrate complicated functions by using easier to handle polynomials. We also discuss the harmonic series, arguably the most interesting divergent series.

Sums and Series

An **infinite series** is a sum of infinitely many terms and is written in the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots.$$

But what does this mean? We cannot add an infinite number of terms like we can add a finite number of terms. Instead, the value of an infinite series is defined in terms of the limit of partial sums. A **partial sum** of an infinite series is a finite sum of the form

$$\sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k.$$

To see how we use partial sums to evaluate infinite series, consider the following example. Suppose oil is seeping into a lake such that 1000 gallons enters the lake the first week. During the second week, an additional 500 gallons of oil enters the lake. The third week, 250 more gallons enters the lake. Assume this pattern continues so that half as much oil enters the lake each week as did the previous week. If this continues forever, what can we say about the amount of oil in the lake? Will the amount of oil continue to get arbitrarily large, or is it possible that it approaches some finite amount? To answer this question, we look at the amount of oil in the lake after k weeks. Letting S_k denote the amount of oil in the lake (measured in thousands of gallons) after k weeks, we see that

$$\begin{aligned}
S_1 &= 1 \\
S_2 &= 1 + 0.5 = 1 + \frac{1}{2} \\
S_3 &= 1 + 0.5 + 0.25 = 1 + \frac{1}{2} + \frac{1}{4} \\
S_4 &= 1 + 0.5 + 0.25 + 0.125 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \\
S_5 &= 1 + 0.5 + 0.25 + 0.125 + 0.0625 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}
\end{aligned}$$

Looking at this pattern, we see that the amount of oil in the lake (in thousands of gallons) after k weeks is

$$S_k = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^{k-1}} = \sum_{n=1}^{k} \left(\frac{1}{2}\right)^{n-1}.$$

We are interested in what happens as $k \to \infty$. Symbolically, the amount of oil in the lake as $k \to \infty$ is given by the infinite series

$$\lim_{k\to\infty} \sum_{n=1}^{k} \left(\frac{1}{2}\right)^{n-1} \equiv \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots.$$

Before we continue, it is important to note that we are *defining* $\lim\limits_{k\to\infty}\sum\limits_{n=1}^{k}\left(\frac{1}{2}\right)^{n-1}$ with the more succinct (and slightly less rigorous) notation $\sum\limits_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-1}$. With that being cleared up, we can get back to the example.

At the same time, as $k\to\infty$, the amount of oil in the lake can be calculated by evaluating $\lim\limits_{k\to\infty}S_k$. Therefore, the behavior of the infinite series can be determined by looking at the behavior of the sequence of partial sums $\{S_k\}$. If the sequence of partial sums, $\{S_k\}$, converges, we say that the infinite series converges. Its sum is given by $\lim\limits_{k\to\infty}S_k$. If the sequence $\{S_k\}$ diverges, we say the infinite series diverges. We now turn our attention to determining the limit of this sequence $\{S_k\}$.

First, simplifying some of these partial sums, we see that

$$\begin{aligned} S_1 &= 1 \\ S_2 &= 1+\frac{1}{2} &&= \frac{3}{2} \\ S_3 &= 1+\frac{1}{2}+\frac{1}{4} &&= \frac{7}{4} \\ S_4 &= 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8} &&= \frac{15}{8} \\ S_5 &= 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16} &&= \frac{31}{16} \end{aligned}$$

Plotting some of these values in Figure 4.2.1, it appears that the sequence $\{S_k\}$ could be approaching 2.

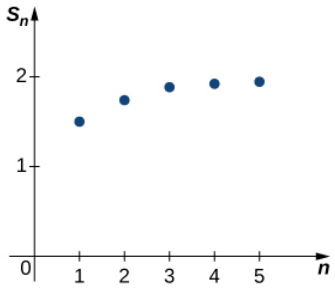


Figure 4.2.1: The graph shows the sequence of partial sums $\{S_k\}$. The sequence appears to be approaching the value 2.

Let's look for more convincing evidence. In the following table, we list the values of S_k for several values of k.

k	5	10	15	20
S_k	1.9375	1.998	1.999939	1.999998

These data supply more evidence suggesting that the sequence $\{S_k\}$ converges to 2. Later, we will provide an analytic argument that can be used to prove that $\lim\limits_{k\to\infty}S_k=2$. For now, we rely on the numerical and graphical data to convince ourselves that the sequence of partial sums does converge to 2. Since this sequence of partial sums converges to 2, we say the infinite series converges to 2 and write

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-1}=2.$$

Returning to the question about the oil in the lake, since this infinite series converges to 2, we conclude that the amount of oil in the lake will get arbitrarily close to 2000 gallons as the amount of time gets sufficiently large.

This series is an example of a geometric series. We discuss geometric series in more detail later in this section. First, we summarize what it means for an infinite series to converge.

Definition: Infinite Series

An **infinite series** is an expression of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots.$$

For each positive integer, k, the sum

$$S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k$$

is called the kth **partial sum** of the infinite series. The partial sums form a sequence $\{S_k\}$. The infinite series converges if the sequence of partial sums converges to a real number S. If we can describe the **convergence** of a series to S, we call S the **sum of the series**, and we write

$$\sum_{n=1}^{\infty} a_n = S.$$

If the sequence of partial sums diverges, we say the series **diverges** (or is **divergent**).

Note that the index for a series need not begin with $n = 1$ but can begin with any value. For example, the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1}$$

can also be written as

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n} \text{ or } \sum_{n=5}^{\infty} \left(\frac{1}{2}\right)^{n-5}.$$

Often, it is convenient for the index to begin at 1. If, for some reason, it begins at a different value, we can **re-index** by changing variables. For example, consider the series

$$\sum_{n=2}^{\infty} \frac{1}{n^2}.$$

By introducing the variable $m = n - 1$ so that $n = m + 1$, we can rewrite the series as

$$\sum_{m=1}^{\infty} \frac{1}{(m+1)^2}.$$

Example 4.2.1: Evaluating Limits of Sequences of Partial Sums

For each of the following series, use the sequence of partial sums to determine whether the series converges or diverges.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}$

b. $\displaystyle\sum_{n=1}^{\infty} (-1)^n$

c. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

Solutions

a. The sequence of partial sums $\{S_k\}$ satisfies

$$\begin{aligned} S_1 &= \frac{1}{2} \\ S_2 &= \frac{1}{2}+\frac{2}{3} \\ S_3 &= \frac{1}{2}+\frac{2}{3}+\frac{3}{4} \\ S_4 &= \frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5} \end{aligned}$$

Notice that each term added is greater than $1/2$. As a result, we see that

$$\begin{aligned} S_1 &= \frac{1}{2} \\ S_2 &= \frac{1}{2}+\frac{2}{3} &&> \frac{1}{2}+\frac{1}{2} &&= 2\left(\frac{1}{2}\right) \\ S_3 &= \frac{1}{2}+\frac{2}{3}+\frac{3}{4} &&> \frac{1}{2}+\frac{1}{2}+\frac{1}{2} &&= 3\left(\frac{1}{2}\right) \\ S_4 &= \frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5} &&> \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2} &&= 4\left(\frac{1}{2}\right) \end{aligned}$$

From this pattern we can see that $S_k > k\left(\frac{1}{2}\right)$ for every integer k. Therefore, $\{S_k\}$ is unbounded and, consequently, diverges. Hence, the infinite series $\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges.

b. The sequence of partial sums $\{S_k\}$ satisfies

$$\begin{aligned} S_1 &= -1 \\ S_2 &= -1+1 &&= 0 \\ S_3 &= -1+1-1 &&= -1 \\ S_4 &= -1+1-1+1 &&= 0 \end{aligned}$$

From this pattern, we can see the sequence of partial sums is

$$\{S_k\} = -1, 0, -1, 0, \ldots.$$

Since this sequence diverges, the infinite series $\sum_{n=1}^{\infty} (-1)^n$ diverges.

c. The sequence of partial sums $\{S_k\}$ satisfies

$$\begin{aligned} S_1 &= \frac{1}{1\cdot 2} &&= \frac{1}{2} \\ S_2 &= \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3} &&= \frac{2}{3} \\ S_3 &= \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4} &&= \frac{3}{4} \\ S_4 &= \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5} &&= \frac{4}{5} \\ S_5 &= \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\frac{1}{5\cdot 6} &&= \frac{5}{6} \end{aligned}$$

From this pattern, we can see that the kth partial sum is given by the explicit formula

$$S_k = \frac{k}{k+1}.$$

Since $k/(k+1) \to 1$, we conclude that the sequence of partial sums converges, and therefore the infinite series converges to 1. We have

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

? Checkpoint 4.2.1

Determine whether the series $\sum_{n=1}^{\infty} \frac{n+1}{n}$ converges or diverges.

Answer

The series diverges because the kth partial sum $S_k > k$.

The Harmonic Series

A useful series to know about is the **harmonic series.** The harmonic series is defined as

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots.$$

This series is interesting because it diverges, but it diverges *very* slowly. By this, we mean that the terms in the sequence of partial sums $\{S_k\}$ approach infinity but do so very slowly. We will show that the series diverges, but first, we illustrate the slow growth of the terms in the sequence $\{S_k\}$ in the following table.

k	10	100	1000	10,00	100,000	1,000,000
S_k	2.92897	5.18738	7.48547	9.78761	12.09015	14.39273

Even after $1,000,000$ terms, the partial sum is still relatively small. From this table, it is not clear that this series diverges. However, we can show analytically that the sequence of partial sums diverges, and therefore the series diverges.

To show that the sequence of partial sums diverges, we show that the sequence of partial sums is unbounded. We begin by writing the first several partial sums:

$$\begin{aligned} S_1 &= 1 \\ S_2 &= 1 + \frac{1}{2} \\ S_3 &= 1 + \frac{1}{2} + \frac{1}{3} \\ S_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \end{aligned}$$

Notice that for the last two terms in S_4,

$$\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4}.$$

Therefore, we conclude that

$$S_4 > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{1}{2} + \frac{1}{2} = 1 + 2\left(\frac{1}{2}\right).$$

Using the same idea for S_8 we see that

$$\begin{aligned} S_8 &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8} \\ &> 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right) \\ &= 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2} \\ &= 1+3\left(\frac{1}{2}\right) \end{aligned}$$

From this pattern, we see that $S_1 = 1$, $S_2 = 1 + 1/2$, $S_4 > 1 + 2(1/2)$, and $S_8 > 1 + 3(1/2)$. More generally, it can be shown that $S_{2^j} > 1 + j(1/2)$ for all $j > 1$. Since $1 + j(1/2) \to \infty$, we conclude that the sequence, $\{S_k\}$, is unbounded. In the previous section, we stated that convergent sequences are bounded. Consequently, it diverges since $\{S_k\}$ is unbounded. Thus, the harmonic series diverges.

Algebraic Properties of Convergent Series

Since the sum of a convergent infinite series is defined as a limit of a sequence, the algebraic properties for series listed below follow directly from the Algebraic Properties for Sequences.

Theorem: Algebraic Properties of Convergent Series

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent series. Then, the following algebraic properties hold.

i. The series $\sum_{n=1}^{\infty}(a_n + b_n)$ converges, and $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$. (Sum Rule for Convergent Series)

ii. The series $\sum_{n=1}^{\infty}(a_n - b_n)$ converges, and $\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$. (Difference Rule for Convergent Series)

iii. For any real number c, the series $\sum_{n=1}^{\infty} ca_n$ converges, and $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$. (Constant Multiple Rule for Convergent Series)

Example 4.2.2: Using Algebraic Properties of Convergent Series

Evaluate $\sum_{n=1}^{\infty}\left[\frac{3}{n(n+1)} + \left(\frac{1}{2}\right)^{n-2}\right]$.

Solution

We showed earlier that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

and

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = 2.$$

Since both of those series converge, we can apply the Algebraic Properties of Convergent Series to evaluate

$$\sum_{n=1}^{\infty}\left[\frac{3}{n(n+1)} + \left(\frac{1}{2}\right)^{n-2}\right].$$

Using the Sum Rule for Convergent Series, write

$$\sum_{n=1}^{\infty}\left[\frac{3}{n(n+1)} + \left(\frac{1}{2}\right)^{n-2}\right] = \sum_{n=1}^{\infty} \frac{3}{n(n+1)} + \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-2}.$$

Then, using the Constant Multiple Rule for Convergent Series and the sums above, we can conclude that

$$\sum_{n=1}^{\infty}\frac{3}{n(n+1)}+\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-2}=3\sum_{n=1}^{\infty}\frac{1}{n(n+1)}+\left(\frac{1}{2}\right)^{-1}\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-1}=3(1)+\left(\frac{1}{2}\right)^{-1}(2)=3+2(2)=7.$$

Checkpoint 4.2.2

Evaluate $\sum_{n=1}^{\infty}\frac{5}{2^{n-1}}$.

Answer

10

Geometric Series

A **geometric series** is any series that we can write in the form

$$a+ar+ar^2+ar^3+\cdots=\sum_{n=1}^{\infty}ar^{n-1}.$$

Because the ratio of each term in this series to the previous term is r, the number r is called the **common ratio**. We refer to a as the initial term because it is the first term in the series. For example, the series

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-1}=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$$

is a geometric series with initial term $a=1$ and common ratio $r=1/2$.

In general, when does a geometric series converge? Consider the geometric series

$$\sum_{n=1}^{\infty}ar^{n-1}$$

when $a>0$. Its sequence of partial sums $\{S_k\}$ is given by

$$S_k=\sum_{n=1}^{k}ar^{n-1}=a+ar+ar^2+\cdots+ar^{k-1}. \tag{4.2.1}$$

Consider the case when $r=1$. In that case,

$$S_k=a+a(1)+a(1)^2+\cdots+a(1)^{k-1}=ak.$$

Since $a>0$, we know $ak\to\infty$ as $k\to\infty$. Therefore, the sequence of partial sums is unbounded and thus diverges. Consequently, the infinite series diverges for $r=1$. For $r\neq 1$, to find the limit of $\{S_k\}$, multiply Equation 4.2.1 by $1-r$. Doing so, we see that

$$\begin{aligned}(1-r)S_k&=a(1-r)(1+r+r^2+r^3+\cdots+r^{k-1})\\&=a[(1+r+r^2+r^3+\cdots+r^{k-1})-(r+r^2+r^3+\cdots+r^k)]=a(1-r^k).\end{aligned}$$

All the other terms cancel out.

Therefore,

$$S_k=\frac{a(1-r^k)}{1-r}\text{ for }r\neq 1.$$

From our discussion in the previous section, we know that the geometric sequence $r^k\to 0$ if $|r|<1$, and that r^k diverges if $|r|>1$ or $r=\pm 1$. Therefore, for $|r|<1$, $S_k\to\frac{a}{1-r}$ and we have

$$\sum_{n=1}^{\infty}ar^{n-1}=\frac{a}{1-r}\text{ if }|r|<1.$$

If $|r|\geq 1$, S_k diverges, and therefore

$$\sum_{n=1}^{\infty} ar^{n-1} \text{ diverges if } |r| \geq 1.$$

Theorem: Convergence of Geometric Series

A **geometric series** is a series of the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots. \tag{4.2.2}$$

If $|r| < 1$ the series **converges**, and

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \text{ for } |r| < 1.$$

If $|r| \geq 1$ the series **diverges**.

Geometric series sometimes appear in slightly different forms. For example, sometimes the index begins at a value other than $n = 1$, or the exponent involves a linear expression for n other than $n - 1$. As long as we can rewrite the series in the form given by Equation 4.2.2, it is a geometric series. For example, consider the series

$$\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{n+2}.$$

To see that this is a geometric series, we write out the first several terms:

$$\begin{aligned} \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{n+2} &= \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^4 + \cdots \\ &= \frac{4}{9} + \frac{4}{9} \cdot \left(\frac{2}{3}\right) + \frac{4}{9} \cdot \left(\frac{2}{3}\right)^2 + \cdots \end{aligned}$$

We see that the initial term is $a = 4/9$ and the ratio is $r = 2/3$. Therefore, the series can be written as

$$\sum_{n=1}^{\infty} \frac{4}{9} \cdot \left(\frac{2}{3}\right)^{n-1}.$$

Since $|r| = |2/3| = 2/3 < 1$, this series converges, and its sum is given by

$$\sum_{n=1}^{\infty} \frac{4}{9} \cdot \left(\frac{2}{3}\right)^{n-1} = \frac{4/9}{1 - 2/3} = \frac{4}{3}.$$

Example 4.2.3: Determining Convergence or Divergence of a Geometric Series

Determine whether each of the following geometric series converges or diverges, and if it converges, find its sum.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{4^{n-1}}$

b. $\displaystyle\sum_{n=1}^{\infty} e^{2n}$

Solutions

a. Writing out the first several terms in the series, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-3)^{n+1}}{4^{n-1}} &= \frac{(-3)^2}{4^0}+\frac{(-3)^3}{4}+\frac{(-3)^4}{4^2}+\cdots \\ &= (-3)^2+(-3)^2\cdot\left(\frac{-3}{4}\right)+(-3)^2\cdot\left(\frac{-3}{4}\right)^2+\cdots \\ &= 9+9\cdot\left(\frac{-3}{4}\right)+9\cdot\left(\frac{-3}{4}\right)^2+\cdots\end{aligned}$$

The initial term is $a=9$ and the ratio is $r=-3/4$. Since $|r|=3/4<1$, the series converges to

$$\frac{9}{1-(-3/4)}=\frac{9}{7/4}=\frac{36}{7}.$$

b. Writing this series as

$$e^2\sum_{n=1}^{\infty}(e^2)^{n-1},$$

we can see that this is a geometric series where $|r|=e^2>1$. Therefore, the series diverges.

Checkpoint 4.2.3

Determine whether the series $\sum_{n=1}^{\infty}\left(\frac{-2}{5}\right)^{n-1}$ converges or diverges. If it converges, find its sum.

Answer

$5/7$

We now turn our attention to a nice application of geometric series. We show how they can be used to write repeating decimals as fractions of integers.

Example 4.2.4: Writing Repeating Decimals as Fractions of Integers

Use a geometric series to write $3.\overline{26}$ as a fraction of integers.

Solution

Since $3.\overline{26}=3.262626\ldots$, first we write

$$\begin{aligned}3.262626\ldots &= 3+\frac{26}{100}+\frac{26}{1000}+\frac{26}{100,000}+\cdots \\ &= 3+\frac{26}{10^2}+\frac{26}{10^4}+\frac{26}{10^6}+\cdots\end{aligned}$$

Ignoring the term 3, the rest of this expression is a geometric series with initial term $a=26/10^2$ and ratio $r=1/10^2$. Therefore, the sum of this series is

$$\frac{26/10^2}{1-(1/10^2)}=\frac{26/10^2}{99/10^2}=\frac{26}{99}.$$

Thus,

$$3.262626\ldots=3+\frac{26}{99}=\frac{323}{99}.$$

https://math.libretexts.org/@go/page/168439

Checkpoint 4.2.4

Write $5.2\overline{7}$ as a fraction of integers.

Answer

$475/90$

Example 4.2.5: Finding the Area of the Koch Snowflake

Define a sequence of figures $\{F_n\}$ recursively as follows (Figure 4.2.2). Let F_0 be an equilateral triangle with sides of length 1. For $n \geq 1$, let F_n be the curve created by removing the middle third of each side of F_{n-1} and replacing it with an equilateral triangle pointing outward. The limiting figure as $n \to \infty$ is known as **Koch's snowflake**.

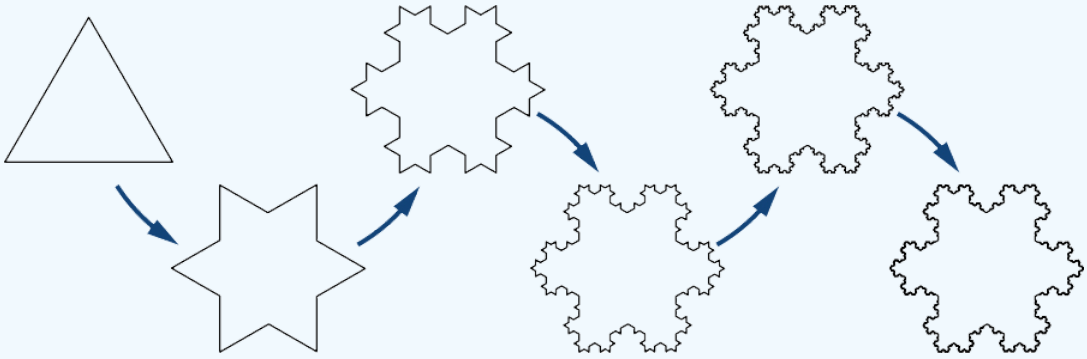


Figure 4.2.2: The first four figures, F_0, F_1, F_2 and F_3 in the construction of the Koch snowflake.

a. Find the length L_n of the perimeter of F_n. Evaluate $\lim_{n\to\infty} L_n$ to find the length of the perimeter of Koch's snowflake.

b. Find the area A_n of figure F_n. Evaluate $\lim_{n\to\infty} A_n$ to find the area of Koch's snowflake.

Solutions

a. Let N_n denote the number of sides of figure F_n. Since F_0 is a triangle, $N_0 = 3$. Let ln denote the length of each side of F_n. Since F_0 is an equilateral triangle with sides of length $l_0 = 1$, we now need to determine N_1 and l_1. Since F_1 is created by removing the middle third of each side and replacing that line segment with two line segments, for each side of F_0, we get four sides in F_1. Therefore, the number of sides for F_1 is

$$N_1 = 4 \cdot 3.$$

Since the length of each of these new line segments is $1/3$ the length of the line segments in F_0, the length of the line segments for F_1 is given by

$$l_1 = \frac{1}{3} \cdot 1 = \frac{1}{3}.$$

Similarly, for F_2, since the middle third of each side of F_1 is removed and replaced with two line segments, the number of sides in F_2 is given by

$$N_2 = 4N_1 = 4(4 \cdot 3) = 4^2 \cdot 3.$$

Since the length of each of these sides is $1/3$ the length of the sides of F_1, the length of each side of figure F_2 is given by

$$l_2 = \frac{1}{3} \cdot l_1 = \frac{1}{3} \cdot \frac{1}{3} = \left(\frac{1}{3}\right)^2.$$

More generally, since F_n is created by removing the middle third of each side of F_{n-1} and replacing that line segment with two line segments of length $\frac{1}{3}|_{n-1}$ in the shape of an equilateral triangle, we know that $N_n = 4N_{n-1}$ and $l_n = \frac{l_{n-1}}{3}$. Therefore, the number of sides of figure F_n is

$$N_n = 4^n \cdot 3$$

and the length of each side is

$$l_n = \left(\frac{1}{3}\right)^n.$$

Therefore, to calculate the perimeter of F_n, we multiply the number of sides N_n and the length of each side l_n. We conclude that the perimeter of F_n is given by

$$L_n = N_n \cdot l_n = 3 \cdot \left(\frac{4}{3}\right)^n$$

Therefore, the length of the perimeter of Koch's snowflake is

$$L = \lim_{n\to\infty} L_n = \infty.$$

b. Let T_n denote the area of each new triangle created when forming F_n. For $n = 0$, T_0 is the area of the original equilateral triangle. Therefore, $T_0 = A_0 = \sqrt{3}/4$. For $n \geq 1$, since the lengths of the sides of the new triangle are $1/3$ the length of the sides of F_{n-1}, we have

$$T_n = \left(\frac{1}{3}\right)^2 \cdot T_{n-1} = \frac{1}{9} \cdot T_{n-1}.$$

Therefore, $T_n = \left(\frac{1}{9}\right)^n \cdot \frac{\sqrt{3}}{4}$. Since a new triangle is formed on each side of F_{n-1},

$$\begin{aligned} A_n &= A_{n-1} + N_{n-1} \cdot T_n \\ &= A_{n-1} + (3 \cdot 4_{n-1}) \cdot \left(\frac{1}{9}\right)^n \cdot \frac{\sqrt{3}}{4} \\ &= A_{n-1} + \frac{3}{4} \cdot \left(\frac{4}{9}\right)^n \cdot \frac{\sqrt{3}}{4} \end{aligned}$$

Writing out the first few terms, we see that

$$\begin{aligned} A_0 &= \frac{\sqrt{3}}{4} \\ A_1 &= A_0 + \frac{3}{4} \cdot \left(\frac{4}{9}\right) \cdot \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{4} + \frac{3}{4} \cdot \left(\frac{4}{9}\right) \cdot \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{4}\left[1 + \frac{3}{4} \cdot \left(\frac{4}{9}\right)\right] \\ A_2 &= A_1 + \frac{3}{4} \cdot (\frac{4}{9})^2 \cdot \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{4}\left[1 + \frac{3}{4} \cdot \left(\frac{4}{9}\right)\right] + \frac{3}{4} \cdot \left(\frac{4}{9}\right)^2 \cdot \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{4}[1 + \frac{3}{4} \cdot (\frac{4}{9}) + \frac{3}{4} \cdot \left(\frac{4}{9}\right)^2] \end{aligned}$$

More generally,

$$A_n = \frac{\sqrt{3}}{4}\left[1 + \frac{3}{4}\left(\frac{4}{9} + \left(\frac{4}{9}\right)^2 + \cdots + \left(\frac{4}{9}\right)^n\right)\right].$$

Factoring $4/9$ out of each term inside the inner parentheses, we rewrite our expression as

$$A_n = \frac{\sqrt{3}}{4}\left[1 + \frac{1}{3}\left(1 + \frac{4}{9} + \left(\frac{4}{9}\right)^2 + \cdots + \left(\frac{4}{9}\right)^{n-1}\right)\right].$$

The expression $1 + \left(\frac{4}{9}\right) + \left(\frac{4}{9}\right)^2 + \cdots + \left(\frac{4}{9}\right)^{n-1}$ is a geometric sum. As shown earlier, this sum satisfies

$$1 + \frac{4}{9} + \left(\frac{4}{9}\right)^2 + \cdots + \left(\frac{4}{9}\right)^{n-1} = \frac{1 - (4/9)^n}{1 - (4/9)}.$$

Substituting this expression into the expression above and simplifying, we conclude that

$$A_n = \frac{\sqrt{3}}{4}\left[1 + \frac{1}{3}(\frac{1 - (4/9)^n}{1 - (4/9)})\right] = \frac{\sqrt{3}}{4}\left[\frac{8}{5} - \frac{3}{5}\left(\frac{4}{9}\right)^n\right].$$

Therefore, the area of Koch's snowflake is

$$A = \lim_{n\to\infty} A_n = \frac{2\sqrt{3}}{5}.$$

Analysis

The Koch snowflake is interesting because it has a finite area yet an infinite perimeter. Although this may seem impossible initially, recall that you have seen similar examples earlier in the text. For example, consider the region bounded by the curve $y = 1/x^2$ and the x-axis on the interval $[1, \infty)$. Since the improper integral

$$\int_1^\infty \frac{1}{x^2}\,dx$$

converges, the area of this region is finite, even though the perimeter is infinite.

Telescoping Series

Consider the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. We discussed this series in Example 4.2.1, showing that the series converges by writing out the first several partial sums S_1, S_2, ..., S_6, and noticing that they are all of the form $S_k = \frac{k}{k+1}$. Here, we use a different technique to show that this series converges. By using partial fractions, we can write

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

Therefore, the series can be written as

$$\sum_{n=1}^{\infty}\left[\frac{1}{n} - \frac{1}{n+1}\right] = \left(1 + \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots.$$

Writing out the first several terms in the sequence of partial sums $\{S_k\}$, we see that

$$\begin{aligned} S_1 &= 1 - \frac{1}{2} \\ S_2 &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) &&= 1 - \frac{1}{3} \\ S_3 &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) &&= 1 - \frac{1}{4} \end{aligned}$$

In general,

$$S_k = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{k} - \frac{1}{k+1}\right) = 1 - \frac{1}{k+1}.$$

We notice that the middle terms cancel each other out, leaving only the first and last terms. The series collapses like a spyglass with tubes that disappear into each other to shorten the telescope. For this reason, we call a series with this property a **telescoping series**. For this series, since $S_k = 1 - 1/(k+1)$ and $1/(k+1) \to 0$ as $k \to \infty$, the sequence of partial sums converges to 1, and therefore the series converges to 1.

Definition: Telescoping Series

A **telescoping series** is a series in which most of the terms cancel in each of the partial sums, leaving only some of the first terms and some of the last terms.

For example, any series of the form

$$\sum_{n=1}^{\infty}[b_n - b_{n+1}] = (b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + \cdots$$

is a telescoping series. We can see this by writing out some of the partial sums. In particular, we see that

$$\begin{aligned} S_1 &= b_1 - b_2 \\ S_2 &= (b_1 - b_2) + (b_2 - b_3) &&= b_1 - b_3 \\ S_3 &= (b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) &&= b_1 - b_4. \end{aligned}$$

In general, the kth partial sum of this series is

$$S_k = b_1 - b_{k+1}.$$

Since the kth partial sum can be simplified to the difference of these two terms, the sequence of partial sums $\{S_k\}$ will converge if and only if the sequence $\{b_{k+1}\}$ converges. Moreover, if the sequence b_{k+1} converges to some finite number B then the sequence of partial sums converges to $b_1 - B$ and therefore

$$\sum_{n=1}^{\infty}[b_n - b_{n+1}] = b_1 - B.$$

In the next example, we show how to use these ideas to analyze a telescoping series of this form.

Example 4.2.6: Evaluating a Telescoping Series

Determine whether the telescoping series

$$\sum_{n=1}^{\infty}\left[\cos\left(\frac{1}{n}\right) - \cos\left(\frac{1}{n+1}\right)\right]$$

converges or diverges. If it converges, find its sum.

Solution

By writing out terms in the sequence of partial sums, we can see that

$$\begin{aligned} S_1 &= \cos(1) - \cos\left(\frac{1}{2}\right) \\ S_2 &= \left(\cos(1) - \cos\left(\frac{1}{2}\right)\right) + \left(\cos\left(\frac{1}{2}\right) - \cos\left(\frac{1}{3}\right)\right) &&= \cos(1) - \cos\left(\frac{1}{3}\right) \\ S_3 &= \left(\cos(1) - \cos\left(\frac{1}{2}\right)\right) + \left(\cos\left(\frac{1}{2}\right) - \cos\left(\frac{1}{3}\right)\right) + \left(\cos\left(\frac{1}{3}\right) - \cos\left(\frac{1}{4}\right)\right) &&= \cos(1) - \cos\left(\frac{1}{4}\right) \end{aligned}$$

In general,

$$S_k = \cos(1) - \cos\left(\frac{1}{k+1}\right).$$

Since $1/(k+1) \to 0$ as $k \to \infty$ and $\cos x$ is a continuous function, $\cos(1/(k+1)) \to \cos(0) = 1$. Therefore, we conclude that $S_k \to \cos(1) - 1$. The telescoping series converges and the sum is given by

$$\sum_{n=1}^{\infty}\left[\cos\left(\frac{1}{n}\right) - \cos\left(\frac{1}{n+1}\right)\right] = \cos(1) - 1.$$

Checkpoint 4.2.5

Determine whether $\sum_{n=1}^{\infty}[e^{1/n} - e^{1/(n+1)}]$ converges or diverges. If it converges, find its sum.

Answer

$e - 1$

Euler's Constant

We have shown that the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges. Here, we investigate the behavior of the partial sums S_k as $k \to \infty$. In particular, we show that they behave like the natural logarithm function by showing that there exists a constant γ such that\[\sum_{n=1}^k\left(\dfrac{1}{n}−\ln k\right) \to \gamma \) as $k \to \infty$.

This constant, γ, is known as **Euler's constant.**

1. Let $T_k = \sum_{n=1}^{k} \left(\frac{1}{n} - \ln k\right)$. Evaluate T_k for various values of k.
2. For T_k as defined in part 1. show that the sequence T_k converges by using the following steps.
   a. Show that the sequence $\{T_k\}$ is monotone decreasing. (Hint: Show that $\ln(1 + 1/k > 1/(k+1))$
   b. Show that the sequence $\{T_k\}$ is bounded below by zero. (Hint: Express $\ln k$ as a definite integral.)
   c. Use the Monotone Convergence Theorem to conclude that the sequence $\{T_k\}$ converges. The limit, γ, is Euler's constant.
3. Now, estimate how far T_k is from γ for a given integer k. Prove that for $k \geq 1, 0 < T_k - \gamma \leq 1/k$ by using the following steps.
   a. Show that $\ln(k+1) - \ln k < 1/k$.
   b. Use the result from part a. to show that for any integer k

$$T_k - T_{k+1} < \frac{1}{k} - \frac{1}{k+1}.$$

   c. For any integers k and j such that $j > k$ express $T_k - T_j$ as a telescoping sum by writing

$$T_k - T_j = (T_k - T_{k+1}) + (T_{k+1} - T_{k+2}) + (T_{k+2} - T_{k+3}) + \cdots + (T_{j-1} - T_j).$$

   Use the result from part b. combined with this telescoping sum to conclude that

$$T_k - T_j < \frac{1}{k} - \frac{1}{j}.$$

   d. Apply the limit to both sides of the inequality in part c. to conclude that

$$T_k - \gamma \leq \frac{1}{k}.$$

   e. Estimate γ to an accuracy of within 0.001.

Key Concepts

- Given the infinite series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

  and the corresponding sequence of partial sums $\{S_k\}$ where

$$S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k$$

  The series converges if and only if the sequence $\{S_k\}$ converges.
- The geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ converges if $|r| < 1$, and diverges if $|r| \geq 1$. For $|r| < 1$,

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}.$$

- The harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

  diverges.
- A series of the form $\sum_{n=1}^{\infty} [b_n - b_{n+1}] = [b_1 - b_2] + [b_2 - b_3] + [b_3 - b_4] + \cdots + [b_n - b_{n+1}] + \cdots$ is a telescoping series. The kth partial sum of this series is given by $S_k = b_1 - b_{k+1}$. The series will converge if and only if $\lim_{k \to \infty} b_{k+1}$ exists. In that case,

$$\sum_{n=1}^{\infty}[b_n - b_{n+1}] = b_1 - \lim_{k\to\infty}(b_{k+1}).$$

Key Equations

- **Harmonic series**

$$\sum_{n=1}^{\infty}\frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

- **Sum of a geometric series**

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \text{ for } |r| < 1$$

Glossary

convergence of a series

a series converges if the sequence of partial sums for that series converges

divergence of a series

a series diverges if the sequence of partial sums for that series diverges

geometric series

a geometric series is a series that can be written in the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots$$

harmonic series

the harmonic series takes the form

$$\sum_{n=1}^{\infty}\frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

infinite series

an infinite series is an expression of the form

$$a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n$$

partial sum

the kth partial sum of the infinite series $\sum_{n=1}^{\infty} a_n$ is the finite sum

$$S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k$$

telescoping series

a telescoping series is one in which most of the terms cancel in each of the partial sums

4.2E: Exercises

In exercises 1 - 4, use sigma notation to write each expressions as an infinite series.

1) $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$

Answer

$$\sum_{n=1}^{\infty}\frac{1}{n}$$

2) $1-1+1-1+\cdots$

3) $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+...$

Answer

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$$

4) $\sin 1+\sin\frac{1}{2}+\sin\frac{1}{3}+\sin\frac{1}{4}+\cdots$

In exercises 5 - 8, compute the first four partial sums $S_1,\ldots,S_4$ for the series having n^{th} term a_n starting with $n=1$ as follows.

5) $a_n=n$

Answer

$1,3,6,10$

6) $a_n=1/n$

7) $a_n=\sin\frac{n\pi}{2}$

Answer

$1,1,0,0$

8) $a_n=(-1)^n$

In exercises 9 - 12, compute the general term a_n of the series with the given partial sum S_n. If the sequence of partial sums converges, find its limit S.

9) $S_n=1-\frac{1}{n},\quad n\geq 2$

Answer

$a_n=S_n-S_{n-1}=\frac{1}{n-1}-\frac{1}{n}$. Since $S=\lim\limits_{n\to\infty}S_n=\lim\limits_{n\to\infty}\left(1-\frac{1}{n}\right)=1$, the series converges to $S=1$.

10) $S_n=\frac{n(n+1)}{2},\quad n\geq 1$

11) $S_n=\sqrt{n},\quad n\geq 2$

Answer

$a_n=S_n-S_{n-1}=\sqrt{n}-\sqrt{n-1}=\frac{1}{\sqrt{n-1}+\sqrt{n}}$.

The series diverges because the partial sums are unbounded.

That is, $\lim\limits_{n\to\infty}S_n=\lim\limits_{n\to\infty}\sqrt{n}=\infty$.

12) $S_n=2-\frac{n+2}{2^n},\quad n\geq 1$

For each series in exercises 13 - 16, use the sequence of partial sums to determine whether the series converges or diverges.

13) $\sum_{n=1}^{\infty} \frac{n}{n+2}$

Answer

$S_1 = 1/3,$
$S_2 = 1/3 + 2/4 > 1/3 + 1/3 = 2/3,$
$S_3 = 1/3 + 2/4 + 3/5 > 3 \cdot (1/3) = 1.$
In general $S_k > k/3,$ so the series diverges.
Note that the nth Term Test for Divergence could also be used to prove that this series diverges.

14) $\sum_{n=1}^{\infty}(1-(-1)^n))$

15) $\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$

(Hint: Use a partial fraction decomposition like that for $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.)

Answer

$S_1 = 1/(2 \cdot 3) = 1/6 = 2/3 - 1/2,$

$S_2 = 1/(2 \cdot 3) + 1/(3 \cdot 4) = 2/12 + 1/12 = 1/4 = 3/4 - 1/2,$

$S_3 = 1/(2 \cdot 3) + 1/(3 \cdot 4) + 1/(4 \cdot 5) = 10/60 + 5/60 + 3/60 = 3/10 = 4/5 - 1/2,$

$S_4 = 1/(2 \cdot 3) + 1/(3 \cdot 4) + 1/(4 \cdot 5) + 1/(5 \cdot 6) = 10/60 + 5/60 + 3/60 + 2/60 = 1/3 = 5/6 - 1/2.$

The pattern is $S_k = \frac{k+1}{k+2} - \frac{1}{2}.$
Then $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(\frac{k+1}{k+2} - \frac{1}{2}\right) = \frac{1}{2}$, so the series converges to $1/2$.

16) $\sum_{n=1}^{\infty} \frac{1}{2n+1}$

(Hint: Follow the reasoning for $\sum_{n=1}^{\infty} \frac{1}{n}$.)

Suppose that $\sum_{n=1}^{\infty} a_n = 1$, that $\sum_{n=1}^{\infty} b_n = -1$, that $a_1 = 2$, and $b_1 = -3$. Use this information to find the sum of the indicated series in exercises 17 - 20.

17) $\sum_{n=1}^{\infty}(a_n + b_n)$

Answer

$$\sum_{n=1}^{\infty}(a_n + b_n) \quad = \quad \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n \quad = \quad 1 + (-1) \quad = \quad 0$$

18) $\sum_{n=1}^{\infty}(a_n - 2b_n)$

19) $\sum_{n=2}^{\infty}(a_n - b_n)$

Answer

$$\sum_{n=2}^{\infty}(a_n - b_n) \quad = \quad \sum_{n=2}^{\infty} a_n - \sum_{n=2}^{\infty} b_n \quad = \quad \left(\sum_{n=1}^{\infty} a_n - a_1\right) - \left(\sum_{n=1}^{\infty} b_n - b_1\right) \quad = \quad (1-2) - (-1-(-3)) = -1 - 2 \quad = \quad -3$$

20) $\sum_{n=1}^{\infty}(3a_{n+1} - 4b_{n+1})$

In exercises 21 - 26, state whether the given series converges or diverges and explain why.

21) $\sum_{n=1}^{\infty} \frac{1}{n+1000}$ (Hint: Rewrite using a change of index.)

Answer

The series diverges, $\sum_{n=1001}^{\infty} \frac{1}{n}$

22) $\sum_{n=1}^{\infty} \frac{1}{n+10^{80}}$ (Hint: Rewrite using a change of index.)

23) $1+\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\cdots$

Answer

This is a convergent geometric series, since $r=\frac{1}{10}<1$

24) $1+\frac{e}{\pi}+\frac{e^2}{\pi^2}+\frac{e^3}{\pi^3}+\cdots$

25) $1+\frac{\pi}{e^2}+\frac{\pi^2}{e^4}+\frac{\pi^3}{e^6}+\frac{\pi^4}{e^8}+\cdots$

Answer

This is a convergent geometric series, since $r=\pi/e^2<1$

26) $1-\sqrt{\frac{\pi}{3}}+\sqrt{\frac{\pi^2}{9}}-\sqrt{\frac{\pi^3}{27}}+\cdots$

For each a_n in exercises 27 - 30, write its sum as a geometric series of the form $\sum_{n=1}^{\infty} ar^n$. State whether the series converges and if it does, find the exact value of its sum.

27) $a_1=-1$ and $\frac{a_n}{a_{n+1}}=-5$ for $n\geq 1$.

Answer

$\sum_{n=1}^{\infty} 5\cdot(-1/5)^n$, converges to $-5/6$

28) $a_1=2$ and $\frac{a_n}{a_{n+1}}=1/2$ for $n\geq 1$.

29) $a_1=10$ and $\frac{a_n}{a_{n+1}}=10$ for $n\geq 1$.

Answer

$\sum_{n=1}^{\infty} 100\cdot(1/10)^n$, converges to $\frac{100}{9}$

30) $a_1=\frac{1}{10}$ and $a_n/a_{n+1}=-10$ for $n\geq 1$.

In exercises 31 - 34, use the identity $\frac{1}{1-y}=\sum_{n=0}^{\infty} y^n$ (which is true for $|y|<1$) to express each function as a geometric series in the indicated term.

31) $\frac{x}{1+x}$ in x

Answer

$x\sum_{n=0}^{\infty}(-x)^n=\sum_{n=1}^{\infty}(-1)^{n-1}x^n$

32) $\frac{\sqrt{x}}{1-x^{3/2}}$ in $\sqrt{x}$

33) $\frac{1}{1+\sin^2 x}$ in $\sin x$

Answer

$$\sum_{n=0}^{\infty}(-1)^n \sin^{2n}(x)$$

34) $\sec^2 x$ in $\sin x$

In exercises 35 - 38, evaluate the telescoping series or state whether the series diverges.

35) $\sum_{n=1}^{\infty} 2^{1/n} - 2^{1/(n+1)}$

Answer

$S_k = 2 - 2^{1/(k+1)} \to 1$ as $k \to \infty$.

36) $\sum_{n=1}^{\infty} \frac{1}{n^{13}} - \frac{1}{(n+1)^{13}}$

37) $\sum_{n=1}^{\infty} (\sqrt{n} - \sqrt{n+1})$

Answer

$S_k = 1 - \sqrt{k+1}$ diverges

38) $\sum_{n=1}^{\infty} (\sin n - \sin(n+1))$

Express each series in exercises 39 - 42 as a telescoping sum and evaluate its nth partial sum.

39) $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$

Answer

$\sum_{n=1}^{\infty} [\ln n - \ln(n+1)]$,
$S_k = -\ln(k+1)$

40) $\sum_{n=1}^{\infty} \frac{2n+1}{(n^2+n)^2}$ (Hint: Factor denominator and use partial fractions.)

41) $\sum_{n=2}^{\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{(\ln n)\ln(n+1)}$

Answer

$a_n = \frac{1}{\ln n} - \frac{1}{\ln(n+1)}$ and $S_k = \frac{1}{\ln(2)} - \frac{1}{\ln(k+1)} \to \frac{1}{\ln(2)}$

42) $\sum_{n=1}^{\infty} \frac{(n+2)}{n(n+1)2^{n+1}}$ (Hint: Look at $1/(n2^n)$.

A general telescoping series is one in which all but the first few terms cancel out after summing a given number of successive terms.

43) Let $a_n = f(n) - 2f(n+1) + f(n+2)$, in which $f(n) \to 0$ as $n \to \infty$. Find $\sum_{n=1}^{\infty} a_n$.

Answer

$$\sum_{n=1}^{\infty} a_n = f(1) - f(2)$$

44) $a_n = f(n) - f(n+1) - f(n+2) + f(n+3)$, in which $f(n) \to 0$ as $n \to \infty$. Find $\sum_{n=1}^{\infty} a_n$.

45) Suppose that $a_n = c_0 f(n) + c_1 f(n+1) + c_2 f(n+2) + c_3 f(n+3) + c_4 f(n+4)$, where $f(n) \to 0$ as $n \to \infty$. Find a condition on the coefficients $c_0, \ldots, c_4$ that make this a general telescoping series.

Answer

$c_0 + c_1 + c_2 + c_3 + c_4 = 0$

46) Evaluate $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$ (Hint: $\frac{1}{n(n+1)(n+2)} = \frac{1}{2n} - \frac{1}{n+1} + \frac{1}{2(n+2)}$)

47) Evaluate $\sum_{n=2}^{\infty} \frac{2}{n^3 - n}$.

Answer

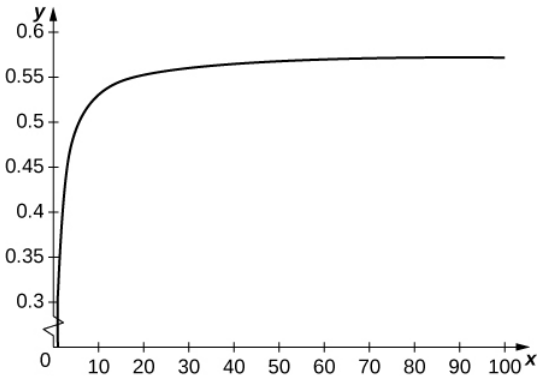
$\frac{2}{n^3-1} = \frac{1}{n-1} - \frac{2}{n} + \frac{1}{n+1}$,
$S_n = (1 - 1 + 1/3) + (1/2 - 2/3 + 1/4) + (1/3 - 2/4 + 1/5) + (1/4 - 2/5 + 1/6) + \cdots = 1/2$

48) Find a formula for $\sum_{n=1}^{\infty} \frac{1}{n(n+N)}$ where N is a positive integer.

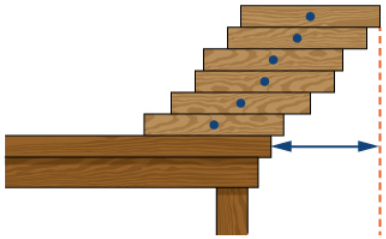
49) [Technology Required] Define a sequence $t_k = \sum_{n=1}^{k-1} (1/k) - \ln k$. Use the graph of $1/x$ to verify that t_k is increasing. Plot t_k for $k = 1 \ldots 100$ and state whether it appears that the sequence converges.

Answer

t_k converges to $0.57721 \ldots t_k$ is a sum of rectangles of height $1/k$ over the interval $[k, k+1]$ which lie above the graph of $1/x$.



50) [Technology Required] Suppose that N equal uniform rectangular blocks are stacked one on top of the other, allowing for some overhang. Archimedes' law of the lever implies that the stack of N blocks is stable as long as the center of mass of the top $(N-1)$ blocks lies at the edge of the bottom block. Let x denote the position of the edge of the bottom block, and think of its position as relative to the center of the next-to-bottom block. This implies that $(N-1)x = \left(\frac{1}{2} - x\right)$ or $x = 1/(2N)$. Use this expression to compute the maximum overhang (the position of the edge of the top block over the edge of the bottom block.) See the following figure.



Each of the following infinite series converges to the given multiple of π or $1/\pi$.

In each case, find the minimum value of N such that the Nth partial sum of the series accurately approximates the left-hand side to the given number of decimal places, and give the desired approximate value. Up to 15 decimals place, $\pi = 3.141592653589793....$

51) [Technology Required] $\pi = -3 + \sum_{n=1}^{\infty} \frac{n2^n n!^2}{(2n)!}$, error < 0.0001

Answer

$N = 22,$

$S_N = 6.1415$

52) [Technology Required] $\frac{\pi}{2} = \sum_{k=0}^{\infty} \frac{k!}{(2k+1)!!} = \sum_{k=0}^{\infty} \frac{2^k k!^2}{(2k+1)!}$, error $< 10^{-4}$

53) [Technology Required] $\frac{9801}{2\pi} = \frac{4}{9801} \sum_{k=0}^{\infty} \frac{(4k)!(1103+26390k)}{(k!)^4 396^{4k}}$, error $< 10^{-12}$

Answer

$N = 3,$

$S_N = 1.559877597243667...$

54) [Technology Required] $\frac{1}{12\pi} = \sum_{k=0}^{\infty} \frac{(-1)^k (6k)!(13591409+545140134k)}{(3k)!(k!)^3 640320^{3k+3/2}}$, error $< 10^{-15}$

55) [Technology Required] A fair coin is one that has probability $1/2$ of coming up heads when flipped.

a. What is the probability that a fair coin will come up tails n times in a row?

b. Find the probability that a coin comes up heads for the first time after an even number of coin flips.

Answer

a. The probability of any given ordered sequence of outcomes for n coin flips is $1/2^n$.

b. The probability of coming up heads for the first time on the n th flip is the probability of the sequence $TT...TH$ which is $1/2^n$. The probability of coming up heads for the first time on an even flip is $\sum_{n=1}^{\infty} 1/2^{2n}$ or $1/3$.

56) [Technology Required] Find the probability that a fair coin is flipped a multiple of three times before coming up heads.

57) [Technology Required] Find the probability that a fair coin will come up heads for the second time after an even number of flips.

Answer

$5/9$

58) [Technology Required] Find a series that expresses the probability that a fair coin will come up heads for the second time on a multiple of three flips.

59) [Technology Required] The expected number of times that a fair coin will come up heads is defined as the sum over $n = 1, 2, ...$ of n times the probability that the coin will come up heads exactly n times in a row, or $\frac{n}{2^{n+1}}$. Compute the expected number of consecutive times that a fair coin will come up heads.

Answer

$E = \sum_{n=1}^{\infty} \frac{n}{2^{n+1}} = 1,$ as can be shown using summation by parts

60) [Technology Required] A person deposits $\$10$ at the beginning of each quarter into a bank account that earns 4 annual interest compounded quarterly (four times a year).

a. Show that the interest accumulated after n quarters is $\$10(\frac{1.01^{n+1}-1}{0.01} - n)$.

b. Find the first eight terms of the sequence.

c. How much interest has accumulated after 2 years?

61) [Technology Required] Suppose that the amount of a drug in a patient's system diminishes by a multiplicative factor $r < 1$ each hour. Suppose that a new dose is administered every N hours. Find an expression that gives the amount $A(n)$ in the patient's system after n hours for each n in terms of the dosage d and the ratio r. (Hint: Write $n = mN + k$, where $0 \le k < N$, and sum over values from the different doses administered.)

Answer

The part of the first dose after n hours is dr^n, the part of the second dose is dr^{n-N}, and, in general, the part remaining of the mth dose is dr^{n-mN}, so $A(n) = \sum_{l=0}^{m} dr^{n-lN} = \sum_{l=0}^{m} dr^{k+(m-l)N} = \sum_{q=0}^{m} dr^{k+qN} = dr^k \sum_{q=0}^{m} r^{Nq} = dr^k \frac{1-r^{(m+1)N}}{1-r^N}$, where $n = k + mN$.

62) [Technology Required] A certain drug is effective for an average patient only if there is at least 1 mg per kg in the patient's system, while it is safe only if there is at most 2 mg per kg in an average patient's system. Suppose that the amount in a patient's system diminishes by a multiplicative factor of 0.9 each hour after a dose is administered. Find the maximum interval N of hours between doses, and corresponding dose range d (in mg/kg) for this N that will enable use of the drug to be both safe and effective in the long term.

63) Suppose that $a_n \ge 0$ is a sequence of numbers. Explain why the sequence of partial sums of a_n is increasing.

Answer

$S_{N+1} = a_{N+1} + S_N \ge S_N$

64) [Technology Required] Suppose that a_n is a sequence of positive numbers and the sequence S_n of partial sums of a_n is bounded above. Explain why $\sum_{n=1}^{\infty} a_n$ converges. Does the conclusion remain true if we remove the hypothesis $a_n \ge 0$?

65) [Technology Required] Suppose that $a_1 = S_1 = 1$ and that, for given numbers $S > 1$ and $0 < k < 1$, one defines $a_{n+1} = k(S - S_n)$ and $S_{n+1} = a_{n+1} + S_n$. Does S_n converge? If so, to what? (Hint: First argue that $S_n < S$ for all n and S_n is increasing.)

Answer

Since $S > 1, a_2 > 0$, and since $k < 1, S_2 = 1 + a_2 < 1 + (S - 1) = S$. If $S_n > S$ for some n, then there is a smallest n. For this $n, S > S_{n-1}$, so $S_n = S_{n-1} + k(S - S_{n-1}) = kS + (1 - k)S_{n-1} < S$, a contradiction. Thus $S_n < S$ and $a_{n+1} > 0$ for all n, so S_n is increasing and bounded by S. Let $S_* = \lim S_n$. If $S_* < S$, then $\Delta = k(S - S_*) > 0$, but we can find n such that $S_* - S_n < \Delta/2$, which implies that $S_{n+1} = S_n + k(S - S_n) > S_* + \Delta/2$, contradicting that Sn is increasing to S_*. Thus $S_n \to S$.

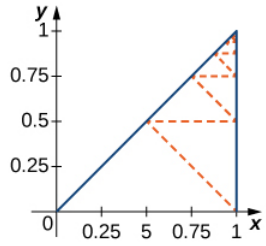
66) [Technology Required] A version of **von Bertalanffy growth** can be used to estimate the age of an individual in a homogeneous species from its length if the annual increase in year $n + 1$ satisfies $a_{n+1} = k(S - S_n)$, with S_n as the length at year n, S as a limiting length, and k as a relative growth constant. If $S_1 = 3, S = 9$, and $k = 1/2$, numerically estimate the smallest value of n such that $S_n \ge 8$. Note that $S_{n+1} = S_n + a_{n+1}$. Find the corresponding n when $k = 1/4$.

67) [Technology Required] Suppose that $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive terms. Explain why $\lim_{N\to\infty} \sum_{n=N+1}^{\infty} a_n = 0$.

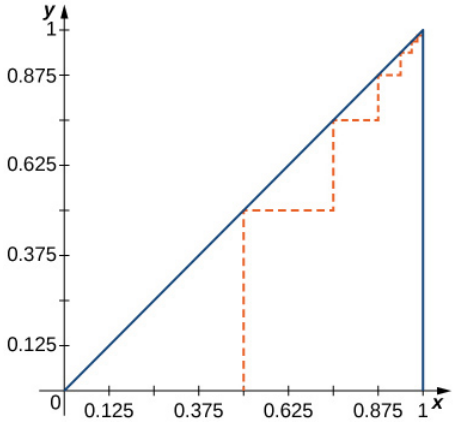
Answer

Let $S_k = \sum_{n=1}^{k} a_n$ and $S_k \to L$. Then S_k eventually becomes arbitrarily close to L, which means that $L - S_N = \sum_{n=N+1}^{\infty} a_n$ becomes arbitrarily small as $N \to \infty$.

68) [Technology Required] Find the length of the dashed zig-zag path in the following figure.



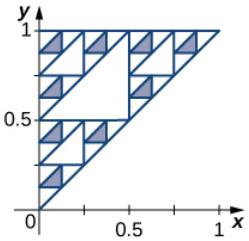
69) [Technology Required] Find the total length of the dashed path in the following figure.



Answer

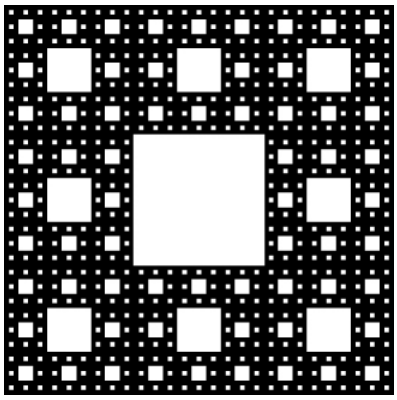
$$L = \left(1 + \frac{1}{2}\right) \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{3}{2}.$$

70) [Technology Required] The **Sierpinski triangle** is obtained from a triangle by deleting the middle fourth as indicated in the first step, by deleting the middle fourths of the remaining three congruent triangles in the second step, and in general deleting the middle fourths of the remaining triangles in each successive step. Assuming that the original triangle is shown in the figure, find the areas of the remaining parts of the original triangle after N steps and find the total length of all of the boundary triangles after N steps.



71) [Technology Required] The Sierpinski gasket is obtained by dividing the unit square into nine equal sub-squares, removing the middle square, then doing the same at each stage to the remaining sub-squares. The figure shows the remaining set after four iterations. Compute the total area removed after N stages, and compute the length the total perimeter of the remaining set after N stages.

https://math.libretexts.org/@go/page/168440



Answer

At stage one a square of area $1/9$ is removed, at stage 2 one removes 8 squares of area $1/9^2$, at stage three one removes 8^2 squares of area $1/9^3$, and so on. The total removed area after N stages is $\sum_{n=0}^{N-1} \frac{8^N}{9^{N+1}} = \frac{1}{8} \cdot \frac{1-(8/9)^N}{1-8/9} \to 1$ as $N \to \infty$. The total perimeter is $4 + 4\sum_{n=0}^{\infty} \frac{8^N}{3^{N+1}} \to \infty$.

This page titled 4.2E: Exercises is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Roy Simpson.

4.3: The Divergence and Integral Tests

Learning Objectives

- Use the Divergence Test to determine whether a series converges or diverges.
- Use the Integral Test to determine the convergence of a series.
- Estimate the value of a series by finding bounds on its remainder term.

In the previous section, we determined the convergence or divergence of several series by explicitly calculating the limit of the sequence of partial sums $\{S_k\}$. In practice, explicitly calculating this limit can be difficult or impossible. Luckily, several tests exist that allow us to determine convergence or divergence for many types of series. This section discusses two tests: the Divergence Test and the Integral Test. We will examine several other tests in the rest of this chapter and then summarize how and when to use them.

Divergence Test

We begin our exploration of tests for the convergence or divergence of series with the test that is the quickest to use.

Theorem: Divergence Test

If $\lim\limits_{n\to\infty} a_n \neq 0$, then the series $\sum\limits_{n=1}^{\infty} a_n$ diverges.

Proof by Contraposition[1]

Suppose $\sum\limits_{n=1}^{\infty} a_n$ *converges*. Let $S_n = a_1 + a_2 + \cdots + a_n$. Then $a_n = S_n - S_{n-1}$.[2] Since we have assumed $\sum\limits_{n=1}^{\infty} a_n$ converges, the sequence of partial sums, $\{S_n\}$ must be convergent. Let $\lim\limits_{n\to\infty} S_n = S$. Since $n-1 \to \infty$ as $n \to \infty$, we also have $\lim\limits_{n\to\infty} S_{n-1} = S$. Therefore

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0.$$

Hence, the limit $\lim_{n\to\infty} a_n$ exists *and* is 0.[3]

As with all of the proofs in Calculus, it is worthwhile to read through the proof of the Divergence Test and try to understand it. The method (proof by contraposition) is a wonderful indirect proof style.

Caution: The Converse in NOT True

It is important to note that the converse of the Divergence Test is *not true*. That is, if $\lim\limits_{n\to\infty} a_n = 0$, we cannot make any conclusion about the convergence of $\sum\limits_{n=1}^{\infty} a_n$.

To illustrate that cautionary statement, consider the harmonic series $\sum\limits_{n=1}^{\infty} \frac{1}{n}$.

$\lim\limits_{n\to 0} \frac{1}{n} = 0$, but the harmonic series $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges. From this point forward, we show many more examples of such series. Consequently, although we can use the Divergence Test to show that a series diverges, we cannot use it to prove that a series converges. Specifically, if $a_n \to 0$, the Divergence Test is inconclusive.

Example 4.3.1: Using the Divergence Test

For each of the following series, apply the Divergence Test. If the Divergence Test proves that the series diverges, state so. Otherwise, indicate that the Divergence Test is inconclusive.

a. $\sum\limits_{n=1}^{\infty} \frac{n}{2n-1}$

b. $\sum_{n=1}^{\infty} \frac{1}{n^3}$

c. $\sum_{n=1}^{\infty} e^{1/n^2}$

Solutions

a. Since $\lim_{n\to\infty} \frac{n}{3n-1} = \frac{1}{3} \neq 0$, by the Divergence Test, we can conclude that $\sum_{n=1}^{\infty} \frac{n}{3n-1}$ diverges.

b. Since $\lim_{n\to\infty} \frac{1}{n^3} = 0$, the Divergence Test is inconclusive.

c. Since $\lim_{n\to\infty} e^{1/n^2} = 1 \neq 0$, by the Divergence Test, the series $\sum_{n=1}^{\infty} e^{1/n^2}$ diverges.

? Checkpoint 4.3.1

What does the Divergence Test tell us about the series $\sum_{n=1}^{\infty} \cos(1/n^2)$?

Answer

The series diverges.

Integral Test

In the previous section, we proved that the harmonic series diverges by looking at the sequence of partial sums $\{S_k\}$ and showing that $S_{2^k} > 1 + k/2$ for all $k \in \mathbb{N}$. In this section, we use a different technique to prove the divergence of the harmonic series. This technique is important because it is used to prove the divergence or convergence of many other series. This test, called the **Integral Test**, compares an infinite sum to an improper integral. It is important to note that this test can only be applied when considering a series with positive terms.

To illustrate how the Integral Test works, use the harmonic series as an example. In Figure 4.3.1, we depict the harmonic series by sketching a sequence of rectangles with areas $1, 1/2, 1/3, 1/4, \ldots$ along with the function $f(x) = 1/x$.

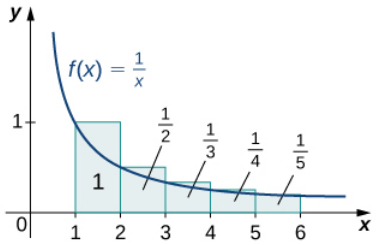


Figure 4.3.1: The sum of the areas of the rectangles is greater than the area between the curve $f(x) = 1/x$ and the x-axis for $x \geq 1$. Since the area bounded by the curve is infinite (as calculated by an improper integral), the sum of the areas of the rectangles is also infinite.

From the graph, we see that

$$\sum_{n=1}^{k} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} > \int_1^{k+1} \frac{1}{x}\, dx.$$

Therefore, for each k, the kth partial sum S_k satisfies

$$\begin{aligned} S_k &= \sum_{n=1}^{k} \frac{1}{n} \\ &> \int_1^{k+1} \frac{1}{x}\,dx \\ &= \ln x\big|_1^{k+1} \\ &= \ln(k+1) - \ln(1) \\ &= \ln(k+1). \end{aligned}$$

Since $\lim_{k\to\infty} \ln(k+1) = \infty$, we see that the sequence of partial sums $\{S_k\}$ is unbounded. Therefore, $\{S_k\}$ diverges, and, consequently, the series $\sum_{n=1}^{\infty} \frac{1}{n}$ also diverges.

Now consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. We show how an integral can be used to prove that this series converges. In Figure 4.3.2, we sketch a sequence of rectangles with areas $1, 1/2^2, 1/3^2, \ldots$ along with the function $f(x) = \frac{1}{x^2}$.

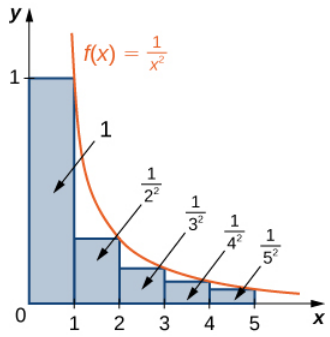


Figure 4.3.2: The sum of the areas of the rectangles is less than the sum of the area of the first rectangle and the area between the curve $f(x) = 1/x^2$ and the x-axis for $x \geq 1$. Since the area bounded by the curve is finite, the sum of the areas of the rectangles is also finite.

From the graph we see that

$$\sum_{n=1}^{k} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} < 1 + \int_1^k \frac{1}{x^2}\,dx.$$

Therefore, for each k, the kth partial sum S_k satisfies

$$\begin{aligned} S_k &= \textstyle\sum_{n=1}^{k} \frac{1}{n^2} \\ &< 1 + \textstyle\int_1^k \frac{1}{x^2}\,dx \\ &= 1 - \frac{1}{x}\Big|_1^k \\ &= 1 - \frac{1}{k} + 1 \\ &= 2 - \frac{1}{k} \\ &< 2. \end{aligned}$$

We conclude that the sequence of partial sums $\{S_k\}$ is bounded. Moreover, since

$$S_k = S_{k-1} + \frac{1}{k^2}$$

for $k \geq 2$, the sequence of partial sums is also increasing. Therefore, by the Monotone Convergence Theorem, it converges. Thus, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

We can extend this idea to prove convergence or divergence for many different series. Suppose $\sum_{n=1}^{\infty} a_n$ is a series with positive terms a_n such that there exists a continuous, positive, decreasing function f where $f(n) = a_n$ for all positive integers. Then, as in Figure 4.3.3*a*, for any integer k, the kth partial sum S_k satisfies

$$S_k = a_1 + a_2 + a_3 + \cdots + a_k < a_1 + \int_1^k f(x)\,dx < 1 + \int_1^{\infty} f(x)\,dx.$$

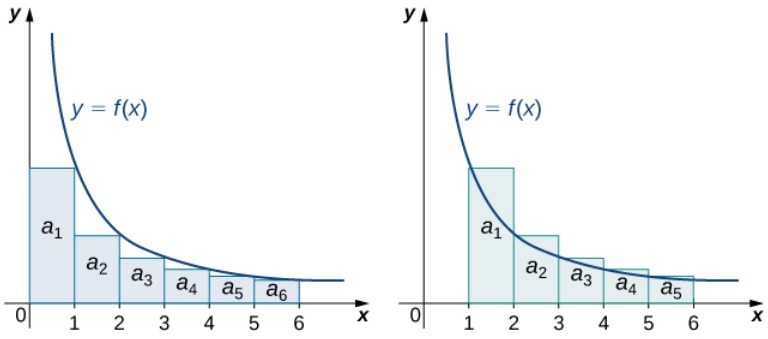


Figure 4.3.3*a*: If we can inscribe rectangles inside a region bounded by a curve $y = f(x)$ and the x-axis, and the area bounded by those curves for $x \geq 1$ is finite, then the sum of the areas of the rectangles is also finite.

Figure 4.3.3*b*: If a set of rectangles circumscribes the region bounded by $y = f(x)$ and the x axis for $x \geq 1$ and the region has infinite area, then the sum of the areas of the rectangles is also infinite.

Therefore, if $\int_1^{\infty} f(x)\,dx$ converges, then the sequence of partial sums $\{S_k\}$ is bounded. Since $\{S_k\}$ is an increasing sequence, it converges by the Monotone Convergence Theorem if it is also a bounded sequence. We conclude that if $\int_1^{\infty} f(x)\,dx$ converges, then the series $\sum_{n=1}^{\infty} a_n$ also converges.

On the other hand, from Figure 4.3.3*b*, for any integer k, the kth partial sum S_k satisfies

$$S_k = a_1 + a_2 + a_3 + \cdots + a_k > \int_1^{k+1} f(x)\,dx.$$

If

$$\lim_{k\to\infty} \int_1^{k+1} f(x)\,dx = \infty,$$

then $\{S_k\}$ is an unbounded sequence and therefore diverges. As a result, the series $\sum_{n=1}^{\infty} a_n$ also diverges. Since f is a positive function, if $\int_1^{\infty} f(x)\,dx$ diverges, then

$$\lim_{k\to\infty} \int_1^{k+1} f(x)\,dx = \infty.$$

We conclude that if $\int_1^{\infty} f(x)\,dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Theorem: Integral Test

Suppose $\sum_{n=1}^{\infty} a_n$ is a series with positive terms a_n. Suppose there exists a function f and a positive integer N such that the following three conditions are satisfied:

i. f is continuous,
ii. f is decreasing, and
iii. $f(n) = a_n$ for all integers $n \geq N$.

Then

$$\sum_{n=1}^{\infty} a_n$$

and

$$\int_N^{\infty} f(x)\,dx$$

both converge or both diverge (Figure 4.3.3).

Caution: The Sequence Must EVENTUALLY be Non-Negative

The base sequence, $\{a_n\}$, in the Integral Test must *eventually* consist of non-negative terms only. That is, to be able to use the Integral Test, there must exist N such that $f(n) = a_n \geq 0$ for all $n \geq N$. This (along with the need to evaluate an improper integral) is a weakness of the Integral Test and a reason we will eventually need to consider other tests for convergence.

Although the convergence of $\int_N^{\infty} f(x)\,dx$ implies convergence of the related series $\sum_{n=1}^{\infty} a_n$, it does not imply that the value of the integral and the series are the same. They may be different and often are. For example,

$$\sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n = \frac{1}{e} + \left(\frac{1}{e}\right)^2 + \left(\frac{1}{e}\right)^3 + \cdots$$

is a geometric series with initial term $a = 1/e$ and ratio $r = 1/e$, which converges to

$$\frac{1/e}{1-(1/e)} = \frac{1/e}{(e-1)/e} = \frac{1}{e-1}.$$

However, the related integral $\int_1^{\infty} (1/e)^x\,dx$ satisfies

$$\int_1^{\infty} \left(\frac{1}{e}\right)^x dx = \int_1^{\infty} e^{-x}\,dx = \lim_{b\to\infty} \int_1^b e^{-x}\,dx = \lim_{b\to\infty} -e^{-x}\Big|_1^b = \lim_{b\to\infty} [-e^{-b} + e^{-1}] = \frac{1}{e}.$$

Example 4.3.2: Using the Integral Test

For each of the following series, use the Integral Test to determine whether the series converges or diverges.

a. $\sum_{n=1}^{\infty} \frac{1}{n^3}$
b. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n-1}}$

Solutions

a. Compare

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \text{ and } \int_1^{\infty} \frac{1}{x^3}\,dx.$$

We have

$$\begin{aligned}\int_1^\infty \frac{1}{x^3}\,dx &= \lim_{b\to\infty}\int_1^b \frac{1}{x^3}\,dx \\ &= \lim_{b\to\infty}\left[-\frac{1}{2x^2}\Big|_1^b\right] \\ &= \lim_{b\to\infty}\left[-\frac{1}{2b^2}+\frac{1}{2}\right] \\ &= \frac{1}{2}.\end{aligned}$$

Thus the integral $\int_1^\infty \frac{1}{x^3}\,dx$ converges, and therefore so does the series

$$\sum_{n=1}^\infty \frac{1}{n^3}.$$

b. Compare

$$\sum_{n=1}^\infty \frac{1}{\sqrt{2n-1}} \text{ and } \int_1^\infty \frac{1}{\sqrt{2x-1}}\,dx.$$

Since

$$\begin{aligned}\int_1^\infty \frac{1}{\sqrt{2x-1}}\,dx &= \lim_{b\to\infty}\int_1^b \frac{1}{\sqrt{2x-1}}\,dx \\ &= \lim_{b\to\infty}\sqrt{2x-1}\Big|_1^b \\ &= \lim_{b\to\infty}\left[\sqrt{2b-1}-1\right] \\ &= \infty,\end{aligned}$$

the integral $\int_1^\infty \frac{1}{\sqrt{2x-1}}\,dx$ diverges, and therefore

$$\sum_{n=1}^\infty \frac{1}{\sqrt{2n-1}}$$

diverges.

? Checkpoint 4.3.2

Use the Integral Test to determine whether the series $\sum_{n=1}^\infty \frac{n}{3n^2+1}$ converges or diverges.

Answer

The series diverges.

The p-Series

The harmonic series $\sum_{n=1}^\infty 1/n$ and the series $\sum_{n=1}^\infty 1/n^2$ are both examples of a type of series called a p-series.

Definition: p-series

For any real number p, the series

$$\sum_{n=1}^\infty \frac{1}{n^p}$$

is called a p-**series**.

We know the p-series converges if $p = 2$ and diverges if $p = 1$. What about other values of p? In general, it is difficult, if not impossible, to compute the exact value of most p-series. However, we can use the tests presented thus far to prove whether a p-series converges or diverges.

If $p < 0$, then $1/n^p \to \infty$, and if $p = 0$, then $1/n^p \to 1$. Therefore, by the Divergence Test,

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

diverges if $p \leq 0$.

If $p > 0$, then $f(x) = 1/x^p$ is a positive, continuous, decreasing function. Therefore, for $p > 0$, we use the Integral Test, comparing

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ and } \int_1^{\infty} \frac{1}{x^p}\, dx.$$

We have already considered the case when $p = 1$. We consider the case when $p > 0, p \neq 1$. For this case,

$$\begin{aligned} \int_1^{\infty} \frac{1}{x^p}\, dx &= \lim_{b\to\infty} \int_1^b \frac{1}{x^p}\, dx \\ &= \lim_{b\to\infty} \frac{1}{1-p} x^{1-p}\Big|_1^b \\ &= \lim_{b\to\infty} \frac{1}{1-p}[b^{1-p} - 1]. \end{aligned}$$

Because $b^{1-p} \to 0$ if $p > 1$ and $b^{1-p} \to \infty$ if $p < 1$, we conclude that

$$\int_1^{\infty} \frac{1}{x^p}\, dx = \begin{cases} \frac{1}{p-1}, \text{if } p > 1 \\ \infty, \text{if } p < 1. \end{cases}$$

Therefore, $\sum_{n=1}^{\infty} 1/n^p$ converges if $p > 1$ and diverges if $0 < p < 1$. We summarize this as a theorem.

Theorem: p-series Test

The p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if and only if $p > 1$. It diverges otherwise.

Example 4.3.3: Testing for Convergence of p-series

For each of the following series, determine whether it converges or diverges.

a. $\sum_{n=1}^{\infty} \frac{1}{n^4}$

b. $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$

Solutions

a. This is a p-series with $p = 4 > 1$, so the series converges.

b. Since $p = 2/3 < 1$, the series diverges.

? Checkpoint 4.3.3

Does the series $\sum_{n=1}^{\infty} \frac{1}{n^{5/4}}$ converge or diverge?

Answer

The series converges.

Estimating the Value of a Series

Suppose we know that a series $\sum_{n=1}^{\infty} a_n$ converges, and we want to estimate the sum of that series. Certainly we can approximate that sum using any finite sum $\sum_{n=1}^{N} a_n$ where $N \in \mathbb{N}$. The question we address here is, for a convergent series $\sum_{n=1}^{\infty} a_n$, how good is the approximation $\sum_{n=1}^{N} a_n$?

More specifically, if we let

$$R_N = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n$$

be the **remainder** when the sum of an infinite series is approximated by the Nth partial sum, how large is R_N? For some types of series, we can use the ideas from the Integral Test to estimate R_N.

Theorem: Remainder Estimate from the Integral Test

Suppose $\sum_{n=1}^{\infty} a_n$ is a *convergent* series with *positive* terms.[4] Also suppose there exists a function f satisfying the following three conditions:

i. f is continuous,
ii. f is decreasing, and
iii. $f(n) = a_n$ for all integers $n \geq 1$.

Let S_N be the Nth partial sum of $\sum_{n=1}^{\infty} a_n$. For all $N \in \mathbb{N}$,

$$S_N + \int_{N+1}^{\infty} f(x)\,dx < \sum_{n=1}^{\infty} a_n < S_N + \int_{N}^{\infty} f(x)\,dx.$$

In other words, the remainder $R_N = \sum_{n=1}^{\infty} a_n - S_N = \sum_{n=N+1}^{\infty} a_n$ satisfies the following estimate:

$$\int_{N+1}^{\infty} f(x)\,dx < R_N < \int_{N}^{\infty} f(x)\,dx.$$

This is known as the **remainder estimate**.

The remainder estimate can be considered the error in approximating the series value.

We illustrate this theorem in Figure 4.3.4.

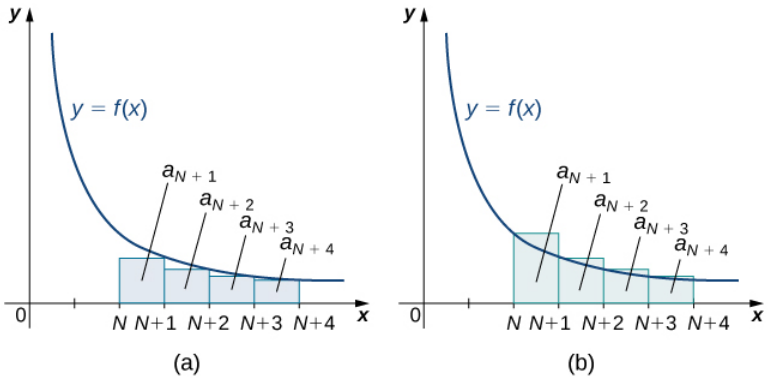


Figure 4.3.4: Given a continuous, positive, decreasing function f and a sequence of positive terms a_n such that $a_n = f(n)$ for all $n \in \mathbb{N}$, (a) the areas $a_{N+1} + a_{N+2} + a_{N+3} + \cdots < \int_N^\infty f(x)\,dx$, or (b) the areas $a_{N+1} + a_{N+2} + a_{N+3} + \cdots > \int_{N+1}^\infty f(x)\,dx$. Therefore, the integral is either an overestimate or an underestimate of the error.

In particular, by representing the remainder $R_N = a_{N+1} + a_{N+2} + a_{N+3} + \cdots$ as the sum of areas of rectangles, we see that the area of those rectangles is bounded above by $\int_N^\infty f(x)\,dx$ and bounded below by $\int_{N+1}^\infty f(x)\,dx$. In other words,

$$R_N = a_{N+1} + a_{N+2} + a_{N+3} + \cdots > \int_{N+1}^\infty f(x)\,dx$$

and

$$R_N = a_{N+1} + a_{N+2} + a_{N+3} + \cdots < \int_N^\infty f(x)\,dx.$$

We conclude that

$$\int_{N+1}^\infty f(x)\,dx < R_N < \int_N^\infty f(x)\,dx.$$

Since

$$\sum_{n=1}^\infty a_n = S_N + R_N,$$

where S_N is the Nth partial sum, we conclude that

$$S_N + \int_{N+1}^\infty f(x)\,dx < \sum_{n=1}^\infty a_n < S_N + \int_N^\infty f(x)\,dx.$$

The ability to approximate a series's value is important, and we will use different approximation methods for different convergence tests as we move forward.

Example 4.3.4: Estimating the Value of a Series

Consider the series $\sum_{n=1}^\infty \frac{1}{n^3}$.

a. Calculate $S_{10} = \sum_{n=1}^{10} \frac{1}{n^3}$ and estimate the error.

b. Determine the least value of N necessary such that S_N will estimate $\sum_{n=1}^\infty \frac{1}{n^3}$ to within 0.001.

Solutions

a. Using technology, we have

$$S_{10} = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots + \frac{1}{10^3} \approx 1.19753.$$

By the remainder estimate, we know

$$R_N < \int_N^\infty \frac{1}{x^3}\,dx.$$

We have

$$\begin{aligned}\int_{10}^\infty \frac{1}{x^3}\,dx &= \lim_{b\to\infty}\int_{10}^b \frac{1}{x^3}\,dx \\ &= \lim_{b\to\infty}\left[-\frac{1}{2x^2}\right]_N^b \\ &= \lim_{b\to\infty}\left[-\frac{1}{2b^2}+\frac{1}{2N^2}\right] \\ &= \frac{1}{2N^2}.\end{aligned}$$

Therefore, the error is $R_{10} < \frac{1}{2(10)^2} = 0.005$.

b. We want to find N such that $R_N < 0.001$. In part a. we showed that $R_N < 1/2N^2$. Therefore, the remainder $R_N < 0.001$ as long as $\frac{1}{2N^2} < 0.001$. That is, we need $2N^2 > 1000$. Solving this inequality for N, we see that we need $N > 22.36$. We need to round up to the nearest integer to ensure the remainder is within the desired amount. Therefore, the minimum necessary value is $N = 23$.

? Checkpoint 4.3.4

For $\sum_{n=1}^\infty \frac{1}{n^4}$, calculate S_5 and estimate the error R_5.

Answer

$S_5 \approx 1.09035, R_5 < 0.00267$

Footnotes

[1] Most theorems can be stated symbolically as the conditional proposition$P \implies Q$, where P is a statement called the **antecedent**, and Q is a statement called the **consequent**. Instead of directly proving $P \implies Q$, it is sometimes easier to prove it indirectly. There are two kinds of indirect proofs: **proof by contradiction**, and **proof by contraposition**.

You have already seen proofs by contradiction (see the uses of Rolle's Theorem in Calculus I for proving that a given equation has *exactly* one root). In a proof by contraposition, we use the fact that an implication, "if P, then Q," is equivalent to its contrapositive, "if *not* Q, then *not* P" (the notation, $\sim P$, is called the **negation** of P). Therefore, instead of proving $P \implies Q$, we may prove its contrapositive $\sim Q \implies \sim P$. Since it is an implication, we could use a direct proof at this point:

Assume $\sim Q$ is true (hence, assume Q is false). Show that $\sim P$ is true (that is, show that P is false).

[2] It can be useful at times to remember that $S_n = S_{n-1} + a_n$.

[3] The negation of "A *or* B" is "not A *and* not B."

[4] Remember, when dealing with the Integral Test, the terms of your base sequence, $\{a_n\}$ must *eventually* be non-negative.

Key Concepts

- If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^\infty a_n$ diverges.
- If $\lim_{n\to\infty} a_n = 0$, the series $\sum_{n=1}^\infty a_n$ may converge or diverge.

- If $\sum_{n=1}^{\infty} a_n$ is a series with positive terms a_n and f is a continuous, decreasing function such that $f(n) = a_n$ for all positive integers n, then

$$\sum_{n=1}^{\infty} a_n$$

and

$$\int_1^{\infty} f(x)\,dx$$

either both converge or both diverge. Furthermore, if $\sum_{n=1}^{\infty} a_n$ converges, then the Nth partial sum approximation S_N is accurate up to an error R_N where $\int_{N+1}^{\infty} f(x)\,dx < R_N < \int_N^{\infty} f(x)\,dx$.
- The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Key Equations

- **Divergence Test**

  If $a_n \nrightarrow 0$ as $n \to \infty$, $\sum_{n=1}^{\infty} a_n$ diverges.
- **p-series**

  $$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \begin{cases} \text{converges, if } p > 1 \\ \text{diverges, if } p \leq 1 \end{cases}$$
- **Remainder estimate from the Integral Test**

  $$\int_{N+1}^{\infty} f(x)\,dx < R_N < \int_N^{\infty} f(x)\,dx$$

Glossary

Divergence Test

if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges

Integral Test

for a series $\sum_{n=1}^{\infty} a_n$ with positive terms a_n, if there exists a continuous, decreasing function f such that $f(n) = a_n$ for all positive integers n, then

$$\sum_{n=1}^{\infty} a_n$$

and

$$\int_1^{\infty} f(x)\,dx$$

either both converge or both diverge

***p*-series**

a series of the form $\sum_{n=1}^{\infty} 1/n^p$

remainder estimate

for a series $\sum_{n=}^{\infty} 1 a_n$ with positive terms a_n and a continuous, decreasing function f such that $f(n) = a_n$ for all positive integers n, the remainder $R_N = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n$ satisfies the following estimate:

$$\int_{N+1}^{\infty} f(x)\,dx < R_N < \int_{N}^{\infty} f(x)\,dx$$

https://math.libretexts.org/@go/page/168441

4.3E: Exercises

Divergence Test Problems

Consider the sequence for each series in exercises 1 - 14, if the Divergence Test applies, either state that $\lim_{n\to\infty} a_n$ does not exist or find $\lim_{n\to\infty} a_n$. If the Divergence Test does not apply, state why.

1) $\sum_{n=1}^{\infty} \frac{n}{n+2}$

2) $\sum_{n=1}^{\infty} \frac{n}{5n^2-3}$

Answer

$\lim_{n\to\infty} a_n = 0$. The Divergence Test does not apply.

3) $\sum_{n=1}^{\infty} \frac{n}{\sqrt{3n^2+2n+1}}$

4) $\sum_{n=1}^{\infty} \frac{(2n+1)(n-1)}{(n+1)^2}$

Answer

$\lim_{n\to\infty} a_n = 2$. So the series diverges by the nth-Term Test for Divergence.

5) $\sum_{n=1}^{\infty} \frac{(2n+1)^{2n}}{(3n^2+1)^n}$

6) $\sum_{n=1}^{\infty} \frac{2^n}{3^{n/2}}$

Answer

$\lim_{n\to\infty} a_n = \infty$ (does not exist). So the series diverges by the nth-Term Test for Divergence.

7) $\sum_{n=1}^{\infty} \frac{2^n+3^n}{10^{n/2}}$

8) $\sum_{n=1}^{\infty} e^{-2/n}$

Answer

$\lim_{n\to\infty} a_n = 1$. So the series diverges by the nth-Term Test for Divergence.

9) $\sum_{n=1}^{\infty} \cos n$

10) $\sum_{n=1}^{\infty} \tan n$

Answer

$\lim_{n\to\infty} a_n$ does not exist. So the series diverges by the nth-Term Test for Divergence.

11) $\sum_{n=1}^{\infty} \frac{1-\cos^2(1/n)}{\sin^2(2/n)}$

12) $\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^{2n}$

Answer

$\lim_{n\to\infty} a_n = 1/e^2$. So the series diverges by the nth-Term Test for Divergence.

13) $\sum_{n=1}^{\infty} \frac{\ln n}{n}$

14) $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{\sqrt{n}}$

Answer

$\lim_{n\to\infty} a_n = 0$. The Divergence Test does not apply.

p-Series Problems & Integral Test Problems

In exercises 15 - 20, state whether the given p-series converges.

15) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

16) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$

Answer

The series converges, since $p = 3/2 > 1$.

17) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2}}$

18) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^4}}$

Answer

The series converges, since $p = 4/3 > 1$.

19) $\sum_{n=1}^{\infty} \frac{n^e}{n^\pi}$

20) $\sum_{n=1}^{\infty} \frac{n^\pi}{n^{2e}}$

Answer

The series converges, since $p = 2e - \pi > 1$.

In exercises 21 - 27, use the Integral Test to determine whether the following sums converge.

21) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+5}}$

22) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n+5}}$

Answer

The series diverges by the Integral Test since $\int_1^{\infty} \frac{dx}{(x+5)^{1/3}}$ can be shown to diverge.

23) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

24) $\sum_{n=1}^{\infty} \frac{n}{1+n^2}$

Answer

The series diverges by the Integral Test since $\int_1^{\infty} \frac{x}{1+x^2}\,dx$ can be shown to diverge.

25) $\sum_{n=1}^{\infty} \frac{e^n}{1+e^{2n}}$

26) $\sum_{n=1}^{\infty} \frac{2n}{1+n^4}$

Answer

The series converges by the Integral Test since $\int_1^{\infty} \frac{2x}{1+x^4}\,dx$ converges.

27) $\sum_{n=2}^{\infty} \frac{1}{n \ln^2 n}$

Express the sums in exercises 28 - 31 as p-series and determine whether each converges.

28) $\sum_{n=1}^{\infty} 2^{-\ln n}$

(Hint: $2^{-\ln n} = \frac{1}{n^{\ln 2}}$.)

Answer

$2^{-\ln n} = 1/n^{\ln 2}$. Since $p = \ln 2 < 1$, this series diverges by the p-series test.

29) $\sum_{n=1}^{\infty} 3^{-\ln n}$

(Hint: $3^{-\ln n} = \frac{1}{n^{\ln 3}}$.)

30) $\sum_{n=1}^{n} 2^{-2\ln n}$

Answer

$2^{-2\ln n} = 1/n^{2\ln 2}$. Since $p = 2\ln 2 - 1 < 1$, this series diverges by the p-series test.

31) $\sum_{n=1}^{\infty} n3^{-2\ln n}$

In exercises 32 - 35, use the estimate $R_N \le \int_N^{\infty} f(t)\,dt$ to find a bound for the remainder $R_N = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n$ where $a_n = f(n)$.

32) $\sum_{n=1}^{1000} \frac{1}{n^2}$

Answer

$$R_{1000} \le \int_{1000}^{\infty} \frac{dt}{t^2} = \lim_{b\to\infty} -\frac{1}{t}\Big|_{1000}^{b} = \lim_{b\to\infty} \left(-\frac{1}{b} + \frac{1}{1000}\right) = 0.001$$

33) $\sum_{n=1}^{1000} \frac{1}{n^3}$

34) $\sum_{n=1}^{1000} \frac{1}{1+n^2}$

Answer

$$R_{1000} \le \int_{1000}^{\infty} \frac{dt}{1+t^2} = \lim_{b\to\infty} \left(\tan^{-1} b - \tan^{-1}(1000)\right) = \pi/2 - \tan^{-1}(1000) \approx 0.000999$$

35) $\sum_{n=1}^{100} \frac{n}{2^n}$

[Technology Required] In exercises 36 - 40, find the minimum value of N such that the remainder estimate $\int_{N+1}^{\infty} f(x)\,dx < R_N < \int_N^{\infty} f(x)\,dx$ guarantees that $\sum_{n=1}^{N} a_n$ estimates $\sum_{n=1}^{\infty} a_n$, accurate to within the given error.

36) $a_n = \frac{1}{n^2}$, error $< 10^{-4}$

Answer

$R_N < \int_N^\infty \frac{dx}{x^2} = 1/N$, for $N > 10^4$

37) $a_n = \frac{1}{n^{1.1}}$, error $< 10^{-4}$

38) $a_n = \frac{1}{n^{1.01}}$, error $< 10^{-4}$

Answer

$R_N < \int_N^\infty \frac{dx}{x^{1.01}} = 100N^{-0.01}$, for $N > 10^{600}$

39) $a_n = \frac{1}{n \ln^2 n}$, error $< 10^{-3}$

40) $a_n = \frac{1}{1+n^2}$, error $< 10^{-3}$

Answer

$R_N < \int_N^\infty \frac{dx}{1+x^2} = \pi/2 - \tan^{-1}(N)$, for $N > \tan(\pi/2 - 10^{-3}) \approx 1000$

In exercises 41 - 45, find a value of N such that R_N is smaller than the desired error. Compute the corresponding sum $\sum_{n=1}^{N} a_n$ and compare it to the given estimate of the infinite series.

41) $a_n = \frac{1}{n^{11}}$, error $< 10^{-4}$, $\sum_{n=1}^{\infty} \frac{1}{n^{11}} = 1.000494\ldots$

42) $a_n = \frac{1}{e^n}$, error $< 10^{-5}$, $\sum_{n=1}^{\infty} \frac{1}{e^n} = \frac{1}{e-1} = 0.581976\ldots$

Answer

$R_N < \int_N^\infty \frac{dx}{e^x} = e^{-N}$, for $N > 5\ln(10)$, okay if $N = 12$; $\sum_{n=1}^{12} e^{-n} = 0.581973\ldots$. Estimate agrees with $1/(e-1)$ to five decimal places.

43) $a_n = \frac{1}{e^{n^2}}$, error $< 10^{-5}$, $\sum_{n=1}^{\infty} \frac{1}{e^{n^2}} = 0.40488139857\ldots$

44) $a_n = \frac{1}{n^4}$, error $< 10^{-4}$, $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} = 1.08232\ldots$

Answer

$R_N < \int_N^\infty \frac{dx}{x^4} = 4/N^3$, for $N > (4.10^4)^{1/3}$, okay if $N = 35$; $\sum_{n=1}^{35} \frac{1}{n^4} = 1.08231\ldots$. Estimate agrees with the sum to four decimal places.

45) $a_n = \frac{1}{n^6}$, error $< 10^{-6}$, $\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945} = 1.01734306\ldots$,

46) Find the limit as $n \to \infty$ of $\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n}$.
(Hint: Compare to $\int_n^{2n} \frac{1}{t}\, dt$.)

Answer

$\ln(2)$

47) Find the limit as $n \to \infty$ of $\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{3n}$

The next few exercises are intended to give a sense of applications in which partial sums of the harmonic series arise.

48) In certain applications of probability, such as the so-called Watterson estimator for predicting mutation rates in population genetics, it is important to have an accurate estimate of the number $H_k = (1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k})$. Recall that $T_k = H_k - \ln k$ is decreasing. Compute $T = \lim_{k \to \infty} T_k$ to four decimal places.

$\left(\text{Hint: } \frac{1}{k+1} < \int_k^{k+1} \frac{1}{x}\, dx.\right)$

Answer

$T = 0.5772...$

49) [Technology Required] Complete sampling with replacement, sometimes called the **coupon collector's problem**, is phrased as follows: Suppose you have N unique items in a bin. At each step, an item is chosen at random, identified, and put back in the bin. The problem asks what is the expected number of steps $E(N)$ that it takes to draw each unique item at least once. It turns out that $E(N) = N.H_N = N\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N}\right)$. Find $E(N)$ for $N = 10, 20,$ and 50.

50) [Technology Required] The simplest way to shuffle cards is to take the top card and insert it at a random place in the deck, called top random insertion, and then repeat. We will consider a deck to be randomly shuffled once enough top random insertions have been made that the card originally at the bottom has reached the top and then been randomly inserted. If the deck has n cards, then the probability that the insertion will be below the card initially at the bottom (call this card B) is $1/n$. Thus the expected number of top random insertions before B is no longer at the bottom is n. Once one card is below B, there are two places below B and the probability that a randomly inserted card will fall below B is $2/n$. The expected number of top random insertions before this happens is $n/2$. The two cards below B are now in random order. Continuing this way, find a formula for the expected number of top random insertions needed to consider the deck to be randomly shuffled.

Answer

The expected number of random insertions to get B to the top is $n + n/2 + n/3 + \cdots + n/(n-1)$. Then one more insertion puts B back in at random. Thus, the expected number of shuffles to randomize the deck is $n(1 + 1/2 + \cdots + 1/n)$.

51) Suppose a scooter can travel 100 km on a full tank of fuel. Assuming that fuel can be transferred from one scooter to another but can only be carried in the tank, present a procedure that will enable one of the scooters to travel $100H_N$ km, where $H_N = 1 + 1/2 + \cdots + 1/N$.

52) Show that for the remainder estimate to apply on $[N, \infty)$ it is sufficient that $f(x)$ be decreasing on $[N, \infty)$, but f need not be decreasing on $[1, \infty)$.

Answer

Set $b_n = a_{n+N}$ and $g(t) = f(t + N)$ such that f is decreasing on $[t, \infty)$.

53) [Technology Required] Use the remainder estimate and Integration by Parts to approximate $\sum_{n=1}^{\infty} \frac{n}{e^n}$ within an error smaller than 0.0001.

54) Does $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converge if p is large enough? If so, for which p?

Answer

The series converges for $p > 1$ by Integral Test using change of variable.

55) [Technology Required] Suppose a computer can sum one million terms per second of the divergent series $\sum_{n=1}^{N} \frac{1}{n}$. Use the Integral Test to approximate how many seconds it will take to add up enough terms for the partial sum to exceed 100.

56) [Technology Required] A fast computer can sum one million terms per second of the divergent series $\sum_{n=2}^{N} \frac{1}{n \ln n}$. Use the Integral Test to approximate how many seconds it will take to add up enough terms for the partial sum to exceed **100**.

Answer

$N = e^{e^{100}} \approx e^{10^{43}}$ terms are needed.

This page titled 4.3E: Exercises is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Roy Simpson.

4.4: The Comparison Tests

Learning Objectives

- Use the Series Comparison Test to test a series for convergence.
- Use the Limit Comparison Test to determine the convergence of a series.

We have seen that the Integral Test allows us to determine the convergence or divergence of a series by comparing it to a related improper integral. In this section, we show how to use Comparison Tests to determine the convergence or divergence of a series by comparing it to a series whose convergence or divergence is known. Typically, these tests determine the convergence of series similar to geometric series or p-series.

Series Comparison Test

In the preceding two sections, we discussed two large classes of series: geometric series and p-series. We know precisely when these series converge and when they diverge. Here, we show how to use the convergence or divergence of these series to prove convergence or divergence for other series using a method called the **Series Comparison Test**.

For example, consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}.$$

This series looks similar to the convergent series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Since the terms in each of the series are positive, the sequence of partial sums for each series is monotonically increasing. Furthermore, since

$$0 < \frac{1}{n^2+1} < \frac{1}{n^2}$$

for all $n \in \mathbb{N}$, the kth partial sum S_k of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ satisfies

$$S_k = \sum_{n=1}^{k} \frac{1}{n^2+1} < \sum_{n=1}^{k} \frac{1}{n^2} < \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

(See Figure 4.4.1a and Table 4.4.1 below.) Since the series on the right converges, the sequence $\{S_k\}$ is bounded above. We conclude that $\{S_k\}$ is a monotonically increasing sequence that is bounded above. Therefore, by the Monotone Convergence Theorem, $\{S_k\}$ converges. Thus,

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

converges.

Similarly, consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n-1/2}.$$

This series looks similar to the divergent series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

The sequence of partial sums for each series is monotonically increasing and

$$\frac{1}{n-1/2} > \frac{1}{n} > 0$$

for every $n \in \mathbb{N}$. Therefore, the kth partial sum S_k of

$$\sum_{n=1}^{\infty} \frac{1}{n-1/2}$$

satisfies

$$S_k = \sum_{n=1}^{k} \frac{1}{n-1/2} > \sum_{n=1}^{k} \frac{1}{n}.$$

(See Figure 4.4.1*b* and Table 4.4.2 below). Since the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to infinity, the sequence of partial sums $\sum_{n=1}^{k} \frac{1}{n}$ is unbounded. Consequently, $\{S_k\}$ is an unbounded sequence that diverges. We conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n-1/2}$$

diverges.

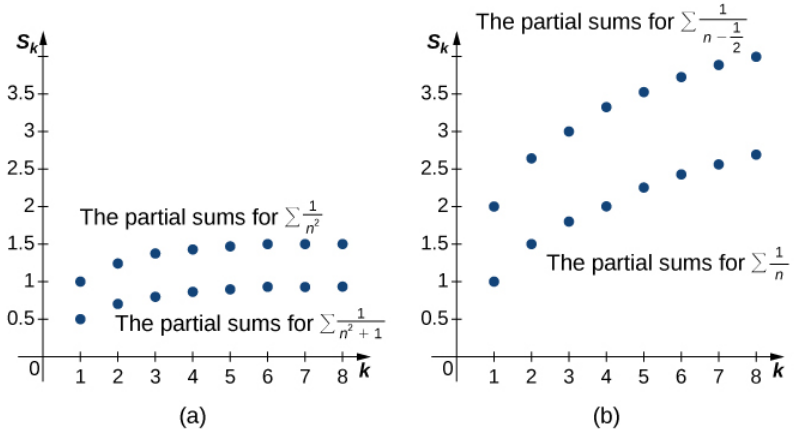


Figure 4.4.1*a*: Each of the partial sums for the given series is less than the corresponding partial sum for the converging $p-$ *series*.
Figure 4.4.1*b*: Each partial sum for the given series is greater than the corresponding partial sum for the diverging harmonic series.

Table 4.4.1: Comparing a series with a p-series ($p = 2$)

k	1	2	3	4	5	6	7	8
$\sum_{n=1}^{k} \frac{1}{n^2+1}$	0.5	0.7	0.8	0.8588	0.8973	0.9243	0.9443	0.9597
$\sum_{n=1}^{k} \frac{1}{n^2}$	1	1.25	1.3611	1.4236	1.4636	1.4914	1.5118	1.5274

Table 4.4.2: Comparing a series with the harmonic series

k	1	2	3	4	5	6	7	8
$\sum_{n=1}^{k} \frac{1}{n-1/2}$	2	2.6667	3.0667	3.3524	3.5746	3.7564	3.9103	4.0436
$\sum_{n=1}^{k} \frac{1}{n}$	1	1.5	1.8333	2.0933	2.2833	2.45	2.5929	2.7179

Theorem: Series Comparison Test

i. Suppose there exists an integer N such that $0 \le a_n \le b_n$ for all $n \ge N$. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

ii. Suppose there exists an integer N such that $a_n \ge b_n \ge 0$ for all $n \ge N$. If $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof

We prove part i (the proof of part ii is the contrapositive of part i). Let $\{S_k\}$ be the sequence of partial sums associated with $\sum_{n=1}^{\infty} a_n$, and let $L = \sum_{n=1}^{\infty} b_n$. Since the terms $a_n \ge 0$,

$$S_k = a_1 + a_2 + \cdots + a_k \le a_1 + a_2 + \cdots + a_k + a_{k+1} = S_{k+1}.$$

Therefore, the sequence of partial sums is increasing. Further, since $a_n \le b_n$ for all $n \ge N$,

$$\sum_{n=N}^{k} a_n \le \sum_{n=N}^{k} b_n \le \sum_{n=1}^{\infty} b_n = L.$$

Therefore, for all $k \ge 1$,

$$S_k = (a_1 + a_2 + \cdots + a_{N-1}) + \sum_{n=N}^{k} a_n \le (a_1 + a_2 + \cdots + a_{N-1}) + L.$$

Since $a_1 + a_2 + \cdots + a_{N-1}$ is a finite number, we conclude that the sequence S_k is bounded above. Therefore, $\{S_k\}$ is an increasing sequence that is bounded above. By the Monotone Convergence Theorem, we conclude that $\{S_k\}$ converges, and therefore the series $\sum_{n=1}^{\infty} a_n$ converges.

Caution: The Terms Must be Positive

You must be sure that the terms of the original series and the series you are using to compare are *positive*! This is critical because the Series Comparison Test fails to work otherwise.

To use the Series Comparison Test (also known as the **SCT**) to determine the convergence or divergence of a series $\sum_{n=1}^{\infty} a_n$, it is necessary to find a suitable series with which to compare it. Since we know the convergence properties of geometric series and p-series, these series are often used. If there exists an integer N such that for all $n \ge N$, each term a_n is less than each corresponding term of a known convergent series, then $\sum_{n=1}^{\infty} a_n$ converges. Similarly, if there exists an integer N such that for all $n \ge N$, each term a_n is greater than each corresponding term of a known divergent series, then $\sum_{n=1}^{\infty} a_n$ diverges.

Example 4.4.1: Using the Comparison Test

For each of the following series, use the Comparison Test to determine whether the series converges or diverges.

a. $\sum_{n=1}^{\infty} \frac{1}{n^3 + 3n + 1}$

b. $\sum_{n=1}^{\infty} \frac{1}{2^n + 1}$

c. $\sum_{n=2}^{\infty} \frac{1}{\ln n}$

Solutions

a. Compare to $\sum_{n=1}^{\infty} \frac{1}{n^3}$. Note that the terms of the original and compared series are positive. Since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a p-series with $p = 3$, it converges. Further,

$$n^3+3n+1>n^3>0 \implies 0<\frac{1}{n^3+3n+1}<\frac{1}{n^3}$$

for every $n \in \mathbb{N}$. Therefore, we can conclude that $\sum_{n=1}^{\infty} \frac{1}{n^3+3n+1}$ converges.

b. Compare to $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$. Again, the terms of both series are positive. Since $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ is a geometric series with $r=\frac{1}{2}$ and $\left|\frac{1}{2}\right|<1$, it converges. Also,

$$2^n+1>2^n>0 \implies 0<\frac{1}{2^n+1}<\frac{1}{2^n}$$

for every $n \in \mathbb{N}$. Therefore, we see that $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ converges.

c. Compare to $\sum_{n=2}^{\infty} \frac{1}{n}$. The terms of both series are positive.

If we let $f(x)=x$ and $g(x)=\ln(x)$, we know that $f(1)=1>0=g(1)$. Moreover, for $x \geq 1$, $f'(x)=1 \geq g'(x)=\frac{1}{x}$. Hence, by the Racetrack Principle (from Calculus I), $f(x)=x>\ln(x)=g(x)$ for $x \geq 1$. Thus,

$$0<\ln(n)<n \implies \frac{1}{\ln(n)}>\frac{1}{n}>0$$

for $n \geq 2$. Since $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges, we have that $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges.

? Checkpoint 4.4.1

Use the Comparison Test to determine if the series $\sum_{n=1}^{\infty} \frac{n}{n^3+n+1}$ converges or diverges.

Answer

The series converges.

Limit Comparison Test

The Series Comparison Test works nicely if we can find a comparable series satisfying the hypothesis of the test. However, sometimes finding an appropriate series can be difficult. Consider the series

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1}.$$

It is natural to compare this series with the convergent series

$$\sum_{n=2}^{\infty} \frac{1}{n^2}.$$

However, this series does not satisfy the hypothesis necessary to use the Series Comparison Test because

$$\frac{1}{n^2-1}>\frac{1}{n^2}$$

for all integers $n \geq 2$. Although we could look for a different series with which to compare $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$, instead we show how we can use the **Limit Comparison Test** (also called the **LCT**) to compare

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1}$$

and

$$\sum_{n=2}^{\infty} \frac{1}{n^2}.$$

Let us examine the idea behind the Limit Comparison Test. Consider two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ with *positive* terms a_n and b_n and evaluate

$$\lim_{n\to\infty} \frac{a_n}{b_n}.$$

If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L \neq 0,$$

then, for n sufficiently large, $a_n \approx Lb_n$. Therefore, either both series converge or both series diverge. For the series $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ and $\sum_{n=2}^{\infty} \frac{1}{n^2}$, we see that

$$\lim_{n\to\infty} \frac{1/(n^2-1)}{1/n^2} = \lim_{n\to\infty} \frac{n^2}{n^2-1} = 1.$$

Since $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges, we conclude that

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1}$$

converges.

The Limit Comparison Test can be used in two other cases. Suppose

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0.$$

You can conceptually think of this as saying b_n will eventually be much greater than a_n. In this case, $\{a_n/b_n\}$ is a bounded sequence. As a result, there exists a constant M such that $a_n \leq Mb_n$. Therefore, if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. On the other hand, suppose

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \infty.$$

Again, conceptually this tells us that a_n is eventually much greater than b_n. In this case, $\{a_n/b_n\}$ is an unbounded sequence. Therefore, for every constant M there exists an integer N such that $a_n \geq Mb_n$ for all $n \geq N$. Hence, if $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges as well.

We summarize these findings in the following theorem.

Theorem: Limit Comparison Test

Let $a_n, b_n \geq 0$ for all $n \geq 1$.

i. If $\lim_{n\to\infty} \frac{a_n}{b_n} = L \neq 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.

ii. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

iii. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Note that if $\frac{a_n}{b_n} \to 0$ and $\sum_{n=1}^{\infty} b_n$ diverges, the Limit Comparison Test gives no information. Similarly, if $\frac{a_n}{b_n} \to \infty$ and $\sum_{n=1}^{\infty} b_n$ converges, the test also provides no information.

For example, consider the two series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$. These series are both p-series with $p = \frac{1}{2}$ and $p = 2$, respectively. Since $p = \frac{1}{2} < 1$, the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges. On the other hand, since $p = 2 > 1$, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. However, suppose we attempted to apply the Limit Comparison Test, using the convergent p–series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ as our comparison series. First, we see that

$$\frac{1/\sqrt{n}}{1/n^3} = \frac{n^3}{\sqrt{n}} = n^{5/2} \to \infty \text{ as } n \to \infty.$$

Similarly, we see that

$$\frac{1/n^2}{1/n^3} = n \to \infty \text{ as } n \to \infty.$$

Therefore, if $\frac{a_n}{b_n} \to \infty$ when $\sum_{n=1}^{\infty} b_n$ converges, we do not gain any information on the convergence or divergence of $\sum_{n=1}^{\infty} a_n$.

Caution: The Terms Must be Positive

Just as in the Series Comparison Test, the terms of the series used for the Limit Comparison Test must be positive.

Example 4.4.2: Using the Limit Comparison Test

For each of the following series, use the Limit Comparison Test to determine whether the series converges or diverges. If the test does not apply, say so.

a. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1}$
b. $\sum_{n=1}^{\infty} \frac{2^n+1}{3^n}$
c. $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$

Solutions

a. Compare this series to $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$. Note that both series have only positive terms. Calculate

$$\lim_{n\to\infty} \frac{1/(\sqrt{n}+1)}{1/\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}+1} = \lim_{n\to\infty} \frac{1/\sqrt{n}}{1+1/\sqrt{n}} = 1.$$

By the Limit Comparison Test, since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges, then $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1}$ diverges.

b. Compare this series to $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$. Again, both series have only positive terms. We see that

$$\lim_{n\to\infty} \frac{(2^n+1)/3^n}{2^n/3^n} = \lim_{n\to\infty} \frac{2^n+1}{3^n} \cdot \frac{3^n}{2^n} = \lim_{n\to\infty} \frac{2^n+1}{2^n} = \lim_{n\to\infty} \left[1 + \left(\tfrac{1}{2}\right)^n\right] = 1.$$

Therefore,

$$\lim_{n\to\infty} \frac{(2^n+1)/3^n}{2^n/3^n} = 1.$$

Since $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ converges, we conclude that $\sum_{n=1}^{\infty} \frac{2^n+1}{3^n}$ converges.

c. Since $\ln n < n$ (see Example 4.4.1c), compare with $\sum_{n=1}^{\infty} \frac{1}{n}$. We see that

$$\lim_{n\to\infty} \frac{\ln n/n^2}{1/n} = \lim_{n\to\infty} \frac{\ln n}{n^2} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{\ln n}{n}.$$

In order to evaluate $\lim_{n\to\infty} \ln n/n$, evaluate the limit as $x \to \infty$ of the real-valued function $\ln(x)/x$. These two limits are equal, and making this change allows us to use l'Hospital's Rule.[1] We obtain

$$\lim_{x\to\infty} \frac{\ln(x)}{x} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

Therefore, $\lim_{n\to\infty} \frac{\ln n}{n} = 0$, and, consequently,

$$\lim_{n\to\infty} \frac{(\ln n)/n^2}{1/n} = 0.$$

Since the limit is 0 but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, the Limit Comparison Test does not provide any information.

Compare with $\sum_{n=1}^{\infty} \frac{1}{n^2}$ instead. In this case,

$$\lim_{n\to\infty} \frac{(\ln n)/n^2}{1/n^2} = \lim_{n\to\infty} \frac{\ln n}{n^2} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \ln n = \infty.$$

Since the limit is ∞ but $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, the test still does not provide any information.

So now we try a series between the two we already tried. Choosing the series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, we see that

$$\lim_{n\to\infty} \frac{(\ln n)/n^2}{1/n^{3/2}} = \lim_{n\to\infty} \frac{\ln n}{n^2} \cdot \frac{n^{3/2}}{1} = \lim_{n\to\infty} \frac{\ln n}{\sqrt{n}}.$$

As above, in order to evaluate $\lim_{n\to\infty} \frac{\ln n}{\sqrt{n}}$, evaluate the limit as $x \to \infty$ of the real-valued function $\frac{\ln n}{\sqrt{n}}$. Using l'Hospital's Rule,

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{2\sqrt{x}}{x} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0.$$

Since the limit is 0 and $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges, we can conclude that $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ converges.

? Checkpoint 4.4.2

Use the Limit Comparison Test to determine whether the series $\sum_{n=1}^{\infty} \frac{5^n}{3^n + 2}$ converges or diverges.

Answer

The series diverges.

Footnotes

[1] The move to considering the limit of the real-valued function $\ln(x)/x$ is purposeful and showcases the need for rigor in Calculus. l'Hospital's Rule is only defined for *continuous* functions - not discontinuous functions like sequences. As such, we could not use l'Hospital's Rule on $\lim_{n\to\infty} \ln(n)/n$.

Key Concepts

- The Comparison Tests determine the convergence or divergence of series with positive terms.
- When using the Comparison Tests, a series $\sum_{n=1}^{\infty} a_n$ is often compared to a geometric or p-series.

Glossary

Comparison Test

If $0 \le a_n \le b_n$ for all $n \ge N$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges; if $a_n \ge b_n \ge 0$ for all $n \ge N$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Limit Comparison Test

Suppose $a_n, b_n \ge 0$ for all $n \ge 1$. If $\lim_{n\to\infty} a_n/b_n \to L \ne 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge; if $\lim_{n\to\infty} a_n/b_n \to 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. If $\lim_{n\to\infty} a_n/b_n \to \infty$, and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

4.4E: Exercises

Use the Series Comparison Test to determine whether each series in exercises 1 - 13 converges or diverges.

1) $\sum_{n=1}^{\infty} a_n$ where $a_n = \dfrac{2}{n(n+1)}$

2) $\sum_{n=1}^{\infty} a_n$ where $a_n = \dfrac{1}{n(n+1/2)}$

Answer

Converges by comparison with $\dfrac{1}{n^2}$.

3) $\sum_{n=1}^{\infty} \dfrac{1}{2(n+1)}$

4) $\sum_{n=1}^{\infty} \dfrac{1}{2n-1}$

Answer

Diverges by comparison with harmonic series, since $2n-1 \geq n$.

5) $\sum_{n=2}^{\infty} \dfrac{1}{(n \ln n)^2}$

6) $\sum_{n=1}^{\infty} \dfrac{n!}{(n+2)!}$

Answer

$a_n = 1/(n+1)(n+2) < 1/n^2$. Converges by comparison with p-series, $p = 2 > 1$.

7) $\sum_{n=1}^{\infty} \dfrac{1}{n!}$

8) $\sum_{n=1}^{\infty} \dfrac{\sin(1/n)}{n}$

Answer

$\sin(1/n) \leq 1/n$, so converges by comparison with p-series, $p = 2 > 1$.

9) $\sum_{n=1}^{\infty} \dfrac{\sin^2 n}{n^2}$

10) $\sum_{n=1}^{\infty} \dfrac{\sin(1/n)}{\sqrt{n}}$

Answer

$\sin(1/n) \leq 1$, so converges by comparison with p-series, $p = 3/2 > 1$.

11) $\sum_{n=1}^{\infty} \dfrac{n^{1.2}-1}{n^{2.3}+1}$

12) $\sum_{n=1}^{\infty} \dfrac{\sqrt{n+1}-\sqrt{n}}{n}$

Answer

Since $\sqrt{n+1}-\sqrt{n} = 1/(\sqrt{n+1}+\sqrt{n}) \leq 2/\sqrt{n}$, series converges by comparison with p-series for $p = 1.5 > 1$.

13) $\sum_{n=1}^{\infty} \dfrac{\sqrt[4]{n}}{\sqrt[3]{n^4+n^2}}$

Use the Limit Comparison Test to determine whether each series in exercises 14 - 28 converges or diverges.

14) $\sum_{n=1}^{\infty}\left(\frac{\ln n}{n}\right)^2$

Answer

Converges by limit comparison with p-series for $p>1$.

15) $\sum_{n=1}^{\infty}\left(\frac{\ln n}{n^{0.6}}\right)^2$

16) $\sum_{n=1}^{\infty}\frac{\ln\left(1+\frac{1}{n}\right)}{n}$

Answer

Converges by limit comparison with p-series, $p=2>1$.

17) $\sum_{n=1}^{\infty}\ln\left(1+\frac{1}{n^2}\right)$

18) $\sum_{n=1}^{\infty}\frac{1}{4^n-3^n}$

Answer

Converges by limit comparison with 4^{-n}.

19) $\sum_{n=1}^{\infty}\frac{1}{n^2-n\sin n}$

20) $\sum_{n=1}^{\infty}\frac{1}{e^{(1.1)n}-3^n}$

Answer

Converges by limit comparison with $1/e^{1.1n}$.

21) $\sum_{n=1}^{\infty}\frac{1}{e^{(1.01)n}-3^n}$

22) $\sum_{n=1}^{\infty}\frac{1}{n^{1+1/n}}$

Answer

Diverges by limit comparison with harmonic series.

23) $\sum_{n=1}^{\infty}\frac{1}{2^{1+1/n}}n^{1+1/n}$

24) $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\sin\left(\frac{1}{n}\right)\right)$

Answer

Converges by limit comparison with p-series, $p=3>1$.

25) $\sum_{n=1}^{\infty}\left(1-\cos\left(\frac{1}{n}\right)\right)$

26) $\sum_{n=1}^{\infty}\frac{1}{n}\left(\tan^{-1}n-\frac{\pi}{2}\right)$

Answer

Converges by limit comparison with p-series, $p=3>1$.

27) $\sum_{n=1}^{\infty}\left(1-\frac{1}{n}\right)^{n \cdot n}$ (Hint: $\left(1-\frac{1}{n}\right)^{n} \to 1/e$.)

28) $\sum_{n=1}^{\infty}\left(1-e^{-1/n}\right)$ (Hint: $1/e \approx (1-1/n)^n$, so $1-e^{-1/n} \approx 1/n$.)

Answer

Diverges by limit comparison with $1/n$.

29) Does $\sum_{n=2}^{\infty}\frac{1}{(\ln n)^p}$ converge if p is large enough? If so, for which p?

30) Does $\sum_{n=1}^{\infty}\left(\frac{\ln n}{n}\right)^p$ converge if p is large enough? If so, for which p?

Answer

Converges for $p > 1$ by comparison with a p series for slightly smaller p.

31) For which p does the series $\sum_{n=1}^{\infty} 2^{pn}/3^n$ converge?

32) For which $p > 0$ does the series $\sum_{n=1}^{\infty}\frac{n^p}{2^n}$ converge?

Answer

Converges for all $p > 0$.

33) For which $r > 0$ does the series $\sum_{n=1}^{\infty}\frac{r^{n^2}}{2^n}$ converge?

34) For which $r > 0$ does the series $\sum_{n=1}^{\infty}\frac{2^n}{r^{n^2}}$ converge?

Answer

Converges for all $r > 1$. If $r > 1$ then $r^n > 4$, say, once $n > \ln(2)/\ln(r)$ and then the series converges by limit comparison with a geometric series with ratio $1/2$.

35) Find all values of p and q such that $\sum_{n=1}^{\infty}\frac{n^p}{(n!)^q}$ converges.

36) Does $\sum_{n=1}^{\infty}\frac{\sin^2(nr/2)}{n}$ converge or diverge? Explain.

Answer

The numerator is equal to 1 when n is odd and 0 when n is even, so the series can be rewritten $\sum_{n=1}^{\infty}\frac{1}{2n+1}$, which diverges by limit comparison with the harmonic series.

37) Explain why, for each n, at least one of $|\sin n|, |\sin(n+1)|, ..., |\sin(n+6)|$ is larger than $1/2$. Use this relation to test convergence of $\sum_{n=1}^{\infty}\frac{|\sin n|}{\sqrt{n}}$.

38) Suppose that $a_n \geq 0$ and $b_n \geq 0$ and that $\sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} b_n^2$ converge. Prove that $\sum_{n=1}^{\infty} a_n b_n$ converges and $\sum_{n=1}^{\infty} a_n b_n \leq \frac{1}{2}\left(\sum_{n=1}^{\infty} a_n^2 + \sum_{n=1}^{\infty} b_n^2\right)$.

Answer

$(a-b)^2 = a^2 - 2ab + b^2$ or $a^2 + b^2 \geq 2ab$, so convergence follows from comparison of $2a_nb_n$ with $a_n^2 + b_n^2$. Since the partial sums on the left are bounded by those on the right, the inequality holds for the infinite series.

39) Does $\sum_{n=1}^{\infty} 2^{-\ln\ln n}$ converge? (Hint: Write $2^{\ln\ln n}$ as a power of $\ln n$.)

40) Does $\sum_{n=1}^{\infty} (\ln n)^{-\ln n}$ converge? (Hint: Use $t = e^{\ln(t)}$ to compare to a p–series.)

Answer

$(\ln n)^{-\ln n} = e^{-\ln(n)\ln\ln(n)}$. If n is sufficiently large, then $\ln\ln n > 2$, so $(\ln n)^{-\ln n} < 1/n^2$, and the series converges by comparison to a p–series.

41) Does $\sum_{n=2}^{\infty} (\ln n)^{-\ln\ln n}$ converge? (Hint: Compare a_n to $1/n$.)

42) Show that if $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n^2$ converges. If $\sum_{n=1}^{\infty} a_n^2$ converges, does $\sum_{n=1}^{\infty} a_n$ necessarily converge?

Answer

$a_n \to 0$, so $a_n^2 \leq |a_n|$ for large n. Convergence follows from limit comparison. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, but $\sum_{n=1}^{\infty} \frac{1}{n}$ does not, so the fact that $\sum_{n=1}^{\infty} a_n^2$ converges does not imply that $\sum_{n=1}^{\infty} a_n$ converges.

43) Suppose that $a_n > 0$ for all n and that $\sum_{n=1}^{\infty} a_n$ converges. Suppose that b_n is an arbitrary sequence of zeros and ones. Does $\sum_{n=1}^{\infty} a_n b_n$ necessarily converge?

44) Suppose that $a_n > 0$ for all n and that $\sum_{n=1}^{\infty} a_n$ diverges. Suppose that b_n is an arbitrary sequence of zeros and ones with infinitely many terms equal to one. Does $\sum_{n=1}^{\infty} a_n b_n$ necessarily diverge?

Answer

No. $\sum_{n=1}^{\infty} 1/n$ diverges. Let $b_k = 0$ unless $k = n^2$ for some n. Then $\sum_k b_k/k = \sum 1/k^2$ converges.

45) Complete the details of the following argument: If $\sum_{n=1}^{\infty} \frac{1}{n}$ converges to a finite sum s, then $\frac{1}{2}s = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots$ and $s - \frac{1}{2}s = 1 + \frac{1}{3} + \frac{1}{5} + \cdots$. Why does this lead to a contradiction?

46) Show that if $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n^2$ converges, then $\sum_{n=1}^{\infty} \sin^2(a_n)$ converges.

47) Suppose that $a_n/b_n \to 0$ in the Limit Comparison Test, where $a_n \geq 0$ and $b_n \geq 0$. Prove that if $\sum b_n$ converges, then $\sum a_n$ converges.

48) Let b_n be an infinite sequence of zeros and ones. What is the largest possible value of $x = \sum_{n=1}^{\infty} b_n/2^n$?

Answer

By the Limit Comparison Test, $x = \sum_{n=1}^{\infty} b_n/2^n \leq \sum_{n=1}^{\infty} 1/2^n = 1$.

49) Let d_n be an infinite sequence of digits, meaning d_n takes values in $\{0, 1, \ldots, 9\}$. What is the largest possible value of $x = \sum_{n=1}^{\infty} d_n/10^n$ that converges?

50) Explain why, if $x > 1/2$, then x cannot be written $x = \sum_{n=2}^{\infty} \frac{b_n}{2^n}$ $(b_n = 0 \text{ or } 1,\ b_1 = 0)$.

Answer

If $b_1 = 0$, then, by comparison, $x \leq \sum_{n=2}^{\infty} 1/2^n = 1/2$.

51) [Technology Required] Evelyn has a perfect balancing scale, an unlimited number of 1-kg weights, and one each of $1/2$-kg, $1/4$-kg, $1/8$-kg, and so on weights. She wishes to weigh a meteorite of unspecified origin to arbitrary precision. Assuming the scale is big enough, can she do it? What does this have to do with infinite series?

52) [Technology Required] Robert wants to know his body mass to arbitrary precision. He has a big balancing scale that works perfectly, an unlimited collection of 1-kg weights, and nine each of 0.1-kg, 0.01-kg, 0.001-kg, and so on weights. Assuming the scale is big enough, can he do this? What does this have to do with infinite series?

Answer

Yes. Keep adding 1-kg weights until the balance tips to the side with the weights. If it balances perfectly, with Robert standing on the other side, stop. Otherwise, remove one of the 1-kg weights, and add 0.1-kg weights one at a time. If it balances after adding some of these, stop. Otherwise if it tips to the weights, remove the last 0.1-kg weight. Start adding 0.01-kg weights. If it balances, stop. If it tips to the side with the weights, remove the last 0.01-kg weight that was added. Continue in this way for the 0.001-kg weights, and so on. After a finite number of steps, one has a finite series of the form $A + \sum_{n=1}^{N} s_n/10^n$ where A is the number of full kg weights and d_n is the number of $1/10^n$-kg weights that were added. If at some state this series is Robert's exact weight, the process will stop. Otherwise it represents the Nth partial sum of an infinite series that gives Robert's exact weight, and the error of this sum is at most $1/10^N$.

53) The series $\sum_{n=1}^{\infty} \frac{1}{2n}$ is half the harmonic series and hence diverges. It is obtained from the harmonic series by deleting all terms in which n is odd. Let $m > 1$ be fixed. Show, more generally, that deleting all terms $1/n$ where $n = mk$ for some integer k also results in a divergent series.

54) In view of the previous exercise, it may be surprising that a subseries of the harmonic series in which about one in every five terms is deleted might converge. A depleted harmonic series is a series obtained from $\sum_{n=1}^{\infty} \frac{1}{n}$ by removing any term $1/n$ if a given digit, say 9, appears in the decimal expansion of n. Argue that this depleted harmonic series converges by answering the following questions.

a. How many whole numbers n have d digits?

b. How many d-digit whole numbers $h(d)$. do not contain 9 as one or more of their digits?

c. What is the smallest d-digit number $m(d)$?

d. Explain why the deleted harmonic series is bounded by $\sum_{d=1}^{\infty} \frac{h(d)}{m(d)}$.

e. Show that $\sum_{d=1}^{\infty} \frac{h(d)}{m(d)}$ converges.

Answer

a. $10^d - 10^{d-1} < 10^d$

b. $h(d) < 9^d$

c. $m(d) = 10^{d-1} + 1$

d. Group the terms in the deleted harmonic series together by number of digits. $h(d)$ bounds the number of terms, and each term is at most $\frac{1}{m(d)}$.

Then $\sum_{d=1}^{\infty} h(d)/m(d) \leq \sum_{d=1}^{\infty} 9^d/(10)^{d-1} \leq 90$. One can actually use comparison to estimate the value to smaller than 80. The actual value is smaller than 23.

55) Suppose that a sequence of numbers $a_n > 0$ has the property that $a_1 = 1$ and $a_{n+1} = \frac{1}{n+1} S_n$, where $S_n = a_1 + \cdots + a_n$. Can you determine whether $\sum_{n=1}^{\infty} a_n$ converges? (Hint: S_n is monotone.)

56) Suppose that a sequence of numbers $a_n > 0$ has the property that $a_1 = 1$ and $a_{n+1} = \frac{1}{(n+1)^2} S_n$, where $S_n = a_1 + \cdots + a_n$. Can you determine whether $\sum_{n=1}^{\infty} a_n$ converges? (Hint: $S_2 = a_2 + a_1 = a_2 + S_1 = a_2 + 1 = 1 + 1/4 = (1 + 1/4)S_1, S_3 = \frac{1}{3^2} S_2 + S_2 = (1 + 1/9)S_2 = (1 + 1/9)(1 + 1/4)S_1$, etc. Look at $\ln(S_n)$, and use $\ln(1 + t) \leq t, t > 0$.)

Answer

Continuing the hint gives $S_N = (1 + 1/N^2)(1 + 1/(N-1)^2 \ldots (1 + 1/4))$. Then $\ln(S_N) = \ln(1 + 1/N^2) + \ln(1 + 1/(N-1)^2) + \cdots + \ln(1 + 1/4)$. Since $\ln(1 + t)$ is bounded by a constant times t, when $0 < t < 1$ one has $\ln(S_N) \leq C \sum_{n=1}^{N} \frac{1}{n^2}$, which converges by comparison to the p-series for $p = 2$.

This page titled 4.4E: Exercises is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Roy Simpson.

4.5: The Alternating Series Test

Learning Objectives

- Use the Alternating Series Test to test an alternating series for convergence.
- Estimate the sum of an alternating series.
- Explain the meaning of absolute convergence and conditional convergence.

So far in this chapter, we have primarily discussed series with positive terms. In this section, we introduce alternating series, which are series whose terms alternate in sign. We will show in a later chapter that these series often arise when studying **power series**. After defining alternating series, we introduce the Alternating Series Test to determine whether such a series converges.

The Alternating Series Test

A series whose terms alternate between positive and negative values is an **alternating series.** For example, the series

$$\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^n = -\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-... \tag{4.5.1}$$

and

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+... \tag{4.5.2}$$

are both alternating series.

Definition: Alternating Series

Any series whose terms alternate between positive and negative values is called an **alternating series**. An alternating series can be written in the form

$$\sum_{n=1}^{\infty}(-1)^{n+1}b_n = b_1-b_2+b_3-b_4+... \tag{4.5.3}$$

or

$$\sum_{n-1}^{\infty}(-1)^{n}b_n = -b_1+b_2-b_3+b_4-... \tag{4.5.4}$$

Where $b_n \geq 0$ for all $n \in \mathbb{N}$.

The series shown in Equation 4.5.1 is a geometric series. Since $|r| = |-\frac{1}{2}| < 1$, the series converges. The series shown in Equation 4.5.2 is called the **alternating harmonic series**. Whereas the harmonic series diverges, the alternating harmonic series converges.

Theorem: Convergence of the Alternating Harmonic Series

The alternating harmonic series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+...$$

converges.

Proof

Consider the odd terms S_{2k+1} for $k \geq 0$. Since $\frac{1}{2k+1} < \frac{1}{2k}$,

$$S_{2k+1} = S_{2k-1}-\frac{1}{2k}+\frac{1}{2k+1} < S_{2k-1}.$$

Therefore, $\{S_{2k+1}\}$ is a decreasing sequence. Also,

$$S_{2k+1} = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{2k-1} - \frac{1}{2k}\right) + \frac{1}{2k+1} > 0.$$

Therefore, $\{S_{2k+1}\}$ is bounded below. Since $\{S_{2k+1}\}$ is a decreasing sequence that is bounded below, by the Monotone Convergence Theorem $\{S_{2k+1}\}$ converges. Similarly, the even terms $\{S_{2k}\}$ form an increasing sequence that is bounded above because

$$S_{2k} = S_{2k-2} + \frac{1}{2k-1} - \frac{1}{2k} > S_{2k-2}$$

and

$$S_{2k} = 1 + \left(-\frac{1}{2} + \frac{1}{3}\right) + \ldots + \left(-\frac{1}{2k-2} + \frac{1}{2k-1}\right) - \frac{1}{2k} < 1.$$

Therefore, by the Monotone Convergence Theorem the sequence $\{S_{2k}\}$ also converges. Since

$$S_{2k+1} = S_{2k} + \frac{1}{2k+1},$$

we know that

$$\lim_{k\to\infty} S_{2k+1} = \lim_{k\to\infty} S_{2k} + \lim_{k\to\infty} \frac{1}{2k+1}.$$

Letting $S = \lim_{k\to\infty} S_{2k+1}$ and using the fact that $\frac{1}{2k+1} \to 0$, we conclude that $\lim_{k\to\infty} S_{2k} = S$. Since the odd and even terms in the sequence of partial sums converge to the same limit S, it can be shown that the sequence of partial sums converges to S. Therefore, the alternating harmonic series converges to S.

It can also be shown that the value of the alternating harmonic series is $S = \ln 2$.

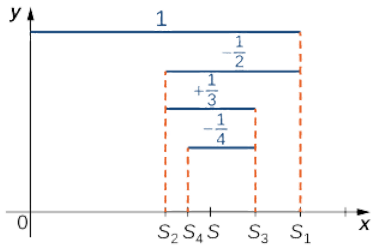


Figure 4.5.1: For the alternating harmonic series, the odd terms S_{2k+1} in the sequence of partial sums are decreasing and bounded below. The even terms S_{2k} are increasing and bounded above.

More generally, any alternating series having the form of either Equation 4.5.3 or Equation 4.5.4 converges as long as $b_1 \geq b_2 \geq b_3 \geq \cdots$ and $b_n \to 0$ (see Figure 4.5.2). This is called the **Alternating Series Test**. The proof is similar to the proof for the alternating harmonic series.

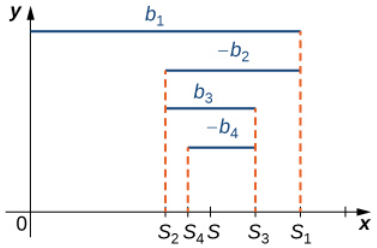


Figure 4.5.2: For an alternating series $b_1 - b_2 + b_3 - \cdots$ in which $b_1 > b_2 > b_3 > \cdots$, the odd terms S_{2k+1} in the sequence of partial sums are decreasing and bounded below. The even terms S_{2k} are increasing and bounded above.

Theorem: Alternating Series Test

An alternating series of the form

$$\sum_{n=1}^{\infty}(-1)^{n+1}b_n$$

or

$$\sum_{n=1}^{\infty}(-1)^{n}b_n$$

converges if

i. $0 \le b_{n+1} \le b_n$ for all $n \ge 1$ and
ii. $\lim\limits_{n\to\infty} b_n = 0$.

This theorem is true more generally as long as there exists some $N \in \mathbb{N}$ such that $0 \le b_{n+1} \le b_n$ for all $n \ge N$.

Example 4.5.1: Convergence of Alternating Series

For each alternating series, determine whether the series converges or diverges.

a. $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{n^2}$
b. $\sum\limits_{n=1}^{\infty}(-1)^{n+1}\dfrac{n}{n+1}$

Solutions

a. Since $\frac{1}{(n+1)^2} < \frac{1}{n^2}$ and $\frac{1}{n^2} \to 0$, the series converges.
b. Since $\frac{n}{n+1}$ does not approach 0 as $n \to \infty$, we cannot apply the Alternating Series Test. Instead, we use the Divergence Test. Since $\lim\limits_{n\to\infty}\dfrac{n}{n+1} = 1 \ne 0$, the series diverges.

Checkpoint 4.5.1

Determine whether the series $\sum\limits_{n=1}^{\infty}(-1)^{n+1}\dfrac{n}{2^n}$ converges or diverges.

Answer

The series converges.

Remainder of an Alternating Series

It is difficult to explicitly calculate the sum of most alternating series, so typically, the sum is approximated using a partial sum. When doing so, we are interested in the amount of error in our approximation.

Theorem: Remainder Estimate from the Alternating Series Test

Consider an alternating series of the form

$$\sum_{n=1}^{\infty}(-1)^{n+1}b_n$$

or

$$\sum_{n=1}^{\infty}(-1)^{n}b_n$$

that satisfies the hypotheses of the Alternating Series Test. Let S denote the sum of the series and S_N denote the Nth partial sum. For any $N \in \mathbb{N}$, the remainder $R_N = S - S_N$ satisfies

$$|R_N| \leq b_{N+1}.$$

Visual Justification

A visualization of the proof can be found in Figure 4.5.2. For example, $|R_2| = |S - S_2|$; however, the distance between S and S_2 can easily be seen to be less than b_3.

In other words, if the conditions of the Alternating Series Test apply, then the error in approximating the infinite series by the Nth partial sum S_N is (in magnitude) at most the size of the next term b_{N+1}.

Example 4.5.2: Estimating the Remainder of an Alternating Series

Consider the alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$

Use the remainder estimate to determine a bound on the error R_{10} if we approximate the sum of the series by the partial sum S_{10}.

Solution

From the theorem stated above,

$$|R_{10}| \leq b_{11} = \frac{1}{11^2} \approx 0.008265.$$

Checkpoint 4.5.2

Find a bound for R_{20} when approximating $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ by S_{20}.

Answer

0.04762

Absolute and Conditional Convergence

Consider a series $\sum_{n=1}^{\infty} a_n$ and the related series $\sum_{n=1}^{\infty} |a_n|$. Here, we discuss possibilities for the relationship between the convergence of these two series. For example, consider the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. The series whose terms are the absolute value of these terms is the harmonic series, since $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$. Since the alternating harmonic series converges, but the harmonic series diverges, we say the alternating harmonic series exhibits **conditional convergence**.

By comparison, consider the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$. The series whose terms are the absolute values of the terms of this series is the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Since both of these series converge, we say the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ exhibits **absolute convergence**.

Definition: Absolute Convergence and Conditional Convergence

A series $\sum_{n=1}^{\infty} a_n$ exhibits **absolute convergence** if $\sum_{n=1}^{\infty} |a_n|$ converges. A series $\sum_{n=1}^{\infty} a_n$ exhibits **conditional convergence** if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

As shown by the alternating harmonic series, a series $\sum_{n=1}^{\infty} a_n$ may converge, but $\sum_{n=1}^{\infty} |a_n|$ may diverge. In the following theorem, however, we show that if $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Theorem: Absolute Convergence Implies Convergence

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Proof

Suppose that $\sum_{n=1}^{\infty} |a_n|$ converges. We know that $a_n = |a_n|$ or $a_n = -|a_n|$. Therefore $|a_n| + a_n = 2|a_n|$ or $|a_n| + a_n = 0$. Hence, $0 \le |a_n| + a_n \le 2|a_n|$. Consequently, by the Series Comparison Test, since $2\sum_{n=1}^{\infty} |a_n|$ converges, the series

$$\sum_{n=1}^{\infty} (|a_n| + a_n)$$

converges. By using the Algebraic Properties for Convergent Series, we conclude that

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (|a_n| + a_n) - \sum_{n=1}^{\infty} |a_n|$$

converges.

Example 4.5.3: Absolute versus Conditional Convergence

For each of the following series, determine whether the series converges absolutely, converges conditionally, or diverges.

a. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3n+1}$

b. $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$

Solutions

a. We can see that

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{3n+1} \right| = \sum_{n=1}^{\infty} \frac{1}{3n+1}$$

diverges by using the Limit Comparison Test with the harmonic series. In fact,

$$\lim_{n\to\infty} \frac{1/(3n+1)}{1/n} = \frac{1}{3}.$$

Therefore, the series does not converge absolutely. However, since

$$3(n+1)+1 > 3n+1 \implies \frac{1}{3(n+1)+1} < \frac{1}{3n+1}, \text{ and } \frac{1}{3n+1} \to 0,$$

the series converges. We can conclude that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3n+1}$ converges conditionally.

b. Noting that $|\cos n| \le 1$, to determine whether the series converges absolutely, compare

$$\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right|$$

with the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, by the Series Comparison Test, $\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right|$ converges, and therefore $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ converges absolutely.

? Checkpoint 4.5.3

Determine whether the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{2n^3+1}$ converges absolutely, converges conditionally, or diverges.

Answer

The series converges absolutely.

To see the difference between absolute and conditional convergence, look at what happens when we *rearrange* the terms of the alternating harmonic series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$. We show that we can rearrange the terms so the new series diverges. Certainly, if we rearrange the terms of a finite sum, the sum will not change. However, exciting things can happen when we work with an infinite sum.

Begin by adding enough positive terms to produce a sum larger than some real number M. For example, let $M = 10$, and find an integer k such that

$$1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2k-1} > 10.$$

(We can do this because the series $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ diverges to infinity.) Now subtract $1/2$. Then, add more positive terms until the sum reaches 100. That is, find another integer $j > k$ such that

$$(1 + \frac{1}{3} + \cdots + \frac{1}{2k-1} - \frac{1}{2} + \frac{1}{2k+1} + \cdots + \frac{1}{2j+1} > 100.$$

After we find this value of j, subtract $1/4$ from the summation. Continuing in this way, we have found a way of rearranging the terms in the alternating harmonic series so that the sequence of partial sums for the rearranged series is unbounded and, therefore, diverges. That is,

$$\overbrace{\overbrace{1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2k-1}}^{\text{slightly less than 10}} - \frac{1}{2} + \frac{1}{2k+1} + \cdots + \frac{1}{2j+1}}^{\text{slightly less than 100}} + \cdots$$

grows without bound.

The terms in the alternating harmonic series can also be rearranged so that the new series converges to a different value. In Example 4.5.4, we show how to rearrange the terms to create a new series that converges to $3\ln(2)/2$. We point out that the alternating harmonic series can be rearranged to create a series that converges to any real number r; however, the proof of that fact is beyond the scope of this text.

In general, any series $\sum_{n=1}^{\infty} a_n$ that converges conditionally can be rearranged so that the new series diverges or converges to a different real number. A series that converges absolutely does not have this property. For any series $\sum_{n=1}^{\infty} a_n$ that converges absolutely, the value of $\sum_{n=1}^{\infty} a_n$ is the same for any rearrangement of the terms. This result is known as the **Riemann Rearrangement Theorem**, which (again) is beyond the scope of this book.

Example 4.5.4: Rearranging Series

Use the fact that

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots=\ln 2$$

to rearrange the terms in the alternating harmonic series so the sum of the rearranged series is $3\ln(2)/2$.

Solution

Let

$$\sum_{n=1}^{\infty}a_n=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\cdots.$$

Since $\sum_{n=1}^{\infty}a_n=\ln(2)$, by the Algebraic Properties of Convergent Series,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{2}a_n &= \frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\cdots\\ &= \frac{1}{2}\sum_{n=1}^{\infty}a_n\\ &= \frac{\ln 2}{2}.\end{aligned}$$

Now introduce the series $\sum_{n=1}^{\infty}b_n$ such that for all $n\geq 1$, $b_{2n-1}=0$ and $b_{2n}=a_n/2$. Then

$$\sum_{n=1}^{\infty}b_n=0+\frac{1}{2}+0-\frac{1}{4}+0+\frac{1}{6}+0-\frac{1}{8}+\cdots=\frac{\ln 2}{2}.$$

Then using the Algebraic Properties of Convergent Series, since $\sum_{n=1}^{\infty}a_n$ and $\sum_{n=1}^{\infty}b_n$ converge, the series $\sum_{n=1}^{\infty}(a_n+b_n)$ converges and

$$\sum_{n=1}^{\infty}(a_n+b_n)=\sum_{n=1}^{\infty}a_n+\sum_{n=1}^{\infty}b_n=\ln 2+\frac{\ln 2}{2}=\frac{3\ln 2}{2}.$$

Now adding the corresponding terms, a_n and b_n, we see that

$$\begin{aligned}\sum_{n=1}^{\infty}(a_n+b_n) &= (1+0)+\left(-\frac{1}{2}+\frac{1}{2}\right)+\left(\frac{1}{3}+0\right)+\left(-\frac{1}{4}-\frac{1}{4}\right)+\\ &\quad \left(\frac{1}{5}+0\right)+\left(-\frac{1}{6}+\frac{1}{6}\right)+\left(\frac{1}{7}+0\right)+\left(\frac{1}{8}-\frac{1}{8}\right)+\cdots\\ &= 1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\cdots.\end{aligned}$$

We notice that the series on the right side of the equal sign is a rearrangement of the alternating harmonic series. Since $\sum_{n=1}^{\infty}(a_n+b_n)=3\ln(2)/2$, we conclude that

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\cdots=\frac{3\ln(2)}{2}.$$

Therefore, we have found a rearrangement of the alternating harmonic series having the desired property.

Media

Videos

- Proof of the Alternating Series Test

Key Concepts

- For an alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$, if $b_{k+1} \leq b_k$ for all k and $b_k \to 0$ as $k \to \infty$, the alternating series converges.
- If $\sum_{n=1}^{\infty}|a_n|$ converges, then $\sum_{n=1}^{\infty}a_n$ converges.

Key Equations

- **Alternating series**

$$\sum_{n=1}^{\infty}(-1)^{n+1}b_n = b_1 - b_2 + b_3 - b_4 + \cdots \text{ or}$$

$$\sum_{n=1}^{\infty}(-1)^{n}b_n = -b_1 + b_2 - b_3 + b_4 - \cdots$$

Glossary

absolute convergence

if the series $\sum_{n=1}^{\infty}|a_n|$ converges, the series $\sum_{n=1}^{\infty}a_n$ is said to converge absolutely

alternating series

a series of the form $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ or $\sum_{n=1}^{\infty}(-1)^{n}b_n$, where $b_n \geq 0$, is called an alternating series

Alternating Series Test

for an alternating series of either form, if $b_{n+1} \leq b_n$ for all integers $n \geq 1$ and $b_n \to 0$, then an alternating series converges

conditional convergence

if the series $\sum_{n=1}^{\infty}a_n$ converges, but the series $\sum_{n=1}^{\infty}|a_n|$ diverges, the series $\sum_{n=1}^{\infty}a_n$ is said to converge conditionally

4.5E: Exercises

In exercises 1 - 30, state whether each of the following series converges absolutely, conditionally, or not at all.

1) $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{n}{n+3}$

2) $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{\sqrt{n}+1}{\sqrt{n}+3}$

Answer

This series diverges by the Divergence Test. Terms do not tend to zero.

3) $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{1}{\sqrt{n+3}}$

4) $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{\sqrt{n+3}}{n}$

Answer

Converges conditionally by alternating series test, since $\sqrt{n+3}/n$ is decreasing and its limit is 0. Does not converge absolutely by comparison with p-series, $p=1/2$.

5) $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{1}{n!}$

6) $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{3^n}{n!}$

Answer

Converges absolutely by limit comparison to $3^n/4^n$, for example.

7) $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\dfrac{n-1}{n}\right)^n$

8) $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\dfrac{n+1}{n}\right)^n$

Answer

Diverges by Divergence Test since $\lim_{n\to\infty}|a_n|=e$ and not 0.

9) $\sum_{n=1}^{\infty}(-1)^{n+1}\sin^2 n$

10) $\sum_{n=1}^{\infty}(-1)^{n+1}\cos^2 n$

Answer

Diverges by the Divergence Test, since its terms do not tend to zero. The limit of the sequence of its terms does not exist.

11) $\sum_{n=1}^{\infty}(-1)^{n+1}\sin^2(1/n)$

12) $\sum_{n=1}^{\infty}(-1)^{n+1}\cos^2(1/n)$

Answer

$\lim_{n\to\infty}\cos^2(1/n)=1$. Diverges by Divergence Test.

13) $\sum_{n=1}^{\infty}(-1)^{n+1}\ln(1/n)$

14) $\sum_{n=1}^{\infty}(-1)^{n+1}\ln(1+\frac{1}{n})$

Answer

Converges by alternating series test.

15) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{1+n^4}$

16) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^e}{1+n^\pi}$

Answer

Converges conditionally by alternating series test. Does not converge absolutely by limit comparison with p-series, $p=\pi-e$

Solution:

17) $\sum_{n=1}^{\infty}(-1)^{n+1}2^{1/n}$

18) $\sum_{n=1}^{\infty}(-1)^{n+1}n^{1/n}$

Answer

Diverges; terms do not tend to zero.

19) $\sum_{n=1}^{\infty}(-1)^{n}(1-n^{1/n})$ (Hint: $n^{1/n}\approx 1+\ln(n)/n$ for large n.)

20) $\sum_{n=1}^{\infty}(-1)^{n+1}n\left(1-\cos\left(\frac{1}{n}\right)\right)$ (Hint: $\cos(1/n)\approx 1-1/n^2$ for large n.)

Answer

Converges by alternating series test. Does not converge absolutely by limit comparison with harmonic series.

21) $\sum_{n=1}^{\infty}(-1)^{n+1}(\sqrt{n+1}-\sqrt{n})$ (Hint: Rationalize the numerator.)

22) $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)$ (Hint: Cross-multiply then rationalize numerator.)

Answer

Converges absolutely by limit comparison with p-series, $p=3/2$, after applying the hint.

23) $\sum_{n=1}^{\infty}(-1)^{n+1}(\ln(n+1)-\ln n)$

24) $\sum_{n=1}^{\infty}(-1)^{n+1}n(\tan^{-1}(n+1)-\tan^{-1}n)$ (Hint: Use Mean Value Theorem.)

Answer

Converges by alternating series test since $n(\tan^{-1}(n+1)-\tan^{-1}n)$ is decreasing to zero for large n.Does not converge absolutely by limit comparison with harmonic series after applying hint.

25) $\sum_{n=1}^{\infty}(-1)^{n+1}((n+1)^2-n^2)$

26) $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{n}-\frac{1}{n+1}\right)$

Answer

Converges absolutely, since $a_n=\frac{1}{n}-\frac{1}{n+1}$ are terms of a telescoping series.

27) $\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n}$

28) $\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n^{1/n}}$

Answer

Terms do not tend to zero. Series diverges by Divergence Test.

29) $\sum_{n=1}^{\infty}\frac{1}{n}\sin(\frac{n\pi}{2})$

30) $\sum_{n=1}^{\infty}\sin(n\pi/2)\sin(1/n)$

Answer

Converges by alternating series test. Does not converge absolutely by limit comparison with harmonic series.

In exercises 31 - 36, use the estimate $|R_N|\le b_{N+1}$ to find a value of N that guarantees that the sum of the first N terms of the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ differs from the infinite sum by at most the given error. Calculate the partial sum S_N for this N.

31) [Technology Required] $b_n=1/n$, error $<10^{-5}$

32) [Technology Required] $b_n=1/\ln(n), n\ge 2$, error $<10^{-1}$

Answer

$\ln(N+1)>10, N+1>e^{10}, N\ge 22026; S_{22026}=0.0257\ldots$

33) [Technology Required] $b_n=1/\sqrt{n}$, error $<10^{-3}$

34) [Technology Required] $b_n=1/2^n$, error $<10^{-6}$

Answer

$2^{N+1}>10^6$ or $N+1>6\ln(10)/\ln(2)=19.93.$ or $N\ge 19; S_{19}=0.333333969\ldots$

35) [Technology Required] $b_n=ln(1+\frac{1}{n})$, error $<10^{-3}$

36) [Technology Required] $b_n=1/n^2$, error $<10^{-6}$

Answer

$(N+1)^2>10^6$ or $N>999; S_{1000}\approx 0.822466.$

For exercises 37 - 45, indicate whether each of the following statements is true or false. If the statement is false, provide an example in which it is false.

37) If $b_n\ge 0$ is decreasing and $\lim_{n\to\infty}b_n=0$, then $\sum_{n=1}^{\infty}(b_{2n-1}-b_{2n})$ converges absolutely.

38) If $b_n\ge 0$ is decreasing, then $\sum_{n=1}^{\infty}(b_{2n-1}-b_{2n})$ converges absolutely.

Answer

True. b_n need not tend to zero since if $c_n = b_n - \lim b_n$, then $c_{2n-1} - c_{2n} = b_{2n-1} - b_{2n}$.

39) If $b_n \geq 0$ and $\lim_{n\to\infty} b_n = 0$ then $\sum_{n=1}^{\infty}(\frac{1}{2}(b_{3n-2} + b_{3n-1}) - b_{3n})$ converges.

40) If $b_n \geq 0$ is decreasing and $\sum_{n=1}^{\infty}(b_{3n-2} + b_{3n-1} - b_{3n})$ converges then $\sum_{n=1}^{\infty} b_{3n-2}$ converges.

Answer

True. $b_{3n-1} - b_{3n} \geq 0$, so convergence of $\sum b_{3n-2}$ follows from the Comparison Test.

41) If $b_n \geq 0$ is decreasing and $\sum_{n=1}^{\infty}(-1)^{n-1} b_n$ converges conditionally but not absolutely, then b_n does not tend to zero.

42) Let $a_n^+ = a_n$ if $a_n \geq 0$ and $a_n^- = -a_n$ if $a_n < 0$. (Also, $a_n^+ = 0$ if $a_n < 0$ and $a_n^- = 0$ if $a_n \geq 0$.) If $\sum_{n=1}^{\infty} a_n$ converges conditionally but not absolutely, then neither $\sum_{n=1}^{\infty} a_n^+$ nor $\sum_{n=1}^{\infty} a_n^-$ converge.

Answer

True. If one converges, then so must the other, implying absolute convergence.

43) Suppose that a_n is a sequence of positive real numbers and that $\sum_{n=1}^{\infty} a_n$ converges.

44) Suppose that b_n is an arbitrary sequence of ones and minus ones. Does $\sum_{n=1}^{\infty} a_n b_n$ necessarily converge?

45) Suppose that a_n is a sequence such that $\sum_{n=1}^{\infty} a_n b_n$ converges for every possible sequence b_n of zeros and ones. Does $\sum_{n=1}^{\infty} a_n$ converge absolutely?

Answer

Yes. Take $b_n = 1$ if $a_n \geq 0$ and $b_n = 0$ if $a_n < 0$. Then $\sum_{n=1}^{\infty} a_n b_n = \sum_{n:a_n \geq 0} a_n$ converges. Similarly, one can show $\sum_{n:a_n<0} a_n$ converges. Since both series converge, the series must converge absolutely.

In exercises 46 - 49, the series do not satisfy the hypotheses of the alternating series test as stated. In each case, state which hypothesis is not satisfied. State whether the series converges absolutely.

46) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin^2 n}{n}$

47) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos^2 n}{n}$

Answer

Not decreasing. Does not converge absolutely.

48) $1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \cdots$

49) $1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + \cdots$

Answer

Not alternating. Can be expressed as $\sum_{n=1}^{\infty}\left(\frac{1}{3n-2} + \frac{1}{3n-1} - \frac{1}{3n}\right)$, which diverges by comparison with $\sum_{n=1}^{\infty}\frac{1}{3n-2}$.

50) Show that the alternating series $1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} + \frac{1}{4} - \frac{1}{8} + \cdots$ does not converge. What hypothesis of the alternating series test is not met?

51) Suppose that $\sum a_n$ converges absolutely. Show that the series consisting of the positive terms a_n also converges.

Answer

Let $a_n^+ = a_n$ if $a_n \geq 0$ and $a_n^+ = 0$ if $a_n < 0$. Then $a_n^+ \leq |a_n|$ for all n so the sequence of partial sums of a_n^+ is increasing and bounded above by the sequence of partial sums of $|a_n|$, which converges; hence, $\sum_{n=1}^{\infty} a_n^+$ converges.

52) Show that the alternating series $\frac{2}{3} - \frac{3}{5} + \frac{4}{7} - \frac{5}{9} + \cdots$ does not converge. What hypothesis of the alternating series test is not met?

53) The formula $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$ will be derived in the next chapter. Use the remainder $|R_N| \leq b_{N+1}$ to find a bound for the error in estimating $\cos\theta$ by the fifth partial sum $1 - \theta^2/2! + \theta^4/4! - \theta^6/6! + \theta^8/8!$ for $\theta = 1, \theta = \pi/6$, and $\theta = \pi$.

Answer

For $N = 5$ one has $| R_N | b_6 = \theta^{10}/10!$. When $\theta = 1, R_5 \leq 1/10! \approx 2.75 \times 10^{-7}$. When $\theta = \pi/6$, $R_5 \leq (\pi/6)^{10}/10! \approx 4.26 \times 10^{-10}$. When $\theta = \pi, R_5 \leq \pi^{10}/10! = 0.0258$.

54) The formula $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$ will be derived in the next chapter. Use the remainder $|R_N| \leq b_{N+1}$ to find a bound for the error in estimating $\sin\theta$ by the fifth partial sum $\theta - \theta^3/3! + \theta^5/5! - \theta^7/7! + \theta^9/9!$ for $\theta = 1, \theta = \pi/6$, and $\theta = \pi$.

55) How many terms in $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$ are needed to approximate $\cos 1$ accurate to an error of at most 0.00001?

Answer

Let $b_n = 1/(2n-2)!$. Then $R_N \leq 1/(2N)! < 0.00001$ when $(2N)! > 10^5$ or $N = 5$ and $1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \frac{1}{8!} = 0.540325\ldots$, whereas $\cos 1 = 0.5403023\ldots$

56) How many terms in $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$ are needed to approximate $\sin 1$ accurate to an error of at most 0.00001?

57) Sometimes the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ converges to a certain fraction of an absolutely convergent series $\sum_{n=1}^{\infty} b_n$ at a faster rate. Given that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, find $S = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$. Which of the series $6\sum_{n=1}^{\infty} \frac{1}{n^2}$ and $S\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ gives a better estimation of π^2 using 1000 terms?

Answer

Let $T = \sum \frac{1}{n^2}$. Then $T - S = \frac{1}{2}T$, so $S = T/2$. $\sqrt{6 \times \sum_{n=1}^{1000} 1/n^2} = 3.140638\ldots$; $\sqrt{\frac{1}{2} \times \sum_{n=1}^{1000} (-1)^{n-1}/n^2} = 3.141591\ldots$; $\pi = 3.141592\ldots$. The alternating series is more accurate for 1000 terms.

The alternating series in exercises 58 & 59 converge to given multiples of π. Find the value of N predicted by the remainder estimate such that the Nth partial sum of the series accurately approximates the left-hand side to within the given error. Find the minimum N for which the error bound holds, and give the desired approximate value in each case. Up to 15 decimals places, $\pi = 3.141592653589793\ldots$.

58) [Technology Required] $\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$, error < 0.0001

59) [Technology Required] $\frac{\pi}{\sqrt{12}} = \sum_{k=0}^{\infty} \frac{(-3)^{-k}}{2k+1}$, error < 0.0001

Answer

$N = 6, S_N = 0.9068$

60) [Technology Required] The series $\sum_{n=0}^{\infty} \frac{\sin(x+\pi n)}{x+\pi n}$ plays an important role in signal processing. Show that $\sum_{n=0}^{\infty} \frac{\sin(x+\pi n)}{x+\pi n}$ converges whenever $0 < x < \pi$. (Hint: Use the formula for the sine of a sum of angles.)

61) [Technology Required] If $\sum_{n=1}^{N} (-1)^{n-1} \frac{1}{n} \to ln2$, what is $1+\frac{1}{3}+\frac{1}{5}-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\frac{1}{8}-\frac{1}{10}-\frac{1}{12}+\cdots$?

Answer

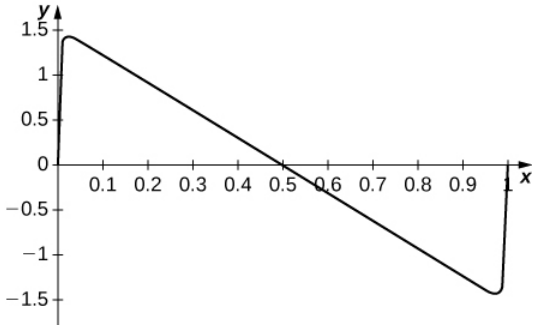
$\ln(2)$. The nth partial sum is the same as that for the alternating harmonic series.

62) [Technology Required] Plot the series $\sum_{n=1}^{100} \frac{\cos(2\pi nx)}{n}$ for $0 \le x < 1$. Explain why $\sum_{n=1}^{100} \frac{\cos(2\pi nx)}{n}$ diverges when $x = 0, 1$. How does the series behave for other x?

63) [Technology Required] Plot the series $\sum_{n=1}^{100} \frac{\sin(2\pi nx)}{n}$ for $0 \le x < 1$ and comment on its behavior

Answer

The series jumps rapidly near the endpoints. For x away from the endpoints, the graph looks like $\pi(1/2 - x)$.

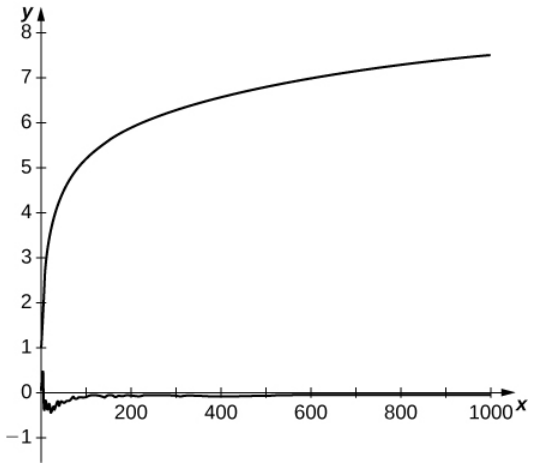


64) [Technology Required] Plot the series $\sum_{n=1}^{100} \frac{\cos(2\pi nx)}{n^2}$ for $0 \le x < 1$ and describe its graph.

65) [Technology Required] The alternating harmonic series converges because of cancelation among its terms. Its sum is known because the cancelation can be described explicitly. A random harmonic series is one of the form $\sum_{n=1}^{\infty} \frac{S_n}{n}$, where s_n is a randomly generated sequence of $\pm 1's$ in which the values ± 1 are equally likely to occur. Use a random number generator to produce 1000 random $\pm 1's$ and plot the partial sums $S_N = \sum_{n=1}^{N} \frac{s_n}{n}$ of your random harmonic sequence for $N = 1$ to 1000. Compare to a plot of the first 1000 partial sums of the harmonic series.

Answer

Here is a typical result. The top curve consists of partial sums of the harmonic series. The bottom curve plots partial sums of a random harmonic series.



66) [Technology Required] Estimates of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ can be accelerated by writing its partial sums as $\sum_{n=1}^{N} \frac{1}{n^2} = \sum_{n=1}^{N} \frac{1}{n(n+1)} + \sum_{n=1}^{N} \frac{1}{n^2(n+1)}$ and recalling that $\sum_{n=1}^{N} \frac{1}{n(n+1)} = 1 - \frac{1}{N+1}$ converges to one as $N \to \infty$. Compare the estimate of $\pi^2/6$ using the sums $\sum_{n=1}^{1000} \frac{1}{n^2}$ with the estimate using $1 + \sum_{n=1}^{1000} \frac{1}{n^2(n+1)}$.

67) [Technology Required] The **Euler transform** rewrites $S = \sum_{n=0}^{\infty} (-1)^n b_n$ as $S = \sum_{n=0}^{\infty} (-1)^n 2^{-n-1} \sum_{m=0}^{n} \binom{n}{m} b_{n-m}$. For the alternating harmonic series, it takes the form $\ln(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \sum_{n=1}^{\infty} \frac{1}{n2^n}$. Compute partial sums of $\sum_{n=1}^{\infty} \frac{1}{n2^n}$ until they approximate $\ln(2)$ accurate to within 0.0001. How many terms are needed? Compare this answer to the number of terms of the alternating harmonic series are needed to estimate $\ln(2)$.

Answer

By the alternating series test, $|S_n - S| \le b_{n+1}$, so one needs 10^4 terms of the alternating harmonic series to estimate $\ln(2)$ to within 0.0001. The first 10 partial sums of the series $\sum_{n=1}^{\infty} \frac{1}{n2^n}$ are (up to four decimals)

$0.5000, 0.6250, 0.6667, 0.6823, 0.6885, 0.6911, 0.6923, 0.6928, 0.6930, 0.6931$ and the tenth partial sum is within 0.0001 of $\ln(2) = 0.6931\ldots$.

68) [Technology Required] In the text it was stated that a conditionally convergent series can be rearranged to converge to any number. Here is a slightly simpler, but similar, fact. If $a_n \ge 0$ is such that $a_n \to 0$ as $n \to \infty$ but $\sum_{n=1}^{\infty} a_n$ diverges, then, given any number A there is a sequence s_n of $\pm 1's$ such that $\sum_{n=1}^{\infty} a_n s_n \to A$. Show this for $A > 0$ as follows.

a. Recursively define s_n by $s_n = 1$ if $S_{n-1} = \sum_{k=1}^{n-1} a_k s_k < A$ and $s_n = -1$ otherwise.

b. Explain why eventually $S_n \ge A$, and for any m larger than this n, $A - a_m \le S_m \le A + a_m$.

c. Explain why this implies that $S_n \to A$ as $n \to \infty$.

This page titled 4.5E: Exercises is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Roy Simpson.

4.6: The Ratio and Root Tests

Learning Objectives

- Use the Ratio Test to determine the absolute convergence of a series.
- Use the Root Test to determine the absolute convergence of a series.
- Describe a strategy for testing the convergence of a given series.

In this section, we prove the last two series convergence tests: the Ratio Test and the Root Test. These tests are particularly nice because they do not require us to find a comparable series. The Ratio Test will be especially useful in discussing power series in the next chapter. Throughout this chapter, we have seen that no single convergence test works for all series. Therefore, at the end of this section, we discuss a strategy for choosing which convergence test to use for a given series.

Ratio Test

Consider a series $\sum_{n=1}^{\infty} a_n$. From our earlier discussion and examples, we know that $\lim_{n\to\infty} a_n = 0$ is not a sufficient condition for the series to converge. Not only do we need $a_n \to 0$, but we need $a_n \to 0$ quickly enough. For example, consider the series $\sum_{n=1}^{\infty} \frac{1}{n}$ and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. We know that $\frac{1}{n} \to 0$ and $\frac{1}{n^2} \to 0$. However, only the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. The series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges because the terms in the sequence $\{\frac{1}{n}\}$ do not approach zero fast enough as $n \to \infty$. Here, we introduce the **Ratio Test**, which provides a way of measuring how fast the terms of a series approach zero.

Theorem: Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be a series with nonzero terms. Let

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

i. If $0 \le \rho < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

ii. If $\rho > 1$ (or $\rho = \infty$), then $\sum_{n=1}^{\infty} a_n$ diverges.

iii. If $\rho = 1$, the test does not provide any information.

Proof

Let $\sum_{n=1}^{\infty} a_n$ be a series with nonzero terms.

We begin with the proof of part i. In this case, $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$. Since $0 \le \rho < 1$, there exists R such that $0 \le \rho < R < 1$. Let $\epsilon = R - \rho > 0$. By the definition of the limit of a sequence, there exists some integer N such that

$$\left|\left|\frac{a_{n+1}}{a_n}\right| - \rho\right| < \epsilon, \text{ for all } n \ge N.$$

This means

$$-\epsilon < \left|\frac{a_{n+1}}{a_n}\right| - \rho < \epsilon, \text{ for all } n \ge N$$

or, written differently,

$$\left|\frac{a_{n+1}}{a_n}\right| - \rho < \epsilon, \text{ for all } n \ge N.$$

Therefore,

$$\left|\frac{a_{n+1}}{a_n}\right| < \rho + \epsilon = R, \text{ for all } n \geq N.$$

Thus,

$$\begin{aligned}
|a_{N+1}| &< R|a_N| \\
|a_{N+2}| &< R|a_{N+1}| < R^2|a_N| \\
|a_{N+3}| &< R|a_{N+2}| < R^2|a_{N+1}| < R^3|a_N| \\
|a_{N+4}| &< R|a_{N+3}| < R^2|a_{N+2}| < R^3|a_{N+1}| < R^4|a_N| \\
&\vdots
\end{aligned}$$

Since $R < 1$, the geometric series

$$\sum_{n=1}^{\infty} R^n|a_N| = R|a_N| + R^2|a_N| + R^3|a_N| + \cdots$$

converges. Since we have shown that $|a_{N+k}| < R^k|a_N|$, we can apply the Series Comparison Test and conclude that the series

$$\sum_{n=1}^{\infty} |a_{N+n}| = |a_{N+1}| + |a_{N+2}| + |a_{N+3}| + |a_{N+4}| + \cdots$$

converges. Therefore, since

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{N} |a_n| + \sum_{n=N+1}^{\infty} |a_n|,$$

where $\sum_{n=1}^{N} |a_n|$ is a finite sum and $\sum_{n=N+1}^{\infty} |a_n|$ converges, we conclude that $\sum_{n=1}^{\infty} |a_n|$ converges.

For part ii,

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1.$$

Therefore, there must exist N such that

$$\left|\frac{a_{n+1}}{a_n}\right| > 1$$

for all $n \geq N$. For these same values of n, $|a_{n+1}| > |a_n|$. Since $\sum_{n=1}^{\infty} a_n$ is a series with nonzero terms, $|a_{n+1}| > |a_n| > 0$. Hence, the sequence, $\{|a_n|\}$ is a monotonically increasing sequence of positive terms for all $n \geq N$. Thus,

$$\lim_{n\to\infty} |a_n| \neq 0.$$

This result implies

$$\lim_{n\to\infty} a_n \neq 0.$$

Hence, the series $\sum_{n=1}^{\infty} a_n$ diverges by the Divergence Test.

For part iii, we show that the test does not provide any information if $\rho = 1$ by considering the p−series $\sum_{n=1}^{\infty} \frac{1}{n^p}$. For any real number p,

$$\rho = \lim_{n\to\infty} \frac{1/(n+1)^p}{1/n^p} = \lim_{n\to\infty} \frac{n^p}{(n+1)^p} = 1.$$

However, we know that if $p \leq 1$, the p−series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges, whereas $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$.

The Ratio Test is handy for series whose terms contain factorials or exponential, where the ratio of terms simplifies the expression. The Ratio Test is convenient because it does not require us to find a comparative series. The drawback is that the test sometimes does not provide any information regarding convergence.

Example 4.6.1: Using the Ratio Test

For each of the following series, use the Ratio Test to determine whether the series converges or diverges.

a. $\sum_{n=1}^{\infty} \frac{2^n}{n!}$

b. $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

c. $\sum_{n=1}^{\infty} \frac{(-1)^n (n!)^2}{(2n)!}$

Solutions

a. From the Ratio Test, we can see that

$$\rho = \lim_{n\to\infty} \frac{2^{n+1}/(n+1)!}{2^n/n!} = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n}.$$

Since $(n+1)! = (n+1)\cdot n!$,

$$\rho = \lim_{n\to\infty} \frac{2}{n+1} = 0.$$

Since $\rho < 1$, the series converges (absolutely).

b. We can see that

$$\rho = \lim_{n\to\infty} \frac{(n+1)^{n+1}/(n+1)!}{n^n/n!} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \lim_{n\to\infty} (\frac{n+1}{n})^n = \lim_{n\to\infty} (1+\frac{1}{n})^n = e.$$

Since $\rho > 1$, the series diverges.

c. Since

$$|\frac{(-1)^{n+1}((n+1)!)^2/(2(n+1))!}{(-1)^n (n!)^2/(2n)!}| = \frac{(n+1)!(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{n!n!} = \frac{(n+1)(n+1)}{(2n+2)(2n+1)}$$

we see that

$$\rho = \lim_{n\to\infty} \frac{(n+1)(n+1)}{(2n+2)(2n+1)} = \frac{1}{4}.$$

Since $\rho < 1$, the series is absolutely convergent.

Checkpoint 4.6.1

Use the Ratio Test to determine whether the series $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$ converges or diverges.

Answer

The series is absolutely convergent.

Root Test

The approach of the **Root Test** is similar to that of the Ratio Test. Consider a series $\sum_{n=1}^{\infty} a_n$ such that $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \rho$ for some real number ρ. Then for N sufficiently large, $|a_N| \approx \rho^N$. Therefore, we can approximate $\sum_{n=N}^{\infty} |a_n|$ by writing

$$|a_N| + |a_{N+1}| + |a_{N+2}| + \cdots \approx \rho^N + \rho^{N+1} + \rho^{N+2} + \cdots.$$

The expression on the right-hand side is a geometric series. As in the Ratio Test, the series $\sum_{n=1}^{\infty} a_n$ converges absolutely if $0 \le \rho < 1$ and the series diverges if $\rho \ge 1$. If $\rho = 1$, the test provides no information. For example, for any p-series, $\sum_{n=1}^{\infty} \frac{1}{n^p}$, we see that

$$\rho = \lim_{n\to\infty} \sqrt[n]{\left|\frac{1}{n^p}\right|} = \lim_{n\to\infty} \frac{1}{n^{p/n}}.$$

To evaluate this limit, we use the natural logarithm function. Doing so, we see that

$$\begin{aligned} \ln \rho &= \ln\left(\lim_{n\to\infty} \frac{1}{n^{p/n}}\right) \\ &= \lim_{n\to\infty} \ln\left(\frac{1}{n}\right)^{p/n} \\ &= \lim_{n\to\infty} \frac{p}{n} \cdot \ln\left(\frac{1}{n}\right) \\ &= \lim_{n\to\infty} \frac{p\ln(1/n)}{n}. \end{aligned}$$

Using l'Hospital's Rule, it follows that $\ln \rho = 0$, and therefore $\rho = 1$ for all p. However, we know that the p-series only converges if $p > 1$ and diverges if $p \le 1$.

Theorem: Root Test

Consider the series $\sum_{n=1}^{\infty} a_n$. Let

$$\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|}$$

.

i. If $0 \le \rho < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

ii. If $\rho > 1$ (or $\rho = \infty$), then $\sum_{n=1}^{\infty} a_n$ diverges.

iii. If $\rho = 1$, the test does not provide any information.

The Root Test is useful for series whose terms involve exponentials. In particular, for a series whose terms a_n satisfy $|a_n| = (b_n)^n$, then $\sqrt[n]{|a_n|} = b_n$ and we need only evaluate $\lim_{n\to\infty} b_n$.

Example 4.6.2: Using the Root Test

For each of the following series, use the Root Test to determine whether the series converges or diverges.

a. $\sum_{n=1}^{\infty} \frac{(n^2+3n)^n}{(4n^2+5)^n}$

b. $\sum_{n=1}^{\infty} \frac{n^n}{(\ln(n))^n}$

Solutions

a. To apply the Root Test, we compute

$$\rho = \lim_{n\to\infty} \sqrt[n]{\frac{(n^2+3n)^n}{(4n^2+5)^n}} = \lim_{n\to\infty} \frac{n^2+3n}{4n^2+5} = \frac{1}{4}.$$

Since $\rho < 1$, the series converges absolutely.

https://math.libretexts.org/@go/page/168447

b. We have

$$\rho = \lim_{n\to\infty} \sqrt[n]{\frac{n^n}{(\ln n)^n}} = \lim_{n\to\infty} \frac{n}{\ln n} = \infty \quad \text{by l'Hospital's Rule.}$$

Since $\rho = \infty$, the series diverges.

Checkpoint 4.6.2

Use the Root Test to determine whether the series $\sum_{n=1}^{\infty} 1/n^n$ converges or diverges.

Answer

The series converges absolutely.

Choosing a Convergence Test

At this point, we have a long list of convergence tests. However, not all tests can be used for all series. When given a series, we must determine which test is best. Here is a strategy for finding the best test to apply.

Problem-Solving Strategy: Choosing a Convergence Test for a Series

Consider a series $\sum_{n=1}^{\infty} a_n$. In the steps below, we outline a strategy for determining whether the series converges.

1. Is $\sum_{n=1}^{\infty} a_n$ a familiar series? For example, is it the harmonic series (which diverges) or the alternating harmonic series (which converges)? Is it a p−series or geometric series? If so, check the power p or the ratio r to determine if the series converges.
2. Is it an alternating series? Are we interested in absolute convergence or just convergence? If we are just interested in whether the series converges, apply the Alternating Series Test. If we are interested in absolute convergence, proceed to step **3**, considering the series of absolute values $\sum_{n=1}^{\infty} |a_n|$.
3. Is the series similar to a p−series or geometric series? If so, try the Series Comparison Test or Limit Comparison Test.
4. Do the terms in the series contain a factorial or power? If the terms are powers such that $a_n = (b_n)^n$, try the Root Test first. Otherwise, try the Ratio Test first.
5. Use the Divergence Test. If this test does not provide any information, try the Integral Test.

Example 4.6.3: Using Convergence Tests

For each of the following series, determine which convergence test is best and explain why. Then, determine if the series converges or diverges. If the series is alternating, determine whether it converges absolutely, converges conditionally, or diverges.

a. $\sum_{n=1}^{\infty} \frac{n^2+2n}{n^3+3n^2+1}$

b. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(3n+1)}{n!}$

c. $\sum_{n=1}^{\infty} \frac{e^n}{n^3}$

d. $\sum_{n=1}^{\infty} \frac{3^n}{(n+1)^n}$

Solutions

a. This is not a familiar series, and it is neither a p–series nor a geometric series. It is also not alternating; however, for large values of n, we approximate the series by the expression

$$\frac{n^2+2n}{n^3+3n^2+1} \approx \frac{n^2}{n^3} = \frac{1}{n}.$$

Therefore, it seems reasonable to apply the Series Comparison Test or Limit Comparison Test using the series $\sum_{n=1}^{\infty} 1/n$.

Using the Limit Comparison Test, we see that

$$\lim_{n\to\infty} \frac{(n^2+2n)/(n^3+3n^2+1)}{1/n} = \lim_{n\to\infty} \frac{n^3+2n^2}{n^3+3n^2+1} = 1.$$

Since the series $\sum_{n=1}^{\infty} 1/n$ diverges, this series diverges as well.

b. This series is not familiar; however, it is alternating. Since we are interested in absolute convergence, consider the series

$$\sum_{n=1}^{\infty} \frac{3n}{(n+1)!}.$$

The resulting series is not similar to a p-series or a geometric series. Still, since each term contains a factorial, we are inclined to apply the Ratio Test. We see that

$$\lim_{n\to\infty} \frac{(3(n+1))/(n+1)!}{(3n+1)/n!} = \lim_{n\to\infty} \frac{3n+3}{(n+1)!} \cdot \frac{n!}{3n+1} = \lim_{n\to\infty} \frac{3n+3}{(n+1)(3n+1)} = 0.$$

Therefore, this series converges, and we conclude that the original series converges absolutely (and, thus, converges).

c. The given series is not a familiar series, an alternating series, or a series with which there is an obvious series to compare to. Moreover, there is no factorial, and the power present is not ideal for the Root Test. Let's see about the Divergence Test.

$$\lim_{n\to\infty} \frac{e^n}{n^3} = \infty.$$

Therefore, by the Divergence Test, the series diverges.

d. This series is not a familiar series or an alternating one. There is no obvious series with which to compare this series; however, since each term is a power of n, we can apply the Root Test. Since

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{3}{n+1}\right)^n} = \lim_{n\to\infty} \frac{3}{n+1} = 0,$$

by the Root Test, we conclude that the series converges.

? Checkpoint 4.6.3

For the series $\sum_{n=1}^{\infty} \frac{2^n}{3^n+n}$, determine which convergence test is the best to use and explain why.

Answer

The Series Comparison Test because $\frac{2^n}{3^n+n} < \frac{2^n}{3^n}$ for all $n \in \mathbb{N}$. The Limit Comparison Test could also be used.

We summarize the convergence tests and when each can be applied in the table below. Note that while the Series Comparison Test, Limit Comparison Test, and Integral Test require the series $\sum_{n=1}^{\infty} a_n$ to have nonnegative terms, if $\sum_{n=1}^{\infty} a_n$ has negative terms, these tests can be applied to $\sum_{n=1}^{\infty} |a_n|$ to test for absolute convergence.

Summary of Convergence Tests

Series or Test	Conclusions	Comments
Divergence Test For any series $\sum_{n=1}^{\infty} a_n$, evaluate $\lim_{n\to\infty} a_n$.	• If $\lim_{n\to\infty} a_n = 0$, the test is inconclusive. • If $\lim_{n\to\infty} a_n \neq 0$, the series diverges.	This test cannot prove convergence of a series.
Geometric Series $\sum_{n=1}^{\infty} ar^{n-1}$	• If $\lvert r\rvert < 1$, the series converges to $a/(1-r)$. • If $\lvert r\rvert \geq 1$, the series diverges.	Any geometric series can be reindexed to be written in the form $a + ar + ar^2 + \cdots$, where a is the initial term and r is the common ratio.
p-Series $\sum_{n=1}^{\infty} \frac{1}{n^p}$	• If $p > 1$, the series converges. • If $p \leq 1$, the series diverges.	For $p = 1$, we have the harmonic series $\sum_{n=1}^{\infty} 1/n$.
Series Comparison Test For $\sum_{n=1}^{\infty} a_n$ with nonnegative terms, compare with a known series $\sum_{n=1}^{\infty} b_n$.	• If $a_n \leq b_n$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. • If $a_n \geq b_n$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.	Typically used for a series similar to a geometric or p-series. It can sometimes be difficult to find an appropriate series.
Limit Comparison Test For $\sum_{n=1}^{\infty} a_n$ with positive terms, compare with a series $\sum_{n=1}^{\infty} b_n$ by evaluating $L = \lim_{n\to\infty} \frac{a_n}{b_n}$.	• If L is a real number and $L \neq 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge. • If $L = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. • If $L = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.	Typically used for a series similar to a geometric or p-series. Often easier to apply than the Series Comparison Test.
Integral Test If there exists a positive, continuous, decreasing function f such that $a_n = f(n)$ for all $n \geq N$, evaluate $\int_N^{\infty} f(x)dx$.	• $\int_N^{\infty} f(x)dx$ and $\sum_{n=1}^{\infty} a_n$ both converge or both diverge.	Limited to those series for which the corresponding function f can be easily integrated.
Alternating Series Test $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ or $\sum_{n=1}^{\infty} (-1)^n b_n$	• If $b_{n+1} \leq b_n$ for all $n \geq 1$ and $b_n \to 0$, then the series converges.	Only applies to alternating series.
Ratio Test For any series $\sum_{n=1}^{\infty} a_n$ with nonzero terms, let $\rho = \lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert$	• If $0 \leq \rho < 1$, the series converges absolutely. • If $\rho > 1$ or $\rho = \infty$, the series diverges. • If $\rho = 1$, the test is inconclusive.	Often used for series involving factorials or exponentials.
Root Test For any series $\sum_{n=1}^{\infty} a_n$, let $\rho = \lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert}$.	• If $0 \leq \rho < 1$, the series converges absolutely. • If $\rho > 1$ or $\rho = \infty$, the series diverges. • If $\rho = 1$, the test is inconclusive.	Often used for series where $\lvert a_n\rvert = (b_n)^n$.

Series Converging to π and $1/\pi$

Dozens of series exist that converge to π or an algebraic expression containing π. Here, we look at several examples and compare their rates of convergence. By the rate of convergence, we mean the number of terms necessary for a partial sum to be within a certain amount of the actual value. The series representations of π in the first two examples can be explained using **Maclaurin series**, which are discussed in the next chapter. The third example relies on material beyond the scope of this text.

1. The series

$$\pi = 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \cdots$$

was discovered by Gregory and Leibniz in the late $1600s$. This result follows from the Maclaurin series for $f(x) = \tan^{-1} x$. We will discuss this series in the next chapter.

   a. Prove that this series converges.
   b. Evaluate the partial sums S_n for $n = 10, 20, 50, 100$.
   c. Use the remainder estimate for alternating series to get a bound on the error R_n.
   d. What is the smallest value of N that guarantees $|R_N| < 0.01$? Evaluate S_N.

2. The series

$$\pi = 6\sum_{n=0}^{\infty} \frac{(2n)!}{2^{4n+1}(n!)^2(2n+1)}$$
$$= 6\left(\frac{1}{2} + \frac{1}{2\cdot 3}\left(\frac{1}{2}\right)^3 + \frac{1\cdot 3}{2\cdot 4\cdot 5}\cdot\left(\frac{1}{2}\right)^5 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}\left(\frac{1}{2}\right)^7 + \cdots\right)$$

has been attributed to Newton in the late $1600s$. The proof of this result uses the Maclaurin series for $f(x) = \sin^{-1} x$.

   a. Prove that the series converges.
   b. Evaluate the partial sums S_n for $n = 5, 10, 20$.
   c. Compare S_n to π for $n = 5, 10, 20$ and discuss the number of correct decimal places.

3. The series

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801}\sum_{n=0}^{\infty} \frac{(4n)!(1103 + 26390n)}{(n!)^4 396^{4n}}$$

was discovered by **Ramanujan** in the early $1900s$. William Gosper, Jr. used this series to calculate π to an accuracy of more than 17 million digits in the mid-1980s. At the time, that was a world record. Since then, this series and others by Ramanujan have led mathematicians to find many other series representations for π and $1/\pi$.

   a. Prove that this series converges.
   b. Evaluate the first term in this series. Compare this number with the value of π from a calculating utility. To how many decimal places do these two numbers agree? What if we add the first two terms in the series?
   c. Investigate the life of Srinivasa Ramanujan (1887–1920) and write a brief summary. Ramanujan is one of the most fascinating stories in the history of mathematics. He was self-taught, with no formal training in mathematics. Yet, he contributed in highly original ways to many advanced areas of mathematics.

Key Concepts

- For the Ratio Test, we consider

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

If $\rho < 1$, the series $\sum_{n=1}^{\infty} a_n$ converges absolutely. The series diverges if $\rho > 1$. If $\rho = 1$, the test provides no information. This test is useful for series whose terms involve factorials.

- For the Root Test, we consider

$$\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|}.$$

If $\rho < 1$, the series $\sum_{n=1}^{\infty} a_n$ converges absolutely. The series diverges if $\rho > 1$. If $\rho = 1$, the test provides no information. The Root Test is useful for series whose terms involve powers.

- For a series that is similar to a geometric series or **p−series,** consider one of the Comparison Tests.

Glossary

Ratio Test

for a series $\sum_{n=1}^{\infty} a_n$ with nonzero terms, let $\rho = \lim_{n\to\infty} |a_{n+1}/a_n|$; if $0 \le \rho < 1$, the series converges absolutely; if $\rho > 1$, the series diverges; if $\rho = 1$, the test is inconclusive

Root Test

for a series $\sum_{n=1}^{\infty} a_n$, let $\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|}$; if $0 \le \rho < 1$, the series converges absolutely; if $\rho > 1$, the series diverges; if $\rho = 1$, the test is inconclusive

4.6E: Exercises

In exercises 1 - 11, use the Ratio Test to determine whether each series $\sum_{n=1}^{\infty} a_n$ converges or diverges. State if the Ratio Test is inconclusive.

1) $\sum_{n=1}^{\infty} \frac{1}{n!}$

Answer

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0$. Converges by the Ratio Test.

2) $\sum_{n=1}^{\infty} \frac{10^n}{n!}$

3) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

Answer

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{2}\left(\frac{n+1}{n}\right)^2 = \frac{1}{2} < 1$. Converges by the Ratio Test.

4) $\sum_{n=1}^{\infty} \frac{n^{10}}{2^n}$

5) $\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!}$

Answer

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{27} < 1$. Converges by the Ratio Test.

6) $\sum_{n=1}^{\infty} \frac{2^{3n}(n!)^3}{(3n)!}$

7) $\sum_{n=1}^{\infty} \frac{(2n)!}{n^{2n}}$

Answer

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{4}{e^2} < 1$. Converges by the Ratio Test.

8) $\sum_{n=1}^{\infty} \frac{(2n)!}{(2n)^n}$

9) $\sum_{n=1}^{\infty} \frac{n!}{(n/e)^n}$

Answer

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$. Ratio Test is inconclusive.

10) $\sum_{n=1}^{\infty} \frac{(2n)!}{(n/e)^{2n}}$

11) $\sum_{n=1}^{\infty} \frac{(2^n n!)^2}{(2n)^{2n}}$

Answer

$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \frac{1}{e^2} < 1$. Converges by the Ratio Test.

In exercises 12 - 21, use the Root Test to determine whether $\sum_{n=1}^{\infty} a_n$ converges, where a_n is as follows.

12) $a_k = \left(\frac{k-1}{2k+3}\right)^k$

13) $a_k = \left(\frac{2k^2-1}{k^2+3}\right)^k$

Answer

$\lim_{k\to\infty} (a_k)^{1/k} = 2 > 1$. Diverges by the Root Test.

14) $a_n = \frac{(\ln n)^{2n}}{n^n}$

15) $a_n = n/2^n$

Answer

$\lim_{n\to\infty} (a_n)^{1/n} = 1/2 < 1$. Converges by the Root Test.

16) $a_n = n/e^n$

17) $a_k = \frac{k^e}{e^k}$

Answer

$\lim_{k\to\infty} (a_k)^{1/k} = 1/e < 1$. Converges by the Root Test.

18) $a_k = \frac{\pi^k}{k^\pi}$

19) $a_n = \left(\frac{1}{e} + \frac{1}{n}\right)^n$

Answer

$\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \frac{1}{e} + \frac{1}{n} = \frac{1}{e} < 1$. Converges by the Root Test.

20) $a_k = \frac{1}{(1+\ln k)^k}$

21) $a_n = \frac{(\ln(1+\ln n))^n}{(\ln n)^n}$

Answer

$\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \frac{(\ln(1+\ln n))}{(\ln n)} = 0$ by L'Hôpital's rule. Converges by the Root Test.

In exercises 22 - 28, use either the Ratio Test or the Root Test as appropriate to determine whether the series $\sum_{k=1}^{\infty} a_k$ with given terms a_k converges, or state if the test is inconclusive.

22) $a_k = \frac{k!}{1\cdot 3\cdot 5\cdots(2k-1)}$

23) $a_k = \frac{2\cdot 4\cdot 6\cdots 2k}{(2k)!}$

Answer

$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{1}{2k+1} = 0$. Converges by the Ratio Test.

24) $a_k = \frac{1\cdot 4\cdot 7\cdots(3k-2)}{3^k k!}$

25) $a_n = \left(1 - \frac{1}{n}\right)^{n^2}$

Answer

$\lim_{n\to\infty}(a_n)^{1/n}=1/e$. Converges by the Root Test.

26) $a_k=\left(\frac{1}{k+1}+\frac{1}{k+2}+\cdots+\frac{1}{2k}\right)^k$ (Hint: Compare $a_k^{1/k}$ to $\int_k^{2k}\frac{dt}{t}$.)

27) $a_k=\left(\frac{1}{k+1}+\frac{1}{k+2}+\cdots+\frac{1}{3k}\right)^k$

Answer

$\lim_{k\to\infty}a_k^{1/k}=\ln(3)>1$. Diverges by the Root Test.

28) $a_n=\left(n^{1/n}-1\right)^n$

In exercises 29 - 30, use the Ratio Test to determine whether $\sum_{n=1}^{\infty}a_n$ converges, or state if the Ratio Test is inconclusive.

29) $\sum_{n=1}^{\infty}\frac{3^{n^2}}{2^{n^3}}$

Answer

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{3^{2n+1}}{2^{3n^2+3n+1}}=0$. Converges by the Ratio Test.

30) $\sum_{n=1}^{\infty}\frac{2^{n^2}}{n^n n!}$

In exercises 31, use the root and limit Comparison Tests to determine whether $\sum_{n=1}^{\infty}a_n$ converges.

31) $\sum_{n=1}^{\infty}\frac{1}{x_n^n}$ where $x_{n+1}=\frac{1}{2}x_n+\frac{1}{x_n}, x_1=1$ (Hint: Find limit of x_n.)

Answer

Converges by the Root Test and Limit Comparison Test since $\lim_{n\to\infty}x_n=\sqrt{2}$.

In exercises 32 - 43, use an appropriate test to determine whether the series converges.

32) $\sum_{n=1}^{\infty}\frac{n+1}{n^3+n^2+n+1}$

33) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(n+1)}{n^3+3n^2+3n+1}$

Answer

Converges absolutely by limit comparison with **p–series, $p=2$.**

34) $\sum_{n=1}^{\infty}\frac{(n+1)^2}{n^3+(1.1)^n}$

35) $\sum_{n=1}^{\infty}\frac{(n-1)^n}{(n+1)^n}$

Answer

$\lim_{n\to\infty}a_n=1/e^2\neq 0$. Series diverges by the Divergence Test.

36) $a_n=\left(1+\frac{1}{n^2}\right)^n$ (Hint: $\left(1+\frac{1}{n^2}\right)^{n^2}\approx e$.)

37) $a_k=1/2^{\sin^2 k}$

Answer

Terms do not tend to zero: $a_k \geq 1/2$, since $\sin^2 x \leq 1$.

38) $a_k = 2^{-\sin(1/k)}$

39) $a_n = 1/\binom{n+2}{n}$ where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Answer

$a_n = \dfrac{2}{(n+1)(n+2)}$, which converges by comparison with p**−series** for $p = 2$.

40) $a_k = 1/\binom{2k}{k}$

41) $a_k = 2^k/\binom{3k}{k}$

Answer

$a_k = \dfrac{2^k 1 \cdot 2 \cdots k}{(2k+1)(2k+2)\cdots 3k} \leq (2/3)^k$ converges by comparison with geometric series.

42) $a_k = \left(\dfrac{k}{k+\ln k}\right)^k$ $\left(\text{Hint: } a_k = \left(1+\dfrac{\ln k}{k}\right)^{-(k/\ln k)\ln k} \approx e^{-\ln k}.\right)$

43) $a_k = \left(\dfrac{k}{k+\ln k}\right)^{2k}$ $\left(\text{Hint: } a_k = \left(1+\dfrac{\ln k}{k}\right)^{-(k/\ln k)\ln k^2}.\right)$

Answer

$a_k \approx e^{-\ln k^2} = 1/k^2$. Series converges by limit comparison with p**−series**, $p = 2$.

The series in exercises 44 - 47 converge by the Ratio Test. Use summation by parts, $\sum_{k=1}^{n} a_k(b_{k+1} - b_k) = [a_{n+1}b_{n+1} - a_1 b_1] - \sum_{k=1}^{n} b_{k+1}(a_{k+1} - a_k)$**, to find the sum of the given series.**

44) $\sum_{k=1}^{\infty} \dfrac{k}{2^k}$ (Hint: Take $a_k = k$ and $b_k = 2^{1-k}$.)

45) $\sum_{k=1}^{\infty} \dfrac{k}{c^k}$, where $c > 1$ (Hint: Take $a_k = k$ and $b_k = c^{1-k}/(c-1)$.)

Answer

If $b_k = c^{1-k}/(c-1)$ and $a_k = k$, then $b_{k+1} - b_k = -c^{-k}$ and $\sum_{n=1}^{\infty} \dfrac{k}{c^k} = a_1 b_1 + \dfrac{1}{c-1}\sum_{k=1}^{\infty} c^{-k} = \dfrac{c}{(c-1)^2}$.

46) $\sum_{n=1}^{\infty} \dfrac{n^2}{2^n}$

47) $\sum_{n=1}^{\infty} \dfrac{(n+1)^2}{2^n}$

Answer

$6 + 4 + 1 = 11$

The kth **term of each of the following series has a factor** x^k**. Find the range of** x **for which the Ratio Test implies that the series converges.**

48) $\sum_{k=1}^{\infty} \dfrac{x^k}{k^2}$

49) $\sum_{k=1}^{\infty} \dfrac{x^{2k}}{k^2}$

Answer

$|x| \leq 1$

50) $\sum_{k=1}^{\infty} \frac{x^{2k}}{3^k}$

51) $\sum_{k=1}^{\infty} \frac{x^k}{k!}$

Answer

$|x| < \infty$

52) Does there exist a number p such that $\sum_{n=1}^{\infty} \frac{2^n}{n^p}$ converges?

53) Let $0 < r < 1$. For which real numbers p does $\sum_{n=1}^{\infty} n^p r^n$ converge?

Answer

All real numbers p by the Ratio Test.

54) Suppose that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = p$. For which values of p must $\sum_{n=1}^{\infty} 2^n a_n$ converge?

55) Suppose that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = p$. For which values of $r > 0$ is $\sum_{n=1}^{\infty} r^n a_n$ guaranteed to converge?

Answer

$r < 1/p$

56) Suppose that $\left|\frac{a_{n+1}}{a_n}\right| \le (n+1)^p$ for all $n = 1, 2, \ldots$ where p is a fixed real number. For which values of p is $\sum_{n=1}^{\infty} n! a_n$ guaranteed to converge?

57) For which values of $r > 0$, if any, does $\sum_{n=1}^{\infty} r^{\sqrt{n}}$ converge? $\left(\text{Hint: } sum_{n=1}^{\infty} a_n = \sum_{k=1}^{\infty} \sum_{n=k^2}^{(k+1)^2-1} a_n.\right)$

Answer

$0 < r < 1$. Note that the ratio and Root Tests are inconclusive. Using the hint, there are $2k$ terms $r^{\sqrt{n}}$ for $k^2 \le n < (k+1)^2$, and for $r < 1$ each term is at least r^k. Thus, $\sum_{n=1}^{\infty} r^{\sqrt{n}} = \sum_{k=1}^{\infty} \sum_{n=k^2}^{(k+1)^2-1} r^{\sqrt{n}} \ge \sum_{k=1}^{\infty} 2k r^k$, which converges by the Ratio Test for $r < 1$. For $r \ge 1$ the series diverges by the Divergence Test.

58) Suppose that $\left|\frac{a_{n+2}}{a_n}\right| \le r < 1$ for all n. Can you conclude that $\sum_{n=1}^{\infty} a_n$ converges?

59) Let $a_n = 2^{-[n/2]}$ where $[x]$ is the greatest integer less than or equal to x. Determine whether $\sum_{n=1}^{\infty} a_n$ converges and justify your answer.

Answer

One has $a_1 = 1, a_2 = a_3 = 1/2, \ldots a_{2n} = a_{2n+1} = 1/2^n$. The Ratio Test does not apply because $a_{n+1}/a_n = 1$ if n is even. However, $a_{n+2}/a_n = 1/2$, so the series converges according to the previous exercise. Of course, the series is just a duplicated geometric series.

The following *advanced exercises* use a generalized Ratio Test to determine convergence of some series that arise in particular applications when tests in this chapter, including the ratio and Root Test, are not powerful enough to determine their convergence. The test states that if $\lim_{n\to\infty} \frac{a_{2n}}{a_n} < 1/2$, then $\sum a_n$ converges, while if $\lim_{n\to\infty} \frac{a_{2n+1}}{a_n} > 1/2$, then $\sum a_n$ diverges.

60) Let $a_n = \frac{1}{4}\frac{3}{6}\frac{5}{8}\cdots\frac{2n-1}{2n+2} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n(n+1)!}$. Explain why the Ratio Test cannot determine convergence of $\sum_{n=1}^{\infty} a_n$. Use the fact that $1-1/(4k)$ is increasing k to estimate $\lim_{n\to\infty}\frac{a_{2n}}{a_n}$.

61) Let $a_n = \frac{1}{1+x}\frac{2}{2+x}\cdots\frac{n}{n+x}\frac{1}{n} = \frac{(n-1)!}{(1+x)(2+x)\cdots(n+x).}$ Show that $a_{2n}/a_n \le e^{-x/2}/2$. For which $x>0$ does the generalized Ratio Test imply convergence of $\sum_{n=1}^{\infty} a_n$? (Hint: Write $2a_{2n}/a_n$ as a product of n factors each smaller than $1/(1+x/(2n))$.)

Answer

$a_{2n}/a_n = \frac{1}{2}\cdot\frac{n+1}{n+1+x}\frac{n+2}{n+2+x}\cdots\frac{2n}{2n+x}$. The inverse of the kth factor is $(n+k+x)/(n+k) > 1+x/(2n)$ so the product is less than $(1+x/(2n))^{-n} \approx e^{-x/2}$. Thus for $x>0, \frac{a_{2n}}{a_n} \le \frac{1}{2}e^{-x/2}$. The series converges for $x>0$.

62) Let $a_n = \frac{n^{\ln n}}{(\ln n)^n}$. Show that $\frac{a_{2n}}{a_n} \to 0$ as $n\to\infty$.

This page titled 4.6E: Exercises is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Roy Simpson.

4.7: Chapter 4 Review Exercises

In exercises 1 - 2, evaluate the integrals, if possible.

1) $\int_1^\infty \frac{1}{x^n}\,dx$, for what values of n does this integral converge or diverge?

2) $\int_1^\infty \frac{e^{-x}}{x}\,dx$

Answer

approximately 0.2194

In exercises 3 - 4, consider the gamma function given by $\gamma(a) = \int_0^\infty e^{-y}y^{a-1}\,dy$.

3) Show that $\gamma(a) = (a-1)\gamma(a-1)$.

4) Extend to show that $\gamma(a) = (a-1)!$, assuming a is a positive integer.

***True or False?* Justify your answer with a proof or a counterexample.**

5) If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

Answer

false

6) If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

7) If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Answer

true

8) If $\sum_{n=1}^{\infty} 2^n a_n$ converges, then $\sum_{n=1}^{\infty} (-2)^n a_n$ converges.

Is the sequence bounded, monotone, and convergent or divergent? If it is convergent, find the limit.

9) $a_n = \frac{3+n^2}{1-n}$

Answer

unbounded, not monotone, divergent

10) $a_n = \ln\left(\frac{1}{n}\right)$

11) $a_n = \frac{\ln(n+1)}{\sqrt{n+1}}$

Answer

bounded, monotone, convergent, **0**

12) $a_n = \frac{2^{n+1}}{5^n}$

13) $a_n = \frac{\ln(\cos n)}{n}$

Answer

unbounded, not monotone, divergent

Is the series convergent or divergent?

14) $\sum_{n=1}^{\infty} \frac{1}{n^2+5n+4}$

15) $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$

Answer

diverges

16) $\sum_{n=1}^{\infty} \frac{2^n}{n^4}$

17) $\sum_{n=1}^{\infty} \frac{e^n}{n!}$

Answer

converges

18) $\sum_{n=1}^{\infty} n^{-(n+1/n)}$

Is the series convergent or divergent? If convergent, is it absolutely convergent?

19) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

Answer

converges, but not absolutely

20) $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{3^n}$

21) $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{n^n}$

Answer

converges absolutely

22) $\sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{2}\right)$

23) $\sum_{n=1}^{\infty} \cos(\pi n) e^{-n}$

Answer

converges absolutely

Evaluate.

24) $\sum_{n=1}^{\infty} \frac{2^{n+4}}{7^n}$

25) $\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$

Answer

$\frac{1}{2}$

26) A legend from India tells that a mathematician invented chess for a king. The king enjoyed the game so much he allowed the mathematician to demand any payment. The mathematician asked for one grain of rice for the first square on the chessboard, two

grains of rice for the second square on the chessboard, and so on. Find an exact expression for the total payment (in grains of rice) requested by the mathematician. Assuming there are $30,000$ grains of rice in 1 pound, and 2000 pounds in 1 ton, how many tons of rice did the mathematician attempt to receive?

The following problems consider a simple population model of the housefly, which can be exhibited by the recursive formula $x_{n+1} = bx_n$, where x_n is the population of houseflies at generation n, and b is the average number of offspring per housefly who survive to the next generation. Assume a starting population x_0.

27) Find $\lim_{n\to\infty} x_n$ if $b > 1,\ b < 1$, and $b = 1$.

Answer

$\infty,\ 0,\ x_0$

28) Find an expression for $S_n = \sum_{i=0}^{n} x_i$ in terms of b and x_0. What does it physically represent?

29) If $b = \frac{3}{4}$ and $x_0 = 100$, find S_{10} and $\lim_{n\to\infty} S_n$

Answer

$S_{10} \approx 383, \quad \lim_{n\to\infty} S_n = 400$

30) For what values of b will the series converge and diverge? What does the series converge to?

CHAPTER OVERVIEW

5: Power Series

A power series (in one variable) is an infinite series. Any polynomial can be easily expressed as a power series around any center c, although most of the coefficients will be zero since a power series has infinitely many terms by definition. One can view power series as being like "polynomials of infinite degree," although power series are not polynomials. The content in this Textmap's chapter is complemented by Guichard's Calculus Textmap.

5.1: Power Series and Functions

5.1E: Exercises

5.2: Properties of Power Series

5.2E: Exercises

5.3: Taylor and Maclaurin Series

5.3E: Exercises

5.4: Working with Taylor Series

5.4E: Exercises

5.5: Chapter 5 Review Exercises

This page titled 5: Power Series is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Roy Simpson.

5.1: Power Series and Functions

Learning Objectives

- Identify a power series and provide examples of them.
- Determine the radius of convergence and interval of convergence of a power series.
- Use a power series to represent a function.

A power series is a series with terms involving a variable. More specifically, if the variable is x, then all the terms of the series involve powers of x. As a result, a power series can be thought of as an infinite polynomial. Power series are used to represent common functions and also to define new functions. In this section, we define power series and show how to determine when a power series converges and when it diverges. We also show how to represent certain functions using power series.

Form of a Power Series

Definition: Power Series

A series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

is called a **power series centered at $x = 0$**. A series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots$$

is a **power series centered at $x = a$**.

To make this definition precise, we stipulate that $x^0 = 1$ and $(x-a)^0 = 1$ even when $x = 0$ and $x = a$, respectively.[1]

The series

$$1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n,$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$$

and

$$\sum_{n=0}^{\infty} n! x^n = 1 + x + 2! x^2 + 3! x^3 + \cdots$$

are all examples of power series centered at $x = 0$. The first of these is a geometric series with common ratio $r = |x|$, and we know that it converges if $|x| < 1$ and diverges if $|x| \geq 1$.

The series

$$\sum_{n=0}^{\infty} \frac{(x-2)^n}{(n+1)3^n} = 1 + \frac{x-2}{2 \cdot 3} + \frac{(x-2)^2}{3 \cdot 3^2} + \frac{(x-2)^3}{4 \cdot 3^3} + \cdots$$

is a power series centered at $x = 2$.

Convergence of a Power Series

Since the terms in a power series involve a variable x, the series may converge for certain values of x and diverge for other values of x. For a power series centered at $x = a$, the value of the series at $x = a$ is given by c_0. Therefore, *a power series always converges at its center*. Some power series converge only at that value of x. Most power series, however, converge for more than one value of

x. In that case, the power series either converges for all real numbers x or converges for all x in a finite interval. For example, the geometric series $\sum_{n=0}^{\infty} x^n$ converges for all x in the interval $(-1,1)$, but diverges for all x outside that interval. We now summarize these three possibilities for a general power series.

Theorem: Convergence of a Power Series

The power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ satisfies exactly one of the following properties:

i. The series converges at $x=a$ and diverges for all $x \neq a$.
ii. The series converges for all real numbers x.
iii. There exists a real number $R>0$ such that the series converges if $|x-a|<R$ and diverges if $|x-a|>R$. At the values x where $|x-a|=R$, the series could converge or diverge.

Proof

Suppose that the power series is centered at $a=0$.[2] We must first prove the following fact:

If there exists a real number $d \neq 0$ such that $\sum_{n=0}^{\infty} c_n d^n$ converges, then the series $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely for all x such that $|x|<|d|$.

Suppose, then, that $\sum_{n=0}^{\infty} c_n d^n$ converges. Then the nth term $c_n d^n \to 0$ as $n \to \infty$. Therefore, there exists an integer N such that $|c_n d^n| \leq 1$ for all $n \geq N$. Writing

$$|c_n x^n| = |c_n d^n| \left|\frac{x}{d}\right|^n,$$

we conclude that, for all $n \geq N$,

$$|c_n x^n| \leq \left|\frac{x}{d}\right|^n.$$

The series

$$\sum_{n=N}^{\infty} \left|\frac{x}{d}\right|^n$$

is a geometric series that converges if $\left|\frac{x}{d}\right|<1$. Therefore, by the Series Comparison Test, we conclude that $\sum_{n=N}^{\infty} c_n x^n$ also converges for $|x|<|d|$. Since we can add a finite number of terms to a convergent series, we conclude that $\sum_{n=0}^{\infty} c_n x^n$ converges for $|x|<|d|$.

With this result, we can now prove the theorem. Consider the series

$$\sum_{n=0}^{\infty} a_n x^n$$

and let S be the set of real numbers for which the series converges. Suppose that the set contains only the center. That is, $S=\{0\}$. Then the series falls under case i.

Suppose that the set S is the set of all real numbers. Then the series falls under case ii.

Suppose that $S \neq 0$ and S is not the set of real numbers. Then there exists a real number $x^* \neq 0$ such that the series does not converge. Thus, the series cannot converge for any x such that $|x|>|x^*|$. Therefore, the set S must be a bounded set, meaning it must have a smallest upper bound.[3] Call that smallest upper bound R. Since $S \neq \{0\}$, the number $R>0$. Therefore, the series converges for all x such that $|x|<R$, and the series falls into case iii.

If a series $\sum_{n=0}^{\infty} c_n(x-a)^n$ falls into case iii., then the series converges for all x such that $|x-a| < R$ for some $R > 0$, and diverges for all x such that $|x-a| > R$. The series may converge or diverge at the values x where $|x-a| = R$. The set of values x for which the series $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges is known as the **interval of convergence**. Since the series diverges for all values x where $|x-a| > R$, the length of the interval of convergence is $2R$, and therefore, the radius of the interval is R. The value R is called the **radius of convergence**.

For example, since the series $\sum_{n=0}^{\infty} x^n$ converges for all values x in the interval $(-1,1)$ and diverges for all values x such that $|x| \geq 1$, the interval of convergence of this series is $(-1,1)$. Since the length of the interval is 2, the radius of convergence is 1.

We formalize these statements in the following definition.

Definition: Radius of Convergence

Consider the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$. The set of real numbers x where the series converges is called the **interval of convergence**. If there exists a real number $R > 0$ such that the series converges for $|x-a| < R$ and diverges for $|x-a| > R$, then R is called the **radius of convergence**. If the series converges only at $x = a$, we say the radius of convergence is $R = 0$. If the series converges for all real numbers x, we say the radius of convergence is $R = \infty$ (see Figure 5.1.1).

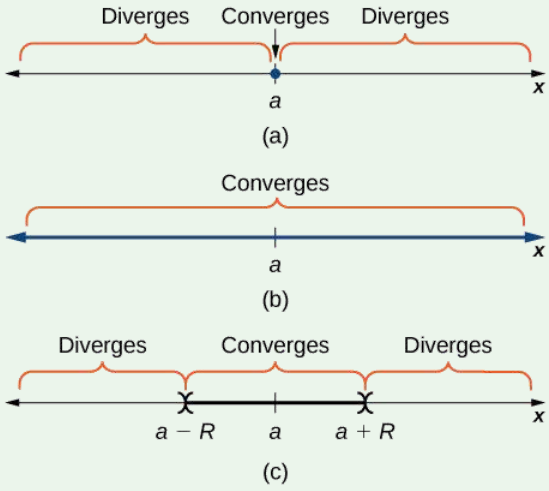


Figure 5.1.1: For a series $\sum_{n=0}^{\infty} c_n(x-a)^n$

Figure 5.1.1A: A radius of convergence of $R = 0$

Figure 5.1.1B: A radius of convergence of $R = \infty$

Figure 5.1.1C: A radius of convergence of R, where the series may or may not converge at the endpoints $x = a + R$ and $x = a - R$.

To determine the interval of convergence for a power series, we typically apply the Ratio Test. In Example 5.1.1, we show the three different possibilities illustrated in Figure 5.1.1.

Example 5.1.1: Finding the Interval and Radius of Convergence

For each of the following series, find the interval and radius of convergence.

a. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

b. $\sum_{n=0}^{\infty} n! x^n$

c. $\sum_{n=0}^{\infty} \frac{(x-2)^n}{(n+1)3^n}$

Solutions

a. To check for convergence, apply the Ratio Test. We have

$$\begin{aligned}\rho &= \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right| \\ &= \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)\cdot n!}\cdot\frac{n!}{x^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{x}{n+1}\right| \\ &= |x|\lim_{n\to\infty}\frac{1}{n+1} \\ &= 0 \\ &< 1\end{aligned}$$

for all values of x. Therefore, the series converges for all real numbers x. The interval of convergence is $(-\infty,\infty)$ and the radius of convergence is $R=\infty$.

b. Apply the Ratio Test. For $x\neq 0$, we see that

$$\begin{aligned}\rho &= \lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| \\ &= \lim_{n\to\infty}|(n+1)x| \\ &= |x|\lim_{n\to\infty}(n+1) \\ &= \infty.\end{aligned}$$

Therefore, the series diverges for all $x\neq 0$. Since the series is centered at $x=0$, it must converge there, so it converges only for $x=0$. The interval of convergence is the single value $x=0$ and the radius of convergence is $R=0$.

c. In order to apply the Ratio Test, consider

$$\begin{aligned}\rho &= \lim_{n\to\infty}\left|\frac{\frac{(x-2)^{n+1}}{(n+2)3^{n+1}}}{\frac{(x-2)^n}{(n+1)3^n}}\right| \\ &= \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(n+2)3^{n+1}}\cdot\frac{(n+1)3^n}{(x-2)^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{(x-2)(n+1)}{3(n+2)}\right| \\ &= \frac{|x-2|}{3}.\end{aligned}$$

The ratio $\rho<1$ if $|x-2|<3$. Since $|x-2|<3$ implies that $-3<x-2<3$, the series converges absolutely if $-1<x<5$. The ratio $\rho>1$ if $|x-2|>3$. Therefore, the series diverges if $x<-1$ or $x>5$. The Ratio Test is inconclusive if $\rho=1$. The ratio $\rho=1$ if and only if $x=-1$ or $x=5$. We need to test these values of x separately. For $x=-1$, the series is given by

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots.$$

Since this is the Alternating Harmonic Series, it converges. Thus, the series converges at $x=-1$. For $x=5$, the series is given by

$$\sum_{n=0}^{\infty} \frac{1}{n+1} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots.$$

This is the Harmonic Series, which is divergent. Therefore, the power series diverges at $x = 5$. We conclude that the interval of convergence is $[-1, 5)$ and the radius of convergence is $R = 3$.

? Checkpoint 5.1.1

Find the interval and radius of convergence for the series

$$\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}.$$

Answer

The interval of convergence is $[-1, 1)$. The radius of convergence is $R = 1$.

Representing Functions as Power Series

Being able to represent a function by an "infinite polynomial" is a powerful tool. Polynomial functions are the most manageable functions to analyze since they only involve the basic arithmetic operations of addition, subtraction, multiplication, and division. If we can represent a complicated function by an infinite polynomial, we can use the polynomial representation to differentiate or integrate it. In addition, we can use a truncated version of the polynomial expression to approximate values of the function. So, the question is:

When can we represent a function by a power series?

Consider again the geometric series

$$1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n. \tag{5.1.1}$$

Recall that the geometric series

$$a + ar + ar^2 + ar^3 + \cdots$$

converges if and only if $|r| < 1$. In that case, it converges to $\frac{a}{1-r}$. Therefore, if $|x| < 1$, the series in Equation 5.1.1 converges to $\frac{1}{1-x}$ and we write

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \text{ for } |x| < 1.$$

As a result, we can represent the function $f(x) = \frac{1}{1-x}$ by the power series

$$1 + x + x^2 + x^3 + \cdots \text{ when } |x| < 1.$$

We now show graphically how this series provides a representation for the function $f(x) = \frac{1}{1-x}$ by comparing the graph of f with the graphs of several of the partial sums of this infinite series.

✓ Example 5.1.2: Graphing a Function and Partial Sums of its Power Series

Sketch a graph of $f(x) = \frac{1}{1-x}$ and the graphs of the corresponding partial sums $S_N(x) = \sum_{n=0}^{N} x^n$ for $N = 2, 4, 6$ on the interval $(-1, 1)$. Comment on the approximation S_N as N increases.

Solution

From the graph in Figure 5.1.2, you see that as N increases, S_N becomes a better approximation for $f(x) = \frac{1}{1-x}$ for x in the interval $(-1, 1)$.

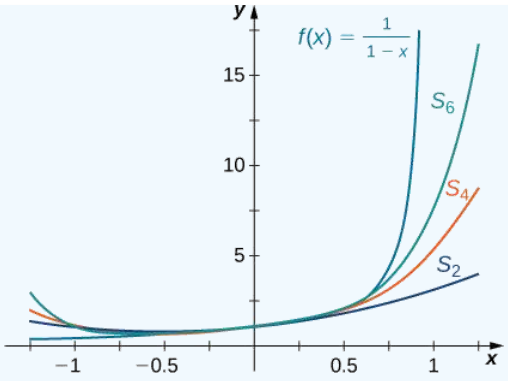
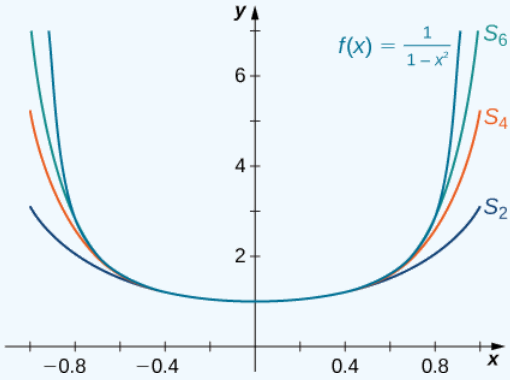


Figure 5.1.2: The graph shows a function and three approximations by partial sums of a power series.

Checkpoint 5.1.2

Sketch a graph of $f(x) = \frac{1}{1-x^2}$ and the corresponding partial sums $S_N(x) = \sum_{n=0}^{N} x^{2n}$ for $N = 2, 4, 6$ on the interval $(-1, 1)$.

Answer



Next, we consider functions involving an expression similar to the sum of a geometric series and show how to represent these functions using power series.

Example 5.1.3: Representing a Function with a Power Series

Use a power series to represent each of the following functions f. Find the interval of convergence.

a. $f(x) = \frac{1}{1+x^3}$
b. $f(x) = \frac{x^2}{4-x^2}$

Solutions

a. You should recognize this function f as the sum of a geometric series, because

$$\frac{1}{1+x^3} = \frac{1}{1-(-x^3)}.$$

Using the fact that, for $|r| < 1$, $\frac{a}{1-r}$ is the sum of the geometric series

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots,$$

we see that, for $|-x^3| < 1$,

$$\begin{aligned}\frac{1}{1+x^3} &= \frac{1}{1-(-x^3)} \\ &= \sum_{n=0}^{\infty}(-x^3)^n \\ &= 1-x^3+x^6-x^9+\cdots.\end{aligned}$$

Since this series converges if and only if $|-x^3|<1$, the interval of convergence is $(-1,1)$, and we have

$$\frac{1}{1+x^3}=1-x^3+x^6-x^9+\cdots \text{ for } |x|<1.$$

b. This function is not in the exact form of a sum of a geometric series. However, we can relate f to a geometric series with a little algebraic manipulation. By factoring 4 out of the two terms in the denominator, we obtain

$$\begin{aligned}\frac{x^2}{4-x^2} &= \frac{x^2}{4(\frac{1-x^2}{4})} \\ &= \frac{x^2}{4(1-(\frac{x}{2})^2)}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\frac{x^2}{4-x^2} &= \frac{x^2}{4(1-(\frac{x}{2})^2)} \\ &= \frac{\frac{x^2}{4}}{1-(\frac{x}{2})^2} \\ &= \sum_{n=0}^{\infty}\frac{x^2}{4}\left(\frac{x}{2}\right)^{2n}.\end{aligned}$$

The series converges as long as $|(\frac{x}{2})^2|<1$ (note that when $|(\frac{x}{2})^2|=1$ the series does not converge). Solving this inequality, we conclude that the interval of convergence is $(-2,2)$ and

$$\begin{aligned}\frac{x^2}{4-x^2} &= \sum_{n=0}^{\infty}\frac{x^{2n+2}}{4^{n+1}} \\ &= \frac{x^2}{4}+\frac{x^4}{4^2}+\frac{x^6}{4^3}+\cdots\end{aligned}$$

for $|x|<2$.

? Checkpoint 5.1.3

Represent the function $f(x)=\frac{x^3}{2-x}$ using a power series and find the interval of convergence.

Answer

$\sum_{n=0}^{\infty}\frac{x^{n+3}}{2^{n+1}}$ with interval of convergence $(-2,2)$

In the remaining sections of this chapter, we will show ways of deriving power series representations for many other functions. We also show how we can use these representations to evaluate, differentiate, and integrate various functions.

Footnotes

[1] It is *incredibly* important to understand that 0^0 is still considered indeterminate for all other purposes within Mathematics. Defining $x^0=1$ even when $x=0$ is a useful but unique convention for power series.

[2] For a series centered at a value of a other than zero, the result follows by letting $y=x-a$ and considering the series

$$\sum_{n=0}^{\infty} c_n y^n.$$

[3] This fact follows from the **Least Upper Bound Property** for the real numbers, which is beyond the scope of this text and is covered in real analysis courses.

Key Concepts

- For a power series centered at $x = a$, one of the following three properties hold:
  - i. The power series converges only at $x = a$. In this case, we say that the radius of convergence is $R = 0$.
  - ii. The power series converges for all real numbers x. In this case, we say that the radius of convergence is $R = \infty$.
  - iii. There is a real number R such that the series converges for $|x - a| < R$ and diverges for $|x - a| > R$. In this case, the radius of convergence is R.
- If a power series converges on a finite interval, the series may or may not converge at the endpoints.
- The Ratio Test may often be used to determine the radius of convergence.
- The geometric series $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x| < 1$ allows us to represent certain functions using geometric series.

Key Equations

- **Power series centered at $x = 0$**

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots n$$

- **Power series centered at $x = a$**

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \ldots$$

Glossary

interval of convergence

the set of real numbers x for which a power series converges

power series

a series of the form $\sum_{n=0}^{\infty} c_n x^n$ is a power series centered at $x = 0$; a series of the form $\sum_{n=0}^{\infty} c_n (x-a)^n$ is a power series centered at $x = a$

radius of convergence

if there exists a real number $R > 0$ such that a power series centered at $x = a$ converges for $|x - a| < R$ and diverges for $|x - a| > R$, then R is the radius of convergence; if the power series only converges at $x = a$, the radius of convergence is $R = 0$; if the power series converges for all real numbers x, the radius of convergence is $R = \infty$

5.1E: Exercises

In exercises 1 - 4, state whether each statement is true, or give an example to show that it is false.

1) If $\sum_{n=1}^{\infty} a_n x^n$ converges, then $a_n x^n \to 0$ as $n \to \infty$.

Answer

True. If a series converges then its terms tend to zero.

2) $\sum_{n=1}^{\infty} a_n x^n$ converges at $x = 0$ for any real numbers a_n.

3) Given any sequence a_n, there is always some $R > 0$, possibly very small, such that $\sum_{n=1}^{\infty} a_n x^n$ converges on $(-R, R)$.

Answer

False. It would imply that $a_n x^n \to 0$ for $|x| < R$. If $a_n = n^n$, then $a_n x^n = (nx)^n$ does not tend to zero for any $x \neq 0$.

4) If $\sum_{n=1}^{\infty} a_n x^n$ has radius of convergence $R > 0$ and if $|b_n| \leq |a_n|$ for all n, then the radius of convergence of $\sum_{n=1}^{\infty} b_n x^n$ is greater than or equal to R.

5) Suppose that $\sum_{n=0}^{\infty} a_n (x-3)^n$ converges at $x = 6$. At which of the following points must the series also converge? Use the fact that if $\sum a_n (x-c)^n$ converges at x, then it converges at any point closer to c than x.

a. $x = 1$

b. $x = 2$

c. $x = 3$

d. $x = 0$

e. $x = 5.99$

f. $x = 0.000001$

Answer

It must converge on $(0, 6]$ and hence at: a. $x = 1$; b. $x = 2$; c. $x = 3$; d. $x = 0$; e. $x = 5.99$; and f. $x = 0.000001$.

6) Suppose that $\sum_{n=0}^{\infty} a_n (x+1)^n$ converges at $x = -2$. At which of the following points must the series also converge? Use the fact that if $\sum a_n (x-c)^n$ converges at x, then it converges at any point closer to c than x.

a. $x = 2$

b. $x = -1$

c. $x = -3$

d. $x = 0$

e. $x = 0.99$

f. $x = 0.000001$

In the following exercises, suppose that $\left|\frac{a_{n+1}}{a_n}\right| \to 1$ as $n \to \infty$. Find the radius of convergence for each series.

7) $\sum_{n=0}^{\infty} a_n 2^n x^n$

Answer

$$\left|\frac{a_{n+1}2^{n+1}x^{n+1}}{a_n 2^n x^n}\right| = 2|x|\left|\frac{a_{n+1}}{a_n}\right| \to 2|x| \text{ so } R = \tfrac{1}{2}$$

8) $\sum_{n=0}^{\infty} \frac{a_n x^n}{2^n}$

9) $\sum_{n=0}^{\infty} \frac{a_n \pi^n x^n}{e^n}$

Answer

$$\left|\frac{a_{n+1}(\frac{\pi}{e})^{n+1}x^{n+1}}{a_n(\frac{\pi}{e})^n x^n}\right| = \frac{\pi|x|}{e}\left|\frac{a_{n+1}}{a_n}\right| \to \frac{\pi|x|}{e} \text{ so } R = \tfrac{e}{\pi}$$

10) $\sum_{n=0}^{\infty} \frac{a_n(-1)^n x^n}{10^n}$

11) $\sum_{n=0}^{\infty} a_n(-1)^n x^{2n}$

Answer

$$\left|\frac{a_{n+1}(-1)^{n+1}x^{2n+2}}{a_n(-1)^n x^{2n}}\right| = |x^2|\left|\frac{a_{n+1}}{a_n}\right| \to |x^2| \text{ so } R = 1$$

12) $\sum_{n=0}^{\infty} a_n(-4)^n x^{2n}$

In exercises 13 - 22, find the radius of convergence R and interval of convergence for $\sum a_n x^n$ with the given coefficients a_n.

13) $\sum_{n=1}^{\infty} \frac{(2x)^n}{n}$

Answer

$a_n = \frac{2^n}{n}$ so $\frac{a_{n+1}x}{a_n} \to 2x$. so $R = \frac{1}{2}$. When $x = \frac{1}{2}$ the series is harmonic and diverges. When $x = -\frac{1}{2}$ the series is alternating harmonic and converges. The interval of convergence is $I = \left[-\frac{1}{2}, \frac{1}{2}\right)$.

14) $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{\sqrt{n}}$

15) $\sum_{n=1}^{\infty} \frac{nx^n}{2^n}$

Answer

$a_n = \frac{n}{2^n}$ so $\frac{a_{n+1}x}{a_n} \to \frac{x}{2}$ so $R = 2$. When $x = \pm 2$ the series diverges by the Divergence Test. The interval of convergence is $I = (-2, 2)$.

16) $\sum_{n=1}^{\infty} \frac{nx^n}{e^n}$

17) $\sum_{n=1}^{\infty} \frac{n^2 x^n}{2^n}$

Answer

$a_n = \frac{n^2}{2^n}$ so $R = 2$. When $x = \pm 2$ the series diverges by the Divergence Test. The interval of convergence is $I = (-2, 2)$.

18) $\sum_{k=1}^{\infty} \frac{k^e x^k}{e^k}$

19) $\sum_{k=1}^{\infty} \frac{\pi^k x^k}{k^\pi}$

Answer

$a_k = \frac{\pi^k}{k^\pi}$ so $R = \frac{1}{\pi}$. When $x = \pm\frac{1}{\pi}$ the series is an absolutely convergent p-series. The interval of convergence is $I = \left[-\frac{1}{\pi}, \frac{1}{\pi}\right]$.

20) $\sum_{n=1}^{\infty} \frac{x^n}{n!}$

21) $\sum_{n=1}^{\infty} \frac{10^n x^n}{n!}$

Answer

$a_n = \frac{10^n}{n!}, \frac{a_{n+1}x}{a_n} = \frac{10x}{n+1} \to 0 < 1$ so the series converges for all x by the Ratio Test and $I = (-\infty, \infty)$.

22) $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{\ln(2n)}$

In exercises 23 - 28, find the radius of convergence of each series.

23) $\sum_{k=1}^{\infty} \frac{(k!)^2 x^k}{(2k)!}$

Answer

$a_k = \frac{(k!)^2}{(2k)!}$ so $\frac{a_{k+1}}{a_k} = \frac{(k+1)^2}{(2k+2)(2k+1)} \to \frac{1}{4}$ so $R = 4$

24) $\sum_{n=1}^{\infty} \frac{(2n)! x^n}{n^{2n}}$

25) $\sum_{k=1}^{\infty} \frac{k!}{1 \cdot 3 \cdot 5 \cdots (2k-1)} x^k$

Answer

$a_k = \frac{k!}{1 \cdot 3 \cdot 5 \cdots (2k-1)}$ so $\frac{a_{k+1}}{a_k} = \frac{k+1}{2k+1} \to \frac{1}{2}$ so $R = 2$

26) $\sum_{k=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \cdots 2k}{(2k)!} x^k$

27) $\sum_{n=1}^{\infty} \frac{x^n}{\binom{2n}{n}}$ where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Answer

$a_n = \frac{1}{\binom{2n}{n}}$ so $\frac{a_{n+1}}{a_n} = \frac{\left((n+1)!\right)^2}{(2n+2)!} \frac{2n!}{(n!)^2} = \frac{(n+1)^2}{(2n+2)(2n+1)} \to \frac{1}{4}$ so $R = 4$

28) $\sum_{n=1}^{\infty} \sin^2 n x^n$

In exercises 29 - 32, use the Ratio Test to determine the radius of convergence of each series.

29) $\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!} x^n$

Answer

$\frac{a_{n+1}}{a_n} = \frac{(n+1)^3}{(3n+3)(3n+2)(3n+1)} \to \frac{1}{27}$ so $R = 27$

30) $\sum_{n=1}^{\infty} \frac{2^{3n}(n!)^3}{(3n)!} x^n$

31) $\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$

Answer

$a_n = \frac{n!}{n^n}$ so $\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{n!} \frac{n^n}{(n+1)^{n+1}} = (\frac{n}{n+1})^n \to \frac{1}{e}$ so $R = e$

32) $\sum_{n=1}^{\infty} \frac{(2n)!}{n^{2n}} x^n$

In the following exercises, given that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ with convergence in $(-1,1)$, find the power series for each function with the given center a, and identify its interval of convergence.

33) $f(x) = \frac{1}{x}; a = 1$ (Hint: $\frac{1}{x} = \frac{1}{1-(1-x)}$)

Answer

$f(x) = \sum_{n=0}^{\infty} (1-x)^n$ on $I = (0,2)$

34) $f(x) = \frac{1}{1-x^2}; a = 0$

35) $f(x) = \frac{x}{1-x^2}; a = 0$

Answer

$\sum_{n=0}^{\infty} x^{2n+1}$ on $I = (-1,1)$

36) $f(x) = \frac{1}{1+x^2}; a = 0$

37) $f(x) = \frac{x^2}{1+x^2}; a = 0$

Answer

$\sum_{n=0}^{\infty} (-1)^n x^{2n+2}$ on $I = (-1,1)$

38) $f(x) = \frac{1}{2-x}; a = 1$

39) $f(x) = \frac{1}{1-2x}; a = 0.$

Answer

$\sum_{n=0}^{\infty} 2^n x^n$ on $(-\frac{1}{2}, \frac{1}{2})$

40) $f(x) = \frac{1}{1-4x^2}; a = 0$

41) $f(x) = \frac{x^2}{1-4x^2}; a = 0$

Answer

$\sum_{n=0}^{\infty} 4^n x^{2n+2}$ on $(-\frac{1}{2}, \frac{1}{2})$

42) $f(x) = \frac{x^2}{5-4x+x^2}; a = 2$

Use the result of exercise 43 to find the radius of convergence of the given series in the subsequent exercises (44 - 47).

43) Explain why, if $|a_n|^{1/n} \to r > 0$, then $|a_n x^n|^{1/n} \to |x|r < 1$ whenever $|x| < \frac{1}{r}$ and, therefore, the radius of convergence of $\sum_{n=1}^{\infty} a_n x^n$ is $R = \frac{1}{r}$.

Answer

$|a_n x^n|^{1/n} = |a_n|^{1/n}|x| \to |x|r$ as $n \to \infty$ and $|x|r < 1$ when $|x| < \frac{1}{r}$. Therefore, $\sum_{n=1}^{\infty} a_n x^n$ converges when $|x| < \frac{1}{r}$ by the nth Root Test.

44) $\sum_{n=1}^{\infty} \frac{x^n}{n^n}$

45) $\sum_{k=1}^{\infty} \left(\frac{k-1}{2k+3}\right)^k x^k$

Answer

$a_k = \left(\frac{k-1}{2k+3}\right)^k$ so $(a_k)^{1/k} \to \frac{1}{2} < 1$ so $R = 2$

46) $\sum_{k=1}^{\infty} (\frac{2k^2-1}{k^2+3})^k x^k$

47) $\sum_{n=1}^{\infty} a_n = (n^{1/n} - 1)^n x^n$

Answer

$a_n = (n^{1/n} - 1)^n$ so $(a_n)^{1/n} \to 0$ so $R = \infty$

48) Suppose that $p(x) = \sum_{n=0}^{\infty} a_n x^n$ such that $a_n = 0$ if n is even. Explain why $p(x) = p(-x)$.

49) Suppose that $p(x) = \sum_{n=0}^{\infty} a_n x^n$ such that $a_n = 0$ if n is odd. Explain why $p(x) = -p(-x)$.

Answer

We can rewrite $p(x) = \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1}$ and $p(x) = p(-x)$ since $x^{2n+1} = -(-x)^{2n+1}$.

50) Suppose that $p(x) = \sum_{n=0}^{\infty} a_n x^n$ converges on $(-1, 1]$. Find the interval of convergence of $p(Ax)$.

51) Suppose that $p(x) = \sum_{n=0}^{\infty} a_n x^n$ converges on $(-1, 1]$. Find the interval of convergence of $p(2x - 1)$.

Answer

If $x \in [0, 1]$, then $y = 2x - 1 \in [-1, 1]$ so $p(2x - 1) = p(y) = \sum_{n=0}^{\infty} a_n y^n$ converges.

In the following exercises, suppose that $p(x) = \sum_{n=0}^{\infty} a_n x^n$ satisfies $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ where $a_n \geq 0$ for each n. State whether each series converges on the full interval $(-1, 1)$, or if there is not enough information to draw a conclusion. Use the Comparison Test when appropriate.

52) $\sum_{n=0}^{\infty} a_n x^{2n}$

53) $\sum_{n=0}^{\infty} a_{2n} x^{2n}$

Answer

Converges on $(-1, 1)$ by the Ratio Test

54) $\sum_{n=0}^{\infty} a_{2n}x^n$ (Hint:$x = \pm\sqrt{x^2}$)

55) $\sum_{n=0}^{\infty} a_{n^2}x^{n^2}$ (Hint: Let $b_k = a_k$ if $k = n^2$ for some n, otherwise $b_k = 0$.)

Answer

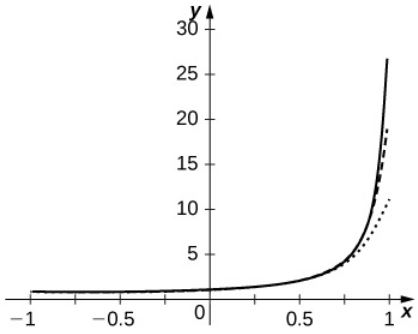
Consider the series $\sum b_k x^k$ where $b_k = a_k$ if $k = n^2$ and $b_k = 0$ otherwise. Then $b_k \le a_k$ and so the series converges on $(-1, 1)$ by the Comparison Test.

56) Suppose that $p(x)$ is a polynomial of degree N. Find the radius and interval of convergence of $\sum_{n=1}^{\infty} p(n)x^n$.

57) [Technology Required] Plot the graphs of $\frac{1}{1-x}$ and of the partial sums $S_N = \sum_{n=0}^{N} x^n$ for $n = 10, 20, 30$ on the interval $[-0.99, 0.99]$. Comment on the approximation of $\frac{1}{1-x}$ by S_N near $x = -1$ and near $x = 1$ as N increases.

Answer

The approximation is more accurate near $x = -1$. The partial sums follow $\frac{1}{1-x}$ more closely as N increases but are never accurate near $x = 1$ since the series diverges there.

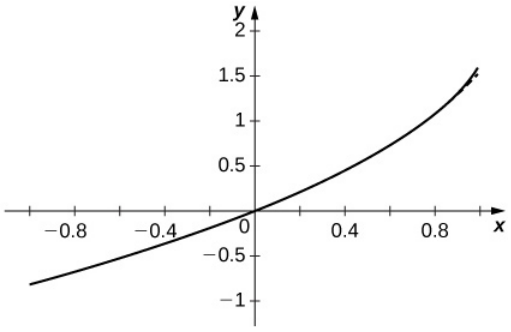


58) [Technology Required] Plot the graphs of $-\ln(1-x)$ and of the partial sums $S_N = \sum_{n=1}^{N} \frac{x^n}{n}$ for $n = 10, 50, 100$ on the interval $[-0.99, 0.99]$. Comment on the behavior of the sums near $x = -1$ and near $x = 1$ as N increases.

59) [Technology Required] Plot the graphs of the partial sums $S_n = \sum_{n=1}^{N} \frac{x^n}{n^2}$ for $n = 10, 50, 100$ on the interval $[-0.99, 0.99]$. Comment on the behavior of the sums near $x = -1$ and near $x = 1$ as N increases.

Answer

The approximation appears to stabilize quickly near both $x = \pm 1$.



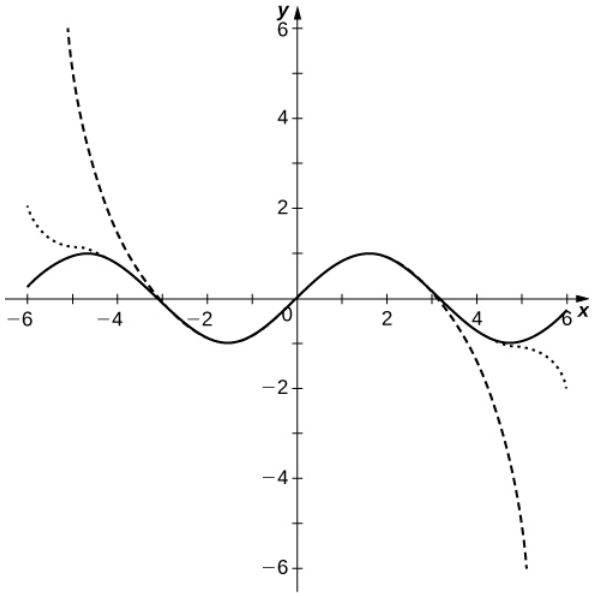
https://math.libretexts.org/@go/page/168452

60) [Technology Required] Plot the graphs of the partial sums $S_N = \sum_{n=1}^{N} (\sin n) x^n$ for $n = 10, 50, 100$ on the interval $[-0.99, 0.99]$. Comment on the behavior of the sums near $x = -1$ and near $x = 1$ as N increases.

61) [Technology Required] Plot the graphs of the partial sums $S_N = \sum_{n=0}^{N} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for $n = 3, 5, 10$ on the interval $[-2\pi, 2\pi]$. Comment on how these plots approximate $\sin x$ as N increases.

Answer

The polynomial curves have roots close to those of $\sin x$ up to their degree and then the polynomials diverge from $\sin x$.



62) [Technology Required] Plot the graphs of the partial sums $S_N = \sum_{n=0}^{N} (-1)^n \frac{x^{2n}}{(2n)!}$ for $n = 3, 5, 10$ on the interval $[-2\pi, 2\pi]$. Comment on how these plots approximate $\cos x$ as N increases.

Contributors

Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

This page titled 5.1E: Exercises is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Gilbert Strang & Edwin “Jed” Herman via source content that was edited to the style and standards of the LibreTexts platform.

- **10.1E: Exercises for Section 10.1** by Edwin “Jed” Herman, Gilbert Strang is licensed CC BY-NC-SA 4.0. Original source: https://openstax.org/details/books/calculus-volume-1.

5.2: Properties of Power Series

Learning Objectives

- Combine power series by addition or subtraction.
- Create a new power series by multiplying by a power of the variable, multiplying by a constant, or by substitution.
- Multiply two power series together.
- Differentiate and integrate power series term-by-term.

In the preceding section on power series and functions, we showed how to represent certain functions using power series. This section discusses how power series can be combined, differentiated, or integrated to create new power series. This is particularly useful for a couple of reasons. First, it allows us to find power series representations for certain elementary functions by writing those functions in terms of functions with known power series. For example, given the power series representation for $f(x) = \frac{1}{1-x}$, we can find a power series representation for $f'(x) = \frac{1}{(1-x)^2}$. Second, creating a power series allows us to define new functions that cannot be written in terms of elementary functions. This capability is handy for solving differential equations for which there is no solution in terms of elementary functions.

Combining Power Series

If we have two power series with the same interval of convergence, we can add or subtract the two series to create a new one with the same interval of convergence. Similarly, we can multiply a power series by a power of x or evaluate a power series at x^m for a positive integer m to create a new power series (we did this a little bit in the previous section). Being able to do this allows us to find power series representations for certain functions by using power series representations of other functions. For example, since we know the power series representation for $f(x) = \frac{1}{1-x}$, we can find power series representations for related functions, such as

$$y = \frac{3x}{1-x^2}$$

and

$$y = \frac{1}{(x-1)(x-3)}.$$

In the theorem below, we state results regarding the addition or subtraction of two power series, compositions involving power series, and multiplication of a power series by a power of the variable. For simplicity, we state the theorem for power series centered at $x = 0$. Similar results hold for power series centered at $x = a$.

Theorem: Combining Power Series

Suppose that the two power series $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$ converge to the functions f and g, respectively, on a common interval I. Then, the following statements are true.

i. The power series $\sum_{n=0}^{\infty} (c_n x^n \pm d_n x^n)$ converges to $f \pm g$ on I.

ii. For any integer $m \geq 0$ and any real number b, the power series $\sum_{n=0}^{\infty} b x^m c_n x^n$ converges to $b x^m f(x)$ on I.

iii. For any integer $m \geq 0$ and any real number b, the series $\sum_{n=0}^{\infty} c_n (bx^m)^n$ converges to $f(bx^m)$ for all x such that bx^m is in I.

Proof

We prove i for the case of the series $\sum_{n=0}^{\infty} (c_n x^n + d_n x^n)$. Suppose that $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$ converge to the functions f and g, respectively, on the interval I. Let x be a point in I and let $S_N(x)$ and $T_N(x)$ denote the Nth partial sums of the series

$\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$, respectively. Then the sequence $\{S_N(x)\}$ converges to $f(x)$ and the sequence $\{T_N(x)\}$ converges to $g(x)$. Furthermore, the Nth partial sum of $\sum_{n=0}^{\infty}(c_n x^n + d_n x^n)$ is

$$\begin{aligned}\sum_{n=0}^{N}(c_n x^n + d_n x^n) &= \sum_{n=0}^{N} c_n x^n + \sum_{n=0}^{N} d_n x^n \\ &= S_N(x) + T_N(x).\end{aligned}$$

Because

$$\begin{aligned}\lim_{N\to\infty}(S_N(x) + T_N(x)) &= \lim_{N\to\infty} S_N(x) + \lim_{N\to\infty} T_N(x) \\ &= f(x) + g(x),\end{aligned}$$

we conclude that the series $\sum_{n=0}^{\infty}(c_n x^n + d_n x^n)$ converges to $f(x) + g(x)$.

We examine products of power series in a later theorem.

Caution: Common Interval of Convergence Required

As with all of the theorems in this section, the two series in question must share a common interval of convergence; **however**, these theorems hold over the *intersection* of the respective intervals of convergence for the given series. Thus, if one series converges on $(-4, 7)$ and the other converges on $[-1, 10)$, the theorems from this section would apply to the common interval of convergence, $[-1, 7)$.

We begin by showing several applications of the previous theorem and how to find the interval of convergence of a power series given the interval of convergence of a related power series.

Example 5.2.1: Combining Power Series

Suppose that $\sum_{n=0}^{\infty} a_n x^n$ is a power series whose interval of convergence is $(-1, 1)$, and suppose that $\sum_{n=0}^{\infty} b_n x^n$ is a power series whose interval of convergence is $(-2, 2)$

a. Find the interval of convergence of the series $\sum_{n=0}^{\infty}(a_n x^n + b_n x^n)$.

b. Find the interval of convergence of the series $\sum_{n=0}^{\infty} a_n 3^n x^n$.

Solutions

a. Since the interval $(-1, 1)$ is a common interval of convergence of the series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$, the interval of convergence of the series $\sum_{n=0}^{\infty}(a_n x^n + b_n x^n)$ is $(-1, 1)$.

b. Since $\sum_{n=0}^{\infty} a_n x^n$ is a power series centered at zero with radius of convergence 1, it converges for all x in the interval $(-1, 1)$. By the previous theorem, the series

$$\sum_{n=0}^{\infty} a_n 3^n x^n = \sum_{n=0}^{\infty} a_n (3x)^n$$

converges if $3x$ is in the interval $(-1, 1)$. Hence, the series converges for all x in the interval $\left(-\frac{1}{3}, \frac{1}{3}\right)$.

? Checkpoint 5.2.1

Suppose that $\sum_{n=0}^{\infty} a_n x^n$ has an interval of convergence of $(-1,1)$. Find the interval of convergence of $\sum_{n=0}^{\infty} a_n \left(\frac{x}{2}\right)^n$.

Answer

Interval of convergence is $(-2,2)$.

In the next example, we show how to construct power series for functions related to f. Specifically, we consider functions related to the function $f(x) = \frac{1}{1-x}$ and we use the fact that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

for $|x| < 1$.

✓ Example 5.2.2: Constructing Power Series from Known Power Series

Use the power series representation for $f(x) = \frac{1}{1-x}$ to construct a power series for each of the following functions. Find the interval of convergence of the power series.

a. $f(x) = \frac{3x}{1+x^2}$
b. $f(x) = \frac{1}{(x-1)(x-3)}$

Solutions

a. First write $f(x)$ as

$$f(x) = 3x\left(\frac{1}{1-(-x^2)}\right).$$

Using the power series representation for $f(x) = \frac{1}{1-x}$ and parts ii. and iii. of the previous theorem, we find that a power series representation for f is given by

$$\sum_{n=0}^{\infty} 3x(-x^2)^n = \sum_{n=0}^{\infty} 3(-1)^n x^{2n+1}.$$

Since the interval of convergence of the series for $\frac{1}{1-x}$ is $(-1,1)$, the interval of convergence for this new series is the set of real numbers x such that $|x^2| < 1$. Therefore, the interval of convergence is $(-1,1)$.

b. To find the power series representation, use partial fractions to write $f(x) = \frac{1}{(x-1)(x-3)}$ as the sum of two fractions. We have

$$\frac{1}{(x-1)(x-3)} = \frac{-1/2}{x-1} + \frac{1/2}{x-3} = \frac{1/2}{1-x} - \frac{1/2}{3-x} = \frac{1/2}{1-x} - \frac{1/6}{1-\frac{x}{3}}.$$

First, we obtain

$$\frac{1/2}{1-x} = \sum_{n=0}^{\infty} \frac{1}{2} x^n \text{ for } |x| < 1.$$

Then, we have

$$\frac{1/6}{1-x/3} = \sum_{n=0}^{\infty} \frac{1}{6}\left(\frac{x}{3}\right)^n \text{ for } |x| < 3.$$

Since we are combining these two power series, the interval of convergence of the difference must be the smaller of these two intervals. Using this fact, we have

$$\frac{1}{(x-1)(x-3)} = \sum_{n=0}^{\infty} \left(\frac{1}{2} - \frac{1}{6 \cdot 3^n}\right) x^n$$

where the interval of convergence is $(-1,1)$.

Checkpoint 5.2.2

Use the series for $f(x)=\frac{1}{1-x}$ on $|x|<1$ to construct a series for $\frac{1}{(1-x)(x-2)}$. Determine the interval of convergence.

Answer

$\sum_{n=0}^{\infty}\left(-1+\frac{1}{2^{n+1}}\right)x^n$. The interval of convergence is $(-1,1)$.

In Example 5.2.2, we showed how to find power series for specific functions. In Example 5.2.3, we show how to do the opposite: given a power series, determine which function it represents.

Example 5.2.3: Finding the Function Represented by a Given Power Series

Consider the power series $\sum_{n=0}^{\infty}2^nx^n$. Find the function f represented by this series. Determine the interval of convergence of the series.

Solution

Writing the given series as

$$\sum_{n=0}^{\infty}2^nx^n=\sum_{n=0}^{\infty}(2x)^n,$$

we can recognize this series as the power series for

$$f(x)=\frac{1}{1-2x}.$$

Since this is a geometric series, the series converges if and only if $|2x|<1$. Therefore, the interval of convergence is $\left(-\frac{1}{2},\frac{1}{2}\right)$.

Checkpoint 5.2.3

Find the function represented by the power series $\sum_{n=0}^{\infty}\frac{1}{3^n}x^n$. Determine its interval of convergence.

Answer

$f(x)=\frac{3}{3-x}$. The interval of convergence is $(-3,3)$.

When winning the lottery, an individual can sometimes receive winnings in one lump sum or smaller payments over fixed intervals. For example, you could receive 20 million dollars today or 1.5 million each year for the next 20 years. Which is the better deal?

Luckily, our new knowledge will allow us to use series to compare values of payments over time with a lump sum payment today. We will compute how much future payments are worth in today's dollars, assuming we can invest winnings and earn interest. The value of future payments in today's dollars is known as the **present value** of those payments.

Example 5.2.4: Present Value of Future Winnings

Suppose you win the lottery and are given the following three options:

- Receive 20 million dollars today;
- Receive 1.5 million dollars per year over the next 20 years; or
- Receive 1 million dollars per year indefinitely (being passed on to your heirs).

Which is the best deal, assuming the annual interest rate is 5%? We answer this by working through the following sequence of questions.

a. How much is the 1.5 million dollars received annually over 20 years worth in terms of today's dollars, assuming an annual interest rate of 5%?
b. Use the answer to part a. to find a general formula for the **present value** of payments of C dollars received each year over the next n years, assuming an average annual interest rate r.
c. Find a formula for the present value if annual payments of C dollars continue indefinitely, assuming an average annual interest rate r.
d. Use the answer to part c. to determine the present value of 1 million dollars paid annually indefinitely.
e. Use your answers to parts a. and d. to determine which of the three options is best.



Figure 5.2.1: (credit: modification of work by Robert Huffstutter, Flickr)

Solutions

a. Consider the payment of 1.5 million dollars made at the end of the first year. If you receive that payment today instead of one year from now, you could invest that money and earn 5% interest. Therefore, the present value of that money P_1 satisfies $P_1(1+0.05) = 1.5$ million dollars. We conclude that

$$P_1 = \frac{1.5}{1.05} = 1.429 \text{ million dollars.}$$

Similarly, consider the payment of 1.5 million dollars made at the end of the second year. If you receive that payment today, you could invest that money for two years, earning 5% interest compounded annually. Therefore, the present value of that money P_2 satisfies $P_2(1+0.05)^2 = 1.5$ million dollars. We conclude that

$$P_2 = 1.5(1.05)^2 = 1.361 \text{ million dollars.}$$

The value of the future payments today is the sum of the present values P_1, P_2, ..., P_{20} of each of those annual payments. The present value P_k satisfies

$$P_k = \frac{1.5}{(1.05)^k}.$$

Therefore,

$$P = \frac{1.5}{1.05} + \frac{1.5}{(1.05)^2} + \ldots + \frac{1.5}{(1.05)^{20}} = 18.693 \text{ million dollars.}$$

b. Using the result from part a., we see that the present value P of C dollars paid annually for n years, assuming an annual interest rate r, is given by

$$P = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \cdots + \frac{C}{(1+r)^n} \text{ dollars.}$$

c. Using the result from part b. we see that the present value of an annuity that continues indefinitely is given by the infinite series

$$P = \sum_{n=0}^{\infty} \frac{C}{(1+r)^{n+1}}.$$

We can view the present value as a power series in r, which converges as long as $\left|\frac{1}{1+r}\right| < 1$. Since $r > 0$, this series converges. Rewriting the series as

$$P = \frac{C}{(1+r)} \sum_{n=0}^{\infty} \left(\frac{1}{1+r}\right)^n,$$

we recognize this series as the power series for

$$f(r) = \frac{1}{1-\left(\frac{1}{1+r}\right)} = \frac{1}{\left(\frac{r}{1+r}\right)} = \frac{1+r}{r}.$$

We conclude that the present value of this annuity is

$$P = \frac{C}{1+r} \cdot \frac{1+r}{r} = \frac{C}{r}.$$

d. From the result to part c., we conclude that the present value P of $C = 1$ million dollars paid out every year indefinitely, assuming an annual interest rate $r = 0.05$, is given by

$$P = \frac{1}{0.05} = 20$$

million dollars.

e. From part a., we see that receiving 1.5 million dollars over 20 years is worth 18.693 million dollars in today's dollars. From part d., we see that indefinitely receiving $1 million annually is worth $20 million in today's dollars. Therefore, either receiving a lump sum payment of $20 million today or receiving $1 million indefinitely have the same present value.

Multiplication of Power Series

We can also create new power series by multiplying power series. Multiplying two power series provides another way of finding power series representations for functions. The way we multiply them is similar to how we multiply polynomials. For example, suppose we want to multiply

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

and

$$\sum_{n=0}^{\infty} d_n x^n = d_0 + d_1 x + d_2 x^2 + \cdots.$$

It appears that the product should satisfy

$$\begin{aligned}\left(\sum_{n=0}^{\infty} c_n x^n\right)\left(\sum_{n=-0}^{\infty} d_n x^n\right) &= (c_0 + c_1 x + c_2 x^2 + \cdots) \cdot (d_0 + d_1 x + d_2 x^2 + \cdots) \\ &= c_0 d^0 + (c_1 d^0 + c_0 d^1)x + (c_2 d^0 + c_1 d^1 + c_0 d^2)x^2 + \cdots.\end{aligned}$$

In the following theorem, we state the main result regarding multiplying power series, showing that if $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$ converge on a common interval I, then we can multiply the series in this way, and the resulting series also converges on the interval I.

Theorem: Product of Power Series

Suppose that the power series $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$ converge to f and g, respectively, on a common interval I. Let

$$e_n = c_0 d_n + c_1 d_{n-1} + c_2 d_{n-2} + \cdots + c_{n-1} d_1 + c_n d_0 = \sum_{k=0}^{n} c_k d_{n-k}.$$

Then

$$\left(\sum_{n=0}^{\infty} c_n x^n\right)\left(\sum_{n=0}^{\infty} d_n x^n\right) = \sum_{n=0}^{\infty} e_n x^n$$

and

$$\sum_{n=0}^{\infty} e_n x^n \text{ converges to } f(x)\cdot g(x) \text{ on } I.$$

The series $\sum_{n=0}^{\infty} e_n x^n$ is known as the **Cauchy product** of the series $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$.[1]

We omit the proof of this theorem as it is beyond the level of this text and is typically covered in a more advanced course. We now provide an example of this theorem by finding the power series representation for

$$f(x) = \frac{1}{(1-x)(1-x^2)}$$

using the power series representations for

$$y = \frac{1}{1-x} \text{ and } y = \frac{1}{1-x^2}.$$

Example 5.2.5: Multiplying Power Series

Multiply the power series representation

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

for $|x| < 1$ with the power series representation

$$\frac{1}{1-x^2} = \sum_{n=0}^{\infty} (x^2)^n = 1 + x^2 + x^4 + x^6 + \cdots$$

for $|x| < 1$ to construct a power series for $f(x) = \frac{1}{(1-x)(1-x^2)}$ on the interval $(-1, 1)$.

Solution

We need to multiply

$$(1 + x + x^2 + x^3 + \cdots)(1 + x^2 + x^4 + x^6 + \cdots).$$

We note that this product will have a constant term and terms for x, x^2, x^3, and so on. Hence, we see that the product is given by

$$\begin{aligned}(1 + x + x^2 + x^3 + \cdots)(1 + x^2 + x^4 + x^6 + \cdots) &= 1 + x + (1+1)x^2 + (1+1)x^3 + (1+1+1)x^4 + (1+1+1)x^5 + \cdots \\ &= 1 + x + 2x^2 + 2x^3 + 3x^4 + 3x^5 + \cdots.\end{aligned}$$

Since the series for $y = \frac{1}{1-x}$ and $y = \frac{1}{1-x^2}$ both converge on the interval $(-1, 1)$, the series for the product also converges on the interval $(-1, 1)$.

Note: The Expansion of Products of Power Series

As you progress in Mathematics, there will be a few moments (specifically, in Differential Equations) when you must find the product of two power series. It's a common convention to list the expansion's first three or four nonzero terms.

? Checkpoint 5.2.5

Multiply the series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ by itself to construct a series for $\frac{1}{(1-x)(1-x)}$.

Answer

$1 + 2x + 3x^2 + 4x^3 + \cdots$

Differentiating and Integrating Power Series

Consider a power series $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$ that converges on some interval I, and let f be the function defined by this series. Here we address two questions about f:

1. Is f differentiable, and if so, how do we determine the derivative f'?
2. How do we evaluate the indefinite integral $\int f(x)\,dx$?

We know we can evaluate the derivative by differentiating each term separately for a polynomial with a finite number of terms. Similarly, we can evaluate the indefinite integral by integrating each term separately. We show we can do the same for *convergent* power series. That is, if

$$f(x) = c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

converges on some interval I, then

$$f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + \cdots$$

and

$$\int f(x)\,dx = C + c_0 x + c_1 \frac{x^2}{2} + c_2 \frac{x^3}{3} + \cdots.$$

Evaluating the derivative and indefinite integral in this way is called **term-by-term differentiation of a power series** and **term-by-term integration of a power series**, respectively.

The ability to differentiate and integrate power series term-by-term also allows us to use known power series representations to find power series representations for other functions. For example, given the power series for $f(x) = \frac{1}{1-x}$, we can differentiate term-by-term to find the power series for $f'(x) = \frac{1}{(1-x)^2}$. Similarly, using the power series for $g(x) = \frac{1}{1+x}$, we can integrate term-by-term to find the power series for $G(x) = \ln(1+x)$, an antiderivative of g. We show how to do this in Example 5.2.6 and Example 5.2.7. First, we state the following theorem, which provides the main result regarding power series differentiation and integration.

Theorem: Term-by-Term Differentiation and Integration for Power Series

Suppose that the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges on the interval $(a-R, a+R)$ for some $R > 0$. Let f be the function defined by the series

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + c_3 (x-a)^3 + \cdots$$

for $|x-a| < R$. Then f is differentiable on the interval $(a-R, a+R)$ and we can find f' by differentiating the series term-by-term:

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1} = c_1 + 2c_2 (x-a) + 3c_3 (x-a)^2 + \cdots$$

for $|x-a| < R$. Also, to find $\int f(x)\,dx$, we can integrate the series term-by-term. The resulting series converges on $(a-R, a+R)$, and we have

$$\int f(x)\,dx = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots$$

for $|x-a| < R$.

This result's proof is omitted and beyond the text's scope. Note that although this theorem guarantees the same radius of convergence when a power series is differentiated or integrated term-by-term, it says nothing about what happens at the endpoints. The differentiated and integrated power series may behave differently at the endpoints than the original series. We see this behavior in the next examples.

Example 5.2.6: Differentiating Power Series

a. Use the power series representation

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

for $|x| < 1$ to find a power series representation for

$$g(x) = \frac{1}{(1-x)^2}$$

on the interval $(-1, 1)$. Determine whether the resulting series converges at the endpoints.

b. Use the result of part a. to evaluate the sum of the series $\sum_{n=0}^{\infty} \frac{n+1}{4^n}$.

Solutions

a. Since $g(x) = \frac{1}{(1-x)^2}$ is the derivative of $f(x) = \frac{1}{1-x}$, we can find a power series representation for g by differentiating the power series for f term-by-term. The result is

$$\begin{aligned} g(x) &= \frac{1}{(1-x)^2} \\ &= \frac{d}{dx}\left(\frac{1}{1-x}\right) \\ &= \sum_{n=0}^{\infty} \frac{d}{dx}(x^n) \\ &= \frac{d}{dx}(1 + x + x^2 + x^3 + \cdots) \\ &= 0 + 1 + 2x + 3x^2 + 4x^3 + \cdots \\ &= \sum_{n=0}^{\infty} (n+1)x^n \end{aligned}$$

for $|x| < 1$. Theorem 5.2.1 does not guarantee anything about the behavior of this series at the endpoints. Testing the endpoints by using the Divergence Test, we find that the series diverges at both endpoints $x = \pm 1$.

b. From part a. we know that

$$\sum_{n=0}^{\infty} (n+1)x^n = \frac{1}{(1-x)^2}.$$

Therefore,

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{n+1}{4^n} &= \textstyle\sum_{n=0}^{\infty}(n+1)\left(\frac{1}{4}\right)^n \\ &= \frac{1}{\left(1-\frac{1}{4}\right)^2} \\ &= \frac{1}{\left(\frac{3}{4}\right)^2} \\ &= \frac{16}{9}\end{aligned}$$

? Checkpoint 5.2.6

Differentiate the series $\frac{1}{(1-x)^2}=\sum_{n=0}^{\infty}(n+1)x^n$ term-by-term to find a power series representation for $\frac{2}{(1-x)^3}$ on the interval $(-1,1)$.

Answer

$$\sum_{n=0}^{\infty}(n+2)(n+1)x^n$$

✓ Example 5.2.7: Integrating Power Series

For each of the following functions f, find a power series representation for f by integrating the power series for f' and find its interval of convergence.

a. $f(x)=\ln(1+x)$
b. $f(x)=\tan^{-1}x$

Solutions

a. For $f(x)=\ln(1+x)$,the derivative is $f'(x)=\frac{1}{1+x}$. We know that

$$\frac{1}{1+x}=\frac{1}{1-(-x)}=\sum_{n=0}^{\infty}(-x)^n=1-x+x^2-x^3+\cdots$$

for $|x|<1$. To find a power series for $f(x)=\ln(1+x)$, we integrate the series term-by-term.

$$\int f'(x)\,dx=\int(1-x+x^2-x^3+\cdots)\,dx=C+x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$$

Since $f(x)=\ln(1+x)$ is an antiderivative of $\frac{1}{1+x}$, it remains to solve for the constant C. Since $\ln(1+0)=0$, we have $C=0$. Therefore, a power series representation for $f(x)=\ln(1+x)$ is

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}\text{ for }|x|<1.$$

Theorem 5.2.1 does not guarantee anything about the behavior of this power series at the endpoints. However, checking the endpoints, we find that at $x=1$, the series is the Alternating Harmonic Series, which converges. Also, at $x=-1$, the series is the Harmonic Series, which diverges. It is important to note that, even though this series converges at $x=1$, we cannot guarantee that the series converges to $\ln(2)$. In fact, the series converges $\ln(2)$, but showing this fact requires more advanced techniques.[2] The interval of convergence is $(-1,1]$.

b. The derivative of $f(x)=\tan^{-1}x$ is $f'(x)=\frac{1}{1+x^2}$. We know that

$$\frac{1}{1+x^2}=\frac{1}{1-(-x^2)}=\sum_{n=0}^{\infty}(-x^2)^n=1-x^2+x^4-x^6+\cdots$$

for $|x|<1$. To find a power series for $f(x)=\tan^{-1}x$, we integrate this series term-by-term.

$$\int f'(x)\,dx = \int (1 - x^2 + x^4 - x^6 + \cdots)\,dx = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots.$$

Since $\tan^{-1}(0) = 0$, we have $C = 0$. Therefore, a power series representation for $f(x) = \tan^{-1} x$ is

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

for $|x| < 1$. Again, Theorem 5.2.1 does not guarantee anything about the convergence of this series at the endpoints. However, checking the endpoints and using the Alternating Series Test, we find that the series converges at $x = 1$ and $x = -1$. As discussed in the footnote to part a., using Abel's Theorem, it can be shown that the series converges to $\tan^{-1}(1)$ and $\tan^{-1}(-1)$ at $x = 1$ and $x = -1$, respectively. Thus, the interval of convergence is $[-1, 1]$.

? Checkpoint 5.2.7

Integrate the power series $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$ term-by-term to evaluate $\int \ln(1+x)\,dx$.

Answer

$$\sum_{n=2}^{\infty} \frac{(-1)^n x^n}{n(n-1)}$$

Up to this point, we have shown several techniques for finding power series representations for functions. However, how do we know that these power series are unique? That is, given a function f and a power series for f at a, is it possible that there is a different power series for f at a that we could have found if we had used a different technique? The answer to this question is no. This fact should not seem surprising if we consider power series as polynomials with infinite terms. Intuitively, if

$$c_0 + c_1 x + c_2 x^2 + \cdots = d_0 + d_1 x + d_2 x^2 + \cdots$$

for all values x in some open interval I about zero, then the coefficients c_n should equal d_n for $n \geq 0$. We now state this result formally.

Theorem: Uniqueness of Power Series

Let $\sum_{n=0}^{\infty} c_n (x-a)^n$ and $\sum_{n=0}^{\infty} d_n (x-a)^n$ be two convergent power series such that

$$\sum_{n=0}^{\infty} c_n (x-a)^n = \sum_{n=0}^{\infty} d_n (x-a)^n$$

for all x in an open interval containing a. Then $c_n = d_n$ for all $n \geq 0$.

Proof

Let

$$\begin{aligned} f(x) &= c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots \\ &= d_0 + d_1(x-a) + d_2(x-a)^2 + d_3(x-a)^3 + \cdots. \end{aligned}$$

Then $f(a) = c_0 = d_0$. By Theorem 5.2.3, we can differentiate both series term-by-term. Therefore,

$$\begin{aligned} f'(x) &= c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots \\ &= d_1 + 2d_2(x-a) + 3d_3(x-a)^2 + \cdots, \end{aligned}$$

and thus, $f'(a) = c_1 = d_1$. Similarly,

$$\begin{aligned} f''(x) &= 2c_2 + 3 \cdot 2c_3(x-a) + \cdots \\ &= 2d_2 + 3 \cdot 2d_3(x-a) + \cdots \end{aligned}$$

implies that $f''(a) = 2c_2 = 2d_2$, and therefore, $c_2 = d_2$. More generally, for any integer $n \geq 0$, $f^{(n)}(a) = n!c_n = n!d_n$, and consequently, $c_n = d_n$ for all $n \geq 0$.

In this section, we have shown how to find power series representations for certain functions using various algebraic operations, differentiation, or integration. However, we are still limited in the functions for which we can find power series representations. Next, by introducing the Taylor series, we show how to find power series representations for many more functions.

Footnotes

[1] This theorem is included for completeness; however, your instructor might "approximate" this theorem (which is perfectly fine) by stating that the product of two convergent power series is best computed by finding the first three or four lowest powers of the product. That is,

$$\begin{aligned} \left(\sum_{n=0}^{\infty} c_n x^n\right)\left(\sum_{n=0}^{\infty} d_n x^n\right) &= \left(c_0 + c_1 x + c_2 x^2 + \cdots\right)\left(d_0 + d_1 x + d_2 x^2 + \cdots\right) \\ &= c_0 d_0 + \left(c_0 d_1 + c_1 d_0\right) x + \left(c_0 d_2 + c_1 d_1 + c_2 d_0\right) x^2 + \cdots \end{aligned}$$

[2] Abel's Theorem, covered in more advanced texts, deals with this more technical point.

Key Concepts

- Given two power series $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$ that converge to functions f and g on a common interval I, the sum and difference of the two series converge to $f \pm g$, respectively, on I. In addition, for any real number b and integer $m \geq 0$, the series $\sum_{n=0}^{\infty} b x^m c_n x^n$ converges to $b x^m f(x)$ and the series $\sum_{n=0}^{\infty} c_n (b x^m)^n$ converges to $f(b x^m)$ whenever $b x^m$ is in the interval I.
- Given two power series that converge on an interval $(-R, R)$, the Cauchy product of the two power series converges on the interval $(-R, R)$.
- Given a power series that converges to a function f on an interval $(-R, R)$, the series can be differentiated term-by-term and the resulting series converges to f' on $(-R, R)$. The series can also be integrated term-by-term and the resulting series converges to $\int f(x)\,dx$ on $(-R, R)$.

Glossary

term-by-term differentiation of a power series

a technique for evaluating the derivative of a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ by considering the derivative of each term separately to create the new power series $\sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$

term-by-term integration of a power series

a technique for integrating a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ by integrating each term separately to create the new power series $C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$

5.2E: Exercises

1) If $f(x)=\sum_{n=0}^{\infty}\frac{x^n}{n!}$ and $g(x)=\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n!}$, find the power series of $\frac{1}{2}\big(f(x)+g(x)\big)$ and of $\frac{1}{2}\big(f(x)-g(x)\big)$.

Answer

$$\frac{1}{2}\big(f(x)+g(x)\big)=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!} \text{ and } \frac{1}{2}\big(f(x)-g(x)\big)=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}.$$

2) If $C(x)=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$ and $S(x)=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$, find the power series of $C(x)+S(x)$ and of $C(x)-S(x)$.

In exercises 3 - 6, use partial fractions to find the power series of each function.

3) $\frac{4}{(x-3)(x+1)}$

Answer

$$\frac{4}{(x-3)(x+1)}=\frac{1}{x-3}-\frac{1}{x+1}=-\frac{1}{3(1-\frac{x}{3})}-\frac{1}{1-(-x)}=-\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n-\sum_{n=0}^{\infty}(-1)^n x^n=\sum_{n=0}^{\infty}\left((-1)^{n+1}-\frac{1}{3n+1}\right)x^n$$

4) $\frac{3}{(x+2)(x-1)}$

5) $\frac{5}{(x^2+4)(x^2-1)}$

Answer

$$\frac{5}{(x^2+4)(x^2-1)}=\frac{1}{x^2-1}-\frac{1}{4}\frac{1}{1+(\frac{x}{2})^2}=-\sum_{n=0}^{\infty}x^{2n}-\frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\left(\frac{x}{2}\right)^n=\sum_{n=0}^{\infty}\left((-1)+(-1)^{n+1}\frac{1}{2^{n+2}}\right)x^{2n}$$

6) $\frac{30}{(x^2+1)(x^2-9)}$

In exercises 7 - 10, express each series as a rational function.

7) $\sum_{n=1}^{\infty}\frac{1}{x^n}$

Answer

$$\frac{1}{x}\sum_{n=0}^{\infty}\frac{1}{x^n}=\frac{1}{x}\cdot\frac{1}{1-\frac{1}{x}}=\frac{1}{x-1}$$

8) $\sum_{n=1}^{\infty}\frac{1}{x^{2n}}$

9) $\sum_{n=1}^{\infty}\frac{1}{(x-3)^{2n-1}}$

Answer

$$\frac{1}{x-3}\cdot\frac{1}{1-\frac{1}{(x-3)^2}}=\frac{x-3}{(x-3)^2-1}$$

10) $\sum_{n=1}^{\infty}\left(\frac{1}{(x-3)^{2n-1}}-\frac{1}{(x-2)^{2n-1}}\right)$

Exercises 11 - 16 explore applications of *annuities*.

11) Calculate the present values P of an annuity in which \$10,000 is to be paid out annually for a period of 20 years, assuming interest rates of $r=0.03$, $r=0.05$, and $r=0.07$.

Answer

$P = P_1 + \cdots + P_{20}$ where $P_k = 10,000\frac{1}{(1+r)^k}$. Then $P = 10,000\sum_{k=1}^{20}\frac{1}{(1+r)^k} = 10,000\frac{1-(1+r)^{-20}}{r}$. When $r = 0.03$, $P \approx 10,000 \times 14.8775 = 148,775$. When $r = 0.05$, $P \approx 10,000 \times 12.4622 = 124,622$. When $r = 0.07$, $P \approx 105,940$.

12) Calculate the present values P of annuities in which \$9,000 is to be paid out annually perpetually, assuming interest rates of $r = 0.03$, $r = 0.05$ and $r = 0.07$.

13) Calculate the annual payouts C to be given for 20 years on annuities having present value \$100,000 assuming respective interest rates of $r = 0.03$, $r = 0.05$, and $r = 0.07$.

Answer

In general, $P = \frac{C(1-(1+r)^{-N})}{r}$ for N years of payouts, or $C = \frac{Pr}{1-(1+r)^{-N}}$. For $N = 20$ and $P = 100,000$, one has $C = 6721.57$ when $r = 0.03$; $C = 8024.26$ when $r = 0.05$; and $C \approx 9439.29$ when $r = 0.07$.

14) Calculate the annual payouts C to be given perpetually on annuities having present value \$100,000 assuming respective interest rates of $r = 0.03$, $r = 0.05$, and $r = 0.07$.

15) Suppose that an annuity has a present value $P = 1$ **million dollars.** What interest rate r would allow for perpetual annual payouts of \$50,000?

Answer

In general, $P = \frac{C}{r}$. Thus, $r = \frac{C}{P} = 5 \times \frac{10^4}{10^6} = 0.05$.

16) Suppose that an annuity has a present value $P = 10$ **million dollars.** What interest rate r would allow for perpetual annual payouts of \$100,000?

In exercises 17 - 20, express the sum of each power series in terms of geometric series, and then express the sum as a rational function.

17) $x + x^2 - x^3 + x^4 + x^5 - x^6 + \cdots$ (Hint: Group powers x^{3k}, x^{3k-1}, and x^{3k-2}.)

Answer

$(x + x^2 - x^3)(1 + x^3 + x^6 + \cdots) = \frac{x + x^2 - x^3}{1 - x^3}$

18) $x + x^2 - x^3 - x^4 + x^5 + x^6 - x^7 - x^8 + \cdots$ (Hint: Group powers x^{4k}, x^{4k-1}, etc.)

19) $x - x^2 - x^3 + x^4 - x^5 - x^6 + x^7 - \cdots$ (Hint: Group powers x^{3k}, x^{3k-1}, and x^{3k-2}.)

Answer

$(x - x^2 - x^3)(1 + x^3 + x^6 + \cdots) = \frac{x - x^2 - x^3}{1 - x^3}$

20) $\frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \frac{x^4}{16} + \frac{x^5}{32} - \frac{x^6}{64} + \cdots$ (Hint: Group powers $\left(\frac{x}{2}\right)^{3k}$, $\left(\frac{x}{2}\right)^{3k-1}$, and $\left(\frac{x}{2}\right)^{3k-2}$.)

In exercises 21 - 24, find the power series of $f(x)g(x)$ given f and g as defined.

21) $f(x) = 2\sum_{n=0}^{\infty} x^n, g(x) = \sum_{n=0}^{\infty} nx^n$

Answer

$a_n = 2$, $b_n = n$ so $c_n = \sum_{k=0}^{n} b_k a_{n-k} = 2\sum_{k=0}^{n} k = (n)(n+1)$ and $f(x)g(x) = \sum_{n=1}^{\infty} n(n+1)x^n$

22) $f(x) = \sum_{n=1}^{\infty} x^n$, $g(x) = \sum_{n=1}^{\infty} \frac{1}{n}x^n$. Express the coefficients of $f(x)g(x)$ in terms of $H_n = \sum_{k=1}^{n} \frac{1}{k}$.

23) $f(x) = g(x) = \sum_{n=1}^{\infty} \left(\frac{x}{2}\right)^n$

Answer

$a_n = b_n = 2^{-n}$ so $c_n = \sum_{k=1}^{n} b_k a_{n-k} = 2^{-n} \sum_{k=1}^{n} 1 = \frac{n}{2^n}$ and $f(x)g(x) = \sum_{n=1}^{\infty} n \left(\frac{x}{2}\right)^n$

24) $f(x) = g(x) = \sum_{n=1}^{\infty} nx^n$

In exercises 25 - 26, differentiate the given series expansion of f term-by-term to obtain the corresponding series expansion for the derivative of f.

25) $f(x) = \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$

Answer

The derivative of f is $-\frac{1}{(1+x)^2} = -\sum_{n=0}^{\infty} (-1)^n (n+1) x^n$.

26) $f(x) = \frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n}$

In exercises 27 - 28, integrate the given series expansion of f term-by-term from zero to x to obtain the corresponding series expansion for the indefinite integral of f.

27) $f(x) = \frac{2x}{(1+x^2)^2} = \sum_{n=1}^{\infty} (-1)^n (2n) x^{2n-1}$

Answer

The indefinite integral of f is $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$.

28) $f(x) = \frac{2x}{1+x^2} = 2\sum_{n=0}^{\infty} (-1)^n x^{2n+1}$

In exercises 29 - 32, evaluate each infinite series by identifying it as the value of a derivative or integral of geometric series.

29) Evaluate $\sum_{n=1}^{\infty} \frac{n}{2^n}$ as $f'\left(\frac{1}{2}\right)$ where $f(x) = \sum_{n=0}^{\infty} x^n$.

Answer

$$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x};\ f'(\frac{1}{2}) = \sum_{n=1}^{\infty} \frac{n}{2^{n-1}} = \frac{d}{dx}(1-x)^{-1}\Big|_{x=1/2} = \frac{1}{(1-x)^2}\Big|_{x=1/2} = 4 \text{ so } \sum_{n=1}^{\infty} \frac{n}{2^n} = 2.$$

30) Evaluate $\sum_{n=1}^{\infty} \frac{n}{3^n}$ as $f'\left(\frac{1}{3}\right)$ where $f(x) = \sum_{n=0}^{\infty} x^{6n}$.

31) Evaluate $\sum_{n=2}^{\infty} \frac{n(n-1)}{2^n}$ as $f''\left(\frac{1}{2}\right)$ where $f(x) = \sum_{n=0}^{\infty} x^n$.

Answer

$$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x};\ f''\left(\frac{1}{2}\right) = \sum_{n=2}^{\infty} \frac{n(n-1)}{2^{n-2}} = \frac{d^2}{dx^2}(1-x)^{-1}\Big|_{x=1/2} = \frac{2}{(1-x)^3}\Big|_{x=1/2} = 16 \text{ so } \sum_{n=2}^{\infty} n\frac{(n-1)}{2^n} = 4.$$

32) Evaluate $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ as $\int_0^1 f(t)\,dt$ where $f(x) = \sum_{n=0}^{\infty} (-1)^n x^{2n} = \frac{1}{1+x^2}$.

In exercises 33 - 39, given that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, use term-by-term differentiation or integration to find power series for each function centered at the given point.

33) $f(x) = \ln x$ centered at $x = 1$ (Hint: $x = 1 - (1-x)$)

Answer

$$\int \sum (1-x)^n\,dx = \int \sum (-1)^n (x-1)^n\,dx = \sum \frac{(-1)^n (x-1)^{n+1}}{n+1}$$

34) $\ln(1-x)$ at $x=0$

35) $\ln(1-x^2)$ at $x=0$

Answer

$$-\int_{t=0}^{x^2} \frac{1}{1-t}\,dt = -\sum_{n=0}^{\infty} \int_0^{x^2} t^n dx - \sum_{n=0}^{\infty} \frac{x^{2(n+1)}}{n+1} = -\sum_{n=1}^{\infty} \frac{x^{2n}}{n}$$

36) $f(x) = \dfrac{2x}{(1-x^2)^2}$ at $x=0$

37) $f(x) = \tan^{-1}(x^2)$ at $x=0$

Answer

$$\int_0^{x^2} \frac{dt}{1+t^2} = \sum_{n=0}^{\infty} (-1)^n \int_0^{x^2} t^{2n}dt = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{2n+1}\,|_{t=0}^{x^2} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1}$$

38) $f(x) = \ln(1+x^2)$ at $x=0$

39) $f(x) = \displaystyle\int_0^x \ln t\,dt$ where $\ln(x) = \displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-1)^n}{n}$

Answer

Term-by-term integration gives $\displaystyle\int_0^x \ln t\,dt = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-1)^{n+1}}{n(n+1)} = \sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{1}{n} - \frac{1}{n+1}\right)(x-1)^{n+1} = (x-1)\ln x + \displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{(x-1)^n}{n} = x\ln x - x.$

40) [Technology Required] Evaluate the power series expansion $\ln(1+x) = \displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ at $x=1$ to show that $\ln(2)$ is the sum of the alternating harmonic series. Use the alternating series test to determine how many terms of the sum are needed to estimate $\ln(2)$ accurate to within 0.001, and find such an approximation.

41) [Technology Required] Subtract the infinite series of $\ln(1-x)$ from $\ln(1+x)$ to get a power series for $\ln\left(\dfrac{1+x}{1-x}\right)$. Evaluate at $x = \frac{1}{3}$. What is the smallest N such that the Nth partial sum of this series approximates $\ln(2)$ with an error less than 0.001?

Answer

We have $\ln(1-x) = -\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$ so $\ln(1+x) = \displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$. Thus, $\ln\left(\dfrac{1+x}{1-x}\right) = \displaystyle\sum_{n=1}^{\infty} \left(1+(-1)^{n-1}\right)\frac{x^n}{n} = 2\sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1}$.

When $x = \frac{1}{3}$ we obtain $\ln(2) = 2\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^{2n-1}(2n-1)}$. We have $2\displaystyle\sum_{n=1}^{3} \frac{1}{3^{2n-1}(2n-1)} = 0.69300\ldots$, while $2\displaystyle\sum_{n=1}^{4} \frac{1}{3^{2n-1}(2n-1)} = 0.69313\ldots$ and $\ln(2) = 0.69314\ldots$; therefore, $N=4$.

In exercises 42 - 45, using a substitution if indicated, express each series in terms of elementary functions and find the radius of convergence of the sum.

42) $\displaystyle\sum_{k=0}^{\infty} (x^k - x^{2k+1})$

43) $\displaystyle\sum_{k=1}^{\infty} \frac{x^{3k}}{6k}$

Answer

$\displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k} = -\ln(1-x)$ so $\displaystyle\sum_{k=1}^{\infty} \frac{x^{3k}}{6k} = -\frac{1}{6}\ln(1-x^3)$. The radius of convergence is equal to 1 by the Ratio Test.

44) $\sum_{k=1}^{\infty}(1+x^2)^{-k}$ using $y=\dfrac{1}{1+x^2}$

45) $\sum_{k=1}^{\infty}2^{-kx}$ using $y=2^{-x}$

Answer

If $y=2^{-x}$, then $\sum_{k=1}^{\infty}y^k=\dfrac{y}{1-y}=\dfrac{2^{-x}}{1-2^{-x}}=\dfrac{1}{2^x-1}$. If $a_k=2^{-kx}$, then $\dfrac{a_{k+1}}{a_k}=2^{-x}<1$ when $x>0$. So the series converges for all $x>0$.

46) Show that, up to powers x^3 and y^3, $E(x)=\sum_{n=0}^{\infty}\dfrac{x^n}{n!}$ satisfies $E(x+y)=E(x)E(y)$.

47) Differentiate the series $E(x)=\sum_{n=0}^{\infty}\dfrac{x^n}{n!}$ term-by-term to show that $E(x)$ is equal to its derivative.

Answer

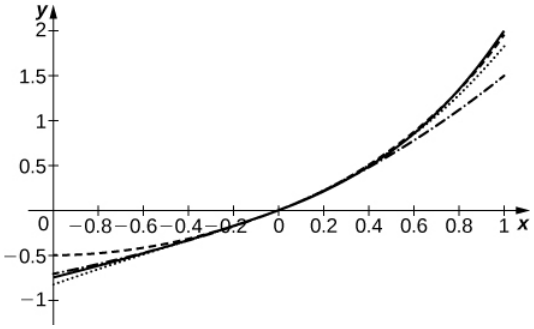
Answers will vary.

48) Show that if $f(x)=\sum_{n=0}^{\infty}a_nx^n$ is a sum of even powers, that is, $a_n=0$ if n is odd, then $F=\int_0^x f(t)\,dt$ is a sum of odd powers, while if I is a sum of odd powers, then F is a sum of even powers.

49) [Technology Required] Suppose that the coefficients an of the series $\sum_{n=0}^{\infty}a_nx^n$ are defined by the recurrence relation $a_n=\dfrac{a_{n-1}}{n}+\dfrac{a_{n-2}}{n(n-1)}$. For $a_0=0$ and $a_1=1$, compute and plot the sums $S_N=\sum_{n=0}^{N}a_nx^n$ for $N=2,3,4,5$ on $[-1,1]$.

Answer

The solid curve is S_5. The dashed curve is S_2, dotted is S_3, and dash-dotted is S_4



50) [Technology Required] Suppose that the coefficients an of the series $\sum_{n=0}^{\infty}a_nx^n$ are defined by the recurrence relation $a_n=\dfrac{a_{n-1}}{\sqrt{n}}-\dfrac{a_{n-2}}{\sqrt{n(n-1)}}$. For $a_0=1$ and $a_1=0$, compute and plot the sums $S_N=\sum_{n=0}^{N}a_nx^n$ for $N=2,3,4,5$ on $[-1,1]$.

51) [Technology Required] Given the power series expansion $\ln(1+x)=\sum_{n=1}^{\infty}(-1)^{n-1}\dfrac{x^n}{n}$, determine how many terms N of the sum evaluated at $x=-1/2$ are needed to approximate $\ln(2)$ accurate to within $1/1000$. Evaluate the corresponding partial sum $\sum_{n=1}^{N}(-1)^{n-1}\dfrac{x^n}{n}$.

Answer

When $x=-\dfrac{1}{2}$, $-\ln(2)=\ln\left(\dfrac{1}{2}\right)=-\sum_{n=1}^{\infty}\dfrac{1}{n2^n}$. Since $\sum_{n=11}^{\infty}\dfrac{1}{n2^n}<\sum_{n=11}^{\infty}\dfrac{1}{2^n}=\dfrac{1}{2^{10}}$, one has $\sum_{n=1}^{10}\dfrac{1}{n2^n}=0.69306\ldots$ whereas $\ln(2)=0.69314\ldots$; therefore, $N=10$.

52) [Technology Required] Given the power series expansion $\tan^{-1}(x) = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{2k+1}$, use the alternating series test to determine how many terms N of the sum evaluated at $x = 1$ are needed to approximate $\tan^{-1}(1) = \frac{\pi}{4}$ accurate to within $1/1000$. Evaluate the corresponding partial sum $\sum_{k=0}^{N}(-1)^k \frac{x^{2k+1}}{2k+1}$.

53) [Technology Required] Recall that $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$. Assuming an exact value of $\frac{1}{\sqrt{3}}$), estimate $\frac{\pi}{6}$ by evaluating partial sums $S_N\left(\frac{1}{\sqrt{3}}\right)$ of the power series expansion $\tan^{-1}(x) = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{2k+1}$ at $x = \frac{1}{\sqrt{3}}$. What is the smallest number N such that $6S_N\left(\frac{1}{\sqrt{3}}\right)$ approximates π accurately to within 0.001? How many terms are needed for accuracy to within 0.00001?

Answer

$6S_N\left(\frac{1}{\sqrt{3}}\right) = 2\sqrt{3}\sum_{n=0}^{N}(-1)^n \frac{1}{3^n(2n+1)}$. One has $\pi - 6S_4\left(\frac{1}{\sqrt{3}}\right) = 0.00101\ldots$ and $\pi - 6S_5\left(\frac{1}{\sqrt{3}}\right) = 0.00028\ldots$ so $N = 5$ is the smallest partial sum with accuracy to within 0.001. Also, $\pi - 6S_7\left(\frac{1}{\sqrt{3}}\right) = 0.00002\ldots$ while $\pi - 6S_8\left(\frac{1}{\sqrt{3}}\right) = -0.000007\ldots$ so $N = 8$ is the smallest N to give accuracy to within 0.00001.

Contributors

Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

This page titled 5.2E: Exercises is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Gilbert Strang & Edwin “Jed” Herman via source content that was edited to the style and standards of the LibreTexts platform.

- **10.2E: Exercises for Section 10.2** by Edwin “Jed” Herman, Gilbert Strang is licensed CC BY-NC-SA 4.0. Original source: https://openstax.org/details/books/calculus-volume-1.

5.3: Taylor and Maclaurin Series

Learning Objectives

- Describe the procedure for finding a Taylor polynomial of a given order for a function.
- Explain the meaning and significance of Taylor's theorem with remainder.
- Estimate the remainder for a Taylor series approximation of a given function.

In the previous two sections, we discussed how to find power series representations for certain types of functions - specifically, functions related to geometric series. Here, we discuss power series representations for other types of functions. In particular, we address the following questions: Which functions can be represented by power series, and how do we find such representations? Suppose we can find a power series representation for a particular function f, and the series converges on some interval. How do we prove that the series converges to f?

Overview of Taylor and Maclaurin Series

Consider a function f that has a power series representation at $x = a$. Then the series has the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots. \tag{5.3.1}$$

What should the coefficients be? For now, we ignore convergence issues but instead focus on what the series should be if one exists. We return to discuss convergence later in this section. If the series in Equation 5.3.1 is a representation for f at $x = a$, we certainly want the series to equal $f(a)$ at $x = a$. Evaluating the series at $x = a$, we see that

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(a-a) + c_2(a-a)^2 + \cdots = c_0. \tag{5.3.2}$$

Thus, the series equals $f(a)$ if the coefficient $c_0 = f(a)$. In addition, we would like the first derivative of the power series to equal $f'(a)$ at $x = a$. Differentiating Equation 5.3.2 term-by-term, we see that

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots. \tag{5.3.3}$$

Therefore, at $x = a$, the derivative is

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) = c_1 + 2c_2(a-a) + 3c_3(a-a)^2 + \cdots = c_1. \tag{5.3.4}$$

Hence, the derivative of the series equals $f'(a)$ if the coefficient $c_1 = f'(a)$. Continuing in this way, we look for coefficients c_n such that all the derivatives of the power series in Equation 5.3.4 will agree with all the corresponding derivatives of f at $x = a$. The second and third derivatives of Equation 5.3.3 are given by

$$\frac{d^2}{dx^2}\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) = 2c_2 + 3\cdot 2c_3(x-a) + 4\cdot 3c_4(x-a)^2 + \cdots$$

and

$$\frac{d^3}{dx^3}\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) = 3\cdot 2c_3 + 4\cdot 3\cdot 2c_4(x-a) + 5\cdot 4\cdot 3c_5(x-a)^2 + \cdots.$$

Therefore, at $x = a$, the second and third derivatives

$$\frac{d^2}{dx^2}\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) = 2c_2 + 3\cdot 2c_3(a-a) + 4\cdot 3c_4(a-a)^2 + \cdots = 2c_2$$

and

$$\frac{d^3}{dx^3}\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) = 3\cdot 2c_3 + 4\cdot 3\cdot 2c_4(a-a) + 5\cdot 4\cdot 3c_5(a-a)^2 + \cdots = 3\cdot 2c_3$$

equal $f''(a)$ and $f'''(a)$, respectively, if $c_2 = \frac{f''(a)}{2}$ and $c_3 = \frac{f'''(a)}{3\cdot 2}$. More generally, we see that if f has a power series representation at $x = a$, then the coefficients should be given by $c_n = \frac{f^{(n)}(a)}{n!}$. That is, the series should be

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

This power series for f is known as the **Taylor series for f at a**. If $x = 0$, then this series is known as the **Maclaurin series** for f. We formalize these statements using the following definition.

Definition: Maclaurin and Taylor series

If f has derivatives of all orders at $x = a$, then the **Taylor series for the function f at a** is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

The Taylor series for f at 0 is known as the **Maclaurin series** for f.

Later in this section, we will show examples of finding Taylor series and discuss conditions under which the Taylor series for a function will converge to that function. Here, we state an important result. Recall that power series representations are unique. Therefore, if a function f has a power series at a, then it must be the Taylor series for f at a.

Theorem: Uniqueness of Taylor Series

If a function f has a power series at a that converges to f on some open interval containing a, then that power series is the Taylor series for f at a.

The proof follows directly from that which was discussed previously.

To determine if a Taylor series converges, we need to look at its sequence of partial sums. These partial sums are finite polynomials, known as **Taylor polynomials**. These are the focus of the next subsection.

Taylor Polynomials

The nth partial sum of the Taylor series for a function f at a is known as the **n^{th}-degree Taylor polynomial**. For example, the 0th, 1st, 2nd, and 3rd partial sums of the Taylor series are given by

$$\begin{aligned}
p_0(x) &= f(a)\\
p_1(x) &= f(a) + f'(a)(x-a)\\
p_2(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2\\
p_3(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3
\end{aligned}$$

respectively. These partial sums are known as the 0th, 1st, 2nd, and 3rd degree Taylor polynomials of f at a, respectively. If $x = 0$, then these polynomials are known as **Maclaurin polynomials** for f. Below, we formally define Taylor and Maclaurin polynomials for a function f.

Definition: Taylor and Maclaurin Polynomials

If f has n derivatives at $x = a$, then the **n^{th}-degree Taylor polynomial** of f at a is

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

The nth-degree Taylor polynomial for f at 0 is known as the **n^{th}-degree Maclaurin polynomial** for f.

We now show how to use this definition to find several Taylor polynomials for $f(x) = \ln x$ at $x = 1$.

Example 5.3.1: Finding Taylor Polynomials

Find the Taylor polynomials p_0, p_1, p_2, and p_3 for $f(x) = \ln x$ at $x = 1$. Use graphing technology to compare the graph of f with the graphs of p_0, p_1, p_2, and p_3.

Solution

To find these Taylor polynomials, we need to evaluate f and its first three derivatives at $x = 1$.

$$\begin{aligned} f(x) &= \ln x \implies f(1) = 0 \\ f'(x) &= \frac{1}{x} \implies f'(1) = 1 \\ f''(x) &= -\frac{1}{x^2} \implies f''(1) = -1 \\ f'''(x) &= \frac{2}{x^3} \implies f'''(1) = 2 \end{aligned}$$

Therefore,

$$\begin{aligned} p_0(x) &= f(1) &&= 0, \\ p_1(x) &= f(1) + f'(1)(x-1) &&= x - 1, \\ p_2(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 &&= (x-1) - \frac{1}{2}(x-1)^2 \\ p_3(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 &&= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 \end{aligned}$$

The graphs of $y = f(x)$ and the first three Taylor polynomials are shown in Figure 5.3.1.

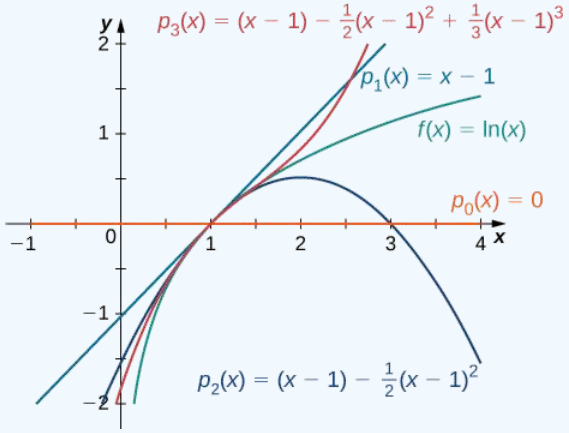


Figure 5.3.1: The function $y = \ln x$ and the Taylor polynomials p_0, p_1, p_2 and p_3 at $x = 1$ are plotted on this graph.

Checkpoint 5.3.1

Find the Taylor polynomials p_0, p_1, p_2, and p_3 for $f(x) = \frac{1}{x^2}$ at $x = 1$.

Answer

$$\begin{aligned} p_0(x) &= 1 \\ p_1(x) &= 1-2(x-1) \\ p_2(x) &= 1-2(x-1)+3(x-1)^2 \\ p_3(x) &= 1-2(x-1)+3(x-1)^2-4(x-1)^3 \end{aligned}$$

We now show how to find Maclaurin polynomials for e^x, $\sin x$, and $\cos x$. As stated above, Maclaurin polynomials are Taylor polynomials centered at zero.

Example 5.3.2: Finding Maclaurin Polynomials

For each of the following functions, find formulas for the Maclaurin polynomials p_0, p_1, p_2, and p_3. Find a formula for the nth-degree Maclaurin polynomial and write it using sigma notation. Use graphing technology to compare the graphs of p_0, p_1, p_2, and p_3 with f.

a. $f(x)=e^x$
b. $f(x)=\sin x$
c. $f(x)=\cos x$

Solutions

a. Since $f(x)=e^x$,we know that $f(x)=f'(x)=f''(x)=\cdots=f^{(n)}(x)=e^x$ for all positive integers n. Therefore,

$$f(0)=f'(0)=f''(0)=\cdots=f^{(n)}(0)=1$$

for all positive integers n. Therefore, we have

$$\begin{aligned} p_0(x) &= f(0) &&= 1, \\ p_1(x) &= f(0)+f'(0)x &&= 1+x, \\ p_2(x) &= f(0)+f'(0)x+\frac{f''(0)}{2!}x^2 &&= 1+x+\frac{1}{2}x^2 \\ p_3(x) &= f(0)+f'(0)x+\frac{f''(0)}{2}x^2+\frac{f'''(0)}{3!}x^3 &&= 1+x+\frac{1}{2}x^2+\frac{1}{3!}x^3, \\ &\vdots \quad \vdots \quad \vdots && \vdots \quad \vdots \\ p_n(x) &= f(0)+f'(0)x+\frac{f''(0)}{2}x^2+\frac{f'''(0)}{3!}x^3+\cdots+\frac{f^{(n)}(0)}{n!}x^n &&= 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!} \end{aligned}$$

This last line could be written as

$$p_n(x)=\sum_{k=0}^{n}\frac{x^k}{k!}.$$

The function and the first three Maclaurin polynomials are shown in Figure 5.3.2.

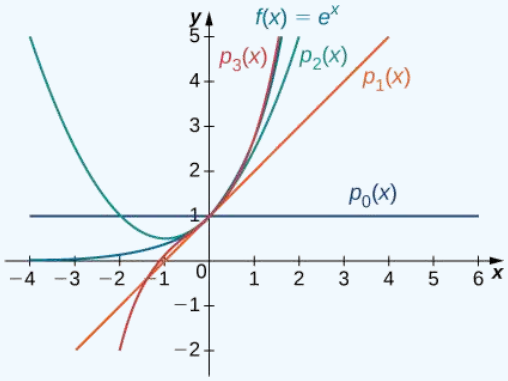


Figure 5.3.2: *The graph shows the function $y=e^x$ and the Maclaurin polynomials p_0, p_1, p_2, and p_3.*

b. For $f(x) = \sin x$, the values of the function and its first four derivatives at $x = 0$ are given as follows:

$$\begin{aligned} f(x) &= \sin x \implies f(0) = 0 \\ f'(x) &= \cos x \implies f'(0) = 1 \\ f''(x) &= -\sin x \implies f''(0) = 0 \\ f'''(x) &= -\cos x \implies f'''(0) = -1 \\ f^{(4)}(x) &= \sin x \implies f^{(4)}(0) = 0. \end{aligned}$$

Since the fourth derivative is $\sin x$, the pattern repeats. That is, $f^{(2m)}(0) = 0$ and $f^{(2m+1)}(0) = (-1)^m$ for $m \geq 0$. Thus, we have

$$\begin{aligned} p_0(x) &= 0, \\ p_1(x) &= 0 + x = x, \\ p_2(x) &= 0 + x + 0 = x, \\ p_3(x) &= 0 + x + 0 - \frac{1}{3!}x^3 = x - \frac{x^3}{3!}, \\ p_4(x) &= 0 + x + 0 - \frac{1}{3!}x^3 + 0 = x - \frac{x^3}{3!}, \\ p_5(x) &= 0 + x + 0 - \frac{1}{3!}x^3 + 0 + \frac{1}{5!}x^5 = x - \frac{x^3}{3!} + \frac{x^5}{5!}, \end{aligned}$$

and for $m \geq 0$,

$$p_{2m+1}(x) = p_{2m+2}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^m \frac{x^{2m+1}}{(2m+1)!} = \sum_{k=0}^{m} (-1)^k \frac{x^{2k+1}}{(2k+1)!}.$$

Graphs of the function and its Maclaurin polynomials are shown in Figure 5.3.3.

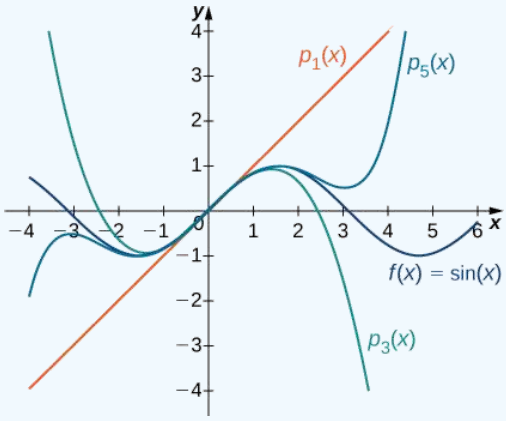


Figure 5.3.3: *The graph shows the function $y = \sin x$ and the Maclaurin polynomials p_1, p_3, and p_5.*

c. For $f(x) = \cos x$, the values of the function and its first four derivatives at $x = 0$ are given as follows:

$$\begin{aligned} f(x) &= \cos x \implies f(0) = 1 \\ f'(x) &= -\sin x \implies f'(0) = 0 \\ f''(x) &= -\cos x \implies f''(0) = -1 \\ f'''(x) &= \sin x \implies f'''(0) = 0 \\ f^{(4)}(x) &= \cos x \implies f^{(4)}(0) = 1. \end{aligned}$$

Since the fourth derivative is $\sin x$, the pattern repeats. In other words, $f^{(2m)}(0) = (-1)^m$ and $f^{(2m+1)} = 0$ for $m \geq 0$. Therefore,

$$
\begin{aligned}
p_0(x) &&&= 1,\\
p_1(x) &= 1+0 &&= 1,\\
p_2(x) &= 1+0-\frac{1}{2!}x^2 &&= 1-\frac{x^2}{2!},\\
p_3(x) &= 1+0-\frac{1}{2!}x^2+0 &&= 1-\frac{x^2}{2!},\\
p_4(x) &= 1+0-\frac{1}{2!}x^2+0+\frac{1}{4!}x^4 &&= 1-\frac{x^2}{2!}+\frac{x^4}{4!},\\
p_5(x) &= 1+0-\frac{1}{2!}x^2+0+\frac{1}{4!}x^4+0 &&= 1-\frac{x^2}{2!}+\frac{x^4}{4!},
\end{aligned}
$$

and for $n \geq 0$,

$$p_{2m}(x) = p_{2m+1}(x) = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots+(-1)^m\frac{x^{2m}}{(2m)!} = \sum_{k=0}^{m}(-1)^k\frac{x^{2k}}{(2k)!}.$$

Graphs of the function and the Maclaurin polynomials appear in Figure 5.3.4.

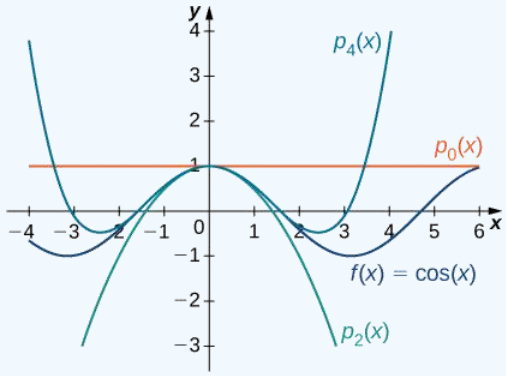


Figure 5.3.4: *The function $y = \cos x$ and the Maclaurin polynomials p_0, p_2, and p_4 are plotted on this graph.*

? Checkpoint 5.3.2

Find formulas for the Maclaurin polynomials p_0, p_1, p_2, and p_3 for $f(x) = \frac{1}{1+x}$.

Find a formula for the nth-degree Maclaurin polynomial. Write your answer using sigma notation.

Answer

$$p_0(x) = 1;\ p_1(x) = 1-x;\ p_2(x) = 1-x+x^2;\ p_3(x) = 1-x+x^2-x^3;\ p_n(x) = 1-x+x^2-x^3+\cdots+(-1)^n x^n = \sum_{k=0}^{n}(-1)^k x^k$$

Taylor's Theorem with Remainder

Recall that the nth-degree Taylor polynomial for a function f at a is the nth partial sum of the Taylor series for f at a. Therefore, to determine if the Taylor series converges, we need to determine whether the sequence of Taylor polynomials $\{p_n\}$ converges. However, not only do we want to know if the sequence of Taylor polynomials converges, we want to know if *it converges to* f. To answer this question, we define the remainder $R_n(x)$ as

$$R_n(x) = f(x) - p_n(x).$$

For the sequence of Taylor polynomials to converge to f, we need the remainder, R_n, to converge to zero. To determine if R_n converges to zero, we introduce **Taylor's Theorem with Remainder** (also known as **Taylor's Remainder Theorem**). Not only is this theorem useful in proving that a Taylor series converges to its related function, but it will also allow us to quantify how well the nth-degree Taylor polynomial approximates the function.

Here we look for a bound on $|R_n|$. Consider the simplest case: $n = 0$. Let p_0 be the 0th-degree Taylor polynomial at a for a function f. The remainder R_0 satisfies

$$R_0(x) = f(x) - p_0(x) = f(x) - f(a).$$

If f is differentiable on an interval I containing a and x, then by the Mean Value Theorem there exists a real number c between a and x such that $f(x) - f(a) = f'(c)(x - a)$. Therefore,

$$R_0(x) = f'(c)(x - a).$$

Using the Mean Value Theorem in a similar argument, we can show that if f is n times differentiable on an interval I containing a and x, then the nth remainder R_n satisfies

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - a)^{n+1}$$

for some real number c between a and x. It is important to note that the value c in the numerator above is not the center a, but rather an unknown value c between a and x. This formula allows us to get a bound on the remainder R_n. If we happen to know that $|f^{(n+1)}(x)|$ is bounded by some real number M on this interval I, then

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x - a|^{n+1}$$

for all x in the interval I.

We now state Taylor's theorem, which provides the formal relationship between a function f and its nth-degree Taylor polynomial $p_n(x)$. This theorem allows us to bound the error when using a Taylor polynomial to approximate a function value. It will be important in proving that a Taylor series for f converges to f.

Theorem: Taylor's Theorem with Remainder

Let f be a function that can be differentiated $n + 1$ times on an interval I containing the real number a. Let p_n be the nth-degree Taylor polynomial of f at a and let

$$R_n(x) = f(x) - p_n(x)$$

be the nth remainder. Then for each x in the interval I, there exists a real number c between a and x such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - a)^{n+1}.$$

If there exists a real number M such that $|f^{(n+1)}(x)| \leq M$ for all $x \in I$, then

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x - a|^{n+1}$$

for all x in I.

Proof

Fix a point $x \in I$ and introduce the function g such that

$$g(t) = f(x) - f(t) - f'(t)(x - t) - \frac{f''(t)}{2!}(x - t)^2 - \cdots - \frac{f^{(n)}(t)}{n!}(x - t)^n - R_n(x)\frac{(x - t)^{n+1}}{(x - a)^{n+1}}.$$

We claim that g satisfies the criteria of Rolle's Theorem. Since g is a polynomial function (in t), it is a differentiable function. Also, g is zero at $t = a$ and $t = x$ because

$$\begin{aligned} g(a) &= f(x) - f(a) - f'(a)(x-a) - \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n - R_n(x) \\ &= f(x) - p_n(x) - R_n(x) \\ &= 0, \\ g(x) &= f(x) - f(x) - 0 - \cdots - 0 \\ &= 0. \end{aligned}$$

Therefore, g satisfies Rolle's Theorem, and consequently, there exists c between a and x such that $g'(c) = 0$. We now calculate g'. Using the Product Rule, we note that

$$\frac{d}{dt}\left[\frac{f^{(n)}(t)}{n!}(x-t)^n\right] = -\frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} + \frac{f^{(n+1)}(t)}{n!}(x-t)^n.$$

Consequently,

$$\begin{aligned} g'(t) &= -f'(t) + [f'(t) - f''(t)(x-t)] + \left[f''(t)(x-t) - \frac{f'''(t)}{2!}(x-t)^2\right] + \cdots \\ &\quad + \left[\frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} - \frac{f^{(n+1)}(t)}{n!}(x-t)^n\right] + (n+1)R_n(x)\frac{(x-t)^n}{(x-a)^{n+1}} \end{aligned}$$

Notice that there is a telescoping effect. Therefore,

$$g'(t) = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n + (n+1)R_n(x)\frac{(x-t)^n}{(x-a)^{n+1}}.$$

By Rolle's Theorem, we conclude that there exists a number c between a and x such that $g'(c) = 0$. Since

$$g'(c) = -\frac{f^{(n+1)}(c)}{n!}(x-c)^n + (n+1)R_n(x)\frac{(x-c)^n}{(x-a)^{n+1}}$$

we conclude that

$$-\frac{f^{(n+1)}(c)}{n!}(x-c)^n + (n+1)R_n(x)\frac{(x-c)^n}{(x-a)^{n+1}} = 0.$$

Adding the first term on the left-hand side to both sides of the equation and dividing both sides of the equation by $n+1$, we conclude that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

as desired. From this fact, it follows that if there exists M such that $|f^{(n+1)}(x)| \leq M$ for all x in I, then

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}.$$

Not only does Taylor's Theorem allow us to prove that a Taylor series converges to a function, but it also allows us to estimate the accuracy of Taylor polynomials in approximating function values. We begin by looking at linear and quadratic approximations of $f(x) = \sqrt[3]{x}$ at $x = 8$ and determine how accurate these approximations are at estimating $\sqrt[3]{11}$.

Example 5.3.3: Using Linear and Quadratic Approximations to Estimate Function Values

Consider the function $f(x) = \sqrt[3]{x}$.

a. Find the first and second Taylor polynomials for f at $x = 8$. Use a graphing utility to compare these polynomials with f near $x = 8$.
b. Use these two polynomials to estimate $\sqrt[3]{11}$.
c. Use Taylor's theorem to bound the error.

Solutions

a. For $f(x) = \sqrt[3]{x}$, the values of the function and its first two derivatives at $x = 8$ are as follows:

$$\begin{aligned} f(x) &= \sqrt[3]{x}, \quad \Rightarrow \quad f(8) = 2 \\ f'(x) &= \frac{1}{3x^{2/3}}, \quad \Rightarrow \quad f'(8) = \frac{1}{12} \\ f''(x) &= \frac{-2}{9x^{5/3}}, \quad \Rightarrow \quad f''(8) = -\frac{1}{144}. \end{aligned}$$

Thus, the first and second Taylor polynomials at $x = 8$ are given by

$$\begin{aligned} p_1(x) &= f(8) + f'(8)(x-8) \\ &= 2 + \frac{1}{12}(x-8) \\ p_2(x) &= f(8) + f'(8)(x-8) + \frac{f''(8)}{2!}(x-8)^2 \\ &= 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2. \end{aligned}$$

The function and the Taylor polynomials are shown in Figure 5.3.5.

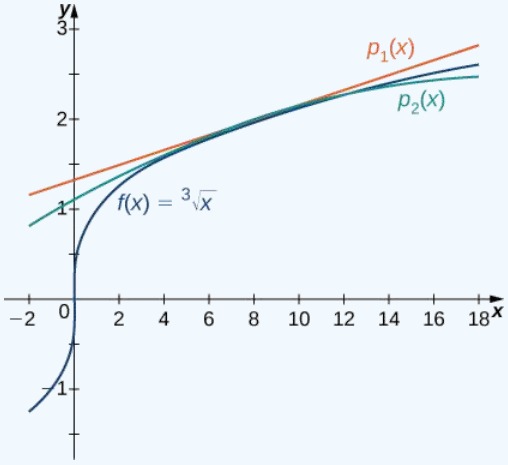


Figure 5.3.5: *The graphs of $f(x) = \sqrt[3]{x}$ and the linear and quadratic approximations $p_1(x)$ and $p_2(x)$*

b. Using the first Taylor polynomial at $x = 8$, we can estimate

$$\sqrt[3]{11} \approx p_1(11) = 2 + \frac{1}{12}(11-8) = 2.25.$$

Using the second Taylor polynomial at $x = 8$, we obtain

$$\sqrt[3]{11} \approx p_2(11) = 2 + \frac{1}{12}(11-8) - \frac{1}{288}(11-8)^2 = 2.21875.$$

c. There exists a c in the interval $(8, 11)$ such that the remainder when approximating $\sqrt[3]{11}$ by the first Taylor polynomial satisfies

$$R_1(11) = \frac{f''(c)}{2!}(11-8)^2.$$

We do not know the exact value of c, so we find an upper bound on $R_1(11)$ by determining the maximum value of f'' on the interval $(8, 11)$. Since $f''(x) = -\frac{2}{9x^{5/3}}$, the largest value for $|f''(x)|$ on that interval occurs at $x = 8$. Using the fact that $f''(8) = -\frac{1}{144}$, we obtain

$$|R_1(11)| \leq \frac{1}{144 \cdot 2!}(11-8)^2 = 0.03125.$$

Similarly, to estimate $R_2(11)$, we use the fact that

$$R_2(11) = \frac{f'''(c)}{3!}(11-8)^3.$$

Since

$$f'''(x) = \frac{10}{27x^{8/3}},$$

the maximum value of f''' on the interval $(8, 11)$ is $f'''(8) \approx 0.0014468$. Therefore, we have

$$|R_2(11)| \leq \frac{0.0011468}{3!}(11-8)^3 \approx 0.0065104.$$

Checkpoint 5.3.3

Find the first and second Taylor polynomials for $f(x) = \sqrt{x}$ at $x = 4$. Use these polynomials to estimate $\sqrt{6}$. Use Taylor's theorem to bound the error.

Answers

$p_1(x) = 2 + \frac{1}{4}(x-4); p_2(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2; p_1(6) = 2.5; p_2(6) = 2.4375;$

$|R_1(6)| \leq 0.0625; |R_2(6)| \leq 0.015625$

Example 5.3.4: Approximating $\sin x$ Using Maclaurin Polynomials

From Example 5.3.2*b*, the Maclaurin polynomials for $\sin x$ are given by

$$p_{2m+1}(x) = p_{2m+2}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^m \frac{x^{2m+1}}{(2m+1)!}$$

for $m = 0, 1, 2, \ldots$.

a. Use the fifth Maclaurin polynomial for $\sin x$ to approximate $\sin\left(\frac{\pi}{18}\right)$ and bound the error.
b. For what values of x does the fifth Maclaurin polynomial approximate $\sin x$ to within 0.0001?

Solutions

a. The fifth Maclaurin polynomial is

$$p_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}.$$

Using this polynomial, we can estimate as follows:

$$\sin\left(\frac{\pi}{18}\right) \approx p_5\left(\frac{\pi}{18}\right) = \frac{\pi}{18} - \frac{1}{3!}\left(\frac{\pi}{18}\right)^3 + \frac{1}{5!}\left(\frac{\pi}{18}\right)^5 \approx 0.173648.$$

To estimate the error, use the fact that the sixth Maclaurin polynomial is $p_6(x) = p_5(x)$ and calculate a bound on $R_6(\frac{\pi}{18})$. By Note, the remainder is

$$R_6\left(\frac{\pi}{18}\right) = \frac{f^{(7)}(c)}{7!}\left(\frac{\pi}{18}\right)^7$$

for some c between 0 and $\frac{\pi}{18}$. Using the fact that $|f^{(7)}(x)| \leq 1$ for all x, we find that the magnitude of the error is at most

$$\frac{1}{7!} \cdot \left(\frac{\pi}{18}\right)^7 \leq 9.8 \times 10^{-10}.$$

b. We need to find the values of x such that

$$\frac{1}{7!}|x|^7 \leq 0.0001.$$

Solving this inequality for x, we have that the fifth Maclaurin polynomial gives an estimate to within 0.0001 as long as $|x| < 0.907$.

Checkpoint 5.3.4

Use the fourth Maclaurin polynomial for $\cos x$ to approximate $\cos\left(\frac{\pi}{12}\right)$.

Answer

0.96593

Now that we can bound the remainder $R_n(x)$, we can use this bound to prove that a Taylor series for f at a converges to f.

Representing Functions with Taylor and Maclaurin Series

We now discuss issues of convergence for Taylor series. We begin by showing how to find a Taylor series for a function and how to find its interval of convergence.

Example 5.3.5: Finding a Taylor Series

Find the Taylor series for $f(x) = \frac{1}{x}$ at $x = 1$. Determine the interval of convergence.

Solution

For $f(x) = \frac{1}{x}$, the values of the function and its first four derivatives at $x = 1$ are

$$\begin{aligned} f(x) &= \frac{1}{x} \quad \Rightarrow \quad f(1) = 1 \\ f'(x) &= -\frac{1}{x^2} \quad \Rightarrow \quad f'(1) = -1 \\ f''(x) &= \frac{2}{x^3} \quad \Rightarrow \quad f''(1) = 2! \\ f'''(x) &= -\frac{3 \cdot 2}{x^4} \quad \Rightarrow \quad f'''(1) = -3! \\ f^{(4)}(x) &= \frac{4 \cdot 3 \cdot 2}{x^5} \quad \Rightarrow \quad f^{(4)}(1) = 4!. \end{aligned}$$

That is, we have $f^{(n)}(1) = (-1)^n n!$ for all $n \geq 0$. Therefore, the Taylor series for f at $x = 1$ is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = \sum_{n=0}^{\infty} (-1)^n (x-1)^n.$$

To find the interval of convergence, we use the Ratio Test. We find that

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|\,(-1)^{n+1}(x-1)^{n+1}\,|}{|(-1)^n(x-1)^n|} = |x-1|.$$

Thus, the series converges if $|x-1| < 1$. That is, the series converges for $0 < x < 2$. Next, we need to check the endpoints. At $x = 2$, we see that

$$\sum_{n=0}^{\infty} (-1)^n (2-1)^n = \sum_{n=0}^{\infty} (-1)^n$$

diverges by the Divergence Test. Similarly, at $x = 0$,

$$\sum_{n=0}^{\infty} (-1)^n (0-1)^n = \sum_{n=0}^{\infty} (-1)^{2n} = \sum_{n=0}^{\infty} 1$$

diverges. Therefore, the interval of convergence is $(0, 2)$.

Checkpoint 5.3.5

Find the Taylor series for $f(x) = \frac{1}{2x}$ at $x = 2$ and determine its interval of convergence.

Answer

$\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{2-x}{2}\right)^n$. The interval of convergence is $(0, 4)$.

We know that the Taylor series found in this example converges on the interval $(0, 2)$, but how do we know it converges to f? We consider this question in more generality in a moment. Still, for this example, we can answer this question by writing

$$f(x) = \frac{1}{x} = \frac{1}{1-(1-x)}.$$

That is, f can be represented by the geometric series $\sum_{n=0}^{\infty}(1-x)^n$. Since this is a geometric series, it converges to $\frac{1}{x}$ as long as $|1 - x| < 1$. Therefore, the Taylor series found in Example 5.3.5 does converge to $f(x) = \frac{1}{x}$ on $(0, 2)$.

We now consider the more general question: if a Taylor series for a function f converges on some interval, how can we determine if it converges to f? To answer this question, recall that a series converges to a particular value if and only if its sequence of partial sums converges to that value. Given a Taylor series for f at a, the nth partial sum is given by the nth-degree Taylor polynomial p_n. Therefore, to determine if the Taylor series converges to f, we need to determine whether

$$\lim_{n\to\infty} p_n(x) = f(x).$$

Since the remainder $R_n(x) = f(x) - p_n(x)$, the Taylor series converges to f if and only if

$$\lim_{n\to\infty} R_n(x) = 0.$$

We now state this theorem formally.

Theorem: Convergence of Taylor Series

Suppose that f has derivatives of all orders on an interval I containing a. Then the Taylor series

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

converges to $f(x)$ for all x in I if and only if

$$\lim_{n\to\infty} R_n(x) = 0$$

for all x in I.

With this theorem, we can prove that a Taylor series for f at a converges to f if we can prove that the remainder $R_n(x) \to 0$. To prove that $R_n(x) \to 0$, we typically use the bound

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$$

from Taylor's Theorem with Remainder.

In the next example, we find the Maclaurin series for e^x and $\sin x$ and show that these series converge to the corresponding functions for all real numbers by proving that the remainders $R_n(x) \to 0$ for all real numbers x.

Example 5.3.6: Finding Maclaurin Series

For each of the following functions, find the Maclaurin series and its interval of convergence. Use the Convergence of Taylor Series theorem to prove that the Maclaurin series for f converges to f on that interval.

a. e^x

b. $\sin x$

Solutions

a. a. Using the nth-degree Maclaurin polynomial for e^x found in Example 5.3.2a., we find that the Maclaurin series for e^x is given by

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

To determine the interval of convergence, we use the Ratio Test. Since

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|x|^{n+1}}{(n+1)!} \cdot \frac{n!}{|x|^n} = \frac{|x|}{n+1},$$

we have

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x|}{n+1} = 0$$

for all x. Therefore, the series converges absolutely for all x, and thus, the interval of convergence is $(-\infty, \infty)$. To show that the series converges to e^x for all x, we use the fact that $f^{(n)}(x) = e^x$ for all $n \geq 0$ and e^x is an increasing function on $(-\infty, \infty)$. Therefore, for any real number b, the maximum value of e^x for all $|x| \leq b$ is e^b. Thus,

$$|R_n(x)| \leq \frac{e^b}{(n+1)!}|x|^{n+1}.$$

Since we just showed that

$$\sum_{n=0}^{\infty} \frac{|x|^n}{n!}$$

converges for all x, by the Divergence Test, we know that

$$\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$$

for any real number x. By combining this fact with the Squeeze Theorem, the result is $\lim_{n\to\infty} R_n(x) = 0$.

b. Using the nth-degree Maclaurin polynomial for $\sin x$ found in Example 5.3.2b., we find that the Maclaurin series for $\sin x$ is given by

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

In order to apply the Ratio Test, consider

$$\begin{aligned} \frac{|a_{n+1}|}{|a_n|} &= \frac{|x|^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{|x|^{2n+1}} \\ &= \frac{|x|^2}{(2n+3)(2n+2)}. \end{aligned}$$

Since

$$\lim_{n\to\infty} \frac{|x|^2}{(2n+3)(2n+2)} = 0$$

for all x, we obtain the interval of convergence as $(-\infty, \infty)$. To show that the Maclaurin series converges to $\sin x$, look at $R_n(x)$. For each x there exists a real number c between 0 and x such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}.$$

Since $|f^{(n+1)}(c)| \leq 1$ for all integers n and all real numbers c, we have

$$|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}$$

for all real numbers x. Using the same idea as in part a., the result is $\lim_{n\to\infty} R_n(x) = 0$ for all x, and therefore, the Maclaurin series for $\sin x$ converges to $\sin x$ for all real x.

? Checkpoint 5.3.6

Find the Maclaurin series for $f(x) = \cos x$. Use the Ratio Test to show that the interval of convergence is $(-\infty, \infty)$. Show that the Maclaurin series converges to $\cos x$ for all real numbers x.

Answer

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

By the Ratio Test, the interval of convergence is $(-\infty, \infty)$. Since $|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}$, the series converges to $\cos x$ for all real x.

Deriving New Maclaurin Series from Known Series

At this point, we have derived Maclaurin series for exponential, trigonometric, and logarithmic functions. In Table **5.3.1**, we summarize the results of these series. We remark that the convergence of the Maclaurin series for $f(x) = \ln(1+x)$ at the endpoint $x = 1$ and the Maclaurin series for $f(x) = \tan^{-1} x$ at the endpoints $x = 1$ and $x = -1$ relies on a more advanced theorem than we present here. (Refer to Abel's Theorem for a discussion of this more technical point.)

Table **5.3.1**: Maclaurin Series for Common Functions

Function	Maclaurin Series	Interval of Convergence
$f(x) = \frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$-1 < x < 1$
$f(x) = e^x$	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$-\infty < x < \infty$
$f(x) = \sin x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	$-\infty < x < \infty$
$f(x) = \cos x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	$-\infty < x < \infty$
$f(x) = \ln(1+x)$	$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$	$-1 < x \leq 1$
$f(x) = \tan^{-1} x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	$-1 \leq x \leq 1$

Earlier in the chapter, we showed how you could combine power series to create new power series. Here we use these properties, combined with the Maclaurin series in Table **5.3.1**, to create Maclaurin series for other functions.

Example 5.3.7: Deriving Maclaurin Series from Known Series

Find the Maclaurin series of each of the following functions by using one of the series listed in Table 5.3.1.

a. $f(x) = \cos\sqrt{x}$
b. $f(x) = \sinh x$

Solutions

a. Using the Maclaurin series for $\cos x$ we find that the Maclaurin series for $\cos\sqrt{x}$ is given by

$$\sum_{n=0}^{\infty} \frac{(-1)^n(\sqrt{x})^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!} = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \frac{x^4}{8!} - \cdots.$$

This series converges to $\cos\sqrt{x}$ for all x in the domain of $\cos\sqrt{x}$; that is, for all $x \geq 0$.

b. To find the Maclaurin series for $\sinh x$, we use the fact that

$$\sinh x = \frac{e^x - e^{-x}}{2}.$$

Using the Maclaurin series for e^x, we see that the nth term in the Maclaurin series for $\sinh x$ is given by

$$\frac{x^n}{n!} - \frac{(-x)^n}{n!}.$$

For n even, this term is zero. For n odd, this term is $\frac{2x^n}{n!}$. Therefore, the Maclaurin series for $\sinh x$ has only odd-order terms and is given by

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots.$$

Exercise 5.3.7

Find the Maclaurin series for $\sin(x^2)$.

Answer

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}$$

Revisiting Limits

At this point, many students feel a sense of frustration at not knowing why Maclaurin and Taylor series are a required investigation in Calculus. In truth, the functions encountered in applications of science and engineering are often not the "nice" functions you have been dealing with since Algebra. It is more often the case that you are not going to be dealing with explicit functions at all and, instead, will only have datasets and regressions (neither of which are the focus of a traditional Calculus course).

To illustrate some of the flexibility we gain by having Taylor (and Maclaurin) series, we reach back to our first investigation in Differential Calculus - limits.

Example 5.3.8: Determining the Value of a Limit

Evaluate the limit using a series.

$$\lim_{x \to 0} \frac{\sin(x) - x + \frac{1}{6}x^3}{x^5}$$

Solution

Since this is an indeterminate limit of the form $0/0$, we could easily use l'Hospital's Rule to find its value; however, using a mathematical tool (like l'Hospital's Rule) simply because we have "always done it that way" closes our minds to the possibilities of other, more efficient, methods. Recall, from Example **5.3.6***b*, that

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

for all $x \in \mathbb{R}$. Therefore,

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin(x) - x - \frac{1}{6}x^3}{x^5} &= \lim_{x\to 0} \frac{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} - x + \frac{1}{6}x^3}{x^5} \\
&= \lim_{x\to 0} \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots\right) - x + \frac{1}{6}x^3}{x^5} \\
&= \lim_{x\to 0} \frac{\cancel{x} - \cancel{\frac{1}{6}x^3} + \frac{x^5}{120} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots - \cancel{x} + \cancel{\frac{1}{6}x^3}}{x^5} \\
&= \lim_{x\to 0} \frac{\frac{x^5}{120} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots}{x^5} \\
&= \lim_{x\to 0} \frac{1}{120} - \frac{x^2}{7!} + \frac{x^4}{9!} - \cdots \\
&= \frac{1}{120}
\end{aligned}$$

The lazy student might look at the amount of work performed in Example **5.3.8** and feel inclined to use l'Hospital's Rule instead. In doing so, they miss a golden opportunity to "streamline" their Calculus skills. In truth, the savvy student (once familiarized with the Maclaurin series of the sine) would immediately recognize the polynomial in the numerator of the limit as the opposite of the first two terms of the Maclaurin series expansion of the sine. Knowing the remaining terms of said expansion are $\frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$, the student would visualize the cancellation of the x^5 between numerator and denominator - leaving $\frac{1}{5!} - (\text{some number})x^2 + (\text{some number})x^4 - \cdots$. All the variable terms would tend to zero leaving $\frac{1}{5!} = \frac{1}{120}$.

? Exercise 5.3.8

Evaluate the limit using a series.

$$\lim_{x\to 0} \frac{1 + x + \frac{1}{2}x^2 - e^x}{x^3}$$

Answer

$-\frac{1}{6}$

Revisiting Integration

In terms of complexity, evaluating limits and differentiating functions is much easier than integration. For example, the limit

$$\lim_{x\to\infty} x^2 e^{-x^2} = 0$$

and the derivative

$$\frac{d}{dx} x^2 e^{-x^2} = 2xe^{-x^2} - 2x^3 e^{-x^2} = 2xe^{-x^2}(1-x)(1+x)$$

are simple affairs at this point. However, up to this point, evaluating the definite integral

$$\int_0^{0.5} x^2 e^{-x^2}\, dx$$

is a disaster without using numerical integration techniques.

Truly, it is for integration that Taylor and Maclaurin series are profoundly useful. Specifically, power series can be used to evaluate integrals involving functions whose antiderivatives cannot be expressed using elementary functions. By **elementary function**, we mean a function that can be written using a finite number of algebraic combinations or compositions of exponential, logarithmic, trigonometric, or power functions. We remark that the term "elementary function" is not synonymous with noncomplicated function. For example, the function $f(x) = \sqrt{x^2 - 3x} + e^{x^3} - \sin(5x + 4)$ is an elementary function, although not a particularly simple-looking function. Any integral of the form $\int f(x)\,dx$ where the antiderivative of f cannot be written as an elementary function is considered a **non-elementary integral**.

Non-elementary integrals cannot be evaluated using the basic integration techniques discussed earlier. One way to evaluate such integrals is by expressing the integrand as a power series and integrating term by term. We demonstrate this technique by considering $\int x^2 e^{-x^2}\,dx$.

Example 5.3.9: Integration Involving Maclaurin Series

a. Represent $\int x^2 e^{-x^2}\,dx$ as an infinite series.

b. Evaluate the definite integral to within 0.0000001.

$$\int_0^{0.5} x^2 e^{-x^2}\,dx$$

Solution

a. We begin by noting that

$$x^2 e^{-x^2} = x^2 \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n!}.$$

Hence,

$$\begin{aligned} \int x^2 e^{-x^2}\,dx &= \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n!} \\ &= C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{n!(2n+3)} \end{aligned}$$

b. Using the result from part a, we get

$$\begin{aligned} \int_0^{0.5} x^2 e^{-x^2}\,dx &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{n!(2n+3)}\Bigg|_0^{0.5} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}\right)^{2n+3}}{n!(2n+3)} - \sum_{n=0}^{\infty} \frac{(-1)^n (0)^{2n+3}}{n!(2n+3)} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+3)2^{2n+3}} \end{aligned}$$

If we let $b_n = \frac{1}{n!(2n+3)2^{2n+3}}$, then

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n!(2n+3)2^{2n+3}} = 0.$$

Moreover,

$$(n+1)!(2(n+1)+3)2^{2(n+1)+3} = (n+1)!(2n+5)2^{2n+5} > n!(2n+3)2^{2n+3}$$

and so

$$\frac{1}{(n+1)!(2(n+1)+3)2^{2(n+1)+3}} < \frac{1}{n!(2n+3)2^{2n+3}} \implies b_{n+1} < b_n.$$

Hence, the sequence, $\{b_n\}$, is decreasing. As such, the series converges by the Alternating Series Test. The error in approximating the value of this series is bounded by b_{n+1}. Therefore, we want

$$b_{n+1} < 0.0000001 \implies \frac{1}{(n+1)!(2(n+1)+3)2^{2(n+1)+3}} < 0.0000001$$

Sadly, the best way to determine the value of n satisfying this inequality is to try $n = 1, 2, 3, \ldots$ until we get the left side to be less than $0.0000001 = 10^{-7}$. Using the **TABLE** feature in Desmos, we see that $n = 4$ will work fine.

x_1	$\frac{1}{(x_1+1)!(2x_1+5)2^{(2x_1+5)}}$
1	5.580357×10^{-4}
2	3.616898×10^{-5}
3	1.84955×10^{-6}
4	7.82502×10^{-8}

Hence,

$$\int_0^{0.5} x^2 e^{-x^2}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+3)2^{2n+3}} \approx \sum_{n=0}^{4} \frac{(-1)^n}{n!(2n+3)2^{2n+3}} \approx 0.03594038$$

? Exercise 5.3.9

a. Express $\int e^{-x^2}\,dx$ as an infinite series.

b. Evaluate $\int_0^1 e^{-x^2}\,dx$ to within an error of 0.01.

Answer

a. $$C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)n!} = C + x - \frac{x^3}{3} + \frac{x^5}{5(2!)} - \frac{x^7}{7(3!)} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)n!} + \cdots$$

b. Using the result from part a. we have

$$\int_0^1 e^{-x^2}\,dx = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \cdots.$$

The sum of the first four terms is approximately 0.74. By the Alternating Series Test, this estimate is accurate to within an error of less than $\frac{1}{216} \approx 0.0046296 < 0.01$.

Identifying Series

Once we get used to the *forms* of specific Maclaurin series (see Table **5.3.1**), we can sometimes worked backwards and identify a function given its power series. To be clear, this requires some comfort with the Maclaurin series we have derived thus far.

✓ Example 5.3.10

Find the sum of the series.

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}\pi^{2n+1}}{7 \cdot 3^{2n}(2n)!}$$

Solution

We begin by noting that the given series is very close to the form of the Maclaurin series for the cosine,

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$

Performing some manipulations (specifically, to obtain the necessary powers in the Maclaurin series for the cosine), we get

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-1)^{n+1}\pi^{2n+1}}{7 \cdot 3^{2n}(2n)!} &= -\frac{\pi}{7} \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{3^{2n}(2n)!} \\ &= -\frac{\pi}{7} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{3}\right)^{2n}}{(2n)!} \\ &= -\frac{\pi}{7} \cos\left(\tfrac{\pi}{3}\right) \\ &= -\frac{\pi}{14} \end{aligned}$$

? Exercise 5.3.10

Find the sum of the series.

$$\sum_{n=0}^{\infty} \frac{2^{n-1}}{3^{n+1} n!}$$

Answer

$\frac{1}{6} e^{2/3}$

Proving that e is Irrational

In this project, we use the Maclaurin polynomials for e^x to prove that e is irrational. The proof relies on supposing that e is rational and arriving at a contradiction. Therefore, in the following steps, we suppose $e = r/s$ for some integers r and s where $s \neq 0$.

1. Write the Maclaurin polynomials $p_0(x), p_1(x), p_2(x), p_3(x), p_4(x)$ for e^x. Evaluate $p_0(1), p_1(1), p_2(1), p_3(1), p_4(1)$ to estimate e.
2. Let $R_n(x)$ denote the remainder when using $p_n(x)$ to estimate e^x. Therefore, $R_n(x) = e^x - p_n(x)$, and $R_n(1) = e - p_n(1)$. Assuming that $e = \frac{r}{s}$ for integers r and s, evaluate $R_0(1), R_1(1), R_2(1), R_3(1), R_4(1)$.
3. Using the results from part 2, show that for each remainder $R_0(1), R_1(1), R_2(1), R_3(1), R_4(1)$, we can find an integer k such that $kR_n(1)$ is an integer for $n = 0, 1, 2, 3, 4$.
4. Write down the formula for the nth-degree Maclaurin polynomial $p_n(x)$ for e^x and the corresponding remainder $R_n(x)$. Show that $sn!R_n(1)$ is an integer.
5. Use Taylor's theorem to write down an explicit formula for $R_n(1)$. Conclude that $R_n(1) \neq 0$, and therefore, $sn!R_n(1) \neq 0$.
6. Use Taylor's theorem to find an estimate on $R_n(1)$. Use this estimate combined with the result from part 5 to show that $|sn!R_n(1)| < \frac{se}{n+1}$. Conclude that if n is large enough, then $|sn!R_n(1)| < 1$. Therefore, $sn!R_n(1)$ is an integer with magnitude less than 1. Thus, $sn!R_n(1) = 0$. But from part 5, we know that $sn!R_n(1) \neq 0$. We have arrived at a contradiction, and consequently, the original supposition that e is rational must be false.

Key Concepts

- Taylor polynomials are used to approximate functions near a value $x = a$. Maclaurin polynomials are Taylor polynomials at $x = 0$.

- The nth-degree Taylor polynomials for a function f are the partial sums of the Taylor series for f.
- If a function f has a power series representation at $x = a$, then it is given by its Taylor series at $x = a$.
- A Taylor series for f converges to f if and only if $\lim_{n\to\infty} R_n(x) = 0$ where $R_n(x) = f(x) - p_n(x)$.
- The Taylor series for e^x, $\sin x$, and $\cos x$ converge to the respective functions for all real x.

Key Equations

- **Taylor series for the function f at the point $x = a$**

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

Glossary

Maclaurin polynomial

a Taylor polynomial centered at 0; the nth-degree Taylor polynomial for f at 0 is the nth-degree Maclaurin polynomial for f

Maclaurin series

a Taylor series for a function f at $x = 0$ is known as a Maclaurin series for f

Taylor polynomials

the nth-degree Taylor polynomial for f at $x = a$ is $p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$

Taylor series

a power series at a that converges to a function f on some open interval containing a.

Taylor's theorem with remainder

for a function f and the nth-degree Taylor polynomial for f at $x = a$, the remainder $R_n(x) = f(x) - p_n(x)$ satisfies $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$

for some c between x and a; if there exists an interval I containing a and a real number M such that $|f^{(n+1)}(x)| \leq M$ for all x in I, then $|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$

5.3E: Exercises

Taylor Polynomials

In exercises 1 - 8, find the Taylor polynomials of degree two approximating the given function centered at the given point.

1) $f(x) = 1 + x + x^2$ at $a = 1$

2) $f(x) = 1 + x + x^2$ at $a = -1$

Answer

$f(-1) = 1;\ f'(-1) = -1;\ f''(-1) = 2; \quad p_2(x) = 1 - (x+1) + (x+1)^2$

3) $f(x) = \cos(2x)$ at $a = \pi$

4) $f(x) = \sin(2x)$ at $a = \frac{\pi}{2}$

Answer

$f'(x) = 2\cos(2x);\ f''(x) = -4\sin(2x); \quad p_2(x) = -2(x - \frac{\pi}{2})$

5) $f(x) = \sqrt{x}$ at $a = 4$

6) $f(x) = \ln x$ at $a = 1$

Answer

$f'(x) = \dfrac{1}{x};\ f''(x) = -\dfrac{1}{x^2}; \quad p_2(x) = 0 + (x-1) - \frac{1}{2}(x-1)^2$

7) $f(x) = \dfrac{1}{x}$ at $a = 1$

8) $f(x) = e^x$ at $a = 1$

Answer

$p_2(x) = e + e(x-1) + \dfrac{e}{2}(x-1)^2$

Taylor Remainder Theorem

In exercises 9 - 14, verify that the given choice of n in the remainder estimate $|R_n| \le \dfrac{M}{(n+1)!}(x-a)^{n+1}$, where M is the maximum value of $| f^{(n+1)}(z) |$ on the interval between a and the indicated point, yields $|R_n| \le \frac{1}{1000}$. Find the value of the Taylor polynomial p_n of f at the indicated point.

9) [Technology Required] $\sqrt{10};\ a = 9,\ n = 3$

10) [Technology Required] $(28)^{1/3};\ a = 27,\ n = 1$

Answer

$\dfrac{d^2}{dx^2}x^{1/3} = -\dfrac{2}{9x^{5/3}} \ge -0.00092\ldots$ when $x \ge 28$ so the remainder estimate applies to the linear approximation $x^{1/3} \approx p_1(27) = 3 + \dfrac{x-27}{27}$, which gives $(28)^{1/3} \approx 3 + \frac{1}{27} = 3.0\overline{37}$, while $(28)^{1/3} \approx 3.03658$.

11) [Technology Required] $\sin(6);\ a = 2\pi,\ n = 5$

12) [Technology Required] $e^2;\ a = 0,\ n = 9$

Answer

Using the estimate $\dfrac{2^{10}}{10!} < 0.000283$ we can use the Taylor expansion of order 9 to estimate e^x at $x = 2$. as $e^2 \approx p_9(2) = 1 + 2 + \frac{2^2}{2} + \frac{2^3}{6} + \cdots + \frac{2^9}{9!} = 7.3887$ \ldots whereas $e^2 \approx 7.3891$.

13) [Technology Required] $\cos(\frac{\pi}{5});\ a=0,\ n=4$

14) [Technology Required] $\ln(2);\ a=1,\ n=1000$

Answer

Since $\frac{d^n}{dx^n}(\ln x)=(-1)^{n-1}\frac{(n-1)!}{x^n}, R_{1000}\approx\frac{1}{1001}$. One has $p_{1000}(1)=\sum_{n=1}^{1000}\frac{(-1)^{n-1}}{n}\approx 0.6936$ whereas $\ln(2)\approx 0.6931\cdots$.

Approximating Definite Integrals Using Taylor Series

15) Integrate the approximation $\sin t\approx t-\frac{t^3}{6}+\frac{t^5}{120}-\frac{t^7}{5040}$ evaluated at πt to approximate $\int_0^1\frac{\sin\pi t}{\pi t}\,dt$.

16) Integrate the approximation $e^x\approx 1+x+\frac{x^2}{2}+\cdots+\frac{x^6}{720}$ evaluated at $-x^2$ to approximate $\int_0^1 e^{-x^2}\,dx$.

Answer

$\int_0^1\left(1-x^2+\frac{x^4}{2}-\frac{x^6}{6}+\frac{x^8}{24}-\frac{x^{10}}{120}+\frac{x^{12}}{720}\right)dx=1-\frac{1^3}{3}+\frac{1^5}{10}-\frac{1^7}{42}+\frac{1^9}{9\cdot 24}-\frac{1^{11}}{120\cdot 11}+\frac{1^{13}}{720\cdot 13}\approx 0.74683$ whereas $\int_0^1 e^{-x^2}dx\approx 0.74682.$

More Taylor Remainder Theorem Problems

In exercises 17 - 20, find the smallest value of n such that the remainder estimate $|R_n|\le\frac{M}{(n+1)!}(x-a)^{n+1}$, where M is the maximum value of $|\,f^{(n+1)}(z)\,|$ on the interval between a and the indicated point, yields $|R_n|\le\frac{1}{1000}$ on the indicated interval.

17) $f(x)=\sin x$ on $[-\pi,\pi],\ a=0$

18) $f(x)=\cos x$ on $[-\frac{\pi}{2},\frac{\pi}{2}],\ a=0$

Answer

Since $f^{(n+1)}(z)$ is $\sin z$ or $\cos z$, we have $M=1$. Since $|x-0|\le\frac{\pi}{2}$, we seek the smallest n such that $\frac{\pi^{n+1}}{2^{n+1}(n+1)!}\le 0.001$. The smallest such value is $n=7$. The remainder estimate is $R_7\le 0.00092$.

19) $f(x)=e^{-2x}$ on $[-1,1], a=0$

20) $f(x)=e^{-x}$ on $[-3,3], a=0$

Answer

Since $f^{(n+1)}(z)=\pm e^{-z}$ one has $M=e^3$. Since $|x-0|\le 3$, one seeks the smallest n such that $\frac{3^{n+1}e^3}{(n+1)!}\le 0.001$. The smallest such value is $n=14$. The remainder estimate is $R_{14}\le 0.000220$.

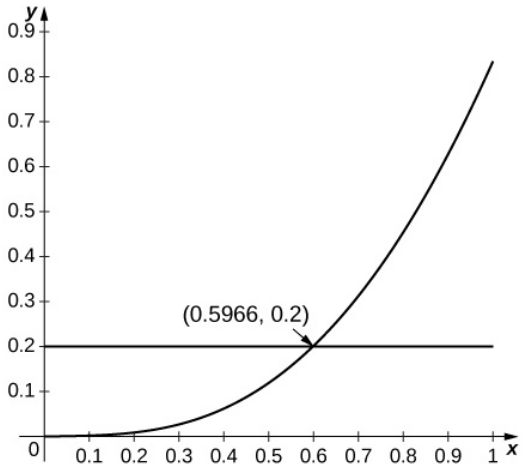
In exercises 21 - 24, the maximum of the right-hand side of the remainder estimate $|R_1|\le\frac{max|f''(z)|}{2}R^2$ on $[a-R,a+R]$ occurs at a or $a\pm R$. Estimate the maximum value of R such that $\frac{max|f''(z)|}{2}R^2\le 0.1$ on $[a-R,a+R]$ by plotting this maximum as a function of R.

21) [Technology Required] e^x approximated by $1+x,\ a=0$

22) [Technology Required] $\sin x$ approximated by $x,\ a=0$

Answer

Since $\sin x$ is increasing for small x and since $\frac{d^2}{dx^2}(\sin x)=-\sin x$, the estimate applies whenever $R^2\sin(R)\le 0.2$, which applies up to $R=0.596$.

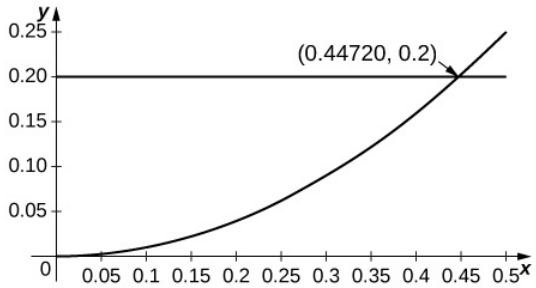


23) [Technology Required] $\ln x$ approximated by $x-1,\ a=1$

24) [Technology Required] $\cos x$ approximated by $1,\ a=0$

Answer

Since the second derivative of $\cos x$ is $-\cos x$ and since $\cos x$ is decreasing away from $x=0$, the estimate applies when $R^2 \cos R \leq 0.2$ or $R \leq 0.447$.



Taylor Series

In exercises 25 - 35, find the Taylor series of the given function centered at the indicated point.

25) $f(x)=x^4$ at $a=-1$

26) $f(x)=1+x+x^2+x^3$ at $a=-1$

Answer

$(x+1)^3-2(x+1)^2+2(x+1)$

27) $f(x)=\sin x$ at $a=\pi$

28) $f(x)=\cos x$ at $a=2\pi$

Answer

Values of derivatives are the same as for $x=0$ so $\cos x=\sum_{n=0}^{\infty}(-1)^n\frac{(x-2\pi)^{2n}}{(2n)!}$

29) $f(x)=\sin x$ at $x=\frac{\pi}{2}$

30) $f(x)=\cos x$ at $x=\frac{\pi}{2}$

Answer

$\cos(\frac{\pi}{2}) = 0,\ -\sin(\frac{\pi}{2}) = -1$ so $\cos x = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-\frac{\pi}{2})^{2n+1}}{(2n+1)!}$, which is also $-\cos(x-\frac{\pi}{2})$.

31) $f(x) = e^x$ at $a = -1$

32) $f(x) = e^x$ at $a = 1$

Answer

The derivatives are $f^{(n)}(1) = e$, so $e^x = e\sum_{n=0}^{\infty} \frac{(x-1)^n}{n!}$.

33) $f(x) = \frac{1}{(x-1)^2}$ at $a = 0$ (Hint: Differentiate the Taylor Series for $\frac{1}{1-x}$.)

34) $f(x) = \frac{1}{(x-1)^3}$ at $a = 0$

Answer

$$\frac{1}{(x-1)^3} = -\frac{1}{2}\frac{d^2}{dx^2}\left(\frac{1}{1-x}\right) = -\sum_{n=0}^{\infty}\left(\frac{(n+2)(n+1)x^n}{2}\right)$$

35) $F(x) = \int_0^x \cos(\sqrt{t})\,dt;$ where $f(t) = \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{(2n)!}$ at a=0 (Note: f is the Taylor series of $\cos(\sqrt{t})$.)

In exercises 36 - 44, compute the Taylor series of each function around $x = 1$.

36) $f(x) = 2 - x$

Answer

$2 - x = 1 - (x-1)$

37) $f(x) = x^3$

38) $f(x) = (x-2)^2$

Answer

$((x-1)-1)^2 = (x-1)^2 - 2(x-1) + 1$

39) $f(x) = \ln x$

40) $f(x) = \frac{1}{x}$

Answer

$$\frac{1}{1-(1-x)} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$$

41) $f(x) = \frac{1}{2x - x^2}$

42) $f(x) = \frac{x}{4x - 2x^2 - 1}$

Answer

$$x\sum_{n=0}^{\infty} 2^n (1-x)^{2n} = \sum_{n=0}^{\infty} 2^n (x-1)^{2n+1} + \sum_{n=0}^{\infty} 2^n (x-1)^{2n}$$

43) $f(x) = e^{-x}$

44) $f(x) = e^{2x}$

Answer

$$e^{2x} = e^{2(x-1)+2} = e^2 \sum_{n=0}^{\infty} \frac{2^n (x-1)^n}{n!}$$

[Technology Required] In exercises 45 - 48, identify the value of x such that the given series $\sum_{n=0}^{\infty} a_n$ is the value of the Maclaurin series of $f(x)$ at x. Approximate the value of $f(x)$ using $S_{10} = \sum_{n=0}^{10} a_n$.

45) $\sum_{n=0}^{\infty} \frac{1}{n!}$

46) $\sum_{n=0}^{\infty} \frac{2^n}{n!}$

Answer

$x = e^2; \quad S_{10} = \frac{34,913}{4725} \approx 7.3889947$

47) $\sum_{n=0}^{\infty} \frac{(-1)^n (2\pi)^{2n}}{(2n)!}$

48) $\sum_{n=0}^{\infty} \frac{(-1)^n (2\pi)^{2n+1}}{(2n+1)!}$

Answer

$\sin(2\pi) = 0; \quad S_{10} = 8.27 \times 10^{-5}$

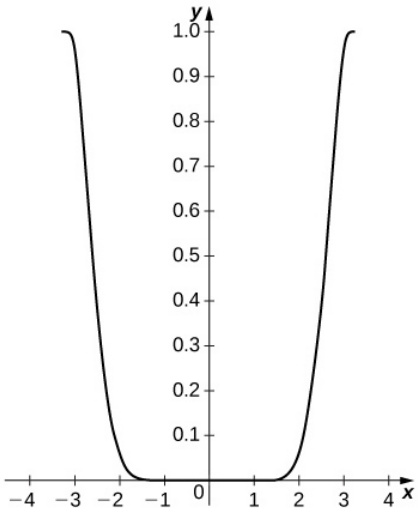
In exercises 49 - 52 use the functions $S_5(x) = x - \frac{x^3}{6} + \frac{x^5}{120}$ and $C_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$ on $[-\pi, \pi]$.

49) [Technology Required] Plot $\sin^2 x - (S_5(x))^2$ on $[-\pi, \pi]$. Compare the maximum difference with the square of the Taylor remainder estimate for $\sin x$.

50) [Technology Required] Plot $\cos^2 x - (C_4(x))^2$ on $[-\pi, \pi]$. Compare the maximum difference with the square of the Taylor remainder estimate for $\cos x$.

Answer

The difference is small on the interior of the interval but approaches 1 near the endpoints. The remainder estimate is $|R_4| = \frac{\pi^5}{120} \approx 2.552$.

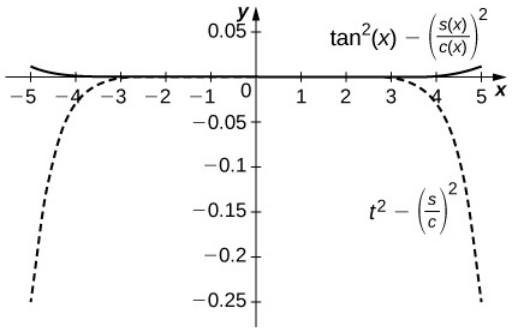


51) [Technology Required] Plot $|2S_5(x)C_4(x) - \sin(2x)|$ on $[-\pi, \pi]$.

52) [Technology Required] Compare $\frac{S_5(x)}{C_4(x)}$ on $[-1,1]$ to $\tan x$. Compare this with the Taylor remainder estimate for the approximation of $\tan x$ by $x+\frac{x^3}{3}+\frac{2x^5}{15}$.

Answer

The difference is on the order of 10^{-4} on $[-1,1]$ while the Taylor approximation error is around 0.1 near ± 1. The top curve is a plot of $\tan^2 x - \left(\frac{S_5(x)}{C_4(x)}\right)^2$ and the lower dashed plot shows $t^2 - \left(\frac{S_5}{C_4}\right)^2$.



53) [Technology Required] Plot $e^x - e_4(x)$ where $e_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$ on $[0,2]$. Compare the maximum error with the Taylor remainder estimate.

54) (Taylor approximations and root finding.) Recall that Newton's method $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ approximates solutions of $f(x) = 0$ near the input x_0.

a. If f and g are inverse functions, explain why a solution of $g(x) = a$ is the value $f(a)$ of f.

b. Let $p_N(x)$ be the Nth degree Maclaurin polynomial of e^x. Use Newton's method to approximate solutions of $p_N(x) - 2 = 0$ for $N = 4, 5, 6$.

c. Explain why the approximate roots of $p_N(x) - 2 = 0$ are approximate values of $\ln(2)$.

Answer

a. Answers will vary.

b. The following are the x_n values after 10 iterations of Newton's method to approximation a root of $p_N(x) - 2 = 0$: for $N = 4, x = 0.6939...$; for $N = 5, x = 0.6932...$; for $N = 6, x = 0.69315...$; . (Note: $\ln(2) = 0.69314...$)

c. Answers will vary.

Evaluating Limits using Taylor Series

In exercises 55 - 58, use the fact that if $q(x) = \sum_{n=1}^{\infty} a_n (x-c)^n$ converges in an interval containing c, then $\lim_{x \to c} q(x) = a_0$ to evaluate each limit using Taylor series.

55) $\lim_{x \to 0} \frac{\cos x - 1}{x^2}$

56) $\lim_{x \to 0} \frac{\ln(1-x^2)}{x^2}$

Answer

$\frac{\ln(1-x^2)}{x^2} \to -1$

57) $\lim_{x \to 0} \frac{e^{x^2} - x^2 - 1}{x^4}$

58) $\lim_{x \to 0^+} \frac{\cos(\sqrt{x}) - 1}{2x}$

Answer

$$\frac{\cos(\sqrt{x})-1}{2x} \approx \frac{(1-\frac{x}{2}+\frac{x^2}{4!}-\cdots)-1}{2x} \to -\frac{1}{4}$$

This page titled 5.3E: Exercises is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Gilbert Strang & Edwin "Jed" Herman via source content that was edited to the style and standards of the LibreTexts platform.

5.4: Working with Taylor Series

Learning Objectives

- Write the terms of the binomial series.
- Recognize the Taylor series expansions of common functions.
- Recognize and apply techniques to find the Taylor series for a function.
- Use Taylor series to evaluate non-elementary integrals.

In the preceding section, we defined Taylor series and showed how to find the Taylor series for several common functions by explicitly calculating the coefficients of the Taylor polynomials. We also showed how to use those Taylor series to derive Taylor series for other functions. We then showed how power series can be used to evaluate integrals when the antiderivative of the integrand cannot be expressed in terms of elementary functions. We now venture into the final discussion surrounding Taylor series - expansions of powers of expressions involving two terms.

The Binomial Series

Our first goal in this section is to determine the Maclaurin series for the function $f(x) = (1+x)^r$ for all real numbers r. The Maclaurin series for this function is known as the **binomial series**. We begin by considering the simplest case: r is a non-negative integer. We recall that, for $r = 0, 1, 2, 3, 4,\ f(x) = (1+x)^r$ can be written as

$$\begin{aligned} f(x) &= (1+x)^0 = 1, \\ f(x) &= (1+x)^1 = 1+x, \\ f(x) &= (1+x)^2 = 1+2x+x^2, \\ f(x) &= (1+x)^3 = 1+3x+3x^2+x^3, \\ f(x) &= (1+x)^4 = 1+4x+6x^2+4x^3+x^4. \end{aligned}$$

The expressions on the right-hand side are known as **binomial expansions** and the coefficients are known as **binomial coefficients**. More generally, for any nonnegative integer r, the binomial coefficient of x^n in the binomial expansion of $(1+x)^r$ is given by

$$\binom{r}{n} = \frac{r!}{n!(r-n)!} \tag{5.4.1}$$

and

$$\begin{aligned} f(x) &= (1+x)^r \\ &= \binom{r}{0} + \binom{r}{1}x + \binom{r}{2}x^2 + \binom{r}{3}x^3 + \cdots + \binom{r}{r-1}x^{r-1} + \binom{r}{r}x^r \\ &= \sum_{n=0}^{r} \binom{r}{n} x^n. \end{aligned} \tag{5.4.2}$$

For example, using this formula for $r = 5$, we see that

$$\begin{aligned} f(x) &= (1+x)^5 \\ &= \binom{5}{0}1 + \binom{5}{1}x + \binom{5}{2}x^2 + \binom{5}{3}x^3 + \binom{5}{4}x^4 + \binom{5}{5}x^5 \\ &= \frac{5!}{0!5!}1 + \frac{5!}{1!4!}x + \frac{5!}{2!3!}x^2 + \frac{5!}{3!2!}x^3 + \frac{5!}{4!1!}x^4 + \frac{5!}{5!0!}x^5 \\ &= 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5. \end{aligned}$$

We now consider the case when the exponent r is any real number, not necessarily a non-negative integer. If r is *not* a nonnegative integer, then $f(x) = (1+x)^r$ cannot be written as a finite polynomial. However, we can find a power series for f. Specifically, we

look for the Maclaurin series for f. To do this, we find the derivatives of f and evaluate them at $x = 0$.

$$\begin{aligned}
f(x) &= (1+x)^r && \Longrightarrow && f(0) &= 1\\
f'(x) &= r(1+x)^{r-1} && \Longrightarrow && f'(0) &= r\\
f''(x) &= r(r-1)(1+x)^{r-2} && \Longrightarrow && f''(0) &= r(r-1)\\
f'''(x) &= r(r-1)(r-2)(1+x)^{r-3} && \Longrightarrow && f'''(0) &= r(r-1)(r-2)\\
f^{(n)}(x) &= r(r-1)(r-2)\cdots(r-n+1)(1+x)^{r-n} && \Longrightarrow && f^{(n)}(0) &= r(r-1)(r-2)\cdots(r-n+1)
\end{aligned}$$

We conclude that the coefficients in the binomial series are given by

$$\frac{f^{(n)}(0)}{n!} = \frac{r(r-1)(r-2)\cdots(r-n+1)}{n!}. \tag{5.4.3}$$

We note that if r is a non-negative integer, then the $(r+1)^{\text{st}}$ derivative $f^{(r+1)}$ is the zero function, and the series terminates. In addition, if r is a non-negative integer, then the coefficients in Equation 5.4.3 agree with those from Equation 5.4.1, and the formula for the binomial series agrees with Equation 5.4.2 for the finite binomial expansion. More generally, to denote the binomial coefficients for any real number r, we define

$$\binom{r}{n} = \frac{(r-1)(r-2)\cdots(r-n+1)}{n!}.$$

With this notation, we can write the binomial series for $(1+x)^r$ as

$$\sum_{n=0}^{\infty}\binom{r}{n}x^n = 1 + rx + \frac{r(r-1)}{2!}x^2 + \cdots + \frac{r(r-1)\cdots(r-n+1)}{n!}x^n + \cdots. \tag{5.4.4}$$

We now need to determine the interval of convergence for the binomial series Equation 5.4.4. To do so, we apply the Ratio Test. Consequently, we consider

$$\begin{aligned}
\frac{|a_{n+1}|}{|a_n|} &= \frac{|r(r-1)(r-2)\cdots(r-n)||x|^{n+1}}{(n+1)!}\cdot\frac{n}{|r(r-1)(r-2)\cdots(r-n+1)||x|^n}\\
&= \frac{|r-n||x|}{|n+1|}.
\end{aligned}$$

Since

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = |x| < 1$$

if and only if $|x| < 1$, we conclude that the interval of convergence for the binomial series is $(-1, 1)$. The behavior at the endpoints depends on r. It can be shown that, for $r \geq 0$, the series converges at both endpoints; for $-1 < r < 0$, the series converges at $x = 1$ and diverges at $x = -1$; and for $r < -1$, the series diverges at both endpoints. The binomial series does converge to $(1+x)^r$ in $(-1, 1)$ for all real numbers r, but proving this fact by showing that the remainder $R_n(x) \to 0$ is difficult.

It's time to summarize this in a formal definition.

Definition: Binomial Series

For any real number r, the Maclaurin series for $f(x) = (1+x)^r$ is called the **binomial series**. It converges to f for $|x| < 1$, and we write

$$(1+x)^r = \sum_{n=0}^{\infty}\binom{r}{n}x^n = 1 + rx + \frac{r(r-1)}{2!}x^2 + \cdots + r\frac{(r-1)\cdots(r-n+1)}{n!}x^n + \cdots$$

for $|x| < 1$.

We can use this definition to find the binomial series for $f(x) = \sqrt{1+x}$ and use the series to approximate $\sqrt{1.5}$.

Example 5.4.1: Finding Binomial Series

a. Find the binomial series for $f(x) = \sqrt{1+x}$.

b. Use the third-order Maclaurin polynomial $p_3(x)$ to estimate $\sqrt{1.5}$. Use Taylor's Theorem to bound the error. Use graphing technology to compare the graphs of f and p_3.

Solutions

a. Here $r = \frac{1}{2}$. Using the definition for the binomial series, we obtain

$$\begin{aligned}\sqrt{1+x} &= 1 + \frac{1}{2}x + \frac{(1/2)(-1/2)}{2!}x^2 + \frac{(1/2)(-1/2)(-3/2)}{3!}x^3 + \cdots \\ &= 1 + \frac{1}{2}x - \frac{1}{2!}\frac{1}{2^2}x^2 + \frac{1}{3!}\frac{1\cdot 3}{2^3}x^3 - \cdots + \frac{(-1)^{n+1}}{n!}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n}x^n + \cdots \\ &= 1 + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n!}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n}x^n.\end{aligned}$$

b. From the result in part a., the third-order Maclaurin polynomial is

$$p_3(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3.$$

Therefore,

$$\sqrt{1.5} = \sqrt{1+0.5} \approx 1 + \frac{1}{2}(0.5) - \frac{1}{8}(0.5)^2 + \frac{1}{16}(0.5)^3 \approx 1.2266.$$

From Taylor's Theorem, the error satisfies

$$R_3(0.5) = \frac{f^{(4)}(c)}{4!}(0.5)^4$$

for some c between 0 and 0.5. Since $f^{(4)}(x) = -\frac{15}{2^4(1+x)^{7/2}}$, and the maximum value of $|f^{(4)}(x)|$ on the interval $(0, 0.5)$ occurs at $x = 0$, we have

$$|R_3(0.5)| \le \frac{15}{4!2^4}(0.5)^4 \approx 0.00244.$$

The function and the Maclaurin polynomial p_3 are graphed in Figure 5.4.1.

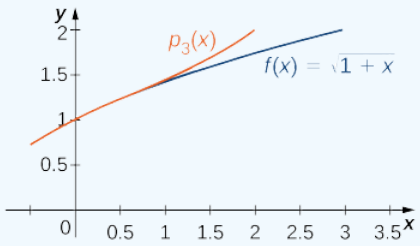


Figure 5.4.1: The third-order Maclaurin polynomial $p_3(x)$ provides a good approximation for $f(x) = \sqrt{1+x}$ for x near zero.

Checkpoint 5.4.1

Find the binomial series for $f(x) = \frac{1}{(1+x)^2}$.

Answer

$$\sum_{n=0}^{\infty}(-1)^n(n+1)x^n$$

A Summary of Common Functions Expressed as Taylor Series

As in Section 4.3, we summarize the common Maclaurin series we have derived up to this point.

Table 5.4.1: Maclaurin Series for Common Functions

Function	Maclaurin Series	Interval of Convergence
$f(x)=\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$-1<x<1$
$f(x)=e^x$	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$-\infty<x<\infty$
$f(x)=\sin x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	$-\infty<x<\infty$
$f(x)=\cos x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	$-\infty<x<\infty$
$f(x)=\ln(1+x)$	$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$	$-1<x\leq 1$
$f(x)=\tan^{-1} x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	$-1\leq x\leq 1$
$f(x)=(1+x)^r$	$\sum_{n=0}^{\infty} \binom{r}{n} x^n$	$-1<x<1$

We showed previously in this chapter how power series can be differentiated term-by-term to create a new power series. In Example 5.4.2, we differentiate the binomial series for $\sqrt{1+x}$ term by term to find the binomial series for $\frac{1}{\sqrt{1+x}}$. Note that we could construct the binomial series for $\frac{1}{\sqrt{1+x}}$ directly from the definition, but differentiating the binomial series for $\sqrt{1+x}$ is an easier calculation.

Example 5.4.2: Differentiating a Series to Find a New Series

Use the binomial series for $\sqrt{1+x}$ to find the binomial series for $\frac{1}{\sqrt{1+x}}$.

Solution

The two functions are related by

$$\frac{d}{dx}\sqrt{1+x}=\frac{1}{2\sqrt{1+x}},$$

so the binomial series for $\frac{1}{\sqrt{1+x}}$ is given by

$$\frac{1}{\sqrt{1+x}}=2\frac{d}{dx}\sqrt{1+x}=1+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n}x^n.$$

Checkpoint 5.4.2

Find the binomial series for $f(x)=\frac{1}{(1+x)^{3/2}}$

Answer

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n}x^n$$

In this example, we differentiated a known Taylor series to construct a Taylor series for another function. The ability to differentiate power series term-by-term makes them a powerful tool for solving differential equations.

Applications Involving Taylor Series

Probability

Taylor and Maclaurin series are used frequently when encountering non-elementary integrals. For example, the integral $\int e^{-x^2}\,dx$ arises often in probability theory. Specifically, it is used when studying data sets that are normally distributed, meaning the data values lie under a bell-shaped curve. For example, if a set of data values is normally distributed with mean μ and standard deviation σ, then the probability that a randomly chosen value lies between $x = a$ and $x = b$ is given by

$$\frac{1}{\sigma\sqrt{2\pi}}\int_a^b e^{-(x-\mu)^2/(2\sigma^2)}\,dx. \tag{5.4.5}$$

(See Figure 5.4.2.)

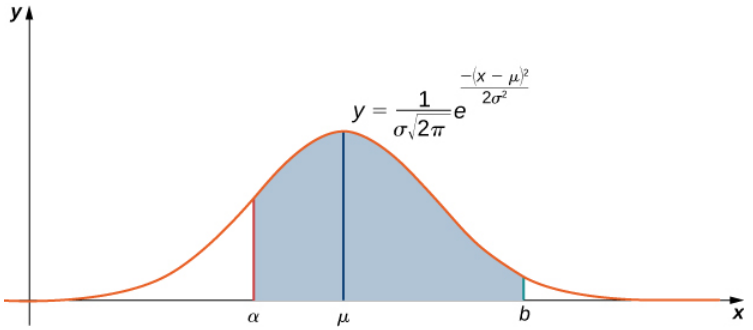


Figure 5.4.2: If data values are normally distributed with mean μ and standard deviation σ, the probability that a randomly selected data value is between a and b is the area under the curve $y = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/(2\sigma^2)}$ between $x = a$ and $x = b$.

To simplify this integral, we typically let $z = \frac{x-\mu}{\sigma}$. This quantity z is the data value's z score. With this simplification, integral Equation 5.4.5 becomes

$$\frac{1}{\sqrt{2\pi}}\int_{(a-\mu)/\sigma}^{(b-\mu)/\sigma} e^{-z^2/2}\,dz.$$

In Example 5.4.3, we show how we can use this integral in calculating probabilities.

Example 5.4.3: Using Maclaurin Series to Approximate a Probability

Suppose a set of standardized test scores are normally distributed with mean $\mu = 100$ and standard deviation $\sigma = 50$. Use Equation 5.4.5 and the first six terms in the Maclaurin series for $e^{-x^2/2}$ to approximate the probability that a randomly selected test score is between $x = 100$ and $x = 200$. Use the Alternating Series Test to determine how accurate your approximation is.

Solution

Since $\mu = 100$, $\sigma = 50$, and we are trying to determine the area under the curve from $a = 100$ to $b = 200$, integral Equation 5.4.5 becomes

$$\frac{1}{\sqrt{2\pi}}\int_0^2 e^{-z^2/2}\,dz.$$

The Maclaurin series for $e^{-x^2/2}$ is given by

$$\begin{aligned} e^{-x^2/2} &= \sum_{n=0}^{\infty}\frac{\left(-\frac{x^2}{2}\right)^n}{n!} \\ &= 1-\frac{x^2}{2^1\cdot 1!}+\frac{x^4}{2^2\cdot 2!}-\frac{x^6}{2^3\cdot 3!}+\cdots+(-1)^n\frac{x^{2n}}{2^n\cdot n}!+\cdots \\ &= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{2^n\cdot n!}. \end{aligned}$$

Therefore,

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int e^{-z^2/2}\,dz &= \frac{1}{\sqrt{2\pi}}\int\left(1-\frac{z^2}{2^1\cdot 1!}+\frac{z^4}{2^2\cdot 2!}-\frac{z^6}{2^3\cdot 3!}+\cdots+(-1)^n\frac{z^{2n}}{2^n\cdot n!}+\cdots\right)dz\\ &= \frac{1}{\sqrt{2\pi}}\left(C+z-\frac{z^3}{3\cdot 2^1\cdot 1!}+\frac{z^5}{5\cdot 2^2\cdot 2!}-\frac{z^7}{7\cdot 2^3\cdot 3!}+\cdots+(-1)^n\frac{z^{2n+1}}{(2n+1)2^n\cdot n!}+\cdots\right)\end{aligned}$$

$$\frac{1}{\sqrt{2\pi}}\int_0^2 e^{-z^2/2}\,dz = \frac{1}{\sqrt{2\pi}}\left(2-\frac{8}{6}+\frac{32}{40}-\frac{128}{336}+\frac{512}{3456}-\frac{2^{11}}{11\cdot 2^5\cdot 5!}+\cdots\right)$$

Using the first five terms, we estimate that the probability is approximately 0.4922. By the Alternating Series Test, we see that this estimate is accurate to within

$$\frac{1}{\sqrt{2\pi}}\frac{2^{13}}{13\cdot 2^6\cdot 6!}\approx 0.00546.$$

Analysis

The probability that a data value is within two standard deviations of the mean is approximately 95%. Here, we calculated the probability that a data value is between the mean and two standard deviations above the mean, so the estimate should be around 47.5%. The estimate, combined with the bound on the accuracy, falls within this range.

? Checkpoint 5.4.3

Use the first five terms of the Maclaurin series for $e^{-x^2/2}$ to estimate the probability that a randomly selected test score is between 100 and 150. Use the Alternating Series Test to determine the accuracy of this estimate.

Answer

The estimate is approximately 0.3414. This estimate is accurate to within 0.0000094.

Elliptical Integrals

Another application in which a non-elementary integral arises involves the period of a pendulum. The integral is

$$\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}.$$

An integral of this form is known as an **elliptic integral** of the first kind. Elliptic integrals originally arose when trying to calculate the arc length of an ellipse. We now show how to use power series to approximate this integral.

✓ Example 5.4.4: Period of a Pendulum

The period of a pendulum is the time it takes for a pendulum to make one complete back-and-forth swing. For a pendulum with length L that makes a maximum angle θ_{max} with the vertical, its period T is given by

$$T=4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

where g is the acceleration due to gravity and $k=\sin\left(\frac{\theta_{max}}{2}\right)$ (see Figure 5.4.3). (We note that this formula for the period arises from a non-linearized model of a pendulum. In some cases, for simplification, a linearized model is used, and $\sin\theta$ is approximated by θ.)

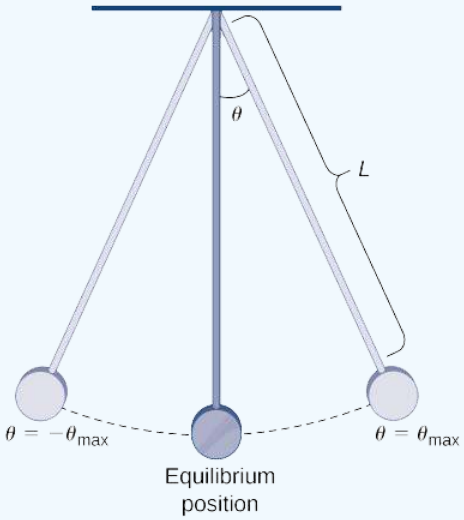


Figure 5.4.3: This pendulum has length L and makes a maximum angle θ_{max} with the vertical.

Use the binomial series

$$\frac{1}{\sqrt{1+x}} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n} x^n$$

to estimate the period of this pendulum. Specifically, approximate the period of the pendulum if

a. you use only the first term in the binomial series, and
b. you use the first two terms in the binomial series.

Solution

We use the binomial series, replacing x with $-k^2 \sin^2 \theta$. Then we can write the period as

$$T = 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} \left(1 + \frac{1}{2}k^2 \sin^2\theta + \frac{1 \cdot 3}{2!2^2}k^4 \sin^4\theta + \cdots\right) d\theta.$$

a. Using just the first term in the integrand, the first-order estimate is

$$T \approx 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} d\theta = 2\pi\sqrt{\frac{L}{g}}.$$

If θ_{max} is small, then $k = \sin\left(\frac{\theta_{max}}{2}\right)$ is small. This is a good estimate when k is small. To justify this claim, consider

$$\int_0^{\pi/2} \left(1 + \frac{1}{2}k^2 \sin^2\theta + \frac{1 \cdot 3}{2!2^2}k^4 \sin^4\theta + \cdots\right) d\theta.$$

Since $|\sin x| \leq 1$, this integral is bounded by

$$\int_0^{\pi/2} \left(\frac{1}{2}k^2 + \frac{1.3}{2!2^2}k^4 + \cdots\right) d\theta < \frac{\pi}{2}\left(\frac{1}{2}k^2 + \frac{1 \cdot 3}{2!2^2}k^4 + \cdots\right).$$

Furthermore, it can be shown that each coefficient on the right-hand side is less than 1 and, therefore, that this expression is bounded by

$$\frac{\pi k^2}{2}(1 + k^2 + k^4 + \cdots) = \frac{\pi k^2}{2} \cdot \frac{1}{1-k^2},$$

which is small for k small.

b. For larger values of θ_{max}, we can approximate T by using more terms in the integrand. By using the first two terms in the integral, we arrive at the estimate

$$T \approx 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\left(1+\frac{1}{2}k^2\sin^2\theta\right)d\theta = 2\pi\sqrt{\frac{L}{g}}\left(1+\frac{k^2}{4}\right).$$

Recentering Polynomials

One of the most useful applications of Taylor series occurs in Differential Equations. A polynomial centered at 0 can always be **recentered** at another point using Taylor series. To showcase how this is done, let

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

By computing the derivatives of P at a, we get the recentered polynomial

$$P(x) = \sum_{n=0}^{\infty}\frac{P^{(n)}(a)}{n!}(x-a)^n = P(a) + P'(a)(x-a) + \frac{P''(a)}{2!}(x-a)^2 + \cdots.$$

The usefulness of recentering polynomials cannot be overstated; however, to describe the reasons at this point is moot (it suffices to say that it is important). Let's see how this works in practice.

Example 5.4.5

Recenter the polynomial at $a = -3$.

$$P(x) = x^6 - x^4 + x^3 + x - 2.$$

Solution

We begin by making a table of derivatives and their values at $a = -3$.

$$\begin{aligned}
P(-3) &= 616\\
P'(x) &= 6x^5 - 4x^3 + 3x^2 + 1 \implies P'(-3) = -1322\\
P''(x) &= 30x^4 - 12x^2 + 6x \implies P''(-3) = 2304\\
P'''(x) &= 120x^3 - 24x + 6 \implies P'''(-3) = -3162\\
P^{(4)}(x) &= 360x^2 - 24 \implies P^{(4)}(-3) = 3216\\
P^{(5)}(x) &= 720x \implies P^{(5)}(-3) = -2160\\
P^{(6)}(x) &= 720 \implies P^{(6)}(-3) = 720\\
P^{(n)}(x) &= 0 \quad \text{for} \quad n \geq 7
\end{aligned}$$

Therefore,

$$\begin{aligned}
P(x) &= \sum_{n=0}^{\infty}\frac{P^{(n)}}{n!}(x-a)^n\\
&= 616 - 1322(x+3) + \frac{2304}{2!}(x+3)^2 - \frac{3162}{3!}(x+3)^3 + \frac{3216}{4!}(x+3)^4 - \frac{2160}{5!}(x+3)^5 + \frac{720}{6!}(x+3)^6\\
&= 616 - 1322(x+3) + 1152(x+3)^2 - 527(x+3)^3 + 134(x+3)^4 - 18(x+3)^5 + (x+3)^6
\end{aligned}$$

Since P is a polynomial with a finite number of terms, the radius of convergence is always ∞.

Exercise 5.4.5

Recenter

$$P(x) = x^5 + 2x^3 + x$$

at $a = 2$.

Answer

$P(x) = 50 + 105(x-2) + 92(x-2)^2 + 42(x-2)^3 + 10(x-2)^4 + (x-2)^5$.

Building Data Models

Our final application involves data given at a single point. If we know the quantity of something at a given time, how quickly that quantity changes at that moment, how quickly that change changes, and so on, we can use Taylor series to build a polynomial model of the underlying dataset. Let's transition that language into Mathematics.

If we know $P(a)$, $P'(a)$, $P''(a)$, and so on, then we can build a model of the data (that works locally) using the Taylor series

$$P(x) = \sum_{n=0}^{\infty} \frac{P^{(n)}(a)}{n!}(x-a)^n = P(a) + P'(a)(x-a) + P''(a)(x-a)^2 + \cdots.$$

The fact that we will only have a finite amount of information will keep this polynomial restricted to a finite number of terms. Moreover, this limitation means that our model will only be predictive in "real-world" situations for a short time; however, this still can be very useful.

Example 5.4.6

The price of a certain stock is currently (at this very moment) at \$23.10. If, based on computer models, we know this price is increasing at a rate of \$12.93 per day, this rate is increasing at a rate of \$0.50 per day (squared), and the rate of this is decreasing at \$0.24 per day (cubed), build a model that can be used to predict the price of the stock in 5 minutes.

Solution

We are given a single data point and three of its derivatives. That is, if we let $p(t)$ be the price of the stock at time t, we were given

$$\begin{aligned} p(0) &= 23.10 \\ p'(0) &= 12.93 \\ p''(0) &= 0.50 \\ p'''(0) &= -0.24 \end{aligned}$$

Therefore, we can build a model for the price using a Taylor polynomial.

$$P_3(t) = \sum_{n=0}^{3} \frac{p^{(n)}(0)}{n!}(t-0)^n = 23.10 + 12.93t + \frac{0.50}{2!}t^2 - \frac{0.24}{3!}t^3 = 23.10 + 12.93t + 0.25t^2 - 0.04t^3$$

Since t is in terms of days, the predicted value of the stock in 5 minutes using this model is

$$P_3\left(\frac{5}{1440}\right) \approx 23.14$$

Thus, we can expect the stock to rise by \$0.04 in the next five minutes.

To be honest, Example 5.4.6 might be a little too immersed in fantasy to be realistic; however, the impact of being able to predict future values given "dense" information at a single point (a value and the values of several derivatives at that point) should imply many uses.

The applications of Taylor series in this section are intended to highlight their importance. In general, Taylor series are helpful because they allow us to represent known functions using polynomials, thus providing a tool for approximating function values and estimating complicated integrals. In addition, they will enable us to define new functions as power series, thus providing us with a powerful tool for solving differential equations.

Key Concepts

- The binomial series is the Maclaurin series for $f(x) = (1+x)^r$. It converges for $|x| < 1$.
- Taylor series for functions can often be derived by algebraic operations with a known Taylor series or by differentiating or integrating a known Taylor series.
- Power series can be used to solve differential equations.
- Taylor series can be used to help approximate integrals that cannot be evaluated by other means.

Glossary

binomial series

the Maclaurin series for $f(x) = (1+x)^r$; it is given by $(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n = 1 + rx + \frac{r(r-1)}{2!}x^2 + \cdots + \frac{r(r-1)\cdots(r-n+1)}{n!}x^n + \cdots$ for $|x| < 1$

non-elementary integral

an integral for which the antiderivative of the integrand cannot be expressed as an elementary function

5.4E: Exercises

In exercises 1 - 4, use appropriate substitutions to write down the Maclaurin series for the given binomial.

1) $(1-x)^{1/3}$

2) $(1+x^2)^{-1/3}$

Answer

$$(1+x^2)^{-1/3}=\sum_{n=0}^{\infty}\left(n^{-\frac{1}{3}}\right)x^{2n}$$

3) $(1-x)^{1.01}$

4) $(1-2x)^{2/3}$

Answer

$$(1-2x)^{2/3}=\sum_{n=0}^{\infty}(-1)^n 2^n\left(n^{\frac{2}{3}}\right)x^n$$

In exercises 5 - 12, use the substitution $(b+x)^r=(b+a)^r\left(1+\dfrac{x-a}{b+a}\right)^r$ in the binomial expansion to find the Taylor series of each function with the given center.

5) $\sqrt{x+2}$ at $a=0$

6) $\sqrt{x^2+2}$ at $a=0$

Answer

$$\sqrt{2+x^2}=\sum_{n=0}^{\infty}2^{(1/2)-n}\left(n^{\frac{1}{2}}\right)x^{2n};(|x^2|<2)$$

7) $\sqrt{x+2}$ at $a=1$

8) $\sqrt{2x-x^2}$ at $a=1$ (Hint: $2x-x^2=1-(x-1)^2$)

Answer

$$\sqrt{2x-x^2}=\sqrt{1-(x-1)^2}\text{ so }\sqrt{2x-x^2}=\sum_{n=0}^{\infty}(-1)^n\left(n^{\frac{1}{2}}\right)(x-1)^{2n}$$

9) $(x-8)^{1/3}$ at $a=9$

10) $\sqrt{x}$ at $a=4$

Answer

$$\sqrt{x}=2\sqrt{1+\tfrac{x-4}{4}}\text{ so }\sqrt{x}=\sum_{n=0}^{\infty}2^{1-2n}\left(n^{\frac{1}{2}}\right)(x-4)^n$$

11) $x^{1/3}$ at $a=27$

12) $\sqrt{x}$ at $x=9$

Answer

$$\sqrt{x}=\sum_{n=0}^{\infty}3^{1-3n}\left(n^{\frac{1}{2}}\right)(x-9)^n$$

In exercises 13 - 14, use the binomial theorem to estimate each number, computing enough terms to obtain an estimate accurate to an error of at most $1/1000$.

13) [Technology Required] $(15)^{1/4}$ using $(16-x)^{1/4}$

14) [Technology Required] $(1001)^{1/3}$ using $(1000+x)^{1/3}$

Answer

$10(1+\frac{x}{1000})^{1/3}=\sum_{n=0}^{\infty}10^{1-3n}\binom{\frac{1}{3}}{n}x^n$. Using, for example, a fourth-degree estimate at $x=1$ gives $(1001)^{1/3}$ $\approx 10\left(1+\left(1\frac{1}{3}\right)10^{-3}+\left(2\frac{1}{3}\right)10^{-6}+\left(3\frac{1}{3}\right)10^{-9}+\left(3\frac{1}{3}\right)10^{-12}\right)=10\left(1+\frac{1}{3.10^3}-\frac{1}{9.10^6}+\frac{5}{81.10^9}-\frac{10}{243.10^{12}}\right)=10.00333222...$ whereas $(1001)^{1/3}=10.00332222839093...$. Two terms would suffice for three-digit accuracy.

In exercises 15 - 18, use the binomial approximation $\sqrt{1-x}\approx 1-\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{16}-\frac{5x^4}{128}-\frac{7x^5}{256}$ for $|x|<1$ to approximate each number. Compare this value to the value given by a scientific calculator.

15) [Technology Required] $\frac{1}{\sqrt{2}}$ using $x=\frac{1}{2}$ in $(1-x)^{1/2}$

16) [Technology Required] $\sqrt{5}=5\times\frac{1}{\sqrt{5}}$ using $x=\frac{4}{5}$ in $(1-x)^{1/2}$

Answer

The approximation is 2.3152; the CAS value is $2.23...$.

17) [Technology Required] $\sqrt{3}=\frac{3}{\sqrt{3}}$ using $x=\frac{2}{3}$ in $(1-x)^{1/2}$

18) [Technology Required] $\sqrt{6}$ using $x=\frac{5}{6}$ in $(1-x)^{1/2}$

Answer

The approximation is $2.583...$; the CAS value is $2.449...$.

19) Integrate the binomial approximation of $\sqrt{1-x}$ to find an approximation of $\int_0^x\sqrt{1-t}\,dt$.

20) [Technology Required] Recall that the graph of $\sqrt{1-x^2}$ is an upper semicircle of radius 1. Integrate the binomial approximation of $\sqrt{1-x^2}$ up to order 8 from $x=-1$ to $x=1$ to estimate $\frac{\pi}{2}$.

Answer

$\sqrt{1-x^2}=1-\frac{x^2}{2}-\frac{x^4}{8}-\frac{x^6}{16}-\frac{5x^8}{128}+\cdots$. Thus $\int_{-1}^{1}\sqrt{1-x^2}\,dx=\left[x-\frac{x^3}{6}-\frac{x^5}{40}-\frac{x^7}{7\cdot 16}-\frac{5x^9}{9\cdot 128}+\cdots\right]\Big|_{-1}^{1}\approx 2-\frac{1}{3}-\frac{1}{20}-\frac{1}{56}-\frac{10}{9\cdot 128}+error=1.590...$ whereas $\frac{\pi}{2}=1.570...$

In exercises 21 - 24, use the expansion $(1+x)^{1/3}=1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3-\frac{10}{243}x^4+\cdots$ to write the first five terms (not necessarily a quartic polynomial) of each expression.

21) $(1+4x)^{1/3};\ a=0$

22) $(1+4x)^{4/3};\ a=0$

Answer

$(1+x)^{4/3}=(1+x)(1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3-\frac{10}{243}x^4+\cdots)=1+\frac{4x}{3}+\frac{2x^2}{9}-\frac{4x^3}{81}+\frac{5x^4}{243}+\cdots$

23) $(3+2x)^{1/3};\ a=-1$

24) $(x^2+6x+10)^{1/3};\ a=-3$

Answer

$(1+(x+3)^2)^{1/3}=1+\frac{1}{3}(x+3)^2-\frac{1}{9}(x+3)^4+\frac{5}{81}(x+3)^6-\frac{10}{243}(x+3)^8+\cdots$

25) Use $(1+x)^{1/3}=1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3-\frac{10}{243}x^4+\cdots$ with $x=1$ to approximate $2^{1/3}$.

26) Use the approximation $(1-x)^{2/3}=1-\frac{2x}{3}-\frac{x^2}{9}-\frac{4x^3}{81}-\frac{7x^4}{243}-\frac{14x^5}{729}+\cdots$ for $|x|<1$ to approximate $2^{1/3}=2.2^{-2/3}$.

Answer

Twice the approximation is $1.260...$ whereas $2^{1/3}=1.2599...$.

27) Find the 25th derivative of $f(x) = (1+x^2)^{13}$ at $x = 0$.

28) Find the 99th derivative of $f(x) = (1+x^4)^{25}$.

Answer

$f^{(99)}(0) = 0$

In exercises 29 - 36, find the Maclaurin series of each function.

29) $f(x) = xe^{2x}$

30) $f(x) = 2^x$

Answer

$$\sum_{n=0}^{\infty} \frac{(\ln(2)x)^n}{n!}$$

31) $f(x) = \dfrac{\sin x}{x}$

32) $f(x) = \dfrac{\sin(\sqrt{x})}{\sqrt{x}}, (x > 0),$

Answer

For $x > 0$, $\sin(\sqrt{x}) = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{(2n+1)/2}}{\sqrt{x}(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(2n+1)!}.$

33) $f(x) = \sin(x^2)$

34) $f(x) = e^{x^3}$

Answer

$$e^{x^3} = \sum_{n=0}^{\infty} \frac{x^{3n}}{n!}$$

35) $f(x) = \cos^2 x$ using the identity $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x)$

36) $f(x) = \sin^2 x$ using the identity $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$

Answer

$$\sin^2 x = -\sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k-1} x^{2k}}{(2k)!}$$

In exercises 37 - 44, find the Maclaurin series of $F(x) = \displaystyle\int_0^x f(t)\,dt$ by integrating the Maclaurin series of f term by term. If f is not strictly defined at zero, you may substitute the value of the Maclaurin series at zero.

37) $F(x) = \displaystyle\int_0^x e^{-t^2}\,dt;\ f(t) = e^{-t^2} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{n!}$

38) $F(x) = \tan^{-1} x;\ f(t) = \dfrac{1}{1+t^2} = \displaystyle\sum_{n=0}^{\infty} (-1)^n t^{2n}$

Answer

$$\tan^{-1} x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}$$

39) $F(x) = \tanh^{-1} x;\ f(t) = \dfrac{1}{1-t^2} = \displaystyle\sum_{n=0}^{\infty} t^{2n}$

40) $F(x) = \sin^{-1} x;\ f(t) = \dfrac{1}{\sqrt{1-t^2}} = \displaystyle\sum_{k=0}^{\infty} \left(k^{\frac{1}{2}}\right) \frac{t^{2k}}{k!}$

Answer

$$\sin^{-1} x = \sum_{n=0}^{\infty} \left(n^{\frac{1}{2}}\right) \frac{x^{2n+1}}{(2n+1)n!}$$

41) $F(x) = \int_0^x \frac{\sin t}{t}\, dt;\ f(t) = \frac{\sin t}{t} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n+1)!}$

42) $F(x) = \int_0^x \cos\left(\sqrt{t}\right)\, dt;\ f(t) = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(2n)!}$

Answer

$$F(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{(n+1)(2n)!}$$

43) $F(x) = \int_0^x \frac{1-\cos t}{t^2}\, dt;\ f(t) = \frac{1-\cos t}{t^2} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n+2)!}$

44) $F(x) = \int_0^x \frac{\ln(1+t)}{t}\, dt;\ f(t) = \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{n+1}$

Answer

$$F(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n^2}$$

In exercises 45 - 52, compute at least the first three nonzero terms (not necessarily a quadratic polynomial) of the Maclaurin series of f.

45) $f(x) = \sin\left(x + \frac{\pi}{4}\right) = \sin x \cos\left(\frac{\pi}{4}\right) + \cos x \sin\left(\frac{\pi}{4}\right)$

46) $f(x) = \tan x$

Answer

$$x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$$

47) $f(x) = \ln(\cos x)$

48) $f(x) = e^x \cos x$

Answer

$$1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \cdots$$

49) $f(x) = e^{\sin x}$

50) $f(x) = \sec^2 x$

Answer

$$1 + x^2 + \frac{2x^4}{3} + \frac{17x^6}{45} + \cdots$$

51) $f(x) = \tanh x$

52) $f(x) = \frac{\tan\sqrt{x}}{\sqrt{x}}$ (see expansion for $\tan x$)

Answer

Using the expansion for $\tan x$ gives $1 + \frac{x}{3} + \frac{2x^2}{15}$.

In exercises 53 - 56, find the radius of convergence of the Maclaurin series of each function.

53) $\ln(1+x)$

54) $\frac{1}{1+x^2}$

Answer

$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$ so $R = 1$ by the Ratio Test.

55) $\tan^{-1} x$

56) $\ln(1+x^2)$

Answer

$\ln(1+x^2) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^{2n}$ so $R = 1$ by the Ratio Test.

57) Find the Maclaurin series of $\sinh x = \frac{e^x - e^{-x}}{2}$.

58) Find the Maclaurin series of $\cosh x = \frac{e^x + e^{-x}}{2}$.

Answer

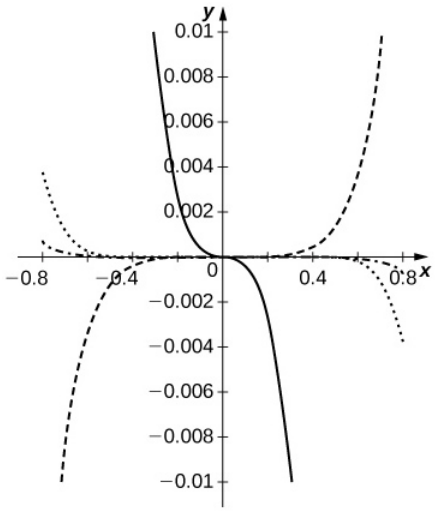
Add series of e^x and e^{-x} term by term. Odd terms cancel and $\cosh x = \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$.

59) Differentiate term by term the Maclaurin series of $\sinh x$ and compare the result with the Maclaurin series of $\cosh x$.

60) [Technology Required] Let $S_n(x) = \sum_{k=0}^{n}(-1)^k\frac{x^{2k+1}}{(2k+1)!}$ and $C_n(x) = \sum_{n=0}^{n}(-1)^k\frac{x^{2k}}{(2k)!}$ denote the respective Maclaurin polynomials of degree $2n+1$ of $\sin x$ and degree $2n$ of $\cos x$. Plot the errors $\frac{S_n(x)}{C_n(x)} - \tan x$ for $n = 1, .., 5$ and compare them to $x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} - \tan x$ on $(-\frac{\pi}{4}, \frac{\pi}{4})$.

Answer

The ratio $\frac{S_n(x)}{C_n(x)}$ approximates $\tan x$ better than does $p_7(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315}$ for $N \geq 3$. The dashed curves are $\frac{S_n}{C_n} - \tan x$ for $n = 1, 2$. The dotted curve corresponds to $n = 3$, and the dash-dotted curve corresponds to $n = 4$. The solid curve is $p_7 - \tan x$.



61) Use the identity $2\sin x \cos x = \sin(2x)$ to find the power series expansion of $\sin^2 x$ at $x = 0$. (Hint: Integrate the Maclaurin series of $\sin(2x)$ term by term.)

https://math.libretexts.org/@go/page/168458

62) If $y = \sum_{n=0}^{\infty} a_n x^n$, find the power series expansions of xy' and x^2y''.

Answer

By the term-by-term differentiation theorem, $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ so $y' = \sum_{n=1}^{\infty} n a_n x^{n-1} xy' = \sum_{n=1}^{\infty} n a_n x^n$, whereas $y' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$ so $xy'' = \sum_{n=2}^{\infty} n(n-1) a_n x^n$.

63) [Technology Required] Suppose that a set of standardized test scores is normally distributed with mean $\mu = 100$ and standard deviation $\Sigma = 10$. Set up an integral that represents the probability that a test score will be between 90 and 110 and use the integral of the degree 10 Maclaurin polynomial of $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ to estimate this probability.

Answer

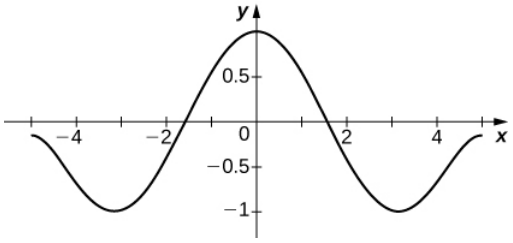
The probability is $p = \frac{1}{\sqrt{2\pi}} \int_{(a-\mu)/\Sigma}^{(b-\mu)/\Sigma} e^{-x^2/2}\, dx$ where $a = 90$ and $b = 100$, that is, $p = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-x^2/2}\, dx$
$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \sum_{n=0}^{5} (-1)^n \frac{x^{2n}}{2^n n!}\, dx = \frac{2}{\sqrt{2\pi}} \sum_{n=0}^{5} (-1)^n \frac{1}{(2n+1)2^n n!} \approx 0.6827.$

64) [Technology Required] Suppose that a set of standardized test scores is normally distributed with mean $\mu = 100$ and standard deviation $\Sigma = 10$. Set up an integral that represents the probability that a test score will be between 70 and 130 and use the integral of the degree 50 Maclaurin polynomial of $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ to estimate this probability.

65) [Technology Required] Suppose that $\sum_{n=0}^{\infty} a_n x^n$ converges to a function $f(x)$ such that $f(0) = 1$, $f'(0) = 0$, and $f''(x) = -f(x)$. Find a formula for a_n and plot the partial sum S_N for $N = 20$ on $[-5, 5]$.

Answer

As in the previous problem one obtains $a_n = 0$ if n is odd and $a_n = -(n+2)(n+1)a_{n+2}$ if n is even, so $a_0 = 1$ leads to $a_{2n} = \frac{(-1)^n}{(2n)!}$.



66) [Technology Required] Suppose that $\sum_{n=0}^{\infty} a_n x^n$ converges to a function $f(x)$ such that $f(0) = 0$, $f'(0) = 1$, and $f''(x) = -f(x)$. Find a formula for an and plot the partial sum S_N for $N = 10$ on $[-5, 5]$.

The error in approximating the integral $\int_a^b f(t)\, dt$ by that of a Taylor approximation $\int_a^b Pn(t)\, dt$ is at most $\int_a^b R_n(t)\, dt$. In exercises 67 - 68, the Taylor remainder estimate $R_n \le \frac{M}{(n+1)!}|x-a|^{n+1}$ guarantees that the integral of the Taylor polynomial of the given order approximates the integral of f with an error less than $\frac{1}{10}$.

a. Evaluate the integral of the appropriate Taylor polynomial and verify that it approximates the CAS value with an error less than $\frac{1}{100}$.

b. Compare the accuracy of the polynomial integral estimate with the remainder estimate.

67) [Technology Required] $\int_0^{\pi} \frac{\sin t}{t}\, dt$; $P_s = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!}$ (You may assume that the absolute value of the ninth derivative of $\frac{\sin t}{t}$ is bounded by 0.1.)

Answer

a. (Proof)

b. We have $R_s \le \frac{0.1}{(9)!}\pi^9 \approx 0.0082 < 0.01$. We have $\int_0^{\pi}\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\frac{x^8}{9!}\right)dx = \pi - \frac{\pi^3}{3\cdot 3!}+\frac{\pi^5}{5\cdot 5!}-\frac{\pi^7}{7\cdot 7!}+\frac{\pi^9}{9\cdot 9!}$ $= 1.852...$, whereas $\int_0^{\pi}\frac{\sin t}{t}\,dt = 1.85194...$, so the actual error is approximately 0.00006.

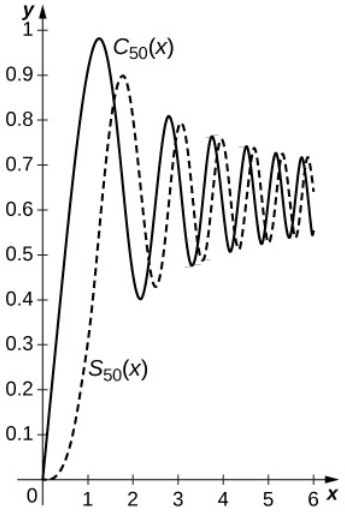
68) [Technology Required] $\int_0^2 e^{-x^2}\,dx;\ p_{11} = 1-x^2+\frac{x^4}{2}-\frac{x^6}{3!}+\cdots-\frac{x^{22}}{11!}$ (You may assume that the absolute value of the 23rd derivative of e^{-x^2} is less than 2×10^{14}.)

The following exercises (69-70) deal with *Fresnel integrals*.

69) The Fresnel integrals are defined by $C(x) = \int_0^x \cos(t^2)\,dt$ and $S(x) = \int_0^x \sin(t^2)\,dt$. Compute the power series of $C(x)$ and $S(x)$ and plot the sums $C_N(x)$ and $S_N(x)$ of the first $N = 50$ nonzero terms on $[0, 2\pi]$.

Answer

Since $\cos(t^2) = \sum_{n=0}^{\infty}(-1)^n\frac{t^{4n}}{(2n)!}$ and $\sin(t^2) = \sum_{n=0}^{\infty}(-1)^n\frac{t^{4n+2}}{(2n+1)!}$, one has $S(x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+3}}{(4n+3)(2n+1)!}$ and $C(x)$ $= \sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+1}}{(4n+1)(2n)!}$. The sums of the first 50 nonzero terms are plotted below with $C_{50}(x)$ the solid curve and $S_{50}(x)$ the dashed curve.



70) [Technology Required] The Fresnel integrals are used in design applications for roadways and railways and other applications because of the curvature properties of the curve with coordinates $(C(t), S(t))$. Plot the curve (C_{50}, S_{50}) for $0 \le t \le 2\pi$, the coordinates of which were computed in the previous exercise.

71) Estimate $\int_0^{1/4}\sqrt{x-x^2}\,dx$ by approximating $\sqrt{1-x}$ using the binomial approximation $1-\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{16}-\frac{5x^4}{2128}-\frac{7x^5}{256}$.

Answer

$\int_0^{1/4}\sqrt{x}\left(1-\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{16}-\frac{5x^4}{128}-\frac{7x^5}{256}\right)dx = \frac{2}{3}2^{-3}-\frac{1}{2}\frac{2}{5}2^{-5}-\frac{1}{8}\frac{2}{7}2^{-7}-\frac{1}{16}\frac{2}{9}2^{-9}-\frac{5}{128}\frac{2}{11}2^{-11}-\frac{7}{256}\frac{2}{13}2^{-13}$ $= 0.0767732...$ whereas $\int_0^{1/4}\sqrt{x-x^2}\,dx = 0.076773$.

72) [Technology Required] Use Newton's approximation of the binomial $\sqrt{1-x^2}$ to approximate π as follows. The circle centered at $(\frac{1}{2}, 0)$ with radius $\frac{1}{2}$ has upper semicircle $y = \sqrt{x}\sqrt{1-x}$. The sector of this circle bounded by the x-axis between $x = 0$ and $x = \frac{1}{2}$ and by the line joining $(\frac{1}{4}, \frac{\sqrt{3}}{4})$ corresponds to $\frac{1}{6}$ of the circle and has area $\frac{\pi}{24}$. This sector is the union of a right triangle with height $\frac{\sqrt{3}}{4}$ and base $\frac{1}{4}$ and the region below the graph between $x = 0$ and $x = \frac{1}{4}$. To find the area of this region you can write y

$= \sqrt{x}\sqrt{1-x} = \sqrt{x} \times (\text{binomial expansion of} \sqrt{1-x})$ and integrate term by term. Use this approach with the binomial approximation from the previous exercise to estimate π.

73) Use the approximation $T \approx 2\pi\sqrt{\frac{L}{g}}(1+\frac{k^2}{4})$ to approximate the period of a pendulum having length 10 meters and maximum angle $\theta_{max} = \frac{\pi}{6}$ where $k = \sin\left(\frac{\theta_{max}}{2}\right)$. Compare this with the small angle estimate $T \approx 2\pi\sqrt{\frac{L}{g}}$.

Answer

$T \approx 2\pi\sqrt{\frac{10}{9.8}}\left(1+\frac{\sin^2(\theta/12)}{4}\right) \approx 6.453$ seconds. The small angle estimate is $T \approx 2\pi\sqrt{\frac{10}{9.8}} \approx 6.347$. The relative error is around 2 percent.

74) Suppose that a pendulum is to have a period of 2 seconds and a maximum angle of $\theta_{max} = \frac{\pi}{6}$. Use $T \approx 2\pi\sqrt{\frac{L}{g}}\left(1+\frac{k^2}{4}\right)$ to approximate the desired length of the pendulum. What length is predicted by the small angle estimate $T \approx 2\pi\sqrt{\frac{L}{g}}$?

75) Evaluate $\int_0^{\pi/2} \sin^4\theta \; d\theta$ in the approximation $T = 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\left(1+\frac{1}{2}k^2\sin^2\theta+\frac{3}{8}k^4\sin^4\theta+\cdots\right) d\theta$ to obtain an improved estimate for T.

Answer

$\int_0^{\pi/2} \sin^4\theta \, d\theta = \frac{3\pi}{16}$. Hence $T \approx 2\pi\sqrt{\frac{L}{g}}\left(1+\frac{k^2}{4}+\frac{9}{256}k^4\right)$.

76) [Technology Required] An equivalent formula for the period of a pendulum with amplitude θ_{max} is $T(\theta_{max}) = 2\sqrt{2}\sqrt{\frac{L}{g}}\int_0^{\theta_{max}} \frac{d\theta}{\sqrt{\cos\theta-\cos(\theta_{max})}}$ where L is the pendulum length and g is the gravitational acceleration constant. When $\theta_{max} = \frac{\pi}{3}$ we get $\frac{1}{\sqrt{\cos t - 1/2}} \approx \sqrt{2}\left(1+\frac{t^2}{2}+\frac{t^4}{3}+\frac{181t^6}{720}\right)$. Integrate this approximation to estimate $T(\frac{\pi}{3})$ in terms of L and g. Assuming $g = 9.806$ meters per second squared, find an approximate length L such that $T(\frac{\pi}{3}) = 2$ seconds.

Contributors

Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

This page titled 5.4E: Exercises is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Roy Simpson.

- **10.4E: Exercises for Section 10.4** by Edwin “Jed” Herman, Gilbert Strang is licensed CC BY-NC-SA 4.0. Original source: https://openstax.org/details/books/calculus-volume-1.

5.5: Chapter 5 Review Exercises

True or False? In exercises 1 - 4, justify your answer with a proof or a counterexample.

1) If the radius of convergence for a power series $\sum_{n=0}^{\infty} a_n x^n$ is 5, then the radius of convergence for the series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ is also 5.

Answer

True

2) Power series can be used to show that the derivative of e^x is e^x. (Hint: Recall that $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$.)

3) For small values of x, $\sin x \approx x$.

Answer

True

4) The radius of convergence for the Maclaurin series of $f(x) = 3^x$ is 3.

In exercises 5 - 8, find the radius of convergence and the interval of convergence for the given series.

5) $\sum_{n=0}^{\infty} n^2 (x-1)^n$

Answer

ROC: 1; IOC: $(0, 2)$

6) $\sum_{n=0}^{\infty} \frac{x^n}{n^n}$

7) $\sum_{n=0}^{\infty} \frac{3n x^n}{12^n}$

Answer

ROC: 12; IOC: $(-16, 8)$

8) $\sum_{n=0}^{\infty} \frac{2^n}{e^n} (x-e)^n$

In exercises 9 - 10, find the power series representation for the given function. Determine the radius of convergence and the interval of convergence for that series.

9) $f(x) = \frac{x^2}{x+3}$

Answer

$\sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} x^n$; ROC: 3; IOC: $(-3, 3)$

10) $f(x) = \frac{8x+2}{2x^2 - 3x + 1}$

In exercises 11 - 12, find the power series for the given function using term-by-term differentiation or integration.

11) $f(x) = \tan^{-1}(2x)$

Answer

integration: $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} (2x)^{2n+1}$

12) $f(x) = \dfrac{x}{(2+x^2)^2}$

In exercises 13 - 14, evaluate the Taylor series expansion of degree four for the given function at the specified point. What is the error in the approximation?

13) $f(x) = x^3 - 2x^2 + 4, \quad a = -3$

Answer

$p_4(x) = (x+3)^3 - 11(x+3)^2 + 39(x+3) - 41$; exact

14) $f(x) = e^{1/(4x)}, \quad a = 4$

In exercises 15 - 16, find the Maclaurin series for the given function.

15) $f(x) = \cos(3x)$

Answer

$$\sum_{n=0}^{\infty} \frac{(-1)^n (3x)^{2n}}{2n!}$$

16) $f(x) = \ln(x+1)$

In exercises 17 - 18, find the Taylor series at the given value.

17) $f(x) = \sin x, \quad a = \frac{\pi}{2}$

Answer

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(x - \frac{\pi}{2}\right)^{2n}$$

18) $f(x) = \dfrac{3}{x}, \quad a = 1$

In exercises 19 - 20, find the Maclaurin series for the given function.

19) $f(x) = e^{-x^2} - 1$

Answer

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} x^{2n}$$

20) $f(x) = \cos x - x \sin x$

In exercises 21 - 23, find the Maclaurin series for $F(x) = \int_0^x f(t)\, dt$ by integrating the Maclaurin series of $f(x)$ term by term.

21) $f(x) = \dfrac{\sin x}{x}$

Answer

$$F(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!} x^{2n+1}$$

22) $f(x) = 1 - e^x$

23) Use power series to prove **Euler's formula:** $e^{ix} = cosx + isinx$

Answer

Answers may vary.

Exercises 24 - 26 consider problems of *annuity payments*.

24) For annuities with a present value of $1 million, calculate the annual payouts given over 25 years assuming interest rates of $1\%, 5\%$, and 10%.

25) A lottery winner has an annuity that has a present value of $10 million. What interest rate would they need to live on perpetual annual payments of $\$250,000$?

Answer

2.5%

26) Calculate the necessary present value of an annuity in order to support annual payouts of $\$15,000$ given over 25 years assuming interest rates of $1\%, 5\%$,and 10%.

This page titled 5.5: Chapter 5 Review Exercises is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Gilbert Strang & Edwin "Jed" Herman via source content that was edited to the style and standards of the LibreTexts platform.

- **10.R: Chapter 10 Review Exercises** by Edwin "Jed" Herman, Gilbert Strang is licensed CC BY-NC-SA 4.0. Original source: https://openstax.org/details/books/calculus-volume-1.

CHAPTER OVERVIEW

6: Parametric Equations and Polar Coordinates

Parametric equations define a group of quantities as functions of one or more independent variables called parameters. Parametric equations are commonly used to express the coordinates of the points that make up a geometric object such as a curve or surface, in which case the equations are collectively called a parametric representation or parameterization. The polar coordinate system is a two-dimensional coordinate system in which each point on a plane is determined by a distance from a reference point and an angle from a reference direction. The reference point (analogous to the origin of a Cartesian system) is called the pole, and the ray from the pole in the reference direction is the polar axis. The distance from the pole is called the radial coordinate or radius, and the angle is called the angular coordinate, polar angle, or azimuth

6.1: Parametric Equations

6.1E: Exercises

6.2: Calculus of Parametric Curves

6.2E: Exercises

6.3: Polar Coordinates

6.3E: Exercises

6.4: Area and Arc Length in Polar Coordinates

6.4E: Exercises

6.5: Conic Sections

6.5E: Exercises

6.6: Chapter 6 Review Exercises

This page titled 6: Parametric Equations and Polar Coordinates is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Roy Simpson.

6.1: Parametric Equations

Learning Objectives

- Plot a curve described by parametric equations.
- Convert the parametric equations of a curve into the form $y = f(x)$.
- Recognize the parametric equations of basic curves, such as a line and a circle.
- Recognize the parametric equations of a cycloid.

This section examines parametric equations and their graphs. In the two-dimensional coordinate system, parametric equations are useful for describing curves that are not necessarily functions. The parameter is an independent variable that both x and y depend on. As the parameter increases, the values of x and y trace a path along a plane curve. For example, if the parameter is t (a common choice), then t might represent time. Then, x and y are defined as functions of time, and $(x(t), y(t))$ can describe the position in the plane of a given object as it moves along a curved path.

Parametric Equations and Their Graphs

Consider the orbit of Earth around the Sun. Our year lasts approximately 365.25 days, but we will use 365 days for this discussion. On January 1 of each year, the physical location of Earth with respect to the Sun is nearly the same, except for leap years, when the lag introduced by the extra $\frac{1}{4}$ day of orbiting time is built into the calendar. We call January 1 "day 1" of the year. Then, for example, day 31 is January 31, day 59 is February 28, and so on.

The day number in a year can be considered a variable that determines Earth's position in its orbit. As Earth revolves around the Sun, its physical location changes relative to the Sun. After one full year, we are back where we started, and a new year begins. According to Kepler's laws of planetary motion, the shape of the orbit is elliptical, with the Sun at one focus of the ellipse. We study this idea in more detail in Conic Sections.

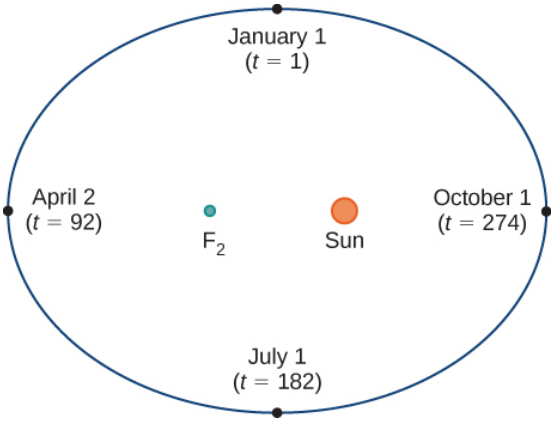


Figure 6.1.1: Earth's orbit around the Sun in one year.

Figure 6.1.1 depicts Earth's orbit around the Sun during one year. The point labeled F_2 is one of the ellipse's foci; the other focus is occupied by the Sun. Suppose we superimpose coordinate axes over this graph. In that case, we can assign ordered pairs to each point on the ellipse (Figure 6.1.2). Then, each x- and y-value on the graph is a value of position as a function of time. Therefore, each point on the graph corresponds to a value of Earth's position as a function of time.

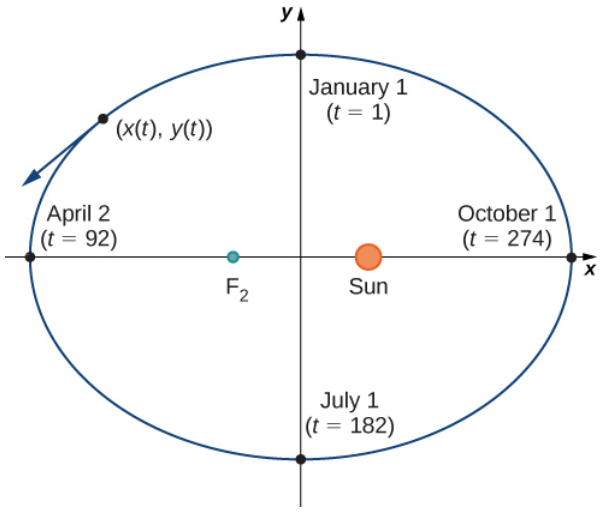


Figure **6.1.2**: Coordinate axes superimposed on the orbit of Earth.

We can determine the functions for $x(t)$ and $y(t)$, thereby **parametrizing** (also called **parameterizing**) the orbit of Earth around the Sun. The variable t is called an **independent parameter** and, in this context, represents time relative to the beginning of each year.

Any curve in the (x, y) plane can be represented parametrically. The equations that are used to define the curve are called **parametric equations**.

Definition: Parametric Equations

If x and y are continuous functions of t on an interval I, then the equations

$$x = x(t)$$

and

$$y = y(t)$$

are called **parametric equations** and t is called the **parameter**. The set of points (x, y) obtained as t varies over the interval I is called the graph of the **parametric equations**. The graph of parametric equations is called a **parametric curve**, or **plane curve**, and is denoted by C.

Notice in this definition that x and y are used in two ways. The first is as functions of the independent variable t. As t varies over the interval I, the functions $x(t)$ and $y(t)$ generate a set of ordered pairs (x, y). This set of ordered pairs generates the graph of the parametric equations. This usage of x and y as coordinates in a set of ordered pairs is the second interpretation of x and y. It is important to distinguish the variables x and y from the functions $x(t)$ and $y(t)$.

Example 6.1.1: Graphing a Parametrically Defined Curve

Sketch the curves described by the following parametric equations:

a. $x(t) = t - 1, \quad y(t) = 2t + 4, \quad \text{for } -3 \le t \le 2$
b. $x(t) = t^2 - 3, \quad y(t) = 2t + 1, \quad \text{for } -2 \le t \le 3$
c. $x(t) = 4\cos t, \quad y(t) = 4\sin t, \quad \text{for } 0 \le t \le 2\pi$

Solutions

a. To create a graph of this curve, first set up a table of values. Since the independent variable in both $x(t)$ and $y(t)$ is t, let t appear in the first column. Then $x(t)$ and $y(t)$ will appear in the second and third columns of the table.

t	$x(t)$	$y(t)$
−3	−4	−2
−2	−3	0
−1	−2	2
0	−1	4
1	0	6
2	1	8

The second and third columns in this table provide a set of points to be plotted. The graph of these points appears in Figure **6.1.3**. The arrows on the graph indicate the **orientation** of the graph. This is the direction that a point moves on the graph as t varies from -3 to 2.

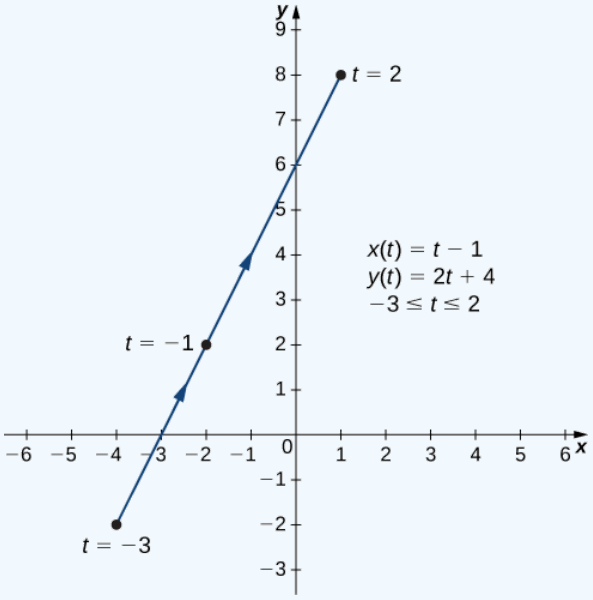


Figure **6.1.3**: Graph of the plane curve described by the parametric equations in part a.

b. To create a graph of this curve, again set up a table of values.

t	$x(t)$	$y(t)$
−2	1	−3
−1	−2	−1
0	−3	1
1	−2	3
2	1	5
3	6	7

The second and third columns in this table give a set of points to be plotted (Figure **6.1.4**). The first point on the graph (corresponding to $t = -2$) has coordinates $(1, -3)$, and the last point (corresponding to $t = 3$) has coordinates $(6, 7)$. As t progresses from -2 to 3, the point on the curve travels along a parabola. The direction the point moves is again called orientation and is indicated on the graph.

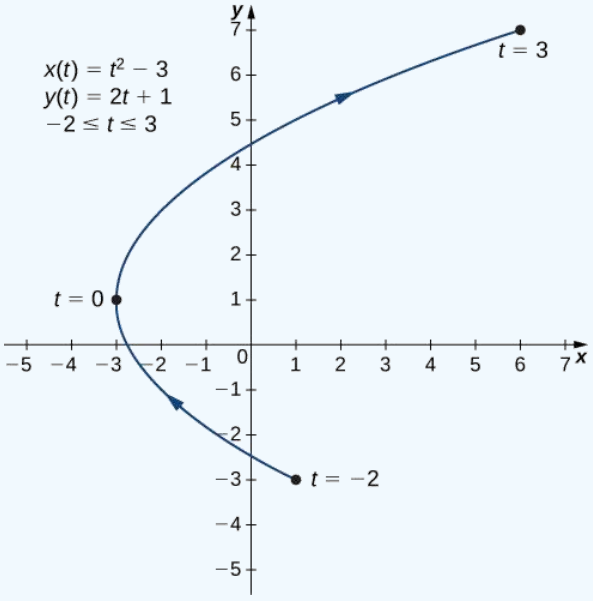


Figure 6.1.4: Graph of the plane curve described by the parametric equations in part b.

c. In this case, use multiples of $\pi/6$ for t and create another table of values:

t	$x(t)$	$y(t)$	t	$x(t)$	$y(t)$
0	4	0	$\frac{7\pi}{6}$	$-2\sqrt{3} \approx -3.5$	-2
$\frac{\pi}{6}$	$2\sqrt{3} \approx 3.5$	2	$\frac{4\pi}{3}$	-2	$-2\sqrt{3} \approx -3.5$
$\frac{\pi}{3}$	2	$2\sqrt{3} \approx 3.5$	$\frac{3\pi}{2}$	0	-4
$\frac{\pi}{2}$	0	4	$\frac{5\pi}{3}$	2	$-2\sqrt{3} \approx -3.5$
$\frac{2\pi}{3}$	-2	$2\sqrt{3} \approx 3.5$	$\frac{11\pi}{6}$	$2\sqrt{3} \approx 3.5$	-2
$\frac{5\pi}{6}$	$-2\sqrt{3} \approx -3.5$	2	2π	4	0
π	-4	0			

The graph of this plane curve appears in the following graph.

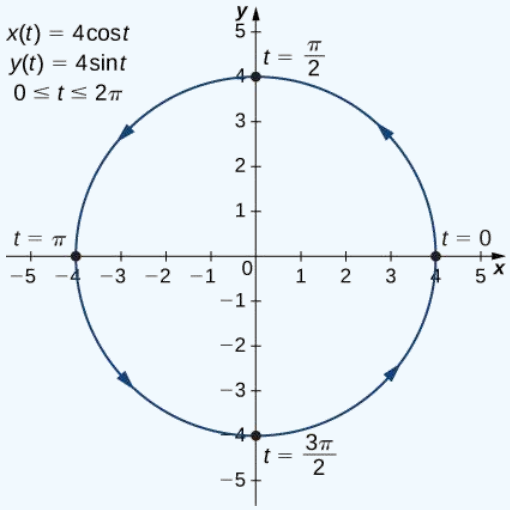


Figure 6.1.5: Graph of the plane curve described by the parametric equations in part c.

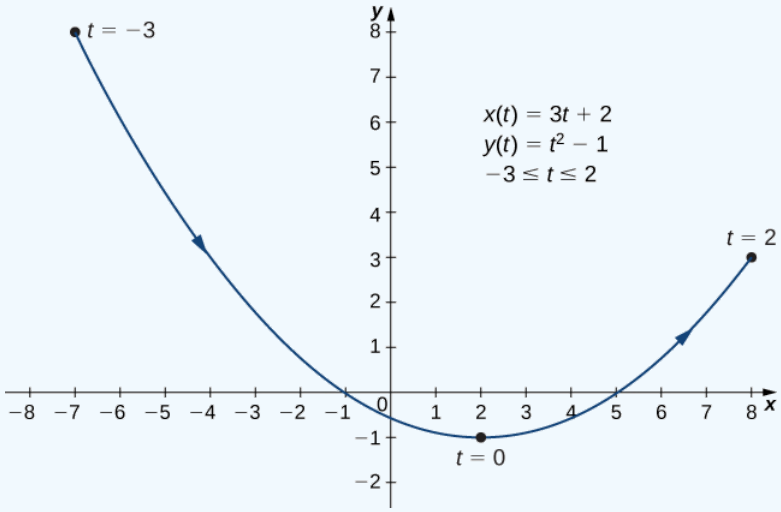
This is the graph of a circle with radius 4 centered at the origin, with a counterclockwise orientation. The starting point and ending points of the curve both have coordinates $(4,0)$.

? Checkpoint 6.1.1

Sketch the curve described by the parametric equations

$$x(t) = 3t + 2, \quad y(t) = t^2 - 1, \quad \text{for } -3 \le t \le 2.$$

Answer



Eliminating the Parameter

To better understand the graph of a curve represented parametrically, it is helpful to rewrite the two equations as a single equation relating the variables x and y. Then, we can apply any previous knowledge of equations of curves in the plane to identify the curve. For example, the equations describing the plane curve in Example $6.1.1b$ are

$$x(t) = t^2 - 3 \tag{6.1.1}$$

$$y(t) = 2t + 1 \tag{6.1.2}$$

over the region $-2 \le t \le 3$.

Solving Equation 6.1.2 for t gives

$$t = \frac{y-1}{2}.$$

This can be substituted into Equation 6.1.1:

$$x = \left(\frac{y-1}{2}\right)^2 - 3 \tag{6.1.3}$$

$$= \frac{y^2 - 2y + 1}{4} - 3 \tag{6.1.4}$$

$$= \frac{y^2 - 2y - 11}{4}. \tag{6.1.5}$$

Equation 6.1.5 describes x as a function of y. These steps give an example of eliminating the parameter. The graph of this function is a parabola opening to the right (Figure $6.1.4$). Recall that the plane curve started at $(1,-3)$ and ended at $(6,7)$. These terminations were due to the restriction on the parameter t.

Example 6.1.2: Eliminating the Parameter

Eliminate the parameter for each plane curve described by the following parametric equations and describe the resulting graph.

a. $x(t) = \sqrt{2t+4}, \quad y(t) = 2t+1, \quad \text{for } -2 \le t \le 6$

b. $x(t) = 4\cos t, \quad y(t) = 3\sin t, \quad \text{for } 0 \le t \le 2\pi$

Solution

a. To eliminate the parameter, we can solve either of the equations for t. For example, solving the first equation for t gives

$$\begin{aligned} x &= \sqrt{2t+4} \\ \Rightarrow \quad x^2 &= 2t+4 \\ \Rightarrow \quad x^2-4 &= 2t \\ \Rightarrow \quad t &= \frac{x^2-4}{2}. \end{aligned}$$

Note that when we square both sides, it is important to observe that $x \ge 0$. Substituting $t = \frac{x^2-4}{2}$ into $y(t)$ yields

$$\begin{aligned} y(t) &= 2t+1 \\ \Rightarrow \quad y &= 2\left(\frac{x^2-4}{2}\right)+1 \\ \Rightarrow \quad y &= x^2-4+1 \\ \Rightarrow \quad y &= x^2-3. \end{aligned}$$

This is the equation of a parabola opening upward. There is, however, a domain restriction because of the limits on the parameter t. When $t = -2$, $x = \sqrt{2(-2)+4} = 0$, and when $t = 6$, $x = \sqrt{2(6)+4} = 4$. The graph of this plane curve follows.

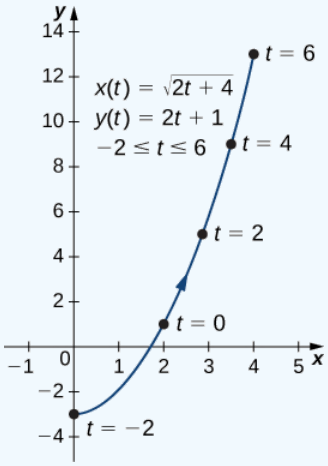


Figure 6.1.6: Graph of the plane curve described by the parametric equations in part a.

b. Sometimes, it is necessary to be creative in eliminating the parameter. The parametric equations for this example are

$$x(t) = 4\cos t$$

and

$$y(t) = 3\sin t.$$

Solving either equation for t directly is not advisable because sine and cosine are not one-to-one functions. However, dividing the first equation by 4 and the second equation by 3 (and suppressing the t) gives us

$$\cos t = \frac{x}{4}$$

and

$$\sin t = \frac{y}{3}.$$

Now use the Pythagorean Identity $\cos^2 t + \sin^2 t = 1$ and replace the expressions for $\sin t$ and $\cos t$ with the equivalent expressions in terms of x and y. This gives

$$\begin{aligned} \left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 &= 1 \\ \Rightarrow \qquad \frac{x^2}{16} + \frac{y^2}{9} &= 1. \end{aligned}$$

This is the equation of a horizontal ellipse centered at the origin, with semi-major axis 4 and semi-minor axis 3, as shown in the following graph.

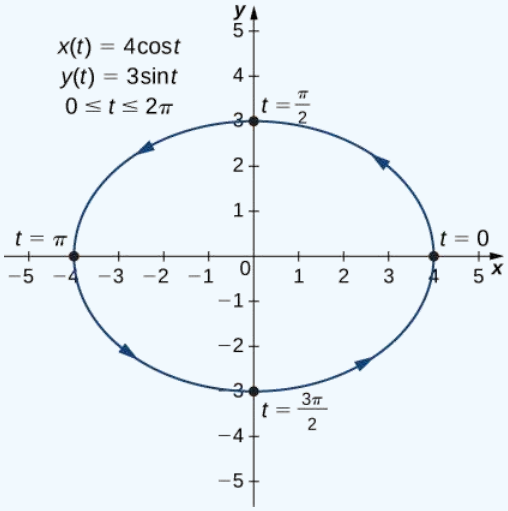


Figure 6.1.7: Graph of the plane curve described by the parametric equations in part b.

As t progresses from 0 to 2π, a point on the curve traverses the ellipse once in a counterclockwise direction. Recall from the section opener that the orbit of Earth around the Sun is also elliptical. This is a perfect example of using parametrized curves to model real-world phenomena.

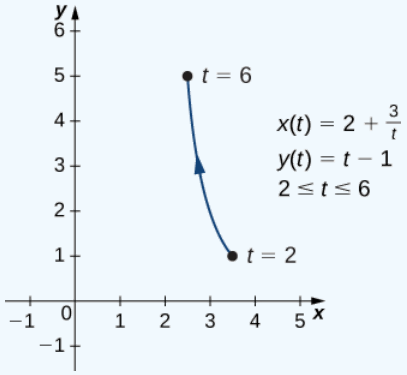
? Checkpoint 6.1.2

Eliminate the parameter for the plane curve defined by the following parametric equations and describe the resulting graph.

$$x(t) = 2 + \frac{3}{t}, \quad y(t) = t - 1, \quad \text{for } 2 \le t \le 6$$

Answer

$x = 2 + \frac{3}{y+1}$, or $y = -1 + \frac{3}{x-2}$. This equation describes a portion of a rectangular hyperbola centered at $(2, -1)$.



So far, we have seen the method of eliminating the parameter, assuming we know a set of parametric equations that describe a plane curve. What if we start with the equation of a curve and determine a pair of parametric equations for that curve? This can be done in many ways for a given curve. The process is known as **parametrization of a curve**.

Example 6.1.3: Parametrizing a Curve

Find two different pairs of parametric equations to represent the graph of $y = 2x^2 - 3$.

Solution

First, it is always possible to parametrize a curve by defining $x(t) = t$, then replacing x with t in the equation for $y(t)$. This gives the parametrization

$$x(t) = t, \quad y(t) = 2t^2 - 3.$$

Since there is no restriction on the domain in the original graph, there is no restriction on the values of t.

We have complete freedom in choosing the second parametrization. For example, we can choose $x(t) = 3t - 2$. The only thing we need to check is that there are no restrictions imposed on x; that is, the range of $x(t)$ is all real numbers. This is the case for $x(t) = 3t - 2$. Now since $y = 2x^2 - 3$, we can substitute $x(t) = 3t - 2$ for x. This gives

$$y(t) = 2(3t - 2)^2 - 2 = 2(9t^2 - 12t + 4) - 2 = 18t^2 - 24t + 8 - 2 = 18t^2 - 24t + 6.$$

Therefore, a second parametrization of the curve can be written as

$$x(t) = 3t - 2 \text{ and } y(t) = 18t^2 - 24t + 6.$$

Checkpoint 6.1.3

Find two sets of parametric equations to represent the graph of $y = x^2 + 2x$.

Answer

One possibility is $x(t) = t, \quad y(t) = t^2 + 2t$. Another possibility is $x(t) = 2t - 3, \quad y(t) = (2t - 3)^2 + 2(2t - 3) = 4t^2 - 8t + 3$. There are, in fact, an infinite number of possibilities.

Cycloids and Other Parametric Curves

Imagine going on a bicycle ride through the country. The tires stay in contact with the road and rotate in a predictable pattern. Suppose a very determined ant is tired after a long day and wants to get home. So he hangs onto the side of the tire and gets a free ride. The path that this ant travels down a straight road is called a cycloid (Figure 6.1.8). A cycloid generated by a circle (or bicycle wheel) of radius a is given by the parametric equations

$$x(t) = a(t - \sin t), \quad y(t) = a(1 - \cos t).$$

To see why this is true, consider the path that the center of the wheel takes. The center moves along the x-axis at a constant height equal to the wheel's radius. If the radius is a, then the coordinates of the center can be given by the equations

$$x(t) = at, \quad y(t) = a$$

for any value of t. Next, consider the ant, which rotates along a circular path around the center. If the bicycle moves from left to right, the wheels spin clockwise. A possible parametrization of the circular motion of the ant (relative to the center of the wheel) is given by

$$\begin{aligned} x(t) &= -a\sin t \\ y(t) &= -a\cos t. \end{aligned}$$

(The negative sign is needed to reverse the orientation of the curve. If the negative sign were not there, we would have to imagine the wheel rotating counterclockwise.) Adding these equations together gives the equations for the cycloid.

$$\begin{aligned} x(t) &= a(t - \sin t) \\ y(t) &= a(1 - \cos t) \end{aligned}$$

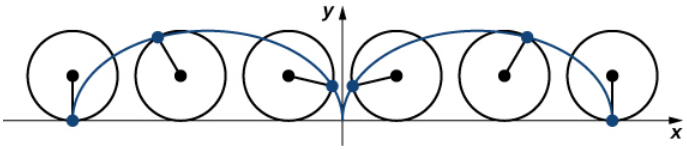


Figure 6.1.8: A wheel traveling along a road without slipping; the point on the edge of the wheel traces out a cycloid.

Now suppose that the bicycle wheel doesn't travel along a straight road but instead moves along the inside of a larger wheel, as in Figure 6.1.9. In this graph, the green circle travels around the blue circle in a counterclockwise direction. A point on the edge of the green circle traces out the red graph, which is called a **hypocycloid**.

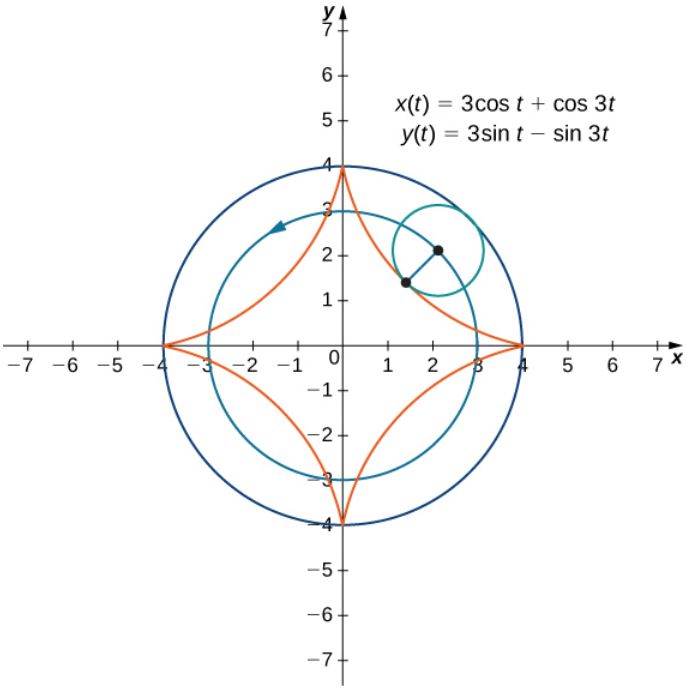


Figure 6.1.9: Graph of the hypocycloid described by the parametric equations shown.

The general parametric equations for a hypocycloid are

$$x(t) = (a-b)\cos t + b\cos\left(\frac{a-b}{b}\right)t$$

$$y(t) = (a-b)\sin t - b\sin\left(\frac{a-b}{b}\right)t.$$

These equations are more complicated. Still, the derivation is somewhat similar to the equations for the cycloid. In this case, we assume the radius of the larger circle is a and the radius of the smaller circle is b. Then, the center of the wheel travels along a circle of radius $a-b$. This fact explains the first term in each equation above. The period of the second trigonometric function in both $x(t)$ and $y(t)$ is equal to $\frac{2\pi b}{a-b}$.

The ratio $\frac{a}{b}$ is related to the number of **cusps** on the graph (cusps are the corners or pointed ends of the graph), as illustrated in Figure **6.1.10**. This ratio can lead to some fascinating graphs, depending on whether or not the ratio is rational. Figure **6.1.9** corresponds to $a = 4$ and $b = 1$. The result is a hypocycloid with four cusps. Figure **6.1.10** shows other possibilities. The last two hypocycloids have irrational values for $\frac{a}{b}$. In these cases, the hypocycloids have an infinite number of cusps, so they never return to their starting point. These are examples of what are known as **space-filling curves.**

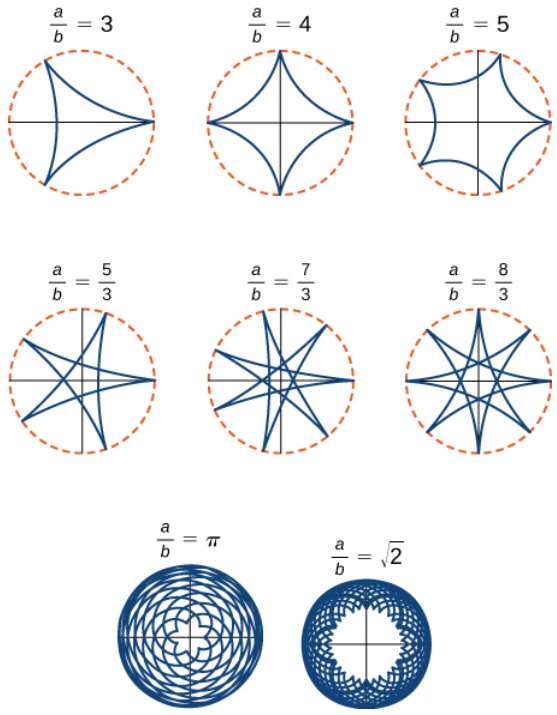


Figure **6.1.10**: Graph of various hypocycloids corresponding to different values of a/b.

The Witch of Agnesi

Many plane curves in mathematics are named after the people who first investigated them, like the **Folium of Descartes** or the **Spiral of Archimedes**. However, arguably, the strangest name for a curve is the **Witch of Agnesi.** Why a witch?

Maria Gaetana Agnesi (1718–1799) was one of eighteenth-century Italy's few recognized women mathematicians. She wrote a popular book on Analytic Geometry, published in 1748, which included an interesting curve studied by Fermat in 1630. In 1703, the mathematician Guido Grandi showed how to construct this curve, which he later called the "versoria," a Latin term for a rope used in sailing. Agnesi used the Italian term for this rope, "versiera," but in Latin, this same word means a "female goblin." When Agnesi's book was translated into English in 1801, the translator used "witch" for the curve instead of rope. The name "witch of Agnesi" has stuck ever since.

The witch of Agnesi is a curve defined as follows: Start with a circle of radius a so that the points $(0,0)$ and $(0,2a)$ are points on the circle (Figure **6.1.11**). Let O denote the origin. Choose any other point A on the circle, and draw the secant line OA. Let B denote the point at which the line OA intersects the horizontal line through $(0,2a)$. The vertical line through B intersects the

horizontal line through A at the point P. As the point A varies, the path that the point P travels is the witch of Agnesi curve for the given circle.

Witch of Agnesi curves have applications in physics, including modeling water waves and distributions of spectral lines. In probability theory, the curve describes the probability density function of the **Cauchy distribution**. In this project, you will parametrize these curves.

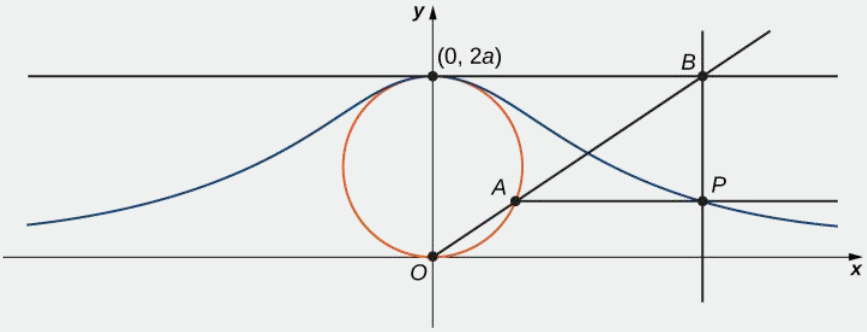


Figure 6.1.11: As the point A moves around the circle, the point P traces out the witch of Agnesi curve for the given circle.

1. On the figure, label the following points, lengths, and angle:
   a. C is the point on the x-axis with the same x-coordinate as A.
   b. x is the x-coordinate of P, and y is the y-coordinate of P.
   c. E is the point $(0, a)$.
   d. F is the point on the line segment OA such that the line segment EF is perpendicular to the line segment OA.
   e. b is the distance from O to F.
   f. c is the distance from F to A.
   g. d is the distance from O to C.
   h. θ is the measure of angle $\angle COA$.

This project aims to parametrize the witch using θ as a parameter. To do this, write equations for x and y in terms of only θ.

2. Show that $d = \frac{2a}{\sin\theta}$.
3. Note that $x = d\cos\theta$. Show that $x = 2a\cot\theta$. When you do this, you will have parametrized the x-coordinate of the curve with respect to θ. If you can get a similar equation for y, you will have parametrized the curve.
4. In terms of θ, what is the angle $\angle EOA$?
5. Show that $b + c = 2a\cos\left(\frac{\pi}{2} - \theta\right)$.
6. Show that $y = 2a\cos\left(\frac{\pi}{2} - \theta\right)\sin\theta$.
7. Show that $y = 2a\sin^2\theta$. You have now parametrized the y-coordinate of the curve with respect to θ.
8. Conclude that a parametrization of the given witch curve is

$$x = 2a\cot\theta, \quad y = 2a\sin^2\theta, \quad \text{for } -\infty < \theta < \infty.$$

9. Use your parametrization to show that the given witch curve is the graph of the function $f(x) = \frac{8a^3}{x^2+4a^2}$.

Travels with My Ant: The Curtate and Prolate Cycloids

Earlier in this section, we looked at the parametric equations for a cycloid, the path a point on the edge of a wheel traces as the wheel rolls along a straight path. In this project, we look at two different variations of the cycloid, called the **curtate** and **prolate cycloids**.

First, let's revisit the derivation of the parametric equations for a cycloid. Recall that we considered a tenacious ant trying to get home by hanging onto the edge of a bicycle tire. We assumed the ant climbed onto the tire at the very edge, where the tire touched the ground. As the wheel rolls, the ant moves with the edge of the tire (Figure 6.1.12).

As we have discussed, we have a lot of flexibility when parametrizing a curve. In this case, we let our parameter t represent the angle the tire has rotated through. Looking at Figure 6.1.12, we see that after the tire has rotated through an angle of t, the

position of the center of the wheel, $C = (x_C, y_C)$, is given by

$$x_C = at \quad \text{and} \quad y_C = a.$$

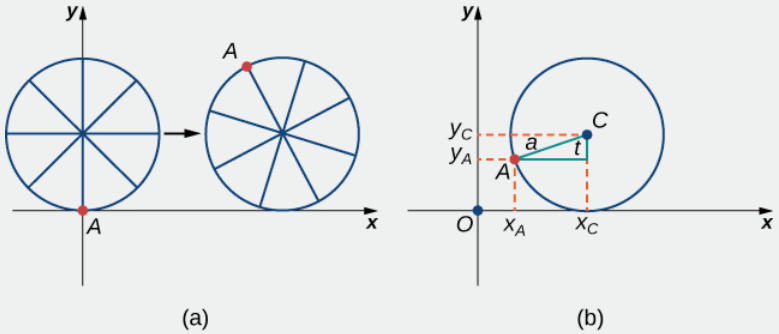
Furthermore, letting $A = (x_A, y_A)$ denote the position of the ant, we note that

$$x_C - x_A = a\sin t \quad \text{and} \quad y_C - y_A = a\cos t.$$

Then

$$x_A = x_C - a\sin t = at - a\sin t = a(t - \sin t)$$

$$y_A = y_C - a\cos t = a - a\cos t = a(1 - \cos t).$$



(a) (b)

Figure **6.1.12***a*: The ant clings to the edge of the bicycle tire as the tire rolls along the ground.

Figure **6.1.12***b*: Using Geometry to determine the position of the ant after the tire has rotated through an angle of t.

Note that these are the same parametric representations we had before, but we have now assigned a physical meaning to the parametric variable t.

After a while, the ant gets dizzy from going round and round on the edge of the tire. So he climbs up one of the spokes toward the center of the wheel. By climbing toward the center of the wheel, the ant has changed his path of motion. The new path has less up-and-down motion and is called a **curtate cycloid** (Figure **6.1.13**). As shown in the figure, we let b denote the distance along the spoke from the center of the wheel to the ant. As before, we let t represent the angle the tire has rotated through. Additionally, we let $C = (x_C, y_C)$ represent the position of the center of the wheel and $A = (x_A, y_A)$ represent the position of the ant.

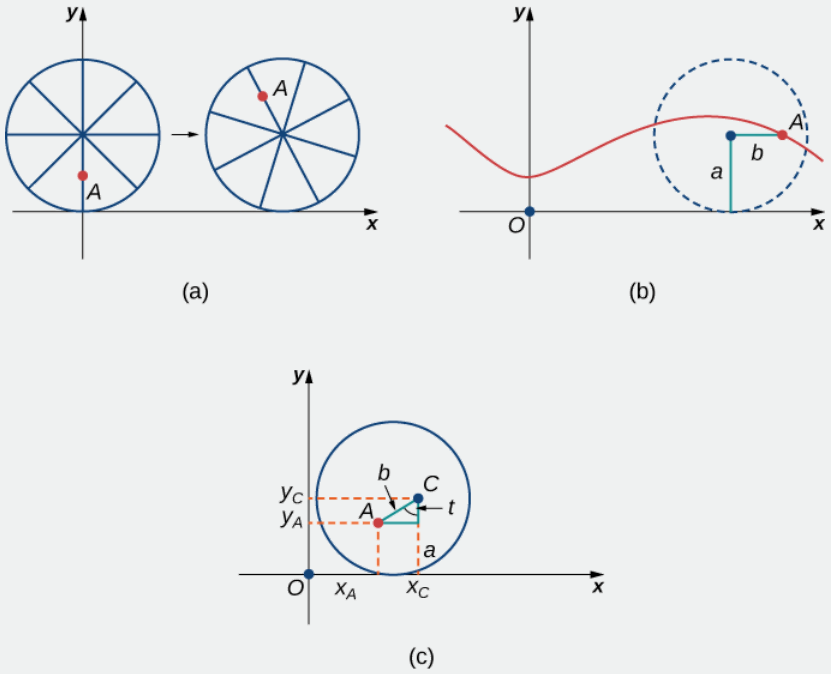


Figure **6.1.13***a*: The ant climbs up one of the spokes toward the center of the wheel.
Figure **6.1.13***b*: The ant's path of motion after climbing closer to the wheel's center. This is called a curtate cycloid.
Figure **6.1.13***c*: The new setup, now that the ant has moved closer to the center of the wheel.

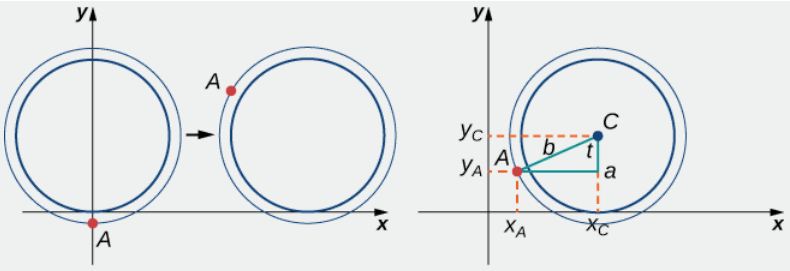
1. What is the position of the center of the wheel after the tire has rotated through an angle of t?
2. Use Geometry to find expressions for $x_C - x_A$ and for $y_C - y_A$.
3. Based on your answers to parts 1 and 2, what parametric equations represent the curtate cycloid?

Once the ant's head clears, he realizes that the bicyclist has made a turn and is now traveling away from his home. So he drops off the bicycle tire and looks around. Fortunately, a set of train tracks is nearby, headed back in the right direction. So the ant heads over to the train tracks to wait. After a while, a train goes by, heading in the right direction, and he manages to jump up and catch the edge of the train wheel (without getting squished!).

The ant is still worried about getting dizzy, but the train wheel is slippery and has no spokes to climb, so he decides to hang on to the edge of the wheel and hope for the best. Now, train wheels have a flange to keep the wheel running on the tracks. So, in this case, since the ant is hanging on to the very edge of the flange, the distance from the center of the wheel to the ant is greater than the radius of the wheel (Figure **6.1.14**).

The setup here is essentially the same as when the ant climbed up the spoke on the bicycle wheel. We let b denote the distance from the center of the wheel to the ant, and we let t represent the angle the tire has rotated through. Additionally, we let $C = (x_C, y_C)$ represent the position of the center of the wheel and $A = (x_A, y_A)$ represent the position of the ant (Figure **6.1.14**).

When the distance from the center of the wheel to the ant is greater than the wheel's radius, his path of motion is called a prolate cycloid. A graph of a **prolate cycloid** is shown in the figure.



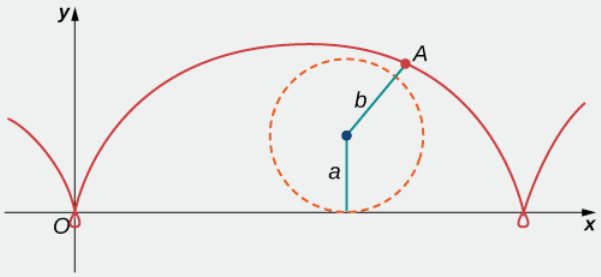


Figure **6.1.14*a***: The ant is hanging onto the flange of the train wheel.
Figure **6.1.14*b***: The new setup, now that the ant has jumped onto the train wheel.
Figure **6.1.14*c***: The ant travels along a prolate cycloid.

4. Using the same approach you used in parts 1– 3, find the parametric equations for the path of motion of the ant.
5. What do you notice about your answer to part 3 and your answer to part 4?

Notice that the ant is traveling backward at times (the "loops" in the graph), even though the train continues to move forward. He is probably going to be dizzy by the time he gets home!

Key Concepts

- Parametric equations provide a convenient way to describe a curve. A parameter can represent time or some other meaningful quantity.
- It is often possible to eliminate the parameter in a parametrized curve to obtain a function or relation describing that curve.
- There is always more than one way to parametrize a curve.
- Parametric equations can describe complicated curves that are difficult or impossible to describe using rectangular coordinates.

Glossary

cycloid
the curve traced by a point on the rim of a circular wheel as the wheel rolls along a straight line without slippage

cusp
a pointed end or part where two curves meet

orientation
the direction that a point moves on a graph as the parameter increases

parameter
an independent variable that both x and y depend on in a parametric curve; usually represented by the variable t

parametric curve

the graph of the parametric equations $x(t)$ and $y(t)$ over an interval $a \leq t \leq b$ combined with the equations

parametric equations

the equations $x = x(t)$ and $y = y(t)$ that define a parametric curve

parametrization of a curve

rewriting the equation of a curve defined by a function $y = f(x)$ as parametric equations

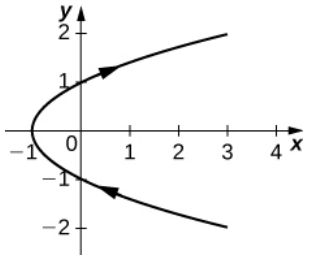
6.1E: EXERCISES

In exercises 1 - 4, sketch the curves below by eliminating the parameter t. Give the orientation of the curve.

1) $x = t^2 + 2t, \quad y = t + 1$

Answer

Orientation: bottom to top

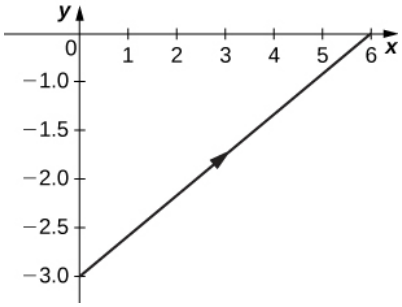


2) $x = \cos(t), \quad y = \sin(t), \quad \text{for } (0, 2\pi]$

3) $x = 2t + 4, \quad y = t - 1$

Answer

Orientation: left to right



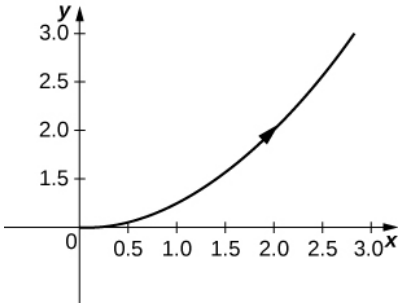
4) $x = 3 - t, \quad y = 2t - 3, \quad \text{for } 1.5 \le t \le 3$

In exercise 5, eliminate the parameter and sketch the graph.

5) $x = 2t^2, \quad y = t^4 + 1$

Answer

$y = \dfrac{x^2}{4} + 1$

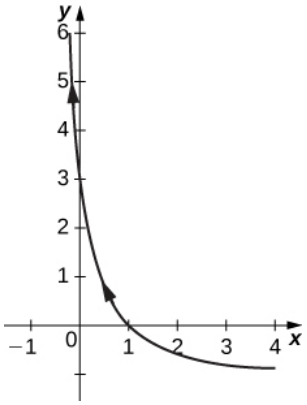


In exercises 6 - 9, use technology (CAS or calculator) to sketch the parametric equations.

6) [Technology Required] $x = t^2 + t, \quad y = t^2 - 1$

7) [Technology Required] $x = e^{-t}, \quad y = e^{2t} - 1$

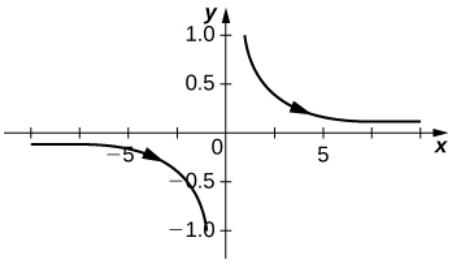
Answer



8) [Technology Required] $x = 3\cos t, \quad y = 4\sin t$

9) [Technology Required] $x = \sec t, \quad y = \cos t$

Answer

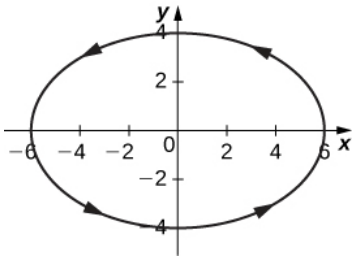


In exercises 10 - 20, sketch the parametric equations by eliminating the parameter. Indicate any asymptotes of the graph.

10) $x = e^t, \quad y = e^{2t} + 1$

11) $x = 6\sin(2\theta), \quad y = -4\cos(2\theta)$

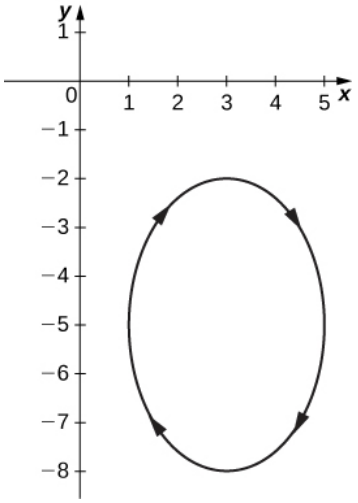
Answer



12) $x = \cos\theta, \quad y = 2\sin(2\theta)$

13) $x = 3 - 2\cos\theta, \quad y = -5 + 3\sin\theta$

Answer

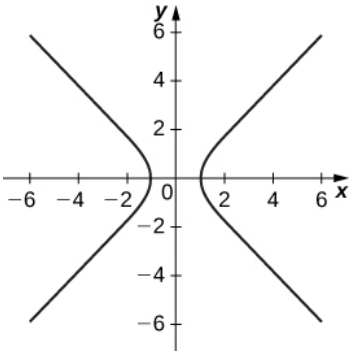


14) $x = 4 + 2\cos\theta, \quad y = -1 + \sin\theta$

15) $x = \sec t, \quad y = \tan t$

Answer

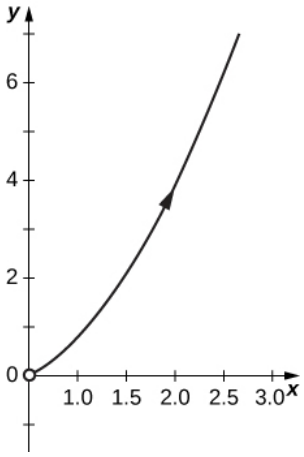
Asymptotes are $y = x$ and $y = -x$



16) $x = \ln(2t), \quad y = t^2$

17) $x = e^t, \quad y = e^{2t}$

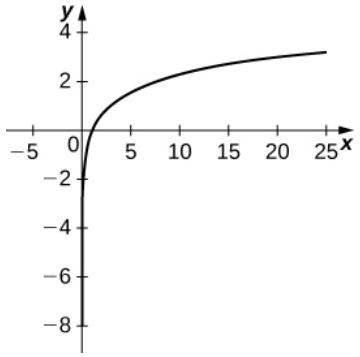
Answer



18) $x = e^{-2t}, \quad y = e^{3t}$

19) $x = t^3, \quad y = 3\ln t$

Answer



20) $x = 4\sec\theta, \quad y = 3\tan\theta$

In exercises 21 - 38, convert the parametric equations of a curve into rectangular form. No sketch is necessary. State the domain of the rectangular form.

21) $x = t^2 - 1, \quad y = \dfrac{t}{2}$

Answer

$x = 4y^2 - 1$; domain: $x \in [1, \infty)$.

22) $x = \dfrac{1}{\sqrt{t+1}}, \quad y = \dfrac{t}{1+t}, \quad \text{for } t > -1$

23) $x = 4\cos\theta, \quad y = 3\sin\theta, \quad \text{for } t \in (0, 2\pi]$

Answer

$\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$; domain $x \in [-4, 4]$.

24) $x = \cosh t, \quad y = \sinh t$

25) $x = 2t - 3, \quad y = 6t - 7$

Answer

$y = 3x + 2$; domain: all real numbers.

26) $x = t^2, \quad y = t^3$

27) $x = 1 + \cos t, \quad y = 3 - \sin t$

Answer

$(x-1)^2 + (y-3)^2 = 1$; domain: $x \in [0, 2]$.

28) $x = \sqrt{t}, \quad y = 2t + 4$

29) $x = \sec t, \quad y = \tan t, \quad \text{for } \pi \le t < \frac{3\pi}{2}$

Answer

$y = \sqrt{x^2 - 1}$; domain: $x \in (-\infty, -1]$.

30) $x = 2\cosh t, \quad y = 4\sinh t$

31) $x = \cos(2t), \quad y = \sin t$

Answer

$y^2 = \dfrac{1-x}{2}$; domain: $x \in [-1, 1]$.

32) $x = 4t + 3, \quad y = 16t^2 - 9$

33) $x = t^2, \quad y = 2\ln t, \quad \text{for } t \ge 1$

Answer

$y = \ln x$; domain: $x \in [1, \infty)$.

34) $x = t^3, \quad y = 3 \ln t, \quad \text{for } t \geq 1$

35) $x = t^n, \quad y = n \ln t, \quad \text{for } t \geq 1$, where n is a natural number

Answer

$y = \ln x$; domain: $x \in (0, \infty)$.

36) $x = \ln(5t), \quad y = \ln(t^2)$ where $1 \leq t \leq e$

37) $x = 2 \sin(8t), \quad y = 2 \cos(8t)$

Answer

$x^2 + y^2 = 4$; domain: $x \in [-2, 2]$.

38) $x = \tan t, \quad y = \sec^2 t - 1$

In exercises 39 - 48, the pairs of parametric equations represent lines, parabolas, circles, ellipses, or hyperbolas. Name the type of basic curve that each pair of equations represents.

39) $x = 3t + 4, \quad y = 5t - 2$

Answer

line

40) $x - 4 = 5t, \quad y + 2 = t$

41) $x = 2t + 1, \quad y = t^2 - 3$

Answer

parabola

42) $x = 3 \cos t, \quad y = 3 \sin t$

43) $x = 2 \cos(3t), \quad y = 2 \sin(3t)$

Answer

circle

44) $x = \cosh t, \quad y = \sinh t$

45) $x = 3 \cos t, \quad y = 4 \sin t$

Answer

ellipse

46) $x = 2 \cos(3t), \quad y = 5 \sin(3t)$

47) $x = 3 \cosh(4t) \quad y = 4 \sinh(4t)$

Answer

the right branch of a horizontally opening hyperbola

48) $x = 2 \cosh t, \quad y = 2 \sinh t$

49) Show that $x = h + r \cos \theta, \quad y = k + r \sin \theta$ represents the equation of a circle.

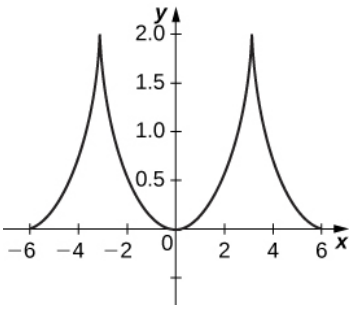
50) Use the equations in the preceding problem to find a set of parametric equations for a circle whose radius is 5 and whose center is $(-2, 3)$.

In exercises 51 - 53, use a graphing utility to graph the curve represented by the parametric equations and identify the curve from its equation.

51) [Technology Required] $x = \theta + \sin \theta, \quad y = 1 - \cos \theta$

Answer

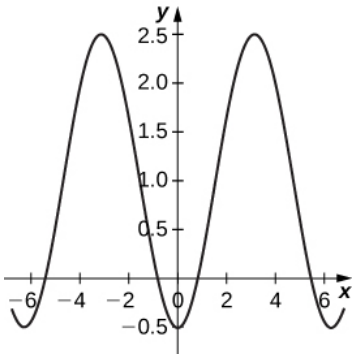
The equations represent a cycloid.



52) [Technology Required] $x = 2t - 2\sin t, \quad y = 2 - 2\cos t$

53) [Technology Required] $x = t - 0.5\sin t, \quad y = 1 - 1.5\cos t$

Answer



54) An airplane traveling horizontally at 100 m/s over flat ground at an elevation of 4000 meters must drop an emergency package on a target on the ground. The trajectory of the package is given by $x = 100t, \quad y = -4.9t^2 + 4000, \quad \text{where } t \geq 0$ where the origin is the point on the ground directly beneath the plane at the moment of release. How many horizontal meters before the target should the package be released in order to hit the target?

55) The trajectory of a bullet is given by $x = v_0(\cos\alpha)t, \quad y = v_0(\sin\alpha)t - \frac{1}{2}gt^2$ where $v_0 = 500$ m/s, $g = 9.8 = 9.8\ \text{m/s}^2$, and $\alpha = 30$ degrees. When will the bullet hit the ground? How far from the gun will the bullet hit the ground?

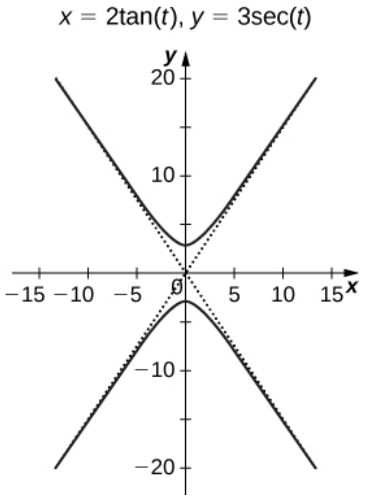
Answer

22,092 meters at approximately 51 seconds.

56) [Technology Required] Use technology to sketch the curve represented by $x = \sin(4t), \quad y = \sin(3t), \quad \text{for } 0 \leq t \leq 2\pi$.

57) [Technology Required] Use technology to sketch $x = 2\tan(t), \quad y = 3\sec(t), \quad \text{for } -\pi < t < \pi$.

Answer



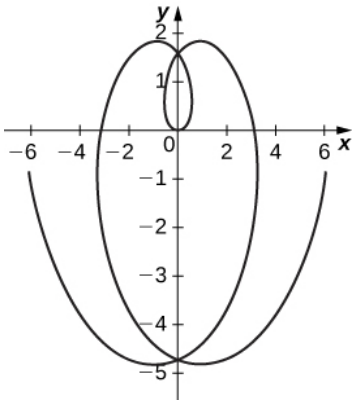
58) Sketch the curve known as an **epitrochoid**, which gives the path of a point on a circle of radius b as it rolls on the outside of a circle of radius a. The equations are

$x = (a+b)\cos t - c \cdot \cos\left[\frac{(a+b)t}{b}\right], \quad y = (a+b)\sin t - c \cdot \sin\left[\frac{(a+b)t}{b}\right].$

Let $a = 1,\ b = 2,\ c = 1.$

59) [Technology Required] Use technology to sketch the spiral curve given by $x = t\cos(t), \quad y = t\sin(t)$ for $-2\pi \le t \le 2\pi.$

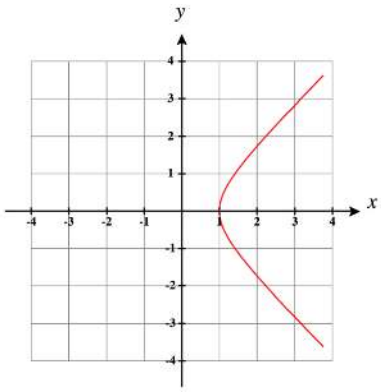
Answer



60) [Technology Required] Use technology to graph the curve given by the parametric equations $x = 2\cot(t), \quad y = 1 - \cos(2t), \quad \text{for } -\pi/2 \le t \le \pi/2.$ This curve is known as the witch of Agnesi.

61) [Technology Required] Sketch the curve given by parametric equations $x = \cosh(t), \quad y = \sinh(t)$, for $-2 \le t \le 2.$

Answer



This page titled 6.1E: Exercises is shared under a not declared license and was authored, remixed, and/or curated by Roy Simpson.

- **11.1E: Exercises for Section 11.1** is licensed CC BY-NC-SA 4.0.

6.2: Calculus of Parametric Curves

Learning Objectives

- Determine derivatives and equations of tangents for parametric curves.
- Find the area under a parametric curve.
- Use the equation for the arc length of a parametric curve.
- Apply the formula for surface area to a volume generated by a parametric curve.

Now that we have introduced the concept of a parametrized curve, our next step is to learn how to work with this concept in the context of Calculus. For example, if we know a parametrization of a given curve, is it possible to calculate the slope of a tangent line to the curve? How about the arc length of the curve? Or the area under the curve?

Another scenario: Suppose we would like to represent the location of a baseball after the ball leaves a pitcher's hand. If the position of the baseball is represented by the plane curve $(x(t), y(t))$, then we should be able to use Calculus to find the speed of the ball at any given time. Furthermore, we should be able to calculate how far that ball has traveled as a function of time.

Derivatives of Parametric Equations

We start by asking how to calculate the slope of a line tangent to a parametric curve at a point. Consider the plane curve defined by the parametric equations

$$x(t) = 2t + 3 \tag{6.2.1}$$

$$y(t) = 3t - 4 \tag{6.2.2}$$

within $-2 \leq t \leq 3$.

The graph of this curve appears in Figure 6.2.1. It is a line segment starting at $(-1, -10)$ and ending at $(9, 5)$.

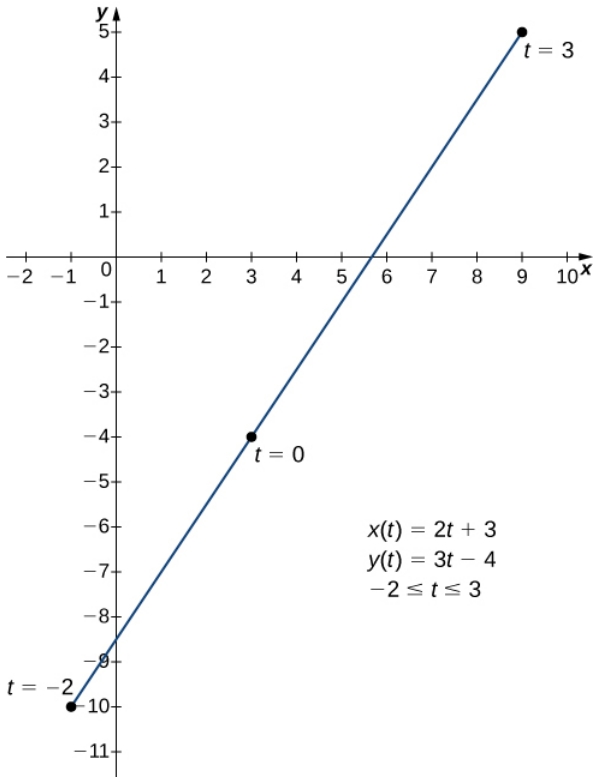


Figure 6.2.1: Graph of the line segment described by the given parametric equations.

We can eliminate the parameter by first solving Equation 6.2.1 for t:

$$\begin{aligned} x(t) &= 2t+3 \\ \Rightarrow \quad x-3 &= 2t \\ \Rightarrow \quad t &= \frac{x-3}{2}. \end{aligned}$$

Substituting this into $y(t)$ (Equation 6.2.2), we obtain

$$\begin{aligned} y(t) &= 3t-4 \\ \Rightarrow \quad y &= 3\left(\frac{x-3}{2}\right)-4 \\ \Rightarrow \quad y &= \frac{3x}{2}-\frac{9}{2}-4 \\ \Rightarrow \quad y &= \frac{3x}{2}-\frac{17}{2}. \end{aligned}$$

The slope of this line is given by $\frac{dy}{dx}=\frac{3}{2}$. Next we calculate $x'(t)$ and $y'(t)$. This gives $x'(t)=2$ and $y'(t)=3$. Notice that

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{3}{2}.$$

This is no coincidence, as outlined in the following theorem.

Theorem: Derivative of Parametric Equations

Consider the plane curve defined by the parametric equations $x=x(t)$ and $y=y(t)$. Suppose that $x'(t)$ and $y'(t)$ exist, and assume that $x'(t)\neq 0$. Then the derivative $\frac{dy}{dx}$ is given by

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{y'(t)}{x'(t)}. \tag{6.2.3}$$

Proof

This theorem can be proven using the Chain Rule. In particular, assume that the parameter t can be eliminated, yielding a differentiable function $y=F(x)$. Then $y(t)=F(x(t))$. Differentiating both sides of this equation using the Chain Rule yields

$$y'(t)=F'(x(t))\,x'(t),$$

so

$$F'(x(t))=\frac{y'(t)}{x'(t)}.$$

But $F'(x(t))=\frac{dy}{dx}$, which proves the theorem.

Equation 6.2.3 can be used to calculate derivatives of plane curves, as well as critical points. Recall that a critical point of a differentiable function $y=f(x)$ is any point $x=x_0$ such that either $f'(x_0)=0$ or $f'(x_0)$ does not exist. Equation 6.2.3 gives a formula for the slope of a tangent line to a curve defined parametrically regardless of whether the curve can be described by a function $y=f(x)$ or not.

Example 6.2.1: Finding the Derivative of a Parametric Curve

Calculate the derivative $\frac{dy}{dx}$ for each of the following parametrically-defined plane curves, and locate any critical points on their respective graphs.

a. $x(t)=t^2-3, \quad y(t)=2t-1, \quad \text{for } -3\leq t\leq 4$
b. $x(t)=2t+1, \quad y(t)=t^3-3t+4, \quad \text{for } -2\leq t\leq 2$
c. $x(t)=5\cos t, \quad y(t)=5\sin t, \quad \text{for } 0\leq t\leq 2\pi$

Solutions

a. To apply Equation 6.2.3, first calculate $x'(t)$ and $y'(t)$:

$$\begin{aligned} x'(t) &= 2t \\ y'(t) &= 2. \end{aligned}$$

Next substitute these into the equation:

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy/dt}{dx/dt} \\ \Rightarrow \quad \frac{dy}{dx} &= \frac{2}{2t} \\ \Rightarrow \quad \frac{dy}{dx} &= \frac{1}{t}. \end{aligned}$$

This derivative is undefined when $t = 0$. Calculating $x(0)$ and $y(0)$ gives $x(0) = (0)^2 - 3 = -3$ and $y(0) = 2(0) - 1 = -1$, which corresponds to the point $(-3, -1)$ on the graph. The graph of this curve is a parabola opening to the right, and the point $(-3, -1)$ is its vertex, as shown.

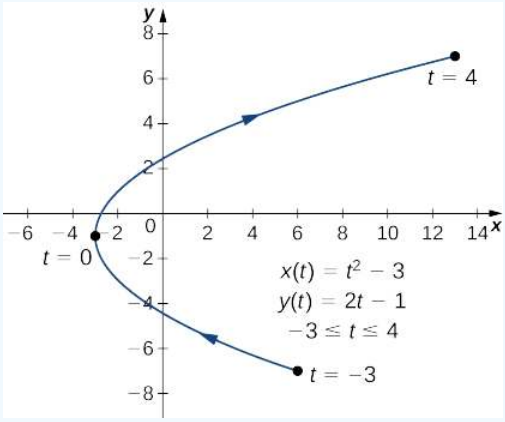


Figure 6.2.2: Graph of the parabola described by parametric equations in part a.

b. To apply Equation 6.2.3, first calculate $x'(t)$ and $y'(t)$:

$$\begin{aligned} x'(t) &= 2 \\ y'(t) &= 3t^2 - 3. \end{aligned}$$

Next substitute these into the equation:

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy/dt}{dx/dt} \\ \frac{dy}{dx} &= \frac{3t^2 - 3}{2}. \end{aligned}$$

This derivative is zero when $t = \pm 1$. When $t = -1$ we have

$$x(-1) = 2(-1) + 1 = -1 \text{ and } y(-1) = (-1)^3 - 3(-1) + 4 = -1 + 3 + 4 = 6,$$

which corresponds to the point $(-1, 6)$ on the graph. When $t = 1$ we have

$$x(1) = 2(1) + 1 = 3 \text{ and } y(1) = (1)^3 - 3(1) + 4 = 1 - 3 + 4 = 2,$$

which corresponds to the point $(3, 2)$ on the graph. The point $(3, 2)$ is a relative minimum, and the point $(-1, 6)$ is a relative maximum, as seen in the following graph.

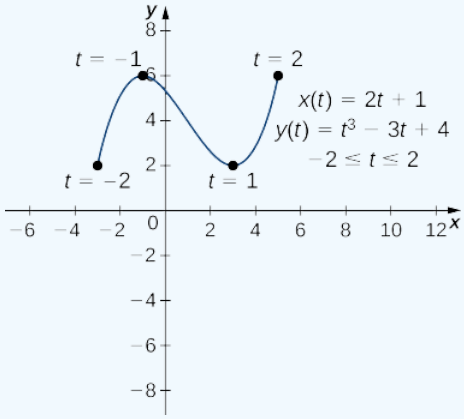


Figure 6.2.3: Graph of the curve described by parametric equations in part b.

c. To apply Equation 6.2.3, first calculate $x'(t)$ and $y'(t)$:

$$\begin{aligned} x'(t) &= -5\sin t \\ y'(t) &= 5\cos t. \end{aligned}$$

Next substitute these into the equation:

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy/dt}{dx/dt} \\ \Rightarrow \quad \frac{dy}{dx} &= \frac{5\cos t}{-5\sin t} \\ \Rightarrow \quad \frac{dy}{dx} &= -\cot t. \end{aligned}$$

This derivative is zero when $\cos t = 0$ and is undefined when $\sin t = 0$. This gives $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and 2π as critical points for t. Substituting each of these into $x(t)$ and $y(t)$, we obtain

t	$x(t)$	$y(t)$
0	5	0
$\frac{\pi}{2}$	0	5
π	−5	0
$\frac{3\pi}{2}$	0	−5
2π	5	0

These points correspond to the sides, top, and bottom of the circle that is represented by the parametric equations (Figure 6.2.4). The derivative is undefined on the left and right edges of the circle. On the top and bottom, the derivative equals zero.

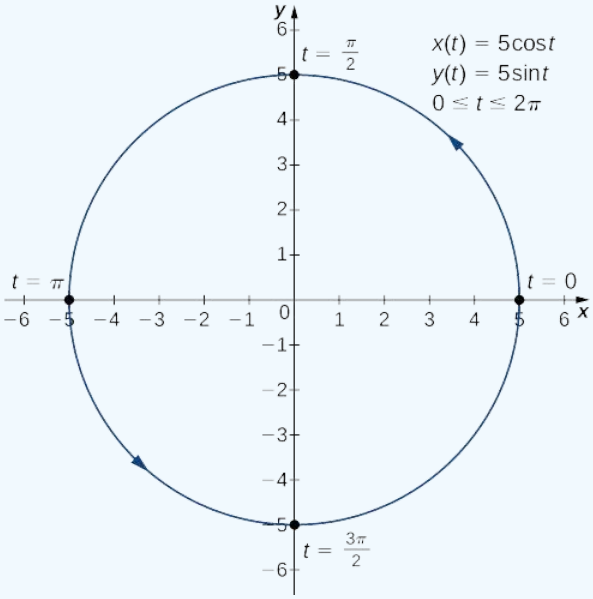


Figure **6.2.4**: Graph of the curve described by parametric equations in part c.

? Checkpoint 6.2.1

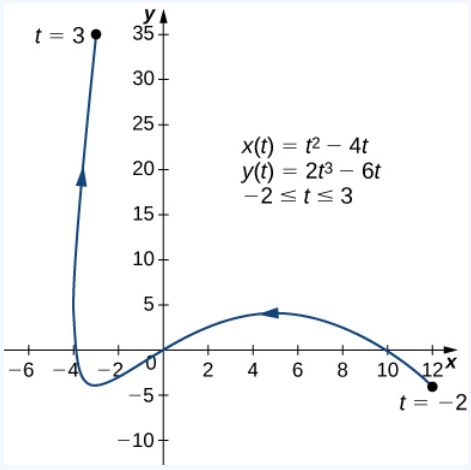
Calculate the derivative dy/dx for the plane curve defined by the equations

$$x(t) = t^2 - 4t, \quad y(t) = 2t^3 - 6t, \quad \text{for } -2 \le t \le 3$$

and locate any critical points on its graph.

Answer

$x'(t) = 2t - 4$ and $y'(t) = 6t^2 - 6$, so $\frac{dy}{dx} = \frac{6t^2-6}{2t-4} = \frac{3t^2-3}{t-2}$. This expression is undefined when $t = 2$ and equal to zero when $t = \pm 1$.



✓ Example 6.2.2: Finding a Tangent Line

Find the equation of the tangent line to the curve defined by the equations

$$x(t) = t^2 - 3, \quad y(t) = 2t - 1, \quad \text{for } -3 \le t \le 4$$

when $t = 2$.

Solution

First find the slope of the tangent line using Equation 6.2.3, which means calculating $x'(t)$ and $y'(t)$:

$$\begin{aligned} x'(t) &= 2t \\ y'(t) &= 2. \end{aligned}$$

Next substitute these into the equation:

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy/dt}{dx/dt} \\ \Rightarrow \quad \frac{dy}{dx} &= \frac{2}{2t} \\ \Rightarrow \quad \frac{dy}{dx} &= \frac{1}{t}. \end{aligned}$$

When $t = 2$, $\frac{dy}{dx} = \frac{1}{2}$, so this is the slope of the tangent line. Calculating $x(2)$ and $y(2)$ gives

$$x(2) = (2)^2 - 3 = 1 \text{ and } y(2) = 2(2) - 1 = 3,$$

which corresponds to the point $(1, 3)$ on the graph (Figure 6.2.5). Now use the point-slope form of the equation of a line to find the equation of the tangent line:

$$\begin{aligned} y - y_0 &= m(x - x_0) \\ \Rightarrow \quad y - 3 &= \frac{1}{2}(x - 1) \\ \Rightarrow \quad y - 3 &= \frac{1}{2}x - \frac{1}{2} \\ \Rightarrow \quad y &= \frac{1}{2}x + \frac{5}{2}. \end{aligned}$$

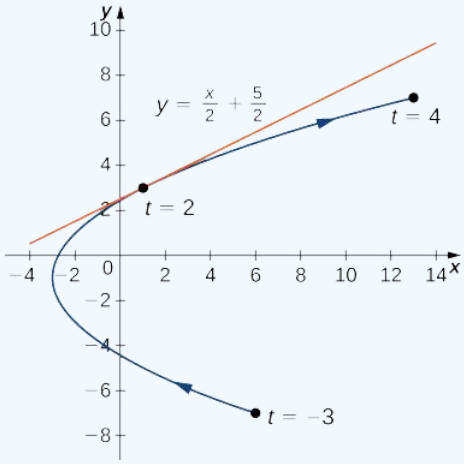


Figure 6.2.5: Tangent line to the parabola described by the given parametric equations when $t = 2$.

? Checkpoint 6.2.2

Find the equation of the tangent line to the curve defined by the equations

$$x(t) = t^2 - 4t, \quad y(t) = 2t^3 - 6t, \quad \text{for } -2 \le t \le 6$$

when $t = 5$.

Answer

The equation of the tangent line is $y = 24x + 100$.

Second-Order Derivatives

Our next goal is to see how to take the second derivative of a function defined parametrically. The second derivative of a function $y = f(x)$ is defined to be the derivative of the first derivative; that is,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right]. \tag{6.2.4}$$

Since

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt},$$

we can replace the y on both sides of Equation 6.2.4 with $\frac{dy}{dx}$. This gives us

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{(d/dt)(dy/dx)}{dx/dt}. \tag{6.2.5}$$

If we know dy/dx as a function of t, then this formula is straightforward to apply

Example 6.2.3: Finding a Second Derivative

Calculate the second derivative d^2y/dx^2 for the plane curve defined by the parametric equations $x(t) = t^2 - 3, \quad y(t) = 2t - 1,$ for $-3 \le t \le 4$.

Solution

From Example 6.2.1 we know that $\frac{dy}{dx} = \frac{2}{2t} = \frac{1}{t}$. Using Equation 6.2.5, we obtain

$$\frac{d^2y}{dx^2} = \frac{(d/dt)(dy/dx)}{dx/dt} = \frac{(d/dt)(1/t)}{2t} = \frac{-t^{-2}}{2t} = -\frac{1}{2t^3}.$$

Checkpoint 6.2.3

Calculate the second derivative d^2y/dx^2 for the plane curve defined by the equations

$$x(t) = t^2 - 4t, \quad y(t) = 2t^3 - 6t, \quad \text{for } -2 \le t \le 3$$

and locate any critical points on its graph.

Answer

$\frac{d^2y}{dx^2} = \frac{3t^2-12t+3}{2(t-2)^3}$. Critical points $(5, 4)$, $(-3, -4)$,and $(-4, 6)$.

Integrals Involving Parametric Equations

Now that we have seen how to calculate the derivative of a plane curve, we naturally ask ourselves how to find the area under a curve defined parametrically. Recall the cycloid defined by these parametric equations

$$\begin{aligned} x(t) &= t - \sin t \\ y(t) &= 1 - \cos t. \end{aligned} \tag{6.2.6}$$

Suppose we want to find the area of the shaded region in the following graph.

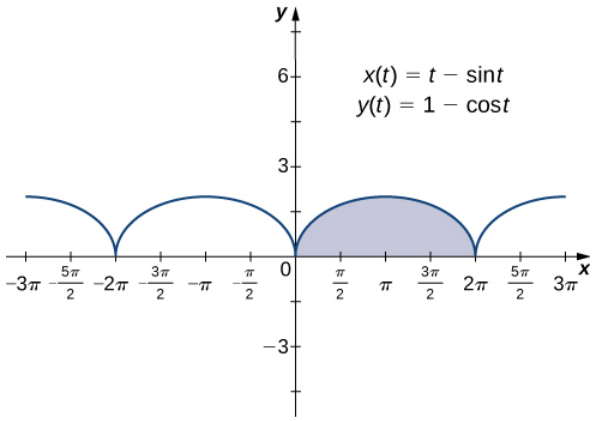


Figure 6.2.6: Graph of a cycloid with the arch over $[0, 2\pi]$ highlighted.

To derive a formula for the area under the curve defined by the functions

$$\begin{aligned} x &= x(t) \\ y &= y(t) \end{aligned} \tag{6.2.7}$$

where $a \le t \le b$, we assume that $x(t)$ is differentiable and start with an equal partition of the interval $a \le t \le b$. Suppose $t_0 = a < t_1 < t_2 < \cdots < t_n = b$ and consider the following graph.

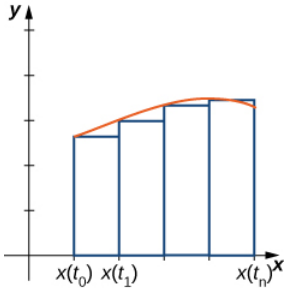


Figure 6.2.7: Approximating the area under a parametrically defined curve.

We use rectangles to approximate the area under the curve. The height of a typical rectangle in this parametrization is $y(x(\bar{t}_i))$ for some value $\bar{t}_i$ in the ith subinterval, and the width can be calculated as $x(t_i) - x(t_{i-1})$. Thus the area of the ith rectangle is given by

$$A_i = y(x(\bar{t}_i))(x(t_i) - x(t_{i-1})).$$

Then a Riemann sum for the area is

$$A_n = \sum_{i=1}^{n} y(x(\bar{t}_i))(x(t_i) - x(t_{i-1})).$$

Multiplying and dividing each area by $t_i - t_{i-1}$ gives

$$\begin{aligned} A_n &= \sum_{i=1}^{n} y(x(\bar{t}_i)) \left(\frac{x(t_i) - x(t_{i-1})}{t_i - t_{i-1}} \right) (t_i - t_{i-1}) \\ &= \sum_{i=1}^{n} y(x(\bar{t}_i)) \left(\frac{x(t_i) - x(t_{i-1})}{\Delta t} \right) \Delta t. \end{aligned}$$

Taking the limit as n approaches infinity gives

$$A = \lim_{n \to \infty} A_n = \int_a^b y(t) x'(t)\, dt.$$

This leads to the following theorem.

Theorem: Area under a Parametric Curve

Consider the non-self-intersecting plane curve defined by the parametric equations

$$x = x(t), \quad y = y(t), \quad \text{for } a \le x \le b$$

and assume that $x(t)$ is differentiable. The area under this curve is given by

$$A = \int_{t_a}^{t_b} y(t)x'(t)\,dt, \tag{6.2.8}$$

where t_a is the time-value associated with $x = a$ and t_b is the time-value associated with $x = b$.

If $x(t)$ is invertible, then $t_a = x^{-1}(a)$ and $t_b = x^{-1}(b)$.

As with our previous work, we require Δx from our partition of the interval $[a, b]$ to be positive. That is, when building the approximating rectangles from the Riemann sums, we get $x_0 = a$, $x_1 = a + i\Delta x$, ..., $x_n = b$. These values imply the computed width of each rectangle is positive. This creates a need to be careful when computing areas under parametric curves. It is often the case that a is *not* associated with the smallest value of t and b is *not* associated with the largest value of t. Therefore, it is best to *initially* think of the area integral for parametric curves as going from $x = a$ to $x = b$.

For example, when finding the area under the parametric curve defined by

$$x(t) = e^{-t}, \quad y(t) = t^2 + 3t + 4,$$

where $0 \le t \le 1$, we start by thinking in terms of how we normally find areas under curves:

$$A = \int f(x)\,dx.$$

Of course, this is an indefinite integral, but it is a placeholder for our thought-process. We know that $f(x)$ is a representation of the y-values for the curve and, in this case, $y(t) = t^2 + 3t + 4$. Moreover, $dx = x'(t)\,dt = -e^{-t}\,dt$. Thus, our "placeholder" integral becomes

$$A = \int f(x)\,dx = \int y(t)x'(t)\,dt = \int (t^2 + 3t + 4)(-e^{-t})\,dt = -\int (t^2 + 3t + 4)e^{-t}\,dt.$$

The only thing we have left to consider are the limits of integration.

Assuming the lower limit is $t = 0$ and the upper is $t = 1$ would be incorrect. Why? Since $x(0) = 1 > e^{-1} = x(1)$, our geometric derivation of the Riemann sum (from Calculus I) tells us that the lower limit is the time value corresponding to $x = e^{-1}$ and the upper limit is the time value corresponding to $x = 1$. Thus, the true integral representing the area under the curve from $t = 0$ to $t = 1$ is

$$-\int_{t=1}^{t=0} (t^2 + 3t + 4)e^{-t}\,dt.$$

The following Cautionary statement summarizes our concerns.

Caution: Areas Are Computed from x_{low} to x_{high}

When computing the area under a curve (parametric or not), we integrate from $x = a$ to $x = b$, where $a \le x \le b$. Any other method will result in an incorrect area.

Example 6.2.4: Finding the Area under a Parametric Curve

Find the area under the curve of the cycloid defined by the equations

$$x(t) = t - \sin t, \quad y(t) = 1 - \cos t, \quad \text{for } 0 \le t \le 2\pi.$$

Solution

We start by making sure we are integrating in the proper direction.

$$x(0) = 0 \quad \text{and} \quad x(2\pi) = 2\pi.$$

This seems convincing that our time-values are "flowing" from a lower x-value to a higher one, but just to be sure the parametric curve doesn't "double-back" on itself, let's compute $x'(t)$ to see if it is always increasing (we have to compute x' in any case, so why not get it out of the way right now?).

$$x'(t) = 1 - \cos t,$$

which is non-negative for all t. Thus, our x-values are *never* decreasing with respect to t. Using Equation 6.2.8, we have

$$\begin{aligned} A &= \int_{t_a}^{t_b} y(t)x'(t)\,dt \\ &= \int_0^{2\pi} (1-\cos t)(1-\cos t)\,dt \\ &= \int_0^{2\pi} (1 - 2\cos t + \cos^2 t)\,dt \\ &= \int_0^{2\pi} \left(1 - 2\cos t + \frac{1+\cos(2t)}{2}\right) dt \\ &= \int_0^{2\pi} \left(\frac{3}{2} - 2\cos t + \frac{\cos(2t)}{2}\right) dt \\ &= \frac{3t}{2} - 2\sin t + \frac{\sin(2t)}{4}\Big|_0^{2\pi} \\ &= 3\pi \end{aligned}$$

? Checkpoint 6.2.4

Find the area under the curve of the hypocycloid defined by the equations

$$x(t) = 3\cos t + \cos(3t), \quad y(t) = 3\sin t - \sin(3t), \quad \text{for } 0 \le t \le \pi.$$

Answer

$A = 3\pi$ (Note that this is one of those cases where the limits of integration go from high time-value to low. This is because $x(t)$ is a decreasing function over the interval $[0, \pi]$; that is, the curve is traced from right to left.)

Arc Length of a Parametric Curve

Besides finding the area under a parametric curve, we sometimes need to find the arc length of a parametric curve. In the case of a line segment, the arc length is the same as the distance between the endpoints. If a particle travels from point A to point B along a curve, then the distance that particle travels is the arc length. To develop a formula for arc length, we start with an approximation by line segments, as shown in the following graph.

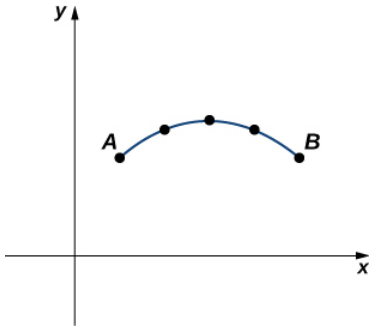


Figure 6.2.7: Approximation of a curve by line segments.

Given a plane curve defined by the functions $x = x(t)$, $y = y(t)$, for $a \le t \le b$, we start by partitioning the interval $[a, b]$ into n equal subintervals: $t_0 = a < t_1 < t_2 < \cdots < t_n = b$. The width of each subinterval is given by $\Delta t = (b-a)/n$. We can calculate the length of each line segment:

$$\begin{aligned} d_1 &= \sqrt{(x(t_1) - x(t_0))^2 + (y(t_1) - y(t_0))^2} \\ d_2 &= \sqrt{(x(t_2) - x(t_1))^2 + (y(t_2) - y(t_1))^2} \\ &\vdots \quad \vdots \quad \vdots \end{aligned}$$

Then add these up. We let s denote the exact arc length and s_n denote the approximation by n line segments:

$$s \approx \sum_{k=1}^{n} s_k = \sum_{k=1}^{n} \sqrt{(x(t_k) - x(t_{k-1}))^2 + (y(t_k) - y(t_{k-1}))^2}. \tag{6.2.9}$$

If we assume that $x(t)$ and $y(t)$ are differentiable functions of t, then the Mean Value Theorem applies, so in each subinterval $[t_{k-1}, t_k]$ there exist $\hat{t_k}$ and $\tilde{t_k}$ such that

$$\begin{aligned} x(t_k) - x(t_{k-1}) &= x'(\hat{t_k})(t_k - t_{k-1}) = x'(\hat{t_k})\,\Delta t \\ y(t_k) - y(t_{k-1}) &= y'(\tilde{t_k})(t_k - t_{k-1}) = y'(\tilde{t_k})\,\Delta t. \end{aligned}$$

Therefore Equation 6.2.9 becomes

$$\begin{aligned} s &\approx \sum_{k=1}^{n} s_k \\ &= \sum_{k=1}^{n} \sqrt{(x'(\hat{t_k})\Delta t)^2 + (y'(\tilde{t_k})\Delta t)^2} \\ &= \sum_{k=1}^{n} \sqrt{(x'(\hat{t_k}))^2(\Delta t)^2 + (y'(\tilde{t_k}))^2(\Delta t)^2} \\ &= \sum_{k=1}^{n} \sqrt{(x'(\hat{t_k}))^2 + (y'(\tilde{t_k}))^2}\Delta t. \end{aligned}$$

This is a Riemann sum that approximates the arc length over a partition of the interval $[a, b]$. If we further assume that the derivatives are continuous and let the number of points in the partition increase without bound, the approximation approaches the exact arc length. This gives

$$\begin{aligned} s &= \lim_{n\to\infty} \sum_{k=1}^{n} s_k \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{(x'(\hat{t_k}))^2 + (y'(\tilde{t_k}))^2}\Delta t \\ &= \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt. \end{aligned}$$

When taking the limit, the values of $\hat{t_k}$ and $\tilde{t_k}$ are both contained within the same ever-shrinking interval of width Δt, so they must converge to the same value.

We can summarize this method in the following theorem.

Theorem: Arc Length of a Parametric Curve

Consider the plane curve defined by the parametric equations

$$x = x(t), \quad y = y(t), \quad \text{for } t_1 \le t \le t_2$$

and assume that $x(t)$ and $y(t)$ are differentiable functions of t. Then the arc length of this curve is given by

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt. \tag{6.2.10}$$

Since the arc length formula involves a square root (which is always non-negative), the value of dt must also be non-negative for the length of an arc to be non-negative. Hence, our integrals for arc length will *always* go from a lower value of t to a higher value of t. This is summarized in the following Note.

Note: Arc Lengths Are Computed from t_{low} to t_{high}

Unlike areas under parametric curves, arc lengths of parametric curves are strictly computed from the lowest time value to the highest.

At this point, a side derivation leads to a previous formula for arc length. In particular, suppose the parameter can be eliminated, leading to a function $y = F(x)$. Then $y(t) = F(x(t))$ and the Chain Rule gives

$$y'(t) = F'\left(x(t)\right) x'(t).$$

Substituting this into Equation 6.2.10 gives

$$\begin{aligned} s &= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(F'(x)\frac{dx}{dt}\right)^2}\, dt \\ &= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 \left(1 + \left(F'(x)\right)^2\right)}\, dt \\ &= \int_{t_1}^{t_2} x'(t)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dt. \end{aligned}$$

Here we have assumed that $x'(t) > 0$, which is a reasonable assumption. The Chain Rule gives $dx = x'(t)\, dt$, and letting $a = x(t_1)$ and $b = x(t_2)$ we obtain the formula

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx,$$

which is the formula for arc length obtained in the Introduction to the Applications of Integration.

Example 6.2.5: Finding the Arc Length of a Parametric Curve

Find the arc length of the semicircle defined by the equations

$$x(t) = 3\cos t, \quad y(t) = 3\sin t, \quad \text{for } 0 \le t \le \pi.$$

Solution

The values $t = 0$ to $t = \pi$ trace out the blue curve in Figure 6.2.8. To determine its length, use Equation 6.2.10:

$$\begin{aligned} s &= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_0^{\pi} \sqrt{(-3\sin t)^2 + (3\cos t)^2}\, dt \\ &= \int_0^{\pi} \sqrt{9\sin^2 t + 9\cos^2 t}\, dt \\ &= \int_0^{\pi} \sqrt{9(\sin^2 t + \cos^2 t)}\, dt \\ &= \int_0^{\pi} 3\, dt \\ &= 3t\Big|_0^{\pi} \\ &= 3\pi \text{ units.} \end{aligned}$$

Note that the formula for the arc length of a semicircle is πr, and the radius of this circle is 3. This is a great example of using Calculus to derive a known formula for a geometric quantity.

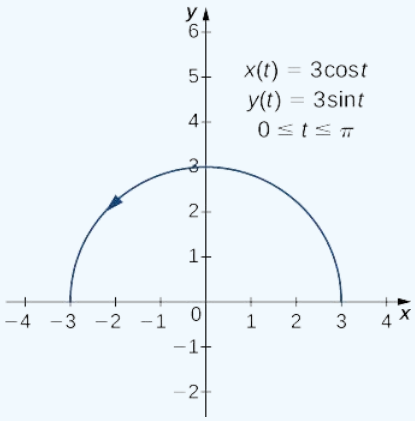


Figure 6.2.8: The arc length of the semicircle is equal to its radius times π.

? Checkpoint 6.2.5

Find the arc length of the curve defined by the equations

$$x(t) = 3t^2, \quad y(t) = 2t^3, \quad \text{for } 1 \le t \le 3.$$

Answer

$s = 2(10^{3/2} - 2^{3/2}) \approx 57.589$ units

We now return to the problem at the beginning of the section about a baseball leaving a pitcher's hand. Ignoring the effect of air resistance (unless it is a curve ball!), the ball travels a parabolic path. Assuming the pitcher's hand is at the origin, and the ball travels left to right in the direction of the positive x-axis, the parametric equations for this curve can be written as

$$x(t) = 140t, \quad y(t) = -16t^2 + 2t$$

where t represents time. We first calculate the distance the ball travels as a function of time. The arc length represents this distance. We can modify the arc length formula slightly. First, rewrite the functions $x(t)$ and $y(t)$ using v as an independent variable to eliminate any confusion with the parameter t:

$$x(v) = 140v, \quad y(v) = -16v^2 + 2v.$$

Then we write the arc length formula as follows:

$$\begin{aligned} s(t) &= \int_0^t \sqrt{(\frac{dx}{dv})^2 + (\frac{dy}{dv})^2}\, dv \\ &= \int_0^t \sqrt{140^2 + (-32v+2)^2}\, dv \end{aligned}$$

The variable v acts as a dummy variable that disappears after integration, leaving the arc length as a function of time t. Using our previous integration techniques,

$$\int \sqrt{a^2+u^2}\, du = \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln|u+\sqrt{a^2+u^2}| + C.$$

We set $a = 140$ and $u = -32v+2$. This gives $du = -32\, dv$, so $dv = -\frac{1}{32}\, du$. Therefore,

$$\begin{aligned} \int \sqrt{140^2 + (-32v+2)^2}\, dv &= -\frac{1}{32}\int \sqrt{a^2+u^2}\, du \\ &= -\frac{1}{32}\left[\frac{(-32v+2)}{2}\sqrt{140^2+(-32v+2)^2}\right. \\ &\quad \left. + \frac{140^2}{2}\ln|(-32v+2)+\sqrt{140^2+(-32v+2)^2}| + C\right] \end{aligned}$$

and

$$\begin{aligned} s(t) &= -\frac{1}{32}\left[\frac{(-32t+2)}{2}\sqrt{140^2+(-32t+2)^2} + \frac{140^2}{2}\ln\left|(-32t+2)+\sqrt{140^2+(-32t+2)^2}\right|\right] \\ &\quad + \frac{1}{32}\left[\sqrt{140^2+2^2} + \frac{140^2}{2}\ln\left|2+\sqrt{140^2+2^2}\right|\right] \\ &= \left(\frac{t}{2}-\frac{1}{32}\right)\sqrt{1024t^2-128t+19604} - \frac{1225}{4}\ln\left|(-32t+2)+\sqrt{1024t^2-128t+19604}\right| \\ &\quad + \frac{\sqrt{19604}}{32} + \frac{1225}{4}\ln(2+\sqrt{19604}). \end{aligned}$$

This function represents the distance traveled by the ball as a function of time. To calculate the speed, take the derivative of this function with respect to t. While this may seem like a daunting task, it is possible to obtain the answer directly from the Fundamental Theorem of Calculus:

$$\frac{d}{dx}\int_a^x f(u)\, du = f(x).$$

Therefore

$$\begin{aligned} s'(t) &= \frac{d}{dt}\left[s(t)\right] \\ &= \frac{d}{dt}\left[\int_0^t \sqrt{140^2+(-32v+2)^2}\, dv\right] \\ &= \sqrt{140^2+(-32t+2)^2} \\ &= \sqrt{1024t^2-128t+19604} \\ &= 2\sqrt{256t^2-32t+4901}. \end{aligned}$$

One-third of a second after the ball leaves the pitcher's hand, the distance it travels is equal to

$$
\begin{aligned}
s\left(\frac{1}{3}\right) &= \left(\frac{1/3}{2}-\frac{1}{32}\right)\sqrt{1024\left(\frac{1}{3}\right)^2-128\left(\frac{1}{3}\right)+19604} \\
&\quad -\frac{1225}{4}\ln\left|\left(-32\left(\frac{1}{3}\right)+2\right)+\sqrt{1024\left(\frac{1}{3}\right)^2-128\left(\frac{1}{3}\right)+19604}\right| \\
&\quad +\frac{\sqrt{19604}}{32}+\frac{1225}{4}\ln(2+\sqrt{19604}) \\
&\approx 46.69 \text{ feet.}
\end{aligned}
$$

This value is just over three-quarters of the way to home plate. The speed of the ball is

$$s'\left(\frac{1}{3}\right)=2\sqrt{256\left(\frac{1}{3}\right)^2-32\left(\frac{1}{3}\right)+4901}\approx 140.27 \text{ ft/s.}$$

This speed translates to approximately 95 mph—a major-league fastball.

Surface Area Generated by a Parametric Curve

Recall the problem of finding the surface area of a volume of revolution. Earlier in the text, we derived a formula for finding the surface area of a volume generated by a function $y=f(x)$ from $x=a$ to $x=b$, revolved around the x-axis:

$$S=2\pi\int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx.$$

We now consider a volume of revolution generated by revolving a parametrically defined curve $x=x(t)$, $y=y(t)$, for $a\le t\le b$ around the x-axis as shown in Figure 6.2.9.

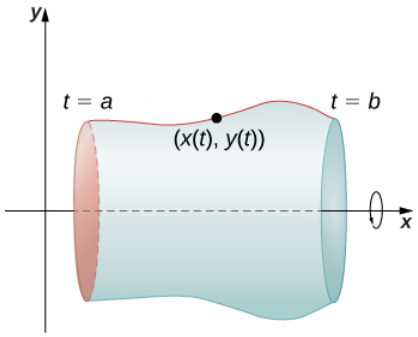


Figure 6.2.9: A surface of revolution generated by a parametrically defined curve.

The analogous formula for a parametrically defined curve is

$$S=2\pi\int_a^b y(t)\sqrt{(x'(t))^2+(y'(t))^2}\,dt \tag{6.2.11}$$

provided that $y(t)$ is not negative on $[a,b]$.

Example 6.2.6: Finding Surface Area

Find the surface area of a sphere of radius r centered at the origin.

Solution

We start with the curve defined by the equations

$$x(t)=r\cos t,\quad y(t)=r\sin t,\quad \text{for } 0\le t\le\pi.$$

This generates an upper semicircle of radius r centered at the origin, as shown in the following graph.

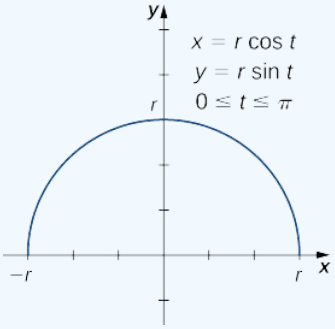


Figure 6.2.10: A semicircle generated by parametric equations.

When this curve is revolved around the x-axis, it generates a sphere of radius r. To calculate the surface area of the sphere, we use Equation 6.2.11:

$$\begin{aligned} S &= 2\pi \int_a^b y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\, dt \\ &= 2\pi \int_0^\pi r\sin t\sqrt{(-r\sin t)^2 + (r\cos t)^2}\, dt \\ &= 2\pi \int_0^\pi r\sin t\sqrt{r^2\sin^2 t + r^2\cos^2 t}\, dt \\ &= 2\pi \int_0^\pi r\sin t\sqrt{r^2(\sin^2 t + \cos^2 t)}\, dt \\ &= 2\pi \int_0^\pi r^2\sin t\, dt \\ &= 2\pi r^2\left(-\cos t\Big|_0^\pi\right) \\ &= 2\pi r^2(-\cos\pi + \cos 0) \\ &= 4\pi r^2 \text{ units}^2. \end{aligned}$$

This is, in fact, the formula for the surface area of a sphere.

? Checkpoint 6.2.6

Find the surface area generated when the plane curve defined by the equations

$$x(t) = t^3, \quad y(t) = t^2, \quad \text{for } 0 \le t \le 1$$

is revolved around the x-axis.

Answer

$A = \frac{\pi(494\sqrt{13}+128)}{1215}$ units^2

Key Concepts

- The derivative of the parametrically defined curve $x = x(t)$ and $y = y(t)$ can be calculated using the formula $\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$. Using the derivative, we can find the equation of a tangent line to a parametric curve.
- The area between a parametric curve and the x-axis can be determined by using the formula $A = \int_{t_1}^{t_2} y(t)x'(t)\, dt$.
- The arc length of a parametric curve can be calculated by using the formula

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

- The surface area of a volume of revolution revolved around the x-axis is given by

$$S = 2\pi \int_a^b y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\, dt.$$

- If the curve is revolved around the y-axis, then the formula is

$$S = 2\pi \int_a^b x(t)\sqrt{(x'(t))^2 + (y'(t))^2}\, dt.$$

Key Equations

- **Derivative of parametric equations**

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}$$

- **Second-order derivative of parametric equations**

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{(d/dt)(dy/dx)}{dx/dt}$$

- **Area under a parametric curve**

$$A = \int_a^b y(t)x'(t)\, dt$$

- **Arc length of a parametric curve**

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

- **Surface area generated by a parametric curve**

$$S = 2\pi \int_a^b y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

6.2E: EXERCISES

In exercises 1 - 4, each set of parametric equations represents a line. Without eliminating the parameter, find the slope of each line.

1) $x = 3 + t, \quad y = 1 - t$

2) $x = 8 + 2t, \quad y = 1$

Answer

$m = 0$

3) $x = 4 - 3t, \quad y = -2 + 6t$

4) $x = -5t + 7, \quad y = 3t - 1$

Answer

$m = -\frac{3}{5}$

In exercises 5 - 9, determine the slope of the tangent line, then find the equation of the tangent line at the given value of the parameter.

5) $x = 3\sin t, \quad y = 3\cos t, \quad \text{for } t = \frac{\pi}{4}$

6) $x = \cos t, \quad y = 8\sin t, \quad \text{for } t = \frac{\pi}{2}$

Answer

Slope $= 0; y = 8.$

7) $x = 2t, \quad y = t^3, \quad \text{for } t = -1$

8) $x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}, \quad \text{for } t = 1$

Answer

Slope is undefined; $x = 2$.

9) $x = \sqrt{t}, \quad y = 2t, \quad \text{for } t = 4$

In exercises 10 - 13, find all points on the curve that have the given slope.

10) $x = 4\cos t, \quad y = 4\sin t$, slope = 0.5

Solution

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4\cos t}{-4\sin t} = -\cot t.$

Setting this derivative equal to 0.5, we obtain the equation, $\tan t = -2$.

$\tan t = -2 \implies \frac{y}{x} = -2 \implies y = -2x.$

Note also that this pair of parametric equations represents the circle $x^2 + y^2 = 16$.

By substitution, we find that this curve has a slope of 0.5 at the points:

$\left(\frac{4\sqrt{5}}{5}, \frac{-8\sqrt{5}}{5}\right)$ and $\left(\frac{-4\sqrt{5}}{5}, \frac{8\sqrt{5}}{5}\right)$.

11) $x = 2\cos t, \quad y = 8\sin t$, slope= -1

12) $x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}$, slope= 1

Answer

No points possible; undefined expression.

13) $x = 2 + \sqrt{t}, \quad y = 2 - 4t$, slope= 0

In exercises 14 - 16, write the equation of the tangent line in Cartesian coordinates for the given parameter t.

14) $x = e^{\sqrt{t}}, \quad y = 1 - \ln t^2, \quad \text{for } t = 1$

Answer

$y = -\left(\frac{4}{e}\right)x + 5$

15) $x = t \ln t, \quad y = \sin^2 t, \quad \text{for } t = \frac{\pi}{4}$

16) $x = e^t, \quad y = (t-1)^2$, at $(1, 1)$

Answer

$y = -2x + 3$

17) For $x = \sin(2t), \quad y = 2\sin t$ where $0 \le t < 2\pi$. Find all values of t at which a horizontal tangent line exists.

18) For $x = \sin(2t), \quad y = 2\sin t$ where $0 \le t < 2\pi$. Find all values of t at which a vertical tangent line exists.

Answer

A vertical tangent line exists at $t = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$

19) Find all points on the curve $x = 4\cos(t), \quad y = 4\sin(t)$ that have the slope of $\frac{1}{2}$.

20) Find $\frac{dy}{dx}$ for $x = \sin(t), \quad y = \cos(t)$.

Answer

$\frac{dy}{dx} = -\tan(t)$

21) Find the equation of the tangent line to $x = \sin(t), \quad y = \cos(t)$ at $t = \frac{\pi}{4}$.

22) For the curve $x = 4t, \quad y = 3t - 2$, find the slope and concavity of the curve at $t = 3$.

Answer

$\frac{dy}{dx} = \frac{3}{4}$ and $\frac{d^2y}{dx^2} = 0$, so the curve is neither concave up nor concave down at $t = 3$. Therefore the graph is linear and has a constant slope but no concavity.

23) For the parametric curve whose equation is $x = 4\cos\theta, \quad y = 4\sin\theta$, find the slope and concavity of the curve at $\theta = \frac{\pi}{4}$.

24) Find the slope and concavity for the curve whose equation is $x = 2 + \sec\theta, \quad y = 1 + 2\tan\theta$ at $\theta = \frac{\pi}{6}$.

Answer

$\frac{dy}{dx} = 4, \quad \frac{d^2y}{dx^2} = -4\sqrt{3}$; the curve is concave down at $\theta = \frac{\pi}{6}$.

25) Find all points on the curve $x = t + 4, \quad y = t^3 - 3t$ at which there are vertical and horizontal tangents.

26) Find all points on the curve $x = \sec\theta, \quad y = \tan\theta$ at which horizontal and vertical tangents exist.

Answer

No horizontal tangents. Vertical tangents at $(1, 0)$ and $(-1, 0)$.

In exercises 27 - 29, find d^2y/dx^2.

27) $x = t^4 - 1, \quad y = t - t^2$

28) $x = \sin(\pi t), \quad y = \cos(\pi t)$

Answer

$d^2y/dx^2 = -\sec^3(\pi t)$

29) $x = e^{-t}, \quad y = te^{2t}$

In exercises 30 - 31, find points on the curve at which tangent line is horizontal or vertical.

30) $x = t(t^2 - 3), \quad y = 3(t^2 - 3)$

Answer

Horizontal $(0, -9)$;
Vertical $(\pm 2, -6)$.

31) $x = \dfrac{3t}{1+t^3}, \quad y = \dfrac{3t^2}{1+t^3}$

In exercises 32 - 34, find dy/dx at the value of the parameter.

32) $x = \cos t, y = \sin t, \quad \text{for } t = \frac{3\pi}{4}$

Answer

$dy/dx = 1$

33) $x = \sqrt{t}, \quad y = 2t + 4, t = 9$

34) $x = 4\cos(2\pi s), \quad y = 3\sin(2\pi s), \quad \text{for } s = -\frac{1}{4}$

Answer

$dy/dx = 0$

In exercises 35 - 36, find d^2y/dx^2 at the given point without eliminating the parameter.

35) $x = \frac{1}{2}t^2, \quad y = \frac{1}{3}t^3, \quad \text{for } t = 2$

36) $x = \sqrt{t}, \quad y = 2t + 4, \quad \text{for } t = 1$

Answer

$d^2y/dx^2 = 4$

37) Find intervals for t on which the curve $x = 3t^2, \quad y = t^3 - t$ is concave up as well as concave down.

38) Determine the concavity of the curve $x = 2t + \ln t, \quad y = 2t - \ln t$.

Answer

Concave up on $t > 0$.

39) Sketch and find the area under one arch of the cycloid $x = r(\theta - \sin\theta), \quad y = r(1 - \cos\theta)$.

40) Find the area bounded by the curve $x = \cos t, \quad y = e^t, \quad \text{for } 0 \le t \le \frac{\pi}{2}$ and the lines $y = 1$ and $x = 0$.

Answer

1 unit^2

41) Find the area enclosed by the ellipse $x = a\cos\theta, \quad y = b\sin\theta, \quad \text{for } 0 \le \theta < 2\pi$.

42) Find the area of the region bounded by $x = 2\sin^2\theta, \quad y = 2\sin^2\theta\tan\theta$, for $0 \le \theta \le \frac{\pi}{2}$.

Answer

$\frac{3\pi}{2} \text{ units}^2$

In exercises 43 - 46, find the area of the regions bounded by the parametric curves and the indicated values of the parameter.

43) $x = 2\cot\theta, \quad y = 2\sin^2\theta, \quad \text{for } 0 \le \theta \le \pi$

44) [Technology Required] $x = 2a\cos t - a\cos(2t), \quad y = 2a\sin t - a\sin(2t), \quad \text{for } 0 \le t < 2\pi$

Answer

$6\pi a^2 \text{ units}^2$

45) [Technology Required] $x = a\sin(2t), \quad y = b\sin(t), \quad \text{for } 0 \le t < 2\pi$ (the “hourglass”)

46) [Technology Required] $x = 2a\cos t - a\sin(2t), \quad y = b\sin t, \quad \text{for } 0 \le t < 2\pi$ (the “teardrop”)

Answer

$2\pi ab \text{ units}^2$

In exercises 47 - 52, find the arc length of the curve on the indicated interval of the parameter.

47) $x = 4t + 3, \quad y = 3t - 2, \quad \text{for } 0 \le t \le 2$

48) $x = \frac{1}{3}t^3, \quad y = \frac{1}{2}t^2, \quad \text{for } 0 \le t \le 1$

Answer

$s = \frac{1}{3}(2\sqrt{2} - 1)$ units

49) $x = \cos(2t), \quad y = \sin(2t), \quad \text{for } 0 \le t \le \frac{\pi}{2}$

50) $x = 1 + t^2, \quad y = (1 + t)^3, \quad \text{for } 0 \le t \le 1$

Answer

$s = 7.075$ units

51) $x = e^t \cos t, \quad y = e^t \sin t, \quad \text{for } 0 \le t \le \frac{\pi}{2}$ (express answer as a decimal rounded to three places)

52) $x = a\cos^3\theta, \quad y = a\sin^3\theta$ on the interval $[0, 2\pi)$ (the hypocycloid)

Answer

$s = 6a$ units

53) Find the length of one arch of the cycloid $x = 4(t - \sin t), \quad y = 4(1 - \cos t)$.

54) Find the distance traveled by a particle with position (x, y) as t varies in the given time interval: $x = \sin^2 t, \quad y = \cos^2 t, \quad \text{for } 0 \le t \le 3\pi$.

Answer

$6\sqrt{2}$ units

55) Find the length of one arch of the cycloid $x = \theta - \sin\theta, \quad y = 1 - \cos\theta$.

56) Show that the total length of the ellipse $x = 4\sin\theta, \quad y = 3\cos\theta$ is $L = 16\int_0^{\pi/2} \sqrt{1 - e^2\sin^2\theta}\, d\theta$, where $e = \frac{c}{a}$ and $c = \sqrt{a^2 - b^2}$.

57) Find the length of the curve $x = e^t - t, \quad y = 4e^{t/2}, \quad \text{for } -8 \le t \le 3$.

In exercises 58 - 59, find the area of the surface obtained by rotating the given curve about the x-axis.

58) $x = t^3, \quad y = t^2, \quad \text{for } 0 \le t \le 1$

Answer

$\frac{2\pi(247\sqrt{13} + 64)}{1215}$ units^2

59) $x = a\cos^3\theta, \quad y = a\sin^3\theta, \quad \text{for } 0 \le \theta \le \frac{\pi}{2}$

60) [Technology Required] Use a CAS to find the area of the surface generated by rotating $x = t + t^3, \quad y = t - \frac{1}{t^2}, \quad \text{for } 1 \le t \le 2$ about the x -axis. (Answer to three decimal places.)

Answer

59.101 units^2

61) Find the surface area obtained by rotating $x = 3t^2, \quad y = 2t^3, \quad \text{for } 0 \le t \le 5$ about the y-axis.

62) Find the area of the surface generated by revolving $x = t^2, \quad y = 2t, \quad \text{for } 0 \le t \le 4$ about the x-axis.

Answer

$\frac{8\pi}{3}(17\sqrt{17} - 1)$ units^2

63) Find the surface area generated by revolving $x = t^2, \quad y = 2t^2, \quad \text{for } 0 \le t \le 1$ about the y-axis.

This page titled 6.2E: Exercises is shared under a not declared license and was authored, remixed, and/or curated by Roy Simpson.

- **11.2E: Exercises for Section 11.2** is licensed CC BY-NC-SA 4.0.

6.3: Polar Coordinates

Learning Objectives

- Locate points in a plane by using polar coordinates.
- Convert points between rectangular and polar coordinates.
- Sketch polar curves from given equations.
- Convert equations between rectangular and polar coordinates.
- Identify symmetry in polar curves and equations.

The rectangular coordinate system (or Cartesian plane) maps points to ordered pairs and ordered pairs to points. This is called a **one-to-one mapping** from points in the plane to ordered pairs. The polar coordinate system provides an alternative method of mapping points to ordered pairs. This section shows that polar coordinates can be more useful in some circumstances than rectangular coordinates.

Defining Polar Coordinates

To find the coordinates of a point in the polar coordinate system, consider Figure 6.3.1. The point P has Cartesian coordinates (x, y). The line segment connecting the origin to the point P measures the distance from the origin to P and has length r. The angle between the positive x-axis and the line segment has measure θ. This observation suggests a natural correspondence between the coordinate pair (x, y) and the values r and θ. This correspondence is the basis of the **polar coordinate system**. Every point in the Cartesian plane has two values (hence the term ordered pair) associated with it. In the polar coordinate system, each point also has two values associated with it: r and θ.

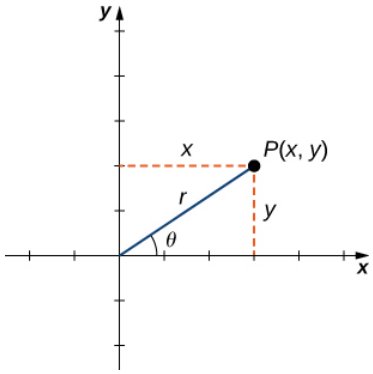


Figure 6.3.1: An arbitrary point in the Cartesian plane.

Using Right Triangle Trigonometry, the following equations are valid for the point P:

$$\cos\theta = \frac{x}{r} \quad \text{so} \quad x = r\cos\theta$$

$$\sin\theta = \frac{y}{r} \quad \text{so} \quad y = r\sin\theta$$

Furthermore,

$$r^2 = x^2 + y^2$$

and

$$\tan\theta = \frac{y}{x}.$$

Each point (x, y) in the Cartesian coordinate system can therefore be represented as an ordered pair (r, θ) in the polar coordinate system. The first coordinate is called the **radial coordinate** and the second coordinate is called the **angular coordinate**. Every point in the plane can be represented in this form.

The equation $\tan\theta = y/x$ has an infinite number of solutions for any ordered pair (x, y). However, if we restrict the solutions to values between 0 and 2π, we can assign a unique solution to the quadrant in which the original point (x, y) is located. Then the corresponding value of r is positive, so $r^2 = x^2 + y^2$.

Theorem: Converting Points between Coordinate Systems

Given a point P in the plane with Cartesian coordinates (x, y) and polar coordinates (r, θ), the following conversion formulas hold true:

$$x = r\cos\theta \tag{6.3.1}$$

$$y = r\sin\theta \tag{6.3.2}$$

and

$$r^2 = x^2 + y^2 \tag{6.3.3}$$

$$\tan\theta = \frac{y}{x} \tag{6.3.4}$$

These formulas can be used to convert from rectangular to polar or from polar to rectangular coordinates. Notice that Equation 6.3.3 is the Pythagorean Theorem. (Figure 6.3.1).

Example 6.3.1: Converting between Rectangular and Polar Coordinates

Convert each of the following points into polar coordinates.

a. $(1, 1)$
b. $(-3, 4)$
c. $(0, 3)$
d. $(5\sqrt{3}, -5)$

Convert each of the following points into rectangular coordinates.

e. $(3, \pi/3)$
f. $(2, 3\pi/2)$
g. $(6, -5\pi/6)$

Solutions

a. Use $x = 1$ and $y = 1$ in Equation 6.3.3:

$$\begin{aligned} r^2 &= x^2 + y^2 \\ &= 1^2 + 1^2 \\ \Rightarrow \quad r &= \sqrt{2} \end{aligned}$$

and via Equation 6.3.4

$$\begin{aligned} \tan\theta &= \frac{y}{x} \\ &= \frac{1}{1} \\ &= 1 \\ \Rightarrow \quad \theta &= \frac{\pi}{4}. \end{aligned}$$

Therefore this point can be represented as $(\sqrt{2}, \frac{\pi}{4})$ in polar coordinates.

b. Use $x = -3$ and $y = 4$ in Equation 6.3.3:

$$\begin{aligned} r^2 &= x^2 + y^2 \\ &= (-3)^2 + (4)^2 \\ \Rightarrow \quad r &= 5 \end{aligned}$$

and via Equation 6.3.4

$$\begin{aligned} \tan\theta &= \frac{y}{x} \\ &= -\frac{4}{3} \\ \Rightarrow \quad \theta &= \arctan\left(-\frac{4}{3}\right) + \pi \\ &\approx 2.21 \end{aligned}$$

Therefore this point can be represented as $(5, 2.21)$ in polar coordinates.

c. Use $x = 0$ and $y = 3$ in Equation 6.3.3:

$$r^2 = x^2 + y^2 = (3)^2 + (0)^2 = 9 + 0 \implies r = 3$$

and via Equation 6.3.4

$$\tan\theta = \frac{y}{x} = \frac{3}{0}.$$

Direct application of the second equation leads to division by zero. Graphing the point $(0, 3)$ on the rectangular coordinate system reveals that the point is located on the positive y-axis. The angle between the positive x-axis and the positive y-axis is $\frac{\pi}{2}$. Therefore this point can be represented as $(3, \frac{\pi}{2})$ in polar coordinates.

d. Use $x = 5\sqrt{3}$ and $y = -5$ in Equation 6.3.3:

$$r^2 = x^2 + y^2 = (5\sqrt{3})^2 + (-5)^2 = 75 + 25 \implies r = 10$$

and via Equation 6.3.4

$$\begin{aligned} \tan\theta &= \frac{y}{x} \\ &= \frac{-5}{5\sqrt{3}} \\ &= -\frac{\sqrt{3}}{3} \\ \Rightarrow \quad \theta &= -\frac{\pi}{6} \end{aligned}$$

Therefore this point can be represented as $(10, -\frac{\pi}{6})$ in polar coordinates.

e. Use $r = 3$ and $\theta = \frac{\pi}{3}$ in Equation 6.3.1:

$$x = r\cos\theta = 3\cos(\frac{\pi}{3}) = 3(\frac{1}{2}) = \frac{3}{2}$$

and

$$y = r\sin\theta = 3\sin\left(\frac{\pi}{3}\right) = 3\left(\frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{2}.$$

Therefore this point can be represented as $\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$ in rectangular coordinates.

f. Use $r = 2$ and $\theta = \frac{3\pi}{2}$ in Equation 6.3.1:

$$x = r\cos\theta = 2\cos\left(\frac{3\pi}{2}\right) = 2(0) = 0$$

and

$$y = r\sin\theta = 2\sin\left(\frac{3\pi}{2}\right) = 2(-1) = -2.$$

Therefore this point can be represented as $(0, -2)$ in rectangular coordinates.

g. Use $r = 6$ and $\theta = -\frac{5\pi}{6}$ in Equation 6.3.1:

$$x = r\cos\theta = 6\cos\left(-\frac{5\pi}{6}\right) = 6\left(-\frac{\sqrt{3}}{2}\right) = -3\sqrt{3}$$

and

$$y = r\sin\theta = 6\sin\left(-\frac{5\pi}{6}\right) = 6\left(-\frac{1}{2}\right) = -3.$$

Therefore this point can be represented as $(-3\sqrt{3}, -3)$ in rectangular coordinates.

? Checkpoint 6.3.1

Convert $(-8, -8)$ into polar coordinates and $\left(4, \frac{2\pi}{3}\right)$ into rectangular coordinates.

Answer

$\left(8\sqrt{2}, \frac{5\pi}{4}\right)$ and $(-2, 2\sqrt{3})$

The polar representation of a point is not unique. For example, the polar coordinates $\left(2, \frac{\pi}{3}\right)$ and $\left(2, \frac{7\pi}{3}\right)$ both represent the point $(1, \sqrt{3})$ in the rectangular system. Also, the value of r can be negative. Therefore, the point with polar coordinates $\left(-2, \frac{4\pi}{3}\right)$ also represents the point $(1, \sqrt{3})$ in the rectangular system, as we can see by using Equation 6.3.1:

$$x = r\cos\theta = -2\cos\left(\frac{4\pi}{3}\right) = -2\left(-\frac{1}{2}\right) = 1$$

and

$$y = r\sin\theta = -2\sin\left(\frac{4\pi}{3}\right) = -2\left(-\frac{\sqrt{3}}{2}\right) = \sqrt{3}.$$

Every point in the plane has infinite representations in polar coordinates. However, each point in the plane has only one representation in the rectangular coordinate system.

Note that the polar representation of a point in the plane also has a visual interpretation. In particular, r is the directed distance that the point lies from the origin, and θ measures the angle that the line segment from the origin to the point makes with the positive x-axis. Positive angles are measured in a counterclockwise direction, and negative angles are measured in a clockwise direction. The polar coordinate system appears in Figure 6.3.2.

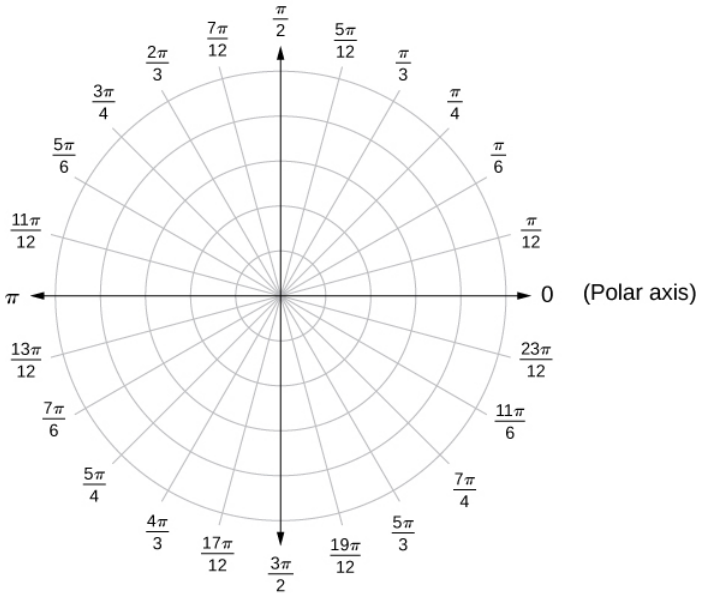


Figure 6.3.2: The polar coordinate system.

The line segment starting from the center of the graph going to the right (called the positive x-axis in the Cartesian system) is the **polar axis**. The center point is the **pole**, or **origin**, of the coordinate system, and corresponds to $r = 0$. The innermost circle shown in Figure 6.3.2 contains all points a distance of 1 unit from the pole and is represented by the equation $r = 1$. Then $r = 2$ is the set of points 2 units from the pole, and so on. The line segments emanating from the pole correspond to fixed angles. To plot a point in the polar coordinate system, start with the angle. If the angle is positive, measure the angle from the polar axis in a counterclockwise direction. If it is negative, then measure it clockwise. If the value of r is positive, move that distance along the terminal ray of the angle. If it is negative, move along the ray opposite the given angle's terminal ray.

Example 6.3.2: Plotting Points in the Polar Plane

Plot each of the following points on the polar plane.

a. $(2, \frac{\pi}{4})$
b. $(-3, \frac{2\pi}{3})$
c. $(4, \frac{5\pi}{4})$

Solutions

The three points are plotted in Figure 6.3.3.

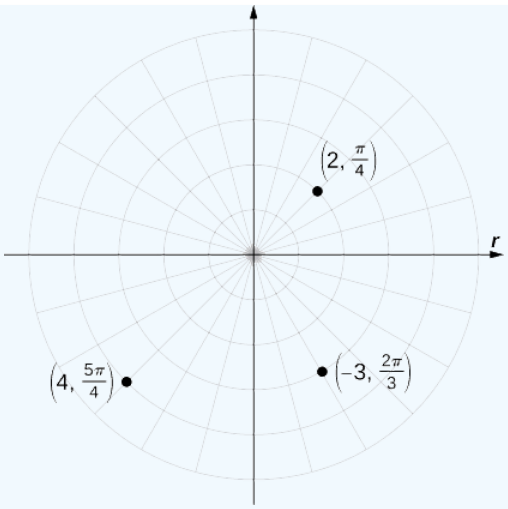
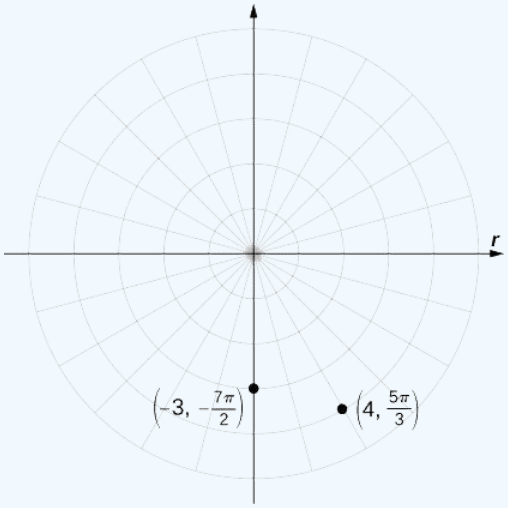


Figure **6.3.3**: Three points plotted in the polar coordinate system.

? Checkpoint 6.3.2

Plot $\left(4, \frac{5\pi}{3}\right)$ and $\left(-3, -\frac{7\pi}{2}\right)$ on the polar plane.

Answer



Polar Curves

Now that we know how to plot points in the polar coordinate system, we can discuss how to plot curves. In the rectangular coordinate system, we can graph a function $y = f(x)$ and create a curve in the Cartesian plane. Similarly, we can graph a curve that is generated by a function $r = f(\theta)$.

The general idea behind graphing a function in polar coordinates is the same as graphing a function in rectangular coordinates. Start with a list of values for the independent variable (θ in this case) and calculate the corresponding values of the dependent variable r. This process generates a list of ordered pairs, which can be plotted in the polar coordinate system. Finally, connect the points and take advantage of any patterns. The function may be periodic, for example, which indicates that only a limited number of values for the independent variable are needed.

Problem-Solving Strategy: Plotting a Curve in Polar Coordinates

1. Create a table with two columns. The first column is for θ, and the second is for r.
2. Create a list of values for θ.
3. Calculate the corresponding r values for each θ.
4. Plot each ordered pair (r, θ) on the coordinate axes.
5. Connect the points and look for a pattern.

Example 6.3.3: Graphing a Function in Polar Coordinates

Graph the curve defined by the function $r = 4\sin\theta$. Identify the curve and rewrite the equation in rectangular coordinates.

Solution

Because the function is a multiple of a sine function, it is periodic with period 2π, so use values for θ between 0 and 2π. The results of steps 1–3 appear in the following table. Figure 6.3.4 shows the graph based on this table.

θ	$r = 4\sin\theta$	θ	$r = 4\sin\theta$
0	0	π	0
$\frac{\pi}{6}$	2	$\frac{7\pi}{6}$	-2
$\frac{\pi}{4}$	$2\sqrt{2} \approx 2.8$	$\frac{5\pi}{4}$	$-2\sqrt{2} \approx -2.8$
$\frac{\pi}{3}$	$2\sqrt{3} \approx 3.4$	$\frac{4\pi}{3}$	$-2\sqrt{3} \approx -3.4$
$\frac{\pi}{2}$	4	$\frac{3\pi}{2}$	-4
$\frac{2\pi}{3}$	$2\sqrt{3} \approx 3.4$	$\frac{5\pi}{3}$	$-2\sqrt{3} \approx -3.4$
$\frac{3\pi}{4}$	$2\sqrt{2} \approx 2.8$	$\frac{7\pi}{4}$	$-2\sqrt{2} \approx -2.8$
$\frac{5\pi}{6}$	2	$\frac{11\pi}{6}$	-2
		2π	0

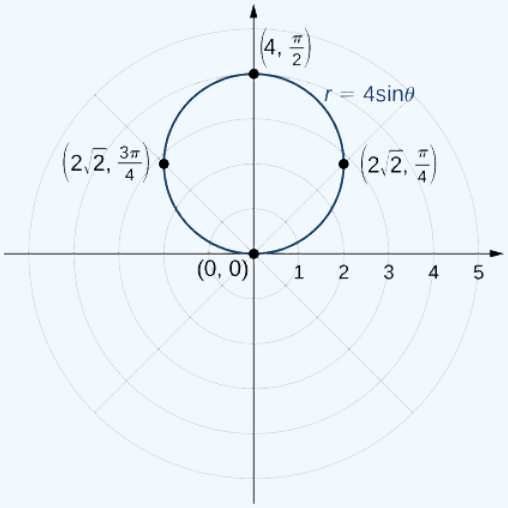


Figure 6.3.4: The graph of the function $r = 4\sin\theta$ is a circle.

This is the graph of a circle. The equation $r = 4\sin\theta$ can be converted into rectangular coordinates by multiplying both sides by r. This gives the equation $r^2 = 4r\sin\theta$. Next use the facts that $r^2 = x^2 + y^2$ and $y = r\sin\theta$. This gives $x^2 + y^2 = 4y$. To put this equation into standard form, subtract $4y$ from both sides of the equation and complete the square:

$$\begin{aligned} x^2+y^2-4y &= 0 \\ \Rightarrow \quad x^2+(y^2-4y) &= 0 \\ \Rightarrow \quad x^2+(y^2-4y+4) &= 0+4 \\ \Rightarrow \quad x^2+(y-2)^2 &= 4 \end{aligned}$$

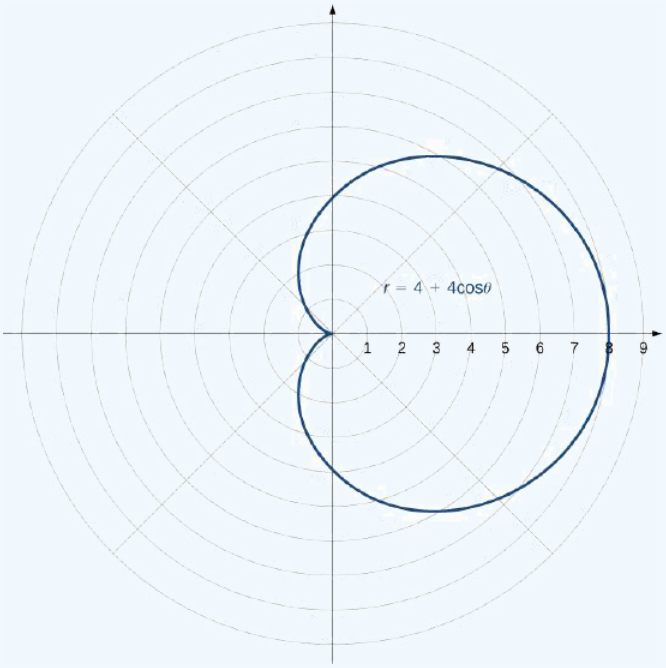
This is the equation of a circle with radius 2 and center $(0,2)$ in the rectangular coordinate system.

? Checkpoint 6.3.3

Create a graph of the curve defined by the function $r=4+4\cos\theta$.

Answer

The name of this shape is a cardioid, which we will study further later in this section.



The graph in Example **6.3.3** was that of a circle. The equation of the circle can be transformed into rectangular coordinates using the coordinate transformation formulas in Equation 6.3.1. Example **6.3.4** gives some more examples of functions for transforming from polar to rectangular coordinates.

✓ Example 6.3.4: Transforming Polar Equations to Rectangular Coordinates

Rewrite the following equations in rectangular coordinates and identify the graph.

a. $\theta=\frac{\pi}{3}$
b. $r=3$
c. $r=6\cos\theta-8\sin\theta$

Solutions

a. Take the tangent of both sides. This gives $\tan\theta=\tan(\pi/3)=\sqrt{3}$. Since $\tan\theta=y/x$, we can replace the left-hand side of this equation with y/x. This gives $y/x=\sqrt{3}$, which can be rewritten as $y=x\sqrt{3}$. This is the equation of a straight line passing through the origin with slope $\sqrt{3}$. In general, any polar equation of the form $\theta=K$ represents a straight line through the pole with slope equal to $\tan K$.

b. First, square both sides of the equation. This gives $r^2 = 9$. Next replace r^2 with $x^2 + y^2$. This gives the equation $x^2 + y^2 = 9$, which is the equation of a circle centered at the origin with radius 3. In general, any polar equation of the form $r = k$ where k is a positive constant represents a circle of radius k centered at the origin. (Note: when squaring both sides of an equation, it is possible to introduce new points unintentionally. This should always be taken into consideration. However, in this case, we do not introduce new points. For example, $\left(-3, \frac{\pi}{3}\right)$ is the same point as $\left(3, \frac{4\pi}{3}\right)$.)

c. Multiply both sides of the equation by r. This leads to $r^2 = 6r\cos\theta - 8r\sin\theta$. Next, use the formulas

$$r^2 = x^2 + y^2, x = r\cos\theta, y = r\sin\theta.$$

This gives

$$\begin{aligned} r^2 &= 6(r\cos\theta) - 8(r\sin\theta) \\ \implies x^2 + y^2 &= 6x - 8y \end{aligned}$$

To put this equation into standard form, move the variables from the right-hand side of the equation to the left-hand side, then complete the square.

$$\begin{aligned} x^2 + y^2 &= 6x - 8y \\ \implies x^2 - 6x + y^2 + 8y &= 0 \\ \implies (x^2 - 6x) + (y^2 + 8y) &= 0 \\ \implies (x^2 - 6x + 9) + (y^2 + 8y + 16) &= 9 + 16 \\ \implies (x-3)^2 + (y+4)^2 &= 25. \end{aligned}$$

This is the equation of a circle with center at $(3, -4)$ and radius 5. Notice that the circle passes through the origin since the center is 5 units away.

? Checkpoint 6.3.4

Rewrite the equation $r = \sec\theta\tan\theta$ in rectangular coordinates and identify its graph.

Answer

$y = x^2$, which is the equation of a parabola opening upward.

We have now seen several examples of drawing graphs of curves defined by **polar equations**. A summary of some common curves is given in the tables below. In each equation, a and b are arbitrary constants.

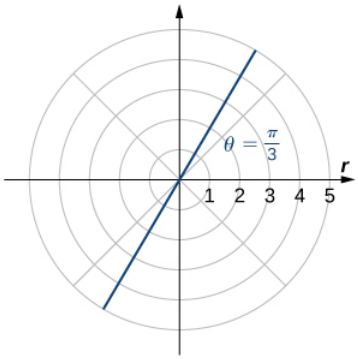
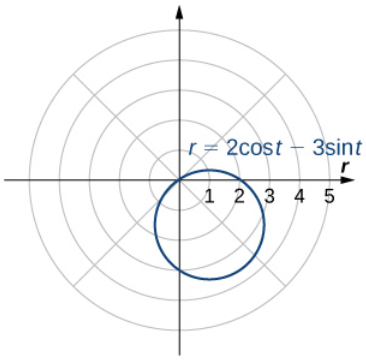
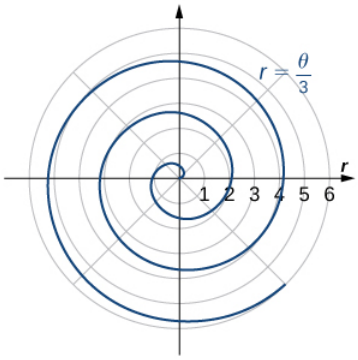
Name	Equation	Example
Line passing through the pole with slope tan K	$\theta = K$	
Circle	$r = a\cos\theta + b\sin\theta$	
Spiral	$r = a + b\theta$	

Figure 6.3.5

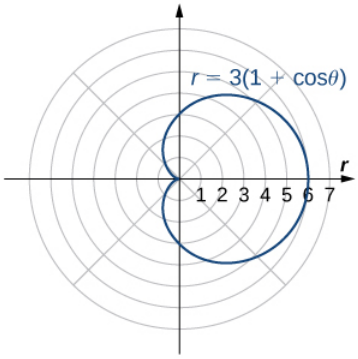
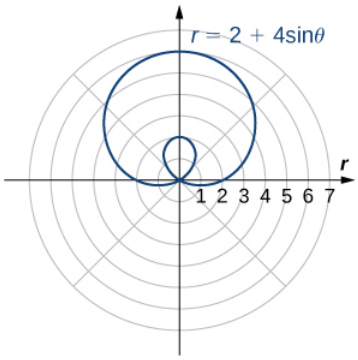
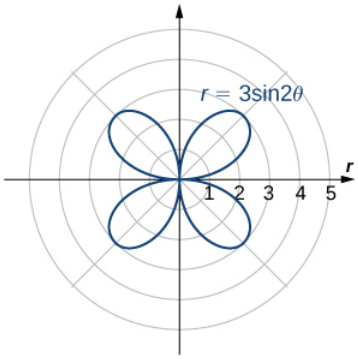
Name	Equation	Example
Cardioid	$r = a(1 + \cos\theta)$ $r = a(1 - \cos\theta)$ $r = a(1 + \sin\theta)$ $r = a(1 - \sin\theta)$	
Limaçon	$r = a\cos\theta + b$ $r = a\sin\theta + b$	
Rose	$r = a\cos(b\theta)$ $r = a\sin(b\theta)$	

Figure 6.3.6

A **cardioid** is a special case of a **limaçon** (pronounced “lee-mah-son”), in which $a = b$ or $a = -b$. The **rose** is an interesting curve. Notice that the graph of $r = 3\sin 2\theta$ has four petals. However, the graph of $r = 3\sin 3\theta$ has three petals.

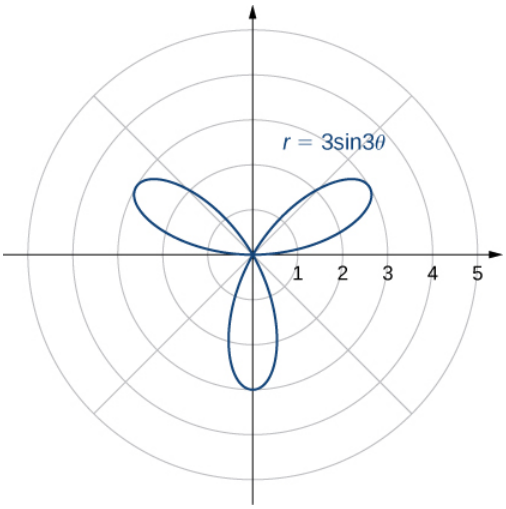


Figure 6.3.7: Graph of $r = 3\sin 3\theta$.

If the coefficient of θ is even, the graph has twice as many petals as the coefficient. If the coefficient of θ is odd, then the number of petals equals the coefficient. You are encouraged to explore why this happens. Even more interesting graphs emerge when the coefficient of θ is not an integer. For example, if it is rational, then the curve is closed; that is, it eventually ends where it started (Figure 6.3.8*a*). However, if the coefficient is irrational, then the curve never closes (Figure 6.3.8*b*). Although it may appear that the curve is closed, a closer examination reveals that the petals just above the positive x axis are slightly thicker. This is because the petal does not quite match up with the starting point.

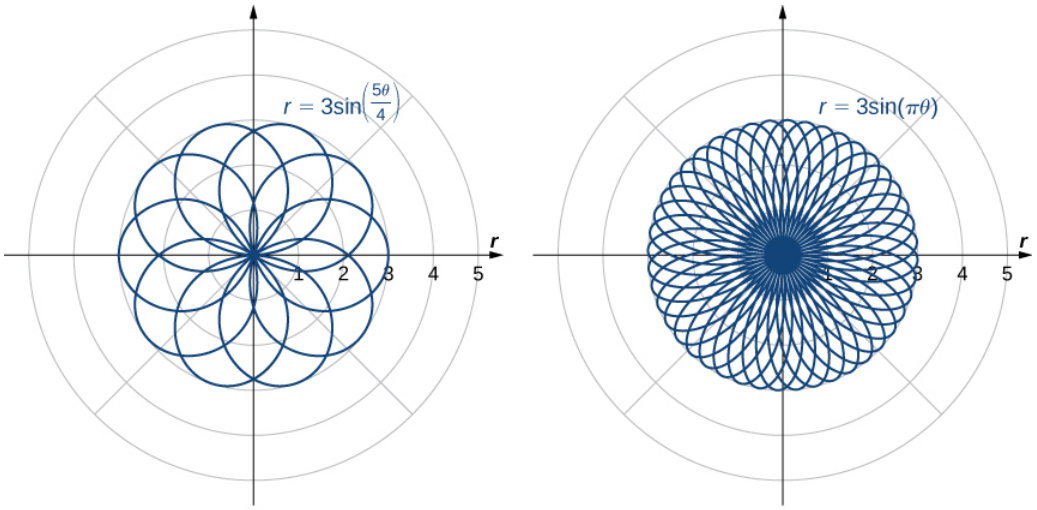


Figure 6.3.8: Polar rose graphs of functions with (a) rational coefficient and (b) irrational coefficient. Note that the rose in part (b) would fill the entire circle if plotted in full.

Since the curve defined by the graph of $r = 3\sin(\pi\theta)$ never closes, the curve depicted in Figure 6.3.8*b* is only a partial depiction. This is an example of a **space-filling curve**. A space-filling curve occupies a two-dimensional subset of the real plane. In this case, the curve occupies the circle of radius 3 centered at the origin.

✔ Example 6.3.5: Describing a Spiral

The **chambered nautilus** displays a spiral when half the outer shell is cut away. It is possible to describe a spiral using rectangular coordinates. Figure 6.3.9 shows a spiral in rectangular coordinates. How can we describe this curve mathematically?

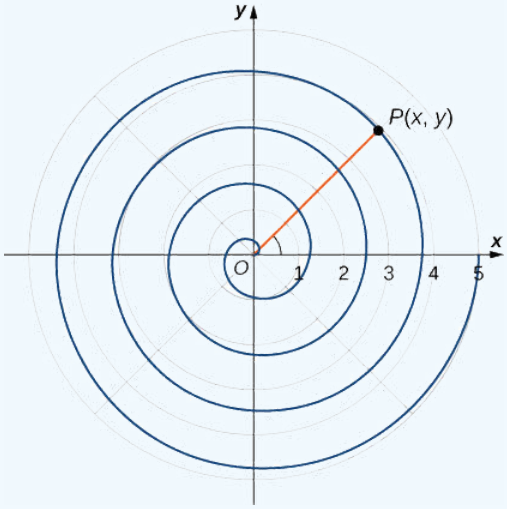


Figure **6.3.9**: How can we describe a spiral graph mathematically?

Solution

As the point P travels around the spiral in a counterclockwise direction, its distance d from the origin increases. Assume that the distance d is a constant multiple k of the angle θ that the line segment OP makes with the positive x-axis. Therefore $d(P,O) = k\theta$, where O is the origin. Now use the Distance Formula and some Trigonometry:

$$\begin{aligned} & d(P,O) = k\theta \\ \Rightarrow \quad & \sqrt{(x-0)^2+(y-0)^2} = k \arctan\left(\frac{y}{x}\right) \\ \Rightarrow \quad & \sqrt{x^2+y^2} = k \arctan\left(\frac{y}{x}\right) \\ \Rightarrow \quad & \arctan\left(\frac{y}{x}\right) = \frac{\sqrt{x^2+y^2}}{k} \\ \Rightarrow \quad & y = x \tan\left(\frac{\sqrt{x^2+y^2}}{k}\right). \end{aligned}$$

Although this equation describes the spiral, it is not possible to solve it directly for either x or y. However, if we use polar coordinates, the equation becomes much simpler. In particular, $d(P,O) = r$, and θ is the second coordinate. Therefore, the equation for the spiral becomes $r = k\theta$. Note that when $\theta = 0$, we also have $r = 0$, so the spiral emanates from the origin. We can remove this restriction by adding a constant to the equation. Then, the equation for the spiral becomes $r = a + k\theta$ for arbitrary constants a and k. This is referred to as an **Archimedean spiral**, after the Greek mathematician Archimedes.

Another type of spiral is the **logarithmic spiral**, described by the function $r = a \cdot b^{\theta}$. A graph of the function $r = 1.2(1.25^{\theta})$ is given in Figure **6.3.10**. This spiral describes the shell shape of the chambered nautilus.

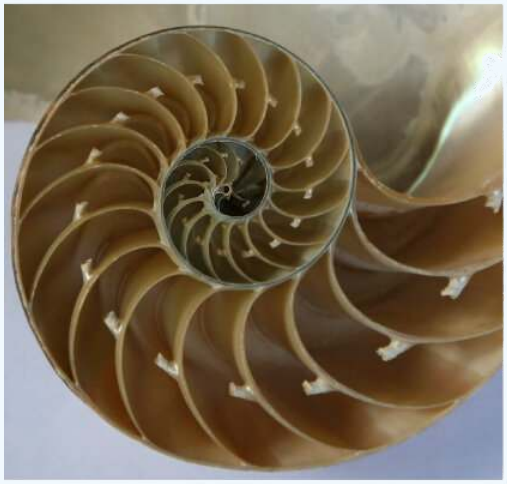
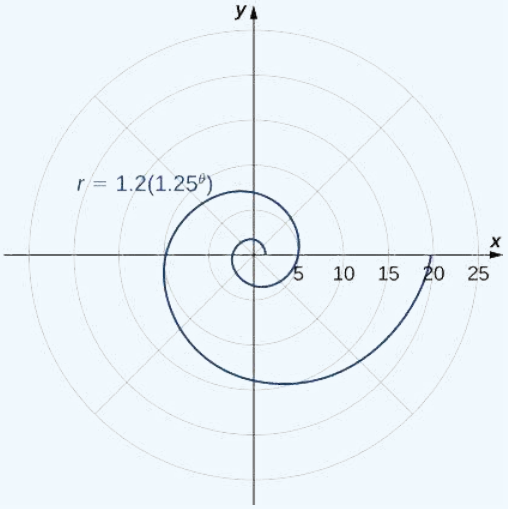



Figure **6.3.10**: A logarithmic spiral is similar to the shape of the chambered nautilus shell. (credit: modification of work by Jitze Couperus, Flickr)

Suppose a curve is described in the polar coordinate system via the function $r = f(\theta)$. Since we have conversion formulas from polar to rectangular coordinates given by

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

it is possible to rewrite these formulas using the function

$$\begin{aligned} x &= f(\theta)\cos\theta \\ y &= f(\theta)\sin\theta \end{aligned}$$

This step gives a parametrization of the curve in rectangular coordinates using θ as the parameter. For example, the spiral formula $r = a + b\theta$ from Figure becomes

$$\begin{aligned} x &= (a + b\theta)\cos\theta \\ y &= (a + b\theta)\sin\theta \end{aligned}$$

Letting θ range from $-\infty$ to ∞ generates the entire spiral.

Symmetry in Polar Coordinates

When studying **symmetry** of functions in rectangular coordinates (i.e., in the form $y = f(x)$), we talk about symmetry with respect to the y-axis and symmetry with respect to the origin. In particular, if $f(-x) = f(x)$ for all x in the domain of f, then f is an even function and its graph is symmetric with respect to the y-axis. If $f(-x) = -f(x)$ for all x in the domain of f, then f is an odd function and its graph is symmetric with respect to the origin. By determining which types of symmetry a graph exhibits, we can learn more about the shape and appearance of the graph. Symmetry can also reveal other properties of the function that generates the graph. Symmetry in polar curves works similarly.

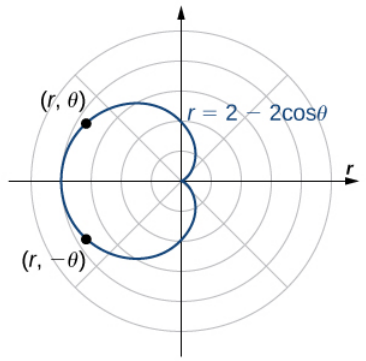
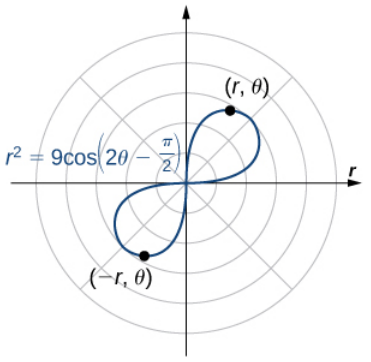
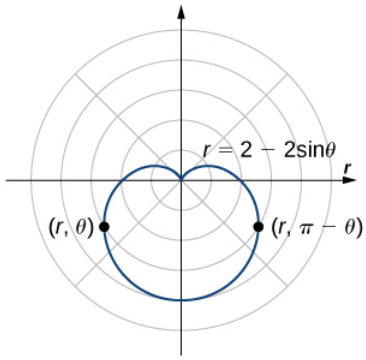
Symmetry in Polar Curves and Equations

Consider a curve generated by the function $r = f(\theta)$ in polar coordinates.

i. The curve is symmetric about the polar axis if for every point (r, θ) on the graph, the point $(r, -\theta)$ is also on the graph. Similarly, the equation $r = f(\theta)$ is unchanged by replacing θ with $-\theta$.
ii. The curve is symmetric about the pole if for every point (r, θ) on the graph, the point $(r, \pi + \theta)$ is as well. Similarly, the equation $r = f(\theta)$ is unchanged when replacing r with $-r$, or θ with $\pi + \theta$.

iii. The curve is symmetric about the vertical line $\theta = \frac{\pi}{2}$ if for every point (r, θ) on the graph, the point $(r, \pi - \theta)$ is also on the graph. Similarly, the equation $r = f(\theta)$ is unchanged when θ is replaced by $\pi - \theta$.

The following table shows examples of each type of symmetry.

Symmetry with respect to the polar axis: For every point (r, θ) on the graph, there is also a point reflected directly across the horizontal (polar) axis.	
Symmetry with respect to the pole: For every point (r, θ) on the graph, there is also a point on the graph that is reflected through the pole as well.	
Symmetry with respect to the vertical line $\theta = \frac{\pi}{2}$: For every point (r, θ) on the graph, there is also a point reflected directly across the vertical axis.	

Example 6.3.6: Using Symmetry to Graph a Polar Equation

Find the symmetry of the rose defined by the equation $r = 3\sin(2\theta)$ and create a graph.

Solution

Suppose the point (r, θ) is on the graph of $r = 3\sin(2\theta)$.

i. To test for symmetry about the polar axis, first try replacing θ with $-\theta$. This gives $r = 3\sin(2(-\theta)) = -3\sin(2\theta)$. Since this changes the original equation, this test is not satisfied. However, returning to the original equation and replacing r with $-r$ and θ with $\pi - \theta$ yields

$$\begin{aligned} -r &= 3\sin(2(\pi-\theta)) \\ \Rightarrow \quad -r &= 3\sin(2\pi-2\theta) \\ \Rightarrow \quad -r &= 3\sin(-2\theta) \\ \Rightarrow \quad -r &= -3\sin 2\theta. \end{aligned}$$

Multiplying both sides of this equation by -1 gives $r = 3\sin 2\theta$, which is the original equation. This demonstrates that the graph is symmetric with respect to the polar axis.

ii. To test for symmetry with respect to the pole, first replace r with $-r$, which yields $-r = 3\sin(2\theta)$. Multiplying both sides by -1 gives $r = -3\sin(2\theta)$, which does not agree with the original equation. Therefore, the equation does not pass the test for this symmetry. However, returning to the original equation and replacing θ with $\theta + \pi$ gives

$$\begin{aligned} r &= 3\sin(2(\theta+\pi)) \\ &= 3\sin(2\theta+2\pi) \\ &= 3(\sin 2\theta\cos 2\pi + \cos 2\theta\sin 2\pi) \\ &= 3\sin 2\theta. \end{aligned}$$

Since this agrees with the original equation, the graph is symmetric about the pole.

iii. To test for symmetry with respect to the vertical line $\theta = \frac{\pi}{2}$, first replace both r with $-r$ and θ with $-\theta$.

$$\begin{aligned} -r &= 3\sin(2(-\theta)) \\ \Rightarrow \quad -r &= 3\sin(-2\theta) \\ \Rightarrow \quad -r &= -3\sin 2\theta. \end{aligned}$$

Multiplying both sides of this equation by -1 gives $r = 3\sin 2\theta$, which is the original equation. Therefore the graph is symmetric about the vertical line $\theta = \frac{\pi}{2}$.

This graph has symmetry with respect to the polar axis, the origin, and the vertical line going through the pole. To graph the function, tabulate values of θ between 0 and $\pi/2$ and then reflect the resulting graph.

0	0
$\frac{\pi}{6}$	$\frac{3\sqrt{3}}{2} \approx 2.6$
$\frac{\pi}{4}$	3
$\frac{\pi}{3}$	$\frac{3\sqrt{3}}{2} \approx 2.6$
$\frac{\pi}{2}$	0

This gives one petal of the rose, as shown in the following graph.

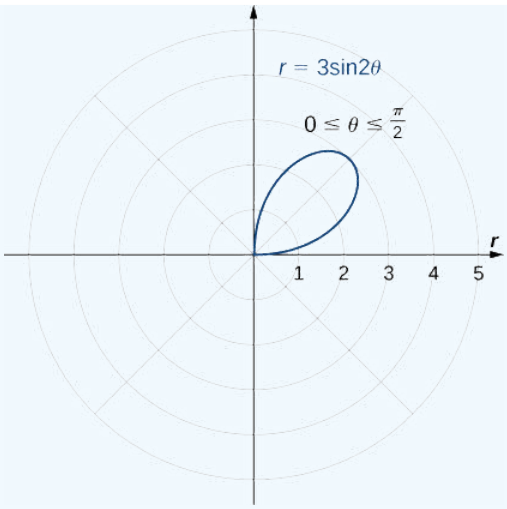


Figure **6.3.11**: The graph of the equation between $\theta = 0$ and $\theta = \pi/2$.

Reflecting this image into the other three quadrants shows the entire graph.

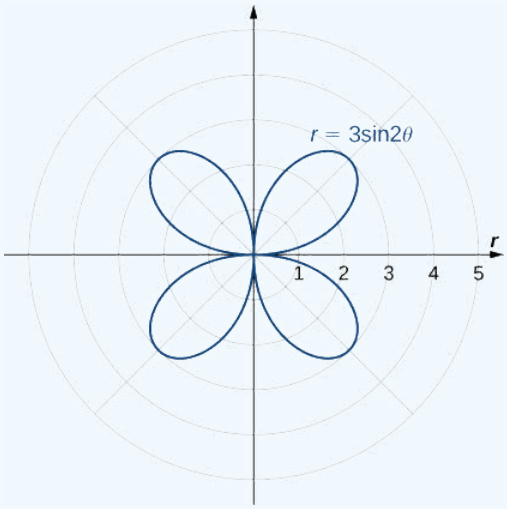


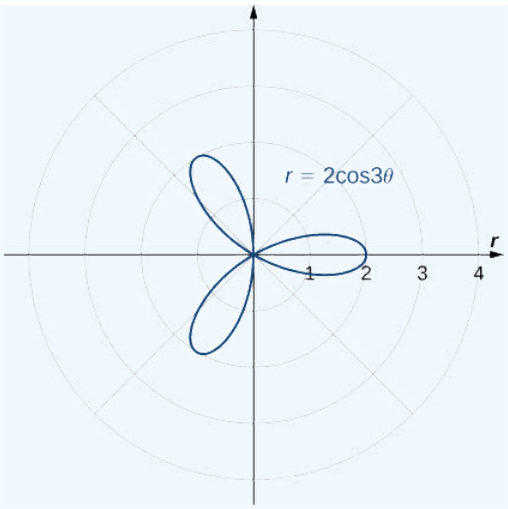
Figure **6.3.12**: The entire graph of the equation is called a four-petaled rose.

? Checkpoint 6.3.6: Symmetry

Determine the symmetry of the graph determined by the equation $r = 2\cos(3\theta)$ and create a graph.

Answer

Symmetric with respect to the polar axis.



Key Concepts

- The polar coordinate system provides an alternative way to locate points in the plane.
- Convert points between rectangular and polar coordinates using the formulas

$$x = r\cos\theta \text{ and } y = r\sin\theta$$

and

$$r = \sqrt{x^2 + y^2} \text{ and } \tan\theta = \frac{y}{x}.$$

- To sketch a polar curve from a given polar function, make a table of values and take advantage of periodic properties.
- Use the conversion formulas to convert equations between rectangular and polar coordinates.
- Identify symmetry in polar curves, which can occur through the pole, the horizontal axis, or the vertical axis.

Glossary

angular coordinate

θ the angle formed by a line segment connecting the origin to a point in the polar coordinate system with the positive radial (x) axis, measured counterclockwise

cardioid

a plane curve traced by a point on the perimeter of a circle that is rolling around a fixed circle of the same radius; the equation of a cardioid is $r = a(1 + \sin\theta)$ or $r = a(1 + \cos\theta)$

limaçon

the graph of the equation $r = a + b\sin\theta$ or $r = a + b\cos\theta$. If $a = b$ then the graph is a cardioid

polar axis

the horizontal axis in the polar coordinate system corresponding to $r \geq 0$

polar coordinate system

a system for locating points in the plane. The coordinates are r, the radial coordinate, and θ, the angular coordinate

polar equation

an equation or function relating the radial coordinate to the angular coordinate in the polar coordinate system

pole

the central point of the polar coordinate system, equivalent to the origin of a Cartesian system

radial coordinate

r the coordinate in the polar coordinate system that measures the distance from a point in the plane to the pole

rose

graph of the polar equation $r = a\cos 2\theta$ or $r = a\sin 2\theta$ for a positive constant a

space-filling curve

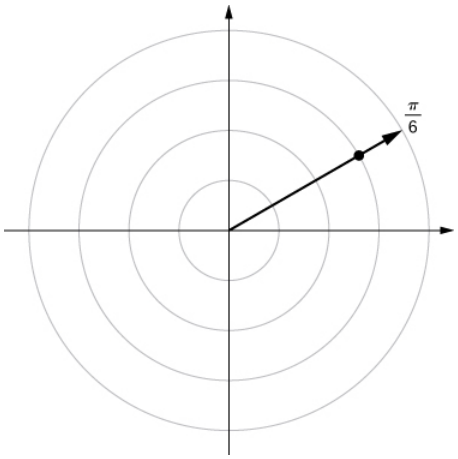
a curve that completely occupies a two-dimensional subset of the real plane

6.3E: Exercises

In exercises 1 - 7, plot the point whose polar coordinates are given by first constructing the angle θ and then marking off the distance r along the ray.

1) $(3, \frac{\pi}{6})$

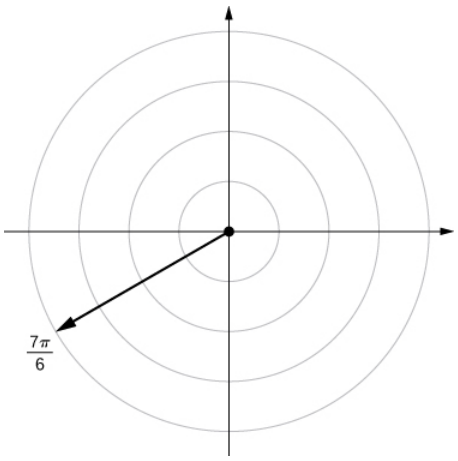
Answer



2) $(-2, \frac{5\pi}{3})$

3) $(0, \frac{7\pi}{6})$

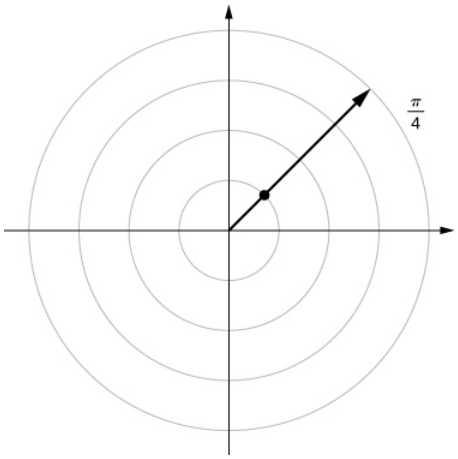
Answer



4) $(-4, \frac{3\pi}{4})$

5) $(1, \frac{\pi}{4})$

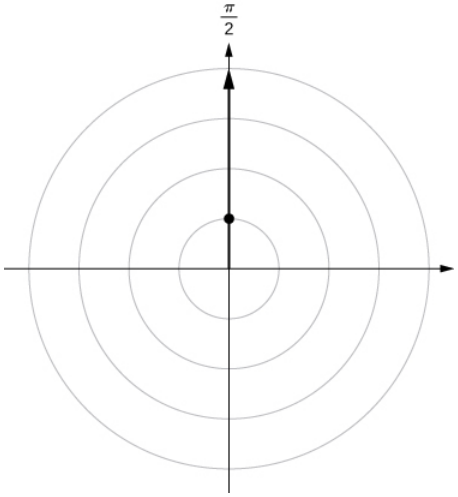
Answer



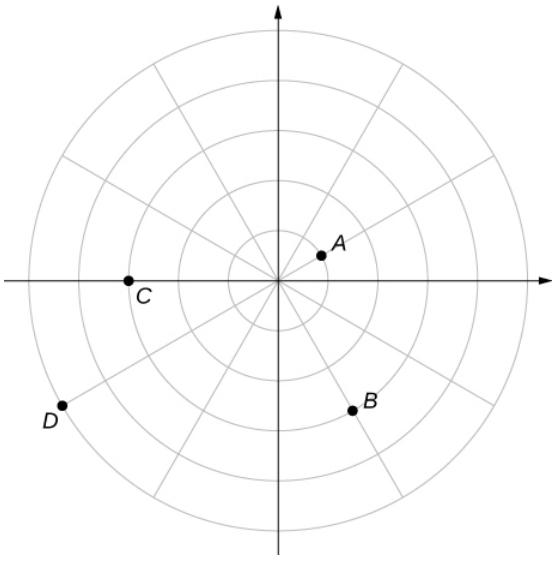
6) $(2, \frac{5\pi}{6})$

7) $(1, \frac{\pi}{2})$

Answer



In exercises 8 - 11, consider the polar graph below. Give two sets of polar coordinates for each point.



8) Coordinates of point A.

9) Coordinates of point B.

Answer

$B\left(3, \frac{-\pi}{3}\right) B\left(-3, \frac{2\pi}{3}\right)$

10) Coordinates of point C.

11) Coordinates of point D.

Answer

$D\left(5, \frac{7\pi}{6}\right) D\left(-5, \frac{\pi}{6}\right)$

In exercises 12 - 17, the rectangular coordinates of a point are given. Find two sets of polar coordinates for the point in $(0, 2\pi]$. Round to three decimal places.

12) $(2, 2)$

13) $(3, -4)$

Answer

$(5, -0.927),\ (-5, -0.927 + \pi)$

14) $(8, 15)$

15) $(-6, 8)$

Answer

$(10, -0.927),\ (-10, -0.927 + \pi)$

16) $(4, 3)$

17) $(3, -\sqrt{3})$

Answer

$(2\sqrt{3}, -0.524),\ (-2\sqrt{3}, -0.524 + \pi)$

In exercises 18 - 24, find rectangular coordinates for the given point in polar coordinates.

18) $\left(2, \frac{5\pi}{4}\right)$

19) $\left(-2, \frac{\pi}{6}\right)$

Answer

$(-\sqrt{3}, -1)$

20) $\left(5, \frac{\pi}{3}\right)$

21) $\left(1, \frac{7\pi}{6}\right)$

Answer

$\left(-\frac{\sqrt{3}}{2}, \frac{-1}{2}\right)$

22) $\left(-3, \frac{3\pi}{4}\right)$

23) $\left(0, \frac{\pi}{2}\right)$

Answer

$(0, 0)$

24) $(-4.5, 6.5)$

In exercises 25 - 29, determine whether the graphs of the polar equation are symmetric with respect to the x-axis, the y -axis, or the origin.

25) $r = 3\sin(2\theta)$

Answer

Symmetry with respect to the *x*-axis, *y*-axis, and origin.

26) $r^2 = 9\cos\theta$

27) $r = \cos\left(\frac{\theta}{5}\right)$

Answer

Symmetric with respect to *x*-axis only.

28) $r = 2\sec\theta$

29) $r = 1 + \cos\theta$

Answer

Symmetry with respect to *x*-axis only.

In exercises 30 - 33, describe the graph of each polar equation. Confirm each description by converting into a rectangular equation.

30) $r = 3$

31) $\theta = \frac{\pi}{4}$

Answer

Line $y = x$

32) $r = \sec\theta$

33) $r = \csc\theta$

Answer

$y = 1$

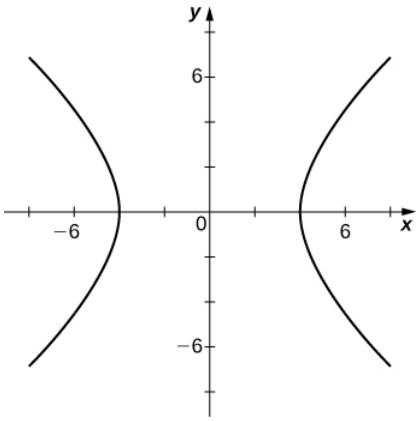
In exercises 34 - 36, convert the rectangular equation to polar form and sketch its graph.

34) $x^2 + y^2 = 16$

35) $x^2 - y^2 = 16$

Answer

Hyperbola; polar form $r^2\cos(2\theta) = 16$ or $r^2 = 16\sec\theta$.



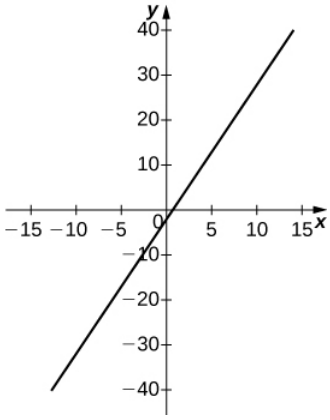
36) $x = 8$

In exercises 37 - 38, convert the rectangular equation to polar form and sketch its graph.

37) $3x - y = 2$

Answer

$r = \frac{2}{3\cos\theta - \sin\theta}$



38) $y^2 = 4x$

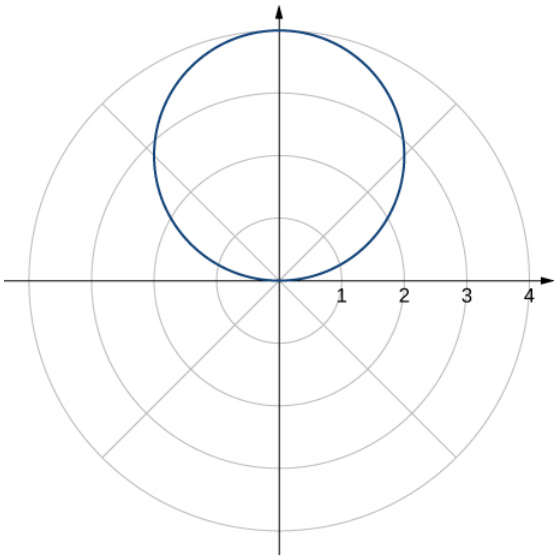
In exercises 39 - 43, convert the polar equation to rectangular form and sketch its graph.

39) $r = 4\sin\theta$

40) $x^2 + y^2 = 4y$

Answer

https://math.libretexts.org/@go/page/168466

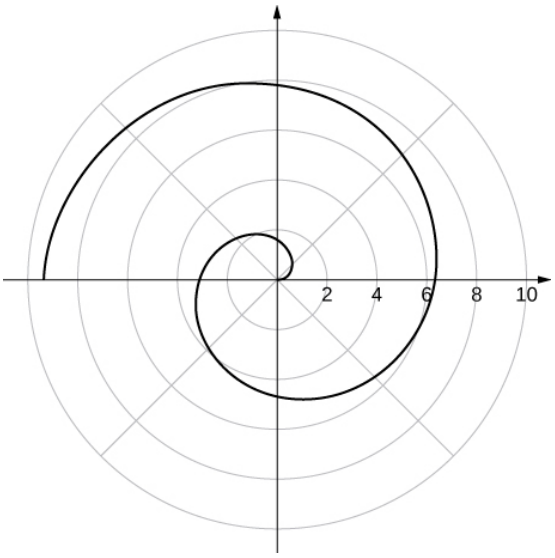


41) $r = 6\cos\theta$

42) $r = \theta$

Answer

$x \tan \sqrt{x^2 + y^2} = y$



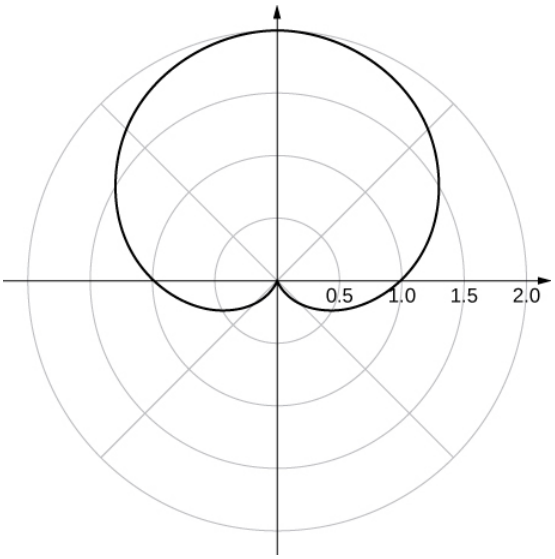
43) $r = \cot\theta \csc\theta$

In exercises 44 - 54, sketch a graph of the polar equation and identify any symmetry.

44) $r = 1 + \sin\theta$

Answer

y-axis symmetry

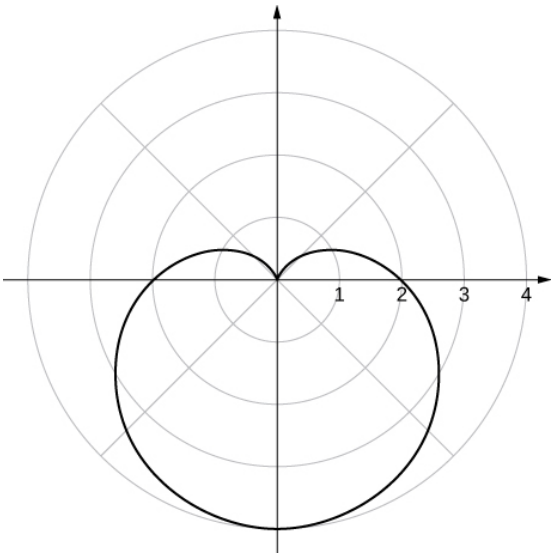


45) $r = 3 - 2\cos\theta$

46) $r = 2 - 2\sin\theta$

Answer

y-axis symmetry

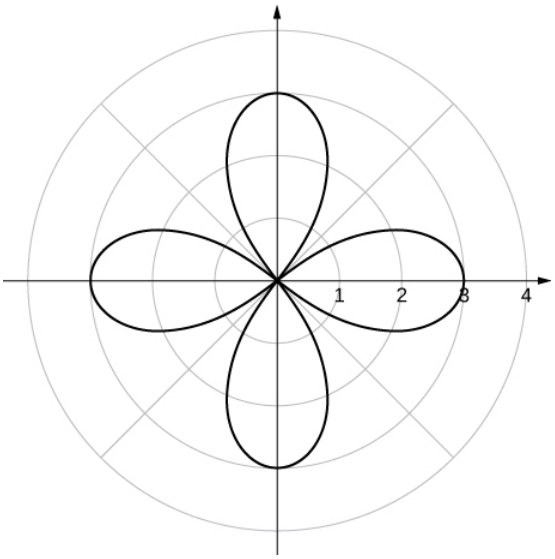


47) $r = 5 - 4\sin\theta$

48) $r = 3\cos(2\theta)$

Answer

x-and y-axis symmetry and symmetry about the pole

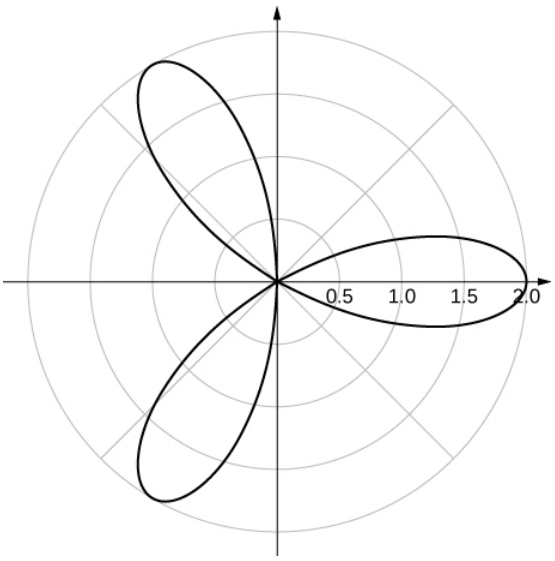


49) $r = 3\sin(2\theta)$

50) $r = 2\cos(3\theta)$

Answer

x-axis symmetry

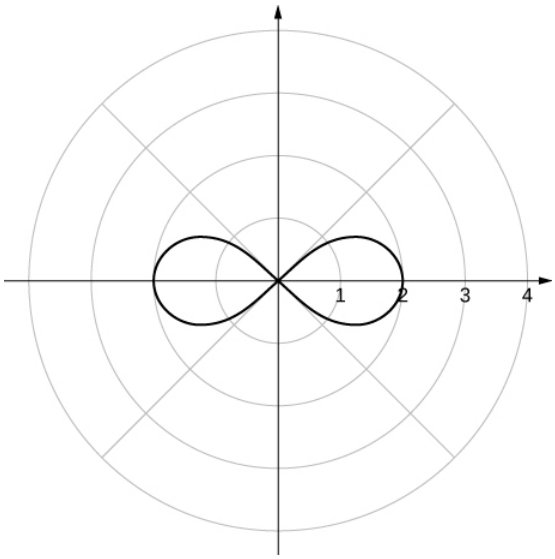


51) $r = 3\cos\left(\frac{\theta}{2}\right)$

52) $r^2 = 4\cos\left(\frac{2}{\theta}\right)$

Answer

x-and y-axis symmetry and symmetry about the pole

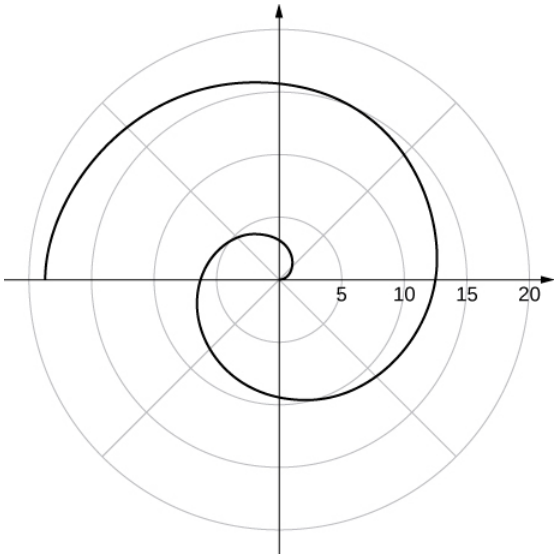


53) $r^2 = 4\sin\theta$

54) $r = 2\theta$

Answer

no symmetry

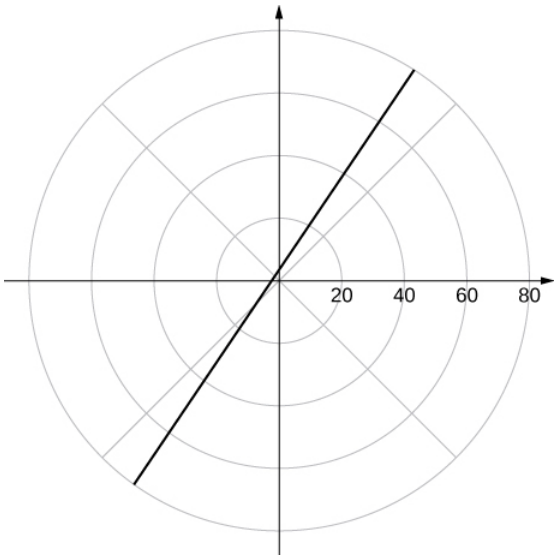


55) [Technology Required] The graph of $r = 2\cos(2\theta)\sec(\theta)$. is called a strophoid. Use a graphing utility to sketch the graph, and, from the graph, determine the asymptote.

56) [Technology Required] Use a graphing utility and sketch the graph of $r = \dfrac{6}{2\sin\theta - 3\cos\theta}$.

Answer

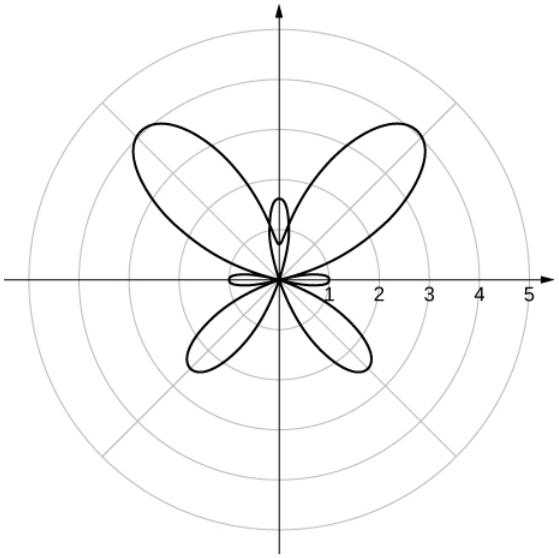
a line



57) [Technology Required] Use a graphing utility to graph $r = \frac{1}{1-\cos\theta}$.

58) [Technology Required] Use technology to graph $r = e^{\sin(\theta)} - 2\cos(4\theta)$.

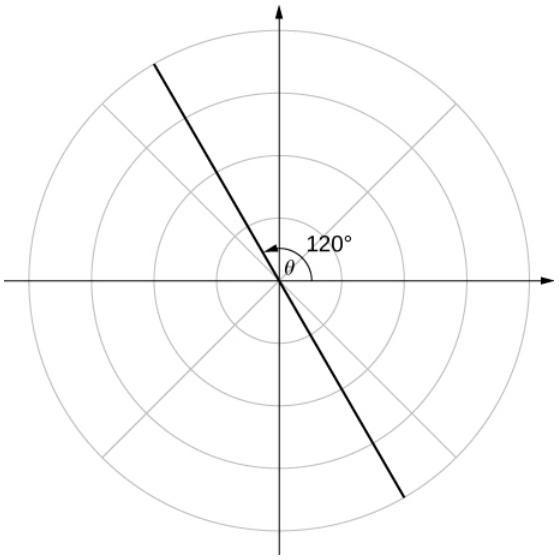
Answer



59) [Technology Required] Use technology to plot $r = \sin(\frac{3\theta}{7})$ (use the interval $0 \leq \theta \leq 14\pi$).

60) Without using technology, sketch the polar curve $\theta = \frac{2\pi}{3}$.

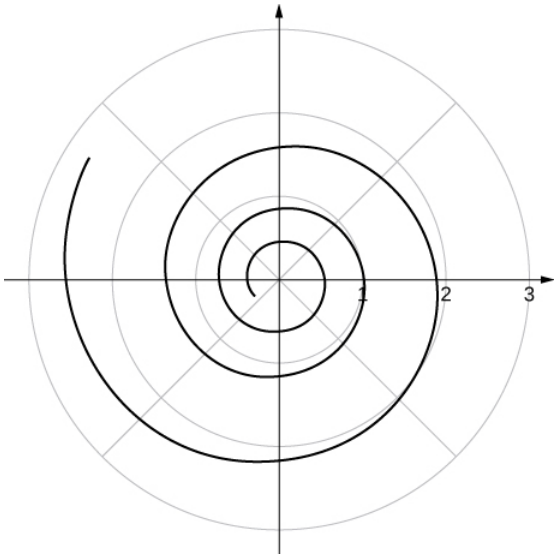
Answer



61) [Technology Required] Use a graphing utility to plot $r = \theta \sin \theta$ for $-\pi \leq \theta \leq \pi$.

62) [Technology Required] Use technology to plot $r = e^{-0.1\theta}$ for $-10 \leq \theta \leq 10$.

Answer



63) [Technology Required] There is a curve known as the "Black Hole." Use technology to plot $r = e^{-0.01\theta}$ for $-100 \leq \theta \leq 100$.

64) [Technology Required] Use the results of the preceding two problems to explore the graphs of $r = e^{-0.001\theta}$ and $r = e^{-0.0001\theta}$ for $|\theta| > 100$.

Answer

Answers vary. One possibility is the spiral lines become closer together and the total number of spirals increases.

This page titled 6.3E: Exercises is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Roy Simpson.

- **11.3E: Exercises for Section 11.3** by Edwin "Jed" Herman, Gilbert Strang is licensed CC BY-NC-SA 4.0. Original source: https://openstax.org/details/books/calculus-volume-1.

6.4: Area and Arc Length in Polar Coordinates

Learning Objectives

- Apply the formula for the area of a region in polar coordinates.
- Determine the arc length of a polar curve.

In the rectangular coordinate system, the definite integral provides a way to calculate the area under a curve. In particular, if we have a function $y = f(x)$ defined from $x = a$ to $x = b$ where $f(x) > 0$ on this interval, the area between the curve and the x-axis is given by

$$A = \int_a^b f(x)dx.$$

This fact, along with the formula for evaluating this integral, is summarized in the Fundamental Theorem of Calculus. Similarly, the arc length of this curve is given by

$$L = \int_a^b \sqrt{1 + (f'(x))^2}dx.$$

In this section, we study analogous formulas for area and arc length in the polar coordinate system.

Areas of Regions Bounded by Polar Curves

We have studied the formulas for the area under a curve defined in rectangular coordinates and parametrically defined curves. Now, we focus on deriving a formula for the area of a region bounded by a polar curve. Recall that the proof of the Fundamental Theorem of Calculus used the concept of a Riemann sum to approximate the area under a curve by using rectangles. For polar curves, we use the Riemann sum again, but sectors of a circle replace the rectangles.

Consider a curve defined by the function $r = f(\theta)$, where $\alpha \leq \theta \leq \beta$. Our first step is to partition the interval $[\alpha, \beta]$ into n equal-width subintervals. The width of each subinterval is given by the formula $\Delta\theta = (\beta - \alpha)/n$, and the ith partition point θ_i is given by the formula $\theta_i = \alpha + i\Delta\theta$. Each partition point $\theta = \theta_i$ defines a line with slope $\tan\theta_i$ passing through the pole as shown in the following graph.

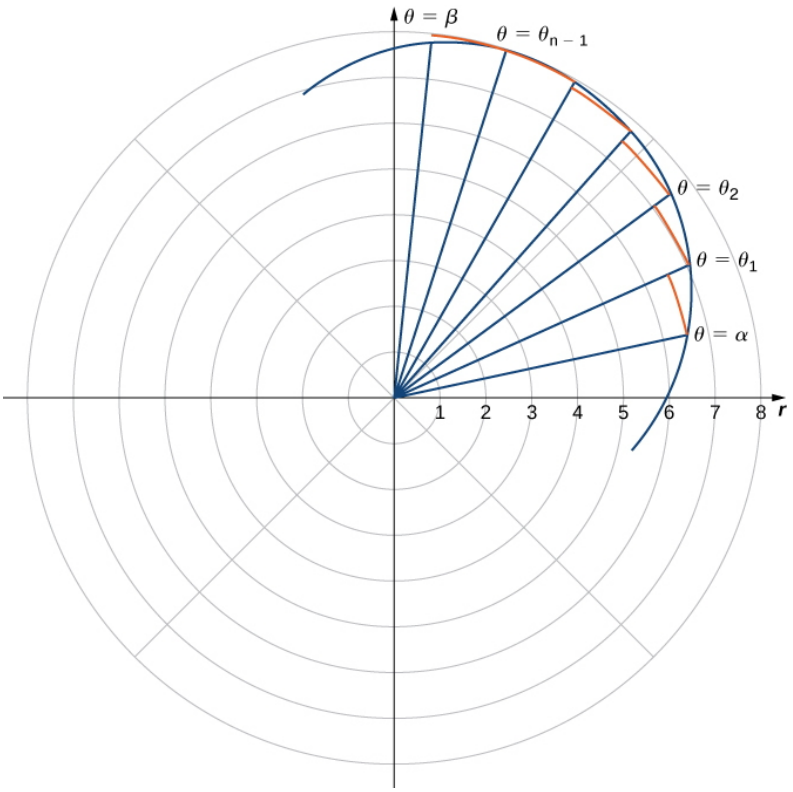


Figure 6.4.1: A partition of a typical curve in polar coordinates.

The line segments are connected by arcs of constant radius. This defines sectors whose areas can be calculated by using a geometric formula. The area of each sector is then used to approximate the area between successive line segments. We then sum the areas of the sectors to approximate the total area. This approach gives a Riemann sum approximation for the total area. The formula for the area of a sector of a circle is illustrated in the following figure.

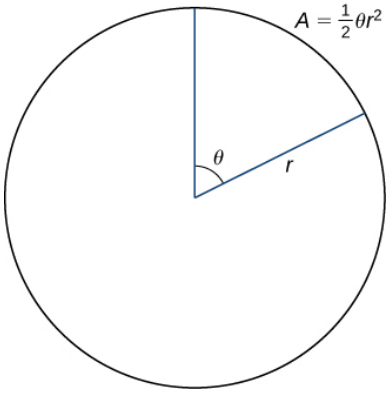


Figure 6.4.2: The area of a sector of a circle is given by $A = \frac{1}{2}\theta r^2$.

Recall that the area of a circle is $A = \pi r^2$. When measuring angles in radians, 360 degrees is equal to 2π radians. Therefore, a fraction of a circle can be measured by the central angle θ. The fraction of the circle is given by $\frac{\theta}{2\pi}$, so the area of the sector is this fraction multiplied by the total area:

$$A = \left(\frac{\theta}{2\pi}\right)\pi r^2 = \frac{1}{2}\theta r^2.$$

Since the radius of a typical sector in Figure 6.4.1 is given by $r_i = f(\theta_i)$, the area of the ith sector is given by

$$A_i = \frac{1}{2}(\Delta\theta)(f(\theta_i))^2.$$

Therefore, a Riemann sum that approximates the area is given by

$$A_n = \sum_{i=1}^{n} A_i \approx \sum_{i=1}^{n} \frac{1}{2}(\Delta\theta)(f(\theta_i))^2.$$

We take the limit as $n \to \infty$ to get the exact area:

$$A = \lim_{n\to\infty} A_n = \frac{1}{2}\int_{\alpha}^{\beta} (f(\theta))^2 d\theta.$$

This gives the following theorem.

Theorem: Area of a Region Bounded by a Polar Curve

Suppose f is continuous and nonnegative on the interval $\alpha \le \theta \le \beta$ with $0 < \beta - \alpha \le 2\pi$. The area of the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 d\theta \tag{6.4.1}$$

$$= \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta. \tag{6.4.2}$$

Example 6.4.1: Finding an Area of a Polar Region

Find the area of one petal of the rose defined by the equation $r = 3\sin(2\theta)$.

Solution

The graph of $r = 3\sin(2\theta)$ follows.

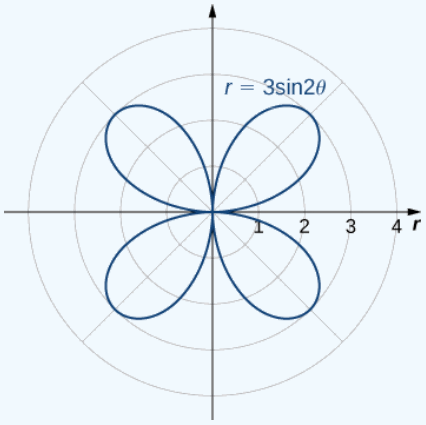


Figure 6.4.3: The graph of $r = 3\sin(2\theta)$.

When $\theta = 0$ we have $r = 3\sin(2(0)) = 0$. The next value for which $r = 0$ is $\theta = \pi/2$. This can be seen by solving the equation $3\sin(2\theta) = 0$ for θ. Therefore the values $\theta = 0$ to $\theta = \pi/2$ trace out the first petal of the rose. To find the area inside this petal, use Equation 6.4.2 with $f(\theta) = 3\sin(2\theta), \alpha = 0$, and $\beta = \pi/2$:

$$\begin{aligned} A &= \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2\, d\theta \\ &= \frac{1}{2}\int_{0}^{\pi/2} [3\sin(2\theta)]^2\, d\theta \\ &= \frac{1}{2}\int_{0}^{\pi/2} 9\sin^2(2\theta)\, d\theta. \end{aligned}$$

To evaluate this integral, use the formula $\sin^2\alpha = (1 - \cos(2\alpha))/2$ with $\alpha = 2\theta$:

$$
\begin{aligned}
A &= \frac{1}{2}\int_0^{\pi/2} 9\sin^2(2\theta)\,d\theta \\
&= \frac{9}{2}\int_0^{\pi/2} \frac{(1-\cos(4\theta))}{2}\,d\theta \\
&= \frac{9}{4}\left(\int_0^{\pi/2} 1-\cos(4\theta)\,d\theta\right) \\
&= \frac{9}{4}\left(\theta - \frac{\sin(4\theta)}{4}\right|_0^{\pi/2} \\
&= \frac{9}{4}\left(\frac{\pi}{2} - \frac{\sin 2\pi}{4}\right) - \frac{9}{4}\left(0 - \frac{\sin 4(0)}{4}\right) \\
&= \frac{9\pi}{8}
\end{aligned}
$$

? Checkpoint 6.4.1

Find the area inside the cardioid defined by the equation $r = 1 - \cos\theta$.

Answer

$A = 3\pi/2$

Example **6.4.1** involved finding the area inside one curve. We can also use Equation 6.4.2 to find the area between two polar curves. However, we often need to find the points of intersection of the curves and determine which function defines the outer or inner curves between these two points.

✓ Example 6.4.2: Finding the Area between Two Polar Curves

Find the area outside the cardioid $r = 2 + 2\sin\theta$ and inside the circle $r = 6\sin\theta$.

Solution

First, draw a graph containing both curves as shown.

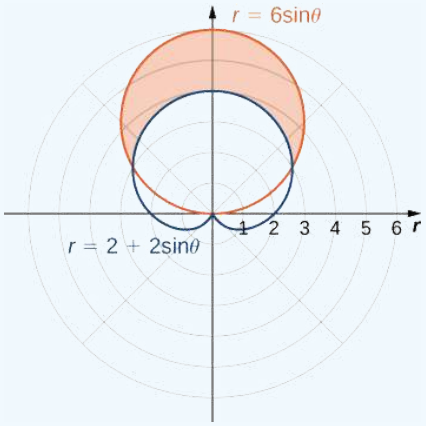


Figure **6.4.4**: The region between the curves $r = 2 + 2\sin\theta$ and $r = 6\sin\theta$.

To determine the limits of integration, first find the points of intersection by setting the two functions equal to each other and solving for θ:

$$
\begin{aligned}
6\sin\theta &= 2 + 2\sin\theta \\
\Rightarrow\quad 4\sin\theta &= 2 \\
\Rightarrow\quad \sin\theta &= \frac{1}{2}
\end{aligned}
$$

This gives the solutions $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$, which are the limits of integration. The circle $r = 3\sin\theta$ is the red graph, which is the outer function, and the cardioid $r = 2 + 2\sin\theta$ is the blue graph, which is the inner function. To calculate the area between the curves, start with the area inside the circle between $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$, then subtract the area inside the cardioid between $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$:

$$\begin{aligned} A &= \text{circle} - \text{cardioid} \\ &= \frac{1}{2}\int_{\pi/6}^{5\pi/6} [6\sin\theta]^2\, d\theta - \frac{1}{2}\int_{\pi/6}^{5\pi/6} [2+2\sin\theta]^2\, d\theta \\ &= \frac{1}{2}\int_{\pi/6}^{5\pi/6} 36\sin^2\theta\, d\theta - \frac{1}{2}\int_{\pi/6}^{5\pi/6} 4 + 8\sin\theta + 4\sin^2\theta\, d\theta \\ &= 18\int_{\pi/6}^{5\pi/6} \frac{1-\cos(2\theta)}{2}\, d\theta - 2\int_{\pi/6}^{5\pi/6} 1 + 2\sin\theta + \frac{1-\cos(2\theta)}{2}\, d\theta \\ &= 9\left[\theta - \frac{\sin(2\theta)}{2}\right]_{\pi/6}^{5\pi/6} - 2\left[\frac{3\theta}{2} - 2\cos\theta - \frac{\sin(2\theta)}{4}\right]_{\pi/6}^{5\pi/6} \\ &= 9\left(\frac{5\pi}{6} - \frac{\sin(10\pi/6)}{2}\right) - 9\left(\frac{\pi}{6} - \frac{\sin(2\pi/6)}{2}\right) - \left(3\left(\frac{5\pi}{6}\right) - 4\cos\frac{5\pi}{6} - \frac{\sin(10\pi/6)}{2}\right) + \left(3\left(\frac{\pi}{6}\right) - 4\cos\frac{\pi}{6} - \frac{\sin(2\pi/6)}{2}\right) \\ &= 4\pi \end{aligned}$$

? Checkpoint 6.4.2

Find the area inside the circle $r = 4\cos\theta$ and outside the circle $r = 2$.

Answer

$A = \frac{4\pi}{3} + 2\sqrt{3}$

In Example **6.4.2**, we found the area inside the circle and outside the cardioid by first finding their intersection points. Notice that solving the equation directly for θ yielded two solutions: $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$. However, in the graph, there are three intersection points. The third intersection point is the origin. This point did not appear as a solution because the origin is on both graphs but for different values of θ. For example, for the cardioid we get

$$\begin{aligned} 2 + 2\sin\theta &= 0 \\ \Rightarrow \quad \sin\theta &= -1, \end{aligned}$$

so the values for θ that solve this equation are $\theta = \frac{3\pi}{2} + 2n\pi$, where n is any integer. For the circle, we get

$$6\sin\theta = 0.$$

The solutions to this equation are of the form $\theta = n\pi$ for any integer value of n. These two solution sets have no points in common. Regardless of this fact, the curves intersect at the origin. This case must always be taken into consideration.

Arc Length in Polar Curves

Here, we derive a formula for the arc length of a curve defined in polar coordinates. In rectangular coordinates, the arc length of a parameterized curve $(x(t), y(t))$ for $a \le t \le b$ is given by

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

In polar coordinates, we define the curve by the equation $r = f(\theta)$, where $\alpha \le \theta \le \beta$. To adapt the arc length formula for a polar curve, we use the equations

$$x = r\cos\theta = f(\theta)\cos\theta$$

and

$$y = r\sin\theta = f(\theta)\sin\theta,$$

and we replace the parameter t by θ. Then

$$\frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta$$

$$\frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta.$$

We replace dt by $d\theta$, and the lower and upper limits of integration are α and β, respectively. Then the arc length formula becomes

$$\begin{aligned} L &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta \\ &= \int_\alpha^\beta \sqrt{(f'(\theta)\cos\theta - f(\theta)\sin\theta)^2 + (f'(\theta)\sin\theta + f(\theta)\cos\theta)^2}\, d\theta \\ &= \int_\alpha^\beta \sqrt{(f'(\theta))^2(\cos^2\theta + \sin^2\theta) + (f(\theta))^2(\cos^2\theta + \sin^2\theta)}\, d\theta \\ &= \int_\alpha^\beta \sqrt{(f'(\theta))^2 + (f(\theta))^2}\, d\theta \\ &= \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \end{aligned}$$

This gives us the following theorem.

Theorem: Arc Length of a Curve Defined by a Polar Function

Let f be a function whose derivative is continuous on an interval $\alpha \le \theta \le \beta$. The length of the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is

$$L = \int_\alpha^\beta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta \tag{6.4.3}$$

$$= \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta. \tag{6.4.4}$$

Example 6.4.3: Finding the Arc Length of a cardioid

Find the arc length of the cardioid $r = 2 + 2\cos\theta$.

Solution

When $\theta = 0, r = 2 + 2\cos 0 = 4$. Furthermore, as θ goes from 0 to 2π, the cardioid is traced out exactly once. Therefore, these are the limits of integration. Using $f(\theta) = 2 + 2\cos\theta, \alpha = 0$, and $\beta = 2\pi$, Equation 6.4.3 becomes

$$\begin{aligned} L &= \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta \\ &= \int_0^{2\pi} \sqrt{[2+2\cos\theta]^2 + [-2\sin\theta]^2}\, d\theta \\ &= \int_0^{2\pi} \sqrt{4 + 8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}\, d\theta \\ &= \int_0^{2\pi} \sqrt{4 + 8\cos\theta + 4(\cos^2\theta + \sin^2\theta)}\, d\theta \\ &= \int_0^{2\pi} \sqrt{8 + 8\cos\theta}\, d\theta \\ &= 2\int_0^{2\pi} \sqrt{2 + 2\cos\theta}\, d\theta. \end{aligned}$$

Next, using the identity $\cos(2\alpha) = 2\cos^2\alpha - 1$, add 1 to both sides and multiply by 2. This gives $2 + 2\cos(2\alpha) = 4\cos^2\alpha$. Substituting $\alpha = \theta/2$ gives $2 + 2\cos\theta = 4\cos^2(\theta/2)$, so the integral becomes

$$\begin{aligned} L &= 2\int_0^{2\pi} \sqrt{2 + 2\cos\theta}\, d\theta \\ &= 2\int_0^{2\pi} \sqrt{4\cos^2(\frac{\theta}{2})}\, d\theta \\ &= 4\int_0^{2\pi} \left|\cos(\frac{\theta}{2})\right|\, d\theta. \end{aligned}$$

The absolute value is necessary because the cosine is negative for some values in its domain. To resolve this issue, change the limits from 0 to π and double the answer. This strategy works because cosine is positive between 0 and $\frac{\pi}{2}$. Thus,

$$\begin{aligned} L &= 4\int_0^{2\pi} \left|\cos(\frac{\theta}{2})\right|\, d\theta \\ &= 8\int_0^{\pi} \cos\left(\frac{\theta}{2}\right)\, d\theta \\ &= 8\left(2\sin\left(\frac{\theta}{2}\right)\right|_0^{\pi} \\ &= 16 \end{aligned}$$

? Checkpoint 6.4.3

Find the total arc length of $r = 3\sin\theta$.

Answer

$s = 3\pi$

Key Concepts

- The area of a region in polar coordinates defined by the equation $r = f(\theta)$ with $\alpha \le \theta \le \beta$ is given by the integral $A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 d\theta$.
- To find the area between two curves in the polar coordinate system, first find the points of intersection, then subtract the corresponding areas.
- The arc length of a polar curve defined by the equation $r = f(\theta)$ with $\alpha \le \theta \le \beta$ is given by the integral $L = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} d\theta$ $= \int_{\alpha}^{\beta} \sqrt{r^2 + (\frac{dr}{d\theta})^2} d\theta$.

Key Equations

- **Area of a region bounded by a polar curve**

$$A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 d\theta = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta$$

- **Arc length of a polar curve**

$$L = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} d\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + (\frac{dr}{d\theta})^2} d\theta$$

6.4E: Exercises

In exercises 1 -13, determine a definite integral that represents the area.

1) Region enclosed by $r = 4$

2) Region enclosed by $r = 3\sin\theta$

Answer

$$\frac{9}{2}\int_0^{\pi} \sin^2\theta\, d\theta$$

3) Region in the first quadrant within the cardioid $r = 1 + \sin\theta$

4) Region enclosed by one petal of $r = 8\sin(2\theta)$

Answer

$$\frac{3}{2}\int_0^{\pi/2} \sin^2(2\theta)\, d\theta$$

5) Region enclosed by one petal of $r = cos(3\theta)$

6) Region below the polar axis and enclosed by $r = 1 - \sin\theta$

Answer

$$\frac{1}{2}\int_{\pi}^{2\pi} (1 - \sin\theta)^2\, d\theta$$

7) Region in the first quadrant enclosed by $r = 2 - \cos\theta$

8) Region enclosed by the inner loop of $r = 2 - 3\sin\theta$

Answer

$$\int_{\sin^{-1}(2/3)}^{\pi/2} (2 - 3\sin\theta)^2\, d\theta$$

9) Region enclosed by the inner loop of $r = 3 - 4\cos\theta$

10) Region enclosed by $r = 1 - 2\cos\theta$ and outside the inner loop

Answer

$$\int_0^{\pi} (1 - 2\cos\theta)^2\, d\theta - \int_0^{\pi/3} (1 - 2\cos\theta)^2\, d\theta$$

11) Region common to $r = 3\sin\theta$ and $r = 2 - \sin\theta$

12) Region common to $r = 2$ and $r = 4\cos\theta$

Answer

$$4\int_0^{\pi/3} d\theta + 16\int_{\pi/3}^{\pi/2} (\cos^2\theta)\, d\theta$$

13) Region common to $r = 3\cos\theta$ and $r = 3\sin\theta$

In exercises 14 -26, find the area of the described region.

14) Enclosed by $r = 6\sin\theta$

Answer

9π units^2

15) Above the polar axis enclosed by $r = 2 + \sin\theta$

16) Below the polar axis and enclosed by $r = 2 - \cos\theta$

Answer

$\frac{9\pi}{4}$ units^2

17) Enclosed by one petal of $r = 4\cos(3\theta)$

18) Enclosed by one petal of $r = 3\cos(2\theta)$

Answer

$\frac{9\pi}{8}$ units^2

19) Enclosed by $r = 1 + \sin\theta$

20) Enclosed by the inner loop of $r = 3 + 6\cos\theta$

Answer

$\frac{18\pi - 27\sqrt{3}}{2}$ units^2

21) Enclosed by $r = 2 + 4\cos\theta$ and outside the inner loop

22) Common interior of $r = 4\sin(2\theta)$ and $r = 2$

Answer

$\frac{4}{3}(4\pi - 3\sqrt{3})$ units^2

23) Common interior of $r = 3 - 2\sin\theta$ and $r = -3 + 2\sin\theta$

24) Common interior of $r = 6\sin\theta$ and $r = 3$

Answer

$\frac{3}{2}(4\pi - 3\sqrt{3})$ units^2

25) Inside $r = 1 + \cos\theta$ and outside $r = \cos\theta$

26) Common interior of $r = 2 + 2\cos\theta$ and $r = 2\sin\theta$

Answer

$(2\pi - 4)$ units^2

In exercises 27 - 30, find a definite integral that represents the arc length.

27) $r = 4\cos\theta$ on the interval $0 \le \theta \le \frac{\pi}{2}$

28) $r = 1 + \sin\theta$ on the interval $0 \le \theta \le 2\pi$

Answer

$$\int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + \cos^2\theta}\, d\theta$$

29) $r = 2\sec\theta$ on the interval $0 \le \theta \le \frac{\pi}{3}$

30) $r = e^\theta$ on the interval $0 \le \theta \le 1$

Answer

$$\sqrt{2}\int_0^1 e^\theta\, d\theta$$

In exercises 31 - 35, find the length of the curve over the given interval.

31) $r = 6$ on the interval $0 \le \theta \le \frac{\pi}{2}$

32) $r = e^{3\theta}$ on the interval $0 \le \theta \le 2$

Answer

$\frac{\sqrt{10}}{3}(e^6 - 1)$ units

33) $r = 6\cos\theta$ on the interval $0 \le \theta \le \frac{\pi}{2}$

34) $r = 8 + 8\cos\theta$ on the interval $0 \le \theta \le \pi$

Answer

32 units

35) $r = 1 - \sin\theta$ on the interval $0 \le \theta \le 2\pi$

In exercises 36 - 40, use the integration capabilities of a calculator to approximate the length of the curve.

36) [Technology Required] $r = 3\theta$ on the interval $0 \le \theta \le \frac{\pi}{2}$

Answer

6.238 units

37) [Technology Required] $r = \dfrac{2}{\theta}$ on the interval $\pi \le \theta \le 2\pi$

38) [Technology Required] $r = \sin^2\left(\frac{\theta}{2}\right)$ on the interval $0 \le \theta \le \pi$

Answer

2 units

39) [Technology Required] $r = 2\theta^2$ on the interval $0 \le \theta \le \pi$

40) [Technology Required] $r = \sin(3\cos\theta)$ on the interval $0 \le \theta \le \pi$

Answer

4.39 units

In exercises 41 - 43, use the familiar formula from geometry to find the area of the region described and then confirm by using the definite integral.

41) $r = 3\sin\theta$ on the interval $0 \le \theta \le \pi$

42) $r = \sin\theta + \cos\theta$ on the interval $0 \le \theta \le \pi$

Answer

$A = \pi\left(\frac{\sqrt{2}}{2}\right)^2 = \frac{\pi}{2}$ units^2 and $\frac{1}{2}\int_0^{\pi}(1 + 2\sin\theta\cos\theta)\,d\theta = \frac{\pi}{2}$ units^2

43) $r = 6\sin\theta + 8\cos\theta$ on the interval $0 \le \theta \le \pi$

In exercises 44 - 46, use the familiar formula from geometry to find the length of the curve and then confirm using the definite integral.

44) $r = 3\sin\theta$ on the interval $0 \le \theta \le \pi$

Answer

$C = 2\pi\left(\frac{3}{2}\right) = 3\pi$ units and $\int_0^{\pi} 3\,d\theta = 3\pi$ units

45) $r = \sin\theta + \cos\theta$ on the interval $0 \le \theta \le \pi$

46) $r = 6\sin\theta + 8\cos\theta$ on the interval $0 \le \theta \le \pi$

Answer

$C = 2\pi(5) = 10\pi$ units and $\int_0^{\pi} 10\,d\theta = 10\pi$ units

47) Verify that if $y = r\sin\theta = f(\theta)\sin\theta$ then $\frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta$.

In exercises 48 - 56, find the slope of a tangent line to a polar curve $r = f(\theta)$. Let $x = r\cos\theta = f(\theta)\cos\theta$ and $y = r\sin\theta = f(\theta)\sin\theta$, so the polar equation $r = f(\theta)$ is now written in parametric form.

48) Use the definition of the derivative $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$ and the Product Rule to derive the derivative of a polar equation.

Answer

$$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

49) $r = 1 - \sin\theta;\ \left(\frac{1}{2}, \frac{\pi}{6}\right)$

50) $r = 4\cos\theta;\ \left(2, \frac{\pi}{3}\right)$

Answer

The slope is $\frac{1}{\sqrt{3}}$.

51) $r = 8\sin\theta;\ \left(4, \frac{5\pi}{6}\right)$

52) $r = 4 + \sin\theta;\ \left(3, \frac{3\pi}{2}\right)$

Answer

The slope is 0.

53) $r = 6 + 3\cos\theta;\ (3, \pi)$

54) $r = 4\cos(2\theta)$; tips of the leaves

Answer

At $(4, 0)$, the slope is undefined. At $\left(-4, \frac{\pi}{2}\right)$, the slope is 0.

55) $r = 2\sin(3\theta)$; tips of the leaves

56) $r = 2\theta;\ \left(\frac{\pi}{2}, \frac{\pi}{4}\right)$

Answer

The slope is undefined at $\theta = \frac{\pi}{4}$.

57) Find the points on the interval $-\pi \le \theta \le \pi$ at which the cardioid $r = 1 - \cos\theta$ has a vertical or horizontal tangent line.

58) For the cardioid $r = 1 + \sin\theta$, find the slope of the tangent line when $\theta = \frac{\pi}{3}$.

Answer

Slope = −1.

In exercises 59 - 62, find the slope of the tangent line to the given polar curve at the point given by the value of θ.

59) $r = 3\cos\theta,\ \theta = \frac{\pi}{3}$

60) $r = \theta,\ \theta = \frac{\pi}{2}$

Answer

Slope is $\frac{-2}{\pi}$.

https://math.libretexts.org/@go/page/168468

61) $r = \ln\theta,\ \theta = e$

62) [Technology Required] Use technology: $r = 2 + 4\cos\theta$ at $\theta = \frac{\pi}{6}$

Answer

Calculator answer: -0.836.

In exercises 63 - 66, find the points at which the following polar curves have a horizontal or vertical tangent line.

63) $r = 4\cos\theta$

64) $r^2 = 4\cos(2\theta)$

Answer

Horizontal tangent at $\left(\pm\sqrt{2}, \frac{\pi}{6}\right)$, $\left(\pm\sqrt{2}, -\frac{\pi}{6}\right)$.

65) $r = 2\sin(2\theta)$

66) The cardioid $r = 1 + \sin\theta$

Answer

Horizontal tangents at $\frac{\pi}{2}$, $\frac{7\pi}{6}$, $\frac{11\pi}{6}$.
Vertical tangents at $\frac{\pi}{6}$, $\frac{5\pi}{6}$ and also at the pole $(0, 0)$.

67) Show that the curve $r = \sin\theta\tan\theta$ (called a **cissoid of Diocles**) has the line $x = 1$ as a vertical asymptote.

This page titled 6.4E: Exercises is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Roy Simpson.

- **11.4E: Exercises for Section 11.4** by Edwin “Jed” Herman, Gilbert Strang is licensed CC BY-NC-SA 4.0. Original source: https://openstax.org/details/books/calculus-volume-1.

6.5: Conic Sections

Learning Objectives

- Identify the equation of a parabola in standard form with given focus and directrix.
- Identify the equation of an ellipse in standard form with given foci.
- Identify the equation of a hyperbola in standard form with given foci.
- Recognize a parabola, ellipse, or hyperbola from its eccentricity value.
- Write the polar equation of a conic section with eccentricity e.
- Identify when a general equation of degree two is a parabola, ellipse, or hyperbola.

Conic sections have been studied since the time of the ancient Greeks, and were considered to be an important mathematical concept. As early as 320 BCE, such Greek mathematicians as Menaechmus, Appollonius, and Archimedes were fascinated by these curves. Appollonius wrote an entire eight-volume treatise on conic sections in which he was, for example, able to derive a specific method for identifying a conic section through the use of geometry. Since then, important applications of conic sections have arisen (for example, in astronomy), and the properties of conic sections are used in radio telescopes, satellite dish receivers, and even architecture. In this section we discuss the three basic conic sections, some of their properties, and their equations.

Conic sections get their name because they can be generated by intersecting a plane with a cone. A cone has two identically shaped parts called **nappes.** One nappe is what most people mean by "cone," having the shape of a party hat. A right circular cone can be generated by revolving a line passing through the origin around the y-axis as shown in Figure 6.5.1.

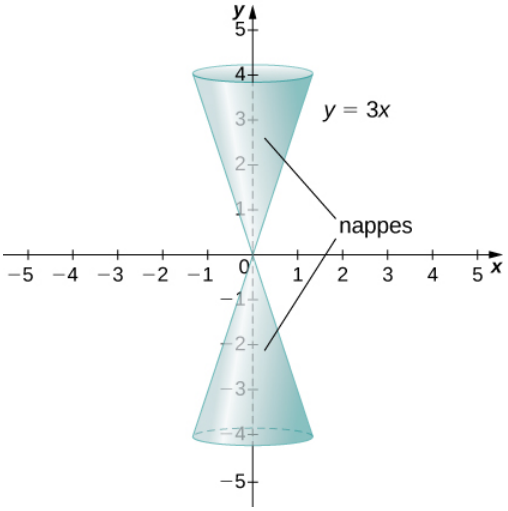


Figure 6.5.1: A cone generated by revolving the line $y = 3x$ around the y-axis.

Conic sections are generated by the intersection of a plane with a cone (Figure 6.5.2). If the plane is parallel to the axis of revolution (the y-axis), then the **conic section** is a hyperbola. If the plane is parallel to the generating line, the conic section is a parabola. If the plane is perpendicular to the axis of revolution, the conic section is a circle. If the plane intersects one nappe at an angle to the axis (other than **90^{\circ}**), then the conic section is an ellipse.

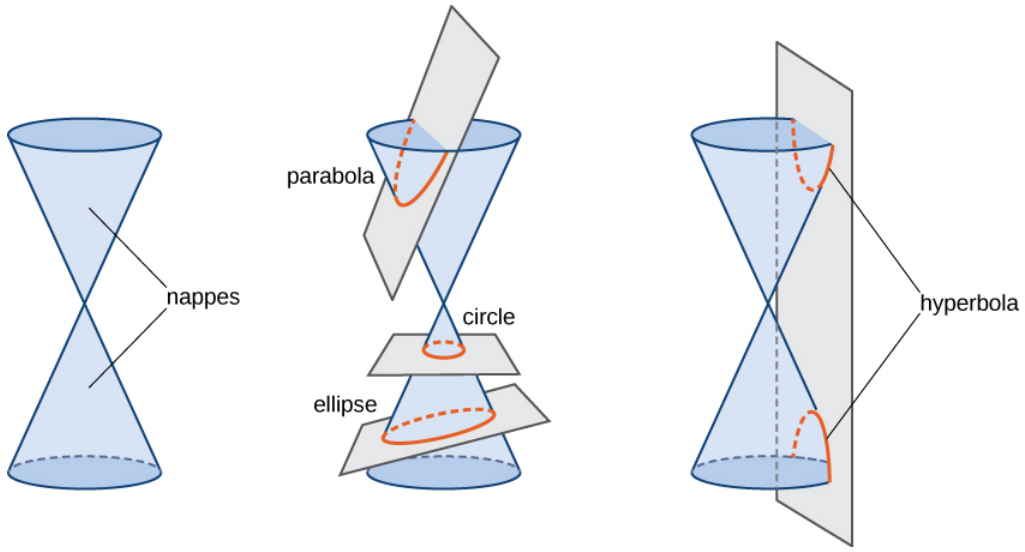


Figure **6.5.2**: The four conic sections. Each conic is determined by the angle the plane makes with the axis of the cone.

Parabolas

A parabola is generated when a plane intersects a cone parallel to the generating line. In this case, the plane intersects only one of the nappes. A parabola can also be defined in terms of distances.

Definitions: The Focus, Directrix and Vertex

A parabola is the set of all points whose distance from a fixed point, called the *focus*, is equal to the distance from a fixed line, called the *directrix*. The point halfway between the focus and the directrix is called the *vertex* of the parabola.

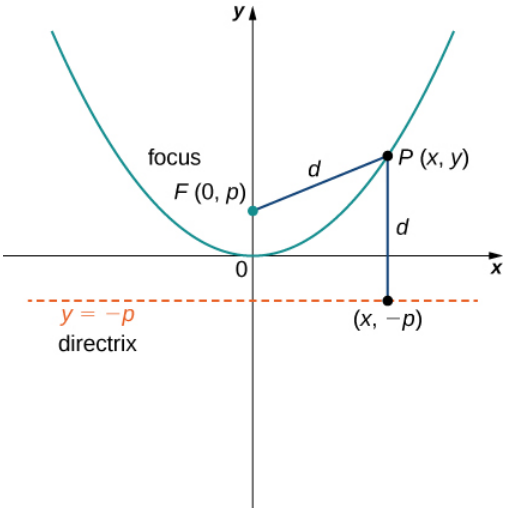


Figure **6.5.3**: A typical parabola in which the distance from the focus to the vertex is represented by the variable p.

A graph of a typical parabola appears in Figure **6.5.3**. Using this diagram in conjunction with the distance formula, we can derive an equation for a parabola. Recall the distance formula: Given point P with coordinates (x_1, y_1) and point Q with coordinates (x_2, y_2), the distance between them is given by the formula

$$d(P,Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Then from the definition of a parabola and Figure **6.5.3**, we get

$$d(F,P) = d(P,Q)$$

$$\sqrt{(0-x)^2+(p-y)^2}=\sqrt{(x-x)^2+(-p-y)^2}.$$

Squaring both sides and simplifying yields

$$x^2+(p-y)^2=0^2+(-p-y)^2 \tag{6.5.1}$$
$$x^2+p^2-2py+y^2=p^2+2py+y^2 \tag{6.5.2}$$
$$x^2-2py=2py \tag{6.5.3}$$
$$x^2=4py. \tag{6.5.4}$$

Now suppose we want to relocate the vertex. We use the variables (h,k) to denote the coordinates of the vertex. Then if the focus is directly above the vertex, it has coordinates $(h,k+p)$ and the directrix has the equation $y=k-p$. Going through the same derivation yields the formula $(x-h)^2=4p(y-k)$. Solving this equation for y leads to the following theorem.

Equations for Parabolas: standard form

Given a parabola opening upward with vertex located at (h,k) and focus located at $(h,k+p)$, where p is a constant, the equation for the parabola is given by

$$y=\frac{1}{4p}(x-h)^2+k.$$

This is the *standard form* of a parabola.

We can also study the cases when the parabola opens down or to the left or the right. The equation for each of these cases can also be written in standard form as shown in the following graphs.

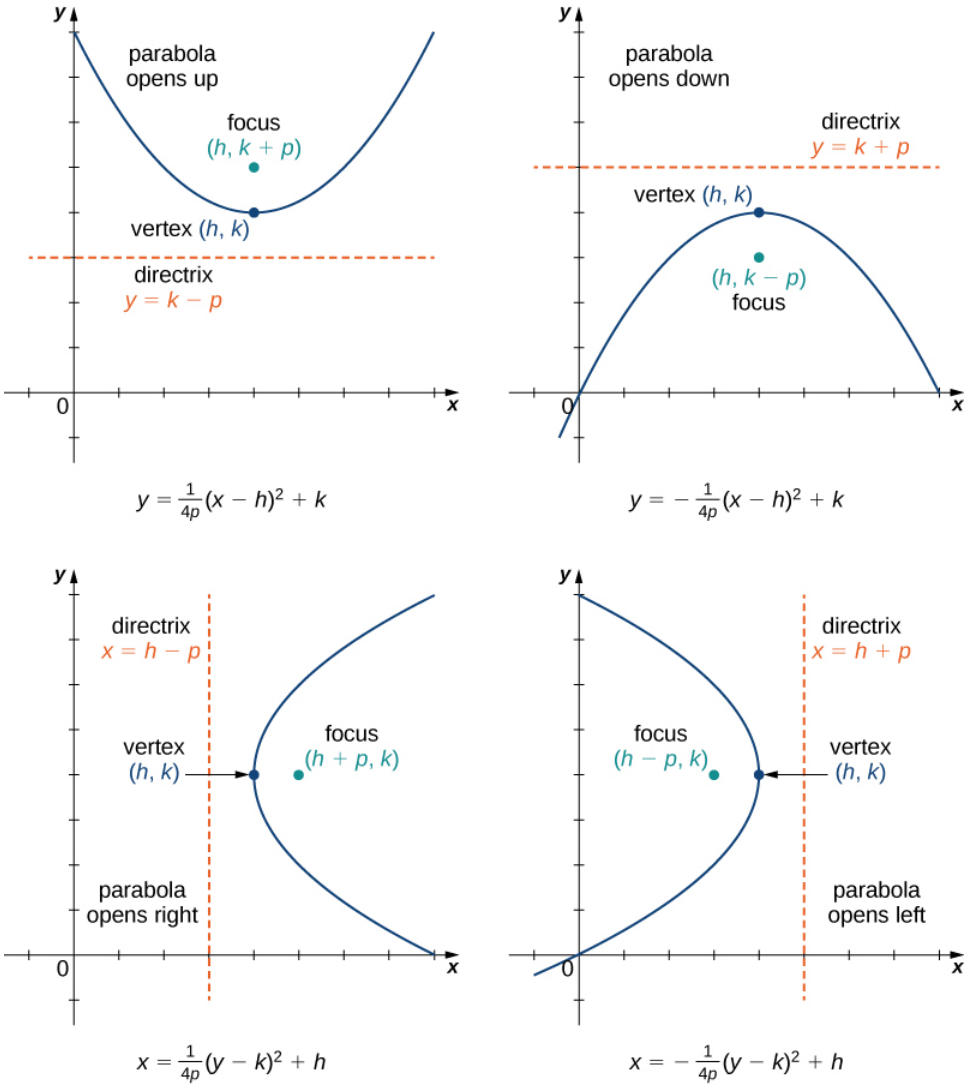


Figure 6.5.4: Four parabolas, opening in various directions, along with their equations in standard form.

In addition, the equation of a parabola can be written in the **general form**, though in this form the values of h, k, and p are not immediately recognizable. The general form of a parabola is written as

$$ax^2 + bx + cy + d = 0 \tag{6.5.5}$$

or

$$ay^2 + bx + cy + d = 0. \tag{6.5.6}$$

Equation 6.5.5 represents a parabola that opens either up or down. Equation 6.5.6 represents a parabola that opens either to the left or to the right. To put the equation into standard form, use the method of completing the square.

Example 6.5.1: Converting the Equation of a Parabola from General into Standard Form

Put the equation

$$x^2 - 4x - 8y + 12 = 0$$

into standard form and graph the resulting parabola.

Solution

Since y is not squared in this equation, we know that the parabola opens either upward or downward. Therefore we need to solve this equation for y, which will put the equation into standard form. To do that, first add $8y$ to both sides of the equation:

$$8y = x^2 - 4x + 12.$$

The next step is to complete the square on the right-hand side. Start by grouping the first two terms on the right-hand side using parentheses:

$$8y = (x^2 - 4x) + 12.$$

Next determine the constant that, when added inside the parentheses, makes the quantity inside the parentheses a perfect square trinomial. To do this, take half the coefficient of x and square it. This gives $(\frac{-4}{2})^2 = 4$. Add 4 inside the parentheses and subtract 4 outside the parentheses, so the value of the equation is not changed:

$$8y = (x^2 - 4x + 4) + 12 - 4.$$

Now combine like terms and factor the quantity inside the parentheses:

$$8y = (x - 2)^2 + 8.$$

Finally, divide by 8:

$$y = \frac{1}{8}(x - 2)^2 + 1.$$

This equation is now in standard form. Comparing this to Equation gives $h = 2, k = 1$, and $p = 2$. The parabola opens up, with vertex at $(2, 1)$, focus at $(2, 3)$, and directrix $y = -1$. The graph of this parabola appears as follows.

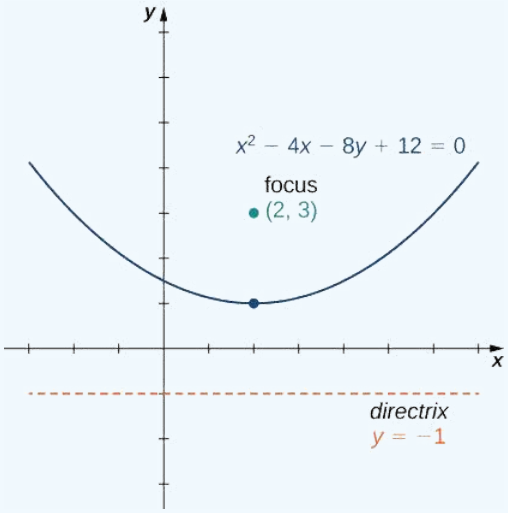


Figure 6.5.5: The parabola in Example 6.5.1.

? Exercise 6.5.1

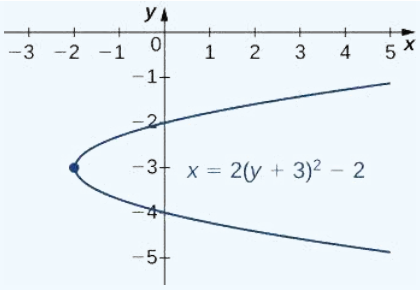
Put the equation $2y^2 - x + 12y + 16 = 0$ into standard form and graph the resulting parabola.

Hint

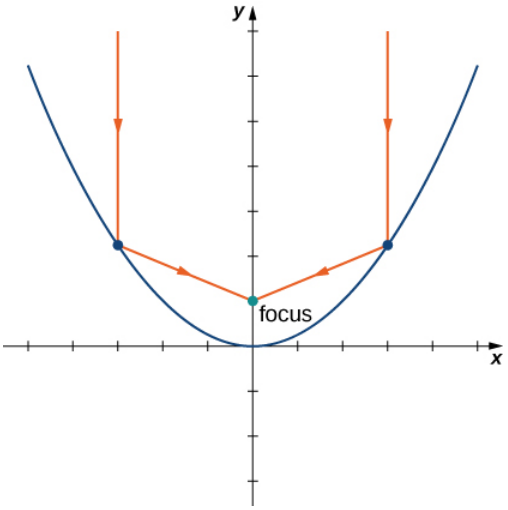
Solve for x. Check which direction the parabola opens.

Answer

$$x = 2(y + 3)^2 - 2$$



The axis of symmetry of a vertical (opening up or down) parabola is a vertical line passing through the vertex. The parabola has an interesting reflective property. Suppose we have a satellite dish with a parabolic cross section. If a beam of electromagnetic waves, such as light or radio waves, comes into the dish in a straight line from a satellite (parallel to the axis of symmetry), then the waves reflect off the dish and collect at the focus of the parabola as shown.



Consider a parabolic dish designed to collect signals from a satellite in space. The dish is aimed directly at the satellite, and a receiver is located at the focus of the parabola. Radio waves coming in from the satellite are reflected off the surface of the parabola to the receiver, which collects and decodes the digital signals. This allows a small receiver to gather signals from a wide angle of sky. Flashlights and headlights in a car work on the same principle, but in reverse: the source of the light (that is, the light bulb) is located at the focus and the reflecting surface on the parabolic mirror focuses the beam straight ahead. This allows a small light bulb to illuminate a wide angle of space in front of the flashlight or car.

Ellipses

An ellipse can also be defined in terms of distances. In the case of an ellipse, there are two foci (plural of focus), and two directrices (plural of directrix). We look at the directrices in more detail later in this section.

Definition: Ellipse

An ellipse is the set of all points for which the sum of their distances from two fixed points (the foci) is constant.

A graph of a typical ellipse is shown in Figure **6.5.6**. In this figure the foci are labeled as F and F'. Both are the same fixed distance from the origin, and this distance is represented by the variable c. Therefore the coordinates of F are $(c, 0)$ and the coordinates of F' are $(-c, 0)$. The points P and P' are located at the ends of the **major axis** of the ellipse, and have coordinates $(a, 0)$ and $(-a, 0)$, respectively. The major axis is always the longest distance across the ellipse, and can be horizontal or vertical. Thus, the length of the major axis in this ellipse is $2a$. Furthermore, P and P' are called the vertices of the ellipse. The points Q and Q' are located at the ends of the **minor axis** of the ellipse, and have coordinates $(0, b)$ and $(0, -b)$, respectively. The minor axis is the shortest distance across the ellipse. The minor axis is perpendicular to the major axis.

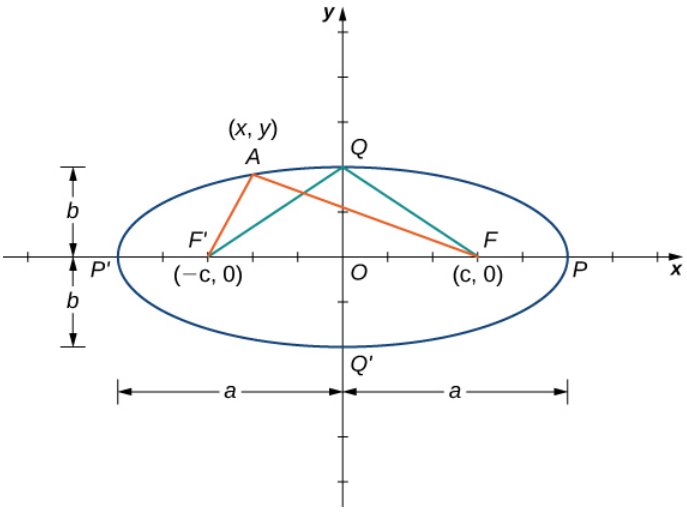


Figure **6.5.6**: A typical ellipse in which the sum of the distances from any point on the ellipse to the foci is constant.

According to the definition of the ellipse, we can choose any point on the ellipse and the sum of the distances from this point to the two foci is constant. Suppose we choose the point P. Since the coordinates of point P are $(a, 0)$, the sum of the distances is

$$d(P, F) + d(P, F') = (a - c) + (a + c) = 2a.$$

Therefore the sum of the distances from an arbitrary point A with coordinates (x, y) is also equal to $2a$. Using the distance formula, we get

$$d(A, F) + d(A, F') = 2a.$$

$$\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a$$

Subtract the second radical from both sides and square both sides:

$$\sqrt{(x-c)^2 + y^2} = 2a - \sqrt{(x+c)^2 + y^2}$$

$$(x-c)^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + (x+c)^2 + y^2$$

$$x^2 - 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + x^2 + 2cx + c^2 + y^2$$

$$-2cx = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + 2cx.$$

Now isolate the radical on the right-hand side and square again:

$$-2cx = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + 2cx$$

$$4a\sqrt{(x+c)^2 + y^2} = 4a^2 + 4cx$$

$$\sqrt{(x+c)^2 + y^2} = a + \frac{cx}{a}$$

$$(x+c)^2 + y^2 = a^2 + 2cx + \frac{c^2x^2}{a^2}$$

$$x^2 + 2cx + c^2 + y^2 = a^2 + 2cx + \frac{c^2x^2}{a^2}$$

$$x^2 + c^2 + y^2 = a^2 + \frac{c^2x^2}{a^2}.$$

Isolate the variables on the left-hand side of the equation and the constants on the right-hand side:

$$x^2 - \frac{c^2x^2}{a^2} + y^2 = a^2 - c^2$$

$$\frac{(a^2 - c^2)x^2}{a^2} + y^2 = a^2 - c^2.$$

Divide both sides by $a^2 - c^2$. This gives the equation

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1.$$

If we refer back to Figure 6.5.6, then the length of each of the two green line segments is equal to a. This is true because the sum of the distances from the point Q to the foci F and F' is equal to $2a$, and the lengths of these two line segments are equal. This line segment forms a right triangle with hypotenuse length a and leg lengths b and c. From the Pythagorean Theorem, $b^2 + c^2 = a^2$ and $b^2 = a^2 - c^2$. Therefore the equation of the ellipse becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Finally, if the center of the ellipse is moved from the origin to a point (h, k), we have the following standard form of an ellipse.

Equation of an Ellipse in Standard Form

Consider the ellipse with center (h, k), a horizontal major axis with length $2a$, and a vertical minor axis with length $2b$. Then the equation of this ellipse in standard form is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \tag{6.5.7}$$

and the foci are located at $(h \pm c, k)$, where $c^2 = a^2 - b^2$. The equations of the directrices are $x = h \pm \frac{a^2}{c}$.

If the major axis is vertical, then the equation of the ellipse becomes

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1 \tag{6.5.8}$$

and the foci are located at $(h, k \pm c)$, where $c^2 = a^2 - b^2$. The equations of the directrices in this case are $y = k \pm \frac{a^2}{c}$.

If the major axis is horizontal, then the ellipse is called horizontal, and if the major axis is vertical, then the ellipse is called vertical. The equation of an ellipse is in general form if it is in the form

$$Ax^2 + By^2 + Cx + Dy + E = 0,$$

where A and B are either both positive or both negative. To convert the equation from general to standard form, use the method of **completing the square**.

Example 6.5.2: Finding the Standard Form of an Ellipse

Put the equation

$$9x^2 + 4y^2 - 36x + 24y + 36 = 0$$

into standard form and graph the resulting ellipse.

Solution

First subtract 36 from both sides of the equation:

$$9x^2 + 4y^2 - 36x + 24y = -36.$$

Next group the x terms together and the y terms together, and factor out the common factor:

$$(9x^2 - 36x) + (4y^2 + 24y) = -36$$

$$9(x^2-4x)+4(y^2+6y)=-36.$$

We need to determine the constant that, when added inside each set of parentheses, results in a perfect square. In the first set of parentheses, take half the coefficient of x and square it. This gives $(\frac{-4}{2})^2=4$. In the second set of parentheses, take half the coefficient of y and square it. This gives $(\frac{6}{2})^2=9$. Add these inside each pair of parentheses. Since the first set of parentheses has a 9 in front, we are actually adding 36 to the left-hand side. Similarly, we are adding 36 to the second set as well. Therefore the equation becomes

$$9(x^2-4x+4)+4(y^2+6y+9)=-36+36+36$$

$$9(x^2-4x+4)+4(y^2+6y+9)=36.$$

Now factor both sets of parentheses and divide by 36:

$$9(x-2)^2+4(y+3)^2=36$$

$$\frac{9(x-2)^2}{36}+\frac{4(y+3)^2}{36}=1$$

$$\frac{(x-2)^2}{4}+\frac{(y+3)^2}{9}=1.$$

The equation is now in standard form. Comparing this to Equation 6.5.8 gives $h=2, k=-3, a=3,$ and $b=2$. This is a vertical ellipse with center at $(2,-3)$, major axis 6, and minor axis 4. The graph of this ellipse appears as follows.

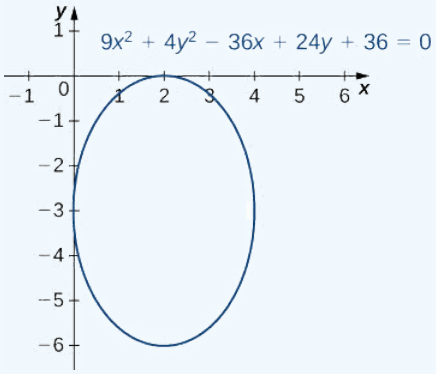


Figure 6.5.7: The ellipse in Example 6.5.2.

Exercise 6.5.2

Put the equation

$$9x^2+16y^2+18x-64y-71=0$$

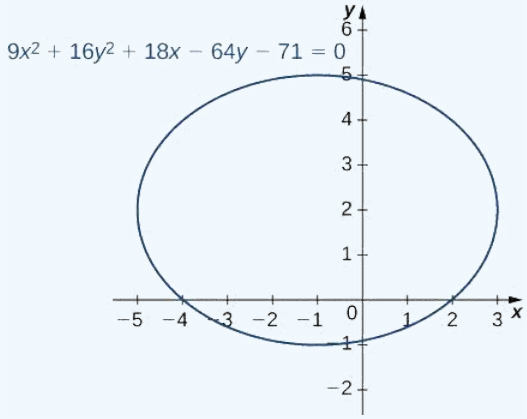
into standard form and graph the resulting ellipse.

Hint

Move the constant over and complete the square.

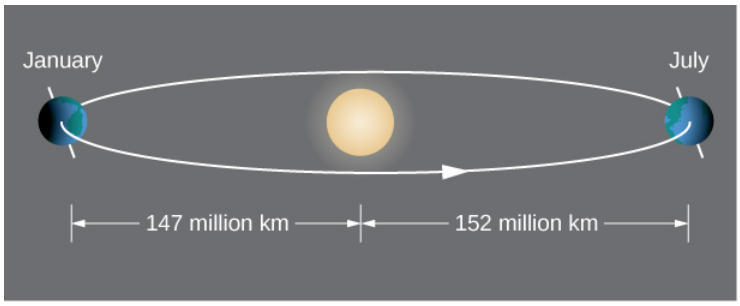
Answer

$$\frac{(x+1)^2}{16}+\frac{(y-2)^2}{9}=1$$



According to Kepler's first law of planetary motion, the orbit of a planet around the Sun is an ellipse with the Sun at one of the foci as shown in Figure **6.5.8*A***. Because Earth's orbit is an ellipse, the distance from the Sun varies throughout the year. A commonly held misconception is that Earth is closer to the Sun in the summer. In fact, in summer for the northern hemisphere, Earth is farther from the Sun than during winter. The difference in season is caused by the tilt of Earth's axis in the orbital plane. Comets that orbit the Sun, such as Halley's Comet, also have elliptical orbits, as do moons orbiting the planets and satellites orbiting Earth.

Ellipses also have interesting reflective properties: A light ray emanating from one focus passes through the other focus after mirror reflection in the ellipse. The same thing occurs with a sound wave as well. The National Statuary Hall in the U.S. Capitol in Washington, DC, is a famous room in an elliptical shape as shown in Figure **6.5.8*B***. This hall served as the meeting place for the U.S. House of Representatives for almost fifty years. The location of the two foci of this semi-elliptical room are clearly identified by marks on the floor, and even if the room is full of visitors, when two people stand on these spots and speak to each other, they can hear each other much more clearly than they can hear someone standing close by. Legend has it that John Quincy Adams had his desk located on one of the foci and was able to eavesdrop on everyone else in the House without ever needing to stand. Although this makes a good story, it is unlikely to be true, because the original ceiling produced so many echoes that the entire room had to be hung with carpets to dampen the noise. The ceiling was rebuilt in 1902 and only then did the now-famous whispering effect emerge. Another famous whispering gallery—the site of many marriage proposals—is in Grand Central Station in New York City.



(a)



(b)

Figure **6.5.8**: (a) Earth's orbit around the Sun is an ellipse with the Sun at one focus. (b) Statuary Hall in the U.S. Capitol is a whispering gallery with an elliptical cross section.

Hyperbolas

A hyperbola can also be defined in terms of distances. In the case of a hyperbola, there are two foci and two directrices. Hyperbolas also have two asymptotes.

Definition: hyperbola

A hyperbola is the set of all points where the difference between their distances from two fixed points (the foci) is constant.

A graph of a typical hyperbola appears as follows.

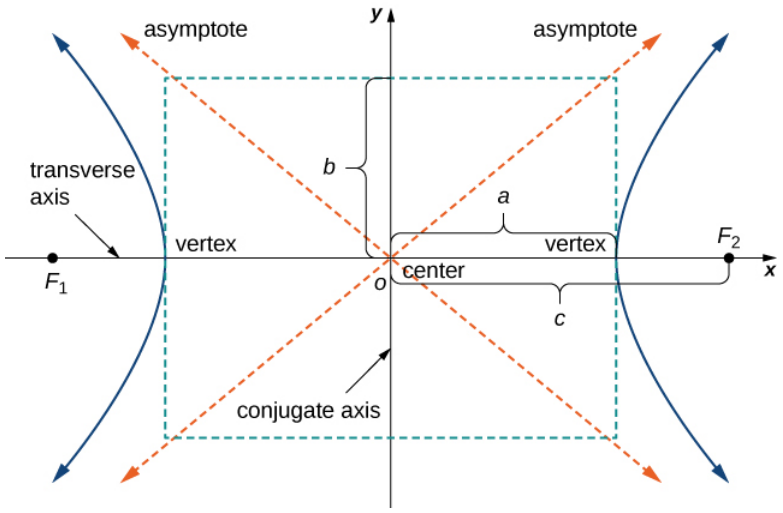


Figure **6.5.9**: A typical hyperbola in which the difference of the distances from any point on the hyperbola to the foci is constant. The transverse axis is also called the major axis, and the conjugate axis is also called the minor axis.

The derivation of the equation of a hyperbola in standard form is virtually identical to that of an ellipse. One slight hitch lies in the definition: The difference between two numbers is always positive. Let P be a point on the hyperbola with coordinates (x, y). Then the definition of the hyperbola gives $|d(P, F_1) - d(P, F_2)| = constant$. To simplify the derivation, assume that P is on the right branch of the hyperbola, so the absolute value bars drop. If it is on the left branch, then the subtraction is reversed. The vertex of the right branch has coordinates $(a, 0)$, so

$$d(P, F_1) - d(P, F_2) = (c + a) - (c - a) = 2a.$$

This equation is therefore true for any point on the hyperbola. Returning to the coordinates (x, y) for P:

$$d(P, F_1) - d(P, F_2) = 2a$$

$$\sqrt{(x + c)^2 + y^2} - \sqrt{(x - c)^2 + y^2} = 2a.$$

Isolate the second radical and square both sides:

$$\sqrt{(x - c)^2 + y^2} = -2a + \sqrt{(x + c)^2 + y^2}$$

$$(x - c)^2 + y^2 = 4a^2 - 4a\sqrt{(x + c)^2 + y^2} + (x + c)^2 + y^2$$

$$x^2 - 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x + c)^2 + y^2} + x^2 + 2cx + c^2 + y^2$$

$$-2cx = 4a^2 - 4a\sqrt{(x + c)^2 + y^2} + 2cx.$$

Now isolate the radical on the right-hand side and square again:

$$-2cx = 4a^2 - 4a\sqrt{(x + c)^2 + y^2} + 2cx$$

$$-4a\sqrt{(x + c)^2 + y^2} = -4a^2 - 4cx$$

$$-\sqrt{(x + c)^2 + y^2} = -a - \frac{cx}{a}$$

$$(x + c)^2 + y^2 = a^2 + 2cx + \frac{c^2x^2}{a^2}$$

$$x^2 + 2cx + c^2 + y^2 = a^2 + 2cx + \frac{c^2x^2}{a^2}$$

$$x^2 + c^2 + y^2 = a^2 + \frac{c^2x^2}{a^2}.$$

Isolate the variables on the left-hand side of the equation and the constants on the right-hand side:

$$x^2 - \frac{c^2x^2}{a^2} + y^2 = a^2 - c^2$$

$$\frac{(a^2 - c^2)x^2}{a^2} + y^2 = a^2 - c^2.$$

Finally, divide both sides by $a^2 - c^2$. This gives the equation

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1.$$

We now define b so that $b^2 = c^2 - a^2$. This is possible because $c > a$. Therefore the equation of the hyperbola becomes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Finally, if the center of the hyperbola is moved from the origin to the point (h, k), we have the following standard form of a hyperbola.

Equation of a Hyperbola in Standard Form

Consider the hyperbola with center (h, k), a horizontal major axis, and a vertical minor axis. Then the equation of this hyperbola is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \tag{6.5.9}$$

and the foci are located at $(h \pm c, k)$, where $c^2 = a^2 + b^2$. The equations of the asymptotes are given by $y = k \pm \frac{b}{a}(x - h)$. The equations of the directrices are

$$x = h \pm \frac{a^2}{\sqrt{a^2 + b^2}} = h \pm \frac{a^2}{c}$$

If the major axis is vertical, then the equation of the hyperbola becomes

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

and the foci are located at $(h, k \pm c)$, where $c^2 = a^2 + b^2$. The equations of the asymptotes are given by $y = k \pm \frac{a}{b}(x - h)$. The equations of the directrices are

$$y = k \pm \frac{a^2}{\sqrt{a^2 + b^2}} = k \pm \frac{a^2}{c}.$$

If the major axis (transverse axis) is horizontal, then the hyperbola is called horizontal, and if the major axis is vertical then the hyperbola is called vertical. The equation of a hyperbola is in general form if it is in the form

$$Ax^2 + By^2 + Cx + Dy + E = 0,$$

where A and B have opposite signs. In order to convert the equation from general to standard form, use the method of completing the square.

Example 6.5.3: Finding the Standard Form of a Hyperbola

Put the equation $9x^2 - 16y^2 + 36x + 32y - 124 = 0$ into standard form and graph the resulting hyperbola. What are the equations of the asymptotes?

Solution

First add 124 to both sides of the equation:

$$9x^2 - 16y^2 + 36x + 32y = 124.$$

Next group the *x* terms together and the *y* terms together, then factor out the common factors:

$$(9x^2 + 36x) - (16y^2 - 32y) = 124$$

$$9(x^2 + 4x) - 16(y^2 - 2y) = 124.$$

We need to determine the constant that, when added inside each set of parentheses, results in a perfect square. In the first set of parentheses, take half the coefficient of x and square it. This gives $(\frac{4}{2})^2 = 4$. In the second set of parentheses, take half the coefficient of y and square it. This gives $(\frac{-2}{2})^2 = 1$. Add these inside each pair of parentheses. Since the first set of parentheses has a 9 in front, we are actually adding 36 to the left-hand side. Similarly, we are subtracting 16 from the second set of parentheses. Therefore the equation becomes

$$9(x^2 + 4x + 4) - 16(y^2 - 2y + 1) = 124 + 36 - 16$$

$$9(x^2 + 4x + 4) - 16(y^2 - 2y + 1) = 144.$$

Next factor both sets of parentheses and divide by 144:

$$9(x + 2)^2 - 16(y - 1)^2 = 144$$

$$\frac{9(x + 2)^2}{144} - \frac{16(y - 1)^2}{144} = 1$$

$$\frac{(x + 2)^2}{16} - \frac{(y - 1)^2}{9} = 1.$$

The equation is now in standard form. Comparing this to Equation 6.5.9 gives $h = -2, k = 1, a = 4$, and $b = 3$. This is a horizontal hyperbola with center at $(-2, 1)$ and asymptotes given by the equations $y = 1 \pm \frac{3}{4}(x + 2)$. The graph of this hyperbola appears in Figure 6.5.10.

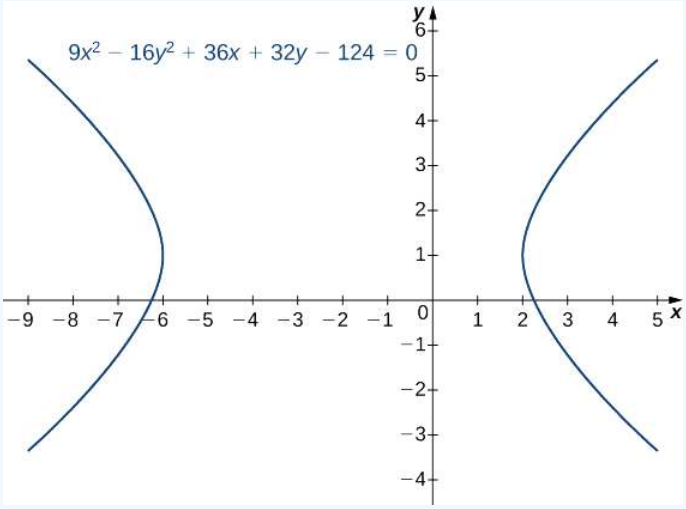


Figure 6.5.10: Graph of the hyperbola in Example 6.5.3.

? Exercise 6.5.3

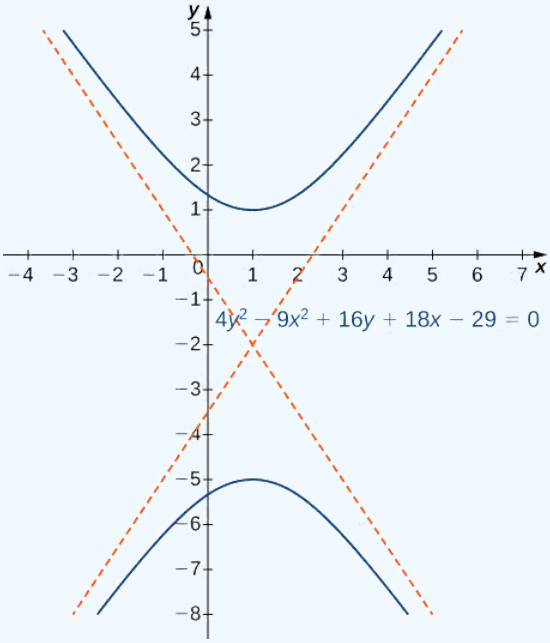
Put the equation $4y^2 - 9x^2 + 16y + 18x - 29 = 0$ into standard form and graph the resulting hyperbola. What are the equations of the asymptotes?

Hint

Move the constant over and complete the square. Check which direction the hyperbola opens

Answer

$\frac{(y + 2)^2}{9} - \frac{(x - 1)^2}{4} = 1$. This is a vertical hyperbola. Asymptotes $y = -2 \pm \frac{3}{2}(x - 1)$.



Hyperbolas also have interesting reflective properties. A ray directed toward one focus of a hyperbola is reflected by a hyperbolic mirror toward the other focus. This concept is illustrated in Figure **6.5.11**.

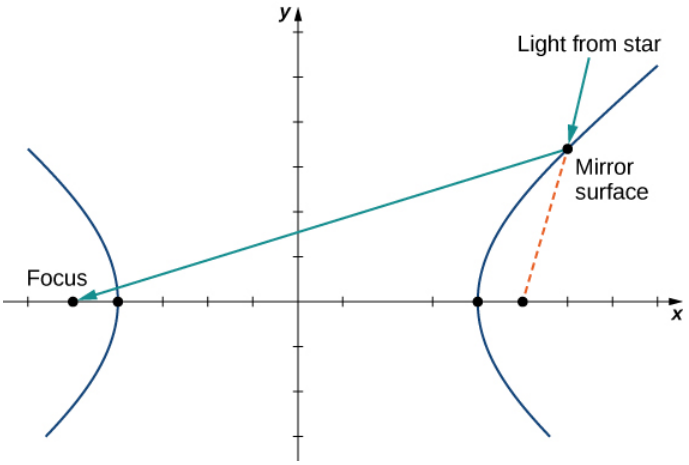


Figure **6.5.11**: A hyperbolic mirror used to collect light from distant stars.

This property of the hyperbola has important applications. It is used in radio direction finding (since the difference in signals from two towers is constant along hyperbolas), and in the construction of mirrors inside telescopes (to reflect light coming from the parabolic mirror to the eyepiece). Another interesting fact about hyperbolas is that for a comet entering the solar system, if the speed is great enough to escape the Sun's gravitational pull, then the path that the comet takes as it passes through the solar system is hyperbolic.

Eccentricity and Directrix

An alternative way to describe a conic section involves the directrices, the foci, and a new property called eccentricity. We will see that the value of the eccentricity of a conic section can uniquely define that conic.

Definition: Eccentricity and Directrices

The *eccentricity* e of a conic section is defined to be the distance from any point on the conic section to its focus, divided by the perpendicular distance from that point to the nearest directrix. This value is constant for any conic section, and can define

the conic section as well:

1. If $e = 1$, the conic is a parabola.
2. If $e < 1$, it is an ellipse.
3. If $e > 1$, it is a hyperbola.

The eccentricity of a circle is zero. The *directrix* of a conic section is the line that, together with the point known as the focus, serves to define a conic section. Hyperbolas and noncircular ellipses have two foci and two associated directrices. Parabolas have one focus and one directrix.

The three conic sections with their directrices appear in Figure 6.5.12.

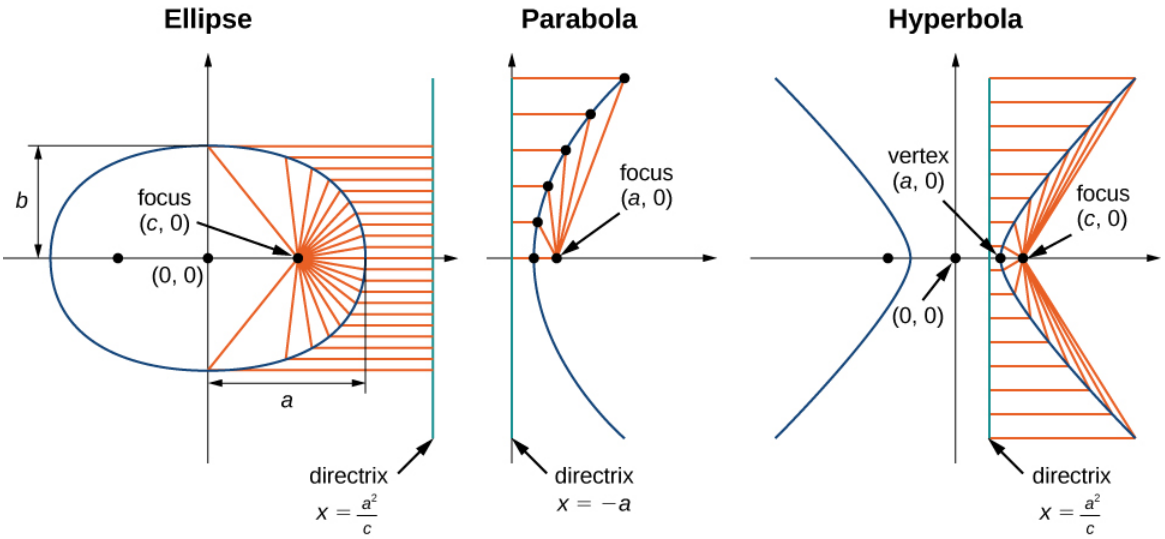


Figure 6.5.12: The three conic sections with their foci and directrices.

Recall from the definition of a parabola that the distance from any point on the parabola to the focus is equal to the distance from that same point to the directrix. Therefore, by definition, the eccentricity of a parabola must be 1. The equations of the directrices of a horizontal ellipse are $x = \pm\frac{a^2}{c}$. The right vertex of the ellipse is located at $(a, 0)$ and the right focus is $(c, 0)$. Therefore the distance from the vertex to the focus is $a - c$ and the distance from the vertex to the right directrix is $\frac{a^2}{c} - c$. This gives the eccentricity as

$$e = \frac{a-c}{\frac{a^2}{c} - a} = \frac{c(a-c)}{a^2 - ac} = \frac{c(a-c)}{a(a-c)} = \frac{c}{a}.$$

Since $c < a$, this step proves that the eccentricity of an ellipse is less than 1. The directrices of a horizontal hyperbola are also located at $x = \pm\frac{a^2}{c}$, and a similar calculation shows that the eccentricity of a hyperbola is also $e = \frac{c}{a}$. However in this case we have $c > a$, so the eccentricity of a hyperbola is greater than 1.

Example 6.5.4: Determining Eccentricity of a Conic Section

Determine the eccentricity of the ellipse described by the equation

$$\frac{(x-3)^2}{16} + \frac{(y+2)^2}{25} = 1.$$

Solution

From the equation we see that $a = 5$ and $b = 4$. The value of c can be calculated using the equation $a^2 = b^2 + c^2$ for an ellipse. Substituting the values of a and b and solving for c gives $c = 3$. Therefore the eccentricity of the ellipse is $e = \frac{c}{a} = \frac{3}{5} = 0.6$.

Exercise 6.5.4

Determine the eccentricity of the hyperbola described by the equation

$$\frac{(y-3)^2}{49}-\frac{(x+2)^2}{25}=1.$$

Hint

First find the values of a and b, then determine c using the equation $c^2=a^2+b^2$.

Answer

$e=\frac{c}{a}=\frac{\sqrt{74}}{7}\approx 1.229$

Polar Equations of Conic Sections

Sometimes it is useful to write or identify the equation of a conic section in polar form. To do this, we need the concept of the focal parameter. The **focal parameter** of a conic section p is defined as the distance from a focus to the nearest directrix. The following table gives the focal parameters for the different types of conics, where a is the length of the semi-major axis (i.e., half the length of the major axis), c is the distance from the origin to the focus, and e is the eccentricity. In the case of a parabola, a represents the distance from the vertex to the focus.

Table 6.5.1: Eccentricities and Focal Parameters of the Conic Sections

Conic	e	p
Ellipse	$0<e<1$	$\frac{a^2-c^2}{c}=\frac{a(1-e^2)}{c}$
Parabola	$e=1$	$2a$
Hyperbola	$e>1$	$\frac{c^2-a^2}{c}=\frac{a(e^2-1)}{c}$

Using the definitions of the focal parameter and eccentricity of the conic section, we can derive an equation for any conic section in polar coordinates. In particular, we assume that one of the foci of a given conic section lies at the pole. Then using the definition of the various conic sections in terms of distances, it is possible to prove the following theorem.

Polar Equation of Conic Sections

The polar equation of a conic section with focal parameter p is given by

$$r=\frac{ep}{1\pm e\cos\theta} \text{ or } r=\frac{ep}{1\pm e\sin\theta}.$$

In the equation on the left, the major axis of the conic section is horizontal, and in the equation on the right, the major axis is vertical. To work with a conic section written in polar form, first make the constant term in the denominator equal to 1. This can be done by dividing both the numerator and the denominator of the fraction by the constant that appears in front of the plus or minus in the denominator. Then the coefficient of the sine or cosine in the denominator is the eccentricity. This value identifies the conic. If cosine appears in the denominator, then the conic is horizontal. If sine appears, then the conic is vertical. If both appear then the axes are rotated. The center of the conic is not necessarily at the origin. The center is at the origin only if the conic is a circle (i.e., $e=0$).

Example 6.5.5: Graphing a Conic Section in Polar Coordinates

Identify and create a graph of the conic section described by the equation

$$r=\frac{3}{1+2\cos\theta}.$$

Solution

The constant term in the denominator is 1, so the eccentricity of the conic is 2. This is a hyperbola. The focal parameter p can be calculated by using the equation $ep = 3$. Since $e = 2$, this gives $p = \frac{3}{2}$. The cosine function appears in the denominator, so the hyperbola is horizontal. Pick a few values for θ and create a table of values. Then we can graph the hyperbola (Figure 6.5.13).

θ	r	θ	r
0	1	π	-3
$\frac{\pi}{4}$	$\frac{3}{1+\sqrt{2}} \approx 1.2426$	$\frac{5\pi}{4}$	$\frac{3}{1-\sqrt{2}} \approx -7.2426$
$\frac{\pi}{2}$	3	$\frac{3\pi}{2}$	3
$\frac{3\pi}{4}$	$\frac{3}{1-\sqrt{2}} \approx -7.2426$	$\frac{7\pi}{4}$	$\frac{3}{1+\sqrt{2}} \approx 1.2426$

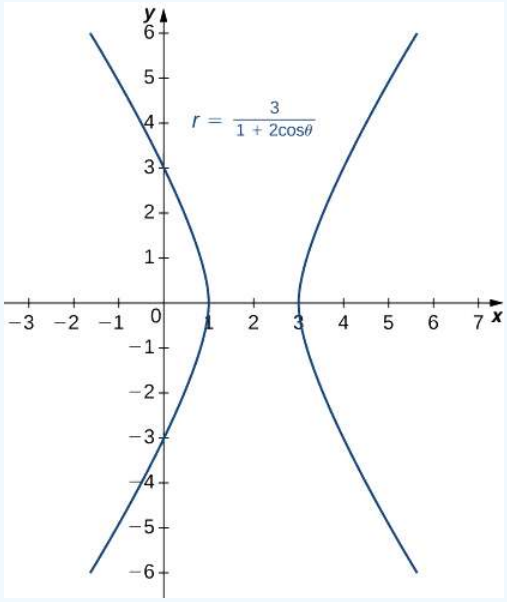


Figure 6.5.13: Graph of the hyperbola described in Example 6.5.5.

Exercise 6.5.5

Identify and create a graph of the conic section described by the equation

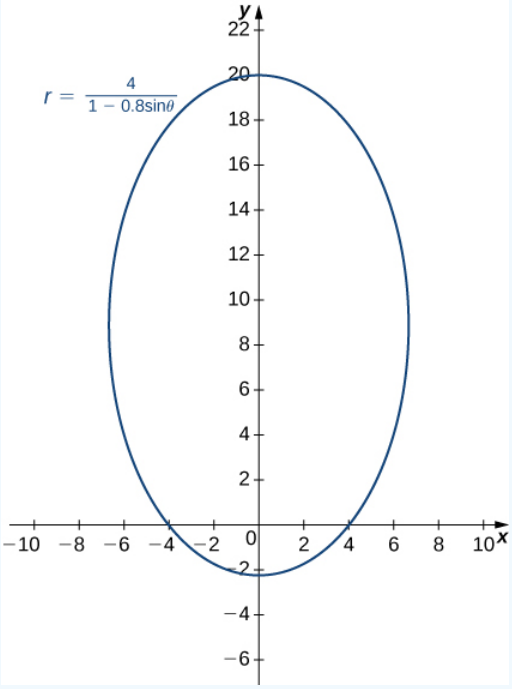
$$r = \frac{4}{1 - 0.8\sin\theta}.$$

Hint

First find the values of *e* and *p*, and then create a table of values.

Answer

Here $e = 0.8$ and $p = 5$. This conic section is an ellipse.



General Equations of Degree Two

A general equation of degree two can be written in the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

The graph of an equation of this form is a conic section. If $B \neq 0$ then the coordinate axes are rotated. To identify the conic section, we use the discriminant of the conic section $4AC - B^2$.

Identifying the Conic Section

One of the following cases must be true:

1. $4AC - B^2 > 0$. If so, the graph is an ellipse.
2. $4AC - B^2 = 0$. If so, the graph is a parabola.
3. $4AC - B^2 < 0$. If so, the graph is a hyperbola.

The simplest example of a second-degree equation involving a cross term is $xy = 1$. This equation can be solved for y to obtain $y = \frac{1}{x}$. The graph of this function is called a rectangular hyperbola as shown.

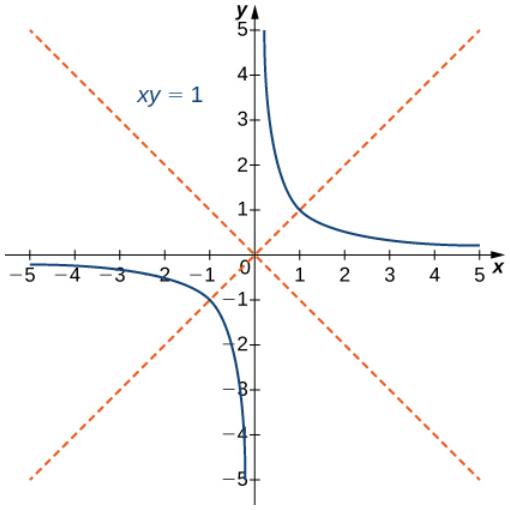


Figure 6.5.14: Graph of the equation $xy = 1$; The red lines indicate the rotated axes.

The asymptotes of this hyperbola are the x and y coordinate axes. To determine the angle \theta of rotation of the conic section, we use the formula $\cot 2\theta = \frac{A-C}{B}$. In this case $A = C = 0$ and $B = 1$, so $\cot 2\theta = (0-0)/1 = 0$ and $\theta = 45°$. The method for graphing a conic section with rotated axes involves determining the coefficients of the conic in the rotated coordinate system. The new coefficients are labeled A', B', C', D', E', and F', and are given by the formulas

$$A' = A\cos^2\theta + B\cos\theta\sin\theta + C\sin^2\theta \tag{6.5.10}$$
$$B' = 0 \tag{6.5.11}$$
$$C' = A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta \tag{6.5.12}$$
$$D' = D\cos\theta + E\sin\theta \tag{6.5.13}$$
$$E' = -D\sin\theta + E\cos\theta \tag{6.5.14}$$
$$F' = F. \tag{6.5.15}$$

Procedure: graphing a rotated conic

The procedure for graphing a rotated conic is the following:

1. Identify the conic section using the discriminant $4AC - B^2$.
2. Determine θ using the formula

$$\cot 2\theta = \frac{A-C}{B}. \tag{6.5.16}$$

3. Calculate A', B', C', D', E',and F'.
4. Rewrite the original equation using A', B', C', D', E',and F'.
5. Draw a graph using the rotated equation.

Example 6.5.6: Identifying a Rotated Conic

Identify the conic and calculate the angle of rotation of axes for the curve described by the equation

$$13x^2 - 6\sqrt{3}xy + 7y^2 - 256 = 0.$$

Solution

In this equation, $A = 13, B = -6\sqrt{3}, C = 7, D = 0, E = 0$, and $F = -256$. The discriminant of this equation is

$$4AC - B^2 = 4(13)(7) - (-6\sqrt{3})^2 = 364 - 108 = 256.$$

Therefore this conic is an ellipse.

To calculate the angle of rotation of the axes, use Equation 6.5.16

$$\cot 2\theta = \frac{A-C}{B}.$$

This gives

$$\cot 2\theta = \frac{A-C}{B} = \frac{13-7}{-6\sqrt{3}} = -\frac{\sqrt{3}}{3}.$$

Therefore $2\theta = 120^o$ and $\theta = 60^o$, which is the angle of the rotation of the axes.

To determine the rotated coefficients, use the formulas given above:

$$\begin{aligned} A' &= A\cos^2\theta + B\cos\theta\sin\theta + C\sin^2\theta \\ &= 13\cos^2 60 + (-6\sqrt{3})\cos 60\sin 60 + 7\sin^2 60 \\ &= 13(\tfrac{1}{2})^2 - 6\sqrt{3}(\tfrac{1}{2})(\tfrac{\sqrt{3}}{2}) + 7(\tfrac{\sqrt{3}}{2})^2 \\ &= 4, \end{aligned}$$

$$B' = 0$$

$$\begin{aligned} C' &= A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta \\ &= 13\sin^2 60 + (6\sqrt{3})\sin 60\cos 60 + 7\cos^2 60 \\ &= 13(\tfrac{\sqrt{3}}{2})^2 + 6\sqrt{3}(\tfrac{\sqrt{3}}{2})(\tfrac{1}{2}) + 7(\tfrac{1}{2})^2 \\ &= 16, \end{aligned}$$

$$\begin{aligned} D' &= D\cos\theta + E\sin\theta \\ &= (0)\cos 60 + (0)\sin 60 \\ &= 0, \end{aligned}$$

$$\begin{aligned} E' &= -D\sin\theta + E\cos\theta \\ &= -(0)\sin 60 + (0)\cos 60 \\ &= 0 \end{aligned}$$

$$\begin{aligned} F' &= F \\ &= -256. \end{aligned}$$

The equation of the conic in the rotated coordinate system becomes

$$4(x')^2 + 16(y')^2 = 256$$

$$\frac{(x')^2}{64} + \frac{(y')^2}{16} = 1.$$

A graph of this conic section appears as follows.

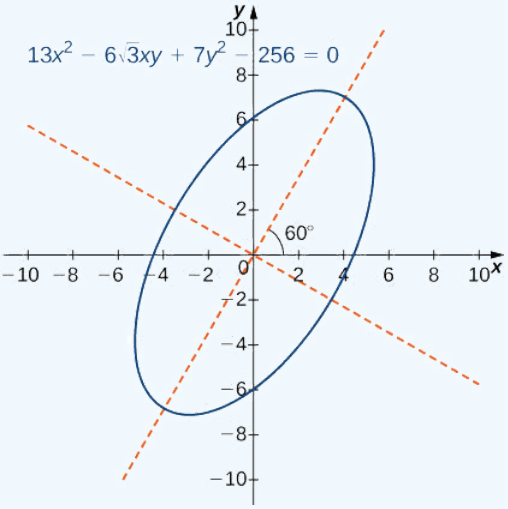


Figure 6.5.15: Graph of the ellipse described by the equation $13x^2 - 6\sqrt{3}xy + 7y^2 - 256 = 0$. The axes are rotated 60°. The red dashed lines indicate the rotated axes.

? Exercise 6.5.6

Identify the conic and calculate the angle of rotation of axes for the curve described by the equation

$$3x^2 + 5xy - 2y^2 - 125 = 0.$$

Hint

Follow steps 1 and 2 of the five-step method outlined above

Answer

The conic is a hyperbola and the angle of rotation of the axes is $\theta = 22.5°$.

Key Concepts

- The equation of a vertical parabola in standard form with given focus and directrix is $y = \frac{1}{4p}(x-h)^2 + k$ where p is the distance from the vertex to the focus and (h, k) are the coordinates of the vertex.
- The equation of a horizontal ellipse in standard form is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ where the center has coordinates (h, k), the major axis has length $2a$, the minor axis has length $2b$, and the coordinates of the foci are $(h \pm c, k)$, where $c^2 = a^2 - b^2$.
- The equation of a horizontal hyperbola in standard form is $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ where the center has coordinates (h, k), the vertices are located at $(h \pm a, k)$, and the coordinates of the foci are $(h \pm c, k)$, where $c^2 = a^2 + b^2$.
- The eccentricity of an ellipse is less than 1, the eccentricity of a parabola is equal to 1, and the eccentricity of a hyperbola is greater than 1. The eccentricity of a circle is 0.
- The polar equation of a conic section with eccentricity e is $r = \frac{ep}{1 \pm e\cos\theta}$ or $r = \frac{ep}{1 \pm e\sin\theta}$, where p represents the focal parameter.
- To identify a conic generated by the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$,first calculate the discriminant $D = 4AC - B^2$. If $D > 0$ then the conic is an ellipse, if $D = 0$ then the conic is a parabola, and if $D < 0$ then the conic is a hyperbola.

Glossary

conic section

a conic section is any curve formed by the intersection of a plane with a cone of two nappes

directrix

a directrix (plural: directrices) is a line used to construct and define a conic section; a parabola has one directrix; ellipses and hyperbolas have two

discriminant

the value $4AC - B^2$, which is used to identify a conic when the equation contains a term involving xy, is called a discriminant

focus

a focus (plural: foci) is a point used to construct and define a conic section; a parabola has one focus; an ellipse and a hyperbola have two

eccentricity

the eccentricity is defined as the distance from any point on the conic section to its focus divided by the perpendicular distance from that point to the nearest directrix

focal parameter

the focal parameter is the distance from a focus of a conic section to the nearest directrix

general form

an equation of a conic section written as a general second-degree equation

major axis

the major axis of a conic section passes through the vertex in the case of a parabola or through the two vertices in the case of an ellipse or hyperbola; it is also an axis of symmetry of the conic; also called the transverse axis

minor axis

the minor axis is perpendicular to the major axis and intersects the major axis at the center of the conic, or at the vertex in the case of the parabola; also called the conjugate axis

nappe

a nappe is one half of a double cone

standard form

an equation of a conic section showing its properties, such as location of the vertex or lengths of major and minor axes

vertex

a vertex is an extreme point on a conic section; a parabola has one vertex at its turning point. An ellipse has two vertices, one at each end of the major axis; a hyperbola has two vertices, one at the turning point of each branch

---

This page titled 6.5: Conic Sections is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Gilbert Strang & Edwin “Jed” Herman via source content that was edited to the style and standards of the LibreTexts platform.

- **11.5: Conic Sections** by Edwin “Jed” Herman, Gilbert Strang is licensed CC BY-NC-SA 4.0. Original source: https://openstax.org/details/books/calculus-volume-1.

6.5E: Exercises

In exercises 1 - 8, determine the equation of the parabola using the information given.

1) Focus $(4,0)$ and directrix $x=-4$

Answer

$y^2=16x$

2) Focus $(0,-3)$ and directrix $y=3$

3) Focus $(0,0.5)$ and directrix $y=-0.5$

Answer

$x^2=2y$

4) Focus $(2,3)$ and directrix $x=-2$

5) Focus $(0,2)$ and directrix $y=4$

Answer

$x^2=-4(y-3)$

6) Focus $(-1,4)$ and directrix $x=5$

7) Focus $(-3,5)$ and directrix $y=1$

Answer

$(x+3)^2=8(y-3)$

8) Focus $\left(\frac{5}{2},-4\right)$ and directrix $x=\frac{7}{2}$

In exercises 9 - 16, determine the equation of the ellipse using the information given.

9) Endpoints of major axis at $(4,0),\ (-4,0)$ and foci located at $(2,0),\ (-2,0)$

Answer

$\frac{x^2}{16}+\frac{y^2}{12}=1$

10) Endpoints of major axis at $(0,5),\ (0,-5)$ and foci located at $(0,3),\ (0,-3)$

11) Endpoints of major axis at $(0,2),\ (0,-2)$ and foci located at $(3,0),\ (-3,0)$

Answer

$\frac{x^2}{13}+\frac{y^2}{4}=1$

12) Endpoints of major axis at $(-3,3),\ (7,3)$ and foci located at $(-2,3),\ (6,3)$

13) Endpoints of major axis at $(-3,5),\ (-3,-3)$ and foci located at $(-3,3),\ (-3,-1)$

Answer

$\frac{(y-1)^2}{16}+\frac{(x+3)^2}{12}=1$

14) Endpoints of major axis at $(0,0),\ (0,4)$ and foci located at $(5,2),\ (-5,2)$

15) Foci located at $(2,0),\ (-2,0)$ and eccentricity of $\frac{1}{2}$

Answer

$$\frac{x^2}{16}+\frac{y^2}{12}=1$$

16) Foci located at $(0,-3),\ (0,3)$ and eccentricity of $\frac{3}{4}$

In exercises 17 - 24, determine the equation of the hyperbola using the information given.

17) Vertices located at $(5,0),\ (-5,0)$ and foci located at $(6,0),\ (-6,0)$

Answer

$\frac{x^2}{25}-\frac{y^2}{11}=1$

18) Vertices located at $(0,2),\ (0,-2)$ and foci located at $(0,3),\ (0,-3)$

19) Endpoints of the conjugate axis located at $(0,3),\ (0,-3)$ and foci located $(4,0),\ (-4,0)$

Answer

$$\frac{x^2}{7}-\frac{y^2}{9}=1$$

20) Vertices located at $(0,1),\ (6,1)$ and focus located at $(8,1)$

21) Vertices located at $(-2,0),\ (-2,-4)$ and focus located at $(-2,-8)$

Answer

$$\frac{(y+2)^2}{4}-\frac{(x+2)^2}{32}=1$$

22) Endpoints of the conjugate axis located at $(3,2),\ (3,4)$ and focus located at $(3,7)$

23) Foci located at $(6,-0),\ (6,0)$ and eccentricity of 3

Answer

$$\frac{x^2}{4}-\frac{y^2}{32}=1$$

24) $(0,10),\ (0,-10)$ and eccentricity of 2.5

In exercises 25 - 30, consider the following polar equations of conics. Determine the eccentricity and identify the conic.

25) $r=\frac{-1}{1+\cos\theta}$

Answer

$e=1$, parabola

26) $r=\frac{8}{2-\sin\theta}$

27) $r=\frac{5}{2+\sin\theta}$

Answer

$e=\frac{1}{2}$, ellipse

28) $r=\frac{5}{-1+2\sin\theta}$

29) $r=\frac{3}{2-6\sin\theta}$

Answer

$e=3$, hyperbola

30) $r=\frac{3}{-4+3\sin\theta}$

In exercises 31 - 34, find a polar equation of the conic with focus at the origin and eccentricity and directrix as given.

31) Directrix: $x = 4;\ e = \frac{1}{5}$

Answer

$$r = \frac{4}{5 + \cos\theta}$$

32) Directrix: $x = -4;\ e = 5$

33) Directrix: $y = 2;\ e = 2$

Answer

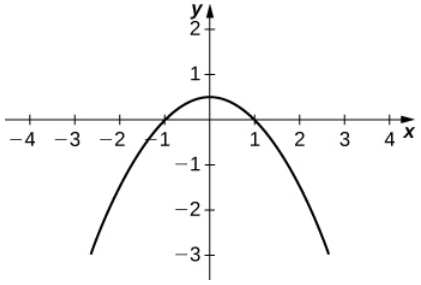
$$r = \frac{4}{1 + 2\sin\theta}$$

34) Directrix: $y = -2;\ e = \frac{1}{2}$

In exercises 35 - 51, sketch the graph of each conic.

35) $r = \frac{1}{1 + \sin\theta}$

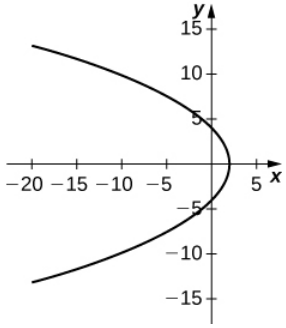
Answer



36) $r = \frac{1}{1 - \cos\theta}$

37) $r = \frac{4}{1 + \cos\theta}$

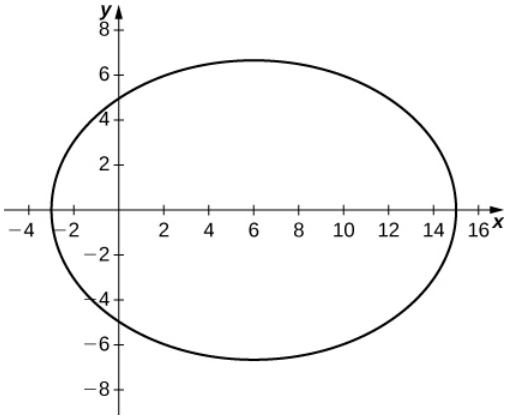
Answer



38) $r = \frac{10}{5 + 4\sin\theta}$

39) $r = \frac{15}{3 - 2\cos\theta}$

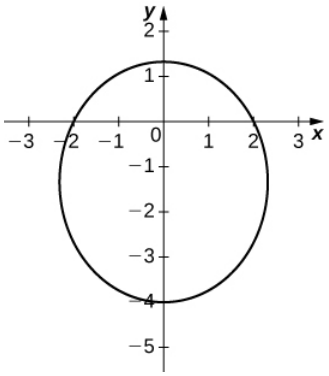
Answer



40) $r = \dfrac{32}{3 + 5\sin\theta}$

41) $r(2 + \sin\theta) = 4$

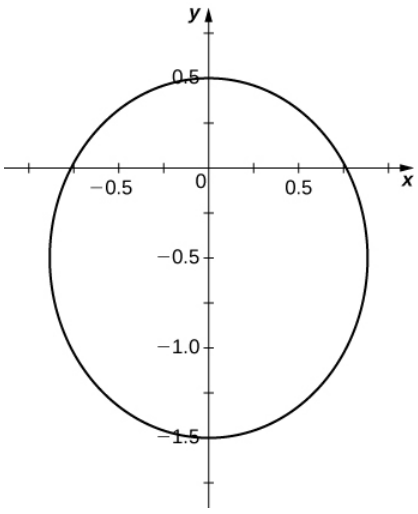
Answer



42) $r = \dfrac{3}{2 + 6\sin\theta}$

43) $r = \dfrac{3}{-4 + 2\sin\theta}$

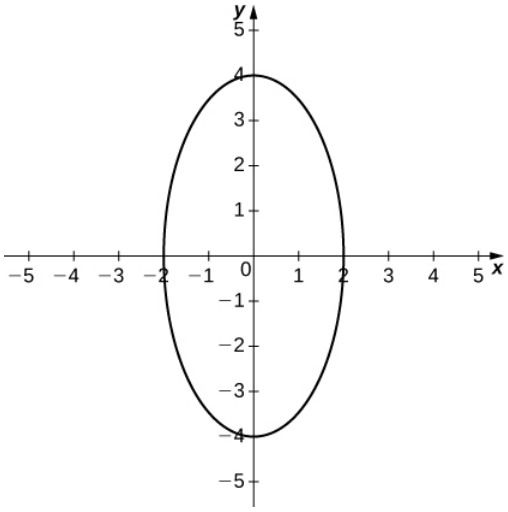
Answer



44) $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

45) $\frac{x^2}{4}+\frac{y^2}{16}=1$

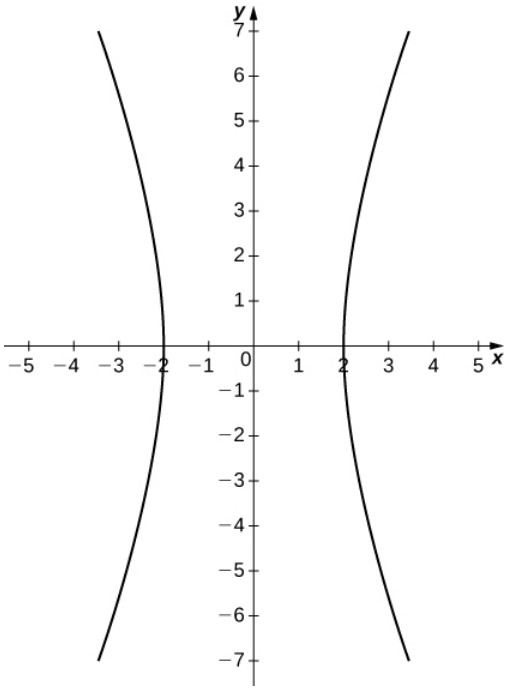
Answer



46) $4x^2+9y^2=36$

47) $25x^2-4y^2=100$

Answer

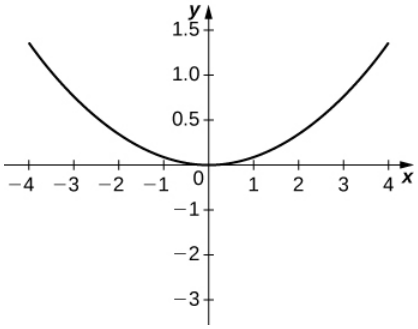


48) $\frac{x^2}{16}-\frac{y^2}{9}=1$

49) $x^2=12y$

Answer

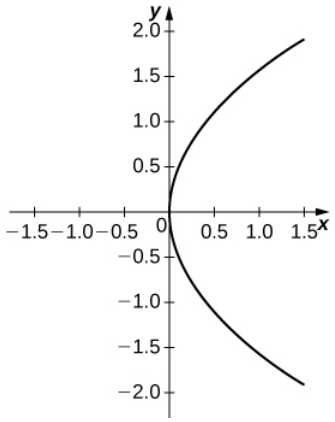
https://math.libretexts.org/@go/page/168470



50) $y^2 = 20x$

51) $12x = 5y^2$

Answer



For the equations in exercises 52 - 57, determine which of the conic sections is described.

52) $xy = 4$

53) $x^2 + 4xy - 2y^2 - 6 = 0$

Answer

Hyperbola

54) $x^2 + 2\sqrt{3}xy + 3y^2 - 6 = 0$

55) $x^2 - xy + y^2 - 2 = 0$

Answer

Ellipse

56) $34x^2 - 24xy + 41y^2 - 25 = 0$

57) $52x^2 - 72xy + 73y^2 + 40x + 30y - 75 = 0$

Answer

Ellipse

58) The mirror in an automobile headlight has a parabolic cross section, with the lightbulb at the focus. On a schematic, the equation of the parabola is given as $x^2 = 4y$. At what coordinates should you place the lightbulb?

59) A satellite dish is shaped like a paraboloid of revolution. The receiver is to be located at the focus. If the dish is 12 feet across at its opening and 4 feet deep at its center, where should the receiver be placed?

Answer

At the point 2.25 feet above the vertex.

60) Consider the satellite dish of the preceding problem. If the dish is 8 feet across at the opening and 2 feet deep, where should we place the receiver?

61) A searchlight is shaped like a paraboloid of revolution. A light source is located 1 foot from the base along the axis of symmetry. If the opening of the searchlight is 3 feet across, find the depth.

Answer

0.5625 feet

62) Whispering galleries are rooms designed with elliptical ceilings. A person standing at one focus can whisper and be heard by a person standing at the other focus because all the sound waves that reach the ceiling are reflected to the other person. If a whispering gallery has a length of 120 feet and the foci are located 30 feet from the center, find the height of the ceiling at the center.

63) A person is standing 8 feet from the nearest wall in a whispering gallery. If that person is at one focus and the other focus is 80 feet away, what is the length and the height at the center of the gallery?

Answer

Length is 96 feet and height is approximately 26.53 feet.

In exercises 64 - 67, determine the polar equation form of the orbit given the length of the major axis and eccentricity for the orbits of the comets or planets. Distance is given in astronomical units (AU).

64) Halley's Comet: length of major axis = 35.88, eccentricity = 0.967

65) Hale-Bopp Comet: length of major axis = 525.91, eccentricity = 0.995

Answer

$$r = \frac{2.616}{1 + 0.995 \cos \theta}$$

66) Mars: length of major axis = 3.049, eccentricity = 0.0934

67) Jupiter: length of major axis = 10.408, eccentricity = 0.0484

Answer

$$r = \frac{5.192}{1 + 0.0484 \cos \theta}$$

This page titled 6.5E: Exercises is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Gilbert Strang & Edwin "Jed" Herman via source content that was edited to the style and standards of the LibreTexts platform.

- **11.5E: Exercises for Section 11.5** by Edwin "Jed" Herman, Gilbert Strang is licensed CC BY-NC-SA 4.0. Original source: https://openstax.org/details/books/calculus-volume-1.

6.6: Chapter 6 Review Exercises

***True or False?* Justify your answer with a proof or a counterexample.**

1) The rectangular coordinates of the point $\left(4, \frac{5\pi}{6}\right)$ are $\left(2\sqrt{3}, -2\right)$.

2) The equations $x = \cosh(3t),\ y = 2\sinh(3t)$ represent a hyperbola.

Answer

True

3) The arc length of the spiral given by $r = \dfrac{\theta}{2}$ for $0 \le \theta \le 3\pi$ is $\frac{9}{4}\pi^3$ units.

4) Given $x = f(t)$ and $y = g(t)$, if $\dfrac{dx}{dy} = \dfrac{dy}{dx}$, then $f(t) = g(t) + C$, where C is a constant.

Answer

False. Imagine $y = t + 1,\ x = -t + 1$.

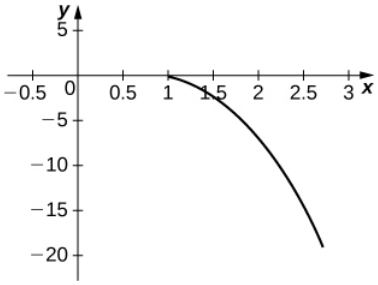
In exercises 5 -8, sketch the parametric curve and eliminate the parameter to find the Cartesian equation of the curve.

5) $x = 1 + t,\ y = t^2 - 1, \quad -1 \le t \le 1$

6) $x = e^t,\ y = 1 - e^{3t}, \quad 0 \le t \le 1$

Answer

$y = 1 - x^3$

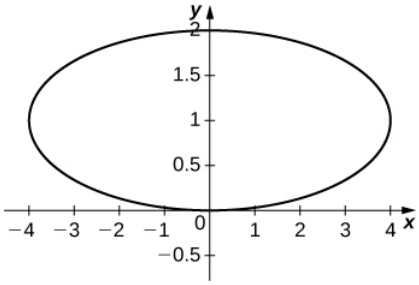


7) $x = \sin\theta,\ y = 1 - \csc\theta, \quad 0 \le \theta \le 2\pi$

8) $x = 4\cos\phi,\ y = 1 - \sin\phi, \quad 0 \le \phi \le 2\pi$

Answer

$\dfrac{x^2}{16} + (y-1)^2 = 1$



In exercises 9 - 10, sketch the polar curve and determine what type of symmetry exists, if any.

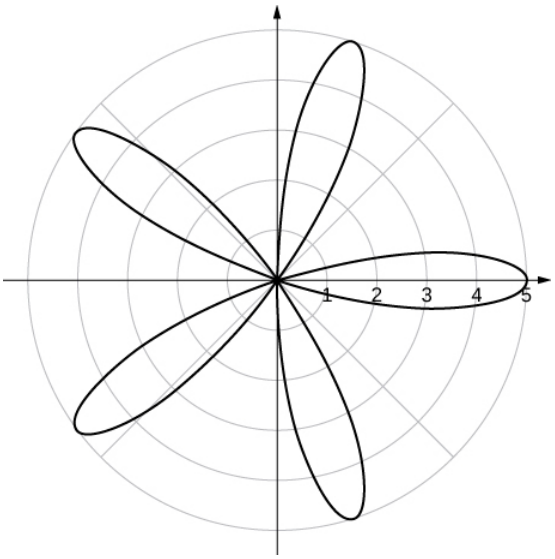
9) $r = 4\sin\left(\frac{\theta}{3}\right)$

https://math.libretexts.org/@go/page/168471

10) $r = 5\cos(5\theta)$

Answer

Symmetric about polar axis



In exercises 11 - 12, find the polar equation for the curve given as a Cartesian equation.

11) $x + y = 5$

12) $y^2 = 4 + x^2$

Answer

$$r^2 = \frac{4}{\sin^2\theta - \cos^2\theta}$$

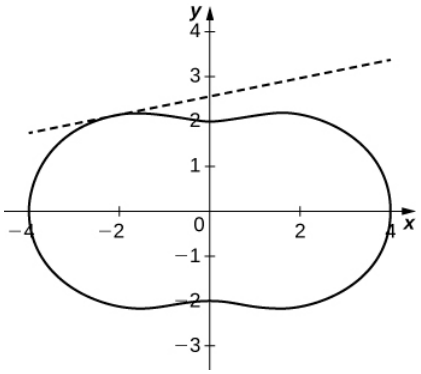
In exercises 13 - 14, find the equation of the tangent line to the given curve. Graph both the function and its tangent line.

13) $x = \ln(t),\ y = t^2 - 1,\ t = 1$

14) $r = 3 + \cos(2\theta),\ \theta = \frac{3\pi}{4}$

Answer

$y = \frac{3\sqrt{2}}{2} + \frac{1}{5}\left(x + \frac{3\sqrt{2}}{2}\right)$



15) Find $\frac{dy}{dx}$, $\frac{dx}{dy}$, and $\frac{d^2x}{dy^2}$ of $y = (2 + e^{-t}),\ x = 1 - \sin t$

In exercises 16 -17, find the area of the region.

16) $x = t^2,\ y = \ln(t),\quad 0 \le t \le e$

Answer

$\frac{e^2}{2}$ units^2

17) $r = 1 - \sin\theta$ in the first quadrant

In exercises 18 - 19, find the arc length of the curve over the given interval.

18) $x = 3t + 4,\ y = 9t - 2,\quad 0 \le t \le 3$

Answer

$9\sqrt{10}$ units

19) $r = 6\cos\theta,\quad 0 \le \theta \le 2\pi$. Check your answer by geometry.

In exercises 20 - 22, find the Cartesian equation describing the given shapes.

20) A parabola with focus $(2, -5)$ and directrix $x = 6$

Answer

$(y+5)^2 = -8x + 32$

21) An ellipse with a major axis length of 10 and foci at $(-7, 2)$ and $(1, 2)$

22) A hyperbola with vertices at $(3, -2)$ and $(-5, -2)$ and foci at $(-2, -6)$ and $(-2, 4)$

Answer

$\frac{(y+1)^2}{16} - \frac{(x+2)^2}{9} = 1$

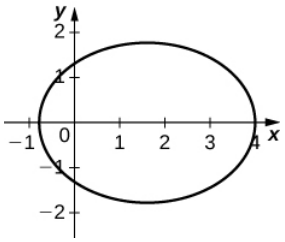
In exercises 23 - 25, determine the eccentricity and identify the conic. Sketch the conic.

23) $r = \frac{6}{1 + 3\cos\theta}$

24) $r = \frac{4}{3 - 2\cos\theta}$

Answer

$e = \frac{2}{3}$, ellipse



25) $r = \frac{7}{5 - 5\cos\theta}$

26) Determine the Cartesian equation describing the orbit of Pluto, the most eccentric orbit around the Sun. The length of the major axis is 39.26 AU and minor axis is 38.07 AU. What is the eccentricity?

Answer

$\frac{y^2}{19.03^2} + \frac{x^2}{19.63^2} = 1,\quad e = 0.2447$

27) The C/1980 E1 comet was observed in 1980. Given an eccentricity of 1.057 and a perihelion (point of closest approach to the Sun) of 3.364 AU, find the Cartesian equations describing the comet's trajectory. Are we guaranteed to see this comet again? (Hint: Consider the Sun at point $(0,0)$.)

This page titled 6.6: Chapter 6 Review Exercises is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Gilbert Strang & Edwin "Jed" Herman via source content that was edited to the style and standards of the LibreTexts platform.

- **11.R: Chapter 11 Review Exercises** by Edwin "Jed" Herman, Gilbert Strang is licensed CC BY-NC-SA 4.0. Original source: https://openstax.org/details/books/calculus-volume-1.

CHAPTER OVERVIEW

7: Introduction to Differential Equations

Many real-world phenomena can be modeled mathematically by using differential equations. Population growth, radioactive decay, predator-prey models, and spring-mass systems are four examples of such phenomena. In this chapter we study some of these applications. A goal of this chapter is to develop solution techniques for different types of differential equations. As the equations become more complicated, the solution techniques also become more complicated, and in fact an entire course could be dedicated to the study of these equations. In this chapter we study several types of differential equations and their corresponding methods of solution.

7.1: Basics of Differential Equations

7.1E: Exercises for Section 8.1

7.2: Direction Fields and Numerical Methods

7.2E: Exercises for Section 8.2

7.3: Separable Equations

7.3E: Exercises for Section 8.3

7.4: The Logistic Equation

7.4E: Exercises for Section 8.4

7.5: First-order Linear Equations

7.5E: Exercises for Section 8.5

7.6: Chapter 7 Review Exercises

This page titled 7: Introduction to Differential Equations is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Gilbert Strang & Edwin “Jed” Herman (OpenStax) via source content that was edited to the style and standards of the LibreTexts platform.

7.1: Basics of Differential Equations

Learning Objectives

- Identify the order of a differential equation.
- Explain what is meant by a solution to a differential equation.
- Distinguish between the general solution and a particular solution of a differential equation.
- Identify an initial-value problem.
- Identify whether a given function is a solution to a differential equation or an initial-value problem.

Calculus is the mathematics of change, and rates of change are expressed by derivatives. Thus, one of the most common ways to use calculus is to set up an equation containing an unknown function $y = f(x)$ and its derivative, known as a differential equation. Solving such equations often provides information about how quantities change and frequently provides insight into how and why the changes occur.

Techniques for solving differential equations can take many different forms, including direct solution, use of graphs, or computer calculations. We introduce the main ideas in this chapter and describe them in a little more detail later in the course. In this section we study what differential equations are, how to verify their solutions, some methods that are used for solving them, and some examples of common and useful equations.

General Differential Equations

Consider the equation $y' = 3x^2$, which is an example of a differential equation because it includes a derivative. There is a relationship between the variables x and y : y is an unknown function of x. Furthermore, the left-hand side of the equation is the derivative of y. Therefore we can interpret this equation as follows: Start with some function $y = f(x)$ and take its derivative. The answer must be equal to $3x^2$. What function has a derivative that is equal to $3x^2$? One such function is $y = x^3$, so this function is considered a *solution* to a differential equation.

Definition: differential equation

A *differential equation* is an equation involving an unknown function $y = f(x)$ and one or more of its derivatives. A solution to a differential equation is a function $y = f(x)$ that satisfies the differential equation when f and its derivatives are substituted into the equation.

Go to this website to explore more on this topic.

Some examples of differential equations and their solutions appear in Table 7.1.1.

Table 7.1.1: Examples of Differential Equations and Their Solutions

Equation	Solution
$y' = 2x$	$y = x^2$
$y' + 3y = 6x + 11$	$y = e^{-3x} + 2x + 3$
$y'' - 3y' + 2y = 24e^{-2x}$	$y = 3e^x - 4e^{2x} + 2e^{-2x}$

Note that a solution to a differential equation is not necessarily unique, primarily because the derivative of a constant is zero. For example, $y = x^2 + 4$ is also a solution to the first differential equation in Table 7.1.1. We will return to this idea a little bit later in this section. For now, let's focus on what it means for a function to be a solution to a differential equation.

Example 7.1.1: Verifying Solutions of Differential Equations

Verify that the function $y = e^{-3x} + 2x + 3$ is a solution to the differential equation $y' + 3y = 6x + 11$.

Solution

To verify the solution, we first calculate y' using the chain rule for derivatives. This gives $y' = -3e^{-3x} + 2$. Next we substitute y and y' into the left-hand side of the differential equation:

$$(-3e^{-2x} + 2) + 3(e^{-2x} + 2x + 3).$$

The resulting expression can be simplified by first distributing to eliminate the parentheses, giving

$$-3e^{-2x} + 2 + 3e^{-2x} + 6x + 9.$$

Combining like terms leads to the expression $6x + 11$, which is equal to the right-hand side of the differential equation. This result verifies that $y = e^{-3x} + 2x + 3$ is a solution of the differential equation.

Exercise 7.1.1

Verify that $y = 2e^{3x} - 2x - 2$ is a solution to the differential equation $y' - 3y = 6x + 4$.

Hint

First calculate y' then substitute both y' and y into the left-hand side.

It is convenient to define characteristics of differential equations that make it easier to talk about them and categorize them. The most basic characteristic of a differential equation is its order.

Definition: order of a differential equation

The *order of a differential equation* is the highest order of any derivative of the unknown function that appears in the equation.

Example 7.1.2: Identifying the Order of a Differential Equation

The highest derivative in the equation is y',

What is the order of each of the following differential equations?

a. $y' - 4y = x^2 - 3x + 4$
b. $x^2y''' - 3xy'' + xy' - 3y = \sin x$
c. $\frac{4}{x}y^{(4)} - \frac{6}{x^2}y'' + \frac{12}{x^4}y = x^3 - 3x^2 + 4x - 12$

Solution

a. The highest derivative in the equation is y',so the order is 1.
b. The highest derivative in the equation is y''', so the order is 3.
c. The highest derivative in the equation is $y^{(4)}$, so the order is 4.

Exercise 7.1.2

What is the order of the following differential equation?

$$(x^4 - 3x)y^{(5)} - (3x^2 + 1)y' + 3y = \sin x \cos x$$

Hint

What is the highest derivative in the equation?

Answer

5

General and Particular Solutions

We already noted that the differential equation $y' = 2x$ has at least two solutions: $y = x^2$ and $y = x^2 + 4$. The only difference between these two solutions is the last term, which is a constant. What if the last term is a different constant? Will this expression still be a solution to the differential equation? In fact, any function of the form $y = x^2 + C$, where C represents any constant, is a solution as well. The reason is that the derivative of $x^2 + C$ is $2x$, regardless of the value of C. It can be shown that any solution of this differential equation must be of the form $y = x^2 + C$. This is an example of a **general solution** to a differential equation. A graph of some of these solutions is given in Figure 7.1.1. (Note: in this graph we used even integer values for C ranging between -4 and 4. In fact, there is no restriction on the value of C; it can be an integer or not.)

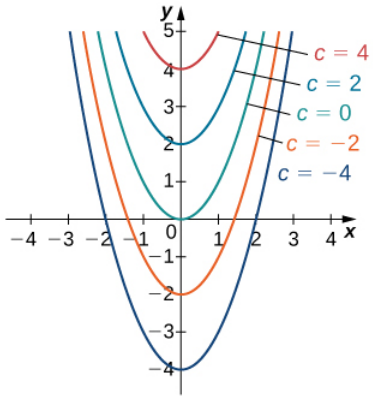


Figure 7.1.1: Family of solutions to the differential equation $y' = 2x$.

In this example, we are free to choose any solution we wish; for example, $y = x^2 - 3$ is a member of the family of solutions to this differential equation. This is called a particular solution to the differential equation. A particular solution can often be uniquely identified if we are given additional information about the problem.

Example 7.1.3: Finding a Particular Solution

Find the particular solution to the differential equation $y' = 2x$ passing through the point $(2, 7)$.

Solution

Any function of the form $y = x^2 + C$ is a solution to this differential equation. To determine the value of C, we substitute the values $x = 2$ and $y = 7$ into this equation and solve for C:

$$\begin{aligned} y &= x^2 + C \\ 7 &= 2^2 + C \\ &= 4 + C \\ C &= 3. \end{aligned}$$

Therefore the particular solution passing through the point $(2, 7)$ is $y = x^2 + 3$.

Exercise 7.1.3

Find the particular solution to the differential equation

$$y' = 4x + 3$$

passing through the point $(1, 7)$, given that $y = 2x^2 + 3x + C$ is a general solution to the differential equation.

Hint

First substitute $x = 1$ and $y = 7$ into the equation, then solve for C.

Answer

$$y = 2x^2 + 3x + 2$$

Initial-Value Problems

Usually a given differential equation has an infinite number of solutions, so it is natural to ask which one we want to use. To choose one solution, more information is needed. Some specific information that can be useful is an *initial value*, which is an ordered pair that is used to find a particular solution.

A differential equation together with one or more initial values is called an *initial-value problem*. The general rule is that the number of initial values needed for an initial-value problem is equal to the order of the differential equation. For example, if we have the differential equation $y' = 2x$, then $y(3) = 7$ is an initial value, and when taken together, these equations form an initial-value problem. The differential equation $y'' - 3y' + 2y = 4e^x$ is second order, so we need two initial values. With initial-value problems of order greater than one, the same value should be used for the independent variable. An example of initial values for this second-order equation would be $y(0) = 2$ and $y'(0) = -1$. These two initial values together with the differential equation form an initial-value problem. These problems are so named because often the independent variable in the unknown function is t, which represents time. Thus, a value of $t = 0$ represents the beginning of the problem.

Example 7.1.4: Verifying a Solution to an Initial-Value Problem

Verify that the function $y = 2e^{-2t} + e^t$ is a solution to the initial-value problem

$$y' - y = -6e^{-2t}, \quad y(0) = 3.$$

Solution

For a function to satisfy an initial-value problem, it must satisfy both the differential equation and the initial condition. To show that y satisfies the differential equation, we start by calculating y'. This gives $y' = -4e^{-2t} + e^t$. Next we substitute both y and y' into the left-hand side of the differential equation and simplify:

$$\begin{aligned} y' - y &= (-4e^{-2t} + e^t) - (2e^{-2t} + e^t) \\ &= -4e^{-2t} + e^t - 2e^{-2t} - e^t = -6e^{-2t}. \end{aligned}$$

This is equal to the right-hand side of the differential equation, so $y = 2e^{-2t} + e^t$ solves the differential equation. Next we calculate $y(0)$:

$$y(0) = 2e^{-2(0)} + e^0 = 2 + 1 = 3.$$

This result verifies the initial value. Therefore the given function satisfies the initial-value problem.

Exercise 7.1.4

Verify that $y = 3e^{2t} + 4\sin t$ is a solution to the initial-value problem

$$y' - 2y = 4\cos t - 8\sin t, y(0) = 3.$$

Hint

First verify that y solves the differential equation. Then check the initial value.

In Example 7.1.4, the initial-value problem consisted of two parts. The first part was the differential equation $y' - y = -6e^{-2t}$, and the second part was the initial value $y(0) = 3$. These two equations together formed the initial-value problem.

The same is true in general. An initial-value problem will consists of two parts: the differential equation and the initial condition. The differential equation has a family of solutions, and the initial condition determines the value of C. The family of solutions to the differential equation in Example 7.1.4 is given by $y = 2e^{-2t} + Ce^t$. This family of solutions is shown in Figure 7.1.2, with the particular solution $y = 2e^{-2t} + e^t$ labeled.

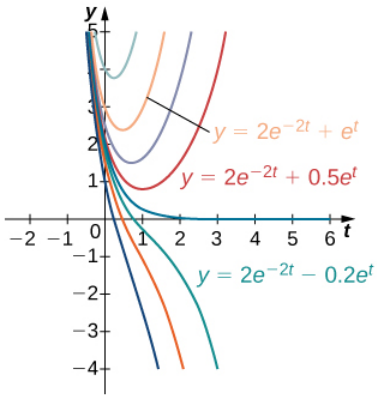


Figure 7.1.2: A family of solutions to the differential equation $y'-y=-6e^{-2t}$. The particular solution $y=2e^{-2t}+e^t$ is labeled.

Example 7.1.5: Solving an Initial-value Problem

Solve the following initial-value problem:

$$y'=3e^x+x^2-4, y(0)=5.$$

Solution

The first step in solving this initial-value problem is to find a general family of solutions. To do this, we find an antiderivative of both sides of the differential equation

$$\int y'\,dx=\int(3e^x+x^2-4)\,dx,$$

namely,

$$y+C_1=3e^x+\tfrac{1}{3}x^3-4x+C_2.$$

We are able to integrate both sides because the y term appears by itself. Notice that there are two integration constants: C_1 and C_2. Solving this equation for y gives

$$y=3e^x+\tfrac{1}{3}x^3-4x+C_2-C_1.$$

Because C_1 and C_2 are both constants, C_2-C_1 is also a constant. We can therefore define $C=C_2-C_1$, which leads to the equation

$$y=3e^x+\tfrac{1}{3}x^3-4x+C.$$

Next we determine the value of C. To do this, we substitute $x=0$ and $y=5$ into this equation and solve for C:

$$\begin{aligned}5&=3e^0+\frac{1}{3}0^3-4(0)+C\\5&=3+C\\C&=2\end{aligned}$$

Now we substitute the value $C=2$ into the general equation. The solution to the initial-value problem is $y=3e^x+\tfrac{1}{3}x^3-4x+2$.

Analysis

The difference between a general solution and a particular solution is that a general solution involves a family of functions, either explicitly or implicitly defined, of the independent variable. The initial value or values determine which particular solution in the family of solutions satisfies the desired conditions.

? Exercise 7.1.5

Solve the initial-value problem

$$y' = x^2 - 4x + 3 - 6e^x, y(0) = 8.$$

Hint

First take the antiderivative of both sides of the differential equation. Then substitute $x = 0$ and $y = 8$ into the resulting equation and solve for C.

Answer

$y = \frac{1}{3}x^3 - 2x^2 + 3x - 6e^x + 14$

Applications

In physics and engineering applications, we often consider the forces acting upon an object, and use this information to understand the resulting motion that may occur. For example, if we start with an object at Earth's surface, the primary force acting upon that object is gravity. Physicists and engineers can use this information, along with Newton's **second law of motion** (in equation form $F = ma$, where F represents force, m represents mass, and a represents acceleration), to derive an equation that can be solved.

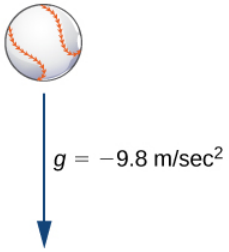


Figure 7.1.3: For a baseball falling in air, the only force acting on it is gravity (neglecting air resistance).

In Figure 7.1.3 we assume that the only force acting on a baseball is the force of gravity. This assumption ignores air resistance. (The force due to air resistance is considered in a later discussion.) The acceleration due to gravity at Earth's surface, g, is approximately $9.8\ \text{m/s}^2$. We introduce a frame of reference, where Earth's surface is at a height of 0 meters. Let $v(t)$ represent the velocity of the object in meters per second. If $v(t) > 0$, the ball is rising, and if $v(t) < 0$, the ball is falling (Figure).



Figure 7.1.4: Possible velocities for the rising/falling baseball.

Our goal is to solve for the velocity $v(t)$ at any time t. To do this, we set up an initial-value problem. Suppose the mass of the ball is m, where m is measured in kilograms. We use Newton's second law, which states that the force acting on an object is equal to its mass times its acceleration ($F = ma$). Acceleration is the derivative of velocity, so $a(t) = v'(t)$. Therefore the force acting on the baseball is given by $F = mv'(t)$. However, this force must be equal to the force of gravity acting on the object, which (again using Newton's second law) is given by $F_g = -mg$, since this force acts in a downward direction. Therefore we obtain the equation $F = F_g$, which becomes $mv'(t) = -mg$. Dividing both sides of the equation by m gives the equation

$$v'(t) = -g.$$

Notice that this differential equation remains the same regardless of the mass of the object.

We now need an initial value. Because we are solving for velocity, it makes sense in the context of the problem to assume that we know the **initial velocity**, or the velocity at time $t = 0$. This is denoted by $v(0) = v_0$.

Example 7.1.6: Velocity of a Moving Baseball

A baseball is thrown upward from a height of 3 meters above Earth's surface with an initial velocity of 10 m/s, and the only force acting on it is gravity. The ball has a mass of 0.15 kg at Earth's surface.

a. Find the velocity $v(t)$ of the basevall at time t.
b. What is its velocity after 2 seconds?

Solution

a. From the preceding discussion, the differential equation that applies in this situation is

$$v'(t) = -g,$$

where $g = 9.8\ \text{m/s}^2$. The initial condition is $v(0) = v_0$, where $v_0 = 10$ m/s. Therefore the initial-value problem is $v'(t) = -9.8\ \text{m/s}^2$, $v(0) = 10$ m/s.

The first step in solving this initial-value problem is to take the antiderivative of both sides of the differential equation. This gives

$$\int v'(t)\,dt = \int -9.8\,dt$$

$$v(t) = -9.8t + C.$$

The next step is to solve for C. To do this, substitute $t = 0$ and $v(0) = 10$:

$$\begin{aligned} v(t) &= -9.8t + C \\ v(0) &= -9.8(0) + C \\ 10 &= C. \end{aligned}$$

Therefore $C = 10$ and the velocity function is given by $v(t) = -9.8t + 10$.

b. To find the velocity after 2 seconds, substitute $t = 2$ into $v(t)$.

$$\begin{aligned} v(t) &= -9.8t + 10 \\ v(2) &= -9.8(2) + 10 \\ v(2) &= -9.6 \end{aligned}$$

The units of velocity are meters per second. Since the answer is negative, the object is falling at a speed of 9.6 m/s.

Exercise 7.1.6

Suppose a rock falls from rest from a height of 100 meters and the only force acting on it is gravity. Find an equation for the velocity $v(t)$ as a function of time, measured in meters per second.

Hint

What is the initial velocity of the rock? Use this with the differential equation in Example 7.1.6 to form an initial-value problem, then solve for $v(t)$.

Answer

$v(t) = -9.8t$

A natural question to ask after solving this type of problem is how high the object will be above Earth's surface at a given point in time. Let $s(t)$ denote the height above Earth's surface of the object, measured in meters. Because velocity is the derivative of position (in this case height), this assumption gives the equation $s'(t) = v(t)$. An initial value is necessary; in this case the initial height of the object works well. Let the initial height be given by the equation $s(0) = s_0$. Together these assumptions give the initial-value problem

$$s'(t) = v(t), s(0) = s_0.$$

If the velocity function is known, then it is possible to solve for the position function as well.

Example 7.1.7: Height of a Moving Baseball

A baseball is thrown upward from a height of 3 meters above Earth's surface with an initial velocity of $10m/s$, and the only force acting on it is gravity. The ball has a mass of 0.15 kilogram at Earth's surface.

a. Find the position $s(t)$ of the baseball at time t.
b. What is its height after 2 seconds?

Solution

We already know the velocity function for this problem is $v(t) = -9.8t + 10$. The initial height of the baseball is 3 meters, so $s_0 = 3$. Therefore the initial-value problem for this example is

To solve the initial-value problem, we first find the antiderivatives:

$$\int s'(t)\,dt = \int (-9.8t + 10)\,dt$$

$$s(t) = -4.9t^2 + 10t + C.$$

Next we substitute $t = 0$ and solve for C:

$$s(t) = -4.9t^2 + 10t + C$$

$$s(0) = -4.9(0)^2 + 10(0) + C$$

$$3 = C.$$

Therefore the position function is $s(t) = -4.9t^2 + 10t + 3$.

b. The height of the baseball after 2 sec is given by $s(2)$:

$$s(2) = -4.9(2)^2 + 10(2) + 3 = -4.9(4) + 23 = 3.4.$$

Therefore the baseball is 3.4 meters above Earth's surface after 2 seconds. It is worth noting that the mass of the ball cancelled out completely in the process of solving the problem.

Key Concepts

- A differential equation is an equation involving a function $y = f(x)$ and one or more of its derivatives. A solution is a function $y = f(x)$ that satisfies the differential equation when f and its derivatives are substituted into the equation.
- The order of a differential equation is the highest order of any derivative of the unknown function that appears in the equation.
- A differential equation coupled with an initial value is called an initial-value problem. To solve an initial-value problem, first find the general solution to the differential equation, then determine the value of the constant. Initial-value problems have many applications in science and engineering.

Glossary

differential equation
an equation involving a function $y = y(x)$ and one or more of its derivatives

general solution (or family of solutions)

the entire set of solutions to a given differential equation

initial value(s)
a value or set of values that a solution of a differential equation satisfies for a fixed value of the independent variable

initial velocity
the velocity at time $t = 0$

initial-value problem
a differential equation together with an initial value or values

order of a differential equation
the highest order of any derivative of the unknown function that appears in the equation

particular solution
member of a family of solutions to a differential equation that satisfies a particular initial condition

solution to a differential equation
a function $y = f(x)$ that satisfies a given differential equation

---

This page titled 7.1: Basics of Differential Equations is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Gilbert Strang & Edwin "Jed" Herman (OpenStax) via source content that was edited to the style and standards of the LibreTexts platform.

- **8.1: Basics of Differential Equations** by Edwin "Jed" Herman, Gilbert Strang is licensed CC BY-NC-SA 4.0. Original source: https://openstax.org/details/books/calculus-volume-1.

7.1E: Exercises for Section 8.1

In exercises 1 - 7, determine the order of each differential equation.

1) $y'+y=3y^2$

Answer

1st-order

2) $(y')^2=y'+2y$

3) $y'''+y''y'=3x^2$

Answer

3rd-order

4) $y'=y''+3t^2$

5) $\frac{dy}{dt}=t$

Answer

1st-order

6) $\frac{dy}{dx}+\frac{d^2y}{dx^2}=3x^4$

7) $\left(\frac{dy}{dt}\right)^2+8\frac{dy}{dt}+3y=4t$

Answer

1st-order

In exercises 8 - 17, verify that the given function is a solution to the given differential equation.

8) $y=\frac{x^3}{3}$ solves $y'=x^2$

9) $y=2e^{-x}+x-1$ solves $y'=x-y$

10) $y=e^{3x}-\frac{e^x}{2}$ solves $y'=3y+e^x$

11) $y=\frac{1}{1-x}$ solves $y'=y^2$

12) $y=\frac{e^{x^2}}{2}$ solves $y'=2xy$

13) $y=4+\ln x$ solves $xy'=1$

14) $y=3-x+x\ln x$ solves $y'=\ln x$

15) $y=2e^x-x-1$ solves $y'=y+x$

16) $y=e^x+\frac{\sin x}{2}-\frac{\cos x}{2}$ solves $y'=\cos x+y$

17) $y=\pi e^{-\cos x}$ solves $y'=y\sin x$

In exercises 18 - 27, verify the given general solution and find the particular solution.

18) Find the particular solution to the differential equation $y'=4x^2$ that passes through $(-3,-30)$, given that $y=C+\frac{4x^3}{3}$ is a general solution.

19) Find the particular solution to the differential equation $y'=3x^3$ that passes through $(1,4.75)$, given that $y=C+\frac{3x^4}{4}$ is a general solution.

Answer

$y = 4 + \frac{3x^4}{4}$

20) Find the particular solution to the differential equation $y' = 3x^2y$ that passes through $(0, 12)$, given that $y = Ce^{x^3}$ is a general solution.

21) Find the particular solution to the differential equation $y' = 2xy$ that passes through $\left(0, \frac{1}{2}\right)$, given that $y = Ce^{x^2}$ is a general solution.

Answer

$y = \frac{1}{2}e^{x^2}$

22) Find the particular solution to the differential equation $y' = \left(2xy\right)^2$ that passes through $\left(1, -\frac{1}{2}\right)$, given that $y = -\frac{3}{C + 4x^3}$ is a general solution.

23) Find the particular solution to the differential equation $y'x^2 = y$ that passes through $\left(1, \frac{2}{e}\right)$, given that $y = Ce^{-1/x}$ is a general solution.

Answer

$y = 2e^{-1/x}$

24) Find the particular solution to the differential equation $8\frac{dx}{dt} = -2\cos(2t) - \cos(4t)$ that passes through (π, π), given that $x = C - \frac{1}{8}\sin(2t) - \frac{1}{32}\sin(4t)$ is a general solution.

25) Find the particular solution to the differential equation $\frac{du}{dt} = \tan u$ that passes through $\left(1, \frac{\pi}{2}\right)$, given that $u = \sin^{-1}\left(e^{C+t}\right)$ is a general solution.

Answer

$u = \sin^{-1}\left(e^{-1+t}\right)$

26) Find the particular solution to the differential equation $\frac{dy}{dt} = e^{t+y}$ that passes through $(1, 0)$, given that $y = -\ln(C - e^t)$ is a general solution.

27) Find the particular solution to the differential equation $y'(1 - x^2) = 1 + y$ that passes through $(0, -2)$, given that $y = C\frac{\sqrt{x+1}}{\sqrt{1-x}} - 1$ is a general solution.

Answer

$y = -\frac{\sqrt{x+1}}{\sqrt{1-x}} - 1$

In exercises 28 - 37, find the general solution to the differential equation.

28) $y' = 3x + e^x$

29) $y' = \ln x + \tan x$

Answer

$y = C - x + x\ln x - \ln(\cos x)$

30) $y' = \sin x e^{\cos x}$

31) $y' = 4^x$

Answer

$y = C + \frac{4^x}{\ln 4}$

32) $y' = \sin^{-1}(2x)$

33) $y' = 2t\sqrt{t^2+16}$

Answer

$y = \frac{2}{3}\sqrt{t^2+16}\left(t^2+16\right) + C$

34) $x' = \coth t + \ln t + 3t^2$

35) $x' = t\sqrt{4+t}$

Answer

$x = \frac{2}{15}\sqrt{4+t}\left(3t^2+4t-32\right) + C$

36) $y' = y$

37) $y' = \frac{y}{x}$

Answer

$y = Cx$

In exercises 38 - 42, solve the initial-value problems starting from $y(t=0) = 1$ and $y(t=0) = -1$. Draw both solutions on the same graph.

38) $\frac{dy}{dt} = 2t$

39) $\frac{dy}{dt} = -t$

Answer

$y = 1 - \frac{t^2}{2}$, and $y = -\frac{t^2}{2} - 1$

40) $\frac{dy}{dt} = 2y$

41) $\frac{dy}{dt} = -y$

Answer

$y = e^{-t}$ and $y = -e^{-t}$

42) $\frac{dy}{dt} = 2$

In exercises 43 - 47, solve the initial-value problems starting from $y_0 = 10$. At what time does y increase to 100 or drop to 1?

43) $\frac{dy}{dt} = 4t$

Answer

$y = 2(t^2+5)$, When $t = 3\sqrt{5}$, y will increase to 100.

44) $\frac{dy}{dt} = 4y$

45) $\frac{dy}{dt} = -2y$

Answer

$y = 10e^{-2t}$, When $t = -\frac{1}{2}\ln\left(\frac{1}{10}\right)$, y will decrease to 1.

46) $\frac{dy}{dt} = e^{4t}$

47) $\frac{dy}{dt} = e^{-4t}$

Answer

$y = \frac{1}{4}(41 - e^{-4t})$, Neither condition will ever happen.

Recall that a family of solutions includes solutions to a differential equation that differ by a constant. For exercises 48 - 52, use your calculator to graph a family of solutions to the given differential equation. Use initial conditions from $y(t = 0) = -10$ to $y(t = 0) = 10$ increasing by 2. Is there some critical point where the behavior of the solution begins to change?

48) [T] $y' = y(x)$

49) [T] $xy' = y$

Answer

Solution changes from increasing to decreasing at $y(0) = 0$.

50) [T] $y' = t^3$

51) [T] $y' = x + y$ (Hint: $y = Ce^x - x - 1$ is the general solution)

Answer

Solution changes from increasing to decreasing at $y(0) = 0$.

52) [T] $y' = x \ln x + \sin x$

53) Find the general solution to describe the velocity of a ball of mass 1 lb that is thrown upward at a rate of a ft/sec.

Answer

$v(t) = -32t + a$

54) In the preceding problem, if the initial velocity of the ball thrown into the air is $a = 25$ ft/s, write the particular solution to the velocity of the ball. Solve to find the time when the ball hits the ground.

55) You throw two objects with differing masses m_1 and m_2 upward into the air with the same initial velocity of a ft/s. What is the difference in their velocity after 1 second?

Answer

0 ft/s

56) [T] You throw a ball of mass 1 kilogram upward with a velocity of $a = 25$ m/s on Mars, where the force of gravity is $g = -3.711$ m/s^2. Use your calculator to approximate how much longer the ball is in the air on Mars.

57) [T] For the previous problem, use your calculator to approximate how much higher the ball went on Mars.

Answer

4.86 meters

58) [T] A car on the freeway accelerates according to $a = 15\cos(\pi t)$, where t is measured in hours. Set up and solve the differential equation to determine the velocity of the car if it has an initial speed of 51 mph. After 40 minutes of driving, what is the driver's velocity?

59) [T] For the car in the preceding problem, find the expression for the distance the car has traveled in time t, assuming an initial distance of 0. How long does it take the car to travel 100 miles? Round your answer to hours and minutes.

Answer

$x = 50t - \frac{15}{\pi^2}\cos(\pi t) + \frac{3}{\pi^2}$, 2 hours 1 minute

60) [T] For the previous problem, find the total distance traveled in the first hour.

61) Substitute $y = Be^{3t}$ into $y' - y = 8e^{3t}$ to find a particular solution.

Answer

$y = 4e^{3t}$

62) Substitute $y = a\cos(2t) + b\sin(2t)$ into $y' + y = 4\sin(2t)$ to find a particular solution.

63) Substitute $y = a + bt + ct^2$ into $y' + y = 1 + t^2$ to find a particular solution.

Answer

$y = 1 - 2t + t^2$

64) Substitute $y = ae^t \cos t + be^t \sin t$ into $y' = 2e^t \cos t$ to find a particular solution.

65) Solve $y' = e^{kt}$ with the initial condition $y(0) = 0$ and solve $y' = 1$ with the same initial condition. As k approaches 0, what do you notice?

Answer

$y = \frac{1}{k}(e^{kt} - 1)$ and $y = t$

This page titled 7.1E: Exercises for Section 8.1 is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Gilbert Strang & Edwin “Jed” Herman (OpenStax) via source content that was edited to the style and standards of the LibreTexts platform.

- **8.1E: Exercises for Section 8.1** by Edwin “Jed” Herman, Gilbert Strang is licensed CC BY-NC-SA 4.0. Original source: https://openstax.org/details/books/calculus-volume-1.

7.2: Direction Fields and Numerical Methods

Learning Objectives

- Draw the direction field for a given first-order differential equation.
- Use a direction field to draw a solution curve of a first-order differential equation.
- Use Euler's Method to approximate the solution to a first-order differential equation.

For the rest of this chapter we will focus on various methods for solving differential equations and analyzing the behavior of the solutions. In some cases it is possible to predict properties of a solution to a differential equation without knowing the actual solution. We will also study numerical methods for solving differential equations, which can be programmed by using various computer languages or even by using a spreadsheet program, such as Microsoft Excel.

Creating Direction Fields

Direction fields (also called slope fields) are useful for investigating first-order differential equations. In particular, we consider a first-order differential equation of the form

$$y' = f(x, y).$$

An applied example of this type of differential equation appears in Newton's law of cooling, which we will solve explicitly later in this chapter. First, though, let us create a direction field for the differential equation

$$T'(t) = -0.4(T - 72).$$

Here $T(t)$ represents the temperature (in degrees Fahrenheit) of an object at time t, and the ambient temperature is $72°F$. Figure 7.2.1 shows the direction field for this equation.

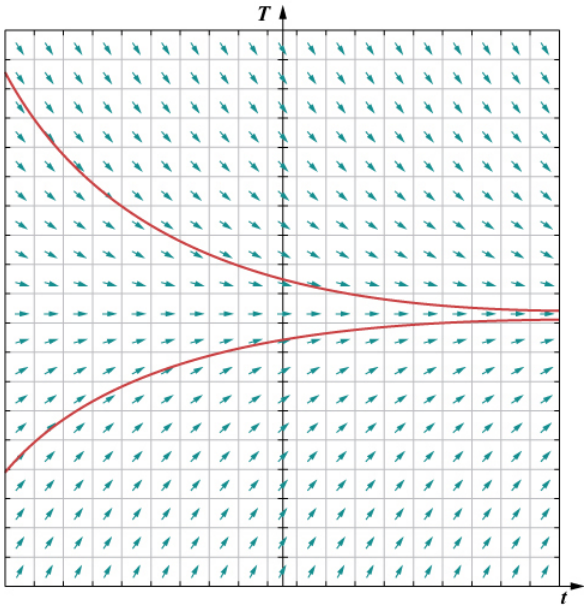


Figure 7.2.1: Direction field for the differential equation $T'(t) = -0.4(T - 72)$. Two solutions are plotted: one with initial temperature less than $72°F$ and the other with initial temperature greater than $72°F$.

The idea behind a direction field is the fact that the derivative of a function evaluated at a given point is the slope of the tangent line to the graph of that function at the same point. Other examples of differential equations for which we can create a direction field include

$$y' = 3x + 2y - 4$$

$$y' = x^2 - y^2$$

$$y' = \frac{2x+4}{y-2}.$$

To create a direction field, we start with the first equation: $y' = 3x + 2y - 4$. We let (x_0, y_0) be any ordered pair, and we substitute these numbers into the right-hand side of the differential equation. For example, if we choose $x = 1$ and $y = 2$, substituting into the right-hand side of the differential equation yields

$$y' = 3x + 2y - 4 = 3(1) + 2(2) - 4 = 3.$$

This tells us that if a solution to the differential equation $y' = 3x + 2y - 4$ passes through the point $(1, 2)$, then the slope of the solution at that point must equal 3. To start creating the direction field, we put a short line segment at the point $(1, 2)$ having slope 3. We can do this for any point in the domain of the function $f(x, y) = 3x + 2y - 4$, which consists of all ordered pairs (x, y) in R^2. Therefore any point in the Cartesian plane has a slope associated with it, assuming that a solution to the differential equation passes through that point. The direction field for the differential equation $y' = 3x + 2y - 4$ is shown in Figure 7.2.2.

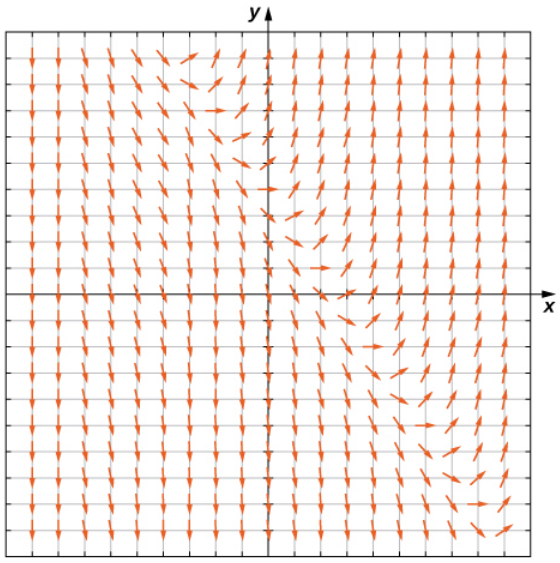


Figure 7.2.2: Direction field for the differential equation $y' = 3x + 2y - 4$.

We can generate a direction field of this type for any differential equation of the form $y' = f(x, y)$.

Definition: Direction Field (Slope Field)

A direction field (slope field) is a mathematical object used to graphically represent solutions to a first-order differential equation. At each point in a direction field, a line segment appears whose slope is equal to the slope of a solution to the differential equation passing through that point.

Using Direction Fields

We can use a direction field to predict the behavior of solutions to a differential equation without knowing the actual solution. For example, the direction field in Figure 7.2.3 serves as a guide to the behavior of solutions to the differential equation $y' = 3x + 2y - 4$.

To use a direction field, we start by choosing any point in the field. The line segment at that point serves as a signpost telling us what direction to go from there. For example, if a solution to the differential equation passes through the point $(0, 1)$, then the slope of the solution passing through that point is given by $y' = 3(0) + 2(1) - 4 = -2$. Now let x increase slightly, say to $x = 0.1$. Using the method of linear approximations gives a formula for the approximate value of y for $x = 0.1$. In particular,

$$L(x) = y_0 + f'(x_0)(x - x_0) = 1 - 2(x_0 - 0) = 1 - 2x_0.$$

Substituting $x_0 = 0.1$ into $L(x)$ gives an approximate y value of 0.8.

At this point the slope of the solution changes (again according to the differential equation). We can keep progressing, recalculating the slope of the solution as we take small steps to the right, and watching the behavior of the solution. Figure 7.2.3 shows a graph of the solution passing through the point $(0,1)$.

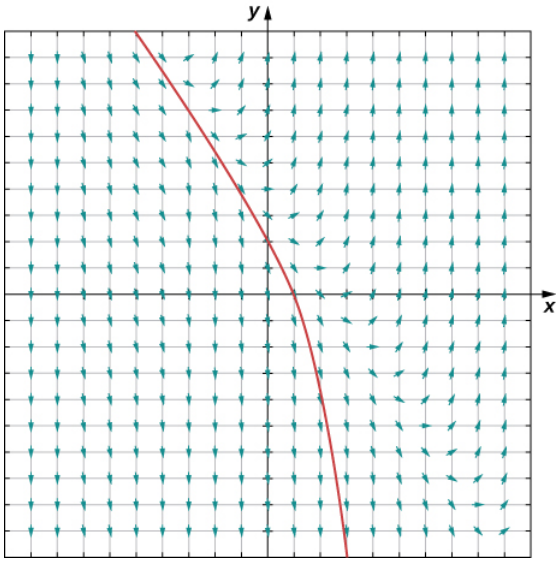


Figure 7.2.3: Direction field for the differential equation $y' = 3x + 2y - 4$ with the solution passing through the point $(0,1)$.

The curve is the graph of the solution to the initial-value problem

$$y' = 3x + 2y - 4,\ y(0) = 1.$$

This curve is called a solution curve passing through the point $(0,1)$. The exact solution to this initial-value problem is

$$y = -\frac{3}{2}x + \frac{5}{4} - \frac{1}{4}e^{2x},$$

and the graph of this solution is identical to the curve in Figure 7.2.3.

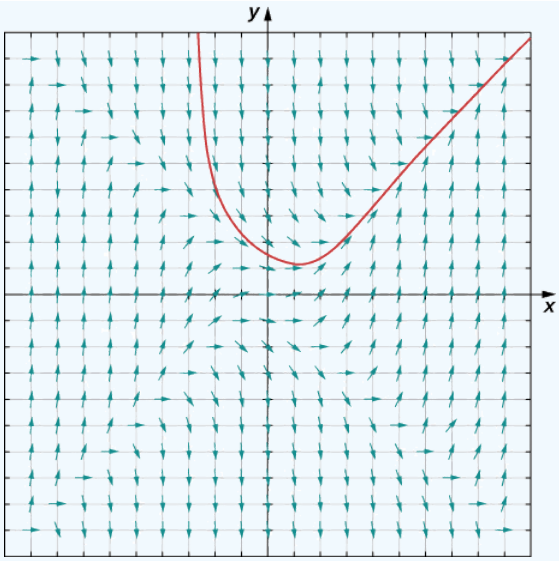
? Exercise 7.2.1

Create a direction field for the differential equation $y' = x^2 - y^2$ and sketch a solution curve passing through the point $(-1,2)$.

Hint

Use x and y values ranging from -5 to 5. For each coordinate pair, calculate y' using the right-hand side of the differential equation.

Answer



Now consider the direction field for the differential equation $y' = (x-3)(y^2-4)$, shown in Figure 7.2.4. This direction field has several interesting properties. First of all, at $y = -2$ and $y = 2$, horizontal dashes appear all the way across the graph. This means that if $y = -2$, then $y' = 0$. Substituting this expression into the right-hand side of the differential equation gives

$$\begin{aligned}(x-3)(y^2-4) &= (x-3)((-2)^2-4)\\ &= (x-3)(0)\\ &= 0\\ &= y'.\end{aligned}$$

Therefore $y = -2$ is a solution to the differential equation. Similarly, $y = 2$ is a solution to the differential equation. These are the only constant-valued solutions to the differential equation, as we can see from the following argument. Suppose $y = k$ is a constant solution to the differential equation. Then $y' = 0$. Substituting this expression into the differential equation yields $0 = (x-3)(k^2-4)$. This equation must be true for all values of x, so the second factor must equal zero. This result yields the equation $k^2 - 4 = 0$. The solutions to this equation are $k = -2$ and $k = 2$, which are the constant solutions already mentioned. These are called the equilibrium solutions to the differential equation.

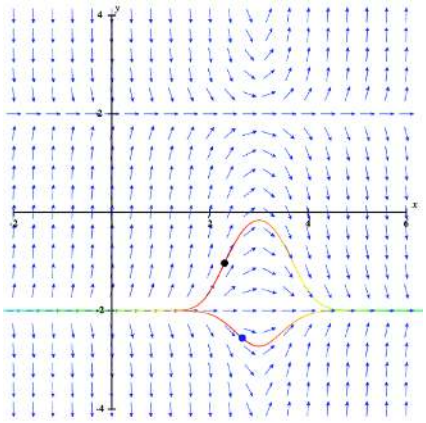


Figure 7.2.4: Direction field for the differential equation $y' = (x-3)(y^2-4)$ showing two solutions. These solutions are very close together, but one is barely above the equilibrium solution $y = -2$ and the other is barely below the same equilibrium solution.

Definition: Equilibrium Solutions

Consider the differential equation $y' = f(x, y)$. An *equilibrium solution* is any solution to the differential equation of the form $y = c$, where c is a constant.

To determine the equilibrium solutions to the differential equation $y' = f(x, y)$, set the right-hand side equal to zero. An equilibrium solution of the differential equation is any function of the form $y = k$ such that $f(x, k) = 0$ for all values of x in the domain of f.

An important characteristic of equilibrium solutions concerns whether or not they approach the line $y = k$ as an asymptote for large values of x.

Definition: asymptotically Stable, Unstable and Semi-Stable Solutions

Consider the differential equation $y' = f(x, y)$, and assume that all solutions to this differential equation are defined for $x \geq x_0$. Let $y = k$ be an equilibrium solution to the differential equation.

1. $y = k$ is an *asymptotically stable solution* to the differential equation if there exists $\varepsilon > 0$ such that for any value $c \in (k - \varepsilon, k + \varepsilon)$ the solution to the initial-value problem $y' = f(x, y)$, $y(x_0) = c$ approaches k as x approaches infinity.
2. $y = k$ is an *asymptotically unstable solution* to the differential equation if there exists $\varepsilon > 0$ such that for any value $c \in (k - \varepsilon, k + \varepsilon)$ the solution to the initial-value problem $y' = f(x, y)$, $y(x_0) = c$ never approaches k as x approaches infinity.
3. $y = k$ is an *asymptotically semi-stable solution* to the differential equation if it is neither asymptotically stable nor asymptotically unstable.

Now we return to the differential equation $y' = (x - 3)(y^2 - 4)$, with the initial condition $y(0) = 0.5$. The direction field for this initial-value problem, along with the corresponding solution, is shown in Figure 7.2.5.

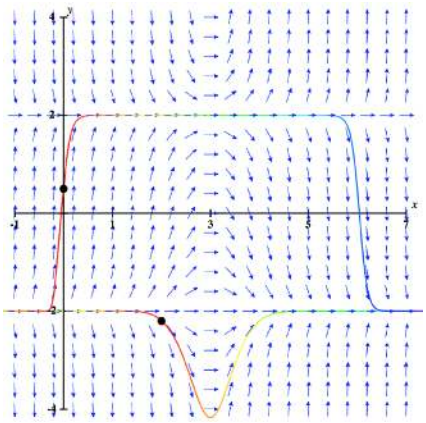


Figure 7.2.5: Direction field for the initial-value problem $y' = (x - 3)(y^2 - 4)$, $y(0) = 0.5$. A second solution curve is graphed with initial condition $(2, -2.2)$.

The values of the solution to this initial-value problem stay between $y = -2$ and $y = 2$, which are the equilibrium solutions to the differential equation. Furthermore, as x approaches infinity, although they initially appear to approach the line y = 2, the y-coordinates clearly approach -2. The behavior of solutions is similar if the initial value is below -2, for example, $y(2) = -2.2$. In this case, the solutions increase and approach $y = -2$ as x approaches infinity. Therefore $y = -2$ is an asymptotically stable solution to this differential equation.

What happens when the initial value is above $y = 2$? This scenario is illustrated in Figure 7.2.6, with the initial value $y(0) = 3$.

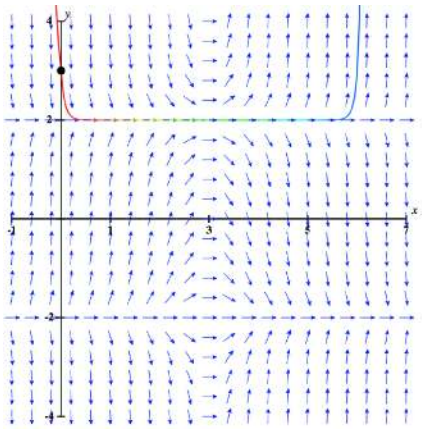


Figure 7.2.6: Direction field for the initial-value problem $y' = (x-3)(y^2-4),\ y(0) = 3$.

The solution increases rapidly toward positive infinity as x approaches infinity. Furthermore, if the initial value is slightly below 2, then the solution approaches -2, which is the other equilibrium solution. Therefore in neither case does the solution approach $y = 2$, so $y = 2$ is called an asymptotically unstable, or unstable, equilibrium solution.

Example 7.2.1: Stability of an Equilibrium Solution

Create a direction field for the differential equation $y' = (y-3)^2(y^2+y-2)$ and identify any equilibrium solutions. Classify each of the equilibrium solutions as stable, unstable, or semi-stable.

Solution

The direction field is shown in Figure 7.2.7.

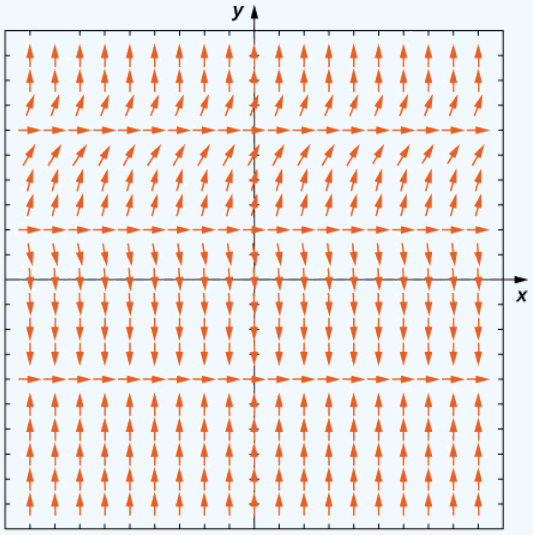


Figure 7.2.7: Direction field for the differential equation $y' = (y-3)^2(y^2+y-2)$.

The equilibrium solutions are $y = -2$, $y = 1$, and $y = 3$. To classify each of the solutions, look at an arrow directly above or below each of these values. For example, at $y = -2$ the arrows directly below this solution point up, and the arrows directly above the solution point down. Therefore all initial conditions close to $y = -2$ approach $y = -2$, and the solution is stable. For the solution $y = 1$, all initial conditions above and below $y = 1$ are repelled (pushed away) from $y = 1$, so this solution is unstable. The solution $y = 3$ is semi-stable, because for initial conditions slightly greater than 3, the solution approaches infinity, and for initial conditions slightly less than 3, the solution approaches $y = 1$.

Analysis

It is possible to find the equilibrium solutions to the differential equation by setting the right-hand side equal to zero and solving for y. This approach gives the same equilibrium solutions as those we saw in the direction field.

? Exercise 7.2.2

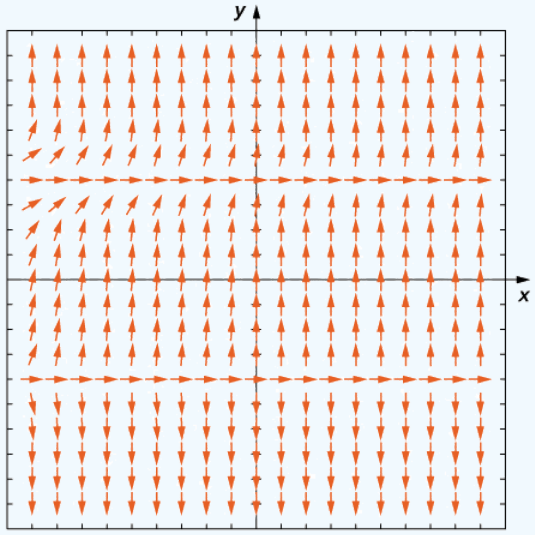
Create a direction field for the differential equation $y' = (x+5)(y+2)(y^2-4y+4)$ and identify any equilibrium solutions. Classify each of the equilibrium solutions as stable, unstable, or semi-stable.

Hint

First create the direction field and look for horizontal dashes that go all the way across. Then examine the slope lines directly above and below the equilibrium solutions.

Answer

The equilibrium solutions are $y = -2$ and $y = 2$. For this equation, $y = -2$ is an unstable equilibrium solution, and $y = 2$ is a semi-stable equilibrium solution.



Euler's Method

Consider the initial-value problem

$$y' = 2x - 3,\ y(0) = 3.$$

Integrating both sides of the differential equation gives $y = x^2 - 3x + C$, and solving for C yields the particular solution $y = x^2 - 3x + 3$. The solution for this initial-value problem appears as the parabola in Figure 7.2.8.

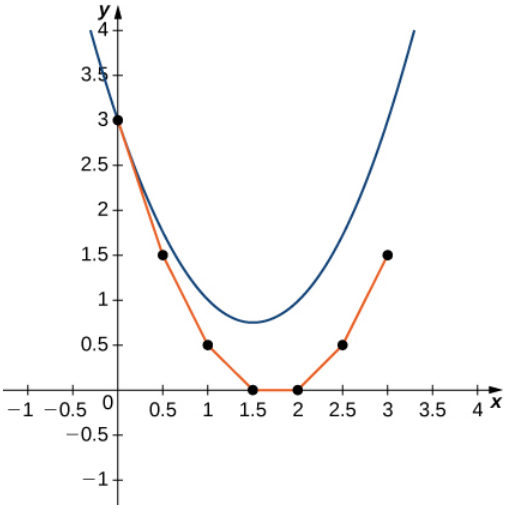


Figure 7.2.8: Euler's Method for the initial-value problem $y' = 2x - 3, y(0) = 3$.

The red graph consists of line segments that approximate the solution to the initial-value problem. The graph starts at the same initial value of $(0, 3)$. Then the slope of the solution at any point is determined by the right-hand side of the differential equation, and the length of the line segment is determined by increasing the x value by 0.5 each time (the step size). This approach is the basis of Euler's Method.

Before we state Euler's Method as a theorem, let's consider another initial-value problem:

$$y' = x^2 - y^2,\ y(-1) = 2.$$

The idea behind direction fields can also be applied to this problem to study the behavior of its solution. For example, at the point $(-1, 2)$, the slope of the solution is given by $y' = (-1)^2 - 2^2 = -3$, so the slope of the tangent line to the solution at that point is also equal to -3. Now we define $x_0 = -1$ and $y_0 = 2$. Since the slope of the solution at this point is equal to -3, we can use the method of linear approximation to approximate y near $(-1, 2)$.

$$L(x) = y_0 + f'(x_0)(x - x_0).$$

Here $x_0 = -1, y_0 = 2$, and $f'(x_0) = -3$, so the linear approximation becomes

$$L(x) = 2 - 3(x - (-1)) = 2 - 3x - 3 = -3x - 1.$$

Now we choose a **step size**. The step size is a small value, typically 0.1 or less, that serves as an increment for x; it is represented by the variable h. In our example, let $h = 0.1$. Incrementing x_0 by h gives our next x value:

$$x_1 = x_0 + h = -1 + 0.1 = -0.9.$$

We can substitute $x_1 = -0.9$ into the linear approximation to calculate y_1.

$$y_1 = L(x_1) = -3(-0.9) - 1 = 1.7.$$

Therefore the approximate y value for the solution when $x = -0.9$ is $y = 1.7$. We can then repeat the process, using $x_1 = -0.9$ and $y_1 = 1.7$ to calculate x_2 and y_2. The new slope is given by $y' = (-0.9)^2 - (1.7)^2 = -2.08$. First, $x_2 = x_1 + h = -0.9 + 0.1 = -0.8$. Using linear approximation gives

$$\begin{aligned} L(x) &= y_1 + f'(x_1)(x - x_1) \\ &= 1.7 - 2.08(x - (-0.9)) \\ &= 1.7 - 2.08x - 1.872 \\ &= -2.08x - 0.172. \end{aligned}$$

Finally, we substitute $x_2 = -0.8$ into the linear approximation to calculate y_2.

$$\begin{aligned} y_2 &= L(x_2) \\ &= -2.08x_2 - 0.172 \\ &= -2.08(-0.8) - 0.172 \\ &= 1.492. \end{aligned}$$

Therefore the approximate value of the solution to the differential equation is $y = 1.492$ when $x = -0.8$.

What we have just shown is the idea behind **Euler's Method**. Repeating these steps gives a list of values for the solution. These values are shown in the table, rounded off to four decimal places.

Table 7.2.1: Using Euler's Method to Approximate Solutions to a Differential Equation

n	0	1	2	3	4	5
x_n	−1	−0.9	−0.8	−0.7	−0.6	−0.5
y_n	2	1.7	1.492	1.3334	1.2046	1.0955
n	6	7	8	9	10	
x_n	−0.4	−0.3	−0.2	−0.1	0	
y_n	1.0004	1.9164	1.8414	1.7746	1.7156	

Euler's method

Consider the initial-value problem

$$y' = f(x, y),\ y(x_0) = y_0.$$

To approximate a solution to this problem using Euler's method, define

$$x_n = x_0 + nh$$

$$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1}).$$

Here $h > 0$ represents the step size and n is an integer, starting with 1. The number of steps taken is counted by the variable n.

Typically h is a small value, say 0.1 or 0.05. The smaller the value of h, the more calculations are needed. The higher the value of h, the fewer calculations are needed. However, the tradeoff results in a lower degree of accuracy for larger step size, as illustrated in Figure 7.2.9.

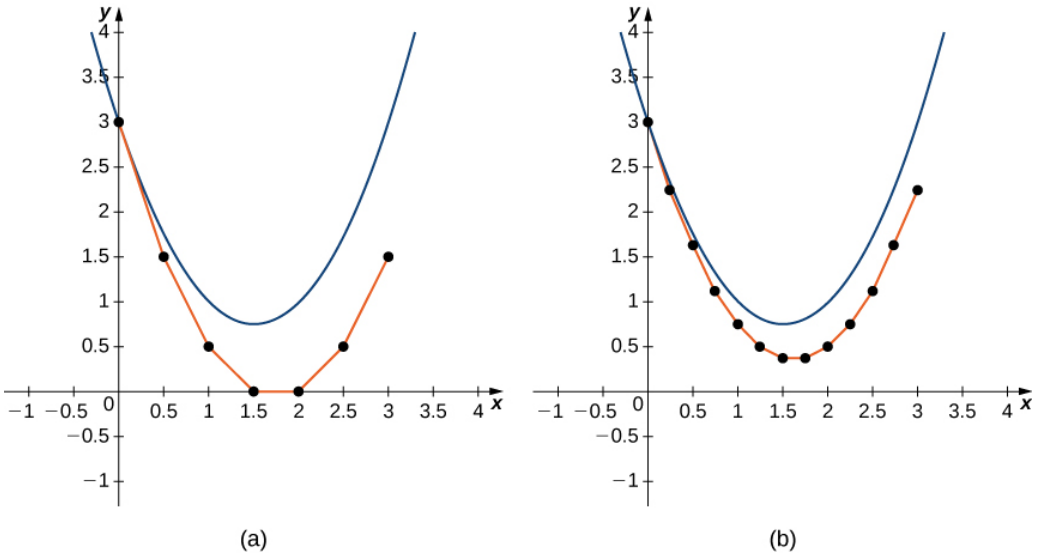


Figure 7.2.9: Euler's method for the initial-value problem $y' = 2x - 3,\ y(0) = 3$ with (a) a step size of $h = 0.5$; and (b) a step size of $h = 0.25$.

Example 7.2.2: Using Euler's Method

Consider the initial-value problem

$$y' = 3x^2 - y^2 + 1,\ y(0) = 2.$$

Use Euler's method with a step size of 0.1 to generate a table of values for the solution for values of x between 0 and 1.

Solution

We are given $h = 0.1$ and $f(x, y) = 3x^2 - y^2 + 1$. Furthermore, the initial condition $y(0) = 2$ gives $x_0 = 0$ and $y_0 = 2$. Using Equation with $n = 0$, we can generate this table.

Table 7.2.2: Using Euler's Method to Approximate Solutions to a Differential Equation

n	x_n	$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$
0	0	2
1	0.1	$y_1 = y_0 + hf(x_0, y_0) = 1.7$
2	0.2	$y_2 = y_1 + hf(x_1, y_1) = 1.514$
3	0.3	$y_3 = y_2 + hf(x_2, y_2) = 1.3968$
4	0.4	$y_4 = y_3 + hf(x_3, y_3) = 1.3287$
5	0.5	$y_5 = y_4 + hf(x_4, y_4) = 1.3001$
6	0.6	$y_6 = y_5 + hf(x_5, y_5) = 1.3061$
7	0.7	$y_7 = y_6 + hf(x_6, y_6) = 1.3435$
8	0.8	$y_8 = y_7 + hf(x_7, y_7) = 1.4100$
9	0.9	$y_9 = y_8 + hf(x_8, y_8) = 1.5032$
10	1.0	$y_{10} = y_9 + hf(x_9, y_9) = 1.6202$

With ten calculations, we are able to approximate the values of the solution to the initial-value problem for values of x between 0 and 1.

Go to this website for a chance to visually explore Euler's method.

Exercise 7.2.3

Consider the initial-value problem

$$y' = x^3 + y^2,\ y(1) = -2.$$

Using a step size of 0.1, generate a table with approximate values for the solution to the initial-value problem for values of x between 1 and 2.

Hint

Start by identifying the value of h, then figure out what $f(x, y)$ is. Then use the formula for Euler's Method to calculate y_1, y_2, and so on.

Answer

Table 7.2.3: Using Euler's Method to approximate solutions to the differential equation in Exercise 7.2.3.

n	\(x_n)	$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$
0	1	-2
1	1.1	$y_1 = y_0 + hf(x_0, y_0) = -1.5$
2	1.2	$y_2 = y_1 + hf(x_1, y_1) = -1.1419$
3	1.3	$y_3 = y_2 + hf(x_2, y_2) = -0.8387$
4	1.4	$y_4 = y_3 + hf(x_3, y_3) = -0.5487$
5	1.5	$y_5 = y_4 + hf(x_4, y_4) = -0.2442$
6	1.6	$y_6 = y_5 + hf(x_5, y_5) = 0.0993$
7	1.7	$y_7 = y_6 + hf(x_6, y_6) = 0.5099$
8	1.8	$y_8 = y_7 + hf(x_7, y_7) = 1.0272$
9	1.9	$y_9 = y_8 + hf(x_8, y_8) = 1.7159$
10	2	$y_{10} = y_9 + hf(x_9, y_9) = 2.6962$

Key Concepts

- A direction field is a mathematical object used to graphically represent solutions to a first-order differential equation.
- Euler's Method is a numerical technique that can be used to approximate solutions to a differential equation.

Key Equations

- **Euler's Method**

  $x_n = x_0 + nh$

  $y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$, where h is the step size

Glossary

asymptotically semi-stable solution

$y = k$ if it is neither asymptotically stable nor asymptotically unstable

asymptotically stable solution

$y = k$ if there exists $\varepsilon > 0$ such that for any value $c \in (k - \varepsilon,\ k + \varepsilon)$ the solution to the initial-value problem $y' = f(x, y),\ y(x_0) = c$ approaches k as x approaches infinity

asymptotically unstable solution

$y = k$ if there exists $\varepsilon > 0$ such that for any value $c \in (k - \varepsilon,\ k + \varepsilon)$ the solution to the initial-value problem $y' = f(x, y),\ y(x_0) = c$ never approaches k as $x$approaches infinity

direction field (slope field)

a mathematical object used to graphically represent solutions to a first-order differential equation; at each point in a direction field, a line segment appears whose slope is equal to the slope of a solution to the differential equation passing through that point

equilibrium solution

any solution to the differential equation of the form $y = c$, where c is a constant

Euler's Method

a numerical technique used to approximate solutions to an initial-value problem

solution curve

a curve graphed in a direction field that corresponds to the solution to the initial-value problem passing through a given point in the direction field

step size

the increment hh that is added to the xx value at each step in Euler's Method

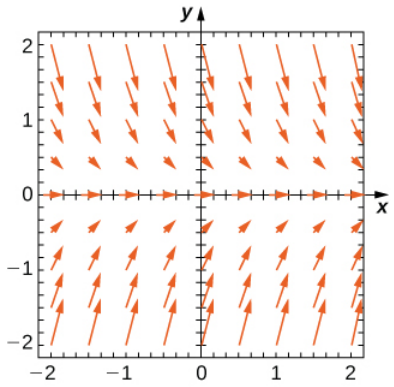
- Gilbert Strang (MIT) and Edwin "Jed" Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.
- Paul Seeburger (Monroe Community College) edited this section to adjust the explanation of equilibrium points in the example shown in Figures **7.2.4 - 7.2.6**. He also created these figures.

This page titled 7.2: Direction Fields and Numerical Methods is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Gilbert Strang & Edwin "Jed" Herman (OpenStax) via source content that was edited to the style and standards of the LibreTexts platform.

- **8.2: Direction Fields and Numerical Methods** by Edwin "Jed" Herman, Gilbert Strang is licensed CC BY-NC-SA 4.0. Original source: https://openstax.org/details/books/calculus-volume-1.

7.2E: Exercises for Section 8.2

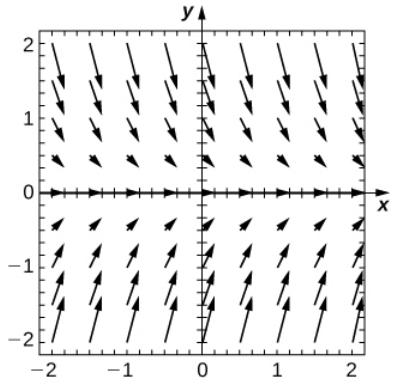
For exercises 1 - 3, use the direction field below from the differential equation $y' = -2y$. Sketch the graph of the solution for the given initial conditions.



1) $y(0) = 1$

2) $y(0) = 0$

Answer



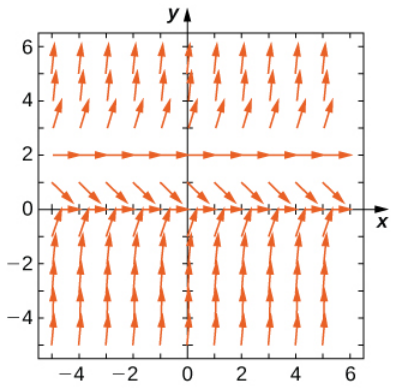
3) $y(0) = -1$

4) Are there any equilibria among the solutions of the differential equation from exercises 1 - 3? List any equilibria along with their stabilities.

Answer

$y = 0$ is a stable equilibrium

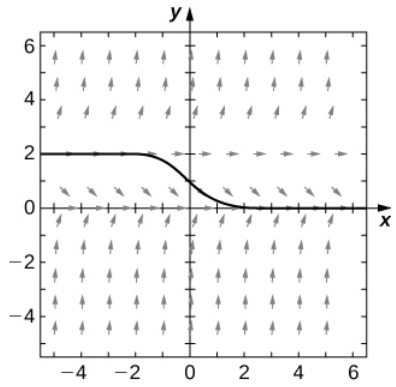
For exercises 5 - 7, use the direction field below from the differential equation $y' = y^2 - 2y$. Sketch the graph of the solution for the given initial conditions.



5) $y(0) = 3$

6) $y(0) = 1$

Answer



7) $y(0) = -1$

8) Are there any equilibria among the solutions of the differential equation from exercises 5 - 7? List any equilibria along with their stabilities.

Answer

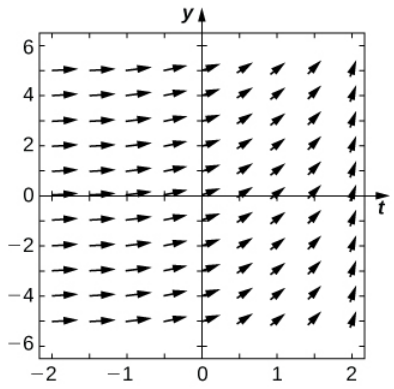
$y = 0$ is a stable equilibrium and $y = 2$ is unstable

In exercises 9 - 13, draw the direction field for the following differential equations, then solve the differential equation. Draw your solution on top of the direction field. Does your solution follow along the arrows on your direction field?

9) $y' = t^3$

10) $y' = e^t$

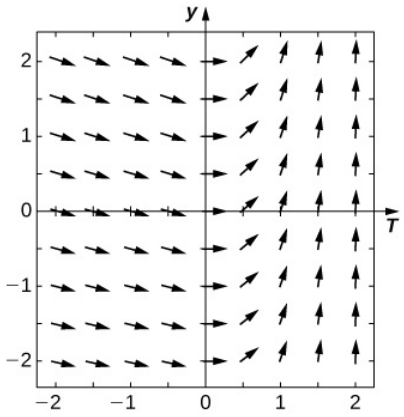
Answer



11) $\frac{dy}{dx} = x^2 \cos x$

12) $\frac{dy}{dt} = te^t$

Answer



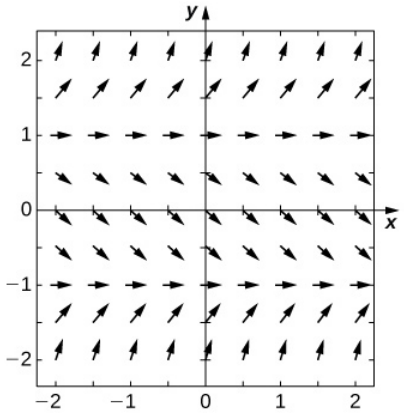
13) $\frac{dx}{dt} = \cosh(t)$

In exercises 14 - 18, draw the directional field for the following differential equations. What can you say about the behavior of the solution? Are there equilibria? What stability do these equilibria have?

14) $y' = y^2 - 1$

Answer

There appear to be equlibria at $y = -1$ (stable) and $y = 1$ (unstable).

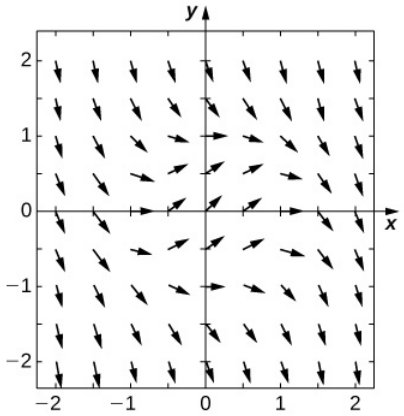


15) $y' = y - x$

16) $y' = 1 - y^2 - x^2$

Answer

There do not appear to be any equilibria.

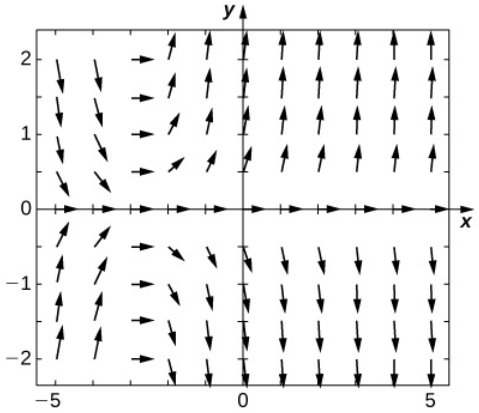


17) $y' = t^2 \sin y$

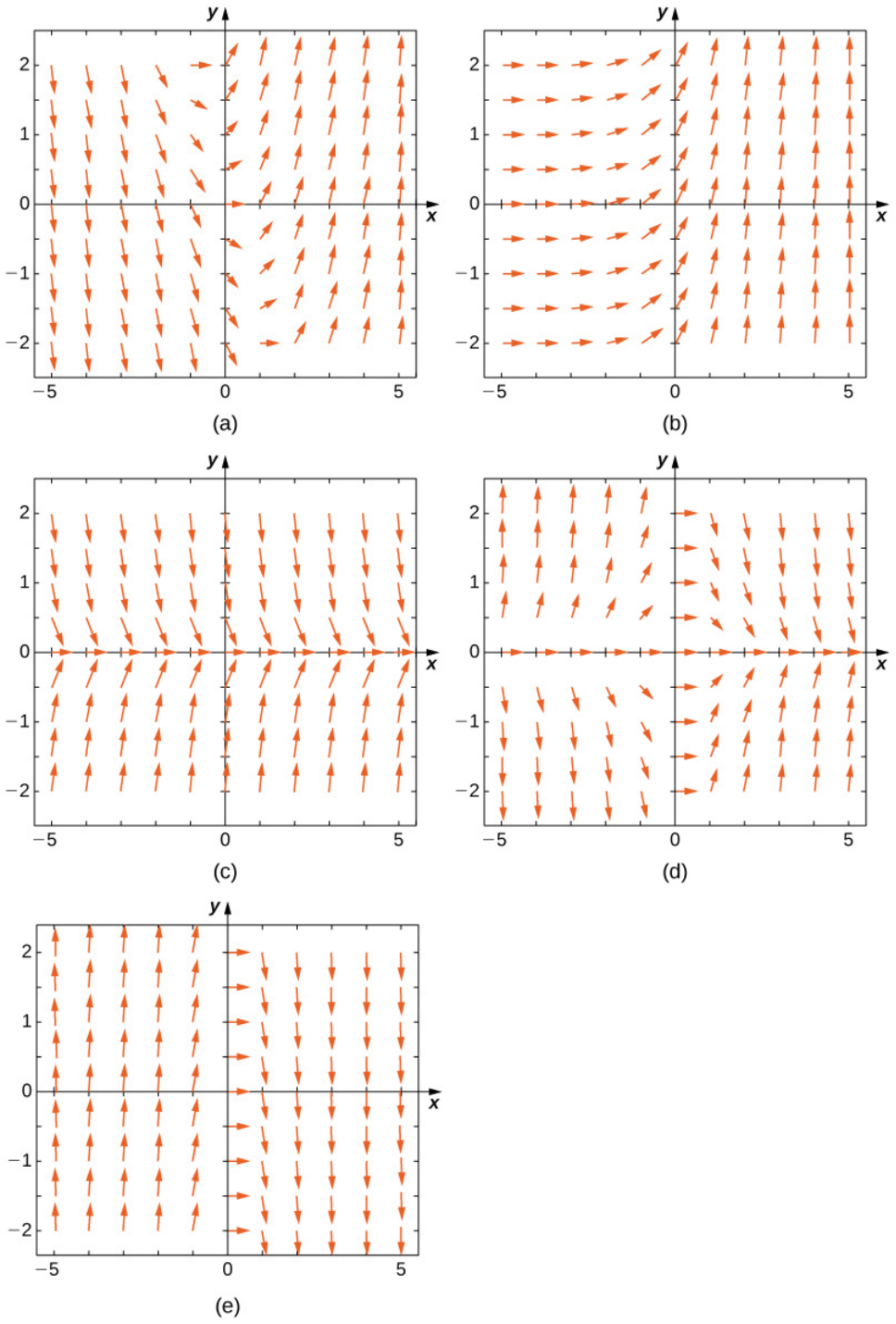
18) $y' = 3y + xy$

Answer

There appears to be an unstable equilibrium at $y = 0$.



Match the direction field with the given differential equations. Explain your selections.



19) $y' = -3y$

20) $y' = -3t$

Answer

E

21) $y' = e^t$

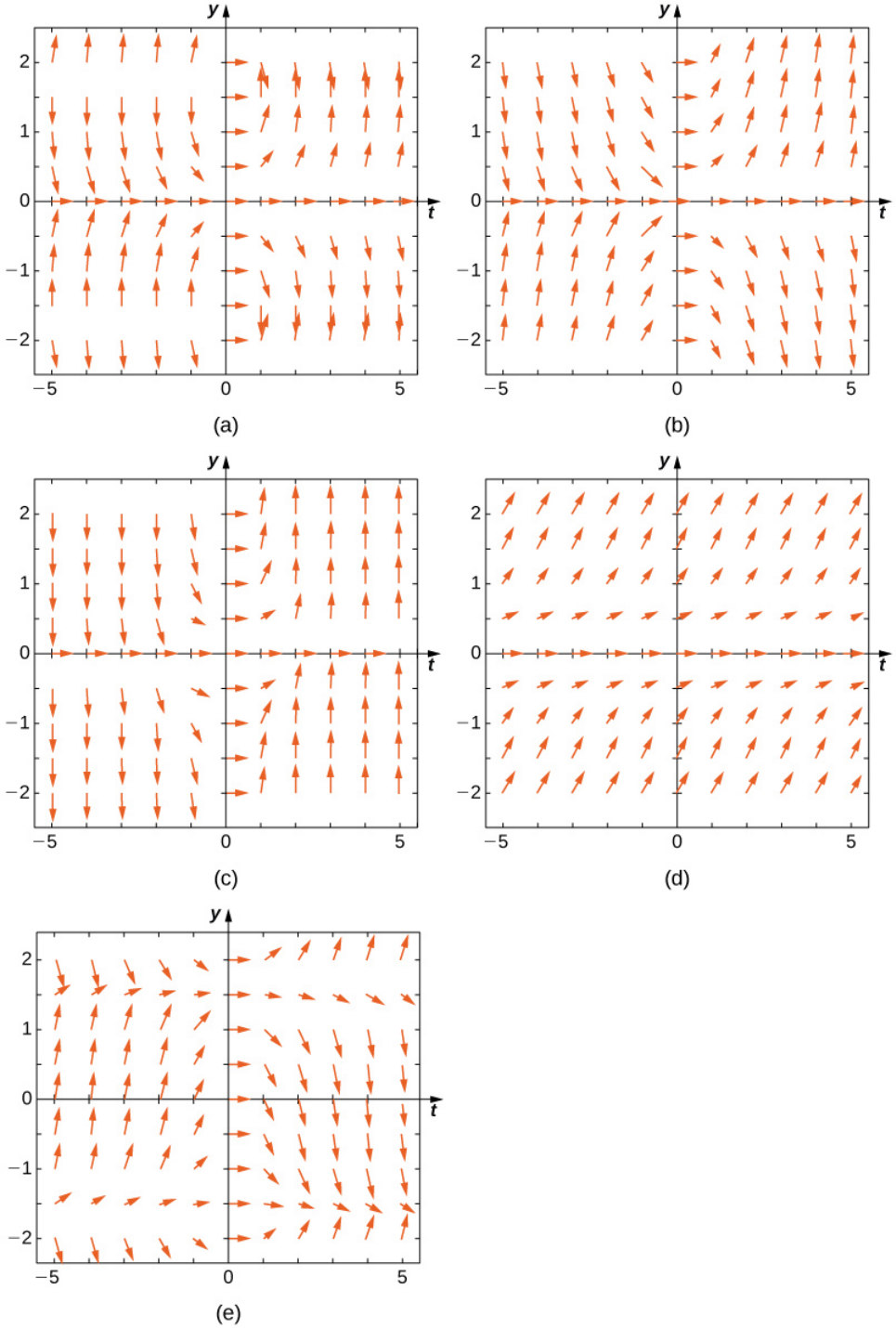
22) $y' = \frac{1}{2}y + t$

Answer

A

23) $y' = -ty$

Match the direction field with the given differential equations. Explain your selections.



24) $y' = t \sin y$

Answer

B

25) $y' = -t\cos y$

26) $y' = t\tan y$

Answer

A

27) $y' = \sin^2 y$

28) $y' = y^2 t^3$

Answer

C

Estimate the following solutions using Euler's method with $n = 5$ steps over the interval $t = [0, 1]$. If you are able to solve the initial-value problem exactly, compare your solution with the exact solution. If you are unable to solve the initial-value problem, the exact solution will be provided for you to compare with Euler's method. How accurate is Euler's method?

29) $y' = -3y, \quad y(0) = 1$

30) $y' = t^2, \quad y(0) = 1$

Answer

1.24, exact: $\frac{4}{3} \approx 1.33$

Solution:

31) $y' = 3t - y, \quad y(0) = 1$. Exact solution is $y = 3t + 4e^{-t} - 3$

32) $y' = y + t^2, \quad y(0) = 3$. Exact solution is $y = 5e^t - 2 - t^2 - 2t$

Answer

7.739364, exact: $5(e - 1)$

33) $y' = 2t, \quad y(0) = 0$

34) [T] $y' = e^{x+y}, y(0) = -1$. Exact solution is $y = -\ln(e + 1 - e^x)$

Answer

-0.2535, exact: 0

35) $y' = y^2 \ln(x + 1), \quad y(0) = 1$. Exact solution is $y = -\dfrac{1}{(x+1)(\ln(x+1) - 1)}$

36) $y' = 2^x, \quad y(0) = 0$. Exact solution is $y = \dfrac{2^x - 1}{\ln 2}$

Answer

1.345, exact: $\frac{1}{\ln(2)}$

37) $y' = y, \quad y(0) = -1$. Exact solution is $y = -e^x$.

38) $y' = -5t, \quad y(0) = -2$. Exact solution is $y = -\frac{5}{2}t^2 - 2$

Answer

-4, exact: $-9/2 = -4.5$

Differential equations can be used to model **disease epidemics.** In the next set of problems, we examine the change of size of two sub-populations of people living in a city: individuals who are infected and individuals who are susceptible to infection. S represents the size of the susceptible population, and I represents the size of the infected population. We assume that if a

susceptible person interacts with an infected person, there is a probability c that the susceptible person will become infected. Each infected person recovers from the infection at a rate r and becomes susceptible again. We consider the case of influenza, where we assume that no one dies from the disease, so we assume that the total population size of the two sub-populations is a constant number, N. The differential equations that model these population sizes are

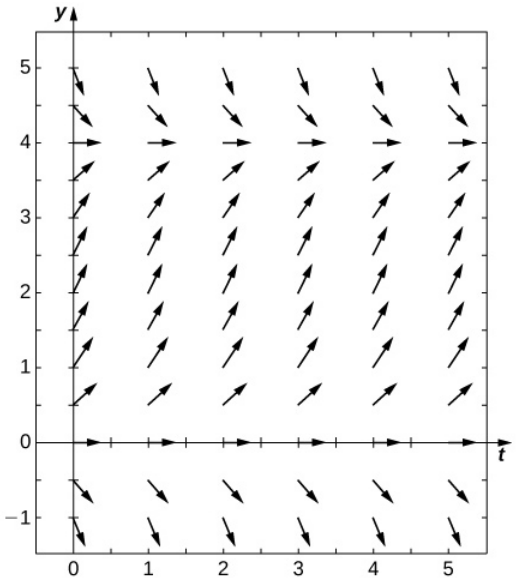
$$S' = rI - cSI \text{ and } I' = cSI - rI.$$

Here c represents the contact rate and r is the recovery rate.

39) Show that, by our assumption that the total population size is constant $(S + I = N)$, you can reduce the system to a single differential equation in I : $I' = c(N - I)I - rI$.

40) Assuming the parameters are $c = 0.5, N = 5,$ and $r = 0.5$, draw the resulting directional field.

Answer



41) [T] Use computational software or a calculator to compute the solution to the initial-value problem $y' = ty, y(0) = 2$ using Euler's Method with the given step size h. Find the solution at $t = 1$. For a hint, here is "pseudo-code" for how to write a computer program to perform Euler's Method for $y' = f(t, y), y(0) = 2$:

Create function $f(t, y)$

Define parameters $y(1) = y_0, t(0) = 0,$ step size h, and total number of steps, N

Write a for-loop:

for $k = 1$ to N

$fn = f(t(k), y(k))$

$y(k + 1) = y(k) + h * fn$

$t(k + 1) = t(k) + h$

42) Solve the initial-value problem for the exact solution.

Answer

$y' = 2e^{t^2/2}$

43) Draw the directional field

44) $h = 1$

Answer

2

45) [T] $h = 10$

46) [T] $h = 100$

Answer

3.2756

47) [T] $h = 1000$

48) [T] Evaluate the exact solution at $t = 1$. Make a table of errors for the relative error between the Euler's method solution and the exact solution. How much does the error change? Can you explain?

Answer

Exact solution: y =$2\sqrt{e}$.

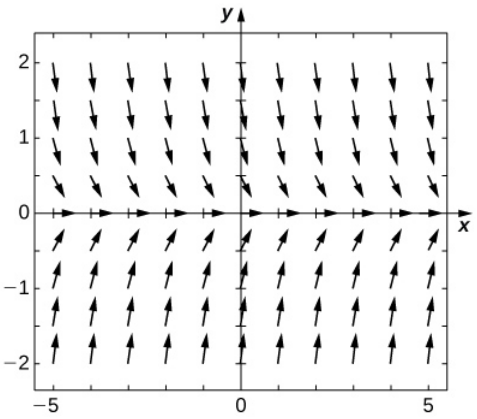
Step Size	Error
$h = 1$	0.3935
$h = 10$	0.06163
$h = 100$	0.006612
$h = 10000$	0.0006661

For exercises 49 - 53, consider the initial-value problem $y' = -2y$, with $y(0) = 2$.

49) Show that $y = 2e^{-2x}$ solves this initial-value problem.

50) Draw the directional field of this differential equation.

Answer



51) [T] By hand or by calculator or computer, approximate the solution using Euler's Method at $t = 10$ using $h = 5$.

52) [T] By calculator or computer, approximate the solution using Euler's Method at $t = 10$ using $h = 100$.

Answer

$4.0741e^{-10}$

53) [T] Plot exact answer and each Euler approximation (for $h = 5$ and $h = 100$) at each h on the directional field. What do you notice?

This page titled 7.2E: Exercises for Section 8.2 is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Gilbert Strang & Edwin “Jed” Herman (OpenStax) via source content that was edited to the style and standards of the LibreTexts platform.

7.3: Separable Equations

Learning Objectives

- Use separation of variables to solve a differential equation.
- Solve applications using separation of variables.

We now examine a solution technique for finding exact solutions to a class of differential equations known as separable differential equations. These equations are common in a wide variety of disciplines, including physics, chemistry, and engineering. We illustrate a few applications at the end of the section.

Separation of Variables

We start with a definition and some examples.

Definition: Separable Differential Equations

A *separable differential equation* is any equation that can be written in the form

$$y' = f(x)g(y). \tag{7.3.1}$$

The term 'separable' refers to the fact that the right-hand side of Equation 7.3.1 can be separated into a function of x times a function of y. Examples of separable differential equations include

$$y' = (x^2 - 4)(3y + 2) \tag{7.3.2}$$
$$y' = 6x^2 + 4x \tag{7.3.3}$$
$$y' = \sec y + \tan y \tag{7.3.4}$$
$$y' = xy + 3x - 2y - 6. \tag{7.3.5}$$

Equation 7.3.3 is separable with $f(x) = 6x^2 + 4x$ and $g(y) = 1$, Equation 7.3.4 is separable with $f(x) = 1$ and $g(y) = \sec y + \tan y$, and the right-hand side of Equation 7.3.5 can be factored as $(x + 3)(y - 2)$, so it is separable as well. Equation 7.3.4 is also called an *autonomous differential equation* because the right-hand side of the equation is a function of y alone. If a differential equation is separable, then it is possible to solve the equation using the method of *separation of variables.*

Problem-Solving Strategy: Separation of Variables

1. Check for any values of y that make $g(y) = 0$. These correspond to constant solutions.
2. Rewrite the differential equation in the form
$$\frac{dy}{g(y)} = f(x)dx.$$
3. Integrate both sides of the equation.
4. Solve the resulting equation for y if possible.
5. If an initial condition exists, substitute the appropriate values for x and y into the equation and solve for the constant.

Note that Step 4 states "Solve the resulting equation for y if possible." It is not always possible to obtain y as an explicit function of x. Quite often we have to be satisfied with finding y as an implicit function of x.

Example 7.3.1: Using Separation of Variables

Find a general solution to the differential equation $y' = (x^2 - 4)(3y + 2)$ using the method of separation of variables.

Solution

Follow the five-step method of separation of variables.

1. In this example, $f(x) = x^2 - 4$ and $g(y) = 3y + 2$. Setting $g(y) = 0$ gives $y = -\frac{2}{3}$ as a constant solution.

2. Rewrite the differential equation in the form

$$\frac{dy}{3y+2} = (x^2 - 4)\,dx.$$

3. Integrate both sides of the equation:

$$\int \frac{dy}{3y+2} = \int (x^2 - 4)\,dx.$$

Let $u = 3y + 2$. Then $du = 3\frac{dy}{dx}\,dx$, so the equation becomes

$$\frac{1}{3}\int \frac{1}{u}\,du = \frac{1}{3}x^3 - 4x + C$$

$$\frac{1}{3}\ln|u| = \frac{1}{3}x^3 - 4x + C$$

$$\frac{1}{3}\ln|3y+2| = \frac{1}{3}x^3 - 4x + C.$$

4. To solve this equation for y, first multiply both sides of the equation by 3.

$$\ln|3y+2| = x^3 - 12x + 3C$$

Now we use some logic in dealing with the constant C. Since C represents an arbitrary constant, $3C$ also represents an arbitrary constant. If we call the second arbitrary constant C_1, where $C_1 = 3C$, the equation becomes

$$\ln|3y+2| = x^3 - 12x + C_1.$$

Now exponentiate both sides of the equation (i.e., make each side of the equation the exponent for the base e).

$$e^{\ln|3y+2|} = e^{x^3 - 12x + C_1}$$
$$|3y+2| = e^{C_1}e^{x^3 - 12x}$$

Again define a new constant $C_2 = e^{C_1}$ (note that $C_2 > 0$):

$$|3y+2| = C_2 e^{x^3 - 12x}.$$

Because of the absolute value on the left side of the equation, this corresponds to two separate equations:

$$3y + 2 = C_2 e^{x^3 - 12x}$$

and

$$3y + 2 = -C_2 e^{x^3 - 12x}.$$

The solution to either equation can be written in the form

$$y = \frac{-2 \pm C_2 e^{x^3 - 12x}}{3}.$$

Since $C_2 > 0$, it does not matter whether we use plus or minus, so the constant can actually have either sign. Furthermore, the subscript on the constant C is entirely arbitrary, and can be dropped. Therefore the solution can be written as

$$y = \frac{-2 + Ce^{x^3 - 12x}}{3}, \text{ where } C = \pm C_2 \text{ or } C = 0.$$

Note that in writing a single general solution in this way, we are also allowing C to equal 0. This gives us the singular solution, $y = -\frac{2}{3}$, for the given differential equation. Check that this is indeed a solution of this differential equation!

5. No initial condition is imposed, so we are finished.

? Exercise 7.3.1

Use the method of separation of variables to find a general solution to the differential equation

$$y' = 2xy + 3y - 4x - 6.$$

Hint

First factor the right-hand side of the equation by grouping, then use the five-step strategy of separation of variables.

Answer

$$y = 2 + Ce^{x^2+3x}$$

✓ Example 7.3.2: Solving an Initial-Value Problem

Using the method of separation of variables, solve the initial-value problem

$$y' = (2x+3)(y^2-4), \quad y(0) = -1.$$

Solution

Follow the five-step method of separation of variables.

1. In this example, $f(x) = 2x + 3$ and $g(y) = y^2 - 4$. Setting $g(y) = 0$ gives $y = \pm 2$ as constant solutions.

2. Divide both sides of the equation by $y^2 - 4$ and multiply by dx. This gives the equation

$$\frac{dy}{y^2-4} = (2x+3)\,dx.$$

3. Next integrate both sides:

$$\int \frac{1}{y^2-4}dy = \int (2x+3)\,dx. \tag{7.3.6}$$

To evaluate the left-hand side, use the method of partial fraction decomposition. This leads to the identity

$$\frac{1}{y^2-4} = \frac{1}{4}\left(\frac{1}{y-2} - \frac{1}{y+2}\right).$$

Then Equation 7.3.6 becomes

$$\frac{1}{4}\int \left(\frac{1}{y-2} - \frac{1}{y+2}\right)dy = \int (2x+3)\,dx$$

$$\frac{1}{4}\left(\ln|y-2| - \ln|y+2|\right) = x^2 + 3x + C.$$

Multiplying both sides of this equation by 4 and replacing $4C$ with C_1 gives

$$\ln|y-2| - \ln|y+2| = 4x^2 + 12x + C_1$$

$$\ln\left|\frac{y-2}{y+2}\right| = 4x^2 + 12x + C_1.$$

4. It is possible to solve this equation for y. First exponentiate both sides of the equation and define $C_2 = e^{C_1}$:

$$\left|\frac{y-2}{y+2}\right| = C_2 e^{4x^2+12x}.$$

Next we can remove the absolute value and let a new constant C_3 be positive, negative, or zero, i.e., $C_3 = \pm C_2$ or $C_3 = 0$.

Then multiply both sides by $y+2$.

$$y - 2 = C_3(y+2)e^{4x^2+12x}$$

$$y - 2 = C_3 y e^{4x^2+12x} + 2C_3 e^{4x^2+12x}.$$

Now collect all terms involving y on one side of the equation, and solve for y:

$$y - C_3 y e^{4x^2+12x} = 2 + 2C_3 e^{4x^2+12x}$$

$$y\left(1 - C_3 e^{4x^2+12x}\right) = 2 + 2C_3 e^{4x^2+12x}$$

$$y = \frac{2 + 2C_3 e^{4x^2+12x}}{1 - C_3 e^{4x^2+12x}}.$$

5. To determine the value of C_3, substitute $x = 0$ and $y = -1$ into the general solution. Alternatively, we can put the same values into an earlier equation, namely the equation $\frac{y-2}{y+2} = C_3 e^{4x^2+12}$. This is much easier to solve for C_3:

$$\frac{y-2}{y+2} = C_3 e^{4x^2+12x}$$

$$\frac{-1-2}{-1+2} = C_3 e^{4(0)^2+12(0)}$$

$$C_3 = -3.$$

Therefore the solution to the initial-value problem is

$$y = \frac{2 - 6e^{4x^2+12x}}{1 + 3e^{4x^2+12x}}.$$

A graph of this solution appears in Figure **7.3.1**.

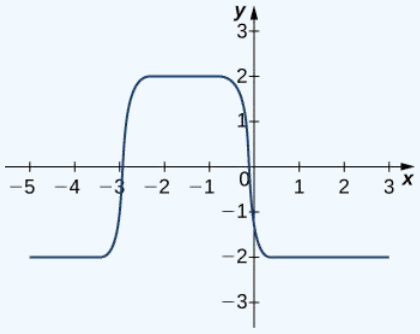


Figure **7.3.1**: Graph of the solution to the initial-value problem $y' = (2x+3)(y^2-4), y(0) = -1$.

? Exercise 7.3.2

Find the solution to the initial-value problem

$$6y' = (2x+1)(y^2 - 2y - 8)$$

with $y(0) = -3$ using the method of separation of variables.

Hint

Follow the steps for separation of variables to solve the initial-value problem.

Answer

$$y = \frac{4 + 14e^{x^2+x}}{1 - 7e^{x^2+x}}$$

Applications of Separation of Variables

Many interesting problems can be described by separable equations. We illustrate two types of problems: solution concentrations and Newton's law of cooling.

Solution concentrations

Consider a tank being filled with a salt solution. We would like to determine the amount of salt present in the tank as a function of time. We can apply the process of separation of variables to solve this problem and similar problems involving *solution concentrations*.

Example 7.3.3: Determining Salt Concentration over Time

A tank containing 100 L of a brine solution initially has 4 kg of salt dissolved in the solution. At time $t = 0$, another brine solution flows into the tank at a rate of 2 L/min. This brine solution contains a concentration of 0.5 kg/L of salt. At the same time, a stopcock is opened at the bottom of the tank, allowing the combined solution to flow out at a rate of 2 L/min, so that the level of liquid in the tank remains constant (Figure 7.3.2). Find the amount of salt in the tank as a function of time (measured in minutes), and find the limiting amount of salt in the tank, assuming that the solution in the tank is well mixed at all times.

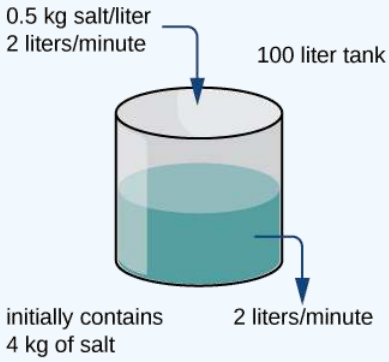


Figure 7.3.2: A brine tank with an initial amount of salt solution accepts an input flow and delivers an output flow. How does the amount of salt change with time?

Solution

First we define a function $u(t)$ that represents the amount of salt in kilograms in the tank as a function of time. Then $\frac{du}{dt}$ represents the rate at which the amount of salt in the tank changes as a function of time. Also, $u(0)$ represents the amount of salt in the tank at time $t = 0$, which is 4 kilograms.

The general setup for the differential equation we will solve is of the form

$$\frac{du}{dt} = \text{INFLOW RATE} - \text{OUTFLOW RATE}.$$

INFLOW RATE represents the rate at which salt enters the tank, and OUTFLOW RATE represents the rate at which salt leaves the tank. Because solution enters the tank at a rate of 2 L/min, and each liter of solution contains 0.5 kilogram of salt, every minute $2(0.5) = 1$ kilogram of salt enters the tank. Therefore INFLOW RATE = 1.

To calculate the rate at which salt leaves the tank, we need the concentration of salt in the tank at any point in time. Since the actual amount of salt varies over time, so does the concentration of salt. However, the volume of the solution remains fixed at 100 liters. The number of kilograms of salt in the tank at time t is equal to $u(t)$. Thus, the concentration of salt is $\frac{u(t)}{100}$ kg/L, and the solution leaves the tank at a rate of 2 L/min. Therefore salt leaves the tank at a rate of $\frac{u(t)}{100} \cdot 2 = \frac{u(t)}{50}$ kg/min, and OUTFLOW RATE is equal to $\frac{u(t)}{50}$. Therefore the differential equation becomes $\frac{du}{dt} = 1 - \frac{u}{50}$, and the initial condition is $u(0) = 4$. The initial-value problem to be solved is

$$\frac{du}{dt} = 1 - \frac{u}{50}, u(0) = 4.$$

The differential equation is a separable equation, so we can apply the five-step strategy for solution.

Step 1. Setting $1 - \frac{u}{50} = 0$ gives $u = 50$ as a constant solution. Since the initial amount of salt in the tank is 4 kilograms, this solution does not apply.

Step 2. Rewrite the equation as

$$\frac{du}{dt} = \frac{50-u}{50}.$$

Then multiply both sides by dt and divide both sides by $50-u$:

$$\frac{du}{50-u} = \frac{dt}{50}.$$

Step 3. Integrate both sides:

$$\int \frac{du}{50-u} = \int \frac{dt}{50}$$
$$-\ln|50-u| = \frac{t}{50} + C.$$

Step 4. Solve for $u(t)$:

$$\ln|50-u| = -\frac{t}{50} - C$$

$$e^{\ln|50-u|} = e^{-(t/50)-C}$$

$$|50-u| = C_1 e^{-t/50}, \text{ where } C_1 = e^{-C}.$$

Eliminate the absolute value by allowing the constant to be positive, negative, or zero, i.e., $C_1 = \pm e^{-C}$ or $C_1 = 0$:

$$50-u = C_1 e^{-t/50}.$$

Finally, solve for $u(t)$:

$$u(t) = 50 - C_1 e^{-t/50}.$$

Step 5. Solve for C_1:

$$u(0) = 50 - C_1 e^{-0/50}$$
$$4 = 50 - C_1$$
$$C_1 = 46.$$

The solution to the initial value problem is $u(t) = 50 - 46e^{-t/50}$. To find the limiting amount of salt in the tank, take the limit as t approaches infinity:

$$\lim_{t\to\infty} u(t) = 50 - 46e^{-t/50}$$
$$= 50 - 46(0) = 50.$$

Note that this was the constant solution to the differential equation. If the initial amount of salt in the tank is 50 kilograms, then it remains constant. If it starts at less than 50 kilograms, then it approaches 50 kilograms over time.

? Exercise 7.3.3

A tank contains 3 kilograms of salt dissolved in 75 liters of water. A salt solution of 0.4 kg salt/L is pumped into the tank at a rate of 6 L/min and is drained at the same rate. Solve for the salt concentration at time t. Assume the tank is well mixed at all times.

Hint

Follow the steps in Example $7.3.3$ and determine an expression for INFLOW and OUTFLOW. Formulate an initial-value problem, and then solve it.

Initial value problem:

$$\frac{du}{dt} = 2.4 - \frac{2u}{25},\ u(0) = 3$$

Answer

$$u(t) = 30 - 27e^{-t/50}$$

Newton's Law of Cooling

Newton's law of cooling states that the rate of change of an object's temperature is proportional to the difference between its own temperature and the ambient temperature (i.e., the temperature of its surroundings). If we let $T(t)$ represent the temperature of an object as a function of time, then $\frac{dT}{dt}$ represents the rate at which that temperature changes. The temperature of the object's surroundings can be represented by T_s. Then Newton's law of cooling can be written in the form

$$\frac{dT}{dt} = k(T(t) - T_s)$$

or simply

$$\frac{dT}{dt} = k(T - T_s).$$

The temperature of the object at the beginning of any experiment is the initial value for the initial-value problem. We call this temperature T_0. Therefore the initial-value problem that needs to be solved takes the form

$$\frac{dT}{dt} = k(T - T_s) \tag{7.3.7}$$

with $T(0) = T_0$, where k is a constant that needs to be either given or determined in the context of the problem. We use these equations in Example 7.3.4.

Example 7.3.4: Waiting for a Pizza to Cool

A pizza is removed from the oven after baking thoroughly, and the temperature of the oven is $350°F$. The temperature of the kitchen is $75°F$, and after 5 minutes the temperature of the pizza is $340°F$. We would like to wait until the temperature of the pizza reaches $300°F$ before cutting and serving it (Figure 7.3.3). How much longer will we have to wait?

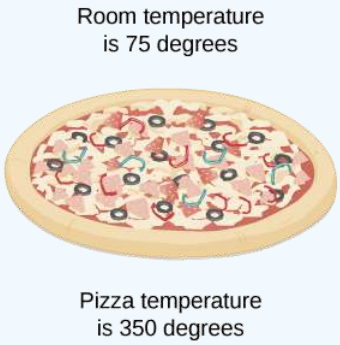


Figure 7.3.3: From Newton's law of cooling, if the pizza cools $10°F$ in 5 minutes, how long before it cools to $300°F$?

Solution

The ambient temperature (surrounding temperature) is $75°F$, so $T_s = 75$. The temperature of the pizza when it comes out of the oven is $350°F$, which is the initial temperature (i.e., initial value), so $T_0 = 350$. Therefore Equation 7.3.7 becomes

$$\frac{dT}{dt} = k(T - 75)$$

with $T(0) = 350$.

To solve the differential equation, we use the five-step technique for solving separable equations.

1. Setting the right-hand side equal to zero gives $T = 75$ as a constant solution. Since the pizza starts at $350°F$, this is not the solution we are seeking.

2. Rewrite the differential equation by multiplying both sides by dt and dividing both sides by $T - 75$:

$$\frac{dT}{T-75} = k\,dt.$$

3. Integrate both sides:

$$\int \frac{dT}{T-75} = \int k\,dt$$
$$\ln|T-75| = kt + C.$$

4. Solve for T by first exponentiating both sides:

$$\begin{aligned} e^{\ln|T-75|} &= e^{kt+C} \\ |T-75| &= C_1 e^{kt}, && \text{where } C_1 = e^C. \\ T-75 &= \pm C_1 e^{kt} \\ T-75 &= Ce^{kt}, && \text{where } C = \pm C_1 \text{ or } C = 0. \\ T(t) &= 75 + Ce^{kt}. \end{aligned}$$

5. Solve for C by using the initial condition $T(0) = 350$:

$$\begin{aligned} T(t) &= 75 + Ce^{kt} \\ T(0) &= 75 + Ce^{k(0)} \\ 350 &= 75 + C \\ C &= 275. \end{aligned}$$

Therefore the solution to the initial-value problem is

$$T(t) = 75 + 275e^{kt}.$$

To determine the value of k, we need to use the fact that after 5 minutes the temperature of the pizza is $340°F$. Therefore $T(5) = 340$. Substituting this information into the solution to the initial-value problem, we have

$$T(t) = 75 + 275e^{kt}$$

$$T(5) = 340 = 75 + 275e^{5k}$$

$$265 = 275e^{5k}$$

$$e^{5k} = \frac{53}{55}$$

$$\ln e^{5k} = \ln\left(\frac{53}{55}\right)$$

$$5k = \ln\left(\frac{53}{55}\right)$$

$$k = \frac{1}{5}\ln\left(\frac{53}{55}\right) \approx -0.007408.$$

So now we have $T(t) = 75 + 275e^{-0.007048t}$. When is the temperature $300°F$? Solving for t, we find

$$T(t) = 75 + 275e^{-0.007048t}$$

$$300 = 75 + 275e^{-0.007048t}$$

$$225 = 275e^{-0.007048t}$$

$$e^{-0.007048t} = \frac{9}{11}$$

$$\ln e^{-0.007048t} = \ln\frac{9}{11}$$

$$-0.007048t = \ln\frac{9}{11}$$

$$t = -\frac{1}{0.007048}\ln\frac{9}{11} \approx 28.5.$$

Therefore we need to wait an additional 23.5 minutes (after the temperature of the pizza reached $340°F$). That should be just enough time to finish this calculation.

? Exercise 7.3.4

A cake is removed from the oven after baking thoroughly, and the temperature of the oven is $450°F$. The temperature of the kitchen is $70°F$, and after 10 minutes the temperature of the cake is $430°F$.

a. Write the appropriate initial-value problem to describe this situation.
b. Solve the initial-value problem for $T(t)$.
c. How long will it take until the temperature of the cake is within $5°F$ of room temperature?

Hint

Determine the values of T_s and T_0 then use Equation 7.3.7.

Answer a

Initial-value problem

$$\frac{dT}{dt} = k(T - 70), \quad T(0) = 450$$

Answer b

$$T(t) = 70 + 380e^{kt}$$

Answer c

Approximately 114 minutes.

Key Concepts

- A separable differential equation is any equation that can be written in the form $y' = f(x)g(y)$.
- The method of separation of variables is used to find the general solution to a separable differential equation.

Key Equations

- **Separable differential equation**

  $y' = f(x)g(y)$

- **Solution concentration**

  $\frac{du}{dt} = \text{INFLOW RATE} - \text{OUTFLOW RATE}$

- **Newton's law of cooling**

  $\frac{dT}{dt} = k(T - T_s)$

Glossary

autonomous differential equation
an equation in which the right-hand side is a function of y alone

separable differential equation
any equation that can be written in the form $y' = f(x)g(y)$

separation of variables
a method used to solve a separable differential equation

This page titled 7.3: Separable Equations is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Gilbert Strang & Edwin “Jed” Herman (OpenStax) via source content that was edited to the style and standards of the LibreTexts platform.

- **8.3: Separable Equations** by Edwin “Jed” Herman, Gilbert Strang is licensed CC BY-NC-SA 4.0. Original source: https://openstax.org/details/books/calculus-volume-1.

7.3E: Exercises for Section 8.3

In exercises 1 - 4, solve the following initial-value problems with the initial condition $y_0 = 0$ and graph the solution.

1) $\frac{dy}{dt} = y + 1$

Answer

$y = e^t - 1$

2) $\frac{dy}{dt} = y - 1$

3) $\frac{dy}{dt} = -y + 1$

Answer

$y = 1 - e^{-t}$

4) $\frac{dy}{dt} = -y - 1$

In exercises 5 - 14, find the general solution to the differential equation.

5) $x^2y' = (x + 1)y$

Answer

$y = Cxe^{-1/x}$

6) $y' = \tan(y)x$

7) $y' = 2xy^2$

Answer

$y = \frac{1}{C - x^2}$

8) $\frac{dy}{dt} = y\cos(3t + 2)$

9) $2x\frac{dy}{dx} = y^2$

Answer

$y = -\frac{2}{C + \ln|x|}$

10) $y' = e^y x^2$

11) $(1 + x)y' = (x + 2)(y - 1)$

Answer

$y = Ce^x(x + 1) + 1$

12) $\frac{dx}{dt} = 3t^2(x^2 + 4)$

13) $t\frac{dy}{dt} = \sqrt{1 - y^2}$

Answer

$y = \sin(\ln|t| + C)$

14) $y' = e^x e^y$

In exercises 15 - 24, find the solution to the initial-value problem.

15) $y' = e^{y-x}, \quad y(0) = 0$

Answer

$y = -\ln(e^{-x})$ which simplifies to $y = x$

16) $y' = y^2(x+1), \quad y(0) = 2$

17) $\frac{dy}{dx} = y^3 x e^{x^2}, \quad y(0) = 1$

Answer

$$y = \frac{1}{\sqrt{2 - e^{x^2}}}$$

18) $\frac{dy}{dt} = y^2 e^x \sin(3x), \quad y(0) = 1$

19) $y' = \frac{x}{\operatorname{sech}^2 y}, \quad y(0) = 0$

Answer

$$y = \tanh^{-1}\left(\frac{x^2}{2}\right)$$

20) $y' = 2xy(1+2y), \quad y(0) = -1$

21) $\frac{dx}{dt} = \ln(t)\sqrt{1-x^2}, \quad x(1) = 0$

Answer

$x = \sin(1 - t + t\ln t)$

22) $y' = 3x^2(y^2+4), \quad y(0) = 0$

23) $y' = e^y 5^x, \quad y(0) = \ln(\ln(5))$

Answer

$y = \ln(\ln(5)) - \ln(2 - 5^x)$

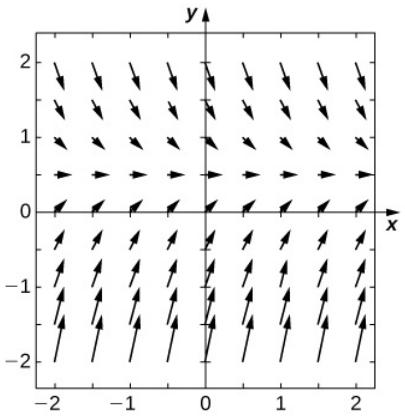
24) $y' = -2x\tan(y), \quad y(0) = \frac{\pi}{2}$

For problems 25 - 29, use a software program or your calculator to generate the directional fields. Solve explicitly and draw solution curves for several initial conditions. Are there some critical initial conditions that change the behavior of the solution?

25) [T] $y' = 1 - 2y$

Answer

$$y = Ce^{-2x} + \frac{1}{2}$$

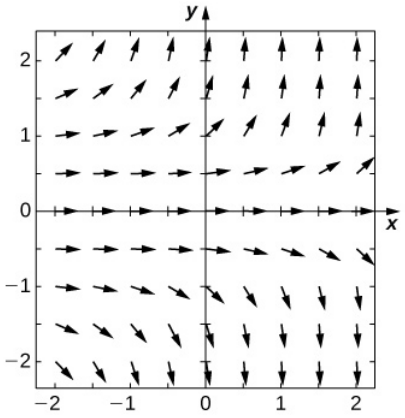


26) [T] $y' = y^2x^3$

27) [T] $y' = y^3e^x$

Answer

$$y = \frac{1}{\sqrt{2}\sqrt{C - e^x}}$$

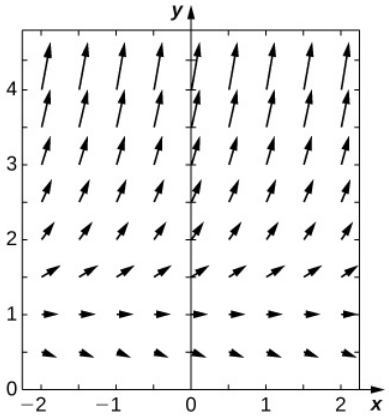


28) [T] $y' = e^y$

29) [T] $y' = y\ln(x)$

Answer

$$y = Ce^{-x}x^x$$



30) Most **drugs in the bloodstream** decay according to the equation $y' = cy$, where y is the concentration of the drug in the bloodstream. If the half-life of a drug is 2 hours, what fraction of the initial dose remains after 6 hours?

31) A drug is administered intravenously to a patient at a rate r mg/h and is cleared from the body at a rate proportional to the amount of drug still present in the body, d. Set up and solve the differential equation, assuming there is no drug initially present in the body.

Answer

$y = \frac{r}{d}(1 - e^{-dt})$

32) [T] How often should a drug be taken if its dose is 3 mg, it is cleared at a rate $c = 0.1$ mg/h, and 1 mg is required to be in the bloodstream at all times?

33) A tank contains 1 kilogram of salt dissolved in 100 liters of water. A salt solution of 0.1 kg salt/L is pumped into the tank at a rate of 2 L/min and is drained at the same rate. Solve for the salt concentration at time t. Assume the tank is well mixed.

Answer

Since the amount of salt at time t is given by the equation $A(t) = 10 - 9e^{-t/50}$, and the tank volume stays constant at 100 liters, the concentration at time t would be $c(t) = \frac{1}{100}\left(10 - 9e^{-t/50}\right)$.

34) A tank containing 10 kilograms of salt dissolved in 1000 liters of water has two salt solutions pumped in. The first solution of 0.2 kg salt/L is pumped in at a rate of 20 L/min and the second solution of 0.05 kg salt/L is pumped in at a rate of 5 L/min. The tank drains at 25 L/min. Assume the tank is well mixed. Solve for the salt concentration at time t.

35) [T] For the preceding problem, find how much salt is in the tank 1 hour after the process begins.

Answer

134.3 kilograms

36) Torricelli's law states that for a water tank with a hole in the bottom that has a cross-section of A and with a height of water h above the bottom of the tank, the rate of change of volume of water flowing from the tank is proportional to the square root of the height of water, according to $\frac{dV}{dt} = -A\sqrt{2gh}$, where g is the acceleration due to gravity. Note that $\frac{dV}{dt} = A\frac{dh}{dt}$. Solve the resulting initial-value problem for the height of the water, assuming a tank with a hole of radius 2 ft. The initial height of water is 100 ft.

37) For the preceding problem, determine how long it takes the tank to drain.

Answer

720 seconds

For problems 38 - 44, use Newton's law of cooling.

38) The liquid base of an ice cream has an initial temperature of $200°F$ before it is placed in a freezer with a constant temperature of $0°F$. After 1 hour, the temperature of the ice-cream base has decreased to $140°F$. Formulate and solve the initial-value problem to determine the temperature of the ice cream.

39) [T] The liquid base of an ice cream has an initial temperature of $210°F$ before it is placed in a freezer with a constant temperature of $20°F$. After 2 hours, the temperature of the ice-cream base has decreased to $170°F$. At what time will the ice cream be ready to eat? (Assume $30°F$ is the optimal eating temperature.)

Answer

24 hours 55 minutes

40) [T] You are organizing an ice cream social. The outside temperature is $80°F$ and the ice cream is at $10°F$. After 10 minutes, the ice cream temperature has risen by $10°F$. How much longer can you wait before the ice cream melts at $40°F$?

41) You have a cup of coffee at temperature $70°C$ and the ambient temperature in the room is $20°C$. Assuming a cooling rate k of 0.125, write and solve the differential equation to describe the temperature of the coffee with respect to time.

Answer

$T(t) = 20 + 50e^{-0.125t}$

42) [T] You have a cup of coffee at temperature $70°C$ that you put outside, where the ambient temperature is $0°C$. After 5 minutes, how much colder is the coffee?

43) You have a cup of coffee at temperature $70°C$ and you immediately pour in 1 part milk to 5 parts coffee. The milk is initially at temperature $1°C$. Write and solve the differential equation that governs the temperature of this coffee.

Answer

$T(t) = 20 + 38.5e^{-0.125t}$

44) You have a cup of coffee at temperature $70°C$, which you let cool 10 minutes before you pour in the same amount of milk at $1°C$ as in the preceding problem. How does the temperature compare to the previous cup after 10 minutes?

45) Solve the generic problem $y' = ay + b$ with initial condition $y(0) = c$.

Answer

$y = (c + ba)e^{ax} - \frac{b}{a}$

46) Prove the basic continual compounded interest equation. Assuming an initial deposit of P_0 and an interest rate of r, set up and solve an equation for continually compounded interest.

47) Assume an initial nutrient amount of I kilograms in a tank with L liters. Assume a concentration of c kg/L being pumped in at a rate of r L/min. The tank is well mixed and is drained at a rate of r L/min. Find the equation describing the amount of nutrient in the tank.

Answer

$y(t) = cL + (I - cL)e^{-rt/L}$

48) Leaves accumulate on the forest floor at a rate of 2 g/cm^2/yr and also decompose at a rate of 90% per year. Write a differential equation governing the number of grams of leaf litter per square centimeter of forest floor, assuming at time 0 there is no leaf litter on the ground. Does this amount approach a steady value? What is that value?

49) Leaves accumulate on the forest floor at a rate of 4 g/cm^2/yr. These leaves decompose at a rate of 10% per year. Write a differential equation governing the number of grams of leaf litter per square centimeter of forest floor. Does this amount approach a steady value? What is that value?

Answer

Differential Equation: $\frac{dy}{dt} = 4 - 0.1y$
Solution, the model for this situation: $y = 40(1 - e^{-0.1t})$,
Amount approaches a steady value of 40 g/cm^2

This page titled 7.3E: Exercises for Section 8.3 is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Gilbert Strang & Edwin “Jed” Herman (OpenStax) via source content that was edited to the style and standards of the LibreTexts platform.

7.4: The Logistic Equation

Learning Objectives

- Describe the concept of environmental carrying capacity in the logistic model of population growth.
- Draw a direction field for a logistic equation and interpret the solution curves.
- Solve a logistic equation and interpret the results.

Differential equations can be used to represent the size of a population as it varies over time. We saw this in an earlier chapter in the section on exponential growth and decay, which is the simplest model. A more realistic model includes other factors that affect the growth of the population. In this section, we study the logistic differential equation and see how it applies to the study of population dynamics in the context of biology.

Population Growth and Carrying Capacity

To model population growth using a differential equation, we first need to introduce some variables and relevant terms. The variable t. will represent time. The units of time can be hours, days, weeks, months, or even years. Any given problem must specify the units used in that particular problem. The variable P will represent population. Since the population varies over time, it is understood to be a function of time. Therefore we use the notation $P(t)$ for the population as a function of time. If $P(t)$ is a differentiable function, then the first derivative $\frac{dP}{dt}$ represents the instantaneous rate of change of the population as a function of time.

In Exponential Growth and Decay, we studied the exponential growth and decay of populations and radioactive substances. An example of an exponential growth function is $P(t) = P_0e^{rt}$. In this function, $P(t)$ represents the population at time t, P_0 represents the **initial population** (population at time $t = 0$), and the constant $r > 0$ is called the **growth rate**. Figure 7.4.1 shows a graph of $P(t) = 100e^{0.03t}$. Here $P_0 = 100$ and $r = 0.03$.

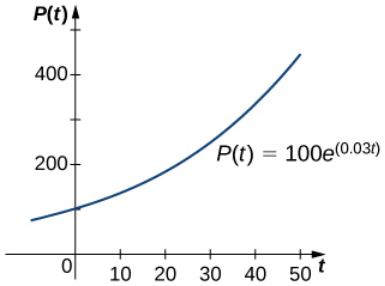


Figure 7.4.1: An exponential growth model of population.

We can verify that the function $P(t) = P_0e^{rt}$ satisfies the initial-value problem

$$\frac{dP}{dt} = rP$$

with $P(0) = P_0$.

This differential equation has an interesting interpretation. The left-hand side represents the rate at which the population increases (or decreases). The right-hand side is equal to a positive constant multiplied by the current population. Therefore the differential equation states that the rate at which the population increases is proportional to the population at that point in time. Furthermore, it states that the constant of proportionality never changes.

One problem with this function is its prediction that as time goes on, the population grows without bound. This is unrealistic in a real-world setting. Various factors limit the rate of growth of a particular population, including birth rate, death rate, food supply, predators, and so on. The growth constant r usually takes into consideration the birth and death rates but none of the other factors, and it can be interpreted as a net (birth minus death) percent growth rate per unit time. A natural question to ask is whether the population growth rate stays constant, or whether it changes over time. Biologists have found that in many biological systems, the population grows until a certain steady-state population is reached. This possibility is not taken into account with exponential

growth. However, the concept of carrying capacity allows for the possibility that in a given area, only a certain number of a given organism or animal can thrive without running into resource issues.

Definition: Carrying Capacity

The *carrying capacity* of an organism in a given environment is defined to be the maximum population of that organism that the environment can sustain indefinitely.

We use the variable K to denote the carrying capacity. The growth rate is represented by the variable r. Using these variables, we can define the logistic differential equation.

Definition: Logistic Differential Equation

Let K represent the carrying capacity for a particular organism in a given environment, and let r be a real number that represents the growth rate. The function $P(t)$ represents the population of this organism as a function of time t, and the constant P_0 represents the initial population (population of the organism at time $t = 0$). Then the *logistic differential equation* is

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right). \tag{7.4.1}$$

The logistic equation was first published by Pierre Verhulst in 1845. This differential equation can be coupled with the initial condition $P(0) = P_0$ to form an initial-value problem for $P(t)$.

Suppose that the initial population is small relative to the carrying capacity. Then $\frac{P}{K}$ is small, possibly close to zero. Thus, the quantity in parentheses on the right-hand side of Equation 7.4.1 is close to 1, and the right-hand side of this equation is close to rP. If $r > 0$, then the population grows rapidly, resembling exponential growth.

However, as the population grows, the ratio $\frac{P}{K}$ also grows, because K is constant. If the population remains below the carrying capacity, then $\frac{P}{K}$ is less than 1, so $1 - \frac{P}{K} > 0$. Therefore the right-hand side of Equation 7.4.1 is still positive, but the quantity in parentheses gets smaller, and the growth rate decreases as a result. If $P = K$ then the right-hand side is equal to zero, and the population does not change.

Now suppose that the population starts at a value higher than the carrying capacity. Then $\frac{P}{K} > 1$, and $1 - \frac{P}{K} < 0$. Then the right-hand side of Equation 7.4.1 is negative, and the population decreases. As long as $P > K$, the population decreases. It never actually reaches K because $\frac{dP}{dt}$ will get smaller and smaller, but the population approaches the carrying capacity as t approaches infinity. This analysis can be represented visually by way of a phase line. A **phase line** describes the general behavior of a solution to an autonomous differential equation, depending on the initial condition. For the case of a carrying capacity in the logistic equation, the phase line is as shown in Figure 7.4.2.



Figure 7.4.2: A phase line for the differential equation $\frac{dP}{dt} = rP(1 - \frac{P}{K})$.

This phase line shows that when P is less than zero or greater than K, the population decreases over time. When P is between 0 and K, the population increases over time.

Example 7.4.1: Examining the Carrying Capacity of a Deer Population

Let's consider the population of white-tailed deer (*Odocoileus virginianus*) in the state of Kentucky. The Kentucky Department of Fish and Wildlife Resources (KDFWR) sets guidelines for hunting and fishing in the state. Before the hunting season of 2004, it estimated a population of 900,000 deer. Johnson notes: "A deer population that has plenty to eat and is not hunted by humans or other predators will double every three years." (George Johnson, "The Problem of Exploding Deer Populations Has No Attractive Solutions," January 12,2001, accessed April 9, 2015)



Figure 7.4.3: (credit: modification of work by Rachel Kramer, Flickr)

This observation corresponds to a rate of increase $r = \frac{\ln(2)}{3} = 0.2311$, so the approximate growth rate is 23.11% per year. (This assumes that the population grows exponentially, which is reasonable—at least in the short term—with plentiful food supply and no predators.) The KDFWR also reports deer population densities for 32 counties in Kentucky, the average of which is approximately 27 deer per square mile. Suppose this is the deer density for the whole state (39,732 square miles). The carrying capacity K is 39,732 square miles times 27 deer per square mile, or 1,072,764 deer.

a. For this application, we have $P_0 = 900,000, K = 1,072,764$, and $r = 0.2311$. Substitute these values into Equation 7.4.1 and form the initial-value problem.
b. Solve the initial-value problem from part a.
c. According to this model, what will be the population in 3 years? Recall that the doubling time predicted by Johnson for the deer population was 3 years. How do these values compare?

Suppose the population managed to reach **1,200,000** What does the logistic equation predict will happen to the population in this scenario?

Solution

a. The initial value problem is

$$\frac{dP}{dt} = 0.2311P\left(1 - \frac{P}{1,072,764}\right), \; P(0) = 900,000.$$

b. The logistic equation is an autonomous differential equation, so we can use the method of separation of variables.

Step 1: Setting the right-hand side equal to zero gives $P = 0$ and $P = 1,072,764$. This means that if the population starts at zero it will never change, and if it starts at the carrying capacity, it will never change.

Step 2: Rewrite the differential equation and multiply both sides by:

$$\frac{dP}{dt} = 0.2311P\left(\frac{1,072,764 - P}{1,072,764}\right)$$

$$dP = 0.2311P\left(\frac{1,072,764 - P}{1,072,764}\right)dt$$

$$\frac{dP}{P(1,072,764 - P)} = \frac{0.2311}{1,072,764}dt.$$

Step 3: Integrate both sides of the equation using partial fraction decomposition:

$$\int \frac{dP}{P(1,072,764-P)} = \int \frac{0.2311}{1,072,764}dt$$
$$\frac{1}{1,072,764}\int \left(\frac{1}{P}+\frac{1}{1,072,764-P}\right)dP = \frac{0.2311t}{1,072,764}+C$$
$$\frac{1}{1,072,764}\left(\ln|P|-\ln|1,072,764-P|\right) = \frac{0.2311t}{1,072,764}+C.$$

Step 4: Multiply both sides by 1,072,764 and use the quotient rule for logarithms:

$$\ln\left|\frac{P}{1,072,764-P}\right| = 0.2311t + C_1.$$

Here $C_1 = 1,072,764C$. Next exponentiate both sides and eliminate the absolute value:

$$e^{\ln\left|\frac{P}{1,072,764-P}\right|} = e^{0.2311t+C_1}$$
$$\left|\frac{P}{1,072,764-P}\right| = C_2e^{0.2311t}$$
$$\frac{P}{1,072,764-P} = C_2e^{0.2311t}.$$

Here $C_2 = e^{C_1}$ but after eliminating the absolute value, it can be negative as well. Now solve for:

$$P = C_2e^{0.2311t}(1,072,764-P)$$
$$P = 1,072,764C_2e^{0.2311t} - C_2Pe^{0.2311t}$$
$$P + C_2Pe^{0.2311t} = 1,072,764C_2e^{0.2311t}$$
$$P(1+C_2e^{0.2311t} = 1,072,764C_2e^{0.2311t}$$
$$P(t) = \frac{1,072,764C_2e^{0.2311t}}{1+C_2e^{0.2311t}}.$$

Step 5: To determine the value of C_2, it is actually easier to go back a couple of steps to where C_2 was defined. In particular, use the equation

$$\frac{P}{1,072,764-P} = C_2e^{0.2311t}.$$

The initial condition is $P(0) = 900,000$. Replace P with $900,000$ and t with zero:

$$\frac{P}{1,072,764-P} = C_2e^{0.2311t}$$
$$\frac{900,000}{1,072,764-900,000} = C_2e^{0.2311(0)}$$
$$\frac{900,000}{172,764} = C_2$$
$$C_2 = \frac{25,000}{4,799}$$
$$\approx 5.209.$$

Therefore

$$P(t) = \frac{1,072,764\left(\frac{25000}{4799}\right)e^{0.2311t}}{1+\left(\frac{25000}{4799}\right)e^{0.2311t}}$$
$$= \frac{1,072,764(25000)e^{0.2311t}}{4799+25000e^{0.2311t}.}$$

Dividing the numerator and denominator by **25,000** gives

$$P(t) = \frac{1,072,764e^{0.2311t}}{0.19196 + e^{0.2311t}}.$$

Figure **7.4.4** is a graph of this equation.

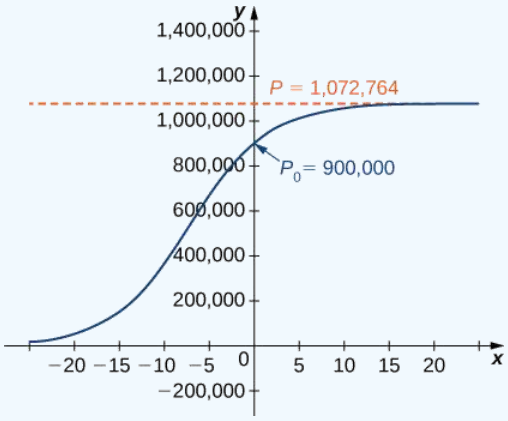


Figure **7.4.4**: Logistic curve for the deer population with an initial population of **900,000** deer.

c. Using this model we can predict the population in **3** years.

$$P(3) = \frac{1,072,764e^{0.2311(3)}}{0.19196 + e^{0.2311(3)}} \approx 978,830\, deer$$

This is far short of twice the initial population of $900,000$. Remember that the doubling time is based on the assumption that the growth rate never changes, but the logistic model takes this possibility into account.

d. If the population reached **1,200,000** deer, then the new initial-value problem would be

$$\frac{dP}{dt} = 0.2311P\left(1 - \frac{P}{1,072,764}\right),\ P(0) = 1,200,000.$$

The general solution to the differential equation would remain the same.

$$P(t) = \frac{1,072,764C_2e^{0.2311t}}{1 + C_2e^{0.2311t}}$$

To determine the value of the constant, return to the equation

$$\frac{P}{1,072,764 - P} = C_2e^{0.2311t}.$$

Substituting the values $t = 0$ and $P = 1,200,000$, you get

$$C_2e^{0.2311(0)} = \frac{1,200,000}{1,072,764 - 1,200,000}$$

$$C_2 = -\frac{100,000}{10,603} \approx -9.431.$$

Therefore

$$P(t) = \frac{1,072,764C_2e^{0.2311t}}{1 + C_2e^{0.2311t}}$$

$$= \frac{1,072,764\left(-\frac{100,000}{10,603}\right)e^{0.2311t}}{1 + \left(-\frac{100,000}{10,603}\right)e^{0.2311t}}$$

$$= -\frac{107,276,400,000e^{0.2311t}}{100,000e^{0.2311t} - 10,603}$$

$$\approx \frac{10,117,551e^{0.2311t}}{9.43129e^{0.2311t} - 1}$$

This equation is graphed in Figure 7.4.5.

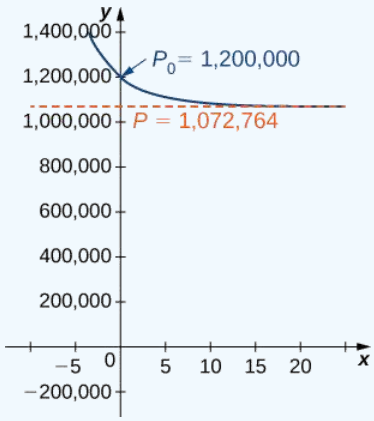


Figure 7.4.5: Logistic curve for the deer population with an initial population of **1,200,000** deer.

Solving the Logistic Differential Equation

The logistic differential equation is an autonomous differential equation, so we can use separation of variables to find the general solution, as we just did in Example 7.4.1.

Step 1: Setting the right-hand side equal to zero leads to $P = 0$ and $P = K$ as constant solutions. The first solution indicates that when there are no organisms present, the population will never grow. The second solution indicates that when the population starts at the carrying capacity, it will never change.

Step 2: Rewrite the differential equation in the form

$$\frac{dP}{dt} = \frac{rP(K-P)}{K}.$$

Then multiply both sides by dt and divide both sides by $P(K-P)$. This leads to

$$\frac{dP}{P(K-P)} = \frac{r}{K}dt.$$

Multiply both sides of the equation by K and integrate:

$$\int \frac{K}{P(K-P)}dP = \int r dt. \tag{7.4.2}$$

The left-hand side of this equation can be integrated using partial fraction decomposition. We leave it to you to verify that

$$\frac{K}{P(K-P)} = \frac{1}{P} + \frac{1}{K-P}.$$

Then the Equation 7.4.2 becomes

$$\int \frac{1}{P} + \frac{1}{K-P}dP = \int r dt$$

$$\ln|P| - \ln|K-P| = rt + C$$

$$\ln\left|\frac{P}{K-P}\right| = rt + C.$$

Now exponentiate both sides of the equation to eliminate the natural logarithm:

$$e^{\ln\left|\frac{P}{K-P}\right|} = e^{rt+C}$$

$$\left|\frac{P}{K-P}\right| = e^C e^{rt}.$$

We define $C_1 = e^c$ so that the equation becomes

$$\frac{P}{K-P} = C_1 e^{rt}. \tag{7.4.3}$$

To solve this equation for $P(t)$, first multiply both sides by $K-P$ and collect the terms containing P on the left-hand side of the equation:

$$\begin{aligned} P &= C_1 e^{rt}(K-P) \\ &= C_1 K e^{rt} - C_1 P e^{rt} \\ P + C_1 P e^{rt} &= C_1 K e^{rt}. \end{aligned}$$

Next, factor P from the left-hand side and divide both sides by the other factor:

$$\begin{aligned} P(1 + C_1 e^{rt}) &= C_1 K e^{rt} \\ P(t) &= \frac{C_1 K e^{rt}}{1 + C_1 e^{rt}}. \end{aligned}$$

The last step is to determine the value of C_1. The easiest way to do this is to substitute $t = 0$ and P_0 in place of P in Equation and solve for C_1:

$$\begin{aligned} \frac{P}{K-P} &= C_1 e^{rt} \\ \frac{P_0}{K-P_0} &= C_1 e^{r(0)} \\ C_1 &= \frac{P_0}{K-P_0}. \end{aligned}$$

Finally, substitute the expression for C_1 into Equation 7.4.3:

$$P(t) = \frac{C_1 K e^{rt}}{1 + C_1 e^{rt}} = \frac{\frac{P_0}{K-P_0} K e^{rt}}{1 + \frac{P_0}{K-P_0} e^{rt}}$$

Now multiply the numerator and denominator of the right-hand side by $(K-P_0)$ and simplify:

$$\begin{aligned} P(t) &= \frac{\frac{P_0}{K-P_0} K e^{rt}}{1 + \frac{P_0}{K-P_0} e^{rt}} \\ &= \frac{\frac{P_0}{K-P_0} K e^{rt}}{1 + \frac{P_0}{K-P_0} e^{rt}} \cdot \frac{K-P_0}{K-P_0} \\ &= \frac{P_0 K e^{rt}}{(K-P_0) + P_0 e^{rt}}. \end{aligned}$$

We state this result as a theorem.

Solution of the Logistic Differential Equation

Consider the logistic differential equation subject to an initial population of P_0 with carrying capacity K and growth rate r. The solution to the corresponding initial-value problem is given by

$$P(t) = \frac{P_0 K e^{rt}}{(K-P_0) + P_0 e^{rt}}$$

.

Now that we have the solution to the initial-value problem, we can choose values for P_0, r, and K and study the solution curve. For example, in Example we used the values $r = 0.2311, K = 1,072,764,$ and an initial population of $900,000$ deer. This leads to the

solution

$$P(t) = \frac{P_0 K e^{rt}}{(K - P_0) + P_0 e^{rt}}$$
$$= \frac{900,000(1,072,764)e^{0.2311t}}{(1,072,764 - 900,000) + 900,000e^{0.2311t}}$$
$$= \frac{900,000(1,072,764)e^{0.2311t}}{172,764 + 900,000e^{0.2311t}}.$$

Dividing top and bottom by $900,000$ gives

$$P(t) = \frac{1,072,764e^{0.2311t}}{0.19196 + e^{0.2311t}}.$$

This is the same as the original solution. The graph of this solution is shown again in blue in Figure $7.4.6$, superimposed over the graph of the exponential growth model with initial population $900,000$ and growth rate 0.2311 (appearing in green). The red dashed line represents the carrying capacity, and is a horizontal asymptote for the solution to the logistic equation.

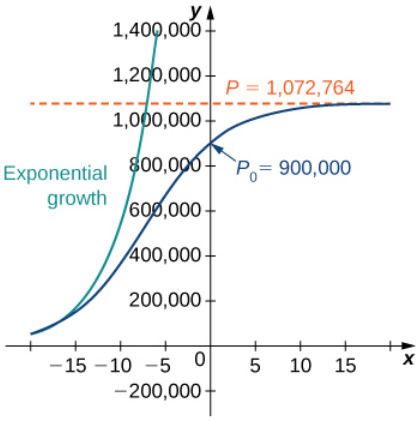


Figure $7.4.6$: A comparison of exponential versus logistic growth for the same initial population of $900,000$ organisms and growth rate of 23.11

Working under the assumption that the population grows according to the logistic differential equation, this graph predicts that approximately 20 years earlier (1984), the growth of the population was very close to exponential. The net growth rate at that time would have been around 23.1 per year. As time goes on, the two graphs separate. This happens because the population increases, and the logistic differential equation states that the growth rate decreases as the population increases. At the time the population was measured (2004), it was close to carrying capacity, and the population was starting to level off.

The solution to the logistic differential equation has a point of inflection. To find this point, set the second derivative equal to zero:

$$P(t) = \frac{P_0 K e^{rt}}{(K - P_0) + P_0 e^{rt}}$$
$$P'(t) = \frac{rP_0 K(K - P_0)e^{rt}}{((K - P_0) + P_0 e^{rt})^2}$$
$$P''(t) = \frac{r^2 P_0 K(K - P_0)^2 e^{rt} - r^2 P_0^2 K(K - P_0)e^{2rt}}{((K - P_0) + P_0 e^{rt})^3}$$
$$= \frac{r^2 P_0 K(K - P_0)e^{rt}((K - P_0) - P_0 e^{rt})}{((K - P_0) + P_0 e^{rt})^3}.$$

Setting the numerator equal to zero,

$$r^2 P_0 K(K - P_0)e^{rt}((K - P_0) - P_0 e^{rt}) = 0.$$

As long as $P_0 \neq K$, the entire quantity before and including $e^{rt}$is nonzero, so we can divide it out:

$$(K - P_0) - P_0 e^{rt} = 0.$$

Solving for t,

$$P_0 e^{rt} = K - P_0$$

$$e^{rt} = \frac{K - P_0}{P_0}$$

$$\ln e^{rt} = \ln \frac{K - P_0}{P_0}$$

$$rt = \ln \frac{K - P_0}{P_0}$$

$$t = \frac{1}{r} \ln \frac{K - P_0}{P_0}.$$

Notice that if $P_0 > K$, then this quantity is undefined, and the graph does not have a point of inflection. In the logistic graph, the point of inflection can be seen as the point where the graph changes from concave up to concave down. This is where the "leveling off" starts to occur, because the net growth rate becomes slower as the population starts to approach the carrying capacity.

Exercise 7.4.1

A population of rabbits in a meadow is observed to be 200 rabbits at time $t = 0$. After a month, the rabbit population is observed to have increased by 4. Using an initial population of 200 and a growth rate of 0.04, with a carrying capacity of 750 rabbits,

a. Write the logistic differential equation and initial condition for this model.
b. Draw a slope field for this logistic differential equation, and sketch the solution corresponding to an initial population of 200 rabbits.
c. Solve the initial-value problem for $P(t)$.
d. Use the solution to predict the population after 1 year.

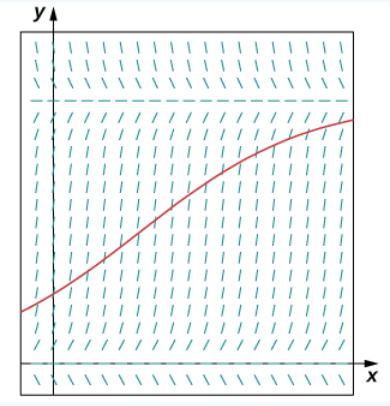
Hint

First determine the values of r, K, and P_0. Then create the initial-value problem, draw the direction field, and solve the problem.

Answer

a. $\frac{dP}{dt} = 0.04(1 - \frac{P}{750}), P(0) = 200$

b.



c. $P(t) = \frac{3000e^{.04t}}{11 + 4e^{.04t}}$

d. After 12 months, the population will be $P(12) \approx 278$ rabbits.

Student Project: Logistic Equation with a Threshold Population

An improvement to the logistic model includes a **threshold population.** The threshold population is defined to be the minimum population that is necessary for the species to survive. We use the variable T to represent the threshold population. A differential equation that incorporates both the threshold population T and carrying capacity K is

$$\frac{dP}{dt} = -rP\left(1 - \frac{P}{K}\right)\left(1 - \frac{P}{T}\right)$$

where r represents the growth rate, as before.

1. The threshold population is useful to biologists and can be utilized to determine whether a given species should be placed on the endangered list. A group of Australian researchers say they have determined the threshold population for any species to survive: 5000 adults. (Catherine Clabby, "A Magic Number," American Scientist 98(1): 24, doi:10.1511/2010.82.24. accessed April 9, 2015, www.americanscientist.org/iss...a-magic-number). Therefore we use $T = 5000$ as the threshold population in this project. Suppose that the environmental carrying capacity in Montana for elk is $25,000$. Set up Equation using the carrying capacity of $25,000$ and threshold population of 5000. Assume an annual net growth rate of 18%.
2. Draw the direction field for the differential equation from step 1, along with several solutions for different initial populations. What are the constant solutions of the differential equation? What do these solutions correspond to in the original population model (i.e., in a biological context)?
3. What is the limiting population for each initial population you chose in step 2? (Hint: use the slope field to see what happens for various initial populations, i.e., look for the horizontal asymptotes of your solutions.)
4. This equation can be solved using the method of separation of variables. However, it is very difficult to get the solution as an explicit function of t. Using an initial population of $18,000$ elk, solve the initial-value problem and express the solution as an implicit function of t, or solve the general initial-value problem, finding a solution in terms of r, K, T, and P_0.

Key Concepts

- When studying population functions, different assumptions—such as exponential growth, logistic growth, or threshold population—lead to different rates of growth.
- The logistic differential equation incorporates the concept of a carrying capacity. This value is a limiting value on the population for any given environment.
- The logistic differential equation can be solved for any positive growth rate, initial population, and carrying capacity.

Key Equations

- **Logistic differential equation and initial-value problem**

  $$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right), \quad P(0) = P_0$$

- **Solution to the logistic differential equation/initial-value problem**

  $$P(t) = \frac{P_0 K e^{rt}}{(K - P_0) + P_0 e^{rt}}$$

- **Threshold population model**

  $$\frac{dP}{dt} = -rP\left(1 - \frac{P}{K}\right)\left(1 - \frac{P}{T}\right)$$

Glossary

carrying capacity
the maximum population of an organism that the environment can sustain indefinitely

growth rate
the constant $r > 0$ in the exponential growth function $P(t) = P_0 e^{rt}$

initial population

the population at time $t = 0$

logistic differential equation

a differential equation that incorporates the carrying capacity K and growth rate rr into a population model

phase line

a visual representation of the behavior of solutions to an autonomous differential equation subject to various initial conditions

threshold population

the minimum population that is necessary for a species to survive

---

This page titled 7.4: The Logistic Equation is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Gilbert Strang & Edwin "Jed" Herman (OpenStax) via source content that was edited to the style and standards of the LibreTexts platform.

- **8.4: The Logistic Equation** by Edwin "Jed" Herman, Gilbert Strang is licensed CC BY-NC-SA 4.0. Original source: https://openstax.org/details/books/calculus-volume-1.

7.4E: Exercises for Section 8.4

Basic Logistic Model

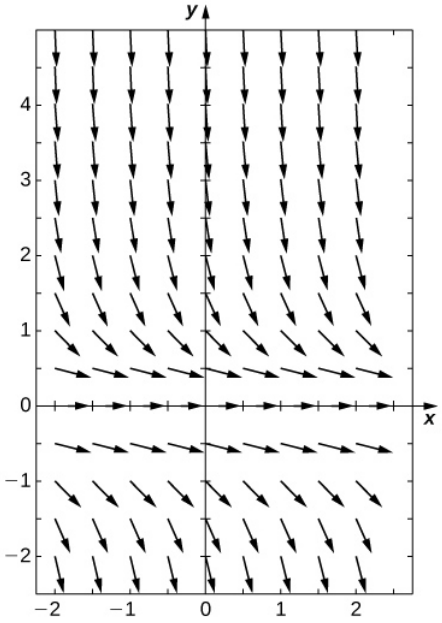
For problems 1 - 11, consider the logistic equation in the form $P' = CP - P^2$. Draw the directional field and find the stability of the equilibria.

1) $C = 3$

2) $C = 0$

Answer

$P = 0$ semi-stable



3) $C = -3$

4) Solve the logistic equation for $C = 10$ and an initial condition of $P(0) = 2$.

Answer

$$P = \frac{10e^{10x}}{e^{10x} + 4}$$

5) Solve the logistic equation for $C = -10$ and an initial condition of $P(0) = 2$.

6) A population of deer inside a park has a carrying capacity of 200 and a growth rate of 2. If the initial population is 50 deer, what is the population of deer at any given time?

Answer

$$P(t) = \frac{10000e^{0.02t}}{150 + 50e^{0.02t}}$$

7) A population of frogs in a pond has a growth rate of 5 If the initial population is 1000 frogs and the carrying capacity is 6000, what is the population of frogs at any given time?

8) [T] Bacteria grow at a rate of 20 per hour in a petri dish. If there is initially one bacterium and a carrying capacity of 1 million cells, how long does it take to reach $500,000$ cells?

Answer

69 hours 5 minutes

9) [T] Rabbits in a park have an initial population of 10 and grow at a rate of 4 per year. If the carrying capacity is 500, at what time does the population reach 100 rabbits?

10) [T] Two monkeys are placed on an island. After 5 years, there are 8 monkeys, and the estimated carrying capacity is 25 monkeys. When does the population of monkeys reach 16 monkeys?

Answer

8 years 11 months

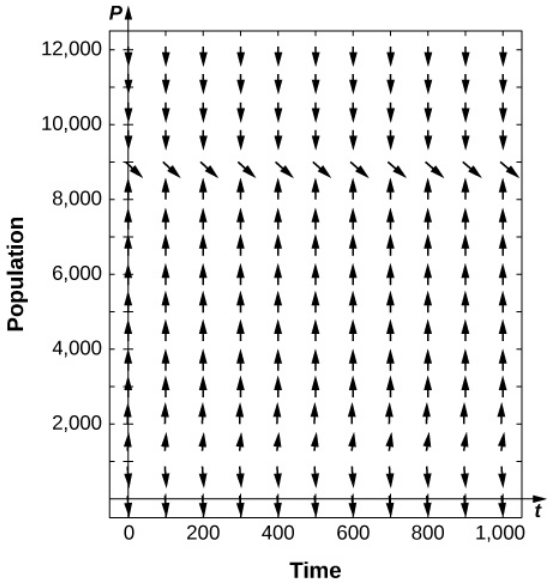
11) [T] A butterfly sanctuary is built that can hold 2000 butterflies, and 400 butterflies are initially moved in. If after 2 months there are now 800 butterflies, when does the population get to 1500 butterflies?

Logistic Population Model with Depletion

The following problems consider the logistic equation with an added term for depletion, either through death or emigration.

12) [T] The population of trout in a pond is given by $P' = 0.4P\left(1 - \frac{P}{10000}\right) - 400$, where 400 trout are caught per year. Use your calculator or computer software to draw a directional field and draw a few sample solutions. What do you expect for the behavior?

Answer

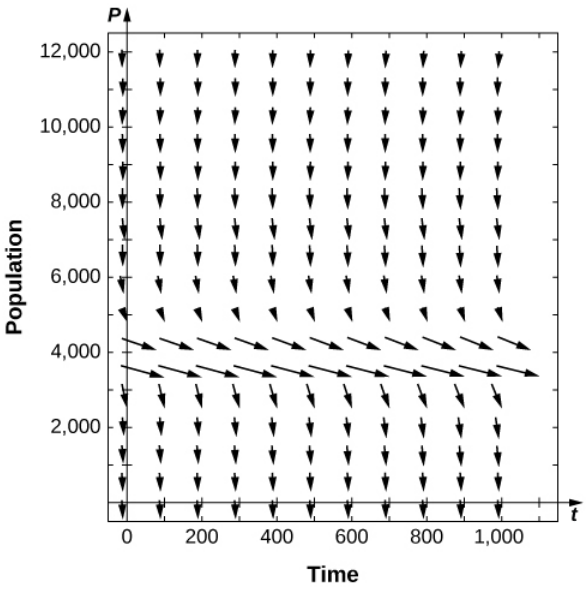


13) In the preceding problem, what are the stabilities of the equilibria $0 < P_1 < P_2$?

14) [T] For the preceding problem, use software to generate a directional field for the value $f = 400$. What are the stabilities of the equilibria?

Answer

P_1 semi-stable

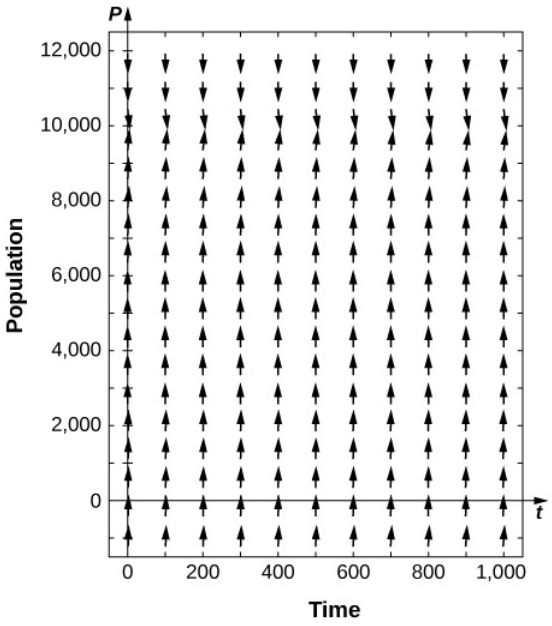


15) [T] For the preceding problems, use software to generate a directional field for the value $f = 600$. What are the stabilities of the equilibria?

16) [T] For the preceding problems, consider the case where a certain number of fish are added to the pond, or $f = -200$. What are the nonnegative equilibria and their stabilities?

Answer

$P_2 > 0$ stable



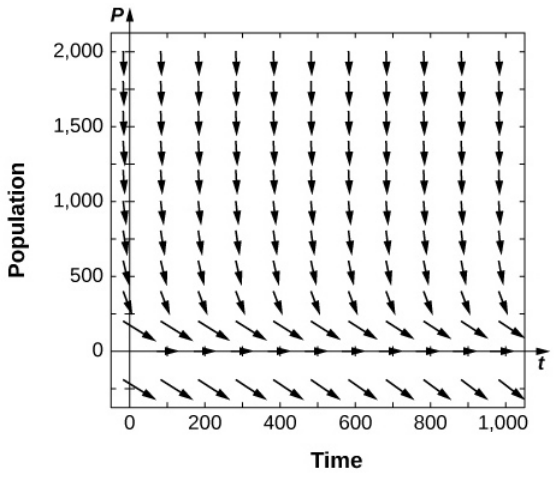
It is more likely that the amount of fishing is governed by the current number of fish present, so instead of a constant number of fish being caught, the rate is proportional to the current number of fish present, with proportionality constant k, as $P' = 0.4P\left(1 - \frac{P}{10000}\right) - kP$.

17) [T] For the previous fishing problem, draw a directional field assuming $k = 0.1$. Draw some solutions that exhibit this behavior. What are the equilibria and what are their stabilities?

18) [T] Use software or a calculator to draw directional fields for $k = 0.4$. What are the nonnegative equilibria and their stabilities?

Answer

$P_1 = 0$ is semi-stable



19) [T] Use software or a calculator to draw directional fields for $k = 0.6$. What are the equilibria and their stabilities?

20) Solve this equation, assuming a value of $k = 0.05$ and an initial condition of 2000 fish.

Answer

$$y = \frac{-20}{4 \times 10^{-6} - 0.002e^{0.01t}}$$

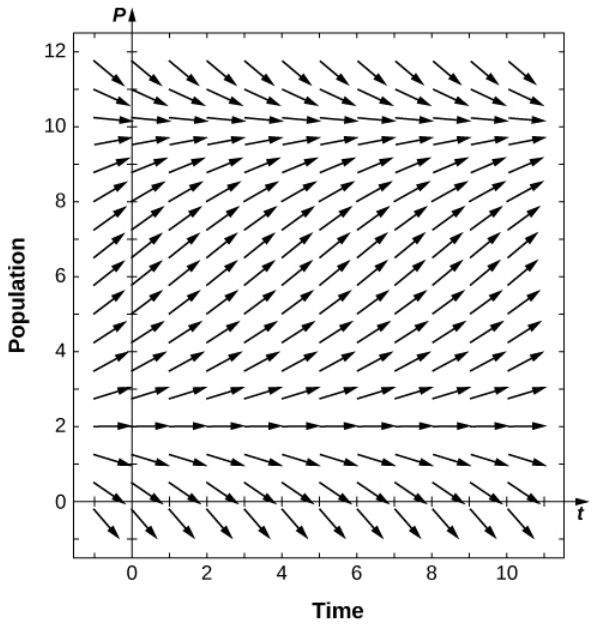
21) Solve this equation, assuming a value of $k = 0.05$ and an initial condition of 5000 fish.

Minimal Sustainable Population Thresholds

The following problems add in a minimal threshold value for the species to survive, T, which changes the differential equation to $P'(t) = rP\left(1 - \frac{P}{K}\right)\left(1 - \frac{T}{P}\right)$.

22) Draw the directional field of the threshold logistic equation, assuming $K = 10, r = 0.1, T = 2$. When does the population survive? When does it go extinct?

Answer



23) For the preceding problem, solve the logistic threshold equation, assuming the initial condition $P(0) = P_0$.

24) Bengal tigers in a conservation park have a carrying capacity of 100 and need a minimum of 10 to survive. If they grow in population at a rate of 1 per year, with an initial population of 15 tigers, solve for the number of tigers present.

Answer

$$P(t) = \frac{850 + 500e^{0.009t}}{85 + 5e^{0.009t}}$$

25) A forest containing ring-tailed lemurs in Madagascar has the potential to support 5000 individuals, and the lemur population grows at a rate of 5 per year. A minimum of 500 individuals is needed for the lemurs to survive. Given an initial population of 600 lemurs, solve for the population of lemurs.

26) The population of mountain lions in Northern Arizona has an estimated carrying capacity of 250 and grows at a rate of 0.25 per year and there must be 25 for the population to survive. With an initial population of 30 mountain lions, how many years will it take to get the mountain lions off the endangered species list (at least 100)?

Answer

13 years months

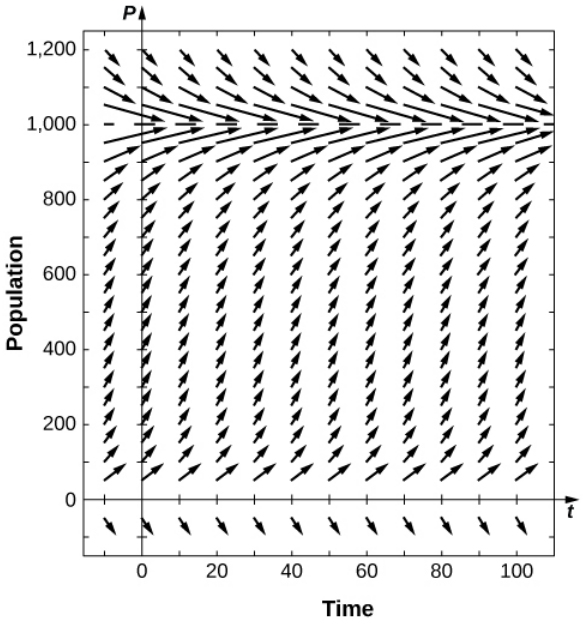
The Gompertz Equation

The following questions consider the Gompertz equation, a modification for logistic growth, which is often used for modeling cancer growth, specifically the number of tumor cells.

27) The Gompertz equation is given by $P(t)' = \alpha \ln\left(\frac{K}{P(t)}\right) P(t)$. Draw the directional fields for this equation assuming all parameters are positive, and given that $K = 1$.

28) Assume that for a population, $K = 1000$ and $\alpha = 0.05$. Draw the directional field associated with this differential equation and draw a few solutions. What is the behavior of the population?

Answer



29) Solve the Gompertz equation for generic α and K and $P(0) = P_0$.

30) [T] The Gompertz equation has been used to model tumor growth in the human body. Starting from one tumor cell on day 1 and assuming $\alpha = 0.1$ and a carrying capacity of 10 million cells, how long does it take to reach "detection" stage at 5 million cells?

Answer

31.465 days

31) [T] It is estimated that the world human population reached 3 billion people in 1959 and 6 billion in 1999. Assuming a carrying capacity of 16 billion humans, write and solve the differential equation for logistic growth, and determine what year the population reached 7 billion.

32) [T] It is estimated that the world human population reached 3 billion people in 1959 and 6 billion in 1999. Assuming a carrying capacity of 16 billion humans, write and solve the differential equation for Gompertz growth, and determine what year the population reached 7 billion. Was logistic growth or Gompertz growth more accurate, considering world population reached 7 billion on October $31, 2011$?

Answer

September 2008

33) Show that the population grows fastest when it reaches half the carrying capacity for the logistic equation $P' = rP\left(1 - \frac{P}{K}\right)$.

34) When does population increase the fastest in the threshold logistic equation $P'(t) = rP\left(1 - \frac{P}{K}\right)\left(1 - \frac{T}{P}\right)$?

Answer

$\frac{K+T}{2}$

35) When does population increase the fastest for the Gompertz equation $P(t)' = \alpha \ln\left(\frac{K}{P(t)}\right) P(t)$?

Below is a table of the populations of whooping cranes in the wild from 1940 to 2000. The population rebounded from near extinction after conservation efforts began. The following problems consider applying population models to fit the data. Assume a carrying capacity of $10,000$ cranes. Fit the data assuming years since 1940 (so your initial population at time 0 would be 22 cranes).

Year (years since conservation began)	Whooping Crane Population
1940(0)	22
1950(10)	31
1960(20)	36
1970(30)	57
1980(40)	91
1990(50)	159
2000(60)	256

Source: www.savingcranes.org/images/...wc_numbers.pdf

36) Find the equation and parameter r that best fit the data for the logistic equation.

Answer

$r = 0.0405$

37) Find the equation and parameters r and T that best fit the data for the threshold logistic equation.

38) Find the equation and parameter α that best fit the data for the Gompertz equation.

Answer

$\alpha = 0.0081$

39) Graph all three solutions and the data on the same graph. Which model appears to be most accurate?

40) Using the three equations found in the previous problems, estimate the population in 2010 (year 70 after conservation). The real population measured at that time was 437. Which model is most accurate?

Answer

Logistic: 361, Threshold: 436, Gompertz: 309.

This page titled 7.4E: Exercises for Section 8.4 is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Gilbert Strang & Edwin “Jed” Herman (OpenStax) via source content that was edited to the style and standards of the LibreTexts platform.

7.5: First-order Linear Equations

Learning Objectives

- Write a first-order linear differential equation in standard form.
- Find an integrating factor and use it to solve a first-order linear differential equation.
- Solve applied problems involving first-order linear differential equations.

Earlier, we studied an application of a first-order differential equation that involved solving for the velocity of an object. In particular, if a ball is thrown upward with an initial velocity of v_0 ft/s, then an initial-value problem that describes the velocity of the ball after t seconds is given by

$$\frac{dv}{dt} = -32$$

with $v(0) = v_0$.

This model assumes that the only force acting on the ball is gravity. Now we add to the problem by allowing for the possibility of **air resistance** acting on the ball.

Air resistance always acts in the direction opposite to motion. Therefore if an object is rising, air resistance acts in a downward direction. If the object is falling, air resistance acts in an upward direction (Figure 7.5.1). There is no exact relationship between the velocity of an object and the air resistance acting on it. For very small objects, air resistance is proportional to velocity; that is, the force due to air resistance is numerically equal to some constant k times v. For larger (e.g., baseball-sized) objects, depending on the shape, air resistance can be approximately proportional to the square of the velocity. In fact, air resistance may be proportional to $v^{1.5}$, or $v^{0.9}$, or some other power of v.

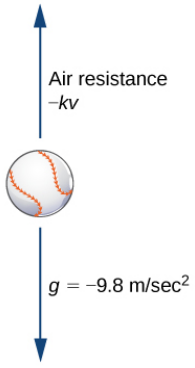


Figure 7.5.1: Forces acting on a moving baseball: gravity acts in a downward direction and air resistance acts in a direction opposite to the direction of motion.

We will work with the linear approximation for air resistance. If we assume $k > 0$, then the expression for the force F_A due to air resistance is given by $F_A = -kv$. Therefore the sum of the forces acting on the object is equal to the sum of the gravitational force and the force due to air resistance. This, in turn, is equal to the mass of the object multiplied by its acceleration at time t(Newton's second law). This gives us the differential equation

$$m\frac{dv}{dt} = -kv - mg.$$

Finally, we impose an initial condition $v(0) = v_0$, where v_0 is the initial velocity measured in meters per second. This makes $g = 9.8m/s^2$. The initial-value problem becomes

$$m\frac{dv}{dt} = -kv - mg$$

with $v(0) = v_0$.

The differential equation in this initial-value problem is an example of a first-order linear differential equation. (Recall that a differential equation is first-order if the highest-order derivative that appears in the equation is 1.) In this section, we study first-order linear equations and examine a method for finding a general solution to these types of equations, as well as solving initial-value problems involving them.

Definition: Linear first-order differential equation

A first-order differential equation is *linear* if it can be written in the form

$$a(x)y'+b(x)y=c(x), \tag{7.5.1}$$

where $a(x), b(x)$, and $c(x)$ are arbitrary functions of x.

Remember that the unknown function y depends on the variable x; that is, x is the independent variable and y is the dependent variable. Some examples of first-order linear differential equations are

$$\begin{aligned}(3x^2-4)y'+(x-3)y&=\sin x\\ (\sin x)y'-(\cos x)y&=\cot x\\ 4xy'+(3\ln x)y&=x^3-4x.\end{aligned}$$

Examples of first-order nonlinear differential equations include

$$\begin{aligned}(y')^4-(y')^3&=(3x-2)(y+4)\\ 4y'+3y^3&=4x-5\\ (y')^2&=\sin y+\cos x.\end{aligned}$$

These equations are nonlinear because of terms like $(y')^4, y^3$, etc. Due to these terms, it is impossible to put these equations into the same form as Equation 7.5.1.

Standard Form

Consider the differential equation

$$(3x^2-4)y'+(x-3)y=\sin x.$$

Our main goal in this section is to derive a solution method for equations of this form. It is useful to have the coefficient of y' be equal to 1. To make this happen, we divide both sides by $3x^2-4$.

$$y'+\left(\frac{x-3}{3x^2-4}\right)y=\frac{\sin x}{3x^2-4}$$

This is called the **standard form** of the differential equation. We will use it later when finding the solution to a general first-order linear differential equation. Returning to Equation 7.5.1, we can divide both sides of the equation by $a(x)$. This leads to the equation

$$y'+\frac{b(x)}{a(x)}y=\frac{c(x)}{a(x)}. \tag{7.5.2}$$

Now define

$$p(x)=\frac{b(x)}{a(x)}$$

and

$$q(x)=\frac{c(x)}{a(x)}$$

Then Equation 7.5.2 becomes

$$y'+p(x)y=q(x).$$

We can write any first-order linear differential equation in this form, and this is referred to as the standard form for a first-order linear differential equation.

Example 7.5.1: Writing First-Order Linear Equations in Standard Form

Put each of the following first-order linear differential equations into standard form. Identify $p(x)$ and $q(x)$ for each equation.

a. $y' = 3x - 4y$

b. $\dfrac{3xy'}{4y-3} = 2$ (here $x > 0$)

c. $y = 3y' - 4x^2 + 5$

Solution

a. Add $4y$ to both sides:

$$y' + 4y = 3x.$$

In this equation, $p(x) = 4$ and $q(x) = 3x$.

b. Multiply both sides by $4y - 3$, then subtract $8y$ from each side:

$$\begin{aligned} \frac{3xy'}{4y-3} &= 2 \\ 3xy' &= 2(4y-3) \\ 3xy' &= 8y - 6 \\ 3xy' - 8y &= -6. \end{aligned}$$

Finally, divide both sides by $3x$ to make the coefficient of y' equal to 1:

$$y' - \frac{8}{3x}y = -\frac{2}{3x}.$$

This is allowable because in the original statement of this problem we assumed that $x > 0$. (If $x = 0$ then the original equation becomes $0 = 2$, which is clearly a false statement.)

In this equation, $p(x) = -\dfrac{8}{3x}$ and $q(x) = -\dfrac{2}{3x}$.

c. Subtract y from each side and add $4x^2 - 5$:

$$3y' - y = 4x^2 - 5.$$

Next divide both sides by 3:

$$y' - \frac{1}{3}y = \frac{4}{3}x^2 - \frac{5}{3}.$$

In this equation, $p(x) = -\dfrac{1}{3}$ and $q(x) = \dfrac{4}{3}x^2 - \dfrac{5}{3}$.

Exercise 7.5.1

Put the equation $\dfrac{(x+3)y'}{2x - 3y - 4} = 5$ into standard form and identify $p(x)$ and $q(x)$.

Hint

Multiply both sides by the common denominator, then collect all terms involving y on one side.

Answer

$$y' + \frac{15}{x+3}y = \frac{10x - 20}{x+3}$$

$$p(x) = \frac{15}{x+3}$$

and

$$q(x) = \frac{10x - 20}{x+3}$$

Integrating Factors

We now develop a solution technique for any first-order linear differential equation. We start with the standard form of a first-order linear differential equation:

$$y' + p(x)y = q(x). \tag{7.5.3}$$

The first term on the left-hand side of Equation 7.5.3 is the derivative of the unknown function, and the second term is the product of a known function with the unknown function. This is somewhat reminiscent of the power rule. If we multiply Equation 7.5.3 by a yet-to-be-determined function $\mu(x)$, then the equation becomes

$$\mu(x)y' + \mu(x)p(x)y = \mu(x)q(x). \tag{7.5.4}$$

The left-hand side Equation 7.5.4 can be matched perfectly to the product rule:

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

Matching term by term gives $y = f(x), g(x) = \mu(x)$, and $g'(x) = \mu(x)p(x)$. Taking the derivative of $g(x) = \mu(x)$ and setting it equal to the right-hand side of $g'(x) = \mu(x)p(x)$ leads to

$$\mu'(x) = \mu(x)p(x).$$

This is a first-order, separable differential equation for $\mu(x)$. We know $p(x)$ because it appears in the differential equation we are solving. Separating variables and integrating yields

$$\begin{aligned}
\frac{\mu'(x)}{\mu(x)} &= p(x) \\
\int \frac{\mu'(x)}{\mu(x)}dx &= \int p(x)dx \\
\ln|\mu(x)| &= \int p(x)dx + C \\
e^{\ln|\mu(x)|} &= e^{\int p(x)dx + C} \\
|\mu(x)| &= C_1 e^{\int p(x)dx} \\
\mu(x) &= C_2 e^{\int p(x)dx}.
\end{aligned}$$

Here C_2 can be an arbitrary (positive or negative) constant. This leads to a general method for solving a first-order linear differential equation. We first multiply both sides of Equation 7.5.3 by this integrating factor by **integrating factor** $\mu(x)$. This gives

$$\mu(x)y' + \mu(x)p(x)y = \mu(x)q(x). \tag{7.5.5}$$

Note that, since the non-zero constant C_2 can be divided out of this equation (as it will appear in every term on both sides), we do not need to use it in this process. That is, we can simply use

$$\mu(x) = e^{\int p(x)dx}.$$

The left-hand side of Equation 7.5.5 can be rewritten as $\frac{d}{dx}(\mu(x)y)$.

$$\frac{d}{dx}(\mu(x)y) = \mu(x)q(x). \tag{7.5.6}$$

Next integrate both sides of Equation 7.5.6 with respect to x.

$$\int \frac{d}{dx}(\mu(x)y)dx = \int \mu(x)q(x)dx$$
$$\mu(x)y = \int \mu(x)q(x)dx$$

Divide both sides of Equation 7.5.6 by $\mu(x)$:

$$y = \frac{1}{\mu(x)}\left[\int \mu(x)q(x)dx + C\right]. \tag{7.5.7}$$

Since $\mu(x)$ was previously calculated, we are now finished. An important note about the integrating constant C: It may seem that we are inconsistent in the usage of the integrating constant. However, the integral involving $p(x)$ is necessary in order to find an integrating factor for Equation 7.5.3. Only one integrating factor is needed in order to solve the equation; therefore, it is safe to assign a value for C for this integral. We chose $C = 0$. When calculating the integral inside the brackets in Equation 7.5.7, it is necessary to keep our options open for the value of the integrating constant, because our goal is to find a general family of solutions to Equation 7.5.3. This integrating factor guarantees just that.

Problem-Solving Strategy: Solving a First-order Linear Differential Equation

1. Put the equation into standard form and identify $p(x)$ and $q(x)$.
2. Calculate the integrating factor

$$\mu(x) = e^{\int p(x)dx}. \tag{7.5.8}$$

3. Multiply both sides of the differential equation by $\mu(x)$.
4. Integrate both sides of the equation obtained in step **3**, and divide both sides by $\mu(x)$.
5. If there is an initial condition, determine the value of C.

Example 7.5.2: Solving a First-order Linear Equation

Find a general solution for the differential equation $xy' + 3y = 4x^2 - 3x$. Assume $x > 0$.

Solution

1. To put this differential equation into standard form, divide both sides by x:

$$y' + \frac{3}{x}y = 4x - 3.$$

Therefore $p(x) = \dfrac{3}{x}$ and $q(x) = 4x - 3$.

2. The integrating factor is $\mu(x) = e^{\int (3/x)dx} = e^{3\ln x} = x^3$.

3. Multiplying both sides of the differential equation by $\mu(x)$ gives us

$$x^3y' + x^3\left(\frac{3}{x}\right)y = x^3(4x - 3)$$
$$x^3y' + 3x^2y = 4x^4 - 3x^3$$
$$\frac{d}{dx}(x^3y) = 4x^4 - 3x^3.$$

4. Integrate both sides of the equation.

$$\int \frac{d}{dx}(x^3y)dx = \int 4x^4 - 3x^3dx$$
$$x^3y = \frac{4x^5}{5} - \frac{3x^4}{4} + C$$
$$y = \frac{4x^2}{5} - \frac{3x}{4} + Cx^{-3}.$$

5. There is no initial value, so the problem is complete.

Analysis

You may have noticed the condition that was imposed on the differential equation; namely, $x > 0$. For any nonzero value of C, the general solution is not defined at $x = 0$. Furthermore, when $x < 0$, the integrating factor changes. The integrating factor is given by Equation 7.5.8 as $\mu(x) = e^{\int p(x)dx}$. For this $p(x)$ we get

$$\begin{aligned} e^{\int p(x)dx} &= e^{\int (3/x)dx} \\ &= e^{3\ln|x|} \\ &= |x|^3 \end{aligned}$$

since $x < 0$. The behavior of the general solution changes at $x = 0$ largely due to the fact that $p(x)$ is not defined there.

Exercise 7.5.2

Find the general solution to the differential equation $(x-2)y' + y = 3x^2 + 2x$. Assume $x > 2$.

Hint

Use the method outlined in the problem-solving strategy for first-order linear differential equations.

Answer

$$y = \frac{x^3 + x^2 + C}{x - 2}$$

Now we use the same strategy to find the solution to an initial-value problem.

Example 7.5.3: A First-order Linear Initial-Value Problem

Solve the initial-value problem

$$y' + 3y = 2x - 1, y(0) = 3.$$

Solution

1. This differential equation is already in standard form with $p(x) = 3$ and $q(x) = 2x - 1$.

2. The integrating factor is $\mu(x) = e^{\int 3dx} = e^{3x}$.

3. Multiplying both sides of the differential equation by $\mu(x)$ gives

$$\begin{aligned} e^{3x}y' + 3e^{3x}y &= (2x-1)e^{3x} \\ \frac{d}{dx}[ye^{3x}] &= (2x-1)e^{3x}. \end{aligned}$$

Integrate both sides of the equation:

$$\begin{aligned} \int \frac{d}{dx}[ye^{3x}]dx &= \int (2x-1)e^{3x}dx \\ ye^{3x} &= \frac{e^{3x}}{3}(2x-1) - \int \frac{2}{3}e^{3x}dx \\ ye^{3x} &= \frac{e^{3x}(2x-1)}{3} - \frac{2e^{3x}}{9} + C \\ y &= \frac{2x-1}{3} - \frac{2}{9} + Ce^{-3x} \\ y &= \frac{2x}{3} - \frac{5}{9} + Ce^{-3x}. \end{aligned}$$

4. Now substitute $x = 0$ and $y = 3$ into the general solution and solve for C:

$$y = \frac{2}{3}x - \frac{5}{9} + Ce^{-3x}$$
$$3 = \frac{2}{3}(0) - \frac{5}{9} + Ce^{-3(0)}$$
$$3 = -\frac{5}{9} + C$$
$$C = \frac{32}{9}.$$

Therefore the solution to the initial-value problem is

$$y = \frac{2}{3}x - \frac{5}{9} + \frac{32}{9}e^{-3x}.$$

Example 7.5.4:

Solve the initial-value problem

$$y' - 2y = 4x + 3y(0) = -2.$$

Solution

$$y = -2x - 4 + 2e^{2x}$$

Applications of First-order Linear Differential Equations

We look at two different applications of first-order linear differential equations. The first involves air resistance as it relates to objects that are rising or falling; the second involves an electrical circuit. Other applications are numerous, but most are solved in a similar fashion.

Free fall with air resistance

We discussed air resistance at the beginning of this section. The next example shows how to apply this concept for a ball in vertical motion. Other factors can affect the force of air resistance, such as the size and shape of the object, but we ignore them here.

Example 7.5.5: A Ball with Air Resistance

A racquetball is hit straight upward with an initial velocity of $2\ \mathrm{m/s}$. The mass of a racquetball is approximately $0.0427\ \mathrm{kg}$. Air resistance acts on the ball with a force numerically equal to $0.5v$, where v represents the velocity of the ball at time t.

a. Find the velocity of the ball as a function of time.
b. How long does it take for the ball to reach its maximum height?
c. If the ball is hit from an initial height of 1 meter, how high will it reach?

Solution

a. The mass $m = 0.0427\ kg, k = 0.5$, and $g = 9.8\ m/s^2$. The initial velocity is $v_0 = 2\ m/s$. Therefore the initial-value problem is

$$0.0427\frac{dv}{dt} = -0.5v - 0.0427(9.8), v_0 = 2.$$

Dividing the differential equation by 0.0427 gives

$$\frac{dv}{dt} = -11.7096v - 9.8, v_0 = 2.$$

The differential equation is linear. Using the problem-solving strategy for linear differential equations:

Step 1. Rewrite the differential equation as $\frac{dv}{dt} + 11.7096v = -9.8$. This gives $p(t) = 11.7096$ and $q(t) = -9.8$

Step 2. The integrating factor is $\mu(t) = e^{\int 11.7096dt} = e^{11.7096t}$.

Step 3. Multiply the differential equation by $\mu(t)$:

$$e^{11.7096t}\frac{dv}{dt} + 11.7096ve^{11.7096t} = -9.8e^{11.7096t}$$
$$\frac{d}{dt}[ve^{11.7096t}] = -9.8e^{11.7096t}.$$

Step 4. Integrate both sides:

$$\int \frac{d}{dt}[ve^{11.7096t}]dt = \int -9.8e^{11.7096t}dt$$
$$ve^{11.7096t} = \frac{-9.8}{11.7096}e^{11.7096t} + C$$
$$v(t) = -0.8369 + Ce^{-11.7096t}.$$

Step 5. Solve for C using the initial condition $v_0 = v(0) = 2$:

$$v(t) = -0.8369 + Ce^{-11.7096t}$$
$$v(0) = -0.8369 + Ce^{-11.7096(0)}$$
$$2 = -0.8369 + C$$
$$C = 2.8369.$$

Therefore the solution to the initial-value problem is

$$v(t) = 2.8369e^{-11.7096t} - 0.8369.$$

b. The ball reaches its maximum height when the velocity is equal to zero. The reason is that when the velocity is positive, it is rising, and when it is negative, it is falling. Therefore when it is zero, it is neither rising nor falling, and is at its maximum height:

$$2.8369e^{-11.7096t} - 0.8369 = 0$$
$$2.8369e^{-11.7096t} = 0.8369$$
$$e^{-11.7096t} = \frac{0.8369}{2.8369} \approx 0.295$$
$$\ln e^{-11.7096t} = \ln 0.295 \approx -1.221$$
$$-11.7096t = -1.221$$
$$t \approx 0.104.$$

Therefore it takes approximately 0.104 second to reach maximum height.

c. To find the height of the ball as a function of time, use the fact that the derivative of position is velocity, i.e., if $h(t)$ represents the height at time t, then $h'(t) = v(t)$. Because we know $v(t)$ and the initial height, we can form an initial-value problem:

$$h'(t) = 2.8369e^{-11.7096t} - 0.8369,\ h(0) = 1.$$

Integrating both sides of the differential equation with respect to t gives

$$\int h'(t)dt = \int 2.8369e^{-11.7096t} - 0.8369\ dt$$
$$h(t) = -\frac{2.8369}{11.7096}e^{-11.7096t} - 0.8369t + C$$
$$h(t) = -0.2423e^{-11.7096t} - 0.8369t + C.$$

Solve for C by using the initial condition:

$$h(t) = -0.2423e^{-11.7096t} - 0.8369t + C$$
$$h(0) = -0.2423e^{-11.7096(0)} - 0.8369(0) + C$$
$$1 = -0.2423 + C$$
$$C = 1.2423.$$

Therefore

$$h(t) = -0.2423e^{-11.7096t} - 0.8369t + 1.2423.$$

After 0.104 second, the height is given by

$$\begin{aligned} h(0.2) &= -0.2423e^{-11.7096t} - 0.8369t + 1.2423 \\ &\approx 1.0836 \text{ meter}. \end{aligned}$$

? Exercise 7.5.3

The weight of a penny is 2.5 grams (United States Mint, "Coin Specifications," accessed April 9, 2015, www.usmint.gov/about_the_mint...specifications), and the upper observation deck of the Empire State Building is 369 meters above the street. Since the penny is a small and relatively smooth object, air resistance acting on the penny is actually quite small. We assume the air resistance is numerically equal to $0.0025v$. Furthermore, the penny is dropped with no initial velocity imparted to it.

a. Set up an initial-value problem that represents the falling penny.
b. Solve the problem for $v(t)$.
c. What is the terminal velocity of the penny (i.e., calculate the limit of the velocity as t approaches infinity)?

Hint

Set up the differential equation the same way as Example **?** 7.5.5. Remember to convert from grams to kilograms.

Answer

a. $\frac{dv}{dt} = -v - 9.8\ v(0) = 0$

b. $v(t) = 9.8(e^{-t} - 1)$

c. $\lim_{t\to\infty} v(t) = \lim_{t\to\infty} (9.8(e^{-t} - 1)) = -9.8 \text{ m/s} \approx -21.922$ mph

Electrical Circuits

A source of electromotive force (e.g., a battery or generator) produces a flow of current in a closed circuit, and this current produces a voltage drop across each resistor, inductor, and capacitor in the circuit. Kirchhoff's Loop Rule states that the sum of the voltage drops across resistors, inductors, and capacitors is equal to the total electromotive force in a closed circuit. We have the following three results:

1. The voltage drop across a resistor is given by

$$E_R = Ri,$$

where R is a constant of proportionality called the resistance, and i is the current.

2. The voltage drop across an inductor is given by

$$E_L = Li',$$

where L is a constant of proportionality called the inductance, and i again denotes the current.

3. The voltage drop across a capacitor is given by

$$E_C = \frac{1}{C}q,$$

where C is a constant of proportionality called the capacitance, and q is the instantaneous charge on the capacitor. The relationship between i and q is $i = q'$.

We use units of volts (V) to measure voltage E, amperes (A) to measure current i, coulombs (C) to measure charge q, ohms (Ω) to measure resistance R, henrys (H) to measure inductance L, and farads (F) to measure capacitance C. Consider the circuit in Figure 7.5.2.

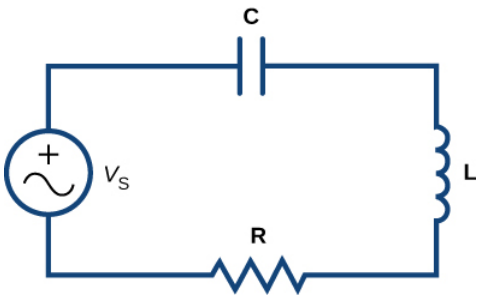


Figure 7.5.2: A typical electric circuit, containing a voltage generator (V_S), capacitor (C), inductor (L), and resistor (R).

Applying Kirchhoff's Loop Rule to this circuit, we let E denote the electromotive force supplied by the voltage generator. Then

$$E_L + E_R + E_C = E.$$

Substituting the expressions for E_L, E_R, and E_C into this equation, we obtain

$$Li' + Ri + \frac{1}{C}q = E. \tag{7.5.9}$$

If there is no capacitor in the circuit, then the equation becomes

$$Li' + Ri = E. \tag{7.5.10}$$

This is a first-order differential equation in i. The circuit is referred to as an **LR circuit**.

Next, suppose there is no inductor in the circuit, but there is a capacitor and a resistor, so $L = 0$, $R \neq 0$, and $C \neq 0$. Then, since $q' = i$, Equation 7.5.9 can be rewritten as

$$Rq' + \frac{1}{C}q = E, \tag{7.5.11}$$

which is a first-order linear differential equation in q. This is referred to as an **RC circuit**. In either case, we can set up and solve an initial-value problem.

Electric Circuit

A circuit has in series an electromotive force given by $E = 50 \sin 20t V$, a resistor of 5Ω, and an inductor of $0.4H$. If the initial current is 0, find the current at time $t > 0$.

Solution

We have a resistor and an inductor in the circuit, so we have an LR circuit and can use Equation 7.5.10. The voltage drop across the resistor is given by $E_R = Ri = 5i$. The voltage drop across the inductor is given by $E_L = Li' = 0.4i'$. The electromotive force becomes the right-hand side of Equation 7.5.10. Therefore Equation 7.5.10 becomes

$$0.4i' + 5i = 50 \sin 20t.$$

Dividing both sides by 0.4 gives the equation

$$i' + 12.5i = 125 \sin 20t.$$

Since the initial current is 0, this result gives an initial condition of $i(0) = 0$. We can solve this initial-value problem using the five-step strategy for solving first-order differential equations.

Step 1. Rewrite the differential equation as $i' + 12.5i = 125 \sin 20t$. This gives $p(t) = 12.5$ and $q(t) = 125 \sin 20t$.

Step 2. The integrating factor is $\mu(t) = e^{\int 12.5dt} = e^{12.5t}$.

Step 3. Multiply the differential equation by $\mu(t)$:

$$e^{12.5t}i' + 12.5e^{12.5t}i = 125e^{12.5t} \sin 20t$$

$$\frac{d}{dt}[ie^{12.5t}] = 125e^{12.5t} \sin 20t.$$

Step 4. Integrate both sides:

$$\int \frac{d}{dt}[ie^{12.5t}]dt = \int 125e^{12.5t}\sin 20t\ dt$$
$$ie^{12.5t} = \left(\frac{250\sin 20t - 400\cos 20t}{89}\right)e^{12.5t} + C$$
$$i(t) = \frac{250\sin 20t - 400\cos 20t}{89} + Ce^{-12.5t}.$$

Step 5. Solve for C using the initial condition $i(0) = 0$:

$$i(t) = \frac{250\sin 20t - 400\cos 20t}{89} + Ce^{-12.5t}$$
$$i(0) = \frac{250\sin 20(0) - 400\cos 20(0)}{89} + Ce^{-12.5(0)}$$
$$0 = -\frac{400}{89} + C$$
$$C = \frac{400}{89}.$$

Therefore the solution to the initial-value problem is

$$i(t) = \frac{250\sin 20t - 400\cos 20t + 400e^{-12.5t}}{89} = \frac{250\sin 20t - 400\cos 20t}{89} + \frac{400e^{-12.5t}}{89}.$$

The first term can be rewritten as a single cosine function. First, multiply and divide by $\sqrt{250^2 + 400^2} = 50\sqrt{89}$:

$$\frac{250\sin 20t - 400\cos 20t}{89} = \frac{50\sqrt{89}}{89}\left(\frac{250\sin 20t - 400\cos 20t}{50\sqrt{89}}\right) = -\frac{50\sqrt{89}}{89}\left(\frac{8\cos 20t}{\sqrt{89}} - \frac{5\sin 20t}{\sqrt{89}}\right).$$

Next, define φ to be an acute angle such that $\cos\varphi = \frac{8}{\sqrt{89}}$. Then $\sin\varphi = \frac{5}{\sqrt{89}}$ and

$$-\frac{50\sqrt{89}}{89}\left(\frac{8\cos 20t}{\sqrt{89}} - \frac{5\sin 20t}{\sqrt{89}}\right) = -\frac{50\sqrt{89}}{89}(\cos\varphi\cos 20t - \sin\varphi\sin 20t) = -\frac{50\sqrt{89}}{89}\cos(20t + \varphi).$$

Therefore the solution can be written as

$$i(t) = -\frac{50\sqrt{89}}{89}cos(20t + \varphi) + \frac{400e^{-12.5t}}{89}.$$

The second term is called the attenuation term, because it disappears rapidly as t grows larger. The phase shift is given by φ, and the amplitude of the steady-state current is given by $\frac{50\sqrt{89}}{89}$. The graph of this solution appears in Figure **7.5.3**:

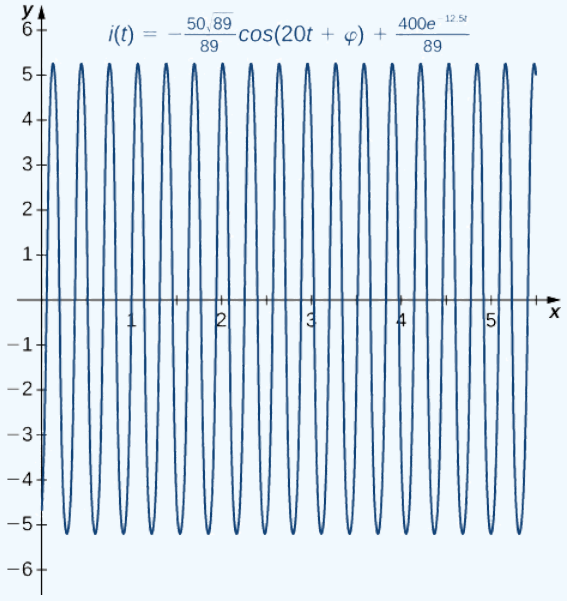


Figure 7.5.3.

? Exercise 7.5.4

A circuit has in series an electromotive force given by $E = 20\sin 5t$ V, a capacitor with capacitance $0.02F$, and a resistor of 8Ω. If the initial charge is $4C$, find the charge at time $t > 0$.

Hint

Use Equation 7.5.11 for an RC circuit to set up an initial-value problem.

Answer

Initial-value problem:

$$8q' + \frac{1}{0.02}q = 20\sin 5t, q(0) = 4$$

$$q(t) = \frac{10\sin 5t - 8\cos 5t + 172e^{-6.25t}}{41}$$

Key Concepts

- Any first-order linear differential equation can be written in the form $y' + p(x)y = q(x)$.
- We can use a five-step problem-solving strategy for solving a first-order linear differential equation that may or may not include an initial value.
- Applications of first-order linear differential equations include determining motion of a rising or falling object with air resistance and finding current in an electrical circuit.

Key Equations

- **standard form**

  $y' + p(x)y = q(x)$

- **integrating factor**

  $\mu(x) = e^{\int p(x)dx}$

Glossary

integrating factor

any function $f(x)$ that is multiplied on both sides of a differential equation to make the side involving the unknown function equal to the derivative of a product of two functions

linear

description of a first-order differential equation that can be written in the form $a(x)y'+b(x)y=c(x)$

standard form

the form of a first-order linear differential equation obtained by writing the differential equation in the form $y'+p(x)y=q(x)$

This page titled 7.5: First-order Linear Equations is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Gilbert Strang & Edwin "Jed" Herman (OpenStax) via source content that was edited to the style and standards of the LibreTexts platform.

- **8.5: First-order Linear Equations** by Edwin "Jed" Herman, Gilbert Strang is licensed CC BY-NC-SA 4.0. Original source: https://openstax.org/details/books/calculus-volume-1.

7.5E: Exercises for Section 8.5

In exercises 1 - 5, state if each of the following differential equations is linear? Explain your reasoning.

1) $\frac{dy}{dx} = x^2y + \sin x$

2) $\frac{dy}{dt} = ty$

Answer

Yes

3) $\frac{dy}{dt} + y^2 = x$

4) $y' = x^3 + e^x$

Answer

Yes

5) $y' = y + e^y$

In exercises 6 - 10, write the following first-order differential equations in standard form.

6) $y' = x^3y + \sin x$

Answer

$y' - x^3y = \sin x$

7) $y' + 3y - \ln x = 0$

8) $-xy' = (3x + 2)y + xe^x$

Answer

$y' + \frac{(3x+2)}{x}y = -e^x$

9) $\frac{dy}{dt} = 4y + ty + \tan t$

10) $\frac{dy}{dt} = yx(x + 1)$

Answer

$\frac{dy}{dt} - yx(x + 1) = 0$

In exercises 11 - 15, state the integrating factors for each of the following differential equations.

11) $y' = xy + 3$

12) $y' + e^xy = \sin x$

Answer

e^x

13) $y' = x\ln(x)y + 3x$

14) $\frac{dy}{dx} = \tanh(x)y + 1$

Answer

$-\ln(\cosh x)$

15) $\frac{dy}{dt} + 3ty = e^t y$

In exercises 16 - 25, solve each differential equation by using integrating factors.

16) $y' = 3y + 2$

Answer

$y = Ce^{3x} - \frac{2}{3}$

17) $y' = 2y - x^2$

18) $xy' = 3y - 6x^2$

Answer

$y = Cx^3 + 6x^2$

19) $(x + 2)y' = 3x + y$

20) $y' = 3x + xy$

Answer

$y = Ce^{x^2/2} - 3$

21) $xy' = x + y$

22) $\sin(x)y' = y + 2x$

Answer

$y = C\tan\left(\frac{x}{2}\right) - 2x + 4\tan\left(\frac{x}{2}\right)\ln\left(\sin(\frac{x}{2})\right)$

23) $y' = y + e^x$

24) $xy' = 3y + x^2$

Answer

$y = Cx^3 - x^2$

25) $y' + \ln x = \frac{y}{x}$

In exercises 26 - 33, solve the given differential equation. Use your calculator to draw a family of solutions. Are there certain initial conditions that change the behavior of the solution?

26) [T] $(x + 2)y' = 2y - 1$

Answer

$y = C(x + 2)^2 + \frac{1}{2}$

27) [T] $y' = 3e^{t/3} - 2y$

28) [T] $xy' + \frac{y}{2} = \sin(3t)$

Answer

$y = \frac{C}{\sqrt{x}} + 2\sin(3t)$

29) [T] $xy' = 2\frac{\cos x}{x} - 3y$

30) [T] $(x + 1)y' = 3y + x^2 + 2x + 1$

Answer

$$y = C(x+1)^3 - x^2 - 2x - 1$$

31) [T] $\sin(x)y' + \cos(x)y = 2x$

32) [T] $\sqrt{x^2+1}y' = y + 2$

Answer

$$y = Ce^{\sinh^{-1}x} - 2$$

33) [T] $x^3y' + 2x^2y = x + 1$

In exercises 34 - 43, solve each initial-value problem by using integrating factors.

34) $y' + y = x, \quad y(0) = 3$

Answer

$$y = x + 4e^x - 1$$

35) $y' = y + 2x^2, \quad y(0) = 0$

36) $xy' = y - 3x^3, \quad y(1) = 0$

Answer

$$y = -\frac{3x}{2}(x^2 - 1)$$

37) $x^2y' = xy - \ln x, \quad y(1) = 1$

38) $(1+x^2)y' = y - 1, \quad y(0) = 0$

Answer

$$y = 1 - e^{\tan^{-1}x}$$

39) $xy' = y + 2x\ln x, \quad y(1) = 5$

40) $(2+x)y' = y + 2 + x, \quad y(0) = 0$

Answer

$$y = (x+2)\ln\left(\frac{x+2}{2}\right)$$

41) $y' = xy + 2xe^x, \quad y(0) = 2$

42) $\sqrt{x}y' = y + 2x, \quad y(0) = 1$

Answer

$$y = 2e^{2\sqrt{x}} - 2x - 2\sqrt{x} - 1$$

43) $y' = 2y + xe^x, \quad y(0) = -1$

44) A falling object of mass m can reach terminal velocity when the drag force is proportional to its velocity, with proportionality constant k. Set up the differential equation and solve for the velocity given an initial velocity of 0.

Answer

$$v(t) = \frac{gm}{k}\left(1 - e^{-kt/m}\right)$$

45) Using your expression from the preceding problem, what is the terminal velocity? (Hint: Examine the limiting behavior; does the velocity approach a value?)

46) [T] Using your equation for terminal velocity, solve for the distance fallen. How long does it take to fall 5000 meters if the mass is 100 kilograms, the acceleration due to gravity is 9.8 m/s^2 and the proportionality constant is 4?

Answer

40.451 seconds

47) A more accurate way to describe terminal velocity is that the drag force is proportional to the square of velocity, with a proportionality constant k. Set up the differential equation and solve for the velocity.

48) Using your expression from the preceding problem, what is the terminal velocity? (Hint: Examine the limiting behavior: Does the velocity approach a value?)

Answer

$\sqrt{\frac{gm}{k}}$

49) [T] Using your equation for terminal velocity, solve for the distance fallen. How long does it take to fall 5000 meters if the mass is 100 kilograms, the acceleration due to gravity is 9.8 m/s^2 and the proportionality constant is 4? Does it take more or less time than your initial estimate?

In exercises 50 - 54, determine how parameter a affects the solution.

50) Solve the generic equation $y' = ax + y$. How does varying a change the behavior?

Answer

$y = Ce^x - a(x+1)$

51) Solve the generic equation $y' = ax + y$. How does varying a change the behavior?

52) Solve the generic equation $y' = ax + xy$. How does varying a change the behavior?

Answer

$y = Ce^{x^2/2} - a$

53) Solve the generic equation $y' = x + axy$. How does varying a change the behavior?

54) Solve $y' - y = e^{kt}$ with the initial condition $y(0) = 0$. As k approaches 1, what happens to your formula?

Answer

$y = \frac{e^{kt} - e^t}{k-1}$

This page titled 7.5E: Exercises for Section 8.5 is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Gilbert Strang & Edwin “Jed” Herman (OpenStax) via source content that was edited to the style and standards of the LibreTexts platform.

- **8.E: Differential Equations (Exercises)** by OpenStax is licensed CC BY-NC-SA 4.0.

7.6: Chapter 7 Review Exercises

True or False? Justify your answer with a proof or a counterexample.

1) The differential equation $y' = 3x^2y - \cos(x)y''$ is linear.

2) The differential equation $y' = x - y$ is separable.

Answer

F

3) You can explicitly solve all first-order differential equations by separation or by the method of integrating factors.

4) You can determine the behavior of all first-order differential equations using directional fields or Euler's method.

Answer

T

For the following problems, find the general solution to the differential equations.

5) $y' = x^2 + 3e^x - 2x$

6) $y' = 2^x + \cos^{-1} x$

Answer

$y(x) = \frac{2^x}{ln(2)} + xcos^{-1}x - \sqrt{1-x^2} + C$

7) $y' = y(x^2 + 1)$

8) $y' = e^{-y} \sin x$

Answer

$y(x) = \ln(C - \cos x)$

9) $y' = 3x - 2y$

10) $y' = y \ln y$

Answer

$y(x) = e^{e^{C+x}}$

For the following problems, find the solution to the initial value problem.

11) $y' = 8x - \ln x - 3x^4, \quad y(1) = 5$

12) $y' = 3x - \cos x + 2, \quad y(0) = 4$

Answer

$y(x) = 4 + \frac{3}{2}x^2 + 2x - \sin x$

13) $xy' = y(x - 2), \quad y(1) = 3$

14) $y' = 3y^2(x + \cos x), \quad y(0) = -2$

Answer

$$y(x) = -\frac{2}{1 + 3(x^2 + 2\sin x)}$$

15) $(x - 1)y' = y - 2, \quad y(0) = 0$

16) $y' = 3y - x + 6x^2, \quad y(0) = -1$

Answer

$y(x) = -2x^2 - 2x - \frac{1}{3} - \frac{2}{3}e^{3x}$

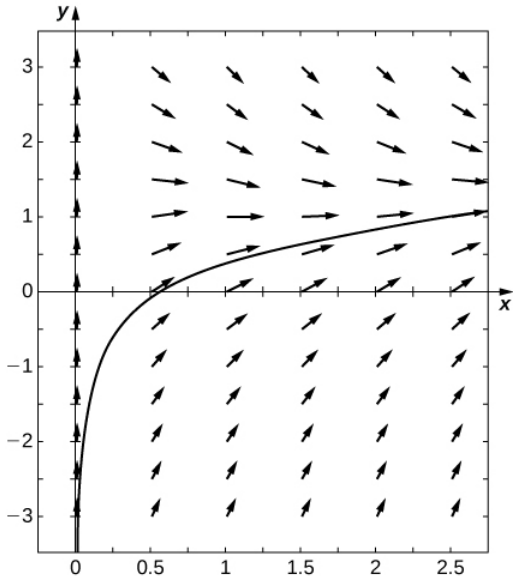
For the following problems, draw the directional field associated with the differential equation, then solve the differential equation. Draw a sample solution on the directional field.

17) $y' = 2y - y^2$

18) $y' = \dfrac{1}{x} + \ln x - y$, for $x > 0$

Answer

$y(x) = Ce^{-x} + \ln x$



For the following problems, use Euler's Method with $n = 5$ steps over the interval $t = [0, 1]$. Then solve the initial-value problem exactly. How close is your Euler's Method estimate?

19) $y' = -4yx, \quad y(0) = 1$

20) $y' = 3^x - 2y, \quad y(0) = 0$

Answer

Euler: 0.6939,

Exact solution: $y(x) = \dfrac{3^x - e^{-2x}}{2 + \ln(3)}$

For the following problems, set up and solve the differential equations.

21) A car drives along a freeway, accelerating according to $a = 5\sin(\pi t)$, where t represents time in minutes. Find the velocity at any time t, assuming the car starts with an initial speed of 60 mph.

22) You throw a ball of mass 2 kilograms into the air with an upward velocity of 8 m/s. Find exactly the time the ball will remain in the air, assuming that gravity is given by $g = 9.8\,\text{m/s}^2$.

Answer

$\frac{40}{49}$ second

23) You drop a ball with a mass of 5 kilograms out an airplane window at a height of 5000 m. How long does it take for the ball to reach the ground?

24) You drop the same ball of mass 5 kilograms out of the same airplane window at the same height, except this time you assume a drag force proportional to the ball's velocity, using a proportionality constant of 3 and the ball reaches terminal velocity. Solve for the distance fallen as a function of time. How long does it take the ball to reach the ground?

Answer

$x(t) = 5000 + \frac{245}{9} - \frac{49}{3}t - \frac{245}{9}e^{-5/3t}, \quad t = 307.8$ seconds

25) A drug is administered to a patient every 24 hours and is cleared at a rate proportional to the amount of drug left in the body, with proportionality constant 0.2. If the patient needs a baseline level of 5 mg to be in the bloodstream at all times, how large should the dose be?

26) A 1000-liter tank contains pure water and a solution of 0.2 kg salt/L is pumped into the tank at a rate of 1 L/min and is drained at the same rate. Solve for total amount of salt in the tank at time t.

Answer

$T(t) = 200\left(1 - e^{-t/1000}\right)$

27) You boil water to make tea. When you pour the water into your teapot, the temperature is $100°C$. After 5 minutes in your $15°C$ room, the temperature of the tea is $85°C$. Solve the equation to determine the temperatures of the tea at time t. How long must you wait until the tea is at a drinkable temperature ($72°C$)?

28) The human population (in thousands) of Nevada in 1950 was roughly 160. If the carrying capacity is estimated at 10 million individuals, and assuming a growth rate of 2% per year, develop a logistic growth model and solve for the population in Nevada at any time (use 1950 as time = 0). What population does your model predict for 2000? How close is your prediction to the true value of $1,998,257$?

Answer

$P(t) = \frac{1600000e^{0.02t}}{9840 + 160e^{0.02t}}$

29) Repeat the previous problem but use Gompertz growth model. Which is more accurate?

This page titled 7.6: Chapter 7 Review Exercises is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Gilbert Strang & Edwin "Jed" Herman (OpenStax) via source content that was edited to the style and standards of the LibreTexts platform.

- **8.R: Chapter 8 Review Exercises** by Edwin "Jed" Herman, Gilbert Strang is licensed CC BY-NC-SA 4.0. Original source: https://openstax.org/details/books/calculus-volume-1.

CHAPTER OVERVIEW

8: Appendices

8.1: Other Strategies for Integration

8.1E: Exercises

8.2: Volumes of Revolution - Cylindrical Shells

8.2E: Exercises for Section 6.3

8.A: Table of Derivatives

8.B: Table of Integrals

This page titled 8: Appendices is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Gilbert Strang & Edwin "Jed" Herman (OpenStax) via source content that was edited to the style and standards of the LibreTexts platform.

8.1: Other Strategies for Integration

Learning Objectives

- Use a table of integrals to solve integration problems.
- Use a computer algebra system (CAS) to solve integration problems.

In addition to the techniques of integration we have already seen, several other tools are widely available to assist with the process of integration. Among these tools are **integration tables**, which are readily available in many books, including the appendices to this one. Also widely available are **computer algebra systems (CAS)**, which are found on calculators and in many campus computer labs, and are free online.

Tables of Integrals

Integration tables, if used in the right manner, can be a handy way either to evaluate or check an integral quickly. The integration tables in this text are in Appendix B. Keep in mind that when using a table to check an answer, it is possible for two completely correct solutions to look very different. For example, in Trigonometric Substitution, we found that, by using the substitution $x = \tan\theta$, we can arrive at

$$\int \frac{dx}{\sqrt{1+x^2}} = \ln\left|x+\sqrt{x^2+1}\right| + C.$$

However, using $x = \sinh\theta$, we obtained a different solution - namely,

$$\int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1} x + C.$$

Luckily, in Differential Calculus (specifically, in Section 1.7 of the Differential Calculus textbook), we showed algebraically that the two solutions are equivalent. That is, we showed that $\sinh^{-1} x = \ln\left|x+\sqrt{x^2+1}\right|$. In this case, the two antiderivatives that we found were actually equal. This need not be the case. However, as long as the difference in the two antiderivatives is a constant, they are equivalent.

Example 8.1.1: Using a Formula from a Table to Evaluate an Integral

Use the table formula

$$\int \frac{\sqrt{a^2-u^2}}{u^2}\,du = -\frac{\sqrt{a^2-u^2}}{u} - \sin^{-1}\frac{u}{a} + C$$

from Appendix B to evaluate $\int \frac{\sqrt{16-e^{2x}}}{e^x}\,dx$.

Solution

If we look at integration tables, we see that several formulas contain expressions of the form $\sqrt{a^2-u^2}$. This expression is actually similar to $\sqrt{16-e^{2x}}$, where $a = 4$ and $u = e^x$. Keep in mind that we must also have $du = e^x\,dx$. Multiplying the numerator and the denominator of the given integral by e^x should help to put this integral in a useful form. Thus, we now have

$$\int \frac{\sqrt{16-e^{2x}}}{e^x}\,dx = \int \frac{\sqrt{16-e^{2x}}}{e^{2x}}e^x\,dx.$$

Substituting $u = e^x$ and $du = e^x\,dx$ produces $\int \frac{\sqrt{a^2-u^2}}{u^2}\,du$. From the integration table (#88 in Appendix B),

$$\int \frac{\sqrt{a^2-u^2}}{u^2}\,du = -\frac{\sqrt{a^2-u^2}}{u} - \sin^{-1}\frac{u}{a} + C.$$

Thus,

$$\begin{aligned}\int \frac{\sqrt{16-e^{2x}}}{e^x}\,dx &= \int \frac{\sqrt{16-e^{2x}}}{e^{2x}}e^x\,dx \\ &= \int \frac{\sqrt{4^2-u^2}}{u^2}\,du \\ &= \frac{\sqrt{4^2-u^2}}{u} - \sin^{-1}\frac{u}{4} + C \\ &= -\frac{\sqrt{16-e^{2x}}}{e^x} - \sin^{-1}\left(\frac{e^x}{4}\right) + C\end{aligned}$$

Computer Algebra Systems

If available, a CAS is a faster alternative to a table for solving an integration problem. Many such systems are widely available and are, in general, quite easy to use.

Example 8.1.2: Using a Computer Algebra System to Evaluate an Integral

Use a computer algebra system to evaluate $\int \frac{dx}{\sqrt{x^2-4}}$. Compare this result with $\ln\left|\frac{\sqrt{x^2-4}}{2}+\frac{x}{2}\right|+C$, a result we might have obtained if we had used Trigonometric Substitution.

Solution

Using Wolfram Alpha, we obtain

$$\int \frac{dx}{\sqrt{x^2-4}} = \ln\left|\sqrt{x^2-4}+x\right|+C.$$

Notice that

$$\ln\left|\frac{\sqrt{x^2-4}}{2}+\frac{x}{2}\right|+C = \ln\left|\frac{\sqrt{x^2-4}+x}{2}\right|+C = \ln\left|\sqrt{x^2-4}+x\right| - \ln 2 + C.$$

Since these two antiderivatives differ by only a constant, the solutions are equivalent. We could have also demonstrated that each of these antiderivatives is correct by differentiating them.

You can access an integral calculator for more examples.

Example 8.1.3: Using a CAS to Evaluate an Integral

Evaluate $\int \sin^3 x\,dx$ using a CAS. Compare the result to $\frac{1}{3}\cos^3 x - \cos x + C$, the result we might have obtained using the technique for integrating odd powers of $\sin x$ discussed earlier in this chapter.

Solution

Using Wolfram Alpha, we obtain

$$\int \sin^3 x\,dx = \frac{1}{12}(\cos(3x) - 9\cos x) + C.$$

This looks quite different from $\frac{1}{3}\cos^3 x - \cos x + C$. To see that these antiderivatives are equivalent, we can make use of a few trigonometric identities:

$$\begin{aligned}\frac{1}{12}(\cos(3x)-9\cos x) &= \frac{1}{12}(\cos(x+2x)-9\cos x)\\ &= \frac{1}{12}(\cos(x)\cos(2x)-\sin(x)\sin(2x)-9\cos x)\\ &= \frac{1}{12}(\cos x(2\cos^2 x-1)-\sin x(2\sin x\cos x)-9\cos x)\\ &= \frac{1}{12}(2\cos^3 x-\cos x-2\cos x(1-\cos^2 x)-9\cos x)\\ &= \frac{1}{12}(4\cos^3 x-12\cos x)\\ &= \frac{1}{3}\cos^3 x-\cos x.\end{aligned}$$

Thus, the two antiderivatives are identical.

We may also use a CAS to compare the graphs of the two functions, as shown in the following figure.

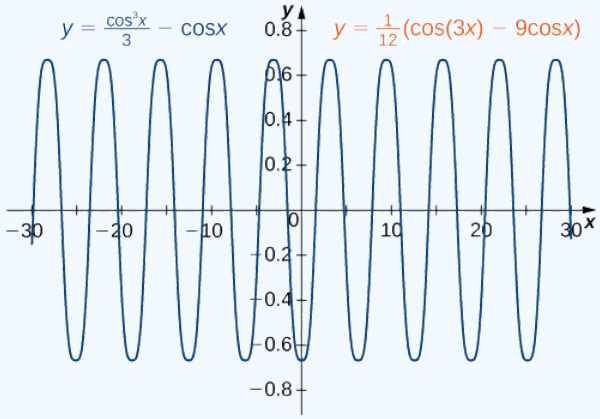


Figure 8.1.1: The graphs of $y=\frac{1}{3}\cos^3 x-\cos x$ and $y=\frac{1}{12}(\cos(3x)-9\cos x)$ are identical.

? Exercise 8.1.3

Use a CAS to evaluate $\int \frac{dx}{\sqrt{x^2+4}}$.

Hint

Answers may vary.

Answer

Possible solutions include $\sinh^{-1}\left(\frac{x}{2}\right)+C$ and $\ln\left|\sqrt{x^2+4}+x\right|+C$.

Key Concepts

- An integration table may be used to evaluate indefinite integrals.
- A CAS (or computer algebra system) may be used to evaluate indefinite integrals.
- It may require some effort to reconcile equivalent solutions obtained using different methods.

Glossary

computer algebra system (CAS)

technology used to perform many mathematical tasks, including integration

integration table

a table that lists integration formulas

Review Topics

Differential Calculus

- hyperbolic functions

This page titled 8.1: Other Strategies for Integration is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by OpenStax.

- **7.5: Other Strategies for Integration** by Edwin “Jed” Herman, Gilbert Strang is licensed CC BY-NC-SA 4.0. Original source: https://openstax.org/details/books/calculus-volume-1.

8.1E: Exercises

Use a table of integrals to evaluate the following integrals.

1) $\int_0^4 \frac{x}{\sqrt{1+2x}}\,dx$

2) $\int \frac{x+3}{x^2+2x+2}\,dx$

Answer

$$\int \frac{x+3}{x^2+2x+2}\,dx = \tfrac{1}{2}\ln|x^2+2x+2| + 2\arctan(x+1) + C$$

3) $\int x^3\sqrt{1+2x^2}\,dx$

4) $\int \frac{1}{\sqrt{x^2+6x}}\,dx$

Answer

$$\int \frac{1}{\sqrt{x^2+6x}}\,dx = \cosh^{-1}\left(\frac{x+3}{3}\right) + C$$

5) $\int \frac{x}{x+1}\,dx$

6) $\int x\cdot 2^{x^2}\,dx$

Answer

$$\int x\cdot 2^{x^2}\,dx = \frac{2^{x^2-1}}{\ln 2} + C$$

7) $\int \frac{1}{4x^2+25}\,dx$

8) $\int \frac{dy}{\sqrt{4-y^2}}$

Answer

$$\int \frac{dy}{\sqrt{4-y^2}} = \arcsin\left(\frac{y}{2}\right) + C$$

9) $\int \sin^3(2x)\cos(2x)\,dx$

10) $\int \csc(2w)\cot(2w)\,dw$

Answer

$$\int \csc(2w)\cot(2w)\,dw = -\tfrac{1}{2}\csc(2w) + C$$

11) $\int 2^y\,dy$

12) $\int_0^1 \frac{3x}{\sqrt{x^2+8}}\,dx$

Answer

$$\int_0^1 \frac{3x}{\sqrt{x^2+8}}\,dx = 9 - 6\sqrt{2}$$

13) $\int_{-1/4}^{1/4} \sec^2(\pi x)\tan(\pi x)\,dx$

14) $\int_0^{\pi/2} \tan^2\left(\frac{x}{2}\right)\,dx$

Answer

$$\int_0^{\pi/2} \tan^2\left(\frac{x}{2}\right)\,dx = 2 - \frac{\pi}{2}$$

15) $\int \cos^3 x\,dx$

16) $\int \tan^5(3x)\,dx$

Answer

$$\int \tan^5(3x)\,dx = \tfrac{1}{12}\tan^4(3x) - \tfrac{1}{6}\tan^2(3x) + \tfrac{1}{3}\ln|\sec 3x| + C$$

17) $\int \sin^2 y \cos^3 y\,dy$

Use a CAS to evaluate the following integrals. Tables can also be used to verify the answers.

18) [Technology Required] $\int \frac{dw}{1+\sec\left(\frac{w}{2}\right)}$

Answer

$$\int \frac{dw}{1+\sec\left(\frac{w}{2}\right)} = 2\cot\left(\tfrac{w}{2}\right) - 2\csc\left(\tfrac{w}{2}\right) + w + C$$

19) [Technology Required] $\int \frac{dw}{1-\cos(7w)}$

20) [Technology Required] $\int_0^t \frac{dt}{4\cos t + 3\sin t}$

Answer

$$\int_0^t \frac{dt}{4\cos t + 3\sin t} = \tfrac{1}{5}\ln\left|\frac{2(5+4\sin t - 3\cos t)}{4\cos t + 3\sin t}\right|$$

21) [Technology Required] $\int \frac{\sqrt{x^2-9}}{3x}\,dx$

22) [Technology Required] $\int \frac{dx}{x^{1/2}+x^{1/3}}$

Answer

$$\int \frac{dx}{x^{1/2}+x^{1/3}} = 6x^{1/6} - 3x^{1/3} + 2\sqrt{x} - 6\ln[1+x^{1/6}] + C$$

23) [Technology Required] $\int \frac{dx}{x\sqrt{x-1}}$

24) [Technology Required] $\int x^3 \sin x\,dx$

Answer

$$\int x^3 \sin x\,dx = -x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x + C$$

25) [Technology Required] $\int x\sqrt{x^4-9}\,dx$

26) [Technology Required] $\int \frac{x}{1+e^{-x^2}}\,dx$

Answer

$$\int \frac{x}{1+e^{-x^2}}\,dx = \tfrac{1}{2}\left(x^2 + \ln|1+e^{-x^2}|\right) + C$$

27) [Technology Required] $\int \frac{\sqrt{3-5x}}{2x}\,dx$

28) [Technology Required] $\int \frac{dx}{x\sqrt{x-1}}$

Answer

$$\int \frac{dx}{x\sqrt{x-1}} = 2\arctan\left(\sqrt{x-1}\right) + C$$

29) [Technology Required] $\int e^x \cos^{-1}(e^x)\,dx$

Use a calculator or CAS to evaluate the following integrals.

30) [Technology Required] $\int_0^{\pi/4} \cos 2x\,dx$

Answer

$$\int_0^{\pi/4} \cos 2x\,dx = 0.5 = \frac{1}{2}$$

31) [Technology Required] $\int_0^1 x \cdot e^{-x^2}\,dx$

32) [Technology Required] $\int_0^8 \frac{2x}{\sqrt{x^2+36}}\,dx$

Answer

$$\int_0^8 \frac{2x}{\sqrt{x^2+36}}\,dx = 8.0$$

33) [Technology Required] $\int_0^{2/\sqrt{3}} \frac{1}{4+9x^2}\,dx$

34) [Technology Required] $\int \frac{dx}{x^2+4x+13}$

Answer

$$\int \frac{dx}{x^2+4x+13} = \tfrac{1}{3}\arctan\left(\tfrac{1}{3}(x+2)\right) + C$$

35) [Technology Required] $\int \frac{dx}{1+\sin x}$

Use tables to evaluate the integrals. You may need to complete the square or change variables to put the integral into a form given in the table.

36) $\int \frac{dx}{x^2+2x+10}$

Answer

$$\int \frac{dx}{x^2+2x+10} = \tfrac{1}{3}\arctan\left(\frac{x+1}{3}\right) + C$$

37) $\int \frac{dx}{\sqrt{x^2-6x}}$

38) $\int \frac{e^x}{\sqrt{e^{2x}-4}}\,dx$

Answer

$$\int \frac{e^x}{\sqrt{e^{2x}-4}}\,dx = \ln\left(e^x + \sqrt{4+e^{2x}}\right) + C$$

39) $\int \frac{\cos x}{\sin^2 x + 2\sin x}\,dx$

40) $\int \frac{\arctan(x^3)}{x^4}\,dx$

Answer

$$\int \frac{\arctan(x^3)}{x^4}\,dx = \ln x - \tfrac{1}{6}\ln(x^6+1) - \frac{\arctan(x^3)}{3x^3} + C$$

41) $\int \frac{\ln|x|\arcsin(\ln|x|)}{x}\,dx$

Use tables to perform the integration.

42) $\int \frac{dx}{\sqrt{x^2+16}}$

Answer

$$\int \frac{dx}{\sqrt{x^2+16}} = \ln|x| + \sqrt{16+x^2}\,| + C$$

43) $\int \frac{3x}{2x+7}\,dx$

44) $\int \frac{dx}{1-\cos 4x}$

Answer

$$\int \frac{dx}{1-\cos 4x} = -\frac{1}{4}\cot 2x + C$$

45) $\int \frac{dx}{\sqrt{4x+1}}$

46) Find the area bounded by $y(4+25x^2)=5,\ x=0,\ y=0,$ and $x=4.$ Use a table of integrals or a CAS.

Answer

$\frac{1}{2}\arctan 10$ units^2

47) The region bounded between the curve $y=\frac{1}{\sqrt{1+\cos x}},\ 0.3\le x\le 1.1,$ and the x-axis is revolved about the x-axis to generate a solid. Use a table of integrals to find the volume of the solid generated. (Round the answer to two decimal places.)

48) Use substitution and a table of integrals to find the area of the surface generated by revolving the curve $y=e^x,\ 0\le x\le 3,$ about the x-axis. (Round the answer to two decimal places.)

Answer

1276.14 units^2

49) [Technology Required] Use an integral table and a calculator to find the area of the surface generated by revolving the curve $y=\frac{x^2}{2},\ 0\le x\le 1,$ about the x-axis. (Round the answer to two decimal places.)

50) [Technology Required] Use a CAS or tables to find the area of the surface generated by revolving the curve $y=\cos x,\ 0\le x\le \frac{\pi}{2},$ about the x-axis. (Round the answer to two decimal places.)

Answer

7.21 units^2

51) Find the length of the curve $y=\frac{x^2}{4}$ over $[0,8]$.

52) Find the length of the curve $y=e^x$ over $[0,\ln(2)]$.

Answer

$\left(\sqrt{5}-\sqrt{2}+\ln\left|\frac{2+2\sqrt{2}}{1+\sqrt{5}}\right|\right)$ units

53) Find the area of the surface formed by revolving the graph of $y = 2\sqrt{x}$ over the interval $[0, 9]$ about the x-axis.

54) Find the average value of the function $f(x) = \dfrac{1}{x^2 + 1}$ over the interval $[-3, 3]$.

Answer

$\frac{1}{3}\arctan(3) \approx 0.416$

55) Approximate the arc length of the curve $y = \tan \pi x$ over the interval $\left[0, \frac{1}{4}\right]$. (Round the answer to three decimal places.)

Contributors

Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

This page titled 8.1E: Exercises is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Gilbert Strang & Edwin “Jed” Herman (OpenStax) via source content that was edited to the style and standards of the LibreTexts platform.

- **7.5E: Exercises for Section 7.5** by Edwin “Jed” Herman, Gilbert Strang is licensed CC BY-NC-SA 4.0. Original source: https://openstax.org/details/books/calculus-volume-1.

8.2: Volumes of Revolution - Cylindrical Shells

This page titled 8.2: Volumes of Revolution - Cylindrical Shells is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Gilbert Strang & Edwin “Jed” Herman (OpenStax) via source content that was edited to the style and standards of the LibreTexts platform.

8.2E: Exercises for Section 6.3

This page titled 8.2E: Exercises for Section 6.3 is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Gilbert Strang & Edwin “Jed” Herman (OpenStax) via source content that was edited to the style and standards of the LibreTexts platform.

8.A: Table of Derivatives

General Formulas

1. $\frac{d}{dx}(c) = 0$
2. $\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$
3. $\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$
4. $\frac{d}{dx}(x^n) = nx^{n-1}, \quad \text{for real numbers } n$
5. $\frac{d}{dx}(cf(x)) = cf'(x)$
6. $\frac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x)$
7. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$
8. $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

Trigonometric Functions

9. $\frac{d}{dx}(\sin x) = \cos x$
10. $\frac{d}{dx}(\tan x) = \sec^2 x$
11. $\frac{d}{dx}(\sec x) = \sec x \tan x$
12. $\frac{d}{dx}(\cos x) = -\sin x$
13. $\frac{d}{dx}(\cot x) = -\csc^2 x$
14. $\frac{d}{dx}(\csc x) = -\csc x \cot x$

Inverse Trigonometric Functions

15. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
16. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
17. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$
18. $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$
19. $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$
20. $\frac{d}{dx}(\csc^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$

Exponential and Logarithmic Functions

21. $\frac{d}{dx}(e^x) = e^x$
22. $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$
23. $\frac{d}{dx}(b^x) = b^x \ln b$
24. $\frac{d}{dx}(\log_b|x|) = \frac{1}{x \ln b}$

Hyperbolic Functions

25. $\frac{d}{dx}(\sinh x) = \cosh x$
26. $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$
27. $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$
28. $\frac{d}{dx}(\cosh x) = \sinh x$
29. $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$
30. $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$

Inverse Hyperbolic Functions

31. $\frac{d}{dx}\left(\sinh^{-1} x\right) = \frac{1}{\sqrt{x^2+1}}$
32. $\frac{d}{dx}\left(\tanh^{-1} x\right) = \frac{1}{1-x^2} \quad (|x| < 1)$
33. $\frac{d}{dx}\left(\operatorname{sech}^{-1} x\right) = \frac{-1}{x\sqrt{1-x^2}} \quad (0 < x < 1)$
34. $\frac{d}{dx}\left(\cosh^{-1} x\right) = \frac{1}{\sqrt{x^2-1}} \quad (x > 1)$
35. $\frac{d}{dx}\left(\coth^{-1} x\right) = \frac{1}{1-x^2} \quad (|x| > 1)$
36. $\frac{d}{dx}\left(\operatorname{csch}^{-1} x\right) = \frac{-1}{|x|\sqrt{1+x^2}} \quad (x \neq 0)$

This page titled 8.A: Table of Derivatives is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Gilbert Strang & Edwin “Jed” Herman (OpenStax) via source content that was edited to the style and standards of the LibreTexts platform.

- **Appendix A: Table of Derivatives** by Edwin “Jed” Herman, Gilbert Strang is licensed CC BY-NC-SA 4.0. Original source: https://openstax.org/details/books/calculus-volume-1.

8.B: Table of Integrals

Basic Integrals

1. $\int u^n\,du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$
2. $\int \frac{du}{u} = \ln|u| + C$
3. $\int e^u\,du = e^u + C$
4. $\int a^u\,du = \frac{a^u}{\ln a} + C$
5. $\int \sin u\,du = -\cos u + C$
6. $\int \cos u\,du = \sin u + C$
7. $\int \sec^2 u\,du = \tan u + C$
8. $\int \csc^2 u\,du = -\cot u + C$
9. $\int \sec u \tan u\,du = \sec u + C$
10. $\int \csc u \cot u\,du = -\csc u + C$
11. $\int \tan u\,du = \ln|\sec u| + C$
12. $\int \cot u\,du = \ln|\sin u| + C$
13. $\int \sec u\,du = \ln|\sec u + \tan u| + C$
14. $\int \csc u\,du = \ln|\csc u - \cot u| + C$
15. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$
16. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$
17. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{|u|}{a} + C$

Trigonometric Integrals

18. $\int \sin^2 u\,du = \frac{1}{2}u - \frac{1}{4}\sin 2u + C$
19. $\int \cos^2 u\,du = \frac{1}{2}u + \frac{1}{4}\sin 2u + C$
20. $\int \tan^2 u\,du = \tan u - u + C$
21. $\int \cot^2 u\,du = -\cot u - u + C$
22. $\int \sin^3 u\,du = -\frac{1}{3}(2 + \sin^2 u)\cos u + C$
23. $\int \cos^3 u\,du = \frac{1}{3}(2 + \cos^2 u)\sin u + C$
24. $\int \tan^3 u\,du = \frac{1}{2}\tan^2 u + \ln|\cos u| + C$
25. $\int \cot^3 u\,du = -\frac{1}{2}\cot^2 u - \ln|\sin u| + C$

26. $\int \sec^3 u\,du = \frac{1}{2}\sec u \tan u + \frac{1}{2}\ln|\sec u + \tan u| + C$

27. $\int \csc^3 u\,du = -\frac{1}{2}\csc u \cot u + \frac{1}{2}\ln|\csc u - \cot u| + C$

28. $\int \sin^n u\,du = \frac{-1}{n}\sin^{n-1} u \cos u + \frac{n-1}{n}\int \sin^{n-2} u\,du$

29. $\int \cos^n u\,du = \frac{1}{n}\cos^{n-1} u \sin u + \frac{n-1}{n}\int \cos^{n-2} u\,du$

30. $\int \tan^n u\,du = \frac{1}{n-1}\tan^{n-1} u - \int \tan^{n-2} u\,du$

31. $\int \cot^n u\,du = \frac{-1}{n-1}\cot^{n-1} u - \int \cot^{n-2} u\,du$

32. $\int \sec^n u\,du = \frac{1}{n-1}\tan u \sec^{n-2} u + \frac{n-2}{n-1}\int \sec^{n-2} u\,du$

33. $\int \csc^n u\,du = \frac{-1}{n-1}\cot u \csc^{n-2} u + \frac{n-2}{n-1}\int \csc^{n-2} u\,du$

34. $\int \sin au \sin bu\,du = \frac{\sin(a-b)u}{2(a-b)} - \frac{\sin(a+b)u}{2(a+b)} + C$

35. $\int \cos au \cos bu\,du = \frac{\sin(a-b)u}{2(a-b)} + \frac{\sin(a+b)u}{2(a+b)} + C$

36. $\int \sin au \cos bu\,du = -\frac{\cos(a-b)u}{2(a-b)} - \frac{\cos(a+b)u}{2(a+b)} + C$

37. $\int u \sin u\,du = \sin u - u\cos u + C$

38. $\int u \cos u\,du = \cos u + u\sin u + C$

39. $\int u^n \sin u\,du = -u^n \cos u + n\int u^{n-1}\cos u\,du$

40. $\int u^n \cos u\,du = u^n \sin u - n\int u^{n-1}\sin u\,du$

41. $$\begin{aligned}\int \sin^n u \cos^m u\,du &= -\frac{\sin^{n-1} u \cos^{m+1} u}{n+m} + \frac{n-1}{n+m}\int \sin^{n-2} u \cos^m u\,du \\ &= \frac{\sin^{n+1} u \cos^{m-1} u}{n+m} + \frac{m-1}{n+m}\int \sin^n u \cos^{m-2} u\,du\end{aligned}$$

Exponential and Logarithmic Integrals

42. $\int ue^{au}\,du = \frac{1}{a^2}(au-1)e^{au} + C$

43. $\int u^n e^{au}\,du = \frac{1}{a}u^n e^{au} - \frac{n}{a}\int u^{n-1}e^{au}\,du$

44. $\int e^{au}\sin bu\,du = \frac{e^{au}}{a^2+b^2}(a\sin bu - b\cos bu) + C$

45. $\int e^{au}\cos bu\,du = \frac{e^{au}}{a^2+b^2}(a\cos bu + b\sin bu) + C$

46. $\int \ln u\,du = u\ln u - u + C$

47. $\int u^n \ln u\,du = \frac{u^{n+1}}{(n+1)^2}[(n+1)\ln u - 1] + C$

48. $\int \frac{1}{u\ln u}\,du = \ln|\ln u| + C$

Hyperbolic Integrals

49. $\int \sinh u\,du = \cosh u + C$

50. $\int \cosh u\,du = \sinh u + C$

51. $\int \tanh u\,du = \ln \cosh u + C$

52. $\int \coth u\,du = \ln|\sinh u| + C$

53. $\int \operatorname{sech} u\,du = \tan^{-1}|\sinh u| + C$

54. $\int \operatorname{csch} u\,du = \ln|\tanh \frac{1}{2}u| + C$

55. $\int \operatorname{sech}^2 u\,du = \tanh u + C$

56. $\int \operatorname{csch}^2 u\,du = -\coth u + C$

57. $\int \operatorname{sech} u \tanh u\,du = -\operatorname{sech} u + C$

58. $\int \operatorname{csch} u \coth u\,du = -\operatorname{csch} u + C$

Inverse Trigonometric Integrals

59. $\int \sin^{-1} u\,du = u\sin^{-1} u + \sqrt{1-u^2} + C$

60. $\int \cos^{-1} u\,du = u\cos^{-1} u - \sqrt{1-u^2} + C$

61. $\int \tan^{-1} u\,du = u\tan^{-1} u - \frac{1}{2}\ln(1+u^2) + C$

62. $\int u\sin^{-1} u\,du = \frac{2u^2-1}{4}\sin^{-1} u + \frac{u\sqrt{1-u^2}}{4} + C$

63. $\int u\cos^{-1} u\,du = \frac{2u^2-1}{4}\cos^{-1} u - \frac{u\sqrt{1-u^2}}{4} + C$

64. $\int u\tan^{-1} u\,du = \frac{u^2+1}{2}\tan^{-1} u - \frac{u}{2} + C$

65. $\int u^n \sin^{-1} u\,du = \frac{1}{n+1}\left[u^{n+1}\sin^{-1} u - \int \frac{u^{n+1}\,du}{\sqrt{1-u^2}}\right], \quad n \neq -1$

66. $\int u^n \cos^{-1} u\,du = \frac{1}{n+1}\left[u^{n+1}\cos^{-1} u + \int \frac{u^{n+1}\,du}{\sqrt{1-u^2}}\right], \quad n \neq -1$

67. $\int u^n \tan^{-1} u\,du = \frac{1}{n+1}\left[u^{n+1}\tan^{-1} u - \int \frac{u^{n+1}\,du}{1+u^2}\right], \quad n \neq -1$

Integrals Involving $a^2 + u^2$, $a > 0$

68. $\int \sqrt{a^2+u^2}\,du = \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln\left(u + \sqrt{a^2+u^2}\right) + C$

69. $\int u^2\sqrt{a^2+u^2}\,du = \frac{u}{8}(a^2+2u^2)\sqrt{a^2+u^2} - \frac{a^4}{8}\ln\left(u + \sqrt{a^2+u^2}\right) + C$

70. $\int \frac{\sqrt{a^2+u^2}}{u}\,du = \sqrt{a^2+u^2} - a\ln\left|\frac{a+\sqrt{a^2+u^2}}{u}\right| + C$

71. $\int \frac{\sqrt{a^2+u^2}}{u^2}\,du = -\frac{\sqrt{a^2+u^2}}{u} + \ln\left(u + \sqrt{a^2+u^2}\right) + C$

72. $\int \frac{du}{\sqrt{a^2+u^2}} = \ln\left(u + \sqrt{a^2+u^2}\right) + C$

73. $\int \frac{u^2}{\sqrt{a^2+u^2}}\,du = \frac{u}{2}\left(\sqrt{a^2+u^2}\right) - \frac{a^2}{2}\ln\left(u + \sqrt{a^2+u^2}\right) + C$

74. $\int \frac{du}{u\sqrt{a^2+u^2}} = \frac{-1}{a}\ln\left|\frac{\sqrt{a^2+u^2}+a}{u}\right| + C$

75. $\int \frac{du}{u^2\sqrt{a^2+u^2}} = -\frac{\sqrt{a^2+u^2}}{a^2 u} + C$

76. $\displaystyle\int \frac{du}{(a^2+u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2+u^2}} + C$

Integrals Involving $u^2 - a^2$, $a > 0$

77. $\displaystyle\int \sqrt{u^2-a^2}\,du = \frac{u}{2}\sqrt{u^2-a^2} - \frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right| + C$

78. $\displaystyle\int u^2\sqrt{u^2-a^2}\,du = \frac{u}{8}(2u^2-a^2)\sqrt{u^2-a^2} - \frac{a^4}{8}\ln\left|u+\sqrt{u^2-a^2}\right| + C$

79. $\displaystyle\int \frac{\sqrt{u^2-a^2}}{u}\,du = \sqrt{u^2-a^2} - a\cos^{-1}\frac{a}{|u|} + C$

80. $\displaystyle\int \frac{\sqrt{u^2-a^2}}{u^2}\,du = -\frac{\sqrt{u^2-a^2}}{u} + \ln\left|u+\sqrt{u^2-a^2}\right| + C$

81. $\displaystyle\int \frac{du}{\sqrt{u^2-a^2}} = \ln\left|u+\sqrt{u^2-a^2}\right| + C$

82. $\displaystyle\int \frac{u^2}{\sqrt{u^2-a^2}}\,du = \frac{u}{2}\sqrt{u^2-a^2} + \frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right| + C$

83. $\displaystyle\int \frac{du}{u^2\sqrt{u^2-a^2}} = \frac{\sqrt{u^2-a^2}}{a^2u} + C$

84. $\displaystyle\int \frac{du}{(u^2-a^2)^{3/2}} = -\frac{u}{a^2\sqrt{u^2-a^2}} + C$

Integrals Involving $a^2 - u^2$, $a > 0$

85. $\displaystyle\int \sqrt{a^2-u^2}\,du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$

86. $\displaystyle\int u^2\sqrt{a^2-u^2}\,du = \frac{u}{8}(2u^2-a^2)\sqrt{a^2-u^2} + \frac{a^4}{8}\sin^{-1}\frac{u}{a} + C$

87. $\displaystyle\int \frac{\sqrt{a^2-u^2}}{u}\,du = \sqrt{a^2-u^2} - a\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$

88. $\displaystyle\int \frac{\sqrt{a^2-u^2}}{u^2}\,du = \frac{-1}{u}\sqrt{a^2-u^2} - \sin^{-1}\frac{u}{a} + C$

89. $\displaystyle\int \frac{u^2}{\sqrt{a^2-u^2}}\,du = \frac{1}{2}\left(-u\sqrt{a^2-u^2} + a^2\sin^{-1}\frac{u}{a}\right) + C$

90. $\displaystyle\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$

91. $\displaystyle\int \frac{du}{u^2\sqrt{a^2-u^2}} = -\frac{1}{a^2u}\sqrt{a^2-u^2} + C$

92. $\displaystyle\int (a^2-u^2)^{3/2}\,du = -\frac{u}{8}\left(2u^2-5a^2\right)\sqrt{a^2-u^2} + \frac{3a^4}{8}\sin^{-1}\frac{u}{a} + C$

93. $\displaystyle\int \frac{du}{(a^2-u^2)^{3/2}} = -\frac{u}{a^2\sqrt{a^2-u^2}} + C$

Integrals Involving $2au - u^2$, $a > 0$

94. $\displaystyle\int \sqrt{2au-u^2}\,du = \frac{u-a}{2}\sqrt{2au-u^2} + \frac{a^2}{2}\cos^{-1}\left(\frac{a-u}{a}\right) + C$

95. $\displaystyle\int \frac{du}{\sqrt{2au-u^2}} = \cos^{-1}\left(\frac{a-u}{a}\right) + C$

96. $\displaystyle\int u\sqrt{2au-u^2}\,du = \frac{2u^2-au-3a^2}{6}\sqrt{2au-u^2} + \frac{a^3}{2}\cos^{-1}\left(\frac{a-u}{a}\right) + C$

97. $\displaystyle\int \frac{du}{u\sqrt{2au-u^2}} = -\frac{\sqrt{2au-u^2}}{au} + C$

Integrals Involving $a+bu$, $a \neq 0$

98. $\displaystyle\int \frac{u}{a+bu}\,du = \frac{1}{b^2}(a+bu-a\ln|a+bu|)+C$

99. $\displaystyle\int \frac{u^2}{a+bu}\,du = \frac{1}{2b^3}\left[(a+bu)^2-4a(a+bu)+2a^2\ln|a+bu|\right]+C$

100. $\displaystyle\int \frac{du}{u(a+bu)} = \frac{1}{a}\ln\left|\frac{u}{a+bu}\right|+C$

101. $\displaystyle\int \frac{du}{u^2(a+bu)} = -\frac{1}{au}+\frac{b}{a^2}\ln\left|\frac{a+bu}{u}\right|+C$

102. $\displaystyle\int \frac{u}{(a+bu)^2}\,du = \frac{a}{b^2(a+bu)}+\frac{1}{b^2}\ln|a+bu|+C$

103. $\displaystyle\int \frac{u}{u(a+bu)^2}\,du = \frac{1}{a(a+bu)}-\frac{1}{a^2}\ln\left|\frac{a+bu}{u}\right|+C$

104. $\displaystyle\int \frac{u^2}{(a+bu)^2}\,du = \frac{1}{b^3}\left(a+bu-\frac{a^2}{a+bu}-2a\ln|a+bu|\right)+C$

105. $\displaystyle\int u\sqrt{a+bu}\,du = \frac{2}{15b^2}(3bu-2a)(a+bu)^{3/2}+C$

106. $\displaystyle\int \frac{u}{\sqrt{a+bu}}\,du = \frac{2}{3b^2}(bu-2a)\sqrt{a+bu}+C$

107. $\displaystyle\int \frac{u^2}{\sqrt{a+bu}}\,du = \frac{2}{15b^3}(8a^2+3b^2u^2-4abu)\sqrt{a+bu}+C$

108. $\displaystyle\int \frac{du}{u\sqrt{a+bu}} = \begin{cases} \dfrac{1}{\sqrt{a}}\ln\left|\dfrac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}\right|+C, & \text{if } a>0 \\ \dfrac{\sqrt{2}}{\sqrt{-a}}\tan^{-1}\sqrt{\dfrac{a+bu}{-a}}+C, & \text{if } a<0 \end{cases}$

109. $\displaystyle\int \frac{\sqrt{a+bu}}{u}\,du = 2\sqrt{a+bu}+a\int \frac{du}{u\sqrt{a+bu}}$

110. $\displaystyle\int \frac{\sqrt{a+bu}}{u^2}\,du = -\frac{\sqrt{a+bu}}{u}+\frac{b}{2}\int \frac{du}{u\sqrt{a+bu}}$

111. $\displaystyle\int u^n\sqrt{a+bu}\,du = \frac{2}{b(2n+3)}\left[u^n(a+bu)^{3/2}-na\int u^{n-1}\sqrt{a+bu}\,du\right]$

112. $\displaystyle\int \frac{u^n}{\sqrt{a+bu}}\,du = \frac{2u^n\sqrt{a+bu}}{b(2n+1)}-\frac{2na}{b(2n+1)}\int \frac{u^{n-1}}{\sqrt{a+bu}}\,du$

113. $\displaystyle\int \frac{du}{u^n\sqrt{a+bu}} = -\frac{\sqrt{a+bu}}{a(n-1)u^{n-1}}-\frac{b(2n-3)}{2a(n-1)}\int \frac{du}{u^{n-1}\sqrt{a+bu}}$

This page titled 8.B: Table of Integrals is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Gilbert Strang & Edwin “Jed” Herman (OpenStax) via source content that was edited to the style and standards of the LibreTexts platform.

- **Appendix B: Table of Integrals** by Edwin “Jed” Herman, Gilbert Strang is licensed CC BY-NC-SA 4.0. Original source: https://openstax.org/details/books/calculus-volume-1.

Index

A

absolute convergence
4.5: The Alternating Series Test
absolute error
3.5: Numerical Integration
air resistance
7.5: First-order Linear Equations
Airy's equation
5.4: Working with Taylor Series
algebraic properties of convergent series
4.2: Infinite Series
Alternating Harmonic Series
4.5: The Alternating Series Test
alternating series
4.5: The Alternating Series Test
Alternating Series Test
4.5: The Alternating Series Test
angular coordinate
6.3: Polar Coordinates
annuli
2.3: Volumes of Revolution - Cylindrical Shells
arc length
2.4: Arc Length of a Curve and Surface Area
arc length function
2.4: Arc Length of a Curve and Surface Area
Arc length of a parametric curve
6.2: Calculus of Parametric Curves
Arc length of a polar curve
6.4: Area and Arc Length in Polar Coordinates
Arccosine Function
1.1: Trigonometry
Archimedean spiral
6.3: Polar Coordinates
Arcsine Function
1.1: Trigonometry
Arctangent Function
1.1: Trigonometry
area between curves
2.1: Areas Between Curves
area between curves (with respect to x)
2.1: Areas Between Curves
area between curves (with respect to y)
2.1: Areas Between Curves
area density
2.7: Moments and Centers of Mass
area of a compound region
2.1: Areas Between Curves
area of a region
2.1: Areas Between Curves
Area of a region bounded by a polar curve
6.4: Area and Arc Length in Polar Coordinates
area of surface of revolution
2.4: Arc Length of a Curve and Surface Area
Area under a parametric curve
6.2: Calculus of Parametric Curves
arithmetic sequence
4.1: Sequences
asymptotically stable solution
7.2: Direction Fields and Numerical Methods
asymptotically unstable solution
7.2: Direction Fields and Numerical Methods
autonomous differential equation
7.3: Separable Equations

B

binomial series
5.4: Working with Taylor Series
bounded above
4.1: Sequences
bounded below
4.1: Sequences
bounded sequence
4.1: Sequences

C

cardioid
6.3: Polar Coordinates
carrying capacity
7.4: The Logistic Equation
catenary
2.4: Arc Length of a Curve and Surface Area
center of mass
2.7: Moments and Centers of Mass
center of mass of the system
2.7: Moments and Centers of Mass
centroid
2.7: Moments and Centers of Mass
change of variables
1.4: Integration by Substitution
comparison test
4.4: The Comparison Tests
Comparison Test for Improper Integrals
3.7: Improper Integrals
computer algebra system
8.1: Other Strategies for Integration
conditional convergence
4.5: The Alternating Series Test
conic sections
6.5: Conic Sections
Contraposition
4.3: The Divergence and Integral Tests
contrapositive
4.3: The Divergence and Integral Tests
convergence of a series
4.2: Infinite Series
convergent
3.7: Improper Integrals
convergent sequence
4.1: Sequences
converges
3.7: Improper Integrals
Cosecant Function
1.1: Trigonometry
Cosine Function
1.1: Trigonometry
Cotangent Function
1.1: Trigonometry
curtate cycloid
6.1: Parametric Equations
cusp
6.1: Parametric Equations
cusps
6.1: Parametric Equations
cycloid
6.1: Parametric Equations
Cycloids
6.1: Parametric Equations
cylindrical shell
2.3: Volumes of Revolution - Cylindrical Shells

D

decreasing sequence
4.1: Sequences
density
2.5: Work
DENSITY FUNCTION
2.5: Work
Derivative of parametric equations
6.2: Calculus of Parametric Curves
derivatives
1.2: Derivatives Review
derivatives of basic functions
1.2: Derivatives Review
differential equation
7.1: Basics of Differential Equations
differential equations
3.7: Improper Integrals
differentiation
1.2: Derivatives Review
directrix
6.5: Conic Sections
discriminant
3.4: Partial Fractions
6.5: Conic Sections
disk method
2.2: Volume by Cross-Sectional Area
distributed mass
2.5: Work
divergence of a series
4.2: Infinite Series
Divergence Test
4.3: The Divergence and Integral Tests
divergent
3.7: Improper Integrals
divergent sequence
4.1: Sequences
diverges
3.7: Improper Integrals

E

eccentricity
6.5: Conic Sections
ellipses
6.5: Conic Sections
end behavior
3.6: Indeterminate Forms and L'Hospital's Rule
English system
2.5: Work
equilibrium
2.5: Work

equilibrium position
2.5: Work
equilibrium solution
7.2: Direction Fields and Numerical Methods
Euler’s Method
7.2: Direction Fields and Numerical Methods
Euler's constant
4.2: Infinite Series
explicit formula for a sequence
4.1: Sequences

F
family of solutions
7.1: Basics of Differential Equations
Fibonacci Numbers
4.1: Sequences
first moment
2.7: Moments and Centers of Mass
first moment of the system
2.7: Moments and Centers of Mass
focal parameter
6.5: Conic Sections
focus
6.5: Conic Sections
force
2.5: Work
frustum
2.4: Arc Length of a Curve and Surface Area
fulcrum
2.7: Moments and Centers of Mass

G
Gabriel's Horn
3.7: Improper Integrals
general form
6.5: Conic Sections
general solution
7.1: Basics of Differential Equations
geometric sequence
4.1: Sequences
geometric series
4.2: Infinite Series
4.5: The Alternating Series Test
growth rate
7.4: The Logistic Equation

H
harmonic series
4.2: Infinite Series
heuristic
3.1: Integration by Parts
Hooke’s law
2.5: Work
horizontal asymptote
3.6: Indeterminate Forms and L’Hospital’s Rule
hydrostatic pressure
2.6: Hydrostatic Force and Pressure
hyperbolas
6.5: Conic Sections
hypocycloid
6.1: Parametric Equations

I
Imperial system
2.5: Work
improper integral
3.7: Improper Integrals
increasing sequence
4.1: Sequences
index variable
4.1: Sequences
infinite limit at infinity
3.6: Indeterminate Forms and L’Hospital’s Rule
infinite sequence
4.1: Sequences
infinite series
4.2: Infinite Series
initial population
7.4: The Logistic Equation
initial value
7.1: Basics of Differential Equations
initial velocity
7.1: Basics of Differential Equations
Integral Test
4.3: The Divergence and Integral Tests
integral transform
3.7: Improper Integrals
integrating factor
7.5: First-order Linear Equations
integration by parts
3.1: Integration by Parts
integration by substitution
1.4: Integration by Substitution
integration table
8.1: Other Strategies for Integration
International System of Units
2.5: Work
interval of convergence
5.1: Power Series and Functions
inverse trigonometric functions
1.1: Trigonometry
irreducible quadratic factors
3.4: Partial Fractions

J
joule
2.5: Work

L
lamina
2.7: Moments and Centers of Mass
Laplace transform
3.7: Improper Integrals
lateral surface area
2.4: Arc Length of a Curve and Surface Area
LIATE
3.1: Integration by Parts
limacon
6.3: Polar Coordinates
limit at infinity
3.6: Indeterminate Forms and L’Hospital’s Rule
Limit Comparison Test
4.4: The Comparison Tests
limit of a sequence
4.1: Sequences
linear
7.5: First-order Linear Equations
linear density
2.7: Moments and Centers of Mass
linear density function
2.7: Moments and Centers of Mass
logistic differential equation
7.4: The Logistic Equation

M
Maclaurin polynomial
5.3: Taylor and Maclaurin Series
Maclaurin series
5.3: Taylor and Maclaurin Series
major axis
6.5: Conic Sections
Mean Value Theorem for Integrals
2.8: The Mean Value Theorem for Integrals
method of cylindrical shells
2.3: Volumes of Revolution - Cylindrical Shells
8.2: Volumes of Revolution - Cylindrical Shells
method of equating coefficients
3.4: Partial Fractions
method of strategic substitution
3.4: Partial Fractions
metric system
2.5: Work
Midpoint Rule
3.5: Numerical Integration
minor axis
6.5: Conic Sections
moment
2.7: Moments and Centers of Mass
moment of the system
2.7: Moments and Centers of Mass
moments
2.7: Moments and Centers of Mass
Moments of Inertia
2.7: Moments and Centers of Mass
Monotone Convergence Theorem
4.1: Sequences
monotone sequence
4.1: Sequences

N
nappe
6.5: Conic Sections
nappes
6.5: Conic Sections
Newton
2.5: Work
Newton’s Law of Cooling
7.3: Separable Equations
Newton's Second Law of Motion
2.5: Work
nonelementary integral
5.4: Working with Taylor Series
nonrepeated irreducible quadratic factors
3.4: Partial Fractions
nonrepeated linear factors
3.4: Partial Fractions
numerical integration
3.5: Numerical Integration

O

oblique asymptote
3.6: Indeterminate Forms and L'Hospital's Rule
order of a differential equation
7.1: Basics of Differential Equations
orientation
6.1: Parametric Equations

P

parabola
6.5: Conic Sections
parameter
6.1: Parametric Equations
parameterization of a curve
6.1: Parametric Equations
parametric curve
6.1: Parametric Equations
parametric equations
6.1: Parametric Equations
partial fraction decomposition
3.4: Partial Fractions
partial sum
4.2: Infinite Series
particular solution
7.1: Basics of Differential Equations
Pascal's Principle
2.6: Hydrostatic Force and Pressure
pascals
2.6: Hydrostatic Force and Pressure
periodic function
1.1: Trigonometry
phase line
7.4: The Logistic Equation
point mass
2.5: Work
2.7: Moments and Centers of Mass
polar axis
6.3: Polar Coordinates
polar coordinate system
6.3: Polar Coordinates
polar equation
6.3: Polar Coordinates
pole
6.3: Polar Coordinates
Power Reduction Formula
3.2: Trigonometric Integrals
power series
5: Power Series
5.1: Power Series and Functions
pressure
2.6: Hydrostatic Force and Pressure
properties of convergent series
4.2: Infinite Series
psi
2.6: Hydrostatic Force and Pressure

R

radial coordinate
6.3: Polar Coordinates
radial density
2.7: Moments and Centers of Mass
radians
1.1: Trigonometry
radical substitution method
3.4: Partial Fractions
radius of convergence
5.1: Power Series and Functions
ratio test
4.6: The Ratio and Root Tests
rationalizing substitution method
3.4: Partial Fractions
recurrence relation
4.1: Sequences
recursion relation
4.1: Sequences
relative error
3.5: Numerical Integration
Remainder Estimate
4.3: The Divergence and Integral Tests
repeated irreducible quadratic factors
3.4: Partial Fractions
repeated linear factors
3.4: Partial Fractions
Riemann Rearrangement Theorem
4.5: The Alternating Series Test
root test
4.6: The Ratio and Root Tests
rose
6.3: Polar Coordinates
rules of differentiation
1.2: Derivatives Review

S

Secant Function
1.1: Trigonometry
separation of variables
7.3: Separable Equations
sequence
4.1: Sequences
sequences
4.1: Sequences
Series Comparison Test
4.4: The Comparison Tests
shell method
2.3: Volumes of Revolution - Cylindrical Shells
SI system
2.5: Work
Simpson's rule
3.5: Numerical Integration
sine function
1.1: Trigonometry
slope field
7.2: Direction Fields and Numerical Methods
smooth curve
2.4: Arc Length of a Curve and Surface Area
smooth function
2.4: Arc Length of a Curve and Surface Area
solid of revolution
2.3: Volumes of Revolution - Cylindrical Shells
8.2: Volumes of Revolution - Cylindrical Shells
solution curve
7.2: Direction Fields and Numerical Methods
solution to a differential equation
7.1: Basics of Differential Equations
specific mass
2.5: Work
specific weight
2.5: Work
spring constant
2.5: Work
Squeeze Theorem for Sequences
4.1: Sequences
standard form
6.5: Conic Sections
step size
7.2: Direction Fields and Numerical Methods
surface area
2.4: Arc Length of a Curve and Surface Area
Surface area generated by a parametric curve
6.2: Calculus of Parametric Curves
surface area of a frustum
2.4: Arc Length of a Curve and Surface Area
surface of revolution
2.4: Arc Length of a Curve and Surface Area
Symmetry Principle
2.7: Moments and Centers of Mass

T

Tangent Function
1.1: Trigonometry
Taylor polynomials
5.3: Taylor and Maclaurin Series
Taylor series
5.3: Taylor and Maclaurin Series
Taylor's theorem with remainder
5.3: Taylor and Maclaurin Series
telescoping series
4.2: Infinite Series
term
4.1: Sequences
The Logistic Equation
7.4: The Logistic Equation
Theorem of Pappus
2.7: Moments and Centers of Mass
Theorem of Pappus for Volume
2.7: Moments and Centers of Mass
threshold population
7.4: The Logistic Equation
Torricelli's Trumpet
3.7: Improper Integrals
trapezoidal rule
3.5: Numerical Integration
trigonometric functions
1.1: Trigonometry
trigonometric identities
1.1: Trigonometry
trigonometric identity
1.1: Trigonometry
trigonometric integral
3.2: Trigonometric Integrals
trigonometric substitution
3.3: Trigonometric Substitution

U

U.S. customary measurement system
2.5: Work
unit circle
1.1: Trigonometry

V

vertex

6.5: Conic Sections

Volume by Shells

8.2: Volumes of Revolution - Cylindrical Shells

volumetric density

2.5: Work

W

washer method

2.2: Volume by Cross-Sectional Area

weight density

2.5: Work

Witch of Agnesi

6.1: Parametric Equations

work

2.5: Work

Glossary

Sample Word 1 | Sample Definition 1

Detailed Licensing

Overview

Title: Calculus II – Integral Calculus

Webpages: 108

Applicable Restrictions: Noncommercial

All licenses found:

- CC BY-NC-SA 4.0: 70.4% (76 pages)
- Undeclared: 20.4% (22 pages)
- CC BY-NC 4.0: 8.3% (9 pages)
- CC BY-NC 3.0: 0.9% (1 page)

By Page

- Calculus II – Integral Calculus - *Undeclared*
  - Front Matter - *Undeclared*
    - TitlePage - *Undeclared*
    - InfoPage - *Undeclared*
    - Table of Contents - *Undeclared*
    - Licensing - *Undeclared*
  - 1: Fast Review of Important Calc I Stuff - *CC BY-NC-SA 4.0*
    - 1.1: Trigonometry - *CC BY-NC-SA 4.0*
      - 1.1E: Exercises - *CC BY-NC-SA 4.0*
    - 1.2: Derivatives Review - *Undeclared*
      - 1.2E: Exercises - *CC BY-NC-SA 4.0*
    - 1.3: Integrals Review - *Undeclared*
      - 1.3.1: Exercises - *Undeclared*
    - 1.4: Integration by Substitution - *CC BY-NC-SA 4.0*
      - 1.4E: Exercises - *Undeclared*
  - 2: Applications of Integration - *CC BY-NC-SA 4.0*
    - 2.1: Areas Between Curves - *CC BY-NC 4.0*
      - 2.1E: Exercises - *CC BY-NC 4.0*
    - 2.2: Volume by Cross-Sectional Area - *CC BY-NC 3.0*
      - 2.2E: Exercises - *CC BY-NC 4.0*
    - 2.3: Volumes of Revolution - Cylindrical Shells - *CC BY-NC 4.0*
      - 2.3E: Exercises - *CC BY-NC 4.0*
    - 2.4: Arc Length of a Curve and Surface Area - *CC BY-NC 4.0*
      - 2.4E: Exercises - *CC BY-NC 4.0*
    - 2.5: Work - *CC BY-NC 4.0*
      - 2.5E: Exercises - *CC BY-NC 4.0*
    - 2.6: Hydrostatic Force and Pressure - *CC BY-NC-SA 4.0*
      - 2.6E: Exercises - *CC BY-NC-SA 4.0*
    - 2.7: Moments and Centers of Mass - *CC BY-NC-SA 4.0*
      - 2.7E: Exercises - *CC BY-NC-SA 4.0*
    - 2.8: The Mean Value Theorem for Integrals - *CC BY-NC-SA 4.0*
      - 2.8E: Exercises - *Undeclared*
    - 2.9: Chapter 2 Review Exercises - *Undeclared*
  - 3: Techniques of Integration - *CC BY-NC-SA 4.0*
    - 3.1: Integration by Parts - *CC BY-NC-SA 4.0*
      - 3.1E: Exercises - *Undeclared*
    - 3.2: Trigonometric Integrals - *CC BY-NC-SA 4.0*
      - 3.2E: Exercises - *CC BY-NC-SA 4.0*
    - 3.3: Trigonometric Substitution - *CC BY-NC-SA 4.0*
      - 3.3E: Exercises - *CC BY-NC-SA 4.0*
    - 3.4: Partial Fractions - *CC BY-NC-SA 4.0*
      - 3.4E: Exercises - *CC BY-NC-SA 4.0*
    - 3.5: Numerical Integration - *CC BY-NC-SA 4.0*
      - 3.5E: Exercises - *CC BY-NC-SA 4.0*
    - 3.6: Indeterminate Forms and L'Hospital's Rule - *CC BY-NC-SA 4.0*
      - 3.6E: Exercises - *Undeclared*
    - 3.7: Improper Integrals - *CC BY-NC-SA 4.0*
      - 3.7E: Exercises - *CC BY-NC-SA 4.0*
    - 3.8: Chapter 3 Review Exercises - *Undeclared*
  - 4: Sequences and Series - *CC BY-NC-SA 4.0*
    - 4.1: Sequences - *CC BY-NC-SA 4.0*
      - 4.1E: Exercises - *CC BY-NC-SA 4.0*

- 4.2: Infinite Series - *CC BY-NC-SA 4.0*
  - 4.2E: Exercises - *CC BY-NC-SA 4.0*
- 4.3: The Divergence and Integral Tests - *CC BY-NC-SA 4.0*
  - 4.3E: Exercises - *CC BY-NC-SA 4.0*
- 4.4: The Comparison Tests - *CC BY-NC-SA 4.0*
  - 4.4E: Exercises - *CC BY-NC-SA 4.0*
- 4.5: The Alternating Series Test - *CC BY-NC-SA 4.0*
  - 4.5E: Exercises - *CC BY-NC-SA 4.0*
- 4.6: The Ratio and Root Tests - *CC BY-NC-SA 4.0*
  - 4.6E: Exercises - *CC BY-NC-SA 4.0*
- 4.7: Chapter 4 Review Exercises - *CC BY-NC-SA 4.0*

5: Power Series - *CC BY-NC-SA 4.0*
- 5.1: Power Series and Functions - *CC BY-NC-SA 4.0*
  - 5.1E: Exercises - *CC BY-NC-SA 4.0*
- 5.2: Properties of Power Series - *CC BY-NC-SA 4.0*
  - 5.2E: Exercises - *CC BY-NC-SA 4.0*
- 5.3: Taylor and Maclaurin Series - *CC BY-NC-SA 4.0*
  - 5.3E: Exercises - *CC BY-NC-SA 4.0*
- 5.4: Working with Taylor Series - *CC BY-NC-SA 4.0*
  - 5.4E: Exercises - *CC BY-NC-SA 4.0*
- 5.5: Chapter 5 Review Exercises - *CC BY-NC-SA 4.0*

6: Parametric Equations and Polar Coordinates - *CC BY-NC-SA 4.0*
- 6.1: Parametric Equations - *CC BY-NC-SA 4.0*
  - 6.1E: Exercises - *Undeclared*
- 6.2: Calculus of Parametric Curves - *Undeclared*
  - 6.2E: Exercises - *Undeclared*
- 6.3: Polar Coordinates - *CC BY-NC-SA 4.0*
  - 6.3E: Exercises - *CC BY-NC-SA 4.0*
- 6.4: Area and Arc Length in Polar Coordinates - *CC BY-NC-SA 4.0*
  - 6.4E: Exercises - *CC BY-NC-SA 4.0*
- 6.5: Conic Sections - *CC BY-NC-SA 4.0*
  - 6.5E: Exercises - *CC BY-NC-SA 4.0*
- 6.6: Chapter 6 Review Exercises - *CC BY-NC-SA 4.0*

7: Introduction to Differential Equations - *CC BY-NC-SA 4.0*
- 7.1: Basics of Differential Equations - *CC BY-NC-SA 4.0*
  - 7.1E: Exercises for Section 8.1 - *CC BY-NC-SA 4.0*
- 7.2: Direction Fields and Numerical Methods - *CC BY-NC-SA 4.0*
  - 7.2E: Exercises for Section 8.2 - *CC BY-NC-SA 4.0*
- 7.3: Separable Equations - *CC BY-NC-SA 4.0*
  - 7.3E: Exercises for Section 8.3 - *CC BY-NC-SA 4.0*
- 7.4: The Logistic Equation - *CC BY-NC-SA 4.0*
  - 7.4E: Exercises for Section 8.4 - *CC BY-NC-SA 4.0*
- 7.5: First-order Linear Equations - *CC BY-NC-SA 4.0*
  - 7.5E: Exercises for Section 8.5 - *CC BY-NC-SA 4.0*
- 7.6: Chapter 7 Review Exercises - *CC BY-NC-SA 4.0*

8: Appendices - *CC BY-NC-SA 4.0*
- 8.1: Other Strategies for Integration - *CC BY-NC-SA 4.0*
  - 8.1E: Exercises - *CC BY-NC-SA 4.0*
- 8.2: Volumes of Revolution - Cylindrical Shells - *CC BY-NC-SA 4.0*
  - 8.2E: Exercises for Section 6.3 - *CC BY-NC-SA 4.0*
- 8.A: Table of Derivatives - *CC BY-NC-SA 4.0*
- 8.B: Table of Integrals - *CC BY-NC-SA 4.0*

Back Matter - *Undeclared*
- Index - *Undeclared*
- Glossary - *Undeclared*
- Detailed Licensing - *Undeclared*