Calculus IV: Multivariable Calculus

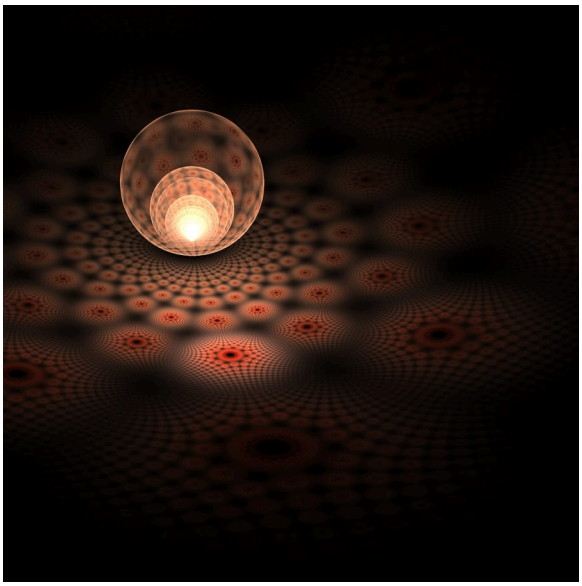
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De Anza College

Mathematics

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Detailed Licensing

Detailed Licensing

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About the Book

Publication Date: August, 2025

This **Calculus textbook** is part of the **De Anza College Zero Textbook Cost (ZTC) Pathways** initiative, supporting both students and instructors with free, adaptable Open Educational Resources (OER).

This series builds upon the OpenStax Calculus foundation and acknowledges the contributions of Lake Tahoe Community College's derivative version, incorporating videos and interactive elements. De Anza's version enhances accessibility, adds multimedia resources, and emphasizes critical thinking and problem-solving for both classroom and independent study.

This fourth volume in a four-part calculus series is designed for both beginners and advanced students, with a focus on developing strong conceptual understanding. It explores essential topics in depth, providing students with a solid foundation in the principles and techniques of calculus. Covering differential and integral calculus in one variable, the construction and use of Taylor series, and vector calculus with its applications, the book connects theory to real-world problems in physics, economics, engineering, and beyond.

With numerous examples, practical applications, and videos for visual demonstrations, the series engages learners and supports multiple modes of understanding. Structured to promote critical thinking and problem-solving skills, it serves as a valuable resource for both independent study and classroom learning. With its blend of open access, practical applications, and rich multimedia support, this textbook reflects De Anza College's commitment to student equity, innovation, and leadership in OER.

In addition to the core text, we provide ancillary materials to enhance teaching and learning.

Included Ancillary Materials

- **PowerPoint Slide Decks**: Accessible chapter-aligned slides for lectures, presentations, or flipped classrooms.
- **Instructor Notes**: Guidance for educators with teaching tips, common misconceptions, and activity ideas.

These ancillaries can be found under **"assets"** on the De Anza Commons page for Calculus IV: Multivariable Calculus.

Online Homework

Aligned online homework is available on **MyOpenMath**. After logging into an instructor account, select *Add New Course → Copy a template or promoted course →* choose *Calculus I: Differential Calculus (De Anza ZTC)*.

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About the Authors

Information about the authors

Author Images	Author Bios
	**Vinh Kha Nguyen** is a dedicated math instructor at De Anza College who is passionate about making mathematics accessible to everyone. With a deep commitment to education, he strives to help students develop a strong foundation and appreciation for the subject. Outside the classroom, Vinh enjoys hiking and traveling with his family, embracing new adventures together. He is especially grateful to his wife for her unwavering support in caring for their child, allowing him the time to focus on completing his Open Educational Resources (OER) textbook to benefit students and educators alike.
	**Fatemeh Yarahmadi** is a dedicated math instructor at De Anza College, whose passion for mathematics extends far beyond the textbook. She believes that math is not just a subject, but a language, a tool for understanding the world, and she is committed to helping every student unlock its power. With a deep commitment to education, she strives to build a nurturing and stimulating learning environment, where students feel empowered to ask questions and explore complex concepts without fear. Her teaching philosophy is rooted in the belief that mathematics is accessible to everyone, regardless of their background or prior experiences. She employs innovative teaching methods, incorporating real-world examples and interactive exercises, to make abstract concepts tangible and relatable. She understands that each student learns differently, and she dedicates herself to providing personalized support, ensuring that no one is left behind.

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1: DIFFERENTIATION OF FUNCTIONS OF SEVERAL VARIABLES

When dealing with a function of more than one independent variable, several questions naturally arise. For example, how do we calculate limits of functions of more than one variable? The definition of derivative we used before involved a limit. Does the new definition of derivative involve limits as well? Do the rules of differentiation apply in this context? Can we find relative extrema of functions using derivatives? All these questions are answered in this chapter.

In Introduction to Applications of Derivatives, we studied how to determine the maximum and minimum of a function of one variable over a closed interval. This function might represent the temperature over a given time interval, the position of a car as a function of time, or the altitude of a jet plane as it travels from New York to San Francisco. In each of these examples, the function has one independent variable.



Figure 1.1: Americans use (and lose) millions of golf balls a year, which keeps golf ball manufacturers in business. In this chapter, we study a profit model and learn methods for calculating optimal production levels for a typical golf ball manufacturing company. (CC-BY-2.0 modification of work by oatsy40, Flickr)

Suppose, however, that we have a quantity that depends on more than one variable. For example, temperature can depend on location and the time of day, or a company's profit model might depend on the number of units sold and the amount of money spent on advertising. In this chapter, we look at a company that produces golf balls. We develop a profit model and, under various restrictions, we find that the optimal level of production and advertising dollars spent determines the maximum possible profit. Depending on the nature of the restrictions, both the method of solution and the solution itself changes.

1.1: Functions of Several Variables

1.1E: Exercises for Section 1.1

1.2: Limits and Continuity

1.2E: Exercises for Section 1.2

1.3: Partial Derivatives

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1.1: Functions of Several Variables

Learning Objectives

- Recognize a function of two variables and identify its domain and range.
- Sketch a graph of a function of two variables.
- Sketch several traces or level curves of a function of two variables.
- Recognize a function of three or more variables and identify its level surfaces.

Our first step is to explain what a function of more than one variable is, starting with functions of two independent variables. This step includes identifying the domain and range of such functions and learning how to graph them. We also examine ways to relate the graphs of functions in three dimensions to graphs of more familiar planar functions.

Functions of Two Variables

The definition of a function of two variables is very similar to the definition for a function of one variable. The main difference is that, instead of mapping values of one variable to values of another variable, we map ordered pairs of variables to another variable.

Definition: Function of Two Variables

A *function of two variables* $z = f(x, y)$ maps each ordered pair (x, y) in a subset D of the real plane R^2 to a unique real number z. The set D is called the domain of the function. The range of f is the set of all real numbers z that has at least one ordered pair $(x, y) \in D$ such that $f(x, y) = z$ as shown in Figure 1.1.1.

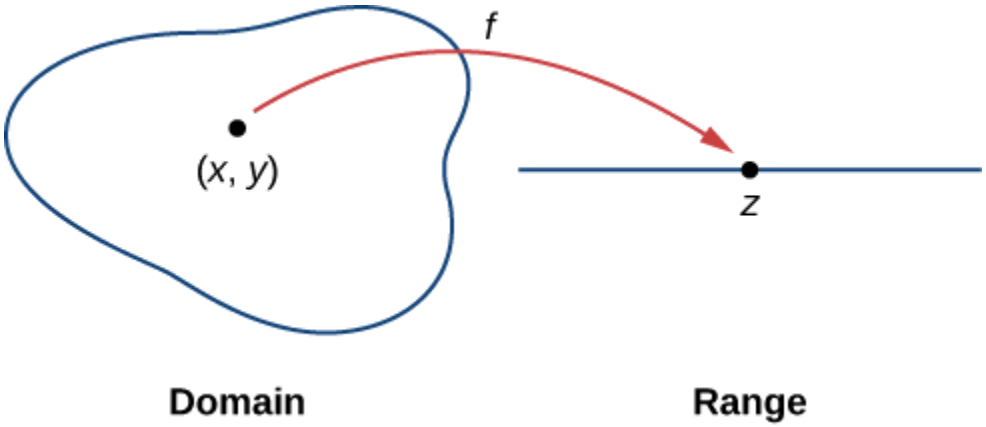


Figure 1.1.1: The image illustrates the concept of a function of two variables and its domain. It contains a closed, irregularly shaped region labeled **Domain**, representing the set of ordered pairs (x, y). Inside this domain, there is a single point (x, y) marked with a black dot. A curved arrow labeled f leads from (x, y) to a point z, illustrating the mapping of the function.

Determining the domain of a function of two variables involves taking into account any domain restrictions that may exist. Let's take a look.

Example 1.1.1: Domains and Ranges for Functions of Two Variables

Find the domain and range of each of the following functions:

a. $f(x, y) = 3x + 5y + 2$

b. $g(x,y)=\sqrt{9-x^2-y^2}$

Solution

a. This is an example of a linear function in two variables. There are no values or combinations of x and y that cause $f(x,y)$ to be undefined, so the domain of f is R^2. To determine the range, first pick a value for z. We need to find a solution to the equation $f(x,y)=z$, or $3x-5y+2=z$. One such solution can be obtained by first setting $y=0$, which yields the equation $3x+2=z$. The solution to this equation is $x=\dfrac{z-2}{3}$, which gives the ordered pair $\left(\dfrac{z-2}{3},0\right)$ as a solution to the equation $f(x,y)=z$ for any value of z. Therefore, the range of the function is all real numbers, or R.

b. For the function $g(x,y)$ to have a real value, the quantity under the square root must be nonnegative:

$$9-x^2-y^2\geq 0.$$

This inequality can be written in the form

$$x^2+y^2\leq 9.$$

Therefore, the domain of $g(x,y)$ is $\{(x,y)\in R^2 \mid x^2+y^2\leq 9\}$. The graph of this set of points can be described as a disk of radius 3 centered at the origin. The domain includes the boundary circle as shown in the following graph.

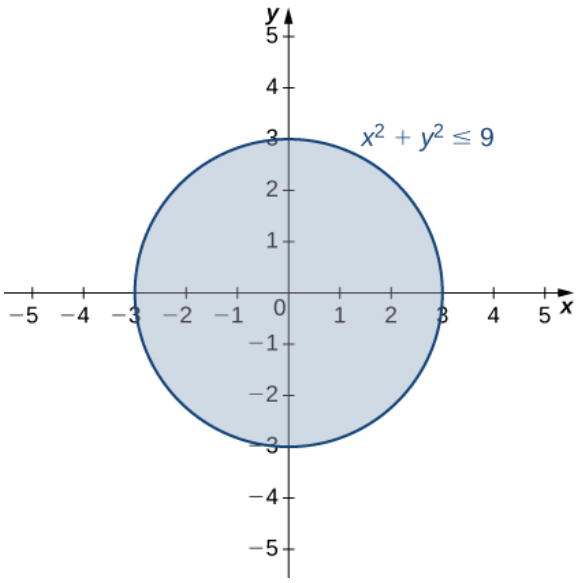


Figure 1.1.2: The image illustrates a shaded circular region centered at the origin in the Cartesian coordinate plane. The circle corresponds to the inequality $x^2+y^2\leq 9$, indicating that all points inside and on the boundary of the circle satisfy the given condition. The coordinate axes are labeled x and y, with tick marks ranging from approximately -5 to 5 on both axes.

To determine the range of $g(x,y)=\sqrt{9-x^2-y^2}$ we start with a point (x_0,y_0) on the boundary of the domain, which is defined by the relation $x^2+y^2=9$. It follows that $x_0^2+y_0^2=9$ and

$$\begin{aligned} g(x_0,y_0) &= \sqrt{9-x_0^2-y_0^2} \\ &= \sqrt{9-(x_0^2+y_0^2)} \\ &= \sqrt{9-9} \\ &= 0. \end{aligned}$$

If $x_0^2+y_0^2=0$ (in other words, $x_0=y_0=0$), then

$$g(x_0, y_0) = \sqrt{9 - x_0^2 - y_0^2}$$
$$= \sqrt{9 - (x_0^2 + y_0^2)}$$
$$= \sqrt{9 - 0} = 3.$$

This is the maximum value of the function. Given any value c between 0 and 3, we can find an entire set of points inside the domain of g such that $g(x, y) = c$:

$$\sqrt{9 - x^2 - y^2} = c$$
$$9 - x^2 - y^2 = c^2$$
$$x^2 + y^2 = 9 - c^2.$$

Since $9 - c^2 \geq 0$, this describes a circle of radius $\sqrt{9 - c^2}$ centered at the origin. Any point on this circle satisfies the equation $g(x, y) = c$. Therefore, the range of this function can be written in interval notation as $[0, 3]$.

Watch the accompanying video for a visual demonstration of the concept.

- **Video length**: 3 minutes 22 seconds.
- **Context**: This video demonstrate how to find the domain and range of a function of two variables.

YouTube Video

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Exercise 1.1.1

Find the domain and range of the function $f(x, y) = \sqrt{36 - 9x^2 - 9y^2}$.

Hint

Determine the set of ordered pairs that do not make the radicand negative.

Solution

The domain is $\{(x, y) | x^2 + y^2 \leq 4\}$ the shaded circle defined by the inequality $x^2 + y^2 \leq 4$, which has a circle of radius 2 as its boundary. The range is $[0, 6]$.

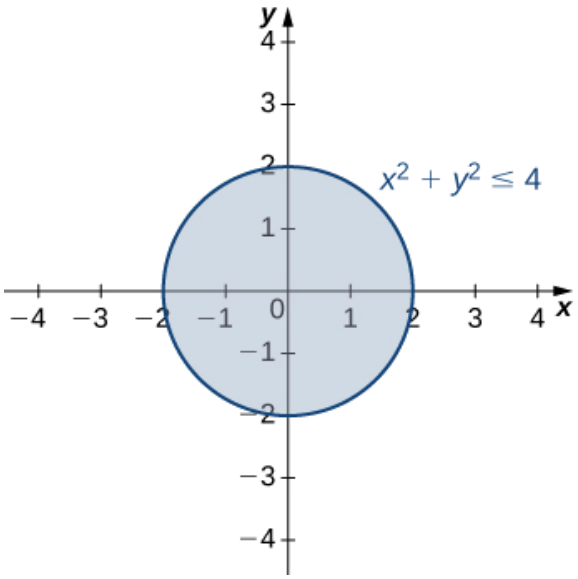


Figure 1.1.3: The image illustrates a shaded circular region centered at the origin in the Cartesian coordinate plane. The circle represents the inequality $x^2 + y^2 \leq 4$, meaning all points inside and on the boundary of the circle satisfy the given condition. The axes are labeled with x and y, with tick marks ranging from approximately -4 to 4 on both axes.

Graphing Functions of Two Variables

Suppose we wish to graph the function $z = (x, y)$. This function has two independent variables (x and y) and one dependent variable (z). When graphing a function $y = f(x)$ of one variable, we use the Cartesian plane. We are able to graph any ordered pair (x, y) in the plane, and every point in the plane has an ordered pair (x, y) associated with it. With a function of two variables, each ordered pair (x, y) in the domain of the function is mapped to a real number z. Therefore, the graph of the function f consists of ordered triples (x, y, z). The graph of a function $z = (x, y)$ of two variables is called a surface.

To understand more completely the concept of plotting a set of ordered triples to obtain a surface in three-dimensional space, imagine the (x, y) coordinate system laying flat. Then, every point in the domain of the function f has a unique *z*-value associated with it. If *z* is positive, then the graphed point is located above the *xy*-plane, if *z* is negative, then the graphed point is located below the *xy*-plane. The set of all the graphed points becomes the two-dimensional surface that is the graph of the function f.

Example 1.1.2: Graphing Functions of Two Variables

Create a graph of each of the following functions:

a. $g(x, y) = \sqrt{9 - x^2 - y^2}$
b. $f(x, y) = x^2 + y^2$

Solution

a. In Example 1.1.2, we determined that the domain of $g(x, y) = \sqrt{9 - x^2 - y^2}$ is $\{(x, y) \in R^2 \mid x^2 + y^2 \leq 9\}$ and the range is $\{z \in R^2 \mid 0 \leq z \leq 3\}$. When $x^2 + y^2 = 9$ we have $g(x, y) = 0$. Therefore any point on the circle of radius 3 centered at the origin in the xy -plane maps to $z = 0$ in R^3. If $x^2 + y^2 = 8$, then $g(x, y) = 1$, so any point on the circle of radius $2\sqrt{2}$ centered at the origin in the xy -plane maps to $z = 1$ in R^3. As $x^2 + y^2$ gets closer to zero, the value of z approaches 3. When $x^2 + y^2 = 0$, then $g(x, y) = 3$. This is the origin in the xy-plane. If $x^2 + y^2$ is equal to any other value between 0 and 9, then $g(x, y)$ equals some other constant between 0 and 3. The surface described by this function is a hemisphere centered at the origin with radius 3 as shown in Figure 1.1.4.

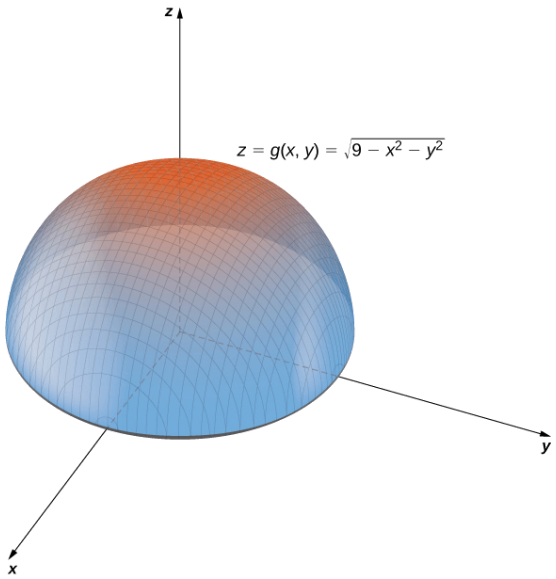


Figure 1.1.4: The image illustrates a three-dimensional graph of a hemisphere representing the function $g(x,y)=\sqrt{9-x^2-y^2}$. The surface is a dome-like shape extending above the xy-plane, where the function is defined. The hemisphere is centered at the origin and has a radius of 3. The coordinate axes are labeled x, y, and z, with the function equation displayed in the upper-right region of the graph.

b. This function also contains the expression x^2+y^2. Setting this expression equal to various values starting at zero, we obtain circles of increasing radius. The minimum value of $f(x,y)=x^2+y^2$ is zero (attained when $x=y=0).When\backslash(x=0$, the function becomes $z=y^2$, and when $y=0$, then the function becomes $z=x^2$. These are cross-sections of the graph, and they are parabolas. Recall from Introduction to Vectors in Space that the name of the graph of $f(x,y)=x^2+y^2$ is a **paraboloid**. The graph of f appears in Figure 1.1.5.

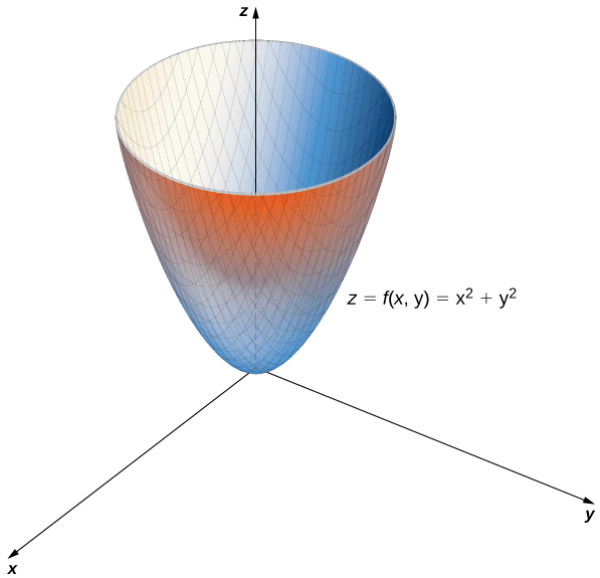


Figure 1.1.5: The image displays a three-dimensional graph of an upward-opening paraboloid representing the function $f(x,y)=x^2+y^2$. The surface is shaped like a smooth, curved bowl with its vertex at the origin, extending upwards. The coordinate axes are labeled x, y, and z, with the z-axis pointing upward.

Example 1.1.3: Nuts and Bolts

A profit function for a hardware manufacturer is given by

$$f(x,y)=16-(x-3)^2-(y-2)^2,$$

where x is the number of nuts sold per month (measured in thousands) and y represents the number of bolts sold per month

(measured in thousands). Profit is measured in thousands of dollars. Sketch a graph of this function.

Solution

This function is a polynomial function in two variables. The domain of f consists of (x, y) coordinate pairs that yield a nonnegative profit:

$$16 - (x-3)^2 - (y-2)^2 \geq 0$$
$$(x-3)^2 + (y-2)^2 \leq 16.$$

This is a disk of radius 4 centered at $(3, 2)$. A further restriction is that both x and y must be nonnegative. When $x = 3$ and $y = 2, f(x, y) = 16$. Note that it is possible for either value to be a noninteger; for example, it is possible to sell 2.5 thousand nuts in a month. The domain, therefore, contains thousands of points, so we can consider all points within the disk. For any $z < 16$, we can solve the equation $f(x, y) = 16$:

$$16 - (x-3)^2 - (y-2)^2 = z$$
$$(x-3)^2 + (y-2)^2 = 16 - z.$$

Since $z < 16$, we know that $16 - z > 0$, so the previous equation describes a circle with radius $\sqrt{16-z}$ centered at the point $(3, 2)$. Therefore, the range of $f(x, y)$ is $\{z \in \mathbb{R} | z \leq 16\}$. The graph of $f(x, y)$ is also a paraboloid, and this paraboloid points downward as shown in Figure $1.1.6$.

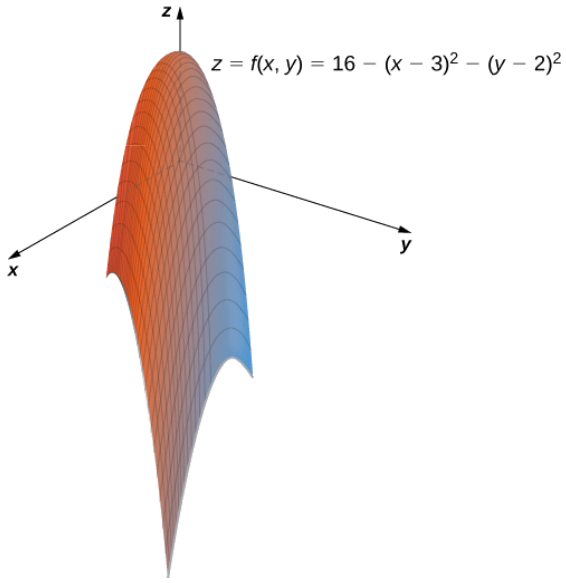


Figure $1.1.6$: The image shows a three-dimensional graph of a downward-opening paraboloid representing the function $f(x, y) = 16 - (x-3)^2 - (y-2)^2$. The surface has a smooth, bowl-like shape that opens downward. The coordinate axes are labeled x, y, and z, with the z-axis pointing upward.

Example $1.1.4$: 3D Solid Defined by an Inequality

Describe the solid that satisfies $x^2 + y^2 + z^2 > -2x - 6y + 4z + 2$

Solution

- **Video length**: 2 minutes 45 seconds.
- **Context**: This video demonstrates how to describe a 3D solid defined by an inequality.

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Level Curves

If hikers walk along rugged trails, they might use a topographical map that shows how steeply the trails change. A *topographical map* contains curved lines called contour lines. Each contour line corresponds to the points on the map that have equal elevation (Figure 1.1.7). A level curve of a function of two variables $f(x, y)$ is completely analogous to a contour line on a topographical map.

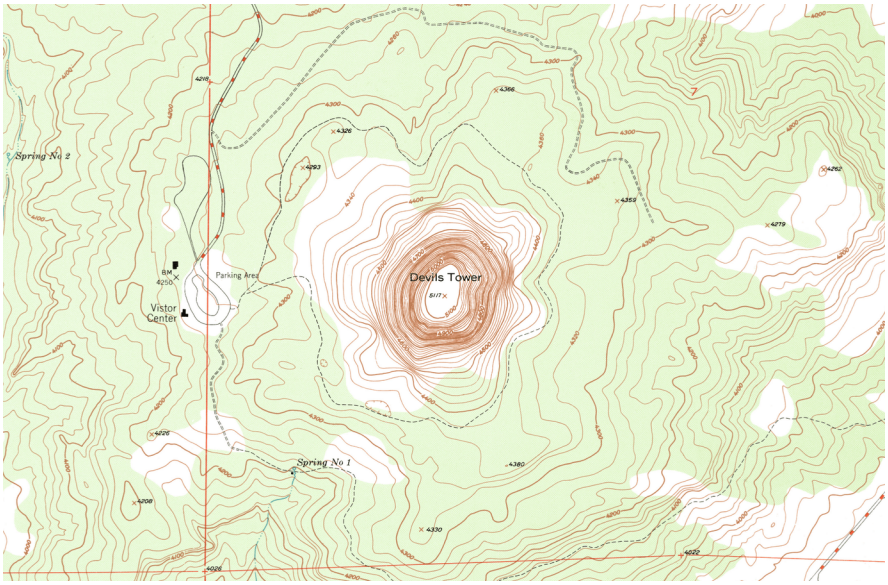


Figure 1.1.7: A topographical map of Devil's Tower, Wyoming. Lines that are close together indicate very steep terrain. (CC BY-NC-SA 4.0, modification of work by Geological Survey)

Definition: Level Curves

Given a function $f(x, y)$ and a number c in the range of f, a level curve of a function of two variables for the value c is defined to be the set of points satisfying the equation $f(x, y) = c$.

Returning to the function $g(x, y) = \sqrt{9 - x^2 - y^2}$, we can determine the level curves of this function. The range of g is the closed interval $[0, 3]$. First, we choose any number in this closed interval—say, $c = 2$. The level curve corresponding to $c = 2$ is described by the equation

$$\sqrt{9 - x^2 - y^2} = 2. \tag{1.1.1}$$

To simplify, square both sides of this equation:

$$9 - x^2 - y^2 = 4. \tag{1.1.2}$$

Now, multiply both sides of the equation by -1 and add $9$to each side:

$$x^2 + y^2 = 5. \tag{1.1.3}$$

This equation describes a circle centered at the origin with radius $\sqrt{5}$. Using values of c between 0 and 3 yields other circles also centered at the origin. If $c = 3$, then the circle has radius 0, so it consists solely of the origin. Figure 1.1.8 is a graph of the

level curves of this function corresponding to $c = 0, 1, 2,$ and 3. Note that in the previous derivation it may be possible that we introduced extra solutions by squaring both sides. This is not the case here because the range of the square root function is nonnegative.

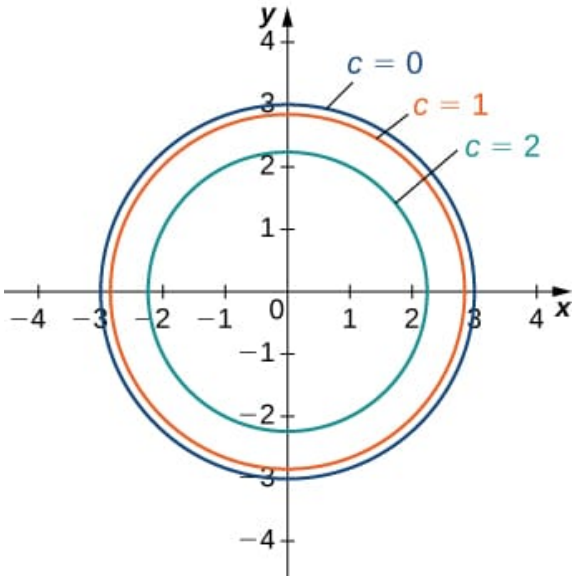


Figure 1.1.8: The image presents a set of level curves for the function $g(x, y) = \sqrt{9 - x^2 - y^2}$. The graph is a two-dimensional coordinate system with labeled x- and y-axes, ranging approximately from -4 to 4. Three concentric circles are drawn, representing level curves for different values of c. The outermost circle, in blue, corresponds to $c = 0$. The middle circle, in orange, represents $c = 1$. The innermost circle, in green, represents $c = 2$. As the value of c increases, the circles shrink towards the origin.

A graph of the various level curves of a function is called a *contour map*.

Example 1.1.5: Making a Contour Map

Given the function $f(x, y) = \sqrt{8 + 8x - 4y - 4x^2 - y^2}$, find the level curve corresponding to $c = 0$. Then create a contour map for this function. What are the domain and range of f?

Solution

To find the level curve for $c = 0$, we set $f(x, y) = 0$ and solve. This gives

$$0 = \sqrt{8 + 8x - 4y - 4x^2 - y^2}.$$

We then square both sides and multiply both sides of the equation by -1:

$$4x^2 + y^2 - 8x + 4y - 8 = 0.$$

Now, we rearrange the terms, putting the x terms together and the y terms together, and add 8 to each side:

$$4x^2 - 8x + y^2 + 4y = 8.$$

Next, we group the pairs of terms containing the same variable in parentheses, and factor 4 from the first pair:

$$4(x^2 - 2x) + (y^2 + 4y) = 8.$$

Then we complete the square in each pair of parentheses and add the correct value to the right-hand side:

$$4(x^2 - 2x + 1) + (y^2 + 4y + 4) = 8 + 4(1) + 4.$$

Next, we factor the left-hand side and simplify the right-hand side:

$$4(x-1)^2+(y+2)^2=16.$$

Last, we divide both sides by 16 :

$$\frac{(x-1)^2}{4}+\frac{(y+2)^2}{16}=1.$$

This equation describes an ellipse centered at $(1,-2)$. The graph of this ellipse appears in Figure $1.1.9$.

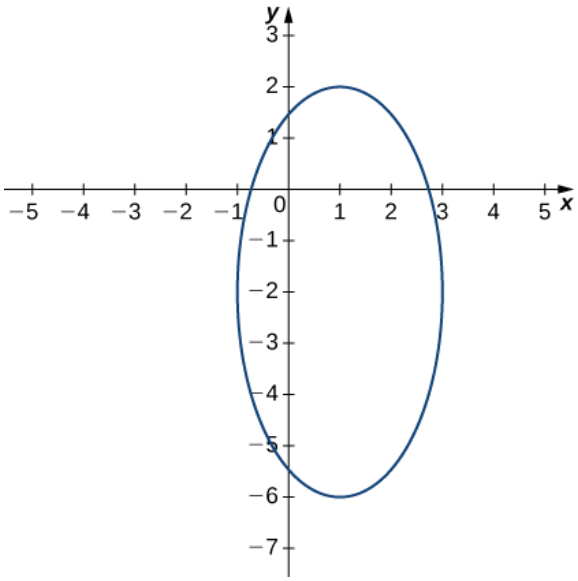


Figure $1.1.9$: Level curve of the function $f(x,y)=\sqrt{8+8x-4y-4x^2-y^2}$ corresponding to $c=0$ is an ellipse with center $(-1,2)$, major axis vertical and of length 8, and minor axis horizontal of length 4.

We can repeat the same derivation for values of c less than 4. Then, Equation $\frac{(x-1)^2}{4}+\frac{(y+2)^2}{16}=1$ becomes

$$\frac{4(x-1)^2}{16-c^2}+\frac{(y+2)^2}{16-c^2}=1$$

for an arbitrary value of c. Figure $1.1.10$ shows a contour map for $f(x,y)$ using the values $c=0,1,2$, and 3. When $c=4$, the level curve is the point $(-1,2)$.

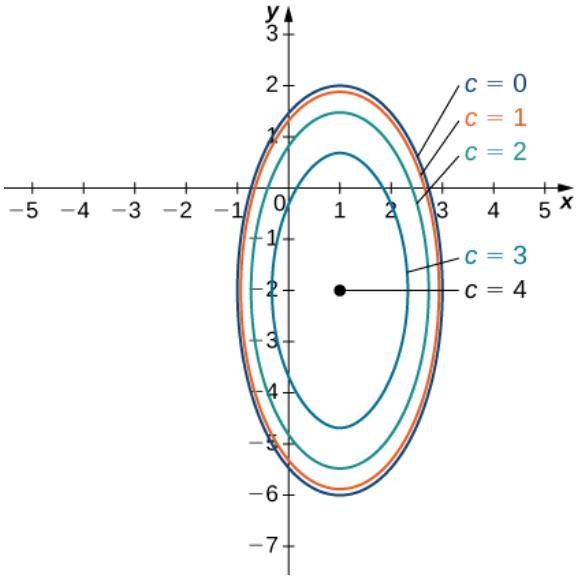


Figure 1.1.10: Contour map for the function $f(x,y) = \sqrt{8 + 8x - 4y - 4x^2 - y^2}$ using the values $c = 0, 1, 2, 3$, and $c = 4$ corresponds to the center $(-1, 2)$.

Watch the accompanying video for a visual demonstration of the concept.

- **Video length**: 4 minutes 15 seconds.
- **Context**: This video demonstrates how to sketch level curves and describe the paths along the surface.

YouTube Video

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Exercise 1.1.4

Find and graph the level curve of the function $g(x,y) = x^2 + y^2 - 6x + 2y$ corresponding to $c = 15$.

Hint

First, set $g(x,y) = 15$ and then complete the square.

Solution

The equation of the level curve can be written as $(x-3)^2 + (y+1)^2 = 25$, which is a circle with radius 5 centered at $(3, -1)$.

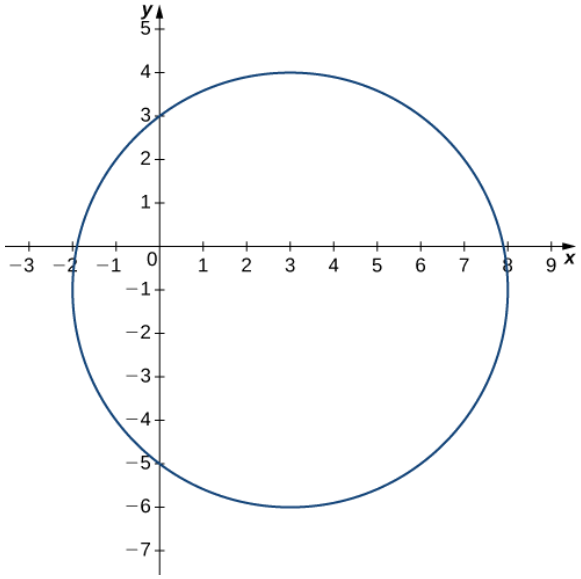


Figure 1.1.11: A circle with radius 5 centered at $(3, -1)$.

Another useful tool for understanding the **graph of a function of two variables** is called a vertical trace. Level curves are always graphed in the $xy-plane$, but as their name implies, vertical traces are graphed in the $xz-$ or $yz-$ planes.

Definition: Vertical Traces

Consider a function $z = f(x, y)$ with domain $D \subseteq \mathbb{R}^2$. A *vertical trace* of the function can be either the set of points that solves the equation $f(a, y) = z$ for a given constant $x = a$ or $f(x, b) = z$ for a given constant $y = b$.

Example 1.1.6: Finding Vertical Traces

Find vertical traces for the function $f(x, y) = \sin x \cos y$ corresponding to $x = -\frac{\pi}{4}, 0$, and $\frac{\pi}{4}$, and $y = -\frac{\pi}{4}, 0$, and $\frac{\pi}{4}$.

Solution

First set $x = -\frac{\pi}{4}$ in the equation $z = \sin x \cos y$:

$$z = \sin(-\frac{\pi}{4}) \cos y = -\frac{\sqrt{2} \cos y}{2} \approx -0.7071 \cos y.$$

This describes a cosine graph in the plane $x = -\frac{\pi}{4}$. The other values of *z* appear in the following table.

Table 1.1.1: *Shows vertical traces parallel to the* xz*-plane for the function* $f(x, y) = \sin x \cos y$.

c	**Vertical Trace for** $x = c$
$-\frac{\pi}{4}$	$z = -\frac{\sqrt{2} \cos y}{2}$
0	$z = 0$
$\frac{\pi}{4}$	$z = \frac{\sqrt{2} \cos y}{2}$

In a similar fashion, we can substitute the $y - values$ in the equation $f(x, y)$ to obtain the traces in the $yz - plane$, as listed in the following table.

Vertical Traces Parallel to the $yz-Plane$ for the Function $f(x,y)=\sin x\cos y$

d	**Vertical Trace for $y=d$**
$\frac{\pi}{4}$	$z=\frac{\sqrt{2}\sin x}{2}$
0	$z=\sin x$
$-\frac{\pi}{4}$	$z=\frac{\sqrt{2}\sin x}{2}$

The three traces in the $xz-plane$ are cosine functions; the three traces in the $yz-plane$ are sine functions. These curves appear in the intersections of the surface with the planes $x=-\frac{\pi}{4}, x=0, x=\frac{\pi}{4}$ and $y=-\frac{\pi}{4}, y=0, y=\frac{\pi}{4}$ as shown in Figure 1.1.12

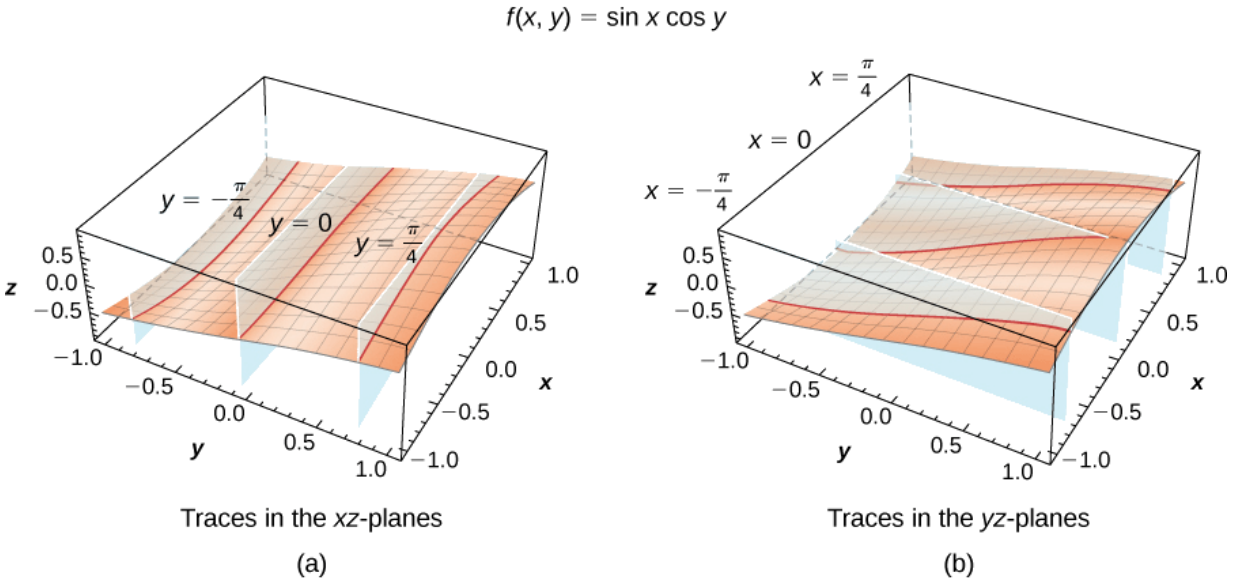


Figure 1.1.12: (a) Vertical traces of the function $f(x,y)$ are cosine curves in the $xz-planes$. (b) Vertical traces of the function $f(x,y)$ are sine curves in the $yz-planes$.

Exercise 1.1.5

Determine the equation of the vertical trace of the function $g(x,y)=-x^2-y^2+2x+4y-1$ corresponding to $y=3$, and describe its graph.

Hint

Set $y=3$ in the equation $z=-x^2-y^2+2x+4y-1$ and complete the square.

Solution

$z=3-(x-1)^2$. This function describes a parabola opening downward in the plane $y=3$.

Functions of two variables can produce some striking-looking surfaces. Figure 1.1.13 shows two examples.

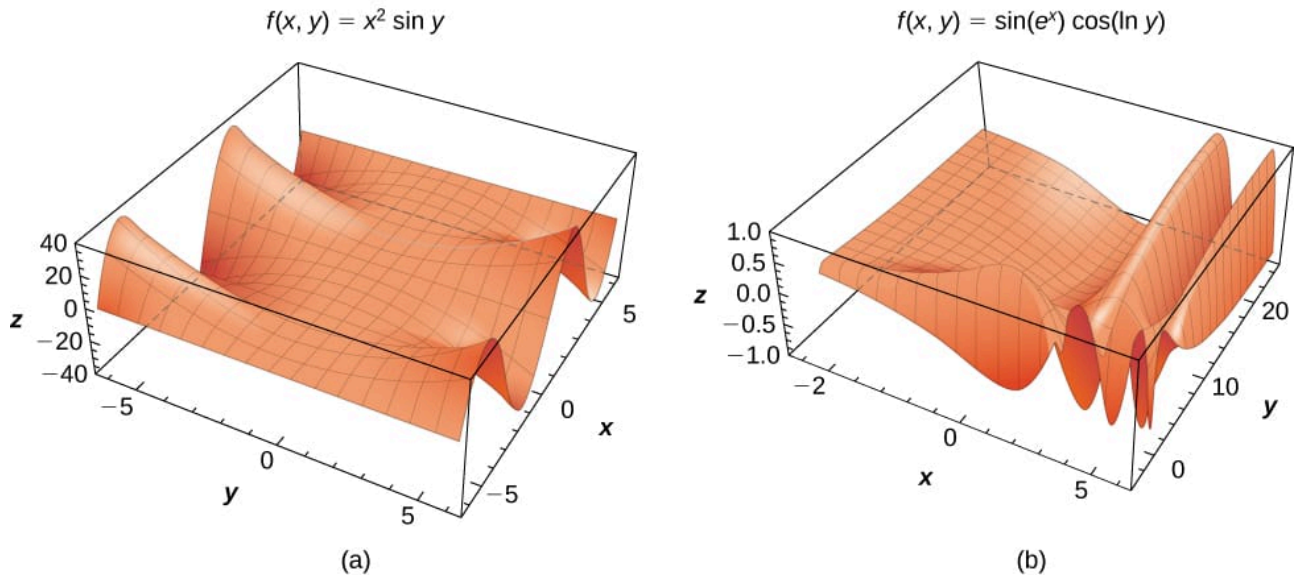


Figure 1.1.13: Examples of surfaces representing functions of two variables: (a) a combination of a power function and a sine function and (b) a combination of trigonometric, exponential, and logarithmic functions.

Functions of More Than Two Variables

So far, we have examined only functions of two variables. However, it is useful to take a brief look at functions of more than two variables. Two such examples are

$$\underbrace{f(x,y,z) = x^2 - 2xy + y^2 + 3yz - z^2 + 4x - 2y + 3x - 6}_{\text{a polynomial in three variables}} \tag{1.1.4}$$

and

$$g(x,y,t) = (x^2 - 4xy + y^2)\sin t - (3x + 5y)\cos t. \tag{1.1.5}$$

In the first function, (x, y, z) represents a point in space, and the function f maps each point in space to a fourth quantity, such as temperature or wind speed. In the second function, (x, y) can represent a point in the plane, and t can represent time. The function might map a point in the plane to a third quantity (for example, pressure) at a given time t. The method for finding the domain of a function of more than two variables is analogous to the method for functions of one or two variables.

Example 1.1.7: Domains for Functions of Three Variables

Find the domain of each of the following functions:

a. $f(x,y,z) = \dfrac{3x - 4y + 2z}{\sqrt{9 - x^2 - y^2 - z^2}}$

b. $g(x,y,t) = \dfrac{\sqrt{2t - 4}}{x^2 - y^2}$

Solution:

a. For the function $f(x,y,z) = \dfrac{3x - 4y + 2z}{\sqrt{9 - x^2 - y^2 - z^2}}$ to be defined (and be a real value), two conditions must hold:

1. The denominator **cannot** be zero.
2. The radicand **cannot** be negative.

Combining these conditions leads to the inequality

$$9 - x^2 - y^2 - z^2 > 0.$$

Moving the variables to the other side and reversing the inequality gives the domain as

$$domain(f) = \{(x, y, z) \in R^3 \mid x^2 + y^2 + z^2 < 9\},$$

which describes a ball of radius 3 centered at the origin. (Note: The surface of the ball is not included in this domain.)

b. For the function $g(x, y, t) = \dfrac{\sqrt{2t - 4}}{x^2 - y^2}$ to be defined (and be a real value), two conditions must hold:

1. The radicand cannot be negative.
2. The denominator cannot be zero.

Since the radicand cannot be negative, this implies $2t - 4 \geq 0$, and therefore that $t \geq 2$. Since the denominator cannot be zero, $x^2 - y^2 \neq 0$, or $x^2 \neq y^2$, Which can be rewritten as $y = \pm x$, which are the equations of two lines passing through the origin. Therefore, the domain of g is

$$domain(g) = \{(x, y, t) | y \neq \pm x, t \geq 2\}.$$

Exercise 1.1.6

Find the domain of the function $h(x, y, t) = (3t - 6)\sqrt{y - 4x^2 + 4}$.

Hint

Check for values that make radicands negative or denominators equal to zero.

Solution

$$domain(h) = \{(x, y, t) \in \mathbb{R}^3 \mid y \geq 4x^2 - 4\}$$

Functions of two variables have level curves, which are shown as curves in the $xy - plane$. However, when the function has three variables, the curves become surfaces, so we can define level surfaces for functions of three variables.

Definition: Level Surface of a Function of Three Variables

Given a function $f(x, y, z)$ and a number c in the range of f, a *level surface of a function of three variables* is defined to be the set of points satisfying the equation $f(x, y, z) = c$.

Example 1.1.8: Finding a Level Surface

Find the level surface for the function $f(x, y, z) = 4x^2 + 9y^2 - z^2$ corresponding to $c = 1$.

Solution

The level surface is defined by the equation $4x^2 + 9y^2 - z^2 = 1$. This equation describes a hyperboloid of one sheet as shown in Figure 1.1.14.

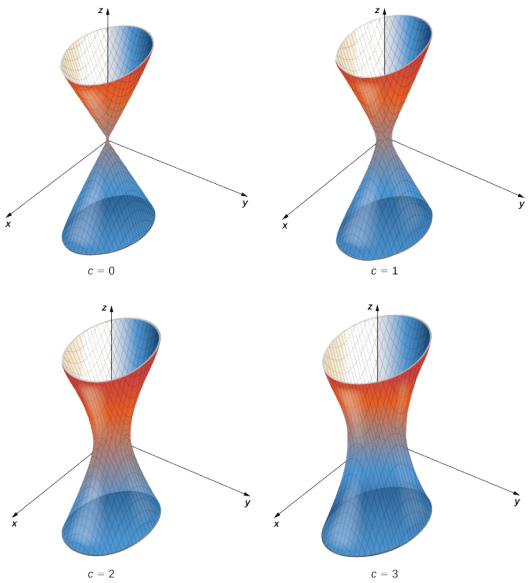


Figure 1.1.14: A hyperboloid of one sheet with four of its level surfaces for $c = 0, 1, 2, 3$. As c increases, the center of the hyperboloid at the origin gets wider.

Exercise 1.1.5

Find the equation of the level surface of the function

$$g(x, y, z) = x^2 + y^2 + z^2 - 2x + 4y - 6z$$

corresponding to $c = 2$, and describe the surface, if possible.

Hint

Set $g(x, y, z) = c$ and complete the square.

Solution

((x–1)^2+(y+2)^2+(z–3)^2=16\) describes a sphere of radius 4 centered at the point $(1, -2, 3)$.

Interactive Element

Using CalcPlot3D to Visualize a Level Surface

Often the level surfaces are too complex to graph enough of them, so using a computer program such as CalcPlot3D is the way to go. If you just set the function equal to a, a slider will automatically be created so you can just dynamically observe the level surfaces. Below is the dynamic computer plot of the level surface for $f(x, y, z) = \sin(3x) + y^2 - z^2$. You can also explore further by changing the equation. The interactive app is unfortunately not accessible.

Summary

- The graph of a function of two variables is a surface in $\mathbb{R}^3$ and can be studied using level curves and vertical traces.
- A set of level curves is called a contour map.

Key Equations

- **Vertical trace**

$f(a, y) = z$ for $x = a$ or $f(x, b) = z$ for $y = b$

- **Level surface of a function of three variables**

  $f(x, y, z) = c$

Glossary

Contour map
: a plot of the various level curves of a given function $f(x, y)$

Function of two variables
: a function $z = f(x, y)$ that maps each ordered pair (x, y) in a subset D of R^2 to a unique real number z

Graph of a function of two variables
: a set of ordered triples (x, y, z) that satisfies the equation $z = f(x, y)$ plotted in three-dimensional Cartesian space

Level curve of a function of two variables
: the set of points satisfying the equation $f(x, y) = c$ for some real number c in the range of f

Level surface of a function of three variables
: the set of points satisfying the equation $f(x, y, z) = c$ for some real number c in the range of f

Surface
: the graph of a function of two variables, $z = f(x, y)$

Vertical trace
: the set of ordered triples (c, y, z) that solves the equation $f(c, y) = z$ for a given constant $x = c$ or the set of ordered triples (x, d, z) that solves the equation $f(x, d) = z$ for a given constant $y = d$

Contributors and Attributions

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1.1E: Exercises for Section 1.1

Exercises 1 - 2

Evaluate each function at the indicated values.

1. $W(x,y) = 4x^2 + y^2$. Find $W(2,-1)$ and $W(-3,6)$.

   Answer

   $W(2,-1) = 17, \quad W(-3,6) = 72$

2. $W(x,y) = 4x^2 + y^2$. Find $W(2+h, 3+h)$.
3. The volume of a right circular cylinder is calculated by a function of two variables, $V(x,y) = \pi x^2 y$, where x is the radius of the right circular cylinder and y represents the height of the cylinder. Evaluate $V(2,5)$ and explain what this means.

   Answer

   $V(2,5) = 20\pi \text{ units}^3$ This is the volume when the radius is 2 and the height is 5.

4. An oxygen tank is constructed of a right cylinder of height y and radius x with two hemispheres of radius x mounted on the top and bottom of the cylinder. Express the volume of the cylinder as a function of two variables, x and y, find $V(10,2)$, and explain what this means.

Exercises 5 - 10

Find the domain and range of the given function. State the domain in set-builder notation and the range in interval notation.

5. $V(x,y) = 4x^2 + y^2$

   Answer

   Domain: $\{(x,y) \mid x \in \mathbb{R}, \mathrm{y} \in \mathbb{R}\}$ That is, all points in the xy-plane
   Range: $[0,\infty)$

6. $f(x,y) = \sqrt{x^2 + y^2 - 4}$

   Answer

   Domain: $\{(x,y) \mid x^2 + y^2 \geq 4\}$
   Range: $[0,\infty)$

7. $f(x,y) = 4\ln(y^2 - x)$

   Answer

   Domain: \(\big\{(x, y) \, | \, x Range: $(-\infty,\infty)$

8. $g(x,y) = \sqrt{16 - 4x^2 - y^2}$

   Answer

   Domain: $\{(x,y) \mid \frac{x^2}{4} + \frac{y^2}{16} \leq 1\}$
   Range: $[0,4]$

9. $z = \arccos(y - x)$

   Answer

   Domain: $\{(x,y) \mid x - 1 \leq y \leq x + 1\}$ That is, all points between the graphs of $y = x - 1$ and $y = x + 1$.

Range: $[0, \pi]$

10. $f(x, y) = \dfrac{y+2}{x^2}$

Answer

Domain: $\{(x, y) \mid x \neq 0\}$
Range: $(-\infty, \infty)$

Exercises 11 - 13

Find the range of the functions.

11. $g(x, y) = \sqrt{16 - 4x^2 - y^2}$

Answer

$\{z \mid 0 \leq z \leq 4\}$ or in interval notation: $[0, 4]$

12. $V(x, y) = 4x^2 + y^2$
13. $z = y^2 - x^2$

Answer

The set $\mathbb{R}$

Exercises 14 - 29

Find the level curves of each function at the indicated values of c to visualize the given function. Sketch a contour plot for those exercises where you are asked for more than 3 values of c.

14. $z(x, y) = y^2 - x^2, \quad c = 1$
15. $z(x, y) = y^2 - x^2, \quad c = 4$

Answer

$y^2 - x^2 = 4$, a hyperbola

16. $g(x, y) = x^2 + y^2; \quad c = 0, 1, 2, 3, 4, 9$
17. $g(x, y) = 4 - x - y; \quad c = 0, 1, 2, 3, 4$

Answer

Level curves are lines with $y = -x + (4 - c)$.
For each value of c these are:
$c = 0: \ y = -x + 4,$
$c = 1: \ y = -x + 3,$
$c = 2: \ y = -x + 2,$
$c = 3: \ y = -x + 1,$
$c = 4: \ y = -x.$
The contour plot consists of a series of parallel lines.

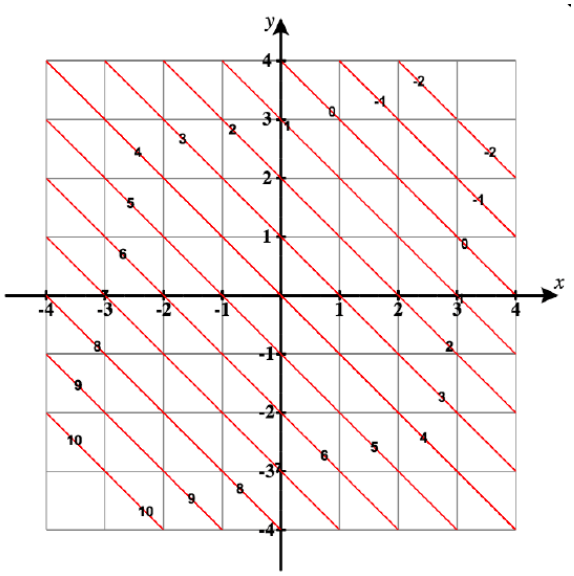


Figure $1.1E.1$: The image displays the contour map of $g(x,y) = 4 - x - y$. The map consists of a series of parallel red lines of slope -1 on the xy-plane.

18. $f(x,y) = xy; c = 1; \quad c = -1$
19. $h(x,y) = 2x - y; \quad c = -2, 0, 2$

Answer

$2x - y = 0, 2x - y = -2, 2x - y = 2$; three lines

20. $f(x,y) = x^2 - y; \quad c = 1, 2$
21. $g(x,y) = \dfrac{x}{x+y}; c = -1, 0, 1, 2$

Answer

Level curves are lines with the form $y = x\left(\dfrac{1-c}{c}\right)$. At $c = 0$, we solve it directly from the equation $\dfrac{x}{x+y} = 0$ to get $x = 0$.
For each value of c these are:
$c = -1: \ y = -2x,$
$c = 0: \ x = 0, \text{ with } y \neq 0,$
$c = 1: \ y = 0, \text{ with } x \neq 0,$
$c = 2: \ y = -\frac{1}{2}x.$

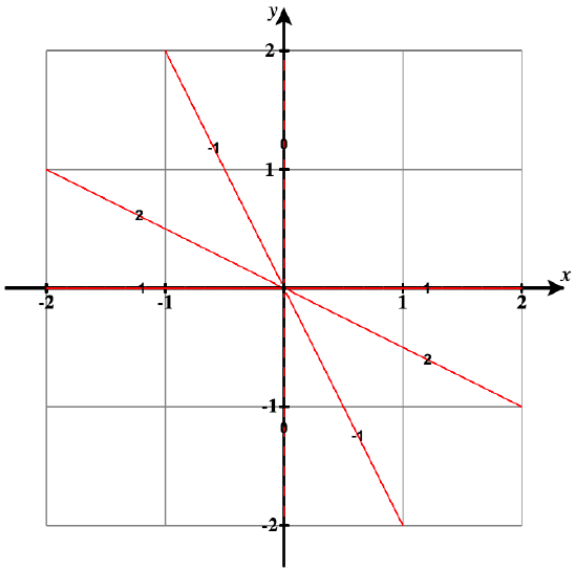


Figure $1.1E.2$: The image displays the contour map of $g(x,y)=\dfrac{x}{x+y}$. It consists of three red lines, one on the x-axis, one with slope of -2 and y-intercept at origin, one with slope of $\dfrac{-1}{2}$ and y-intercept at origin.

22. $g(x,y)=x^3-y; \quad c=-1,0,2$
23. $g(x,y)=e^{xy}; \quad c=\frac{1}{2},3$

Answer

The level curves have the form, $y=\dfrac{\ln c}{x}$.
For each value of c these are:
$c=\frac{1}{2}: \ y=\dfrac{\ln\frac{1}{2}}{x}$ that can be rewritten as, $y=-\dfrac{\ln 2}{x}$

$c=3: \ y=\dfrac{\ln 3}{x}$.

24. $f(x,y)=x^2; \quad c=4,9$
25. $f(x,y)=xy-x; \quad c=-2,0,2$

Answer

Level curves have the form: $y=\dfrac{c}{x}+1$.
Here $y=\dfrac{-2}{x}+1, \quad y=1, \quad y=\dfrac{2}{x}+1$ or $xy-x=-2, \ xy-x=0, \ xy-x=2$

26. $h(x,y)=\ln(x^2+y^2); \quad c=-1,0,1$
27. $g(x,y)=\ln\left(\dfrac{y}{x^2}\right); \quad c=-2,0,2$

Answer

The level curves have the form, $y=e^c x^2$.
For each value of c these are:
$c=-2: \ y=e^{-2}x^2$,
$c=0: \ y=x^2$,
$c=2: \ y=e^2x^2$.

28. $z=f(x,y)=\sqrt{x^2+y^2}, \quad c=3$
29. $f(x,y)=\dfrac{y+2}{x^2}, \quad c=$ any constant

Answer

The level curves are parabolas of the form $y = cx^2 - 2, \text{ with } x \neq 0.$

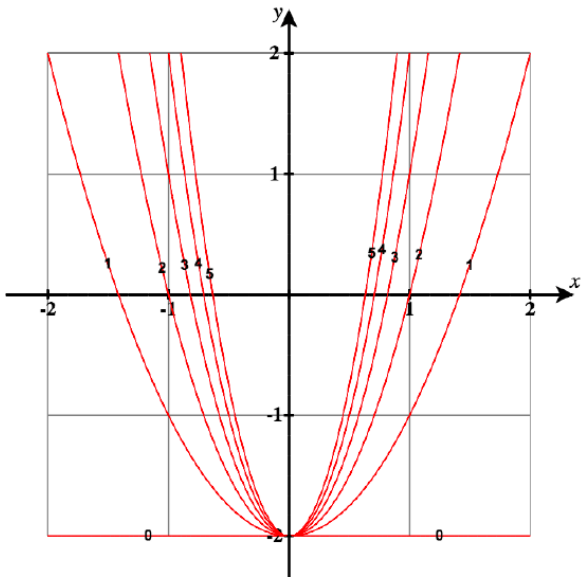


Figure $1.1E.3$: The image displays a contour plot for the function $f(x,y) = \dfrac{y+2}{x^2}$. It consists of multiple parabolas open upward with vertices at the point (0,-2).

Exercises 30 - 32

Find the vertical traces of the functions at the indicated values of x and y, and plot the traces.

30. $z = 4 - x - y, \quad x = 2$
31. $f(x,y) = 3x + y^3, \quad x = 1$

Answer

$z = 3 + y^3$, a curve in the zy-**plane** with rulings parallel to the x-axis

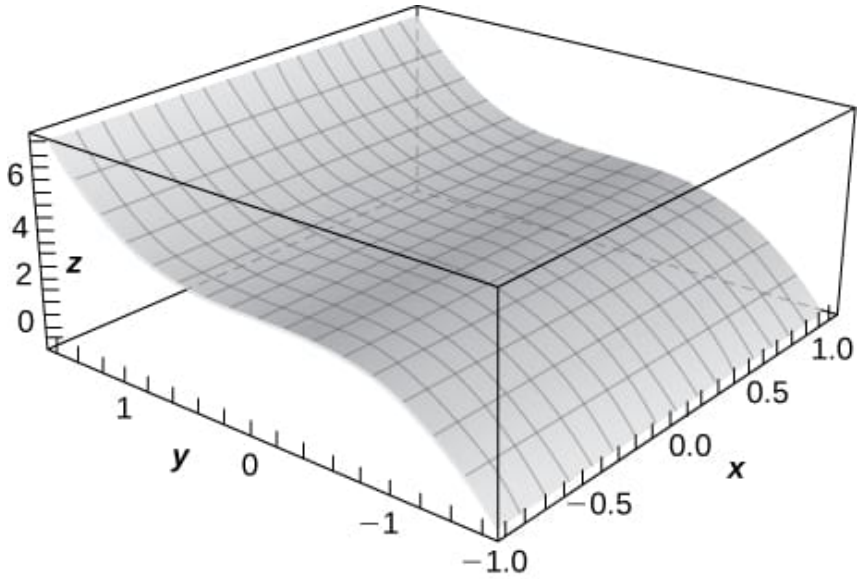


Figure $1.1E.4$: The image displays the plot of the surface $f(x,y) = 3x + y^3$. It is a waving sheet and has rulings parallel to the x-axis.

32. $z = \cos\sqrt{x^2 + y^2}, \quad x = 1$

Exercises 33 - 38

Find the domain and range of each function.

33. $z = \sqrt{100 - 4x^2 - 25y^2}$

Answer

Domain: $\left\{(x, y) \mid \frac{x^2}{25} + \frac{y^2}{4} \leq 1\right\}$
Range: $[0, 10]$

34. $z = \ln(x - y^2)$
35. $f(x, y, z) = \dfrac{1}{\sqrt{36 - 4x^2 - 9y^2 - z^2}}$

Answer

Domain: $\left\{(x, y, z) \mid \frac{x^2}{9} + \frac{y^2}{4} + \frac{z^2}{36} < 1\right\}$
Range: $\left[\frac{1}{6}, \infty\right)$

36. $f(x, y, z) = \sqrt{49 - x^2 - y^2 - z^2}$
37. $f(x, y, z) = \sqrt[3]{16 - x^2 - y^2 - z^2}$

Answer

Domain: All points in xyz-space
Range: $\left(-\infty, \sqrt[3]{16}\ \right]$

38. $f(x, y) = \cos\sqrt{x^2 + y^2}$

Exercises 39 - 40

Plot a graph of the function.

39. $z = f(x, y) = \sqrt{x^2 + y^2}$

Answer

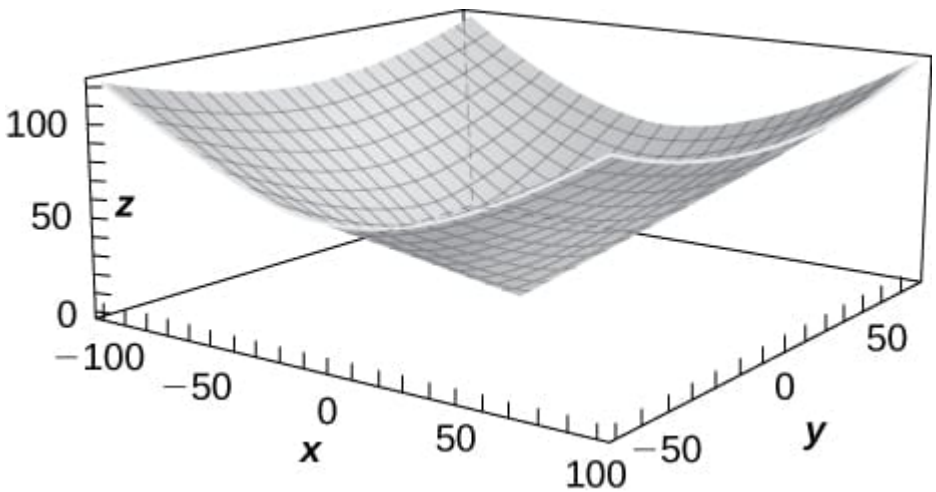


Figure $1.1E.5$: The image displays the plot of the surface $z = f(x, y) = \sqrt{x^2 + y^2}$. It is a cone symmetric to the z-axis.

40. $z = x^2 + y^2$
41. Use technology to graph $z = x^2 y$.

Answer

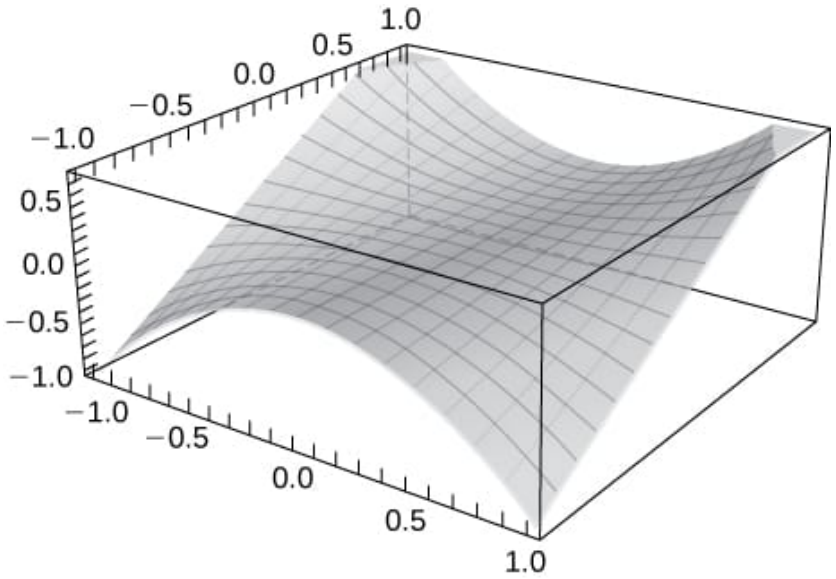


Figure $1.1E.6$: The image displays the plot of the surface $z = x^2y$. It is curving downward at one end and upward at the other end.

Exercises 42 - 46

Sketch the function by finding its level curves. Verify the graph using technology, such as CalcPlot3D.

42. $f(x, y) = \sqrt{4 - x^2 - y^2}$
43. $f(x, y) = 2 - \sqrt{x^2 + y^2}$

Answer

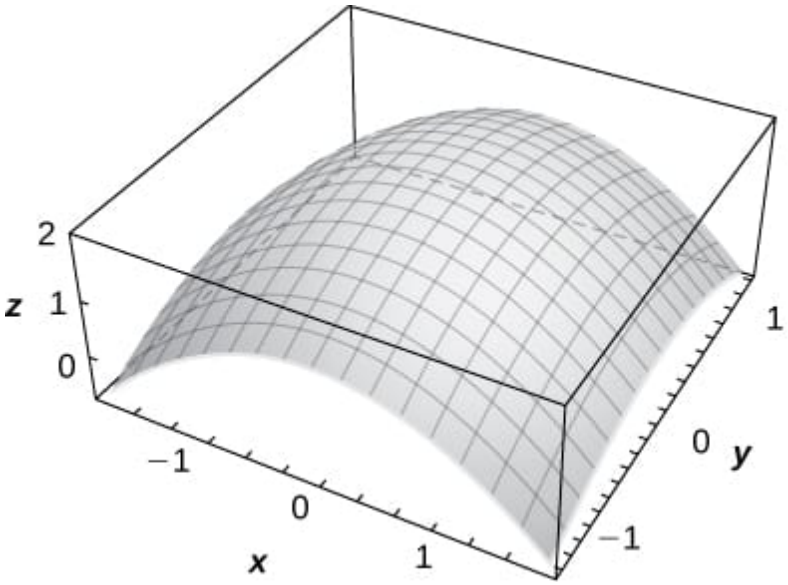


Figure $1.1E.7$: The image displays the plot of the surface $f(x, y) = 2 - \sqrt{x^2 + y^2}$. It looks like a downward cone with vertex at (0,0,2).

44. $z = 1 + e^{-x^2-y^2}$
45. $z = \cos\sqrt{x^2 + y^2}$

Answer

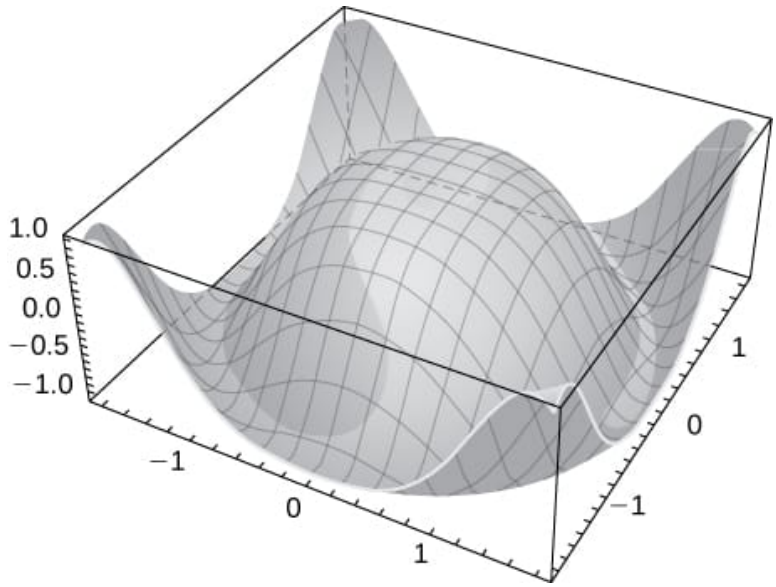


Figure $1.1E.8$: The image displays the plot of the surface $z = \cos\sqrt{x^2 + y^2}$. The surface exhibits wave-like oscillations with peaks and valleys, indicating a periodic behavior.

46. $z = y^2 - x^2$
47. Describe the contour lines for several values of c for $z = x^2 + y^2 - 2x - 2y$.

Answer

The contour lines are concentric circles centered at the point, $(1, 1)$.
You can see this by completing the square after setting this function equal to c.
That is, we write $x^2 - 2x + 1 + y^2 - 2y + 1 = c + 2$ which can be rewritten as, $(x-1)^2 + (y-1)^2 = c + 2$.
This gives us circles centered at the point, $(1, 1)$, each with a radius of $\sqrt{c+2}$.

Exercises 48 - 52

Find the level surface for the given value of c for each function of three variables and describe it.

48. $w(x, y, z) = x - 2y + z, \quad c = 4$
49. $w(x, y, z) = x^2 + y^2 + z^2, \quad c = 9$

Answer

$x^2 + y^2 + z^2 = 9$, a sphere of radius 3

50. $w(x, y, z) = x^2 + y^2 - z^2, \quad c = -4$
51. $w(x, y, z) = x^2 + y^2 - z^2, \quad c = 4$

Answer

$x^2 + y^2 - z^2 = 4$, a hyperboloid of one sheet

52. $w(x, y, z) = 9x^2 - 4y^2 + 36z^2, \quad c = 0$

Exercises 53 - 55

Find an equation of the level curve of f that contains the point P.

53. $f(x, y) = 1 - 4x^2 - y^2, \quad P(0, 1)$

Answer

$$4x^2 + y^2 = 1,$$

54. $g(x, y) = y^2 \arctan x, \quad P(1, 2)$
55. $g(x, y) = e^{xy}(x^2 + y^2), \quad P(1, 0)$

Answer

$$1 = e^{xy}(x^2 + y^2)$$

56. The strength E of an electric field at point (x, y, z) resulting from an infinitely long charged wire lying along the y-axis is given by $E(x, y, z) = k/\sqrt{x^2 + y^2}$, where k is a positive constant. For simplicity, let $k = 1$ and find the equations of the level surfaces for $E = 10$ and $E = 100$.
57. A thin plate made of iron is located in the xy-plane. The temperature T in degrees Celsius at a point $P(x, y)$ is inversely proportional to the square of its distance from the origin. Express T as a function of x and y.

Answer

$$T(x, y) = \frac{k}{x^2 + y^2}$$

58. Refer to the preceding problem. Using the temperature function found there, determine the proportionality constant if the temperature at point $P(1, 2)$ is $50\ ^\circ C$. Use this constant to determine the temperature at point $Q(3, 4)$.
59. Refer to the preceding problem. Find the level curves for $T = 40\ ^\circ C$ and $T = 100\ ^\circ C$, and describe what the level curves represent.

Answer

$x^2 + y^2 = \frac{k}{40}, \quad x^2 + y^2 = \frac{k}{100}$. The level curves represent circles of radii $\sqrt{10k}/20$ and $\sqrt{k}/10$

Contributors

- Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.
- Paul Seeburger (Monroe Community College) added the contour plots to answers for problems 17, 21 and 29.

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1.2: Limits and Continuity

Learning Objectives

- Calculate the limit of a function of two variables.
- Learn how a function of two variables can approach different values at a boundary point, depending on the path of approach.
- State the conditions for continuity of a function of two variables.
- Verify the continuity of a function of two variables at a point.
- Calculate the limit of a function of three or more variables and verify the continuity of the function at a point.

We have now studied functions of more than one variable and seen how to graph them. In this section, we see how to take the limit of a function of more than one variable, and what it means for a function of more than one variable to be continuous at a point in its domain. It turns out these concepts have aspects that just don't occur with functions of one variable.

Limit of a Function of Two Variables

Recall the definition of a limit of a function of one variable:

Let $f(x)$ be defined for all $x \neq a$ in an open interval containing a. Let L be a real number. Then

$$\lim_{x \to a} f(x) = L \tag{1.2.1}$$

if for every $\varepsilon > 0$, there exists a $\delta > 0$, such that if $0 < |x - a| < \varepsilon$

$$|f(x) - L| < \varepsilon. \tag{1.2.2}$$

Before we can adapt this definition to define a limit of a function of two variables, we first need to see how to extend the idea of an open interval in one variable to an open interval in two variables.

Definition: δ Disks

Consider a point $(a, b) \in \mathbb{R}^2$. A δ-**disk** centered at point (a, b) is defined to be an open disk of radius δ centered at point (a, b) —that is,

$$\{(x, y) \in \mathbb{R}^2 \mid (x - a)^2 + (y - b)^2 < \delta^2\} \tag{1.2.3}$$

as shown in Figure 1.2.1.

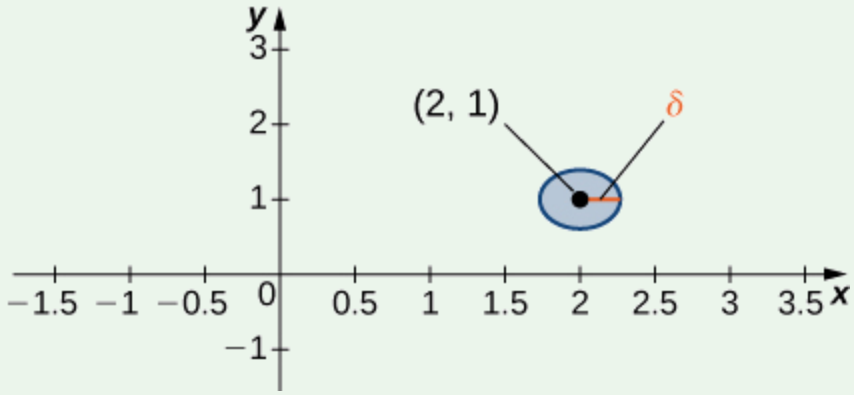


Figure 1.2.1: Image illustrates a δ-disk centered at the point $(2, 1)$ in the first quadrant of the xy-plane. The disk is a small circular region surrounding this point, representing all points within a distance of δ from $(2, 1)$. The xy-plane is displayed with labeled axes, where the x-axis ranges from approximately -1.5 to 3.5 and the y-axis extends from -1 to 3.

The concept of a δ-disk is central to defining the limit of a function of two variables. If δ is small, then all points (x, y) inside the δ -disk are close to (a, b). This is analogous to how, in one dimension, we define a limit by requiring that x be close to a.

In one-dimensional analysis, this closeness is expressed by the inequality:

$$a - \delta < x < a + \delta. \tag{1.2.4}$$

However, in more than one dimension, we generalize this concept using a δ-disk, which consists of all points (x, y) satisfying:

$$\sqrt{(x-a)^2 + (y-b)^2} < \delta. \tag{1.2.5}$$

This defines a circular region centered at (a, b) with radius δ, ensuring that all points inside the disk are within a distance δ from (a, b).

Definition: Limit of a Function of Two Variables

Let f be a function of two variables, x and y. The limit of $f(x, y)$ as (x, y) approaches (a, b) is L, written

$$\lim_{(x,y)\to(a,b)} f(x, y) = L \tag{1.2.6}$$

if for each $\varepsilon > 0$ there exists a small enough $\delta > 0$ such that for all points (x, y) in a δ disk around (a, b), except possibly for (a, b) itself, the value of $f(x, y)$ is no more than ε away from L (Figure 1.2.2).

Using symbols, for any $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$|f(x, y) - L| < \varepsilon \tag{1.2.7}$$

whenever

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta. \tag{1.2.8}$$

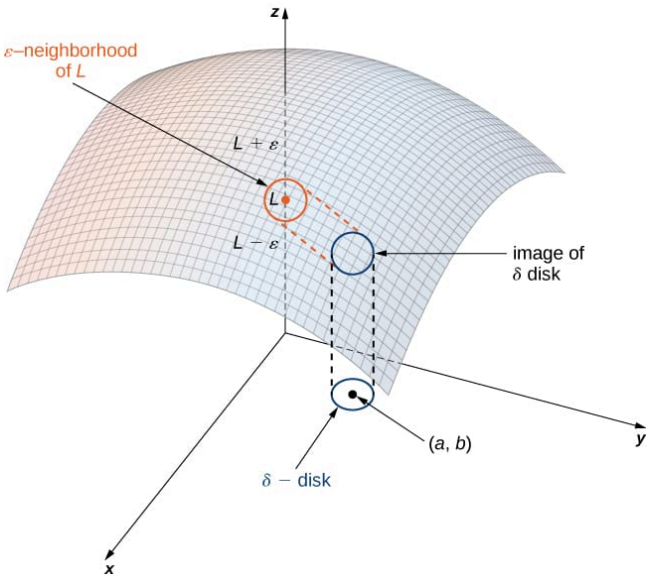


Figure 1.2.2: Image illustrates the ε-δ definition of the limit for a function of two variables. The image shows a curved surface representing the function $f(x, y)$ in three-dimensional space. The xy-plane is depicted below, with a point (a, b) marked as a black dot. A small circular region around (a, b) is labeled as the δ-disk, representing all points within a distance δ from (a, b). The function's value at these points lies within a vertical range around L, the limit value. This range is indicated by an ε-neighborhood, which forms a horizontal band around L with boundaries at $L + \varepsilon$ and $L - \varepsilon$.

Proving that a limit exists using the definition of a **limit of a function of two variables** can be challenging. Instead, we use the following theorem, which gives us shortcuts to finding limits. The formulas in this theorem are an extension of the formulas in the limit laws theorem in The Limit Laws.

Limit Laws for Functions of Two Variables

Let $f(x, y)$ and $g(x, y)$ be defined for all $(x, y) \neq (a, b)$ in a neighborhood around (a, b), and assume the neighborhood is contained completely inside the domain of f. Assume that L and M are real numbers such that

$$\lim_{(x,y)\to(a,b)} f(x, y) = L \tag{1.2.9}$$

and

$$\lim_{(x,y)\to(a,b)} g(x, y) = M, \tag{1.2.10}$$

and let c be a constant. Then each of the following statements holds:

Constant Law:

$$\lim_{(x,y)\to(a,b)} c = c \tag{1.2.11}$$

Identity Laws:

$$\lim_{(x,y)\to(a,b)} x = a \tag{1.2.12}$$

$$\lim_{(x,y)\to(a,b)} y = b \tag{1.2.13}$$

Sum Law:

$$\lim_{(x,y)\to(a,b)} (f(x, y) + g(x, y)) = L + M \tag{1.2.14}$$

Difference Law:

$$\lim_{(x,y)\to(a,b)}(f(x,y)-g(x,y))=L-M \tag{1.2.15}$$

Constant Multiple Law:

$$\lim_{(x,y)\to(a,b)}(cf(x,y))=cL \tag{1.2.16}$$

Product Law:

$$\lim_{(x,y)\to(a,b)}(f(x,y)g(x,y))=LM \tag{1.2.17}$$

Quotient Law:

$$\lim_{(x,y)\to(a,b)}\frac{f(x,y)}{g(x,y)}=\frac{L}{M}\text{ for }M\neq 0 \tag{1.2.18}$$

Power Law:

$$\lim_{(x,y)\to(a,b)}(f(x,y))^n=L^n \tag{1.2.19}$$

for any positive integer n.

Root Law:

$$\lim_{(x,y)\to(a,b)}\sqrt[n]{f(x,y)}=\sqrt[n]{L} \tag{1.2.20}$$

for all L if n is odd and positive, and for $L\geq 0$ if n is even and positive.

The proofs of these properties are similar to those for the limits of functions of one variable. We can apply these laws to finding limits of various functions.

Example 1.2.1: Finding the Limit of a Function of Two Variables

Find each of the following limits:

a. $\lim_{(x,y)\to(2,-1)}(x^2-2xy+3y^2-4x+3y-6)$

b. $\lim_{(x,y)\to(2,-1)}\frac{2x+3y}{4x-3y}$

Solution

a. First use the sum and difference laws to separate the terms:

$$\begin{aligned}\lim_{(x,y)\to(2,-1)}(x^2-2xy+3y^2-4x+3y-6)&=\left(\lim_{(x,y)\to(2,-1)}x^2\right)-\left(\lim_{(x,y)\to(2,-1)}2xy\right)+\left(\lim_{(x,y)\to(2,-1)}3y^2\right)-\left(\lim_{(x,y)\to(2,-1)}4x\right)\\&\quad+\left(\lim_{(x,y)\to(2,-1)}3y\right)-\left(\lim_{(x,y)\to(2,-1)}6\right).\end{aligned}$$

Next, use the constant multiple law on the second, third, fourth, and fifth limits:

$$\begin{aligned}&=\left(\lim_{(x,y)\to(2,-1)}x^2\right)-2\left(\lim_{(x,y)\to(2,-1)}xy\right)+3\left(\lim_{(x,y)\to(2,-1)}y^2\right)-4\left(\lim_{(x,y)\to(2,-1)}x\right)\\&\quad+3\left(\lim_{(x,y)\to(2,-1)}y\right)-\lim_{(x,y)\to(2,-1)}6.\end{aligned}$$

Now, use the power law on the first and third limits, and the product law on the second limit:

$$\left(\lim_{(x,y)\to(2,-1)} x\right)^2 - 2\left(\lim_{(x,y)\to(2,-1)} x\right)\left(\lim_{(x,y)\to(2,-1)} y\right) + 3\left(\lim_{(x,y)\to(2,-1)} y\right)^2 - 4\left(\lim_{(x,y)\to(2,-1)} x\right) + 3\left(\lim_{(x,y)\to(2,-1)} y\right) - \lim_{(x,y)\to(2,-1)} 6.$$

Last, use the identity laws on the first six limits and the constant law on the last limit:

$$\lim_{(x,y)\to(2,-1)} (x^2 - 2xy + 3y^2 - 4x + 3y - 6) = (2)^2 - 2(2)(-1) + 3(-1)^2 - 4(2) + 3(-1) - 6 = -6.$$

b. Before applying the quotient law, we need to verify that the limit of the denominator is nonzero. Using the difference law, constant multiple law, and identity law,

$$\begin{aligned}\lim_{(x,y)\to(2,-1)} (4x - 3y) &= \lim_{(x,y)\to(2,-1)} 4x - \lim_{(x,y)\to(2,-1)} 3y \\ &= 4(\lim_{(x,y)\to(2,-1)} x) - 3(\lim_{(x,y)\to(2,-1)} y) \\ &= 4(2) - 3(-1) = 11.\end{aligned}$$

Since the limit of the denominator is nonzero, the quotient law applies. We now calculate the limit of the numerator using the difference law, constant multiple law, and identity law:

$$\begin{aligned}\lim_{(x,y)\to(2,-1)} (2x + 3y) &= \lim_{(x,y)\to(2,-1)} 2x + \lim_{(x,y)\to(2,-1)} 3y \\ &= 2(\lim_{(x,y)\to(2,-1)} x) + 3(\lim_{(x,y)\to(2,-1)} y) \\ &= 2(2) + 3(-1) = 1.\end{aligned}$$

Therefore, according to the quotient law we have

$$\lim_{(x,y)\to(2,-1)} \frac{2x + 3y}{4x - 3y} = \frac{\lim_{(x,y)\to(2,-1)} (2x + 3y)}{\lim_{(x,y)\to(2,-1)} (4x - 3y)} = \frac{1}{11}.$$

Watch the accompanying video for a visual demonstration of the concept.

- **Video length**: 2 minutes 37 seconds.
- **Context**: This video demonstrates how to find a limit of a function of two variables using polar coordinates.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Exercise 1.2.1:

Evaluate the following limit:

$$\lim_{(x,y)\to(5,-2)} \sqrt[3]{\frac{x^2 - y}{y^2 + x - 1}}.$$

Hint available at the end of this section.

Since we are taking the limit of a function of two variables, the point (a, b) is in $\mathbb{R}^2$, and it is possible to approach this point from an infinite number of directions. Sometimes when calculating a limit, the answer varies depending on the path taken toward (a, b). If this is the case, then the limit fails to exist. In other words, the limit must be unique, regardless of path taken.

Example 1.2.2: Limits That Fail to Exist

Show that neither of the following limits exist:

a. $\lim\limits_{(x,y)\to(0,0)} \dfrac{2xy}{3x^2+y^2}$

b. $\lim\limits_{(x,y)\to(0,0)} \dfrac{4xy^2}{x^2+3y^4}$

Solution

a. The domain of the function $f(x, y) = \dfrac{2xy}{3x^2+y^2}$ consists of all points in the xy-plane except for the point $(0, 0)$ (Figure 1.2.3). To show that the limit does not exist as (x, y) approaches $(0, 0)$, we note that it is impossible to satisfy the definition of a limit of a function of two variables because of the fact that the function takes different values along different lines passing through point $(0, 0)$. First, consider the line $y = 0$ in the xy-plane. Substituting $y = 0$ into $f(x, y)$ gives

$$f(x, 0) = \frac{2x(0)}{3x^2+0^2} = 0$$

for any value of x. Therefore the value of f remains constant for any point on the x-axis, and as y approaches zero, the function remains fixed at zero.

Next, consider the line $y = x$. Substituting $y = x$ into $f(x, y)$ gives

$$f(x, x) = \frac{2x(x)}{3x^2+x^2} = \frac{2x^2}{4x^2} = \frac{1}{2}.$$

This is true for any point on the line $y = x$. If we let x approach zero while staying on this line, the value of the function remains fixed at $\dfrac{1}{2}$, regardless of how small x is.

Choose a value for ε that is less than $\dfrac{1}{2}$ such as $\dfrac{1}{4}$. Then, no matter how small a δ disk we draw around $(0, 0)$, the values of $f(x, y)$ for points inside that δ disk will include both 0 and $\dfrac{1}{2}$. Therefore, the definition of limit at a point is never satisfied and the limit fails to exist.

Interactive Element

Interactive Surface

Below is an interactive surface viewer. You can move the box around to view the surface at different angles. You can also change x-value and the point size. The interactive app is unfortunately not accessible.

Figure 1.2.3: Graph of the function $f(x, y) = \dfrac{2xy}{3x^2+y^2}$. Along the line $y = 0$, the function is equal to zero; along the line $y = x$, the function is equal to $\frac{1}{2}$.

b. In a similar fashion to a., we can approach the origin along any straight line passing through the origin. If we try the x-axis (i.e. $y = 0$), then the function remains fixed at zero. The same is true for the y-axis. Suppose we approach the origin along a straight line of slope k. The equation of this line is $y = kx$. Then the limit becomes

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+3y^4} &= \lim_{(x,y)\to(0,0)} \frac{4x(kx)^2}{x^2+3(kx)^4} \\ &= \lim_{(x,y)\to(0,0)} \frac{4k^2x^3}{x^2+3k^4x^4} \\ &= \lim_{(x,y)\to(0,0)} \frac{4k^2x}{1+3k^4x^2} \\ &= \frac{\lim\limits_{(x,y)\to(0,0)} (4k^2x)}{\lim\limits_{(x,y)\to(0,0)} (1+3k^4x^2)} \\ &= 0.\end{aligned}$$

regardless of the value of k. It would seem that the limit is equal to zero. What if we chose a curve passing through the origin instead? For example, we can consider the parabola given by the equation $x = y^2$. Substituting y^2 in place of x in $f(x, y)$ gives

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+3y^4} &= \lim_{(x,y)\to(0,0)} \frac{4(y^2)y^2}{(y^2)^2+3y^4} \\ &= \lim_{(x,y)\to(0,0)} \frac{4y^4}{y^4+3y^4} \\ &= \lim_{(x,y)\to(0,0)} 1 \\ &= 1.\end{aligned}$$

By the same logic in part a, it is impossible to find a δ disk around the origin that satisfies the definition of the limit for any value of $\varepsilon < 1$. Therefore,

$$\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+3y^4}$$

does not exist.

Watch the accompanying video for a visual demonstration of the concept.

- **Video length**: 3 minutes 16 seconds.
- **Context**: This video demonstrates how to show a limit for a function of two variables does not exist.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Exercise 1.2.2:

Show that

$$\lim_{(x,y)\to(2,1)} \frac{(x-2)(y-1)}{(x-2)^2+(y-1)^2}$$

does not exist.

Hint available at the end of this section.

Interior Points and Boundary Points

To study continuity and differentiability of a function of two or more variables, we first need to learn some new terminology.

Definition: Interior and Boundary Points

Let S be a subset of $\mathbb{R}^2$ (Figure $1.2.4$).

1. A point P_0 is called an *interior point* of S if there is a δ disk centered around P_0 contained completely in S.
2. A point P_0 is called a *boundary point* of S if every δ disk centered around P_0 contains points both inside and outside S.

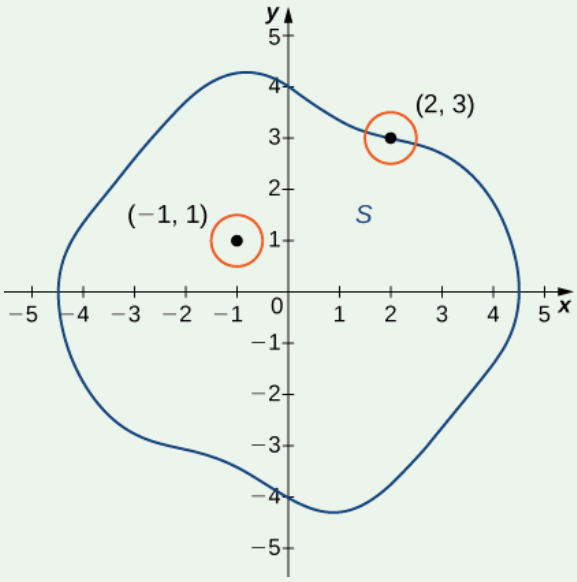


Figure $1.2.4$: Image illustrates the concept of interior and boundary points in a set S. The set S is enclosed by a closed, irregular curve on the xy-plane. In the set S inside the closed curve, $(-1, 1)$ is an interior point and $(2, 3)$ is a boundary point.

Definition: Open and Closed Sets

Let S be a subset of $\mathbb{R}^2$ (Figure $1.2.4$).

1. S is called an *open set* if every point of S is an interior point.
2. S is called a *closed set* if it contains all its boundary points.

An example of an open set is a δ disk. If we include the boundary of the disk, then it becomes a closed set. A set that contains some, but not all, of its boundary points is neither open nor closed. For example if we include half the boundary of a δ disk but not the other half, then the set is neither open nor closed.

Definition: Connected Sets and Regions

Let S be a subset of $\mathbb{R}^2$ (Figure $1.2.4$).

1. An open set S is a *connected set* if it cannot be represented as the union of two or more disjoint, nonempty open subsets.
2. A set S is a *region* if it is open, connected, and nonempty.

The definition of a limit of a function of two variables requires the δ disk to be contained inside the domain of the function. However, if we wish to find the limit of a function at a boundary point of the domain, the δ **disk** is not contained inside the domain. By definition, some of the points of the δ **disk** are inside the domain and some are outside. Therefore, we need only consider

points that are inside both the δ disk and the domain of the function. This leads to the definition of the limit of a function at a boundary point.

Definition: Limit of a Function at a Boundary Point

Let f be a function of two variables, x and y, and suppose (a,b) is on the boundary of the domain of f. Then, the limit of $f(x,y)$ as (x,y) approaches (a,b) is L, written

$$\lim_{(x,y)\to(a,b)} f(x,y) = L, \tag{1.2.21}$$

if for any $\varepsilon > 0$, there exists a number $\delta > 0$ such that for any point (x,y) inside the domain of f and within a suitably small distance positive δ of (a,b), the value of $f(x,y)$ is no more than ε away from L (Figure 1.2.2).

Using symbols, for any $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$|f(x,y) - L| < \varepsilon \tag{1.2.22}$$

whenever

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta. \tag{1.2.23}$$

Example 1.2.3: Limit of a Function at a Boundary Point

Prove

$$\lim_{(x,y)\to(4,3)} \sqrt{25 - x^2 - y^2} = 0.$$

Solution

The domain of the function $f(x,y) = \sqrt{25 - x^2 - y^2}$ is $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 25\}$, which is a circle of radius 5 centered at the origin, along with its interior as shown in Figure 1.2.5.

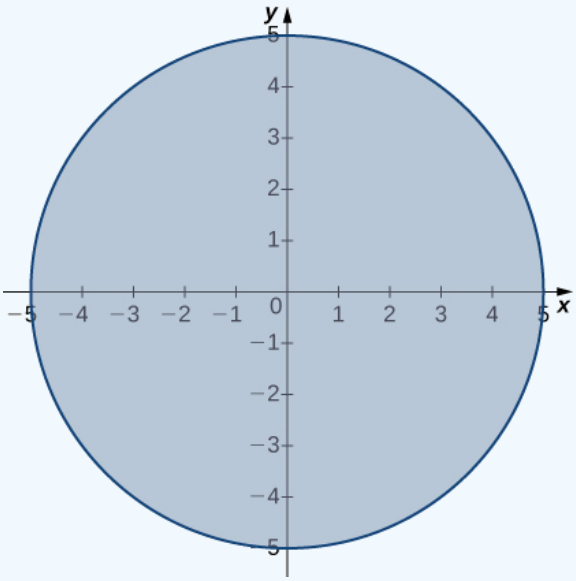


Figure 1.2.5: Domain of the function $f(x,y) = \sqrt{25 - x^2 - y^2}$ is a disk of radius 5 centered at origin.

We can use the limit laws, which apply to limits at the boundary of domains as well as interior points:

$$\lim_{(x,y)\to(4,3)} \sqrt{25-x^2-y^2} = \sqrt{\lim_{(x,y)\to(4,3)} (25-x^2-y^2)}$$
$$= \sqrt{\lim_{(x,y)\to(4,3)} 25 - \lim_{(x,y)\to(4,3)} x^2 - \lim_{(x,y)\to(4,3)} y^2}$$
$$= \sqrt{25-4^2-3^2}$$
$$= 0$$

See Figure 1.2.6.

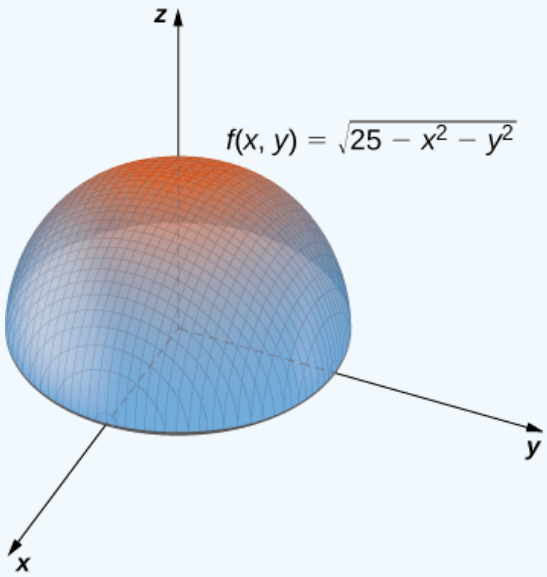


Figure 1.2.6: Graph of the function $f(x,y) = \sqrt{25-x^2-y^2}$ is the upper hemisphere in 3d-space with radius 5 and center at origin.

Exercise 1.2.3

Evaluate the following limit:

$$\lim_{(x,y)\to(5,-2)} \sqrt{29-x^2-y^2}.$$

Hint available at the end of this section.

Continuity of Functions of Two Variables

In Continuity, we defined the continuity of a function of one variable and saw how it relied on the limit of a function of one variable. In particular, three conditions are necessary for $f(x)$ to be continuous at point $x = a$

1. $f(a)$ exists.
2. $\lim_{x\to a} f(x)$ exists.
3. $\lim_{x\to a} f(x) = f(a)$.

These three conditions are necessary for continuity of a function of two variables as well.

Definition: Continuous Functions

A function $f(x,y)$ is continuous at a point (a,b) in its domain if the following conditions are satisfied:

1. $f(a,b)$ exists.
2. $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exists.
3. $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

Example 1.2.4: Demonstrating Continuity for a Function of Two Variables

Show that the function

$$f(x,y) = \frac{3x+2y}{x+y+1}$$

is continuous at point $(5,-3)$.

Solution

There are three conditions to be satisfied, per the definition of continuity. In this example, $a = 5$ and $b = -3$.

1. $f(a,b)$ exists. This is true because the domain of the function f consists of those ordered pairs for which the denominator is nonzero (i.e., $x+y+1 \neq 0$). Point $(5,-3)$ satisfies this condition. Furthermore,

$$f(a,b) = f(5,-3) = \frac{3(5)+2(-3)}{5+(-3)+1} = \frac{15-6}{2+1} = 3.$$

2. $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exists. This is also true:

$$\begin{aligned}\lim_{(x,y)\to(a,b)} f(x,y) &= \lim_{(x,y)\to(5,-3)} \frac{3x+2y}{x+y+1} \\ &= \frac{\lim\limits_{(x,y)\to(5,-3)} (3x+2y)}{\lim\limits_{(x,y)\to(5,-3)} (x+y+1)} \\ &= \frac{15-6}{5-3+1} \\ &= 3.\end{aligned}$$

3. $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$. This is true because we have just shown that both sides of this equation equal three.

Watch the accompanying video for a visual demonstration of the concept.

- **Video length**: 3 minutes 18 seconds.
- **Context**: This video demonstrates how to describe the continuity of a function of two variables.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Exercise 1.2.4

Show that the function

$$f(x,y) = \sqrt{26-2x^2-y^2}$$

is continuous at point $(2, -3)$.

Hint available at the end of this section.

Continuity of a function of any number of variables can also be defined in terms of delta and epsilon. A function of two variables is continuous at a point (x_0, y_0) in its domain if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that, whenever $\sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$ (epsilon–delta definition) of continuity of a function of one variable to prove the following theorems:

Theorem: The Sum of Continuous Functions Is Continuous

If $f(x, y)$ is continuous at (x_0, y_0), and $g(x, y)$ is continuous at (x_0, y_0), then $f(x, y) + g(x, y)$ is continuous at (x_0, y_0).

Theorem: The Product of Continuous Functions Is Continuous

If $g(x)$ is continuous at x_0 and $h(y)$ is continuous at y_0, then $f(x, y) = g(x)h(y)$ is continuous at (x_0, y_0).

Theorem: The Composition of Continuous Functions Is Continuous

Let g be a function of two variables from a domain $D \subseteq \mathbb{R}^2$ to a range $R \subseteq R$. Suppose g is continuous at some point (x_0, y_0) $\in D$ and define $z_0 = g(x_0, y_0)$. Let f be a function that maps R to R such that z_0 is in the domain of f. Last, assume f is continuous at z_0. Then $f \circ g$ is continuous at (x_0, y_0) as shown in Figure 1.2.7.

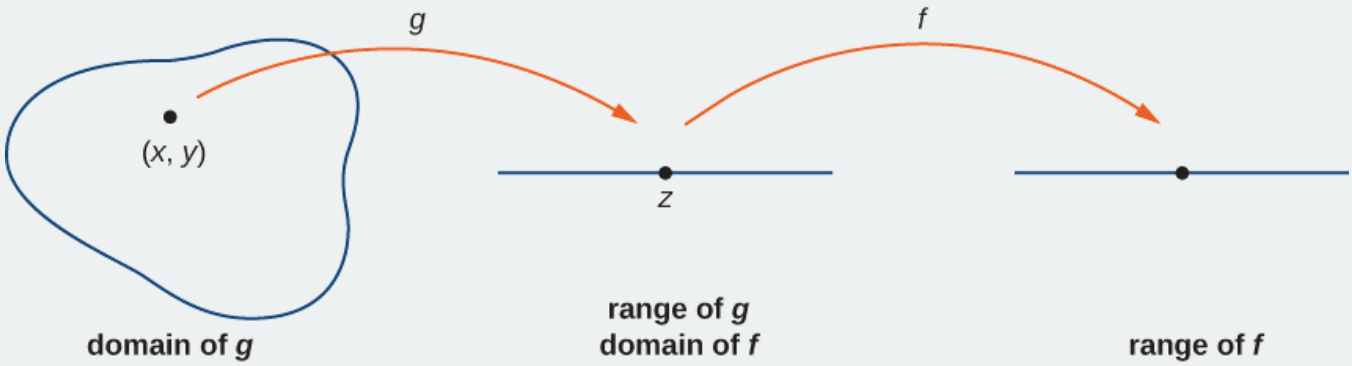


Figure 1.2.7: The composition of two continuous functions is continuous. From left to right, a closed shape is shown labeled the domain of g with point (x, y) inside of it. From the domain of g there is an arrow marked g pointing to the range of function g,which is a straight line with point z on it. The range of function g is also marked the domain of f. Then there is another arrow marked f from this shape to a line marked range of f.

Let's now use the previous theorems to show continuity of functions in the following examples.

Example 1.2.5: More Examples of Continuity of a Function of Two Variables

Show that the functions $f(x, y) = 4x^3y^2$ and $g(x, y) = \cos(4x^3y^2)$ are continuous everywhere.

Solution

The polynomials $g(x) = 4x^3$ and $h(y) = y^2$ are continuous at every real number, and therefore by the product of continuous functions theorem, $f(x, y) = 4x^3y^2$ is continuous at every point (x, y) in the xy-plane. Since $f(x, y) = 4x^3y^2$ is continuous at every point (x, y) in the xy-plane and $g(x) = \cos x$ is continuous at every real number x, the continuity of the composition of functions tells us that $g(x, y) = \cos(4x^3y^2)$ is continuous at every point (x, y) in the xy-plane.

Exercise 1.2.5

Show that the functions $f(x,y) = 2x^2y^3 + 3$ and $g(x,y) = (2x^2y^3 + 3)^4$ are continuous everywhere.

Hint available at the end of this section.

Functions of Three or More Variables

The limit of a function of three or more variables occurs readily in applications. For example, suppose we have a function $f(x,y,z)$ that gives the temperature at a physical location (x,y,z) in three dimensions. Or perhaps a function $g(x,y,z,t)$ can indicate air pressure at a location (x,y,z) at time t. How can we take a limit at a point in $\mathbb{R}^3$? What does it mean to be continuous at a point in four dimensions?

The answers to these questions rely on extending the concept of a δ disk into more than two dimensions. Then, the ideas of the limit of a function of three or more variables and the continuity of a function of three or more variables are very similar to the definitions given earlier for a function of two variables.

Definition: δ-balls

Let (x_0,y_0,z_0) be a point in $\mathbb{R}^3$. Then, a δ-ball in three dimensions consists of all points in $\mathbb{R}^3$ lying at a distance of less than δ from (x_0,y_0,z_0) —that is,

$$\{(x,y,z) \in \mathbb{R}^3 \mid \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} < \delta\}. \tag{1.2.24}$$

To define a δ-ball in higher dimensions, add additional terms under the radical to correspond to each additional dimension. For example, given a point $P = (w_0,x_0,y_0,z_0)$ in $\mathbb{R}^4$, a δ ball around P can be described by

$$\{(w,x,y,z) \in \mathbb{R}^4 \mid \sqrt{(w-w_0)^2 + (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} < \delta\}. \tag{1.2.25}$$

To show that a limit of a function of three variables exists at a point (x_0,y_0,z_0), it suffices to show that for any point in a δ ball centered at (x_0,y_0,z_0), the value of the function at that point is arbitrarily close to a fixed value (the limit value). All the limit laws for functions of two variables hold for functions of more than two variables as well.

Example 1.2.6: Finding the Limit of a Function of Three Variables

Find

$$\lim_{(x,y,z)\to(4,1,-3)} \frac{x^2y - 3z}{2x + 5y - z}.$$

Solution

Before we can apply the quotient law, we need to verify that the limit of the denominator is nonzero. Using the difference law, the identity law, and the constant law,

$$\begin{aligned}\lim_{(x,y,z)\to(4,1,-3)} (2x+5y-z) &= 2(\lim_{(x,y,z)\to(4,1,-3)} x) + 5(\lim_{(x,y,z)\to(4,1,-3)} y) - (\lim_{(x,y,z)\to(4,1,-3)} z) \\ &= 2(4) + 5(1) - (-3) \\ &= 16.\end{aligned}$$

Since this is nonzero, we next find the limit of the numerator. Using the product law, power law, difference law, constant multiple law, and identity law,

$$\begin{aligned}\lim_{(x,y,z)\to(4,1,-3)}(x^2y-3z) &= (\lim_{(x,y,z)\to(4,1,-3)}x)^2(\lim_{(x,y,z)\to(4,1,-3)}y)-3(\lim_{(x,y,z)\to(4,1,-3)}z)\\ &= (4^2)(1)-3(-3)\\ &= 16+9\\ &= 25\end{aligned}$$

Last, applying the quotient law:

$$\lim_{(x,y,z)\to(4,1,-3)}\frac{x^2y-3z}{2x+5y-z} = \frac{\lim\limits_{(x,y,z)\to(4,1,-3)}(x^2y-3z)}{\lim\limits_{(x,y,z)\to(4,1,-3)}(2x+5y-z)} = \frac{25}{16}$$

Exercise 1.2.6

Find

$$\lim_{(x,y,z)\to(4,-1,3)}\sqrt{13-x^2-2y^2+z^2}$$

Hint available at the end of this section.

Key Concepts

- To study limits and continuity for functions of two variables, we use a δ disk centered around a given point.
- A function of several variables has a limit if for any point in a δ ball centered at a point P, the value of the function at that point is arbitrarily close to a fixed value (the limit value).
- The limit laws established for a function of one variable have natural extensions to functions of more than one variable.
- A function of two variables is continuous at a point if the limit exists at that point, the function exists at that point, and the limit and function are equal at that point.

Glossary

Boundary point
: a point P_0 of R is a boundary point if every δ disk centered around P_0 contains points both inside and outside R

Closed set
: a set S that contains all its boundary points

Connected set
: an open set S that cannot be represented as the union of two or more disjoint, nonempty open subsets

δ disk
: an open disk of radius δ centered at point (a,b)

δ ball
: all points in $\mathbb{R}^3$ lying at a distance of less than δ from (x_0,y_0,z_0)

Interior point
: a point P_0 of $\mathbb{R}$ is a boundary point if there is a δ disk centered around P_0 contained completely in $\mathbb{R}$

Open set
: a set S that contains none of its boundary points

Region
: an open, connected, nonempty subset of $\mathbb{R}^2$

Contributors and Attributions

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Answers

Exercise 1.2.1**:**

Use the limit laws.

$$\lim_{(x,y)\to(5,-2)} \sqrt[3]{\frac{x^2-y}{y^2+x-1}} = \frac{3}{2}$$

Exercise 1.2.2**:**

Pick a line with slope k passing through point $(2,1)$.

If $y = k(x-2)+1$, then $\lim_{(x,y)\to(2,1)} \frac{(x-2)(y-1)}{(x-2)^2+(y-1)^2} = \frac{k}{1+k^2}$. Since the answer depends on k, the limit fails to exist.

Exercise 1.2.3

Determine the domain of $f(x,y) = \sqrt{29-x^2-y^2}$.

$$\lim_{(x,y)\to(5,-2)} \sqrt{29-x^2-y^2}$$

Exercise 1.2.4

Use the three-part definition of continuity.

1. The domain of f contains the ordered pair $(2,-3)$ because $f(a,b) = f(2,-3) = \sqrt{16-2(2)^2-(-3)^2} = 3$
2. $\lim_{(x,y)\to(a,b)} f(x,y) = 3$
3. $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b) = 3$

Exercise 1.2.5

Use the continuity of the sum, product, and composition of two functions.

The polynomials $g(x) = 2x^2$ and $h(y) = y^3$ are continuous at every real number; therefore, by the product of continuous functions theorem, $f(x,y) = 2x^2y^3$ is continuous at every point (x,y) in the xy-plane. Furthermore, any constant function is continuous everywhere, so $g(x,y) = 3$ is continuous at every point (x,y) in the xy-plane. Therefore, $f(x,y) = 2x^2y^3+3$ is continuous at every point (x,y) in the xy-plane. Last, $h(x) = x^4$ is continuous at every real number x, so by the continuity of composite functions theorem $g(x,y) = (2x^2y^3+3)^4$ is continuous at every point (x,y) in the xy-plane.

Exercise 1.2.6

Use the limit laws and the continuity of the composition of functions.

$$\lim_{(x,y,z)\to(4,-1,3)} \sqrt{13-x^2-2y^2+z^2} = 2$$

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1.2E: Exercises for Section 1.2

1. Use the limit laws for functions of two variables to evaluate each limit below, given that $\lim_{(x,y)\to(a,b)} f(x,y) = 5$ and $\lim_{(x,y)\to(a,b)} g(x,y) = 2$.
   a. $\lim_{(x,y)\to(a,b)} [f(x,y) + g(x,y)]$
   b. $\lim_{(x,y)\to(a,b)} [f(x,y)g(x,y)]$
   c. $\lim_{(x,y)\to(a,b)} \left[\dfrac{7f(x,y)}{g(x,y)}\right]$
   d. $\lim_{(x,y)\to(a,b)} \left[\dfrac{2f(x,y) - 4g(x,y)}{f(x,y) - g(x,y)}\right]$

   Answer

   a. $\lim_{(x,y)\to(a,b)} [f(x,y) + g(x,y)] = \lim_{(x,y)\to(a,b)} f(x,y) + \lim_{(x,y)\to(a,b)} g(x,y) = 5 + 2 = 7$
   b. $\lim_{(x,y)\to(a,b)} [f(x,y)g(x,y)] = \left(\lim_{(x,y)\to(a,b)} f(x,y)\right)\left(\lim_{(x,y)\to(a,b)} g(x,y)\right) = 5(2) = 10$
   c. $\lim_{(x,y)\to(a,b)} \left[\dfrac{7f(x,y)}{g(x,y)}\right] = \dfrac{7\left(\lim_{(x,y)\to(a,b)} f(x,y)\right)}{\lim_{(x,y)\to(a,b)} g(x,y)} = \dfrac{7(5)}{2} = 17.5$
   d. $\lim_{(x,y)\to(a,b)} \left[\dfrac{2f(x,y) - 4g(x,y)}{f(x,y) - g(x,y)}\right] = \dfrac{2\left(\lim_{(x,y)\to(a,b)} f(x,y)\right) - 4\left(\lim_{(x,y)\to(a,b)} g(x,y)\right)}{\lim_{(x,y)\to(a,b)} f(x,y) - \lim_{(x,y)\to(a,b)} g(x,y)} = \dfrac{2(5) - 4(2)}{5 - 2} = \dfrac{2}{3}$

Exercises 2 - 4

Find the limit of the function.

2. $\lim_{(x,y)\to(1,2)} x$
3. $\lim_{(x,y)\to(1,2)} \dfrac{5x^2y}{x^2 + y^2}$

   Answer

   $$\lim_{(x,y)\to(1,2)} \frac{5x^2y}{x^2 + y^2} = 2$$

4. Show that the limit $\lim_{(x,y)\to(0,0)} \dfrac{5x^2y}{x^2 + y^2}$ exists and is the same along the paths: y-axis and x-axis, and along $y = x$.

Exercises 5 - 19

Evaluate the limits at the indicated values of x and y. If the limit does not exist, state this and explain why the limit does not exist.

5. $\lim_{(x,y)\to(0,0)} \dfrac{4x^2 + 10y^2 + 4}{4x^2 - 10y^2 + 6}$

   Answer

   $$\lim_{(x,y)\to(0,0)} \frac{4x^2 + 10y^2 + 4}{4x^2 - 10y^2 + 6} = \frac{2}{3}$$

6. $\lim_{(x,y)\to(11,13)} \sqrt{\dfrac{1}{xy}}$

7. $\lim_{(x,y)\to(0,1)} \dfrac{y^2 \sin x}{x}$

Answer

$$\lim_{(x,y)\to(0,1)} \frac{y^2 \sin x}{x} = 1$$

8. $\lim_{(x,y)\to(0,0)} \sin\left(\dfrac{x^8+y^7}{x-y+10}\right)$
9. $\lim_{(x,y)\to(\frac{\pi}{4},1)} \dfrac{y\tan x}{y+1}$

Answer

$$\lim_{(x,y)\to(\frac{\pi}{4},1)} \frac{y\tan x}{y+1} = \tfrac{1}{2}$$

10. $\lim_{(x,y)\to(0,\frac{\pi}{4})} \dfrac{\sec x+2}{3x-\tan y}$
11. $\lim_{(x,y)\to(2,5)} \left(\dfrac{1}{x}-\dfrac{5}{y}\right)$

Answer

$$\lim_{(x,y)\to(2,5)} \left(\frac{1}{x}-\frac{5}{y}\right) = -\frac{1}{2}$$

12. $\lim_{(x,y)\to(4,4)} x\ln y$
13. $\lim_{(x,y)\to(4,4)} e^{-x^2-y^2}$

Answer

$$\lim_{(x,y)\to(4,4)} e^{-x^2-y^2} = e^{-32}$$

14. $\lim_{(x,y)\to(0,0)} \sqrt{9-x^2-y^2}$
15. $\lim_{(x,y)\to(1,2)} (x^2y^3-x^3y^2+3x+2y)$

Answer

$$\lim_{(x,y)\to(1,2)} (x^2y^3-x^3y^2+3x+2y) = 11$$

16. $\lim_{(x,y)\to(\pi,\pi)} x\sin\left(\dfrac{x+y}{4}\right)$
17. $\lim_{(x,y)\to(0,0)} \dfrac{xy+1}{x^2+y^2+1}$

Answer

$$\lim_{(x,y)\to(0,0)} \frac{xy+1}{x^2+y^2+1} = 1$$

18. $\lim_{(x,y)\to(0,0)} \dfrac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$
19. $\lim_{(x,y)\to(0,0)} \ln(x^2+y^2)$

Answer

The limit does not exist because when x and y both approach zero, the function approaches $\ln 0$, which is undefined (approaches negative infinity).

Exercises 20 - 21

Complete the statement.

20. A point (x_0, y_0) in a plane region R is an interior point of R if _____.
21. A point (x_0, y_0) in a plane region R is called a boundary point of R if _____.

Answer

Every open disk centered at (x_0, y_0) contains points inside R and outside R.

Exercises 22 - 25

Use algebraic techniques to evaluate the limit.

22. $\lim\limits_{(x,y)\to(2,1)} \dfrac{x-y-1}{\sqrt{x-y}-1}$
23. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^4-4y^4}{x^2+2y^2}$

Answer

$$\lim_{(x,y)\to(0,0)} \frac{x^4-4y^4}{x^2+2y^2} = 0$$

24. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3-y^3}{x-y}$
25. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-xy}{\sqrt{x}-\sqrt{y}}$

Answer

$$\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}} = 0$$

Exercises 26 - 27

Evaluate the limits of the functions of three variables.

26. $\lim\limits_{(x,y,z)\to(1,2,3)} \dfrac{xz^2-y^2z}{xyz-1}$
27. $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{x^2-y^2-z^2}{x^2+y^2-z^2}$

Answer

The limit does not exist.

Exercises 28 - 31

Evaluate the limit of the function by determining the value the function approaches along the indicated paths. If the limit does not exist, explain why not.

28. $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy+y^3}{x^2+y^2}$
    a. Along the x-axis $(y=0)$
    b. Along the y-axis $(x=0)$
    c. Along the path $y=2x$
29. Evaluate $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy+y^3}{x^2+y^2}$ using the results of previous problem.

Answer

The limit does not exist. The function approaches two different values along different paths.

30. $\lim_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^2}$
    a. Along the x-axis $(y=0)$
    b. Along the y-axis $(x=0)$
    c. Along the path $y=x^2$
31. Evaluate $\lim_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^2}$ using the results of previous problem.

Answer

The limit does not exist because the function approaches two different values along the paths.

Exercises 32 - 35

Discuss the continuity of each function. Find the largest region in the xy-plane in which each function is continuous.

32. $f(x,y)=\sin(xy)$
33. $f(x,y)=\ln(x+y)$

Answer

The function f is continuous in the region $y>-x$.

34. $f(x,y)=e^{3xy}$
35. $f(x,y)=\dfrac{1}{xy}$

Answer

The function f is continuous at all points in the xy-plane except at points on the x- and y-axes.

Exercises 36 - 38

Determine the region in which the function is continuous. Explain your answer.

36. $f(x,y)=\dfrac{x^2y}{x^2+y^2}$
37. $f(x,y)=\begin{cases}\dfrac{x^2y}{x^2+y^2} & if(x,y)\neq(0,0)\\ 0 & if(x,y)=(0,0)\end{cases}$

Hint:

Show that the function approaches different values along two different paths.

Answer

The function is continuous at $(0,0)$ since the limit of the function at $(0,0)$ is 0, the same value of $f(0,0)$.

38. $f(x,y)=\dfrac{\sin(x^2+y^2)}{x^2+y^2}$
39. Determine whether $g(x,y)=\dfrac{x^2-y^2}{x^2+y^2}$ is continuous at $(0,0)$.

Answer

The function is discontinuous at $(0,0)$. The limit at $(0,0)$ fails to exist and $g(0,0)$ does not exist.

40. Create a plot using graphing software to determine where the limit does not exist. Determine the region of the coordinate plane in which $f(x,y)=\dfrac{1}{x^2-y}$ is continuous.
41. Determine the region of the xy-plane in which the composite function $g(x,y)=\arctan(\frac{xy^2}{x+y})$ is continuous. Use technology to support your conclusion.

Answer

Since the function $\arctan x$ is continuous over $(-\infty, \infty), g(x, y) = \arctan(\frac{xy^2}{x+y})$ is continuous where $z = \dfrac{xy^2}{x+y}$ is continuous. The inner function z is continuous on all points of the xy-plane except where $y = -x$. Thus, $g(x, y) = \arctan(\frac{xy^2}{x+y})$ is continuous on all points of the coordinate plane except at points at which $y = -x$.

12. Determine the region of the xy-plane in which $f(x, y) = \ln(x^2 + y^2 - 1)$ is continuous. Use technology to support your conclusion. (Hint: Choose the range of values for x and y carefully!)
13. At what points in space is $g(x, y, z) = x^2 + y^2 - 2z^2$ continuous?

Answer

All points $P(x, y, z)$ in space

14. At what points in space is $g(x, y, z) = \dfrac{1}{x^2 + z^2 - 1}$ continuous?
15. Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{1}{x^2 + y^2}$ does not exist at $(0, 0)$ by plotting the graph of the function.

Answer

The graph increases without bound as x and y both approach zero.

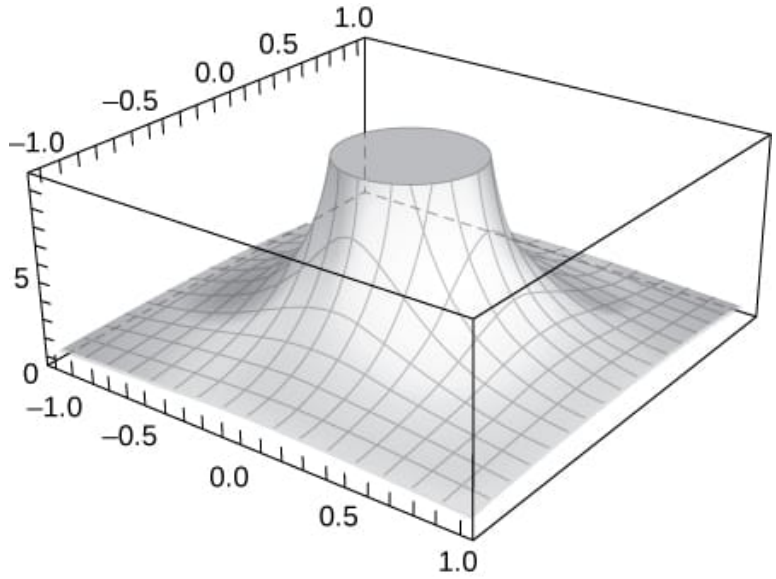


Figure $1.2E.1$: Image illustrates surface of $g(x, y, z) = \dfrac{1}{x^2 + z^2 - 1}$ where the z-coordinate increases without bound as the input point (x, y) approaches the origin.

16. [T] Evaluate $\lim\limits_{(x,y)\to(0,0)} \dfrac{-xy^2}{x^2 + y^4}$ by plotting the function using a CAS. Determine analytically the limit along the path $x = y^2$.
17. [T]
    a. Use a CAS to draw a contour map of $z = \sqrt{9 - x^2 - y^2}$.
    b. What is the name of the geometric shape of the level curves?
    c. Give the general equation of the level curves.
    d. What is the maximum value of z?
    e. What is the domain of the function?
    f. What is the range of the function?

Answer

a.

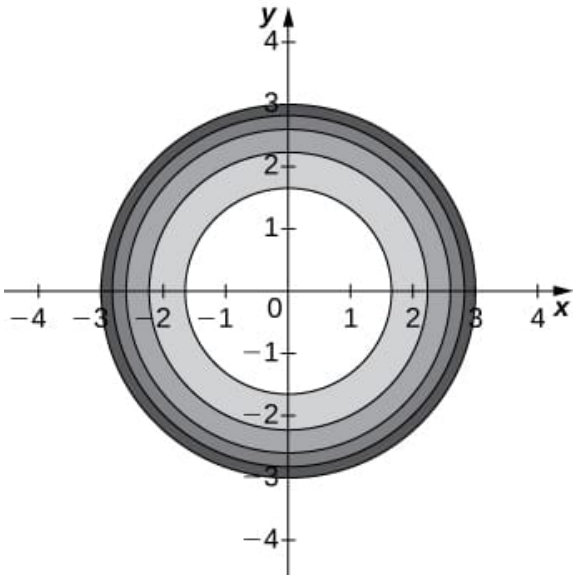


Figure $1.2E.2$: Image illustrates contour map of $z = \sqrt{9 - x^2 - y^2}$. It consists of a series of circles centered at origin.

b. The level curves are circles centered at $(0, 0)$ with radius $9 - c$.
c. $x^2 + y^2 = 9 - c$
d. $z = 3$
e. $\{(x, y) \in R^2 \mid x^2 + y^2 \leq 9\}$
f. $\{z | 0 \leq z \leq 3\}$

48. True or False: If we evaluate $\lim_{(x,y)\to(0,0)} f(x)$ along several paths and each time the limit is 1, we can conclude that $\lim_{(x,y)\to(0,0)} f(x) = 1$.

49. Use polar coordinates to find $\lim_{(x,y)\to(0,0)} \dfrac{\sin\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}$. You can also find the limit using L'Hôpital's rule.

Answer

$$\lim_{(x,y)\to(0,0)} \frac{\sin\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} = 1$$

50. Use polar coordinates to find $\lim_{(x,y)\to(0,0)} \cos(x^2 + y^2)$.

51. Discuss the continuity of $f(g(x, y))$ where $f(t) = 1/t$ and $g(x, y) = 2x - 5y$.

Answer

$f(g(x, y))$ is continuous at all points (x, y) that are not on the line $2x - 5y = 0$.

52. Given $f(x, y) = x^2 - 4y$, find $\lim_{h\to 0} \dfrac{f(x + h, y) - f(x, y)}{h}$.

53. Given $f(x, y) = x^2 - 4y$, find $\lim_{h\to 0} \dfrac{f(1 + h, y) - f(1, y)}{h}$.

Answer

$$\lim_{h\to 0} \frac{f(1 + h, y) - f(1, y)}{h} = 2$$

Contributors

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1.3: Partial Derivatives

Learning Objectives

- Calculate the partial derivatives of a function of two variables.
- Calculate the partial derivatives of a function of more than two variables.
- Determine the higher-order derivatives of a function of two variables.
- Explain the meaning of a partial differential equation and give an example.

Now that we have examined limits and continuity of functions of two variables, we can proceed to study derivatives. Finding derivatives of functions of two variables is the key concept in this chapter, with as many applications in mathematics, science, and engineering as differentiation of single-variable functions. However, we have already seen that limits and continuity of multivariable functions have new issues and require new terminology and ideas to deal with them. This carries over into differentiation as well.

Derivatives of a Function of Two Variables

When studying derivatives of functions of one variable, we found that one interpretation of the derivative is an instantaneous rate of change of y as a function of x. Leibniz notation for the derivative is dy/dx, which implies that y is the dependent variable and x is the independent variable. For a function $z = f(x, y)$ of two variables, x and y are the independent variables and z is the dependent variable. This raises two questions right away: How do we adapt Leibniz notation for functions of two variables? Also, what is an interpretation of the derivative? The answer lies in partial derivatives.

Definition: Partial Derivatives

Let $f(x, y)$ be a function of two variables. Then the *partial derivative* of f with respect to x, written as $\partial f/\partial x$, or f_x, is defined as

$$\frac{\partial f}{\partial x} = f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h} \tag{1.3.1}$$

The partial derivative of f with respect to y, written as $\partial f/\partial y$, or f_y, is defined as

$$\frac{\partial f}{\partial y} = f_y(x, y) = \lim_{k \to 0} \frac{f(x, y + k) - f(x, y)}{k}. \tag{1.3.2}$$

This definition shows two differences already. First, the notation changes, in the sense that we still use a version of Leibniz notation, but the d in the original notation is replaced with the symbol ∂. (This rounded "d" is usually called "partial," so $\partial f/\partial x$ is spoken as the "partial derivative of f with respect to x.") This is the first hint that we are dealing with partial derivatives. Second, we now have two different derivatives we can take, since there are two different independent variables. Depending on which variable we choose, we can come up with different partial derivatives altogether, and often do.

Example 1.3.1: Calculating Partial Derivatives from the Definition

Use the definition of the partial derivative as a limit to calculate $\partial f/\partial x$ and $\partial f/\partial y$ for the function

$$f(x, y) = x^2 - 3xy + 2y^2 - 4x + 5y - 12.$$

Solution

First, calculate $f(x + h, y)$.

$$\begin{aligned} f(x+h,y) &= (x+h)^2 - 3(x+h)y + 2y^2 - 4(x+h) + 5y - 12 \\ &= x^2 + 2xh + h^2 - 3xy - 3hy + 2y^2 - 4x - 4h + 5y - 12. \end{aligned}$$

Next, substitute this into Equation 1.3.1 and simplify:

$$\begin{aligned} \frac{\partial f}{\partial x} &= \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} \\ &= \lim_{h\to 0} \frac{(x^2 + 2xh + h^2 - 3xy - 3hy + 2y^2 - 4x - 4h + 5y - 12) - (x^2 - 3xy + 2y^2 - 4x + 5y - 12)}{h} \\ &= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - 3xy - 3hy + 2y^2 - 4x - 4h + 5y - 12 - x^2 + 3xy - 2y^2 + 4x - 5y + 12}{h} \\ &= \lim_{h\to 0} \frac{2xh + h^2 - 3hy - 4h}{h} \\ &= \lim_{h\to 0} \frac{h(2x + h - 3y - 4)}{h} \\ &= \lim_{h\to 0} (2x + h - 3y - 4) \\ &= 2x - 3y - 4. \end{aligned}$$

To calculate $\dfrac{\partial f}{\partial y}$, first calculate $f(x, y+h)$:

$$\begin{aligned} f(x+h,y) &= x^2 - 3x(y+h) + 2(y+h)^2 - 4x + 5(y+h) - 12 \\ &= x^2 - 3xy - 3xh + 2y^2 + 4yh + 2h^2 - 4x + 5y + 5h - 12. \end{aligned}$$

Next, substitute this into Equation 1.3.2 and simplify:

$$\begin{aligned} \frac{\partial f}{\partial y} &= \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h} \\ &= \lim_{h\to 0} \frac{(x^2 - 3xy - 3xh + 2y^2 + 4yh + 2h^2 - 4x + 5y + 5h - 12) - (x^2 - 3xy + 2y^2 - 4x + 5y - 12)}{h} \\ &= \lim_{h\to 0} \frac{x^2 - 3xy - 3xh + 2y^2 + 4yh + 2h^2 - 4x + 5y + 5h - 12 - x^2 + 3xy - 2y^2 + 4x - 5y + 12}{h} \\ &= \lim_{h\to 0} \frac{-3xh + 4yh + 2h^2 + 5h}{h} \\ &= \lim_{h\to 0} \frac{h(-3x + 4y + 2h + 5)}{h} \\ &= \lim_{h\to 0} (-3x + 4y + 2h + 5) \\ &= -3x + 4y + 5 \end{aligned}$$

Watch the accompanying video for a visual demonstration of the concept.

- **Video length**: 4 minutes 03 seconds.
- **Context**: This video demonstrates how to use the limit definition to find a partial derivative.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Exercise 1.3.1

Use the definition of the partial derivative as a limit to calculate $\partial f/\partial x$ and $\partial f/\partial y$ for the function

$$f(x,y) = 4x^2 + 2xy - y^2 + 3x - 2y + 5.$$

Hint available at the end of this section.

The idea to keep in mind when calculating partial derivatives is to treat all independent variables, other than the variable with respect to which we are differentiating, as constants. Then proceed to differentiate as with a function of a single variable. To see why this is true, first fix y and define $g(x) = f(x, y)$ as a function of x. Then

$$\begin{aligned} g'(x) &= \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h} \\ &= \frac{\partial f}{\partial x}. \end{aligned}$$

The same is true for calculating the partial derivative of f with respect to y. This time, fix x and define $h(y) = f(x, y)$ as a function of y. Then

$$\begin{aligned} h'(x) &= \lim_{k \to 0} \frac{h(x+k) - h(x)}{k} \\ &= \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k} \\ &= \frac{\partial f}{\partial y}. \end{aligned}$$

All differentiation rules apply.

Example 1.3.2: Calculating Partial Derivatives

Calculate $\partial f/\partial x$ and $\partial f/\partial y$ for the following functions by holding the opposite variable constant then differentiating:

a. $f(x, y) = x^2 - 3xy + 2y^2 - 4x + 5y - 12$
b. $g(x, y) = \sin(x^2 y - 2x + 4)$

Solution:

a. To calculate $\partial f/\partial x$, treat the variable y as a constant. Then differentiate $f(x, y)$ with respect to x using the sum, difference, and power rules:

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left[x^2 - 3xy + 2y^2 - 4x + 5y - 12\right] \\ &= \frac{\partial}{\partial x}[x^2] - \frac{\partial}{\partial x}[3xy] + \frac{\partial}{\partial x}[2y^2] - \frac{\partial}{\partial x}[4x] + \frac{\partial}{\partial x}[5y] - \frac{\partial}{\partial x}[12] \\ &= 2x - 3y + 0 - 4 + 0 - 0 \\ &= 2x - 3y - 4. \end{aligned}$$

The derivatives of the third, fifth, and sixth terms are all zero because they do not contain the variable x, so they are treated as constant terms. The derivative of the second term is equal to the coefficient of x, which is $-3y$. Calculating $\partial f/\partial y$:

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left[x^2 - 3xy + 2y^2 - 4x + 5y - 12\right] \\ &= \frac{\partial}{\partial y}[x^2] - \frac{\partial}{\partial y}[3xy] + \frac{\partial}{\partial y}[2y^2] - \frac{\partial}{\partial y}[4x] + \frac{\partial}{\partial y}[5y] - \frac{\partial}{\partial y}[12] \\ &= -3x + 4y - 0 + 5 - 0 \\ &= -3x + 4y + 5. \end{aligned}$$

These are the same answers obtained in Example 1.3.1.

b. To calculate $\partial g/\partial x$, treat the variable *y* as a constant. Then differentiate $g(x, y)$ with respect to x using the chain rule and power rule:

$$\begin{aligned}\frac{\partial g}{\partial x} &= \frac{\partial}{\partial x}\left[\sin(x^2y - 2x + 4)\right] \\ &= \cos(x^2y - 2x + 4)\frac{\partial}{\partial x}[x^2y - 2x + 4] \\ &= (2xy - 2)\cos(x^2y - 2x + 4).\end{aligned}$$

To calculate $\partial g/\partial y$, treat the variable x as a constant. Then differentiate $g(x, y)$ with respect to y using the chain rule and power rule:

$$\begin{aligned}\frac{\partial g}{\partial y} &= \frac{\partial}{\partial y}\left[\sin(x^2y - 2x + 4)\right] \\ &= \cos(x^2y - 2x + 4)\frac{\partial}{\partial y}[x^2y - 2x + 4] \\ &= x^2\cos(x^2y - 2x + 4).\end{aligned}$$

Watch the accompanying video for a visual demonstration of the concept.

- **Video length**: 3 minutes 16 seconds
- **Context**: This video demonstrates how to find the first partial derivatives.

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Exercise 1.3.2

Calculate $\partial f/\partial x$ and $\partial f/\partial y$ for the function

$$f(x, y) = \tan(x^3 - 3x^2y^2 + 2y^4)$$

by holding the opposite variable constant, then differentiating.

Hint available at the end of this section.

How can we interpret these partial derivatives? Recall that the graph of a function of two variables is a surface in R^3. If we remove the limit from the definition of the partial derivative with respect to x, the difference quotient remains:

$$\frac{f(x + h, y) - f(x, y)}{h}. \tag{1.3.3}$$

This resembles the difference quotient for the derivative of a function of one variable, except for the presence of the y variable. Figure 1.3.1 illustrates a surface described by an arbitrary function $z = f(x, y)$.

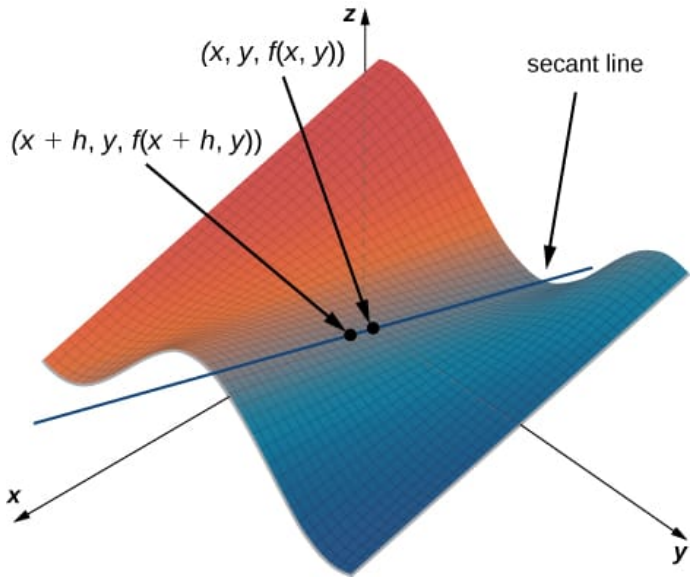


Figure 1.3.1: Image illustrates a 3D surface plot of $z = f(x, y)$ with points $(x, y, f(x, y))$ and $(x + h, y, f(x + h, y))$. A secant line connects these points.

In Figure 1.3.1, the value of h is positive. If we graph $f(x, y)$ and $f(x + h, y)$ for an arbitrary point (x, y), then the slope of the secant line passing through these two points is given by

$$\frac{f(x + h, y) - f(x, y)}{h}. \tag{1.3.4}$$

This line is parallel to the x-axis. Therefore, the slope of the secant line represents an average rate of change of the function f as we travel parallel to the x-axis. As h approaches zero, the slope of the secant line approaches the slope of the tangent line.

If we choose to change y instead of x by the same incremental value h, then the secant line is parallel to the y-axis and so is the tangent line. Therefore, $\partial f/\partial x$ represents the slope of the tangent line passing through the point $(x, y, f(x, y))$ parallel to the x-axis and $\partial f/\partial y$ represents the slope of the tangent line passing through the point $(x, y, f(x, y))$ parallel to the y-axis. If we wish to find the slope of a tangent line passing through the same point in any other direction, then we need what are called directional derivatives.

We now return to the idea of contour maps, which we introduced in Functions of Several Variables. We can use a **contour map** to estimate partial derivatives of a function $g(x, y)$.

Example 1.3.3: Partial Derivatives from a Contour Map

Use a contour map to estimate $\partial g/\partial x$ at the point $(\sqrt{5}, 0)$ for the function

$$g(x, y) = \sqrt{9 - x^2 - y^2}.$$

Solution

Figure 1.3.2 represents a contour map for the function $g(x, y)$.

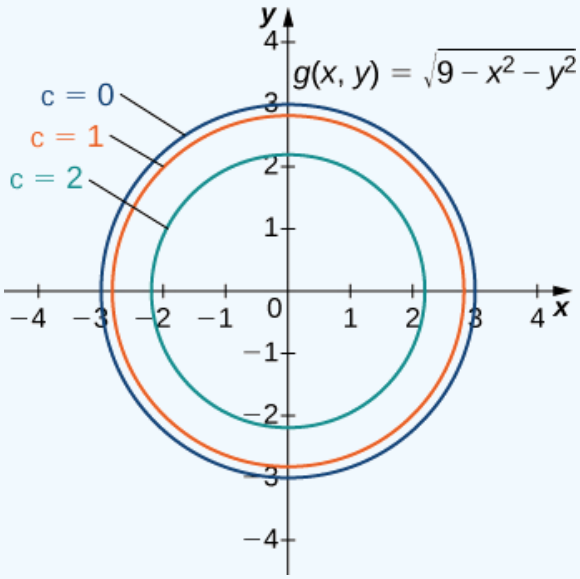


Figure 1.3.2: Contour map for the function $g(x,y)=\sqrt{9-x^2-y^2}$, using $c=0,1,2$, and $3(c=3$ corresponds to the origin).

The inner circle on the contour map corresponds to $c=2$ and the next circle out corresponds to $c=1$. The first circle is given by the equation $2=\sqrt{9-x^2-y^2}$; the second circle is given by the equation $1=\sqrt{9-x^2-y^2}$. The first equation simplifies to $x^2+y^2=5$ and the second equation simplifies to $x^2+y^2=8$. The x-intercept of the first circle is $(\sqrt{5},0)$ and the x-intercept of the second circle is $(2\sqrt{2},0)$. We can estimate the value of $\partial g/\partial x$ evaluated at the point $(\sqrt{5},0)$ using the slope formula:

$$\begin{aligned}\frac{\partial g}{\partial x}\Big|_{(x,y)=(\sqrt{5},0)} &\approx \frac{g(\sqrt{5},0)-g(2\sqrt{2},0)}{\sqrt{5}-2\sqrt{2}}\\ &=\frac{2-1}{\sqrt{5}-2\sqrt{2}}\\ &=\frac{1}{\sqrt{5}-2\sqrt{2}}\approx -1.688.\end{aligned}$$

To calculate the exact value of $\partial g/\partial x$ evaluated at the point $(\sqrt{5},0)$, we start by finding $\partial g/\partial x$ using the chain rule. First, we rewrite the function as

$$g(x,y)=\sqrt{9-x^2-y^2}=(9-x^2-y^2)^{1/2}$$

and then differentiate with respect to x while holding y constant:

$$\begin{aligned}\frac{\partial g}{\partial x}&=\frac{1}{2}(9-x^2-y^2)^{-1/2}(-2x)\\ &=-\frac{x}{\sqrt{9-x^2-y^2}}.\end{aligned}$$

Next, we evaluate this expression using $x=\sqrt{5}$ and $y=0$:

$$\begin{aligned}\frac{\partial g}{\partial x}\Big|_{(x,y)=(\sqrt{5},0)} &=-\frac{\sqrt{5}}{\sqrt{9-(\sqrt{5})^2-(0)^2}}\\ &=-\frac{\sqrt{5}}{\sqrt{4}}\\ &=-\frac{\sqrt{5}}{2}\approx -1.118.\end{aligned}$$

The estimate for the partial derivative corresponds to the slope of the secant line passing through the points $(\sqrt{5}, 0, g(\sqrt{5}, 0))$ and $(2\sqrt{2}, 0, g(2\sqrt{2}, 0))$. It represents an approximation to the slope of the tangent line to the surface through the point $(\sqrt{5}, 0, g(\sqrt{5}, 0))$, which is parallel to the x-axis.

Exercise 1.3.3

Use a contour map to estimate $\partial f/\partial y$ at point $(0, \sqrt{2})$ for the function

$$f(x, y) = x^2 - y^2.$$

Compare this with the exact answer.

Hint available at the end of this section.

Functions of More Than Two Variables

Suppose we have a function of three variables, such as $w = f(x, y, z)$. We can calculate partial derivatives of w with respect to any of the independent variables, simply as extensions of the definitions for partial derivatives of functions of two variables.

Definition: Partial Derivatives

Let $f(x, y, z)$ be a function of three variables. Then, the partial derivative of f with respect to x, written as $\partial f/\partial x$, or f_x, is defined to be

$$\frac{\partial f}{\partial x} = f_x(x, y, z) = \lim_{h \to 0} \frac{f(x + h, y, z) - f(x, y, z)}{h}. \tag{1.3.5}$$

The *partial derivative* of f with *respect* to y, written as $\partial f/\partial y$, or f_y, is defined to be

$$\frac{\partial f}{\partial y} = f_y(x, y, z) = \lim_{k \to 0} \frac{f(x, y + k, z) - f(x, y, z)}{k.} \tag{1.3.6}$$

The *partial derivative* of f with respect to z, written as $\partial f/\partial z$, or f_z, is defined to be

$$\frac{\partial f}{\partial z} = f_z(x, y, z) = \lim_{m \to 0} \frac{f(x, y, z + m) - f(x, y, z)}{m}. \tag{1.3.7}$$

We can calculate a partial derivative of a function of three variables using the same idea we used for a function of two variables. For example, if we have a function f of x, y, and z, and we wish to calculate $\partial f/\partial x$, then we treat the other two independent variables as if they are constants, then differentiate with respect to x.

Example 1.3.4: Calculating Partial Derivatives for a Function of Three Variables

Use the limit definition of partial derivatives to calculate $\partial f/\partial x$ for the function

$$f(x, y, z) = x^2 - 3xy + 2y^2 - 4xz + 5yz^2 - 12x + 4y - 3z.$$

Then, find $\partial f/\partial y$ and $\partial f/\partial z$ by setting the other two variables constant and differentiating accordingly.

Solution:

We first calculate $\partial f/\partial x$ using Equation 1.3.5, then we calculate the other two partial derivatives by holding the remaining variables constant. To use the equation to find $\partial f/\partial x$, we first need to calculate $f(x + h, y, z)$:

$$f(x+h,y,z) = (x+h)^2 - 3(x+h)y + 2y^2 - 4(x+h)z + 5yz^2 - 12(x+h) + 4y - 3z$$
$$= x^2 + 2xh + h^2 - 3xy - 3xh + 2y^2 - 4xz - 4hz + 5yz^2 - 12x - 12h + 4y - 3z$$

and recall that $f(x,y,z) = x^2 - 3xy + 2y^2 - 4zx + 5yz^2 - 12x + 4y - 3z$. Next, we substitute these two expressions into the equation:

$$\frac{\partial f}{\partial x} = \lim_{h\to 0}\left[\frac{x^2 + 2xh + h^2 - 3xy - 3hy + 2y^2 - 4xz - 4hz + 5yz^2 - 12x - 12h + 4y - 3zh - x^2 - 3xy + 2y^2 - 4xz + 5yz^2 - 12x + 4y - 3z}{h}\right]$$
$$= \lim_{h\to 0}\left[\frac{2xh + h^2 - 3hy - 4hz - 12h}{h}\right]$$
$$= \lim_{h\to 0}\left[\frac{h(2x + h - 3y - 4z - 12)}{h}\right]$$
$$= \lim_{h\to 0}(2x + h - 3y - 4z - 12)$$
$$= 2x - 3y - 4z - 12.$$

Then we find $\partial f/\partial y$ by holding x and z constant. Therefore, any term that does not include the variable y is constant, and its derivative is zero. We can apply the sum, difference, and power rules for functions of one variable:

$$\frac{\partial}{\partial y}\left[x^2 - 3xy + 2y^2 - 4xz + 5yz^2 - 12x + 4y - 3z\right]$$
$$= \frac{\partial}{\partial y}[x^2] - \frac{\partial}{\partial y}[3xy] + \frac{\partial}{\partial y}[2y^2] - \frac{\partial}{\partial y}[4xz] + \frac{\partial}{\partial y}[5yz^2] - \frac{\partial}{\partial y}[12x] + \frac{\partial}{\partial y}[4y] - \frac{\partial}{\partial z}[3z]$$
$$= 0 - 3x + 4y - 0 + 5z^2 - 0 + 4 - 0$$
$$= -3x + 4y + 5z^2 + 4.$$

To calculate $\partial f/\partial z$, we hold x and y constant and apply the sum, difference, and power rules for functions of one variable:

$$\frac{\partial}{\partial z}[x^2 - 3xy + 2y^2 - 4xz + 5yz^2 - 12x + 4y - 3z]$$
$$= \frac{\partial}{\partial z}[x^2] - \frac{\partial}{\partial z}[3xy] + \frac{\partial}{\partial z}[2y^2] - \frac{\partial}{\partial z}[4xz] + \frac{\partial}{\partial z}[5yz^2] - \frac{\partial}{\partial z}[12x] + \frac{\partial}{\partial z}[4y] - \frac{\partial}{\partial z}[3z]$$
$$= 0 - 0 + 0 - 4x + 10yz - 0 + 0 - 3$$
$$= -4x + 10yz - 3$$

Exercise 1.3.4

Use the limit definition of partial derivatives to calculate $\partial f/\partial x$ for the function

$$f(x,y,z) = 2x^2 - 4x^2y + 2y^2 + 5xz^2 - 6x + 3z - 8.$$

Then find $\partial f/\partial y$ and $\partial f/\partial z$ by setting the other two variables constant and differentiating accordingly.

Hint available at the end of this section.

Example 1.3.5: Calculating Partial Derivatives for a Function of Three Variables

Calculate the three partial derivatives of the following functions.

a. $f(x,y,z) = \frac{x^2y - 4xz + y^2}{x - 3yz}$
b. $g(x,y,z) = \sin(x^2y - z) + \cos(x^2 - yz)$

Solution

In each case, treat all variables as constants except the one whose partial derivative you are calculating.

a.

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left[\frac{x^2y-4xz+y^2}{x-3yz}\right]\\ &= \frac{\frac{\partial}{\partial x}(x^2y-4xz+y^2)(x-3yz)-(x^2y-4xz+y^2)\frac{\partial}{\partial x}(x-3yz)}{(x-3yz)^2}\\ &= \frac{(2xy-4z)(x-3yz)-(x^2y-4xz+y^2)(1)}{(x-3yz)^2}\\ &= \frac{2x^2y-6xy^2z-4xz+12yz^2-x^2y+4xz-y^2}{(x-3yz)^2}\\ &= \frac{x^2y-6xy^2z-4xz+12yz^2+4xz-y^2}{(x-3yz)^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left[\frac{x^2y-4xz+y^2}{x-3yz}\right]\\ &= \frac{\frac{\partial}{\partial y}(x^2y-4xz+y^2)(x-3yz)-(x^2y-4xz+y^2)\frac{\partial}{\partial y}(x-3yz)}{(x-3yz)^2}\\ &= \frac{(x^2+2y)(x-3yz)-(x^2y-4xz+y^2)(-3z)}{(x-3yz)^2}\\ &= \frac{x^3-3x^2yz+2xy-6y^2z+3x^2yz-12xz^2+3y^2z}{(x-3yz)^2}\\ &= \frac{x^3+2xy-3y^2z-12xz^2}{(x-3yz)^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial z} &= \frac{\partial}{\partial z}\left[\frac{x^2y-4xz+y^2}{x-3yz}\right]\\ &= \frac{\frac{\partial}{\partial z}(x^2y-4xz+y^2)(x-3yz)-(x^2y-4xz+y^2)\frac{\partial}{\partial z}(x-3yz)}{(x-3yz)^2}\\ &= \frac{(-4x)(x-3yz)-(x^2y-4xz+y^2)(-3y)}{(x-3yz)^2}\\ &= \frac{-4x^2+12xyz+3x^2y^2-12xyz+3y^3}{(x-3yz)^2}\\ &= \frac{-4x^2+3x^2y^2+3y^3}{(x-3yz)^2}\end{aligned}$$

b.

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left[\sin(x^2y-z)+\cos(x^2-yz)\right]\\ &= (\cos(x^2y-z))\frac{\partial}{\partial x}(x^2y-z)-(\sin(x^2-yz))\frac{\partial}{\partial x}(x^2-yz)\\ &= 2xy\cos(x^2y-z)-2x\sin(x^2-yz)\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}[\sin(x^2y - z) + \cos(x^2 - yz)] \\ &= (\cos(x^2y - z))\frac{\partial}{\partial y}(x^2y - z) - (\sin(x^2 - yz))\frac{\partial}{\partial y}(x^2 - yz) \\ &= x^2\cos(x^2y - z) + z\sin(x^2 - yz) \\ \frac{\partial f}{\partial z} &= \frac{\partial}{\partial z}[\sin(x^2y - z) + \cos(x^2 - yz)] \\ &= (\cos(x^2y - z))\frac{\partial}{\partial z}(x^2y - z) - (\sin(x^2 - yz))\frac{\partial}{\partial z}(x^2 - yz) \\ &= -\cos(x^2y - z) + y\sin(x^2 - yz)\end{aligned}$$

Exercise 1.3.5

Calculate $\partial f/\partial x, \partial f/\partial y$, and $\partial f/\partial z$ for the function

$$f(x, y, z) = \sec(x^2y) - \tan(x^3yz^2).$$

Hint available at the end of this section.

Higher-Order Partial Derivatives

Consider the function

$$f(x, y) = 2x^3 - 4xy^2 + 5y^3 - 6xy + 5x - 4y + 12. \tag{1.3.8}$$

Its partial derivatives are

$$\frac{\partial f}{\partial x} = 6x^2 - 4y^2 - 6y + 5 \tag{1.3.9}$$

and

$$\frac{\partial f}{\partial y} = -8xy + 15y^2 - 6x - 4. \tag{1.3.10}$$

Each of these partial derivatives is a function of two variables, so we can calculate partial derivatives of these functions. Just as with derivatives of single-variable functions, we can call these *second-order derivatives,* third-order derivatives, and so on. In general, they are referred to as **higher-order partial derivatives.** There are four second-order partial derivatives for any function (provided they all exist):

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] \\ \frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] \\ \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] \\ \frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right].\end{aligned}$$

An alternative notation for each is f_{xx}, f_{xy}, f_{yx}, and f_{yy}, respectively. Higher-order partial derivatives calculated with respect to different variables, such as f_{xy} and f_{yx}, are commonly called **mixed partial derivatives.**

Example 1.3.6: Calculating Second Partial Derivatives

Calculate all four second partial derivatives for the function

$$f(x,y) = xe^{-3y} + \sin(2x - 5y). \tag{1.3.11}$$

Solution:

To calculate $\dfrac{\partial^2 f}{\partial x^2}$ and $\dfrac{\partial^2 f}{\partial y \partial x}$, we first calculate $\partial f/\partial x$:

$$\frac{\partial f}{\partial x} = e^{-3y} + 2\cos(2x - 5y). \tag{1.3.12}$$

To calculate $\dfrac{\partial^2 f}{\partial x^2}$, differentiate $\partial f/\partial x$ (Equation 1.3.12) with respect to x:

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] \\ &= \frac{\partial}{\partial x}[e^{-3y} + 2\cos(2x - 5y)] \\ &= -4\sin(2x - 5y).\end{aligned}$$

To calculate $\dfrac{\partial^2 f}{\partial y \partial x}$, differentiate $\partial f/\partial x$ (Equation 1.3.12) with respect to y:

$$\begin{aligned}\frac{\partial^2 f}{\partial y\,\partial x} &= \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] \\ &= \frac{\partial}{\partial y}[e^{-3y} + 2\cos(2x - 5y)] \\ &= -3e^{-3y} + 10\sin(2x - 5y).\end{aligned}$$

To calculate $\dfrac{\partial^2 f}{\partial x \partial y}$ and $\dfrac{\partial^2 f}{\partial y^2}$, first calculate $\partial f/\partial y$:

$$\frac{\partial f}{\partial y} = -3xe^{-3y} - 5\cos(2x - 5y). \tag{1.3.13}$$

To calculate $\dfrac{\partial^2 f}{\partial x \partial y}$, differentiate $\partial f/\partial y$ (Equation 1.3.13) with respect to x:

$$\begin{aligned}\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] \\ &= \frac{\partial}{\partial x}[-3xe^{-3y} - 5\cos(2x - 5y)] \\ &= -3e^{-3y} + 10\sin(2x - 5y).\end{aligned}$$

To calculate $\dfrac{\partial^2 f}{\partial y^2}$, differentiate $\partial f/\partial y$ (Equation 1.3.13) with respect to y:

$$\begin{aligned}\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right] \\ &= \frac{\partial}{\partial y}[-3xe^{-3y} - 5\cos(2x - 5y)] \\ &= 9xe^{-3y} - 25\sin(2x - 5y).\end{aligned}$$

Exercise 1.3.6

Calculate all four second partial derivatives for the function

$$f(x,y) = \sin(3x - 2y) + \cos(x + 4y).$$

Hint available at the end of this section.

At this point we should notice that, in both Example 1.3.6 and the checkpoint, it was true that $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$. Under certain conditions, this is always true. In fact, it is a direct consequence of the following theorem.

Clairaut's Theorem (Equality of Mixed Partial Derivatives)

Suppose that $f(x,y)$ is defined on an open disk D that contains the point (a,b). If the functions f_{xy} and f_{yx} are continuous on D, then $f_{xy} = f_{yx}$.

Clairaut's theorem guarantees that as long as mixed second-order derivatives are continuous, the order in which we choose to differentiate the functions (i.e., which variable goes first, then second, and so on) does not matter. It can be extended to higher-order derivatives as well. The proof of Clairaut's theorem can be found in most advanced calculus books.

Two other second-order partial derivatives can be calculated for any function $f(x,y)$. The partial derivative f_{xx} is equal to the partial derivative of f_x with respect to x, and f_{yy} is equal to the partial derivative of f_y with respect to y.

Partial Differential Equations

Previously, we studied differential equations in which the unknown function had one independent variable. A partial differential equation is an equation that involves an unknown function of more than one independent variable and one or more of its partial derivatives. Examples of partial differential equations are

$$\underbrace{u_t = c^2(u_{xx} + u_{yy})}_{\text{heat equation in two dimensions}} \tag{1.3.14}$$

$$\underbrace{u_{tt} = c^2(u_{xx} + u_{yy})}_{\text{wave equation in two dimensions}} \tag{1.3.15}$$

$$\underbrace{u_{xx} + u_{yy} = 0}_{\text{Laplace's equation in two dimensions}} \tag{1.3.16}$$

In the heat and wave equations, the unknown function u has three independent variables: t, x, and y with c is an arbitrary constant. The independent variables x and y are considered to be spatial variables, and the variable t represents time. In Laplace's equation, the unknown function u has two independent variables x and y.

Example 1.3.7: A Solution to the Wave Equation

Verify that

$$u(x,y,t) = 5\sin(3\pi x)\sin(4\pi y)\cos(10\pi t) \tag{1.3.17}$$

is a solution to the wave equation

$$u_{tt} = 4(u_{xx} + u_{yy}). \tag{1.3.18}$$

Solution

First, we calculate u_{tt}, u_{xx}, and u_{yy} :

$$\begin{aligned}
u_{tt}(x,y,t) &= \frac{\partial}{\partial t}\left[\frac{\partial u}{\partial t}\right] \\
&= \frac{\partial}{\partial t}[5\sin(3\pi x)\sin(4\pi y)(-10\pi\sin(10\pi t))] \\
&= \frac{\partial}{\partial t}\left[-50\pi\sin(3\pi x)\sin(4\pi y)\sin(10\pi t)\right] \\
&= -500\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t) \\
u_{xx}(x,y,t) &= \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}\right] \\
&= \frac{\partial}{\partial x}\left[15\pi\cos(3\pi x)\sin(4\pi y)\cos(10\pi t)\right] \\
&= -45\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t) \\
u_{yy}(x,y,t) &= \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial y}\right] \\
&= \frac{\partial}{\partial y}\left[5\sin(3\pi x)(4\pi\cos(4\pi y))\cos(10\pi t)\right] \\
&= \frac{\partial}{\partial y}\left[20\pi\sin(3\pi x)\cos(4\pi y)\cos(10\pi t)\right] \\
&= -80\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t).
\end{aligned}$$

Next, we substitute each of these into the right-hand side of Equation 1.3.18 and simplify:

$$\begin{aligned}
4(u_{xx}+u_{yy}) &= 4(-45\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t) + -80\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t)) \\
&= 4(-125\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t)) \\
&= -500\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t) \\
&= u_{tt}.
\end{aligned}$$

This verifies the solution.

Exercise 1.3.7: A Solution to the Heat Equation

Verify that

$$u(x,y,t) = 2\sin\left(\frac{x}{3}\right)\sin\left(\frac{y}{4}\right)e^{-25t/16}$$

is a solution to the heat equation

$$u_t = 9(u_{xx}+u_{yy}).$$

Hint available at the end of this section.

Since the solution to the two-dimensional heat equation is a function of three variables, it is not easy to create a visual representation of the solution. We can graph the solution for fixed values of t, which amounts to snapshots of the heat distributions at fixed times. These snapshots show how the heat is distributed over a two-dimensional surface as time progresses. The graph of the preceding solution at time $t = 0$ appears in Figure 1.3.3. As time progresses, the extremes level out, approaching zero as t approaches infinity.

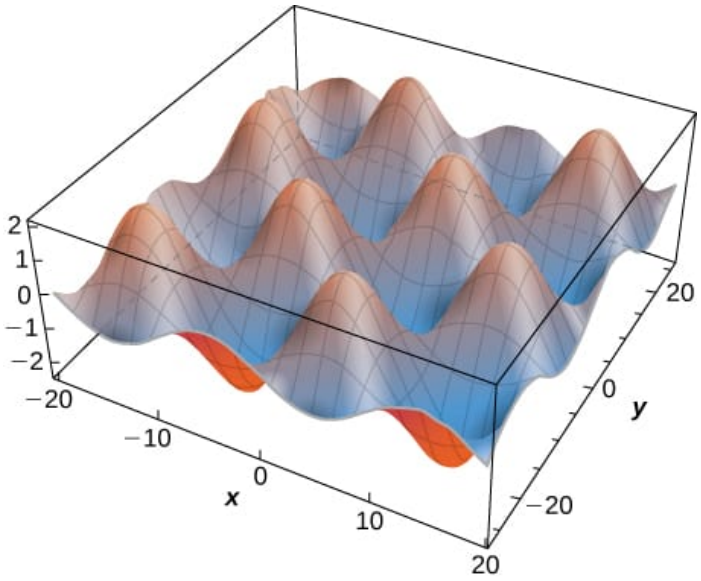


Figure 1.3.3: A 3D surface plot showing the solution to the 2D heat equation at $t = 0$. The plot has a wavy structure, representing temperature distribution over a two-dimensional surface.

If we consider the heat equation in one dimension, then it is possible to graph the solution over time. The heat equation in one dimension becomes

$$u_t = c^2 u_{xx},$$

where c^2 represents the thermal diffusivity of the material in question. A solution of this differential equation can be written in the form

$$u_m(x,t) = e^{-\pi^2 m^2 c^2 t} \sin(m\pi x)$$

where m is any positive integer. A graph of this solution using $m = 1$ appears in Figure 1.3.4, where the initial temperature distribution over a wire of length 1 is given by $u(x,0) = \sin \pi x$. Notice that as time progresses, the wire cools off. This is seen because, from left to right, the highest temperature (which occurs in the middle of the wire) decreases and changes color from red to blue.

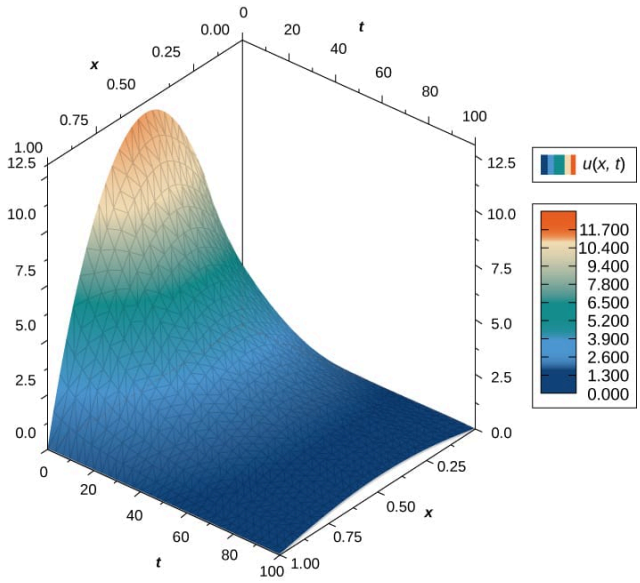


Figure 1.3.4: Graph of a solution of the heat equation in 3d-space over time. a 3D surface plot representing a function $u(x,t)$. The surface rises steeply near $x=0$ and $t=0$ and gradually decreases towards the edge of the plot, where x increases and x progresses forward. A color gradient is applied to the surface, with warmer colors (orange/red) representing higher values of $u(x,t)$, while cooler colors (blue) correspond to lower values.

Example 1.3.8: Function of Two Variables and Laplace's Equation

Show that the function $z = xy - e^x \cos y$ satisfies the Laplace's Equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0 \tag{1.3.19}$$

.

Watch the accompanying video for a visual demonstration of the concept.

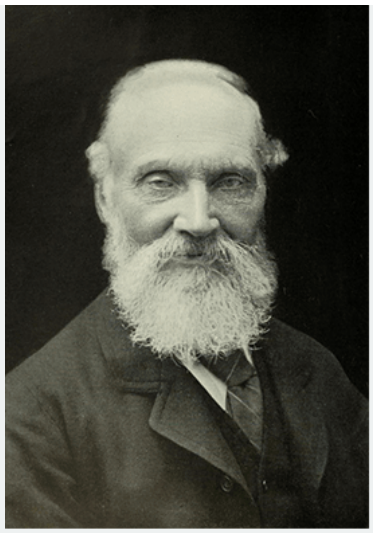
- **Video length**: 3 minutes 45 seconds.
- **Context**: This video demonstrates how to prove a function of two variables satisfies Laplace's Equation.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

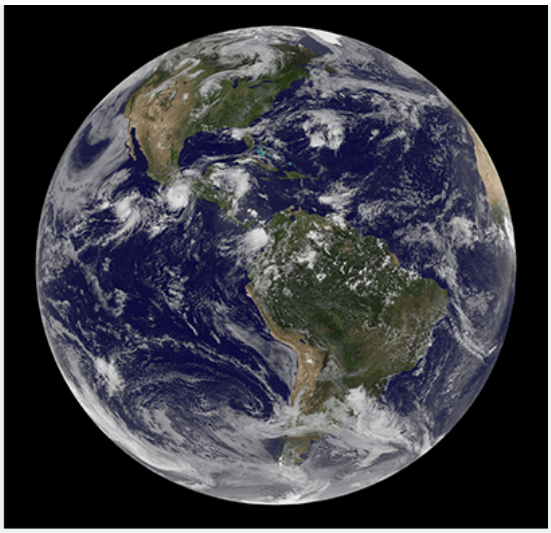
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Lord Kelvin and the Age of Earth

During the late 1800s, the scientists of the new field of geology were coming to the conclusion that Earth must be “millions and millions” of years old. At about the same time, Charles Darwin had published his treatise on evolution. Darwin’s view was that evolution needed many millions of years to take place, and he made a bold claim that the Weald chalk fields, where important fossils were found, were the result of 300 million years of erosion.



(a)



(b)

Figure 1.3.5: (a) William Thomson (Lord Kelvin), 1824-1907, was a British physicist and electrical engineer; (b) Kelvin used the heat diffusion equation to estimate the age of Earth (CC-BY-2.0 modification of work by NASA).

At that time, eminent physicist William Thomson (Lord Kelvin) used an important partial differential equation, known as the heat diffusion equation, to estimate the age of Earth by determining how long it would take Earth to cool from molten rock to what we had at that time. His conclusion was a range of **20** to **400** million years, but most likely about **50** million years. For many decades, the proclamations of this irrefutable icon of science did not sit well with geologists or with Darwin.

- Read Kelvin's paper on estimating the age of the Earth.

Kelvin made reasonable assumptions based on what was known in his time, but he also made several assumptions that turned out to be wrong. One incorrect assumption was that Earth is solid and that the cooling was therefore via conduction only, hence justifying the use of the diffusion equation. But the most serious error was a forgivable one—omission of the fact that Earth contains radioactive elements that continually supply heat beneath Earth's mantle. The discovery of radioactivity came near the end of Kelvin's life and he acknowledged that his calculation would have to be modified.

Kelvin used the simple one-dimensional model applied only to Earth's outer shell, and derived the age from graphs and the roughly known temperature gradient near Earth's surface. Let's take a look at a more appropriate version of the diffusion equation in radial coordinates, which has the form

$$\frac{\partial T}{\partial t} = K\left[\frac{\partial^2 T}{\partial^2 r} + \frac{2}{r}\frac{\partial T}{\partial r}\right] \tag{1.3.20}$$

.

Here, $T(r,t)$ is temperature as a function of r (measured from the center of Earth) and time $t.K$ is the heat conductivity—for molten rock, in this case. The standard method of solving such a partial differential equation is by separation of variables, where we express the solution as the product of functions containing each variable separately. In this case, we would write the temperature as

$$T(r,t) = R(r)f(t). \tag{1.3.21}$$

1. Substitute this form into Equation 1.3.20 and, noting that $f(t)$ is constant with respect to distance (r) and $R(r)$ is constant with respect to time (t), show that

$$\frac{1}{f}\frac{\partial f}{\partial t} = \frac{K}{R}\left[\frac{\partial^2 R}{\partial r^2} + \frac{2}{r}\frac{\partial R}{\partial r}\right]. \tag{1.3.22}$$

2. This equation represents the separation of variables we want. The left-hand side is only a function of t and the right-hand side is only a function of r, and they must be equal for all values of r and t. Therefore, they both must be equal to a constant. Let's call that constant $-\lambda^2$. (The convenience of this choice is seen on substitution.) So, we have

$$\frac{1}{f}\frac{\partial f}{\partial t} = -\lambda^2 \text{and} \frac{K}{R}\left[\frac{\partial^2 R}{\partial r^2} + \frac{2}{r}\frac{\partial R}{\partial r}\right] = -\lambda^2. \quad (1.3.23)$$

3. Now, we can verify through direct substitution for each equation that the solutions are $f(t) = Ae^{-\lambda^2 t}$ and $R(r) = B\left(\frac{\sin \alpha r}{r}\right) + C\left(\frac{\cos \alpha r}{r}\right)$, where $\alpha = \lambda/\sqrt{K}$. Note that $f(t) = Ae^{+\lambda n^2 t}$ is also a valid solution, so we could have chosen $+\lambda^2$for our constant. Can you see why it would not be valid for this case as time increases?
4. Let's now apply boundary conditions.
   a. The temperature must be finite at the center of Earth, $r = 0$. Which of the two constants, B or C, must therefore be zero to keep R finite at $r = 0$? (Recall that $\sin(\alpha r)/r \to \alpha =$ as $r \to 0$, but $\cos(\alpha r)/r$ behaves very differently.)
   b. Kelvin argued that when magma reaches Earth's surface, it cools very rapidly. A person can often touch the surface within weeks of the flow. Therefore, the surface reached a moderate temperature very early and remained nearly constant at a surface temperature T_s. For simplicity, let's set $T = 0$ at $r = R_E$ and find α such that this is the temperature there for all time t. (Kelvin took the value to be $300K \approx 80\ ^\circ F$. We can add this $300K$ constant to our solution later.) For this to be true, the sine argument must be zero at $r = R_E$. Note that α has an infinite series of values that satisfies this condition. Each value of α represents a valid solution (each with its own value for A). The total or general solution is the sum of all these solutions.
   c. At $t = 0$, we assume that all of Earth was at an initial hot temperature T_0 (Kelvin took this to be about $7000K$.) The application of this boundary condition involves the more advanced application of Fourier coefficients. As noted in part b. each value of α_n represents a valid solution, and the general solution is a sum of all these solutions. This results in a series solution:

$$T(r,t) = \left(\frac{T_0 R_E}{\pi}\right)\sum_n \frac{(-1)^{n-1}}{n} e^{-\lambda n^2 t}\frac{\sin(\alpha_n r)}{r}$$

   where $\alpha_n = n\pi/R_E$.

Note how the values of α_n come from the boundary condition applied in part b. The term $\frac{-1^{n-1}}{n}$ is the constant A_n for each term in the series, determined from applying the Fourier method. Letting $\beta = \frac{\pi}{R_E}$, examine the first few terms of this solution shown here and note how λ^2 in the exponential causes the higher terms to decrease quickly as time progresses:

$$T(r,t) = \frac{T_0 R_E}{\pi r}\left(e^{-K\beta^2 t}(\sin \beta r) - \frac{1}{2}e^{-4K\beta^2 t}(\sin 2\beta r) + \frac{1}{3}e^{-9K\beta^2 t}(\sin 3\beta r) - \frac{1}{4}e^{-16K\beta^2 t}(\sin 4\beta r) + \frac{1}{5}e^{-25K\beta^2 t}(\sin 5\beta r)...\right). \quad ($$

Near time $t = 0$, many terms of the solution are needed for accuracy. Inserting values for the conductivity K and $\beta = \pi/R_E$ for time approaching merely thousands of years, only the first few terms make a significant contribution. Kelvin only needed to look at the solution near Earth's surface (Figure 1.3.6) and, after a long time, determine what time best yielded the estimated temperature gradient known during his era ($1\ ^\circ F$ increase per $50ft$). He simply chose a range of times with a gradient close to this value. In Figure 1.3.6, the solutions are plotted and scaled, with the $300 - K$ surface temperature added. Note that the center of Earth would be relatively cool. At the time, it was thought Earth must be solid.

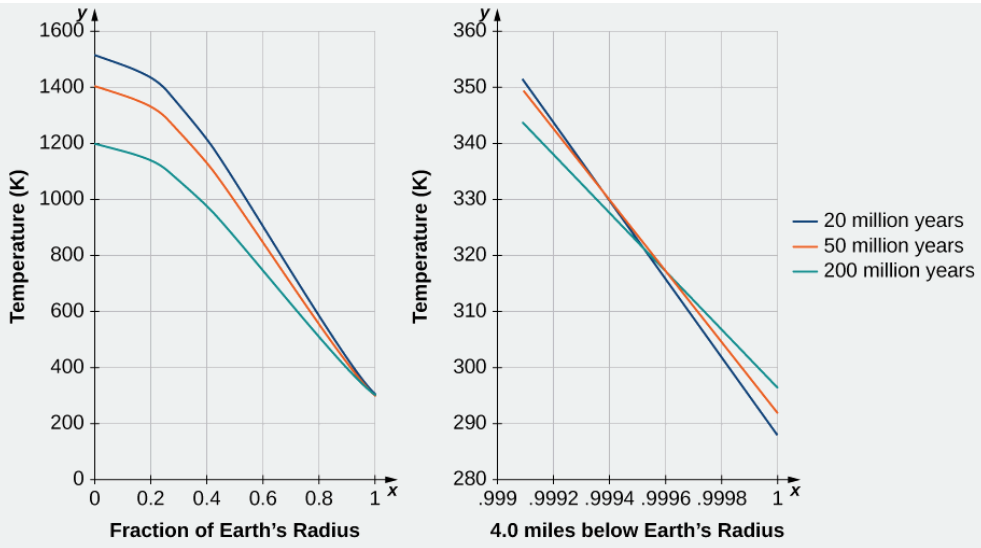


Figure 1.3.6: Temperature on y-axis and radial distance from the center of Earth on x-axis. (a) Kelvin's results, plotted to scale. (b) A close-up of the results at a depth of 4.0 miles below Earth's surface.

Epilog

On May **20, 1904,** physicist **Ernest Rutherford** spoke at the Royal Institution to announce a revised calculation that included the contribution of radioactivity as a source of Earth's heat. In Rutherford's own words:

> "I came into the room, which was half-dark, and presently spotted Lord Kelvin in the audience, and realized that I was in for trouble at the last part of my speech dealing with the age of the Earth, where my views conflicted with his. To my relief, Kelvin fell fast asleep, but as I came to the important point, I saw the old bird sit up, open an eye and cock a baleful glance at me.
>
> Then a sudden inspiration came, and I said Lord Kelvin had limited the age of the Earth, provided no new source [of heat] was discovered. That prophetic utterance referred to what we are now considering tonight, radium! Behold! The old boy beamed upon me."

Rutherford calculated an age for Earth of about **500** million years. Today's accepted value of Earth's age is about **4.6** billion years.

Key Concepts

- A partial derivative is a derivative involving a function of more than one independent variable.
- To calculate a partial derivative with respect to a given variable, treat all the other variables as constants and use the usual differentiation rules.
- Higher-order partial derivatives can be calculated in the same way as higher-order derivatives.

Key Equations

Partial derivative of f with respect to x

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

Partial derivative of f with respect to y

$$\frac{\partial f}{\partial y} = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$$

Glossary

Higher-order partial derivatives
: second-order or higher partial derivatives, regardless of whether they are mixed partial derivatives

Mixed partial derivatives
: second-order or higher partial derivatives, in which at least two of the differentiations are with respect to different variables

Partial derivative
: a derivative of a function of more than one independent variable in which all the variables but one are held constant

Partial differential equation
: an equation that involves an unknown function of more than one independent variable and one or more of its partial derivatives

Contributors and Attributions

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Answers

Exercise 1.3.1

Use Equations 1.3.1 and 1.3.2 from the definition of partial derivatives.

$\frac{\partial f}{\partial x} = 8x + 2y + 3$

$\frac{\partial f}{\partial y} = 2x - 2y - 2$

Exercise 1.3.2

Use Equations 1.3.1 and 1.3.1 from the definition of partial derivatives.

$\frac{\partial f}{\partial x} = (3x^2 - 6xy^2)\sec^2(x^3 - 3x^2y^2 + 2y^4)$

$\frac{\partial f}{\partial y} = (-6x^2y + 8y^3)\sec^2(x^3 - 3x^2y^2 + 2y^4)$

Exercise 1.3.3

Create a contour map for f using values of c from -3 to 3. Which of these curves passes through point $(0, \sqrt{2})$?

Using the curves corresponding to $c = -2$ and $c = -3$, we obtain

$$\begin{aligned}\frac{\partial f}{\partial y}\Big|_{(x,y)=(0,\sqrt{2})} &\approx \frac{f(0,\sqrt{3}) - f(0,\sqrt{2})}{\sqrt{3} - \sqrt{2}} \\ &= \frac{-3 - (-2)}{\sqrt{3} - \sqrt{2}} \cdot \frac{\sqrt{3} + \sqrt{2}}{\sqrt{3} + \sqrt{2}} \\ &= -\sqrt{3} - \sqrt{2} \approx -3.146.\end{aligned}$$

The exact answer is

$$\frac{\partial f}{\partial y}\Big|_{(x,y)=(0,\sqrt{2})} = (-2y|_{(x,y)=(0,\sqrt{2})} = -2\sqrt{2} \approx -2.828.$$

Exercise 1.3.4

Use the strategy in the preceding example.

$\frac{\partial f}{\partial x} = 4x - 8xy + 5z^2 - 6$, $\frac{\partial f}{\partial y} = -4x^2 + 4y$, $\frac{\partial f}{\partial z} = 10xz + 3$

Exercise 1.3.5

Use the strategy in the preceding example.

$\frac{\partial f}{\partial x} = 2xy\sec(x^2y)\tan(x^2y) - 3x^2yz^2\sec^2(x^3yz^2)$

$\frac{\partial f}{\partial y} = x^2\sec(x^2y)\tan(x^2y) - x^3z^2\sec^2(x^3yz^2)$

$\frac{\partial f}{\partial z} = -2x^3yz\sec^2(x^3yz^2)$

Exercise 1.3.6

Follow the same steps as in the previous example.

$\frac{\partial^2 f}{\partial x^2} = -9\sin(3x - 2y) - \cos(x + 4y)$

$$\frac{\partial^2 f}{\partial y \partial x} = 6\sin(3x-2y) - 4\cos(x+4y)$$
$$\frac{\partial^2 f}{\partial x \partial y} = 6\sin(3x-2y) - 4\cos(x+4y)$$
$$\frac{\partial^2 f}{\partial y^2} = -4\sin(3x-2y) - 16\cos(x+4y)$$

Exercise 1.3.7**: A Solution to the Heat Equation**

Calculate the partial derivatives and substitute into the right-hand side.

Since

$$u_t = -\frac{25}{16}u(x,y,t), \tag{1.3.25}$$

and

$$9(u_{xx} + u_{yy}) = -\frac{25}{16}u(x,y,t), \tag{1.3.26}$$

we see that

$$u_t = 9(u_{xx} + u_{yy}), \tag{1.3.27}$$

which verifies that $u(x,y,t)$ is indeed a solution to the heat equation.

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1.3E: Exercises for Section 1.3

Exercise 1

Calculate the partial derivative using the limit definitions only.

1. $\dfrac{\partial z}{\partial y}$ for $z = x^2 - 3xy + y^2$

Answer

$$\frac{\partial z}{\partial y} = -3x + 2y$$

Exercises 2 - 5

Calculate the sign of the partial derivative using the graph of the surface.

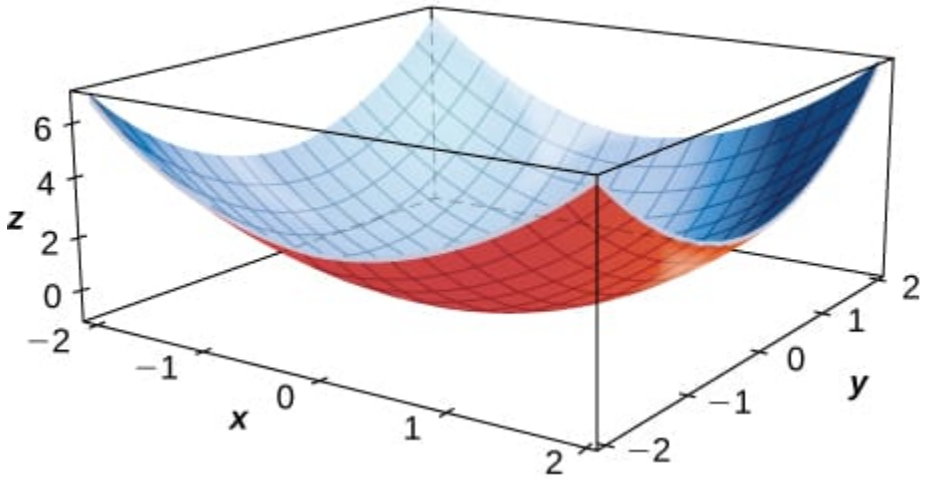


Figure $1.3E.1$: The image illustrates the surface plot of a paraboloid that is concave up and has its vertex at the origin

2. $f_x(1,1)$
3. $f_x(-1,1)$

Answer

The sign is negative.

4. $f_y(1,1)$
5. $f_x(0,0)$

Answer

The partial derivative is zero at the origin.

Exercises 6 - 16

Calculate the requested partial derivatives.

6. $\dfrac{\partial z}{\partial x}$ for $z = \sin(3x)\cos(3y)$

7. $\dfrac{\partial z}{\partial y}$ for $z = \sin(3x)\cos(3y)$

Answer

$$\frac{\partial z}{\partial y} = -3\sin(3x)\sin(3y)$$

8. $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ for $z = x^8 e^3 y$
9. $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ for $z = \ln(x^6 + y^4)$

Answer

$$\frac{\partial z}{\partial x} = \tfrac{6x^5}{x^6+y^4}; \quad \frac{\partial z}{\partial y} = \tfrac{4y^3}{x^6+y^4}$$

10. Find $f_y(x, y)$ for $f(x, y) = e^{xy}\cos(x)\sin(y)$.
11. Let $z = e^{xy}$. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

Answer

$$\frac{\partial z}{\partial x} = ye^{xy}; \quad \frac{\partial z}{\partial y} = xe^{xy}$$

12. Let $z = \ln(\frac{x}{y})$. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.
13. Let $z = \tan(2x - y)$. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

Answer

$$\frac{\partial z}{\partial x} = 2\sec^2(2x - y), \quad \frac{\partial z}{\partial y} = -\sec^2(2x - y)$$

14. Let $z = \sinh(2x + 3y)$. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.
15. Let $f(x, y) = \arctan(\frac{y}{x})$. Evaluate $f_x(2, -2)$ and $f_y(2, -2)$.

Answer

$$f_x(2, -2) = \tfrac{1}{4} = f_y(2, -2)$$

16. Let $f(x, y) = \dfrac{xy}{x - y}$. Find $f_x(2, -2)$ and $f_y(2, -2)$.
17. Find $\dfrac{\partial z}{\partial x}$ at $(0, 1)$ for $z = e^{-x}cos(y)$.

Answer

$$\frac{\partial z}{\partial x} = -\cos(1)$$

18. Given $f(x, y, z) = x^3yz^2$, find $\dfrac{\partial^2 f}{\partial x \partial y}$ and $f_z(1, 1, 1)$.
19. Given $f(x, y, z) = 2\sin(x + y)$, find $f_x(0, \frac{\pi}{2}, -4)$, $f_y(0, \frac{\pi}{2}, -4)$, and $f_z(0, \frac{\pi}{2}, -4)$.

Answer

$$f_x(x, y, z) = 0, \quad f_y(x, y, z) = 0, \quad f_z(x, y, z) = 0$$

20. The area of a parallelogram with adjacent side lengths that are a and b, and in which the angle between these two sides is θ, is given by the function $A(a, b, \theta) = ba\sin(\theta)$.Find the rate of change of the area of the parallelogram with respect to the following:
    a. Side a
    b. Side b
    c. Angle θ
21. Express the volume of a right circular cylinder as a function of two variables

a. Its radius r and its height h.
b. Show that the rate of change of the volume of the cylinder with respect to its radius is the product of its circumference multiplied by its height.
c. Show that the rate of change of the volume of the cylinder with respect to its height is equal to the area of the circular base.

Answer

a. $V(r,h)=\pi r^2 h$
b. $\dfrac{\partial V}{\partial r}=2\pi rh$
c. $\dfrac{\partial V}{\partial h}=\pi r^2$

22. Calculate $\dfrac{\partial w}{\partial z}$ for $w=z\sin(xy^2+2z)$.

Exercises 23 - 39

Find the indicated higher-order partial derivatives.

23. $f_{xy}(x,y)$ for $f(x,y)=\ln(x-y)$

Answer

$f_{xy}(x,y)=\frac{1}{(x-y)^2}$

24. $f_{yx}(x,y)$ for $f(x,y)=\ln(x-y)$
25. Let $z=x^2+3xy+2y^2$. Find $\dfrac{\partial^2 z}{\partial x^2}$ and $\dfrac{\partial^2 z}{\partial y^2}$.

Answer

$$\frac{\partial^2 z}{\partial x^2}=2,\quad \frac{\partial^2 z}{\partial y^2}=4$$

26. Given $z=e^x\tan y$, find $\dfrac{\partial^2 z}{\partial x\partial y}$ and $\dfrac{\partial^2 z}{\partial y\partial x}$.
27. Given $f(x,y,z)=xyz$, find $f_{xyy}(x,y,z)$, $f_{yxy}(x,y,z)$, and $f_{yyx}(x,y,z)$.

Answer

$$f_{xyy}(x,y,z)=f_{yxy}(x,y,z)=f_{yyx}(x,y,z)=0$$

28. Given $f(x,y,z)=e^{-2x}\sin(z^2y)$, show that $f_{xyy}(x,y,z)=f_{yxy}(x,y,z)$.
29. Show that $z=\frac{1}{2}(e^y-e^{-y})\sin x$ is a solution of the differential equation $\dfrac{\partial^2 z}{\partial x^2}+\dfrac{\partial^2 z}{\partial y^2}=0$.

Answer

$$\frac{d^2z}{dx^2}=-\tfrac{1}{2}(e^y-e^{-y})\sin x$$
$$\frac{d^2z}{dy^2}=\tfrac{1}{2}(e^y-e^{-y})\sin x$$
$$\frac{d^2z}{dx^2}+\frac{d^2z}{dy^2}=0$$

30. Find $f_{xx}(x,y)$ for $f(x,y)=\frac{4x^2}{y}+\frac{y^2}{2x}$.
31. Let $f(x,y,z)=x^2y^3z-3xy^2z^3+5x^2z-y^3z$. Find f_{xyz}.

Answer

$$f_{xyz}(x,y,z)=6y^2x-18yz^2$$

32. Let $F(x,y,z)=x^3yz^2-2x^2yz+3xz-2y^3z$. Find $F_{xyz}(x,y,z)$.
33. Given $f(x,y)=x^2+x-3xy+y^3-5$, find all points at which $f_x(x,y)=f_y(x,y)=0$ simultaneously.

Answer

$(\frac{1}{4}, \frac{1}{2}), \quad (1,1)$

34. Given $f(x,y) = 2x^2 + 2xy + y^2 + 2x - 3$, find all points at which $\dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0$ simultaneously.
35. Given $f(x,y) = y^3 - 3yx^2 - 3y^2 - 3x^2 + 1$, find all points on f at which $f_x(x,y) = f_y(x,y) = 0$ simultaneously.

Answer

$(0,0), \quad (0,2), \quad (\sqrt{3}, -1), \quad (-\sqrt{3}, -1)$

36. Given $f(x,y) = 15x^3 - 3xy + 15y^3$, find all points at which $f_x(x,y) = f_y(x,y) = 0$ simultaneously.
37. Show that $z = e^x \sin y$ satisfies the equation $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$.

Answer

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = e^x \sin y - e^x \sin y = 0$$

38. Show that $f(x,y) = \ln(x^2 + y^2)$ solves Laplace's equation $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$.
39. Show that $z = e^{-t}\cos(\frac{x}{c})$ satisfies the heat equation $\dfrac{\partial z}{\partial t} = -e^{-t}\cos(\frac{x}{c})$.

Answer

$$c^2 \frac{\partial^2 z}{\partial x^2} = e^{-t}\cos(\tfrac{x}{c})$$

40. Find $\lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x,y)}{\Delta x}$ for $f(x,y) = -7x - 2xy + 7y$.
41. Find $\lim\limits_{\Delta y \to 0} \dfrac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$ for $f(x,y) = -7x - 2xy + 7y$.

Answer

$$\frac{\partial f}{\partial y} = -2x + 7$$

42. Find $\lim\limits_{\Delta x \to 0} \dfrac{\Delta f}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$ for $f(x,y) = x^2y^2 + xy + y$.
43. Find $\lim\limits_{\Delta x \to 0} \dfrac{\Delta f}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$ for $f(x,y) = \sin(xy)$.

Answer

$$\frac{\partial f}{\partial x} = y\cos xy$$

44. The function $P(T,V) = \dfrac{nRT}{V}$ gives the pressure at a point in a gas as a function of temperature T and volume V. The letters n and R are constants. Find $\dfrac{\partial P}{\partial V}$ and $\dfrac{\partial P}{\partial T}$, and explain what these quantities represent.
45. The equation for heat flow in the xy-plane is $\dfrac{\partial f}{\partial t} = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2}$. Show that $f(x,y,t) = e^{-2t}\sin x \sin y$ is a solution.
46. The basic wave equation is $f_{tt} = f_{xx}$. Verify that $f(x,t) = \sin(x+t)$ and $f(x,t) = \sin(x-t)$ are solutions.
47. The law of cosines can be thought of as a function of three variables. Let x, y, and θ be two sides of any triangle where the angle θ is the included angle between the two sides. Then, $F(x,y,\theta) = x^2 + y^2 - 2xy\cos\theta$ gives the square of the third side of the triangle. Find $\dfrac{\partial F}{\partial \theta}$ and $\dfrac{\partial F}{\partial x}$ when $x = 2,\ y = 3$, and $\theta = \frac{\pi}{6}$.

Answer

$$\frac{\partial F}{\partial \theta} = 6, \quad \frac{\partial F}{\partial x} = 4 - 3\sqrt{3}$$

48. Suppose the sides of a rectangle are changing with respect to time. The first side is changing at a rate of 2in./sec whereas the second side is changing at the rate of 4 in/sec. How fast is the diagonal of the rectangle changing when the first side measures 16 in. and the second side measures 20 in.? (Round answer to three decimal places.)

49. A **Cobb-Douglas production function** is $f(x,y) = 200x^{0.7}y^{0.3}$, where x and y represent the amount of labor and capital available. Let $x = 500$ and $y = 1000$. Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at these values, which represent the marginal productivity of labor and capital, respectively.

Answer

$\dfrac{\partial f}{\partial x}$ at $(500, 1000) = 172.36,$ $\dfrac{\partial f}{\partial y}$ at $(500, 1000) = 36.93$

50. The apparent temperature index is a measure of how the temperature feels, and it is based on two variables: h, which is relative humidity, and t, which is the air temperature.

$A = 0.885t - 22.4h + 1.20th - 0.544$. Find $\dfrac{\partial A}{\partial t}$ and $\dfrac{\partial A}{\partial h}$ when $t = 20\,°F$ and $h = 0.90$.

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1.4: Tangent Planes and Linear Approximations

Learning Objectives

- Determine the equation of a plane tangent to a given surface at a point.
- Use the tangent plane to approximate a function of two variables at a point.
- Explain when a function of two variables is differentiable.
- Use the total differential to approximate the change in a function of two variables.

In this section, we consider the problem of finding the tangent plane to a surface, which is analogous to finding the equation of a tangent line to a curve when the curve is defined by the graph of a function of one variable, $y = f(x)$. The slope of the tangent line at the point $x = a$ is given by $m = f\,'(a)$; what is the slope of a tangent plane? We learned about the equation of a plane in Equations of Lines and Planes in Space; in this section, we see how it can be applied to the problem at hand.

Tangent Planes

Intuitively, it seems clear that, in a plane, only one line can be tangent to a curve at a point. However, in three-dimensional space, many lines can be tangent to a given point. If these lines lie in the same plane, they determine the tangent plane at that point. A more intuitive way to think of a tangent plane is to assume the surface is smooth at that point (no corners). Then, a tangent line to the surface at that point in any direction does not have any abrupt changes in slope because the direction changes smoothly. Therefore, in a small-enough neighborhood around the point, a tangent plane touches the surface at that point only.

Definition: Tangent Lines

Let $P_0 = (x_0, y_0, z_0)$ be a point on a surface S, and let C be any curve passing through P_0 and lying entirely in S. If the tangent lines to all such curves C at P_0 lie in the same plane, then this plane is called the *tangent plane* to S at P_0 (Figure 1.4.1).

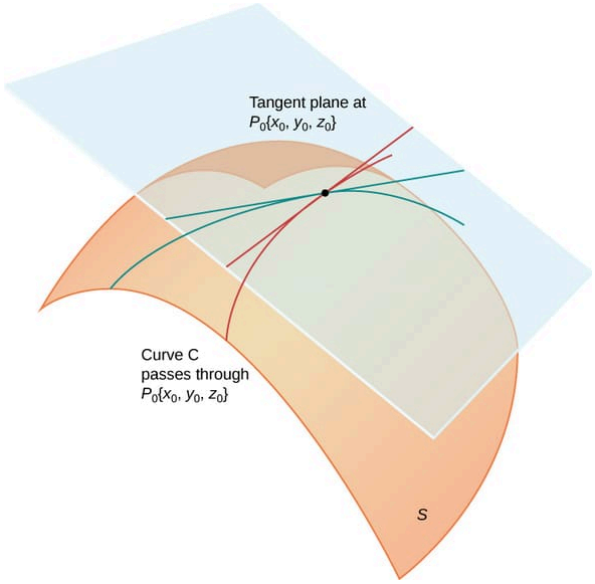


Figure 1.4.1: This image illustrates the tangent plane to a surface at a given point. The diagram shows a curved surface S with a flat plane touching it at a single point $P_0(x_0, y_0, z_0)$. Various curves on the surface pass through this point, and their tangent lines lie in the plane.

Interactive Element

Interactive Tangent Plane to a Surface

Below is an interactive tangent plane viewer. You can move the point's xy-coordinates and see the tangent plane drawn. You can also change the function. The interactive app is unfortunately not accessible.

For a tangent plane to a surface to exist at a point on that surface, it is sufficient for the function that defines the surface to be differentiable at that point. We define the term tangent plane here and then explore the idea intuitively.

Definition: Tangent Planes

Let S be a surface defined by a differentiable function $z = f(x, y)$, and let $P_0 = (x_0, y_0)$ be a point in the domain of f. Then, the equation of the tangent plane to S at P_0 is given by

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0). \tag{1.4.1}$$

To see why this formula is correct, let's first find two tangent lines to the surface S. The equation of the tangent line to the curve that is represented by the intersection of S with the vertical trace given by $x = x_0$ is $z = f(x_0, y_0) + f_y(x_0, y_0)(y - y_0)$. Similarly, the equation of the tangent line to the curve that is represented by the intersection of S with the vertical trace given by $y = y_0$ is $z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0)$. A parallel vector to the first tangent line is $\mathbf{a} = \mathbf{j} + f_y(x_0, y_0)\ \mathbf{k}$; a parallel vector to the second tangent line is $\mathbf{b} = \mathbf{i} + f_x(x_0, y_0)\ \mathbf{k}$. We can take the cross product of these two vectors:

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= (\mathbf{j} + f_y(x_0, y_0)\,\mathbf{k}) \times (\mathbf{i} + f_x(x_0, y_0)\,\mathbf{k}) \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & f_y(x_0, y_0) \\ 1 & 0 & f_x(x_0, y_0) \end{vmatrix} \\
&= f_x(x_0, y_0)\,\mathbf{i} + f_y(x_0, y_0)\,\mathbf{j} - \mathbf{k}.
\end{aligned}$$

This vector is perpendicular to both lines and is therefore perpendicular to the tangent plane. We can use this vector as a normal vector to the tangent plane, along with the point $P_0 = (x_0, y_0, f(x_0, y_0))$ in the equation for a plane:

$$\begin{aligned}
\mathbf{n} \cdot ((x - x_0)\,\mathbf{i} + (y - y_0)\,\mathbf{j} + (z - f(x_0, y_0))\,\mathbf{k}) &= 0 \\
(f_x(x_0, y_0)\,\mathbf{i} + f_y(x_0, y_0)\,\mathbf{j} - \mathbf{k}) \cdot ((x - x_0)\,\mathbf{i} + (y - y_0)\,\mathbf{j} + (z - f(x_0, y_0))\,\mathbf{k}) &= 0 \\
f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - f(x_0, y_0)) &= 0.
\end{aligned}$$

Solving this equation for z gives Equation 1.4.1.

Example 1.4.1: Finding a Tangent Plane

Find the equation of the tangent plane to the surface defined by the function $f(x, y) = 2x^2 - 3xy + 8y^2 + 2x - 4y + 4$ at point $(2, -1)$.

Solution

First, we must calculate $f_x(x, y)$ and $f_y(x, y)$, then use Equation 1.4.1 with $x_0 = 2$ and $y_0 = -1$:

$$\begin{aligned}
f_x(x, y) &= 4x - 3y + 2 \\
f_y(x, y) &= -3x + 16y - 4 \\
f(2, -1) &= 2(2)^2 - 3(2)(-1) + 8(-1)^2 + 2(2) - 4(-1) + 4 = 34 \\
f_x(2, -1) &= 4(2) - 3(-1) + 2 = 13 \\
f_y(2, -1) &= -3(2) + 16(-1) - 4 = -26.
\end{aligned}$$

Then Equation 1.4.1 becomes

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$
$$z = 34 + 13(x - 2) - 26(y - (-1))$$
$$z = 34 + 13x - 26 - 26y - 26$$
$$z = 13x - 26y - 18.$$

(See Figure 1.4.2).

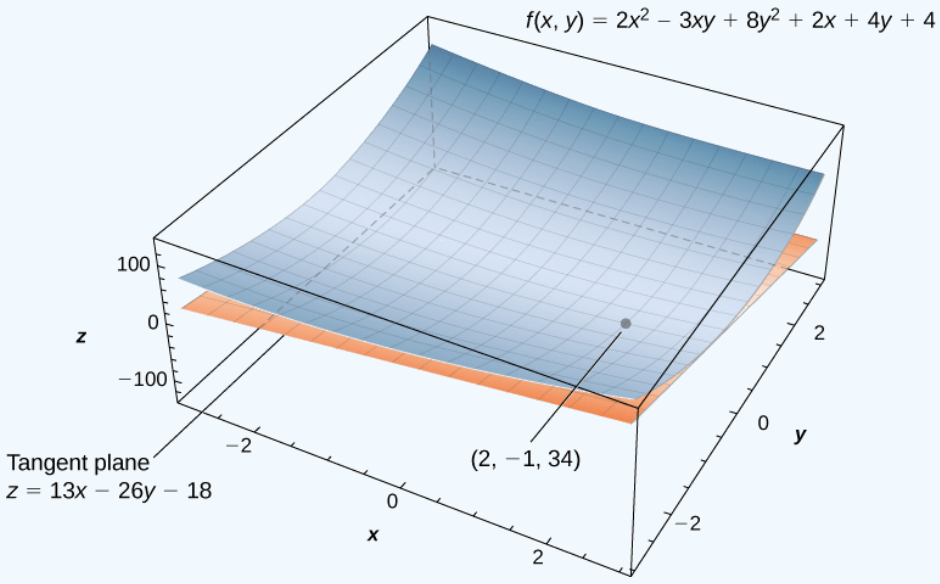


Figure 1.4.2: Tangent plane yellow below the surface $z = 13x - 26y - 18$ to the surface gray $f(x, y) = 2x^2 - 3xy + 8y^2 + 2x - 4y + 4$ at point $(2, -1)$.

Watch the accompanying video for a visual demonstration of the concept.

- **Video length**: 3 minutes 49 seconds.
- **Context**: This video demonstrates how to find the equation of the tangent plane to a surface.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Exercise 1.4.1

Find the equation of the tangent plane to the surface defined by the function $f(x, y) = x^3 - x^2y + y^2 - 2x + 3y - 2$ at point $(-1, 3)$.

Hint available at the end of this section.

Example 1.4.2: Finding Another Tangent Plane

Find the equation of the tangent plane to the surface defined by the function $f(x, y) = \sin(2x)\cos(3y)$ at the point $(\pi/3, \pi/4)$.

Solution

First, calculate $f_x(x,y)$ and $f_y(x,y)$, then use Equation 1.4.1 with $x_0 = \pi/3$ and $y_0 = \pi/4$:

$$f_x(x,y) = 2\cos(2x)\cos(3y)$$
$$f_y(x,y) = -3\sin(2x)\sin(3y)$$
$$f\left(\frac{\pi}{3},\frac{\pi}{4}\right) = \sin\left(2\left(\frac{\pi}{3}\right)\right)\cos\left(3\left(\frac{\pi}{4}\right)\right) = \left(\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{6}}{4}$$
$$f_x\left(\frac{\pi}{3},\frac{\pi}{4}\right) = 2\cos\left(2\left(\frac{\pi}{3}\right)\right)\cos\left(3\left(\frac{\pi}{4}\right)\right) = 2\left(-\frac{1}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}$$
$$f_y\left(\frac{\pi}{3},\frac{\pi}{4}\right) = -3\sin\left(2\left(\frac{\pi}{3}\right)\right)\sin\left(3\left(\frac{\pi}{4}\right)\right) = -3\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = -\frac{3\sqrt{6}}{4}.$$

Then Equation 1.4.1 becomes

$$\begin{aligned} z &= f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \\ &= -\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{2}\left(x-\frac{\pi}{3}\right) - \frac{3\sqrt{6}}{4}\left(y-\frac{\pi}{4}\right) \\ &= \frac{\sqrt{2}}{2}x - \frac{3\sqrt{6}}{4}y - \frac{\sqrt{6}}{4} - \frac{\pi\sqrt{2}}{6} + \frac{3\pi\sqrt{6}}{16} \end{aligned}$$

A tangent plane to a surface does not always exist at every point on the surface. Consider the piecewise function

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}. \tag{1.4.2}$$

See Figure 1.4.3.

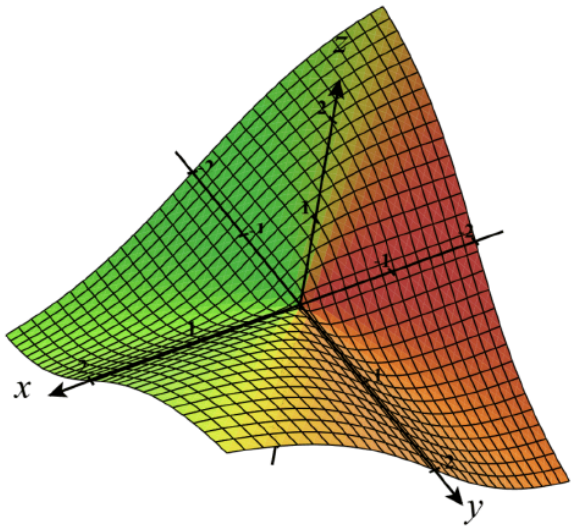


Figure 1.4.3: Graph of a function that does not have a tangent plane at the origin. It has a saddle-like shape near the center where the surface changes curvature. The shape at the origin suggest the function decreases in one direction and increases in another.

If either $x = 0$ or $y = 0$, then $f(x,y) = 0$, so the value of the function does not change on either the x- or y-axis. Therefore, $f_x(x,0) = f_y(0,y) = 0$, so as either x or y approach zero, these partial derivatives stay equal to zero. Substituting them into Equation 1.4.1 gives $z = 0$ as the equation of the tangent line. However, if we approach the origin from a different direction, we get a different story. For example, suppose we approach the origin along the line $y = x$. If we put $y = x$ into the original function, it becomes

$$f(x,x) = \frac{x(x)}{\sqrt{x^2+(x)^2}} = \frac{x^2}{\sqrt{2x^2}} = \frac{|x|}{\sqrt{2}}. \tag{1.4.3}$$

When $x > 0$, the slope of this curve is equal to $\sqrt{2}/2$; when $x < 0$, the slope of this curve is equal to $-(\sqrt{2}/2)$. This presents a problem. In the definition of tangent plane, we presumed that all tangent lines through point P (in this case, the origin) lay in the same plane. This is clearly not the case here. When we study differentiable functions, we will see that this function is not differentiable at the origin.

Linear Approximations

Recall from Linear Approximations and Differentials that the formula for the linear approximation of a function $f(x)$ at the point $x = a$ is given by

$$y \approx f(a) + f'(a)(x-a). \tag{1.4.4}$$

The diagram for the linear approximation of a function of one variable appears in Figure 1.4.4.

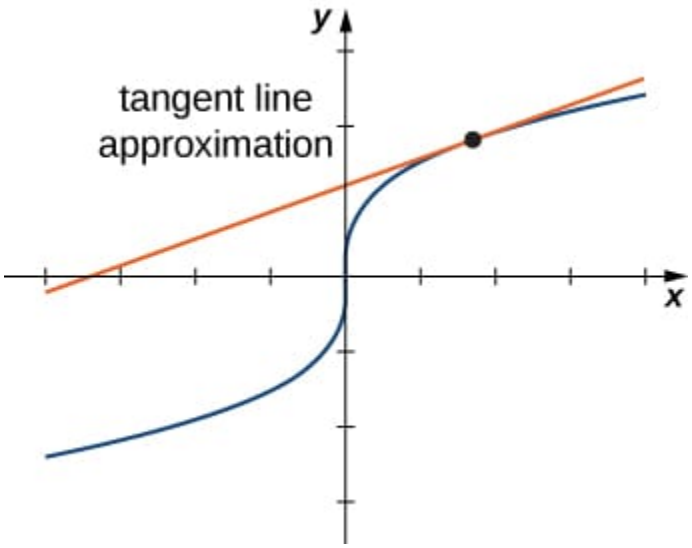


Figure 1.4.4: An illustration of a fundamental concept from calculus, where the tangent line at a specific point serves as a linear approximation to the curve's behavior near that point. A smooth curve is plotted on the xy-plane in blue color, and it has a distinct S-like shape. The orange line represents the tangent to the curve at a specific point. It is a straight line that touches the curve at exactly one point, indicating the slope (derivative) at that location.

The tangent line can be used as an approximation to the function $f(x)$ for values of x reasonably close to $x = a$. When working with a function of two variables, the tangent line is replaced by a tangent plane, but the approximation idea is much the same.

Definition: Linear Approximation

Given a function $z = f(x,y)$ with continuous partial derivatives that exist at the point (x_0, y_0), the linear approximation of f at the point (x_0, y_0) is given by the equation

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0). \tag{1.4.5}$$

Notice that this equation also represents the tangent plane to the surface defined by $z = f(x,y)$ at the point (x_0, y_0). The idea behind using a linear approximation is that, if there is a point (x_0, y_0) at which the precise value of $f(x,y)$ is known, then for values of (x,y) reasonably close to (x_0, y_0), the linear approximation (i.e., tangent plane) yields a value that is also reasonably close to the exact value of $f(x,y)$ (Figure 1.4.5). Furthermore the plane that is used to find the linear approximation is also the tangent plane to the surface at the point (x_0, y_0).

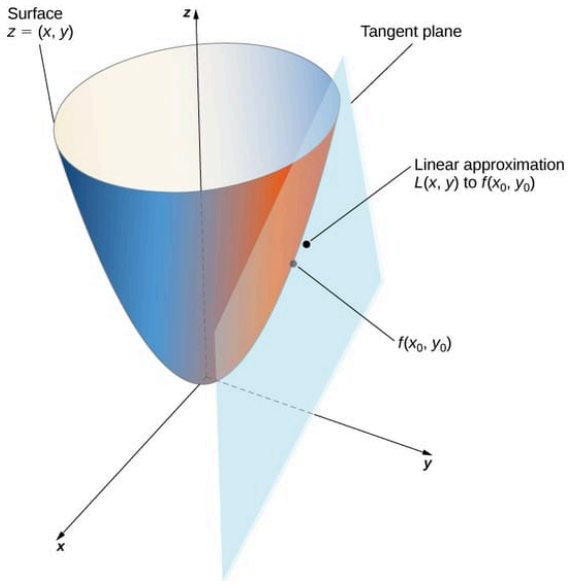


Figure 1.4.5: The surface is an upward paraboloid. There is a tangent plane intersecting the surface at the point $f(x_0, y_0)$. A linear approximation $L(x, y)$ at a point (x_0, y_0) of the surface function $z = f(x, y)$ uses the tangent plane.

Example 1.4.3: Using a Tangent Plane Approximation

Given the function $f(x, y) = \sqrt{41 - 4x^2 - y^2}$, approximate $f(2.1, 2.9)$ using point $(2, 3)$ for (x_0, y_0). What is the approximate value of $f(2.1, 2.9)$ to four decimal places?

Solution

To apply Equation 1.4.5, we first must calculate $f(x_0, y_0), f_x(x_0, y_0)$, and $f_y(x_0, y_0)$ using $x_0 = 2$ and $y_0 = 3$:

$$f(x_0, y_0) = f(2, 3) = \sqrt{41 - 4(2)^2 - (3)^2} = \sqrt{41 - 16 - 9} = \sqrt{16} = 4$$

$$f_x(x, y) = -\frac{4x}{\sqrt{41 - 4x^2 - y^2}} \text{ so } f_x(x_0, y_0) = -\frac{4(2)}{\sqrt{41 - 4(2)^2 - (3)^2}} = -2$$

$$f_y(x, y) = -\frac{y}{\sqrt{41 - 4x^2 - y^2}} \text{ so } f_y(x_0, y_0) = -\frac{3}{\sqrt{41 - 4(2)^2 - (3)^2}} = -\frac{3}{4}.$$

Now we substitute these values into Equation 1.4.5:

$$\begin{aligned} L(x, y) &= f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\ &= 4 - 2(x - 2) - \frac{3}{4}(y - 3) \\ &= \frac{41}{4} - 2x - \frac{3}{4}y. \end{aligned}$$

Last, we substitute $x = 2.1$ and $y = 2.9$ into $L(x, y)$:

$$L(2.1, 2.9) = \frac{41}{4} - 2(2.1) - \frac{3}{4}(2.9) = 10.25 - 4.2 - 2.175 = 3.875.$$

The approximate value of $f(2.1, 2.9)$ to four decimal places is

$$f(2.1, 2.9) = \sqrt{41 - 4(2.1)^2 - (2.9)^2} = \sqrt{14.95} \approx 3.8665,$$

which corresponds to a 0.2 error in approximation.

Exercise 1.4.2

Given the function $f(x,y) = e^{5-2x+3y}$, approximate $f(4.1, 0.9)$ using point $(4,1)$ for (x_0, y_0). What is the approximate value of $f(4.1, 0.9)$ to four decimal places?

Hint available at the end of this section.

Differentiability

When working with a function $y = f(x)$ of one variable, the function is said to be differentiable at a point $x = a$ if $f\,'(a)$ exists. Furthermore, if a function of one variable is differentiable at a point, the graph is "smooth" at that point (i.e., no corners exist) and a tangent line is well-defined at that point.

The idea behind differentiability of a function of two variables is connected to the idea of smoothness at that point. In this case, a surface is considered to be smooth at point P if a tangent plane to the surface exists at that point. If a function is differentiable at a point, then a tangent plane to the surface exists at that point. Recall the formula (Equation 1.4.1) for a tangent plane at a point (x_0, y_0) is given by

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

For a tangent plane to exist at the point (x_0, y_0), the partial derivatives must therefore exist at that point. However, this is not a sufficient condition for smoothness, as was illustrated in Figure. In that case, the partial derivatives existed at the origin, but the function also had a corner on the graph at the origin.

Definition: Differentiable Functions

A function $f(x,y)$ is *differentiable* at a point $P(x_0, y_0)$ if, for all points (x,y) in a δ disk around P, we can write

$$f(x,y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + E(x,y), \tag{1.4.6}$$

where the error term E satisfies

$$\lim_{(x,y)\to(x_0,y_0)} \frac{E(x,y)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0. \tag{1.4.7}$$

The last term in Equation 1.4.6 is known as the **error term** and it represents how closely the tangent plane comes to the surface in a small neighborhood (δ disk) of point P. For the function f to be differentiable at P, the function must be smooth—that is, the graph of f must be close to the tangent plane for points near P.

Example 1.4.4: Demonstrating Differentiability

Show that the function $f(x,y) = 2x^2 - 4y$ is differentiable at point $(2, -3)$.

Solution

First, we calculate $f(x_0, y_0), f_x(x_0, y_0)$, and $f_y(x_0, y_0)$ using $x_0 = 2$ and $y_0 = -3$, then we use Equation 1.4.6:

$$\begin{aligned} f(2,-3) &= 2(2)^2 - 4(-3) = 8 + 12 = 20 \\ f_x(2,-3) &= 4(2) = 8 \\ f_y(2,-3) &= -4. \end{aligned}$$

Therefore $m_1 = 8$ and $m_2 = -4$, and Equation 1.4.6 becomes

$$
\begin{aligned}
f(x,y) &= f(2,-3) + f_x(2,-3)(x-2) + f_y(2,-3)(y+3) + E(x,y)\\
2x^2 - 4y &= 20 + 8(x-2) - 4(y+3) + E(x,y)\\
2x^2 - 4y &= 20 + 8x - 16 - 4y - 12 + E(x,y)\\
2x^2 - 4y &= 8x - 4y - 8 + E(x,y)\\
E(x,y) &= 2x^2 - 8x + 8.
\end{aligned}
$$

Next, we calculate the limit in Equation 1.4.7:

$$
\begin{aligned}
\lim_{(x,y)\to(x_0,y_0)} \frac{E(x,y)}{\sqrt{(x-x+0)^2+(y-y_0)^2}} &= \lim_{(x,y)\to(2,-3)} \frac{2x^2-8x+8}{\sqrt{(x-2)^2+(y+3)^2}}\\
&= \lim_{(x,y)\to(2,-3)} \frac{2(x^2-4x+4)}{\sqrt{(x-2)^2+(y+3)^2}}\\
&= \lim_{(x,y)\to(2,-3)} \frac{2(x-2)^2}{\sqrt{(x-2)^2+(y+3)^2}}\\
&= \lim_{(x,y)\to(2,-3)} \frac{2((x-2)^2+(y+3)^2)}{\sqrt{(x-2)^2+(y+3)^2}}\\
&= \lim_{(x,y)\to(2,-3)} 2\sqrt{(x-2)^2+(y+3)^2}\\
&= 0.
\end{aligned}
$$

Since $E(x,y) \geq 0$ for any value of x or y, the original limit must be equal to zero. Therefore, $f(x,y) = 2x^2 - 4y$ is differentiable at point $(2,-3)$.

Exercise 1.4.3

Show that the function $f(x,y) = 3x - 4y^2$ is differentiable at point $(-1,2)$.

Hint available at the end of this section.

This function from (Equation 1.4.2)

$$
f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}
$$

is not differentiable at the origin (Figure 1.4.3). We can see this by calculating the partial derivatives. This function appeared earlier in the section, where we showed that $f_x(0,0) = f_y(0,0) = 0$. Substituting this information into Equations 1.4.6 and 1.4.7 using $x_0 = 0$ and $y_0 = 0$, we get

$$
\begin{aligned}
f(x,y) &= f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) + E(x,y)\\
E(x,y) &= \frac{xy}{\sqrt{x^2+y^2}}.
\end{aligned}
$$

Calculating

$$
\lim_{(x,y)\to(x_0,y_0)} \frac{E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} \tag{1.4.8}
$$

gives

$$\lim_{(x,y)\to(x_0,y_0)} \frac{E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = \lim_{(x,y)\to(0,0)} \frac{\frac{xy}{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}$$
$$= \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}.$$

Depending on the path taken toward the origin, this limit takes different values. Therefore, the limit does not exist and the function f is not differentiable at the origin as shown in Figure 1.4.6.

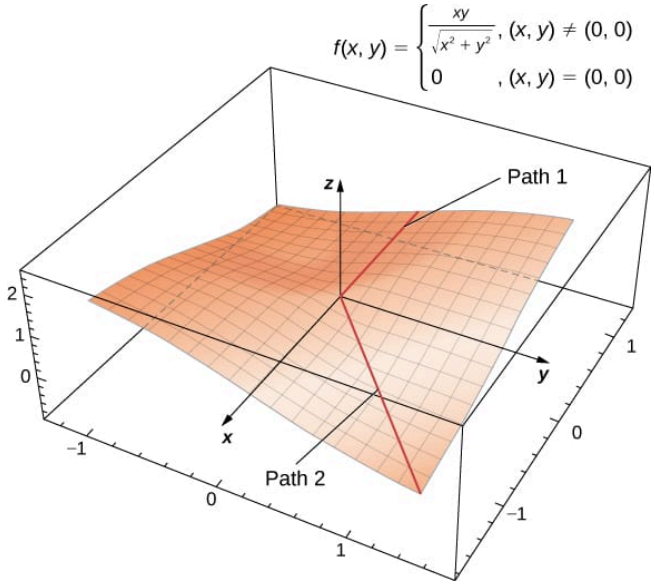


Figure 1.4.6: This function $f(x,y)$ (Equation 1.4.2) is not differentiable at the origin because its values along different paths Path1 and Path2 as they approaches the origin appear to behave differently.

Differentiability and continuity for functions of two or more variables are connected, the same as for functions of one variable. In fact, with some adjustments of notation, the basic theorem is the same.

Theorem: Differentiability Implies Continuity

Let $z = f(x,y)$ be a function of two variables with (x_0,y_0) in the domain of f. If $f(x,y)$ is differentiable at (x_0,y_0), then $f(x,y)$ is continuous at (x_0,y_0).

Note shows that if a function is differentiable at a point, then it is continuous there. However, if a function is continuous at a point, then it is not necessarily differentiable at that point. For example, the function discussed above (Equation 1.4.2)

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

is *continuous* at the origin, but it is *not differentiable* at the origin. This observation is also similar to the situation in single-variable calculus.

We can further explores the connection between continuity and differentiability at a point. This next theorem says that if the function and its partial derivatives are continuous at a point, the function is differentiable.

Theorem: Continuity of First Partials Implies Differentiability

Let $z = f(x,y)$ be a function of two variables with (x_0,y_0) in the domain of f. If $f(x,y)$, $f_x(x,y)$, and $f_y(x,y)$ all exist in a neighborhood of (x_0,y_0) and are continuous at (x_0,y_0), then $f(x,y)$ is differentiable there.

Recall that earlier we showed that the function in Equation 1.4.2 was not differentiable at the origin. Let's calculate the partial derivatives f_x and f_y:

$$\frac{\partial f}{\partial x} = \frac{y^3}{(x^2+y^2)^{3/2}} \tag{1.4.9}$$

and

$$\frac{\partial f}{\partial y} = \frac{x^3}{(x^2+y^2)^{3/2}}. \tag{1.4.10}$$

The contrapositive of the preceding theorem states that if a function is not differentiable, then at least one of the hypotheses must be false. Let's explore the condition that $f_x(0,0)$ must be continuous. For this to be true, it must be true that

$$\lim_{(x,y)\to(0,0)} f_x(x,y) = f_x(0,0) \tag{1.4.11}$$

therefore

$$\lim_{(x,y)\to(0,0)} f_x(x,y) = \lim_{(x,y)\to(0,0)} \frac{y^3}{(x^2+y^2)^{3/2}}. \tag{1.4.12}$$

Let $x = ky$. Then

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{y^3}{(x^2+y^2)^{3/2}} &= \lim_{y\to 0} \frac{y^3}{((ky)^2+y^2)^{3/2}} \\
&= \lim_{y\to 0} \frac{y^3}{(k^2y^2+y^2)^{3/2}} \\
&= \lim_{y\to 0} \frac{y^3}{|y|^3(k^2+1)^{3/2}} \\
&= \frac{1}{(k^2+1)^{3/2}} \lim_{y\to 0} \frac{|y|}{y}.
\end{aligned}$$

If $y > 0$, then this expression equals $1/(k^2+1)^{3/2}$; if $y < 0$, then it equals $-(1/(k^2+1)^{3/2})$. In either case, the value depends on k, so the limit fails to exist.

Differentials

In Linear Approximations and Differentials we first studied the concept of differentials. The differential of y, written dy, is defined as $f\,'(x)dx$. The differential is used to approximate $\Delta y = f(x+\Delta x) - f(x)$, where $\Delta x = dx$. Extending this idea to the linear approximation of a function of two variables at the point (x_0, y_0) yields the formula for the total differential for a function of two variables.

Definition: Total Differential

Let $z = f(x,y)$ be a function of two variables with (x_0, y_0) in the domain of f, and let Δx and Δy be chosen so that $(x_0 + \Delta x, y_0 + \Delta y)$ is also in the domain of f. If f is differentiable at the point (x_0, y_0), then the differentials dx and dy are defined as

$$dx = \Delta x \tag{1.4.13}$$

and

$$dy = \Delta y. \tag{1.4.14}$$

The differential dz, also called the ***total differential*** of $z = f(x,y)$ at (x_0, y_0), is defined as

$$dz = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy. \tag{1.4.15}$$

Notice that the symbol ∂ is not used to denote the total differential; rather, d appears in front of z. Now, let's define $\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$. We use dz to approximate Δz, so

$$\Delta z \approx dz = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy. \tag{1.4.16}$$

Therefore, the differential is used to approximate the change in the function $z = f(x_0, y_0)$ at the point (x_0, y_0) for given values of Δx and Δy. Since $\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$, this can be used further to approximate $f(x + \Delta x, y + \Delta y)$:

$$f(x + \Delta x, y + \Delta y) = f(x, y) + \Delta z \approx f(x, y) + fx(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y. \tag{1.4.17}$$

See Figure 1.4.7.

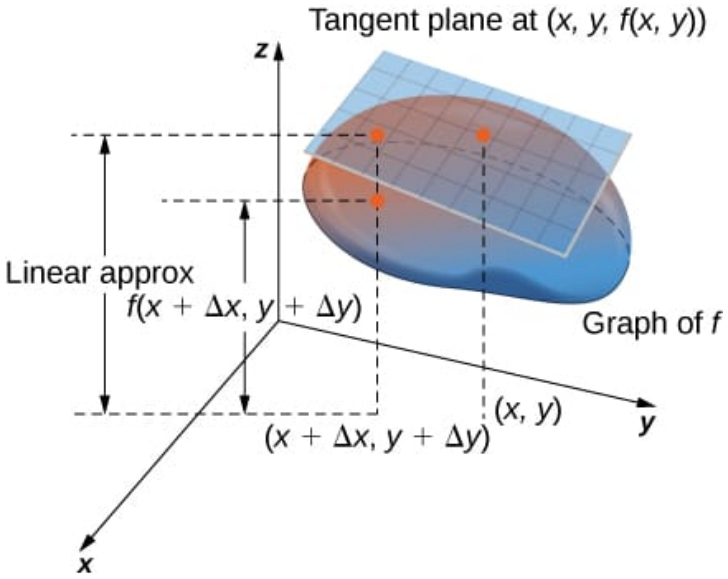


Figure 1.4.7: The linear approximation of a surface at a point is calculated via the formula $f(x + \Delta x, y + \Delta y) \approx f(x, y) + f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$. There is a tangent plane on the top of the surface.

One such application of this idea is to determine error propagation. For example, if we are manufacturing a gadget and are off by a certain amount in measuring a given quantity, the differential can be used to estimate the error in the total volume of the gadget.

Example 1.4.5: Approximation by Differentials

Find the differential dz of the function $f(x, y) = 3x^2 - 2xy + y^2$ and use it to approximate Δz at point $(2, -3)$. Use $\Delta x = 0.1$ and $\Delta y = -0.05$. What is the exact value of Δz?

Solution

First, we must calculate $f(x_0, y_0), f_x(x_0, y_0)$, and $f_y(x_0, y_0)$ using $x_0 = 2$ and $y_0 = -3$:

$$\begin{aligned} f(x_0, y_0) &= f(2, -3) = 3(2)^2 - 2(2)(-3) + (-3)^2 = 12 + 12 + 9 = 33 \\ f_x(x, y) &= 6x - 2y \end{aligned}$$

$$\begin{aligned} f_y(x, y) &= -2x + 2y \\ f_x(x_0, y_0) &= fx(2, -3) \\ &= 6(2) - 2(-3) = 12 + 6 = 18 \end{aligned}$$

$$\begin{aligned} f_y(x_0, y_0) &= f_y(2, -3) \\ &= -2(2) + 2(-3) \\ &= -4 - 6 = -10. \end{aligned}$$

Then, we substitute these quantities into Equation 1.4.15:

$$dz = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy$$
$$dz = 18(0.1) - 10(-0.05) = 1.8 + 0.5 = 2.3.$$

This is the approximation to $\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$. The exact value of Δz is given by

$$\begin{aligned}\Delta z &= f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)\\ &= f(2 + 0.1, -3 - 0.05) - f(2, -3)\\ &= f(2.1, -3.05) - f(2, -3)\\ &= 2.3425.\end{aligned}$$

Watch the accompanying video for a visual demonstration of the concept.

- **Video length**: 3 minutes 29 seconds.
- **Context**: This video demonstrates how to use differentials to approximate percent change in temperature.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Watch the accompanying video for another visual demonstration of the concept.

- **Video length**: 3 minutes 1 second.
- **Context**: This video demonstrates how to use differentials to approximate propagated error.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Exercise 1.4.4

Find the differential dz of the function $f(x, y) = 4y^2 + x^2y - 2xy$ and use it to approximate Δz at point $(1, -1)$. Use $\Delta x = 0.03$ and $\Delta y = -0.02$. What is the exact value of Δz?

Hint available at the end of this section.

Differentiability of a Function of Three Variables

All of the preceding results for differentiability of functions of two variables can be generalized to functions of three variables. First, the definition:

Definition: Differentiability at a Point

A function $f(x, y, z)$ is differentiable at a point $P(x_0, y_0, z_0)$ if for all points (x, y, z) in a δ disk around P we can write

$$f(x,y) = f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0) + E(x,y,z), \tag{1.4.18}$$

where the error term E satisfies

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} \frac{E(x,y,z)}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}} = 0. \tag{1.4.19}$$

If a function of three variables is differentiable at a point (x_0, y_0, z_0), then it is continuous there. Furthermore, continuity of first partial derivatives at that point guarantees differentiability.

Key Concepts

- The analog of a tangent line to a curve is a tangent plane to a surface for functions of two variables.
- Tangent planes can be used to approximate values of functions near known values.
- A function is differentiable at a point if it is "smooth" at that point (i.e., no corners or discontinuities exist at that point).
- The total differential can be used to approximate the change in a function $z = f(x_0, y_0)$ at the point (x_0, y_0) for given values of Δx and Δy.

Key Equations

- **Tangent plane**

  $z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

- **Linear approximation**

  $L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

- **Total differential**

  $dz = f_x(x_0,y_0)dx + f_y(x_0,y_0)dy.$

- **Differentiability (two variables)**

  $f(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + E(x,y),$

  where the error term E satisfies

  $$\lim_{(x,y)\to(x_0,y_0)} \frac{E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0.$$

- **Differentiability (three variables)**

  $f(x,y) = f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0) + E(x,y,z),$

  where the error term E satisfies

  $$\lim_{(x,y,z)\to(x_0,y_0,z_0)} \frac{E(x,y,z)}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}} = 0.$$

Glossary

Differentiable

a function $f(x,y)$ is differentiable at (x_0,y_0) if $f(x,y)$ can be expressed in the form $f(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + E(x,y)$,

where the error term $E(x,y)$ satisfies $\lim_{(x,y)\to(x_0,y_0)} \dfrac{E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0$

Linear approximation
: given a function $f(x,y)$ and a tangent plane to the function at a point (x_0,y_0), we can approximate $f(x,y)$ for points near (x_0,y_0) using the tangent plane formula

Tangent plane
: given a function $f(x,y)$ that is differentiable at a point (x_0,y_0), the equation of the tangent plane to the surface $z = f(x,y)$ is given by $z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

Total differential
: the total differential of the function $f(x,y)$ at (x_0,y_0) is given by the formula $dz = f_x(x_0,y_0)dx + fy(x_0,y_0)dy$

Contributors and Attributions

- Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

Answers

Exercise 1.4.1

First, calculate $f_x(x,y)$ and $f_y(x,y)$, then use Equation 1.4.1.

$z = 7x + 8y - 3$

Exercise 1.4.2

First calculate $f(x_0,y_0)$, $f_x(x_0,y_0)$, and $f_y(x_0,y_0)$ using $x_0 = 4$ and $y_0 = 1$, then use Equation 1.4.5.

$L(x,y) = 6 - 2x + 3y$, so $L(4.1, 0.9) = 6 - 2(4.1) + 3(0.9) = 0.5$ $f(4.1, 0.9) = e^{5-2(4.1)+3(0.9)} = e^{-0.5} \approx 0.6065$.

Exercise 1.4.3

First, calculate $f(x_0,y_0)$, $f_x(x_0,y_0)$, and $f_y(x_0,y_0)$ using $x_0 = -1$ and $y_0 = 2$, then use Equation 1.4.7 to find $E(x,y)$. Last, calculate the limit.

$$f(-1,2) = -19, \quad f_x(-1,2) = 3, \quad f_y(-1,2) = -16, \quad E(x,y) = -4(y-2)^2.$$

$$\begin{aligned}
\lim_{(x,y)\to(x_0,y_0)} \frac{E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} &= \lim_{(x,y)\to(-1,2)} \frac{-4(y-2)^2}{\sqrt{(x+1)^2+(y-2)^2}} \\
&\le \lim_{(x,y)\to(-1,2)} \frac{-4((x+1)^2+(y-2)^2)}{\sqrt{(x+1)^2+(y-2)^2}} \\
&= \lim_{(x,y)\to(-1,2)} -4\sqrt{(x+1)^2+(y-2)^2} \\
&= 0.
\end{aligned}$$

Exercise 1.4.4

First, calculate $f_x(x_0,y_0)$ and $f_y(x_0,y_0)$ using $x_0 = 1$ and $y_0 = -1$, then use Equation 1.4.15.

$dz = 0.18$

$\Delta z = f(1.03, -1.02) - f(1,-1) = 0.180682$

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1.4E: Exercises for Section 1.4

Exercises 1 - 2

Find a unit normal vector to the surface at the indicated point.

1. $f(x,y) = x^3, \quad (2,-1,8)$

   Answer

   $(\frac{\sqrt{145}}{145})(12\hat{\mathbf{i}} - \hat{\mathbf{k}})$

2. $\ln\left(\frac{x}{y-z}\right) = 0$ when $x = y = 1$

Exercises 3 - 7

Find a normal vector and a tangent vector at point P.

3. $x^2 + xy + y^2 = 3, \quad P(-1,-1)$

   Answer

   Normal vector: $\hat{\mathbf{i}} + \hat{\mathbf{j}}$, tangent vector: $\hat{\mathbf{i}} - \hat{\mathbf{j}}$

4. $(x^2+y^2)^2 = 9(x^2-y^2), \quad P(\sqrt{2},1)$
5. $xy^2 - 2x^2 + y + 5x = 6, \quad P(4,2)$

   Answer

   Normal vector: $7\hat{\mathbf{i}} - 17\hat{\mathbf{j}}$, tangent vector: $17\hat{\mathbf{i}} + 7\hat{\mathbf{j}}$

6. $2x^3 - x^2y^2 = 3x - y - 7, \quad P(1,-2)$
7. $ze^{x^2-y^2} - 3 = 0, \quad P(2,2,3)$

   Answer

   Normal vector: $-12\hat{\mathbf{i}} + 12\hat{\mathbf{j}} - \hat{\mathbf{k}}$ or $12\hat{\mathbf{i}} - 12\hat{\mathbf{j}} + \hat{\mathbf{k}}$,
   Tangent vector: $0\hat{\mathbf{i}} + 1\hat{\mathbf{j}} + 12\hat{\mathbf{k}}$ or $1\hat{\mathbf{i}} + 0\hat{\mathbf{j}} - 12\hat{\mathbf{k}}$

Exercises 8 - 19

Find the equation for the tangent plane to the surface at the indicated point. (Hint: If the given function is not already solved for z, start by solving it for z in terms of x and y.)

8. $-8x - 3y - 7z = -19, \quad P(1,-1,2)$
9. $z = -9x^2 - 3y^2, \quad P(2,1,-39)$

   Answer

   $-36x - 6y - z = -39$

10. $x^2 + 10xyz + y^2 + 8z^2 = 0, \quad P(-1,-1,-1)$
11. $z = \ln(10x^2 + 2y^2 + 1), \quad P(0,0,0)$

    Answer

    $z = 0$

12. $z = e^{7x^2+4y^2}, \quad P(0,0,1)$

13. $xy + yz + zx = 11, \quad P(1, 2, 3)$

Answer

$$5x + 4y + 3z - 22 = 0$$

14. $x^2 + 4y^2 = z^2, \quad P(3, 2, 5)$
15. $x^3 + y^3 = 3xyz, \quad P(1, 2, \frac{3}{2})$

Answer

$$4x - 5y + 4z = 0$$

16. $z = axy, \quad P(1, \frac{1}{a}, 1)$
17. $z = \sin x + \sin y + \sin(x + y), \quad P(0, 0, 0)$

Answer

$$2x + 2y - z = 0$$

18. $h(x, y) = \ln \sqrt{x^2 + y^2}, \quad P(3, 4)$
19. $z = x^2 - 2xy + y^2, \quad P(1, 2, 1)$

Answer

$$-2(x - 1) + 2(y - 2) - (z - 1) = 0$$

Exercises 20 - 25

Find parametric equations for the normal line to the surface at the indicated point. (Recall that to find the equation of a line in space, you need a point on the line, $P_0(x_0, y_0, z_0)$, and a vector $\vecs{v} = \langle a, b, c \rangle$ that is parallel to the line. Then the equations of the line are: $x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct.$)

20. $-3x + 9y + 4z = -4, \quad P(1, -1, 2)$
21. $z = 5x^2 - 2y^2, \quad P(2, 1, 18)$

Answer

$$x = 20t + 2, y = -4t + 1, z = -t + 18$$

22. $x^2 - 8xyz + y^2 + 6z^2 = 0, \quad P(1, 1, 1)$
23. $z = \ln(3x^2 + 7y^2 + 1), \quad P(0, 0, 0)$

Answer

$$x = 0, y = 0, z = t$$

24. $z = e^{4x^2 + 6y^2}, \quad P(0, 0, 1)$
25. $z = x^2 - 2xy + y^2$ at point $P(1, 2, 1)$

Answer

$$x - 1 = 2t; y - 2 = -2t; z - 1 = t$$

Exercises 26 - 28

Use Figure $1.4E.1$ **to answer each questions.**

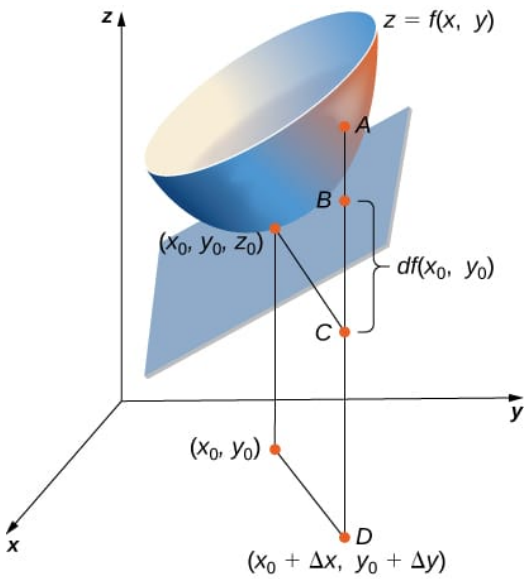


Figure $1.4E.1$: The image illustrates the concept of a 3D surface defined by a function z=f(x,y) and the relationship between a point on the surface and its projection onto the xy-plane. The blue and orange curved surface represents the graph of the function z=f(x,y). The point (x_0, y_0, z_0) lies on the surface, where $z_0 = f(x_0, y_0)$. The point (x_0, y_0) is the projection of (x_0, y_0, z_0) onto the xy-plane. The blue plane passing through (x_0, y_0, z_0) represents the tangent plane at that point. The differential $df(x_0, y_0)$, connects point B to point C. The points A, B, C, and D represent nearby locations in the vicinity of (x_0, y_0), illustrating changes in x, y, and z relative to the origin point. Point D is at $(x_0 + \Delta x, y_0 + \Delta y)$.

26. The length of line segment AC is equal to what mathematical expression?
27. The length of line segment BC is equal to what mathematical expression?

Answer

The differential of the function $z(x, y) = dz = f_x dx + f_y dy$

28. Using the figure, explain what the length of line segment AB represents.
29. Show that $f(x, y) = e^{xy}x$ is differentiable at point $(1, 0)$.

Answer

Using the definition of differentiability, we have $e^{xy}x \approx x + y$.

30. Show that $f(x, y) = x^2 + 3y$ is differentiable at every point. In other words, show that $\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y) = f_x \Delta x + f_y \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$, where both ε_1 and ε_2 approach zero as $(\Delta x, \Delta y)$ approaches $(0, 0)$.

Answer

$\Delta z = 2x\Delta x + 3\Delta y + (\Delta x)^2.(\Delta x)^2 \to 0$ for small Δx and z satisfies the definition of differentiability.

31. Find the total differential of each function:
    a. $z = x^3 + y^3 - 5$
    b. $z = e^{xy}$
    c. $z = y \cos x + \sin y$
    d. $P = t^2 + 3t + tu^3$
    e. $w = e^y \cos(x) + z^2$

32. a. Find the total differential dz of the function $z = \dfrac{xy}{y + x}$
    b. State its value where x changes from 10 to 10.5 and y changes from 15 to 13.

Answers:

a. $dz = 3x^2\ dx + 3y^2\ dy$
b. $dz = ye^{xy}\ dx + xe^{xy}\ dy$
c. $dz = -y \sin x\ dx + (\cos x + \cos y)\ dy$
d. $dP = (2t + 3 + u^3)\ dt + 3tu^2\ du$

e. $dw = -e^y \sin(x)\ dx + e^y \cos(x)\ dy + 2z\ dz$

Answer

a. $dz = \dfrac{y^2}{(x+y)^2}\ dx + \dfrac{x^2}{(x+y)^2}\ dy$

b. $dx = 0.5$ and $dy = -2$ so

$$\begin{aligned} dz &= f_x(10,15)\,dx + f_y(10,15)\,dy \\ &= \frac{15^2}{25^2}\,dx + \frac{10^2}{25^2}\,dy \\ &= \frac{225}{625}\,(0.5) + \frac{100}{625}(-2) \\ &= \frac{9}{25}\left(\frac{1}{2}\right) + \frac{4}{25}(-2) \\ &= \frac{18}{100} - \frac{32}{100} \\ &= .18 - .32 = -0.14 \end{aligned}$$

33. Let $z = f(x,y) = xe^y$. State its total differential. Then compute Δz from $P(1,2)$ to $Q(1.05, 2.1)$ and then find the approximate change in z, dz, from point P to point Q. Recall $\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$, and dz and Δz should be approximately equal, if dx and dy are both reasonably small.

Answer

Total Differential: $dz = e^y\ dx + xe^y\ dy$

$\Delta z \approx 1.185422$ and $dz \approx 1.108$. Note that they are relatively close.

34. The volume of a right circular cylinder is given by $V(r,h) = \pi r^2 h$. Find the differential dV. Interpret the formula geometrically.

Answer

$dV = 2\pi rh\ dr + \pi r^2\ dh$

35. See the preceding problem. Use differentials to estimate the amount of aluminum in an enclosed aluminum can with diameter $8.0cm$ and height $12cm$ if the aluminum is 0.04 cm thick.

Answer

$16\ \text{cm}^3$

36. Use the differential dz to approximate the change in $z = \sqrt{4 - x^2 - y^2}$ as (x,y) moves from point $(1,1)$ to point $(1.01, 0.97)$. Compare this approximation with the actual change in the function.

37. Let $z = f(x,y) = x^2 + 3xy - y^2$. Find the exact change in the function and the approximate change in the function as x changes from 2.00 to 2.05 and y changes from 3.00 to 2.96.

Answer

$\Delta z =$ exact change $= 0.6449$, approximate change is $dz = 0.65$. The two values are close.

38. The centripetal acceleration of a particle moving in a circle is given by $a(r,v) = \frac{v^2}{r}$, where v is the velocity and r is the radius of the circle. Approximate the maximum percent error in measuring the acceleration resulting from errors of 3% in v and 2% in r. (Recall that the percentage error is the ratio of the amount of error over the original amount. So, in this case, the percentage error in a is given by $\frac{da}{a}$.)

39. The radius r and height h of a right circular cylinder are measured with possible errors of 4% and 5%, respectively. Approximate the maximum possible percentage error in measuring the volume (Recall that the percentage error is the ratio of the amount of error over the original amount. So, in this case, the percentage error in V is given by $\frac{dV}{V}$.)

Answer

13% or 0.13

40. The base radius and height of a right circular cone are measured as 10 in. and 25 in., respectively, with a possible error in measurement of as much as 0.1 in. each. Use differentials to estimate the maximum error in the calculated volume of the cone.

41. The **electrical resistance** R produced by wiring resistors R_1 and R_2 in parallel can be calculated from the formula $\frac{1}{R} = \frac{1}{R_1}$

$+\frac{1}{R_2}$. If R_1 and R_2 are measured to be 7Ω and 6Ω, respectively, and if these measurements are accurate to within 0.05Ω, estimate the maximum possible error in computing R. (The symbol Ω represents an ohm, the unit of electrical resistance.)

Answer

0.025

12. The area of an ellipse with axes of length $2a$ and $2b$ is given by the formula $A = \pi ab$. Approximate the percent change in the area when a increases by 2% and b increases by 1.5%.
13. The period T of a **simple pendulum** with small oscillations is calculated from the formula $T = 2\pi\sqrt{\frac{L}{g}}$, where L is the length of the pendulum and g is the acceleration resulting from gravity. Suppose that L and g have errors of, at most, 0.5% and 0.1% , respectively. Use differentials to approximate the maximum percentage error in the calculated value of T.

Answer

0.3%

14. **Electrical power** P is given by $P = \frac{V^2}{R}$, where V is the voltage and R is the resistance. Approximate the maximum percentage error in calculating power if $120V$ is applied to a $2000 - \Omega$ resistor and the possible percent errors in measuring V and R are 3% and 4%, respectively.

Exercises 45 - 49

Find the linear approximation of each function at the indicated point.

15. $f(x, y) = x\sqrt{y}, \quad P(1, 4)$

Answer

$L(x, y) = 2x + \frac{1}{4}y - 1$

16. $f(x, y) = e^x \cos y; \quad P(0, 0)$
17. $f(x, y) = \arctan(x + 2y), \quad P(1, 0)$

Answer

$L(x, y) = \frac{1}{2}x + y + \frac{1}{4}\pi - \frac{1}{2}$

18. $f(x, y) = \sqrt{20 - x^2 - 7y^2}, \quad P(2, 1)$
19. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}, \quad P(3, 2, 6)$

Answer

$L(x, y, z) = \frac{3}{7}x + \frac{2}{7}y + \frac{6}{7}z$

50. [T] Find the equation of the tangent plane to the surface $f(x, y) = x^2 + y^2$ at point $(1, 2, 5)$, and graph the surface and the tangent plane at the point.
51. [T] Find the equation for the tangent plane to the surface at the indicated point, and graph the surface and the tangent plane: $z = \ln(10x^2 + 2y^2 + 1), \quad P(0, 0, 0).$

Answer

$z = 0$

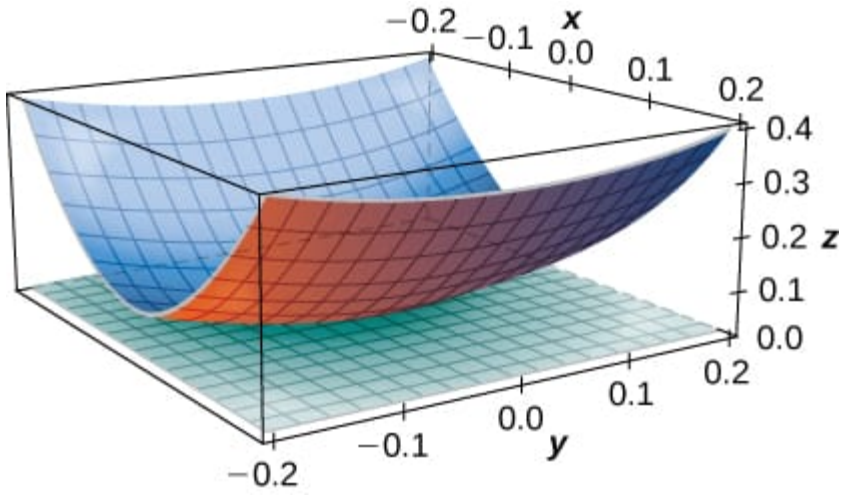


Figure $1.4E.2$: Surface representing the function $z = ln(10x^2 + 2y^2 + 1)$ and its tangent plane at the point $(0, 0)$. The surface concaves up above the tangent plane.

52. [T] Find the equation of the tangent plane to the surface $z = f(x, y) = \sin(x + y^2)$ at point $\left(\frac{\pi}{4}, 0, \frac{\sqrt{2}}{2}\right)$, and graph the surface and the tangent plane.

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- Paul Seeburger (Monroe Community College) created all but part e of exercise 31.

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1.5: The Chain Rule for Multivariable Functions

Learning Objectives

- State the chain rules for one or two independent variables.
- Use tree diagrams as an aid to understanding the chain rule for several independent and intermediate variables.
- Perform implicit differentiation of a function of two or more variables.

In single-variable calculus, we found that one of the most useful differentiation rules is the chain rule, which allows us to find the derivative of the composition of two functions. The same thing is true for multivariable calculus, but this time we have to deal with more than one form of the chain rule. In this section, we study extensions of the chain rule and learn how to take derivatives of compositions of functions of more than one variable.

Chain Rules for One or Two Independent Variables

Recall that the chain rule for the derivative of a composite of two functions can be written in the form

$$\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x). \tag{1.5.1}$$

In this equation, both $f(x)$ and $g(x)$ are functions of one variable. Now suppose that f is a function of two variables and g is a function of one variable. Or perhaps they are both functions of two variables, or even more. How would we calculate the derivative in these cases? The following theorem gives us the answer for the case of one independent variable.

Chain Rule for One Independent Variable

Suppose that $x = g(t)$ and $y = h(t)$ are differentiable functions of t and $z = f(x, y)$ is a differentiable function of x and y. Then $z = f(x(t), y(t))$ is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}, \tag{1.5.2}$$

where the ordinary derivatives are evaluated at t and the partial derivatives are evaluated at (x, y).

Proof

The proof of this theorem uses the definition of differentiability of a function of two variables. Suppose that *f* is differentiable at the point $P(x_0, y_0)$, where $x_0 = g(t_0)$ and $y_0 = h(t_0)$ for a fixed value of t_0. We wish to prove that $z = f(x(t), y(t))$ is differentiable at $t = t_0$ and that Equation 1.5.2 holds at that point as well.

Since f is differentiable at P, we know that

$$z(t) = f(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + E(x, y),$$

where

$$\lim_{(x,y)\to(x_0,y_0)} \frac{E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0.$$

We then subtract $z_0 = f(x_0, y_0)$ from both sides of this equation:

$$z(t) - z(t_0) = f(x(t), y(t)) - f(x(t_0), y(t_0))$$
$$= f_x(x_0, y_0)(x(t) - x(t_0)) + f_y(x_0, y_0)(y(t) - y(t_0)) + E(x(t), y(t)).$$

Next, we divide both sides by $t - t_0$:

$$z(t) - z(t_0)t - t_0 = fx(x_0, y_0)(x(t) - x(t_0)t - t_0) + f_y(x_0, y_0)(y(t) - y(t_0)t - t_0) + E(x(t), y(t))t - t_0.$$

Then we take the limit as t approaches t_0:

$$\lim_{t\to t_0} \frac{z(t) - z(t_0)}{t - t_0} = f_x(x_0, y_0) \lim_{t\to t_0} \left(\frac{x(t) - x(t_0)}{t - t_0} \right)$$
$$+ f_y(x_0, y_0) \lim_{t\to t_0} \left(\frac{y(t) - y(t_0)}{t - t_0} \right)$$
$$+ \lim_{t\to t_0} \frac{E(x(t), y(t))}{t - t_0}.$$

The left-hand side of this equation is equal to dz/dt, which leads to

$$\frac{dz}{dt} = f_x(x_0, y_0)\frac{dx}{dt} + f_y(x_0, y_0)\frac{dy}{dt} + \lim_{t\to t_0} \frac{E(x(t), y(t))}{t - t_0}.$$

The last term can be rewritten as

$$\lim_{t\to t_0} \frac{E(x(t), y(t))}{t - t_0} = \lim_{t\to t_0} \frac{(E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} \frac{\sqrt{(x - x_0)^2 + (y - y_0)^2}}{t - t_0})$$
$$= \lim_{t\to t_0} \left(\frac{E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} \right) \lim_{t\to t_0} \left(\frac{\sqrt{(x - x_0)^2 + (y - y_0)^2}}{t - t_0} \right).$$

As t approaches t_0, $(x(t), y(t))$ approaches $(x(t_0), y(t_0))$, so we can rewrite the last product as

$$\lim_{(x,y)\to(x_0,y_0)} \frac{(E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} \lim_{(x,y)\to(x_0,y_0)} (\frac{\sqrt{(x - x_0)^2 + (y - y_0)^2}}{t - t_0}).$$

Since the first limit is equal to zero, we need only show that the second limit is finite:

$$\lim_{(x,y)\to(x_0,y_0)} \frac{\sqrt{(x - x_0)^2 + (y - y_0)^2}}{t - t_0} = \lim_{(x,y)\to(x_0,y_0)} \sqrt{\frac{(x - x_0)^2 + (y - y_0)^2}{(t - t_0)^2}}$$
$$= \lim_{(x,y)\to(x_0,y_0)} \sqrt{\left(\frac{x - x_0}{t - t_0} \right)^2 + \left(\frac{y - y_0}{t - t_0} \right)^2}$$
$$= \sqrt{\left[\lim_{(x,y)\to(x_0,y_0)} \left(\frac{x - x_0}{t - t_0} \right) \right]^2 + \left[\lim_{(x,y)\to(x_0,y_0)} \left(\frac{y - y_0}{t - t_0} \right) \right]^2}.$$

Since $x(t)$ and $y(t)$ are both differentiable functions of t, both limits inside the last radical exist. Therefore, this value is finite. This proves the chain rule at $t = t_0$; the rest of the theorem follows from the assumption that all functions are differentiable over their entire domains.

□

Closer examination of Equation 1.5.2 reveals an interesting pattern. The first term in the equation is $\frac{\partial f}{\partial x} \cdot \frac{dx}{dt}$ and the second term is $\frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$. Recall that when multiplying fractions, cancelation can be used. If we treat these derivatives as fractions, then each product "simplifies" to something resembling $\partial f/dt$. The variables x and y that disappear in this simplification are often called **intermediate variables**: they are independent variables for the function f, but are dependent variables for the variable t. Two terms appear on the right-hand side of the formula, and f is a function of two variables. This pattern works with functions of more

than two variables as well, as we see later in this section.

Example 1.5.1: Using the Chain Rule

Calculate dz/dt for each of the following functions:

a. $z = f(x,y) = 4x^2 + 3y^2, x = x(t) = \sin t, y = y(t) = \cos t$
b. $z = f(x,y) = \sqrt{x^2 - y^2}, x = x(t) = e^{2t}, y = y(t) = e^{-t}$

Solution

a. To use the chain rule, we need four quantities—$\partial z/\partial x, \partial z/\partial y, dx/dt$, and dy/dt:

- $\dfrac{\partial z}{\partial x} = 8x$
- $\dfrac{dx}{dt} = \cos t$
- $\dfrac{\partial z}{\partial y} = 6y$
- $\dfrac{dy}{dt} = -\sin t$

Now, we substitute each of these into Equation 1.5.2:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} = (8x)(\cos t) + (6y)(-\sin t) = 8x\cos t - 6y\sin t.$$

This answer has three variables in it. To reduce it to one variable, use the fact that $x(t) = \sin t$ and $y(t) = \cos t$. We obtain

$$\frac{dz}{dt} = 8x\cos t - 6y\sin t = 8(\sin t)\cos t - 6(\cos t)\sin t = 2\sin t\cos t.$$

This derivative can also be calculated by first substituting $x(t)$ and $y(t)$ into $f(x,y)$, then differentiating with respect to t:

$$z = f(x,y) = f(x(t), y(t)) = 4(x(t))^2 + 3(y(t))^2 = 4\sin^2 t + 3\cos^2 t.$$

Then

$$\frac{dz}{dt} = 2(4\sin t)(\cos t) + 2(3\cos t)(-\sin t) = 8\sin t\cos t - 6\sin t\cos t = 2\sin t\cos t,$$

which is the same solution. However, it may not always be this easy to differentiate in this form.

b. To use the chain rule, we again need four quantities—$\partial z/\partial x, \partial z/dy, dx/dt$, and dy/dt :

- $\dfrac{\partial z}{\partial x} = \dfrac{x}{\sqrt{x^2 - y^2}}$
- $\dfrac{dx}{dt} = 2e^{2t}$
- $\dfrac{\partial z}{\partial y} = \dfrac{-y}{\sqrt{x^2 - y^2}}$
- $\dfrac{dy}{dt} = -e^{-t}.$

We substitute each of these into Equation 1.5.2:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$
$$= \left(\frac{x}{\sqrt{x^2 - y^2}}\right)(2e^{2t}) + \left(\frac{-y}{\sqrt{x^2 - y^2}}\right)(-e^{-t})$$
$$= \frac{2xe^{2t} - ye^{-t}}{\sqrt{x^2 - y^2}}.$$

To reduce this to one variable, we use the fact that $x(t) = e^{2t}$ and $y(t) = e^{-t}$. Therefore,

$$\frac{dz}{dt} = \frac{2xe^2t + ye^{-t}}{\sqrt{x^2 - y^2}}$$
$$= \frac{2(e^{2t})e^{2t} + (e^{-t})e^{-t}}{\sqrt{e^{4t} - e^{-2t}}}$$
$$= \frac{2e^{4t} + e^{-2t}}{\sqrt{e^{4t} - e^{-2t}}}.$$

To eliminate negative exponents, we multiply the top by e^{2t} and the bottom by $\sqrt{e^{4t}}$:

$$\frac{dz}{dt} = \frac{2e^{4t} + e^{-2t}}{\sqrt{e^{4t} - e^{-2t}}} \cdot \frac{e^{2t}}{\sqrt{e^{4t}}}$$
$$= \frac{2e^{6t} + 1}{\sqrt{e^{8t} - e^{2t}}}$$
$$= \frac{2e^{6t} + 1}{\sqrt{e^{2t}(e^{6t} - 1)}}$$
$$= \frac{2e^{6t} + 1}{e^t\sqrt{e^{6t} - 1}}.$$

Again, this derivative can also be calculated by first substituting $x(t)$ and $y(t)$ into $f(x, y)$, then differentiating with respect to t:

$$z = f(x, y)$$
$$= f(x(t), y(t))$$
$$= \sqrt{(x(t))^2 - (y(t))^2}$$
$$= \sqrt{e^{4t} - e^{-2t}}$$
$$= (e^{4t} - e^{-2t})^{1/2}.$$

Then

$$\frac{dz}{dt} = \frac{1}{2}(e^{4t} - e^{-2t})^{-1/2}\left(4e^{4t} + 2e^{-2t}\right)$$
$$= \frac{2e^{4t} + e^{-2t}}{\sqrt{e^{4t} - e^{-2t}}}.$$

This is the same solution.

Exercise 1.5.1

Calculate dz/dt given the following functions. Express the final answer in terms of t.

$$z = f(x, y) = x^2 - 3xy + 2y^2$$
$$x = x(t) = 3\sin 2t, y = y(t) = 4\cos 2t$$

Hint available at the end of this section.

It is often useful to create a visual representation of Equation for the chain rule. This is called a **tree diagram** for the chain rule for functions of one variable and it provides a way to remember the formula (Figure 1.5.1). This diagram can be expanded for functions of more than one variable, as we shall see very shortly.

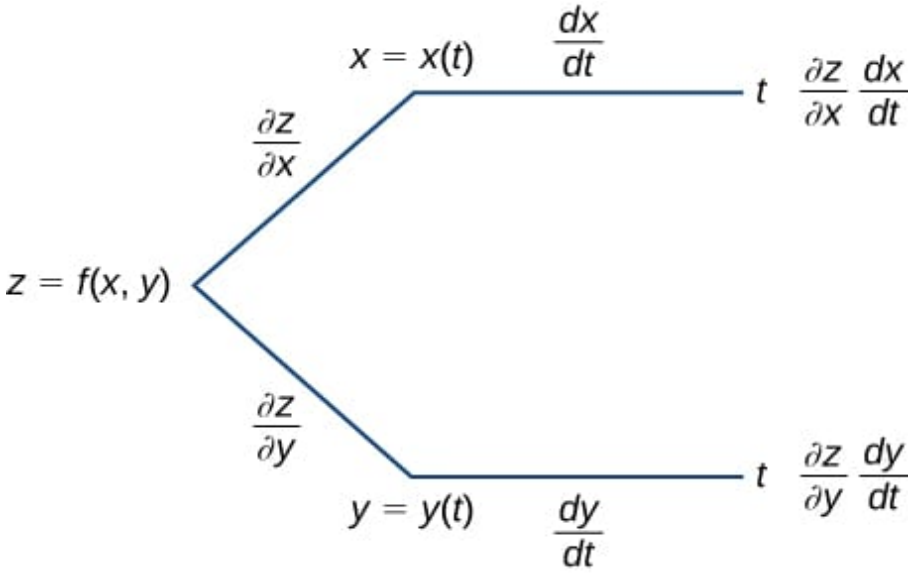


Figure 1.5.1: The top branch is $\frac{\partial z}{\partial x}$ and it has one subbranch $\frac{dx}{dt}$. The bottom branch is $\frac{\partial z}{\partial y}$ and it has one subbranch $\frac{dy}{dt}$. The result is a tree diagram for the case $\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$.

In this diagram, the leftmost corner corresponds to $z = f(x, y)$. Since f has two **independent variables**, there are two lines coming from this corner. The upper branch corresponds to the variable x and the lower branch corresponds to the variable y. Since each of these variables is then dependent on one variable t, one branch then comes from x and one branch comes from y. Last, each of the branches on the far right has a label that represents the path traveled to reach that branch. The top branch is reached by following the x branch, then the t branch; therefore, it is labeled $(\partial z/\partial x) \times (dx/dt)$. The bottom branch is similar: first the y branch, then the t branch. This branch is labeled $(\partial z/\partial y) \times (dy/dt)$. To get the formula for dz/dt, add all the terms that appear on the rightmost side of the diagram. This gives us Equation 1.5.2.

In Note, $z = f(x, y)$ is a function of x and y, and both $x = g(u, v)$ and $y = h(u, v)$ are functions of the independent variables u and v.

Chain Rule for Two Independent Variables

Suppose $x = g(u, v)$ and $y = h(u, v)$ are differentiable functions of u and v, and $z = f(x, y)$ is a differentiable function of x and y. Then, $z = f(g(u, v), h(u, v))$ is a differentiable function of u and v, and

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \tag{1.5.3}$$

and

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}. \tag{1.5.4}$$

We can draw a tree diagram for each of these formulas as well as follows.

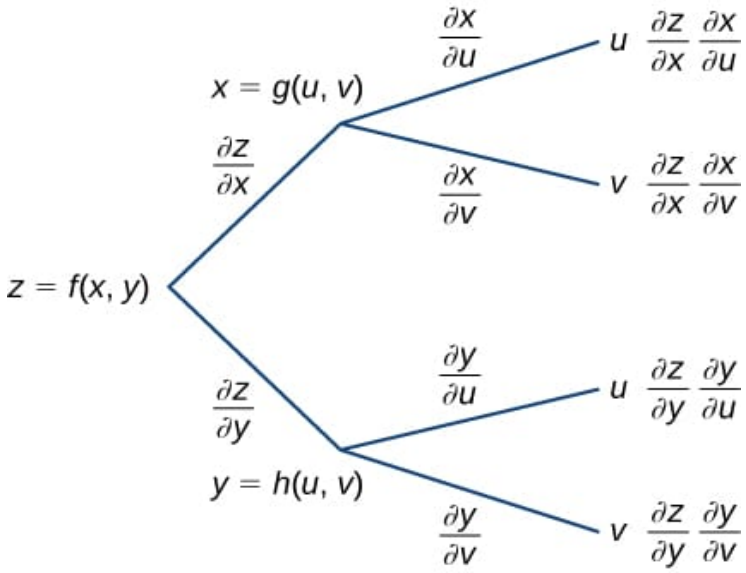


Figure 1.5.2: The top branch is $\frac{\partial z}{\partial x}$ and it has two subbranches $\frac{\partial x}{\partial u}$ and $\frac{\partial x}{\partial v}$. The bottom branch is $\frac{\partial z}{\partial y}$ and it has two subbranches $\frac{\partial y}{\partial u}$ and $\frac{\partial y}{\partial v}$. The results are the tree diagram for $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$ and $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$.

To derive the formula for $\partial z/\partial u$, start from the left side of the diagram, then follow only the branches that end with u and add the terms that appear at the end of those branches. For the formula for $\partial z/\partial v$, follow only the branches that end with v and add the terms that appear at the end of those branches.

There is an important difference between these two chain rule theorems. In Note, the left-hand side of the formula for the derivative is not a partial derivative, but in Note it is. The reason is that, in Note, z is ultimately a function of t alone, whereas in Note, z is a function of both u and v.

Example 1.5.2: Using the Chain Rule for Two Variables

Calculate $\partial z/\partial u$ and $\partial z/\partial v$ using the following functions:

$$z = f(x, y) = 3x^2 - 2xy + y^2,\ x = x(u, v) = 3u + 2v,\ y = y(u, v) = 4u - v.$$

Solution

To implement the chain rule for two variables, we need six partial derivatives—$\partial z/\partial x,\ \partial z/\partial y,\ \partial x/\partial u,\ \partial x/\partial v,\ \partial y/\partial u$, and $\partial y/\partial v$:

$$\frac{\partial z}{\partial x} = 6x - 2y \quad \frac{\partial z}{\partial y} = -2x + 2y$$
$$\frac{\partial x}{\partial u} = 3 \quad \frac{\partial x}{\partial v} = 2$$
$$\frac{\partial y}{\partial u} = 4 \quad \frac{\partial y}{\partial v} = -1.$$

To find $\partial z/\partial u$, we use Equation 1.5.3:

$$\begin{aligned}\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u} \\ &= 3(6x - 2y) + 4(-2x + 2y) \\ &= 10x + 2y.\end{aligned}$$

Next, we substitute $x(u, v) = 3u + 2v$ and $y(u, v) = 4u - v$:

$$\begin{aligned}\frac{\partial z}{\partial u} &= 10x + 2y \\ &= 10(3u + 2v) + 2(4u - v) \\ &= 38u + 18v.\end{aligned}$$

To find $\partial z/\partial v$, we use Equation 1.5.4:

$$\begin{aligned}\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \\ &= 2(6x - 2y) + (-1)(-2x + 2y) \\ &= 14x - 6y.\end{aligned}$$

Then we substitute $x(u, v) = 3u + 2v$ and $y(u, v) = 4u - v$:

$$\begin{aligned}\frac{\partial z}{\partial v} &= 14x - 6y \\ &= 14(3u + 2v) - 6(4u - v) \\ &= 18u + 34v\end{aligned}$$

Watch the accompanying video for a visual demonstration of the concept.

- **Video length**: 2 minutes 39 seconds.
- **Context**: This video demonstrates how to use the chain rule with polar coordinates.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Exercise 1.5.2

Calculate $\partial z/\partial u$ and $\partial z/\partial v$ given the following functions:

$$z = f(x, y) = \frac{2x - y}{x + 3y},\ x(u, v) = e^{2u}\cos 3v,\ y(u, v) = e^{2u}\sin 3v.$$

Hint available at the end of this section.

The Generalized Chain Rule

Now that we've see how to extend the original chain rule to functions of two variables, it is natural to ask: Can we extend the rule to more than two variables? The answer is yes, as the *generalized chain rule* states.

Generalized Chain Rule

Let $w = f(x_1, x_2, \ldots, x_m)$ be a differentiable function of m independent variables, and for each $i \in 1, \ldots, m$, let $x_i = x_i(t_1, t_2, \ldots, t_n)$ be a differentiable function of n independent variables. Then

$$\frac{\partial w}{\partial t_j} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_j} + \cdots + \frac{\partial w}{\partial x_m}\frac{\partial x_m}{\partial t_j} \tag{1.5.5}$$

for any $j \in 1, 2, \ldots, n$.

In the next example we calculate the derivative of a function of three independent variables in which each of the three variables is dependent on two other variables.

Example 1.5.3: Using the Generalized Chain Rule

Calculate $\partial w/\partial u$ and $\partial w/\partial v$ using the following functions:

$$\begin{aligned} w = f(x, y, z) &= 3x^2 - 2xy + 4z^2 \\ x = x(u, v) &= e^u \sin v \\ y = y(u, v) &= e^u \cos v \\ z = z(u, v) &= e^u. \end{aligned}$$

Solution

The formulas for $\partial w/\partial u$ and $\partial w/\partial v$ are

$$\begin{aligned} \frac{\partial w}{\partial u} &= \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial u} \\ \frac{\partial w}{\partial v} &= \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial v}. \end{aligned}$$

Therefore, there are nine different partial derivatives that need to be calculated and substituted. We need to calculate each of them:

$$\begin{aligned} \frac{\partial w}{\partial x} &= 6x - 2y \quad \frac{\partial w}{\partial y} = -2x \quad \frac{\partial w}{\partial z} = 8z \\ \frac{\partial x}{\partial u} &= e^u \sin v \quad \frac{\partial y}{\partial u} = e^u \cos v \quad \frac{\partial z}{\partial u} = e^u \\ \frac{\partial x}{\partial v} &= e^u \cos v \quad \frac{\partial y}{\partial v} = -e^u \sin v \quad \frac{\partial z}{\partial v} = 0. \end{aligned}$$

Now, we substitute each of them into the first formula to calculate $\partial w/\partial u$:

$$\begin{aligned} \frac{\partial w}{\partial u} &= \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial u} \\ &= (6x - 2y)e^u \sin v - 2xe^u \cos v + 8ze^u, \end{aligned}$$

then substitute $x(u, v) = e^u \sin v, y(u, v) = e^u \cos v$, and $z(u, v) = e^u$ into this equation:

$$\begin{aligned} \frac{\partial w}{\partial u} &= (6x - 2y)e^u \sin v - 2xe^u \cos v + 8ze^u \\ &= (6e^u \sin v - 2eu \cos v)e^u \sin v - 2(e^u \sin v)e^u \cos v + 8e^{2u} \\ &= 6e^{2u} \sin^2 v - 4e^{2u} \sin v \cos v + 8e^{2u} \\ &= 2e^{2u}(3 \sin^2 v - 2 \sin v \cos v + 4). \end{aligned}$$

Next, we calculate $\partial w/\partial v$:

$$\begin{aligned} \frac{\partial w}{\partial v} &= \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial v} \\ &= (6x - 2y)e^u \cos v - 2x(-e^u \sin v) + 8z(0), \end{aligned}$$

then we substitute $x(u, v) = e^u \sin v, y(u, v) = e^u \cos v$, and $z(u, v) = e^u$ into this equation:

$$\frac{\partial w}{\partial v} = (6x - 2y)e^u \cos v - 2x(-e^u \sin v)$$
$$= (6e^u \sin v - 2e^u \cos v)e^u \cos v + 2(e^u \sin v)(e^u \sin v)$$
$$= 2e^{2u} \sin^2 v + 6e^{2u} \sin v \cos v - 2e^{2u} \cos^2 v$$
$$= 2e^{2u}(\sin^2 v + \sin v \cos v - \cos^2 v).$$

Exercise 1.5.3

Calculate $\partial w/\partial u$ and $\partial w/\partial v$ given the following functions:

$$w = f(x, y, z) = \frac{x + 2y - 4z}{2x - y + 3z}$$
$$x = x(u, v) = e^{2u} \cos 3v$$
$$y = y(u, v) = e^{2u} \sin 3v$$
$$z = z(u, v) = e^{2u}.$$

Hint available at the end of this section.

Example 1.5.4: Drawing a Tree Diagram

Create a tree diagram for the case when

$$w = f(x, y, z), x = x(t, u, v), y = y(t, u, v), z = z(t, u, v)$$

and write out the formulas for the three partial derivatives of w.

Solution

Starting from the left, the function f has three independent variables: x, y, and z. Therefore, three branches must be emanating from the first node. Each of these three branches also has three branches, for each of the variables t, u, and v.

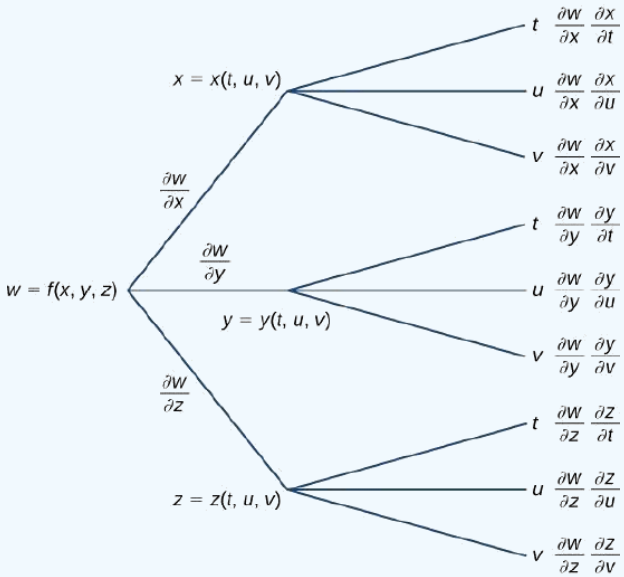


Figure 1.5.3: The top branch is $\frac{\partial w}{\partial x}$ and it has three subbranches $\frac{\partial x}{\partial t}$, $\frac{\partial x}{\partial u}$ and $\frac{\partial x}{\partial v}$. The middle branch is $\frac{\partial w}{\partial y}$ and it has three subbranches $\frac{\partial y}{\partial t}$, $\frac{\partial y}{\partial u}$ and $\frac{\partial y}{\partial v}$. The bottom branch is $\frac{\partial w}{\partial z}$ and it has three subbranches $\frac{\partial z}{\partial t}$, $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$. The results are the tree diagram for $\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$ and $\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$ and $\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$.

The three formulas are

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$
$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$
$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}.$$

Exercise 1.5.4

Create a tree diagram for the case when

$$w = f(x, y), x = x(t, u, v), y = y(t, u, v)$$

and write out the formulas for the three partial derivatives of w.

Hint available at the end of this section.

Implicit Differentiation

Recall from implicit differentiation provides a method for finding dy/dx when y is defined implicitly as a function of x. The method involves differentiating both sides of the equation defining the function with respect to x, then solving for dy/dx. Partial derivatives provide an alternative to this method.

Consider the ellipse defined by the equation $x^2 + 3y^2 + 4y - 4 = 0$ as in Figure 1.5.5.

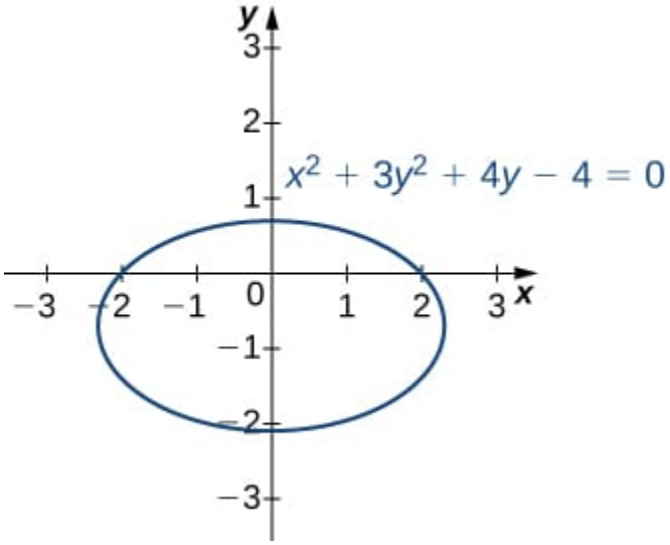


Figure 1.5.5: Graph of the ellipse defined by $x^2 + 3y^2 + 4y - 4 = 0$.

This equation implicitly defines y as a function of x. As such, we can find the derivative dy/dx using the method of implicit differentiation:

$$\frac{d}{dx}(x^2+3y^2+4y-4)=\frac{d}{dx}(0)$$
$$2x+6y\frac{dy}{dx}+4\frac{dy}{dx}=0$$
$$(6y+4)\frac{dy}{dx}=-2x$$
$$\frac{dy}{dx}=-\frac{x}{3y+2}$$

We can also define a function $z=f(x,y)$ by using the left-hand side of the equation defining the ellipse. Then $f(x,y)=x^2+3y^2+4y-4$. The ellipse $x^2+3y^2+4y-4=0$ can then be described by the equation $f(x,y)=0$. Using this function and the following theorem gives us an alternative approach to calculating dy/dx.

Theorem: Implicit Differentiation of a Function of Two or More Variables

Suppose the function $z=f(x,y)$ defines y implicitly as a function $y=g(x)$ of x via the equation $f(x,y)=0$. Then

$$\frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y} \tag{1.5.6}$$

provided $f_y(x,y)\neq 0$.

If the equation $f(x,y,z)=0$ defines z implicitly as a differentiable function of x and y, then

$$\frac{dz}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial z} \text{ and } \frac{dz}{dy}=-\frac{\partial f/\partial y}{\partial f/\partial z} \tag{1.5.7}$$

as long as $f_z(x,y,z)\neq 0$.

Equation 1.5.6 is a direct consequence of Equation 1.5.3. In particular, if we assume that y is defined implicitly as a function of x via the equation $f(x,y)=0$, we can apply the chain rule to find dy/dx :

$$\frac{d}{dx}f(x,y)=\frac{d}{dx}(0)$$
$$\frac{\partial f}{\partial x}\cdot\frac{dx}{dx}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dx}=0$$
$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dx}=0.$$

Solving this equation for dy/dx gives Equation 1.5.6. Equation 1.5.6 can be derived in a similar fashion.

Let's now return to the problem that we started before the previous theorem. Using Note and the function $f(x,y)=x^2+3y^2+4y-4$, we obtain

$$\frac{\partial f}{\partial x}=2x$$
$$\frac{\partial f}{\partial y}=6y+4.$$

Then Equation 1.5.6 gives

$$\frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y}=-\frac{2x}{6y+4}=-\frac{x}{3y+2}, \tag{1.5.8}$$

which is the same result obtained by the earlier use of implicit differentiation.

Example 1.5.5: Implicit Differentiation by Partial Derivatives

a. Calculate dy/dx if y is defined implicitly as a function of x via the equation $3x^2 - 2xy + y^2 + 4x - 6y - 11 = 0$. What is the equation of the tangent line to the graph of this curve at point $(2, 1)$?
b. Calculate $\partial z/\partial x$ and $\partial z/\partial y$, given $x^2e^y - yze^x = 0$.

Solution

a. Set $f(x, y) = 3x^2 - 2xy + y^2 + 4x - 6y - 11 = 0$, then calculate f_x and f_y : $f_x = 6x - 2y + 4$ $f_y = -2x + 2y - 6$.

The derivative is given by

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = \frac{6x - 2y + 4}{-2x + 2y - 6} = \frac{3x - y + 2}{x - y + 3}.$$

The slope of the tangent line at point $(2, 1)$ is given by

$$\frac{dy}{dx}\,|_{(x,y)=(2,1)} = \frac{3(2) - 1 + 2}{2 - 1 + 3} = \frac{7}{4}$$

To find the equation of the tangent line, we use the point-slope form:

$$\begin{aligned} y - y_0 &= m(x - x_0) \\ y - 1 &= \frac{7}{4}(x - 2) \\ y &= \frac{7}{4}x - \frac{7}{2} + 1 \\ y &= \frac{7}{4}x - \frac{5}{2}. \end{aligned}$$

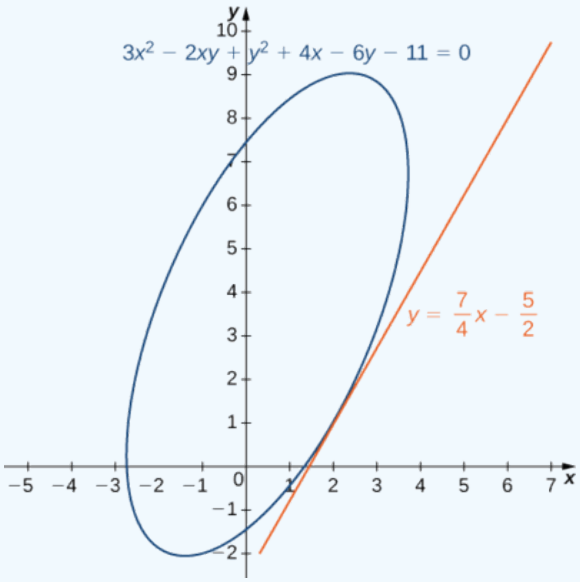


Figure 1.5.6: Graph of the rotated ellipse defined by $3x^2 - 2xy + y^2 + 4x - 6y - 11 = 0$ and its equation of tangent line $y = \frac{7}{4}x - \frac{5}{2}$ at the point $(2, 1)$.

b. We have $f(x, y, z) = x^2e^y - yze^x$. Therefore,

$$\frac{\partial f}{\partial x} = 2xe^y - yze^x$$

$$\frac{\partial f}{\partial y} = x^2e^y - ze^x$$

$$\frac{\partial f}{\partial z} = -ye^x$$

Using Equation 1.5.7,

$$\frac{\partial z}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial y}\frac{\partial z}{\partial y} = -\frac{\partial f/\partial y}{\partial f/\partial z}$$
$$= -\frac{2xe^y - yze^x}{-ye^x}$$
$$= \frac{2xe^y - yze^x}{ye^x}$$

Similarly,

$$\frac{\partial z}{\partial y} = -\frac{x^2e^y - ze^x}{-ye^x}$$
$$= \frac{x^2e^y - ze^x}{ye^x}$$

.

Watch the accompanying video for a visual demonstration of the concept.

- **Video length**: 2 minutes 37 seconds.
- **Context**: This video demonstrates how to evaluate implicit differentiation of a function having four variables.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Exercise 1.5.5

Find dy/dx if y is defined implicitly as a function of x by the equation $x^2 + xy - y^2 + 7x - 3y - 26 = 0$. What is the equation of the tangent line to the graph of this curve at point $(3, -2)$?

Hint available at the end of this section.

Key Concepts

- The chain rule for functions of more than one variable involves the partial derivatives with respect to all the independent variables.
- Tree diagrams are useful for deriving formulas for the chain rule for functions of more than one variable, where each independent variable also depends on other variables.

Key Equations

- **Chain rule, one independent variable**

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

- **Chain rule, two independent variables**

$$\frac{dz}{du} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u} \frac{dz}{dv} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

- **Generalized chain rule**

$$\frac{\partial w}{\partial t_j} = \frac{\partial w}{\partial x_1} \frac{\partial x_1}{\partial t_j} + \frac{\partial w}{\partial x_2} \frac{\partial x_1}{\partial t_j} + \cdots + \frac{\partial w}{\partial x_m} \frac{\partial x_m}{\partial t_j}$$

Glossary

Generalized chain rule
: the chain rule extended to functions of more than one independent variable, in which each independent variable may depend on one or more other variables

Intermediate variable
: given a composition of functions (e.g., $f(x(t), y(t))$), the intermediate variables are the variables that are independent in the outer function but dependent on other variables as well; in the function $f(x(t), y(t))$, the variables x and y are examples of intermediate variables

Tree diagram
: illustrates and derives formulas for the generalized chain rule, in which each independent variable is accounted for

Contributors and Attributions

- Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

Answers

Exercise 1.5.1

Calculate $\partial z/\partial x, \partial z/dy, dx/dt$, and dy/dt, then use Equation 1.5.2.

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\ &= (2x - 3y)(6\cos 2t) + (-3x + 4y)(-8\sin 2t) \\ &= -92\sin 2t\cos 2t - 72(\cos^2 2t - \sin^2 2t) \\ &= -46\sin 4t - 72\cos 4t.\end{aligned}$$

Exercise 1.5.2

Calculate $\partial z/\partial x,\ \partial z/\partial y,\ \partial x/\partial u,\ \partial x/\partial v,\ \partial y/\partial u$, and $\partial y/\partial v$, then use Equation 1.5.3 and Equation 1.5.4.

$$\frac{\partial z}{\partial u} = 0, \frac{\partial z}{\partial v} = \frac{-21}{(3\sin 3v + \cos 3v)^2}$$

Exercise 1.5.3

Calculate nine partial derivatives, then use the same formulas from Example 1.5.3.

$$\frac{\partial w}{\partial u} = 0$$
$$\frac{\partial w}{\partial v} = \frac{15 - 33\sin 3v + 6\cos 3v}{(3 + 2\cos 3v - \sin 3v)^2}$$

Exercise 1.5.4

Determine the number of branches that emanate from each node in the tree.

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}$$
$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u}$$
$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v}$$

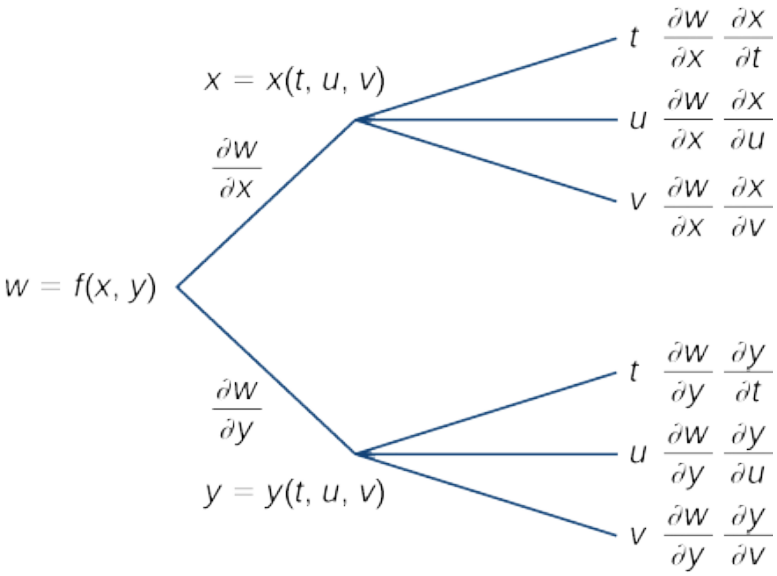


Figure 1.5.4: The top branch is $\frac{\partial w}{\partial x}$ and it has three subbranches $\frac{\partial x}{\partial t}$, $\frac{\partial x}{\partial u}$ and $\frac{\partial x}{\partial v}$. The bottom branch is $\frac{\partial w}{\partial y}$ and it has three subbranches $\frac{\partial y}{\partial t}$, $\frac{\partial y}{\partial u}$ and $\frac{\partial y}{\partial v}$.

Exercise 1.5.5

Calculate $\partial f/\partial x$ and $\partial f/\partial y$, then use Equation 1.5.6.

$$\frac{dy}{dx} = \frac{2x+y+7}{2y-x+3}\Big|_{(3,-2)} = \frac{2(3)+(-2)+7}{2(-2)-(3)+3} = -\frac{11}{4}$$

Equation of the tangent line: $y = -\frac{11}{4}x + \frac{25}{4}$

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1.5E: Exercises for Section 1.5

Exercises 1 - 6

Use the information provided to solve the problem.

1. Let $w(x,y,z) = xy\cos z$, where $x = t, y = t^2$, and $z = \arcsin t$. Find $\dfrac{dw}{dt}$.

   Answer

   $$\frac{dw}{dt} = y\cos z + x\cos z(2t) - \frac{xy\sin z}{\sqrt{1-t^2}}$$

2. Let $w(t,v) = e^{tv}$ where $t = r + s$ and $v = rs$. Find $\dfrac{\partial w}{\partial r}$ and $\dfrac{\partial w}{\partial s}$.
3. If $w = 5x^2 + 2y^2, \quad x = -3u + v$, and $y = u - 4v$, find $\dfrac{\partial w}{\partial u}$ and $\dfrac{\partial w}{\partial v}$.

   Answer

   $$\frac{\partial w}{\partial u} = -30x + 4y \quad = \quad -30(-3u+v) + 4(u-4v) \quad = \quad 90u - 30v + 4u - 16v \quad = \quad 94u - 46v,$$
   $$\frac{\partial w}{\partial v} = 10x - 16y \quad = \quad 10(-3u+v) - 16(u-4v) \quad = \quad -30u + 10v - 16u + 64v \quad = \quad -46u + 74v$$

4. If $w = xy^2, x = 5\cos(2t)$, and $y = 5\sin(2t)$, find $\dfrac{\partial w}{\partial t}$.
5. If $f(x,y) = xy, x = r\cos\theta$, and $y = r\sin\theta$, find $\dfrac{\partial f}{\partial r}$ and express the answer in terms of r and θ.

   Answer

   $$\frac{\partial f}{\partial r} = r\sin(2\theta)$$

6. Suppose $f(x,y) = x + y, u = e^x \sin y, \quad x = t^2$ and $y = \pi t$, where $x = r\cos\theta$ and $y = r\sin\theta$. Find $\dfrac{\partial f}{\partial \theta}$.

Exercises 7 - 12

Find $\dfrac{dz}{dt}$ in two ways, first using the chain rule and then by direct substitution.

7. $z = x^2 + y^2, \quad x = t, y = t^2$

   Answer

   $$\frac{dz}{dt} = 2t + 4t^3$$

8. $z = \sqrt{x^2 + y^2}, \quad y = t^2, x = t$
9. $z = xy, \quad x = 1 - \sqrt{t}, y = 1 + \sqrt{t}$

   Answer

   $$\frac{dz}{dt} = -1$$

10. $z = \frac{x}{y}, \quad x = e^t, y = 2e^t$
11. $z = \ln(x + y), \quad x = e^t, y = e^t$

    Answer

    $$\frac{dz}{dt} = 1$$

12. $z = x^4, \quad x = t, y = t$
13. Let $w(x, y, z) = x^2 + y^2 + z^2, \quad x = cost, y = sint$, and $z = e^t$. Express w as a function of t and find $\frac{dw}{dt}$ directly. Then, find $\frac{dw}{dt}$ using the chain rule.

Answer

$$\frac{dw}{dt} = 2e^{2t} \text{ in both cases}$$

14. Let $z = x^2y$, where $x = t^2$ and $y = t^3$. Find $\frac{dz}{dt}$.
15. Let $u = e^x \sin y$, where $x = -\ln 2t$ and $y = \pi t$. Find $\frac{du}{dt}$ when $x = \ln 2$ and $y = \frac{\pi}{4}$.

Answer

$$\frac{du}{dt} = \sqrt{2}\left(\pi - 4\right)$$

Exercises 16 - 33

Find $\frac{dy}{dx}$ using partial derivatives.

16. $\sin(6x) + \tan(8y) + 5 = 0$
17. $x^3 + y^2x - 3 = 0$

Answer

$$\frac{dy}{dx} = -\frac{3x^2 + y^2}{2xy}$$

18. $\sin(x + y) + \cos(x - y) = 4$
19. $x^2 - 2xy + y^4 = 4$

Answer

$$\frac{dy}{dx} = \frac{y - x}{-x + 2y^3}$$

20. $xe^y + ye^x - 2x^2y = 0$
21. $x^{2/3} + y^{2/3} = a^{2/3}$

Answer

$$\frac{dy}{dx} = -\sqrt[3]{\frac{y}{x}}$$

22. $x\cos(xy) + y\cos x = 2$
23. $e^{xy} + ye^y = 1$

Answer

$$\frac{dy}{dx} = -\frac{ye^{xy}}{xe^{xy} + e^y(1 + y)}$$

24. $x^2y^3 + \cos y = 0$
25. Find $\frac{dz}{dt}$ using the chain rule where $z = 3x^2y^3, \quad x = t^4$, and $y = t^2$.

Answer

$$\frac{dz}{dt} = 42t^{13}$$

26. Let $z = 3\cos x - \sin(xy), x = \frac{1}{t}$, and $y = 3t$. Find $\frac{dz}{dt}$.
27. Let $z = e^{1-xy}, \quad x = t^{1/3}$, and $y = t^3$. Find $\frac{dz}{dt}$.

Answer

$$\frac{dz}{dt} = -\frac{10}{3}t^{7/3} \times e^{1-t^{10/3}}$$

28. Find $\frac{dz}{dt}$ by the chain rule where $z = \cosh^2(xy), \quad x = \frac{1}{2}t$, and $y = e^t$.
29. Let $z = \frac{x}{y}, \quad x = 2\cos u$, and $y = 3\sin v$. Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.

Answer

$$\frac{\partial z}{\partial u} = \frac{-2\sin u}{3\sin v} \text{ and } \frac{\partial z}{\partial v} = \frac{-2\cos u\cos v}{3\sin^2 v}$$

30. Let $z = e^{x^2y}$, where $x = \sqrt{uv}$ and $y = \frac{1}{v}$. Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.
31. If $z = xye^{x/y}, \quad x = r\cos\theta$, and $y = r\sin\theta$, find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$ when $r = 2$ and $\theta = \frac{\pi}{6}$.

Answer

$$\frac{\partial z}{\partial r} = \sqrt{3}e^{\sqrt{3}}, \frac{\partial z}{\partial \theta} = (2 - 4\sqrt{3})e^{\sqrt{3}}$$

32. Find $\frac{\partial w}{\partial s}$ if $w = 4x + y^2 + z^3, \quad x = e^{rs^2}, \quad y = \ln(\frac{r+s}{t})$, and $z = rst^2$.
33. If $w = \sin(xyz), \quad x = 1 - 3t, \quad y = e^{1-t}$, and $z = 4t$, find $\frac{\partial w}{\partial t}$.

Answer

$$\frac{\partial w}{\partial t} = -3yz\cos(xyz) - xze^{1-t}\cos(xyz) + 4xy\cos(xyz)$$

Exercises 34 - 36

Use this information: A function $f(x, y)$ is said to be homogeneous of degree n if $f(tx, ty) = t^n f(x, y)$. For all homogeneous functions of degree n, the following equation is true: $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y)$. Show that the given function is homogeneous and verify that $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y)$.

34. $f(x, y) = 3x^2 + y^2$
35. $f(x, y) = \sqrt{x^2 + y^2}$

Answer

$$f(tx, ty) = \sqrt{t^2x^2 + t^2y^2} = t^1 f(x, y), \qquad \frac{\partial f}{\partial y} = x\frac{1}{2}(x^2 + y^2)^{-1/2} \times 2x + y\frac{1}{2}(x^2 + y^2)^{-1/2} \times 2y = 1f(x, y)$$

36. $f(x, y) = x^2y - 2y^3$
37. The volume of a right circular cylinder is given by $V(x, y) = \pi x^2 y$, where x is the radius of the cylinder and y is the cylinder height. Suppose x and y are functions of t given by $x = \frac{1}{2}t$ and $y = \frac{1}{3}t$ so that x and y are both increasing with time. How fast is the volume increasing when $x = 2$ and $y = 5$? Assume time is measured in seconds.

Answer

$$\ddfrac{dV}{dt} = \frac{34\pi}{3} \text{ units}^3/\text{s}$$

38. The pressure P of a gas is related to the volume and temperature by the formula $PV = kT$, where temperature is expressed in kelvins. Express the pressure of the gas as a function of both V and T. Find $\frac{dP}{dt}$ when $k = 1, \frac{dV}{dt} = 2$ cm^3/min, $\frac{dT}{dt} = 12$ K/min, $V = 20cm^3$, and $T = 20\,°F$.
39. The radius of a right circular cone is increasing at 3 cm/min whereas the height of the cone is decreasing at 2 cm/min. Find the rate of change of the volume of the cone when the radius is 13 cm and the height is 18 cm.

Answer

$$\frac{dV}{dt} = \frac{1066\pi}{3} \text{ cm}^3/\text{min}$$

10. The volume of a frustum of a cone is given by the formula $V = \frac{1}{3}\pi z(x^2 + y^2 + xy)$, where x is the radius of the smaller circle, y is the radius of the larger circle, and z is the height of the frustum (see Figure $1.5E.1$). Find the rate of change of the volume of this frustum when $x = 10$ in., $y = 12$ in., and $z = 18$ in.

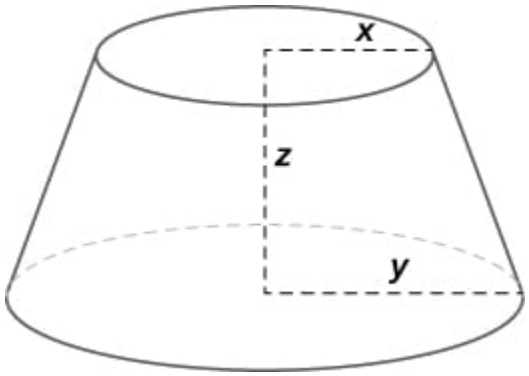


Figure $1.5E.1$: The graph illustrates the volume of a frustum of a cone is given by the formula $V = \frac{1}{3}\pi z(x^2 + y^2 + xy)$, where x is the radius of the smaller circle, y is the radius of the larger circle, and z is the height of the frustum

11. A closed box is in the shape of a rectangular solid with dimensions x, y, and z. (Dimensions are in inches.) Suppose each dimension is changing at the rate of 0.5 in./min. Find the rate of change of the total surface area of the box when $x = 2$ in., $y = 3$ in., and $z = 1$ in.

Answer

$$\frac{dA}{dt} = 12 \text{ in.}^2/\text{min}$$

12. The total resistance in a circuit that has three individual resistances represented by x, y, and z is given by the formula $R(x, y, z) = \dfrac{xyz}{yz + xz + xy}$. Suppose at a given time the x resistance is $100\ \Omega$, the y resistance is $200\ \Omega$, and the z resistance is $300\ \Omega$. Also, suppose the x resistance is changing at a rate of $2\ \Omega/\text{min}$, the y resistance is changing at the rate of $1\ \Omega/\text{min}$, and the z resistance has no change. Find the rate of change of the total resistance in this circuit at this time.
13. The temperature T at a point (x, y) is $T(x, y)$ and is measured using the Celsius scale. A fly crawls so that its position after t seconds is given by $x = \sqrt{1 + t}$ and $y = 2 + \frac{1}{3}t$, where x and y are measured in centimeters. The temperature function satisfies $T_x(2, 3) = 4$ and $T_y(2, 3) = 3$. How fast is the temperature increasing on the fly's path after 3 sec?

Answer

2°C/sec

14. The x and y components of a fluid moving in two dimensions are given by the following functions: $u(x, y) = 2y$ and $v(x, y) = -2x$ with $x \geq 0$ and $y \geq 0$. The speed of the fluid at the point (x, y) is $s(x, y) = \sqrt{u(x, y)^2 + v(x, y)^2}$. Find $\dfrac{\partial s}{\partial x}$ and $\dfrac{\partial s}{\partial y}$ using the chain rule.
15. Let $u = u(x, y, z)$, where $x = x(w, t),\ y = y(w, t),\ z = z(w, t),\ w = w(r, s)$, and $t = t(r, s)$. Use a tree diagram and the chain rule to find an expression for $\dfrac{\partial u}{\partial r}$.

Answer

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\left(\frac{\partial x}{\partial w}\frac{\partial w}{\partial r} + \frac{\partial x}{\partial t}\frac{\partial t}{\partial r}\right) + \frac{\partial u}{\partial y}\left(\frac{\partial y}{\partial w}\frac{\partial w}{\partial r} + \frac{\partial y}{\partial t}\frac{\partial t}{\partial r}\right) + \frac{\partial u}{\partial z}\left(\frac{\partial z}{\partial w}\frac{\partial w}{\partial r} + \frac{\partial z}{\partial t}\frac{\partial t}{\partial r}\right)$$

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1.6: Directional Derivatives and the Gradient

Learning Objectives

- Determine the directional derivative in a given direction for a function of two variables.
- Determine the gradient vector of a given real-valued function.
- Explain the significance of the gradient vector with regard to direction of change along a surface.
- Use the gradient to find the tangent to a level curve of a given function.
- Calculate directional derivatives and gradients in three dimensions.

A function $z = f(x, y)$ has two partial derivatives: $\partial z/\partial x$ and $\partial z/\partial y$. These derivatives correspond to each of the independent variables and can be interpreted as instantaneous rates of change (that is, as slopes of a tangent line). For example, $\partial z/\partial x$ represents the slope of a tangent line passing through a given point on the surface defined by $z = f(x, y)$, assuming the tangent line is parallel to the x**-axis**. Similarly, $\partial z/\partial y$ represents the slope of the tangent line parallel to the y**-axis.** Now we consider the possibility of a tangent line parallel to neither axis.

Directional Derivatives

We start with the graph of a surface defined by the equation $z = f(x, y)$. Given a point (a, b) in the domain of f, we choose a direction to travel from that point. We measure the direction using an angle θ, which is measured counterclockwise in the xy -plane, starting at zero from the positive x-axis (Figure 1.6.1). The distance we travel is h and the direction we travel is given by the unit vector $\mathbf{u} = (\cos\theta)\ \mathbf{i} + (\sin\theta)\ \mathbf{j}$. Therefore, the z-coordinate of the second point on the graph is given by $z = f(a + h\cos\theta, b + h\sin\theta)$.

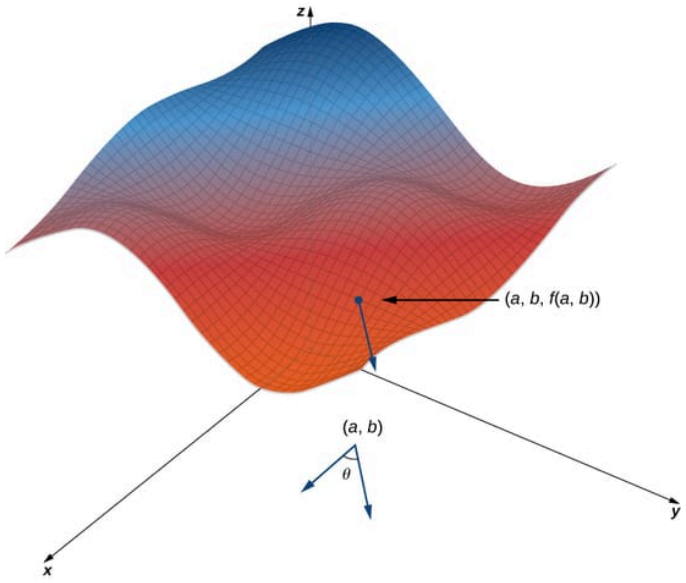


Figure 1.6.1: The image illustrates the graph of $z = f(x, y)$. The slope of the blue arrow on the graph indicates the value of the directional derivative at the point $a, b, f(a, b)$. On the xy-plane, the point (a, b) is marked and an angle θ is made between the projection of the directional derivative onto the plane and a line parallel to the x-axis.

We can calculate the slope of the secant line by dividing the difference in z-values by the length of the line segment connecting the two points in the domain. The length of the line segment is h. Therefore, the slope of the secant line is

$$m_{sec} = \frac{f(a + h\cos\theta, b + h\sin\theta) - f(a, b)}{h} \tag{1.6.1}$$

To find the slope of the tangent line in the same direction, we take the limit as h approaches zero.

Definition: Directional Derivatives

Suppose $z = f(x, y)$ is a function of two variables with a domain of D. Let $(a, b) \in D$ and define $\mathbf{u} = (\cos\theta)\ \mathbf{i} + (\sin\theta)\ \mathbf{j}$. Then the directional derivative of f in the direction of $\mathbf{u}$ is given by

$$D_{\mathbf{u}} f(a, b) = \lim_{h \to 0} \frac{f(a + h\cos\theta, b + h\sin\theta) - f(a, b)}{h} \tag{1.6.2}$$

provided the limit exists.

Equation 1.6.2 provides a formal definition of the directional derivative that can be used in many cases to calculate a directional derivative.

Note that since the point (a, b) is chosen randomly from the domain D of the function f, we can use this definition to find the directional derivative as a function of x and y.

That is,

$$D_{\mathbf{u}} f(x, y) = \lim_{h \to 0} \frac{f(x + h\cos\theta, y + h\sin\theta) - f(x, y)}{h} \tag{1.6.3}$$

Example 1.6.1: Finding a Directional Derivative from the Definition

Let $\theta = \arccos(\frac{3}{5})$. Find the directional derivative $D_{\mathbf{u}} f(x, y)$ of $f(x, y) = x^2 - xy + 3y^2$ in the direction of $\mathbf{u} = (\cos\theta)\ \mathbf{i} + (\sin\theta)\ \mathbf{j}$.

Then determine $D_{\mathbf{u}} f(-1, 2)$.

Solution

First of all, since $\cos\theta = \dfrac{3}{5}$ and θ is acute, this implies

$$\sin\theta = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \sqrt{\frac{16}{25}} = \frac{4}{5}.$$

Using $f(x, y) = x^2 - xy + 3y^2$, we first calculate $f(x + h\cos\theta, y + h\sin\theta)$:

$$\begin{aligned} f(x + h\cos\theta, y + h\sin\theta) &= (x + h\cos\theta)^2 - (x + h\cos\theta)(y + h\sin\theta) + 3(y + h\sin\theta)^2 \\ &= x^2 + 2xh\cos\theta + h^2\cos^2\theta - xy - xh\sin\theta - yh\cos\theta - h^2\sin\theta\cos\theta + 3y^2 + 6yh\sin\theta + 3h^2\sin^2\theta \\ &= x^2 + 2xh(\frac{3}{5}) + \frac{9h^2}{25} - xy - \frac{4xh}{5} - \frac{3yh}{5} - \frac{12h^2}{25} + 3y^2 + 6yh(\frac{4}{5}) + 3h^2(\frac{16}{25}) \\ &= x^2 - xy + 3y^2 + \frac{2xh}{5} + \frac{9h^2}{5} + \frac{21yh}{5}. \end{aligned}$$

We substitute this expression into Equation 1.6.2 with $a = x$ and $b = y$:

$$\begin{aligned} D_{\mathbf{u}} f(x, y) &= \lim_{h \to 0} \frac{f(x + h\cos\theta, y + h\sin\theta) - f(x, y)}{h} \\ &= \lim_{h \to 0} \frac{(x^2 - xy + 3y^2 + \frac{2xh}{5} + \frac{9h^2}{5} + \frac{21yh}{5}) - (x^2 - xy + 3y^2)}{h} \\ &= \lim_{h \to 0} \frac{\frac{2xh}{5} + \frac{9h^2}{5} + \frac{21yh}{5}}{h} \\ &= \lim_{h \to 0} \frac{2x}{5} + \frac{9h}{5} + \frac{21y}{5} \\ &= \frac{2x + 21y}{5}. \end{aligned}$$

To calculate $D_{\mathbf{u}}f(-1,2)$, we substitute $x=-1$ and $y=2$ into this answer (Figure 1.6.2):

$$D_{\mathbf{u}}f(-1,2)=\frac{2(-1)+21(2)}{5}=\frac{-2+42}{5}=8.$$

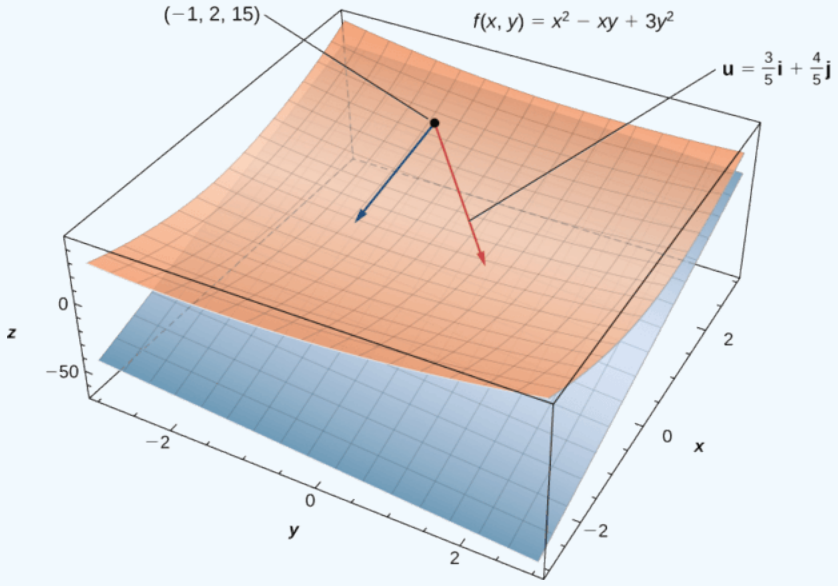


Figure 1.6.2: The surface $f(x,y)=x^2-xy+3y^2$ in orange with tangent plane at point $(-1,2,15)$ in blue. There are two arrows from the point $(-1,2,15)$, one seemingly along the surface and the other in a direction on the plane. The one that corresponds to the plane is marked $\mathbf{u}=\frac{3}{4}\mathbf{i}+\frac{4}{5}\mathbf{j}$.

An easier approach to calculating directional derivatives that involves partial derivatives is outlined in the following theorem.

Directional Derivative of a Function of Two Variables

Let $z=f(x,y)$ be a function of two variables x and y, and assume that f_x and f_y exist. Then the directional derivative of f in the direction of $\mathbf{u}=(\cos\theta)\ \mathbf{i}+(\sin\theta)\ \mathbf{j}$ is given by

$$D_{\mathbf{u}}f(x,y)=f_x(x,y)\cos\theta+f_y(x,y)\sin\theta. \tag{1.6.4}$$

Proof

Applying the definition of a directional derivative stated above in Equation 1.6.2, the directional derivative of f in the direction of $\mathbf{u}=(\cos\theta)\ \mathbf{i}+(\sin\theta)\ \mathbf{j}$ at a point (x_0,y_0) in the domain of f can be written

$$D_{\mathbf{u}}f((x_0,y_0))=\lim_{t\to 0}\frac{f(x_0+t\cos\theta,y_0+t\sin\theta)-f(x_0,y_0)}{t}. \tag{1.6.5}$$

Let $x=x_0+t\cos\theta$ and $y=y_0+t\sin\theta$, and define $g(t)=f(x,y)$. Since f_x and f_y both exist, we can use the chain rule for functions of two variables to calculate $g\,'(t)$:

$$g'(t)=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}=f_x(x,y)\cos\theta+f_y(x,y)\sin\theta. \tag{1.6.6}$$

If $t=0$, then $x=x_0$ and $y=y_0$, so

$$g'(0)=f_x(x_0,y_0)\cos\theta+f_y(x_0,y_0)\sin\theta \tag{1.6.7}$$

By the definition of $g\,'(t)$, it is also true that

$$g'(0) = \lim_{t\to 0} \frac{g(t) - g(0)}{t} = \lim_{t\to 0} \frac{f(x_0 + t\cos\theta, y_0 + t\sin\theta) - f(x_0, y_0)}{t}. \tag{1.6.8}$$

Therefore, $D_{\mathbf{u}} f(x_0, y_0) = f_x(x_0, y_0)\cos\theta + f_y(x_0, y_0)\sin\theta$.

Since the point (x_0, y_0) is an arbitrary point from the domain of f, this result holds for all points in the domain of f for which the partials f_x and f_y exist.

Therefore,

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta. \tag{1.6.9}$$

□

Example 1.6.2: An Alterntive Approach to Finding a Directional Derivative

Let $\theta = \arccos(\frac{3}{5})$. Find the directional derivative $D_{\mathbf{u}} f(x, y)$ of $f(x, y) = x^2 - xy + 3y^2$ in the direction of $\mathbf{u} = (\cos\theta)\ \mathbf{i} + (\sin\theta)\ \mathbf{j}$.

Then determine $D_{\mathbf{u}} f(-1, 2)$.

Solution

First, we must calculate the partial derivatives of f:

$$\begin{aligned} f_x(x, y) &= 2x - y \\ f_y(x, y) &= -x + 6y, \end{aligned}$$

Then we use Equation 1.6.4 with $\theta = \arccos(\frac{3}{5})$:

$$\begin{aligned} D_{\mathbf{u}} f(x, y) &= f_x(x, y)\cos\theta + f_y(x, y)\sin\theta \\ &= (2x - y)\frac{3}{5} + (-x + 6y)\frac{4}{5} \\ &= \frac{6x}{5} - \frac{3y}{5} - \frac{4x}{5} + \frac{24y}{5} \\ &= \frac{2x + 21y}{5}. \end{aligned}$$

To calculate $D_{\mathbf{u}} f(-1, 2)$, let $x = -1$ and $y = 2$:

$$D_{\mathbf{u}} f(-1, 2) = \frac{2(-1) + 21(2)}{5} = \frac{-2 + 42}{5} = 8.$$

This is the same answer obtained in Example 1.6.1.

Watch the accompanying video for a visual demonstration of the concept.

- **Video length**: 4 minutes 5 seconds.
- **Context**: This video demonstrates how to find a directional derivative.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Exercise 1.6.1

Find the directional derivative $D_{\mathbf{u}}f(x,y)$ of $f(x,y) = 3x^2y - 4xy^3 + 3y^2 - 4x$ in the direction of $\mathbf{u} = (\cos\frac{\pi}{3})\ \mathbf{i} + (\sin\frac{\pi}{3})\ \mathbf{j}$ using Equation 1.6.4.

What is $D_{\mathbf{u}}f(3,4)$?

Hint available at the end of this section.

If the vector that is given for the direction of the derivative is not a unit vector, then it is only necessary to divide by the norm of the vector. For example, if we wished to find the directional derivative of the function in Example 1.6.2 in the direction of the vector $\langle -5, 12\rangle$, we would first divide by its magnitude to get $\mathbf{u}$. This gives us $\mathbf{u} = \langle -\frac{5}{13}, \frac{12}{13}\rangle$.

Then

$$\begin{aligned} D_{\mathbf{u}}f(x,y) &= f_x(x,y)\cos\theta + f_y(x,y)\sin\theta \\ &= -\frac{5}{13}(2x-y) + \frac{12}{13}(-x+6y) \\ &= -\frac{22}{13}x + \frac{17}{13}y \end{aligned}$$

Gradient

The right-hand side of Equation 1.6.4 is equal to $f_x(x,y)\cos\theta + f_y(x,y)\sin\theta$, which can be written as the dot product of two vectors. Define the first vector as $\nabla f(x,y) = f_x(x,y)\ \mathbf{i} + f_y(x,y)\ \mathbf{j}$ and the second vector as $\mathbf{u} = (\cos\theta)\ \mathbf{i} + (\sin\theta)\ \mathbf{j}$. Then the right-hand side of the equation can be written as the dot product of these two vectors:

$$D_{\mathbf{u}}f(x,y) = \nabla f(x,y)\cdot\mathbf{u}. \tag{1.6.10}$$

The first vector in Equation 1.6.10 has a special name: the gradient of the function f. The symbol ∇ is called **nabla** and the vector ∇f is read "**del** f."

Definition: The Gradient

Let $z = f(x,y)$ be a function of x and y such that f_x and f_y exist. The vector $\nabla f(x,y)$ is called the **gradient** of f and is defined as

$$\nabla f(x,y) = f_x(x,y)\,\mathbf{i} + f_y(x,y)\,\mathbf{j}. \tag{1.6.11}$$

The vector $\nabla f(x,y)$ is also written as "**grad** f."

Example 1.6.3: Finding Gradients

Find the gradient $\nabla f(x,y)$ of each of the following functions:

a. $f(x,y) = x^2 - xy + 3y^2$
b. $f(x,y) = \sin 3x\cos 3y$

Solution

For both parts a. and b., we first calculate the partial derivatives f_x and f_y, then use Equation 1.6.11.

a. $f_x(x,y) = 2x - y$ and $f_y(x,y) = -x + 6y$, so

$$\begin{aligned}\nabla f(x,y) &= f_x(x,y)\,\mathbf{i} + f_y(x,y)\,\mathbf{j} \\ &= (2x-y)\,\mathbf{i} + (-x+6y)\,\mathbf{j}.\end{aligned}$$

b. $f_x(x,y) = 3\cos 3x \cos 3y$ and $f_y(x,y) = -3\sin 3x \sin 3y$, so

$$\begin{aligned}\nabla f(x,y) &= f_x(x,y)\,\mathbf{i} + f_y(x,y)\,\mathbf{j} \\ &= (3\cos 3x \cos 3y)\,\mathbf{i} - (3\sin 3x \sin 3y)\,\mathbf{j}.\end{aligned}$$

Exercise 1.6.2

Find the gradient $\nabla f(x,y)$ of $f(x,y) = \dfrac{x^2 - 3y^2}{2x+y}$.

Hint available at the end of this section.

The gradient has some important properties. We have already seen one formula that uses the gradient: the formula for the directional derivative. Recall from The Dot Product that if the angle between two vectors $\mathbf{a}$ and $\mathbf{b}$ is φ, then $\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos\varphi$. Therefore, if the angle between $\nabla f(x_0,y_0)$ and $\mathbf{u} = (cos\theta)\ \mathbf{i} + (sin\theta)\ \mathbf{j}$ is φ, we have

$$D_{\mathbf{u}}f(x_0,y_0) = \nabla f(x_0,y_0)\cdot\mathbf{u} = \|\nabla f(x_0,y_0)\|\|\mathbf{u}\|\cos\varphi = \|\nabla f(x_0,y_0)\|\cos\varphi. \tag{1.6.12}$$

The $\|\mathbf{u}\|$ disappears because $\mathbf{u}$ is a unit vector. Therefore, the directional derivative is equal to the magnitude of the gradient evaluated at (x_0,y_0) multiplied by $\cos\varphi$. Recall that $\cos\varphi$ ranges from -1 to 1.

If $\varphi = 0$, then $\cos\varphi = 1$ and $\nabla f(x_0,y_0)$ and $\mathbf{u}$ both point in the same direction.

If $\varphi = \pi$, then $\cos\varphi = -1$ and $\nabla f(x_0,y_0)$ and $\mathbf{u}$ point in opposite directions.

In the first case, the value of $D_{\mathbf{u}}f(x_0,y_0)$ is maximized; in the second case, the value of $D_{\mathbf{u}}f(x_0,y_0)$ is minimized.

We can also see that if $\nabla f(x_0,y_0) = \mathbf{0}$, then

$$D_{\mathbf{u}}f(x_0,y_0) = \nabla f(x_0,y_0)\cdot\mathbf{u} = 0 \tag{1.6.13}$$

for any vector $\mathbf{u}$. These three cases are outlined in the following theorem.

Properties of the Gradient

Suppose the function $z = f(x,y)$ is differentiable at (x_0,y_0) (Figure 1.6.3).

i. If $\nabla f(x_0,y_0) = \mathbf{0}$, then $D_{\mathbf{u}}f(x_0,y_0) = 0$ for any unit vector $\mathbf{u}$.
ii. If $\nabla f(x_0,y_0) \neq \mathbf{0}$, then $D_{\mathbf{u}}f(x_0,y_0)$ is **maximized** when $\mathbf{u}$ points in the same direction as $\nabla f(x_0,y_0)$. The maximum value of $D_{\mathbf{u}}f(x_0,y_0)$ is $\|\nabla f(x_0,y_0)\|$.
iii. If $\nabla f(x_0,y_0) \neq \mathbf{0}$, then $D_{\mathbf{u}}f(x_0,y_0)$ is **minimized** when $\mathbf{u}$ points in the opposite direction from $\nabla f(x_0,y_0)$. The minimum value of $D_{\mathbf{u}}f(x_0,y_0)$ is $-\|\nabla f(x_0,y_0)\|$.

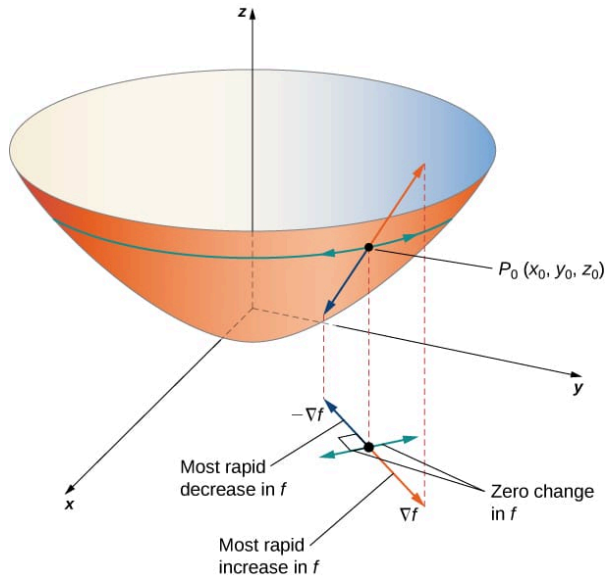


Figure 1.6.3: An upward facing paraboloid with point $P_0(x_0, y_0, z_0)$. From this point, there are arrows going up, down, and around the paraboloid. On the xy-plane, the point (x_0, y_0) is marked, and the corresponding arrows are drawn onto the plane: the down arrow corresponds to $-\nabla f$ (most rapid decrease in f), the up arrow corresponds to ∇f (most rapid increase in f), and the arrows around correspond to no change in f. The up/down arrows are perpendicular to the around arrows in their projection on the plane. The gradient indicates the maximum and minimum values of the directional derivative at a point.

Example 1.6.4: Finding a Maximum Directional Derivative

Find the direction for which the directional derivative of $f(x, y) = 3x^2 - 4xy + 2y^2$ at $(-2, 3)$ is a maximum. What is the maximum value?

Solution

The maximum value of the directional derivative occurs when ∇f and the unit vector point in the same direction. Therefore, we start by calculating $\nabla f(x, y)$:

$$f_x(x, y) = 6x - 4y \text{ and } f_y(x, y) = -4x + 4y$$

so

$$\nabla f(x, y) = f_x(x, y)\,\mathbf{i} + f_y(x, y)\,\mathbf{j} = (6x - 4y)\,\mathbf{i} + (-4x + 4y)\,\mathbf{j}.$$

Next, we evaluate the gradient at $(-2, 3)$:

$$\nabla f(-2, 3) = (6(-2) - 4(3))\,\mathbf{i} + (-4(-2) + 4(3))\,\mathbf{j} = -24\,\mathbf{i} + 20\,\mathbf{j}.$$

We need to find a unit vector that points in the same direction as $\nabla f(-2, 3)$, so the next step is to divide $\nabla f(-2, 3)$ by its magnitude, which is $\sqrt{(-24)^2 + (20)^2} = \sqrt{976} = 4\sqrt{61}$. Therefore,

$$\frac{\nabla f(-2, 3)}{\|\nabla f(-2, 3)\|} = \frac{-24}{4\sqrt{61}}\mathbf{i} + \frac{20}{4\sqrt{61}}\mathbf{j} = -\frac{6\sqrt{61}}{61}\,\mathbf{i} + \frac{5\sqrt{61}}{61}\,\mathbf{j}.$$

This is the unit vector that points in the same direction as $\nabla f(-2, 3)$. To find the angle corresponding to this unit vector, we solve the equations

$$\cos\theta = \frac{-6\sqrt{61}}{61} \text{ and } \sin\theta = \frac{5\sqrt{61}}{61}$$

for θ. Since cosine is negative and sine is positive, the angle must be in the second quadrant. Therefore, $\theta = \pi - \arcsin((5\sqrt{61})/61) \approx 2.45$ rad.

The maximum value of the directional derivative at $(-2,3)$ is $\|\nabla f(-2,3)\| = 4\sqrt{61}$ (Figure 1.6.4).

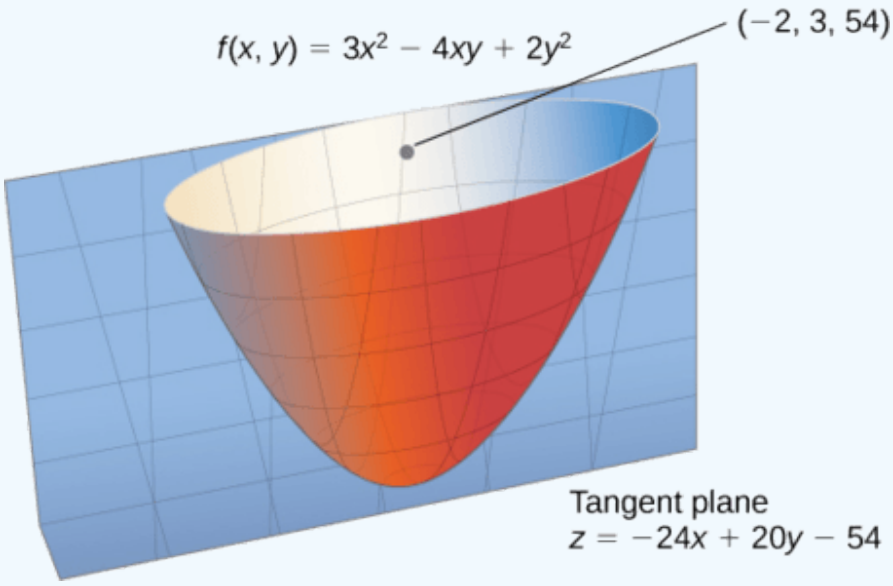


Figure 1.6.4: An upward facing paraboloid $f(x,y) = 3x^2 - 4xy + 2y^2$ with tangent plane at the point $(-2,3,54)$. The tangent plane has equation $z = -24x + 20y - 54$.

Exercise 1.6.3

Find the direction for which the directional derivative of $g(x,y) = 4x - xy + 2y^2$ at $(-2,3)$ is a maximum. What is the maximum value?

Hint available at the end of this section.

Figure 1.6.5 shows a portion of the graph of the function $f(x,y) = 3 + \sin x \sin y$. Given a point (a,b) in the domain of f, the maximum value of the directional derivative at that point is given by $\|\nabla f(a,b)\|$. This would equal the rate of greatest ascent if the surface represented a topographical map. If we went in the opposite direction, it would be the rate of greatest descent.

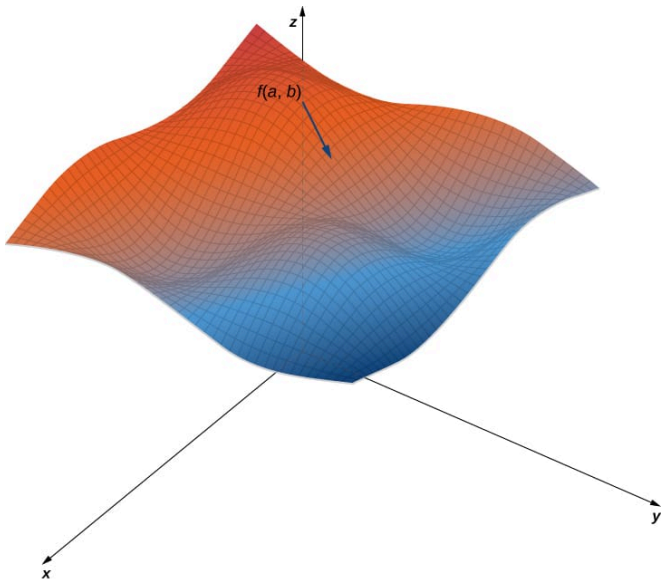


Figure 1.6.5: A portion of the graph of the function $f(x,y) = 3 + \sin x \sin y$. The surface is symmetrical with an arrow pointing down at the point $f(a,b)$.

Interactive Element

Visualizing the Gradient Vector on Its Surface

Below is the graph of the surface $f(x,y) = x - y^2$. You can move the point and see the associated gradient of the function at that point. The interactive app is unfortunately not accessible.

When using a topographical map, the steepest slope is always in the direction where the contour lines are closest together (Figure 1.6.6). This is analogous to the contour map of a function, assuming the level curves are obtained for equally spaced values throughout the range of that function.

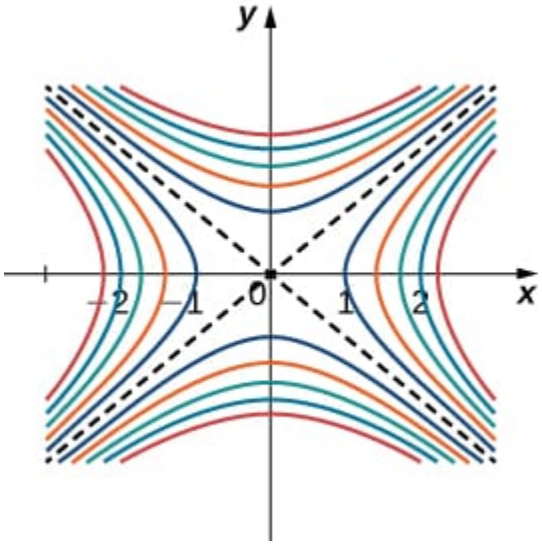


Figure 1.6.6: Contour map for the function $f(x,y) = x^2 - y^2$ using level values between -5 and 5. Two crossing dashed lines that pass through the origin and a series of curved lines approaching the crosses dashed lines as if they are asymptotes.

Gradients and Level Curves

Recall that if a curve is defined parametrically by the function pair $(x(t), y(t))$, then the vector $x'(t)\ \mathbf{i} + y'(t)\ \mathbf{j}$ is tangent to the curve for every value of t in the domain. Now let's assume $z = f(x,y)$ is a differentiable function of x and y, and (x_0, y_0) is in its domain. Let's suppose further that $x_0 = x(t_0)$ and $y_0 = y(t_0)$ for some value of t, and consider the level curve $f(x,y) = k$. Define $g(t) = f(x(t), y(t))$ and calculate $g'(t)$ on the level curve. By the chain Rule,

$$g'(t) = f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t). \tag{1.6.14}$$

But $g'(t) = 0$ because $g(t) = k$ for all t. Therefore, on the one hand,

$$f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t) = 0; \tag{1.6.15}$$

on the other hand,

$$f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t) = \nabla f(x,y) \cdot \langle x'(t), y'(t)\rangle. \tag{1.6.16}$$

Therefore,

$$\nabla f(x,y) \cdot \langle x'(t), y'(t)\rangle = 0. \tag{1.6.17}$$

Thus, the dot product of these vectors is equal to zero, which implies they are orthogonal. However, the second vector is tangent to the level curve, which implies the gradient must be normal to the level curve, which gives rise to the following theorem.

Gradient Is Normal to the Level Curve

Suppose the function $z = f(x, y)$ has continuous first-order partial derivatives in an open disk centered at a point (x_0, y_0). If $\nabla f(x_0, y_0) \neq 0$, then $\nabla f(x_0, y_0)$ is normal to the level curve of f at (x_0, y_0).

We can use this theorem to find tangent and normal vectors to level curves of a function.

Example 1.6.5: Finding Tangent Vector to Level Curve

For the function $f(x, y) = 2x^2 - 3xy + 8y^2 + 2x - 4y + 4$, find a tangent vector to the level curve at point $(-2, 1)$. Graph the level curve corresponding to $f(x, y) = 18$ and draw in $\nabla f(-2, 1)$ and a tangent vector.

Solution

First, we must calculate $\nabla f(x, y)$:

$$f_x(x, y) = 4x - 3y + 2 \text{ and } f_y = -3x + 16y - 4 \text{ so } \nabla f(x, y) = (4x - 3y + 2)\,\mathbf{i} + (-3x + 16y - 4)\,\mathbf{j}.$$

Next, we evaluate $\nabla f(x, y)$ at $(-2, 1)$:

$$\nabla f(-2, 1) = (4(-2) - 3(1) + 2)\,\mathbf{i} + (-3(-2) + 16(1) - 4)\,\mathbf{j} = -9\,\mathbf{i} + 18\,\mathbf{j}.$$

This vector is orthogonal to the curve at point $(-2, 1)$. We can obtain a tangent vector by reversing the components and multiplying either one by -1. Thus, for example, $-18\ \mathbf{i} - 9\ \mathbf{j}$ is a tangent vector (Figure 1.6.7).

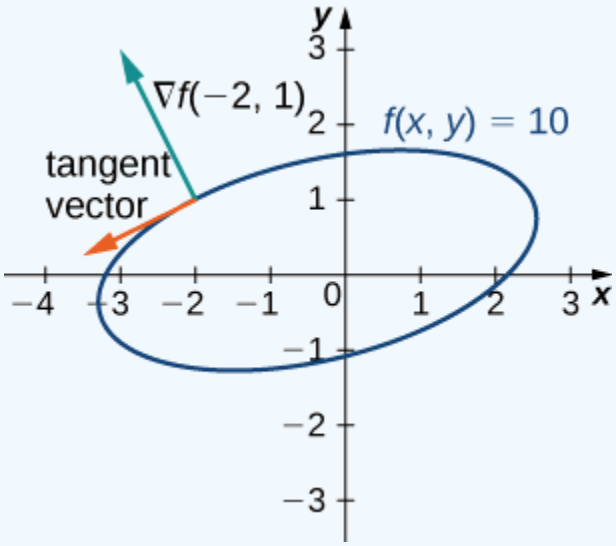


Figure 1.6.7: Tangent vector in orange and normal vector $\nabla f(-2, 1)$ in green to the curve $f(x, y = 10$ at point $(-2, 1)$.

Example 1.6.6: Finding Normal Vector to Level Curve

Find the normal vector to the level curve $f(x, y) = x - y^2 = c$ at $(5, 1)$ with $c = 4$.

Solution

- **Video length**: 1 minute 47 seconds.
- **Context**: This video demonstrates how to find the angle between two vectors.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Exercise 1.6.4

For the function $f(x,y) = x^2 - 2xy + 5y^2 + 3x - 2y + 3$, find the tangent to the level curve at point $(1,1)$. Draw the graph of the level curve corresponding to $f(x,y) = 8$ and draw $\nabla f(1,1)$ and a tangent vector.

Hint available at the end of this section.

Three-Dimensional Gradients and Directional Derivatives

The definition of a gradient can be extended to functions of more than two variables.

Definition: Gradients in 3D

Let $w = f(x,y,z)$ be a function of three variables such that f_x, f_y, and f_z exist. The vector $\nabla f(x,y,z)$ is called the gradient of f and is defined as

$$\nabla f(x,y,z) = f_x(x,y,z)\,\mathbf{i} + f_y(x,y,z)\,\mathbf{j} + f_z(x,y,z)\,\mathbf{k}. \tag{1.6.18}$$

$\nabla f(x,y,z)$ can also be written as **grad** $f(x,y,z)$.

Calculating the gradient of a function in three variables is very similar to calculating the gradient of a function in two variables. First, we calculate the partial derivatives f_x, f_y, and f_z, and then we use Equation 1.6.18.

Example 1.6.7: Finding Gradients in Three Dimensions

Find the gradient $\nabla f(x,y,z)$ of each of the following functions:

a. $f(x,y,z) = 5x^2 - 2xy + y^2 - 4yz + z^2 + 3xz$
b. $f(x,y,z) = e^{-2z}\sin 2x \cos 2y$

Solution

For both parts a. and b., we first calculate the partial derivatives f_x, f_y, and f_z, then use Equation 1.6.18.

a. $f_x(x,y,z) = 10x - 2y + 3z$, $f_y(x,y,z) = -2x + 2y - 4z$, and $f_z(x,y,z) = 3x - 4y + 2z$, so

$$\begin{aligned}\nabla f(x,y,z) &= f_x(x,y,z)\,\mathbf{i} + f_y(x,y,z)\,\mathbf{j} + f_z(x,y,z)\,\mathbf{k} \\ &= (10x - 2y + 3z)\,\mathbf{i} + (-2x + 2y - 4z)\,\mathbf{j} + (3x - 4y + 2z)\,\mathbf{k}.\end{aligned}$$

b. $f_x(x,y,z) = 2e^{-2z}\cos 2x \cos 2y$, $f_y(x,y,z) = -2e^{-2z}\sin 2x \sin 2y$, and $f_z(x,y,z) = -2e^{-2z}\sin 2x \cos 2y$, so

$$\begin{aligned}\nabla f(x,y,z) &= f_x(x,y,z)\,\mathbf{i} + f_y(x,y,z)\,\mathbf{j} + f_z(x,y,z)\,\mathbf{k} \\ &= (2e^{-2z}\cos 2x \cos 2y)\,\mathbf{i} + (-2e^{-2z}\sin 2x \sin 2y)\,\mathbf{j} + (-2e^{-2z}\sin 2x \cos 2y)\,\mathbf{k} \\ &= 2e^{-2z}((\cos 2x \cos 2y)\,\mathbf{i} - (\sin 2x \sin 2y)\,\mathbf{j} - (\sin 2x \cos 2y)\,\mathbf{k}).\end{aligned}$$

Example 1.6.8: Finding Symmetric Equations to a Normal Line to a Surface

Find symmetric equations of the normal line to the surface $xy + z^2 = 11$ at $(1, 2, 3)$.

Solution

- **Video length**: 1 minute 58 seconds.
- **Context**: This video demonstrates how to find symmetric equations to a normal line to a surface.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

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Exercise 1.6.5

Find the gradient $\nabla f(x, y, z)$ of $f(x, y, z) = \dfrac{x^2 - 3y^2 + z^2}{2x + y - 4z.}$

Answer available at the end of this section.

Gradient vector is useful in engineering. Here is a simple video example of what it can do.

Example 1.6.9: Path of a Heat Seeking Missile

Find the path of a heat seeking particle placed at $(2, 1)$ on a metal plate with temperature field $T(x, y) = 40 - x^2 - 4y^2$.

Solution

- **Video length**: 4 minutes 47 seconds.
- **Context**: This video demonstrates how to find a path of a heat seeking missile.

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The directional derivative can also be generalized to functions of three variables. To determine a direction in three dimensions, a vector with three components is needed. This vector is a unit vector, and the components of the unit vector are called *directional cosines***.** Given a three-dimensional unit vector $\mathbf{u}$ in standard form (i.e., the initial point is at the origin), this vector forms three different angles with the positive x-, y-, and z-axes. Let's call these angles α, β, and γ. Then the directional cosines are given by $\cos\alpha, \cos\beta$, and $\cos\gamma$. These are the components of the unit vector $\mathbf{u}$; since $\mathbf{u}$ is a unit vector, it is true that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

Definition: Directional Derivative of a Function of Three variables

Suppose $w = f(x, y, z)$ is a function of three variables with a domain of D. Let $(x_0, y_0, z_0) \in D$ and let $\mathbf{u} = \cos\alpha\ \mathbf{i} + \cos\beta\ \mathbf{j} + \cos\gamma\ \mathbf{k}$ be a unit vector. Then, the directional derivative of f in the direction of u is given by

$$D_{\mathbf{u}}f(x_0,y_0,z_0)=\lim_{t\to 0}\frac{f(x_0+t\cos\alpha,y_0+t\cos\beta,z_0+t\cos\gamma)-f(x_0,y_0,z_0)}{t} \tag{1.6.19}$$

provided the limit exists.

We can calculate the directional derivative of a function of three variables by using the gradient, leading to a formula that is analogous to Equation 1.6.4.

Directional Derivative of a Function of Three Variables

Let $f(x,y,z)$ be a differentiable function of three variables and let $\mathbf{u}=\cos\alpha\ \mathbf{i}+\cos\beta\ \mathbf{j}+\cos\gamma\ \mathbf{k}$ be a unit vector. Then, the directional derivative of f in the direction of $\mathbf{u}$ is given by

$$D_{\mathbf{u}}f(x,y,z)=\nabla f(x,y,z)\cdot\mathbf{u}=f_x(x,y,z)\cos\alpha+f_y(x,y,z)\cos\beta+f_z(x,y,z)\cos\gamma. \tag{1.6.20}$$

The three angles α,β, and γ determine the unit vector $\mathbf{u}$. In practice, we can use an arbitrary (nonunit) vector, then divide by its magnitude to obtain a unit vector in the desired direction.

Example 1.6.10: Finding a Directional Derivative in Three Dimensions

Calculate $D_{\mathbf{v}}f(1,-2,3)$ in the direction of $\mathbf{v}=-\ \mathbf{i}+2\ \mathbf{j}+2\ \mathbf{k}$ for the function

$$f(x,y,z)=5x^2-2xy+y^2-4yz+z^2+3xz.$$

Solution:

First, we find the magnitude of v:

$$\|\mathbf{v}\|=\sqrt{(-1)^2+(2)^2+(2)^2}=\sqrt{9}=3.$$

Therefore, $\dfrac{\mathbf{v}}{\|\mathbf{v}\|}=\dfrac{-\mathbf{i}+2\,\mathbf{j}+2\,\mathbf{k}}{3}=-\dfrac{1}{3}\ \mathbf{i}+\dfrac{2}{3}\ \mathbf{j}+\dfrac{2}{3}\ \mathbf{k}$ is a unit vector in the direction of $\mathbf{v}$, so $\cos\alpha=-\dfrac{1}{3},\cos\beta=\dfrac{2}{3}$, and $\cos\gamma=\dfrac{2}{3}$. Next, we calculate the partial derivatives of f:

$$\begin{aligned}f_x(x,y,z)&=10x-2y+3z\\f_y(x,y,z)&=-2x+2y-4z\\f_z(x,y,z)&=-4y+2z+3x,\end{aligned}$$

then substitute them into Equation 1.6.20:

$$\begin{aligned}D_{\mathbf{v}}f(x,y,z)&=f_x(x,y,z)\cos\alpha+f_y(x,y,z)\cos\beta+f_z(x,y,z)\cos\gamma\\&=(10x-2y+3z)(-\frac{1}{3})+(-2x+2y-4z)(\frac{2}{3})+(-4y+2z+3x)(\frac{2}{3})\\&=-\frac{10x}{3}+\frac{2y}{3}-\frac{3z}{3}-\frac{4x}{3}+\frac{4y}{3}-\frac{8z}{3}-\frac{8y}{3}+\frac{4z}{3}+\frac{6x}{3}\\&=-\frac{8x}{3}-\frac{2y}{3}-\frac{7z}{3}.\end{aligned}$$

Last, to find $D_{\mathbf{v}}f(1,-2,3)$, we substitute $x=1,\ \ y=-2$, and $z=3$:

$$\begin{aligned}D_{\mathbf{v}}f(1,-2,3)&=-\frac{8(1)}{3}-\frac{2(-2)}{3}-\frac{7(3)}{3}\\&=-\frac{8}{3}+\frac{4}{3}-\frac{21}{3}\\&=-\frac{25}{3}.\end{aligned}$$

Exercise 1.6.6

Calculate $D_{\mathbf{v}}f(x,y,z)$ and $D_{\mathbf{v}}f(0,-2,5)$ in the direction of $\mathbf{v}=-3\ \mathbf{i}+12\ \mathbf{j}-4\ \mathbf{k}$ for the function

$$f(x,y,z)=3x^2+xy-2y^2+4yz-z^2+2xz.$$

Hint available at the end of this section.

Summary

- A directional derivative represents a rate of change of a function in any given direction.
- The gradient can be used in a formula to calculate the directional derivative.
- The gradient indicates the direction of greatest change of a function of more than one variable.

Key Equations

- **Directional derivative (two dimensions)**

$$D_{\mathbf{u}}f(a,b)=\lim_{h\to 0}\frac{f(a+h\cos\theta,b+h\sin\theta)-f(a,b)}{h}$$

or

$$D_{\mathbf{u}}f(x,y)=f_x(x,y)\cos\theta+f_y(x,y)\sin\theta$$

- **Gradient (two dimensions)**

$$\nabla f(x,y)=f_x(x,y)\,\mathbf{i}+f_y(x,y)\,\mathbf{j}$$

- **Gradient (three dimensions)**

$$\nabla f(x,y,z)=f_x(x,y,z)\,\mathbf{i}+f_y(x,y,z)\,\mathbf{j}+f_z(x,y,z)\,\mathbf{k}$$

- **Directional derivative (three dimensions)**

$$D_{\mathbf{u}}f(x,y,z)=\nabla f(x,y,z)\cdot\mathbf{u}=f_x(x,y,z)\cos\alpha+f_y(x,y,z)\cos\beta+f_z(x,y,z)\cos\gamma$$

Glossary

Directional derivative

the derivative of a function in the direction of a given unit vector

Gradient

the gradient of the function $f(x,y)$ is defined to be $\nabla f(x,y)=(\partial f/\partial x)\ \mathbf{i}+(\partial f/\partial y)\ \mathbf{j}$, which can be generalized to a function of any number of independent variables

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Answers

Exercise 1.6.1

Calculate the partial derivatives and determine the value of θ.

$$D_{\mathbf{u}}f(x,y)=\frac{(6xy-4y^3-4)(1)}{2}+\frac{(3x^2-12xy^2+6y)\sqrt{3}}{2}$$

$$D_{\mathbf{u}}f(3,4)=\frac{72-256-4}{2}+\frac{(27-576+24)\sqrt{3}}{2}=-94-\frac{525\sqrt{3}}{2}$$

Exercise 1.6.2

Calculate the partial derivatives, then use Equation 1.6.11.

$\nabla f(x,y) = \dfrac{2x^2+2xy+6y^2}{(2x+y)^2}\ \mathbf{i} - \dfrac{x^2+12xy+3y^2}{(2x+y)^2}\ \mathbf{j}$

Exercise 1.6.3

Evaluate the gradient of g at point $(-2,3)$.

The gradient of g at $(-2,3)$ is $\nabla g(-2,3) =\ \mathbf{i} + 14\ \mathbf{j}$. The unit vector that points in the same direction as $\nabla g(-2,3)$ is

$$\frac{\nabla g(-2,3)}{\|\nabla g(-2,3)\|} = \frac{1}{\sqrt{197}}\,\mathbf{i} + \frac{14}{\sqrt{197}}\,\mathbf{j} = \frac{\sqrt{197}}{197}\,\mathbf{i} + \frac{14\sqrt{197}}{197}\,\mathbf{j},$$

which gives an angle of $\theta = \arcsin((14\sqrt{197})/197) \approx 1.499$ rad.

The maximum value of the directional derivative is $\|\nabla g(-2,3)\| = \sqrt{197}$.

Exercise 1.6.4

Calculate the gradient at point $(1,1)$.

$\nabla f(x,y) = (2x-2y+3)\ \mathbf{i} + (-2x+10y-2)\ \mathbf{j}$

$\nabla f(1,1) = 3\ \mathbf{i} + 6\ \mathbf{j}$

Tangent vector: $6\ \mathbf{i} - 3\ \mathbf{j}$ or $-6\ \mathbf{i} + 3\ \mathbf{j}$

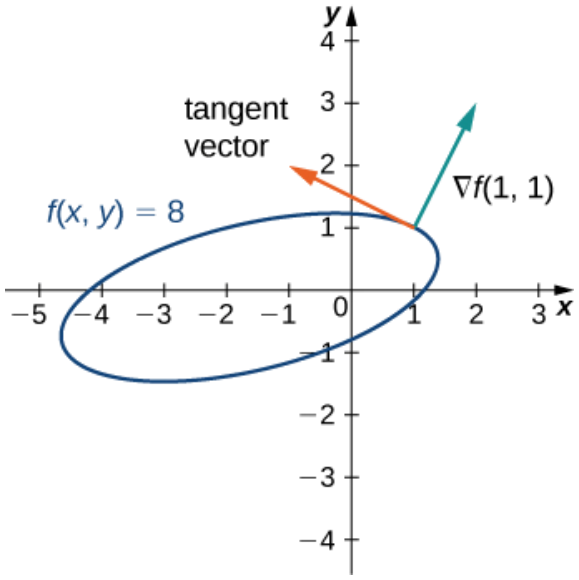


Figure 1.6.8: Tangent vector and normal vector $\nabla f(1,1)$ to $f(x,y) = 8$ at point $(1,1)$.

Exercise 1.6.5

$$\nabla f(x,y,z) = \frac{2x^2+2xy+6y^2-8xz-2z^2}{(2x+y-4z)^2}\,\mathbf{i} - \frac{x^2+12xy+3y^2-24yz+z^2}{(2x+y-4z)^2}\,\mathbf{j} + \frac{4x^2-12y^2-4z^2+4xz+2yz}{(2x+y-4z)^2}\,\mathbf{k}$$

Exercise 1.6.6

First, divide $\mathbf{v}$ by its magnitude, calculate the partial derivatives of f, then use Equation 1.6.20.

$D_{\mathbf{v}} f(x,y,z) = -\dfrac{3}{13}(6x+y+2z) + \dfrac{12}{13}(x-4y+4z) - \dfrac{4}{13}(2x+4y-2z)$

$D_{\mathbf{v}} f(0,-2,5) = \dfrac{384}{13}$

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1.6E: Exercises for Section 1.6

Exercise 1 - 2

Find the directional derivative using the limit definition.

1. Find the directional derivative of the function in the direction of given vector.
   a. $f(x,y) = 5 - 2x^2 - \frac{1}{2}y^2$ at point $P(3,4)$ in the direction of $\mathbf{u} = (\cos\frac{\pi}{4})\ \mathbf{i} + (\sin\frac{\pi}{4})\ \mathbf{j}$

   Answer
   $$D_{\mathbf{u}}f(3,4) = -8\sqrt{2}$$

   b. $f(x,y) = y^2\cos(2x)$ at point $P(\frac{\pi}{3},2)$ in the direction of $\mathbf{u} = (\cos\frac{\pi}{4})\ \mathbf{i} + (\sin\frac{\pi}{4})\ \mathbf{j}$

   Answer
   $$D_{\mathbf{u}}f(\tfrac{\pi}{3},2) = -2\sqrt{6} - \sqrt{2}$$

2. Find the directional derivative of $f(x,y) = y^2\sin(2x)$ at point $P(\frac{\pi}{4},2)$ in the direction of $\mathbf{u} = 5\ \mathbf{i} + 12\ \mathbf{j}$.

Exercises 3 - 13

Find the directional derivative of the function in the direction of v as a function of x and y. Remember that you first need to find a unit vector in the direction of the direction vector. Then find the value of the directional derivative at point P.

3. $f(x,y) = xy, \quad P(-2,0), \quad \mathbf{v} = \frac{1}{2}\ \mathbf{i} + \frac{\sqrt{3}}{2}\ \mathbf{j}$

   Answer
   $$D_{\mathrm{v}}f(x,y) = \tfrac{1}{2}y + \tfrac{\sqrt{3}}{2}x$$
   $$D_{\mathrm{v}}f(-2,0) = -\sqrt{3}$$

4. $h(x,y) = e^x\sin y, \quad P(1,\frac{\pi}{2}), \quad \mathbf{v} = -\ \mathbf{i}$
5. $f(x,y) = x^2y, \quad P(-5,5), \quad \mathbf{v} = 3\ \mathbf{i} - 4\ \mathbf{j}$

   Answer
   $$D_{\mathrm{v}}f(x,y) = \tfrac{6}{5}xy - \tfrac{4}{5}x^2$$
   $$D_{\mathrm{v}}f(-5,5) = -50$$

6. $f(x,y) = xy, \quad P(1,1), \quad \mathbf{u} = \langle\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\rangle$
7. $f(x,y) = x^2 - y^2, \quad P(1,0), \quad \mathbf{u} = \langle\frac{\sqrt{3}}{2},\frac{1}{2}\rangle$

   Answer
   $$D_{\mathbf{u}}f(x,y) = x\sqrt{3} - y$$
   $$D_{\mathbf{u}}f(1,0) = \sqrt{3}$$

8. $f(x,y) = 3x + 4y + 7, \quad P(0,\frac{\pi}{2}), \quad \mathbf{u} = \langle\frac{3}{5},\frac{4}{5}\rangle$
9. $f(x,y) = e^x\cos y, \quad P = (0,\frac{\pi}{2}), \quad \mathbf{u} = \langle 0,5\rangle$

   Answer
   $$D_{\mathbf{u}}f(x,y) = -e^x\sin y$$
   $$D_{\mathbf{u}}f(0,\tfrac{\pi}{2}) = -1$$

10. $f(x,y) = y^{10}, \quad \mathbf{u} = \langle 0,-3\rangle, \quad P = (1,-1)$
11. $f(x,y) = \ln(x^2+y^2), \quad \mathbf{u} = \langle 2,-5\rangle, \quad P(1,2)$

Answer

$$D_{\mathrm{u}}f(x,y) = \frac{\sqrt{29}}{29}\left(\frac{4x-10y}{x^2+y^2}\right)$$
$$D_{\mathrm{u}}f(1,2) = -\frac{16\sqrt{29}}{145}$$

12. $h(x,y,z) = xyz, \quad P(2,1,1), \quad \mathbf{v} = 2\ \mathbf{i} + \ \mathbf{j} - \ \mathbf{k}$

Answer

$$D_{\mathrm{v}}h(x,y,z) = \frac{\sqrt{6}}{6}(2yz + xz - xy)$$
$$D_{\mathrm{v}}h(2,1,1) = \frac{2}{\sqrt{6}} = \frac{\sqrt{6}}{3}$$

13. $f(x,y,z) = y^2 + xz, \quad P(1,2,2), \quad \mathbf{v} = \langle 2,-1,2\rangle$

Answer

$$D_{\mathrm{v}}f(x,y,z) = \frac{2}{3}(z - y + x)$$
$$D_{\mathrm{v}}f(1,2,2) = \frac{2}{3}$$

Exercises 14 - 19

Find the directional derivative of the function in the direction of the unit vector $\mathrm{u} = \cos\theta\ \mathbf{i} + \sin\theta\ \mathbf{j}$.

14. $f(x,y) = x^2 + 2y^2, \quad \theta = \frac{\pi}{6}$
15. $f(x,y) = \dfrac{y}{x+2y}, \quad \theta = -\frac{\pi}{4}$

Answer

$$D_{\mathrm{u}}f(x,y) = \frac{-\sqrt{2}(x+y)}{2(x+2y)^2}$$

16. $f(x,y) = \cos(3x+y), \quad \theta = \frac{\pi}{4}$
17. $w(x,y) = ye^x, \quad \theta = \frac{\pi}{3}$

Answer

$$D_{\mathrm{u}}f(x,y) = \frac{e^x(y+\sqrt{3})}{2}$$

18. $f(x,y) = x\arctan(y), \quad \theta = \frac{\pi}{2}$
19. $f(x,y) = \ln(x+2y), \quad \theta = \frac{\pi}{3}$

Answer

$$D_{\mathrm{u}}f(x,y) = \frac{1+2\sqrt{3}}{2(x+2y)}$$

Exercises 20 - 23

Find the gradient ∇f.

20. Find the gradient of $f(x,y) = 3x^2 + y^3 - 3x + y$. Then find its value at the point $P(2,3)$.
21. Find the gradient of $f(x,y) = \dfrac{14 - x^2 - y^2}{3}$. Then, find the gradient at point $P(1,2)$.

Answer

$$\nabla f(x,y) = -\frac{2}{3}x\ \mathbf{i} - \frac{2}{3}y\ \mathbf{j}$$
$$\nabla f(1,2) = -\frac{2}{3}\ \mathbf{i} - \frac{4}{3}\ \mathbf{j}$$

22. Find the gradient of $f(x,y) = \ln(4x^3 - 3y)$. Then, find the gradient at point $P(1,1)$.
23. Find the gradient of $f(x,y,z) = xy + yz + xz$. Then find the gradient at point $P(1,2,3)$.

Answer

$$\nabla f(x,y,z) = \langle y+z, x+z, y+x \rangle$$
$$\nabla f(1,2,3) = \langle 5,4,3 \rangle$$

Exercises 24 - 25

Find the directional derivative of the function at point P in the direction of Q.

24. $f(x,y) = x^2 + 3y^2, \quad P(1,1), \quad Q(4,5)$
25. $f(x,y,z) = \dfrac{y}{x+z}, \quad P(2,1,-1), \quad Q(-1,2,0)$

Answer

$$D_{\vec{PQ}} f(x,y) = \frac{3}{\sqrt{11}}$$

26. Find the directional derivative of $f(x,y,z))$ at P and in the direction of $\mathbf{u}$: $f(x,y,z) = \ln(x^2+2y^2+3z^2), \quad P(2,1,4), \quad \mathbf{u} = \frac{-3}{13}\ \mathbf{i} - \frac{4}{13}\ \mathbf{j} - \frac{12}{13}\ \mathbf{k}$.

Exercises 27 - 29

Find the directional derivative of the function at P in the direction of $\mathbf{u}$.

27. $f(x,y) = \ln(5x+4y), \quad P(3,9), \quad \mathbf{u} = 6\ \mathbf{i} + 8\ \mathbf{j}$

Answer

$$D_{\mathbf{u}} f(3,9) = \frac{31}{255}$$

28. $f(x,y) = -7x + 2y, \quad P(2,-4), \quad \mathbf{u} = 4\ \mathbf{i} - 3\ \mathbf{j}$
29. $f(x,y,z) = 4x^5y^2z^3, \quad P(2,-1,1), \quad \mathbf{u} = \frac{1}{3}\ \mathbf{i} + \frac{2}{3}\ \mathbf{j} - \frac{2}{3}\ \mathbf{k}$

Answer

$$D_{\mathbf{u}} f(2,-1,1) = -320$$

30. [T] Use technology to sketch the level curve of $f(x,y) = 4x - 2y + 3$ that passes through $P(1,2)$ and draw the gradient vector at P.
31. [T] Use technology to sketch the level curve of $f(x,y) = x^2 + 4y^2$ that passes through $P(-2,0)$ and draw the gradient vector at P.

Answer

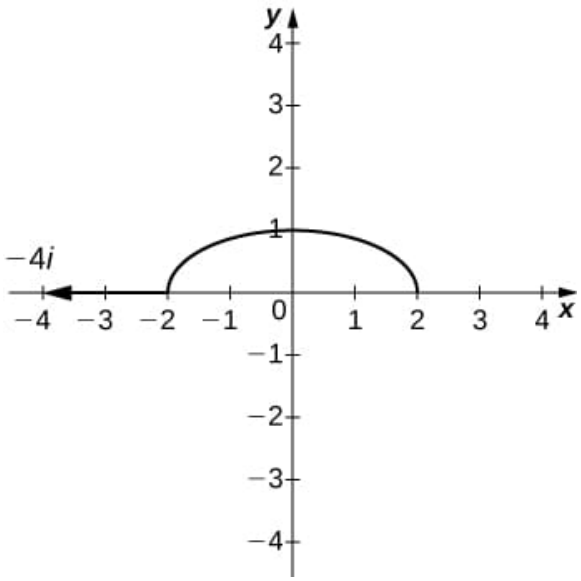


Figure $1.6E.1$: Level curves of a surface consisting of a horizontal line from -2 to $-\infty$ and an upper semi-ellipse from -2 to 2.

Exercises 32 - 35

Find the gradient vector at the indicated point.

32. $f(x,y) = xy^2 - yx^2, \quad P(-1,1)$
33. $f(x,y) = xe^y - \ln(x), \quad P(-3,0)$

Answer

$$\nabla f(-3,0) = \tfrac{4}{3}\ \mathbf{i} - 3\ \mathbf{j}$$

34. $f(x,y,z) = xy - \ln(z), \quad P(2,-2,2)$
35. $f(x,y,z) = x\sqrt{y^2+z^2}, \quad P(-2,-1,-1)$

Answer

$$\nabla f(-2,-1,-1) = \sqrt{2}\ \mathbf{i} + \sqrt{2}\ \mathbf{j} + \sqrt{2}\ \mathbf{k}$$

Exercises 36 - 40

Find the indicated directional derivative of the function.

36. $f(x,y) = x^2 + xy + y^2$ at point $(-5,-4)$ in the direction the function increases most rapidly.
37. $f(x,y) = e^{xy}$ at point $(6,7)$ in the direction the function increases most rapidly.

Answer

$$1.6(10^{19})$$

38. $f(x,y) = \arctan\left(\frac{y}{x}\right)$ at point $(-9,9)$ in the direction the function increases most rapidly.
39. $f(x,y,z) = \ln(xy + yz + zx)$ at point $(-9,-18,-27)$ in the direction the function increases most rapidly.

Answer

$$\tfrac{5\sqrt{2}}{99}$$

40. $f(x,y,z) = \dfrac{x}{y} + \dfrac{y}{z} + \dfrac{z}{x}$ at point $(5,-5,5)$ in the direction the function increases most rapidly.

Exercises 41 - 43

Find the maximum rate of change of f at the given point and the direction in which it occurs.

41. $f(x,y) = xe^{-y}, \quad (-2,0)$

Answer

$\max\{D_{\mathbf{u}}f(-2,0)\} = \sqrt{5}, \quad \langle 1,2\rangle$

Solution:

$\nabla f(x,y) = e^{-y}\hat{\mathbf{i}} - xe^{-y}\hat{\mathbf{j}}$. So $\max\{D_{\mathbf{u}}f(-2,0)\} = \|\nabla f(-2,0)\| = \sqrt{5}$.
The direction in which it occurs will just be the direction of the gradient vector: $\nabla f(-2,0) = \langle 1,2\rangle$.

42. $f(x,y) = \sqrt{x^2+2y}, \quad (4,10)$
43. $f(x,y) = \cos(3x+2y), \quad (\frac{\pi}{6}, -\frac{\pi}{8})$

Answer

$\max\{D_{\mathbf{u}}f\left(\frac{\pi}{6}, -\frac{\pi}{8}\right)\} = \sqrt{\frac{13}{2}} = \frac{\sqrt{26}}{2}, \quad \langle -3,-2\rangle$

Exercises 44 - 47

Find equations of

a. **the tangent plane and**
b. **the normal line to the given surface at the given point.**

44. The level curve $f(x,y,z) = 12$ for $f(x,y,z) = 4x^2 - 2y^2 + z^2$ at point $(2,2,2)$.
45. $f(x,y,z) = xy + yz + xz = 3$ at point $(1,1,1)$

Answer

a. tangent plane equation: $x + y + z = 3$,
b. normal line equations: $x - 1 = y - 1 = z - 1$

46. $f(x,y,z) = xyz = 6$ at point $(1,2,3)$
47. $f(x,y,z) = xe^y \cos z - z = 1$ at point $(1,0,0)$

Answer

a. tangent plane equation: $x + y - z = 1$,
b. normal line equations: $x - 1 = y = -z$

Exercises 48 - 51

Solve the stated problem.

48. The temperature T in a metal sphere is inversely proportional to the distance from the center of the sphere (the origin: $(0,0,0)$). The temperature at point $(1,2,2)$ is 120°C.
    a. Find the rate of change of the temperature at point $(1,2,2)$ in the direction toward point $(2,1,3)$.
    b. Show that, at any point in the sphere, the direction of greatest increase in temperature is given by a vector that points toward the origin.
49. The **electrical potential** (voltage) in a certain region of space is given by the function $V(x,y,z) = 5x^2 - 3xy + xyz$.
50. Find the rate of change of the voltage at point $(3,4,5)$ in the direction of the vector $\langle 1,1,-1\rangle$.
51. In which direction does the voltage change most rapidly at point $(3,4,5)$?
52. What is the maximum rate of change of the voltage at point $(3,4,5)$?

Answer

a. $\frac{32}{\sqrt{3}}$,

b. $\langle 38, 6, 12\rangle$,
c. $2\sqrt{406}$

50. If the electric potential at a point (x, y) in the xy-plane is $V(x, y) = e^{-2x}\cos(2y)$, then the electric intensity vector at (x, y) is $E = -\nabla V(x, y)$.
   a. Find the electric intensity vector at $(\frac{\pi}{4}, 0)$.
   b. Show that, at each point in the plane, the electric potential decreases most rapidly in the direction of the vector E.
51. In two dimensions, the motion of an ideal fluid is governed by a velocity potential φ. The velocity components of the fluid u in the x-direction and v in the y-direction, are given by $\langle u, v\rangle = \nabla\varphi$. Find the velocity components associated with the velocity potential $\varphi(x, y) = \sin\pi x \sin 2\pi y$.

Answer

$$\langle u, v\rangle = \langle \pi\cos(\pi x)\sin(2\pi y), 2\pi\sin(\pi x)\cos(2\pi y)\rangle$$

Contributors

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- Paul Seeburger (Monroe Community College) created problems 20 and 22 and added a solution to problem 41.

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1.7: Maxima/Minima Problems

Learning Objectives

- Use partial derivatives to locate critical points for a function of two variables.
- Apply a second derivative test to identify a critical point as a local maximum, local minimum, or saddle point for a function of two variables.
- Examine critical points and boundary points to find absolute maximum and minimum values for a function of two variables.

One of the most useful applications for derivatives of a function of one variable is the determination of maximum and/or minimum values. This application is also important for functions of two or more variables, but as we have seen in earlier sections of this chapter, the introduction of more independent variables leads to more possible outcomes for the calculations. The main ideas of finding critical points and using derivative tests are still valid, but new wrinkles appear when assessing the results.

Critical Points

For functions of a single variable, we defined critical points as the values of the variable at which the function's derivative equals zero or does not exist. For functions of two or more variables, the concept is essentially the same, except for the fact that we are now working with partial derivatives.

Definition: Critical Points

Let $z = f(x, y)$ be a function of two variables that is differentiable on an open set containing the point (x_0, y_0). The point (x_0, y_0) is called a *critical point* of a function of two variables f if one of the two following conditions holds:

1. $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$
2. Either $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist.

Example 1.7.1: Finding Critical Points

Find the critical points of each of the following functions:

a. $f(x, y) = \sqrt{4y^2 - 9x^2 + 24y + 36x + 36}$
b. $g(x, y) = x^2 + 2xy - 4y^2 + 4x - 6y + 4$

Solution

a. First, we calculate $f_x(x, y)$ and $f_y(x, y)$:

$$f_x(x, y) = \frac{1}{2}(-18x + 36)(4y^2 - 9x^2 + 24y + 36x + 36)^{-1/2}$$
$$= \frac{-9x + 18}{\sqrt{4y^2 - 9x^2 + 24y + 36x + 36}}$$

$$f_y(x, y) = \frac{1}{2}(8y + 24)(4y^2 - 9x^2 + 24y + 36x + 36)^{-1/2}$$
$$= \frac{4y + 12}{\sqrt{4y^2 - 9x^2 + 24y + 36x + 36}}.$$

Next, we set each of these expressions equal to zero:

$$\frac{-9x+18}{\sqrt{4y^2-9x^2+24y+36x+36}}=0$$
$$\frac{4y+12}{\sqrt{4y^2-9x^2+24y+36x+36}}=0.$$

Then, multiply each equation by its common denominator:

$$-9x+18=0$$
$$4y+12=0.$$

Therefore, $x=2$ and $y=-3$, so $(2,-3)$ is a critical point of f.

We must also check for the possibility that the denominator of each partial derivative can equal zero, thus causing the partial derivative not to exist. Since the denominator is the same in each partial derivative, we need only do this once:

$$4y^2-9x^2+24y+36x+36=0. \tag{1.7.1}$$

Equation 1.7.1 represents a hyperbola. We should also note that the domain of f consists of points satisfying the inequality

$$4y^2-9x^2+24y+36x+36\geq 0. \tag{1.7.2}$$

Therefore, any points on the hyperbola are not only critical points, they are also on the boundary of the domain. To put the hyperbola in standard form, we use the method of completing the square:

$$\begin{aligned}4y^2-9x^2+24y+36x+36&=0\\ 4y^2-9x^2+24y+36x&=-36\\ 4y^2+24y-9x^2+36x&=-36\\ 4(y^2+6y)-9(x^2-4x)&=-36\\ 4(y^2+6y+9)-9(x^2-4x+4)&=-36-36+36\\ 4(y+3)^2-9(x-2)^2&=-36.\end{aligned}$$

Dividing both sides by -36 puts the equation in standard form:

$$\frac{4(y+3)^2}{-36}-\frac{9(x-2)^2}{-36}=1$$
$$\frac{(x-2)^2}{4}-\frac{(y+3)^2}{9}=1.$$

Notice that point $(2,-3)$ is the center of the hyperbola.

Thus, the critical points of the function f are $(2,-3)$ and all points on the hyperbola, $\frac{(x-2)^2}{4}-\frac{(y+3)^2}{9}=1$.

b. First, we calculate $g_x(x,y)$ and $g_y(x,y)$:

$$g_x(x,y)=2x+2y+4$$
$$g_y(x,y)=2x-8y-6.$$

Next, we set each of these expressions equal to zero, which gives a system of equations in x and y:

$$2x+2y+4=0$$
$$2x-8y-6=0.$$

Subtracting the second equation from the first gives $10y+10=0$, so $y=-1$. Substituting this into the first equation gives $2x+2(-1)+4=0$, so $x=-1$.

Therefore $(-1,-1)$ is a critical point of g. There are no points in $\mathbb{R}^2$ that make either partial derivative not exist.

Figure 1.7.1 shows the behavior of the surface at the critical point.

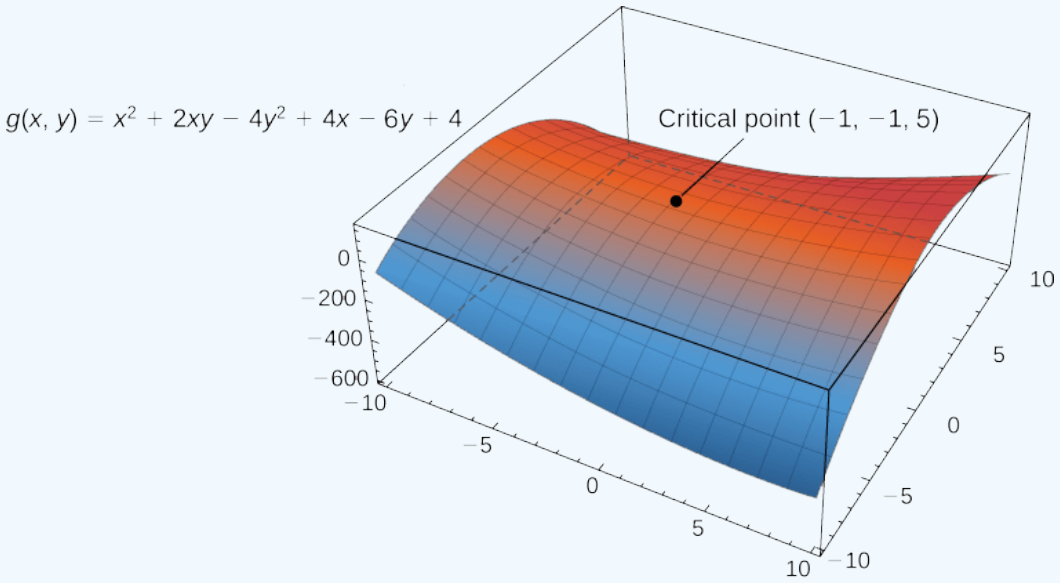


Figure 1.7.1: The graph illustrates the function $g(x, y) = x^2 + 2xy - 4y^2 + 4x - 6y + 4$ with a critical point at $(-1, -1, 5)$. The critical point is located where the derivative in the x and y directions are both zero.

Exercise 1.7.1

Find the critical point of the function $f(x, y) = x^3 + 2xy - 2x - 4y$.

Hint available at the end of this section.

Example 1.7.2: Maximizing Volume With a Fixed Cost

The cost of building the floor and the roof of a rectangular building is twice the cost of the side walls. Find the dimensions of the building with maximum volume that can be built at a fixed cost C.

Solution

- **Video length**: 4 minutes 53 seconds.
- **Context**: This video demonstrates how to maximize volume with a fixed cost.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

The main purpose for determining critical points is to locate relative maxima and minima, as in single-variable calculus. When working with a function of one variable, the definition of a local extremum involves finding an interval around the critical point such that the function value is either greater than or less than all the other function values in that interval. When working with a function of two or more variables, we work with an open disk around the point.

Definition: Global and Local Extrema

Let $z = f(x, y)$ be a function of two variables that is defined and continuous on an open set containing the point (x_0, y_0). Then f has a **local maximum** at (x_0, y_0) if

$$f(x_0, y_0) \geq f(x, y) \tag{1.7.3}$$

for all points (x, y) within some disk centered at (x_0, y_0). The number $f(x_0, y_0)$ is called a local maximum value. If the preceding inequality holds for every point (x, y) in the domain of f, then f has a **global maximum** (also called an absolute maximum) at (x_0, y_0).

The function f has a local minimum at (x_0, y_0) if

$$f(x_0, y_0) \leq f(x, y) \tag{1.7.4}$$

for all points (x, y) within some disk centered at (x_0, y_0). The number $f(x_0, y_0)$ is called a local minimum value. If the preceding inequality holds for every point (x, y) in the domain of f, then f has a **global minimum** (also called an absolute minimum) at (x_0, y_0).

If $f(x_0, y_0)$ is either a local maximum or local minimum value, then it is called a **local extremum** (Figure 1.7.2).

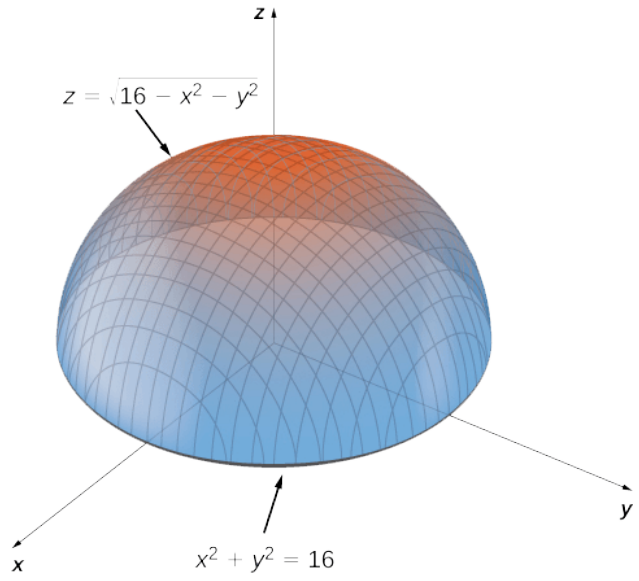


Figure 1.7.2: The graph of $z = \sqrt{16 - x^2 - y^2}$ is an upper hemisphere. It has a maximum value when $(x, y) = (0, 0)$. It attains its minimum value at the boundary of its domain, which is the circle $x^2 + y^2 = 16$.

In Calculus 1, we showed that extrema of functions of one variable occur at critical points. The same is true for functions of more than one variable, as stated in the following theorem.

Fermat's Theorem for Functions of Two Variables

Let $z = f(x, y)$ be a function of two variables that is defined and continuous on an open set containing the point (x_0, y_0). Suppose f_x and f_y each exist at (x_0, y_0). If f has a local extremum at (x_0, y_0), then (x_0, y_0) is a critical point of f.

Second Derivative Test

Consider the function $f(x) = x^3$. This function has a critical point at $x = 0$, since $f'(0) = 3(0)^2 = 0$. However, f does not have an extreme value at $x = 0$. Therefore, the existence of a critical value at $x = x_0$ does not guarantee a local extremum at $x = x_0$. The same is true for a function of two or more variables. One way this can happen is at a **saddle point**. An example of a saddle point

appears in Figure 1.7.3.

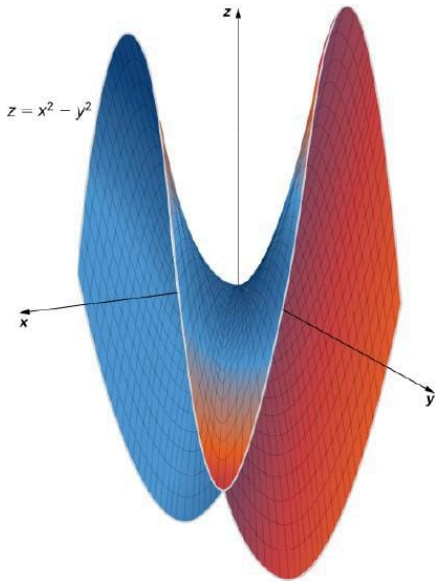


Figure 1.7.3: Graph of the function $z = x^2 - y^2$. This surface looks like a horse saddle, and it has a saddle point at the origin. The origin is both the maximum value of one trace and minimum value of another trace.

In this graph, the origin is a saddle point. This is because the first partial derivatives of f$(x, y) = x^2 - y^2$ are both equal to zero at this point, but it is neither a maximum nor a minimum for the function. Furthermore the vertical trace corresponding to $y = 0$ is $z = x^2$ (a parabola opening upward), but the vertical trace corresponding to $x = 0$ is $z = -y^2$ (a parabola opening downward). Therefore, it is both a global maximum for one trace and a global minimum for another.

Definition: Saddle Point

Given the function $z = f(x, y)$, the point $\big(x_0, y_0, f(x_0, y_0)\big)$ is a saddle point if both $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$, but f does not have a local extremum at (x_0, y_0).

The second derivative test for a function of one variable provides a method for determining whether an extremum occurs at a critical point of a function. When extending this result to a function of two variables, an issue arises related to the fact that there are, in fact, four different second-order partial derivatives, although equality of mixed partials reduces this to three. The second derivative test for a function of two variables, stated in the following theorem, uses a **discriminant** D that replaces $f''(x_0)$ in the second derivative test for a function of one variable.

Second Derivative Test

Let $z = f(x, y)$ be a function of two variables for which the first- and second-order partial derivatives are continuous on some disk containing the point (x_0, y_0). Suppose $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$. Define the quantity

$$D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - \big(f_{xy}(x_0, y_0)\big)^2. \tag{1.7.5}$$

Then:

i. If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then f has a local minimum at (x_0, y_0).
ii. If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then f has a local maximum at (x_0, y_0).
iii. If $D < 0$, then f has a saddle point at (x_0, y_0).
iv. If $D = 0$, then the test is inconclusive.

See Figure 1.7.4.

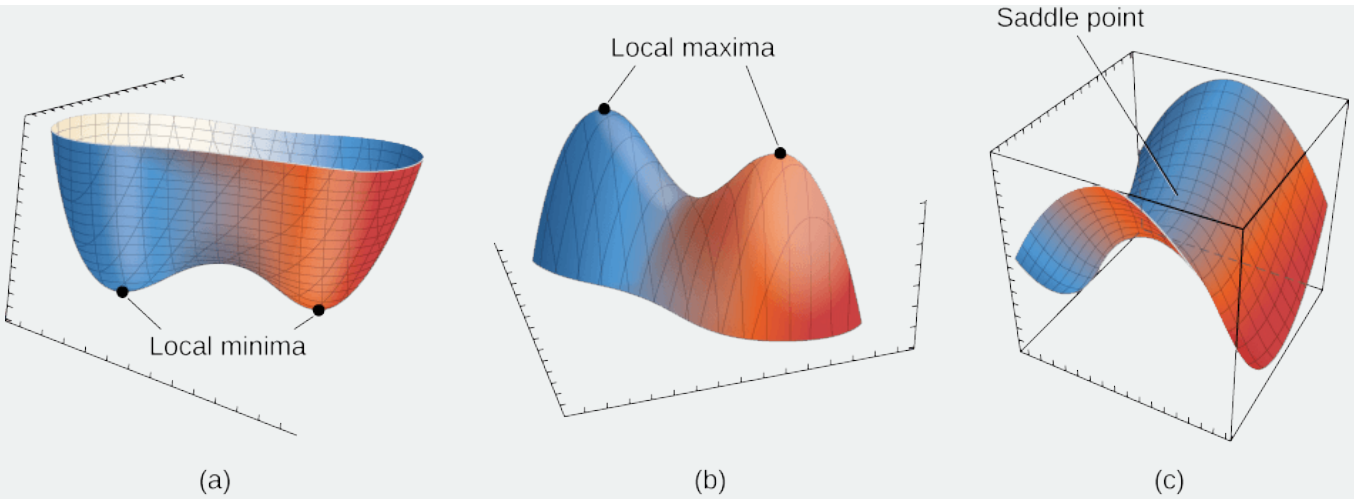


Figure 1.7.4: The second derivative test can often determine whether a function of two variables has local minima (a), local maxima (b), or a saddle point (c). Surface (a) has two bulbous mounds pointing down, and the two extrema are listed as the local minima. Surface (b) has two bulbous mounds pointed up, and the two extrema are listed as the local maxima. Surface (c) is shaped like a saddle, and in the middle of the saddle, a point is marked as the saddle point.

To apply the second derivative test, it is necessary that we first find the critical points of the function. There are several steps involved in the entire procedure, which are outlined in a problem-solving strategy.

Problem-Solving Strategy: Using the Second Derivative Test for Functions of Two Variables

Let $z = f(x, y)$ be a function of two variables for which the first- and second-order partial derivatives are continuous on some disk containing the point (x_0, y_0). To apply the second derivative test to find local extrema, use the following steps:

1. Determine the critical points (x_0, y_0) of the function f where $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$. Discard any points where at least one of the partial derivatives does not exist.
2. Calculate the discriminant $D = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - \left(f_{xy}(x_0, y_0)\right)^2$ for each critical point of f.
3. Apply the four cases of the test to determine whether each critical point is a local maximum, local minimum, or saddle point, or whether the theorem is inconclusive.

Example 1.7.3: Using the Second Derivative Test

Find the critical points for each of the following functions, and use the second derivative test to find the local extrema:

a. $f(x, y) = 4x^2 + 9y^2 + 8x - 36y + 24$
b. $g(x, y) = \frac{1}{3}x^3 + y^2 + 2xy - 6x - 3y + 4$

Solution

a. **Step 1** of the problem-solving strategy involves finding the critical points of f. To do this, we first calculate $f_x(x, y)$ and $f_y(x, y)$, then set each of them equal to zero:

$$f_x(x, y) = 8x + 8$$
$$f_y(x, y) = 18y - 36.$$

Setting them equal to zero yields the system of equations

$$8x + 8 = 0$$
$$18y - 36 = 0.$$

The solution to this system is $x = -1$ and $y = 2$. Therefore $(-1, 2)$ is a critical point of f.

Step 2 of the problem-solving strategy involves calculating D. To do this, we first calculate the second partial derivatives of f:

$$f_{xx}(x, y) = 8$$
$$f_{xy}(x, y) = 0$$
$$f_{yy}(x, y) = 18.$$

Therefore, $D = f_{xx}(-1, 2) f_{yy}(-1, 2) - \left(f_{xy}(-1, 2)\right)^2 = (8)(18) - (0)^2 = 144.$

Step 3 states to apply the four cases of the test to classify the function's behavior at this critical point.

Since $D > 0$ and $f_{xx}(-1, 2) > 0$, this corresponds to case 1. Therefore, f has a local minimum at $(-1, 2)$ as shown in Figure 1.7.5.

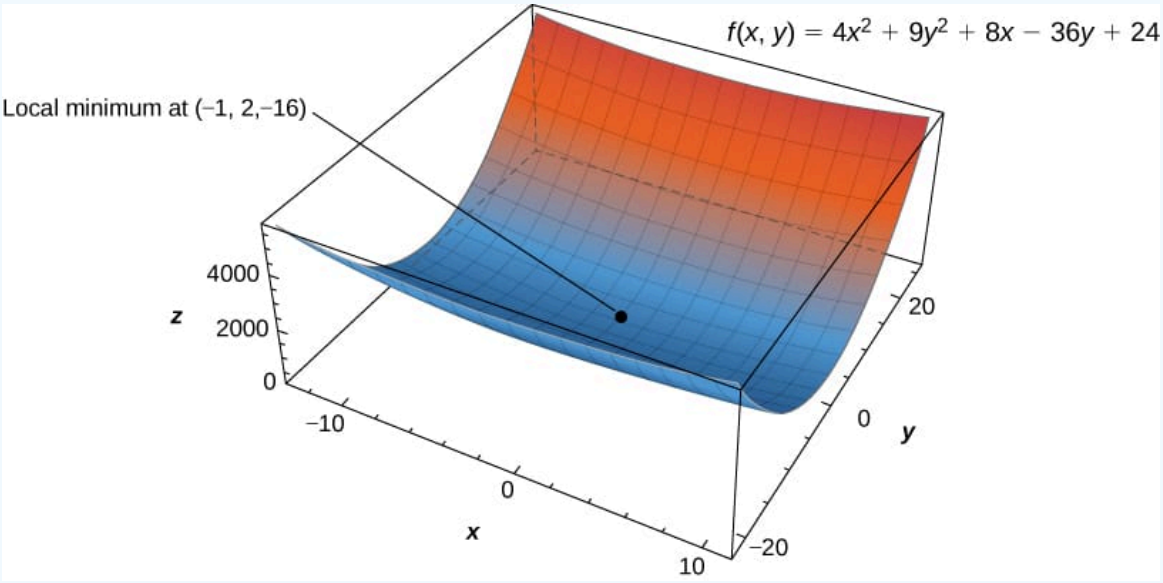


Figure 1.7.5: The function $f(x, y)$ has a local minimum at $(-1, 2, -16)$. The shape is a smooth surface curving up on both ends parallel to the y axis. Note the scale on the y-axis in this plot is in thousands.

b. For **step 1**, we first calculate $g_x(x, y)$ and $g_y(x, y)$, then set each of them equal to zero:

$$g_x(x, y) = x^2 + 2y - 6$$
$$g_y(x, y) = 2y + 2x - 3.$$

Setting them equal to zero yields the system of equations

$$x^2 + 2y - 6 = 0$$
$$2y + 2x - 3 = 0.$$

To solve this system, first solve the second equation for y. This gives $y = \dfrac{3 - 2x}{2}$. Substituting this into the first equation gives

$$x^2 + 3 - 2x - 6 = 0$$
$$x^2 - 2x - 3 = 0$$
$$(x - 3)(x + 1) = 0.$$

Therefore, $x = -1$ or $x = 3$. Substituting these values into the equation $y = \dfrac{3-2x}{2}$ yields the critical points $\left(-1, \frac{5}{2}\right)$ and $\left(3, -\frac{3}{2}\right)$.

Step 2 involves calculating the second partial derivatives of g:

$$\begin{aligned} g_{xx}(x,y) &= 2x \\ g_{xy}(x,y) &= 2 \\ g_{yy}(x,y) &= 2. \end{aligned}$$

Then, we find a general formula for D:

$$\begin{aligned} D(x_0,y_0) &= g_{xx}(x_0,y_0)g_{yy}(x_0,y_0) - \left(g_{xy}(x_0,y_0)\right)^2 \\ &= (2x_0)(2) - 2^2 \\ &= 4x_0 - 4. \end{aligned}$$

Next, we substitute each critical point into this formula:

$$\begin{aligned} D\left(-1, \tfrac{5}{2}\right) &= (2(-1))(2) - (2)^2 = -4 - 4 = -8 \\ D\left(3, -\tfrac{3}{2}\right) &= (2(3))(2) - (2)^2 = 12 - 4 = 8. \end{aligned}$$

In step 3, we note that, $\left(-1, \frac{5}{2}\right)$ leads to case 3, which means that $\left(-1, \frac{5}{2}\right)$ is a saddle point. Applying the theorem to point $\left(3, -\frac{3}{2}\right)$ leads to case 1, which means that $\left(3, -\frac{3}{2}\right)$ corresponds to a local minimum as shown in Figure 1.7.6.

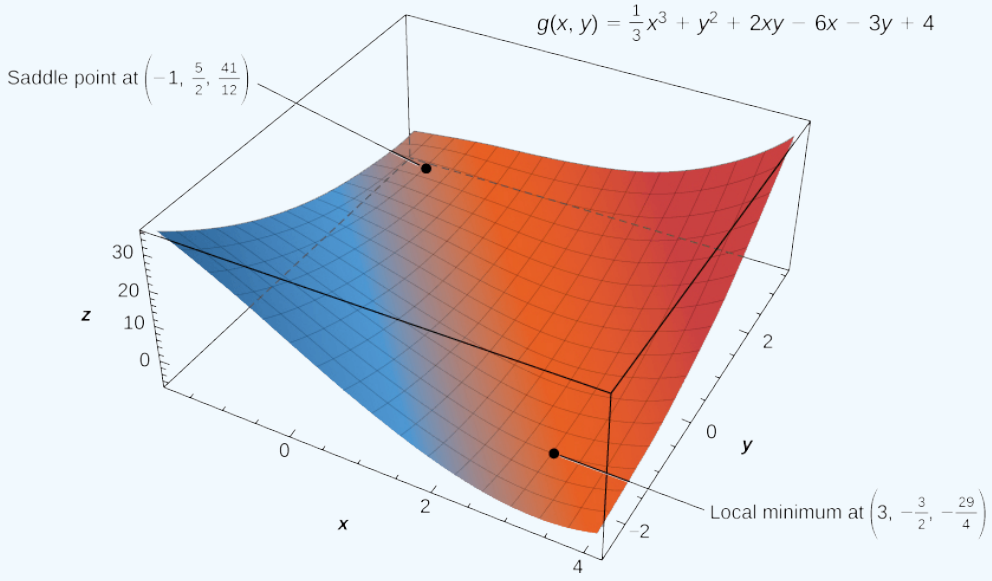


Figure 1.7.6: The function $g(x,y)$ has a local minimum at $(3,-3/2,-29/4)$ and saddle point at $(-1,5/2,41/12)$. The shape is a smooth surface curving up on the corners near $(4,3)$ and $(-2,-2)$.

Watch the accompanying video for a visual demonstration of the concept.

- **Video length**: 6 minutes 29 seconds.
- **Context**: This video demonstrates how to use the second partial derivatives test for a function of two variables.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Exercise 1.7.2

Use the second derivative test to find the local extrema of the function

$$f(x,y) = x^3 + 2xy - 6x - 4y^2.$$

Hint available at the end of this section.

Absolute Maxima and Minima

When finding global extrema of functions of one variable on a closed interval, we start by checking the critical values over that interval and then evaluate the function at the endpoints of the interval. When working with a function of two variables, the closed interval is replaced by a closed, bounded set. A set is bounded if all the points in that set can be contained within a ball (or disk) of finite radius. First, we need to find the critical points inside the set and calculate the corresponding critical values. Then, it is necessary to find the maximum and minimum value of the function on the boundary of the set. When we have all these values, the largest function value corresponds to the global maximum and the smallest function value corresponds to the absolute minimum. First, however, we need to be assured that such values exist. The following theorem does this.

Extreme Value Theorem

A continuous function $f(x,y)$ on a closed and bounded set D in the plane attains an absolute maximum value at some point of D and an absolute minimum value at some point of D.

Now that we know any continuous function f defined on a closed, bounded set attains its extreme values, we need to know how to find them.

Finding Extreme Values of a Function of Two Variables

Assume $z = f(x,y)$ is a differentiable function of two variables defined on a closed, bounded set D. Then f will attain the absolute maximum value and the absolute minimum value, which are, respectively, the largest and smallest values found among the following:

1. The values of f at the critical points of f in D.
2. The values of f on the boundary of D.

The proof of this theorem is a direct consequence of the extreme value theorem and Fermat's theorem. In particular, if either extremum is not located on the boundary of D, then it is located at an interior point of D. But an interior point (x_0, y_0) of D that's an absolute extremum is also a local extremum; hence, (x_0, y_0) is a critical point of f by Fermat's theorem. Therefore the only possible values for the global extrema of f on D are the extreme values of f on the interior or boundary of D.

Problem-Solving Strategy: Finding Absolute Maximum and Minimum Values

Let $z = f(x, y)$ be a continuous function of two variables defined on a closed, bounded set D, and assume f is differentiable on D. To find the absolute maximum and minimum values of f on D, do the following:

1. Determine the critical points of f in D.
2. Calculate f at each of these critical points.
3. Determine the maximum and minimum values of f on the boundary of its domain.
4. The maximum and minimum values of f will occur at one of the values obtained in steps 2 and 3.

Finding the maximum and minimum values of f on the boundary of D can be challenging. If the boundary is a rectangle or set of straight lines, then it is possible to parameterize the line segments and determine the maxima on each of these segments, as seen in Example $1.7.3$. The same approach can be used for other shapes such as circles and ellipses.

If the boundary of the set D is a more complicated curve defined by a function $g(x, y) = c$ for some constant c, and the first-order partial derivatives of g exist, then the method of Lagrange multipliers can prove useful for determining the extrema of f on the boundary which is introduced in Lagrange Multipliers.

Example $1.7.4$: Finding Absolute Extrema

Use the problem-solving strategy for finding absolute extrema of a function to determine the absolute extrema of each of the following functions:

a. $f(x, y) = x^2 - 2xy + 4y^2 - 4x - 2y + 24$ on the domain defined by $0 \leq x \leq 4$ and $0 \leq y \leq 2$
b. $g(x, y) = x^2 + y^2 + 4x - 6y$ on the domain defined by $x^2 + y^2 \leq 16$

Solution

a. Using the problem-solving strategy, step 1 involves finding the critical points of f on its domain. Therefore, we first calculate $f_x(x, y)$ and $f_y(x, y)$, then set them each equal to zero:

$$f_x(x, y) = 2x - 2y - 4$$
$$f_y(x, y) = -2x + 8y - 2.$$

Setting them equal to zero yields the system of equations

$$2x - 2y - 4 = 0$$
$$-2x + 8y - 2 = 0.$$

The solution to this system is $x = 3$ and $y = 1$. Therefore $(3, 1)$ is a critical point of f. Calculating $f(3, 1)$ gives $f(3, 1) = 17.$

The next step involves finding the extrema of f on the boundary of its domain. The boundary of its domain consists of four line segments as shown in Figure $1.7.7$:

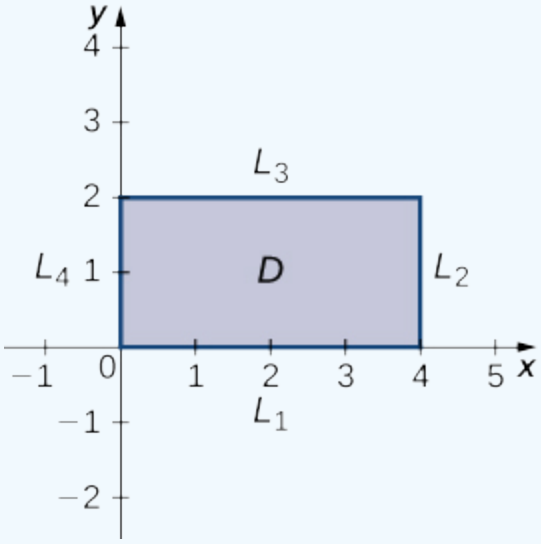


Figure 1.7.7: Graph of the domain of the function $f(x,y) = x^2 - 2xy + 4y^2 - 4x - 2y + 24$. A rectangle is drawn in the first quadrant with one corner at the origin, horizontal length 4, and height 2. This rectangle is marked D, and the sides are marked in counterclockwise order from the side overlapping the x-axis L_1, L_2, L_3, L_4.

L_1 is the line segment connecting $(0,0)$ and $(4,0)$, and it can be parameterized by the equations $x(t) = t, y(t) = 0$ for $0 \le t \le 4$. Define $g(t) = f\big(x(t), y(t)\big)$. This gives $g(t) = t^2 - 4t + 24$. Differentiating g leads to $g\,'(t) = 2t - 4$. Therefore, g has a critical value at $t = 2$, which corresponds to the point $(2,0)$. Calculating $f(2,0)$ gives the z-value 20.

L_2 is the line segment connecting $(4,0)$ and $(4,2)$, and it can be parameterized by the equations $x(t) = 4, y(t) = t$ for $0 \le t \le 2$. Again, define $g(t) = f\big(x(t), y(t)\big)$. This gives $g(t) = 4t^2 - 10t + 24$. Then, $g\,'(t) = 8t - 10$. g has a critical value at $t = \frac{5}{4}$, which corresponds to the point $\left(0, \frac{5}{4}\right)$. Calculating $f\left(0, \frac{5}{4}\right)$ gives the z-value 27.75.

L_3 is the line segment connecting $(0,2)$ and $(4,2)$, and it can be parameterized by the equations $x(t) = t, y(t) = 2$ for $0 \le t \le 4$. Again, define $g(t) = f\big(x(t), y(t)\big)$. This gives $g(t) = t^2 - 8t + 36$. The critical value corresponds to the point $(4,2)$. So, calculating $f(4,2)$ gives the z-value 20.

L_4 is the line segment connecting $(0,0)$ and $(0,2)$, and it can be parameterized by the equations $x(t) = 0, y(t) = t$ for $0 \le t \le 2$. This time, $g(t) = 4t^2 - 2t + 24$ and the critical value $t = \frac{1}{4}$ correspond to the point $\left(0, \frac{1}{4}\right)$. Calculating $f\left(0, \frac{1}{4}\right)$ gives the z-value 23.75.

We also need to find the values of $f(x,y)$ at the corners of its domain. These corners are located at $(0,0), (4,0), (4,2)$ and $(0,2)$:

$$f(0,0) = (0)^2 - 2(0)(0) + 4(0)^2 - 4(0) - 2(0) + 24 = 24$$
$$f(4,0) = (4)^2 - 2(4)(0) + 4(0)^2 - 4(4) - 2(0) + 24 = 24$$
$$f(4,2) = (4)^2 - 2(4)(2) + 4(2)^2 - 4(4) - 2(2) + 24 = 20$$
$$f(0,2) = (0)^2 - 2(0)(2) + 4(2)^2 - 4(0) - 2(2) + 24 = 36.$$

The absolute maximum value is 36, which occurs at $(0,2)$, and the global minimum value is 20, which occurs at both $(4,2)$ and $(2,0)$ as shown in Figure $1.7.8$.

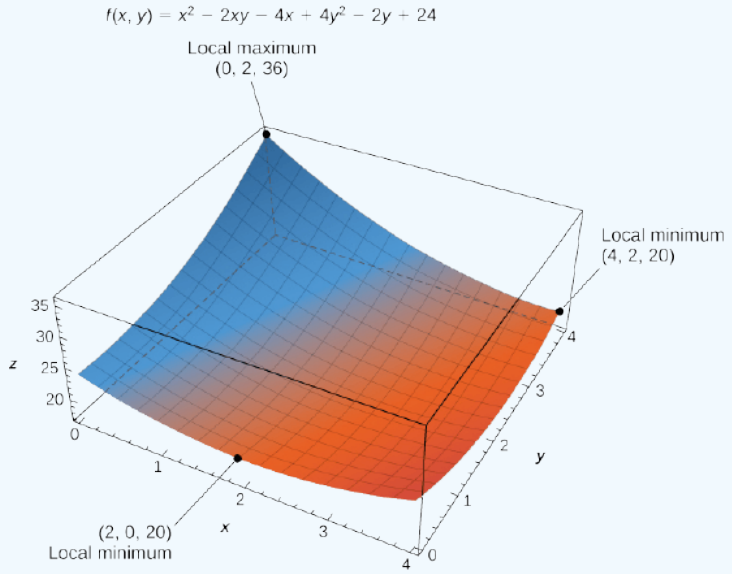


Figure $1.7.8$: The function $f(x,y)$ has two global minima and one global maximum over its domain. Local minima at $(4,2,20)$ and $(2,0,20)$ and local maximum at $(0,2,36)$. The shape is a smooth surface curving up on the corners significantly near $(0,2)$ and slightly less near $(4,0)$.

b. Using the problem-solving strategy, step 1 involves finding the critical points of g on its domain. Therefore, we first calculate $g_x(x,y)$ and $g_y(x,y)$, then set them each equal to zero:

$$g_x(x,y) = 2x + 4$$
$$g_y(x,y) = 2y - 6.$$

Setting them equal to zero yields the system of equations

$$2x + 4 = 0$$
$$2y - 6 = 0.$$

The solution to this system is $x = -2$ and $y = 3$. Therefore, $(-2,3)$ is a critical point of g. Calculating $g(-2,3)$, we get

$$g(-2,3) = (-2)^2 + 3^2 + 4(-2) - 6(3) = 4 + 9 - 8 - 18 = -13. \tag{1.7.6}$$

The next step involves finding the extrema of g on the boundary of its domain. The boundary of its domain consists of a circle of radius 4 centered at the origin as shown in Figure $1.7.9$.

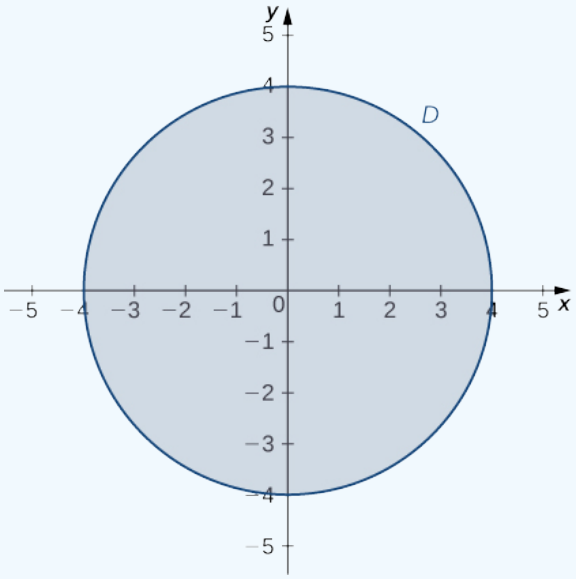


Figure 1.7.9: Graph of the restricted domain of the function $g(x,y)=x^2+y^2+4x-6y$. It is a disk of radius 4 centered at origin.

The boundary of the domain of g can be parameterized using the functions $x(t)=4\cos t,\ \ y(t)=4\sin t$ for $0\le t\le 2\pi$. Define $h(t)=g\big(x(t),y(t)\big)$:

$$\begin{aligned} h(t) &= g\big(x(t),y(t)\big) \\ &= (4\cos t)^2+(4\sin t)^2+4(4\cos t)-6(4\sin t) \\ &= 16\cos^2 t+16\sin^2 t+16\cos t-24\sin t \\ &= 16+16\cos t-24\sin t. \end{aligned}$$

Setting $h\,'(t)=0$ leads to

$$\begin{aligned} -16\sin t-24\cos t &= 0 \\ -16\sin t &= 24\cos t \\ \frac{-16\sin t}{-16\cos t} &= \frac{24\cos t}{-16\cos t} \\ \tan t &= -\frac{3}{2}. \end{aligned}$$

This equation has two solutions over the interval $0\le t\le 2\pi$. One is $t=\pi-\arctan(\frac{3}{2})$ and the other is $t=2\pi-\arctan(\frac{3}{2})$. For the first angle,

$$\sin t=\sin(\pi-\arctan(\tfrac{3}{2}))=\sin(\arctan(\tfrac{3}{2}))=\frac{3\sqrt{13}}{13}$$

$$\cos t=\cos(\pi-\arctan(\tfrac{3}{2}))=-\cos(\arctan(\tfrac{3}{2}))=-\frac{2\sqrt{13}}{13}.$$

Therefore, $x(t)=4\cos t=-\frac{8\sqrt{13}}{13}$ and $y(t)=4\sin t=\frac{12\sqrt{13}}{13}$, so $\left(-\frac{8\sqrt{13}}{13},\frac{12\sqrt{13}}{13}\right)$ is a critical point on the boundary and

$$\begin{aligned} g\left(-\tfrac{8\sqrt{13}}{13},\tfrac{12\sqrt{13}}{13}\right) &= \left(-\tfrac{8\sqrt{13}}{13}\right)^2+\left(\tfrac{12\sqrt{13}}{13}\right)^2+4\left(-\tfrac{8\sqrt{13}}{13}\right)-6\left(\tfrac{12\sqrt{13}}{13}\right) \\ &= \frac{144}{13}+\frac{64}{13}-\frac{32\sqrt{13}}{13}-\frac{72\sqrt{13}}{13} \\ &= \frac{208-104\sqrt{13}}{13}\approx -12.844. \end{aligned}$$

For the second angle,

$$\sin t = \sin(2\pi - \arctan(\tfrac{3}{2})) = -\sin(\arctan(\tfrac{3}{2})) = -\frac{3\sqrt{13}}{13}$$

$$\cos t = \cos(2\pi - \arctan(\tfrac{3}{2})) = \cos(\arctan(\tfrac{3}{2})) = \frac{2\sqrt{13}}{13}.$$

Therefore, $x(t) = 4\cos t = \frac{8\sqrt{13}}{13}$ and $y(t) = 4\sin t = -\frac{12\sqrt{13}}{13}$, so $\left(\frac{8\sqrt{13}}{13}, -\frac{12\sqrt{13}}{13}\right)$ is a critical point on the boundary and

$$\begin{aligned} g\left(\tfrac{8\sqrt{13}}{13}, -\tfrac{12\sqrt{13}}{13}\right) &= \left(\tfrac{8\sqrt{13}}{13}\right)^2 + \left(-\tfrac{12\sqrt{13}}{13}\right)^2 + 4\left(\tfrac{8\sqrt{13}}{13}\right) - 6\left(-\tfrac{12\sqrt{13}}{13}\right) \\ &= \frac{144}{13} + \frac{64}{13} + \frac{32\sqrt{13}}{13} + \frac{72\sqrt{13}}{13} \\ &= \frac{208 + 104\sqrt{13}}{13} \approx 44.844. \end{aligned}$$

The absolute minimum of g is -13, which is attained at the point $(-2, 3)$, which is an interior point of D. The absolute maximum of g is approximately equal to 44.844, which is attained at the boundary point $\left(\frac{8\sqrt{13}}{13}, -\frac{12\sqrt{13}}{13}\right)$. These are the absolute extrema of g on D as shown in Figure 1.7.10.

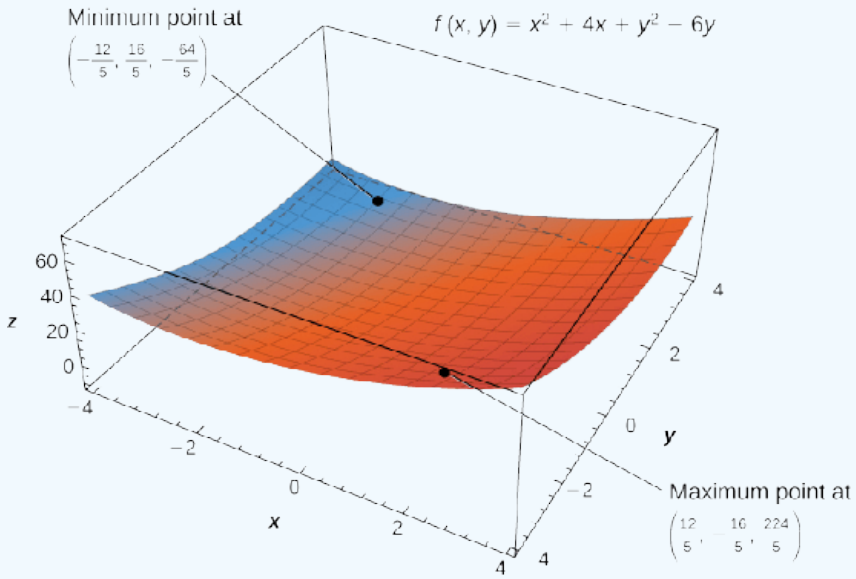


Figure 1.7.10: The function has a local minimum and a local maximum. Local minimum at $(-12/5, 16/5, -64/5)$ and local maximum at $(12/5, -16/5, 224/5)$. The shape is a smooth surface curving up from near $(-4, 4)$ to $(4, -4)$.

Watch the accompanying video for a visual demonstration of the concept.

- **Video length**: 5 minutes 5 seconds.
- **Context**: This video demonstrates how to find the absolute extrema for a function of two variables.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

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Exercise 1.7.3:

Use the problem-solving strategy for finding absolute extrema of a function to find the absolute extrema of the function

$$f(x,y) = 4x^2 - 2xy + 6y^2 - 8x + 2y + 3$$

on the domain defined by $0 \leq x \leq 2$ and $-1 \leq y \leq 3$.

Hint available at the end of this section.

Example 1.7.5: Profitable Golf Balls

Pro-T company has developed a profit model that depends on the number x of golf balls sold per month (measured in thousands), and the number of hours per month of advertising y, according to the function

$$z = f(x,y) = 48x + 96y - x^2 - 2xy - 9y^2, \tag{1.7.7}$$

where z is measured in thousands of dollars. The maximum number of golf balls that can be produced and sold is $50,000$, and the maximum number of hours of advertising that can be purchased is 25. Find the values of x and y that maximize profit, and find the maximum profit.



Figure 1.7.11: A basket full of golf balls. (CC-BY-2.0) modification of work by oatsy40, Flickr)

Solution

Using the problem-solving strategy, step 1 involves finding the critical points of f on its domain. Therefore, we first calculate $f_x(x,y)$ and $f_y(x,y)$, then set them each equal to zero:

$$f_x(x,y) = 48 - 2x - 2y$$
$$f_y(x,y) = 96 - 2x - 18y.$$

Setting them equal to zero yields the system of equations

$$48 - 2x - 2y = 0$$
$$96 - 2x - 18y = 0.$$

The solution to this system is $x = 21$ and $y = 3$. Therefore $(21,3)$ is a critical point of f. Calculating $f(21,3)$ gives $f(21,3) = 48(21) + 96(3) - 21^2 - 2(21)(3) - 9(3)^2 = 648$.

The domain of this function is $0 \leq x \leq 50$ and $0 \leq y \leq 25$ as shown in Figure 1.7.12.

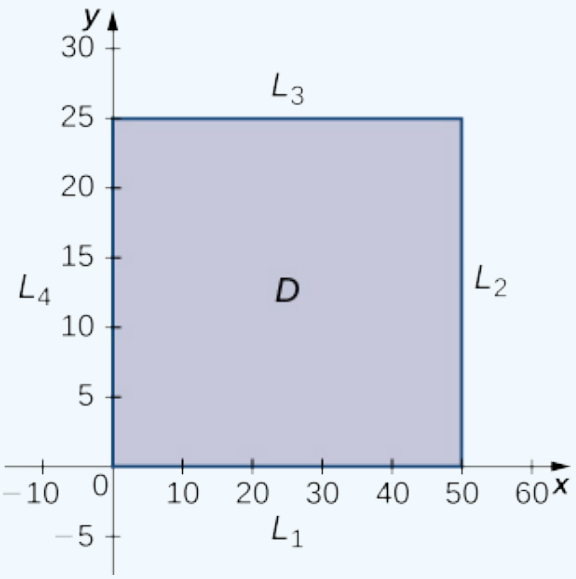


Figure 1.7.12: Graph of the domain of the function $f(x,y) = 48x + 96y - x^2 - 2xy - 9y^2$. A rectangle is drawn in the first quadrant with one corner at the origin, horizontal length 50, and height 25. This rectangle is marked D, and the sides are marked in counterclockwise order from the side overlapping the x-axis L_1, L_2, L_3, L_4.

L_1 is the line segment connecting $(0,0)$ and $(50,0)$, and it can be parameterized by the equations $x(t) = t, y(t) = 0$ for $0 \leq t \leq 50$. We then define $g(t) = f(x(t), y(t))$:

$$\begin{aligned} g(t) &= f(x(t), y(t)) \\ &= f(t, 0) \\ &= 48t + 96(0) - y^2 - 2(t)(0) - 9(0)^2 \\ &= 48t - t^2. \end{aligned}$$

Setting $g\,'(t) = 0$ yields the critical point $t = 24$, which corresponds to the point $(24,0)$ in the domain of f. Calculating $f(24,0)$ gives 576.

L_2 is the line segment connecting $(50,0)$ and $(50,25)$, and it can be parameterized by the equations $x(t) = 50, y(t) = t$ for $0 \leq t \leq 25$. Once again, we define $g(t) = f\big(x(t), y(t)\big)$:

$$\begin{aligned} g(t) &= f\big(x(t), y(t)\big) \\ &= f(50, t) \\ &= 48(50) + 96t - 50^2 - 2(50)t - 9t^2 \\ &= -9t^2 - 4t - 100. \end{aligned}$$

This function has a critical point at $t = -\frac{2}{9}$, which corresponds to the point $(50, -29)$. This point is not in the domain of f.

L_3 is the line segment connecting $(0,25)$ and $(50,25)$, and it can be parameterized by the equations $x(t) = t, y(t) = 25$ for $0 \leq t \leq 50$. We define $g(t) = f\big(x(t), y(t)\big)$:

$$\begin{aligned} g(t) &= f\big(x(t), y(t)\big) \\ &= f(t, 25) \\ &= 48t + 96(25) - t^2 - 2t(25) - 9(25^2) \\ &= -t^2 - 2t - 3225. \end{aligned}$$

This function has a critical point at $t = -1$, which corresponds to the point $(-1, 25)$, which is not in the domain.

L_4 is the line segment connecting $(0,0)$ to $(0,25)$, and it can be parameterized by the equations $x(t)=0, y(t)=t$ for $0 \le t \le 25$. We define $g(t)=f\big(x(t),y(t)\big)$:

$$\begin{aligned} g(t) &= f\big(x(t),y(t)\big) \\ &= f(0,t) \\ &= 48(0)+96t-(0)^2-2(0)t-9t^2 \\ &= 96t-9t^2. \end{aligned}$$

This function has a critical point at $t=\frac{16}{3}$, which corresponds to the point $\left(0,\frac{16}{3}\right)$, which is on the boundary of the domain. Calculating $f\left(0,\frac{16}{3}\right)$ gives 256.

We also need to find the values of $f(x,y)$ at the corners of its domain. These corners are located at $(0,0),(50,0),(50,25)$ and $(0,25)$:

$$\begin{aligned} f(0,0) &= 48(0)+96(0)-(0)^2-2(0)(0)-9(0)^2=0 \\ f(50,0) &= 48(50)+96(0)-(50)^2-2(50)(0)-9(0)^2=-100 \\ f(50,25) &= 48(50)+96(25)-(50)^2-2(50)(25)-9(25)^2=-5825 \\ f(0,25) &= 48(0)+96(25)-(0)^2-2(0)(25)-9(25)^2=-3225. \end{aligned}$$

The maximum value is 648, which occurs at $(21,3)$. Therefore, a maximum profit of $\$648,000$ is realized when $21,000$ golf balls are sold and 3 hours of advertising are purchased per month as shown in Figure $1.7.13$.

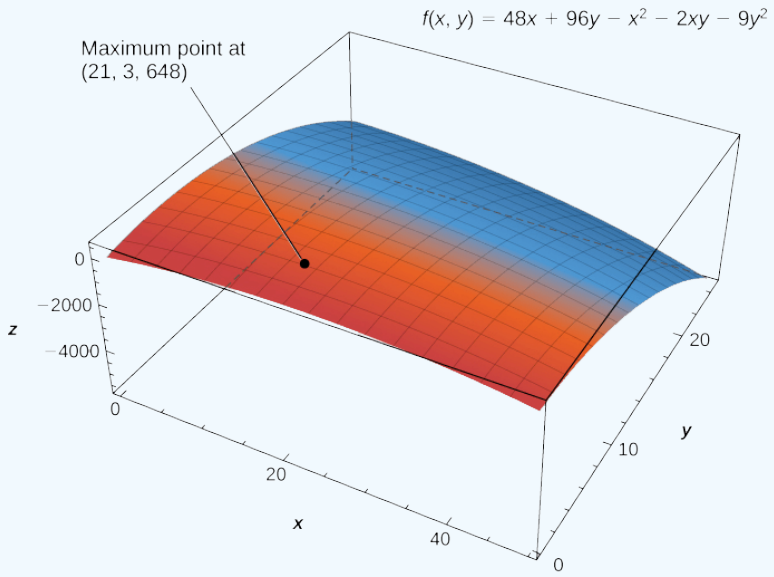


Figure $1.7.13$: The image illustrates the profit function $f(x,y)=48x+96y-x^2-2xy-9y^2$ with a maximum at $(21,3,648)$. The shape is a smooth surface curving from near the origin down to $(50,25)$.

Key Concepts

- A critical point of the function $f(x,y)$ is any point (x_0,y_0) where either $f_x(x_0,y_0)=f_y(x_0,y_0)=0$, or at least one of $f_x(x_0,y_0)$ and $f_y(x_0,y_0)$ does not exist.
- A saddle point is a point (x_0,y_0) where $f_x(x_0,y_0)=f_y(x_0,y_0)=0$, but $f(x_0,y_0)$ is neither a maximum nor a minimum at that point.
- To find extrema of functions of two variables, first find the critical points, then calculate the discriminant and apply the second derivative test.

Key Equations

- **Discriminant**

$$D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2$$

Glossary

Critical point of a function of two variables

the point (x_0, y_0) is called a critical point of $f(x, y)$ if one of the two following conditions holds:

1. $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$
2. At least one of $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ do not exist

Discriminant

the discriminant of the function $f(x, y)$ is given by the formula $D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2$

Saddle point

given the function $z = f(x, y)$, the point $(x_0, y_0, f(x_0, y_0))$ is a saddle point if both $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$, but f does not have a local extremum at (x_0, y_0)

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Answers

Exercise 1.7.1

Calculate $f_x(x, y)$ and $f_y(x, y)$, then set them equal to zero.

The only critical point of f is $(2, -5)$.

Exercise 1.7.2

Follow the problem-solving strategy for applying the second derivative test.

$(\frac{4}{3}, \frac{1}{3})$ is a saddle point, $(-\frac{3}{2}, -\frac{3}{8})$ is a local maximum.

Exercise 1.7.3**:**

Calculate $f_x(x, y)$ and $f_y(x, y)$, and set them equal to zero. Then, calculate f for each critical point and find the extrema of f on the boundary of D.

The absolute minimum occurs at $(1, 0) : f(1, 0) = -1$.

The absolute maximum occurs at $(0, 3) : f(0, 3) = 63$.

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1.7E: Exercises for Section 1.7

Exercises 1 - 5

Find all critical points.

1. $f(x,y) = 1 + x^2 + y^2$

   Answer

   $(0,0)$

2. $f(x,y) = 1 - (x-2)^2 + (y+3)^2$
3. $f(x,y) = (3x-2)^2 + (y-4)^2$

   Answer

   $\left(\frac{2}{3}, 4\right)$

4. $f(x,y) = x^4 + y^4 - 16xy$

   Answer

   $(0,0), \quad (-2,-2), \quad (2,2)$

5. $f(x,y) = 15x^3 - 3xy + 15y^3$

   Answer

   $(0,0), \quad \left(\frac{1}{15}, \frac{1}{15}\right)$

Exercises 6 - 9

Find the critical points of the function and test for extrema or saddle points by using algebraic techniques (completing the square) or by examining the form of the equation. Where possible, verify your results using the Second Partials Test.

6. $f(x,y) = -\sqrt{x^2 + y^2}$

   Answer

   Critical points: $(0,0)$
   Extrema: f has a relative maximum of 0 at $(0,0)$.
   To justify this, consider the fact that the square root function cannot give a negative value, so this function cannot possibly return a positive value. Since its value is 0 at the critical point $(0,0)$, we know it must be the function's **absolute maximum** value.

7. $f(x,y) = -x^2 - 5y^2 + 8x - 10y - 13$

   Answer

   Crit. pts.: $(4,-1)$
   Extrema: f has a relative maximum of 8 at $(4,-1)$.
   To justify this, we complete the square on this function, being careful to factor out the coefficient of the squared terms before completing the square.

$$\begin{aligned} f(x,y) &= -x^2 - 5y^2 + 8x - 10y - 13 \\ &= -(x^2 - 8x \quad\quad) - 5(y^2 + 2y \quad\quad) - 13 \\ &= -(x^2 - 8x + 16) - 5(y^2 + 2y + 1) - 13 + 16 + 5 \\ &= -(x-4)^2 - 5(y+1)^2 + 8 \end{aligned}$$

Note that this quadratic polynomial function takes the form $z = -(x^2 + y^2)$, so we can see that it will have a **relative** (and, in fact, absolute) **maximum** at its vertex (the critical point $(4, -1)$). We can also argue that since we are subtracting squared terms from 8, we cannot possibly obtain a function value larger than 8, and since we do get a value of 8 at the critical point $(4, -1)$, we know it will be the absolute maximum value of this function.

8. $f(x, y) = x^2 + y^2 + 2x - 6y + 6$
9. $f(x, y) = \sqrt{x^2 + y^2} + 1$

Answer

Crit. Pts.: $(0, 0)$
Extrema: f has a relative minimum of 1 at $(0, 0)$.
To justify this, consider the fact that the square root function cannot give a negative value, so this function cannot possibly return a value less than 1. Since its value is 1 at the critical point $(0, 0)$, we know 1 must be the function's **absolute minimum** value.

Exercises 10 - 34

Identify any critical points and use the Second Partials Test to determine the behavior of the function at each critical point, whether there is a maximum, minimum, saddle point, or none of these. If the Second Partials Test fails, determine the function's behavior at the point using another method and justify your answer clearly.

10. $f(x, y) = -x^3 + 4xy - 2y^2 + 1$
11. $f(x, y) = x^2 y^2$

Answer

Crit. pts.: All points on the lines $x = 0$ and $y = 0$ are critical points of this function.
Exrema: The Second Partials Test fails.
Since $x^2 y^2 > 0$ for all x and y different from zero, and $x^2 y^2 = 0$ when either x or y equals zero (or both), then the absolute minimum of 0 occurs at all points on the x- or y-axes, that is, for all points on the lines $x = 0$ and $y = 0$.

12. $f(x, y) = x^2 - 6x + y^2 + 4y - 8$
13. $f(x, y) = 2xy + 3x + 4y$

Answer

Crit. pts.: $\left(-2, -\frac{3}{2}\right)$
Exrema: f has a saddle point at $\left(-2, -\frac{3}{2}, -6\right)$.

14. $f(x, y) = 8xy(x + y) + 7$
15. $f(x, y) = x^2 + 4xy + y^2$

Answer

Crit. pts.: $(0, 0)$
Exrema: f has a saddle point at $(0, 0, 0)$.

16. $f(x, y) = x^3 + y^3 - 300x - 75y - 3$
17. $f(x, y) = 9 - x^4 y^4$

Answer

Crit. pts.: All points on the lines $x = 0$ and $y = 0$ are critical points of this function.
Extrema: The Second Partials Test fails.
Since the term $-x^4 y^4 < 0$ for all x and y different from zero, and $-x^4 y^4 = 0$ when either x or y equals zero (or both), then this function cannot attain a value greater than 9 anywhere, but is 9 at the critical points. Thus f has an absolute maximum of 9 at all points on the x- or y-axes, that is, for all points on the lines $x = 0$ and $y = 0$.

18. $f(x, y) = x^2 + 10xy + y^2$

Answer

Crit. pts.: $(0,0)$
Extrema: f has a saddle point at $(0,0,0)$.

19. $f(x,y) = x^4 + y^2 + 2xy + 3$

Answer

Crit. pts.: $(0,0), \quad \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \quad \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
Extrema: f has a saddle point at $(0,0,3)$,
f has a local minimum of 2.75 at the point $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.
f has a local minimum of 2.75 at the point $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

20. $f(x,y) = 7x^2y + 9xy^2$
21. $f(x,y) = 3x^2 - 2xy + y^2 - 8y$

Answer

Crit. pts.: $(2,6)$
Extrema: f has a relative minimum of -24 located at $(2,6)$.

22. $f(x,y) = 3x^2 + 2xy + y^2$
23. $f(x,y) = y^2 + xy + 3y + 2x + 3$

Answer

Crit. pts.: $(1,-2)$
Extrema: f has a saddle point at $(1,-2,1)$.

24. $f(x,y) = x^2 + xy + y^2 - 3x$
25. $f(x,y) = x^2 + 2y^2 - x^2y$

Answer

Crit. pts.: $(0,0), \quad (-2,1), \quad (2,1)$
Extrema: f has a relative minimum of 0 at $(0,0)$ and saddle points at $(2,1,2)$ and $(-2,1,2)$.

26. $f(x,y) = x^2 + y - e^y$
27. $f(x,y) = e^{-(x^2+y^2+2x)}$

Answer

Crit. pts.: $(-1,0)$
Extrema: f has a relative maximum of e located at $(-1,0)$.
See this problem illustrated in CalcPlot3D(opens in new window).

28. $f(x,y) = x^2 + xy + y^2 - x - y + 1$
29. $f(x,y) = x^2y(9 - x + y)$

Answer

Crit. pts.: $\left(\frac{9}{2}, -\frac{9}{4}\right), \quad (9,0)$, and all points on the line $x = 0$
Extrema: f has a saddle point at $(9,0,0)$ and a relative minimum of -102.515625 at $\left(\frac{9}{2}, -\frac{9}{4}\right)$.
At the critical points on the line $x = 0$, f has neither relative extrema nor saddle points, but they do represent a sort of trough on the surface.

30. $f(x,y) = -x^2 - 5y^2 + 10x - 30y - 62$
31. $f(x,y) = 120x + 120y - xy - x^2 - y^2$

Answer

Crit. pts.: $(40,40)$
Extrema: f has a relative maximum of 4800 located at $(40,40)$.

32. $f(x,y) = 2x^2 + 2xy + y^2 + 2x - 3$

33. $f(x,y) = x^2 + x - 3xy + y^3 - 5$

Answer

Crit. pts.: $\left(\frac{1}{4}, \frac{1}{2}\right)$ and $(1,1)$
Extrema: f has a saddle point at $\left(\frac{1}{4}, \frac{1}{2}, -\frac{79}{16}\right)$ and a relative minimum of -5 at $(1,1)$.

34. $f(x,y) = 2xye^{-x^2-y^2}$

Exercises 35 - 37

Determine the extreme values and the saddle points. Use a CAS to graph the function.

35. [T] $f(x,y) = ye^x - e^y$

Answer

A saddle point is located at $(0,0,-1)$.

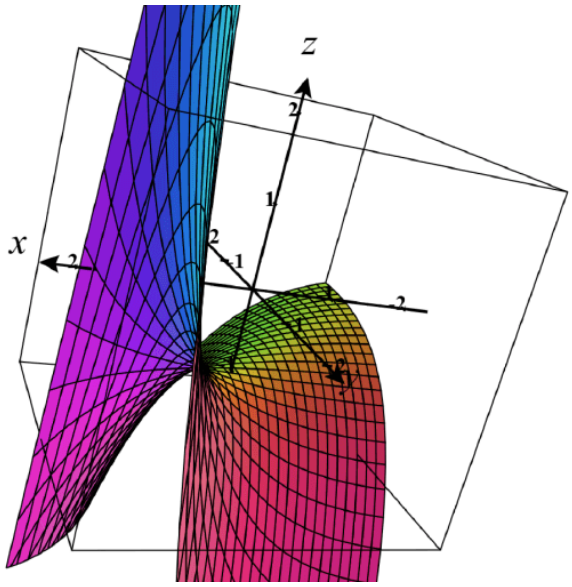


Figure $1.7E.1$: Surface $f(x,y) = ye^x - e^y$ with a saddle point is located at $(0,0,-1)$.

36. [T] $f(x,y) = x\sin(y)$
37. [T] $f(x,y) = \sin(x)\sin(y), \quad x \in (0, 2\pi), \quad y \in (0, 2\pi)$

Answer

There is a saddle point at (π, π), local maxima at $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)$, and local minima at $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ and $\left(\frac{3\pi}{2}, \frac{\pi}{2}\right)$.

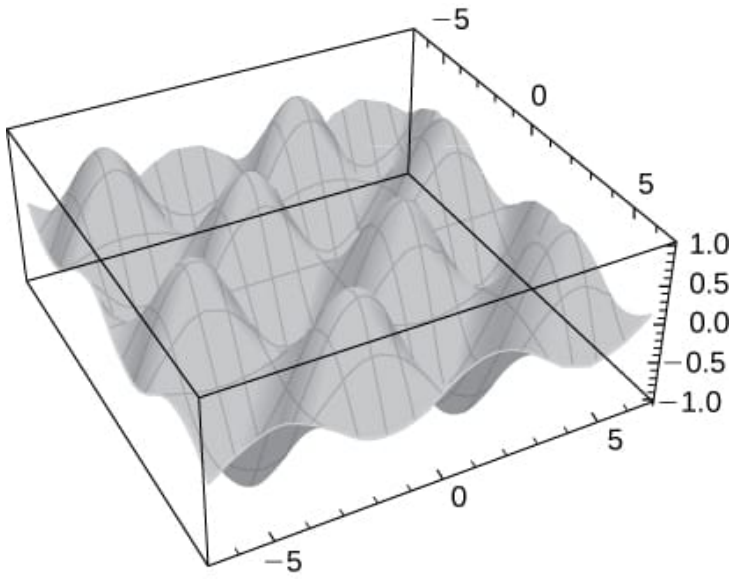


Figure $1.7E.2$: Surface $f(x,y)=\sin(x)\sin(y)$ with symmetrical local minimums and maximums.

Exercises 38 - 41

Find the absolute extrema of the given function on the indicated closed and bounded set R.

38. $f(x,y)=xy-x-3y$ where R is the triangular region with vertices $(0,0),\ (0,4)$, and $(5,0)$.
39. Find the absolute maximum and minimum values of $f(x,y)=x^2+y^2-2y+1$ on the region $R=\{(x,y)\mid x^2+y^2\le 4\}$.

Answer

$(0,1,0)$ is the absolute minimum and $(0,-2,9)$ is the absolute maximum.

40. $f(x,y)=x^3-3xy-y^3$ on $R=\{(x,y)\mid -2\le x\le 2,\ -2\le y\le 2\}$
41. $f(x,y)=\dfrac{-2y}{x^2+y^2+1}$ on $R=\{(x,y)\mid x^2+y^2\le 4\}$

Answer

There is an absolute minimum at $(0,1,-1)$ and an absolute maximum at $(0,-1,1)$.

42. Find three positive numbers the sum of which is 27, such that the sum of their squares is as small as possible.
43. Find the points on the surface $x^2-yz=5$ that are closest to the origin.

Answer

$\left(\sqrt{5},0,0\right),\ \left(-\sqrt{5},0,0\right)$

44. Find the maximum volume of a rectangular box with three faces in the coordinate planes and a vertex in the first octant on the line $x+y+z=1$.
45. The sum of the length and the girth (perimeter of a cross-section) of a package carried by a delivery service cannot exceed 108 in. Find the dimensions of the rectangular package of largest volume that can be sent.

Answer

18 by 36 by 18 in.

46. A cardboard box without a lid is to be made with a volume of 4 ft^3. Find the dimensions of the box that requires the least amount of cardboard.
47. Find the point on the surface $f(x,y)=x^2+y^2+10$ nearest the plane $x+2y-z=0$. Identify the point on the plane.

Answer

$\left(\frac{47}{24}, \frac{47}{12}, \frac{235}{24}\right)$

18. Find the point in the plane $2x - y + 2z = 16$ that is closest to the origin.
19. A company manufactures two types of athletic shoes: jogging shoes and cross-trainers. The total revenue from x units of jogging shoes and y units of cross-trainers is given by $R(x, y) = -5x^2 - 8y^2 - 2xy + 42x + 102y$, where x and y are in thousands of units. Find the values of x and y to maximize the total revenue.

Answer

$x = 3$ and $y = 6$

50. A shipping company handles rectangular boxes provided the sum of the length, width, and height of the box does not exceed 96in. Find the dimensions of the box that meets this condition and has the largest volume.
51. Find the maximum volume of a cylindrical soda can such that the sum of its height and circumference is 120 cm.

Answer

$V = \frac{64,000}{\pi} \approx 20,372 \text{ cm}^3$

Contributors

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- Paul Seeburger (Monroe Community College) created problems 19 and 29, and added dynamic figures for problems 27 and 35.

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1.8: Lagrange Multipliers

Learning Objectives

- Use the method of Lagrange multipliers to solve optimization problems with one constraint.
- Use the method of Lagrange multipliers to solve optimization problems with two constraints.

Solving optimization problems for functions of two or more variables can be similar to solving such problems in single-variable calculus. However, techniques for dealing with multiple variables allow us to solve more varied optimization problems for which we need to deal with additional conditions or constraints. In this section, we examine one of the more common and useful methods for solving optimization problems with constraints.

Lagrange Multipliers

In the previous section, an applied situation was explored involving maximizing a profit function, subject to certain **constraints.** In that example, the constraints involved a maximum number of golf balls that could be produced and sold in 1 month (x), and a maximum number of advertising hours that could be purchased per month (y). Suppose these were combined into a single budgetary constraint, such as $20x + 4y \leq 216$, that took into account both the cost of producing the golf balls and the number of advertising hours purchased per month. The goal is still to maximize profit, but now there is a different type of constraint on the values of x and y. This constraint and the corresponding profit function

$$f(x, y) = 48x + 96y - x^2 - 2xy - 9y^2$$

is an example of an **optimization problem**, and the function $f(x, y)$ is called the **objective function**. A graph of various level curves of the function $f(x, y)$ follows.

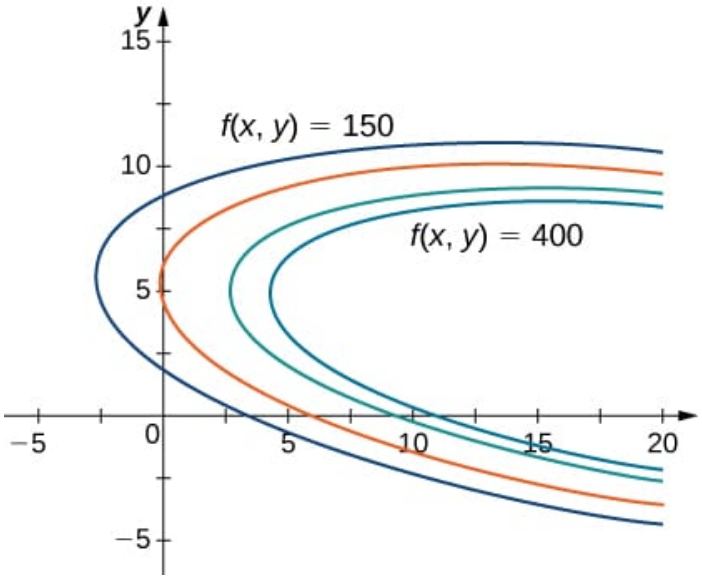


Figure $1.8.1$: Graph showing level curves of the function $f(x, y) = 48x + 96y - x^2 - 2xy - 9y^2$ corresponding to $c = 150, 250, 350,$ and 400. A series of rotated ellipses that become increasingly large. The smallest one is marked $f(x, y) = 400$, and the biggest one is marked $f(x, y) = 150$.

In Figure $1.8.1$, the value c represents different profit levels (i.e., values of the function f). As the value of c increases, the curve shifts to the right. Since our goal is to maximize profit, we want to choose a curve as far to the right as possible. If there were no restrictions on the number of golf balls the company could produce or the number of units of advertising available, then we could produce as many golf balls as we want, and advertise as much as we want, and there would not be a maximum profit for the company. Unfortunately, we have a budgetary constraint that is modeled by the inequality $20x + 4y \leq 216$. To see how this

constraint interacts with the profit function, Figure $1.8.2$ shows the graph of the line $20x + 4y = 216$ superimposed on the previous graph.

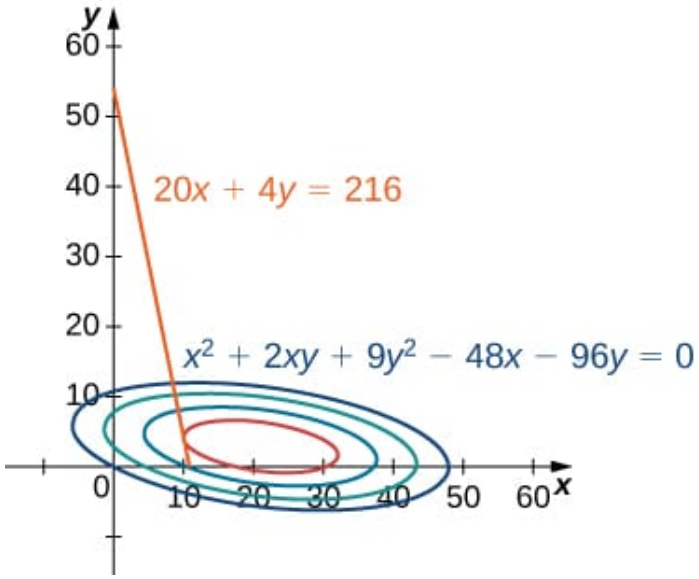


Figure $1.8.2$: Graph of level curves of the function $f(x, y) = 48x + 96y - x^2 - 2xy - 9y^2$ corresponding to $c = 150, 250, 350,$ and 395. A series of rotated ellipses that become increasingly large. On the smallest ellipse, which is red, there is a straight line marked with the constraint equation $20x + 4y = 216$ that appears to touch the ellipse near $(10, 4)$.

As mentioned previously, the maximum profit occurs when the level curve is as far to the right as possible. However, the level of production corresponding to this maximum profit must also satisfy the budgetary constraint, so the point at which this profit occurs must also lie on (or to the left of) the red line in Figure $1.8.2$. Inspection of this graph reveals that this point exists where the line is tangent to the level curve of f. Trial and error reveals that this profit level seems to be around 395, when x and y are both just less than 5. We return to the solution of this problem later in this section. From a theoretical standpoint, at the point where the profit curve is tangent to the constraint line, the gradient of both of the functions evaluated at that point must point in the same (or opposite) direction. Recall that the gradient of a function of more than one variable is a vector. If two vectors point in the same (or opposite) directions, then one must be a constant multiple of the other. This idea is the basis of the **method of Lagrange multipliers.**

Method of Lagrange Multipliers: One Constraint

Theorem $1.8.1$: Let f and g be functions of two variables with continuous partial derivatives at every point of some open set containing the smooth curve $g(x, y) = 0$. Suppose that f, when restricted to points on the curve $g(x, y) = 0$, has a local extremum at the point (x_0, y_0) and that $\nabla g(x_0, y_0) \neq 0$. Then there is a number λ called a **Lagrange multiplier,** for which

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0). \tag{1.8.1}$$

Proof

Assume that a constrained extremum occurs at the point (x_0, y_0). Furthermore, we assume that the equation $g(x, y) = 0$ can be smoothly parametrized as

$$x = x(s) \text{ and } y = y(s)$$

where s is an arc length parameter with reference point (x_0, y_0) at $s = 0$. Therefore, the quantity $z = f(x(s), y(s))$ has a relative maximum or relative minimum at $s = 0$, and this implies that $\dfrac{dz}{ds} = 0$ at that point. From the chain rule,

$$\begin{aligned}\frac{dz}{ds} &= \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s}\\ &= \left(\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}\right)\cdot\left(\frac{\partial x}{\partial s}\mathbf{i}+\frac{\partial y}{\partial s}\mathbf{j}\right)\\ &= 0,\end{aligned}$$

where the derivatives are all evaluated at $s=0$. However, the first factor in the dot product is the gradient of f, and the second factor is the unit tangent vector $\mathbf{T}(0)$ to the constraint curve. Since the point (x_0,y_0) corresponds to $s=0$, it follows from this equation that

$$\boldsymbol{\nabla} f(x_0,y_0)\cdot\mathbf{T}(0)=0,$$

which implies that the gradient is either the zero vector $\mathbf{0}$ or it is normal to the constraint curve at a constrained relative extremum. However, the constraint curve $g(x,y)=0$ is a level curve for the function $g(x,y)$ so that if $\boldsymbol{\nabla} g(x_0,y_0)\neq 0$ then $\boldsymbol{\nabla} g(x_0,y_0)$ is normal to this curve at (x_0,y_0) It follows, then, that there is some scalar λ such that

$$\boldsymbol{\nabla} f(x_0,y_0)=\lambda\boldsymbol{\nabla} g(x_0,y_0)$$

□

To apply Theorem $1.8.1$ to an optimization problem similar to that for the golf ball manufacturer, we need a problem-solving strategy.

Problem-Solving Strategy: Steps for Using Lagrange Multipliers

1. Determine the objective function $f(x,y)$ and the constraint function $g(x,y)$. Does the optimization problem involve maximizing or minimizing the objective function?
2. Set up a system of equations using the following template:
$$\boldsymbol{\nabla} f(x_0,y_0)=\lambda\boldsymbol{\nabla} g(x_0,y_0) \tag{1.8.2}$$
$$g(x_0,y_0)=0. \tag{1.8.3}$$
3. Solve for x_0 and y_0.
4. The largest of the values of f at the solutions found in step 3 maximizes f; the smallest of those values minimizes f.

Example $1.8.1$: Using Lagrange Multipliers

Use the method of Lagrange multipliers to find the minimum value of $f(x,y)=x^2+4y^2-2x+8y$ subject to the constraint $x+2y=7$.

Solution

Let's follow the problem-solving strategy:

1. The objective function is $f(x,y)=x^2+4y^2-2x+8y$. To determine the constraint function, we must first subtract 7 from both sides of the constraint. This gives $x+2y-7=0$. The constraint function is equal to the left-hand side, so $g(x,y)=x+2y-7$. The problem asks us to solve for the minimum value of f, subject to the constraint (Figure $1.8.3$).

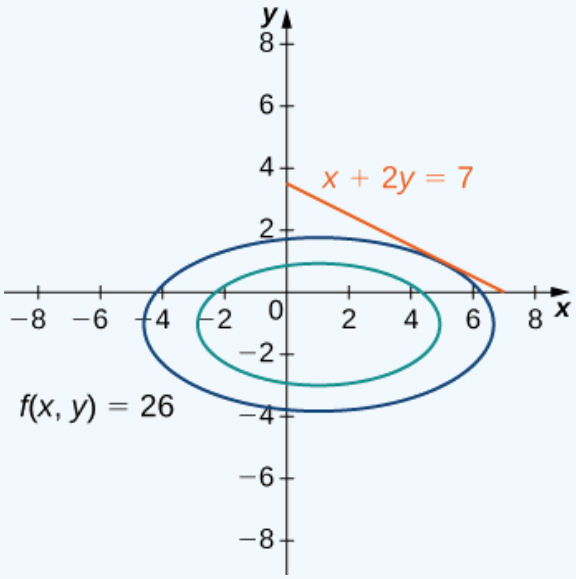


Figure 1.8.3: Graph of two level curves of the function $f(x,y)=x^2+4y^2-2x+8y$ corresponding to $c=10$ and 26, which are two rotated ellipses, one within the other. On the largest ellipse, which is marked $f(x,y)=26$, there is a tangent line marked with equation $x+2y=7$ that appears to touch the ellipse near $(5,1)$.The orange line is the graph of the constraint equation.

2. We then must calculate the gradients of both f and g:

$$\nabla f(x,y)=(2x-2)\,\mathbf{i}+(8y+8)\,\mathbf{j}$$
$$\nabla g(x,y)=\mathbf{i}+2\mathbf{j}.$$

The equation $\nabla f(x_0,y_0)=\lambda\nabla g(x_0,y_0)$ becomes

$$(2x_0-2)\,\mathbf{i}+(8y_0+8)\,\mathbf{j}=\lambda(\mathbf{i}+2\mathbf{j}),$$

which can be rewritten as

$$(2x_0-2)\,\mathbf{i}+(8y_0+8)\,\mathbf{j}=\lambda\mathbf{i}+2\lambda\mathbf{j}.$$

Next, we set the coefficients of $\mathbf{i}$ and $\mathbf{j}$ equal to each other:

$$2x_0-2=\lambda$$
$$8y_0+8=2\lambda.$$

The equation $g(x_0,y_0)=0$ becomes $x_0+2y_0-7=0$. Therefore, the system of equations that needs to be solved is

$$\begin{aligned}2x_0-2&=\lambda\\8y_0+8&=2\lambda\\x_0+2y_0-7&=0.\end{aligned}$$

3. This is a linear system of three equations in three variables. We start by solving the second equation for λ and substituting it into the first equation. This gives $\lambda=4y_0+4$, so substituting this into the first equation gives

$$2x_0-2=4y_0+4.$$

Solving this equation for x_0 gives $x_0=2y_0+3$. We then substitute this into the third equation:

$$\begin{aligned}(2y_0+3)+2y_0-7&=0\\4y_0-4&=0\\y_0&=1.\end{aligned}$$

Since $x_0=2y_0+3$, this gives $x_0=5$.

4. Next, we evaluate $f(x,y)=x^2+4y^2-2x+8y$ at the point $(5,1)$,

$$f(5,1) = 5^2 + 4(1)^2 - 2(5) + 8(1) = 27.$$

To ensure this corresponds to a minimum value on the constraint function, let's try some other points on the constraint from either side of the point $(5,1)$, such as the intercepts of $g(x,y) = 0$, which are $(7,0)$ and $(0,3.5)$.

We get $f(7,0) = 35 > 27$ and $f(0,3.5) = 77 > 27$.

So it appears that f has a relative minimum of 27 at $(5,1)$, subject to the given constraint.

Watch the accompanying video for a visual demonstration of the concept.

- **Video length**: 3 minutes 12 seconds.
- **Context**: This video demonstrates how to use Lagrange multipliers to find a maximum of a function of two variables and one constraint.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

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Exercise 1.8.1

Use the method of Lagrange multipliers to find the maximum value of

$$f(x,y) = 9x^2 + 36xy - 4y^2 - 18x - 8y$$

subject to the constraint $3x + 4y = 32$.

Hint available at the end of this section.

Let's now return to the problem posed at the beginning of the section.

Example 1.8.2: Golf Balls and Lagrange Multipliers

The golf ball manufacturer, Pro-T, has developed a profit model that depends on the number x of golf balls sold per month (measured in thousands), and the number of hours per month of advertising *y*, according to the function

$$z = f(x,y) = 48x + 96y - x^2 - 2xy - 9y^2,$$

where z is measured in thousands of dollars. The budgetary constraint function relating the cost of the production of thousands golf balls and advertising units is given by $20x + 4y = 216$. Find the values of x and y that maximize profit, and find the maximum profit.

Solution:

Again, we follow the problem-solving strategy:

1. The objective function is $f(x,y) = 48x + 96y - x^2 - 2xy - 9y^2$. To determine the constraint function, we first subtract 216 from both sides of the constraint, then divide both sides by 4, which gives $5x + y - 54 = 0$. The constraint function is equal to the left-hand side, so $g(x,y) = 5x + y - 54$. The problem asks us to solve for the maximum value of f, subject to this constraint.
2. So, we calculate the gradients of both f and g:

$$\nabla f(x,y) = (48 - 2x - 2y)\mathbf{i} + (96 - 2x - 18y)\mathbf{j}$$
$$\nabla g(x,y) = 5\mathbf{i} + \mathbf{j}.$$

The equation $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$ becomes

$$(48 - 2x_0 - 2y_0)\mathbf{i} + (96 - 2x_0 - 18y_0)\mathbf{j} = \lambda(5\mathbf{i} + \mathbf{j}),$$

which can be rewritten as

$$(48 - 2x_0 - 2y_0)\mathbf{i} + (96 - 2x_0 - 18y_0)\mathbf{j} = \lambda 5\mathbf{i} + \lambda\mathbf{j}.$$

We then set the coefficients of $\mathbf{i}$ and $\mathbf{j}$ equal to each other:

$$48 - 2x_0 - 2y_0 = 5\lambda$$
$$96 - 2x_0 - 18y_0 = \lambda.$$

The equation $g(x_0, y_0) = 0$ becomes $5x_0 + y_0 - 54 = 0$. Therefore, the system of equations that needs to be solved is

$$48 - 2x_0 - 2y_0 = 5\lambda$$
$$96 - 2x_0 - 18y_0 = \lambda$$
$$5x_0 + y_0 - 54 = 0.$$

3. We use the left-hand side of the second equation to replace λ in the first equation:

$$48 - 2x_0 - 2y_0 = 5(96 - 2x_0 - 18y_0)$$
$$48 - 2x_0 - 2y_0 = 480 - 10x_0 - 90y_0$$
$$8x_0 = 432 - 88y_0$$
$$x_0 = 54 - 11y_0.$$

Then we substitute this into the third equation:

$$5(54 - 11y_0) + y_0 - 54 = 0$$
$$270 - 55y_0 + y_0 - 54 = 0$$
$$216 - 54y_0 = 0$$
$$y_0 = 4.$$

Since $x_0 = 54 - 11y_0$, this gives $x_0 = 10$.

4. We then substitute $(10, 4)$ into $f(x,y) = 48x + 96y - x^2 - 2xy - 9y^2$, which gives

$$\begin{aligned} f(10,4) &= 48(10) + 96(4) - (10)^2 - 2(10)(4) - 9(4)^2 \\ &= 480 + 384 - 100 - 80 - 144 \\ &= 540. \end{aligned}$$

Therefore the maximum profit that can be attained, subject to budgetary constraints, is $\$540,000$ with a production level of $10,000$ golf balls and 4 hours of advertising bought per month. Let's check to make sure this truly is a maximum. The endpoints of the line that defines the constraint are $(10.8, 0)$ and $(0, 54)$ Let's evaluate f at both of these points:

$$\begin{aligned} f(10.8, 0) &= 48(10.8) + 96(0) - 10.8^2 - 2(10.8)(0) - 9(0^2) \\ &= 401.76 \\ f(0, 54) &= 48(0) + 96(54) - 0^2 - 2(0)(54) - 9(54^2) \\ &= -21,060. \end{aligned}$$

The second value represents a loss, since no golf balls are produced. Neither of these values exceed 540, so it seems that our extremum is a maximum value of f, subject to the given constraint.

Exercise 1.8.2: Optimizing the Cobb-Douglas function

A company has determined that its production level is given by the Cobb-Douglas function $f(x,y) = 2.5x^{0.45}y^{0.55}$ where x represents the total number of labor hours in 1 year and y represents the total capital input for the company. Suppose 1 unit of labor costs $\$40$ and 1 unit of capital costs $\$50$. Use the method of Lagrange multipliers to find the maximum value of $f(x,y) = 2.5x^{0.45}y^{0.55}$ subject to a budgetary constraint of $\$500,000$ per year.

Hint available at the end of this section.

In the case of an objective function with three variables and a single constraint function, it is possible to use the method of Lagrange multipliers to solve an optimization problem as well. An example of an objective function with three variables could be the **Cobb-Douglas function** in Exercise 1.8.2: $f(x,y,z) = x^{0.2}y^{0.4}z^{0.4}$, where x represents the cost of labor, y represents capital input, and z represents the cost of advertising. The method is the same as for the method with a function of two variables; the equations to be solved are

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$$
$$g(x,y,z) = 0.$$

Example 1.8.3: Lagrange Multipliers with a Three-Variable objective function

Maximize the function $f(x,y,z) = x^2 + y^2 + z^2$ subject to the constraint $x + y + z = 1$.

Solution

1. The objective function is $f(x,y,z) = x^2 + y^2 + z^2$. To determine the constraint function, we subtract 1 from each side of the constraint: $x + y + z - 1 = 0$ which gives the constraint function as $g(x,y,z) = x + y + z - 1$.

2. Next, we calculate $\nabla f(x,y,z)$ and $\nabla g(x,y,z)$:

$$\nabla f(x,y,z) = \langle 2x, 2y, 2z \rangle$$
$$\nabla g(x,y,z) = \langle 1, 1, 1 \rangle.$$

This leads to the equations

$$\langle 2x_0, 2y_0, 2z_0 \rangle = \lambda \langle 1, 1, 1 \rangle$$
$$x_0 + y_0 + z_0 - 1 = 0$$

which can be rewritten in the following form:

$$2x_0 = \lambda$$
$$2y_0 = \lambda$$
$$2z_0 = \lambda$$
$$x_0 + y_0 + z_0 - 1 = 0.$$

3. Since each of the first three equations has λ on the right-hand side, we know that $2x_0 = 2y_0 = 2z_0$ and all three variables are equal to each other. Substituting $y_0 = x_0$ and $z_0 = x_0$ into the last equation yields $3x_0 - 1 = 0$, so $x_0 = \frac{1}{3}$ and $y_0 = \frac{1}{3}$ and $z_0 = \frac{1}{3}$ which corresponds to a critical point on the constraint curve.

4. Then, we evaluate f at the point $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$:

$$f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 = \frac{3}{9} = \frac{1}{3}$$

Therefore, a possible extremum of the function is $\frac{1}{3}$. To verify it is a minimum, choose other points that satisfy the constraint from either side of the point we obtained above and calculate f at those points. For example,

$$f(1,0,0) = 1^2 + 0^2 + 0^2 = 1$$
$$f(0,-2,3) = 0^2 + (-2)^2 + 3^2 = 13.$$

Both of these values are greater than $\frac{1}{3}$, leading us to believe the extremum is a minimum, subject to the given constraint.

Watch the accompanying video for a visual demonstration of the concept.

- **Video length**: 3 minutes 58 seconds.
- **Context**: This video demonstrates how to use Lagrange multipliers to find the minimum distance from the origin.

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Exercise 1.8.3

Use the method of Lagrange multipliers to find the minimum value of the function

$$f(x, y, z) = x + y + z$$

subject to the constraint $x^2 + y^2 + z^2 = 1$.

Hint available at the end of this section.

Problems with Two Constraints

The method of Lagrange multipliers can be applied to problems with more than one constraint. In this case the objective function, w is a function of three variables:

$$w = f(x, y, z) \tag{1.8.4}$$

and it is subject to two constraints:

$$g(x, y, z) = 0 \text{ and } h(x, y, z) = 0. \tag{1.8.5}$$

There are two Lagrange multipliers, λ_1 and λ_2, and the system of equations becomes

$$\begin{aligned} \nabla f(x_0, y_0, z_0) &= \lambda_1 \nabla g(x_0, y_0, z_0) + \lambda_2 \nabla h(x_0, y_0, z_0) \\ g(x_0, y_0, z_0) &= 0 \\ h(x_0, y_0, z_0) &= 0 \end{aligned}$$

Example 1.8.4: Lagrange Multipliers with Two Constraints

Find the maximum and minimum values of the function

$$f(x, y, z) = x^2 + y^2 + z^2$$

subject to the constraints $z^2 = x^2 + y^2$ and $x + y - z + 1 = 0$.

Solution

Let's follow the problem-solving strategy:

1. The objective function is $f(x, y, z) = x^2 + y^2 + z^2$. To determine the constraint functions, we first subtract z^2 from both sides of the first constraint, which gives $x^2 + y^2 - z^2 = 0$, so $g(x, y, z) = x^2 + y^2 - z^2$. The second constraint function is $h(x, y, z) = x + y - z + 1$.
2. We then calculate the gradients of f, g, and h:

$$\begin{aligned} \nabla f(x, y, z) &= 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \\ \nabla g(x, y, z) &= 2x\mathbf{i} + 2y\mathbf{j} - 2z\mathbf{k} \\ \nabla h(x, y, z) &= \mathbf{i} + \mathbf{j} - \mathbf{k}. \end{aligned}$$

The equation $\nabla f(x_0, y_0, z_0) = \lambda_1 \nabla g(x_0, y_0, z_0) + \lambda_2 \nabla h(x_0, y_0, z_0)$ becomes

$$2x_0\mathbf{i} + 2y_0\mathbf{j} + 2z_0\mathbf{k} = \lambda_1(2x_0\mathbf{i} + 2y_0\mathbf{j} - 2z_0\mathbf{k}) + \lambda_2(\mathbf{i} + \mathbf{j} - \mathbf{k}),$$

which can be rewritten as

$$2x_0\mathbf{i} + 2y_0\mathbf{j} + 2z_0\mathbf{k} = (2\lambda_1 x_0 + \lambda_2)\mathbf{i} + (2\lambda_1 y_0 + \lambda_2)\mathbf{j} - (2\lambda_1 z_0 + \lambda_2)\mathbf{k}.$$

Next, we set the coefficients of $\mathbf{i}$ and $\mathbf{j}$ equal to each other:

$$\begin{aligned} 2x_0 &= 2\lambda_1 x_0 + \lambda_2 \\ 2y_0 &= 2\lambda_1 y_0 + \lambda_2 \\ 2z_0 &= -2\lambda_1 z_0 - \lambda_2. \end{aligned}$$

The two equations that arise from the constraints are $z_0^2 = x_0^2 + y_0^2$ and $x_0 + y_0 - z_0 + 1 = 0$. Combining these equations with the previous three equations gives

$$\begin{aligned} 2x_0 &= 2\lambda_1 x_0 + \lambda_2 \\ 2y_0 &= 2\lambda_1 y_0 + \lambda_2 \\ 2z_0 &= -2\lambda_1 z_0 - \lambda_2 \\ z_0^2 &= x_0^2 + y_0^2 \\ x_0 + y_0 - z_0 + 1 &= 0. \end{aligned}$$

3. The first three equations contain the variable λ_2. Solving the third equation for λ_2 and replacing into the first and second equations reduces the number of equations to four:

$$\begin{aligned} 2x_0 &= 2\lambda_1 x_0 - 2\lambda_1 z_0 - 2z_0 \\ 2y_0 &= 2\lambda_1 y_0 - 2\lambda_1 z_0 - 2z_0 \\ z_0^2 &= x_0^2 + y_0^2 \\ x_0 + y_0 - z_0 + 1 &= 0. \end{aligned}$$

Next, we solve the first and second equation for λ_1. The first equation gives $\lambda_1 = \dfrac{x_0 + z_0}{x_0 - z_0}$, the second equation gives $\lambda_1 = \dfrac{y_0 + z_0}{y_0 - z_0}$. We set the right-hand side of each equation equal to each other and cross-multiply:

$$\begin{aligned} \frac{x_0 + z_0}{x_0 - z_0} &= \frac{y_0 + z_0}{y_0 - z_0} \\ (x_0 + z_0)(y_0 - z_0) &= (x_0 - z_0)(y_0 + z_0) \\ x_0y_0 - x_0z_0 + y_0z_0 - z_0^2 &= x_0y_0 + x_0z_0 - y_0z_0 - z_0^2 \\ 2y_0z_0 - 2x_0z_0 &= 0 \\ 2z_0(y_0 - x_0) &= 0. \end{aligned}$$

Therefore, either $z_0 = 0$ or $y_0 = x_0$. If $z_0 = 0$, then the first constraint becomes $0 = x_0^2 + y_0^2$. The only real solution to this equation is $x_0 = 0$ and $y_0 = 0$, which gives the ordered triple $(0, 0, 0)$. This point does not satisfy the second constraint, so it is not a solution. Next, we consider $y_0 = x_0$, which reduces the number of equations to three:

$$\begin{aligned} y_0 &= x_0 \\ z_0^2 &= x_0^2 + y_0^2 \\ x_0 + y_0 - z_0 + 1 &= 0. \end{aligned}$$

We substitute the first equation into the second and third equations:

$$\begin{aligned} z_0^2 &= x_0^2 + x_0^2 \\ &= x_0 + x_0 - z_0 + 1 \qquad = 0. \end{aligned}$$

Then, we solve the second equation for z_0, which gives $z_0 = 2x_0 + 1$. We then substitute this into the first equation,

$$\begin{aligned} z_0^2 &= 2x_0^2 \\ (2x_0^2 + 1)^2 &= 2x_0^2 \\ 4x_0^2 + 4x_0 + 1 &= 2x_0^2 \\ 2x_0^2 + 4x_0 + 1 &= 0, \end{aligned}$$

and use the quadratic formula to solve for x_0:

$$x_0 = \frac{-4 \pm \sqrt{4^2 - 4(2)(1)}}{2(2)} = \frac{-4 \pm \sqrt{8}}{4} = \frac{-4 \pm 2\sqrt{2}}{4} = -1 \pm \frac{\sqrt{2}}{2}.$$

Recall $y_0 = x_0$, so this solves for y_0 as well. Then, $z_0 = 2x_0 + 1$, so

$$z_0 = 2x_0 + 1 = 2\left(-1 \pm \frac{\sqrt{2}}{2}\right) + 1 = -2 + 1 \pm \sqrt{2} = -1 \pm \sqrt{2}.$$

Therefore, there are two ordered triplet solutions:

$$\left(-1 + \frac{\sqrt{2}}{2}, -1 + \frac{\sqrt{2}}{2}, -1 + \sqrt{2}\right) \text{ and } \left(-1 - \frac{\sqrt{2}}{2}, -1 - \frac{\sqrt{2}}{2}, -1 - \sqrt{2}\right).$$

4. We substitute $\left(-1 + \frac{\sqrt{2}}{2}, -1 + \frac{\sqrt{2}}{2}, -1 + \sqrt{2}\right)$ into $f(x, y, z) = x^2 + y^2 + z^2$, which gives

$$\begin{aligned} f\left(-1 + \frac{\sqrt{2}}{2}, -1 + \frac{\sqrt{2}}{2}, -1 + \sqrt{2}\right) &= \left(-1 + \frac{\sqrt{2}}{2}\right)^2 + \left(-1 + \frac{\sqrt{2}}{2}\right)^2 + (-1 + \sqrt{2})^2 \\ &= \left(1 - \sqrt{2} + \frac{1}{2}\right) + \left(1 - \sqrt{2} + \frac{1}{2}\right) + (1 - 2\sqrt{2} + 2) \\ &= 6 - 4\sqrt{2}. \end{aligned}$$

Then, we substitute $\left(-1 - \frac{\sqrt{2}}{2}, -1 + \frac{\sqrt{2}}{2}, -1 + \sqrt{2}\right)$ into $f(x, y, z) = x^2 + y^2 + z^2$, which gives

$$\begin{aligned} f\left(-1 - \frac{\sqrt{2}}{2}, -1 + \frac{\sqrt{2}}{2}, -1 + \sqrt{2}\right) &= \left(-1 - \frac{\sqrt{2}}{2}\right)^2 + \left(-1 - \frac{\sqrt{2}}{2}\right)^2 + (-1 - \sqrt{2})^2 \\ &= \left(1 + \sqrt{2} + \frac{1}{2}\right) + \left(1 + \sqrt{2} + \frac{1}{2}\right) + (1 + 2\sqrt{2} + 2) \\ &= 6 + 4\sqrt{2}. \end{aligned}$$

$6 + 4\sqrt{2}$ is the maximum value and $6 - 4\sqrt{2}$ is the minimum value of $f(x, y, z)$, subject to the given constraints.

Watch the accompanying video for a visual demonstration of the concept.

- **Video length**: 4 minutes 7 seconds.
- **Context**: This video demonstrates how to use Lagrange multipliers with two constraints.

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Exercise 1.8.4

Use the method of Lagrange multipliers to find the minimum value of the function

$$f(x, y, z) = x^2 + y^2 + z^2$$

subject to the constraints $2x + y + 2z = 9$ and $5x + 5y + 7z = 29$.

Hint available at the end of this section.

Key Concepts

- An objective function combined with one or more constraints is an example of an optimization problem.
- To solve optimization problems, we apply the method of Lagrange multipliers using a four-step problem-solving strategy.

Key Equations

- **Method of Lagrange multipliers, one constraint**

  $$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$$

  $$g(x_0, y_0) = 0$$

- **Method of Lagrange multipliers, two constraints**

  $$\nabla f(x_0, y_0, z_0) = \lambda_1 \nabla g(x_0, y_0, z_0) + \lambda_2 \nabla h(x_0, y_0, z_0)$$

  $$g(x_0, y_0, z_0) = 0$$

  $$h(x_0, y_0, z_0) = 0$$

Glossary

Constraint
: an inequality or equation involving one or more variables that is used in an optimization problem; the constraint enforces a limit on the possible solutions for the problem

Lagrange multiplier
: the constant (or constants) used in the method of Lagrange multipliers; in the case of one constant, it is represented by the variable λ

Method of Lagrange multipliers
: a method of solving an optimization problem subject to one or more constraints

Objective function
: the function that is to be maximized or minimized in an optimization problem

Optimization problem
: calculation of a maximum or minimum value of a function of several variables, often using Lagrange multipliers

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Answers

Exercise $1.8.1$

Use the problem-solving strategy for the method of Lagrange multipliers.

Subject to the given constraint, f has a maximum value of 976 at the point $(8, 2)$.

Exercise $1.8.2$**: Optimizing the Cobb-Douglas function**

Use the problem-solving strategy for the method of Lagrange multipliers.

Subject to the given constraint, a maximum production level of 13890 occurs with 5625 labor hours and $\$5500$ of total capital input.

Exercise $1.8.3$

Use the problem-solving strategy for the method of Lagrange multipliers with an objective function of three variables.

Evaluating f at both points we obtained, gives us,

$$f\left(\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}\right)=\frac{\sqrt{3}}{3}+\frac{\sqrt{3}}{3}+\frac{\sqrt{3}}{3}=\sqrt{3}$$
$$f\left(-\frac{\sqrt{3}}{3},-\frac{\sqrt{3}}{3},-\frac{\sqrt{3}}{3}\right)=-\frac{\sqrt{3}}{3}-\frac{\sqrt{3}}{3}-\frac{\sqrt{3}}{3}=-\sqrt{3}$$

Since the constraint is continuous, we compare these values and conclude that f has a relative minimum of $-\sqrt{3}$ at the point $\left(-\frac{\sqrt{3}}{3},-\frac{\sqrt{3}}{3},-\frac{\sqrt{3}}{3}\right)$, subject to the given constraint.

Exercise $1.8.4$

Use the problem-solving strategy for the method of Lagrange multipliers with two constraints.

$f(2,1,2)=9$ is a minimum value of f, subject to the given constraints.

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1.8E: Exercises for Section 1.8

Exercises 1-15

Use the method of Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraint.

1. Objective function: $f(x,y) = 4xy$ Constraint: $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$

   Answer

   Subject to the given constraint, the function f has a relative minimum of -24 at both $\left(-\frac{3\sqrt{2}}{2}, 2\sqrt{2}\right)$ and $\left(\frac{3\sqrt{2}}{2}, -2\sqrt{2}\right)$ and a relative maximum of 24 at both $\left(\frac{3\sqrt{2}}{2}, 2\sqrt{2}\right)$ and $\left(-\frac{3\sqrt{2}}{2}, -2\sqrt{2}\right)$

2. Objective function: $f(x,y) = x^2y$ Constraint: $x^2 + 2y^2 = 6$
3. Objective function: $f(x,y) = x^2 + y^2 + 2x - 2y + 1$ Constraint: $g(x,y) = x^2 + y^2 = 2$

   Answer

   Subject to the given constraint, f has a relative minimum of -1 at $(-1,1)$ and a relative maximum of 7 at $(1,-1)$.

4. Objective function: $f(x,y) = xy$ Constraint: $4x^2 + 8y^2 = 16$
5. Objective function: $f(x,y) = x^2 + y^2$ Constraint: $xy = 1$

   Answer

   f has a relative minimum of 2 at both $(-1,-1)$ and $(1,1)$, subject to the given constraint.

6. Objective function: $f(x,y) = x^2 - y^2$ Constraint: $x - 2y + 6 = 0$
7. Objective function: $f(x,y) = x^2 + y^2$ Constraint: $x + 2y - 5 = 0$

   Answer

   Subject to the given constraint, f has a relative minimum of $f(1,2) = 5$ at the point $(1,2)$.

8. Objective function: $f(x,y) = x^2 + y^2$ Constraint: $(x-1)^2 + 4y^2 = 4$
9. Objective function: $f(x,y) = 4x^3 + y^2$ Constraint: $2x^2 + y^2 = 1$

   Answer

   Subject to the given constraint, the function f has a relative minimum of $-\sqrt{2}$ at $\left(-\frac{\sqrt{2}}{2}, 0\right)$,
   a relative minimum of $\frac{25}{27}$ at both points $\left(\frac{1}{3}, -\frac{\sqrt{7}}{3}\right)$ and $\left(\frac{1}{3}, \frac{\sqrt{7}}{3}\right)$,
   a relative maximum of $\sqrt{2}$ at $\left(\frac{\sqrt{2}}{2}, 0\right)$, and a relative maximum of 1 at both points $(0,1)$ and $(0,-1)$.

   Solution:

   Let $g(x,y) = 2x^2 + y^2$ be the constraint function. Then:

   $\vecs\nabla f(x,y) = 12x^2\ \hat{\mathbf{i}} + 2y\ \hat{\mathbf{j}}$ and $\vecs\nabla g(x,y) = 4x\ \hat{\mathbf{i}} + 2y\ \hat{\mathbf{j}}$

   Using the Lagrange Multiplier equation,

   $$\vecs\nabla f(x,y) = \lambda \vecs\nabla g(x,y),$$

   we have:

   $$12x^2\,\hat{\mathbf{i}} + 2y\hat{\mathbf{j}} = 4x\lambda\,\hat{\mathbf{i}} + 2y\lambda\hat{\mathbf{j}}$$

giving us the system of equations:

$$12x^2 = 4x\lambda, \quad 2y = 2y\lambda, \quad \text{and the constraint} \quad 2x^2 + y^2 = 1$$

Rewriting the first two equations as zero-products (by moving to one-side and factoring), we get:

$$\begin{array}{ccc} 4x(3x-\lambda) = 0 & \text{and} & 2y(1-\lambda) = 0 \\ x = 0 \quad \text{or} \quad \lambda = 3x & & y = 0 \quad \text{or} \quad \lambda = 1 \end{array}$$

Now we consider the combinations of these solutions to the above two equations and plug each of these into the constraint equation to solve for the corresponding Lagrange points.

The combination $x = 0$ and $y = 0$ produces a contradiction when placed in the constraint equation, since this point is not on the ellipse.

Taking the combination $x = 0$ and $\lambda = 1$, we put 0 in for x in the constraint and solve for y, obtaining: $y = \pm 1$. This gives us two Lagrange points: $(0, 1)$ and $(0, -1)$.

Taking the combination $\lambda = 3x$ and $y = 0$, we put 0 in for y in the constraint and solve for x, obtaining: $x = \pm\frac{\sqrt{2}}{2}$. This gives us two Lagrange points: $\left(-\frac{\sqrt{2}}{2}, 0\right)$ and $\left(\frac{\sqrt{2}}{2}, 0\right)$.

Taking the combination $\lambda = 3x$ and $\lambda = 1$, we substitute 1 into the first equation for λ, giving us $1 = 3x$ so $x = \frac{1}{3}$. Plugging this value in for x in the constraint equation and solving for y, we obtain $y = \pm\frac{\sqrt{7}}{3}$ which gives us the two Lagrange points: $\left(\frac{1}{3}, -\frac{\sqrt{7}}{3}\right)$ and $\left(\frac{1}{3}, \frac{\sqrt{7}}{3}\right)$.

Evaluating the function f at these Lagrange points, we find:

$$\begin{array}{ll} f(0,-1) = 1 & f(0,1) = 1 \\ f\left(-\frac{\sqrt{2}}{2}, 0\right) = \frac{-4(\sqrt{2})^3}{8} = -\sqrt{2} & f\left(\frac{\sqrt{2}}{2}, 0\right) = \frac{4(\sqrt{2})^3}{8} = \sqrt{2} \\ f\left(\frac{1}{3}, -\frac{\sqrt{7}}{3}\right) = \frac{25}{27} & f\left(\frac{1}{3}, \frac{\sqrt{7}}{3}\right) = \frac{25}{27} \end{array}$$

Comparing these values with where the corresponding Lagrange points lie on the constraint curve, we conclude the results stated in the answer above.

10. Objective function: $f(x, y) = 2x^2 + y^2$ Constraint: $g(x, y) = x^2 + y^2 = 1$
11. Objective function: $f(x, y, z) = x + 3y - z$ Constraint: $x^2 + y^2 + z^2 = 4$

Answer

Subject to the given constraint, f has a relative minimum of $-2\sqrt{11}$ at the point $\left(-\frac{2\sqrt{11}}{11}, -\frac{6\sqrt{11}}{11}, \frac{2\sqrt{11}}{11}\right)$ and a relative maximum of $2\sqrt{11}$ at the point $\left(\frac{2\sqrt{11}}{11}, \frac{6\sqrt{11}}{11}, -\frac{2\sqrt{11}}{11}\right)$.

12. Objective function: $f(x, y, z) = x + y + z$ Constraint: $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 1$
13. Objective function: $f(x, y, z) = xyz$ Constraint: $x^2 + 2y^2 + 3z^2 = 6$

Answer

Subject to the given constraint, f has a relative minimum of $-\frac{2\sqrt{3}}{3}$ at $\left(\sqrt{2}, 1, -\frac{\sqrt{6}}{3}\right)$, $\left(\sqrt{2}, -1, \frac{\sqrt{6}}{3}\right)$, $\left(-\sqrt{2}, 1, \frac{\sqrt{6}}{3}\right)$, and $\left(-\sqrt{2}, -1, -\frac{\sqrt{6}}{3}\right)$ and a relative maximum of $\frac{2\sqrt{3}}{3}$ at $\left(\sqrt{2}, 1, \frac{\sqrt{6}}{3}\right)$, $\left(\sqrt{2}, -1, -\frac{\sqrt{6}}{3}\right)$, $\left(-\sqrt{2}, -1, \frac{\sqrt{6}}{3}\right)$, and $\left(-\sqrt{2}, 1, -\frac{\sqrt{6}}{3}\right)$.

14. Objective function: $f(x, y, z) = x^2 + y^2 + z^2$ Constraint: $x^4 + y^4 + z^4 = 1$
15. Objective function: $f(x, y, z) = x^2 + y^2 + z^2$ Constraint: $xyz = 4$

Answer

Subject to the given constraint, f has a relative minimum of $6\sqrt[3]{2}$ at the points $\left(\sqrt[3]{4}, \sqrt[3]{4}, \sqrt[3]{4}\right)$, $\left(\sqrt[3]{4}, -\sqrt[3]{4}, -\sqrt[3]{4}\right)$, $\left(-\sqrt[3]{4}, \sqrt[3]{4}, -\sqrt[3]{4}\right)$, and $\left(-\sqrt[3]{4}, -\sqrt[3]{4}, \sqrt[3]{4}\right)$.
To see a 3D visualization of this problem, see: CalcPlot3D for Problem 15(opens in new window).

Exercises 16-21

Use the method of Lagrange multipliers to find the requested extremum of the given function subject to the given constraint.

16. Maximize $f(x,y) = \sqrt{6 - x^2 - y^2}$ subject to the constraint, $x + y - 2 = 0$.
17. Maximize $f(x,y) = x^2 - y^2$ subject to the constraints, $g(x,y) = y - x^2 = 0, \quad x > 0, \quad y > 0$.

Answer

Subject to the given constraints, f has a relative maximum of $f\left(\frac{\sqrt{2}}{2}, \frac{1}{2}\right) = \frac{1}{4}$ at the point $\left(\frac{\sqrt{2}}{2}, \frac{1}{2}\right)$. If $x > 0$ were not a constraint, there would have been two other Lagrange points with relative extrema of f subject to the other two constraints. These would have been $(0,0)$ and $\left(-\frac{\sqrt{2}}{2}, \frac{1}{2}\right)$.

To verify that f really has a relative **maximum** at the point $\left(\frac{\sqrt{2}}{2}, \frac{1}{2}\right)$, we would need to check the value of f on either side of this point on the constraint curve, $y - x^2 = 0$.

If $x = 0.5$ which is less than $\frac{\sqrt{2}}{2}$, y would be $y = (0.5)^2 = 0.25$.
If $x = 1$ which is greater than $\frac{\sqrt{2}}{2}$, y would be $y = (1)^2 = 1$.

Then we compare the value of f at the Lagrange point, $\left(\frac{\sqrt{2}}{2}, \frac{1}{2}\right)$, with the values of f at these other points on the constraint.
We have $f(0.5, 0.25) = (0.5)^2 - (0.25)^2 = 0.25 - 0.0625 = 0.1875 < \frac{1}{4}$ and $f(1,1) = (1)^2 - (1)^2 = 0 < \frac{1}{4}$.

Therefore, we can conclude that f indeed has a relative maximum of $\frac{1}{4}$ at the point $\left(\frac{\sqrt{2}}{2}, \frac{1}{2}\right)$.

18. Maximize $U(x,y) = 8x^{4/5}y^{1/5}$ subject to the constraint, $4x + 2y = 12$.
19. Minimize $f(x,y,z) = x^2 + y^2 + z^2$ subject to the constraint, $x + y + z = 1$.

Answer

Subject to the given constraint, f has a relative minimum of $f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \frac{1}{3}$ at the point $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

20. Minimize $f(x,y) = xy$ on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.
21. Maximize $f(x,y,z) = 2x + 3y + 5z$ on the sphere $x^2 + y^2 + z^2 = 19$.

Answer

Subject to the given constraint, f has a relative maximum of $19\sqrt{2}$ at the point $\left(\sqrt{2}, \frac{3\sqrt{2}}{2}, \frac{5\sqrt{2}}{2}\right)$.
Note that, subject to this constraint, f also has a relative minimum of $-19\sqrt{2}$ at the point $\left(-\sqrt{2}, -\frac{3\sqrt{2}}{2}, -\frac{5\sqrt{2}}{2}\right)$.
To see a 3D visualization of this problem, see: CalcPlot3D for Problem 21(opens in new window).

Exercises 22-23

Use the method of Lagrange multipliers with two constraints.

22. Optimize $f(x,y,z) = yz + xy$ subject to the constraints: $xy = 1, \quad y^2 + z^2 = 1$.

Answer

maximum: $\frac{3}{2}$, minimum: $\frac{1}{2}$

23. Minimize $f(x,y,z) = x^2 + y^2 + z^2$ when $x + y + z = 9$ and $x + 2y + 3z = 20$.

Answer

minimum: $f(2,3,4) = 29$

Exercises 24-38

Use the method of Lagrange multipliers to solve the following applied problems.

24. A large container in the shape of a rectangular solid must have a volume of 480 m^3. The bottom of the container costs \$5/m^2 to construct whereas the top and sides cost \$3/m^2 to construct. Use Lagrange multipliers to find the dimensions of the container of this size that has the minimum cost.
25. A rectangular box without a top (a topless box) is to be made from 12 ft^2 of cardboard. Find the maximum volume of such a box.

Answer

The maximum volume is 4 ft^3. The dimensions are $1 \times 2 \times 2$ ft.

26. Find the minimum distance from the parabola $y = x^2$ to point $(0,3)$.
27. Find the point on the line $y = 2x + 3$ that is closest to point $(4,2)$.

Answer

$(2/5, 19/5)$

28. Find the minimum distance from point $(0,1)$ to the parabola $x^2 = 4y$.

Answer

1.0 unit

29. Find the minimum and maximum distances between the ellipse $x^2 + xy + 2y^2 = 1$ and the origin.
30. Find the minimum distance from the plane $x + y + z = 1$ to point $(2,1,1)$.

Answer

$\sqrt{3}$ units

31. Find the point on the plane $4x + 3y + z = 2$ that is closest to the point $(1,-1,1)$.
32. Find the point on the surface $x^2 - 2xy + y^2 - x + y = 0$ closest to the point $(1,2,-3)$.

Answer

$(1, \frac{1}{2}, -3)$

33. A pentagon is formed by placing an isosceles triangle on a rectangle, see Figure $1.8E.1$. If the perimeter of the pentagon is 10 in., find the lengths of the sides of the pentagon that will maximize the area of the pentagon.

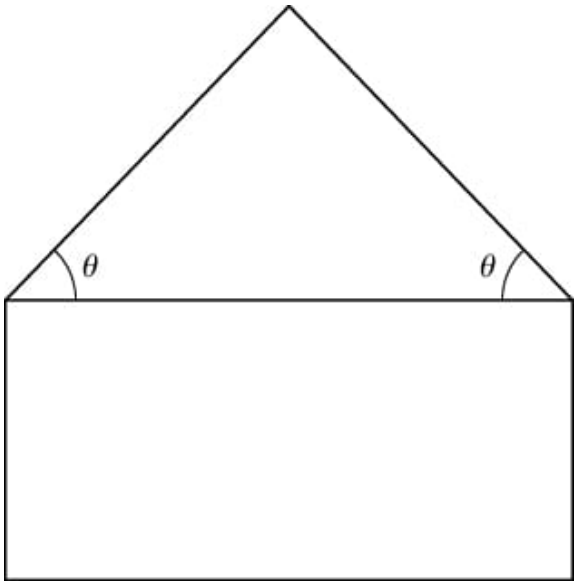


Figure $1.8E.1$: A pentagon composed of a rectangle with an isosceles triangle placed on top of one of its sides.

34. **[T]** By investing x units of labor and y units of capital, a watch manufacturer can produce $P(x,y) = 50x^{0.4}y^{0.6}$ watches. Find the maximum number of watches that can be produced on a budget of $20,000 if labor costs $100/unit and capital costs $200/unit. Use a grapher like CalcPlot3D to sketch a contour plot of the function.

Answer

Roughly 3365 watches at the critical point $(80, 60)$.

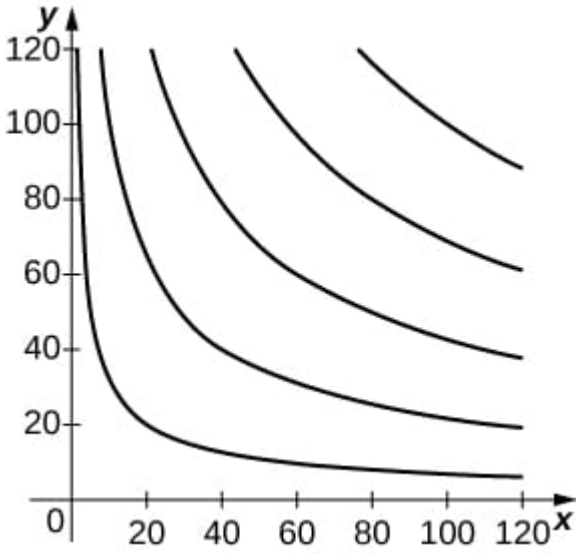


Figure $1.8E.2$: A series of curves in the first quadrant, with the first starting near (2, 120), decreasing sharply to near (20, 20), and then decreasing slowly to (120, 5). The next curve starts near (10, 120), decreases sharply to near (40, 40), and then decreases slowly to (120, 20). The next curve starts near (20, 120), decreases sharply to near (60, 60), and then decreases slowly to (120, 40). The next curve starts near (40, 120), decreases to near (80, 80), and then decreases a little slowly to (120, 60). The last curve starts near (60, 120) and decreases rather evenly through (100, 100) to (120, 90).

35. A rectangular solid is contained within a tetrahedron with vertices at $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, and the origin. The base of the box has dimensions x and y, and the height of the box is z. If the sum of x, y, and z is 1, find the dimensions that maximizes the volume of the rectangular solid.
36. Find the maximum value of $f(x,y) = \sin x \sin y$, where x and y denote the acute angles of a right triangle. Draw the surface plot and contour plot of the function using a CAS.

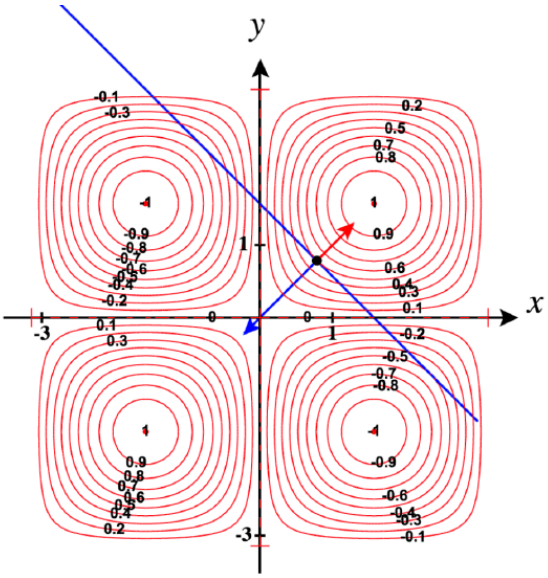


Figure $1.8E.3$: Contour map for $f(x,y) = \sin x \sin y$,. It is a set of 4 symmetrical circular paths with uneven spacing. The first set of circular paths is on the first quadrant of the xy-plane. The second set of circular paths is on the second quadrant of the xy-plane. The third set of circular paths is on the third quadrant of the xy-plane. The last set of circular paths is on the fourth quadrant of the xy-plane.

Answer

Subject to this constraint, f has a relative maximum of $\frac{1}{2}$ when $x = \frac{\pi}{4}$ and $y = \frac{\pi}{4}$.
Surface plot and contour plot for f:

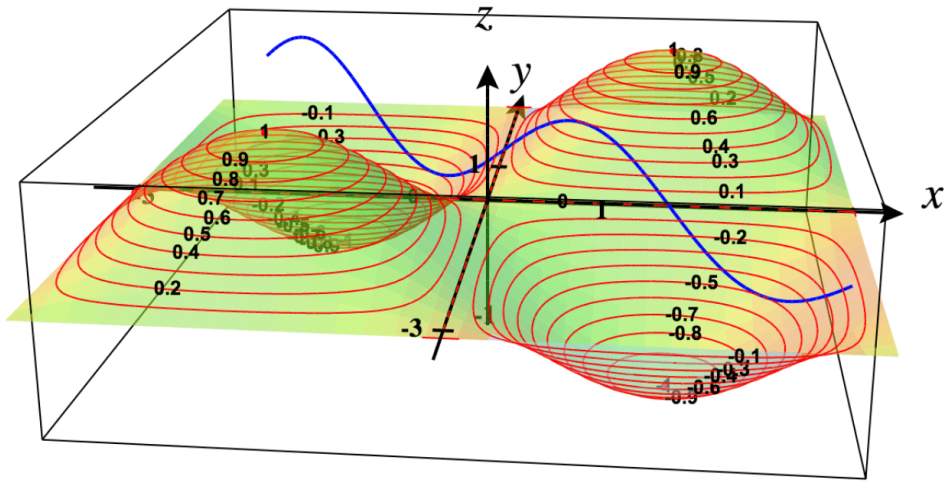


Figure $1.8E.4$: A 3D surface plot of the function f(x,y) = sin(x)sin(y), showing alternating hills and valleys across the xyz-space, representing the function's periodic nature.

37. Show that, of all the triangles inscribed in a circle of radius R (see Figure $1.8E.5$), the equilateral triangle has the largest perimeter.

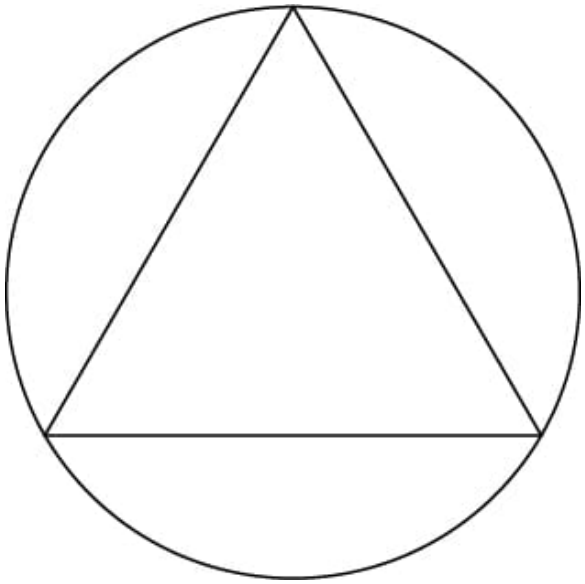


Figure $1.8E.5$: A circle with an inscribed equilateral triangle, where each vertex of the triangle touches the circumference of the circle.

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- Paul Seeburger (Monroe Community College) reordered these problems, adding problems 3 and 10 and answers for problems 15 and 17. He also added a full worked-out solution for problem 9 and a link to CalcPlot3D in problems 15, 21 and 35. He also created new images for Problem 37 and expanded the answers for many problems.

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1.9: Chapter 1 Review Exercises

Exercises 1-4

Determine whether the statement is *true or false*. Justify your answer with a proof or a counterexample.

1. The domain of $f(x,y) = x^3 \arcsin(y)$ is $\{(x,y) \mid x \in \mathbb{R} \text{ and } -\pi \leq y \leq \pi\}$.
2. If the function $f(x,y)$ is continuous everywhere, then $f_{xy}(x,y) = f_{yx}(x,y)$.

Answer

True, by Clairaut's theorem

3. The linear approximation to the function of $f(x,y) = 5x^2 + x\tan y$ at the point $(2,\pi)$ is given by $L(x,y) = 22 + 21(x-2) + (y-\pi)$.
4. $(34, 916)$ is a critical point of $g(x,y) = 4x^3 - 2x^2y + y^2 - 2$.

Answer

False

Exercises 5-6

Sketch the function in one graph and, in a second, sketch several level curves.

5. $f(x,y) = e^{-(x^2+2y^2)}$
6. $f(x,y) = x + 4y^2$

Answer

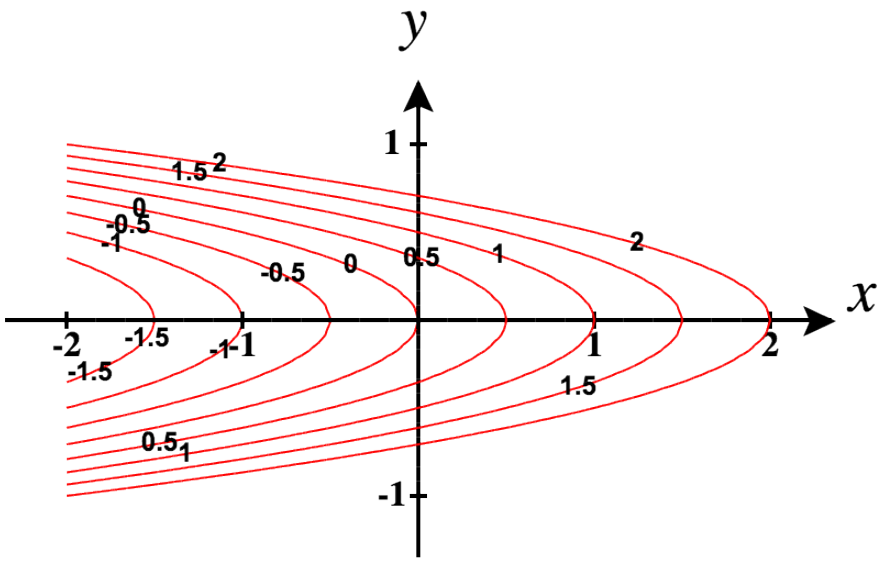


Figure 1.9.1: Level curves of $f(x,y) = x + 4y^2$, showing a series of parabolas opening towards the negative x-axis.

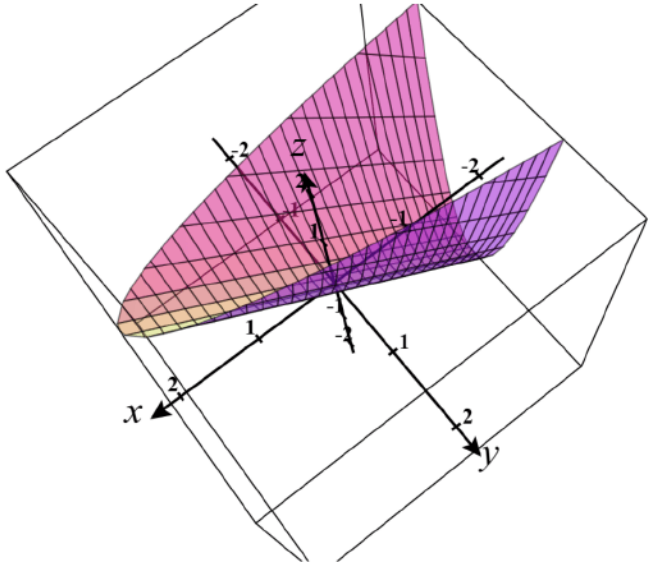


Figure 1.9.2: A 3D surface plot of the function $f(x, y) = x + 4y^2$, illustrating a parabolic trough that opens upward along the z-axis.

Exercises 7-8

Evaluate the following limits, if they exist. If they do not exist, prove it.

7. $\lim_{(x,y)\to(1,1)} \dfrac{4xy}{x - 2y^2}$
8. $\lim_{(x,y)\to(0,0)} \dfrac{4xy}{x - 2y^2}$

Answer

Does not exist.

Exercises 9-10

Find the largest interval of continuity for the function.

9. $f(x, y) = x^3 \arcsin y$
10. $g(x, y) = \ln(4 - x^2 - y^2)$

Answer

Continuous at all points on the xy-plane, except where $x^2 + y^2 > 4$.

Exercises 11-12

Find all first partial derivatives.

11. $f(x, y) = x^2 - y^2$
12. $u(x, y) = x^4 - 3xy + 1$, with $x = 2t$ and $y = t^3$

Answer

$$\frac{\partial u}{\partial x} = 4x^3 - 3y,$$

$$\frac{\partial u}{\partial y} = -3x,$$

$$\frac{dx}{dt} = 2 \text{ and } \frac{dy}{dt} = 3t^2$$

$$\begin{aligned}\frac{du}{dt} &= \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} \\ &= 8x^3 - 6y - 9xt^2 \\ &= 8\Big(2t\Big)^3 - 6(t^3) - 9(2t)t^2 \\ &= 64t^3 - 6t^3 - 18t^3 \\ &= 40t^3\end{aligned}$$

Exercises 13-14

Find all second partial derivatives.

13. $g(t,x) = 3t^2 - \sin(x+t)$
14. $h(x,y,z) = \dfrac{x^3e^{2y}}{z}$

Answer

$$\begin{aligned}h_{xx}(x,y,z) &= \frac{6xe^{2y}}{z}, \\ h_{xy}(x,y,z) &= \frac{6x^2e^{2y}}{z}, \\ h_{xz}(x,y,z) &= -\frac{3x^2e^{2y}}{z^2}, \\ h_{yx}(x,y,z) &= \frac{6x^2e^{2y}}{z}, \\ h_{yy}(x,y,z) &= \frac{4x^3e^{2y}}{z}, \\ h_{yz}(x,y,z) &= -\frac{2x^3e^{2y}}{z^2}, \\ h_{zx}(x,y,z) &= -\frac{3x^2e^{2y}}{z^2}, \\ h_{zy}(x,y,z) &= -\frac{2x^3e^{2y}}{z^2}, \\ h_{zz}(x,y,z) &= \frac{2x^3e^{2y}}{z^3}\end{aligned}$$

Exercises 15-20

Find the equation of the tangent plane to the specified surface at the given point.

15. $z = x^3 - 2y^2 + y - 1$ at point $(1,1,-1)$
16. $z = e^x + \dfrac{2}{y}$ at point $(0,1,3)$

Answer

$$z = x - 2y + 5$$

17. Approximate $f(x,y) = e^{x^2} + \sqrt{y}$ at $(0.1, 9.1)$. Write down your linear approximation function $L(x,y)$. How accurate is the approximation to the exact answer, rounded to four digits?
18. Find the differential dz of $h(x,y) = 4x^2 + 2xy - 3y$ and approximate Δz at the point $(1,-2)$. Let $\Delta x = 0.1$ and $\Delta y = 0.01$.

Answer

$$dz = 4\ dx - dy, \quad dz(0.1, 0.01) = 0.39, \quad \Delta z = 0.432$$

19. Find the directional derivative of $f(x,y) = x^2 + 6xy - y^2$ in the direction $\vec{v} = \hat{\mathbf{i}} + 4\ \hat{\mathbf{j}}$.
20. Find the maximal directional derivative magnitude and direction for the function $f(x,y) = x^3 + 2xy - \cos(\pi y)$ at point $(3,0)$.

Answer

$3\sqrt{85}\langle 27, 6\rangle$

Exercises 21-22

Find the gradient.

21. $c(x,t) = e(t-x)^2 + 3\cos t$
22. $f(x,y) = \dfrac{\sqrt{x}+y^2}{xy}$

Answer

$$\vecs{\nabla} f(x,y) = -\frac{\sqrt{x}+2y^2}{2x^2y}\,\hat{\mathbf{i}} + \left(\frac{1}{x} + \frac{1}{\sqrt{x}y^2}\right)\,\hat{\mathbf{j}}$$

Exercise 23

Find and classify the critical points.

23. $z = x^3 - xy + y^2 - 1$

Exercises 24-27

Use Lagrange multipliers to find the maximum and minimum values for the functions with the given constraints.

24. $f(x,y) = x^2y$, subject to the constraint: $x^2 + y^2 = 4$

Answer

maximum: $\dfrac{16}{3\sqrt{3}}$, minimum: $-\dfrac{16}{3\sqrt{3}}$,

25. $f(x,y) = x^2 - y^2$, subject to the constraint: $x + 6y = 4$
26. A machinist is constructing a right circular cone out of a block of aluminum. The machine gives an error of 5% in height and 2% in radius. Find the maximum error in the volume of the cone if the machinist creates a cone of height 6 cm and radius 2 cm.

Answer

2.3228 cm^3

27. A trash compactor is in the shape of a cuboid. Assume the trash compactor is filled with incompressible liquid. The length and width are decreasing at rates of 2 ft/sec and 3 ft/sec, respectively. Find the rate at which the liquid level is rising when the length is 14 ft, the width is 10 ft, and the height is 4 ft.

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2: MULTIPLE INTEGRATION

In this chapter we extend the concept of a definite integral of a single variable to double and triple integrals of functions of two and three variables, respectively. We examine applications involving integration to compute volumes, masses, and centroids of more general regions. We will also see how the use of other coordinate systems (such as polar, cylindrical, and spherical coordinates) makes it simpler to compute multiple integrals over some types of regions and functions. In the preceding chapter, we discussed differential calculus with multiple independent variables. Now we examine integral calculus in multiple dimensions. Just as a partial derivative allows us to differentiate a function with respect to one variable while holding the other variables constant, we will see that an iterated integral allows us to integrate a function with respect to one variable while holding the other variables constant.

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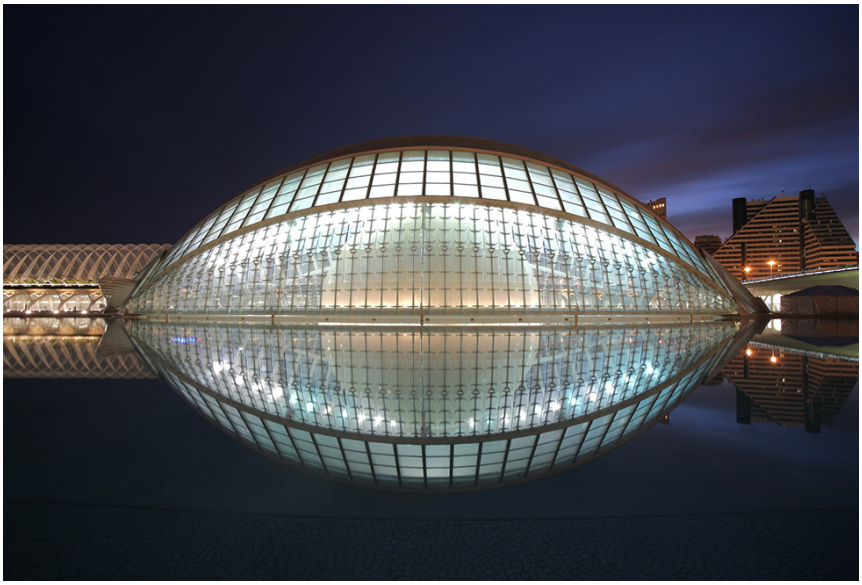


Figure 2.1: The City of Arts and Sciences in Valencia, Spain, has a unique structure along an axis of just two kilometers that was formerly the bed of the River Turia. The l'Hemisfèric has an IMAX cinema with three systems of modern digital projections onto a concave screen of 900 square meters. An oval roof over 100 meters long has been made to look like a huge human eye that comes alive and opens up to the world as the "Eye of Wisdom." (CC-BY-2.0 modification of work by Javier Yaya Tur, Wikimedia Commons)

In the preceding chapter, we discussed differential calculus with multiple independent variables. Now we examine integral calculus in multiple dimensions. Just as a partial derivative allows us to differentiate a function with respect to one variable while holding the other variables constant, we will see that an iterated integral allows us to integrate a function with respect to one variable while holding the other variables constant.

2.1: Double Integrals over Rectangular Regions

2.1E: Exercises

2.2: Double Integrals over General Regions

2.2E: Exercises

2.3: Double Integrals in Polar Coordinates

2.3E: Exercises

2.4: Triple Integrals

2.4E: Exercises

2.5: Cylindrical and Spherical Coordinates

2.5E: Exercises

2.6: Triple Integrals in Cylindrical and Spherical Coordinates

2.6E: Exercises

2.7: Calculating Centers of Mass and Moments of Inertia

2.7E: Exercises

2.8: Change of Variables in Multiple Integrals

2.8E: Exercises

2.9: Chapter 2 Review Exercises

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2.1: Double Integrals over Rectangular Regions

Learning Objectives

- Recognize when a function of two variables is integrable over a rectangular region.
- Recognize and use some of the properties of double integrals.
- Evaluate a double integral over a rectangular region by writing it as an iterated integral.
- Use a double integral to calculate the area of a region, volume under a surface, or average value of a function over a plane region.

In this section we investigate double integrals and show how we can use them to find the volume of a solid over a rectangular region in the xy-plane. Many of the properties of double integrals are similar to those we have already discussed for single integrals.

Volumes and Double Integrals

We begin by considering the space above a rectangular region R. Consider a continuous function $f(x,y) \geq 0$ of two variables defined on the closed rectangle R:

$$R = [a,b] \times [c,d] = \{(x,y) \in \mathbb{R}^2 | \, a \leq x \leq b, \, c \leq y \leq d\}$$

Here $[a,b] \times [c,d]$ denotes the Cartesian product of the two closed intervals $[a,b]$ and $[c,d]$. It consists of rectangular pairs (x,y) such that $a \leq x \leq b$ and $c \leq y \leq d$. The graph of f represents a surface above the xy-plane with equation $z = f(x,y)$ where z is the height of the surface at the point (x,y). Let S be the solid that lies above R and under the graph of f (Figure 2.1.1). The base of the solid is the rectangle R in the xy-plane. We want to find the volume V of the solid S.

Interactive Element

Visualizing the Definition of the Double Integral

Below is the graph of $z = 1 + \sin(x) + \cos(y)$ above the grid on the xy-plane cut up into several pieces. Use the buttons to incrementally increase or decrease the number of pieces along the x and y axes. The interactive app is unfortunately not accessible.

4 Partitions

Decrease Partitions | Increase Partitions

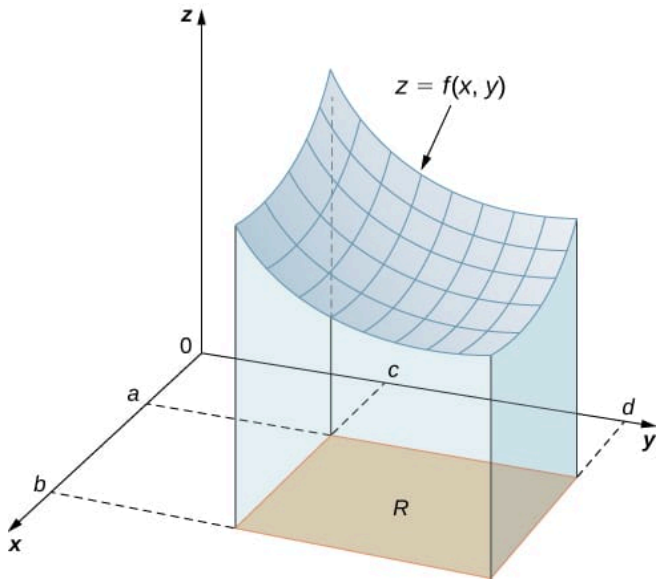


Figure 2.1.1: The graph of $f(x,y)$ *over the rectangle R in the* xy*-plane is a curved surface. On the x-axis, the lines denoting* a *and* b *are drawn; on the y-axis the lines for* c *and* d *are drawn. When the surface is projected onto the xy plane, it forms a rectangle with corners* (a,c)*,* (a,d)*,* (b,c)*, and* (b,d)*.*

We divide the region R into small rectangles R_{ij}, each with area ΔA and with sides Δx and Δy (Figure 2.1.2). We do this by dividing the interval $[a,b]$ into m subintervals and dividing the interval $[c,d]$ into n subintervals. Hence $\Delta x = \frac{b-a}{m}$, $\Delta y = \frac{d-c}{n}$, and $\Delta A = \Delta x \Delta y$.

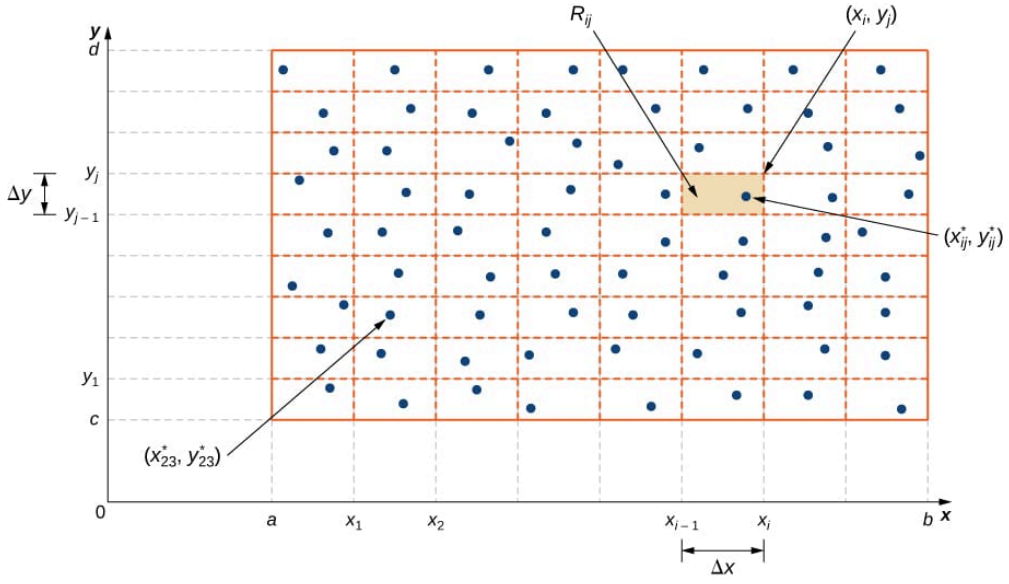


Figure 2.1.2: Rectangle R is divided into small rectangles R_{ij}. Each smaller rectangle has area ΔA which is the product of sides Δx and Δy.

The volume of a thin rectangular box above R_{ij} is $f(x^*_{ij},\ y^*_{ij})\ \Delta A$, where $(x^*_{ij},\ y^*_{ij})$ is an arbitrary sample point in each R_{ij} as shown in Figure 2.1.3, $f(x^*_{ij},\ y^*_{ij})$ is the height of the corresponding thin rectangular box, and ΔA is the area of each rectangle R_{ij}.

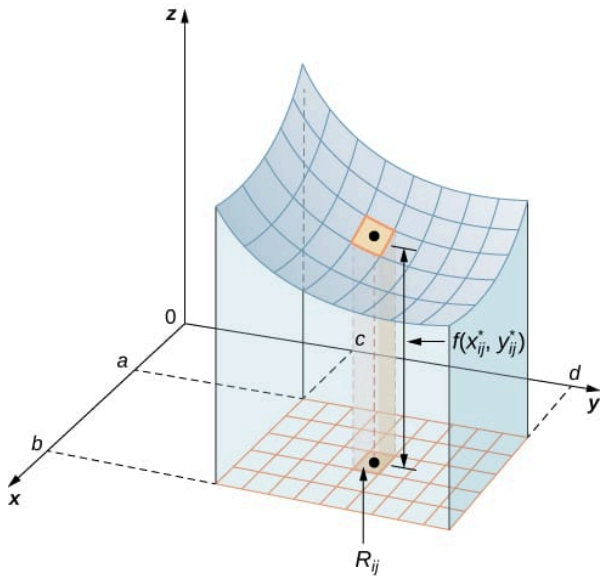


Figure 2.1.3: The same solid as in Figure 2.1.1. The region R on the xy-plane is divided into small rectangles. A thin rectangular box above R_{ij} with height $f(x_{ij}^*,\ y_{ij}^*)$ is shown in the middle of the solid.

Using the same idea for all the subrectangles, we obtain an approximate volume of the solid S as

$$V \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*,\ y_{ij}^*)\Delta A.$$

This sum is known as a **double Riemann sum** and can be used to approximate the value of the volume of the solid. Here the double sum means that for each subrectangle we evaluate the function at the chosen point, multiply by the area of each rectangle, and then add all the results.

As we have seen in the single-variable case, we obtain a better approximation to the actual volume if m and n become larger.

$$V = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*,\ y_{ij}^*)\Delta A$$

or

$$V = \lim_{\Delta x,\ \Delta y\to 0} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*,\ y_{ij}^*)\Delta A.$$

Note that the sum approaches a limit in either case and the limit is the volume of the solid with the base R. Now we are ready to define the double integral.

Definition: Double Integral

The double integral of the function $f(x,\ y)$ over the rectangular region R in the xy-plane is defined as

$$\iint_R f(x,\ y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*,\ y_{ij}^*)\Delta A. \tag{2.1.1}$$

If $f(x,y) \geq 0$, then the volume V of the solid S, which lies above R in the xy-plane and under the graph of f, is the double integral of the function $f(x,y)$ over the rectangle R. If the function is ever negative, then the double integral can be considered a "signed" volume in a manner similar to the way we defined net signed area in The Definite Integral.

Example 2.1.1: Setting up a Double Integral and Approximating It by Double Sums

Consider the function $z = f(x,\ y) = 3x^2 - y$ over the rectangular region $R = [0, 2] \times [0, 2]$ (Figure 2.1.4).

a. Set up a double integral for finding the value of the signed volume of the solid S that lies above R and "under" the graph of f.
b. Divide R into four squares with $m = n = 2$, and choose the sample point as the upper right corner point of each square (1,1),(2,1),(1,2), and (2,2) (Figure 2.1.4) to approximate the signed volume of the solid S that lies above R and "under" the graph of f.
c. Divide R into four squares with $m = n = 2$, and choose the sample point as the midpoint of each square: (1/2, 1/2), (3/2, 1/2), (1/2,3/2), and (3/2, 3/2) to approximate the signed volume.

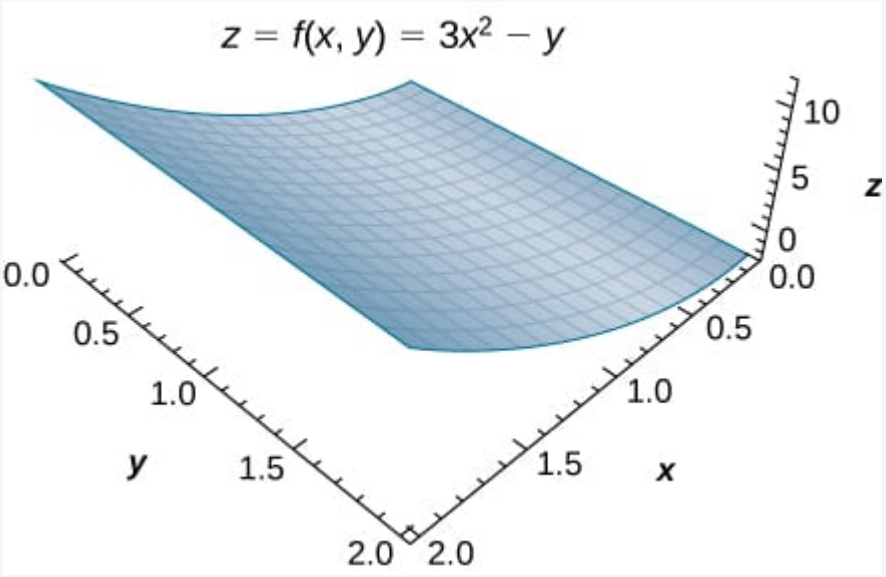


Figure 2.1.4: The function $z = f(x, y)$ graphed over the rectangular region $R = [0, 2] \times [0, 2]$. The corners of the surface are given as $(0, 0, 0)$, $(2, 0, 12)$, $(0, 2, -2)$, and $(2, 2, 10)$. The surface is parabolic along the x-axis.

Solution

a. As we can see, the function $z = f(x, y) = 3x^2 - y$ is above the plane. To find the signed volume of S, we need to divide the region R into small rectangles R_{ij}, each with area ΔA and with sides Δx and Δy, and choose (x^*_{ij}, y^*_{ij}) as sample points in each R_{ij}. Hence, a double integral is set up as

$$V = \iint_R (3x^2 - y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} [3(x^*_{ij})^2 - y^*_{ij}]\Delta A.$$

b. Approximating the signed volume using a Riemann sum with $m = n = 2$ we have $\Delta A = \Delta x \Delta y = 1 \times 1 = 1$. Also, the sample points are (1, 1), (2, 1), (1, 2), and (2, 2) as shown in Figure 2.1.5.

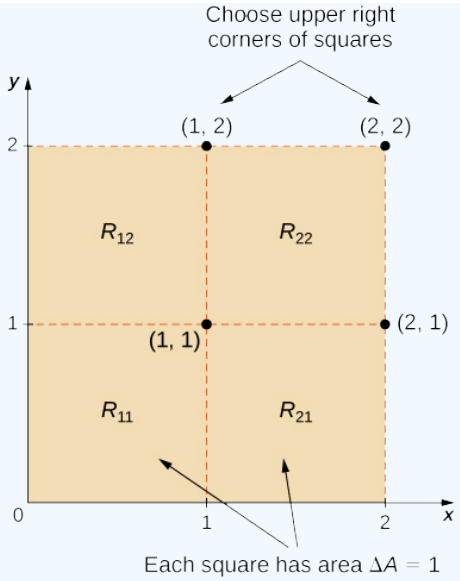


Figure 2.1.5: Sub-rectangles for the rectangular region $R = [0,2] \times [0,2]$. On the xy-plane, the points $(1,1)$, $(1,2)$, $(2,1)$, and $(2,2)$ are marked, and these form the upper right corners of four squares marked $R_{11}, R_{12}, R_{21}, R_{22}$, respectively. Each square has area $\Delta A = 1$.

Hence,

$$\begin{aligned} V &\approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(x_{ij}^*, y_{ij}^*)\Delta A \\ &= \sum_{i=1}^{2}(f(x_{i1}^*, y_{i1}^*) + f(x_{i2}^*, y_{i2}^*))\Delta A \\ &= f(x_{11}^*, y_{11}^*)\Delta A + f(x_{21}^*, y_{21}^*)\Delta A + f(x_{12}^*, y_{12}^*)\Delta A + f(x_{22}^*, y_{22}^*)\Delta A \\ &= f(1,1)(1) + f(2,1)(1) + f(1,2)(1) + f(2,2)(1) \\ &= (3-1)(1) + (12-1)(1) + (3-2)(1) + (12-2)(1) \\ &= 2 + 11 + 1 + 10 = 24. \end{aligned}$$

c. Approximating the signed volume using a Riemann sum with $m = n = 2$ we have$\Delta A = \Delta x \Delta y = 1 \times 1 = 1$. In this case the sample points are (1/2, 1/2), (3/2, 1/2), (1/2, 3/2), and (3/2, 3/2).
Hence,

$$\begin{aligned} V &\approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(x_{ij}^*, y_{ij}^*)\Delta A \\ &= f(x_{11}^*, y_{11}^*)\Delta A + f(x_{21}^*, y_{21}^*)\Delta A + f(x_{12}^*, y_{12}^*)\Delta A + f(x_{22}^*, y_{22}^*)\Delta A \\ &= f(1/2, 1/2)(1) + f(3/2, 1/2)(1) + f(1/2, 3/2)(1) + f(3/2, 3/2)(1) \\ &= \left(\frac{3}{4} - \frac{1}{4}\right)(1) + \left(\frac{27}{4} - \frac{1}{2}\right)(1) + \left(\frac{3}{4} - \frac{3}{2}\right)(1) + \left(\frac{27}{4} - \frac{3}{2}\right)(1) \\ &= \frac{2}{4} + \frac{25}{4} + \left(-\frac{3}{4}\right) + \frac{21}{4} = \frac{45}{4}. \end{aligned}$$

Analysis

Notice that the approximate answers differ due to the choices of the sample points. In either case, we are introducing some error because we are using only a few sample points. Thus, we need to investigate how we can achieve an accurate answer.

Exercise 2.1.1

Use the same function$z = f(x, y) = 3x^2 - y$ over the rectangular region $R = [0, 2] \times [0, 2]$.

Divide R into the same four squares with $m = n = 2$, and choose the sample points as the upper left corner point of each square (0,1), (1,1), (0,2), and (1,2) (Figure 2.1.5) to approximate the signed volume of the solid S that lies above R and "under" the graph of f.

Hint available at the end of this section.

Answer available at the end of this section.

Note that we developed the concept of double integral using a rectangular region R. This concept can be extended to any general region. However, when a region is not rectangular, the subrectangles may not all fit perfectly into R, particularly if the base area is curved. We examine this situation in more detail in the next section, where we study regions that are not always rectangular and subrectangles may not fit perfectly in the region R. Also, the heights may not be exact if the surface $z = f(x, y)$ is curved. However, the errors on the sides and the height where the pieces may not fit perfectly within the solid S approach 0 as m and n approach infinity. Also, the double integral of the function $z = f(x, y)$ exists provided that the function f is not too discontinuous. If the function is bounded and continuous over R except on a finite number of smooth curves, then the double integral exists and we say that ff is integrable over R.

Since $\Delta A = \Delta x \Delta y = \Delta y \Delta x$, we can express dA as $dx\ dy$ or $dy\ dx$. This means that, when we are using rectangular coordinates, the double integral over a region R denoted by

$$\iint_R f(x, y)\, dA$$

can be written as

$$\iint_R f(x, y)\, dx\, dy$$

or

$$\iint_R f(x, y)\, dy\, dx.$$

Now let's list some of the properties that can be helpful to compute double integrals.

Properties of Double Integrals

The properties of double integrals are very helpful when computing them or otherwise working with them. We list here six properties of double integrals. Properties 1 and 2 are referred to as the linearity of the integral, property 3 is the additivity of the integral, property 4 is the monotonicity of the integral, and property 5 is used to find the bounds of the integral. Property 6 is used if $f(x, y)$ is a product of two functions $g(x)$ and $h(y)$.

Properties of Double Integral

Assume that the functions $f(x, y)$ and $g(x, y)$ are integrable over the rectangular region R; S and T are subregions of R; and assume that m and M are real numbers.

i. The sum $f(x, y) + g(x, y)$ is integrable and

$$\iint_R [f(x, y) + g(x, y)]\, dA = \iint_R f(x, y)\, dA + \iint_R g(x, y)\, dA.$$

ii. If c is a constant, then $cf(x,y)$ is integrable and

$$\iint_R cf(x,y)\,dA = c\iint_R f(x,y)\,dA.$$

iii. If $R = S \cup T$ and $S \cap T = \emptyset$ except an overlap on the boundaries, then

$$\iint_R f(x,y)\,dA = \iint_S f(x,y)\,dA + \iint_T f(x,y)\,dA.$$

iv. If $f(x,y) \geq g(x,y)$ for (x,y) in R, then

$$\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA.$$

v. If $m \leq f(x,y) \leq M$ and $A(R) = \text{ the area of } R$, then

$$m \cdot A(R) \leq \iint_R f(x,y)\,dA \leq M \cdot A(R).$$

vi. In the case where $f(x,y)$ can be factored as a product of a function $g(x)$ of x only and a function $h(y)$ of y only, then over the region $R = \{(x,y) \mid a \leq x \leq b,\ c \leq y \leq d\}$, the double integral can be written as

$$\iint_R f(x,y)\,dA = \left(\int_a^b g(x)\,dx\right)\left(\int_c^d h(y)\,dy\right).$$

These properties are used in the evaluation of double integrals, as we will see later. We will become skilled in using these properties once we become familiar with the computational tools of double integrals. So let's get to that now.

Iterated Integrals

So far, we have seen how to set up a double integral and how to obtain an approximate value for it. We can also imagine that evaluating double integrals by using the definition can be a very lengthy process if we choose larger values for m and n.Therefore, we need a practical and convenient technique for computing double integrals. In other words, we need to learn how to compute double integrals without employing the definition that uses limits and double sums.

The basic idea is that the evaluation becomes easier if we can break a double integral into single integrals by integrating first with respect to one variable and then with respect to the other. The key tool we need is called an iterated integral.

Definitions: Iterated Integrals

Assume a, b, c, and d are real numbers. We define an *iterated integral* for a function $f(x,y)$ over the rectangular region $R = [a,b] \times [c,d]$ as

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx \tag{2.1.2}$$

or

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy. \tag{2.1.3}$$

The notation $\int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$ means that we integrate $f(x,y)$ with respect to y while holding x constant. Similarly, the notation $\int_c^d \left[\int_a^b f(x,y)\,dx\right] dy$ means that we integrate $f(x,y)$ with respect to x while holding y constant. The fact that double integrals can be split into iterated integrals is expressed in Fubini's theorem. Think of this theorem as an essential tool for evaluating double integrals.

Fubini's Theorem

Suppose that $f(x,y)$ is a function of two variables that is continuous over a rectangular region $R = \{(x,y) \in \mathbb{R}^2 |\ a \le x \le b,\ c \le y \le d\}$. Then we see from Figure 2.1.6 that the double integral of f over the region equals an iterated integral,

$$\iint_R f(x,y)\,dA = \iint_R f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy. \tag{2.1.4}$$

More generally, Fubini's theorem is true if f is bounded on R and f is discontinuous only on a finite number of continuous curves. In other words, f has to be integrable over R.

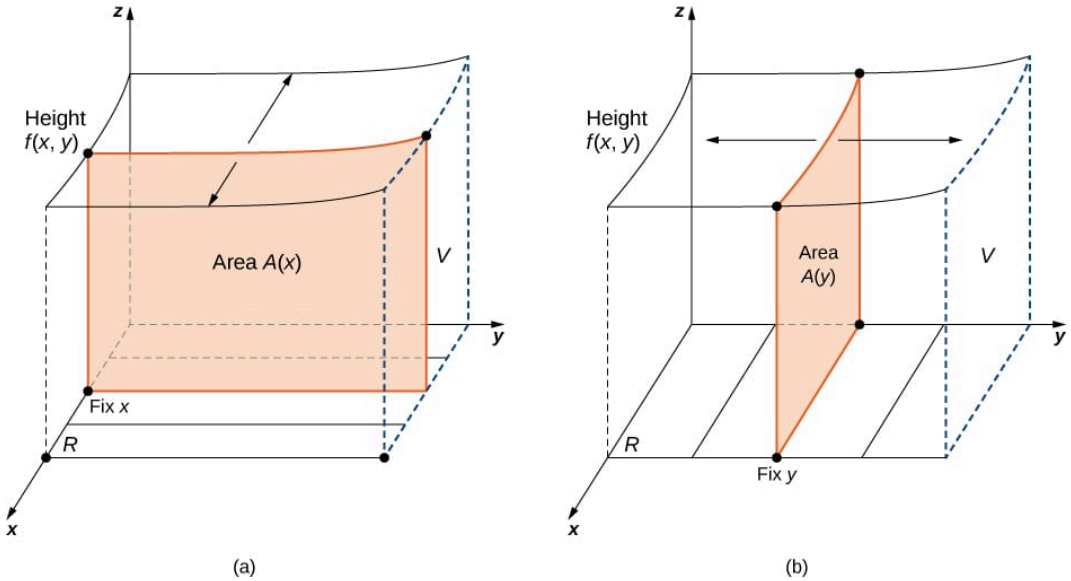


Figure 2.1.6: (a) Integrating first with respect to y and then with respect to x to find the area $A(x)$ and then the volume V. A surface is given by the function $f(x,y)$. A point x is chosen on the x-axis, and at this point, it is written Fix x. From this point, a plane is projected perpendicular to the xy-plane along the line with value x. This plane is marked Area $A(x)$, and the entire space under the surface is marked V. (b) integrating first with respect to x and then with respect to y to find the area $A(y)$ and then the volume V. A surface is given by the function $f(x,y)$. A point y is chosen on the y-axis, and at this point, it is written Fix y. From this point, a plane is projected perpendicular to the xy-plane along the line with value y. This plane is marked Area $A(y)$, and the entire space under the surface is marked V.

Example 2.1.2: Using Fubini's Theorem

Use Fubini's theorem to compute the double integral $\iint_R f(x,y)\ dA$ where $f(x,y) = x$ and $R = [0,2] \times [0,1]$.

Solution

Fubini's theorem offers an easier way to evaluate the double integral by the use of an iterated integral. Note how the boundary values of the region R become the upper and lower limits of integration.

$$\begin{aligned}
\iint_R f(x,y)\,dA &= \iint_R f(x,y)\,dx\,dy \\
&= \int_{y=0}^{y=1} \int_{x=0}^{x=2} x\,dx\,dy \\
&= \int_{y=0}^{y=1} \left[\frac{x^2}{2}\Big|_{x=0}^{x=2}\right] dy \\
&= \int_{y=0}^{y=1} 2\,dy = 2y\Big|_{y=0}^{y=1} = 2
\end{aligned}$$

The double integration in this example is simple enough to use Fubini's theorem directly, allowing us to convert a double integral into an iterated integral. Consequently, we are now ready to convert all double integrals to iterated integrals and demonstrate how the properties listed earlier can help us evaluate double integrals when the function $f(x,y)$ is more complex. Note that the order of integration can be changed (see Figure 2.1.6).

Example 2.1.3: Illustrating Properties i and ii

Evaluate the double integral

$$\iint_R (xy - 3xy^2)\,dA, \text{ where } R = \{(x,y)\,|\,0 \le x \le 2,\ 1 \le y \le 2\}.$$

Solution

This function has two pieces: one piece is xy and the other is $3xy^2$. Also, the second piece has a constant 3. Notice how we use properties i and ii to help evaluate the double integral.

$$\begin{aligned}
\iint_R (xy - 3xy^2)\,dA &= \iint_R xy\,dA + \iint_R (-3xy^2)\,dA && \text{Property i: Integral of a sum is the sum of the integrals.}\\
&= \int_{y=1}^{y=2}\int_{x=0}^{x=2} xy\,dx\,dy - \int_{y=1}^{y=2}\int_{x=0}^{x=2} 3xy^2\,dx\,dy && \text{Convert double integrals to iterated integrals.}\\
&= \int_{y=1}^{y=2}\left(\frac{x^2}{2}y\right)\Big|_{x=0}^{x=2} dy - 3\int_{y=1}^{y=2}\left(\frac{x^2}{2}y^2\right)\Big|_{x=0}^{x=2} dy && \text{Integrate with respect to } x, \text{ holding } y \text{ constant.}\\
&= \int_{y=1}^{y=2} 2y\,dy - \int_{y=1}^{y=2} 6y^2 dy && \text{Property ii: Placing the constant before the integral.}\\
&= 2\int_1^2 y\,dy - 6\int_1^2 y^2\,dy && \text{Integrate with respect to y.}\\
&= 2\frac{y^2}{2}\Big|_1^2 - 6\frac{y^3}{3}\Big|_1^2\\
&= y^2\Big|_1^2 - 2y^3\Big|_1^2\\
&= (4-1) - 2(8-1) = 3 - 2(7) = 3 - 14 = -11.
\end{aligned}$$

Example 2.1.4: Illustrating Property v.

Over the region $R = \{(x,y)\ |\ 1 \le x \le 3,\ \ 1 \le y \le 2\}$, we have $2 \le x^2 + y^2 \le 13$. Find a lower and an upper bound for the integral $\iint_R (x^2 + y^2)\ dA$.

Solution

For a lower bound, integrate the constant function 2 over the region R. For an upper bound, integrate the constant function 13 over the region R.

$$\int_1^2\int_1^3 2\,dx\,dy = \int_1^2 [2x\Big|_1^3]\,dy = \int_1^2 2(2)dy = 4y\Big|_1^2 = 4(2-1) = 4$$

$$\int_1^2\int_1^3 13dx\,dy = \int_1^2 [13x\Big|_1^3]\,dy = \int_1^2 13(2)\,dy = 26y\Big|_1^2 = 26(2-1) = 26.$$

Hence, we obtain $4 \le \iint_R (x^2 + y^2)\ dA \le 26$.

Example 2.1.5: Illustrating Property vi

Evaluate the integral $\iint_R e^y \cos x \; dA$ over the region $R = \{(x, y) \mid 0 \le x \le \frac{\pi}{2}, \; 0 \le y \le 1\}$.

Solution

This is a great example for property vi because the function $f(x, y)$ is clearly the product of two single-variable functions e^y and $\cos x$. Thus we can split the integral into two parts and then integrate each one as a single-variable integration problem.

$$\begin{aligned}\iint_R e^y \cos x \, dA &= \int_0^1 \int_0^{\pi/2} e^y \cos x \, dx \, dy \\ &= \left(\int_0^1 e^y dy\right)\left(\int_0^{\pi/2} \cos x \, dx\right) \\ &= (e^y\Big|_0^1)(\sin x\Big|_0^{\pi/2}) \\ &= e - 1.\end{aligned}$$

Exercise 2.1.2

a. Use the properties of the double integral and Fubini's theorem to evaluate the integral

$$\int_0^1 \int_{-1}^3 (3 - x + 4y) \, dy \, dx.$$

b. Show that $0 \le \iint_R \sin \pi x \;\; \cos \pi y \; dA \le \dfrac{1}{32}$ where $R = [0, \frac{1}{4}] \times [14, \frac{1}{2}]$.

Hint available at the end of this section.

Answer available at the end of this section.

As we mentioned before, when we are using rectangular coordinates, the double integral over a region R denoted by $\iint_R f(x, y) \; dA$ can be written as $\iint_R \; f(x, y) \; dx \; dy$ or $\iint_R \; f(x, y) \; dy \; dx$. The next example shows that the results are the same regardless of which order of integration we choose.

Example 2.1.6: Evaluating an Iterated Integral in Two Ways

Let's return to the function $f(x, y) = 3x^2 - y$ from Example 1, this time over the rectangular region $R = [0, 2] \times [0, 3]$. Use Fubini's theorem to evaluate $\iint_R f(x, y) \; dA$ in two different ways:

a. First integrate with respect to y and then with respect to x;
b. First integrate with respect to x and then with respect to y.

Solution

Figure 2.1.6 shows how the calculation works in two different ways.

a. First integrate with respect to y and then integrate with respect to x:

$$\iint_R f(x,y)\,dA = \int_{x=0}^{x=2}\int_{y=0}^{y=3}(3x^2-y)\,dy\,dx$$
$$= \int_{x=0}^{x=2}\left(\int_{y=0}^{y=3}(3x^2-y)\,dy\right)dx = \int_{x=0}^{x=2}\left[3x^2y-\frac{y^2}{2}\Big|_{y=0}^{y=3}\right]dx$$
$$= \int_{x=0}^{x=2}\left(9x^2-\frac{9}{2}\right)dx = 3x^3-\frac{9}{2}x\Big|_{x=0}^{x=2} = 15.$$

b. First integrate with respect to x and then integrate with respect to y:

$$\iint_R f(x,y)\,dA = \int_{y=0}^{y=3}\int_{x=0}^{x=2}(3x^2-y)\,dx\,dy$$
$$= \int_{y=0}^{y=3}\left(\int_{x=0}^{x=2}(3x^2-y)\,dx\right)dy$$
$$= \int_{y=0}^{y=3}\left[x^3-xy\Big|_{x=0}^{x=2}\right]dy$$
$$= \int_{y=0}^{y=3}(8-2y)\,dy = 8y-y^2\Big|_{y=0}^{y=3} = 15.$$

Analysis

With either order of integration, the double integral gives us an answer of 15. We might wish to interpret this answer as a volume in cubic units of the solid S below the function $f(x,y)=3x^2-y$ over the region $R=[0,2]\times[0,3]$. However, remember that the interpretation of a double integral as a (non-signed) volume works only when the integrand f is a nonnegative function over the base region R.

Exercise 2.1.3

Evaluate

$$\int_{y=-3}^{y=2}\int_{x=3}^{x=5}(2-3x^2+y^2)\,dx\,dy.$$

Hint available at the end of this section.

Answer available at the end of this section.

In the next example we see that it can actually be beneficial to switch the order of integration to make the computation easier. We will come back to this idea several times in this chapter.

Example 2.1.7: Switching the Order of Integration

Consider the double integral $\iint_R x\ \sin(xy)\ dA$ over the region $R=\{(x,y)\mid 0\le x\le\pi,\ 1\le y\le 2\}$ (Figure 2.1.7).

a. Express the double integral in two different ways.
b. Analyze whether evaluating the double integral in one way is easier than the other and why.
c. Evaluate the integral.

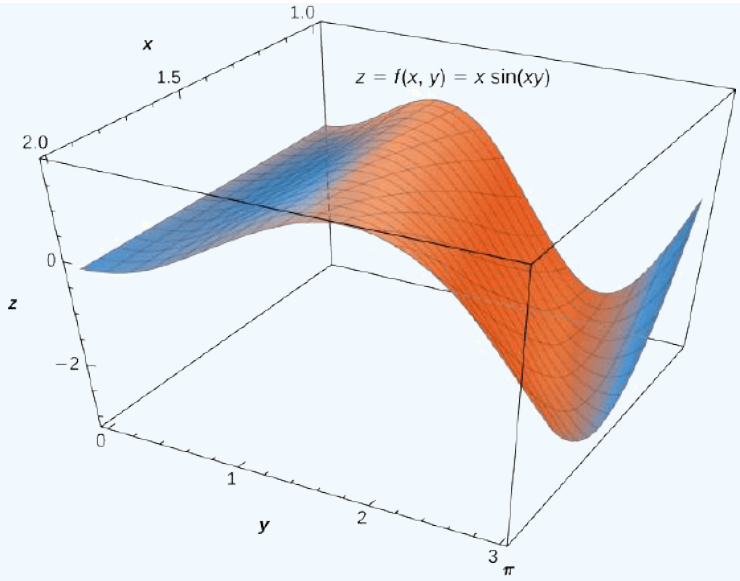


Figure 2.1.7: The function $z = f(x, y) = x\ \sin(xy)$ over the rectangular region $R = [0, \pi] \times [1, 2]$. The graph of the function starts at $z = 0$ then increases roughly as a normal sine function would, but then skews a bit and decreases as x increases after pi/2.

a. We can express $\iint_R x\ \sin(xy)\ dA$ in the following two ways: first by integrating with respect to y and then with respect to x; second by integrating with respect to x and then with respect to y.

$$\iint_R x\ \sin(xy)\, dA = \int_{x=0}^{x=\pi} \int_{y=1}^{y=2} x\ \sin(xy)\, dy\, dx$$

Integrate first with respect to y.

$$= \int_{y=1}^{y=2} \int_{x=0}^{x=\pi} x\ \sin(xy)\, dx\, dy$$

Integrate first with respect to x.

b. If we want to integrate with respect to *y* first and then integrate with respect to x, we see that we can use the substitution $u = xy$, which gives $du = x\ dy$. Hence the inner integral is simply $\int \sin u\ du$ and we can change the limits to be functions of x,

$$\iint_R x\ \sin(xy)\, dA = \int_{x=0}^{x=\pi} \int_{y=1}^{y=2} x\ \sin(xy)\, dy\, dx = \int_{x=0}^{x=\pi} \left[\int_{u=x}^{u=2x} \sin(u)\, du \right] dx.$$

However, integrating with respect to x first and then integrating with respect to y requires integration by parts for the inner integral, with $u = x$ and $dv = \sin(xy)dx$

Then $du = dx$ and $v = -\frac{\cos(xy)}{y}$, so

$$\iint_R x \sin(xy)\, dA = \int_{y=1}^{y=2} \int_{x=0}^{x=\pi} x \sin(xy)\, dx\, dy = \int_{y=1}^{y=2} \left[-\frac{x\ \cos(xy)}{y} \Big|_{x=0}^{x=\pi} + \frac{1}{y} \int_{x=0}^{x=\pi} \cos(xy)\, dx \right] dy.$$

Since the evaluation is getting complicated, we will only do the computation that is easier to do, which is clearly the first method.

c. Evaluate the double integral using the easier way.

$$\begin{aligned}\iint_R x\,\sin(xy)\,dA &= \int_{x=0}^{x=\pi}\int_{y=1}^{y=2} x\,\sin(xy)\,dy\,dx \\ &= \int_{x=0}^{x=\pi}\left[\int_{u=x}^{u=2x}\sin(u)\,du\right]dx = \int_{x=0}^{x=\pi}\left[-\cos u\Big|_{u=x}^{u=2x}\right]dx \\ &= \int_{x=0}^{x=\pi}(-\cos 2x+\cos x)\,dx \\ &= \left(-\frac{1}{2}\sin 2x+\sin x\right)\Big|_{x=0}^{x=\pi} = 0.\end{aligned}$$

Exercise 2.1.4

Evaluate the integral $\iint_R xe^{xy}\,dA$ where $R=[0,1]\times[0,\ln 5]$.

Hint available at the end of this section.

Answer available at the end of this section.

Applications of Double Integrals

Double integrals are very useful for finding the area of a region bounded by curves of functions. We describe this situation in more detail in the next section. However, if the region is a rectangular shape, we can find its area by integrating the constant function $f(x,y)=1$ over the region R.

Definition: Area of the Region

The area of the region R is given by

$$A(R)=\iint_R 1\,dA. \tag{2.1.5}$$

This definition makes sense because using $f(x,y)=1$ and evaluating the integral make it a product of length and width. Let's check this formula with an example and see how this works.

Example 2.1.8: Finding Area Using a Double Integral

Find the area of the region $R=\{\ (x,y)\ |\ 0\le x\le 3,\ \ 0\le y\le 2\}$ by using a double integral, that is, by integrating 1 over the region R.

Solution

The region is rectangular with length 3 and width 2, so we know that the area is 6. We get the same answer when we use a double integral:

$$A(R)=\int_0^2\int_0^3 1\,dx\,dy=\int_0^2\left[x\Big|_0^3\right]dy=\int_0^2 3dy=3\int_0^2 dy=3y\Big|_0^2=3(2)=6\ \text{units}^2.$$

We have already seen how double integrals can be used to find the volume of a solid bounded above by a function $f(x,y)\ge 0$ over a region R provided $f(x,y)\ge 0$ for all (x,y) in R. Here is another example to illustrate this concept.

Example 2.1.9: Volume of an Elliptic Paraboloid

Find the volume V of the solid S that is bounded by the elliptic paraboloid $2x^2 + y^2 + z = 27$, the planes $x = 3$ and $y = 3$, and the three coordinate planes.

Solution

First notice the graph of the surface $z = 27 - 2x^2 - y^2$ in Figure 2.1.8(a) and above the square region $R_1 = [-3, 3] \times [-3, 3]$. However, we need the volume of the solid bounded by the elliptic paraboloid $2x^2 + y^2 + z = 27$, the planes $x = 3$ and $y = 3$, and the three coordinate planes.

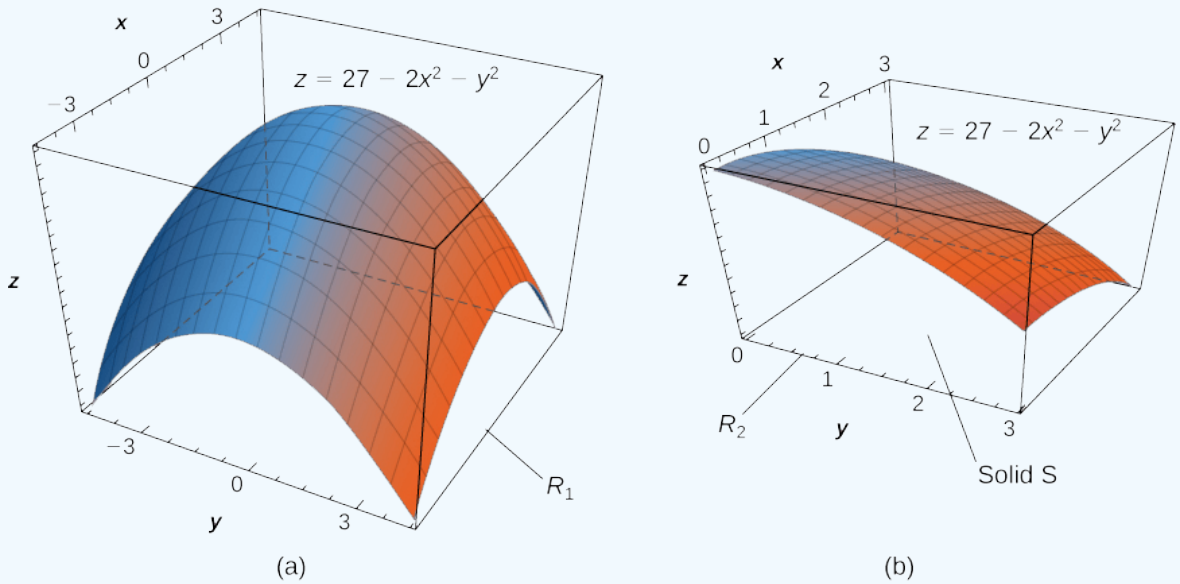


Figure 2.1.8: (a) The surface $z = 27 - 2x^2 - y^2$ above the square region $R1 = [-3, 3] \times [-3, 3]$ is an upside down paraboloid. (b) The solid S lies under the surface $z = 27 - 2x^2 - y^2$ above the square region $R1 = [0, 3] \times [0, 3]$.

Now let's look at the graph of the surface in Figure 2.1.8(b). We determine the volume V by evaluating the double integral over R_2:

$$
\begin{aligned}
V &= \iint_R z\,dA = \iint_R (27 - 2x^2 - y^2)\,dA \\
&= \int_{y=0}^{y=3} \int_{x=0}^{x=3} (27 - 2x^2 - y^2)\,dx\,dy && \text{Convert to iterated integral.} \\
&= \int_{y=0}^{y=3} [27x - \frac{2}{3}x^3 - y^2 x]\Big|_{x=0}^{x=3} dy && \text{Integrate with respect to } x. \\
&= \int_{y=0}^{y=3} (63 - 3y^2)dy = 63y - y^3\Big|_{y=0}^{y=3} = 162.
\end{aligned}
$$

Exercise 2.1.5

Find the volume of the solid bounded above by the graph of $f(x, y) = xy\sin(x^2 y)$ and below by the xy-plane on the rectangular region $R = [0, 1] \times [0, \pi]$.

Hint available at the end of this section.

Answer available at the end of this section.

Recall that we defined the average value of a function of one variable on an interval $[a, b]$ as

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{2.1.6}$$

Similarly, we can define the average value of a function of two variables over a region R. The main difference is that we divide by an area instead of the width of an interval.

Definition: Average Value

The **average value of a function** of two variables over a region R is

$$F_{ave} = \frac{1}{\text{Area of } R}\iint_R f(x,y)\,dx\,dy. \tag{2.1.7}$$

In the next example we find the average value of a function over a rectangular region. This is a good example of obtaining useful information for an integration by making individual measurements over a grid, instead of trying to find an algebraic expression for a function.

Example 2.1.10: Calculating Average Storm Rainfall

The weather map in Figure 2.1.9 shows an unusually moist storm system associated with the remnants of Hurricane Karl, which dumped 4–8 inches (100–200 mm) of rain in some parts of the Midwest on September 22–23, 2010. The area of rainfall measured 300 miles east to west and 250 miles north to south. Estimate the average rainfall over the entire area in those two days.

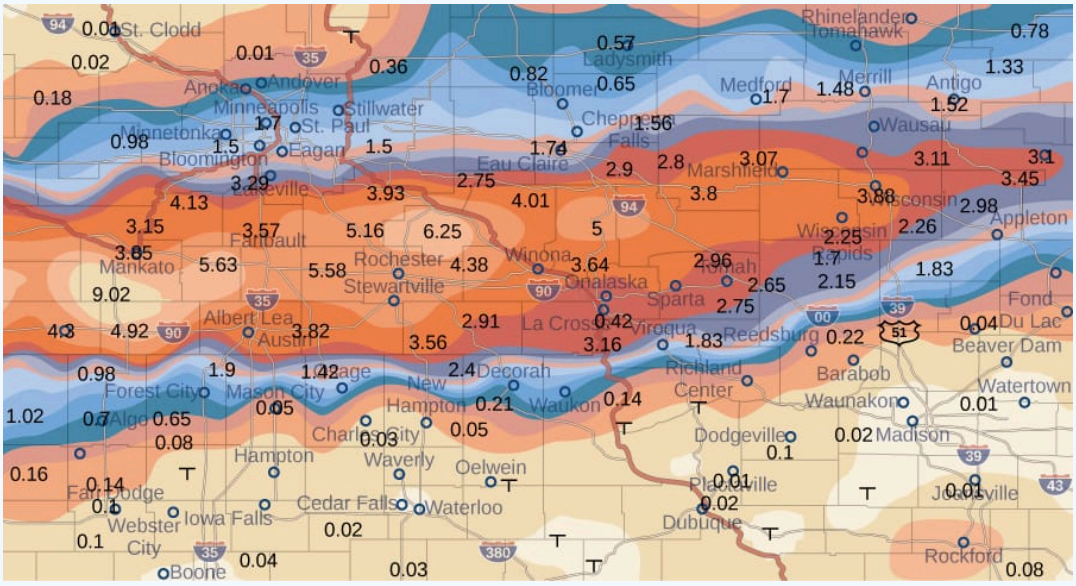


Figure 2.1.9: Effects of Hurricane Karl, which dumped 4–8 inches (100–200 mm) of rain in some parts of southwest Wisconsin, southern Minnesota, and southeast South Dakota over a span of 300 miles east to west and 250 miles north to south. A map of Wisconsin and Minnesota that shows many cities with numbers affixed to them. The highest numbers come in a narrow band, and the map is colored accordingly. The map effectively looks like a contour map, but instead of elevations, it uses numbers.

Solution

Place the origin at the southwest corner of the map so that all the values can be considered as being in the first quadrant and hence all are positive. Now divide the entire map into six rectangles ($m = 2$ and $n = 3$), as shown in Figure 2.1.9. Assume $f(x, y)$ denotes the storm rainfall in inches at a point approximately x miles to the east of the origin and y miles to the north of the origin. Let R represent the entire area of $250 \times 300 = 75000$ square miles. Then the area of each subrectangle is

$$\Delta A = \frac{1}{6}(75000) = 12500.$$

Assume $(x_{ij}*, y_{ij}*)$ are approximately the midpoints of each subrectangle R_{ij}. Note the color-coded region at each of these points, and estimate the rainfall. The rainfall at each of these points can be estimated as:

- At (x_{11}, y_{11}), the rainfall is 0.08.
- At (x_{12}, y_{12}), the rainfall is 0.08.
- At (x_{13}, y_{13}), the rainfall is 0.01.
- At (x_{21}, y_{21}), the rainfall is 1.70.
- At (x_{22}, y_{22}), the rainfall is 1.74.
- At (x_{23}, y_{23}), the rainfall is 3.00.

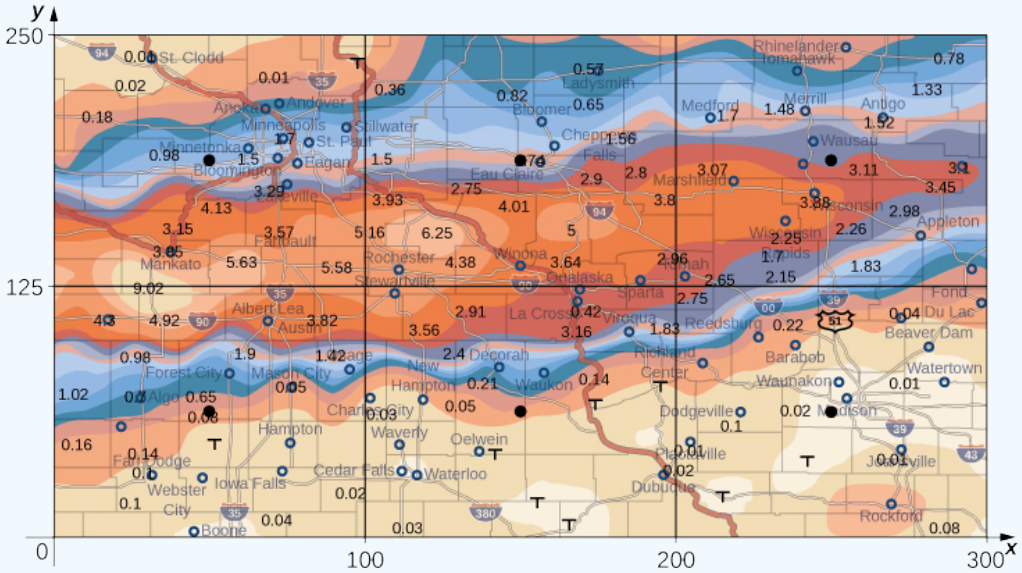


Figure 2.1.10: Storm rainfall with rectangular axes and showing the midpoints of each subrectangle. Another version of the previous storm map, but this time with lines drawn for $x = 100, 200, 300$ and for $y = 125, 250$. There is a dot in the center of each of the resulting rectangles.

According to our definition, the average storm rainfall in the entire area during those two days was

$$
\begin{aligned}
f_{ave} = \frac{1}{Area\ R}\iint_R = f(x,y)\,dx\,dy &= \frac{1}{75000}\iint_R f(x,y)\,dx\,dy \\
&\approx \frac{1}{75000}\sum_{i=1}^{3}\sum_{j=1}^{2} f(x_{ij}^*, y_{ij}^*)\Delta A \\
&= \frac{1}{75000}[f(x_{11}^*, y_{11}^*)\Delta A + f(x_{12}^*, y_{12}^*)\Delta A + f(x_{13}^*, y_{13}^*)\Delta A + f(x_{21}^*, y_{21}^*)\Delta A + f(x_{22}^*, y_{22}^*)\Delta A + f(x_{23}^*, y_{23}^*)\Delta A] \\
&\approx \frac{1}{75000}[0.08 + 0.08 + 0.01 + 1.70 + 1.74 + 3.00]\Delta A \\
&= \frac{1}{75000}[0.08 + 0.08 + 0.01 + 1.70 + 1.74 + 3.00]12500 \\
&= \frac{1}{6}[0.08 + 0.08 + 0.01 + 1.70 + 1.74 + 3.00] \\
&\approx 1.10 \text{ in.}
\end{aligned}
$$

During September 22–23, 2010 this area had an average storm rainfall of approximately 1.10 inches.

Exercise 2.1.6

A contour map is shown for a function $f(x, y)$ on the rectangle $R = [-3, 6] \times [-1, 4]$.

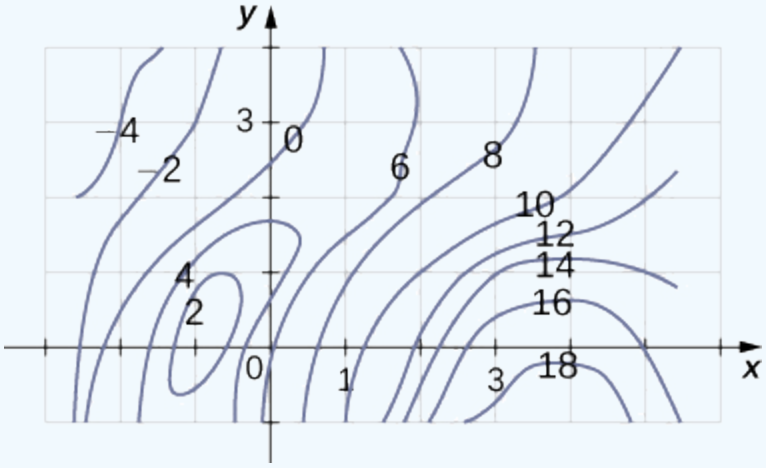


Figure 2.1.11: A contour map is shown with the highest point being about 18 and centered near $(4, -1)$. From this point, the values decrease to 16, 14, 12, 10, 8, and 6 roughly every 0.5 to 1 distance. The lowest point is -4 near $(-3, 4)$. There is a local minimum of 2 near $(-1, 0)$.

a. Use the midpoint rule with $m = 3$ and $n = 2$ to estimate the value of $\iint_R f(x, y)\ dA$.

b. Estimate the average value of the function $f(x, y)$.

Hint available at the end of this section.

Answer available at the end of this section.

Key Concepts

- We can use a double Riemann sum to approximate the volume of a solid bounded above by a function of two variables over a rectangular region. By taking the limit, this becomes a double integral representing the volume of the solid.
- Properties of double integral are useful to simplify computation and find bounds on their values.
- We can use Fubini's theorem to write and evaluate a double integral as an iterated integral.
- Double integrals are used to calculate the area of a region, the volume under a surface, and the average value of a function of two variables over a rectangular region.

Key Equations

- **Double Integral using Riemann Sum**

$$\iint_R f(x, y)\, dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i j*, y_i j*)\, \Delta A$$

- **Double Integral using Fubini's theorem**

$$\int_a^b \int_c^d f(x, y)\, dx\, dy = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx$$

or

$$\int_c^d \int_a^b f(x, y)\, dx\, dy = \int_c^d \left[\int_a^b f(x, y)\, dx \right] dy$$

- **Average Value of a Function of 2 Variables**

$$f_{ave} = \frac{1}{\text{Area of } R} \iint_R f(x, y)\, dx\, dy$$

Glossary

Double integral
of the function $f(x,y)$ over the region R in the xy-plane is defined as the limit of a double Riemann sum,

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A.$$

Double Riemann sum
of the function $f(x,y)$ over a rectangular region R is

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A,$$

where R is divided into smaller subrectangles R_{ij} and (x_{ij}^*, y_{ij}^*) is an arbitrary point in R_{ij}

Fubini's theorem
if $f(x,y)$ is a function of two variables that is continuous over a rectangular region $R = \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b,\ c \le y \le d\}$, then the double integral of f over the region equals an iterated integral,

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dx\,dy = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Iterated integral
for a function $f(x,y)$ over the region R is

a. $\displaystyle\int_a^b \int_c^d f(x,y)\ dx\ dy = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx,$

b. $\displaystyle\int_c^d \int_a^b f(x,y)\ dx\ dy = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy,$

where a, b, c, and d are any real numbers and $R = [a,b] \times [c,d]$

Contributors and Attributions

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Answers

Exercise 2.1.1
Follow the steps of the previous example.

Exercise 2.1.1

$$V \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(x_{ij}^*, y_{ij}^*)\,\Delta A = 0$$

Exercise 2.1.2
Use properties i. and ii. and evaluate the iterated integral, and then use property v.

Exercise 2.1.2
a. 26
b. Answers may vary.

Exercise 2.1.3
Use Fubini's theorem.

Exercise 2.1.3
$-\frac{1340}{3}$

Exercise 2.1.4
Integrate with respect to y first.

Exercise 2.1.4

$\frac{4-\ln 5}{\ln 5}$

Exercise 2.1.5

Graph the function, set up the integral, and use an iterated integral.

Exercise 2.1.5

$\frac{\pi}{2}$

Exercise 2.1.6

Divide the region into six rectangles, and use the contour lines to estimate the values for $f(x, y)$.

Exercise 2.1.6

Answers to both parts a. and b. may vary.

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2.1E: Exercises

Exercises 1 - 2

Use the midpoint rule with $m = 4$ and $n = 2$ to estimate the volume of the solid bounded by the surface $z = f(x, y)$, the vertical planes $x = 1$, $x = 2$, $y = 1$, and $y = 2$, and the horizontal plane $x = 0$.

1. $f(x, y) = 4x + 2y + 8xy$

   Answer

   27

2. $f(x, y) = 16x^2 + \frac{y}{2}$

Exercises 3 - 4

Estimate the volume of the solid under the surface $z = f(x, y)$ and above the rectangular region R by using a Riemann sum with $m = n = 2$ and the sample points to be the lower left corners of the sub-rectangles of the partition.

3. $f(x, y) = \sin x - \cos y$, $R = [0, \pi] \times [0, \pi]$

   Answer

   0

4. $f(x, y) = \cos x + \cos y$, $R = [0, \pi] \times [0, \frac{\pi}{2}]$

Exercises 5 - 12

Follow the instruction for each question.

4. Use the midpoint rule with $m = n = 2$ to estimate $\iint_R f(x, y)\ dA$, where the values of the function *f* on $R = [8, 10] \times [9, 11]$ are given in Table $2.1E.1$.

Table $2.1E.1$: Table with x-values and y-values of function f on $R = [8, 10] \times [9, 11]$.

	y				
x	9	9.5	10	10.5	11
8	9.8	5	6.7	5	5.6
8.5	9.4	4.5	8	5.4	3.4
9	8.7	4.6	6	5.5	3.4
9.5	6.7	6	4.5	5.4	6.7
10	6.8	6.4	5.5	5.7	6.8

   Answer

   21.3

6. The values of the function f on the rectangle $R = [0, 2] \times [7, 9]$ are given in Table $2.1E.2$. Estimate the double integral $\iint_R f(x, y)\ dA$ by using a Riemann sum with $m = n = 2$. Select the sample points to be the upper right corners of the sub-squares of *R*.

Table 2.1E.2: Table with x_0,x_1,x_2 and y_0,y_1,y_2 values of the function f on the rectangle $R = [0, 2] \times [7, 9]$.

	$y_0 = 7$	$y_1 = 8$	$y_2 = 9$
$x_0 = 0$	10.22	10.21	9.85
$x_1 = 1$	6.73	9.75	9.63
$x_2 = 2$	5.62	7.83	8.21

7. The depth of a children's 4-ft by 4-ft swimming pool, measured at 1-ft intervals, is given in Table 2.1E.3.
   a. Estimate the volume of water in the swimming pool by using a Riemann sum with $m = n = 2$. Select the sample points using the midpoint rule on $R = [0, 4] \times [0, 4]$.
   b. Find the average depth of the swimming pool.

Table 2.1E.3: Table with x-values and y-values of a children's 4-ft by 4-ft swimming pool.

	y				
x	0	1	2	3	4
0	1	1.5	2	2.5	3
1	1	1.5	2	2.5	3
2	1	1.5	1.5	2.5	3
3	1	1	1.5	2	2.5
4	1	1	1	1.5	2

Answer

a. 28 ft^3
b. 1.75 ft.

8. The depth of a 3-ft by 3-ft hole in the ground, measured at 1-ft intervals, is given in Table 2.1E.4.
   a. Estimate the volume of the hole by using a Riemann sum with $m = n = 3$ and the sample points to be the upper left corners of the sub-squares of R.
   b. Find the average depth of the hole.

Table 2.1E.4: Table with x-values and y-values of the depth of a 3-ft by 3-ft hole in the ground.

	y			
x	0	1	2	3
0	6	6.5	6.4	6
1	6.5	7	7.5	6.5
2	6.5	6.7	6.5	6
3	6	6.5	5	5.6

9. The level curves $f(x, y) = k$ of the function f are given in Figure 2.1E.1, where k is a constant.
   a. Apply the midpoint rule with $m = n = 2$ to estimate the double integral $\iint_R f(x, y)\ dA$, where $R = [0.2, 1] \times [0, 0.8]$.
   b. Estimate the average value of the function f on R.

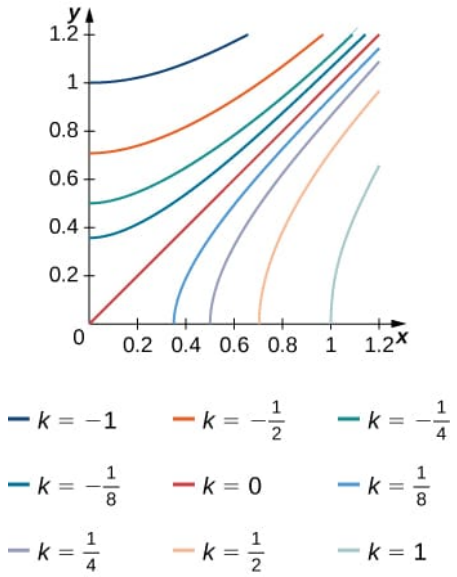


Figure $2.1E.1$: Level curves of a surface in quadrant 1 where the red line in the middle is linear with symmetric curves on both sides curving toward the red line.

Answer

a. 0.112

b. $f_{ave} \simeq 0.175$; here $f(0.4, 0.2) \simeq 0.1$, $f(0.2, 0.6) \simeq -0.2$, $f(0.8, 0.2) \simeq 0.6$, and $f(0.8, 0.6) \simeq 0.2$

10. The level curves $f(x, y) = k$ of the function f are given in the following graph, where k is a constant.
    a. Apply the midpoint rule with $m = n = 2$ to estimate the double integral $\iint_R f(x, y)\ dA$, where $R = [0.1, 0.5] \times [0.1, 0.5]$.
    b. Estimate the average value of the function *f* on R.

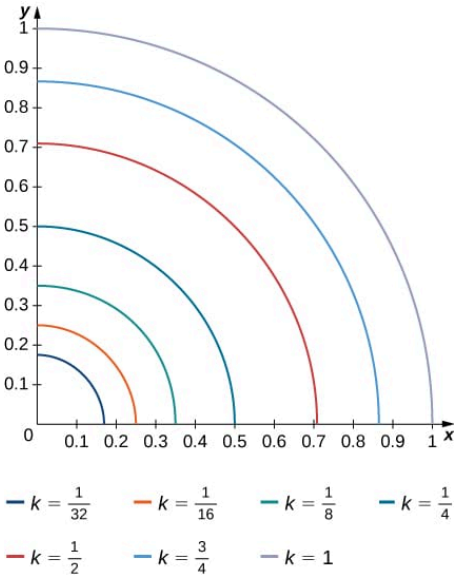


Figure $2.1E.2$: Level curves of a surface in quadrant 1 where each curve appears to be 1/4 of a circle with different radius.

11. The solid lying under the surface $z = \sqrt{4 - y^2}$ and above the rectangular region$R = [0, 2] \times [0, 2]$ is illustrated in Figure $2.1E.3$. Evaluate the double integral $\iint_R f(x, y)$, where $f(x, y) = \sqrt{4 - y^2}$ by finding the volume of the corresponding solid.

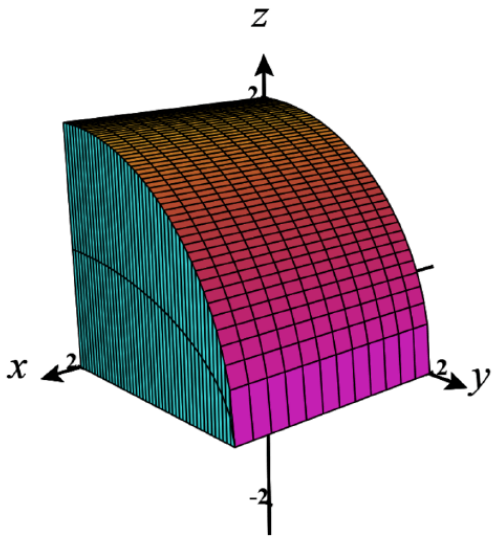


Figure $2.1E.3$: Solid lying under the surface $z = \sqrt{4 - y^2}$ and above the rectangular region $R = [0, 2] \times [0, 2]$. Rotate the image to see different side of the solid.

Answer

2π

12. The solid lying under the plane $z = y + 4$ and above the rectangular region $R = [0, 2] \times [0, 4]$ is illustrated in Figure $2.1E.4$. Evaluate the double integral $\iint_R f(x, y)\ dA$, where $f(x, y) = y + 4$, by finding the volume of the corresponding solid.

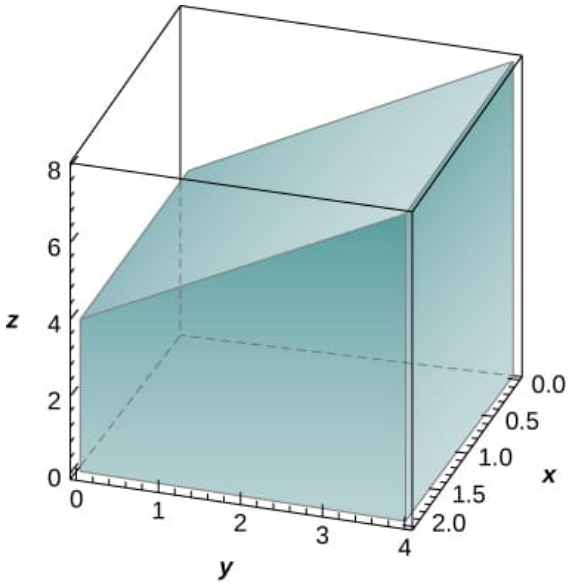


Figure $2.1E.4$: Solid lying under the plane $z = y + 4$ and above the rectangular region $R = [0, 2] \times [0, 4]$.

Exercises 13 - 20

Calculate the integrals by reversing the order of integration.

13. $\int_{-1}^{1} \left(\int_{-2}^{2} (2x + 3y + 5)\, dx \right)\ dy$

Answer

40

14. $\int_0^2 \left(\int_0^1 (x + 2e^y + 3)\, dx \right)\, dy$
15. $\int_1^{27} \left(\int_1^2 (\sqrt[3]{x} + \sqrt[3]{y})\, dy \right)\, dx$

Answer

$\frac{81}{2} + 39\sqrt[3]{2}$

16. $\int_1^{16} \left(\int_1^8 (\sqrt[4]{x} + 2\sqrt[3]{y})\, dy \right)\, dx$
17. $\int_{\ln 2}^{\ln 3} \left(\int_0^1 e^{x+y}\, dy \right)\, dx$

Answer

$e - 1$

18. $\int_0^2 \left(\int_0^1 3^{x+y}\, dy \right)\, dx$
19. $\int_1^6 \left(\int_2^9 \frac{\sqrt{y}}{x^2}\, dy \right)\, dx$

Answer

$15 - \frac{10\sqrt{2}}{9}$

20. $\int_1^9 \left(\int_4^2 \frac{\sqrt{x}}{y^2}\, dy \right)\, dx$

Exercises 21 - 34

Evaluate the iterated integrals by choosing the order of integration.

21. $\int_0^{\pi} \int_0^{\pi/2} \sin(2x)\cos(3y)\ dx\ dy$

Answer

0

22. $\int_{\pi/12}^{\pi/8} \int_{\pi/4}^{\pi/3} [\cot x + \tan(2y)]\ dx\ dy$
23. $\int_1^e \int_1^e \left[\frac{1}{x}\sin(\ln x) + \frac{1}{y}\cos(\ln y) \right]\ dx\ dy$

Answer

$(e-1)(1 + \sin 1 - \cos 1)$

24. $\int_1^e \int_1^e \frac{\sin(\ln x)\cos(\ln y)}{xy}\ dx\ dy$
25. $\int_1^2 \int_1^2 \left(\frac{\ln y}{x} + \frac{x}{2y+1} \right)\ dy\ dx$

Answer

$\frac{3}{4}\ln\left(\frac{5}{3}\right) + 2(\ln 2)^2 - \ln 2$

26. $\int_1^e \int_1^2 x^2 \ln(x)\ dy\ dx$
27. $\int_1^{\sqrt{3}} \int_1^2 y\ \arctan\left(\frac{1}{x}\right)\ dy\ dx$

Answer

$\frac{1}{8}[(2\sqrt{3}-3)\pi+6\ \ln 2]$

28. $\int_0^1 \int_0^{1/2} (\arcsin x + \arcsin y)\ dy\ dx$
29. $\int_0^1 \int_1^2 xe^{x+4y}\ dy\ dx$

Answer

$\frac{1}{4}e^4(e^4-1)$

30. $\int_1^2 \int_0^1 xe^{x-y}\ dy\ dx$
31. $\int_1^e \int_1^e \left(\frac{\ln y}{\sqrt{y}} + \frac{\ln x}{\sqrt{x}}\right)\ dy\ dx$

Answer

$4(e-1)(2-\sqrt{e})$

32. $\int_1^e \int_1^e \left(\frac{x\ \ln y}{\sqrt{y}} + \frac{y\ \ln x}{\sqrt{x}}\right)\ dy\ dx$
33. $\int_0^1 \int_1^2 \left(\frac{x}{x^2+y^2}\right)\ dy\ dx$

Answer

$-\frac{\pi}{4} + \ln\left(\frac{5}{4}\right) - \frac{1}{2}\ln 2 + \arctan 2$

34. $\int_0^1 \int_1^2 \frac{y}{x+y^2}\ dy\ dx$

Exercises 35 - 38

Find the average value of the function over the given rectangles.

35. $f(x,y) = -x + 2y$, $R = [0,1] \times [0,1]$

Answer

$\frac{1}{2}$

36. $f(x,y) = x^4 + 2y^3$, $R = [1,2] \times [2,3]$
37. $f(x,y) = \sinh x + \sinh y$, $R = [0,1] \times [0,2]$

Answer

$\frac{1}{2}(2\ \cosh 1 + \cosh 2 - 3)$.

38. $f(x,y) = \arctan(xy)$, $R = [0,1] \times [0,1]$
39. Let f and g be two continuous functions such that $0 \le m_1 \le f(x) \le M_1$ for any $x \in [a,b]$ and $0 \le m_2 \le g(y) \le M_2$ for any$y \in [c,d]$. Show that the following inequality is true:

$$m_1 m_2 (b-a)(c-d) \le \int_a^b \int_c^d f(x)g(y)\,dydx \le M_1 M_2 (b-a)(c-d).$$

Exercises 40 - 43

Use property v. of double integrals and the answer from the preceding exercise to show that the following inequalities are true.

40. $\frac{1}{e^2} \le \iint_R e^{-x^2-y^2}\ dA \le 1$, where $R = [0,1] \times [0,1]$

41. $\frac{\pi^2}{144} \leq \iint_R \sin x \cos y \; dA \leq \frac{\pi^2}{48}$, where $R = [\frac{\pi}{6}, \frac{\pi}{3}] \times [\frac{\pi}{6}, \frac{\pi}{3}]$
42. $0 \leq \iint_R e^{-y} \cos x \; dA \leq \frac{\pi}{2}$, where $R = [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$
43. $0 \leq \iint_R (\ln x)(\ln y) \; dA \leq (e-1)^2$, where $R = [1, e] \times [1, e]$
44. Let f and g be two continuous functions such that $0 \leq m_1 \leq f(x) \leq M_1$ for any $x \in [a, b]$ and $0 \leq m_2 \leq g(y) \leq M_2$ for any $y \in [c, d]$. Show that the following inequality is true:

$$(m_1 + m_2)(b-a)(c-d) \leq \int_a^b \int_c^d |f(x) + g(y)| \; dy \; dx \leq (M_1 + M_2)(b-a)(c-d)$$

Exercises 45 - 48

Use property v. of double integrals and the answer from the preceding exercise to show that the following inequalities are true.

45. $\frac{2}{e} \leq \iint_R (e^{-x^2} + e^{-y^2}) \; dA \leq 2$, where $R = [0, 1] \times [0, 1]$
46. $\frac{\pi^2}{36} \iint_R (\sin x + \cos y) \; dA \leq \frac{\pi^2\sqrt{3}}{36}$, where $R = [\frac{\pi}{6}, \frac{\pi}{3}] \times [\frac{\pi}{6}, \frac{\pi}{3}]$
47. $\frac{\pi}{2} e^{-\pi/2} \leq \iint_R (\cos x + e^{-y}) \; dA \leq \pi$, where $R = [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]\backslash$
48. $\frac{1}{e} \leq \iint_R (e^{-y} - \ln x) \; dA \leq 2$, where $R = [0, 1] \times [0, 1]$

Exercises 49 - 50

The function f is given in terms of double integrals.

a. Determine the explicit form of the function f.
b. Find the volume of the solid under the surface $z = f(x, y)$ and above the region R.
c. Find the average value of the function f on R.
d. Use a computer algebra system (CAS) to plot $z = f(x, y)$ and $z = f_{ave}$ in the same system of coordinates.

49. **[T]** $f(x, y) = \int_0^y \int_0^x (xs + yt) ds \; dt$, where $(x, y) \in R = [0, 1] \times [0, 1]$

Answer

a. $f(x, y) = \frac{1}{2} xy(x^2 + y^2)$;
b. $V = \int_0^1 \int_0^1 f(x, y) \; dx \; dy = \frac{1}{8}$;
c. $f_{ave} = \frac{1}{8}$;
d.

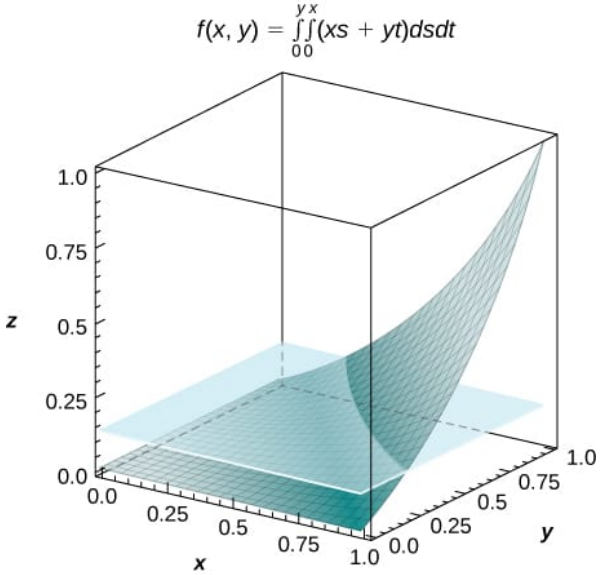


Figure 2.1E.5: Plot of the function $f(x, y) = \int_0^y \int_0^x (xs + yt) ds \; dt$, where $(x, y) \in R = [0, 1] \times [0, 1]$. Surface slowly rise upward until half-way and quickly curves upward afterward . $z = f_{ave}$ looks like a plane about half-way between z=0 and z=0.25.

50. **[T]** $f(x,y) = \int_0^x \int_0^y [\cos(s) + \cos(t)]\ dt\ ds$, where $(x,y) \in R = [0,3] \times [0,3]$
51. Show that if f and g are continuous on $[a,b]$ and $[c,d]$, respectively, then

$$\int_a^b \int_c^d |f(x) + g(y)| dy\ dx = (d-c) \int_a^b f(x)\ dx$$

$$+ \int_a^b \int_c^d g(y)\ dy\ dx = (b-a) \int_c^d g(y)\ dy + \int_c^d \int_a^b f(x)\ dx\ dy.$$

52. Show that $\int_a^b \int_c^d yf(x) + xg(y)\ dy\ dx = \frac{1}{2}(d^2 - c^2) \left(\int_a^b f(x)\, dx \right) + \frac{1}{2}(b^2 - a^2) \left(\int_c^d g(y)\, dy \right)$.
53. **[T]** Consider the function $f(x,y) = e^{-x^2-y^2}$, where $(x,y) \in R = [-1,1] \times [-1,1]$.
    a. Use the midpoint rule with $m = n = 2, 4, \ldots, 10$ to estimate the double integral $I = \iint_R e^{-x^2-y^2} dA$. Round your answers to the nearest hundredths.
    b. For $m = n = 2$, find the average value of *f* over the region *R*. Round your answer to the nearest hundredths.
    c. Use a CAS to graph in the same coordinate system the solid whose volume is given by $\iint_R e^{-x^2-y^2} dA$ and the plane $z = f_{ave}$.

Answer

a. For $m = n = 2$, $I = 4e^{-0.5} \approx 2.43$
b. $f_{ave} = e^{-0.5} \simeq 0.61$;
c.

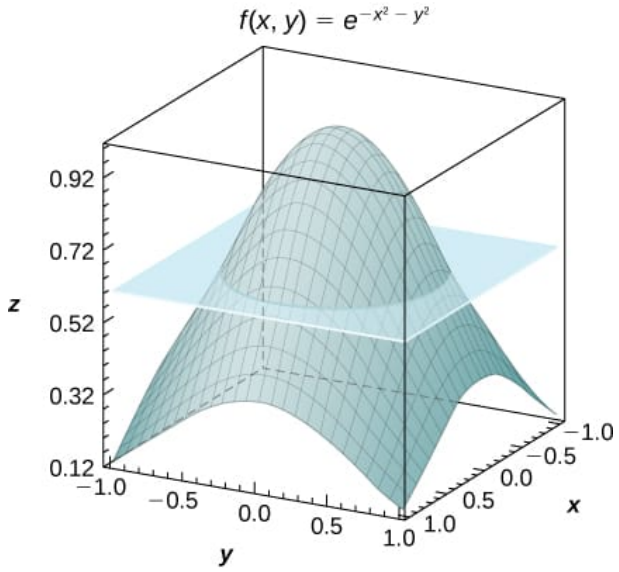


Figure $2.1E.6$: Plot of the function $f(x,y) = e^{-x^2-y^2}$, where $(x,y) \in R = [-1,1] \times [-1,1]$. Surface looks like a symmetric hill . $z = f_{ave}$ looks like a plane about half-way between z=0.52 and z=0.72.

54. **[T]** Consider the function $f(x,y) = \sin(x^2)\ \cos(y^2)$, where $(x, y \in R = [-1,1] \times [-1,1]$.
    a. Use the midpoint rule with $m = n = 2, 4, \ldots, 10$ to estimate the double integral $I = \iint_R \sin(x^2)\cos(y^2)\ dA$. Round your answers to the nearest hundredths.
    b. For $m = n = 2$, find the average value of $f$over the region *R*. Round your answer to the nearest hundredths.
    c. Use a CAS to graph in the same coordinate system the solid whose volume is given by $\iint_R \sin(x^2)\cos(y^2)\ dA$ and the plane $z = f_{ave}$.

Exercises 55 - 56

The functions f_n are given, where $n \geq 1$ is a natural number.

a. Find the volume of the solids S_n under the surfaces $z = f_n(x,y)$ and above the region R.
b. Determine the limit of the volumes of the solids S_n as n increases without bound.

55. $f(x,y) = x^n + y^n + xy,\ (x,y) \in R = [0,1] \times [0,1]$

Answer

a. $\frac{2}{n+1} + \frac{1}{4}$
b. $\frac{1}{4}$

56. $f(x,y) = \frac{1}{x^n} + \frac{1}{y^n},\ (x,y) \in R = [1,2] \times [1,2]$

57. Show that the average value of a function f on a rectangular region $R = [a,b] \times [c,d]$ is $f_{ave} \approx \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)$,where (x_{ij}^*, y_{ij}^*) are the sample points of the partition of R, where $1 \le i \le m$ and $1 \le j \le n$.

58. Use the midpoint rule with $m = n$ to show that the average value of a function f on a rectangular region $R = [a,b] \times [c,d]$ is approximated by

$$f_{ave} \approx \frac{1}{n^2} \sum_{i,j=1}^{n} f\left(\frac{1}{2}(x_{i=1} + x_i),\ \frac{1}{2}(y_{j=1} + y_j)\right).$$

59. An isotherm map is a chart connecting points having the same temperature at a given time for a given period of time. Use the preceding exercise and apply the midpoint rule with $m = n = 2$ to find the average temperature over the region given in Figure $2.1E.7$.

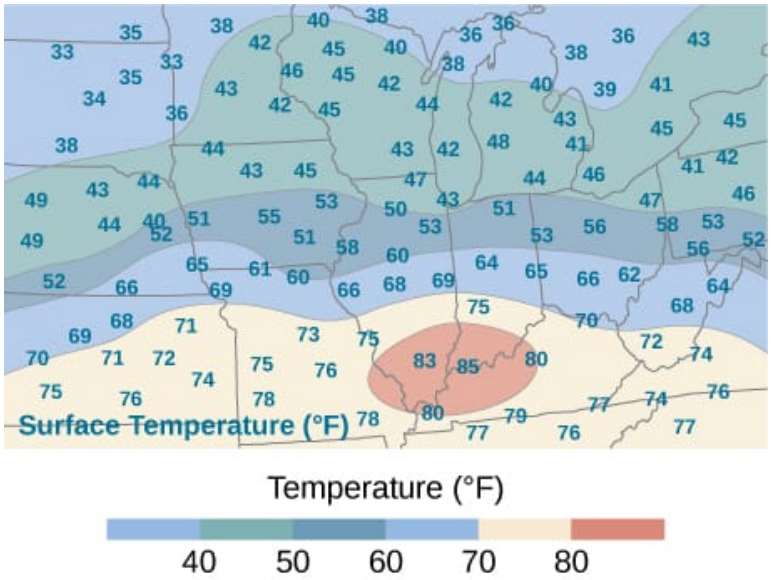


Figure $2.1E.7$: An isotherm map showing temperature contours over a region. Temperature values range from30^0 F to 90^0 F.

Answer

56.5° F; here $f(x_1^*, y_1^*) = 71,\ f(x_2^*, y_1^*) = 72,\ f(x_2^*, y_1^*) = 40,\ f(x_2^*, y_2^*) = 43$, where x_i^* and y_j^* are the midpoints of the subintervals of the partitions of $[a,b]$ and $[c,d]$, respectively.

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2.2: Double Integrals over General Regions

Learning Objectives

- Recognize when a function of two variables is integrable over a general region.
- Evaluate a double integral by computing an iterated integral over a region bounded by two vertical lines and two functions of x, or two horizontal lines and two functions of y.
- Simplify the calculation of an iterated integral by changing the order of integration.
- Use double integrals to calculate the volume of a region between two surfaces or the area of a plane region.
- Solve problems involving double improper integrals.

Previously, we studied the concept of double integrals and examined the tools needed to compute them. We learned techniques and properties to integrate functions of two variables over rectangular regions. We also discussed several applications, such as finding the volume bounded above by a function over a rectangular region, finding area by integration, and calculating the average value of a function of two variables.

In this section we consider double integrals of functions defined over a general bounded region D on the plane. Most of the previous results hold in this situation as well, but some techniques need to be extended to cover this more general case.

General Regions of Integration

An example of a general bounded region D on a plane is shown in Figure 2.2.1. Since D is bounded on the plane, there must exist a rectangular region R on the same plane that encloses the region D that is, a rectangular region R exists such that D is a subset of $R (D \subseteq R)$.

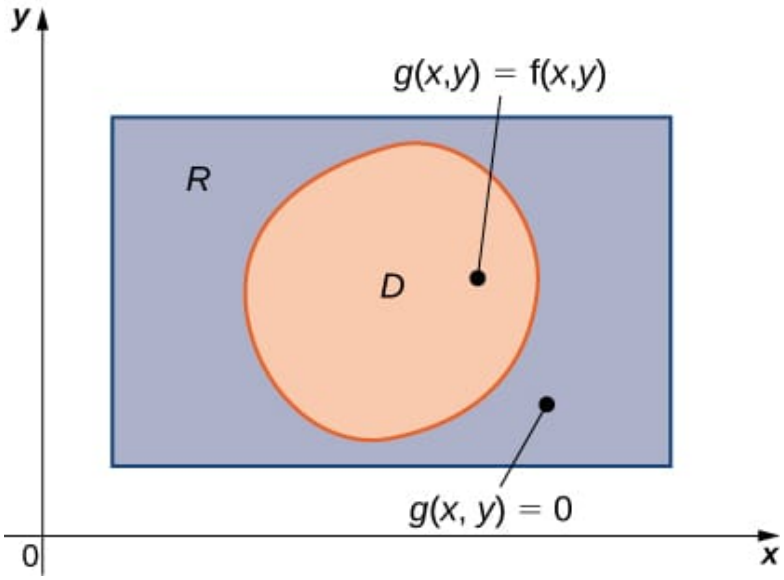


Figure 2.2.1: A rectangle R with a shape D inside of it. Inside D, there is a point labeled $g(x, y) = f(x, y)$. Outside D but still inside R, there is a point labeled $g(x, y) = 0$. For a region D that is a subset of R, we can define a function $g(x, y)$ *to equal* $f(x, y)$ *at every point in* D *and* 0 *at every point of* R *not in* D.

Suppose $z = f(x, y)$ is defined on a general planar bounded region D as in Figure 2.2.1. In order to develop double integrals of f over D we extend the definition of the function to include all points on the rectangular region R and then use the concepts and tools from the preceding section. But how do we extend the definition of f to include all the points on R? We do this by defining a new function $g(x, y)$ on R as follows:

$$g(x,y) = \begin{cases} f(x,y) \text{ if } (x,y) \text{ is in } D \\ 0 \text{ if } (x,y) \text{ is in } R \text{ but not in } D \end{cases} \tag{2.2.1}$$

Note that we might have some technical difficulties if the boundary of D is complicated. So we assume the boundary to be a piecewise smooth and continuous simple closed curve. Also, since all the results developed in the section on Double Integrals over Rectangular Regions used an integrable function $f(x,y)$ we must be careful about $g(x,y)$ and verify that $g(x,y)$ is an integrable function over the rectangular region R. This happens as long as the region D is bounded by simple closed curves. For now we will concentrate on the descriptions of the regions rather than the function and extend our theory appropriately for integration.

We consider two types of planar bounded regions.

Definition: Type I and Type II regions

A region D in the (x,y)-plane is of Type I if it lies between two vertical lines and the graphs of two continuous functions $g_1(x)$ and $g_2(x)$. That is (Figure 2.2.2),

$$D = \{(x,y) \,|\, a \le x \le b,\ g_1(x) \le y \le g_2(x)\}. \tag{2.2.2}$$

A region D in the xy-plane is of Type II if it lies between two horizontal lines and the graphs of two continuous functions $h_1(y)$ and $h_2(y)$. That is (Figure 2.2.3),

$$D = \{(x,y) \,|\, c \le y \le d,\ h_1(y) \le x \le h_2(y)\}. \tag{2.2.3}$$

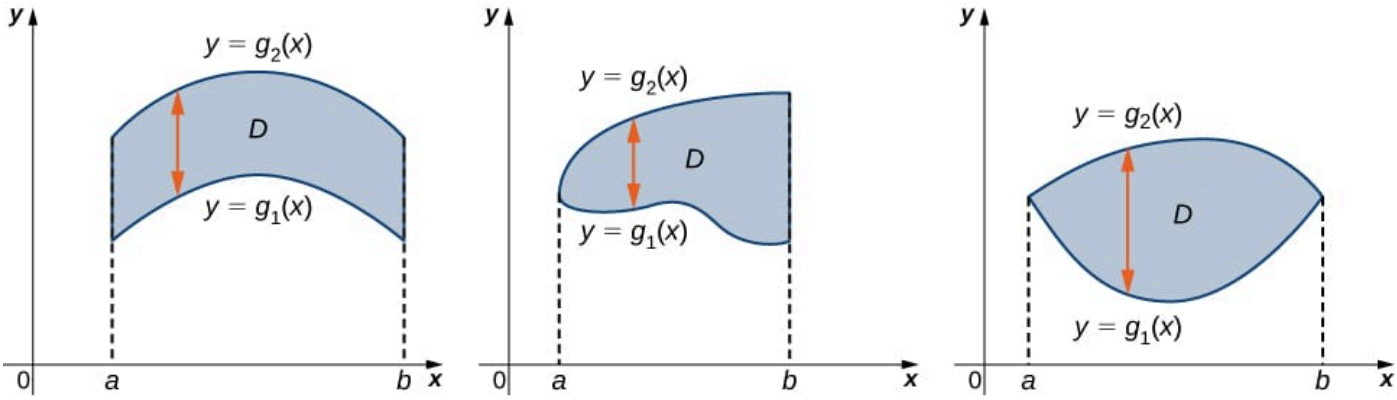


Figure 2.2.2. The graphs showing a region marked D. In all instances, between a and b, there is a shape that is defined by two functions $g_1(x)$ and $g_2(x)$. In one instance, the two functions do not touch; in another instance, they touch at the end point a, and in the last instance they touch at both end points. A Type I region lies between two vertical lines and the graphs of two functions of x.

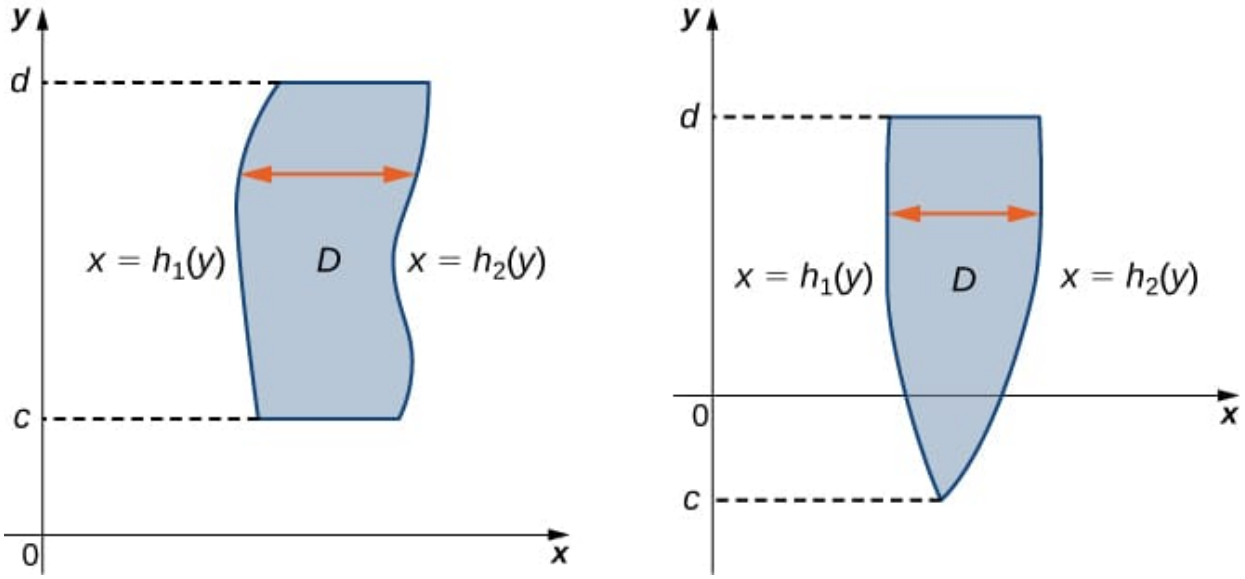


Figure 2.2.3: The graphs show a region marked D. In all instances, between c and d, there is a shape that is defined by two vertically oriented functions $x = h_1(y)$ and $x = h_2(y)$. In one instance, the two functions do not touch; in the other instance, they touch at the end point c. A Type II region lies between two horizontal lines and the graphs of two functions of y.

Example 2.2.1: Describing a Region as Type I and Also as Type II

Consider the region in the first quadrant between the functions $y = \sqrt{x}$ and $y = x^3$ (Figure 2.2.4). Describe the region first as Type I and then as Type II.

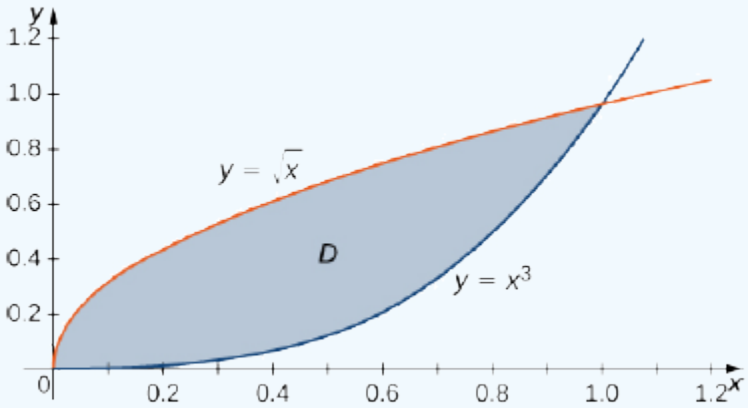


Figure 2.2.4: The region D is drawn between two functions, namely, $y = \sqrt{x}$ and $y = x^3$. This region D can be described as Type I or as Type II.

Solution

When describing a region as Type I, we need to identify the function that lies above the region and the function that lies below the region. Here, region D is bounded above by $y = \sqrt{x}$ and below by $y = x^3$ in the interval for x in $[0, 1]$. Hence, as Type I, D is described as the set $\{(x, y) \mid 0 \le x \le 1,\ x^3 \le y \le \sqrt{x}\}$.

However, when describing a region as Type II, we need to identify the function that lies on the left of the region and the function that lies on the right of the region. Here, the region D is bounded on the left by $x = y^2$ and on the right by $x = \sqrt[3]{y}$ in the interval for y in $[0, 1]$. Hence, as Type II, D is described as the set $\{(x, y) \mid 0 \le y \le 1,\ y^2 \le x \le \sqrt[3]{y}\}$.

Exercise 2.2.1

Consider the region in the first quadrant between the functions $y = 2x$ and $y = x^2$. Describe the region first as Type I and then as Type II.

Hint available at the end of this section.

Answer available at the end of this section.

Double Integrals over Non-rectangular Regions

To develop the concept and tools for evaluation of a double integral over a general, non-rectangular region, we need to first understand the region and be able to express it as Type I or Type II or a combination of both. Without understanding the regions, we will not be able to decide the limits of integrations in double integrals. As a first step, let us look at the following theorem.

Theorem: Double Integrals over Non-rectangular Regions

Suppose $g(x, y)$ is the extension to the rectangle R of the function $f(x, y)$ defined on the regions D and R as shown in Figure 2.2.1 inside R. Then $g(x, y)$ is integrable and we define the double integral of $f(x, y)$ over D by

$$\iint_D f(x, y)\, dA = \iint_R g(x, y)\, dA. \tag{2.2.4}$$

The right-hand side of this equation is what we have seen before, so this theorem is reasonable because R is a rectangle and $\iint_R g(x, y)dA$ has been discussed in the preceding section. Also, the equality works because the values of $g(x, y)$ are 0 for any point (x, y) that lies outside D and hence these points do not add anything to the integral. However, it is important that the rectangle R contains the region D.

As a matter of fact, if the region D is bounded by smooth curves on a plane and we are able to describe it as Type I or Type II or a mix of both, then we can use the following theorem and not have to find a rectangle R containing the region.

Fubini's Theorem (Strong Form)

For a function $f(x, y)$ that is continuous on a region D of Type I, we have

$$\iint_D f(x, y)\, dA = \iint_D f(x, y)\, dy\, dx = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x, y)\, dy \right] dx. \tag{2.2.5}$$

Similarly, for a function $f(x, y)$ that is continuous on a region D of Type II, we have

$$\iint_D f(x, y)\, dA = \iint_D f(x, y)\, dx\, dy = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x, y)\, dx \right] dy. \tag{2.2.6}$$

The integral in each of these expressions is an iterated integral, similar to those we have seen before. Notice that, in the inner integral in the first expression, we integrate $f(x, y)$ with x being held constant and the limits of integration being $g_1(x)$ and $g_2(x)$. In the inner integral in the second expression, we integrate $f(x, y)$ with y being held constant and the limits of integration are $h_1(x)$ and $h_2(x)$.

Example 2.2.2: Evaluating an Iterated Integral over a Type I Region

Evaluate the integral $\iint_D x^2 e^{xy}\ dA$ where D is shown in Figure 2.2.5.

Solution

First construct the region as a Type I region (Figure 2.2.5). Here $D = \{(x, y) \mid 0 \le x \le 2,\ \frac{1}{2}x \le y \le 1\}$. Then we have

$$\iint_D x^2 e^{xy}\, dA = \int_{x=0}^{x=2} \int_{y=1/2x}^{y=1} x^2 e^{xy}\, dy\, dx.$$

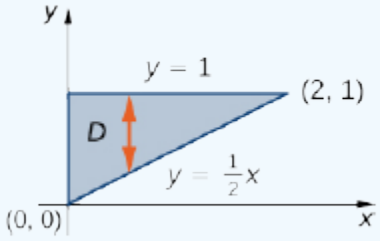


Figure 2.2.5: A triangle marked D drawn with lines $x = 2y$ and $y = 1$, with vertices (0, 0), (2, 1), and (0, 1). Here there is a pair of red arrows reaching horizontally from one edge to the other. We can express region D as a Type I region and integrate from $y = \frac{1}{2}x$ to $y = 1$ between the lines $x = 0$ and $x = 2$.

Therefore, we have

$$\begin{aligned}
\int_{x=0}^{x=2} \int_{y=\frac{1}{2}x}^{y=1} x^2 e^{xy}\, dy\, dx &= \int_{x=0}^{x=2} \left[\int_{y=\frac{1}{2}x}^{y=1} x^2 e^{xy}\, dy\right] dx && \text{Iterated integral for a Type I region.} \\
&= \int_{x=0}^{x=2} \left[x^2 \frac{e^{xy}}{x}\right]\Big|_{y=1/2x}^{y=1} dx && \text{Integrate with respect to } y \\
&= \int_{x=0}^{x=2} \left[xe^x - xe^{x^2/2}\right] dx && \text{Integrate with respect to } x \\
&= \left[xe^x - e^x - e^{\frac{1}{2}x^2}\right]\Big|_{x=0}^{x=2} = 2.
\end{aligned}$$

In Example 2.2.2, we could have looked at the region in another way, such as $D = \{(x, y) \mid 0 \le y \le 1,\ 0 \le x \le 2y\}$ (Figure 2.2.6).

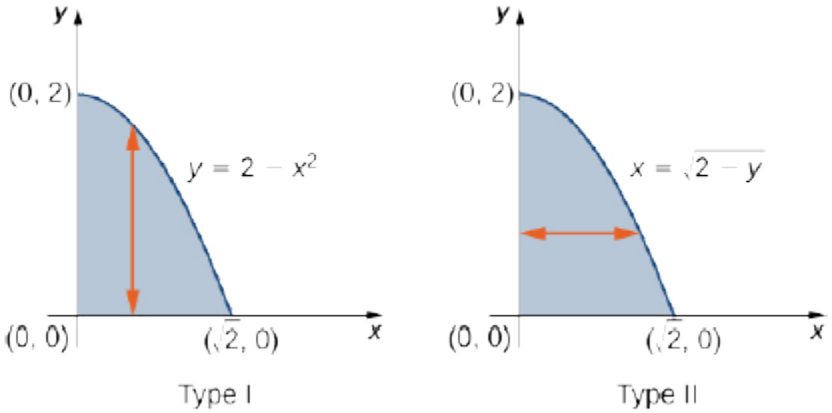


Figure 2.2.6: This figure consists of two figures labeled Type I and Type II. In the Type I figure, a curve is given as $y = 2 - x^2$, which forms a shape with the x and y axes. There is a vertical line with arrows on the end of it within this shape. In the Type II figure, a curve is given as $x = \sqrt{2 - y}$, which forms a shape with the x and y axes. There is a horizontal line with arrows on the end of it within this shape.

This is a Type II region and the integral would then look like

$$\iint_D x^2 e^{xy}\, dA = \int_{y=0}^{y=1} \int_{x=0}^{x=2y} x^2 e^{xy}\, dx\ dy. \tag{2.2.7}$$

However, if we integrate first with respect to x this integral is lengthy to compute because we have to use integration by parts twice.

Example 2.2.3: Evaluating an Iterated Integral over a Type II Region

Evaluate the integral

$$\iint_D (3x^2 + y^2)\, dA$$

where $D = \{(x, y) \mid -2 \le y \le 3,\ y^2 - 3 \le x \le y + 3\}$.

Solution

Notice that D can be seen as either a Type I or a Type II region, as shown in Figure 2.2.7. However, in this case describing D as Type I is more complicated than describing it as Type II. Therefore, we use D as a Type II region for the integration.

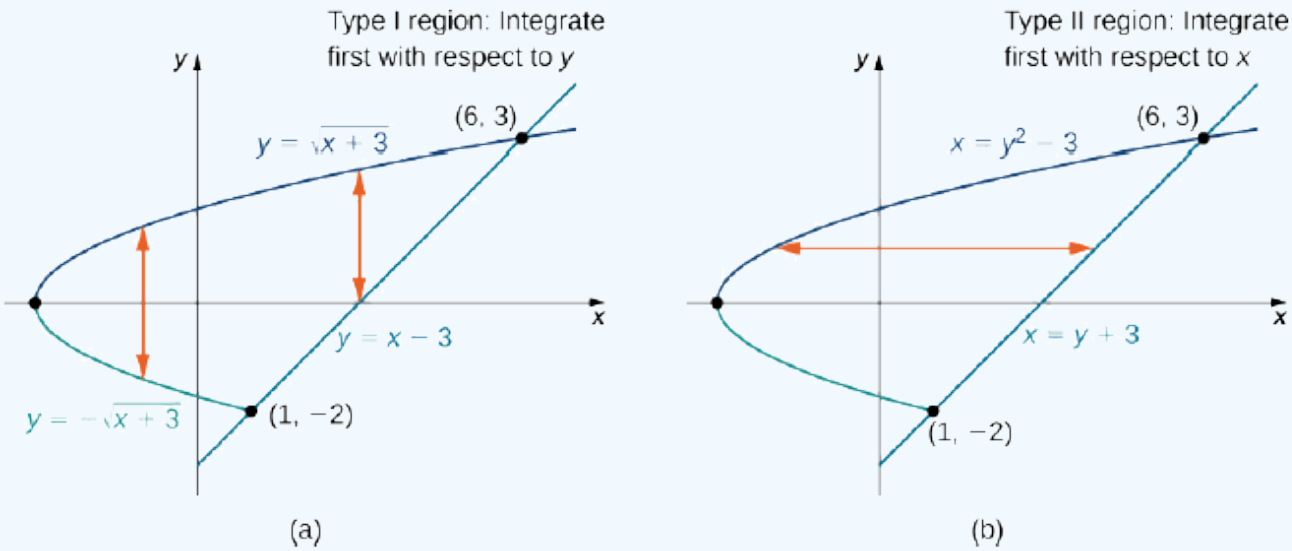


Figure 2.2.7: Two figures labeled a and b. In figure a, a region is bounded by $y = \sqrt{x+3}$, $y = -\sqrt{x+3}$, and $y = x - 3$, which has points of intersection (6, 3), (1, -2), and (0, -3). There are vertical lines in the shape, and it is noted that this is a type I region: integrate first with respect to y. In figure b, a region is bounded by $x = y^2 - 3$ and $x = y + 3$, which has points of intersection (6, 3), (1, -2), and (0, -3). There are horizontal lines in the shape, and it is noted that this is a type II region: integrate first with respect to x. This region D can be either (a) Type I or (b) Type II.

Choosing this order of integration, we have

$$\begin{aligned}
\iint_D (3x^2+y^2)\,dA &= \int_{y=-2}^{y=3}\int_{x=y^2-3}^{x=y+3}(3x^2+y^2)\,dx\,dy \\
&= \int_{y=-2}^{y=3}(x^3+xy^2)\Big|_{y^2-3}^{y+3}\,dy && \text{Iterated integral, Type II region} \\
&= \int_{y=-2}^{y=3}\left((y+3)^3+(y+3)y^2-(y^2-3)y^2\right)\,dy \\
&= \int_{-2}^{3}(54+27y-12y^2+2y^3+8y^4-y^6)\,dy && \text{Integrate with respect to } x. \\
&= \left[54y+\frac{27y^2}{2}-4y^3+\frac{y^4}{2}+\frac{8y^5}{5}-\frac{y^7}{7}\right]_{-2}^{3} \\
&= \frac{2375}{7}.
\end{aligned}$$

Exercise 2.2.2

Sketch the region D and evaluate the iterated integral

$$\iint_D xy\,dy\,dx \tag{2.2.8}$$

where D is the region bounded by the curves $y=\cos\ x$ and $y=\sin\ x$ in the interval $[-3\pi/4,\ \pi/4]$.

Hint available at the end of this section.

Answer available at the end of this section.

Recall from Double Integrals over Rectangular Regions the properties of double integrals. As we have seen from the examples here, all these properties are also valid for a function defined on a non-rectangular bounded region on a plane. In particular, property 3 states:

If $R=S\cup T$ and $S\cap T=0$ except at their boundaries, then

$$\iint_R f(x,y)\,dA=\iint_S f(x,y)\,dA+\iint_T f(x,y)\,dA. \tag{2.2.9}$$

Similarly, we have the following property of double integrals over a non-rectangular bounded region on a plane.

Theorem: Decomposing Regions into Smaller Regions

Suppose the region D can be expressed as $D=D_1\cup D_2$ where D_1 and D_2 do not overlap except at their boundaries. Then

$$\iint_D f(x,y)\,dA=\iint_{D_1} f(x,y)\,dA+\iint_{D_2} f(x,y)\,dA. \tag{2.2.10}$$

This theorem is particularly useful for non-rectangular regions because it allows us to split a region into a union of regions of Type I and Type II. Then we can compute the double integral on each piece in a convenient way, as in the next example.

Example 2.2.4: Decomposing Regions

Express the region D shown in Figure 2.2.8 as a union of regions of Type I or Type II, and evaluate the integral

$$\iint_D (2x + 5y)\, dA.$$

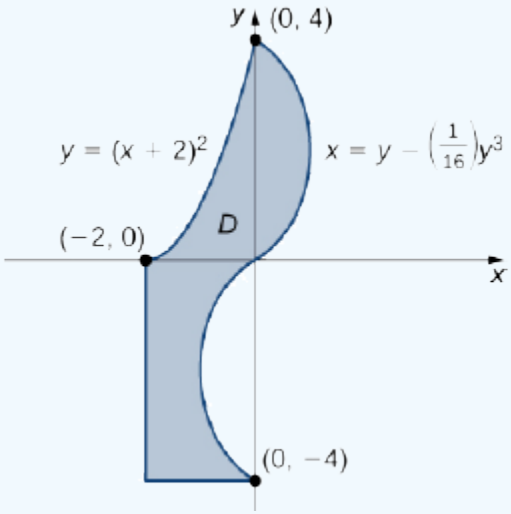


Figure 2.2.8: A complicated shape enclosed by the lines $y = (x + 2)^2$, $x = y - \frac{1}{16}y^3$, x = -2, and y = -4. This graph has intersection points (0, 4), (-2, 0), (0, -4), and (-2, -4). This region can be decomposed into a union of three regions of Type I or Type II.

Solution

The region D is not easy to decompose into any one type; it is actually a combination of different types. So we can write it as a union of three regions D_1, D_2, and D_3 where, $D_1 = \{(x, y) \mid -2 \le x \le 0,\ 0 \le y \le (x + 2)^2\}$, $D_2 = \{(x, y) \mid 0 \le y \le 4,\ 0 \le x \le \left(y - \frac{1}{16}y^3\right)\}$, and $D_3 = \{(x, y) \mid -4 \le y \le 0,\ -2 \le x \le \left(y - \frac{1}{16}y^3\right)\}$. These regions are illustrated more clearly in Figure 2.2.9.

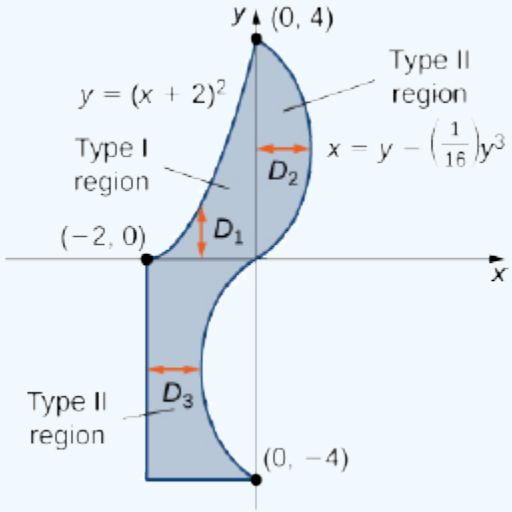


Figure 2.2.9: A complicated shape enclosed by the lines $y = (x + 2)^2$, $x = y - \frac{1}{16}y^3$, x = -2, and y = -4. This graph has intersection points (0, 4), (-2, 0), (0, -4), and (-2, -4). This region can be decomposed into a union of three regions of Type I or Type II. The area in the first quadrant is marked as D2 and a Type II region. The region in the second quadrant is marked as D1 and is a Type I region. The region in the third quadrant is marked as D3 and is a Type II region. Breaking the region into three subregions makes it easier to set up the integration.

Here D_1 is Type I and D_2 and D_3 are both of Type II. Hence,

$$\begin{aligned}\iint_D (2x+5y)\,dA &= \iint_{D_1} (2x+5y)\,dA + \iint_{D_2} (2x+5y)\,dA + \iint_{D_3} (2x+5y)\,dA \\ &= \int_{x=-2}^{x=0}\int_{y=0}^{y=(x+2)^2} (2x+5y)\,dy\,dx + \int_{y=0}^{y=4}\int_{x=0}^{x=y-(1/16)y^3} (2+5y)\,dx\,dy + \int_{y=-4}^{y=0}\int_{x=-2}^{x=y-(1/16)y^3} (2x+5y)\,dx\,dy \\ &= \int_{x=-2}^{x=0}\left[\frac{1}{2}(2+x)^2(20+24x+5x^2)\right]dx + \int_{y=0}^{y=4}\left[\frac{1}{256}y^6 - \frac{7}{16}y^4 + 6y^2\right]dy + \int_{y=-4}^{y=0}\left[\frac{1}{256}y^6 - \frac{7}{16}y^4 + 6y^2 + 10y - 4\right]dy \\ &= \frac{40}{3} + \frac{1664}{35} - \frac{1696}{35} = \frac{1304}{105}.\end{aligned}$$

Watch the accompanying video for a visual demonstration of the concept.

- **Video length**: 4 minutes 28 seconds.
- **Context**: This video demonstrates how to split a double integral into two parts to find area.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Exercise 2.2.3

Consider the region bounded by the curves $y = \ln x$ and $y = e^x$ in the interval $[1, 2]$. Decompose the region into smaller regions of Type II.

Hint available at the end of this section.

Answer available at the end of this section.

Exercise 2.2.4

Redo Example 2.2.4 using a union of two Type II regions.

Hint available at the end of this section.

Answer available at the end of this section.

Changing the Order of Integration

As we have already seen when we evaluate an iterated integral, sometimes one order of integration leads to a computation that is significantly simpler than the other order of integration. Sometimes the order of integration does not matter, but it is important to learn to recognize when a change in order will simplify our work.

Example 2.2.5: Changing the Order of Integration

Reverse the order of integration in the iterated integral

$$\int_{x=0}^{x=\sqrt{2}}\int_{y=0}^{y=2-x^2} xe^{x^2}\,dy\,dx.$$

Then evaluate the new iterated integral.

Solution

The region as presented is of Type I. To reverse the order of integration, we must first express the region as Type II. Refer to Figure 2.2.10.

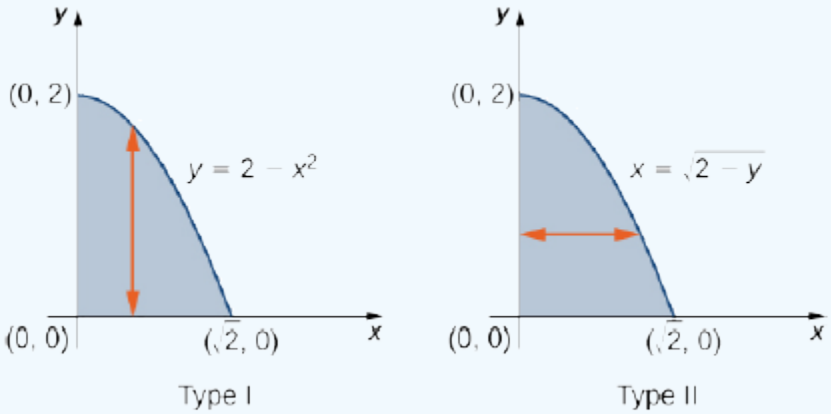


Figure 2.2.10: This figure consists of two figures labeled Type I and Type II. In the Type I figure, a curve is given as $y = 2 - x^2$, which forms a shape with the x and y axes. There is a vertical line with arrows on the end of it within this shape. In the Type II figure, a curve is given as $x = \sqrt{2 - y}$, which forms a shape with the x and y axes. There is a horizontal line with arrows on the end of it within this shape.

We can see from the limits of integration that the region is bounded above by $y = 2 - x^2$ and below by $y = 0$ where x is in the interval $[0, \sqrt{2}]$. By reversing the order, we have the region bounded on the left by $x = 0$ and on the right by $x = \sqrt{2 - y}$ where y is in the interval $[0, 2]$. We solved $y = 2 - x^2$ in terms of x to obtain $x = \sqrt{2 - y}$.

Hence

$$\int_0^{\sqrt{2}} \int_0^{2-x^2} xe^{x^2} dy\, dx = \int_0^2 \int_0^{\sqrt{2-y}} xe^{x^2}\, dx\, dy \qquad \text{Reverse the order of integration then use substitution.}$$

$$= \int_0^2 \left[\frac{1}{2} e^{x^2} \Big|_0^{\sqrt{2-y}} \right] dy = \int_0^2 \frac{1}{2}(e^{2-y} - 1)\, dy$$

$$= -\frac{1}{2}(e^{2-y} + y)\Big|_0^2 = \frac{1}{2}(e^2 - 3).$$

Watch the accompanying video for a visual demonstration of the concept.

- **Video length**: 3 minutes 36 seconds.
- **Context**: This video demonstrates how to switch the order of integration from dydx to dxdy.

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Example 2.2.6: Evaluating an Iterated Integral by Reversing the Order of Integration

Consider the iterated integral

$$\iint_R f(x, y)\, dx\, dy \tag{2.2.11}$$

where $z = f(x, y) = x - 2y$ over a triangular region R that has sides on $x = 0,\ y = 0$, and the line $x + y = 1$. Sketch the region, and then evaluate the iterated integral by

a. integrating first with respect to y and then
b. integrating first with respect to x.

Solution

A sketch of the region appears in Figure 2.2.11.

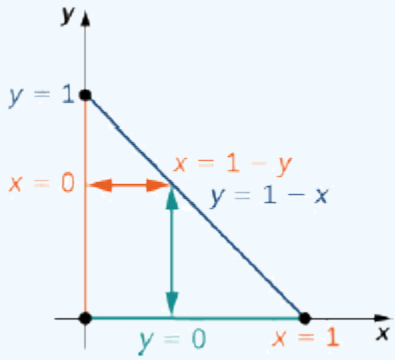


Figure 2.2.11: The line $y = 1 - x$ is drawn, and it is also marked as $x = 1 - y$. There is a shaded region around $x = 0$ that comes from the y-axis, which projects down to make a shaded region marked $y = 0$ from the x-axis. The line $x = 0$, $y = 1 - x$, and $y = 0$ forms a triangular region in quadrant 1 of the xy-plane.

We can complete this integration in two different ways.

a. One way to look at it is by first integrating y from $y = 0$ to $y = 1 - x$ vertically and then integrating x from $x = 0$ to $x = 1$:

$$\iint_R f(x, y)\, dx\ dy = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} (x - 2y)\, dy\ dx = \int_{x=0}^{x=1} \left[xy - y^2\right]_{y=0}^{y=1-x} dx$$

$$\int_{x=0}^{x=1} \left[x(1-x) - (1-x)^2\right]\ dx = \int_{x=0}^{x=1} [-1 + 3x - 2x^2] dx = \left[-x + \frac{3}{2}x^2 - \frac{2}{3}x^3\right]_{x=0}^{x=1} = -\frac{1}{6}.$$

b. The other way to do this problem is by first integrating x from $x = 0$ to $x = 1 - y$ horizontally and then integrating y from $y = 0$ to $y = 1$:

$$\begin{aligned}
\iint_D (3x^2 + y^2)\, dA &= \int_{y=-2}^{y=3} \int_{x=y^2-3}^{x=y+3} (3x^2 + y^2)\, dx\ dy \\
&= \int_{y=-2}^{y=3} (x^3 + xy^2)\Big|_{y^2-3}^{y+3}\, dy && \text{Iterated integral, Type II region} \\
&= \int_{y=-2}^{y=3} \left((y+3)^3 + (y+3)y^2 - (y^2-3)^3 - (y^2-3)y^2\right)\, dy \\
&= \int_{-2}^{3} (54 + 27y - 12y^2 + 2y^3 + 8y^4 - y^6)\, dy && \text{Integrate with respect to } x. \\
&= \left[54y + \frac{27y^2}{2} - 4y^3 + \frac{y^4}{2} + \frac{8y^5}{5} - \frac{y^7}{7}\right]_{-2}^{3} \\
&= \frac{2375}{7}.
\end{aligned}$$

Watch the accompanying video for a visual demonstration of the concept.

- **Video length**: 4 minutes 27 seconds.
- **Context**: This video demonstrates how to switching the order of integration from dydx to dxdy.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

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Exercise 2.2.5

Evaluate the iterated integral $\iint_D (x^2 + y^2)\ dA$ over the region D in the first quadrant between the functions $y = 2x$ and $y = x^2$.

Evaluate the iterated integral by integrating first with respect to y and then integrating first with resect to x.

Hint available at the end of this section.

Answer available at the end of this section.

Calculating Volumes, Areas, and Average Values

We can use double integrals over general regions to compute volumes, areas, and average values. The methods are the same as those in Double Integrals over Rectangular Regions, but without the restriction to a rectangular region, we can now solve a wider variety of problems.

Example 2.2.7: Finding the Volume of a Tetrahedron

Find the volume of the solid bounded by the planes $x = 0,\ y = 0,\ z = 0$, and $2x + 3y + z = 6$.

Solution

The solid is a tetrahedron with the base on the xy-plane and a height $z = 6 - 2x - 3y$. The base is the region D bounded by the lines, $x = 0$, $y = 0$ and $2x + 3y = 6$ where $z = 0$ (Figure 2.2.12). Note that we can consider the region D as Type I or as Type II, and we can integrate in both ways.

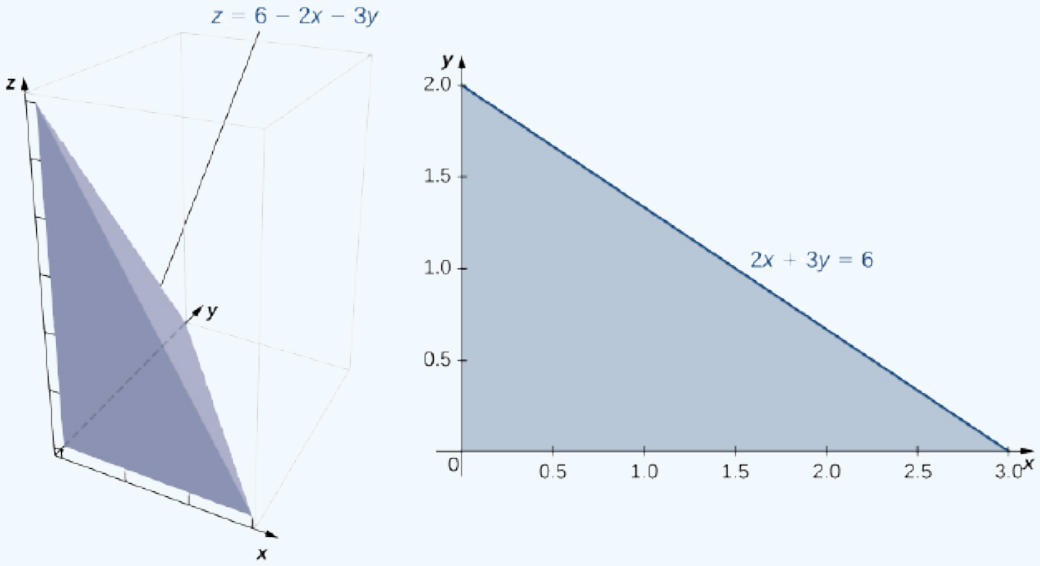


Figure 2.2.12: On the left side is a tetrahedron consisting of the three coordinate planes and the plane $z = 6 - 2x - 3y$, with the base bound by $x = 0,\ y = 0$, and $2x + 3y = 6$ on the left side. On the right side is the region D, and it is a right triangular region bounded by the x-axis, the y-axis, and the line $2x + 3y = 6$.

First, consider D as a Type I region, and hence $D = \{(x, y) \mid 0 \le x \le 3,\ 0 \le y \le 2 - \frac{2}{3}x\}$.

Therefore, the volume is

$$V = \int_{x=0}^{x=3} \int_{y=0}^{y=2-(2x/3)} (6 - 2x - 3y)\, dy\, dx = \int_{x=0}^{x=3} \left[\left(6y - 2xy - \frac{3}{2}y^2 \right) \Big|_{y=0}^{y=2-(2x/3)} \right] dx$$
$$= \int_{x=0}^{x=3} \left[\frac{2}{3}(x-3)^2 \right] dx = 6.$$

Now consider D as a Type II region, so $D = \{(x, y) \mid 0 \le y \le 2,\ 0 \le x \le 3 - \frac{3}{2}y\}$. In this calculation, the volume is

$$V = \int_{y=0}^{y=2} \int_{x=0}^{x=3-(3y/2)} (6 - 2x - 3y)\, dx\, dy = \int_{y=0}^{y=2} \left[(6x - x^2 - 3xy) \Big|_{x=0}^{x=3-(3y/2)} \right] dy$$
$$= \int_{y=0}^{y=2} \left[\frac{9}{4}(y-2)^2 \right] dy = 6.$$

Therefore, the volume is 6 cubic units.

Watch the accompanying video for a visual demonstration of the concept.

- **Video length**: 2 minutes 42 seconds.
- **Context**: This video demonstrates how to use a double integral to find volume.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

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Exercise 2.2.6

Find the volume of the solid bounded above by $f(x, y) = 10 - 2x + y$ over the region enclosed by the curves $y = 0$ and $y = e^x$ where x is in the interval $[0, 1]$.

Hint available at the end of this section.

Finding the area of a rectangular region is easy, but finding the area of a non-rectangular region is not so easy. As we have seen, we can use double integrals to find a rectangular area. As a matter of fact, this comes in very handy for finding the area of a general non-rectangular region, as stated in the next definition.

Definition: Area as Double Integral

The area of a plane-bounded region D is defined as the double integral

$$\iint_D 1\, dA. \tag{2.2.12}$$

We have already seen how to find areas in terms of single integration. Here we are seeing another way of finding areas by using double integrals, which can be very useful, as we will see in the later sections of this chapter.

Example 2.2.8: Finding the Area of a Region

Find the area of the region bounded below by the curve $y = x^2$ and above by the line $y = 2x$ in the first quadrant (Figure 2.2.13).

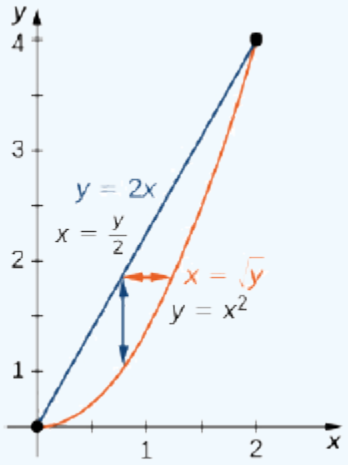


Figure 2.2.13: The region bounded by $y = x^2$ and $y = 2x$. There are vertical and horizontal shadings giving for small stretch of this region, denoting that it can be treated as a Type I or Type II area.

Solution

We just have to integrate the constant function $f(x, y) = 1$ over the region. Thus, the area A of the bounded region is $\int_{x=0}^{x=2} \int_{y=x^2}^{y=2x} dy\ dx$ *or* $\int_{y=0}^{y=4} \int_{x=y/2}^{x=\sqrt{y}} dx\ dy$:

$$\begin{aligned} A &= \iint_D 1\, dx\ dy \\ &= \int_{x=0}^{x=2} \int_{y=x^2}^{y=2x} 1\, dy\ dx \\ &= \int_{x=0}^{x=2} \left[y|_{y=x^2}^{y=2x} \right] dx \\ &= \int_{x=0}^{x=2} (2x - x^2)\, dx \\ &= x^2 - \frac{x^3}{3}\Big|_0^2 = \frac{4}{3}. \end{aligned}$$

Exercise 2.2.7

Find the area of a region bounded above by the curve $y = x^3$ and below by $y = 0$ over the interval $[0, 3]$.

Answer available at the end of this section.

Answer available at the end of this section.

We can also use a double integral to find the average value of a function over a general region. The definition is a direct extension of the earlier formula.

Definition: Average Value and Area

If $f(x,y)$ is integrable over a plane-bounded region D with positive area $A(D)$, then the average value of the function is

$$f_{ave} = \frac{1}{A(D)} \iint_D f(x,y)\, dA. \tag{2.2.13}$$

Note that the area is $A(D) = \iint_D 1 \; dA$.

Example 2.2.9: Finding an Average Value

Find the average value of the function $f(x,y) = 7xy^2$ on the region bounded by the line $x = y$ and the curve $x = \sqrt{y}$ (Figure 2.2.14).

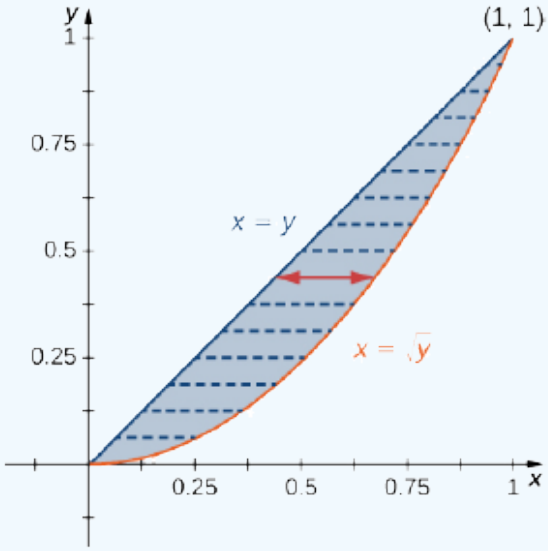


Figure 2.2.14: The region bounded by $x = y$ and $x = \sqrt{y}$. There are horizontal dashed lines marked throughout the region.

Solution

First find the area $A(D)$ where the region D is given by the figure. We have

$$A(D) = \iint_D 1\, dA = \int_{y=0}^{y=1} \int_{x=y}^{x=\sqrt{y}} 1\, dx\; dy = \int_{y=0}^{y=1} \left[x\big|_{x=y}^{x=\sqrt{y}} \right] dy = \int_{y=0}^{y=1} (\sqrt{y} - y)\, dy = \frac{2}{3} y^{3/2} - \frac{y^2}{2} \Big|_0^1 = \frac{1}{6} \tag{2.2.14}$$

Then the average value of the given function over this region is

$$f_{ave} = \frac{1}{A(D)} \iint_D f(x,y)\, dA = \frac{1}{A(D)} \int_{y=0}^{y=1} \int_{x=y}^{x=\sqrt{y}} 7xy^2\, dx\; dy = \frac{1}{1/6} \int_{y=0}^{y=1} \left[\frac{7}{2} x^2 y^2 \Big|_{x=y}^{x=\sqrt{y}} \right] dy$$

$$= 6 \int_{y=0}^{y=1} \left[\frac{7}{2} y^2 (y - y^2) \right] dy = 6 \int_{y=0}^{y=1} \left[\frac{7}{2} (y^3 - y^4) \right] dy = \frac{42}{2} \left(\frac{y^4}{4} - \frac{y^5}{5} \right) \Big|_0^1 = \frac{42}{40} = \frac{21}{20}.$$

Exercise 2.2.8

Find the average value of the function $f(x,y) = xy$ over the triangle with vertices $(0,0)$, $(1,0)$ and $(1,3)$.

Hint available at the end of this section.

Answer available at the end of this section.

Improper Double Integrals

An **improper double integral** is an integral $\iint_D f\ dA$ where either D is an unbounded region or f is an unbounded function. For example, $D = \{(x,y)\ |\ |x-y| \geq 2\}$ is an unbounded region, and the function $f(x,y) = 1/(1-x^2-2y^2)$ over the ellipse $x^2+3y^2 \geq 1$ is an unbounded function. Hence, both of the following integrals are improper integrals:

i. $$\iint_D xy\ dA \text{ where } D = \{(x,y)||\, x-y| \geq 2\}; \tag{2.2.15}$$

ii. $$\iint_D \frac{1}{1-x^2-2y^2}\ dA \text{ where } D = \{(x,y)|\, x^2+3y^2 \leq 1\}. \tag{2.2.16}$$

In this section we would like to deal with improper integrals of functions over rectangles or simple regions such that f has only finitely many discontinuities. Not all such improper integrals can be evaluated; however, a form of Fubini's theorem does apply for some types of improper integrals.

Theorem: Fubini's Theorem for Improper Integrals

If D is a bounded rectangle or simple region in the plane defined by

$\{(x,y)\ : a \leq x \leq b,\ g(x) \leq y \leq h(x)\}$ and also by

$\{(x,y)\ : c \leq y \leq d,\ j(y) \leq x \leq k(y)\}$ and f is a nonnegative function on D with finitely many discontinuities in the interior of D then

$$\iint_D f\ dA = \int_{x=a}^{x=b}\int_{y=g(x)}^{y=h(x)} f(x,y)\,dy\,dx = \int_{y=c}^{y=d}\int_{x=j(y)}^{x=k(y)} f(x,y)\,dx\,dy \tag{2.2.17}$$

It is very important to note that we required that the function be nonnegative on D for the theorem to work. We consider only the case where the function has finitely many discontinuities inside D.

Example 2.2.10: Evaluating a Double Improper Integral

Consider the function $f(x,y) = \frac{e^y}{y}$ over the region $D = \{(x,y)\ : 0 \leq x \leq 1,\ x \leq y \leq \sqrt{x}\}$.

Notice that the function is nonnegative and continuous at all points on D except $(0,0)$. Use Fubini's theorem to evaluate the improper integral.

Solution

First we plot the region D (Figure 2.2.15); then we express it in another way.

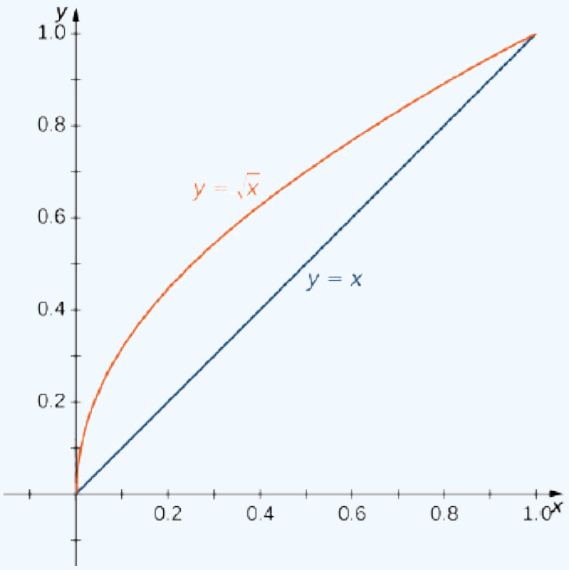


Figure 2.2.15: A region bounded by the intersection of $y = \sqrt{x}$ and $y = x$.

The other way to express the same region D is

$$D = \{(x, y) \,:\, 0 \le y \le 1,\ y^2 \le x \le y\}.$$

Thus we can use Fubini's theorem for improper integrals and evaluate the integral as

$$\int_{y=0}^{y=1} \int_{x=y^2}^{x=y} \frac{e^y}{y}\, dx\ dy.$$

Therefore, we have

$$\int_{y=0}^{y=1} \int_{x=y^2}^{x=y} \frac{e^y}{y}\, dx\ dy = \int_{y=0}^{y=1} \frac{e^y}{y} x \Big|_{x=y^2}^{x=y} dy = \int_{y=0}^{y=1} \frac{e^y}{y}(y - y^2)\, dy = \int_0^1 (e^y - ye^y)\, dy = e - 2. \tag{2.2.18}$$

As mentioned before, we also have an improper integral if the region of integration is unbounded. Suppose now that the function f is continuous in an unbounded rectangle R.

Theorem: Improper Integrals on an Unbounded Region

If R is an unbounded rectangle such as $R = \{(x, y) \,:\, a \le x \le \infty,\ c \le y \le \infty\}$, then when the limit exists, we have

$$\iint_R f(x, y)\, dA = \lim_{(b,d)\to(\infty,\infty)} \int_a^b \left(\int_c^d f(x, y)\, dy \right) dx = \lim_{(b,d)\to(\infty,\infty)} \int_c^d \left(\int_a^b f(x, y)\, dx \right) dy. \tag{2.2.19}$$

The following example shows how this theorem can be used in certain cases of improper integrals.

Example 2.2.11

Evaluate the integral $\iint_R xye^{-x^2-y^2}\ dA$ where R is the first quadrant of the plane.

Solution

The region R is the first quadrant of the plane, which is unbounded. So

$$\iint_R xye^{-x^2-y^2}\,dA = \lim_{(b,d)\to(\infty,\infty)} \int_{x=0}^{x=b} \left(\int_{y=0}^{y=d} xye^{-x^2-y^2} dy \right) dx$$
$$= \lim_{(b,d)\to(\infty,\infty)} \int_{y=0}^{x=b} xye^{-x^2-y^2}\,dy$$
$$= \lim_{(b,d)\to(\infty,\infty)} \frac{1}{4}\left(1-e^{-b^2}\right)\left(1-e^{-d^2}\right) = \frac{1}{4}$$

Thus,

$$\iint_R xye^{-x^2-y^2}\,dA$$

is convergent and the value is $\frac{1}{4}$.

Exercise 2.2.9

$$\iint_D \frac{y}{\sqrt{1-x^2-y^2}} dA \tag{2.2.20}$$

where $D = \left\{(x,y) \ : \ x \geq 0,\ y \geq 0,\ x^2+y^2 \leq 1\right\}$.

Hint available at the end of this section.

Answer available at the end of this section.

In some situations in probability theory, we can gain insight into a problem when we are able to use double integrals over general regions. Before we go over an example with a double integral, we need to set a few definitions and become familiar with some important properties.

Definition: Joint Density Function

Consider a pair of continuous random variables X and Y such as the birthdays of two people or the number of sunny and rainy days in a month. The **joint density function** f of X and Y satisfies the probability that (X,Y) lies in a certain region D :

$$P((X,Y) \in D) = \iint_D f(x,y)\,dA. \tag{2.2.21}$$

Since the probabilities can never be negative and must lie between 0 and 1 the joint density function satisfies the following inequality and equation:

$$f(x,y) \geq 0 \text{ and } \iint_R f(x,y)\,dA = 1. \tag{2.2.22}$$

Definition: Independent Random Variables

The variables X and Y are said to be **independent random variables** if their joint density function is the product of their individual density functions:

$$f(x,y) = f_1(x)f_2(y). \tag{2.2.23}$$

Example 2.2.12: Application to Probability

At Sydney's Restaurant, customers must wait an average of 15 minutes for a table. From the time they are seated until they have finished their meal requires an additional 40 minutes, on average. What is the probability that a customer spends less than an hour and a half at the diner, assuming that waiting for a table and completing the meal are independent events?

Solution

Waiting times are mathematically modeled by exponential density functions, with m being the average waiting time, as

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{m} e^{-t/m} & \text{if } t \geq 0. \end{cases}$$

if X and Y are random variables for 'waiting for a table' and 'completing the meal,' then the probability density functions are, respectively,

$$f_1(x) = \begin{cases} 0 & \text{if } x < 0. \\ \frac{1}{15} e^{-x/15} & \text{if } x \geq 0. \end{cases} \quad \text{and} \quad f_2(y) = \begin{cases} 0 & \text{if } y < 0 \\ \frac{1}{40} e^{-y/40} & \text{if } y \geq 0. \end{cases} \tag{2.2.24}$$

Clearly, the events are independent and hence the joint density function is the product of the individual functions

$$f(x,y) = f_1(x) f_2(y) = \begin{cases} 0 & \text{if } x < 0 \text{ or } y < 0, \\ \frac{1}{600} e^{-x/15} & \text{if } x, y \geq 0 \end{cases}$$

We want to find the probability that the combined time $X + Y$ is less than 90 minutes. In terms of geometry, it means that the region D is in the first quadrant bounded by the line $x + y = 90$ (Figure 2.2.16).

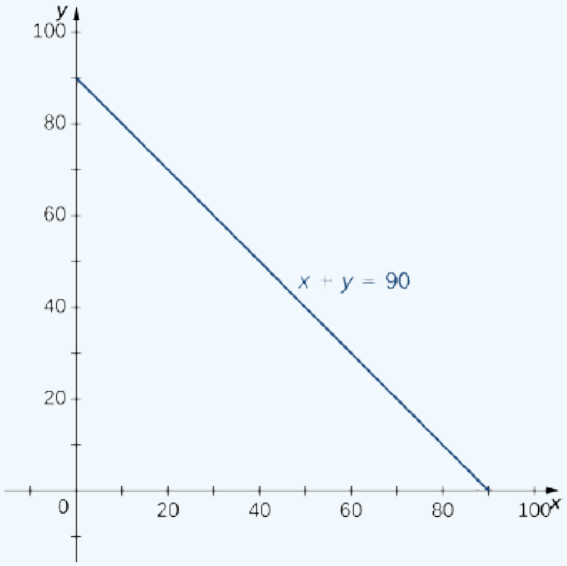


Figure 2.2.16: The line $x + y = 90$ is sketch in the first quadrant of the xy-plane. The region of integration for a joint probability density function.

Hence, the probability that (X, Y) is in the region D is

$$P(X + Y \leq 90) = P((X,Y) \in D) = \iint_D f(x,y)\, dA = \iint_D \frac{1}{600} e^{-x/15} e^{-y/40}\, dA. \tag{2.2.25}$$

Since $x + y = 90$ is the same as $y = 90 - x$, we have a region of Type I, so

$$D = \{(x, y) \mid 0 \le x \le 90,\ 0 \le y \le 90 - x\}, \tag{2.2.26}$$

$$P(X + Y \le 90) = \frac{1}{600} \int_{x=0}^{x=90} \int_{y=0}^{y=90-x} e^{-(/15} e^{-y/40} dx\ dy = \frac{1}{600} \int_{x=0}^{x=90} \int_{y=0}^{y=90-x} e^{-x/15} e^{-y/40} dx\ dy \tag{2.2.27}$$

$$= \frac{1}{600} \int_{x=0}^{x=90} \int_{y=0}^{y=90-x} e^{-(x/15+y/40)} dx\ dy = 0.8328 \tag{2.2.28}$$

Thus, there is an 83.2% chance that a customer spends less than an hour and a half at the restaurant.

Watch the accompanying video for a visual demonstration of the concept.

- **Video length**: 5 minutes and 11 seconds.
- **Context**: This video demonstrates how to show a function of two variables is a joint density function and finding a probability.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Another important application in probability that can involve improper double integrals is the calculation of expected values. First we define this concept and then show an example of a calculation.

Definition: Expected Values

In probability theory, we denote the **expected values** $E(X)$ and $E(Y)$ respectively, as the most likely outcomes of the events. The expected values $E(X)$ and $E(Y)$ are given by

$$E(X) = \iint_S x\, f(x, y)\, dA \text{ and } E(Y) = \iint_S y\, f(x, y)\, dA, \tag{2.2.29}$$

where S is the sample space of the random variables X and Y.

Example 2.2.13: Finding Expected Value

Find the expected time for the events 'waiting for a table' and 'completing the meal' in Example 2.2.12.

Solution

Using the first quadrant of the rectangular coordinate plane as the sample space, we have improper integrals for $E(X)$ and $E(Y)$. The expected time for a table is

$$E(X) = \iint_S x\frac{1}{600}e^{-x/15}e^{-y/40}dA = \frac{1}{600}\int_{x=0}^{x=\infty}\int_{y=0}^{y=\infty} xe^{-x/15}e^{-y/40}dA \tag{2.2.30}$$

$$= \frac{1}{600}\lim_{(a,b)\to(\infty,\infty)}\int_{x=0}^{x=a}\int_{y=0}^{y=b} xe^{-x/15}e^{-y/40}dx\ dy \tag{2.2.31}$$

$$= \frac{1}{600}\left(\lim_{a\to\infty}\int_{x=0}^{x=a} xe^{-x/15}dx\right)\left(\lim_{b\to\infty}\int_{y=0}^{y=b} e^{-y/40}dy\right) \tag{2.2.32}$$

$$\frac{1}{600}\left((\lim_{a\to\infty}(-15e^{-x/15}(x+15)))\Big|_{x=0}^{x=a}\right)\left((\lim_{b\to\infty}(-40e^{-y/40}))\Big|_{y=0}^{y=b}\right) \tag{2.2.33}$$

$$= \frac{1}{600}\left(\lim_{a\to\infty}(-15e^{-a/15}(x+15)+225)\right)\left(\lim_{b\to\infty}(-40e^{-b/40}+40)\right) \tag{2.2.34}$$

$$= \frac{1}{600}(225)(40) = 15. \tag{2.2.35}$$

A similar calculation shows that $E(Y) = 40$. This means that the expected values of the two random events are the average waiting time and the average dining time, respectively.

Exercise 2.2.10

The joint density function for two random variables X and Y is given by

$$f(x,y) = \begin{cases} \frac{1}{600}(x^2+y^2), & \text{if } \le x \le 15,\ 0 \le y \le 10 \\ 0, & \text{otherwise} \end{cases}$$

Find the probability that X is at most 10 and Y is at least 5.

Hint available at the end of this section.

Answer available at the end of this section.

Key Concepts

- A general bounded region D on the plane is a region that can be enclosed inside a rectangular region. We can use this idea to define a double integral over a general bounded region.
- To evaluate an iterated integral of a function over a general non-rectangular region, we sketch the region and express it as a Type I or as a Type II region or as a union of several Type I or Type II regions that overlap only on their boundaries.
- We can use double integrals to find volumes, areas, and average values of a function over general regions, similarly to calculations over rectangular regions.
- We can use Fubini's theorem for improper integrals to evaluate some types of improper integrals.

Key Equations

- **Iterated integral over a Type I region**

$$\iint_D f(x,y)\,dA = \iint_D f(x,y)\,dy\,dx = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\right]dx \tag{2.2.36}$$

- **Iterated integral over a Type II region**

$$\iint_D f(x,y)\,dA = \iint_D (x,y)\,dx\,dy = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\right]dy \tag{2.2.37}$$

Glossary

Improper double integral
: a double integral over an unbounded region or of an unbounded function

Type I region
: a region D in the xy- plane is Type I if it lies between two vertical lines and the graphs of two continuous functions $g_1(x)$ and $g_2(x)$

Type II region
: a region D in the xy-plane is Type II if it lies between two horizontal lines and the graphs of two continuous functions $h_1(y)$ and $h_2(y)$

Contributors and Attributions

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Answers

Exercise 2.2.1

Graph the functions, and draw vertical and horizontal lines.

Exercise 2.2.1

Type I and Type II are expressed as $\{(x,y) \mid 0 \le x \le 2,\ x^2 \le y \le 2x\}$ and $\{(x,y) \mid 0 \le y \le 4,\ \frac{1}{2}y \le x \le \sqrt{y}\}$, respectively.

Exercise 2.2.2

Express D as a Type I region, and integrate with respect to y first.

Exercise 2.2.2

$\frac{\pi}{4}$

Exercise 2.2.3

Sketch the region, and split it into three regions to set it up.

Exercise 2.2.3

$$\{(x,y) \mid 0 \le y \le \ln 2,\ 1 \le x \le e^y\} \cup \{(x,y) \mid \ln 2 \le y \le e,\ 1 \le x \le 2\} \cup \{(x,y) \mid e \le y \le e^2,\ \ln y \le x \le 2\}$$

Exercise 2.2.4

$$\{(x,y) \mid 0 \le y \le 4,\ 2 + \sqrt{y} \le x \le \left(y - \frac{1}{16}y^3\right)\} \cup \{(x,y) \mid -4 \le y \le 0,\ -2 \le x \le \left(y - \frac{1}{16}y^{13}\right)\}$$

Exercise 2.2.4

Same as in the example shown.

Exercise 2.2.5

Sketch the region and follow Example 2.2.6.

Exercise 2.2.5

$\frac{216}{35}$

Exercise 2.2.6

Sketch the region, and describe it as Type I.

$\frac{e^2}{4} + 10e - \frac{49}{4}$ cubic units

Exercise 2.2.7

Hint

Sketch the region.

Exercise 2.2.7

$\frac{81}{4}$ square units

Exercise 2.2.8

Express the line joining $(0,0)$ and $(1,3)$ as a function $y = g(x)$.

Exercise 2.2.8

$\frac{3}{4}$

Exercise 2.2.9

Notice that the integral is nonnegative and discontinuous on $x^2 + y^2 = 1$. Express the region D as $D = \{(x,y) \ :\ 0 \le x \le 1,\ 0 \le y \le \sqrt{1-x^2}\}$ and integrate using the method of substitution.

Exercise 2.2.9

$\frac{\pi}{4}$

Exercise 2.2.10

Compute the probability

$$P(X \leq 10,\ Y \geq 5) = \int_{x=-\infty}^{10} \int_{y=5}^{y=10} \frac{1}{6000}(x^2 + y^2) dy\ dx.$$

Exercise 2.2.10

$\frac{55}{72} \approx 0.7638$

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2.2E: Exercises

Question 1 - 13

Follow the instructions in each question.

1. The region D bounded by $y = x^3,\ y = x^3 + 1,\ x = 0$, and $x = 1$ as given in Figure $2.2E.1$.

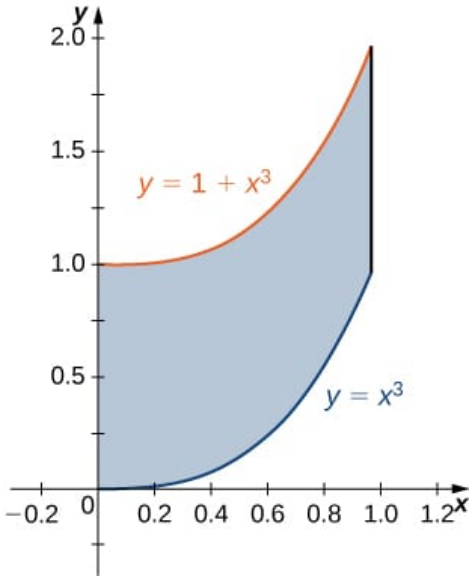


Figure $2.2E.1$: A region is bounded above by $y = 1 + x^3$, bounded below by $y = x^3$, bounded on the left by $x = 0$, and bounded on the right by $x = 1$.

a. Classify this region as vertically simple (Type I) or horizontally simple (Type II).

Type:

Type I but not Type II

b. Find the area of the region D.
c. Find the average value of the function $f(x, y) = 3xy$ on the region graphed in the previous exercise.

Answer

$\frac{27}{20}$

2. The region D bounded by $y = \sin x,\ y = 1 + \sin x,\ x = 0$, and $x = \frac{\pi}{2}$ as given in Figure $2.2E.2$.

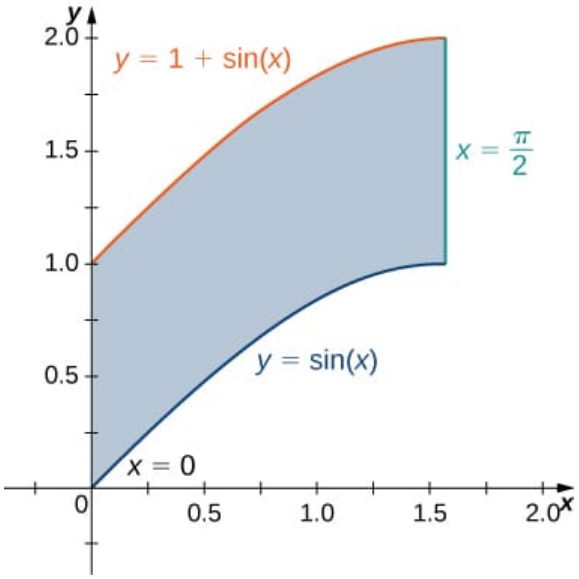


Figure $2.2E.2$: A region is bounded above by $y = 1 + \sin x$, bounded below by $y = \sin x$, bounded on the left by $x = 0$, and bounded on the right by $x = \frac{\pi}{2}$.

a. Classify this region as vertically simple (Type I) or horizontally simple (Type II).

Type:

Type I but not Type II

b. Find the area of the region D.

Answer

$\frac{\pi}{2}$ units^2

c. Find the average value of the function $f(x, y) = \cos x$ on the region D.

3. The region D bounded by $x = y^2 - 1$ and $x = \sqrt{1 - y^2}$ as given in Figure $2.2E.3$.

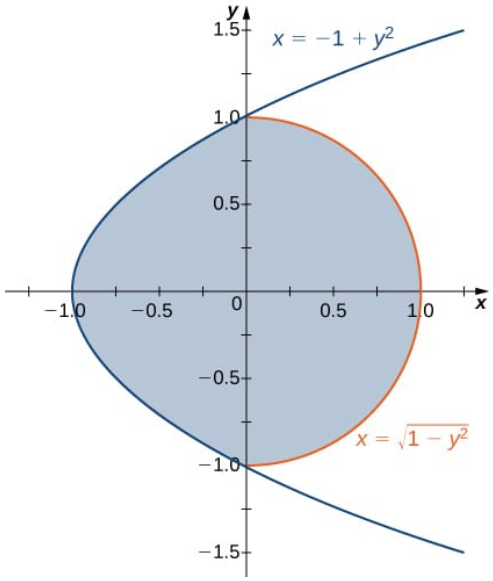


Figure $2.2E.3$: A region is bounded from left by $x = -1 + y^2$ and bounded on the right by $x = \sqrt{1 - y^2}$.

a. Classify this region as vertically simple (Type I) or horizontally simple (Type II).

Type:

Type II but not Type I

b. Find the volume of the solid under the graph of the function $f(x,y) = xy + 1$ and above the region D.

Answer

$\frac{1}{6}(8 + 3\pi)$ units^3

4. The region D bounded by $y = 0,\ x = -10 + y$, and $x = 10 - y$ as given in Figure $2.2E.4$.

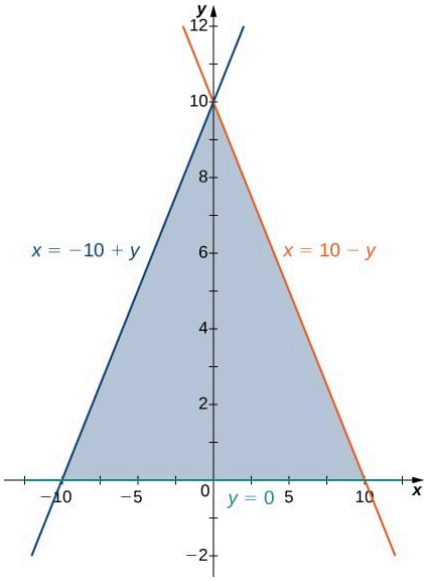


Figure $2.2E.4$: A region is bounded on the left by by $x = -10 + y$, bounded on the right by $x = 10 - y$, and bounded below by $y = 0$.

a. Classify this region as vertically simple (Type I) or horizontally simple (Type II).

Type:

Type II but not Type I

b. Find the volume of the solid under the graph of the function $f(x,y) = x + y$ and above the region in the figure from the previous exercise.

Answer

$\frac{1000}{3}$ units^3

5. The region D bounded by $y = 0,\ x = y - 1,\ x = \frac{\pi}{2}$.

Classify this region as vertically simple (Type I) or horizontally simple (Type II).

Type:

Type I and Type II

6. The region D bounded by $y = 0$ and $y = x^2 - 1$ as given Figure $2.2E.5$.

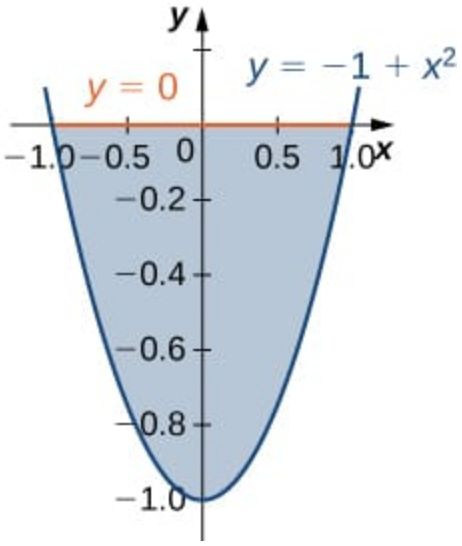


Figure $2.2E.5$: A region is bounded above by $y = 0$ and bounded below by $y = -1 + x^2$.

Classify this region as vertically simple (Type I) or horizontally simple (Type II).

Type:

Type I and Type II

7. Let D be the region bounded by the curves of equations $y = \cos x$ and $y = 4 - x^2$ and the x-axis. Explain why D is neither of Type I nor II.

Answer

The region D is not of Type I: it does not lie between two vertical lines and the graphs of two continuous functions $g_1(x)$ and $g_2(x)$. The region is not of Type II: it does not lie between two horizontal lines and the graphs of two continuous functions $h_1(y)$ and $h_2(y)$.

8. Let D be the region bounded by the curves of equations $y = x,\ y = -x$ and $y = 2 - x^2$. Explain why D is neither of Type I nor II.

Exercises 9 - 14

Evaluate the double integral $\iint_D f(x,y)\ dA$ over the region D.

9. $f(x,y) = 1$ and

$D = \{(x,y)|\ 0 \le x \le \frac{\pi}{2},\ \sin x \le y \le 1 + \sin x\}$

Answer

$\frac{\pi}{2}$

10. $f(x,y) = 2$ and

$D = \{(x,y)|\ 0 \le y \le 1,\ y - 1 \le x \le \arccos y\}$

11. $f(x,y) = xy$ and

$D = \{(x,y)|\ -1 \le y \le 1,\ y^2 - 1 \le x \le \sqrt{1 - y^2}\}$

Answer

0

12. $f(x,y)=\sin y$ and D is the triangular region with vertices $(0,0),\ (0,3)$, and $(3,0)$
13. $f(x,y)=-x+1$ and D is the triangular region with vertices $(0,0),\ (0,2)$, and $(2,2)$

Answer

$\frac{2}{3}$

14. $f(x,y)=2x+4y$ and

$D=\left\{(x,y)|\ 0\le x\le 1,\ x^3\le y\le x^3+1\right\}$

Exercises 15 - 20

Evaluate the iterated integrals.

15. $\int_0^1\int_{2\sqrt{x}}^{2\sqrt{x}+1}(xy+1)\ dy\ dx$

Answer

$\frac{41}{20}$

16. $\int_0^3\int_{2x}^{3x}(x+y^2)\ dy\ dx$
17. $\int_1^2\int_{-u^2-1}^{-u}(8uv)\ dv\ du$

Answer

-63

18. $\int_e^{e^2}\int_{\ln u}^{2}(v+\ln u)\ dv\ du$
19. $\int_0^{1/2}\int_{-\sqrt{1-4y^2}}^{\sqrt{1-4y^2}}4\ dx\ dy$

Answer

π

20. $\int_0^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}(2x+4y^3)\ dx\ dy$
21. Let D be the region bounded by $y=1-x^2,\ y=4-x^2$, and the x- and y-axes.
    a. Show that $\iint_D x\ dA=\int_0^1\int_{1-x^2}^{4-x^2}x\ dy\ dx+\int_1^2\int_0^{4-x^2}x\ dy\ dx$ by dividing the region D into two regions of Type I.
    b. Evaluate the integral $\iint_D x\ dA$.
22. Let D be the region bounded by $y=1,\ y=x,\ y=\ln x$, and the x-axis.
    a. Show that $\iint_D y^2\ dA=\int_{-1}^0\int_{-x}^{2-x^2}y^2dy\ dx+\int_0^1\int_x^{2-x^2}y^2dy\ dx$ by dividing the region D into two regions of Type I, where $D=\left\{(x,y)\ |\ y\ge x, y\ge -x,\ y\le 2-x^2\right\}$.
    b. Evaluate the integral $\iint_D y^2\ dA$.
23. Let D be the region bounded by $y=x^2$, $y=x+2$, and $y=-x$.
    a. Show that $\iint_D x\ dA=\int_0^1\int_{-y}^{\sqrt{y}}x\ dx\ dy+\int_1^4\int_{y-2}^{\sqrt{y}}x\ dx\ dy$ by dividing the region D into two regions of Type II, where $D=\left\{(x,y)\ |\ y\ge x^2,\ y\ge -x,\ y\le x+2\right\}$.
    b. Evaluate the integral $\iint_D x\ dA$.

Answer

a. Answers may vary;

b. $\frac{8}{12}$

24. The region D bounded by $x = 0, y = x^5 + 1$, and $y = 3 - x^2$ is shown in Figure $2.2E.6$. Find the area $A(D)$ of the region D.

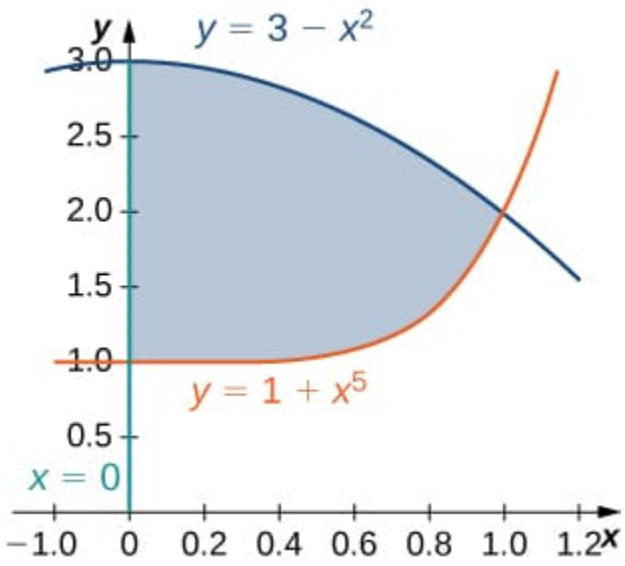


Figure $2.2E.6$: A region is bounded below by $y = 1 + x^5$, bounded above by $y = 3 - x^2$, and bounded on the left by $x = 0$.

25. The region D bounded by $y = \cos x,\ y = 4 + \cos x$, and $x = \pm\frac{\pi}{3}$ is shown in Figure $2.2E.7$. Find the area $A(D)$ of the region D.

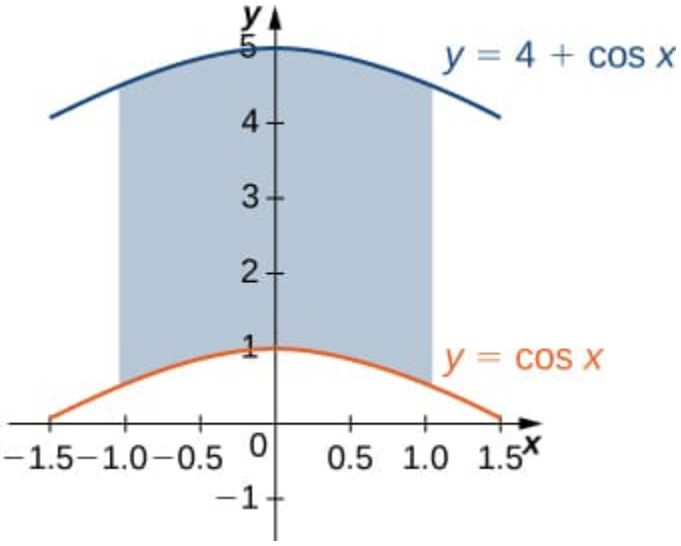


Figure $2.2E.7$: A region is bounded below by $y = \cos x$, bounded above by $y = 4 + \cos x$, bounded from left by $x = -1$, and bounded from right by $x = 1$.

Answer

$\frac{8\pi}{3}$

26. Find the area $A(D)$ of the region $D = \{(x, y)|\ y \geq 1 - x^2, y \leq 4 - x^2,\ y \geq 0,\ x \geq 0\}$.
27. Let D be the region bounded by $y = 1,\ y = x,\ y = \ln x$, and the x-axis. Find the area $A(D)$ of the region D.

Answer

$\left(e - \frac{3}{2}\right)\ \text{units}^2$

28. Find the average value of the function $f(x, y) = \sin y$ on the triangular region with vertices $(0, 0),\ (0, 3)$, and $(3, 0)$.
29. Find the average value of the function $f(x, y) = -x + 1$ on the triangular region with vertices $(0, 0),\ (0, 2)$, and $(2, 2)$.

Answer

The average value of f on this triangular region is $\frac{1}{3}$.

Exercises 30 - 33

Change the order of integration and evaluate the integral.

30. $\int_{-1}^{\pi/2} \int_{0}^{x+1} \sin x \ dy \ dx$
31. $\int_{0}^{1} \int_{x-1}^{1-x} x \ dy \ dx$

Answer

$$\int_0^1 \int_{x-1}^{1-x} x \ dy \ dx = \int_{-1}^0 \int_0^{y+1} x \ dx \ dy + \int_0^1 \int_0^{1-y} x \ dx \ dy = \frac{1}{3}$$

32. $\int_{-1}^{0} \int_{-\sqrt{y+1}}^{\sqrt{y+1}} y^2 dx \ dy$
33. $\int_{-1/2}^{1/2} \int_{-\sqrt{y^2+1}}^{\sqrt{y^2+1}} y \ dx \ dy$

Answer

$$\int_{-1/2}^{1/2} \int_{-\sqrt{y^2+1}}^{\sqrt{y^2+1}} y \ dx \ dy = \int_1^2 \int_{-\sqrt{x^2-1}}^{\sqrt{x^2-1}} y \ dy \ dx = 0$$

34. The region D is shown in Figure $2.2E.8$. Evaluate the double integral $\iint_D (x^2 + y) \ dA$ by using the easier order of integration.

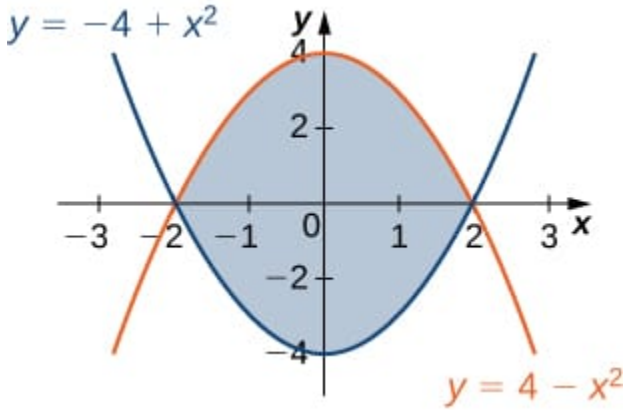


Figure $2.2E.8$: A region is bounded below by $y = -4 + x^2$ and bounded above by $y = 4 - x^2$.

35. The region D is shown in Figure $2.2E.9$. Evaluate the double integral $\iint_D (x^2 - y^2) \ dA$ by using the easier order of integration.

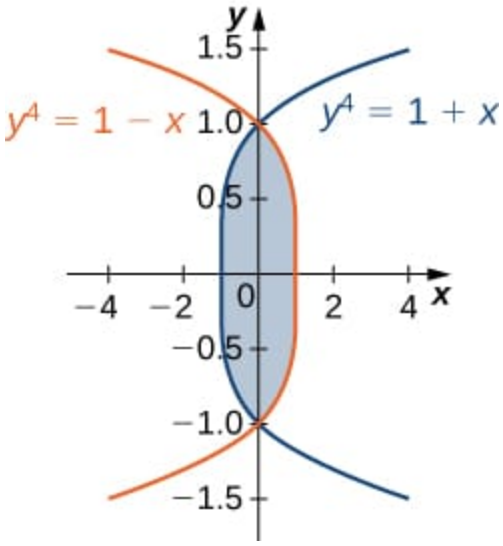


Figure 2.2E.9: A region is bounded by $y^4 = 1 - x$ and $y^4 = 1 + x$.

Answer

$$\iint_D (x^2 - y^2)dA = \int_{-1}^{1} \int_{y^4-1}^{1-y^4} (x^2 - y^2)dx\ dy = \frac{464}{4095}$$

36. Find the volume of the solid under the surface $z = 2x + y^2$ and above the region bounded by $y = x^5$ and $y = x$.
37. Find the volume of the solid under the plane $z = 3x + y$ and above the region determined by $y = x^7$ and $y = x$.

Answer

$\frac{4}{5}$ units^3

38. Find the volume of the solid under the plane $z = 3x + y$ and above the region bounded by $x = \tan y,\ x = -\tan y$, and $x = 1$.
39. Find the volume of the solid under the surface $z = x^3$ and above the plane region bounded by $x = \sin y,\ x = -\sin y$, and $x = 1$.

Answer

$\frac{5\pi}{32}$ units^3

40. Let g be a positive, increasing, and differentiable function on the interval $[a, b]$. Show that the volume of the solid under the surface $z = g'(x)$ and above the region bounded by $y = 0,\ y = g(x),\ x = a$, and $x = b$ is given by $\frac{1}{2}(g^2(b) - g^2(a))$.
41. Let g be a positive, increasing, and differentiable function on the interval $[a, b]$ and let k be a positive real number. Show that the volume of the solid under the surface $z = g'(x)$ and above the region bounded by $y = g(x),\ y = g(x) + k,\ x = a$, and $x = b$ is given by $k(g(b) - g(a))$.
42. Find the volume of the solid situated in the first octant and determined by the planes $z = 2$, $z = 0,\ x + y = 1,\ x = 0$, and $y = 0$.
43. Find the volume of the solid situated in the first octant and bounded by the planes $x + 2y = 1$, $x = 0,\ z = 4$, and $z = 0$.

Answer

1 units^3

44. Find the volume of the solid bounded by the planes $x + y = 1,\ x - y = 1,\ x = 0,\ z = 0$, and $z = 10$.
45. Find the volume of the solid bounded by the planes $x + y = 1,\ x - y = 1,\ x + y = -1\ x - y = -1,\ z = 1$, and $z = 0$

Answer

2 units^3

46. Let S_1 and S_2 be the solids situated in the first octant under the planes $x + y + z = 1$ and $x + y + 2z = 1$ respectively, and let S be the solid situated between $S_1,\ S_2,\ x = 0$, and $y = 0$.
    a. Find the volume of the solid S_1.
    b. Find the volume of the solid S_2.
    c. Find the volume of the solid S by subtracting the volumes of the solids S_1 and S_2.

47. Let S_1 and S_2 be the solids situated in the first octant under the planes $2x + 2y + z = 2$ and $x + y + z = 1$ respectively, and let S be the solid situated between $S_1,\ S_2,\ x = 0$, and $y = 0$.
    a. Find the volume of the solid S_1.
    b. Find the volume of the solid S_2.
    c. Find the volume of the solid S by subtracting the volumes of the solids S_1 and S_2.

    Answer

    a. $\frac{1}{3}$ units^3
    b. $\frac{1}{6}$ units^3
    c. $\frac{1}{6}$ units^3

48. Let S_1 and S_2 be the solids situated in the first octant under the plane $x + y + z = 2$ and under the sphere $x^2 + y^2 + z^2 = 4$, respectively. If the volume of the solid S_2 is $\frac{4\pi}{3}$ determine the volume of the solid S situated between S_1 and S_2 by subtracting the volumes of these solids.

49. Let S_1 and S_2 be the solids situated in the first octant under the plane $x + y + z = 2$ and bounded by the cylinder $x^2 + y^2 = 4$, respectively.
    a. Find the volume of the solid S_1.
    b. Find the volume of the solid S_2.
    c. Find the volume of the solid S situated between S_1 and S_2 by subtracting the volumes of the solids S_1 and S_2.

    Answer

    a. $\frac{4}{3}$ units^3
    b. 2π units^3
    c. $\frac{6\pi - 4}{3}$ units^3

50. [T] Figure $2.2E.10$ shows the region D bounded by the curves $y = \sin x,\ x = 0$, and $y = x^4$. Use a graphing calculator or CAS to find the x-coordinates of the intersection points of the curves and to determine the area of the region D. Round your answers to six decimal places.

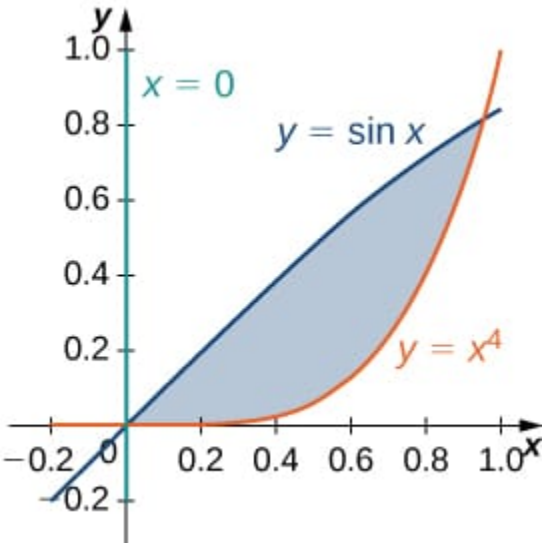


Figure $2.2E.10$: A region is bounded above by $y = \sin x$, bounded below by $y = x^4$, and bounded from left by $x = 0$.

51. [T] The region D bounded by the curves $y = \cos x,\ x = 0$, and $y = x^3$ is shown in Figure $2.2E.11$. Use a graphing calculator or CAS to find the x-coordinates of the intersection points of the curves and to determine the area of the region D. Round your answers to six decimal places.

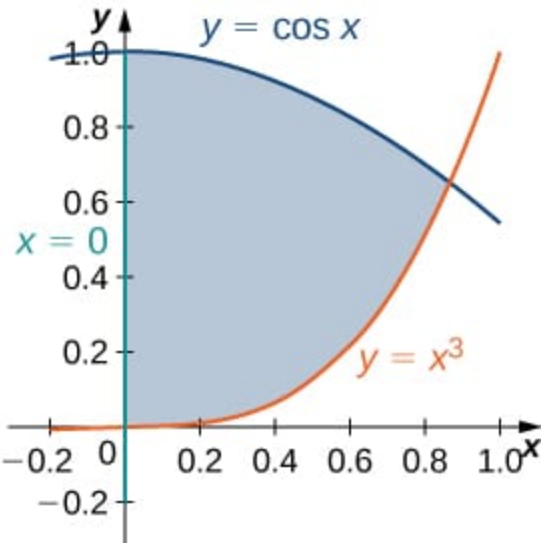


Figure $2.2E.11$: A region is bounded above by $y = \cos x$, bounded below by $y = x^3$, and bounded from left by $x = 0$.

Answer

0 and 0.865474; $A(D) = 0.621135 \text{ units}^3$

52. Suppose that (X, Y) is the outcome of an experiment that must occur in a particular region S in the xy-plane. In this context, the region S is called the sample space of the experiment and X and Y are random variables. If D is a region included in S , then the probability of (X, Y) being in D is defined as $P[(X, Y) \in D] = \iint_D p(x, y)dx\ dy$, where $p(x, y)$ is the joint probability density of the experiment. Here, $p(x, y)$ is a nonnegative function for which $\iint_S p(x, y)dx\ dy = 1$. Assume that a point (X, Y) is chosen arbitrarily in the square $[0, 3] \times [0, 3]$ with the probability density

$$p(x, y) = \frac{1}{9}(x, y) \in [0, 3] \times [0, 3],$$

$$p(x, y) = 0 \text{ otherwise}$$

Find the probability that the point (X, Y) is inside the unit square and interpret the result.

53. Consider X and Y two random variables of probability densities $p_1(x)$ and $p_2(x)$, respectively. The random variables X and Y are said to be independent if their joint density function is given by $p_(x, y) = p_1(x)p_2(y)$. At a drive-thru restaurant, customers spend, on average, 3 minutes placing their orders and an additional 5 minutes paying for and picking up their meals. Assume that placing the order and paying for/picking up the meal are two independent events X and Y. If the waiting times are modeled by the exponential probability densities

$$p_1(x) = \frac{1}{3}e^{-x/3}\ x \geq 0,$$

$$p_1(x) = 0 \text{ otherwise}$$

$$p_2(y) = \frac{1}{5}e^{-y/5}\ y \geq 0$$

$$p_2(y) = 0 \text{ otherwise}$$

respectively, the probability that a customer will spend less than 6 minutes in the drive-thru line is given by $P[X + Y \leq 6] = \iint_D p(x, y)dx\ dy$, where $D = (x, y)|x \geq 0,\ y \geq 0,\ x + y \leq 6$. Find $P[X + Y \leq 6]$ and interpret the result.

Answer

$P[X + Y \leq 6] = 1 + \frac{3}{2e^2} - \frac{5}{e^{6/5}} \approx 0.45$; there is a 45% chance that a customer will spend 6 minutes in the drive-thru line.

54. [T] The Reuleaux triangle consists of an equilateral triangle and three regions, each of them bounded by a side of the triangle and an arc of a circle of radius s centered at the opposite vertex of the triangle. Show that the area of the Reuleaux triangle in Figure $2.2E.12$ of side length s is $\frac{s^2}{2}(\pi - \sqrt{3})$.

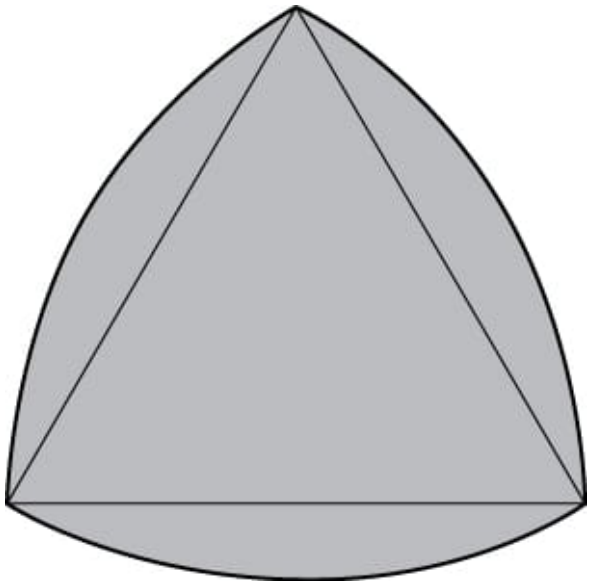


Figure $2.2E.12$: An equilateral triangle with additional regions consisting of three arcs of a circle with radius equal to the length of the side of the triangle. These arcs connect two adjacent vertices, and the radius is taken from the opposite vertex.

55. [T] Show that the area of the lunes of Alhazen, the two blue lunes in the following figure, is the same as the area of the right triangle ABC. The outer boundaries of the lunes are semicircles of diameters AB and AC respectively, and the inner boundaries are formed by the circumcircle of the triangle ABC.

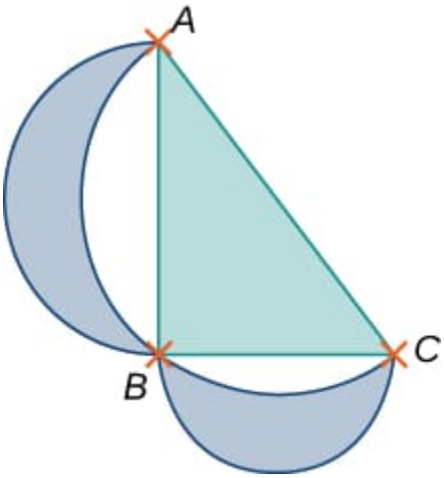


Figure $2.2E.13$: A right triangle with points A, B, and C. Point B has the right angle. There are two lunes drawn from A to B and from B to C with outer diameters AB and AC, respectively, and with the inner boundaries formed by the circumcircle of the triangle ABC.

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2.3: Double Integrals in Polar Coordinates

Learning Objectives

- Recognize the format of a double integral over a polar rectangular region.
- Evaluate a double integral in polar coordinates by using an iterated integral.
- Recognize the format of a double integral over a general polar region.
- Use double integrals in polar coordinates to calculate areas and volumes.

Double integrals are sometimes much easier to evaluate if we change rectangular coordinates to polar coordinates. However, before we describe how to make this change, we need to establish the concept of a double integral in a polar rectangular region.

Polar Rectangular Regions of Integration

When we defined the double integral for a continuous function in rectangular coordinates—say, g over a region R in the xy -plane—we divided R into subrectangles with sides parallel to the coordinate axes. These sides have either constant x-values and/or constant y-values. In polar coordinates, the shape we work with is a polar rectangle, whose sides have constant r-values and/or constant θ-values. This means we can describe a polar rectangle as in Figure 2.3.1a, with $R = \{(r, \theta) \mid a \leq r \leq b, \ \alpha \leq \theta \leq \beta\}$.

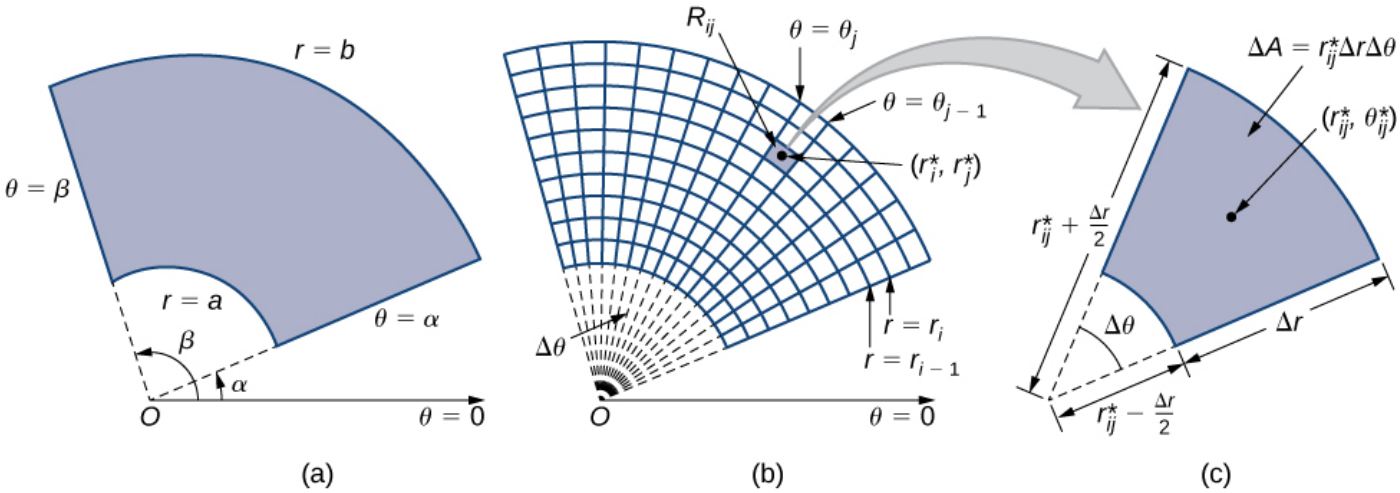


Figure 2.3.1: (a) A polar rectangle R. A sector of an annulus is shown in the polar coordinate plane with radii a and b and angles α and β from the $\theta = 0$ axis. (b) Rectangle R divided into subrectangles R_{ij}. This sector of an annulus is cut up into subsectors in a manner similar to the way in which previous spaces were cut up into subrectangles. (c) Close-up of a subrectangle. One of these subsectors is shown with angle $\Delta\theta$, distance between inner and outer radii Δr, and area $\Delta A = r^*_{ij}\Delta r\Delta\theta$, where the center point is given as $(r^*_{ij}, \theta^*_{ij})$.

In this section, we are looking to integrate over polar rectangles. Consider a function $f(r, \theta)$ over a polar rectangle R. We divide the interval $[a, b]$ into m subintervals $[r_{i-1}, r_i]$ of length $\Delta r = (b - a)/m$ and divide the interval $[\alpha, \beta]$ into n subintervals $[\theta_{j-1}, \theta_j]$ of width $\Delta\theta = (\beta - \alpha)/n$. This means that the circles $r = r_i$ and rays $\theta = \theta_j$ for $1 \leq i \leq m$ and $1 \leq j \leq n$ divide the polar rectangle R into smaller polar subrectangles R_{ij} (Figure 2.3.1b).

As before, we need to find the area ΔA of the polar subrectangle R_{ij} and the "polar" volume of the thin box above R_{ij}. Recall that, in a circle of radius r the length s of an arc subtended by a central angle of θ radians is $s = r\theta$. Notice that the polar rectangle R_{ij} looks a lot like a trapezoid with parallel sides $r_{i-1}\Delta\theta$ and $r_i\Delta\theta$ and with a width Δr. Hence the area of the polar subrectangle R_{ij} is

$$\Delta A = \frac{1}{2}\Delta r(r_{i-1}\Delta\theta + r_i\Delta\theta).$$

Simplifying and letting

$$r_{ij}^* = \frac{1}{2}(r_{i-1} + r_i)$$

we have $\Delta A = r_{ij}^* \Delta r \Delta\theta$.

Therefore, the polar volume of the thin box above R_{ij} (Figure 2.3.2) is

$$f(r_{ij}^*, \theta_{ij}^*)\Delta A = f(r_{ij}^*, \theta_{ij}^*) r_{ij}^* \Delta r \Delta\theta.$$

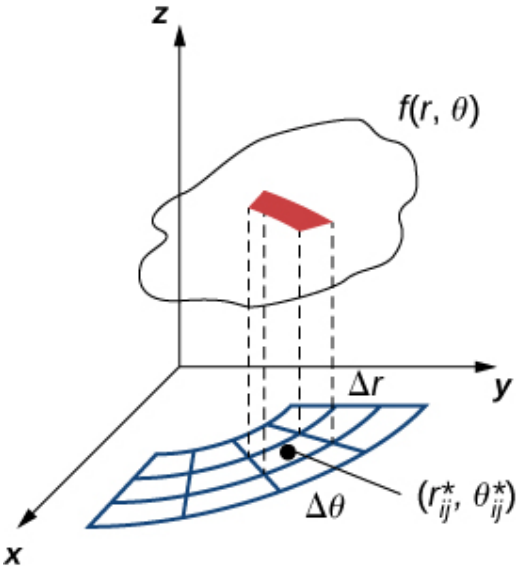


Figure 2.3.2: A surface $f(r, \theta)$. On the xy-plane, a series of subsectors of annuli are drawn as in the previous figure with radius between annuli Δr and angle between subsectors $\Delta\theta$. A subsector from the surface $f(r, \theta)$ is projected down onto one of these subsectors. This subsector has center point marked $(r_{ij}^*, \theta_{ij}^*)$.

Using the same idea for all the subrectangles and summing the volumes of the rectangular boxes, we obtain a double Riemann sum as

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^*, \theta_{ij}^*) r_{ij}^* \Delta r \Delta\theta.$$

As we have seen before, we obtain a better approximation to the polar volume of the solid above the region R when we let m and n become larger. Hence, we define the polar volume as the limit of the double Riemann sum,

$$V = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^*, \theta_{ij}^*) r_{ij}^* \Delta r \Delta\theta.$$

This becomes the expression for the double integral.

Definition: The Double Integral in Polar Coordinates

The double integral of the function $f(r, \theta)$ over the polar rectangular region R in the $r\theta$-plane is defined as

$$\iint_R f(r, \theta) dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^*, \theta_{ij}^*)\Delta A \tag{2.3.1}$$

$$= \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^*, \theta_{ij}^*) r_{ij}^* \Delta r \Delta\theta. \tag{2.3.2}$$

Again, just as in the section on Double Integrals over Rectangular Regions, the double integral over a polar rectangular region can be expressed as an iterated integral in polar coordinates. Hence,

$$\iint_R f(r,\theta)\,dA = \iint_R f(r,\theta)\,r\,dr\,d\theta = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=a}^{r=b} f(r,\theta)\,r\,dr\,d\theta. \tag{2.3.3}$$

Notice that the expression for dA is replaced by $r\ dr\ d\theta$ when working in polar coordinates. Another way to look at the polar double integral is to change the double integral in rectangular coordinates by substitution. When the function f is given in terms of x and y using $x = r\ \cos\ \theta,\ y = r\ \sin\ \theta$, and $dA = r\ dr\ d\theta$ changes it to

$$\iint_R f(x,y)\,dA = \iint_R f(r\ \cos\ \theta,\ r\ \sin\ \theta)\,r\,dr\,d\theta. \tag{2.3.4}$$

Note that all the properties listed in section on Double Integrals over Rectangular Regions for the double integral in rectangular coordinates hold true for the double integral in polar coordinates as well, so we can use them without hesitation.

Example 2.3.1A: Sketching a Polar Rectangular Region

Sketch the polar rectangular region

$$R = \{(r,\theta)\,|\,1 \le r \le 3, 0 \le \theta \le \pi\}.$$

Solution

As we can see from Figure 2.3.3, $r = 1$ and $r = 3$ are circles of radius 1 and 3 and $0 \le \theta \le \pi$ covers the entire top half of the plane. Hence the region R looks like a semicircular band.

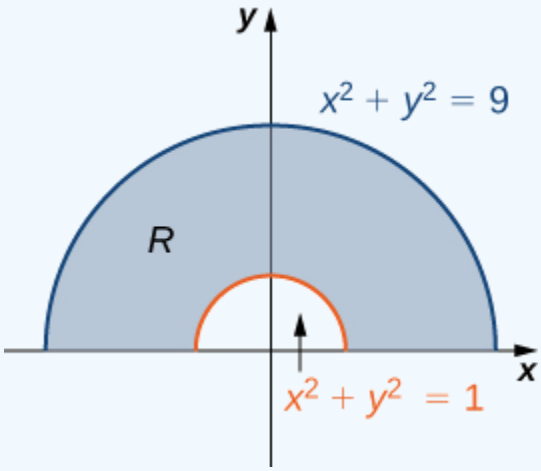


Figure 2.3.3: The polar region R lies between two semicircles. Half an annulus R is drawn with inner semicircle given by $x^2 + y^2 = 1$, whereas the outer semicircle is given by $x^2 + y^2 = 9$.

Now that we have sketched a polar rectangular region, let us demonstrate how to evaluate a double integral over this region by using polar coordinates.

Example 2.3.1B: Evaluating a Double Integral over a Polar Rectangular Region

Evaluate the integral $\iint_R 3x\ dA$ over the region $R = \{(r,\theta)\ |\ 1 \le r \le 2,\ 0 \le \theta \le \pi\}$.

Solution

First we sketch a figure similar to Figure 2.3.3, but with outer radius $r = 2$. From the figure we can see that we have

$$\iint_R 3x\,dA = \int_{\theta=0}^{\theta=\pi}\int_{r=1}^{r=2} 3r\cos\theta\, r\,dr\,d\theta \quad \text{Use an integral with correct limits of integration.}$$
$$= \int_{\theta=0}^{\theta=\pi} \cos\theta\left[r^3\Big|_{r=1}^{r=2}\right] d\theta \quad \text{Integrate first with respect to } r.$$
$$= \int_{\theta=0}^{\theta=\pi} 7\cos\theta\,d\theta$$
$$= 7\sin\theta\Big|_{\theta=0}^{\theta=\pi} = 0.$$

Exercise 2.3.1

Sketch the region $D = \{(r,\theta)|1 \le r \le 2,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\}$, and evaluate $\iint_R x\ dA$.

Hint available at the end of this section.

Answer available at the end of this section.

Example 2.3.2A: Evaluating a Double Integral by Converting from Rectangular Coordinates

Evaluate the integral

$$\iint_R (1-x^2-y^2)\,dA$$

where R is the unit circle on the xy-plane.

Solution

The region R is a unit circle, so we can describe it as $R = \{(r,\theta) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$.

Using the conversion $x = r\ \cos\ \theta,\ y = r\ \sin\ \theta$, and $dA = r\ dr\ d\theta$, we have

$$\iint_R (1-x^2-y^2)\,dA = \int_0^{2\pi}\int_0^1 (1-r^2)\,r\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^1 (r-r^3)\,dr\,d\theta$$
$$= \int_0^{2\pi}\left[\frac{r^2}{2}-\frac{r^4}{4}\right]_0^1 d\theta$$
$$= \int_0^{2\pi}\frac{1}{4}\,d\theta = \frac{\pi}{2}.$$

Watch the accompanying video for a visual demonstration of the concept.

- **Video length**: 3 minutes 40 seconds.
- **Context**: This video demonstrates how to evaluate a double integral by first converting to polar coordinates.

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Example $2.3.2B$: Evaluating a Double Integral by Converting from Rectangular Coordinates

Evaluate the integral

$$\iint_R (x+y)\,dA$$

where $R = \{(x,y) \mid 1 \le x^2+y^2 \le 4,\ x \le 0\}$.

Solution

We can see that R is an annular region that can be converted to polar coordinates and described as $R = \{(r,\theta) \mid 1 \le r \le 2,\ \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}\}$ (see Figure 2.3.4.

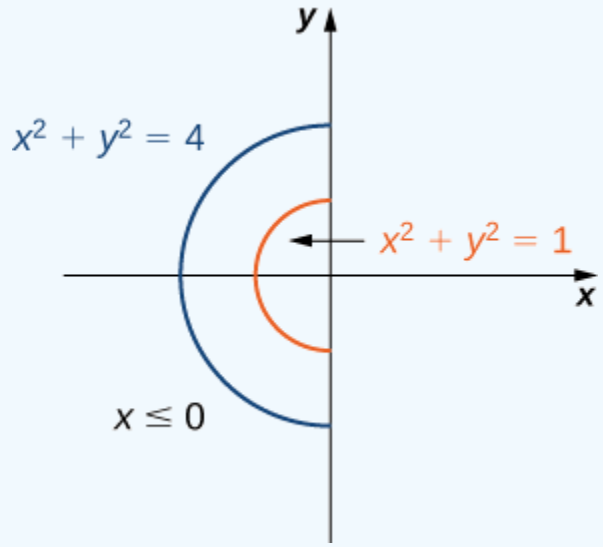


Figure $2.3.4$: The annular region of integration R. Two semicircles are drawn in the second and third quadrants, with equations $x^2+y^2=1$ and $x^2+y^2=4$.

Hence, using the conversion $x = r\ \cos\ \theta,\ y = r\ \sin\ \theta$, and $dA = r\ dr\ d\theta$, we have

$$\begin{aligned}\iint_R (x+y)\,dA &= \int_{\theta=\pi/2}^{\theta=3\pi/2}\int_{r=1}^{r=2}(r\ \cos\ \theta + r\ \sin\ \theta) r\,dr\,d\theta \\ &= \left(\int_{r=1}^{r=2} r^2\,dr\right)\left(\int_{\pi/2}^{3\pi/2}(\cos\ \theta + \sin\ \theta)\,d\theta\right) \\ &= \left[\frac{r^3}{3}\right]_1^2 [\sin\ \theta - \cos\ \theta]\Big|_{\pi/2}^{3\pi/2} \\ &= -\frac{14}{3}.\end{aligned}$$

Watch the accompanying video for a visual demonstration of the concept.

- **Video length**: 3 minutes 17 seconds.
- **Context**: This video demonstrates how to change a double integral from rectangular coordinates to polar coordinates.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

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Exercise 2.3.2

Evaluate the integral

$$\iint_R (4 - x^2 - y^2)\, dA$$

where R is the circle of radius 2 on the xy-plane.

Hint available at the end of this section.

Answer available at the end of this section.

General Polar Regions of Integration

To evaluate the double integral of a continuous function by iterated integrals over general polar regions, we consider two types of regions, analogous to Type I and Type II as discussed for rectangular coordinates in section on Double Integrals over General Regions. It is more common to write polar equations as $r = f(\theta)$ than $\theta = f(r)$, so we describe a general polar region as $R = \{(r, \theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$ (Figure 2.3.5).

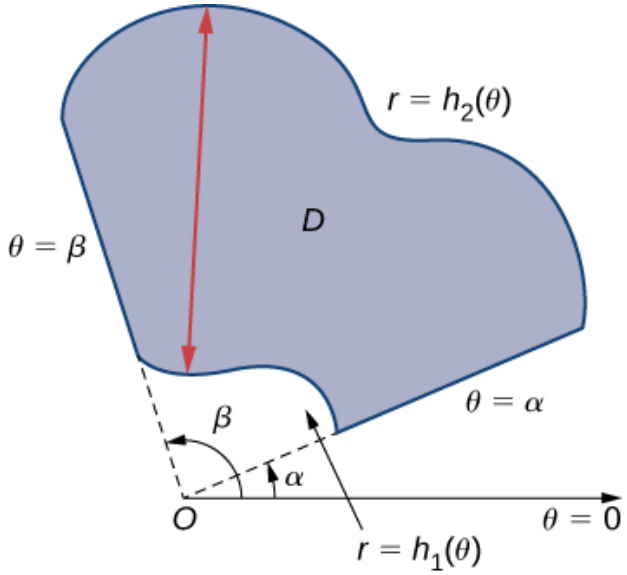


Figure 2.3.5: A general polar region D between $\alpha \le \theta \le \beta$ and $h_1(\theta) \le r \le h_2(\theta)$.

Theorem: Double Integrals over General Polar Regions

If $f(r, \theta)$ is continuous on a general polar region D as described above, then

$$\iint_D f(r, \theta)\, r\, dr\, d\theta = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} f(r, \theta)\, r\, dr\, d\theta. \tag{2.3.5}$$

Example 2.3.3: Evaluating a Double Integral over a General Polar Region

Evaluate the integral

$$\iint_D r^2 \sin\theta\, r\, dr\, d\theta$$

where D is the region bounded by the polar axis and the upper half of the cardioid $r = 1 + \cos\ \theta$.

Solution

We can describe the region D as $\{(r, \theta)\ |\ 0 \le \theta \le \pi,\ 0 \le r \le 1 + \cos\ \theta\}$ as shown in Figure 2.3.6.

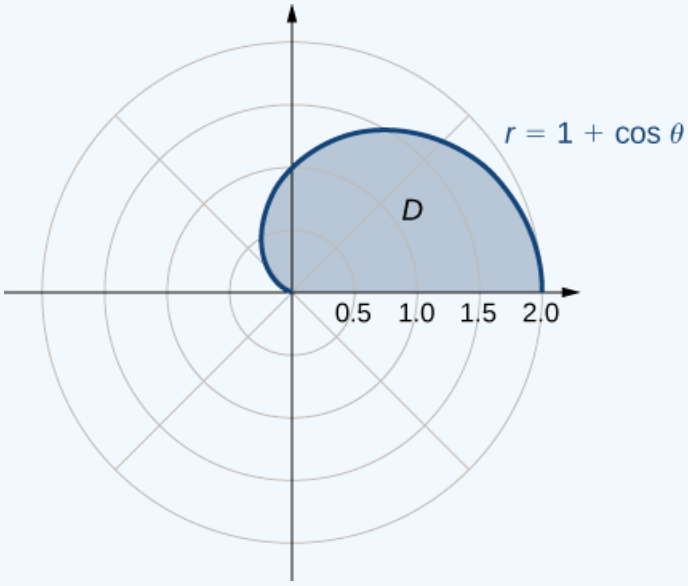


Figure 2.3.6: The region D is the top half of a cardioid $r = 1 + \cos\theta$.

Hence, we have

$$\begin{aligned}\iint_D r^2 \sin\ \theta\, r\, dr\, d\theta &= \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=1+\cos\theta} (r^2 \sin\ \theta)\, r\, dr\, d\theta \\ &= \frac{1}{4} \int_{\theta=0}^{\theta=\pi} [r^4]\Big|_{r=0}^{r=1+\cos\ \theta} \sin\ \theta\, d\theta \\ &= \frac{1}{4} \int_{\theta=0}^{\theta=\pi} (1 + \cos\ \theta)^4 \sin\ \theta\, d\theta \\ &= -\frac{1}{4}\left[\frac{(1 + \cos\ \theta)^5}{5}\right]_0^{\pi} = \frac{8}{5}.\end{aligned}$$

Exercise 2.3.3

Evaluate the integral

$$\iint_D r^2 \sin^2 2\theta\, r\, dr\, d\theta$$

where $D = \left\{(r, \theta)\,|\,0 \le \theta \le \pi,\, 0 \le r \le 2\sqrt{\cos\ 2\theta}\right\}$.

Hint available at the end of this section.

Answer available at the end of this section.

Polar Areas and Volumes

As in rectangular coordinates, if a solid S is bounded by the surface $z = f(r, \theta)$, as well as by the surfaces $r = a,\ r = b,\ \theta = \alpha$, and $\theta = \beta$, we can find the volume V of S by double integration, as

$$V = \iint_R f(r,\theta)\, r\, dr\, d\theta = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=a}^{r=b} f(r,\theta)\, r\, dr\, d\theta. \tag{2.3.6}$$

If the base of the solid can be described as $D = \{(r,\theta) | \alpha \le \theta \le \beta,\ \ h_1(\theta) \le r \le h_2(\theta)\}$, then the double integral for the volume becomes

$$V = \iint_D f(r,\theta)\, r\, dr\, d\theta = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} f(r,\theta)\, r\, dr\, d\theta. \tag{2.3.7}$$

We illustrate this idea with some examples.

Example 2.3.4A: Finding a Volume Using a Double Integral

Find the volume of the solid that lies under the paraboloid $z = 1 - x^2 - y^2$ and above the unit circle on the xy-plane (Figure 2.3.7).

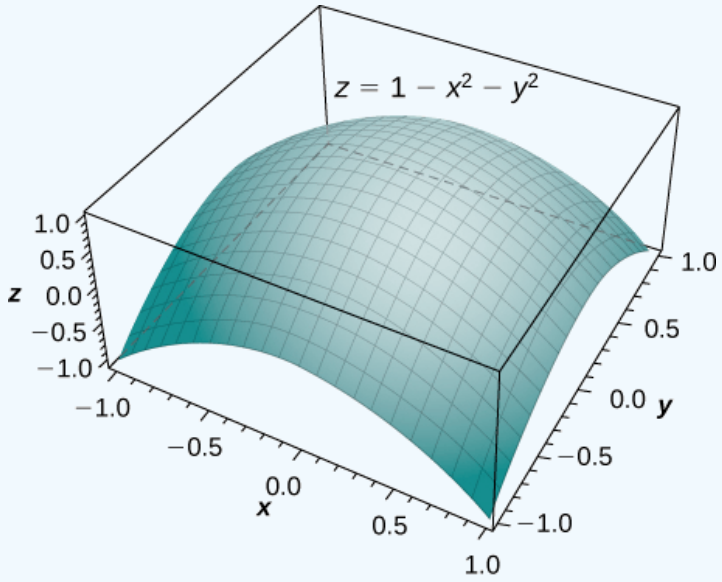


Figure 2.3.7: The paraboloid $z = 1 - x^2 - y^2$, which in this graph looks like a sheet with the middle gently puffed up and the corners anchored.

Solution

By the method of double integration, we can see that the volume is the iterated integral of the form

$$\iint_R (1 - x^2 - y^2)\, dA$$

where $R = \left\{(r,\theta) \mid 0 \le r \le 1,\ \ 0 \le \theta \le 2\pi\right\}$.

This integration was shown before in Example 2.3.2A, so the volume is $\frac{\pi}{2}$ cubic units.

Example 2.3.4B: Finding a Volume Using Double Integration

Find the volume of the solid that lies under the paraboloid $z = 4 - x^2 - y^2$ and above the disk $(x-1)^2 + y^2 = 1$ on the xy -plane. See the paraboloid intersecting the cylinder $(x-1)^2 + y^2 = 1$ above the xy-plane in Figure 2.3.8.

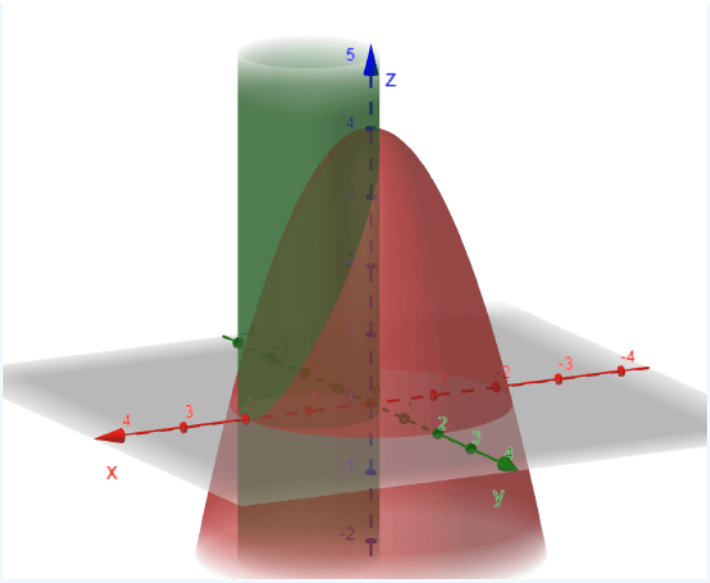


Figure 2.3.8: A paraboloid with equation $z = 4 - x^2 - y^2$ in red is intersected by a cylinder with equation $(x-1)^2 + y^2 = 1$ in green. The intersection region is above the xy-plane. (CC BY-NC-SA 3.0 modified by Vinh Kha Nguyen using Geogebra)

Solution

First change the disk $(x-1)^2 + y^2 = 1$ to polar coordinates. Expanding the square term, we have $x^2 - 2x + 1 + y^2 = 1$. Then simplify to get $x^2 + y^2 = 2x$, which in polar coordinates becomes $r^2 = 2r\ \cos\ \theta$ and then either $r = 0$ or $r = 2\ \cos\ \theta$. Similarly, the equation of the paraboloid changes to $z = 4 - r^2$. Therefore we can describe the disk $(x-1)^2 + y^2 = 1$ on the xy -plane as the region

$$D = \{(r, \theta) \,|\, 0 \le \theta \le \pi,\ 0 \le r \le 2\ \cos\theta\}.$$

Hence the volume of the solid bounded above by the paraboloid $z = 4 - x^2 - y^2$ and below by $r = 2\ \cos\theta$ is

$$\begin{aligned} V &= \iint_D f(r,\theta)\, r\, dr\, d\theta \\ &= \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=2\ \cos\ \theta} (4 - r^2)\, r\, dr\, d\theta \\ &= \int_{\theta=0}^{\theta=\pi} \left[4\frac{r^2}{2} - \frac{r^4}{4}\Big|_0^{2\ \cos\ \theta} \right] d\theta \\ &= \int_0^{\pi} [8\ \cos^2\theta - 4\ \cos^4\theta]\, d\theta \\ &= \left[\frac{5}{2}\theta + \frac{5}{2}\sin\ \theta\ \cos\ \theta - \sin\ \theta \cos^3\theta \right]_0^{\pi} = \frac{5}{2}\pi\ \text{units}^3. \end{aligned}$$

Notice in the next example that integration is not always easy with polar coordinates. Complexity of integration depends on the function and also on the region over which we need to perform the integration. If the region has a more natural expression in polar coordinates or if f has a simpler antiderivative in polar coordinates, then the change in polar coordinates is appropriate; otherwise, use rectangular coordinates.

Example 2.3.5A: Finding a Volume Using a Double Integral

Find the volume of the region that lies under the paraboloid $z = x^2 + y^2$ and above the triangle enclosed by the lines $y = x,\ x = 0$, and $x + y = 2$ in the xy-plane.

Solution

First examine the region over which we need to set up the double integral and the accompanying paraboloid.

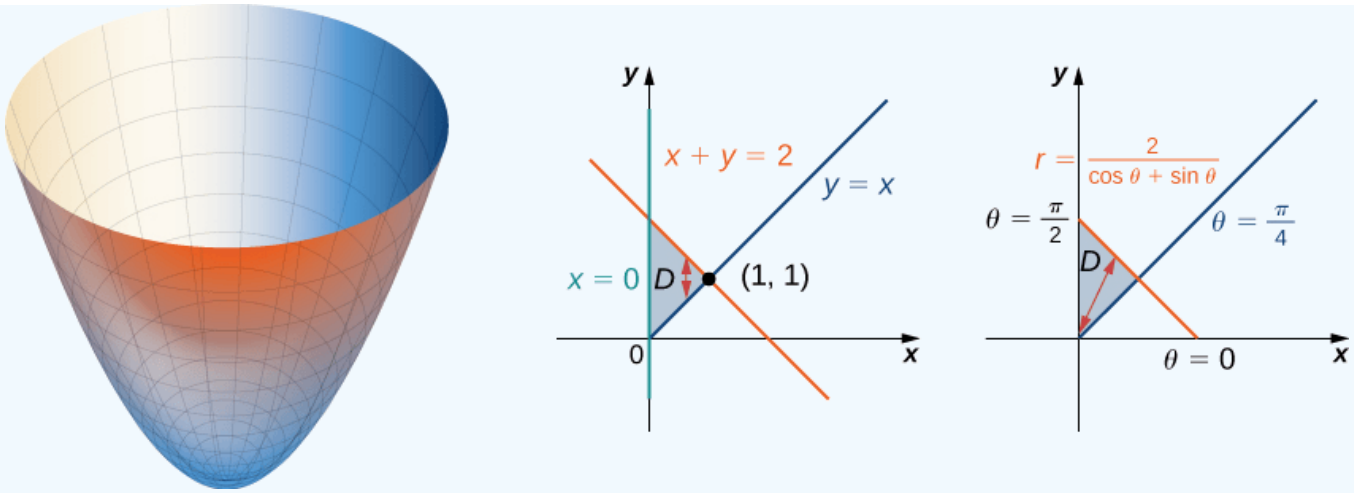


Figure 2.3.9: The first image from left to right is a paraboloid that opens up. The second shows the region D bounded by $9x = 0, y = x$, and $x + y = 2$ with a vertical double-sided arrow within the region. The third shows the same region but in polar coordinates, so the lines bounding D are $\theta = \frac{\pi}{2}$, $r = \frac{2}{\cos\theta + \sin\theta}$, and $\theta = \frac{\pi}{4}$, with a double-sided arrow that has one side pointed at the origin.

The region D is $\{(x, y) \mid 0 \le x \le 1,\ x \le y \le 2 - x\}$. Converting the lines $y = x,\ x = 0$, and $x + y = 2$ in the xy-plane to functions of r and θ we have $\theta = \pi/4,\ \theta = \pi/2$, and $r = 2/(\cos\ \theta + \sin\ \theta)$, respectively. Graphing the region on the xy- plane, we see that it looks like $D = \{(r, \theta) \mid \pi/4 \le \theta \le \pi/2,\ 0 \le r \le 2/(\cos\ \theta + \sin\ \theta)\}$.

Now converting the equation of the surface gives $z = x^2 + y^2 = r^2$. Therefore, the volume of the solid is given by the double integral

$$\begin{aligned} V &= \iint_D f(r, \theta)\, r\, dr\, d\theta \\ &= \int_{\theta=\pi/4}^{\theta=\pi/2} \int_{r=0}^{r=2/(\cos\ \theta+\sin\ \theta)} r^2 r\, dr d\theta \\ &= \int_{\pi/4}^{\pi/2} \left[\frac{r^4}{4}\right]_0^{2/(\cos\ \theta+\sin\ \theta)} d\theta \\ &= \frac{1}{4}\int_{\pi/4}^{\pi/2} \left(\frac{2}{\cos\ \theta + \sin\ \theta}\right)^4 d\theta \\ &= \frac{16}{4}\int_{\pi/4}^{\pi/2} \left(\frac{1}{\cos\ \theta + \sin\ \theta}\right)^4 d\theta \\ &= 4\int_{\pi/4}^{\pi/2} \left(\frac{1}{\cos\ \theta + \sin\ \theta}\right)^4 d\theta. \end{aligned}$$

As you can see, this integral is very complicated. So, we can instead evaluate this double integral in rectangular coordinates as

$$V = \int_0^1 \int_x^{2-x} (x^2 + y^2)\, dy\, dx.$$

Evaluating gives

$$
\begin{aligned}
V &= \int_0^1 \int_x^{2-x} (x^2 + y^2)\, dy\, dx \\
&= \int_0^1 \left[x^2 y + \frac{y^3}{3} \right] \Bigg|_x^{2-x} dx \\
&= \int_0^1 \frac{8}{3} - 4x + 4x^2 - \frac{8x^3}{3}\, dx \\
&= \left[\frac{8x}{3} - 2x^2 + \frac{4x^3}{3} - \frac{2x^4}{3} \right] \Bigg|_0^1 \\
&= \frac{4}{3} \text{ units}^3.
\end{aligned}
$$

To answer the question of how the formulas for the volumes of different standard solids such as a sphere, a cone, or a cylinder are found, we want to demonstrate an example and find the volume of an arbitrary cone.

Example 2.3.5B: Finding a Volume Using a Double Integral

Use polar coordinates to find the volume inside the cone $z = 2 - \sqrt{x^2 + y^2}$ and above the xy-plane.

Solution

The region D for the integration is the base of the cone, which appears to be a circle on the xy-plane (Figure 2.3.10).

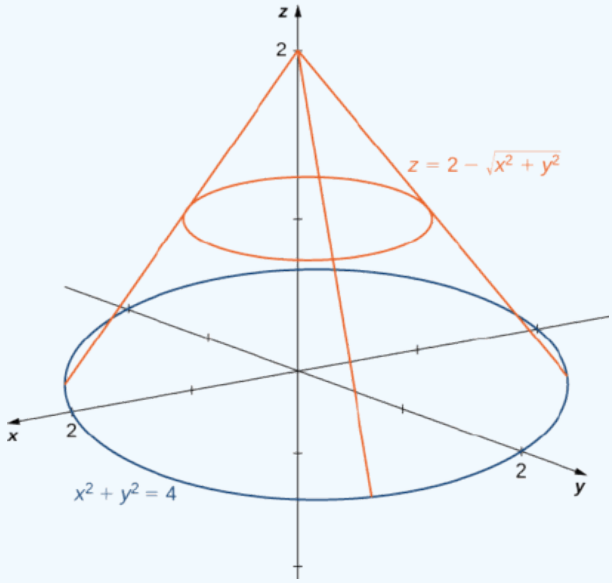


Figure 2.3.10: A cone given by $z = 2 - \sqrt{x^2 + y^2}$ and a circle given by $x^2 + y^2 = 4$. The cone is above the circle in 3d-space.

We find the equation of the circle by setting $z = 0$:

$$
\begin{aligned}
0 &= 2 - \sqrt{x^2 + y^2} \\
2 &= \sqrt{x^2 + y^2} \\
x^2 + y^2 &= 4.
\end{aligned}
$$

This means the radius of the circle is 2 so for the integration we have $0 \le \theta \le 2\pi$ and $0 \le r \le 2$. Substituting $x = r\ \cos\theta$ and $y = r\ \sin\ \theta$ in the equation $z = 2 - \sqrt{x^2 + y^2}$ we have $z = 2 - r$. Therefore, the volume of the cone is

$$
\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} (2 - r)\, r\, dr\, d\theta = 2\pi \frac{4}{3} = \frac{8\pi}{3} \text{ cubic units.}
$$

Analysis

Note that if we were to find the volume of an arbitrary cone with radius α units and height h units, then the equation of the cone would be $z = h - \frac{h}{a}\sqrt{x^2 + y^2}$.

We can still use Figure $2.3.10$ and set up the integral as

$$\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=a} \left(h - \frac{h}{a}r\right) r \, dr \, d\theta.$$

Evaluating the integral, we get $\frac{1}{3}\pi a^2 h$.

Exercise $2.3.5$

Use polar coordinates to find an iterated integral for finding the volume of the solid enclosed by the paraboloids $z = x^2 + y^2$ and $z = 16 - x^2 - y^2$.

Hint available at the end of this section.

Answer available at the end of this section.

As with rectangular coordinates, we can also use polar coordinates to find areas of certain regions using a double integral. As before, we need to understand the region whose area we want to compute. Sketching a graph and identifying the region can be helpful to realize the limits of integration. Generally, the area formula in double integration will look like

$$\text{Area of } A = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} 1 \, r \, dr \, d\theta. \tag{2.3.8}$$

Example $2.3.6A$: Finding an Area Using a Double Integral in Polar Coordinates

Evaluate the area bounded by the curve $r = \cos\ 4\theta$.

Solution

Sketching the graph of the function $r = \cos\ 4\theta$ reveals that it is a polar rose with eight petals (Figure $2.3.11$).

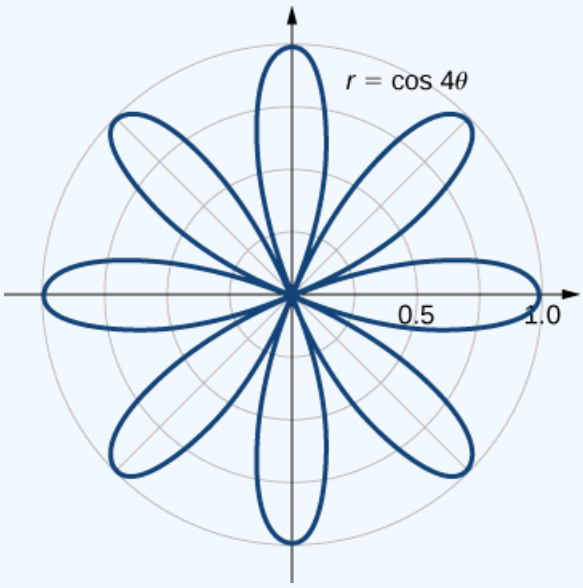


Figure $2.3.11$: A rose with eight petals given by polar equation $r = \cos 4\theta$. Each petal has a length of one unit from origin.

Using symmetry, we can see that we need to find the area of one petal and then multiply it by 8. Notice that the values of θ for which the graph passes through the origin are the zeros of the function $\cos\ 4\theta$, and these are odd multiples of $\pi/8$. Thus, one of the petals corresponds to the values of θ in the interval $[-\pi/8, \pi/8]$. Therefore, the area bounded by the curve $r = \cos\ 4\theta$ is

$$\begin{aligned} A &= 8\int_{\theta=-\pi/8}^{\theta=\pi/8}\int_{r=0}^{r=\cos\ 4\theta} 1\, r\, dr\, d\theta \\ &= 8\int_{\theta=-\pi/8}^{\theta=\pi/8}\left[\frac{1}{2}r^2\Big|_0^{\cos\ 4\theta}\right] d\theta \\ &= 8\int_{-\pi/8}^{\pi/8}\frac{1}{2}\cos^2 4\theta\, d\theta \\ &= 8\left[\frac{1}{4}\theta + \frac{1}{16}\sin\ 4\theta\ \cos\ 4\theta\Big|_{-\pi/8}^{\pi/8}\right] \\ &= 8\left[\frac{\pi}{16}\right] = \frac{\pi}{2}\ \text{units}^2. \end{aligned}$$

Example $2.3.6B$: Finding Area Between Two Polar Curves

Find the area enclosed by the circle $r = 3\ \cos\ \theta$ and the cardioid $r = 1 + \cos\ \theta$.

Solution

First and foremost, sketch the graphs of the region (Figure $2.3.12$).

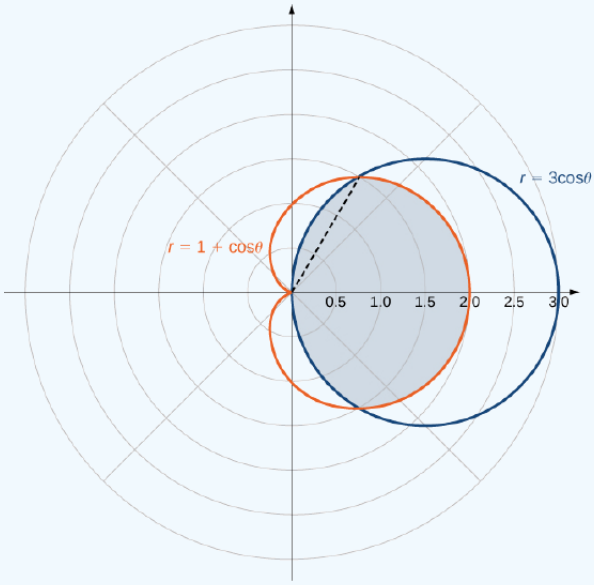


Figure $2.3.12$: Area enclosed by both a circle and a cardioid. A cardioid with equation $r = 1 + \cos\theta$ is shown overlapping a circle given by $r = 3\cos\theta$, which is a circle of radius 3 with center $(1.5, 0)$. The area bounded by the x-axis, the cardioid, and the dashed line connecting the origin to the intersection of the cardioid and circle on the $r = 2$ line is shaded.

We can from see the symmetry of the graph that we need to find the points of intersection. Setting the two equations equal to each other gives

$$3\ \cos\ \theta = 1 + \cos\ \theta.$$

One of the points of intersection is $\theta = \pi/3$. The area above the polar axis consists of two parts, with one part defined by the cardioid from $\theta = 0$ to $\theta = \pi/3$ and the other part defined by the circle from $\theta = \pi/3$ to $\theta = \pi/2$. By symmetry, the total area is twice the area above the polar axis. Thus, we have

$$A = 2\left[\int_{\theta=0}^{\theta=\pi/3}\int_{r=0}^{r=1+\cos\,\theta} 1\,r\,dr\,d\theta + \int_{\theta=\pi/3}^{\theta=\pi/2}\int_{r=0}^{r=3\,\cos\,\theta} 1\,r\,dr\,d\theta\right].$$

Evaluating each piece separately, we find that the area is

$$A = 2\left(\frac{1}{4}\pi + \frac{9}{16}\sqrt{3} + \frac{3}{8}\pi - \frac{9}{16}\sqrt{3}\right) = 2\left(\frac{5}{8}\pi\right) = \frac{5}{4}\pi\,\text{square units}.$$

Watch the accompanying video for a visual demonstration of the concept.

- **Video length**: 3 minutes 34 seconds.
- **Context**: This video demonstrates how to use a double integral to find the area of a region defined in polar coordinates.

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Exercise 2.3.6

Find the area enclosed inside the cardioid $r = 3 - 3\ \sin\theta$ and outside the cardioid $r = 1 + \sin\theta$.

Hint available at the end of this section.

Answer available at the end of this section.

Example 2.3.7: Evaluating an Improper Double Integral in Polar Coordinates

Evaluate the integral

$$\iint_{R^2} e^{-10(x^2+y^2)}\,dx\,dy.$$

Solution

This is an improper integral because we are integrating over an unbounded region R^2. In polar coordinates, the entire plane R^2 can be seen as $0 \le \theta \le 2\pi,\ 0 \le r \le \infty$.

Using the changes of variables from rectangular coordinates to polar coordinates, we have

$$\begin{aligned}
\iint_{R^2} e^{-10(x^2+y^2)}\,dx\,dy &= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=\infty} e^{-10r^2}\,r\,dr\,d\theta = \int_{\theta=0}^{\theta=2\pi}\left(\lim_{a\to\infty}\int_{r=0}^{r=a} e^{-10r^2} r\,dr\right)d\theta \\
&= \left(\int_{\theta=0}^{\theta=2\pi}\right)d\theta\left(\lim_{a\to\infty}\int_{r=0}^{r=a} e^{-10r^2} r\,dr\right) \\
&= 2\pi\left(\lim_{a\to\infty}\int_{r=0}^{r=a} e^{-10r^2} r\,dr\right) \\
&= 2\pi\lim_{a\to\infty}\left(-\frac{1}{20}\right)\left(e^{-10r^2}\Big|_0^a\right) \\
&= 2\pi\left(-\frac{1}{20}\right)\lim_{a\to\infty}\left(e^{-10a^2} - 1\right) \\
&= \frac{\pi}{10}.
\end{aligned}$$

Exercise 2.3.7

Evaluate the integral

$$\iint_{R^2} e^{-4(x^2+y^2)}dx\,dy.$$

Hint available at the end of this section.

Answer available at the end of this section.

Key Concepts

- To apply a double integral to a situation with circular symmetry, it is often convenient to use a double integral in polar coordinates. We can apply these double integrals over a polar rectangular region or a general polar region, using an iterated integral similar to those used with rectangular double integrals.
- The area dA in polar coordinates becomes $r\ dr\ d\theta$.
- Use $x = r\ \cos\ \theta,\ y = r\ \sin\ \theta$, and $dA = r\ dr\ d\theta$ to convert an integral in rectangular coordinates to an integral in polar coordinates.
- Use $r^2 = x^2 + y^2$ and $\theta = tan^{-1}\left(\frac{y}{x}\right)$ to convert an integral in polar coordinates to an integral in rectangular coordinates, if needed.
- To find the volume in polar coordinates bounded above by a surface $z = f(r, \theta)$ over a region on the xy-plane, use a double integral in polar coordinates.

Key Equations

- **Double integral over a polar rectangular region** R

$$\iint_R f(r,\theta)dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^*,\theta_{ij}^*)\Delta A = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^*,\theta_{ij}^*)r_{ij}^*\Delta r\Delta\theta$$

- **Double integral over a general polar region**

$$\iint_D f(r,\theta)\,r\,dr\,d\theta = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=h_1(\theta)}^{r_2(\theta)} f(r,\theta)\,r\,dr\,d\theta$$

Glossary

Polar rectangle
: the region enclosed between the circles $r = a$ and $r = b$ and the angles $\theta = \alpha$ and $\theta = \beta$; it is described as $R = \{(r,\theta) \mid a \le r \le b,\ \alpha \le \theta \le \beta\}$

Contributors and Attributions

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Answers

Exercise 2.3.1

Follow the steps in Example 2.3.1*A*.

Exercise 2.3.1

$\frac{14}{3}$

Exercise 2.3.2

Follow the steps in the previous example.

Exercise 2.3.2

8π

Exercise 2.3.3

Graph the region and follow the steps in the previous example.

Exercise 2.3.3

$\frac{\pi}{2}$

Exercise 2.3.5

Sketching the graphs can help.

Exercise 2.3.5

$$V = \int_0^{2\pi} \int_0^{2\sqrt{2}} (16 - 2r^2)\, r\, dr\, d\theta = 64\pi \text{ cubic units.}$$

Exercise 2.3.6

Sketch the graph, and solve for the points of intersection.

Exercise 2.3.6

$$A = 2 \int_{-\pi/2}^{\pi/6} \int_{1+\sin\,\theta}^{3-3\sin\,\theta} r\, dr\, d\theta = \left(8\pi + 9\sqrt{3}\right) \text{ units}^2$$

Exercise 2.3.7

Convert to the polar coordinate system.

Exercise 2.3.7

$\frac{\pi}{4}$

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2.3E: Exercises

Exercises 1 - 2

Follow the instruction in each question.

1. When evaluating $\int\int_R f(x,y)\ dA$ using polar coordinates, $f(x,y)$ is replaced with _____ and dA is replaced with _____.

   Answer

   $f(x,y)$ is replaced with $f(r\cos\theta, r\sin\theta)$ and dA is replaced with $r\ dr\ d\theta$.

2. Why would one be interested in evaluating a double integral with polar coordinates?

Exercises 3 - 9

Express the region R in polar coordinates.

3. R is the region of the disk of radius 2 centered at the origin that lies in the first quadrant.

   Answer

   $R = \{(r,\theta) \mid 0 \le r \le 2,\ 0 \le \theta \le \frac{\pi}{2}\}$

4. R is the region of the disk of radius 3 centered at the origin.
5. R is the region between the circles of radius 4 and radius 5 centered at the origin that lies in the second quadrant.

   Answer

   $R = \{(r,\theta) \mid 4 \le r \le 5,\ \frac{\pi}{2} \le \theta \le \pi\}$

6. R is the region bounded by the y-axis and $x = \sqrt{1-y^2}$.
7. R is the region bounded by the x-axis and $y = \sqrt{2-x^2}$.

   Answer

   $R = \{(r,\theta) \mid 0 \le r \le \sqrt{2},\ 0 \le \theta \le \pi\}$

8. $R = \{(x,y) \mid x^2+y^2 \le 4x\}$
9. $R = \{(x,y) \mid x^2+y^2 \le 4y\}$

   Answer

   $R = \{(r,\theta) \mid 0 \le r \le 4\ \sin\theta,\ 0 \le \theta \le \pi\}$

Exercises 10 - 15

The graph of the polar rectangular region D is given. Express D in polar coordinates.

10.

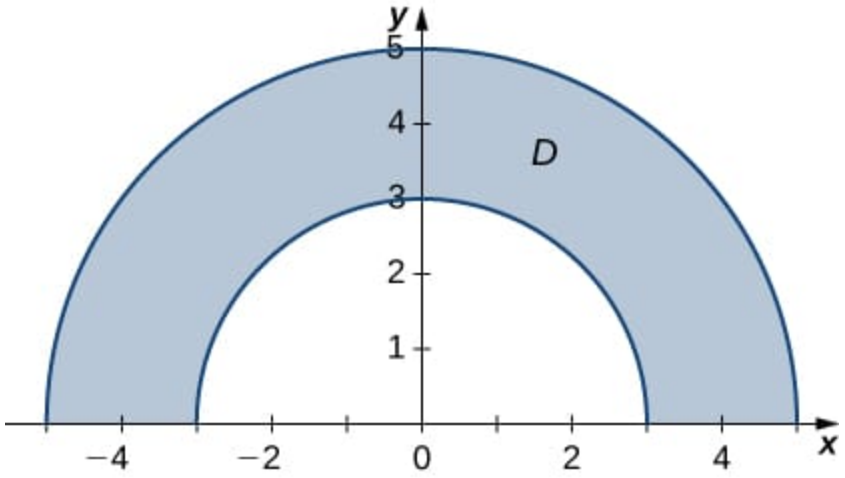


Figure $2.3E.1$: Half an annulus D is drawn in the first and second quadrants with inner radius 3 and outer radius 5.

11.

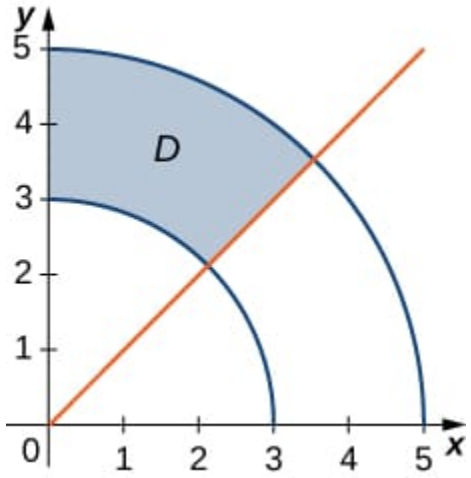


Figure $2.3E.2$: A sector of an annulus D is drawn between theta = pi/4 and theta = pi/2 with inner radius 3 and outer radius 5.

Answer

$$D = \left\{(r, \theta) \mid 3 \leq r \leq 5,\ \tfrac{\pi}{4} \leq \theta \leq \tfrac{\pi}{2}\right\}$$

12.

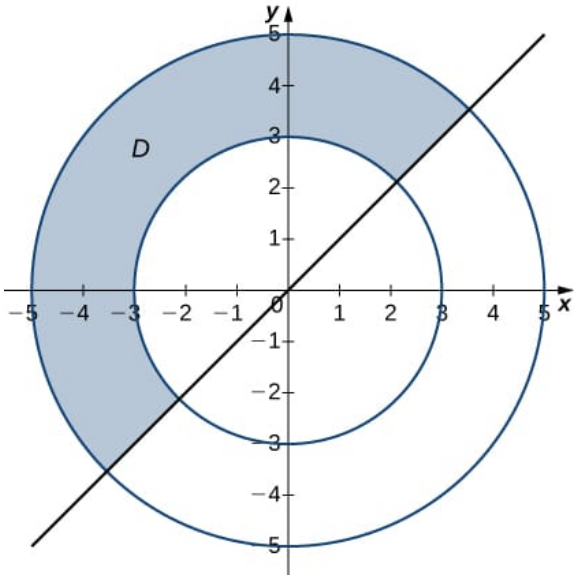


Figure $2.3E.3$: Half of an annulus D is drawn between \(\theta = \frac{\pi}{4}\) and \(\theta = \frac{5\pi}{4}\) with inner radius 3 and outer radius 5.

13.

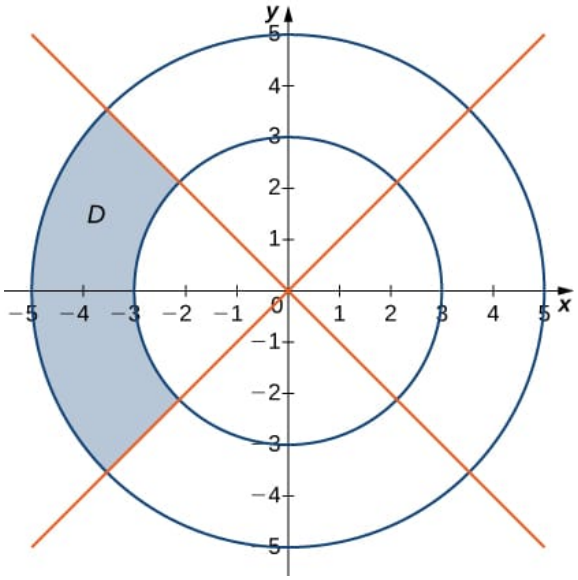


Figure $2.3E.4$: A sector of an annulus D is drawn between \(\theta = \frac{3\pi}{4}\) and \(\theta = \frac{5\pi}{4}\) with inner radius 3 and outer radius 5.

Answer

$$D = \left\{(r, \theta) \mid 3 \leq r \leq 5,\ \tfrac{3\pi}{4} \leq \theta \leq \tfrac{5\pi}{4}\right\}$$

14. In Figure $2.3E.5$, the region D is situated below $y = x$ and is bounded by $x = 1,\ x = 5$, and $y = 0$.

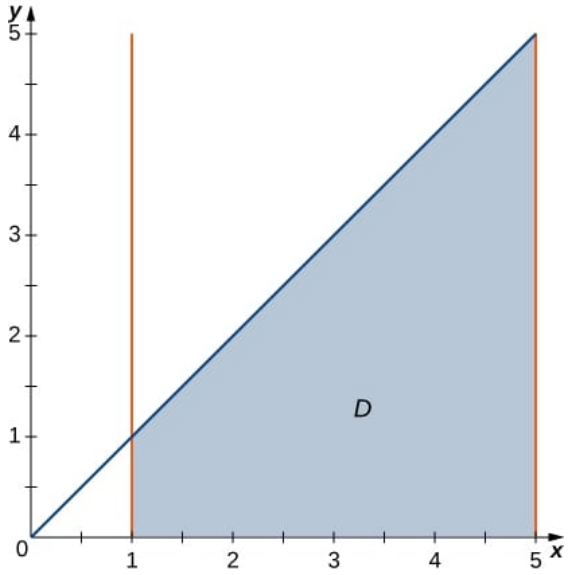


Figure $2.3E.5$: A region D is given that is bounded by $y = 0, x = 1, x = 5$, and $y = x$, that is, a right triangle with a corner cut off.

15. In Figure $2.3E.6$, the region D is bounded by $y = x$ and $y = x^2$.

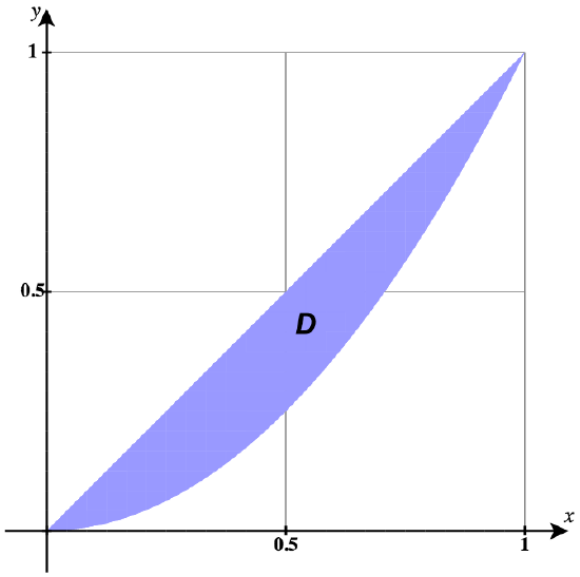


Figure $2.3E.6$: A region D is drawn between $y = x$ and $y = x^2$, which looks like a deformed lens, with the bulbous part below the straight part.

Answer

$$D = \left\{(r, \theta) \mid 0 \leq r \leq \tan\theta \ \sec\theta, \ 0 \leq \theta \leq \tfrac{\pi}{4}\right\}$$

Exercises 16 - 25

Evaluate the double integral $\iint_R f(x, y) \ dA$ over the polar rectangular region R.

16. $f(x, y) = x^2 + y^2$, $R = \left\{(r, \theta) \mid 3 \leq r \leq 5, \ 0 \leq \theta \leq 2\pi\right\}$
17. $f(x, y) = x + y$, $R = \left\{(r, \theta) \mid 3 \leq r \leq 5, \ 0 \leq \theta \leq 2\pi\right\}$

Answer

0

18. $f(x, y) = x^2 + xy, \ R = \left\{(r, \theta) \mid 1 \leq r \leq 2, \ \pi \leq \theta \leq 2\pi\right\}$

19. $f(x,y) = x^4 + y^4,\ R = \left\{(r,\theta) \mid 1 \le r \le 2,\ \frac{3\pi}{2} \le \theta \le 2\pi\right\}$

Answer

$\frac{63\pi}{16}$

20. $f(x,y) = \sqrt[3]{x^2+y^2}$, where $R = \left\{(r,\theta) \mid 0 \le r \le 1,\ \frac{\pi}{2} \le \theta \le \pi\right\}$.
21. $f(x,y) = x^4 + 2x^2y^2 + y^4$, where $R = \left\{(r,\theta) \mid 3 \le r \le 4,\ \frac{\pi}{3} \le \theta \le \frac{2\pi}{3}\right\}$.

Answer

$\frac{3367\pi}{18}$

22. $f(x,y) = \sin(\arctan\frac{y}{x})$, where $R = \left\{(r,\theta) \mid 1 \le r \le 2,\ \frac{\pi}{6} \le \theta \le \frac{\pi}{3}\right\}$
23. $f(x,y) = \arctan\left(\frac{y}{x}\right)$, where $R = \left\{(r,\theta) \mid 2 \le r \le 3,\ \frac{\pi}{4} \le \theta \le \frac{\pi}{3}\right\}$

Answer

$\frac{35\pi^2}{576}$

24. $\iint_R e^{x^2+y^2}\left[1 + 2\ \arctan\left(\frac{y}{x}\right)\right]\ dA,\ R = \left\{(r,\theta) \mid 1 \le r \le 2,\ \frac{\pi}{6} \le \theta \frac{\pi}{3}\right\}$
25. $\iint_R \left(e^{x^2+y^2} + x^4 + 2x^2y^2 + y^4\right)\arctan\left(\frac{y}{x}\right)\ dA,\ R = \left\{(r,\theta) \mid 1 \le r \le 2,\ \frac{\pi}{4} \le \theta \le \frac{\pi}{3}\right\}$

Answer

$\frac{7}{576}\pi^2(21 - e + e^4)$

Exercises 26 - 29

The integrals have been converted to polar coordinates. Verify that the identities are true and choose the easiest way to evaluate the integrals, in rectangular or polar coordinates.

26. $\int_1^2 \int_0^x (x^2+y^2)\ dy\ dx = \int_0^{\frac{\pi}{4}} \int_{\sec\theta}^{2\ \sec\theta} r^3\ dr\ d\theta$
27. $\int_2^3 \int_0^x \frac{x}{\sqrt{x^2+y^2}}\ dy\ dx = \int_0^{\pi/4} \int_{2\sec\theta}^{3\sec\theta} r\ \cos\theta\ dr\ d\theta$

Answer

$\frac{5}{2}\ln(1+\sqrt{2})$

28. $\int_0^1 \int_{x^2}^x \frac{1}{\sqrt{x^2+y^2}}\ dy\ dx = \int_0^{\pi/4} \int_0^{\tan\theta\ \sec\theta} dr\ d\theta$
29. $\int_0^1 \int_{x^2}^x \frac{y}{\sqrt{x^2+y^2}}\ dy\ dx = \int_0^{\pi/4} \int_0^{\tan\theta\ \sec\theta} r\ \sin\theta\ dr\ d\theta$

Answer

$\frac{1}{6}(2-\sqrt{2})$

Exercises 30 - 37

Draw the region of integration, R, labeling all limits of integration, convert the integrals to polar coordinates and evaluate them.

30. $\int_0^3 \int_0^{\sqrt{9-y^2}} \left(x^2+y^2\right)\ dx\ dy$
31. $\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \left(x^2+y^2\right)^2\ dx\ dy$

Answer

$$\int_0^{\pi}\int_0^{2} r^5 \; dr \; d\theta \quad = \quad \frac{32\pi}{3}$$

32. $\int_0^1 \int_0^{\sqrt{1-x^2}} (x+y) \; dy \; dx$
33. $\int_0^4 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \sin(x^2+y^2) \; dy \; dx$

Answer

$$\int_{-\pi/2}^{\pi/2}\int_0^{4} r \; \sin(r^2) \; dr \; d\theta \quad = \quad \pi \; \sin^2 8$$

34. $\int_0^5 \int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} \sqrt{x^2+y^2} \; dy \; dx$
35. $\int_{-4}^4 \int_{-\sqrt{16-y^2}}^{0} (2y-x) \; dx \; dy$

Answer

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\int_0^{4} \left(2r\sin\theta - r\cos\theta\right) \; r \; dr \; d\theta \quad = \quad \frac{128}{3}$$

36. $\int_0^2 \int_y^{\sqrt{8-y^2}} (x+y) \; dx \; dy$
37. $\int_{-2}^{-1} \int_0^{\sqrt{4-x^2}} (x+5) \; dy \; dx + \int_{-1}^{1} \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} (x+5) \; dy \; dx + \int_1^2 \int_0^{\sqrt{4-x^2}} (x+5) \; dy \; dx$

Exercises 38 - 49

Follow the instruction in each question.

Answer

$$\int_0^{\pi}\int_1^{2} \left(r\cos\theta + 5\right) \; r \; dr \; d\theta \quad = \quad \frac{15\pi}{2}$$

38. Evaluate the integral $\iint_D r \; dA$ where D is the region bounded by the polar axis and the upper half of the cardioid $r = 1 + \cos\theta$.
39. Find the area of the region D bounded by the polar axis and the upper half of the cardioid $r = 1 + \cos\theta$.

Answer

$\frac{3\pi}{4}$

40. Evaluate the integral $\iint_D r \; dA$, where D is the region bounded by the part of the four-leaved rose $r = \sin 2\theta$ situated in the first quadrant (see Figure $2.3E.7$).

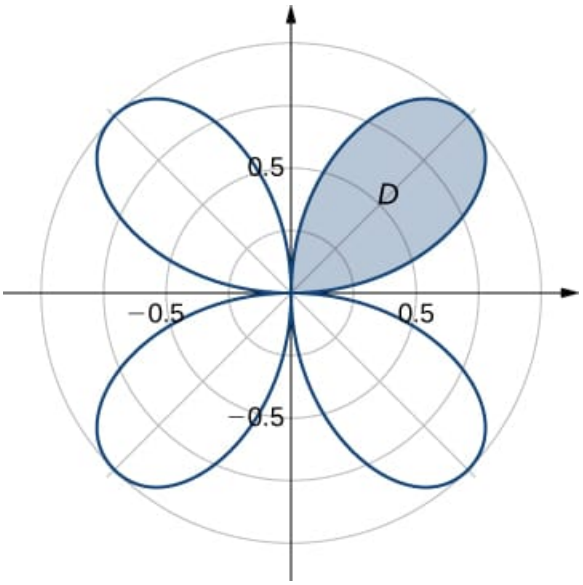


Figure $2.3E.7$: A region D is drawn in the first quadrant petal of the four petal rose given by $r = \sin 2\theta$.

41. Find the total area of the region enclosed by the four-leaved rose $r = \sin 2\theta$ (see the figure in the previous exercise).

Answer

$\frac{\pi}{2}$

42. Find the area of the region D which is the region bounded by $y = \sqrt{4 - x^2}$, $x = \sqrt{3}$, $x = 2$, and $y = 0$.
43. Find the area of the region D, which is the region inside the disk $x^2 + y^2 \leq 4$ and to the right of the line $x = 1$.

Answer

$\frac{1}{3}(4\pi - 3\sqrt{3})$

44. Determine the average value of the function $f(x, y) = x^2 + y^2$ over the region D bounded by the polar curve $r = \cos 2\theta$, where $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$ (see Figure $2.3E.8$).

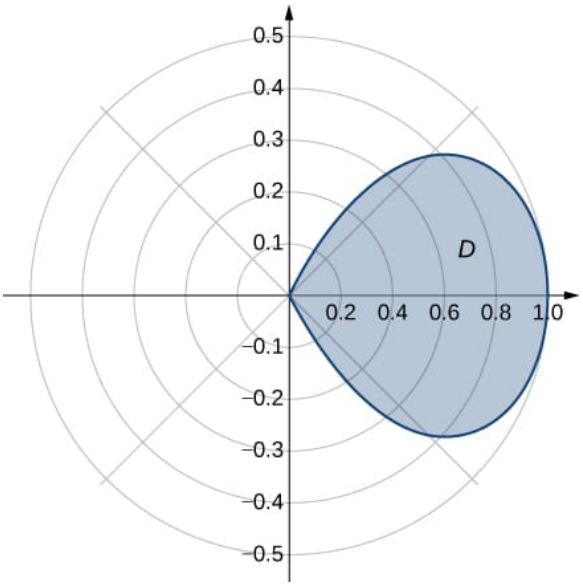


Figure $2.3E.8$: The first and fourth-quadrant petal of the four-petal rose given by $r = \cos 2\theta$.

45. Determine the average value of the function $f(x, y) = \sqrt{x^2 + y^2}$ over the region D bounded by the polar curve $r = 3 \sin 2\theta$, where $0 \leq \theta \leq \frac{\pi}{2}$ (see Figure $2.3E.9$).

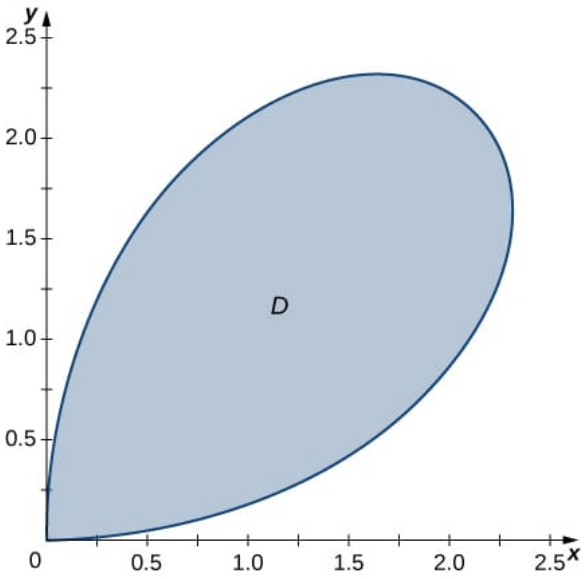


Figure $2.3E.9$: The first-quadrant petal of the four-petal rose given by $r = 3\sin 2\theta$.

Answer

$\frac{16}{3\pi}$

16. Find the volume of the solid situated in the first octant and bounded by the paraboloid $z = 1 - 4x^2 - 4y^2$ and the planes $x = 0,\ y = 0$, and $z = 0$.
17. Find the volume of the solid bounded by the paraboloid $z = 2 - 9x^2 - 9y^2$ and the plane $z = 1$.

Answer

$\frac{\pi}{18}$

18. a. Find the volume of the solid S_1 bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = 0$ and $z = 1$.
    b. Find the volume of the solid S_2 outside the double cone $z^2 = x^2 + y^2$ inside the cylinder $x^2 + y^2 = 1$, and above the plane $z = 0$.
    c. Find the volume of the solid inside the cone $z^2 = x^2 + y^2$ and below the plane $z = 1$ by subtracting the volumes of the solids S_1 and S_2.
19. a. Find the volume of the solid S_1 inside the unit sphere $x^2 + y^2 + z^2 = 1$ and above the plane $z = 0$.
    b. Find the volume of the solid S_2 inside the double cone $(z-1)^2 = x^2 + y^2$ and above the plane $z = 0$.
    c. Find the volume of the solid outside the double cone $(z-1)^2 = x^2 + y^2$ and inside the sphere $x^2 + y^2 + z^2 = 1$.

Answer

a. $\frac{2\pi}{3}$; b. $\frac{\pi}{3}$; c. $\frac{\pi}{3}$

Exercises 50 - 51

Special double integrals are presented that are especially well suited for evaluation in polar coordinates.

50. The surface of a right circular cone with height h and base radius a can be described by the equation $f(x,y) = h - h\sqrt{\frac{x^2}{a^2} + \frac{y^2}{a^2}}$, where the tip of the cone lies at $(0,0,h)$ and the circular base lies in the xy-plane, centered at the origin.

Confirm that the volume of a right circular cone with height h and base radius a is $V = \frac{1}{3}\pi a^2 h$ by evaluating $\int\int_R f(x,y)\ dA$ in polar coordinates.

51. Consider $\int\int_R e^{-(x^2+y^2)}\ dA$.
    a. Why is this integral difficult to evaluate in rectangular coordinates, regardless of the region R?
    b. Let R be the region bounded by the circle of radius a centered at the origin. Evaluate the double integral using polar coordinates.

c. Take the limit of your answer from (b), as $a \to \infty$. What does this imply about the volume under the surface of $e^{-(x^2+y^2)}$ over the entire xy-plane?

Exercises 52 - 53

Consider a spherical ring, which is a sphere with a cylindrical hole cut so that the axis of the cylinder passes through the center of the sphere (see Figure $2.3E.10$).



Figure $2.3E.10$: A sphere with a cylindrical hole going all the way through it. It looks like a spherical ring.

52. If the sphere has radius 4 and the cylinder has radius 2 find the volume of the spherical ring.
53. A cylindrical hole of diameter 6 cm is bored through a sphere of radius 5 cm such that the axis of the cylinder passes through the center of the sphere. Find the volume of the resulting spherical ring.

Answer

$\frac{256\pi}{3}\ \text{cm}^3$

54. Find the volume of the solid that lies under the double cone $z^2 = 4x^2 + 4y^2$, inside the cylinder $x^2 + y^2 = x$, and above the plane $z = 0$.
55. Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, inside the cylinder $x^2 + y^2 = 1$ and above the plane $z = 0$.

Answer

$\frac{3\pi}{32}$

56. Find the volume of the solid that lies under the plane $x + y + z = 10$ and above the disk $x^2 + y^2 = 4x$.
57. Find the volume of the solid that lies under the plane $2x + y + 2z = 8$ and above the unit disk $x^2 + y^2 = 1$.

Answer

4π

58. A radial function f is a function whose value at each point depends only on the distance between that point and the origin of the system of coordinates; that is, $f(x, y) = g(r)$, where $r = \sqrt{x^2 + y^2}$. Show that if f is a continuous radial function, then

$$\iint_D f(x,y)dA = (\theta_2 - \theta_1)[G(R_2) - G(R_1)],\ where\ G'(r) = rg(r)$$

and $(x, y) \in D = \{(r, \theta) \mid R_1 \le r \le R_2,\ 0 \le \theta \le 2\pi\}$, with $0 \le R_1 < R_2$ and $0 \le \theta_1 < \theta_2 \le 2\pi$.

59. Use the information from the preceding exercise to calculate the integral $\iint_D (x^2 + y^2)^3 dA$, where D is the unit disk.

Answer

$\frac{\pi}{4}$

60. Let $f(x,y) = \dfrac{F'(r)}{r}$ be a continuous radial function defined on the annular region $D = \{(r,\theta) \mid R_1 \le r \le R_2,\ 0 \le \theta \le 2\pi\}$, where $r = \sqrt{x^2+y^2}$, $0 < R_1 < R_2$, and F is a differentiable function.

Show that $\iint_D f(x,y)\ dA = 2\pi[F(R_2) - F(R_1)]$.

61. Apply the preceding exercise to calculate the integral $\iint_D \dfrac{e^{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}\ dx\ dy$ where D is the annular region between the circles of radii 1 and 2 situated in the third quadrant.

Answer

$\frac{1}{2}\pi e(e-1)$

62. Let f be a continuous function that can be expressed in polar coordinates as a function of θ only; that is, $f(x,y) = h(\theta)$, where $(x,y) \in D = \{(r,\theta) \mid R_1 \le r \le R_2,\ \theta_1 \le \theta \le \theta_2\}$, with $0 \le R_1 < R_2$ and $0 \le \theta_1 < \theta_2 \le 2\pi$.

Show that $\iint_D f(x,y)\ dA = \dfrac{1}{2}(R_2^2 - R_1^2)[H(\theta_2) - H(\theta_1)]$, where H is an antiderivative of h.

63. Apply the preceding exercise to calculate the integral $\iint_D \dfrac{y^2}{x^2}\ dA$, where $D = \{(r,\theta) \mid 1 \le r \le 2,\ \frac{\pi}{6} \le \theta \le \frac{\pi}{3}\}$.

Answer

$\sqrt{3} - \frac{\pi}{4}$

64. Let f be a continuous function that can be expressed in polar coordinates as a function of θ only; that is $f(x,y) = g(r)h(\theta)$, where $(x,y) \in \{(r,\theta) \mid R_1 \le r \le R_2,\ \theta_1 \le \theta \le \theta_2\}$ with $0 \le R_1 < R_2$ and $0 \le \theta_1 < \theta_2 \le 2\pi$. Show that

$$\iint_D f(x,y)\,dA = [G(R_2) - G(R_1)]\ [H(\theta_2) - H(\theta_1)],$$

where G and H are antiderivatives of g and h, respectively.

65. Evaluate $\iint_D \arctan\left(\dfrac{y}{x}\right)\sqrt{x^2+y^2}\ dA$, where $D = \{(r,\theta) \mid 2 \le r \le 3,\ \frac{\pi}{4} \le \theta \le \frac{\pi}{3}\}$.

Answer

$\frac{133}{864}\pi^2$

66. A spherical cap is the region of a sphere that lies above or below a given plane.
    a. Show that the volume of the spherical cap in Figure $2.3E.11$ is $\frac{1}{6}\pi h(3a^2 + h^2)$.

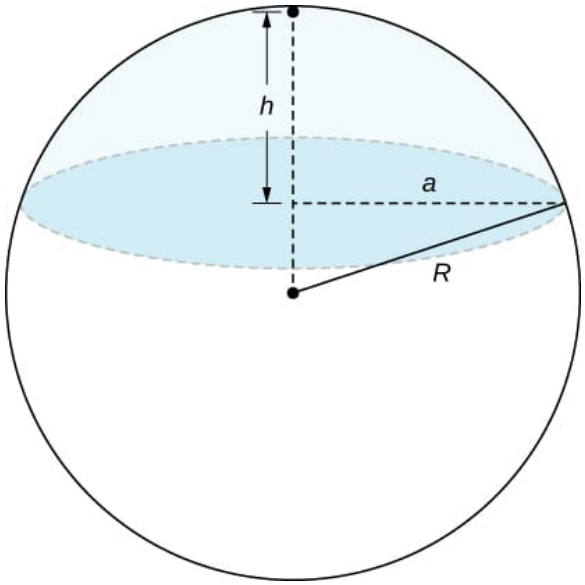


Figure $2.3E.11$: A sphere of radius R has a circle inside of it h units from the top of the sphere. This circle has radius a, which is less than R.

b. A spherical segment is the solid defined by intersecting a sphere with two parallel planes. If the distance between the planes is h show that the volume of the spherical segment in Figure $2.3E.12$ is $\frac{1}{6}\pi h(3a^2+3b^2+h^2)$.

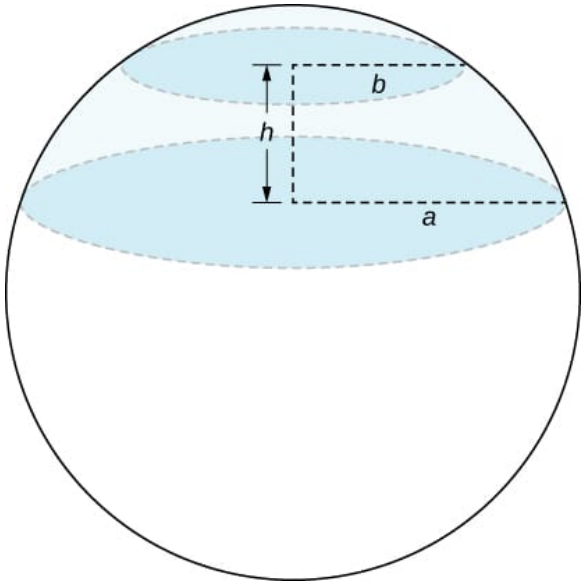


Figure $2.3E.12$: A sphere has two parallel circles inside of it h units apart. The upper circle has radius b, and the lower circle has radius a. Note that a > b.

57. In statistics, the joint density for two independent, normally distributed events with a mean $\mu = 0$ and a standard distribution σ is defined by $p(x,y) = \frac{1}{2\pi\sigma^2}e^{-\frac{x^2+y^2}{2\sigma^2}}$. Consider (X,Y), the Cartesian coordinates of a ball in the resting position after it was released from a position on the *z*-axis toward the xy-plane. Assume that the coordinates of the ball are independently normally distributed with a mean $\mu = 0$ and a standard deviation of σ (in feet). The probability that the ball will stop no more than a feet from the origin is given by

$$P[X^2+Y^2 \le a^2] = \iint_D p(x,y)dy\ dx,$$

where D is the disk of radius a centered at the origin. Show that

$$P[X^2+Y^2 \le a^2] = 1 - e^{-a^2/2\sigma^2}.$$

58. The double improper integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2+y^2/2}\,dy\,dx$$

may be defined as the limit value of the double integrals $\iint_D e^{-x^2+y^2/2}\ dA$ over disks D_a of radii a centered at the origin, as a increases without bound; that is,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2+y^2/2}dy\ dx = \lim_{a\to\infty}\iint_{D_a} e^{-x^2+y^2/2}\,dA.$$

Use polar coordinates to show that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2+y^2/2}\ dy\ dx = 2\pi.$

59. Show that $\int_{-\infty}^{\infty} e^{-x^2/2}\ dx = \sqrt{2\pi}$ by using the relation

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2+y^2/2}\,dy\,dx = \left(\int_{-\infty}^{\infty} e^{-x^2/2}dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2/2}dy\right).$$

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- Problems 1, 2, 34 - 37 and 50 - 51 are from Apex Calculus, Chapter 13.3
- Edited by Paul Seeburger (Monroe Community College)

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2.4: Triple Integrals

Learning Objectives

- Recognize when a function of three variables is integrable over a rectangular box.
- Evaluate a triple integral by expressing it as an iterated integral.
- Recognize when a function of three variables is integrable over a closed and bounded region.
- Simplify a calculation by changing the order of integration of a triple integral.
- Calculate the average value of a function of three variables.

Previously, we discussed the double integral of a function $f(x, y)$ of two variables over a rectangular region in the plane. In this section we define the triple integral of a function $f(x, y, z)$ of three variables over a rectangular solid box in space, $\mathbb{R}^3$. Later in this section we extend the definition to more general regions in $\mathbb{R}^3$.

Integrable Functions of Three Variables

We can define a rectangular box B in $\mathbb{R}^3$ as

$$B = \{(x, y, z) \,|\, a \leq x \leq b,\, c \leq y \leq d,\, e \leq z \leq f\}. \tag{2.4.1}$$

We follow a similar procedure to what we did previously. We divide the interval $[a, b]$ into l subintervals $[x_{i-1}, x_i]$ of equal length Δx with

$$\Delta x = \frac{x_i - x_{i-1}}{l}, \tag{2.4.2}$$

divide the interval $[c, d]$ into m subintervals $[y_{j-1}, y_j]$ of equal length Δy with

$$\Delta y = \frac{y_j - y_{j-1}}{m}, \tag{2.4.3}$$

and divide the interval $[e, f]$ into n subintervals $[z_{k-1}, z_k]$ of equal length Δz with

$$\Delta z = \frac{z_k - z_{k-1}}{n} \tag{2.4.4}$$

Then the rectangular box B is subdivided into lmn subboxes:

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k], \tag{2.4.5}$$

Figure 2.4.1 illustrates the subboxes.

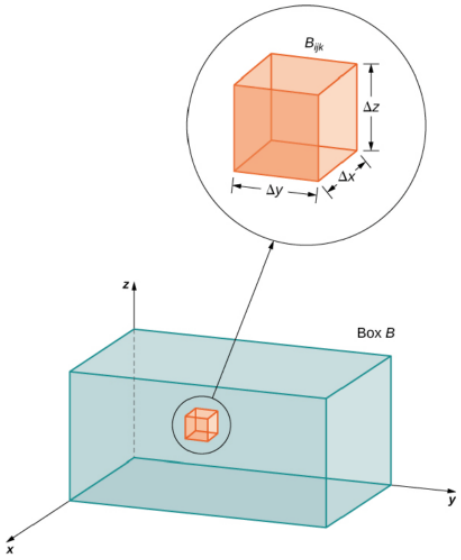


Figure 2.4.1: The subdivision of a rectangular box into smaller sub-boxes using planes parallel to the coordinate planes. For each i, j, and k, consider a sample point $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ in each sub-box B_{ijk}. We see that its volume is $\Delta V = \Delta x \Delta y \Delta z$. Form the triple Riemann sum.

$$\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\,\Delta x \Delta y \Delta z. \tag{2.4.6}$$

We define the triple integral in terms of the limit of a triple Riemann sum, as we did for the double integral in terms of a double Riemann sum.

Definition: The Triple Integral

The triple integral of a function $f(x, y, z)$ over a rectangular box B is defined as

$$\lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\,\Delta x \Delta y \Delta z = \iiint_B f(x, y, z)\,dV \tag{2.4.7}$$

if this limit exists.

When the triple integral exists on B the function $f(x, y, z)$ is said to be integrable on B. Also, the triple integral exists if $f(x, y, z)$ is continuous on B. Therefore, we will use continuous functions for our examples. However, continuity is sufficient but not necessary; in other words, f is bounded on B and continuous except possibly on the boundary of B. The sample point $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ can be any point in the rectangular sub-box B_{ijk} and all the properties of a double integral apply to a triple integral. Just as the double integral has many practical applications, the triple integral also has many applications, which we discuss in later sections.

Now that we have developed the concept of the triple integral, we need to know how to compute it. Just as in the case of the double integral, we can have an iterated triple integral, and consequently, a version of Fubini's Theorem for triple integrals exists.

Fubini's Theorem for Triple Integrals

If $f(x, y, z)$ is continuous on a rectangular box $B = [a, b] \times [c, d] \times [e, f]$, then

$$\iiint_B f(x, y, z)\,dV = \int_e^f \int_c^d \int_a^b f(x, y, z)\,dx\,dy\,dz. \tag{2.4.8}$$

This integral is also equal to any of the other five possible orderings for the iterated triple integral.

For a, b, c, d, e and f real numbers, the iterated triple integral can be expressed in six different orderings:

$$\int_e^f \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz = \int_e^f \left(\int_c^d \left(\int_a^b f(x,y,z)\,dx \right) dy \right) dz \tag{2.4.9}$$

$$= \int_c^d \left(\int_e^f \left(\int_a^b f(x,y,z)\,dx \right) dz \right) dy \tag{2.4.10}$$

$$= \int_a^b \left(\int_e^f \left(\int_c^d f(x,y,z)\,dy \right) dz \right) dx \tag{2.4.11}$$

$$= \int_e^f \left(\int_a^b \left(\int_c^d f(x,y,z)\,dy \right) dx \right) dz \tag{2.4.12}$$

$$= \int_c^d \left(\int_a^b \left(\int_e^f f(x,y,z)\,dz \right) dx \right) dy \tag{2.4.13}$$

$$= \int_a^b \left(\int_c^d \left(\int_e^f f(x,y,z)\,dz \right) dy \right) dx \tag{2.4.14}$$

For a rectangular box, the order of integration does not make any significant difference in the level of difficulty in computation. We compute triple integrals using Fubini's Theorem rather than using the Riemann sum definition. We follow the order of integration in the same way as we did for double integrals (that is, from inside to outside).

Example 2.4.1: Evaluating a Triple Integral

Evaluate the triple integral

$$\int_{z=0}^{z=1} \int_{y=2}^{y=4} \int_{x=-1}^{x=5} (x + yz^2)\,dx\,dy\,dz.$$

Solution

The order of integration is specified in the problem, so integrate with respect to x first, then y, and then z.

$$\begin{aligned}
&\int_{z=0}^{z=1} \int_{y=2}^{y=4} \int_{x=-1}^{x=5} (x + yz^2)\,dx\,dy\,dz \\
&= \int_{z=0}^{z=1} \int_{y=2}^{y=4} \left[\frac{x^2}{2} + xyz^2 \Big|_{x=-1}^{x=5} \right] dy\,dz \\
&= \int_{z=0}^{z=1} \int_{y=2}^{y=4} \left[12 + 6yz^2 \right] dy\,dz \\
&= \int_{z=0}^{z=1} \left[12y + 6\frac{y^2}{2}z^2 \Big|_{y=2}^{y=4} \right] dz \\
&= \int_{z=0}^{z=1} \left[24 + 36z^2 \right] dz \\
&= \left[24z + 36\frac{z^3}{3} \right]_{z=0}^{z=1} \\
&= 36.
\end{aligned}$$

Example 2.4.2: Evaluating a Triple Integral

Evaluate the triple integral

$$\iiint_B x^2yz\,dV \tag{2.4.15}$$

where $B = \{(x,y,z) \mid -2 \le x \le 1,\ 0 \le y \le 3,\ 1 \le z \le 5\}$ as shown in Figure 2.4.2.

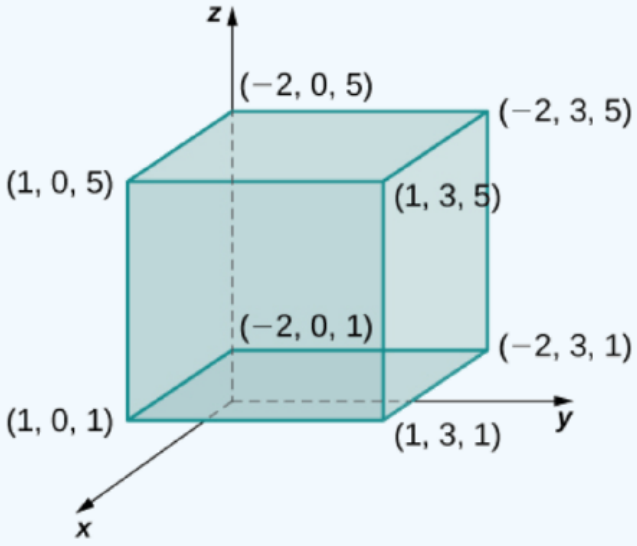


Figure 2.4.2: A rectangular box in 3D space with labeled vertices and dimensions used to evaluate a triple integral.

Solution

The order is not specified, but we can use the iterated integral in any order without changing the level of difficulty. Choose, say, to integrate y first, then x, and then z.

$$\begin{aligned}\iiint_B x^2yz\,dV &= \int_1^5\int_{-2}^1\int_0^3 [x^2yz]\,dy\,dx\,dz \\ &= \int_1^5\int_{-2}^1 \left[x^2\frac{y^2}{2}z\Big|_0^3\right] dx\,dz \\ &= \int_1^5\int_{-2}^1 \frac{y}{2}x^2z\,dx\,dz \\ &= \int_1^5 \left[\frac{9}{2}\frac{x^3}{3}z\Big|_{-2}^1\right] dz = \int_1^5 \frac{27}{2}z\,dz \\ &= \frac{27}{2}\frac{z^2}{2}\Big|_1^5 = 162.\end{aligned}$$

Now try to integrate in a different order just to see that we get the same answer. Choose to integrate with respect to x first, then z, then y

$$\begin{aligned}\iiint_B x^2yz\,dV &= \int_0^3\int_1^5\int_{-2}^1 [x^2yz]\,dx\,dz\,dy \\ &= \int_0^3\int_1^5 \left[\frac{x^3}{3}yz\Big|_{-2}^1\right] dz\,dy \\ &= \int_0^3\int_1^5 3yz\,dz\,dy \\ &= \int_0^3 \left[3y\frac{z^2}{2}\Big|_1^5\right] dy \\ &= \int_0^3 36y\,dy \\ &= 36\frac{y^2}{2}\Big|_0^3 = 18(9-0) = 162.\end{aligned}$$

Exercise 2.4.1

Evaluate the triple integral

$$\iiint z \sin x \cos y \, dV$$

where $B = \left\{(x, y, z) \mid 0 \leq x \leq \pi, \ \frac{3\pi}{2} \leq y \leq 2\pi, \ 1 \leq z \leq 3\right\}$.

Hint available at the end of this section.

Triple Integrals over a General Bounded Region

We now expand the definition of the triple integral to compute a triple integral over a more **general bounded region** E in $\mathbb{R}^3$. The general bounded regions we will consider are of three types. First, let D be the bounded region that is a projection of E onto the xy-plane. Suppose the region E in $\mathbb{R}^3$ has the form

$$E = \left\{(x, y, z) \,|\, (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\right\}. \tag{2.4.16}$$

For two functions $z = u_1(x, y)$ and $u_2(x, y)$, such that $u_1(x, y) \leq u_2(x, y)$ for all (x, y) in D as shown in the figure 2.4.3.

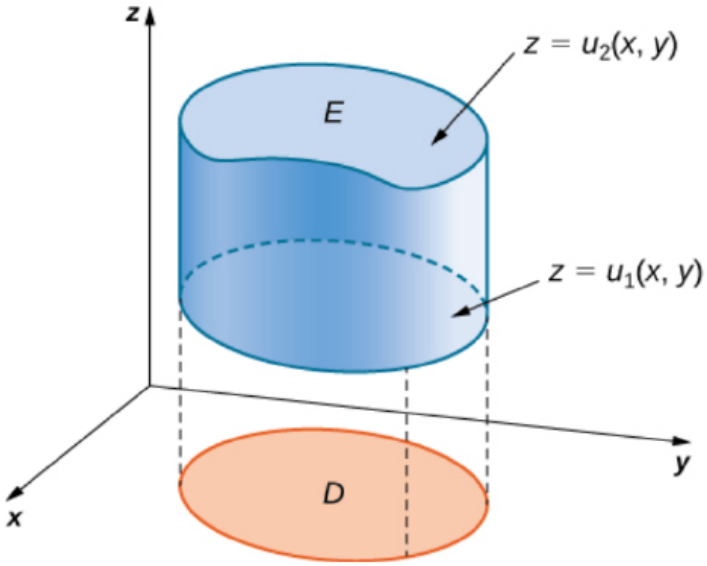


Figure 2.4.3: A general bounded region E in $\mathbb{R}^3$ for triple integrals. Region E is described as the space between two surfaces, $z = u_1(x, y)$ and $z = u_2(x, y)$, above the bounded projection D of E onto the xy-plane.

Triple Integral over a General Region

The triple integral of a continuous function $f(x, y, z)$ over a general three-dimensional region

$$E = \left\{(x, y, z) \,|\, (x, y) \in D, \, u_1(x, y) \leq z \leq u_2(x, y)\right\} \tag{2.4.17}$$

in $\mathbb{R}^3$, where D is the projection of E onto the xy-plane, is

$$\iiint_E f(x, y, z) \, dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z) \, dz\right] dA. \tag{2.4.18}$$

Similarly, we can consider a general bounded region D in the xz-plane and two functions $y = u_1(x, z)$ and $y = u_2(x, z)$ such that

$u_1(x,z) \le u_2(x,z)$ for all (x,z) in D. Then we can describe the solid region E in $\mathbb{R}^3$ as

$$E = \{(x,y,z) \,|\, (x,z) \in D,\ u_1(x,z) \le y \le u_2(x,z)\} \tag{2.4.19}$$

where D is the projection of E onto the xz-plane and the triple integral is

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy\right] dA. \tag{2.4.20}$$

Finally, if D is a general bounded region in the yz-plane and we have two functions $x = u_1(y,z)$ and $x = u_2(y,z)$ such that $u_1(y,z) \le u_2(y,z)$ for all (y,z) in D, then the solid region E in $\mathbb{R}^3$ can be described as

$$E = \{(x,y,z) \,|\, (y,z) \in D,\ u_1(y,z) \le x \le u_2(y,z)\} \tag{2.4.21}$$

where D is the projection of E onto the yz-plane and the triple integral is

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx\right] dA. \tag{2.4.22}$$

Note that the region D in any of the planes may be of Type I or Type II as described previously. If D in the xy-plane is of Type I (Figure 2.4.4), then

$$E = \{(x,y,z) \,|\, a \le x \le b,\ g_1(x) \le y \le g_2(x),\ u_1(x,y) \le z \le u_2(x,y)\}. \tag{2.4.23}$$

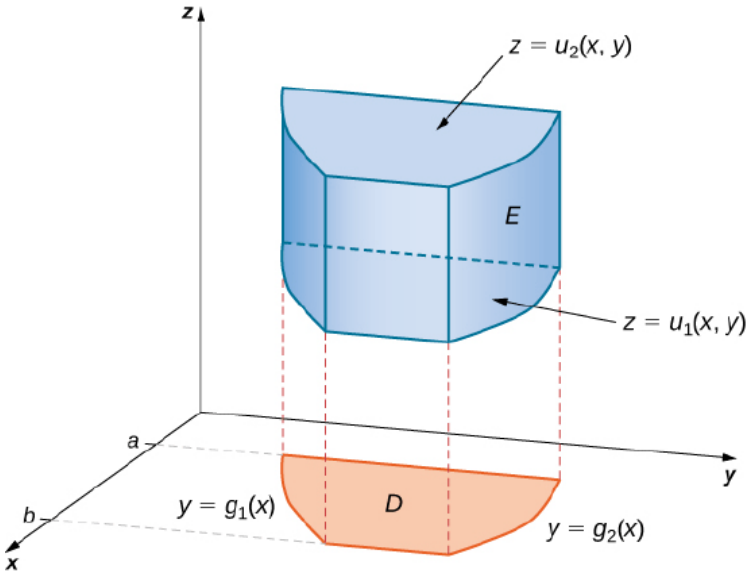


Figure 2.4.4: A 3D solid region E, which is bounded above by the surface $z = u_2(x,y)$ and below by $z = u_1(x,y)$. Its projection onto the xy-plane is a 2D region D, bounded by the curves $y = g_1(x)$ and $y = g_2(x)$.

Then the triple integral becomes

$$\iiint_E f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx. \tag{2.4.24}$$

If D in the xy-plane is of Type II (Figure 2.4.5), then

$$E = \{(x,y,z) \,|\, c \le y \le d, h_1(y) \le x \le h_2(y),\ u_1(x,y) \le z \le u_2(x,y)\}. \tag{2.4.25}$$

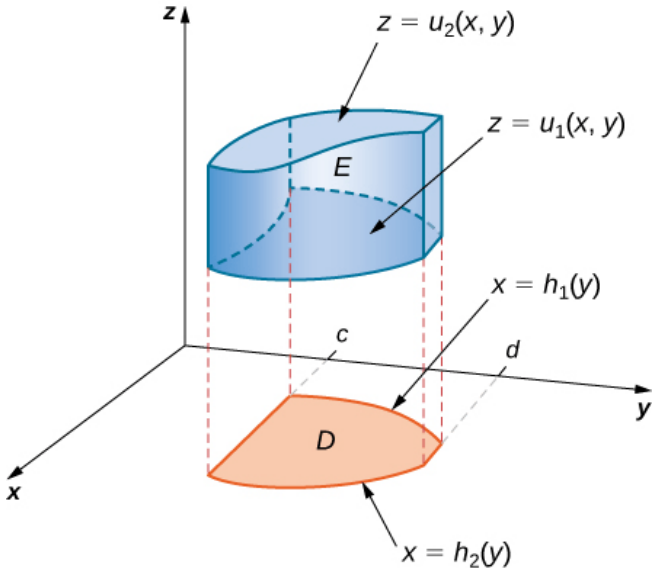


Figure 2.4.5: A 3D solid region E, which is bounded above by the surface $z = u_2(x, y)$ and below by $z = u_1(x, y)$. Its projection onto the xy-plane is a 2D region D, bounded by the curves $y = h_1(x)$ and $y = h_2(x)$.

Then the triple integral becomes

$$\iiint_E f(x, y, z)\, dV = \int_{y=c}^{y=d} \int_{x=h_1(y)}^{x=h_2(y)} \int_{z=u_1(x,y)}^{z=u_2(x,y)} f(x, y, z)\, dz\, dx\, dy. \tag{2.4.26}$$

Example 2.4.3A: Evaluating a Triple Integral over a General Bounded Region

Evaluate the triple integral of the function $f(x, y, z) = 5x - 3y$ over the solid tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

Solution

Figure 2.4.6 shows the solid tetrahedron E and its projection D on the xy-plane.

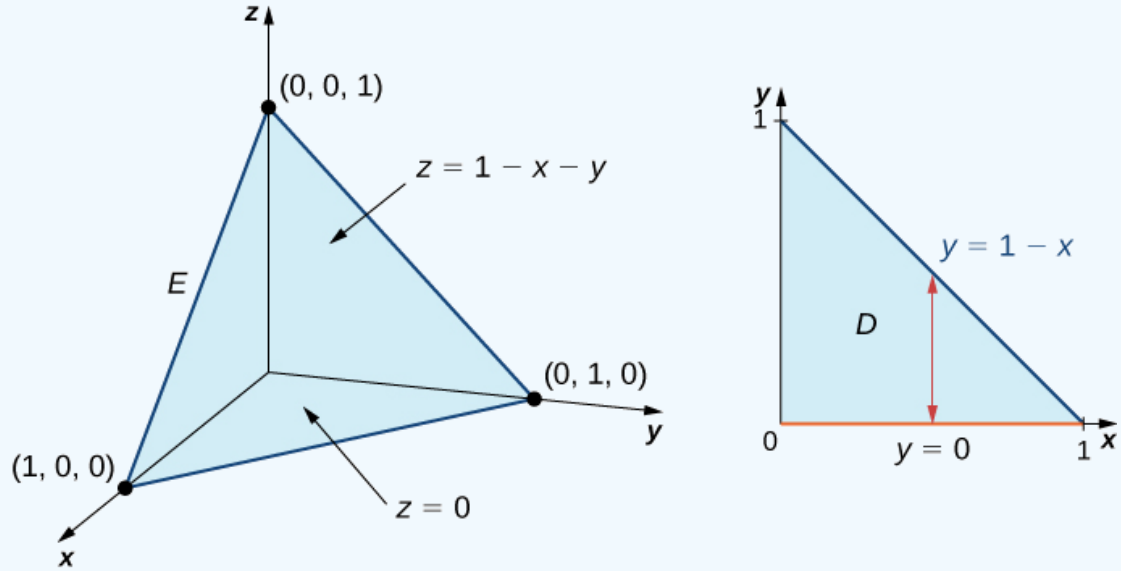


Figure 2.4.6: On the left side is a 3D tetrahedron E with its projection D onto the xy-plane. The tetrahedron is bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$. On the right side is a triangular region D bounded by the x-axis, the y-axis, and the line $y = 1 - x$.

We can describe the solid region tetrahedron as

$$E = \{(x, y, z) \mid 0 \le x \le 1,\ 0 \le y \le 1 - x,\ 0 \le z \le 1 - x - y\}.$$

Hence, the triple integral is

$$\iiint_E f(x, y, z)\, dV = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \int_{z=0}^{z=1-x-y} (5x - 3y)\, dz\, dy\, dx.$$

To simplify the calculation, first evaluate the integral $\int_{z=0}^{z=1-x-y} (5x - 3y)\ dz$. We have

$$\int_{z=0}^{z=1-x-y} (5x - 3y)\, dz = (5x - 3y)z \Big|_{z=0}^{z=1-x-y} = (5x - 3y)(1 - x - y).$$

Now evaluate the integral

$$\int_{y=0}^{y=1-x} (5x - 3y)(1 - x - y)\, dy,$$

obtaining

$$\int_{y=0}^{y=1-x} (5x - 3y)(1 - x - y)\, dy = \frac{1}{2}(x - 1)^2(6x - 1).$$

Finally evaluate

$$\int_{x=0}^{x=1} \frac{1}{2}(x - 1)^2(6x - 1)\, dx = \frac{1}{12}.$$

Putting it all together, we have

$$\iiint_E f(x, y, z)\, dV = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \int_{z=0}^{z=1-x-y} (5x - 3y)\, dz\, dy\, dx = \frac{1}{12}.$$

Just as we used the double integral

$$\iint_D 1\, dA \tag{2.4.27}$$

to find the area of a general bounded region D we can use

$$\iiint_E 1\, dV \tag{2.4.28}$$

to find the volume of a general solid bounded region E. The next example illustrates the method.

Example $2.4.3B$: Finding a Volume by Evaluating a Triple Integral

Find the volume of a right pyramid that has the square base in the xy-plane $[-1, 1] \times [-1, 1]$ and vertex at the point $(0, 0, 1)$ as shown in the figure $2.4.7$.

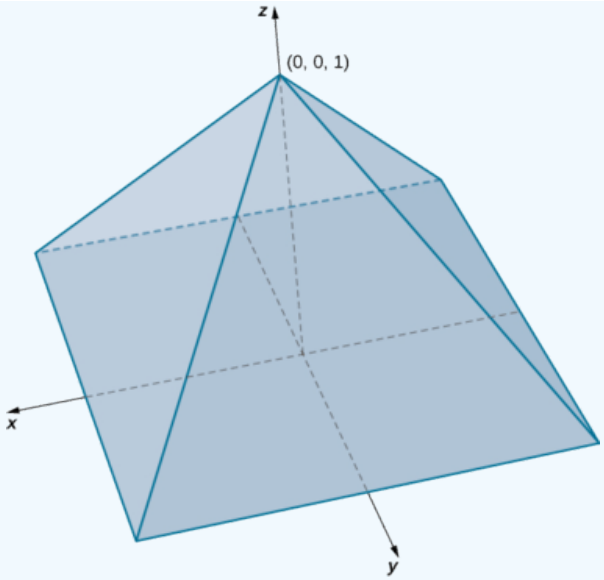


Figure 2.4.7: A 3D right pyramid with a square base. The base lies in the xy-plane with vertices at $(-1,-1)$, $(-1,1)$, $(1,-1)$, and $(1,1)$. The vertex of the pyramid is at the point $(0,0,1)$.

Solution

In this pyramid the value of z changes from 0 to 1 and at each height z the cross section of the pyramid for any value of z is the square

$$[-1+z,\, 1-z] \times [-1+z,\, 1-z].$$

Hence, the volume of the pyramid is

$$\iiint_E 1\, dV$$

where

$$E = \{(x,y,z) \,|\, 0 \le z \le 1,\, -1+z \le y \le 1-z,\, -1+z \le x \le 1-z\}.$$

Thus, we have

$$\begin{aligned} \iiint_E 1\, dV &= \int_{z=0}^{z=1} \int_{y=-1+z}^{y=1-z} \int_{x=-1+z}^{x=1-z} 1\, dx\, dy\, dz \\ &= \int_{z=0}^{z=1} \int_{y=-1+z}^{y=1-z} (2-2z)\, dy\, dz \\ &= \int_{z=0}^{z=1} (2-2z)^2\, dz = \frac{4}{3}. \end{aligned}$$

Hence, the volume of the pyramid is $\frac{4}{3}$ cubic units.

Watch the accompanying video for a visual demonstration of the concept.

- **Video length**: 4 minutes 12 seconds.
- **Context**: This video demonstrates how to a triple integral to find the volume.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Exercise 2.4.3

Consider the solid sphere $E = \{(x, y, z) \mid x^2 + y^2 + z^2 = 9\}$. Write the triple integral

$$\iiint_E f(x, y, z)\, dV$$

for an arbitrary function f as an iterated integral. Then evaluate this triple integral with $f(x, y, z) = 1$. Notice that this gives the volume of a sphere using a triple integral.

Hint available at the end of this section.

Changing the Order of Integration

As we have already seen in double integrals over general bounded regions, changing the order of the integration is done quite often to simplify the computation. With a triple integral over a rectangular box, the order of integration does not change the level of difficulty of the calculation. However, with a triple integral over a general bounded region, choosing an appropriate order of integration can simplify the computation quite a bit. Sometimes making the change to polar coordinates can also be very helpful. We demonstrate two examples here.

Example 2.4.4: Changing the Order of Integration

Consider the iterated integral

$$\int_{x=0}^{x=1} \int_{y=0}^{y=x^2} \int_{z=0}^{z=y} f(x, y, z)\, dz\, dy\, dx. \tag{2.4.29}$$

The order of integration here is first with respect to *z*, then *y*, and then *x*. Express this integral by changing the order of integration to be first with respect to x, then z, and then y. Verify that the value of the integral is the same if we let $f(x, y, z) = xyz$.

Solution

The best way to do this is to sketch the region E and its projections onto each of the three coordinate planes. Thus, let

$$E = \{(x, y, z) \,|\, 0 \le x \le 1,\ 0 \le y \le x^2,\ 0 \le z \le y\}.$$

and

$$\int_{x=0}^{x=1} \int_{y=0}^{y=x^2} \int_{z=0}^{z=x^2} f(x, y, z)\, dz\, dy\, dx = \iiint_E f(x, y, z)\, dV.$$

We need to express this triple integral as

$$\int_{y=c}^{y=d} \int_{z=v_1(y)}^{z=v_2(y)} \int_{x=u_1(y,z)}^{x=u_2(y,z)} f(x, y, z)\, dx\, dz\, dy.$$

Knowing the region E we can draw the following projections (Figure 2.4.8):

on the xy-plane is $D_1 = \{(x, y) \mid 0 \le x \le 1,\ 0 \le y \le x^2\} = \{(x, y) \mid 0 \le y \le 1,\ \sqrt{y} \le x \le 1\}$,

on the yz-plane is $D_2 = \{(y, z) \mid 0 \le y \le 1,\ 0 \le z \le y^2\}$, and

on the xz-plane is $D_3 = \{(x, z) \mid 0 \le x \le 1,\ 0 \le z \le x^2\}$.

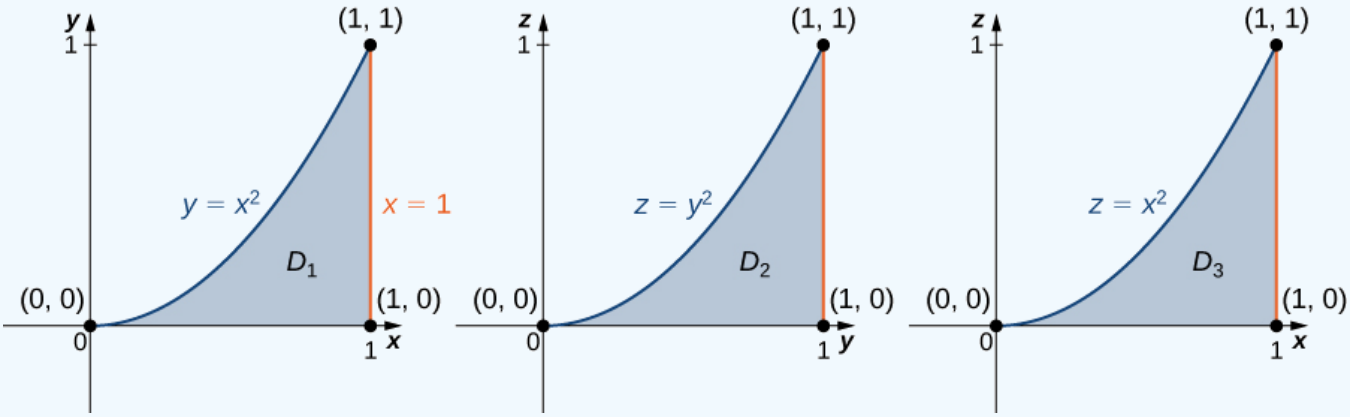


Figure 2.4.8: Three projections of a 3D region E onto the xy-, yz-, and xz-planes. Each projection is a triangle with vertices at $(0, 0)$, $(1, 0)$, and $(1, 1)$.

Now we can describe the same region E as $\{(x, y, z) \mid 0 \le y \le 1,\ 0 \le z \le y^2,\ \sqrt{y} \le x \le 1\}$, and consequently, the triple integral becomes

$$\int_{y=c}^{y=d} \int_{z=v_1(y)}^{z=v_2(y)} \int_{x=u_1(y,z)}^{x=u_2(y,z)} f(x, y, z)\, dx\, dz\, dy = \int_{y=0}^{y=1} \int_{z=0}^{z=x^2} \int_{x=\sqrt{y}}^{x=1} f(x, y, z)\, dx\, dz\, dy \tag{2.4.30}$$

Now assume that $f(x, y, z) = xyz$ in each of the integrals. Then we have

$$\int_{x=0}^{x=1} \int_{y=0}^{y=x^2} \int_{z=0}^{z=y^2} xyz\, dz\, dy\, dx$$

$$= \int_{x=0}^{x=1} \int_{y=0}^{y=x^2} \left[xy \frac{z^2}{2} \Big|_{z=0}^{z=y^2} \right] dy\, dx = \int_{x=0}^{x=1} \int_{y=0}^{y=x^2} \left(x \frac{y^5}{2} \right) dy\, dx = \int_{x=0}^{x=1} \left[x \frac{y^6}{12} \Big|_{y=0}^{y=x^2} \right] dx = \int_{x=0}^{x=1} \frac{x^{13}}{12} dx = \frac{x^{14}}{168} \Big|_{x=0}^{x=1} = \frac{1}{168},$$

$$\int_{y=0}^{y=1} \int_{z=0}^{z=y^2} \int_{x=\sqrt{y}}^{x=1} xyz\, dx\, dz\, dy$$

$$= \int_{y=0}^{y=1} \int_{z=0}^{z=y^2} \left[yz \frac{x^2}{2} \Big|_{\sqrt{y}}^{1} \right] dz\, dy = \int_{y=0}^{y=1} \int_{z=0}^{z=y^2} \left(\frac{yz}{2} - \frac{y^2 z}{2} \right) dz\, dy = \int_{y=0}^{y=1} \left[\frac{yz^2}{4} - \frac{y^2 z^2}{4} \Big|_{z=0}^{z=y^2} \right] dy = \int_{y=0}^{y=1} \left(\frac{y^5}{4} - \frac{y^6}{4} \right) dy = \left(\frac{y^6}{24} - \frac{y^7}{28} \right) \Big|_{y=0}^{y=1} = \frac{1}{168}.$$

The answers match.

Watch the accompanying video for a visual demonstration of the concept.

- **Video length**: 4 minutes 50 seconds.
- **Context**: This video demonstrates how to switch order of integration in triple integral.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Exercise 2.4.4

Write five different iterated integrals equal to the given integral

$$\int_{z=0}^{z=4}\int_{y=0}^{y=4-z}\int_{x=0}^{x=\sqrt{y}} f(x,y,z)\,dx\,dy\,dz.$$

Hint available at the end of this section.

Example 2.4.5: Changing Integration Order and Coordinate Systems

Evaluate the triple integral

$$\iiint_E \sqrt{x^2+z^2}\,dV,$$

where E is the region bounded by the paraboloid $y=x^2+z^2$ (Figure 2.4.9) and the plane $y=4$.

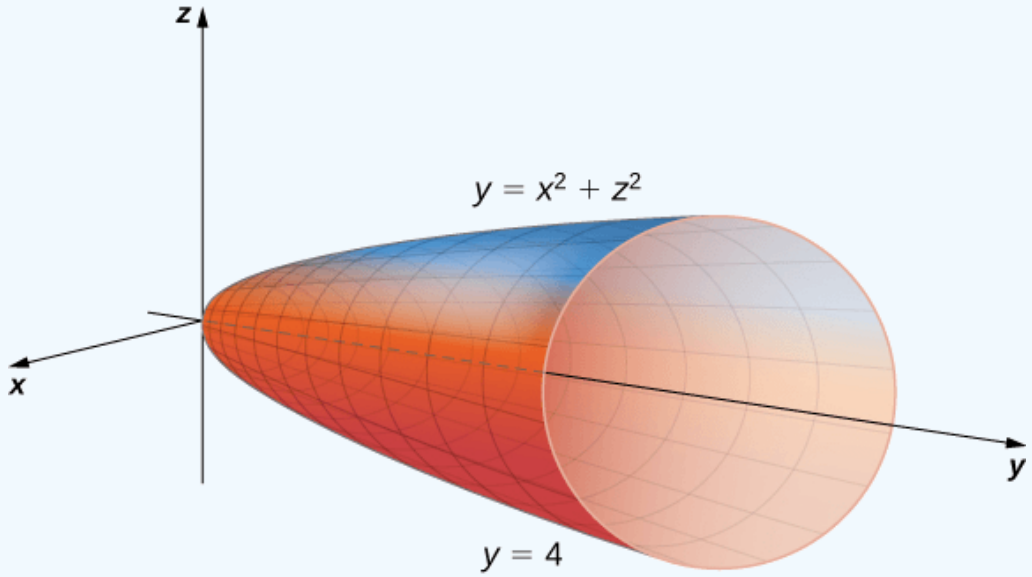


Figure 2.4.9: A 3D paraboloid, opening upwards, with its vertex at the origin and bounded above by the plane $y=4$.

Solution

The projection of the solid region E onto the xy-plane is the region bounded above by $y=4$ and below by the parabola $y=x^2$ as shown.

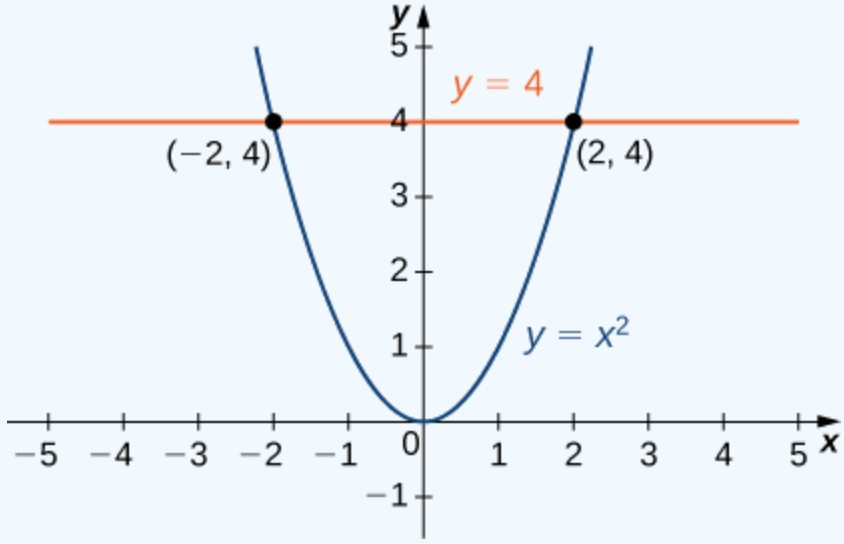


Figure 2.4.10: The projection of the above 3D paraboloid onto the xy-plane, which is a region bounded by the parabola $y = x^2$ and the line $y = 4$.

Thus, we have

$$E = \{(x, y, z) \mid -2 \le x \le 2,\ x^2 \le y \le 4,\ -\sqrt{y - x^2} \le z \le \sqrt{y - x^2}\}.$$

The triple integral becomes

$$\iiint_E \sqrt{x^2 + z^2}\, dV = \int_{x=-2}^{x=2} \int_{y=x^2}^{y=4} \int_{z=-\sqrt{y-x^2}}^{z=\sqrt{y-x^2}} \sqrt{x^2 + z^2}\, dz\, dy\, dx.$$

This expression is difficult to compute, so consider the projection of E onto the xz-plane. This is a circular disc $x^2 + z^2 \le 4$. So we obtain

$$\iiint_E \sqrt{x^2 + z^2}\, dV = \int_{x=-2}^{x=2} \int_{y=x^2}^{y=4} \int_{z=-\sqrt{y-x^2}}^{z=\sqrt{y-x^2}} \sqrt{x^2 + z^2}\, dz\, dy\, dx = \int_{x=-2}^{x=2} \int_{z=-\sqrt{4-x^2}}^{z=\sqrt{4-x^2}} \int_{y=x^2+z^2}^{y=4} \sqrt{x^2 + z^2}\, dy\, dz\, dx.$$

Here the order of integration changes from being first with respect to z then y and then x to being first with respect to y then to z and then to x. It will soon be clear how this change can be beneficial for computation. We have

$$\int_{x=-2}^{x=2} \int_{z=-\sqrt{4-x^2}}^{z=\sqrt{4-x^2}} \int_{y=x^2+z^2}^{y=4} \sqrt{x^2 + z^2}\, dy\, dz\, dx = \int_{x=-2}^{x=2} \int_{z=-\sqrt{4-x^2}}^{z=\sqrt{4-x^2}} (4 - x^2 - z^2)\sqrt{x^2 + z^2}\, dz\, dx.$$

Now use the polar substitution $x = r\ \cos\ \theta,\ z = r\ \sin\ \theta$, and $dz\ dx = r\ dr\ d\theta$ in the xz-plane. This is essentially the same thing as when we used polar coordinates in the xy-plane, except we are replacing y by z. Consequently the limits of integration change and we have, by using $r^2 = x^2 + z^2$,

$$\int_{x=-2}^{x=2} \int_{z=-\sqrt{4-x^2}}^{z=\sqrt{4-x^2}} (4 - x^2 - z^2)\sqrt{x^2 + z^2}\, dz\, dx = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} (4 - r^2) r r\, dr\, d\theta = \int_0^{2\pi} \left[\frac{4r^3}{3} - \frac{r^5}{5}\Big|_0^2\right] d\theta = \int_0^{2\pi} \frac{64}{15}\, d\theta = \frac{128\pi}{15}$$

Average Value of a Function of Three Variables

Recall that we found the average value of a function of two variables by evaluating the double integral over a region on the plane and then dividing by the area of the region. Similarly, we can find the average value of a function in three variables by evaluating the triple integral over a solid region and then dividing by the volume of the solid.

Average Value of a Function of Three Variables

If $f(x,y,z)$ is integrable over a solid bounded region E with positive volume $V\ (E)$, then the average value of the function is

$$f_{ave} = \frac{1}{V\,(E)} \iiint_E f(x,y,z)\,dV. \tag{2.4.31}$$

Note that the volume is

$$V\,(E) = \iiint_E 1\,dV. \tag{2.4.32}$$

Example 2.4.6: Finding an Average Temperature

The temperature at a point (x,y,z) of a solid E bounded by the coordinate planes and the plane $x+y+z=1$ is $T(x,y,z) = (xy+8z+20)$ °C. Find the average temperature over the solid.

Solution

Use the theorem given above and the triple integral to find the numerator and the denominator. Then do the division. Notice that the plane $x+y+z=1$ has intercepts $(1,0,0),\ (0,1,0)$, and $(0,0,1)$. The region E looks like

$$E = \{(x,y,z)\,|\,0 \le x \le 1,\ 0 \le y \le 1-x,\ 0 \le z \le 1-x-y\}.$$

Hence the triple integral of the temperature is

$$\iiint_E f(x,y,z)\,dV = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \int_{z=0}^{z=1-x-y} (xy+8z+20)\,dz\,dy\,dx = \frac{147}{40}.$$

The volume evaluation is

$$V\,(E) = \iiint_E 1\,dV = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \int_{z=0}^{z=1-x-y} 1\,dz\,dy\,dx = \frac{1}{6}.$$

Hence the average value is

$$T_{ave} = \frac{147/40}{1/6} = \frac{6(147)}{40} = \frac{441}{20}\ ^\circ\text{C}$$

.

Exercise 2.4.6

Find the average value of the function $f(x,y,z) = xyz$ over the cube with sides of length 4 units in the first octant with one vertex at the origin and edges parallel to the coordinate axes.

Hint available at the end of this section.

Key Concepts

- To compute a triple integral we use Fubini's theorem, which states that if $f(x,y,z)$ is continuous on a rectangular box $B = [a,b] \times [c,d] \times [e,f]$, then

  $$\iiint_B f(x,y,z)\,dV = \int_e^f \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

  and is also equal to any of the other five possible orderings for the iterated triple integral.

- To compute the volume of a general solid bounded region E we use the triple integral

$$V(E) = \iiint_E 1\, dV.$$

- Interchanging the order of the iterated integrals does not change the answer. As a matter of fact, interchanging the order of integration can help simplify the computation.
- To compute the average value of a function over a general three-dimensional region, we use

$$f_{ave} = \frac{1}{V(E)} \iiint_E f(x,y,z)\, dV.$$

Key Equations

- **Triple integral**

$$\lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\, \Delta x \Delta y \Delta z = \iiint_B f(x,y,z)\, dV$$

Glossary

Triple integral
: the triple integral of a continuous function $f(x,y,z)$ over a rectangular solid box B is the limit of a Riemann sum for a function of three variables, if this limit exists

Contributors and Attributions

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Answers

Exercise 2.4.1

Follow the steps in the previous example.

$$\iiint_B z \sin x \cos y\, dV = 8$$

Exercise 2.4.3

Follow the steps in the previous example. Use symmetry.

$$\iiint_E 1\, dV = 8\int_{x=-3}^{x=3}\int_{y=-\sqrt{9-z^2}}^{y=\sqrt{9-z^2}}\int_{z=-\sqrt{9-x^2-y^2}}^{z=\sqrt{9-x^2-y^2}} 1\, dz\, dy\, dx$$
$$= 36\pi \text{ cubic units.}$$

Exercise 2.4.4

Follow the steps in the previous example, using the region E as $\{(x,y,z) \mid 0 \le z \le 4,\ 0 \le y \le 4-z,\ 0 \le x \le \sqrt{y}\}$, and describe and sketch the projections onto each of the three planes, five different times.

$$(i)\ \int_{z=0}^{z=4}\int_{x=0}^{x=\sqrt{4-z}}\int_{y=x^2}^{y=4-z} f(x,y,z)\, dy\, dx\, dz,\ (ii)\ \int_{y=0}^{y=4}\int_{z=0}^{z=4-y}\int_{x=0}^{x=\sqrt{y}} f(x,y,z)\, dx\, dz\, dy,\ (iii)\ \int_{y=0}^{y=4}\int_{x=0}^{x=\sqrt{y}}\int_{z=0}^{z=4-y} f(x,y,z)\, dz\, dx\, dy,$$
$$(iv)\ \int_{x=0}^{x=2}\int_{y=x^2}^{y=4}\int_{z=0}^{z=4-y} f(x,y,z)\, dz\, dy\, dx,\ (v)\ \int_{x=0}^{x=2}\int_{z=0}^{z=4-x^2}\int_{y=x^2}^{y=4-z} f(x,y,z)\, dy\, dz\, dx$$

Exercise 2.4.6

Follow the steps in the previous example.

$f_{ave} = 8$

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2.4E: Exercises

Exercises 1 - 4

Evaluate the triple integrals over the rectangular solid box B.

1. $\iiint_B (2x + 3y^2 + 4z^3)\ dV$, where $B = \{(x, y, z) \mid 0 \le x \le 1,\ 0 \le y \le 2,\ 0 \le z \le 3\}$

   Answer

   192

2. $\iiint_B (xy + yz + xz)\ dV$, where $B = \{(x, y, z) \mid 1 \le x \le 2,\ 0 \le y \le 2,\ 1 \le z \le 3\}$

3. $\iiint_B (x \cos y + z)\ dV$, where $B = \{(x, y, z) \mid 0 \le x \le 1,\ 0 \le y \le \pi,\ -1 \le z \le 1\}$

   Answer

   0

4. $\iiint_B (z \sin x + y^2)\ dV$, where $B = \{(x, y, z) \mid 0 \le x \le \pi,\ 0 \le y \le 1,\ -1 \le z \le 2\}$

Exercises 5 - 8

Change the order of integration by integrating first with respect to z, then x, then y.

5. $\int_0^1 \int_1^2 \int_2^3 (x^2 + \ln y + z)\ dx\ dy\ dz$

   Answer

   $$\int_0^1 \int_1^2 \int_2^3 (x^2 + \ln y + z)\ dx\ dy\ dz = \frac{35}{6} + 2 \ln 2$$

6. $\int_0^1 \int_{-1}^1 \int_0^3 (ze^x + 2y)\ dx\ dy\ dz$

7. $\int_{-1}^2 \int_1^3 \int_0^4 \left(x^2 z + \frac{1}{y}\right)\ dx\ dy\ dz$

   Answer

   $$\int_{-1}^2 \int_1^3 \int_0^4 \left(x^2 z + \frac{1}{y}\right)\ dx\ dy\ dz = 64 + 12 \ln 3$$

8. $\int_1^2 \int_{-2}^{-1} \int_0^1 \frac{x + y}{z}\ dx\ dy\ dz$

9. Let F, G, and H be continuous functions on $[a, b]$, $[c, d]$, and $[e, f]$, respectively, where $a,\ b,\ c,\ d,\ e$, and f are real numbers such that $a < b,\ c < d$, and $e < f$. Show that

$$\int_a^b \int_c^d \int_e^f F(x)\, G(y)\, H(z)\, dz\, dy\, dx = \left(\int_a^b F(x)\, dx\right) \left(\int_c^d G(y)\, dy\right) \left(\int_e^f H(z)\, dz\right).$$

10. Let F, G, and H be differential functions on $[a, b]$, $[c, d]$, and $[e, f]$, respectively, where $a,\ b,\ c,\ d,\ e$, and f are real numbers such that $a < b,\ c < d$, and $e < f$. Show that

$$\int_a^b \int_c^d \int_e^f F'(x)\,G'(y)\,H'(z)\,dz\,dy\,dx = [F(b)-F(a)]\,[G(d)-G(c)]\,H(f)-H(e)].$$

Exercises 11 - 14

Evaluate the triple integrals over the bounded region.

$E = \{(x,y,z) \mid a \le x \le b,\ h_1(x) \le y \le h_2(x),\ e \le z \le f\}.$

11. $\iiint_E (2x+5y+7z)\ dV$, where $E = \{(x,y,z) \mid 0 \le x \le 1,\ 0 \le y \le -x+1,\ 1 \le z \le 2\}$

Answer

$\frac{77}{12}$

12. $\iiint_E (y \ln x + z)\ dV$, where $E = \{(x,y,z) \mid 1 \le x \le e,\ 0 \le y \ln x,\ 0 \le z \le 1\}$

13. $\iiint_E (\sin x + \sin y)\ dV$, where $E = \{(x,y,z) \mid 0 \le x \le \frac{\pi}{2},\ -\cos x \le y \le \cos x,\ -1 \le z \le 1\}$

Answer

2

14. $\iiint_E (xy+yz+xz)dV$ where $E = \{(x,y,z) \mid 0 \le x \le 1,\ -x^2 \le y \le x^2,\ 0 \le z \le 1\}$

Exercises 15 - 18

Evaluate the triple integrals over the indicated bounded region E.

15. $\iiint_E (x+2yz)\ dV$, where $E = \{(x,y,z) \mid 0 \le x \le 1,\ 0 \le y \le x,\ 0 \le z \le 5-x-y\}$

Answer

$\frac{439}{120}$

16. $\iiint_E (x^3+y^3+z^3)\ dV$, where $E = \{(x,y,z) \mid 0 \le x \le 2,\ 0 \le y \le 2x,\ 0 \le z \le 4-x-y\}$

17. $\iiint_E y\ dV$, where $E = \{(x,y,z) \mid -1 \le x \le 1,\ -\sqrt{1-x^2} \le y \le \sqrt{1-x^2},\ 0 \le z \le 1-x^2-y^2\}$

Answer

0

18. $\iiint_E x\ dV$, where $E = \{(x,y,z) \mid -2 \le x \le 2,\ -4\sqrt{1-x^2} \le y \le \sqrt{4-x^2},\ 0 \le z \le 4-x^2-y^2\}$

Exercises 19 - 22

Evaluate the triple integrals over the bounded region E of the form.

$E = \{(x,y,z) \mid g_1(y) \le x \le g_2(y),\ c \le y \le d,\ e \le z \le f\}.$

19. $\iiint_E x^2\ dV$, where $E = \{(x,y,z) \mid 1-y^2 \le x \le y^2-1,\ -1 \le y \le 1,\ 1 \le z \le 2\}$

Answer

$-\frac{64}{105}$

20. $\iiint_E (y+\sin x)\ dV$, where $E=\{(x,y,z)\mid -y^4\le x\le y^4,\ 0\le y\le 2,\ 0\le z\le 4\}$

21. $\iiint_E (x-yz)\ dV$, where $E=\{(x,y,z)\mid -y^6\le x\le \sqrt{y},\ 0\le y\le 1x,\ -1\le z\le 1\}$

Answer

$\frac{11}{26}$

22. $\iiint_E z\ dV$, where $E=\{(x,y,z)\mid 2-2y\le x\le 2+\sqrt{y},\ 0\le y\le 1x,\ 2\le z\le 3\}$

Exercises 23 - 26

Evaluate the triple integrals over the bounded region.

$E=\{(x,y,z)\mid g_1(y)\le x\le g_2(y),\ c\le y\le d,\ u_1(x,y)\le z\le u_2(x,y)\}$

23. $\iiint_E z\ dV$, where $E=\{(x,y,z)\mid -y\le x\le y,\ 0\le y\le 1,\ 0\le z\le 1-x^4-y^4\}$

Answer

$\frac{113}{450}$

24. $\iiint_E (xz+1)\ dV$, where $E=\{(x,y,z)\mid 0\le x\le \sqrt{y},\ 0\le y\le 2,\ 0\le z\le 1-x^2-y^2\}$

25. $\iiint_E (x-z)\ dV$, where $E=\{(x,y,z)\mid -\sqrt{1-y^2}\le x\le y,\ 0\le y\le \frac{1}{2}x,\ 0\le z\le 1-x^2-y^2\}$

Answer

$\frac{1}{160}(6\sqrt{3}-41)$

26. $\iiint_E (x+y)\ dV$, where $E=\{(x,y,z)\mid 0\le x\le \sqrt{1-y^2},\ 0\le y\le 1x,\ 0\le z\le 1-x\}$

Exercises 27 - 30

Evaluate the triple integrals over the bounded region.

$E=\{(x,y,z)\mid (x,y)\in D,\ u_1(x,y)\le z\le u_2(x,y)\}$**, where** D **is the projection of** E **onto the** xy**-plane.**

27. $\iint_D \left(\int_1^2 (x+y)\,dz\right)\ dA$, where $D=\{(x,y)\mid x^2+y^2\le 1\}$

Answer

0

28. $\iint_D \left(\int_1^3 x(z+1)\,dz\right)\ dA$, where $D=\{(x,y)\mid x^2-y^2\ge 1,\ x\le\sqrt{5}\}$

29. $\iint_D \left(\int_0^{10-x-y} (x+2z)\,dz\right)\ dA$, where $D=\{(x,y)\mid y\ge 0,\ x\ge 0,\ x+y\le 10\}$

Answer

1250

30. $\iint_D \left(\int_0^{4x^2+4y^2} y\,dz \right)\, dA$, where $D = \{(x,y) \mid x^2+y^2 \le 4,\ y \ge 1,\ x \ge 0\}$

31. The solid E bounded by $y^2+z^2=9,\ z=0$, and $x=5$ is shown in the Figure $2.4E.1$. Evaluate the integral $\iiint_E z\ dV$ by integrating first with respect to z, then y, and then x.

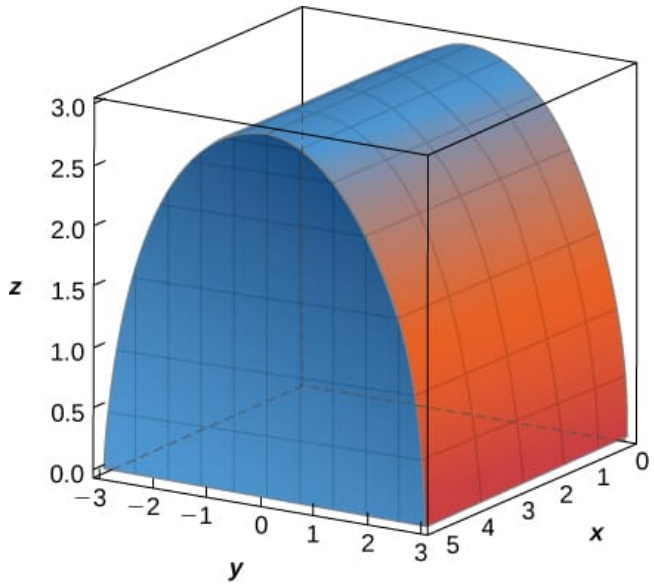


Figure $2.4E.1$: The solid E bounded above by $y^2+z^2=9$, bounded below by $z=0$, and $x=5$.

Answer

$$\int_0^5 \int_{-3}^3 \int_0^{\sqrt{9-y^2}} z\ dz\ dy\ dx = 90$$

32. The solid E bounded by $y=\sqrt{x},\ x=4,\ y=0$, and $z=1$ is given in the Figure $2.4E.2$. Evaluate the integral $\iiint_E xyz\ dV$ by integrating first with respect to x, then y, and then z.

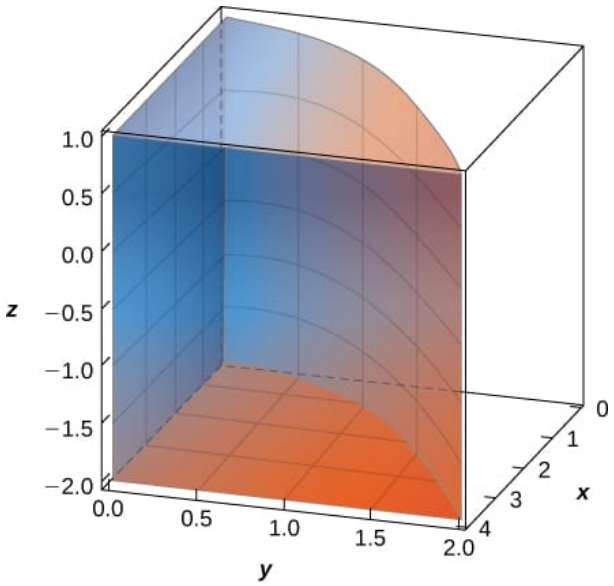


Figure $2.4E.2$: The solid E bounded by $y=\sqrt{x},\ x=4,\ y=0$, and $z=1$.

33. [T] The volume of a solid E is given by the integral $\int_{-2}^0 \int_x^0 \int_0^{x^2+y^2} dz\ dy\ dx$. Use a computer algebra system (CAS) to graph E and find its volume. Round your answer to two decimal places.

Answer

$V \approx 5.33 \text{ units}^3$

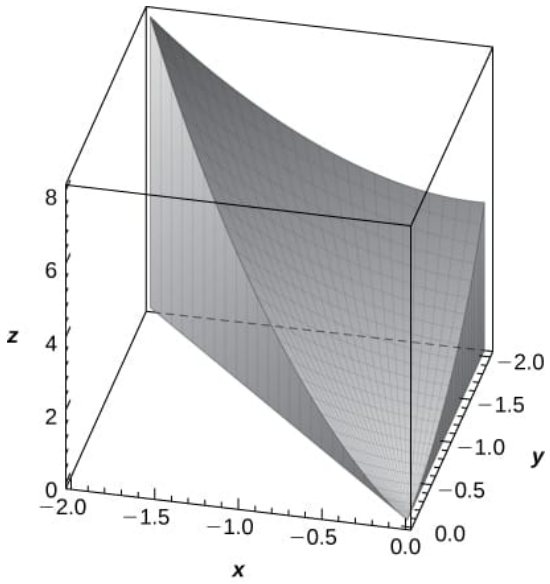


Figure $2.4E.3$: The graph illustrates the solid E bounded by $z = x^2 + y^2$, $z = 0$, $y = x$, $y = 0$, $x = 0$, and $x = 2$.

34. [T] The volume of a solid E is given by the integral $\int_{-1}^{0}\int_{-x^3}^{0}\int_{0}^{1+\sqrt{x^2+y^2}} dz\ dy\ dx$. Use a CAS to graph E and find its volume V. Round your answer to two decimal places.

Exercises 35 - 38

Use two circular permutations of the variables x, y, and z to write new integrals whose values equal the value of the original integral. A circular permutation of $x,\ y$, and z is the arrangement of the numbers in one of the following orders: y, z, and x or z, x, and y.

35. $\int_0^1\int_1^3\int_2^4 (x^2z^2+1)\ dx\ dy\ dz$

Answer

$$\int_0^1\int_1^3\int_2^4 (y^2z^2+1)\ dz\ dx\ dy;$$
$$\int_0^1\int_1^3\int_2^4 (x^2y^2+1)\ dy\ dz\ dx$$

36. $\int_0^3\int_0^1\int_0^{-x+1}(2x+5y+7z)dy\ dx\ dz$

37. $\int_0^1\int_{-y}^{y}\int_0^{1-x^4-y^4} \ln x\ dz\ dx\ dy$

38. $\int_{-1}^1\int_0^1\int_{-y^6}^{\sqrt{y}}(x+yz)\ dx\ dy\ dz$

39. Set up the integral that gives the volume of the solid E bounded by $y^2 = x^2 + z^2$ and $y = a^2$, where $a > 0$.

Answer

$$V = \int_{-a}^{a}\int_{-\sqrt{a^2-z^2}}^{\sqrt{a^2-z^2}}\int_{\sqrt{x^2+z^2}}^{a^2} dy\ dx\ dz$$

40. Set up the integral that gives the volume of the solid E bounded by $x = y^2 + z^2$ and $x = a^2$, where $a > 0$.

Average Value of a Function

41. Find the average value of the function $f(x, y, z) = x + y + z$ over the parallelepiped determined by $x + 0,\ x = 1,\ y = 0,\ y = 3,\ z = 0$, and $z = 5$.

Answer

$\frac{9}{2}$

42. Find the average value of the function $f(x, y, z) = xyz$ over the solid $E = [0, 1] \times [0, 1] \times [0, 1]$ situated in the first octant.

Finding Volumes using Triple Integrals

43. Find the volume of the solid E that lies under the plane $x + y + z = 9$ and whose projection onto the xy-plane is bounded by $x = \sqrt{y - 1},\ x = 0$, and $x + y = 7$.

Answer

$\frac{156}{5}\ \text{units}^3$

44. Find the volume of the solid E that lies under the plane $2x + y + z = 8$ and whose projection onto the xy-plane is bounded by $x = sin^{-1}y,\ y = 0$, and $x = \frac{\pi}{2}$.

45. Consider the pyramid with the base in the xy-plane of $[-2, 2] \times [-2, 2]$ and the vertex at the point $(0, 0, 8)$.

a. Show that the equations of the planes of the lateral faces of the pyramid are $4y + z = 8,\ 4y - z = -8,\ 4x + z = 8$, and $-4x + z = 8$.

b. Find the volume of the pyramid.

Answer

a. Answers may vary;
b. $\frac{128}{3}\ \text{units}^3$

46. Consider the pyramid with the base in the xy-plane of $[-3, 3] \times [-3, 3]$ and the vertex at the point $(0, 0, 9)$.

a. Show that the equations of the planes of the side faces of the pyramid are $3y + z = 9,\ 3y + z = 9,\ y = 0$ and $x = 0$.

b. Find the volume of the pyramid.

47. The solid E bounded by the sphere of equation $x^2 + y^2 + z^2 = r^2$ with $r > 0$ and located in the first octant is represented in the Figure $2.4E.4$.

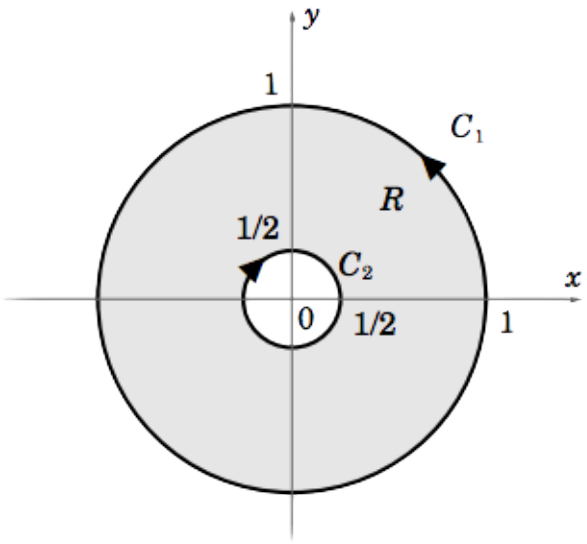


Figure $2.4E.4$: The graph illustrates the annulus.

a. Write the triple integral that gives the volume of E by integrating first with respect to z, then with y, and then with x.

b. Rewrite the integral in part a. as an equivalent integral in five other orders.

Answer

a. $\int_0^4 \int_0^{\sqrt{r^2-x^2}} \int_0^{\sqrt{r^2-x^2-y^2}} dz\ dy\ dx$

b. $\int_0^2 \int_0^{\sqrt{r^2-x^2}} \int_0^{\sqrt{r^2-x^2-y^2}} dz\ dx\ dy,$

$\int_0^r \int_0^{\sqrt{r^2-x^2}} \int_0^{\sqrt{r^2-x^2-y^2}} dy\ dx\ dz,$

$\int_0^r \int_0^{\sqrt{r^2-x^2}} \int_0^{\sqrt{r^2-x^2-y^2}} dy\ dz\ dx,$

$\int_0^r \int_0^{\sqrt{r^2-x^2}} \int_0^{\sqrt{r^2-x^2-y^2}} dx\ dy\ dz,$

$\int_0^r \int_0^{\sqrt{r^2-x^2}} \int_0^{\sqrt{r^2-x^2-y^2}} dx\ dz\ dy$

48. The solid E bounded by the sphere of equation $9x^2 + 4y^2 + z^2 = 1$ and located in the first octant is represented in the Figure $2.4E.5$.

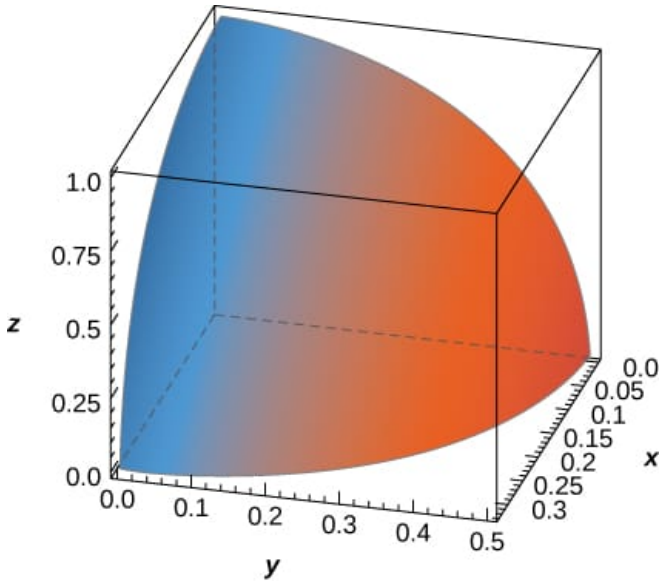


Figure $2.4E.5$: The graph illustrates the solid E ounded by the sphere of equation $9x^2 + 4y^2 + z^2 = 1$ and located in the first octant.

a. Write the triple integral that gives the volume of E by integrating first with respect to z then with y and then with x.

b. Rewrite the integral in part a. as an equivalent integral in five other orders.

49. Find the volume of the prism with vertices $(0, 0, 0),\ (2, 0, 0),\ (2, 3, 0),\ (0, 3, 0),\ (0, 0, 1)$, and $(2, 0, 1)$.

Answer

3 units^3

50. Find the volume of the prism with vertices $(0, 0, 0),\ (4, 0, 0),\ (4, 6, 0),\ (0, 6, 0),\ (0, 0, 1)$, and $(4, 0, 1)$.

51. The solid E bounded by $z = 10 - 2x - y$ and situated in the first octant is given in the Figure $2.4E.6$. Find the volume of the solid.

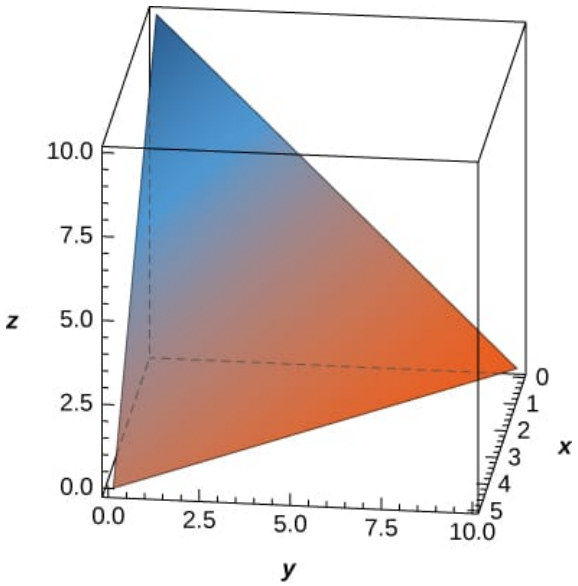


Figure $2.4E.6$: The graph illustrates the solid E bounded by $z = 10 - 2x - y$ and situated in the first octant.

Answer

$\frac{250}{3} \text{ units}^3$

52. The solid E bounded by $z = 1 - x^2$ and situated in the first octant is given in the Figure $2.4E.7$. Find the volume of the solid.

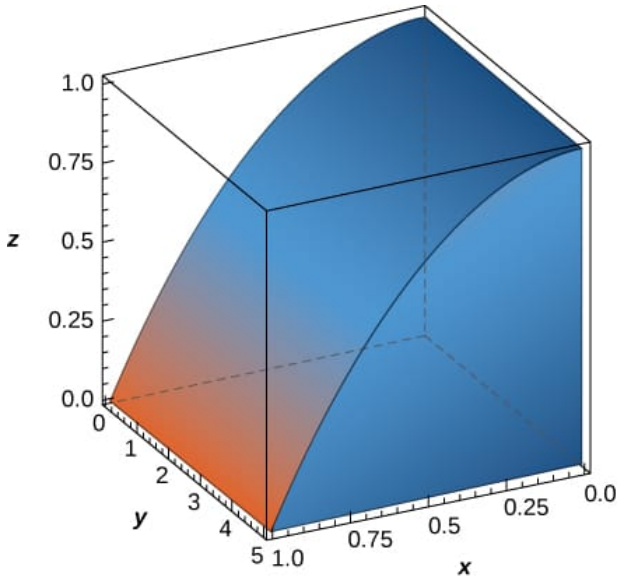


Figure $2.4E.7$: The graph illustrates the solid E bounded by $z = 1 - x^2$ and situated in the first octant.

Approximating Triple Integrals

53. The midpoint rule for the triple integral $\iiint_B f(x, y, z)\ dV$ over the rectangular solid box B is a generalization of the midpoint rule for double integrals. The region B is divided into subboxes of equal sizes and the integral is approximated by the triple Riemann sum

$$\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(\bar{x}_i, \bar{y}_j, \bar{z}_k)\Delta V,$$

where $(\bar{x}_i, \bar{y}_j, \bar{z}_k)$ is the center of the box B_{ijk} and ΔV is the volume of each subbox. Apply the midpoint rule to approximate

$$\iiint_B x^2\, dV$$

over the solid $B = \{(x, y, z) \mid 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$ by using a partition of eight cubes of equal size. Round your answer to three decimal places.

Answer

$$\iiint_B f(x, y, z)\ dV \quad \approx \quad \tfrac{5}{16} \approx 0.313 \text{ units}^3$$

54. [T]

a. Apply the midpoint rule to approximate $\iiint_B e^{-x^2}\ dV$ over the solid $B = \{(x, y, z) \mid 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$ by using a partition of eight cubes of equal size. Round your answer to three decimal places.

b. Use a CAS to improve the above integral approximation in the case of a partition of n^3 cubes of equal size, where $n = 3,\ 4,\ \ldots,\ 10$.

Applications

55. Suppose that the temperature in degrees Celsius at a point (x, y, z) of a solid E bounded by the coordinate planes and the plane $x + y + z = 5$ is given by:

$$T(x, y, z) = xz + 5z + 10$$

Find the average temperature over the solid.

Answer

17.5° C

56. Suppose that the temperature in degrees Fahrenheit at a point (x, y, z) of a solid E bounded by the coordinate planes and the plane $x + y + z = 5$ is given by:

$$T(x, y, z) = x + y + xy$$

Find the average temperature over the solid.

57. Show that the volume of a right square pyramid of height h and side length a is $v = \dfrac{ha^2}{3}$ by using triple integrals.

58. Show that the volume of a regular right hexagonal prism of edge length a is $\dfrac{3a^3\sqrt{3}}{2}$ by using triple integrals.

59. Show that the volume of a regular right hexagonal pyramid of edge length a is $\dfrac{a^3\sqrt{3}}{2}$ by using triple integrals.

60. If the charge density at an arbitrary point (x, y, z) of a solid E is given by the function $\rho(x, y, z)$, then the total charge inside the solid is defined as the triple integral $\iiint_E \rho(x, y, z)\ dV$. Assume that the charge density of the solid E enclosed by the paraboloids $x = 5 - y^2 - z^2$ and $x = y^2 + z^2 - 5$ is equal to the distance from an arbitrary point of E to the origin. Set up the integral that gives the total charge inside the solid E.

Answer

$$\text{Total Charge inside the Solid } E \quad = \quad \int_{-\sqrt{5}}^{\sqrt{5}} \int_{-\sqrt{5-y^2}}^{\sqrt{5-y^2}} \int_{y^2+z^2-5}^{5-y^2-z^2} \sqrt{x^2 + y^2 + z^2}\ dx\ dz\ dy$$

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2.5: Cylindrical and Spherical Coordinates

Learning Objectives

- Convert from cylindrical to rectangular coordinates.
- Convert from rectangular to cylindrical coordinates.
- Convert from spherical to rectangular coordinates.
- Convert from rectangular to spherical coordinates.

The Cartesian coordinate system provides a straightforward way to describe the location of points in space. Some surfaces, however, can be difficult to model with equations based on the Cartesian system. This is a familiar problem; recall that in two dimensions, polar coordinates often provide a useful alternative system for describing the location of a point in the plane, particularly in cases involving circles. In this section, we look at two different ways of describing the location of points in space, both of them based on extensions of polar coordinates. As the name suggests, cylindrical coordinates are useful for dealing with problems involving cylinders, such as calculating the volume of a round water tank or the amount of oil flowing through a pipe. Similarly, spherical coordinates are useful for dealing with problems involving spheres, such as finding the volume of domed structures.

Cylindrical Coordinates

When we expanded the traditional Cartesian coordinate system from two dimensions to three, we simply added a new axis to model the third dimension. Starting with polar coordinates, we can follow this same process to create a new three-dimensional coordinate system, called the cylindrical coordinate system. In this way, cylindrical coordinates provide a natural extension of polar coordinates to three dimensions.

Definition: The Cylindrical Coordinate System

In the cylindrical coordinate system**,** a point in space (Figure $2.5.1$) is represented by the ordered triple (r, θ, z), where

- (r, θ) are the polar coordinates of the point's projection in the xy-plane
- z is the usual z-**coordinate** in the Cartesian coordinate system

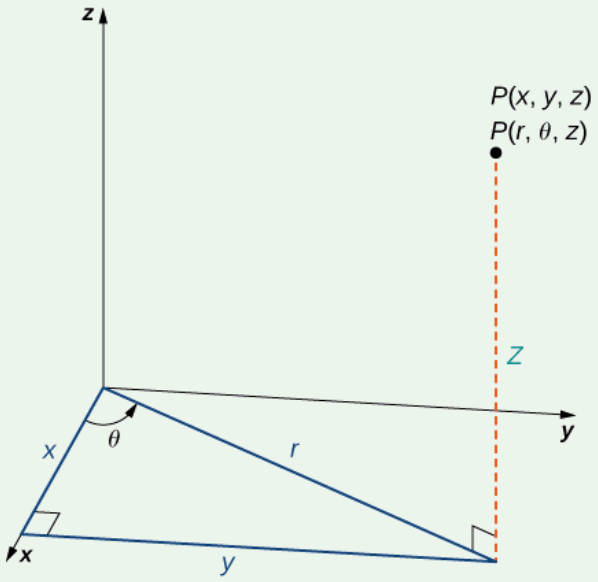


Figure $2.5.1$: The right triangle lies in the xy-plane. The length of the hypotenuse is r and θ is the measure of the angle formed by the positive x-axis and the hypotenuse. The z-coordinate describes the location of the point above or below the xy-plane.

In the xy-plane, the right triangle shown in Figure $2.5.1$ provides the key to transformation between cylindrical and Cartesian, or rectangular, coordinates.

Conversion Between Cylindrical and Cartesian Coordinates

The rectangular coordinates (x, y, z) and the cylindrical coordinates (r, θ, z) of a point are related as follows:

These equations are used to convert from cylindrical coordinates to rectangular coordinates.

- $x = r\cos\theta$
- $y = r\sin\theta$
- $z = z$

These equations are used to convert from rectangular coordinates to cylindrical coordinates

1. $r^2 = x^2 + y^2$
2. $\tan\theta = \dfrac{y}{x}$
3. $z = z$

As when we discussed conversion from rectangular coordinates to polar coordinates in two dimensions, it should be noted that the equation $\tan\theta = \dfrac{y}{x}$ has an infinite number of solutions. However, if we restrict θ to values between 0 and 2π, then we can find a unique solution based on the quadrant of the xy-plane in which original point (x, y, z) is located. Note that if $x = 0$, then the value of θ is either $\dfrac{\pi}{2}, \dfrac{3\pi}{2}$, or 0, depending on the value of y.

Notice that these equations are derived from properties of right triangles. To make this easy to see, consider point P in the xy -plane with rectangular coordinates $(x, y, 0)$ and with cylindrical coordinates $(r, \theta, 0)$, as shown in Figure $2.5.2$.

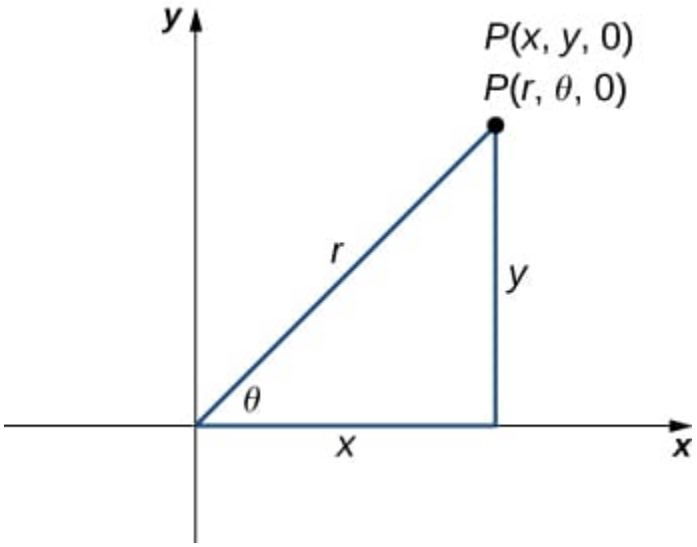


Figure $2.5.2$: The Pythagorean theorem provides equation $r^2 = x^2 + y^2$. Right-triangle relationships tell us that $x = r\cos\theta, y = r\sin\theta$, and $\tan\theta = y/x$.

Let's consider the differences between rectangular and cylindrical coordinates by looking at the surfaces generated when each of the coordinates is held constant. If c is a constant, then in rectangular coordinates, surfaces of the form $x = c, y = c$, or $z = c$ are all planes. Planes of these forms are parallel to the yz-plane, the xz-plane, and the xy-plane, respectively. When we convert to cylindrical coordinates, the z-coordinate does not change. Therefore, in cylindrical coordinates, surfaces of the form $z = c$ are planes parallel to the xy-plane. Now, let's think about surfaces of the form $r = c$. The points on these surfaces are at a fixed distance from the z-axis. In other words, these surfaces are vertical circular cylinders. Last, what about $\theta = c$? The points on a surface of the form $\theta = c$ are at a fixed angle from the x-axis, which gives us a half-plane that starts at the z-axis (Figures $2.5.3$ and $2.5.4$).

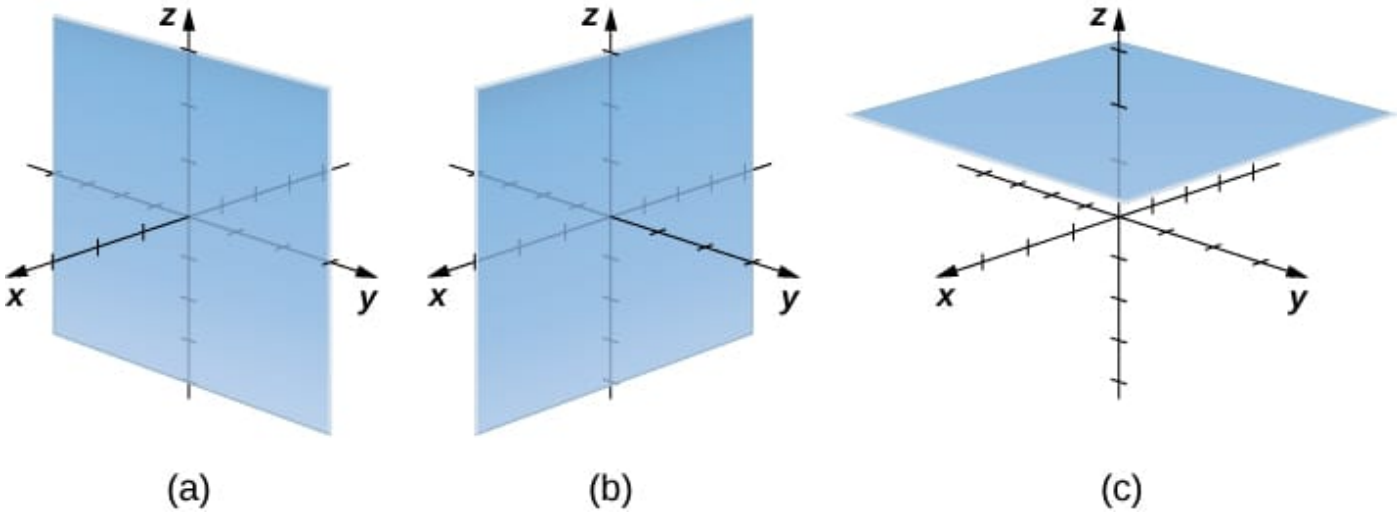


Figure 2.5.3: In rectangular coordinates, (a) surfaces of the form $x = c$ are planes parallel to the yz-plane, (b) surfaces of the form $y = c$ are planes parallel to the xz-plane, and (c) surfaces of the form $z = c$ are planes parallel to the xy-plane.

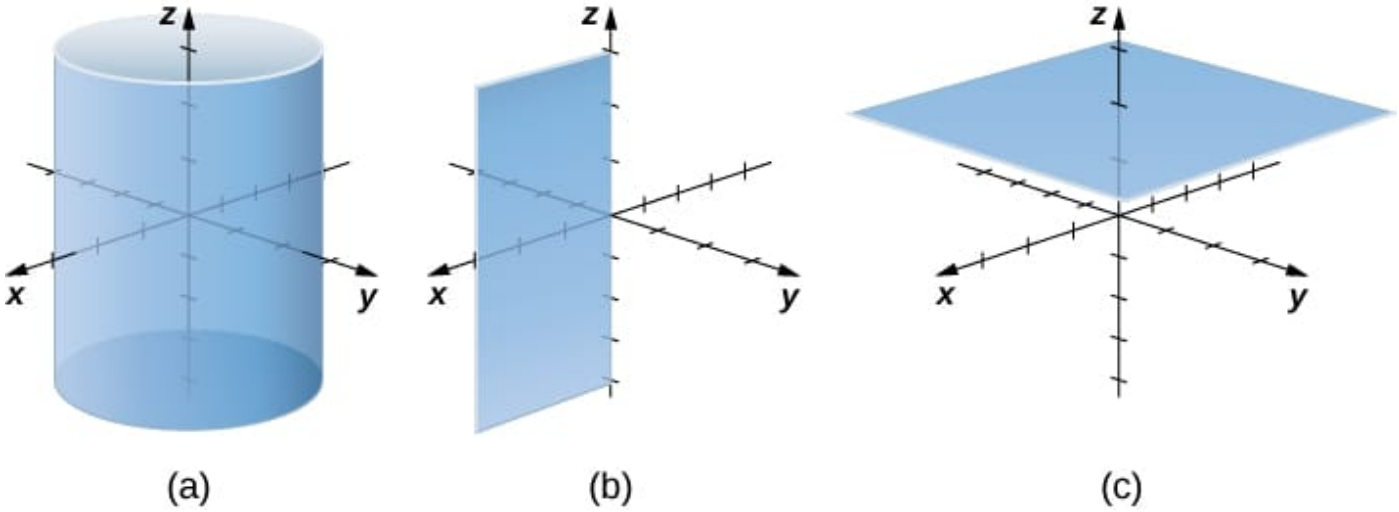


Figure 2.5.4: In cylindrical coordinates, (a) surfaces of the form $r = c$ are vertical cylinders of radius r, (b) surfaces of the form $\theta = c$ are half-planes at angle θ from the x-axis, and (c) surfaces of the form $z = c$ are planes parallel to the xy-plane.

Example 2.5.1: Converting from Cylindrical to Rectangular Coordinates

Plot the point with cylindrical coordinates $(4, \frac{2\pi}{3}, -2)$ and express its location in rectangular coordinates.

Solution

Conversion from cylindrical to rectangular coordinates requires a simple application of the equations listed in above:

$$x = r\cos\theta = 4\cos\frac{2\pi}{3} = -2$$

$$y = r\sin\theta = 4\sin\frac{2\pi}{3} = 2\sqrt{3}.$$

$$z = -2$$

The point with cylindrical coordinates $(4, \frac{2\pi}{3}, -2)$ has rectangular coordinates $(-2, 2\sqrt{3}, -2)$ (Figure 2.5.5).

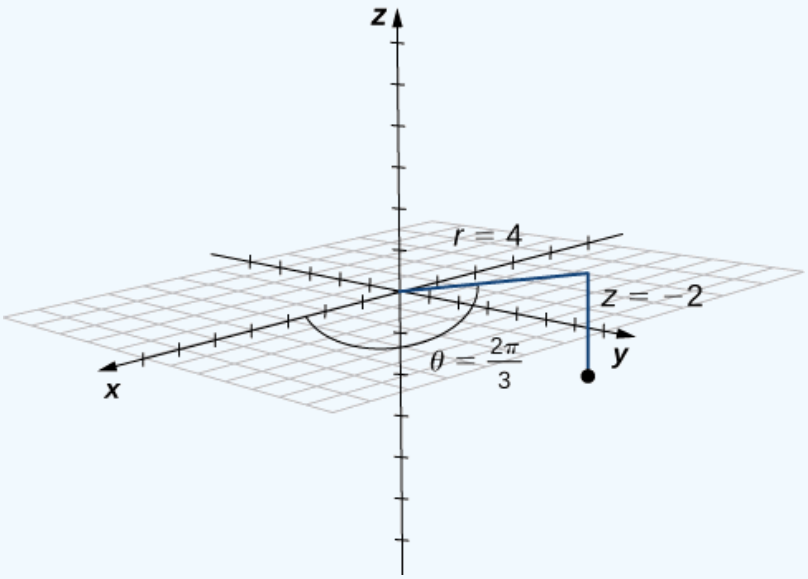


Figure 2.5.5: The projection of the point in the xy-plane is 4 units from the origin. The line from the origin to the point's projection forms an angle of $\frac{2\pi}{3}$ with the positive x-axis. The point lies 2 units below the xy-plane.

Exercise 2.5.1

Point R has cylindrical coordinates $(5, \frac{\pi}{6}, 4)$. Plot R and describe its location in space using rectangular, or Cartesian, coordinates.

Hint available at the end of this section.

If this process seems familiar, it is with good reason. This is exactly the same process that we followed in Introduction to Parametric Equations and Polar Coordinates to convert from polar coordinates to two-dimensional rectangular coordinates.

Example 2.5.2: Converting from Rectangular to Cylindrical Coordinates

Convert the rectangular coordinates $(1, -3, 5)$ to cylindrical coordinates.

Solution

Use the second set of equations from Note to translate from rectangular to cylindrical coordinates:

$$\begin{aligned} r^2 &= x^2 + y^2 \\ r &= \pm\sqrt{1^2 + (-3)^2} \\ &= \pm\sqrt{10}. \end{aligned}$$

We choose the positive square root, so $r = \sqrt{10}$. Now, we apply the formula to find θ. In this case, y is negative and x is positive, which means we must select the value of θ between $\frac{3\pi}{2}$ and 2π:

$$\begin{aligned} \tan\theta &= \frac{y}{x} & &= \frac{-3}{1} \\ \theta &= \arctan(-3) & &\approx 5.03\,\text{rad}. \end{aligned}$$

In this case, the z-coordinates are the same in both rectangular and cylindrical coordinates:

$$z = 5.$$

The point with rectangular coordinates $(1, -3, 5)$ has cylindrical coordinates approximately equal to $(\sqrt{10}, 5.03, 5)$.

Exercise 2.5.2

Convert point $(-8, 8, -7)$ from Cartesian coordinates to cylindrical coordinates.

Hint available at the end of this section.

The use of cylindrical coordinates is common in fields such as physics. Physicists studying electrical charges and the capacitors used to store these charges have discovered that these systems sometimes have a cylindrical symmetry. These systems have complicated modeling equations in the Cartesian coordinate system, which make them difficult to describe and analyze. The equations can often be expressed in more simple terms using cylindrical coordinates. For example, the cylinder described by equation $x^2 + y^2 = 25$ in the Cartesian system can be represented by cylindrical equation $r = 5$.

Example 2.5.3: Identifying Surfaces in the Cylindrical Coordinate System

Describe the surfaces with the given cylindrical equations.

a. $\theta = \dfrac{\pi}{4}$
b. $r^2 + z^2 = 9$
c. $z = r$

Solution

a. When the angle θ is held constant while r and z are allowed to vary, the result is a half-plane (Figure 2.5.7).

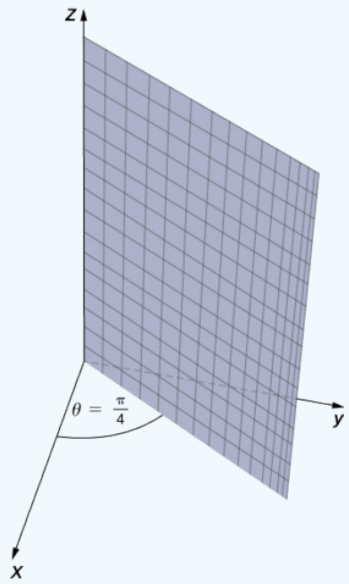


Figure 2.5.7: In polar coordinates, the equation $\theta = \frac{\pi}{4}$ describes the ray extending diagonally through the first quadrant. In three dimensions, this same equation describes a half-plane.

b. Substitute $r^2 = x^2 + y^2$ into equation $r^2 + z^2 = 9$ to express the rectangular form of the equation: $x^2 + y^2 + z^2 = 9$. This equation describes a sphere centered at the origin with radius 3 (Figure 2.5.8).

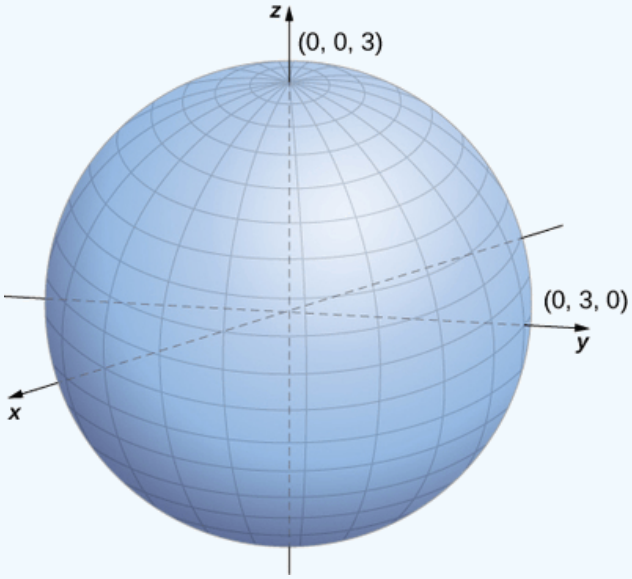


Figure 2.5.8: The sphere centered at the origin with radius 3 can be described by the cylindrical equation $r^2 + z^2 = 9$.

c. To describe the surface defined by equation $z = r$, is it useful to examine traces parallel to the xy-plane. For example, the trace in plane $z = 1$ is circle $r = 1$, the trace in plane $z = 3$ is circle $r = 3$, and so on. Each trace is a circle. As the value of z increases, the radius of the circle also increases. The resulting surface is a cone (Figure 2.5.9).

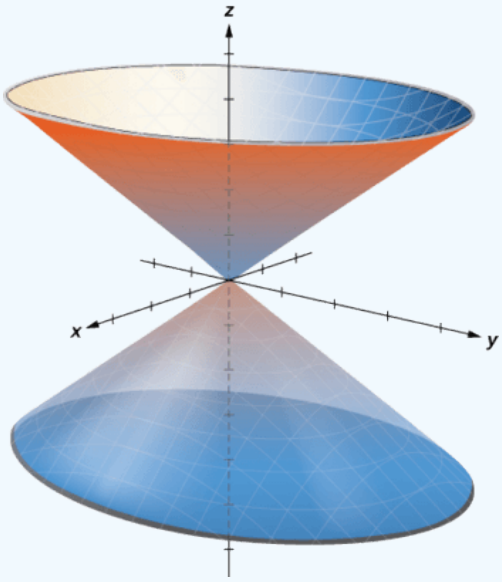


Figure 2.5.9: the surface defined by equation $z = r$ is a cone, one cone above the xy-plane and another below the xy-plane. The traces of the cone $z = r$ in planes parallel to the xy-plane are circles. The radius of the circles increases as z increases.

Exercise 2.5.3

Describe the surface with cylindrical equation $r = 6$.

Hint available at the end of this section.

Spherical Coordinates

In the Cartesian coordinate system, the location of a point in space is described using an ordered triple in which each coordinate represents a distance. In the cylindrical coordinate system, location of a point in space is described using two distances $(r$ and $z)$ and an angle measure (θ). In the spherical coordinate system, we again use an ordered triple to describe the location of a point in space. In this case, the triple describes one distance and two angles. Spherical coordinates make it simple to describe a sphere, just as cylindrical coordinates make it easy to describe a cylinder. Grid lines for spherical coordinates are based on angle measures, like those for polar coordinates.

Definition: Spherical Coordinate System

In the *spherical coordinate system*, a point P in space (Figure $2.5.10$) is represented by the ordered triple (ρ, θ, φ) where

- ρ (the Greek letter rho) is the distance between P and the origin $(\rho \neq 0)$;
- θ is the same angle used to describe the location in cylindrical coordinates;
- φ (the Greek letter phi) is the angle formed by the positive z-axis and line segment $\bar{OP}$, where O is the origin and $0 \leq \varphi \leq \pi$.

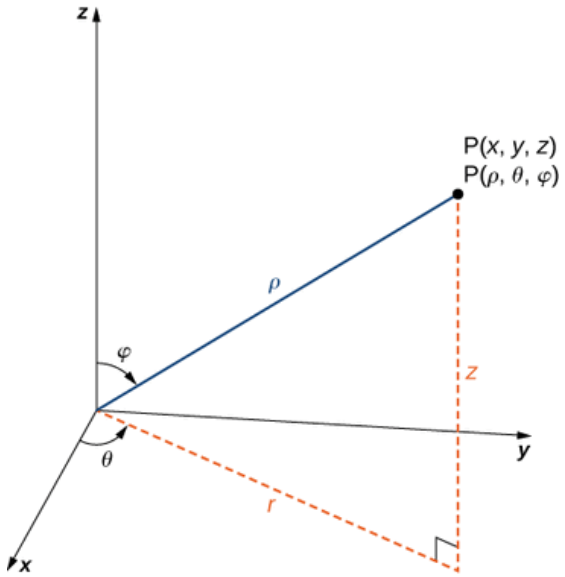


Figure $2.5.10$: This image represents the relationship among spherical coordinates (ρ, θ, φ), rectangular (Cartesian) coordinates (x, y, z), and cylindrical coordinates (r, θ, z).

By convention, the origin is represented as $(0, 0, 0)$ in spherical coordinates.

Converting Among Spherical, Cylindrical, and Rectangular Coordinates

Rectangular coordinates (x, y, z), cylindrical coordinates (r, θ, z), and spherical coordinates (ρ, θ, φ) of a point are related as follows:

Convert from spherical coordinates to rectangular coordinates

These equations are used to convert from spherical coordinates to rectangular coordinates.

- $x = \rho \sin \varphi \cos \theta$
- $y = \rho \sin \varphi \sin \theta$
- $z = \rho \cos \varphi$

Convert from rectangular coordinates to spherical coordinates

These equations are used to convert from rectangular coordinates to spherical coordinates.

- $\rho^2 = x^2 + y^2 + z^2$
- $\tan\theta = \dfrac{y}{x}$
- $\varphi = \arccos\left(\dfrac{z}{\sqrt{x^2+y^2+z^2}}\right)$.

Convert from spherical coordinates to cylindrical coordinates

These equations are used to convert from spherical coordinates to cylindrical coordinates.

- $r = \rho\sin\varphi$
- $\theta = \theta$
- $z = \rho\cos\varphi$

Convert from cylindrical coordinates to spherical coordinates

These equations are used to convert from cylindrical coordinates to spherical coordinates.

- $\rho = \sqrt{r^2+z^2}$
- $\theta = \theta$
- $\varphi = \arccos\left(\dfrac{z}{\sqrt{r^2+z^2}}\right)$

The formulas to convert from spherical coordinates to rectangular coordinates may seem complex, but they are straightforward applications of trigonometry. Looking at Figure $2.5.11$, it is easy to see that $r = \rho\sin\varphi$. Then, looking at the triangle in the xy -plane with r as its hypotenuse, we have $x = r\cos\theta = \rho\sin\varphi\cos\theta$. The derivation of the formula for y is similar. Figure also shows that $\rho^2 = r^2 + z^2 = x^2 + y^2 + z^2$ and $z = \rho\cos\varphi$. Solving this last equation for φ and then substituting $\rho = \sqrt{r^2+z^2}$ (from the first equation) yields $\varphi = \arccos\left(\dfrac{z}{\sqrt{r^2+z^2}}\right)$. Also, note that, as before, we must be careful when using the formula $\tan\theta = \dfrac{y}{x}$ to choose the correct value of θ.

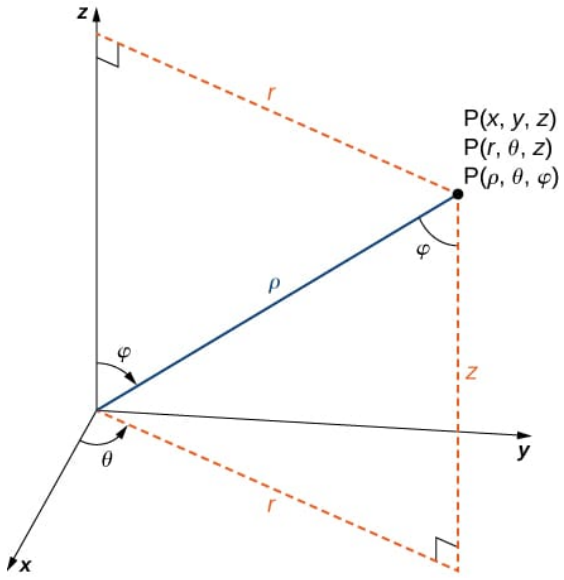


Figure $2.5.11$: The equations that convert from one system to another are derived from right-triangle relationships.

As we did with cylindrical coordinates, let's consider the surfaces that are generated when each of the coordinates is held constant. Let c be a constant, and consider surfaces of the form $\rho = c$. Points on these surfaces are at a fixed distance from the origin and form a sphere. The coordinate θ in the spherical coordinate system is the same as in the cylindrical coordinate system, so surfaces of the form $\theta = c$ are half-planes, as before. Last, consider surfaces of the form $\varphi = 0$. The points on these surfaces are at a fixed angle from the z-axis and form a half-cone (Figure $2.5.12$).

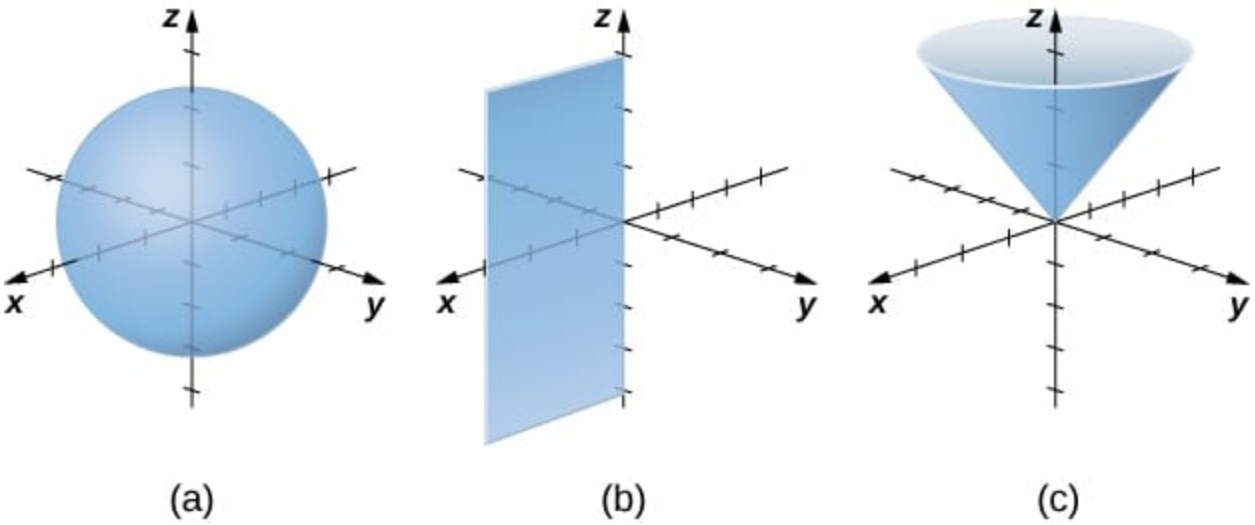


Figure 2.5.12: In spherical coordinates, surfaces of the form $\rho = c$ are spheres of radius ρ (a), surfaces of the form $\theta = c$ are half-planes at an angle θ from the x-axis (b), and surfaces of the form $\phi = c$ are half-cones at an angle ϕ from the z-axis (c).

Example 2.5.4: Converting From Spherical Coordinates

Plot the point with spherical coordinates $(8, \frac{\pi}{3}, \frac{\pi}{6})$ and express its location in both rectangular and cylindrical coordinates.

Solution

Use the equations in Note to translate between spherical and cylindrical coordinates (Figure 2.5.13):

$$\begin{aligned} x &= \rho \sin\varphi \cos\theta \\ &= 8 \sin(\frac{\pi}{6}) \cos(\frac{\pi}{3}) \\ &= 8(\frac{1}{2})\frac{1}{2} \\ &= 2 \\ y &= \rho \sin\varphi \sin\theta \\ &= 8 \sin(\frac{\pi}{6}) \sin(\frac{\pi}{3}) \\ &= 8(\frac{1}{2})\frac{\sqrt{3}}{2} \\ &= 2\sqrt{3} \\ z &= \rho \cos\varphi \\ &= 8 \cos(\frac{\pi}{6}) \\ &= 8(\frac{\sqrt{3}}{2}) \\ &= 4\sqrt{3} \end{aligned}$$

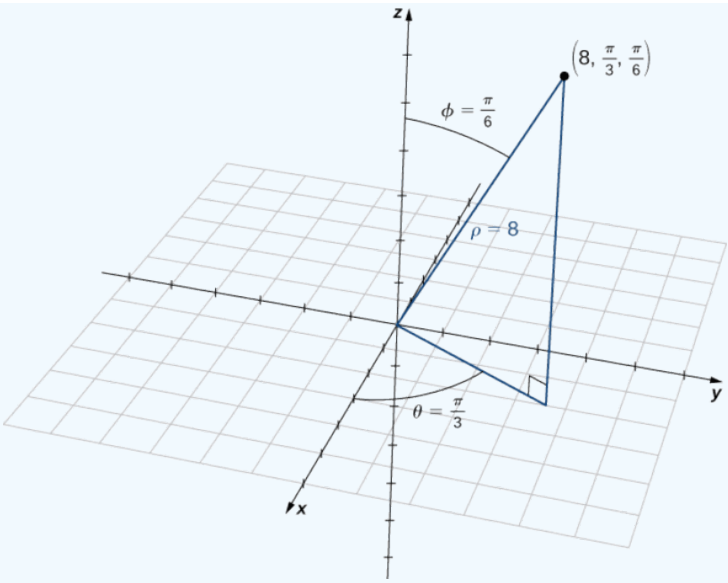


Figure 2.5.13: The projection of the point in the xy-plane is 4 units from the origin. The line from the origin to the point's projection forms an angle of $\pi/3$ with the positive x-axis. The point lies $4\sqrt{3}$ units above the xy-plane.

The point with spherical coordinates $(8, \frac{\pi}{3}, \frac{\pi}{6})$ has rectangular coordinates $(2, 2\sqrt{3}, 4\sqrt{3})$.

Finding the values in cylindrical coordinates is equally straightforward:

$$\begin{aligned} r &= \rho \sin \varphi \\ &= 8 \sin \frac{\pi}{6} \qquad = 4 \\ \theta &= \theta \\ z &= \rho \cos \varphi \\ &= 8 \cos \frac{\pi}{6} \\ &= 4\sqrt{3}. \end{aligned}$$

Thus, cylindrical coordinates for the point are $(4, \frac{\pi}{3}, 4\sqrt{3})$.

Exercise 2.5.4

Plot the point with spherical coordinates $(2, -\frac{5\pi}{6}, \frac{\pi}{6})$ and describe its location in both rectangular and cylindrical coordinates.

Hint available at the end of this section.

Example 2.5.5: Converting From Rectangular Coordinates

Convert the rectangular coordinates $(-1, 1, \sqrt{6})$ to both spherical and cylindrical coordinates.

Solution

Start by converting from rectangular to spherical coordinates:

$$\rho^2 = x^2 + y^2 + z^2 = (-1)^2 + 1^2 + (\sqrt{6})^2 = 8$$

$$\tan\theta = \frac{1}{-1}$$

$$\rho = 2\sqrt{2} \text{ and } \theta = \arctan(-1) = \frac{3\pi}{4}.$$

Because $(x, y) = (-1, 1)$, then the correct choice for θ is $\frac{3\pi}{4}$.

There are actually two ways to identify φ. We can use the equation $\varphi = \arccos(\frac{z}{\sqrt{x^2+y^2+z^2}})$. A more simple approach, however, is to use equation $z = \rho\cos\varphi$. We know that $z = \sqrt{6}$ and $\rho = 2\sqrt{2}$, so

$$\sqrt{6} = 2\sqrt{2}\cos\varphi, \text{ so } \cos\varphi = \frac{\sqrt{6}}{2\sqrt{2}} = \frac{\sqrt{3}}{2}$$

and therefore $\varphi = \frac{\pi}{6}$. The spherical coordinates of the point are $(2\sqrt{2}, \frac{3\pi}{4}, \frac{\pi}{6})$.

To find the cylindrical coordinates for the point, we need only find r:

$$r = \rho\sin\varphi = 2\sqrt{2}\sin(\frac{\pi}{6}) = \sqrt{2}.$$

The cylindrical coordinates for the point are $(\sqrt{2}, \frac{3\pi}{4}, \sqrt{6})$.

Watch the accompanying video for a visual demonstration of the concept.

- **Video length**: 3 minutes 15 seconds.
- **Context**: This video demonstrates how to converting an Equation in Rectangular Coordinates to Cylindrical and Spherical Coordinates.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Example 2.5.6: Identifying Surfaces in the Spherical Coordinate System

Describe the surfaces with the given spherical equations.

a. $\theta = \frac{\pi}{3}$
b. $\varphi = \frac{5\pi}{6}$
c. $\rho = 6$
d. $\rho = \sin\theta\sin\varphi$

Solution

a. The variable θ represents the measure of the same angle in both the cylindrical and spherical coordinate systems. Points with coordinates $(\rho, \frac{\pi}{3}, \varphi)$ lie on the plane that forms angle $\theta = \frac{\pi}{3}$ with the positive x-axis. Because $\rho > 0$, the surface described by equation $\theta = \frac{\pi}{3}$ is the half-plane shown in Figure 2.5.15.

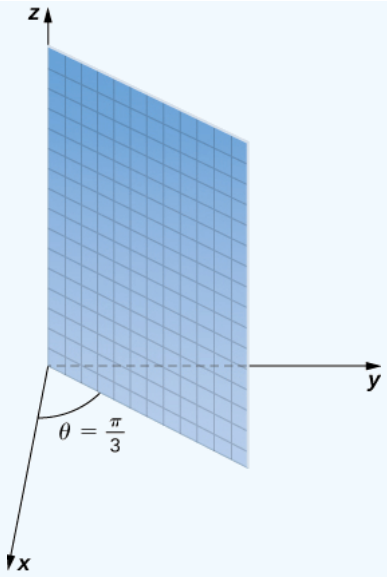


Figure 2.5.15: The surface described by equation $\theta = \frac{\pi}{3}$ in spherical coordinates represents a half-plane extending from the z-axis at an angle of $\theta = \frac{\pi}{3}$ with the positive x-axis.

b. Equation $\varphi = \frac{5\pi}{6}$ describes all points in the spherical coordinate system that lie on a line from the origin forming an angle measuring $\frac{5\pi}{6}$ rad with the positive z-axis. These points form a half-cone (Figure 2.5.16). Because there is only one value for φ that is measured from the positive z-axis, we do not get the full cone (with two pieces).

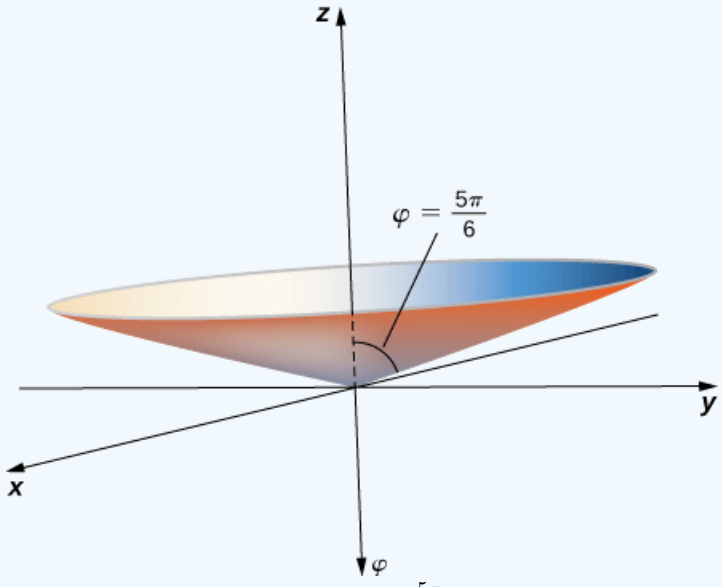


Figure 2.5.16: The equation $\varphi = \frac{5\pi}{6}$ describes a cone.

To find the equation in rectangular coordinates, use equation $\varphi = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$.

$$\frac{5\pi}{6} = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

$$\cos\frac{5\pi}{6} = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$-\frac{\sqrt{3}}{2} = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$\frac{3}{4} = \frac{z^2}{x^2+y^2+z^2}$$

$$\frac{3x^2}{4} + \frac{3y^2}{4} + \frac{3z^2}{4} = z^2$$

$$\frac{3x^2}{4} + \frac{3y^2}{4} - \frac{z^2}{4} = 0.$$

This is the equation of a cone centered on the z-axis.

c. Equation $\rho = 6$ describes the set of all points 6 units away from the origin—a sphere with radius 6 (Figure $2.5.17$).

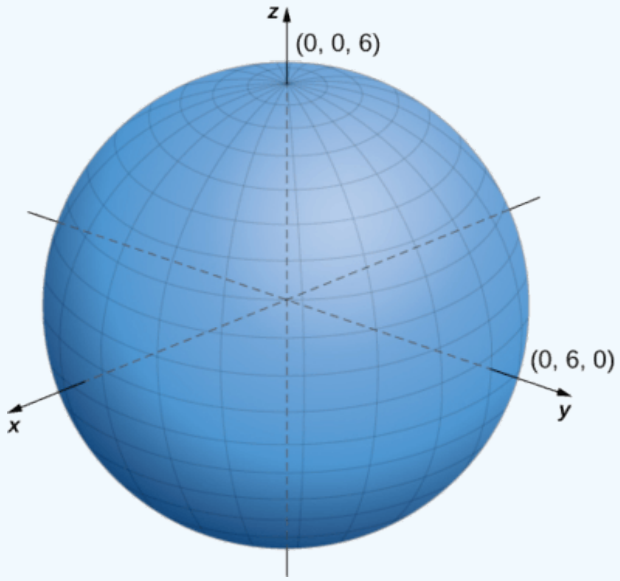


Figure $2.5.17$: Equation $\rho = 6$ describes a sphere with radius 6 centered at origin.

d. To identify this surface, convert the equation from spherical to rectangular coordinates, using equations $y = \rho sin\varphi \sin\theta$ and $\rho^2 = x^2 + y^2 + z^2$:

$\rho = \sin\theta \sin\varphi$

$\rho^2 = \rho \sin\theta \sin\varphi$ Multiply both sides of the equation by ρ.

$x^2 + y^2 + z^2 = y$ Substitute rectangular variables using the equations above.

$x^2 + y^2 - y + z^2 = 0$ Subtract y from both sides of the equation.

$x^2 + y^2 - y + \frac{1}{4} + z^2 = \frac{1}{4}$ Complete the square.

$x^2 + (y - \frac{1}{2})^2 + z^2 = \frac{1}{4}$. Rewrite the middle terms as a perfect square.

The equation describes a sphere centered at point $(0, \frac{1}{2}, 0)$ with radius $\frac{1}{2}$.

Exercise 2.5.5

Describe the surfaces defined by the following equations.

a. $\rho = 13$
b. $\theta = \dfrac{2\pi}{3}$
c. $\varphi = \dfrac{\pi}{4}$

Hint available at the end of this section.

Spherical coordinates are useful in analyzing systems that have some degree of symmetry about a point, such as the volume of the space inside a domed stadium or wind speeds in a planet's atmosphere. A sphere that has Cartesian equation $x^2 + y^2 + z^2 = c^2$ has the simple equation $\rho = c$ in spherical coordinates.

In geography, latitude and longitude are used to describe locations on Earth's surface, as shown in Figure. Although the shape of Earth is not a perfect sphere, we use spherical coordinates to communicate the locations of points on Earth. Let's assume Earth has the shape of a sphere with radius 4000 mi. We express angle measures in degrees rather than radians because latitude and longitude are measured in degrees.

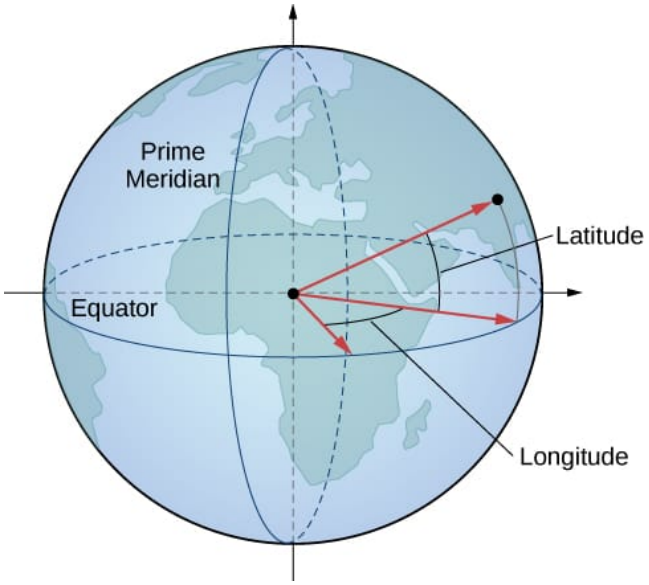


Figure 2.5.18: In the latitude–longitude system, angles describe the location of a point on Earth relative to the equator and the prime meridian.

Let the center of Earth be the center of the sphere, with the ray from the center through the North Pole representing the positive z-axis. The prime meridian represents the trace of the surface as it intersects the xz-plane. The equator is the trace of the sphere intersecting the xy-plane.

Example 2.5.7: Converting Latitude and Longitude to Spherical Coordinates

The latitude of Columbus, Ohio, is $40\,°$ N and the longitude is $83\,°$ W, which means that Columbus is $40\,°$ north of the equator. Imagine a ray from the center of Earth through Columbus and a ray from the center of Earth through the equator directly south of Columbus. The measure of the angle formed by the rays is $40\,°$. In the same way, measuring from the prime meridian, Columbus lies $83\,°$ to the west. Express the location of Columbus in spherical coordinates.

Solution

The radius of Earth is 4000mi, so $\rho = 4000$. The intersection of the prime meridian and the equator lies on the positive x-axis. Movement to the west is then described with negative angle measures, which shows that $\theta = -83\,°$, Because Columbus lies $40\,°$ north of the equator, it lies $50\,°$ south of the North Pole, so $\varphi = 50\,°$. In spherical coordinates, Columbus lies at point $(4000, -83\,°, 50\,°)$.

Exercise 2.5.6

Sydney, Australia is at $34\,°S$ and $151\,°E$. Express Sydney's location in spherical coordinates.

Hint available at the end of this section.

Cylindrical and spherical coordinates give us the flexibility to select a coordinate system appropriate to the problem at hand. A thoughtful choice of coordinate system can make a problem much easier to solve, whereas a poor choice can lead to unnecessarily complex calculations. In the following example, we examine several different problems and discuss how to select the best coordinate system for each one.

Example 2.5.8: Choosing the Best Coordinate System

In each of the following situations, we determine which coordinate system is most appropriate and describe how we would orient the coordinate axes. There could be more than one right answer for how the axes should be oriented, but we select an orientation that makes sense in the context of the problem. Note: There is not enough information to set up or solve these problems; we simply select the coordinate system (Figure 2.5.19).

a. Find the center of gravity of a bowling ball illustrated in Figure 2.5.19: (a)



Figure 2.5.19: (a) Bowlingballs on ball return (CC-BY-2.0 modification of work by scl hua, Wikimedia, Wikimedia Commons.)

b. Determine the velocity of a submarine subjected to an ocean current illustrated in Figure 2.5.19: (b).



Figure 2.5.19: (b) Submarine conducts alpha trials in the Atlantic Ocean (CC-BY-2.0 modification of work by DVIDSHUB, Flickr, Flicker.)

c. Calculate the pressure in a conical water tank illustrated in Figure 2.5.19: (c).



Figure 2.5.19: (c) A 12-inch (30-cm) tall orange traffic cone (CC-BY-2.0 modification of work by Michael Malak, Wikimedia Commons.)

d. Find the volume of oil flowing through a pipeline illustrated in Figure 2.5.19: (d).



Figure 2.5.19: (d) Oil flowing through pipeline (CC BY-SA 4.0 modification of work by Diego Delso, Wikimedia Commons).

e. Determine the amount of leather required to make a football illustrated in Figure 2.5.19: (e).



Figure 2.5.19: (e) The image shows a football, highlighting its shape and material requirements for construction (CC-BY-2.0 modification of work by Elvert Barnes, Flickr).

Solution

a. Clearly, a bowling ball is a sphere, so spherical coordinates would probably work best here. The origin should be located at the physical center of the ball. There is no obvious choice for how the x-, y- and z-axes should be oriented. Bowling balls normally have a weight block in the center. One possible choice is to align the z-axis with the axis of symmetry of the weight block.
b. A submarine generally moves in a straight line. There is no rotational or spherical symmetry that applies in this situation, so rectangular coordinates are a good choice. The z-axis should probably point upward. The x- and y-axes could be aligned to point east and north, respectively. The origin should be some convenient physical location, such as the starting position of the submarine or the location of a particular port.
c. A cone has several kinds of symmetry. In cylindrical coordinates, a cone can be represented by equation $z = kr$, where k is a constant. In spherical coordinates, we have seen that surfaces of the form $\varphi = c$ are half-cones. Last, in rectangular coordinates, elliptic cones are quadric surfaces and can be represented by equations of the form $z^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$. In

this case, we could choose any of the three. However, the equation for the surface is more complicated in rectangular coordinates than in the other two systems, so we might want to avoid that choice. In addition, we are talking about a water tank, and the depth of the water might come into play at some point in our calculations, so it might be nice to have a component that represents height and depth directly. Based on this reasoning, cylindrical coordinates might be the best choice. Choose the z-axis to align with the axis of the cone. The orientation of the other two axes is arbitrary. The origin should be the bottom point of the cone.

d. A pipeline is a cylinder, so cylindrical coordinates would be the best choice. In this case, however, we would likely choose to orient our z-axis with the center axis of the pipeline. The x-axis could be chosen to point straight downward or to some other logical direction. The origin should be chosen based on the problem statement. Note that this puts the z-axis in a horizontal orientation, which is a little different from what we usually do. It may make sense to choose an unusual orientation for the axes if it makes sense for the problem.

e. A football has rotational symmetry about a central axis, so cylindrical coordinates would work best. The z-axis should align with the axis of the ball. The origin could be the center of the ball or perhaps one of the ends. The position of the x-axis is arbitrary.

Exercise 2.5.7

Which coordinate system is most appropriate for creating a star map, as viewed from Earth (see Figure 2.5.20)?

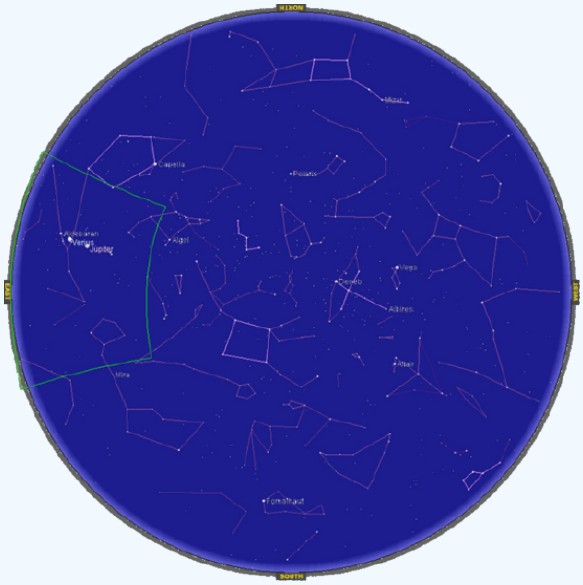


Figure 2.5.20: The star map viewed from Earth.

How should we orient the coordinate axes?

Hint available at the end of this section.

Key Concepts

- In the cylindrical coordinate system, a point in space is represented by the ordered triple (r, θ, z), where (r, θ) represents the polar coordinates of the point's projection in the xy-plane and z represents the point's projection onto the z-axis.
- To convert a point from cylindrical coordinates to Cartesian coordinates, use equations $x = r\cos\theta, y = r\sin\theta$, and $z = z$.
- To convert a point from Cartesian coordinates to cylindrical coordinates, use equations $r^2 = x^2 + y^2, \tan\theta = \dfrac{y}{x}$, and $z = z$.
- In the spherical coordinate system, a point P in space is represented by the ordered triple (ρ, θ, φ), where ρ is the distance between P and the origin $(\rho \neq 0), \theta$ is the same angle used to describe the location in cylindrical coordinates, and φ is the angle formed by the positive z-axis and line segment $\overline{OP}$, where O is the origin and $0 \leq \varphi \leq \pi$.
- To convert a point from spherical coordinates to Cartesian coordinates, use equations $x = \rho\sin\varphi\cos\theta, y = \rho\sin\varphi\sin\theta$, and z

$= \rho\cos\varphi$.

- To convert a point from Cartesian coordinates to spherical coordinates, use equations $\rho^2 = x^2 + y^2 + z^2, \tan\theta = \frac{y}{x}$, and $\varphi = \arccos(\frac{z}{\sqrt{x^2+y^2+z^2}})$.
- To convert a point from spherical coordinates to cylindrical coordinates, use equations $r = \rho\sin\varphi, \theta = \theta$, and $z = \rho\cos\varphi$.
- To convert a point from cylindrical coordinates to spherical coordinates, use equations $\rho = \sqrt{r^2+z^2}, \theta = \theta$, and $\varphi = \arccos(\frac{z}{\sqrt{r^2+z^2}})$.

Glossary

Cylindrical coordinate system
: a way to describe a location in space with an ordered triple (r, θ, z), where (r, θ) represents the polar coordinates of the point's projection in the xy-plane, and z represents the point's projection onto the z-axis

Spherical coordinate system
: a way to describe a location in space with an ordered triple (ρ, θ, φ), where ρ is the distance between P and the origin $(\rho \neq 0)$, θ is the same angle used to describe the location in cylindrical coordinates, and φ is the angle formed by the positive z-axis and line segment $\bar{OP}$, where O is the origin and $0 \leq \varphi \leq \pi$

Contributors and Attributions

- Gilbert Strang (MIT) and Edwin "Jed" Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

- Paul Seeburger edited the LaTeX on the page

Answers

Exercise 2.5.1

The first two components match the polar coordinates of the point in the xy-plane.

The rectangular coordinates of the point are $(\frac{5\sqrt{3}}{2}, \frac{5}{2}, 4)$.

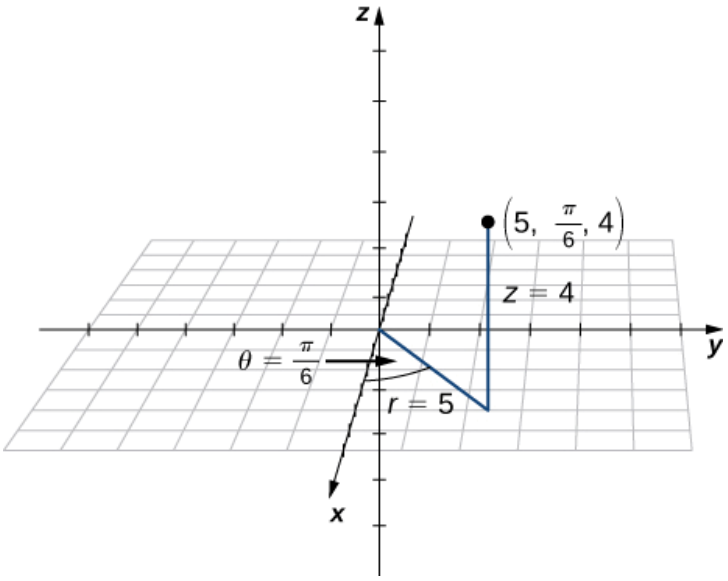


Figure 2.5.6: A plot of the point $(5, \frac{\pi}{6}, 4)$.

Exercise 2.5.2

$r^2 = x^2 + y^2$ and $\tan\theta = \frac{y}{x}$

$(8\sqrt{2}, \frac{3\pi}{4}, -7)$

Exercise 2.5.3

The θ and z components of points on the surface can take any value.

This surface is a cylinder with radius 6.

Exercise $2.5.4$

Converting the coordinates first may help to find the location of the point in space more easily.

Cartesian: $(-\frac{\sqrt{3}}{2}, -\frac{1}{2}, \sqrt{3})$, cylindrical: $(1, -\frac{5\pi}{6}, \sqrt{3})$

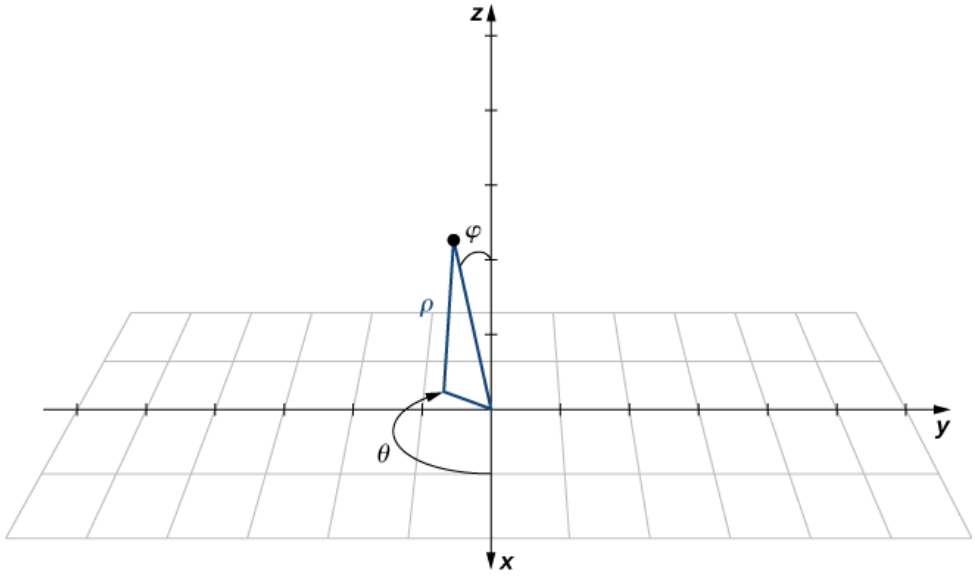


Figure $2.5.14$: Plot of the point $(2, -\frac{5\pi}{6}, \frac{\pi}{6})$.

Exercise $2.5.5$

Think about what each component represents and what it means to hold that component constant.

This is the set of all points 13 units from the origin. This set forms a sphere with radius 13.

This set of points forms a half plane. The angle between the half plane and the positive x-axis is $\theta = \dfrac{2\pi}{3}$.

Let P be a point on this surface. The position vector of this point forms an angle of $\varphi = \dfrac{\pi}{4}$ with the positive z-axis, which means that points closer to the origin are closer to the axis. These points form a half-cone.

Exercise $2.5.6$

Because Sydney lies south of the equator, we need to add $90\,^\circ$ to find the angle measured from the positive z-axis.

$(4000, 151\,^\circ, 124\,^\circ)$

Exercise $2.5.7$

What kinds of symmetry are present in this situation?

Spherical coordinates with the origin located at the center of the earth, the z-axis aligned with the North Pole, and the x-axis aligned with the prime meridian

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2.5E: Exercises

Use Figure $2.5E.1$ as an aid in identifying the relationship between the rectangular, cylindrical, and spherical coordinate systems.

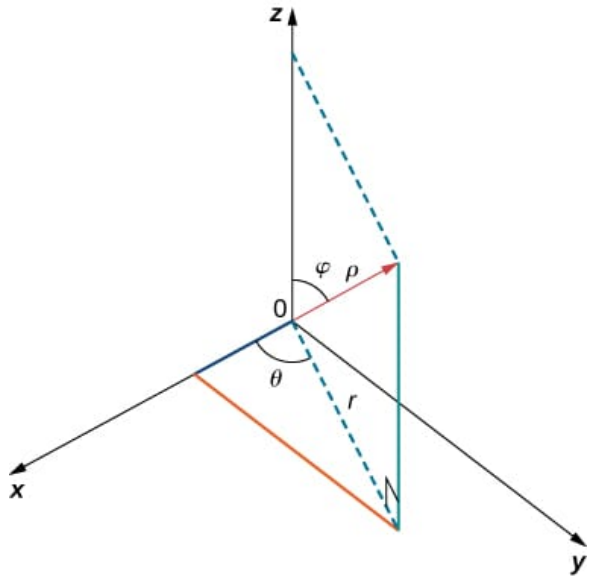


Figure $2.5E.1$: The image illustrates the relationship between rectangular, cylindrical, and spherical coordinate systems.

Exercises 1 - 4

The cylindrical coordinates (r, θ, z) of a point are given. Find the rectangular coordinates (x, y, z) of the point.

1. $\left(4, \frac{\pi}{6}, 3\right)$

   Answer

   $(2\sqrt{3}, 2, 3)$

2. $\left(3, \frac{\pi}{3}, 5\right)$
3. $\left(4, \frac{7\pi}{6}, 3\right)$

   Answer

   $(-2\sqrt{3}, -2, 3)$

4. $(2, \pi, -4)$

Exercises 5 - 8

The rectangular coordinates (x, y, z) of a point are given. Find the cylindrical coordinates (r, θ, z) of the point.

5. $(1, \sqrt{3}, 2)$

   Answer

   $\left(2, \frac{\pi}{3}, 2\right)$

6. $(1, 1, 5)$
7. $(3, -3, 7)$

Answer

$\left(3\sqrt{2}, -\frac{\pi}{4}, 7\right)$

8. $(-2\sqrt{2}, 2\sqrt{2}, 4)$

Exercises 9 - 16

The equation of a surface in cylindrical coordinates is given. Find the equation of the surface in rectangular coordinates. Identify and graph the surface.

9. [T] $r = 4$

Answer

A cylinder of equation $x^2 + y^2 = 16$, with its center at the origin and rulings parallel to the z-axis,

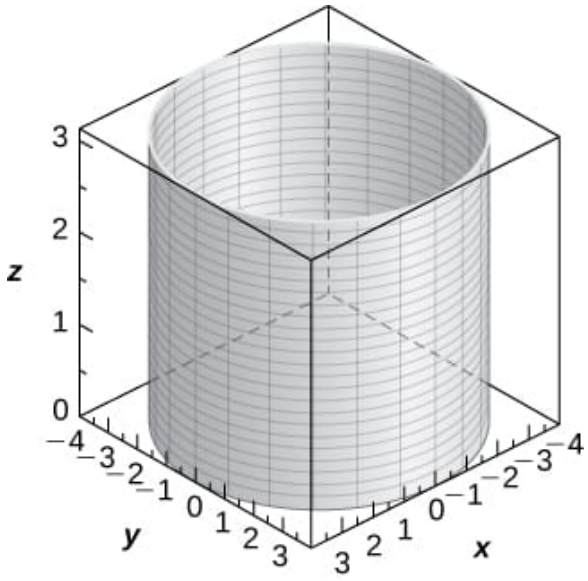


Figure $2.5E.2$: The image shows a cylinder of equation $x^2 + y^2 = 16$,.

10. [T] $z = r^2 \cos^2 \theta$
11. [T] $r^2 \cos(2\theta) + z^2 + 1 = 0$

Answer

Hyperboloid of two sheets of equation $-x^2 + y^2 - z^2 = 1$, with the y-axis as the axis of symmetry,

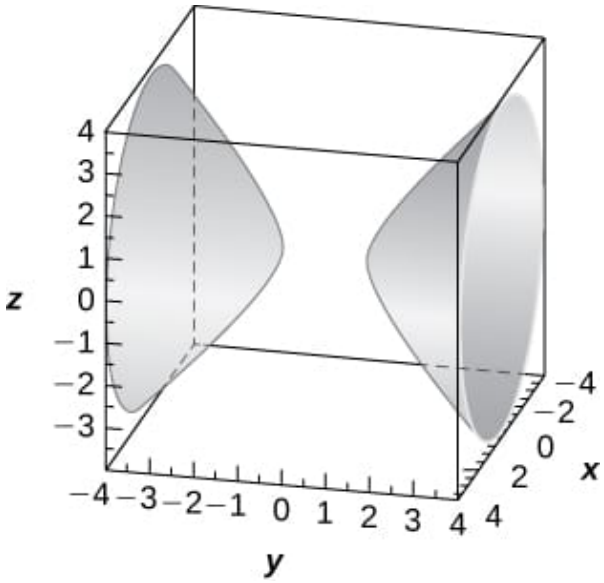


Figure $2.5E.3$: The image shows hyperboloid of two sheets of equation $-x^2 + y^2 - z^2 = 1$,.

12. [T] $r = 3 \sin \theta$
13. [T] $r = 2 \cos \theta$

Answer

Cylinder of equation $x^2 - 2x + y^2 = 0$, with a center at $(1, 0, 0)$ and radius 1, with rulings parallel to the z-axis,

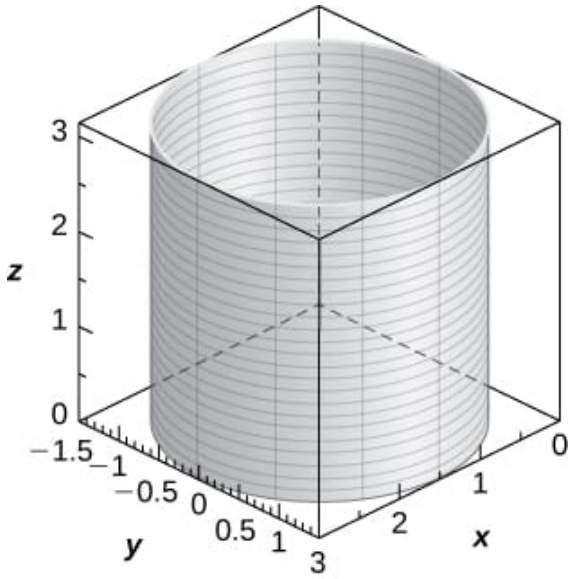


Figure $2.5E.4$: The image shows cylinder of equation $x^2 - 2x + y^2 = 0$,.

14. [T] $r^2 + z^2 = 5$
15. [T] $r = 2 \sec \theta$

Answer

Plane of equation $x = 2$,

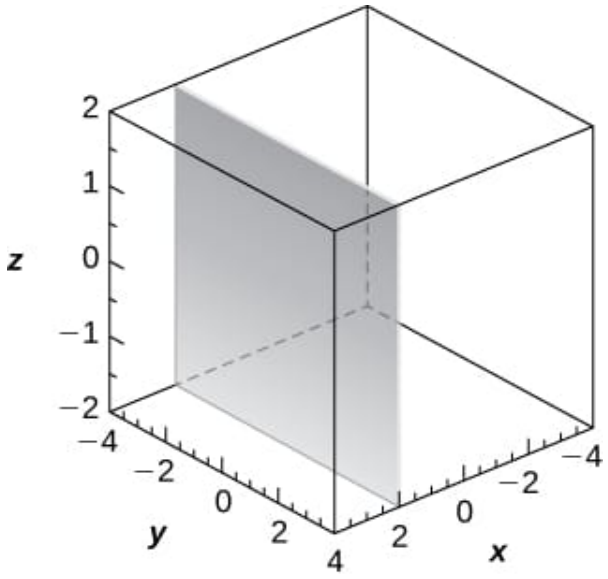


Figure $2.5E.5$: The image shows plane of equation $x = 2$,.

16. [T] $r = 3\csc\theta$

Exercises 17 - 22

The equation of a surface in rectangular coordinates is given. Find the equation of the surface in cylindrical coordinates.

17. $z = 3$

Answer

$z = 3$

18. $x = 6$
19. $x^2 + y^2 + z^2 = 9$

Answer

$r^2 + z^2 = 9$

20. $y = 2x^2$
21. $x^2 + y^2 - 16x = 0$

Answer

$r = 16\cos\theta, \quad r = 0$

22. $x^2 + y^2 - 3\sqrt{x^2 + y^2} + 2 = 0$

Exercises 23 - 26

The spherical coordinates (ρ, θ, φ) of a point are given. Find the rectangular coordinates (x, y, z) of the point.

23. $(3, 0, \pi)$

Answer

$(0, 0, -3)$

24. $\left(1, \frac{\pi}{6}, \frac{\pi}{6}\right)$
25. $\left(12, -\frac{\pi}{4}, \frac{\pi}{4}\right)$

Answer

$(6, -6, 6\sqrt{2})$

26. $\left(3, \frac{\pi}{4}, \frac{\pi}{6}\right)$

Exercises 27 - 30

The rectangular coordinates (x, y, z) of a point are given. Find the spherical coordinates (ρ, θ, φ) of the point. Express the measure of the angles in degrees rounded to the nearest integer.

27. $(4, 0, 0)$

Answer

$(4, 0, 90\,^\circ)$

28. $(-1, 2, 1)$
29. $(0, 3, 0)$

Answer

$(3, 90\,^\circ, 90\,^\circ)$

30. $(-2, 2\sqrt{3}, 4)$

Exercises 31 - 36

The equation of a surface in spherical coordinates is given. Find the equation of the surface in rectangular coordinates. Identify and graph the surface.

31. [T] $\rho = 3$

Answer

Sphere of equation $x^2 + y^2 + z^2 = 9$ centered at the origin with radius 3,

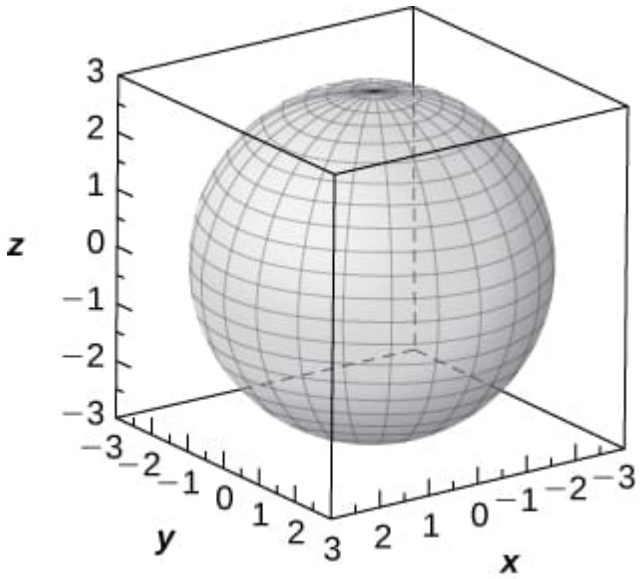


Figure $2.5E.6$: The image shows sphere of equation $x^2 + y^2 + z^2 = 9$.

32. [T] $\varphi = \frac{\pi}{3}$
33. [T] $\rho = 2\cos\varphi$

Answer

Sphere of equation $x^2 + y^2 + (z-1)^2 = 1$ centered at $(0, 0, 1)$ with radius 1,

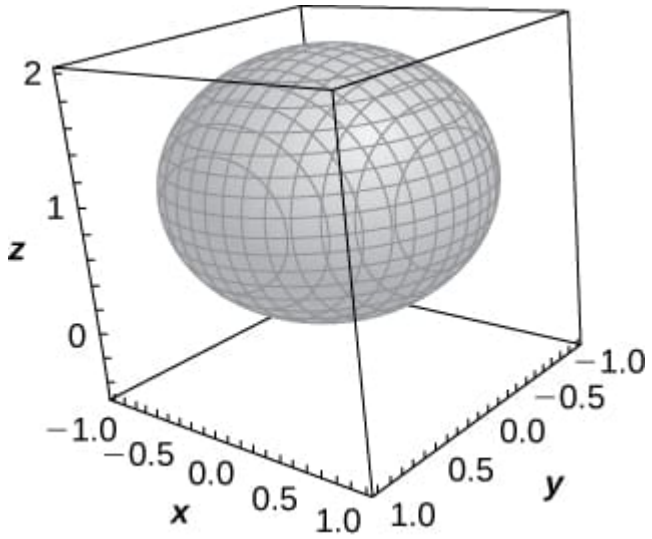


Figure $2.5E.7$: The image shows sphere of equation $x^2 + y^2 + (z-1)^2 = 1$.

34. [T] $\rho = 4 \csc \varphi$
35. [T] $\varphi = \frac{\pi}{2}$

Answer

The xy-plane of equation $z = 0$,

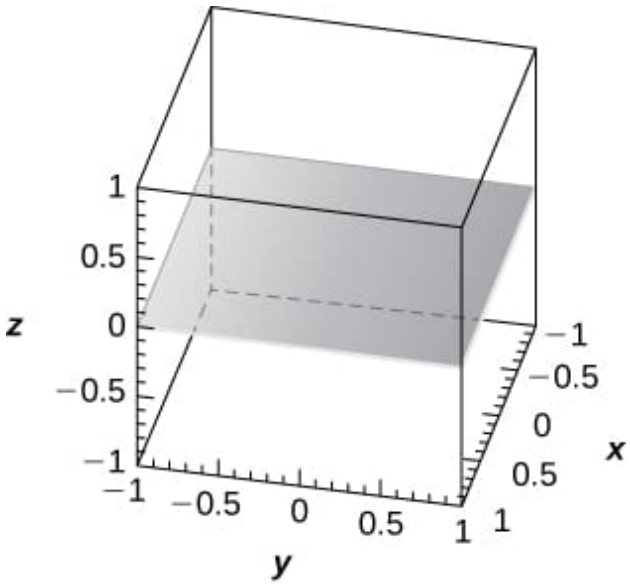


Figure $2.5E.8$: The image shows the xy-plane of equation $z = 0$,.

36. [T] $\rho = 6 \csc \varphi \sec \theta$

Exercises 37 - 40

The equation of a surface in rectangular coordinates is given. Find the equation of the surface in spherical coordinates. Identify the surface.

37. $x^2 + y^2 - 3z^2 = 0, \quad z \neq 0$

Answer

$\varphi = \frac{\pi}{3}$ or $\varphi = \frac{2\pi}{3}$; Elliptic cone

38. $x^2 + y^2 + z^2 - 4z = 0$
39. $z = 6$

Answer

$\rho \cos \varphi = 6$; Plane at $z = 6$

40. $x^2 + y^2 = 9$

Exercises 41 - 44

The cylindrical coordinates of a point are given. Find its associated spherical coordinates, with the measure of the angle φ in radians rounded to four decimal places.

41. [T] $(1, \frac{\pi}{4}, 3)$

Answer

$\left(\sqrt{10}, \frac{\pi}{4}, 0.3218\right)$

42. [T] $(5, \pi, 12)$
43. $(3, \frac{\pi}{2}, 3)$

Answer

$(3\sqrt{2}, \frac{\pi}{2}, \frac{\pi}{4})$

44. $(3, -\frac{\pi}{6}, 3)$

Exercises 45 - 48

The spherical coordinates of a point are given. Find its associated cylindrical coordinates.

45. $(2, -\frac{\pi}{4}, \frac{\pi}{2})$

Answer

$(2, -\frac{\pi}{4}, 0)$

46. $(4, \frac{\pi}{4}, \frac{\pi}{6})$
47. $(8, \frac{\pi}{3}, \frac{\pi}{2})$

Answer

$(8, \frac{\pi}{3}, 0)$

48. $(9, -\frac{\pi}{6}, \frac{\pi}{3})$

Exercises 49 - 52

Find the most suitable system of coordinates to describe the solids.

49. The solid situated in the first octant with a vertex at the origin and enclosed by a cube of edge length a, where $a > 0$

Answer

Cartesian system, $\{(x, y, z) \mid 0 \le x \le a, \ \ 0 \le y \le a, \ \ 0 \le z \le a\}$

50. A spherical shell determined by the region between two concentric spheres centered at the origin, of radii of a and b, respectively, where $b > a > 0$

51. A solid inside sphere $x^2 + y^2 + z^2 = 9$ and outside cylinder $\left(x - \frac{3}{2}\right)^2 + y^2 = \frac{9}{4}$

Answer

Cylindrical system, $\{(r, \theta, z) \mid r^2 + z^2 \le 9, \ \ r \ge 3\cos\theta, \ \ 0 \le \theta \le 2\pi\}$

52. A cylindrical shell of height 10 determined by the region between two cylinders with the same center, parallel rulings, and radii of 2 and 5, respectively

53. [T] Use a CAS or CalcPlot3D to graph in cylindrical coordinates the region between elliptic paraboloid $z = x^2 + y^2$ and cone $x^2 + y^2 - z^2 = 0$.

Answer

The region is described by the set of points $\{(r, \theta, z) \mid 0 \le r \le 1, \ \ 0 \le \theta \le 2\pi, \ \ r^2 \le z \le r\}$.

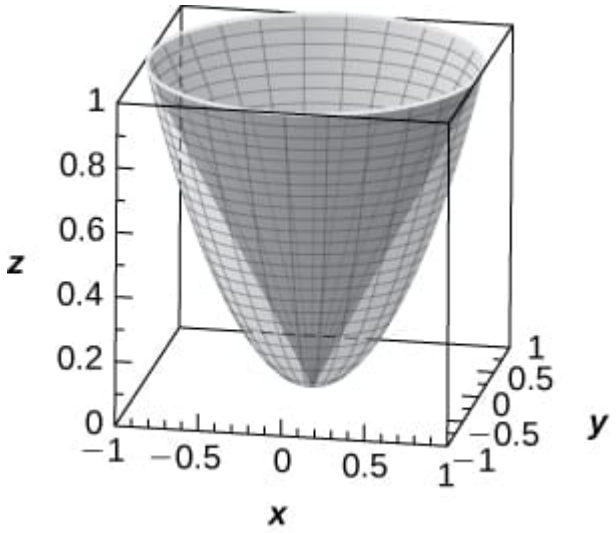


Figure $2.5E.9$: The image shows a region described by the set of points $\{(r, \theta, z) \mid 0 \le r \le 1, \ \ 0 \le \theta \le 2\pi, \ \ r^2 \le z \le r\}$.

54. [T] Use a CAS or CalcPlot3D to graph in spherical coordinates the "ice cream-cone region" situated above the xy-plane between sphere $x^2 + y^2 + z^2 = 4$ and elliptical cone $x^2 + y^2 - z^2 = 0$.

55. Washington, DC, is located at $39\,^\circ$ N and $77\,^\circ$ W (Figure $2.5E.10$). Assume the radius of Earth is 4000 mi. Express the location of Washington, DC, in spherical coordinates.



Figure $2.5E.10$: The image shows the location of Washington D.C. at $39\,^\circ$ N and $77\,^\circ$ W.

Answer

$(4000, -77\,^\circ, 51\,^\circ)$

56. San Francisco is located at $37.78\,^\circ N$ and $122.42\,^\circ W$. Assume the radius of Earth is 4000mi. Express the location of San Francisco in spherical coordinates.
57. Find the latitude and longitude of Rio de Janeiro if its spherical coordinates are $(4000, -43.17\,^\circ, 102.91\,^\circ)$.

Answer

$43.17\,^\circ W, 22.91\,^\circ S$

58. Find the latitude and longitude of Berlin if its spherical coordinates are $(4000, 13.38\,^\circ, 37.48\,^\circ)$.
59. [T] Consider the torus of equation $\left(x^2+y^2+z^2+R^2-r^2\right)^2 = 4R^2(x^2+y^2)$, where $R \geq r > 0$.

a. Write the equation of the torus in spherical coordinates.

b. If $R = r$, the surface is called a horn torus. Show that the equation of a horn torus in spherical coordinates is $\rho = 2R\sin\varphi$.

c. Use a CAS or CalcPlot3D to graph the horn torus with $R = r = 2$ in spherical coordinates.

Answer

a. $\rho = 0, \quad \rho + R^2 - r^2 - 2R\sin\varphi = 0$

c.

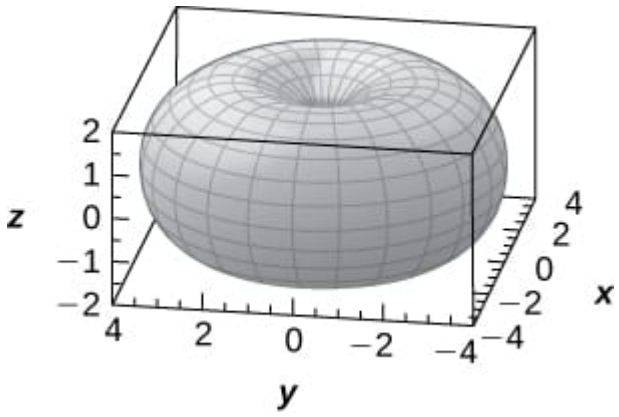


Figure $2.5E.11$: The image shows the torus of equation $\left(x^2+y^2+z^2+R^2-r^2\right)^2=4R^2(x^2+y^2)$, where $R \geq r>0$..

50. [T] The "bumpy sphere" with an equation in spherical coordinates is $\rho=a+b\cos(m\theta)\sin(n\varphi)$, with $\theta \in [0,2\pi]$ and $\varphi \in [0,\pi]$, where a and b are positive numbers and m and n are positive integers, may be used in applied mathematics to model tumor growth.

1. Show that the "bumpy sphere" is contained inside a sphere of equation $\rho=a+b$. Find the values of θ and φ at which the two surfaces intersect.

2. Use a CAS or CalcPlot3D to graph the surface for $a=14, b=2, m=4$, and $n=6$ along with sphere $\rho=a+b$.

3. Find the equation of the intersection curve of the surface at b. with the cone $\varphi=\frac{\pi}{12}$. Graph the intersection curve in the plane of intersection.

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Links to CalcPlot3D added by Paul Seeburger (Monroe Community College).

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2.6: Triple Integrals in Cylindrical and Spherical Coordinates

Learning Objectives

- Evaluate a triple integral by changing to cylindrical coordinates.
- Evaluate a triple integral by changing to spherical coordinates.

Earlier in this chapter we showed how to convert a double integral in rectangular coordinates into a double integral in polar coordinates in order to deal more conveniently with problems involving circular symmetry. A similar situation occurs with triple integrals, but here we need to distinguish between cylindrical symmetry and spherical symmetry. In this section we convert triple integrals in rectangular coordinates into a triple integral in either cylindrical or spherical coordinates.

Also recall the chapter prelude, which showed the opera house l'Hemisphèric in Valencia, Spain. It has four sections with one of the sections being a theater in a five-story-high sphere (ball) under an oval roof as long as a football field. Inside is an IMAX screen that changes the sphere into a planetarium with a sky full of 9000 twinkling stars. Using triple integrals in spherical coordinates, we can find the volumes of different geometric shapes like these.

Review of Cylindrical Coordinates

As we have seen earlier, in two-dimensional space $\mathbb{R}^2$ a point with rectangular coordinates (x, y) can be identified with (r, θ) in polar coordinates and vice versa, where $x = r\ \cos\theta$, $y = r\ \sin\ \theta$, $r^2 = x^2 + y^2$ and $\tan\ \theta = (\frac{y}{x})$ are the relationships between the variables.

In three-dimensional space $\mathbb{R}^3$ a point with rectangular coordinates (x, y, z) can be identified with cylindrical coordinates (r, θ, z) and vice versa. We can use these same conversion relationships, adding z as the vertical distance to the point from the xy-plane as shown in Figure $2.6.1$.

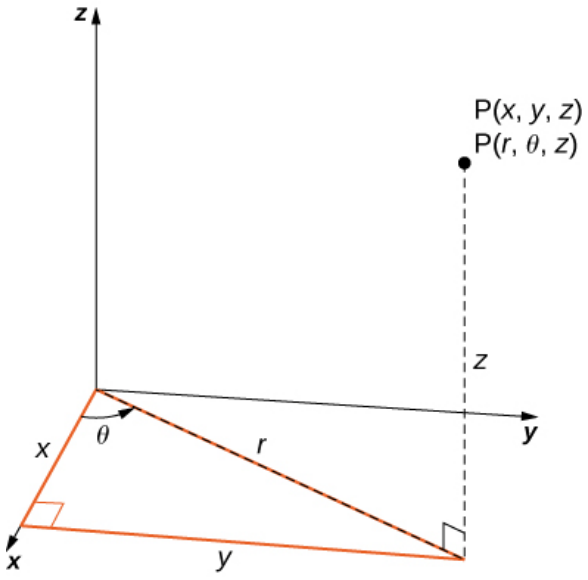


Figure $2.6.1$: Cylindrical coordinates are similar to polar coordinates with a vertical z coordinate added.

To convert from rectangular to cylindrical coordinates, we use the conversion

- $x = r\ \cos\theta$
- $y = r\ \sin\ \theta$
- $z = z$

To convert from cylindrical to rectangular coordinates, we use

- $r^2 = x^2 + y^2$ and
- $\theta = \tan^{-1}\left(\frac{y}{x}\right)$
- $z = z$

Note that z-coordinate remains the same in both cases.

In the two-dimensional plane with a rectangular coordinate system, when we say $x = k$ (constant) we mean an unbounded vertical line parallel to the y-axis and when $y = l$ (constant) we mean an unbounded horizontal line parallel to the x-axis. With the polar coordinate system, when we say $r = c$ (constant), we mean a circle of radius c units and when $\theta = \alpha$ (constant) we mean an infinite ray making an angle α with the positive x-axis.

Similarly, in three-dimensional space with rectangular coordinates (x, y, z) the equations $x = k,\ \ y = l$ and $z = m$ where $k,\ \ l$ and m are constants, represent unbounded planes parallel to the yz-plane, xz-plane and xy-plane, respectively. With cylindrical coordinates (r, θ, z), by $r = c,\ \ \theta = \alpha$, and $z = m$, where c, α, and m are constants, we mean an unbounded vertical cylinder with the z-axis as its radial axis; a plane making a constant angle α with the xy-plane; and an unbounded horizontal plane parallel to the xy-plane, respectively. This means that the circular cylinder $x^2 + y^2 = c^2$ in rectangular coordinates can be represented simply as $r = c$ in cylindrical coordinates. (Refer to Cylindrical and Spherical Coordinates for more review.)

Integration in Cylindrical Coordinates

Triple integrals can often be more readily evaluated by using cylindrical coordinates instead of rectangular coordinates. Some common equations of surfaces in rectangular coordinates along with corresponding equations in cylindrical coordinates are listed in Table 2.6.1. These equations will become handy as we proceed with solving problems using triple integrals.

Table 2.6.1: Equations of Some Common Shapes including Circular cylinder, Circular cone, Sphere, and Paraboloid

	Circular cylinder	Circular cone	Sphere	Paraboloid
Rectangular	$x^2 + y^2 = c^2$	$z^2 = c^2(x^2 + y^2)$	$x^2 + y^2 + z^2 = c^2$	$z = c(x^2 + y^2)$
Cylindrical	$r = c$	$z = cr$	$r^2 + z^2 = c^2$	$z = cr^2$

As before, we start with the simplest bounded region B in $\mathbb{R}^3$ to describe in cylindrical coordinates, in the form of a cylindrical box, $B = \{(r, \theta, z) | a \le r \le b,\ \ \alpha \le \theta \le \beta,\ \ c \le z \le d\}$ (Figure 2.6.2). Suppose we divide each interval into $l,\ \ m$, and n subdivisions such that $\Delta r = \frac{b-a}{l},\ \ \Delta\theta = \frac{\beta-\alpha}{m}$, and $\Delta z = \frac{d-c}{n}$. Then we can state the following definition for a triple integral in cylindrical coordinates.

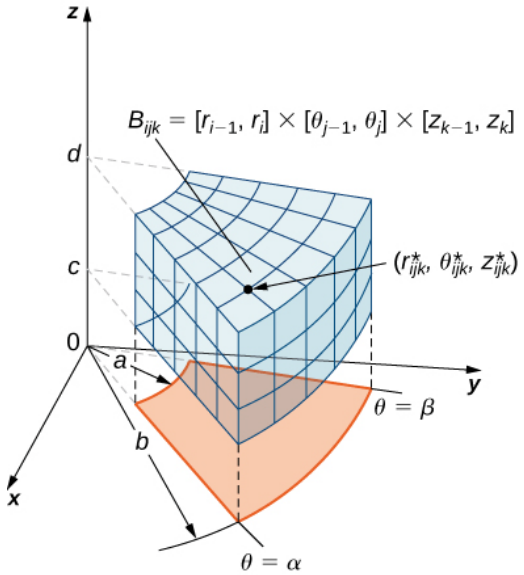


Figure 2.6.2: A cylindrical region divided into a grid, showing its relation to cylindrical coordinates.

Definition: Triple Integral in Cylindrical Coordinates

Consider the cylindrical box (expressed in cylindrical coordinates)

$$B = \{(r, \theta, z) | a \leq r \leq b,\ \alpha \leq \theta \leq \beta,\ c \leq z \leq d\}. \tag{2.6.1}$$

If the function $f(r, \theta, z)$ is continuous on B and if $(r^*_{ijk}, \theta^*_{ijk}, z^*_{ijk})$ is any sample point in the cylindrical subbox $B_{ijk} = |r_{i-1}, r_i| \times |\theta_{j-1}, \theta_j| \times |z_{k-1}, k_i|$ (Figure 2.6.2), then we can define the *triple integral in cylindrical coordinates* as the limit of a triple Riemann sum, provided the following limit exists:

$$\lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(r^*_{ijk}, \theta^*_{ijk}, z^*_{ijk}) \Delta r \Delta\theta \Delta z. \tag{2.6.2}$$

Note that if $g(x, y, z)$ is the function in rectangular coordinates and the box B is expressed in rectangular coordinates, then the triple integral

$$\iiint_B g(x, y, z) dV \tag{2.6.3}$$

is equal to the triple integral

$$\iiint_B g(r\ \cos\theta,\ r\ \sin\ \theta,\ z) r\, dr\, d\theta\, dz \tag{2.6.4}$$

and we have

$$\iiint_B g(x, y, z) dV = \iiint_B g(r\ \cos\theta,\ r\ \sin\ \theta,\ z) r\, dr\, d\theta\, dz = \iiint_B f(r, \theta\, z) r\, dr\, d\theta\, dz. \tag{2.6.5}$$

As mentioned in the preceding section, all the properties of a double integral work well in triple integrals, whether in rectangular coordinates or cylindrical coordinates. They also hold for iterated integrals. To reiterate, in cylindrical coordinates, Fubini's theorem takes the following form:

Theorem: Fubini's Theorem in Cylindrical Coordinates

Suppose that $g(x, y, z)$ is continuous on a rectangular box B which when described in cylindrical coordinates looks like $B = \{(r, \theta, z) | a \leq r \leq b,\ \ \alpha \leq \theta \leq \beta,\ \ c \leq z \leq d\}$.

Then $g(x, y, z) = g(r\ \ \cos\theta, r\ \ \sin\ \ \theta, z) = f(r, \theta, z)$ and

$$\iiint_B g(x, y, z) dV = \int_c^d \int_\beta^\alpha \int_a^b f(r, \theta, z) r\, dr\, d\theta\, dz. \tag{2.6.6}$$

The iterated integral may be replaced equivalently by any one of the other five iterated integrals obtained by integrating with respect to the three variables in other orders.

Cylindrical coordinate systems work well for solids that are symmetric around an axis, such as cylinders and cones. Let us look at some examples before we define the triple integral in cylindrical coordinates on general cylindrical regions.

Example 2.6.1: Evaluating a Triple Integral over a Cylindrical Box

Evaluate the triple integral

$$\iiint_B (zr\ \sin\ \theta) r\, dr\, d\theta\, dz \tag{2.6.7}$$

where the cylindrical box B is $B = \{(r, \theta, z) | 0 \leq r \leq 2,\ 0 \leq \theta \leq \pi/2,\ 0, \leq z \leq 4\}$.

Solution

As stated in Fubini's theorem, we can write the triple integral as the iterated integral

$$\iiint_B (zr \sin \theta) r \, dr \, d\theta \, dz = \int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r=2} \int_{z=0}^{z=4} (zr \sin \theta) r \, dz \, dr \, d\theta. \tag{2.6.8}$$

The evaluation of the iterated integral is straightforward. Each variable in the integral is independent of the others, so we can integrate each variable separately and multiply the results together. This makes the computation much easier:

$$\int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r=2} \int_{z=0}^{z=4} (zr \sin \theta) r \, dz \, dr \, d\theta = \left(\int_0^{\pi/2} \sin \theta \, d\theta \right) \left(\int_0^2 r^2 dr \right) \left(\int_0^4 z \, dz \right) = \left(-\cos \theta \Big|_0^{\pi/2} \right) \left(\frac{r^3}{3} \Big|_0^2 \right) \left(\frac{z^2}{2} \Big|_0^4 \right) = \frac{64}{3}.$$

Watch the accompanying video for a visual demonstration of the concept.

- **Video length**: 1 minute 2 seconds.
- **Context**: This video demonstrates how to use Cylindrical Coordinates to find a mass.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Exercise 2.6.1:

Evaluate the triple integral

$$\int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=1} \int_{z=0}^{z=4} rz \sin \theta r \, dz \, dr \, d\theta. \tag{2.6.10}$$

Hint available at the end of this section.

If the cylindrical region over which we have to integrate is a general solid, we look at the projections onto the coordinate planes. Hence the triple integral of a continuous function $f(r, \theta, z)$ over a general solid region $E = \{(r, \theta, z) | (r, \theta) \in D, u_1(r, \theta) \leq z \leq u_2(r, \theta)\}$ in $\mathbb{R}^3$ where D is the projection of E onto the $r\theta$-plane, is

$$\iiint_E f(r, \theta, z) r \, dr \, d\theta \, dz = \iint_D \left[\int_{u_1(r,\theta)}^{u_2(r,\theta)} f(r, \theta, z) dz \right] r \, dr \, d\theta. \tag{2.6.11}$$

In particular, if $D = \{(r, \theta) | G_1(\theta) \leq r \leq g_2(\theta), \alpha \leq \theta \leq \beta\}$, then we have

$$\iiint_E f(r, \theta, z) r \, dr \, d\theta = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} \int_{z=u_1(r,\theta)}^{z=u_2(r,\theta)} f(r, \theta, z) r \, dz \, dr \, d\theta. \tag{2.6.12}$$

Similar formulas exist for projections onto the other coordinate planes. We can use polar coordinates in those planes if necessary.

Example 2.6.2: Setting up a Triple Integral in Cylindrical Coordinates over a General Region

Consider the region E inside the right circular cylinder with equation $r = 2 \sin \theta$, bounded below by the $r\theta$-plane and bounded above by the sphere with radius 4 centered at the origin. Set up a triple integral over this region with a function $f(r, \theta, z)$ in cylindrical coordinates.

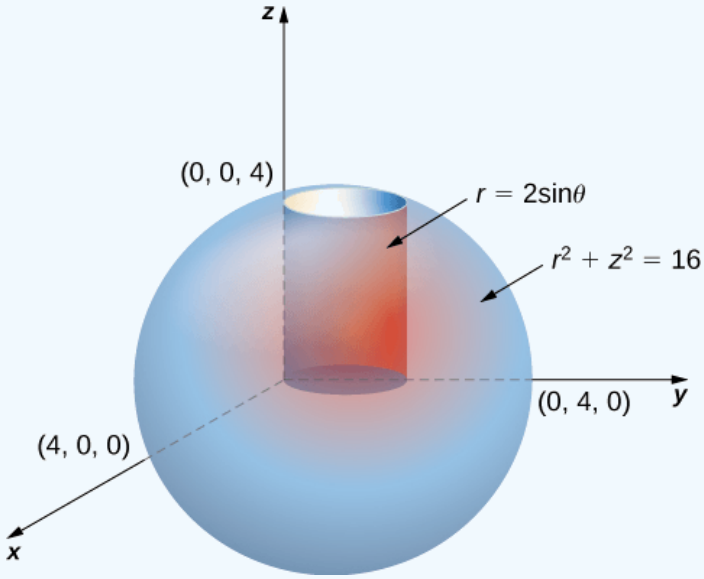


Figure 2.6.3: Region E inside the right circular cylinder with equation $r = 2 \sin \theta$, bounded below by the $r\theta$-plane and bounded above by the sphere with radius 4 centered at the origin.

Solution

First, identify that the equation for the sphere is $r^2 + z^2 = 16$. We can see that the limits for z are from 0 to $z = \sqrt{16 - r^2}$. Then the limits for r are from 0 to $r = 2 \sin \theta$. Finally, the limits for θ are from 0 to π. Hence the region is $E = \{(r, \theta, z) | 0 \leq \theta \leq \pi, \ 0 \leq r \leq 2 \sin \theta, \ 0 \leq z \leq \sqrt{16 - r^2}\}$. Therefore, the triple integral is

$$\iiint_E f(r, \theta, z) r \, dz \, dr \, d\theta = \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=2 \sin \theta} \int_{z=0}^{z=\sqrt{16-r^2}} f(r, \theta, z) r \, dz \, dr \, d\theta. \tag{2.6.13}$$

Exercise 2.6.2:

Consider the region inside the right circular cylinder with equation $r = 2 \sin \theta$ bounded below by the $r\theta$-plane and bounded above by $z = 4 - y$. Set up a triple integral with a function $f(r, \theta, z)$ in cylindrical coordinates.

Hint available at the end of this section.

Example 2.6.3: Setting up a Triple Integral in Two Ways

Let E be the region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the paraboloid $z = 2 - x^2 - y^2$. (Figure 2.6.4). Set up a triple integral in cylindrical coordinates to find the volume of the region, using the following orders of integration:

a. $dz \ dr \ d\theta$

b. $dr \ dz \ d\theta$

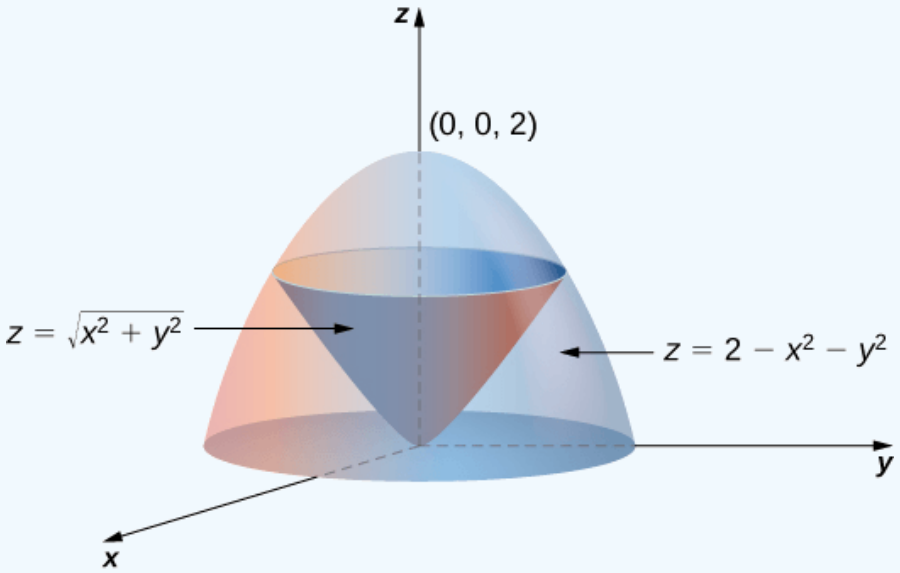


Figure 2.6.4: Region E is bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the paraboloid $z = 2 - x^2 - y^2$.

Solution

a. The cone is of radius 1 where it meets the paraboloid. Since $z = 2 - x^2 - y^2 = 2 - r^2$ and $z = \sqrt{x^2 + y^2} = r$ (assuming r is nonnegative), we have $2 - r^2 = r$. Solving, we have $r^2 + r - 2 = (r + 2)(r - 1) = 0$. Since $r \geq 0$, we have $r = 1$. Therefore $z = 1$. So the intersection of these two surfaces is a circle of radius 1 in the plane $z = 1$. The cone is the lower bound for z and the paraboloid is the upper bound. The projection of the region onto the xy-plane is the circle of radius 1 centered at the origin.

Thus, we can describe the region as $E = \{(r, \theta, z) | 0 \leq \theta \leq 2\pi,\ 0 \leq r \leq 1,\ r \leq z \leq 2 - r^2\}$.

Hence the integral for the volume is

$$V = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \int_{z=r}^{z=2-r^2} r\, dz\, dr\, d\theta. \tag{2.6.14}$$

b. We can also write the cone surface as $r = z$ and the paraboloid as $r^2 = 2 - z$. The lower bound for r is zero, but the upper bound is sometimes the cone and the other times it is the paraboloid. The plane $z = 1$ divides the region into two regions. Then the region can be described as

$$E = \{(r, \theta, z) | 0 \leq \theta \leq 2\pi,\ 0 \leq z \leq 1,\ 0 \leq r \leq z\} \cup \{(r, \theta, z) | 0 \leq \theta \leq 2\pi,\ 1 \leq z \leq 2,\ 0 \leq r \leq \sqrt{2 - z}\}. \tag{2.6.15}$$

Now the integral for the volume becomes

$$V = \int_{\theta=0}^{\theta=2\pi} \int_{z=0}^{z=1} \int_{r=0}^{r=z} r\, dr\, dz\, d\theta + \int_{\theta=0}^{\theta=2\pi} \int_{z=1}^{z=2} \int_{r=0}^{r=\sqrt{2-z}} r\, dr\, dz\, d\theta. \tag{2.6.16}$$

Exercise 2.6.3:

Redo the previous example with the order of integration $d\theta\ dz\ dr$.

Hint available at the end of this section.

Example 2.6.4: Finding a Volume with Triple Integrals in Two Ways

Let *E* be the region bounded below by the $r\theta$-plane, above by the sphere $x^2 + y^2 + z^2 = 4$, and on the sides by the cylinder $x^2 + y^2 = 1$. Set up a triple integral in cylindrical coordinates to find the volume of the region using the following orders of integration, and in each case find the volume and check that the answers are the same:

a. $dz\ dr\ d\theta$
b. $dr\ dz\ d\theta$.

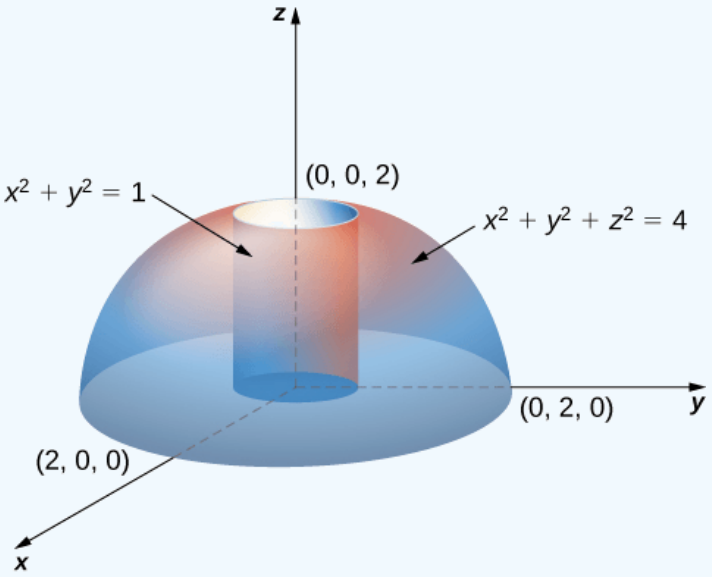


Figure 2.6.5: Region E is bounded below by the $r\theta$-plane, above by the sphere $x^2+y^2+z^2=4$, and on the sides by the cylinder $x^2+y^2=1$.

Solution

a. Note that the equation for the sphere is

$$x^2+y^2+z^2=4 \text{ or } r^2+z^2=4 \tag{2.6.17}$$

and the equation for the cylinder is

$$x^2+y^2=1 \text{ or } r^2=1. \tag{2.6.18}$$

Thus, we have for the region E

$$E=\{(r,\theta,z)|0\le z\le\sqrt{4-r^2},\ 0\le r\le 1,\ 0\le\theta\le 2\pi\} \tag{2.6.19}$$

Hence the integral for the volume is

$$V(E)=\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1}\int_{z=0}^{z=\sqrt{4-r^2}} r\,dz\,dr\,d\theta \tag{2.6.20}$$

$$=\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1}\left[\rangle rz|_{z=0}^{z=\sqrt{4-r^2}}\right]dr\,d\theta=\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1}\left(r\sqrt{4-r^2}\right)dr\,d\theta \tag{2.6.21}$$

$$=\int_0^{2\pi}\left(\frac{8}{3}-\sqrt{3}\right)d\theta=2\pi\left(\frac{8}{3}-\sqrt{3}\right)\text{ cubic units.} \tag{2.6.22}$$

b. Since the sphere is $x^2+y^2+z^2=4$, which is $r^2+z^2=4$, and the cylinder is $x^2+y^2=1$, which is $r^2=1$, we have $1+z^2=4$, that is, $z^2=3$. Thus we have two regions, since the sphere and the cylinder intersect at $(1,\sqrt{3})$ in the rz-plane

$$E_1=\{(r,\theta,z)|0\le r\le\sqrt{4-r^2},\ \sqrt{3}\le z\le 2,\ 0\le\theta\le 2\pi\} \tag{2.6.23}$$

and

$$E_2=\{(r,\theta,z)|0\le r\le 1,\ 0\le z\le\sqrt{3},\ 0\le\theta\le 2\pi\}. \tag{2.6.24}$$

Hence the integral for the volume is

$$V(E) = \int_{\theta=0}^{\theta=2\pi} \int_{z=\sqrt{3}}^{z=2} \int_{r=0}^{r=\sqrt{4-r^2}} r\,dr\,dz\,d\theta + \int_{\theta=0}^{\theta=2\pi} \int_{z=0}^{z=\sqrt{3}} \int_{r=0}^{r=1} r\,dr\,dz\,d\theta \tag{2.6.25}$$

$$= \sqrt{3}\pi + \left(\frac{16}{3} - 3\sqrt{3}\right)\pi = 2\pi\left(\frac{8}{3} - \sqrt{3}\right) \text{ cubic units.} \tag{2.6.26}$$

Exercise 2.6.4

Redo the previous example with the order of integration $d\theta\ dz\ dr$.

Hint available at the end of this section.

Review of Spherical Coordinates

In three-dimensional space $\mathbb{R}^3$ in the spherical coordinate system, we specify a point P by its distance ρ from the origin, the polar angle θ from the positive x-axis (same as in the cylindrical coordinate system), and the angle φ from the positive z-axis and the line OP (Figure 2.6.6). Note that $\rho > 0$ and $0 \le \varphi \le \pi$. Spherical coordinates are useful for triple integrals over regions that are symmetric with respect to the origin.

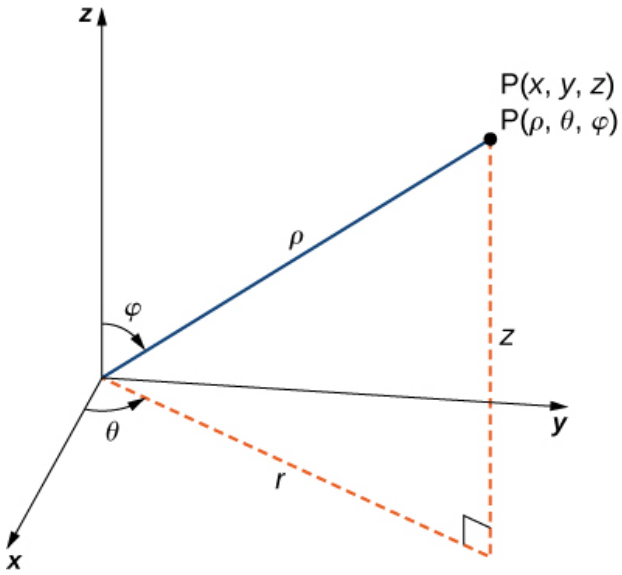


Figure 2.6.6: The spherical coordinate system locates points with two angles and a distance from the origin. In three-dimensional space $\mathbb{R}^3$ in the spherical coordinate system, we specify a point P by its distance ρ from the origin, the polar angle θ from the positive x-axis (same as in the cylindrical coordinate system), and the angle φ from the positive z-axis and the line OP.

The relationships that connect rectangular coordinates with spherical coordinates.

From spherical coordinates to rectangular coordinates:

$$x = \rho \sin \varphi \cos\theta,\ y = \rho \sin \varphi \sin \theta,\ and\ z = \rho \cos \varphi. \tag{2.6.27}$$

From rectangular coordinates to spherical coordinates:

$$\rho^2 = x^2 + y^2 + z^2,\ \tan\theta = \frac{y}{x},\ \varphi = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right). \tag{2.6.28}$$

Other relationships that are important to know for conversions are

- $r = \rho \sin \varphi$
- $\theta = \theta$ These equations are used to convert from spherical coordinates to cylindrical coordinates.

- $z = \rho \cos \varphi$

and

- $\rho = \sqrt{r^2 + z^2}$
- $\theta = \theta$ These equations are used to convert from cylindrical coordinates to spherical coordinates.
- $\varphi = \arccos\left(\frac{z}{\sqrt{r^2+z^2}}\right)$

Figure 2.6.7 shows a few solid regions that are convenient to express in spherical coordinates.

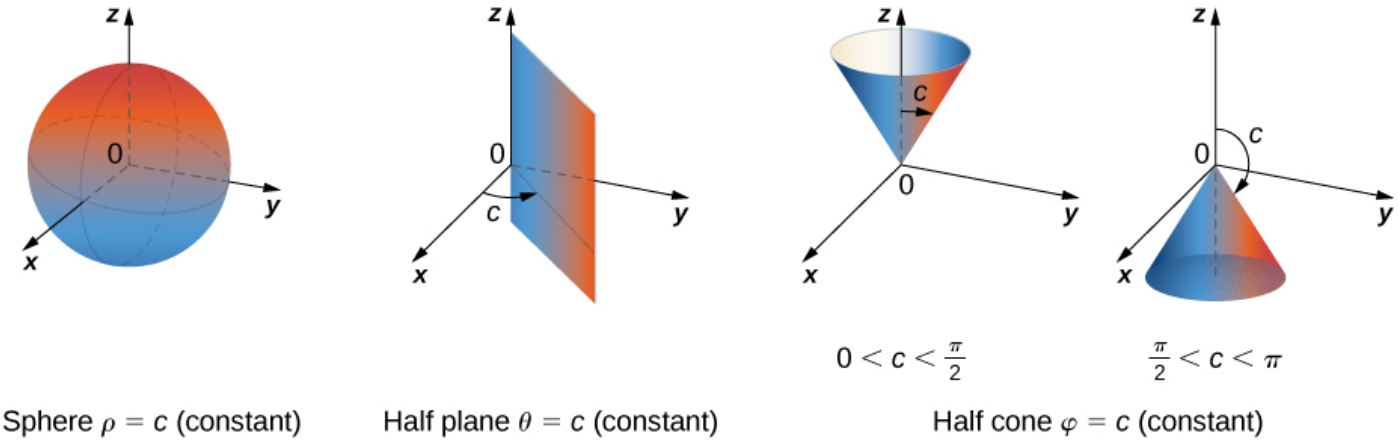


Figure 2.6.7: Surfaces in spherical coordinates including the sphere ($\rho = c$), half-plane ($\theta = c$), and cones ($\varphi = c, \quad 0 < \varphi < \frac{\pi}{2} \quad or \quad \frac{\pi}{2} < \varphi < \pi$), emphasizing their symmetry and convenience for representation in spherical coordinates.

Integration in Spherical Coordinates

We now establish a triple integral in the spherical coordinate system, as we did before in the cylindrical coordinate system. Let the function $f(\rho, \theta, \varphi)$ be continuous in a bounded spherical box, $B = \{(\rho, \theta, \varphi) | a \leq \rho \leq b, \ \alpha \leq \theta \leq \beta, \ \gamma \leq \varphi \leq \psi\}$. We then divide each interval into l, m, n and n subdivisions such that $\Delta\rho = \frac{b-a}{l}, \ \Delta\theta = \frac{\beta-\alpha}{m}. \ \Delta\varphi = \frac{\psi-\gamma}{n}$. Now we can illustrate the following theorem for triple integrals in spherical coordinates with $(\rho^*_{ijk}, \theta^*_{ijk}, \varphi^*_{ijk})$ being any sample point in the spherical subbox B_{ijk}. For the volume element of the subbox ΔV in spherical coordinates, we have $\Delta V = (\Delta\rho) \ (\rho\Delta\varphi) \ (\rho \ \sin \ \varphi \ \Delta\theta)$, as shown in the Figure 2.6.8.

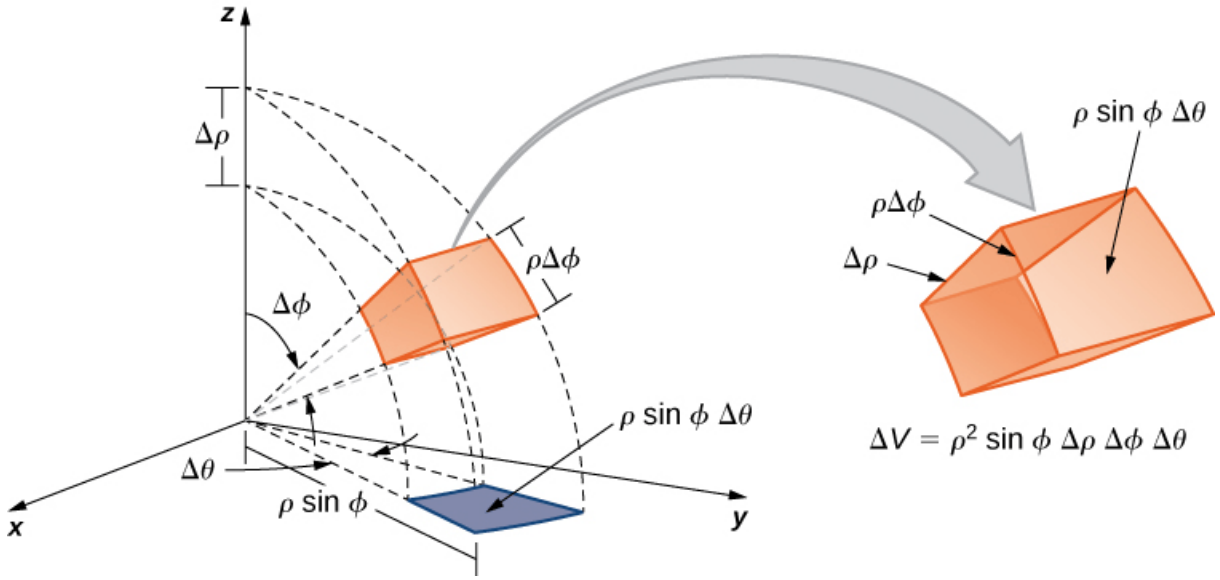


Figure 2.6.8: A volume element in spherical coordinates, which is a small box with sides $\Delta\rho$, $\rho\Delta\varphi$, and $\rho \sin\varphi\Delta\theta$, resulting in a volume of $\Delta V = \rho^2 \sin\varphi\Delta\rho\Delta\varphi\Delta\theta$.

Definition: Triple Integral in Spherical Coordinates

The **triple integral in spherical coordinates** is the limit of a triple Riemann sum,

$$\lim_{l,m,n\to\infty}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(\rho_{ijk}^*,\theta_{ijk}^*,\varphi_{ijk}^*)(\rho_{ijk}^*)^2\sin\,\varphi\Delta\rho\Delta\theta\Delta\varphi \tag{2.6.29}$$

provided the limit exists.

As with the other multiple integrals we have examined, all the properties work similarly for a triple integral in the spherical coordinate system, and so do the iterated integrals. Fubini's theorem takes the following form.

Theorem: Fubini's Theorem for Spherical Coordinates

If $f(\rho,\theta,\varphi)$ is continuous on a spherical solid box $B=[a,b]\times[\alpha,\beta]\times[\gamma,\psi]$, then

$$\iiint_B f(\rho,\theta,\varphi)\,\rho^2\sin\,\varphi d\rho\,d\varphi\,d\theta=\int_{\varphi=\gamma}^{\varphi=\psi}\int_{\theta=\alpha}^{\theta=\beta}\int_{\rho=a}^{\rho=b} f(\rho,\theta,\varphi)\,\rho^2\sin\,\varphi\,d\rho\,d\varphi\,d\theta. \tag{2.6.30}$$

This iterated integral may be replaced by other iterated integrals by integrating with respect to the three variables in other orders.

As stated before, spherical coordinate systems work well for solids that are symmetric around a point, such as spheres and cones. Let us look at some examples before we consider triple integrals in spherical coordinates on general spherical regions.

Example 2.6.5: Evaluating a Triple Integral in Spherical Coordinates

Evaluate the iterated triple integral

$$\int_{\theta=0}^{\theta=2\pi}\int_{\varphi=0}^{\varphi=\pi/2}\int_{\rho=0}^{\rho=1}\rho^2\sin\,\varphi\,d\rho\,d\varphi\,d\theta. \tag{2.6.31}$$

Solution

As before, in this case the variables in the iterated integral are actually independent of each other and hence we can integrate each piece and multiply:

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^1\rho^2\sin\,\varphi\,d\rho\,d\varphi\,d\theta=\int_0^{2\pi}d\theta\int_0^{\pi/2}\sin\,\varphi\,d\varphi\int_0^1\rho^2d\rho=(2\pi)\,(1)\left(\frac{1}{3}\right)=\frac{2\pi}{3} \tag{2.6.32}$$

The concept of triple integration in spherical coordinates can be extended to integration over a general solid, using the projections onto the coordinate planes. Note that dV and dA mean the increments in volume and area, respectively. The variables V and A are used as the variables for integration to express the integrals.

The triple integral of a continuous function $f(\rho,\theta,\varphi)$ over a general solid region

$$E=\{(\rho,\theta,\varphi)|(\rho,\theta)\in D,u_1(\rho,\theta)\le\varphi\le u_2(\rho,\theta)\} \tag{2.6.33}$$

in $\mathbb{R}^3$, where D is the projection of E onto the $\rho\theta$-plane, is

$$\iiint_E f(\rho,\theta,\varphi)dV=\iint_D\left[\int_{u_1(\rho,\theta)}^{u_2(\rho,\theta)} f(\rho,\theta,\varphi)\,d\varphi\right]dA. \tag{2.6.34}$$

In particular, if $D=\{(\rho,\theta)|g_1(\theta)\le\rho\le g_2(\theta),\ \alpha\le\theta\le\beta\}$, the we have

$$\iiint_E f(\rho,\theta,\varphi)dV = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} \int_{u_1(\rho,\theta)}^{u_2(\rho,\theta)} f(\rho,\theta,\varphi)\rho^2 \sin\,\varphi\, d\varphi\, d\rho\, d\theta. \tag{2.6.35}$$

Similar formulas occur for projections onto the other coordinate planes.

Example 2.6.6: Setting up a Triple Integral in Spherical Coordinates

Set up an integral for the volume of the region bounded by the cone $z = \sqrt{3(x^2+y^2)}$ and the hemisphere $z = \sqrt{4-x^2-y^2}$ (Figure 2.6.9 below).

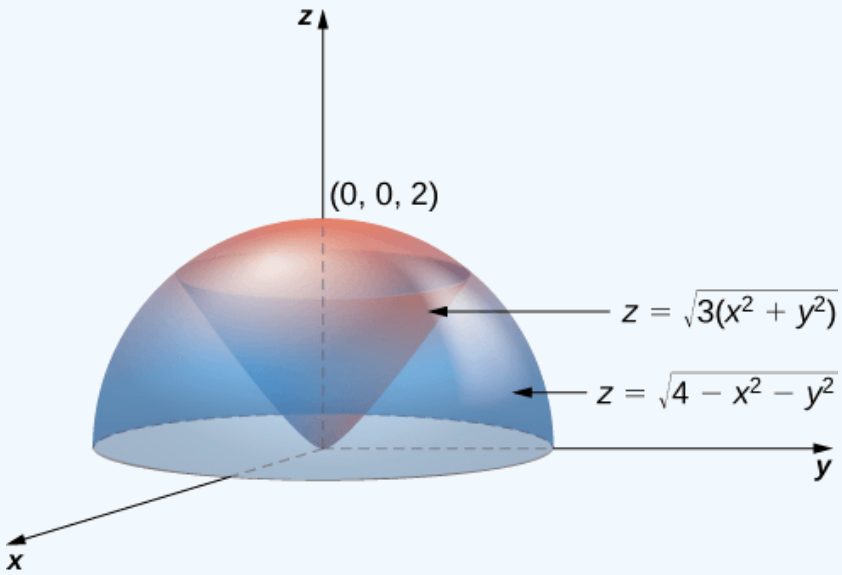


Figure 2.6.9: Region E is bounded below by cone $z = \sqrt{3(x^2+y^2)}$ and above by hemisphere $z = \sqrt{4-x^2-y^2}$.

Solution

Using the conversion formulas from rectangular coordinates to spherical coordinates, we have:

For the cone: $z = \sqrt{3(x^2+y^2)}$ or $\rho\ \cos\ \varphi = \sqrt{3}\rho\ \sin\ \varphi$ or $\tan\ \varphi = \frac{1}{\sqrt{3}}$ or $\varphi = \frac{\pi}{6}$.

For the sphere: $z = \sqrt{4-x^2-y^2}$ or $z^2+x^2+y^2 = 4$ or $\rho^2 = 4$ or $\rho = 2$.

Thus, the triple integral for the volume is

$$V(E) = \int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\pi/6} \int_{\rho=0}^{\rho=2} \rho^2 \sin\,\varphi\, d\rho\, d\varphi\, d\theta. \tag{2.6.36}$$

Watch the accompanying video for a visual demonstration of the concept.

- **Video length**: 3 minutes 17 seconds.
- **Context**: This video demonstrates how to use Spherical Coordinates to find the volume between a cone and a sphere.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Exercise 2.6.5

Set up a triple integral for the volume of the solid region bounded above by the sphere $\rho = 2$ and bounded below by the cone $\varphi = \pi/3$.

Hint available at the end of this section.

Example 2.6.7: Interchanging Order of Integration in Spherical Coordinates

Let E be the region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the sphere $z = x^2 + y^2 + z^2$. Set up a triple integral in spherical coordinates and find the volume of the region using the following orders of integration:

a. $d\rho\ d\phi\ d\theta$
b. $d\varphi\ d\rho\ d\theta$

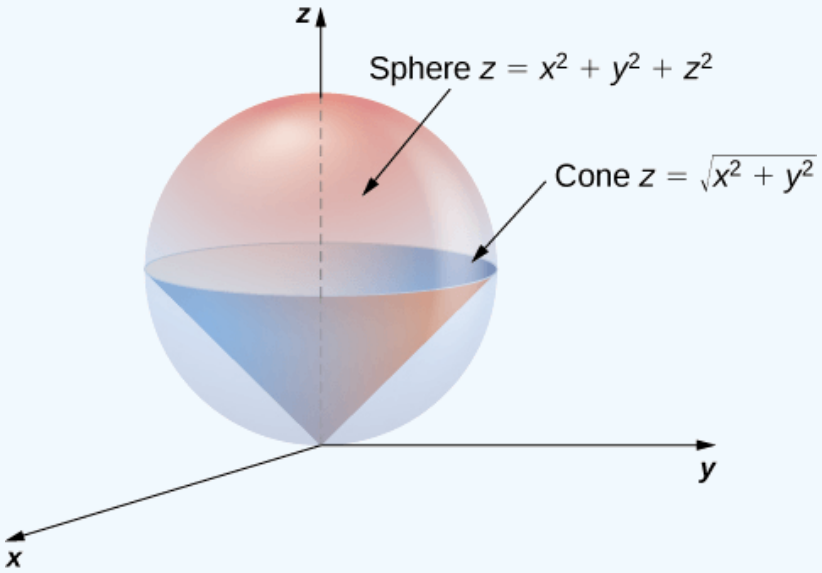


Figure 2.6.10: Region E is bounded below by cone $z = \sqrt{x^2 + y^2}$ and above by sphere $z = x^2 + y^2 + z^2$.

Solution

a. Use the conversion formulas to write the equations of the sphere and cone in spherical coordinates.

For the sphere:

$$x^2 + y^2 + z^2 = z \tag{2.6.37}$$
$$\rho^2 = \rho \cos \varphi \tag{2.6.38}$$
$$\rho = \cos \varphi. \tag{2.6.39}$$

For the cone:

$$z = \sqrt{x^2 + y^2} \tag{2.6.40}$$
$$\rho \cos \varphi = \sqrt{\rho^2 \sin^2 \varphi \cos^2 \theta + \rho^2 \sin^2 \varphi \sin^2 \theta} \tag{2.6.41}$$
$$\rho \cos \varphi = \sqrt{\rho^2 \sin^2 \varphi (\cos^2 \theta + \sin^2 \theta)} \tag{2.6.42}$$
$$\rho \cos \varphi = \rho \sin \varphi \tag{2.6.43}$$
$$\cos \varphi = \sin \varphi \tag{2.6.44}$$
$$\varphi = \pi/4. \tag{2.6.45}$$

Hence the integral for the volume of the solid region E becomes

$$V(E) = \int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\pi/4} \int_{\rho=0}^{\rho=\cos\ \varphi} \rho^2 \sin\ \varphi\, d\rho\, d\varphi\, d\theta. \tag{2.6.46}$$

b. Consider the $\varphi\rho$-plane. Note that the ranges for φ and ρ (from part a.) are

$$0 \leq \rho\sqrt{2}/2 \text{and}\ \sqrt{2} \leq \rho 1 \tag{2.6.47}$$
$$0 \leq \varphi \leq \pi/4 0 \leq \rho \leq \cos\ \varphi \tag{2.6.48}$$

The curve $\rho = \cos\ \varphi$ meets the line $\varphi = \pi/4$ at the point $(\pi/4, \sqrt{2}/2)$. Thus, to change the order of integration, we need to use two pieces:

$$0 \leq \rho \leq \sqrt{2}/2,\ 0 \leq \varphi \leq \pi/4 \tag{2.6.49}$$

and

$$\sqrt{2}/2 \leq \rho \leq 1,\ 0 \leq \varphi \leq \cos^{-1}\rho. \tag{2.6.50}$$

Hence the integral for the volume of the solid region E becomes

$$V(E) = \int_{\theta=0}^{\theta=2\pi} \int_{\rho=0}^{\rho=\sqrt{2}/2} \int_{\varphi=0}^{\varphi=\pi/4} \rho^2 \sin\ \varphi\, d\varphi\, d\rho\, d\theta + \int_{\theta=0}^{\theta=2\pi} \int_{\rho=\sqrt{2}/2}^{\rho=1} \int_{\varphi=0}^{\varphi=\cos^{-1}\rho} \rho^2 \sin\ \varphi\, d\varphi\, d\rho\, d\theta \tag{2.6.51}$$

In each case, the integration results in $V(E) = \frac{\pi}{8}$.

Before we end this section, we present a couple of examples that can illustrate the conversion from rectangular coordinates to cylindrical coordinates and from rectangular coordinates to spherical coordinates.

Example 2.6.8: Converting from Rectangular Coordinates to Cylindrical Coordinates

Convert the following integral into cylindrical coordinates:

$$\int_{y=-1}^{y=1} \int_{x=0}^{x=\sqrt{1-y^2}} \int_{z=x^2+y^2}^{z=\sqrt{x^2+y^2}} xyz\, dz\, dx\, dy. \tag{2.6.52}$$

Solution

The ranges of the variables are

$$-1 \leq y \leq 1 \tag{2.6.53}$$
$$0 \leq x \leq \sqrt{1-y^2} \tag{2.6.54}$$
$$x^2 + y^2 \leq z \leq \sqrt{x^2+y^2}. \tag{2.6.55}$$

The first two inequalities describe the right half of a circle of radius 1. Therefore, the ranges for θ and r are

$$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \text{and}\, 0 \leq r \leq 1. \tag{2.6.56}$$

The limits of z are $r^2 \leq z \leq r$, hence

$$\int_{y=-1}^{y=1} \int_{x=0}^{x=\sqrt{1-y^2}} \int_{z=x^2+y^2}^{z=\sqrt{x^2+y^2}} xyz\, dz\, dx\, dy = \int_{\theta=-\pi/2}^{\theta=\pi/2} \int_{r=0}^{r=1} \int_{z=r^2}^{z=r} r(r\ \cos\theta)\,(r\ \sin\ \theta)\, z\, dz\, dr\, d\theta. \tag{2.6.57}$$

Example 2.6.9: Converting from Rectangular Coordinates to Spherical Coordinates

Convert the following integral into spherical coordinates:

$$\int_{y=0}^{y=3}\int_{x=0}^{x=\sqrt{9-y^2}}\int_{z=\sqrt{x^2+y^2}}^{z=\sqrt{18-x^2-y^2}}(x^2+y^2+z^2)dz\,dx\,dy. \tag{2.6.58}$$

Solution

The ranges of the variables are

$$0 \le y \le 3 \tag{2.6.59}$$
$$0 \le x \le \sqrt{9-y^2} \tag{2.6.60}$$
$$\sqrt{x^2+y^2} \le z \le \sqrt{18-x^2-y^2}. \tag{2.6.61}$$

The first two ranges of variables describe a quarter disk in the first quadrant of the xy-plane. Hence the range for θ is $0 \le \theta \le \frac{\pi}{2}$.

The lower bound $z=\sqrt{x^2+y^2}$ is the upper half of a cone and the upper bound $z=\sqrt{18-x^2-y^2}$ is the upper half of a sphere. Therefore, we have $0 \le \rho \le \sqrt{18}$, which is $0 \le \rho \le 3\sqrt{2}$.

For the ranges of φ we need to find where the cone and the sphere intersect, so solve the equation

$$r^2+z^2=18 \tag{2.6.62}$$
$$(\sqrt{x^2+y^2})^2+z^2=18 \tag{2.6.63}$$
$$z^2+z^2=18 \tag{2.6.64}$$
$$2z^2=18 \tag{2.6.65}$$
$$z^2=9 \tag{2.6.66}$$
$$z=3. \tag{2.6.67}$$

This gives

$$3\sqrt{2}\cos\varphi=3 \tag{2.6.68}$$
$$\cos\varphi=\frac{1}{\sqrt{2}} \tag{2.6.69}$$
$$\varphi=\frac{\pi}{4}. \tag{2.6.70}$$

Putting this together, we obtain

$$\int_{y=0}^{y=3}\int_{x=0}^{x=\sqrt{9-y^2}}\int_{z=\sqrt{x^2+y^2}}^{z=\sqrt{18-x^2-y^2}}(x^2+y^2+z^2)dz\,dx\,dy=\int_{\varphi=0}^{\varphi=\pi/4}\int_{\theta=0}^{\theta=\pi/2}\int_{\rho=0}^{\rho=3\sqrt{2}}\rho^4\sin\varphi\,d\rho\,d\theta\,d\varphi. \tag{2.6.71}$$

Exercise 2.6.6:

Use rectangular, cylindrical, and spherical coordinates to set up triple integrals for finding the volume of the region inside the sphere $x^2+y^2+z^2=4$ but outside the cylinder $x^2+y^2=1$.

Answer: Rectangular available at the end of this section.

Now that we are familiar with the spherical coordinate system, let's find the volume of some known geometric figures, such as spheres and ellipsoids.

Example 2.6.10: Chapter Opener: Finding the Volume of l'Hemisphèric

Find the volume of the spherical planetarium in l'Hemisphèric in Valencia, Spain, which is five stories tall and has a radius of approximately 50 ft, using the equation $x^2 + y^2 + z^2 = r^2$.



Figure 2.6.11: The City of Arts and Sciences in Valencia, Spain, has a unique structure along an axis of just two kilometers that was formerly the bed of the River Turia. The l'Hemisfèric has an IMAX cinema with three systems of modern digital projections onto a concave screen of 900 square meters. An oval roof over 100 meters long has been made to look like a huge human eye that comes alive and opens up to the world as the "Eye of Wisdom." (CC-BY-2.0 modification of work by Javier Yaya Tur, Wikimedia Commons)

Solution

We calculate the volume of the ball in the first octant, where $x \leq 0,\ y \leq 0$, and $z \leq 0$, using spherical coordinates, and then multiply the result by 8 for symmetry. Since we consider the region D as the first octant in the integral, the ranges of the variables are

$$0 \leq \varphi \leq \frac{\pi}{2},\ 0 \leq \rho \leq r,\ 0 \leq \theta \leq \frac{\pi}{2}. \tag{2.6.72}$$

Therefore,

$$V = \iiint_D dx\,dy\,dz = 8\int_{\theta=0}^{\theta=\pi/2}\int_{\rho=0}^{\rho=r}\int_{\varphi=0}^{\varphi=\pi/2} \rho^2 \sin\,\theta\,d\varphi\,d\rho\,d\varphi \tag{2.6.73}$$

$$= 8\int_{\varphi=0}^{\varphi=\pi/2} d\varphi \int_{\rho=0}^{\rho=r} \rho^2 d\rho \int_{\theta=0}^{\theta=\pi/2} \sin\,\theta\,d\theta \tag{2.6.74}$$

$$= 8\left(\frac{\pi}{2}\right)\left(\frac{r^3}{3}\right)(1) \tag{2.6.75}$$

$$= \frac{4}{3}\pi r^3. \tag{2.6.76}$$

This exactly matches with what we knew. So for a sphere with a radius of approximately 50 ft, the volume is $\frac{4}{3}\pi(50)^3 \approx 523,600\ ft^3$.

For the next example we find the volume of an ellipsoid.

Example 2.6.11: Finding the Volume of an Ellipsoid

Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Solution

We again use symmetry and evaluate the volume of the ellipsoid using spherical coordinates. As before, we use the first octant $x \leq 0,\ y \leq 0$, and $z \leq 0$ and then multiply the result by 8.

In this case the ranges of the variables are

$$0 \leq \varphi \leq \frac{\pi}{2}\, 0 \leq \rho \leq 1, \text{ and } 0 \leq \theta \leq \frac{\pi}{2}. \tag{2.6.77}$$

Also, we need to change the rectangular to spherical coordinates in this way:

$$x = a\rho\, \cos\, \varphi\, \sin\, \theta,\ y = b\rho\, \sin\, \varphi\, \sin\, \theta, \text{ and } z = c\rho\, \cos\theta. \tag{2.6.78}$$

Then the volume of the ellipsoid becomes

$$V = \iiint_D dx\, dy\, dz \tag{2.6.79}$$

$$= 8 \int_{\theta=0}^{\theta=\pi/2} \int_{\rho=0}^{\rho=1} \int_{\varphi=0}^{\varphi=\pi/2} abc\, \rho^2 \sin\, \theta\, d\varphi\, d\rho\, d\theta \tag{2.6.80}$$

$$\tag{2.6.81}$$

$$= 8abc \int_{\varphi=0}^{\varphi=\pi/2} d\varphi \int_{\rho=0}^{\rho=1} \rho^2 d\rho \int_{\theta=0}^{\theta=\pi/2} \sin\, \theta\, d\theta \tag{2.6.82}$$

$$= 8abc \left(\frac{\pi}{2}\right) \left(\frac{1}{3}\right) (1) \tag{2.6.83}$$

$$= \frac{4}{3}\pi abc. \tag{2.6.84}$$

Example 2.6.12: Finding the Volume of the Space Inside an Ellipsoid and Outside a Sphere

Find the volume of the space inside the ellipsoid $\frac{x^2}{75^2} + \frac{y^2}{80^2} + \frac{z^2}{90^2} = 1$ and outside the sphere $x^2 + y^2 + z^2 = 50^2$.

Solution

This problem is directly related to the l'Hemisphèric structure. The volume of space inside the ellipsoid and outside the sphere might be useful to find the expense of heating or cooling that space. We can use the preceding two examples for the volume of the sphere and ellipsoid and then substract.

First we find the volume of the ellipsoid using $a = 75$ ft, $b = 80$ ft, and $c = 90$ ft. Hence the volume of the ellipsoid is

$$V_{ellipsoid} = \frac{4}{3}\pi(75)(80)(90) \approx 2,262,000\, ft^3. \tag{2.6.85}$$

The volume of the sphere is

$$V_{sphere} \approx 523,600\, ft^3. \tag{2.6.86}$$

Therefore, the volume of the space inside the ellipsoid $\frac{x^2}{75^2} + \frac{y^2}{80^2} + \frac{z^2}{90^2} = 1$ and outside the sphere $x^2 + y^2 + z^2 = 50^2$ is approximately

$$V_{Hemispheric} = V_{ellipsoid} - V_{sphere} = 1,738,400\, ft^3. \tag{2.6.87}$$

Student Project: Hot Air Balloons

Hot air ballooning is a relaxing, peaceful pastime that many people enjoy. Many balloonist gatherings take place around the world, such as the Albuquerque International Balloon Fiesta. The Albuquerque event is the largest hot air balloon festival in the world, with over 500 balloons participating each year.



Figure 2.6.12: Balloons lift off at the 2001 Albuquerque International Balloon Fiesta. (CC-BY-2.0 work by David Herrera, Flickr).

As the name implies, hot air balloons use hot air to generate lift. (Hot air is less dense than cooler air, so the balloon floats as long as the hot air stays hot.) The heat is generated by a propane burner suspended below the opening of the basket. Once the balloon takes off, the pilot controls the altitude of the balloon, either by using the burner to heat the air and ascend or by using a vent near the top of the balloon to release heated air and descend. The pilot has very little control over where the balloon goes, however—balloons are at the mercy of the winds. The uncertainty over where we will end up is one of the reasons balloonists are attracted to the sport.

In this project we use triple integrals to learn more about hot air balloons. We model the balloon in two pieces. The top of the balloon is modeled by a half sphere of radius 28

feet. The bottom of the balloon is modeled by a frustum of a cone (think of an ice cream cone with the pointy end cut off). The radius of the large end of the frustum is 28 feet and the radius of the small end of the frustum is 28 feet. A graph of our balloon model and a cross-sectional diagram showing the dimensions are shown in the figure 2.6.13.

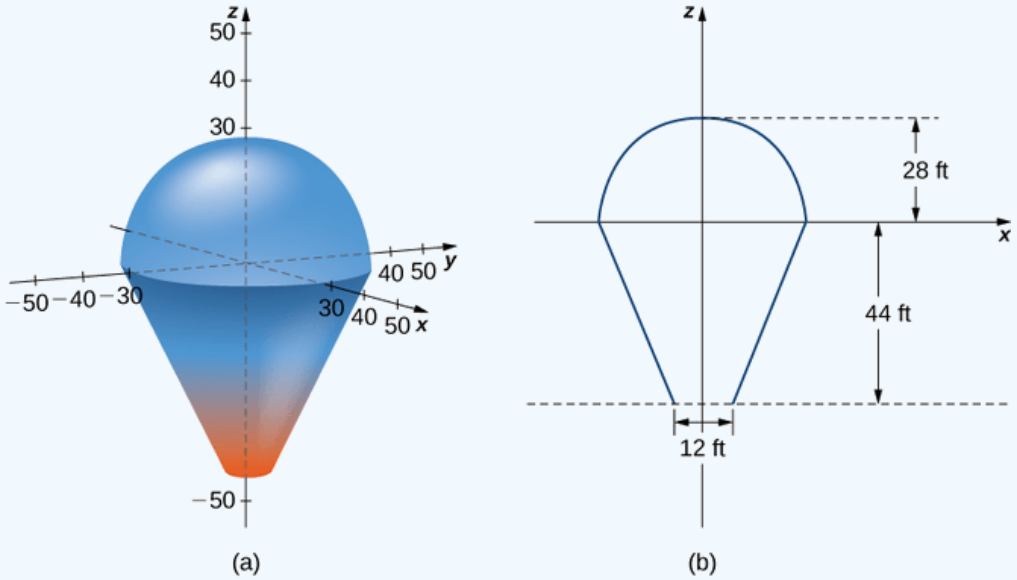


Figure 2.6.13: (a) Use a half sphere to model the top part of the balloon and a frustum of a cone to model the bottom part of the balloon. (b) A cross section of the balloon showing its dimensions.

We first want to find the volume of the balloon. If we look at the top part and the bottom part of the balloon separately, we see that they are geometric solids with known volume formulas. However, it is still worthwhile to set up and evaluate the integrals we would need to find the volume. If we calculate the volume using integration, we can use the known volume formulas to check our answers. This will help ensure that we have the integrals set up correctly for the later, more complicated stages of the project.

1. Find the volume of the balloon in two ways.

a. Use triple integrals to calculate the volume. Consider each part of the balloon separately. (Consider using spherical coordinates for the top part and cylindrical coordinates for the bottom part.)

b. Verify the answer using the formulas for the volume of a sphere, $V = \frac{4}{3}\pi r^3$, and for the volume of a cone, $V = \frac{1}{3}\pi r^2 h$.

In reality, calculating the temperature at a point inside the balloon is a tremendously complicated endeavor. In fact, an entire branch of physics (thermodynamics) is devoted to studying heat and temperature. For the purposes of this project, however, we are going to make some simplifying assumptions about how temperature varies from point to point within the balloon. Assume that just prior to liftoff, the temperature (in degrees Fahrenheit) of the air inside the balloon varies according to the function

$$T_0(r, \theta, z) = \frac{z - r}{10} + 210. \tag{2.6.88}$$

2. What is the average temperature of the air in the balloon just prior to liftoff? (Again, look at each part of the balloon separately, and do not forget to convert the function into spherical coordinates when looking at the top part of the balloon.)

Now the pilot activates the burner for 10 seconds. This action affects the temperature in a 12-foot-wide column 20 feet high, directly above the burner. A cross section of the balloon depicting this column is shown in the figure $2.6.14$.

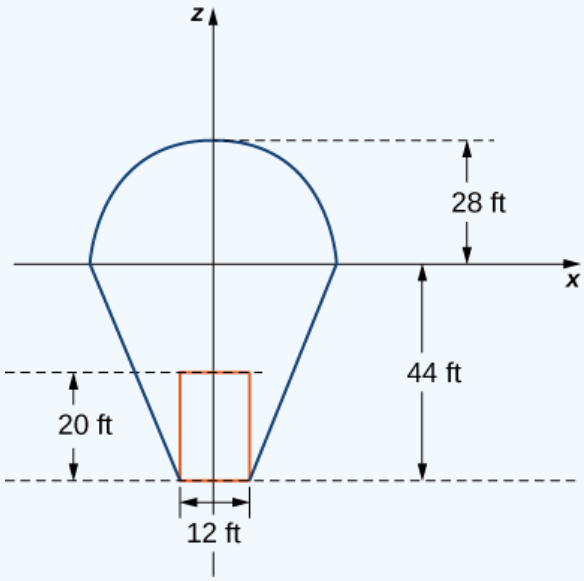


Figure $2.6.14$: Activating the burner heats the air in a 20-foot-high, 12-foot-wide column directly above the burner.

Assume that after the pilot activates the burner for 10 seconds, the temperature of the air in the column described above *increases* according to the formula

$$H(r, \theta, z) = -2z - 48. \tag{2.6.89}$$

Then the temperature of the air in the column is given by

$$T_1(r, \theta, z) = \frac{z - r}{10} + 210 + (-2z - 48), \tag{2.6.90}$$

while the temperature in the remainder of the balloon is still given by

$$T_0(r,\theta,z)=\frac{z-r}{10}+210. \tag{2.6.91}$$

3. Find the average temperature of the air in the balloon after the pilot has activated the burner for 10 seconds.

Key Concepts

- To evaluate a triple integral in cylindrical coordinates, use the iterated integral
$$\int_{\theta=\alpha}^{\theta=\beta}\int_{r=g_1(\theta)}^{r=g_2(\theta)}\int_{z=u_1(r,\theta)}^{u_2(r,\theta)} f(r,\theta,z)r\,dz\,dr\,d\theta.$$
- To evaluate a triple integral in spherical coordinates, use the iterated integral
$$\int_{\theta=\alpha}^{\theta=\beta}\int_{\rho=g_1(\theta)}^{\rho=g_2(\theta)}\int_{\varphi=u_1(r,\theta)}^{u_2(r,\theta)} f(\rho,\theta,\varphi)\,\rho^2\sin\varphi\,d\varphi\,d\rho\,d\theta.$$

Key Equations

- **Triple integral in cylindrical coordinates**
$$\iiint_B g(s,y,z)dV=\iiint_B g(r\,\cos\theta,\ r\,\sin\,\theta,\ z)r\,dr\,d\theta\,dz=\iiint_B f(r,\theta,z)r\,dr\,d\theta\,dz$$
- **Triple integral in spherical coordinates**
$$\iiint_B f(\rho,\theta,\varphi)\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta=\int_{\varphi=\gamma}^{\varphi=\psi}\int_{\theta=\alpha}^{\theta=\beta}\int_{\rho=a}^{\rho=b} f(\rho,\theta,\varphi)\rho^2\sin\,\varphi\,d\rho\,d\varphi\,d\theta$$

Glossary

Triple integral in cylindrical coordinates

the limit of a triple Riemann sum, provided the following limit exists:

$$lim_{l,m,n\to\infty}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(r^*_{ijk},\theta^*_{ijk},s^*_{ijk})r^*_{ijk}\Delta r\Delta\theta\Delta z$$

Triple integral in spherical coordinates

the limit of a triple Riemann sum, provided the following limit exists:

$$lim_{l,m,n\to\infty}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(\rho^*_{ijk},\theta^*_{ijk},\varphi^*_{ijk})(\rho^*_{ijk})^2\sin\,\varphi\Delta\rho\Delta\theta\Delta\varphi$$

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Answers

Exercise $2.6.1$**:**

Follow the same steps as in the previous example.

8

Exercise $2.6.2$**:**

Analyze the region, and draw a sketch.

$$\iiint_E f(r,\theta,z)r\,dz\,dr\,d\theta = \int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=2\sin\theta}\int_{z=0}^{z=4-r\sin\theta} f(r,\theta,z)r\,dz\,dr\,d\theta. \tag{2.6.92}$$

Exercise 2.6.3**:**

Note that θ is independent of r and z.

$E=\{(r,\theta,z)|0\le\theta\le 2\pi,\ 0\le z\le 1,\ 0\le r\le 2-z^2\}$ and

$$V=\int_{r=0}^{r=1}\int_{z=0}^{z=2-r^2}\int_{\theta=0}^{\theta=2\pi} r\,d\theta\,dz\,dr. \tag{2.6.93}$$

Exercise 2.6.4

Note that θ is independent of r and z.

$E_2=\{(r,\theta,z)|0\le\theta\le 2\pi,\ 0\le r\le 1,\ r\le z\le\sqrt{4-r^2}\}$ and

$$V=\int_{r=0}^{r=1}\int_{z=r}^{z=\sqrt{4-r^2}}\int_{\theta=0}^{\theta=2\pi} r\,d\theta\,dz\,dr. \tag{2.6.94}$$

Exercise 2.6.5

Follow the steps of the previous example.

$$V(E)=\int_{\theta=0}^{\theta=2\pi}\int_{\varphi=0}^{\varphi=\pi/3}\int_{\rho=0}^{\rho=2}\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta \tag{2.6.95}$$

Exercise 2.6.6**:**

$$\int_{x=-2}^{x=2}\int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}}\int_{z=-\sqrt{4-x^2-y^2}}^{z=\sqrt{4-x^2-y^2}} dz\,dy\,dx-\int_{x=-1}^{x=1}\int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}}\int_{z=-\sqrt{4-x^2-y^2}}^{z=\sqrt{4-x^2-y^2}} dz\,dy\,dx. \tag{2.6.96}$$

$$\int_{\theta=0}^{\theta=2\pi}\int_{r=1}^{r=2}\int_{z=-\sqrt{4-r^2}}^{z=\sqrt{4-r^2}} r\,dz\,dr\,d\theta. \tag{2.6.97}$$

$$\int_{\varphi=\pi/6}^{\varphi=5\pi/6}\int_{\theta=0}^{\theta=2\pi}\int_{\rho=\csc\varphi}^{\rho=2}\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi. \tag{2.6.98}$$

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2.6E: Exercises

Exercises 1 - 8

Evaluate the triple integrals $\iiint_E f(x,y,z)\ dV$ over the solid B.

1. $f(x,y,z) = z, \quad B = \{(x,y,z) \mid x^2 + y^2 \le 9, \quad x \le 0, \quad y \le 0, \quad 0 \le z \le 1\}$

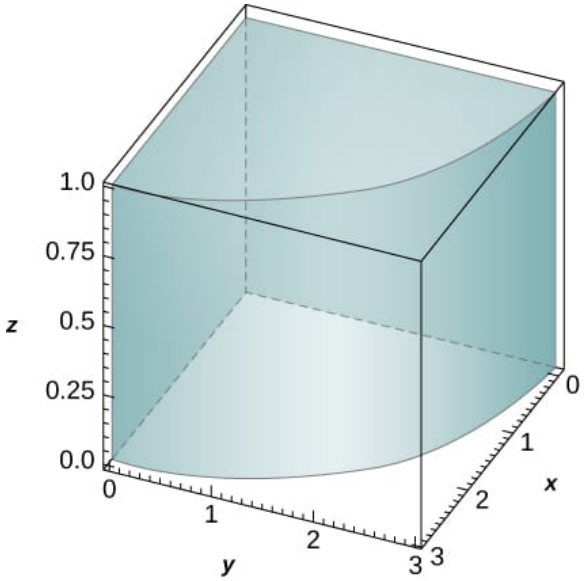


Figure $2.6E.1$: The graph illustrates the solid B bounded by $x^2 + y^2 \le 9, \quad x \le 0, \quad y \le 0, and \quad 0 \le z \le 1.$

Answer

$\frac{9\pi}{8}$

2. $f(x,y,z) = xz^2,\ B = \{(x,y,z) \mid x^2 + y^2 \le 16,\ x \ge 0,\ y \le 0,\ -1 \le z \le 1\}$

3. $f(x,y,z) = xy,\ B = \{(x,y,z) \mid x^2 + y^2 \le 1,\ x \ge 0,\ x \ge y,\ -1 \le z \le 1\}$

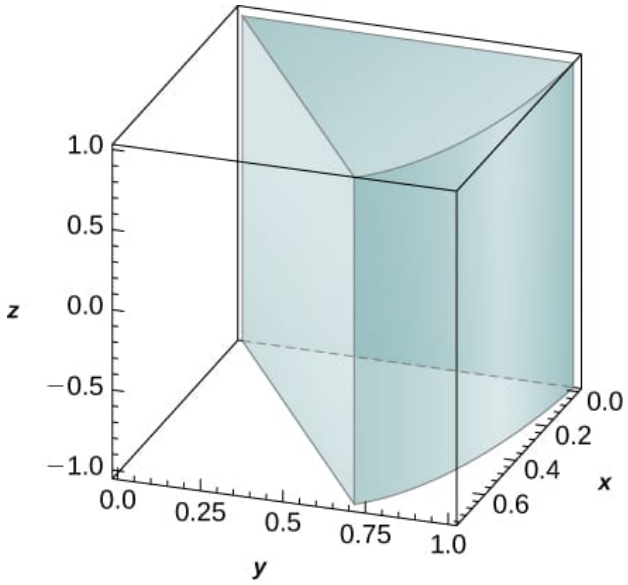


Figure $2.6E.2$: The graph illustrates the solid B bounded by $x^2+y^2 \le 1, \quad x \ge 0, \quad x \ge y, and \quad -1 \le z \le 1.$

Answer

$\frac{1}{8}$

4. $f(x,y,z) = x^2+y^2,\ B = \{(x,y,z) \mid x^2+y^2 \le 4,\ x \ge 0,\ x \le y,\ 0 \le z \le 3\}$

5. $f(x,y,z) = e^{\sqrt{x^2+y^2}},\ B = \{(x,y,z) \mid 1 \le x^2+y^2 \le 4,\ y \le 0,\ x \le y\sqrt{3},\ 2 \le z \le 3\}$

Answer

$\frac{\pi e^2}{6}$

6. $f(x,y,z) = \sqrt{x^2+y^2},\ B = \{(x,y,z) \mid 1 \le x^2+y^2 \le 9,\ y \le 0,\ 0 \le z \le 1\}$

7.1. Let B be a cylindrical shell with inner radius a outer radius b, and height c where $0 < a < b$ and $c > 0$. Assume that a function F defined on B can be expressed in cylindrical coordinates as $F(x,y,z) = f(r) + h(z)$, where f and h are differentiable functions. If $\int_a^b \bar{f}(r)\ dr = 0$ and $\bar{h}(0) = 0$, where $\bar{f}$ and $\bar{h}$ are antiderivatives of f and h, respectively, show that $\iiint_B F(x,y,z)\ dV = 2\pi c(b\bar{f}(b) - a\bar{f}(a)) + \pi(b^2-a^2)\bar{h}(c)$.

7.2. Use the previous result to show that $\iiint_B \left(z + \sin\sqrt{x^2+y^2}\right)\ dx\ dy\ dz = 6\pi^2(\pi-2)$, where B is a cylindrical shell with inner radius π outer radius 2π, and height 2.

8.1. Let B be a cylindrical shell with inner radius a outer radius b and height c where $0 < a < b$ and $c > 0$. Assume that a function F defined on B can be expressed in cylindrical coordinates as $F(x,y,z) = f(r)g(\theta)f(z)$, where $f,\ g$, and h are differentiable functions. If $\int_a^b \tilde{f}(r)\ dr = 0$, where $\tilde{f}$ is an antiderivative of f, show that $\iiint_B F(x,y,z)\ dV = [b\tilde{f}(b) - a\tilde{f}(a)][\tilde{g}(2\pi) - \tilde{g}(0)][\tilde{h}(c) - \tilde{h}(0)]$, where $\tilde{g}$ and $\tilde{h}$ are antiderivatives of g and h, respectively.

8.2. Use the previous result to show that $\iiint_B z\sin\sqrt{x^2+y^2}\ dx\ dy\ dz = -12\pi^2$, where B is a cylindrical shell with inner radius π outer radius 2π, and height 2.

Exercises 9 - 12

The boundaries of the solid E are given in cylindrical coordinates.

1. Express the region E in cylindrical coordinates.

2. Convert the integral $\iiint_E f(x,y,z)\ dV$ to cylindrical coordinates.

9. E is bounded by the right circular cylinder $r = 4\sin\theta$, the $r\theta$-plane, and the sphere $r^2 + z^2 = 16$.

Answer

1. $E = \left\{(r,\theta,z)\ |\ 0 \le \theta \le \pi,\ 0 \le r \le 4\sin\theta,\ 0 \le z \le \sqrt{16-r^2}\right\}$
2. $\int_0^{\pi}\int_0^{4\sin\theta}\int_0^{\sqrt{16-r^2}} f(r,\theta,z)r\ dz\ dr\ d\theta$

10. E is bounded by the right circular cylinder $r = \cos\theta$, the $r\theta$-plane, and the sphere $r^2 + z^2 = 9$.

11. E is located in the first octant and is bounded by the circular paraboloid $z = 9 - 3r^2$, the cylinder $r = \sqrt{3}$, and the plane $r(\cos\theta + \sin\theta) = 10 - z$.

Answer

1. $E = \left\{(r,\theta,z)\ |\ 0 \le \theta \le \frac{\pi}{2},\ 0 \le r \le \sqrt{3},\ 9 - 3r^2 \le z \le 10 - r(\cos\theta + \sin\theta)\right\}$
2. $\int_0^{\pi/2}\int_0^{\sqrt{3}}\int_{9-3r^2}^{10-r(\cos\theta+\sin\theta)} f(r,\theta,z)r\ dz\ dr\ d\theta$

12. E is located in the first octant outside the circular paraboloid $z = 10 - 2r^2$ and inside the cylinder $r = \sqrt{5}$ and is bounded also by the planes $z = 20$ and $\theta = \frac{\pi}{4}$.

Exercises 13 - 16

The function f and region E are given.

1. Express the region E and the function f in cylindrical coordinates.
2. Convert the integral $\iiint_B f(x,y,z)\ dV$ into cylindrical coordinates and evaluate it.

13. $f(x,y,z) = x^2 + y^2$, $E = \left\{(x,y,z)\ |\ 0 \le x^2 + y^2 \le 9,\ x \ge 0,\ y \ge 0,\ 0 \le z \le x + 3\right\}$

Answer

1. $E = \left\{(r,\theta,z)\ |\ 0 \le r \le 3,\ 0 \le \theta \le \frac{\pi}{2},\ 0 \le z \le r\ \cos\theta + 3\right\}$,
$f(r,\theta,z) = r^2$
2. $\int_0^3\int_0^{\pi/2}\int_0^{r\ \cos\theta+3} r(r^2)\ dz\ d\theta\ dr = \frac{243}{5} + \frac{243\pi}{8}$

14. $f(x,y,z) = x^2 + y^2,\ E = \left\{(x,y,z)|0 \le x^2 + y^2 \le 4,\ y \ge 0,\ 0 \le z \le 3 - x\right\}$

15. $f(x,y,z) = x,\ E = \left\{(x,y,z)\ |\ 1 \le y^2 + z^2 \le 9,\ 0 \le x \le 9 - y^2 - z^2\right\}$

Answer

1. $y = r\ \cos\theta,\ z = r\ \sin\theta,\ x = z,\ E = \left\{(r,\theta,z)\ |\ 1 \le r \le 3,\ 0 \le \theta \le 2\pi,\ 0 \le z \le 3 - r^2\right\},\ f(r,\theta,z) = z$;
2. $\int_1^3\int_0^{2\pi}\int_0^{3-r^2} zr\ dz\ d\theta\ dr = \frac{112\pi}{3}$

16. $f(x,y,z) = y,\ E = \left\{(x,y,z)\ |\ 1 \le x^2 + z^2 \le 9,\ 0 \le y \le 1 - x^2 - z^2\right\}$

Exercises 17 - 24

Find the volume of the solid E whose boundaries are given in rectangular coordinates.

17. E is above the xy-plane, inside the cylinder $x^2 + y^2 = 1$, and below the plane $z = 1$.

Answer

π

18. E is below the plane $z = 1$ and inside the paraboloid $z = x^2 + y^2$.

19. E is bounded by the circular cone $z = \sqrt{x^2 + y^2}$ and $z = 1$.

Answer

$\frac{\pi}{3}$

20. E is located above the xy-plane, below $z = 1$, outside the one-sheeted hyperboloid $x^2 + y^2 - z^2 = 1$, and inside the cylinder $x^2 + y^2 = 2$.

21. E is located inside the cylinder $x^2 + y^2 = 1$ and between the circular paraboloids $z = 1 - x^2 - y^2$ and $z = x^2 + y^2$.

Answer

π

22. E is located inside the sphere $x^2 + y^2 + z^2 = 1$, above the xy-plane, and inside the circular cone $z = \sqrt{x^2 + y^2}$.

23. E is located outside the circular cone $x^2 + y^2 = (z - 1)^2$ and between the planes $z = 0$ and $z = 2$.

Answer

$\frac{4\pi}{3}$

24. E is located outside the circular cone $z = 1 - \sqrt{x^2 + y^2}$, above the xy-plane, below the circular paraboloid, and between the planes $z = 0$ and $z = 2$.

25. [T] Use a computer algebra system (CAS) to graph the solid whose volume is given by the iterated integral in cylindrical coordinates $\int_{-\pi/2}^{\pi/2} \int_0^1 \int_{r^2}^r r \; dz \; dr \; d\theta$. Find the volume V of the solid. Round your answer to four decimal places.

Answer

$V = \frac{pi}{12} \approx 0.2618$

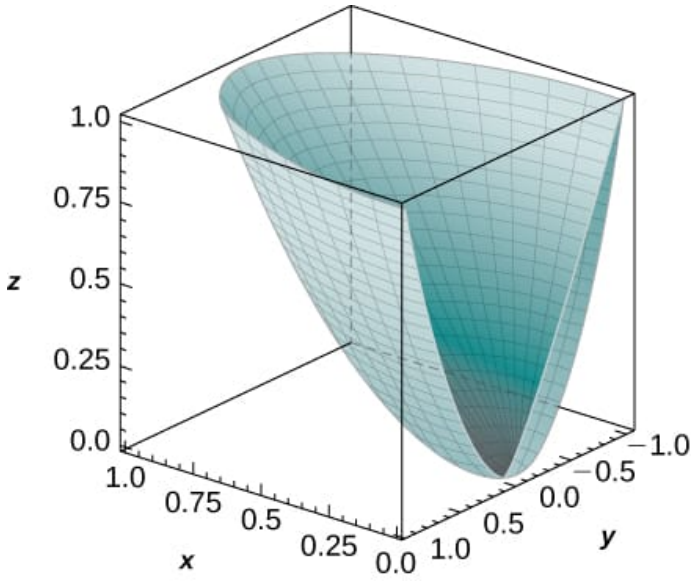


Figure $2.6E.3$: The graph illustrates the solid E.

26. [T] Use a CAS to graph the solid whose volume is given by the iterated integral in cylindrical coordinates $\int_0^{\pi/2}\int_0^1\int_{r^4}^{r} r\ dz\ dr\ d\theta$. Find the volume E of the solid. Round your answer to four decimal places.

27. Convert the integral $\int_0^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{x^2+y^2}^{\sqrt{x^2+y^2}} xz\ dz\ dx\ dy$ into an integral in cylindrical coordinates.

Answer

$$\int_0^1\int_0^{\pi}\int_{r^2}^{r} zr^2\ \cos\theta\ dz\ d\theta\ dr$$

28. Convert the integral $\int_0^2\int_0^y\int_0^1 (xy+z)\ dz\ dx\ dy$ into an integral in cylindrical coordinates.

Exercises 29 - 32

Evaluate the triple integral $\iiint_B f(x,y,z)\ dV$ over the solid B.

29. $f(x,y,z)=1,\ B=\{(x,y,z)\mid x^2+y^2+z^2\le 90,\ z\ge 0\}$

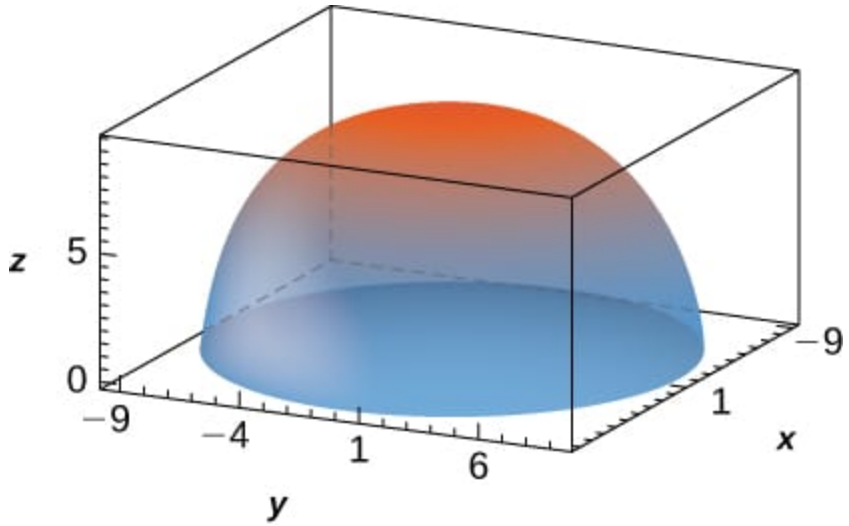


Figure $2.6E.4$: The graph illustrates the solid B bounded by $x^2+y^2+z^2 \le 90, and \quad z \ge 0.$

Answer

$180\pi\sqrt{10}$

30. $f(x,y,z) = 1-\sqrt{x^2+y^2+z^2},\ B = \{(x,y,z) \mid x^2+y^2+z^2 \le 9,\ y \ge 0,\ z \ge 0\}$

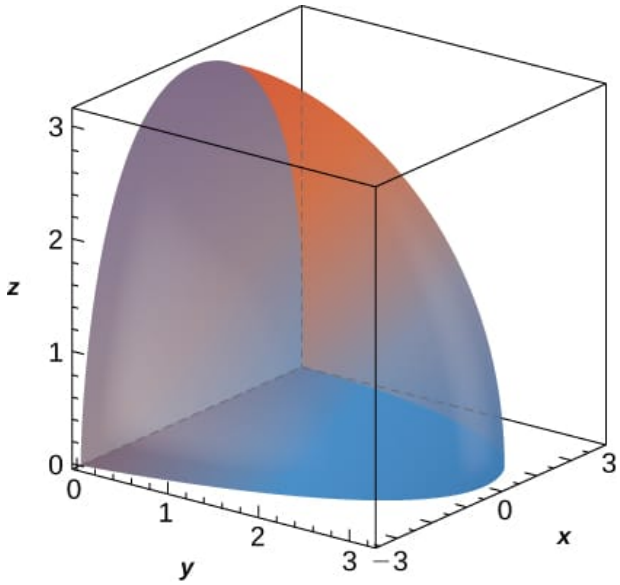


Figure $2.6E.5$: The graph illustrates the solid B bounded by $x^2+y^2+z^2 \le 9, \quad y \ge 0, \quad \text{and} \quad z \ge 0.$

31. $f(x,y,z) = \sqrt{x^2+y^2},\ B$ is bounded above by the half-sphere $x^2+y^2+z^2 = 9$ with $z \ge 0$ and below by the cone $z^2 = x^2+y^2$.

Answer

$\frac{81\pi(\pi-2)}{16}$

32. $f(x,y,z) = \sqrt{x^2+y^2},\ B$ is bounded above by the half-sphere $x^2+y^2+z^2 = 16$ with $z \ge 0$ and below by the cone $2z^2 = x^2+y^2$.

33. Show that if $F(\rho,\theta,\varphi) = f(\rho)g(\theta)h(\varphi)$ is a continuous function on the spherical box $B = \{(\rho,\theta,\varphi) \mid a \le \rho \le b,\ \alpha \le \theta \le \beta,\ \gamma \le \varphi \le \psi\}$, then $\iiint_B F\ dV = \left(\int_a^b \rho^2 f(\rho)\ dr\right)\left(\int_\alpha^\beta g(\theta)\ d\theta\right)\left(\int_\gamma^\psi h(\varphi)\ \sin\varphi\ d\varphi\right)$.

34. A function F is said to have spherical symmetry if it depends on the distance to the origin only, that is, it can be expressed in spherical coordinates as $F(x,y,z) = f(\rho)$, where $\rho = \sqrt{x^2+y^2+z^2}$. Show that $\iiint_B F(x,y,z)\ dV = 2\pi\int_a^b \rho^2 f(\rho)\ d\rho$, where B is

the region between the upper concentric hemispheres of radii a and b centered at the origin, with $0 < a < b$ and F a spherical function defined on B.

Use the previous result to show that $\iiint_B (x^2+y^2+z^2)\sqrt{x^2+y^2+z^2}dV = 21\pi$, where $B = \{(x,y,z) \mid 1 \le x^2+y^2+z^2 \le 2,\ z \ge 0\}$.

35. Let B be the region between the upper concentric hemispheres of radii *a* and *b* centered at the origin and situated in the first octant, where $0 < a < b$. Consider *F* a function defined on *B* whose form in spherical coordinates (ρ, θ, φ) is $F(x,y,z) = f(\rho)\cos\varphi$. Show that if $g(a) = g(b) = 0$ and $\int_a^b h(\rho)\ d\rho = 0$, then $\iiint_B F(x,y,z)\ dV = \frac{\pi^2}{4}[ah(a) - bh(b)]$, where g is an antiderivative of f and h is an antiderivative of g.

Use the previous result to show that $\iiint_B = \frac{z\cos\sqrt{x^2+y^2+z^2}}{\sqrt{x^2+y^2+z^2}}\ dV = \frac{3\pi^2}{2}$, where B is the region between the upper concentric hemispheres of radii π and 2π centered at the origin and situated in the first octant.

Exercises 36 - 39

The function f and region E are given.

1. Express the region E and function f in cylindrical coordinates.
2. Convert the integral $\iiint_B f(x,y,z)\ dV$ into cylindrical coordinates and evaluate it.

36. $f(x,y,z) = z;\ E = \{(x,y,z) \mid 0 \le x^2+y^2+z^2 \le 1,\ z \ge 0\}$

37. $f(x,y,z) = x+y;\ E = \{(x,y,z) \mid 1 \le x^2+y^2+z^2 \le 2,\ z \ge 0,\ y \ge 0\}$

Answer

1. $f(\rho,\theta,\varphi) = \rho\ \sin\varphi\ (\cos\theta + \sin\theta),\ E = \{(\rho,\theta,\varphi) \mid 1 \le \rho \le 2,\ 0 \le \theta \le \pi,\ 0 \le \varphi \le \frac{\pi}{2}\}$;
2. $\int_0^{\pi}\int_0^{\pi/2}\int_1^2 \rho^3\cos\varphi\ \sin\varphi\ d\rho\ d\varphi\ d\theta = \frac{15\pi}{8}$

38. $f(x,y,z) = 2xy;\ E = \{(x,y,z) \mid \sqrt{x^2+y^2} \le z \le \sqrt{1-x^2-y^2},\ x \ge 0,\ y \ge 0\}$

39. $f(x,y,z) = z;\ E = \{(x,y,z) \mid x^2+y^2+z^2-2x \le 0,\ \sqrt{x^2+y^2} \le z\}$

Answer

1. $f(\rho,\theta,\varphi) = \rho\ \cos\varphi;\ E = \{(\rho,\theta,\varphi) \mid 0 \le \rho \le 2\ \cos\varphi,\ 0 \le \theta \le \frac{\pi}{2},\ 0 \le \varphi \le \frac{\pi}{4}\}$;
2. $\int_0^{\pi/2}\int_0^{\pi/4}\int_0^{2\ \cos\varphi} \rho^3\sin\varphi\ \cos\varphi\ d\rho\ d\varphi\ d\theta = \frac{7\pi}{24}$

Exercises 40 - 41

Find the volume of the solid E whose boundaries are given in rectangular coordinates.

40. $E = \{(x,y,z) \mid \sqrt{x^2+y^2} \le z \le \sqrt{16-x^2-y^2},\ x \ge 0,\ y \ge 0\}$

41. $E = \{(x,y,z) \mid x^2+y^2+z^2-2z \le 0,\ \sqrt{x^2+y^2} \le z\}$

Answer

$\frac{\pi}{4}$

42. Use spherical coordinates to find the volume of the solid situated outside the sphere $\rho = 1$ and inside the sphere $\rho = \cos\varphi$, with $\varphi \in [0, \frac{\pi}{2}]$.

43. Use spherical coordinates to find the volume of the ball $\rho \leq 3$ that is situated between the cones $\varphi = \frac{\pi}{4}$ and $\varphi = \frac{\pi}{3}$.

Answer

$9\pi(\sqrt{2} - 1)$

44. Convert the integral $\int_{-4}^{4} \int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}} \int_{-\sqrt{16-x^2-y^2}}^{\sqrt{16-x^2-y^2}} (x^2 + y^2 + z^2)\ dz\ dx\ dy$ into an integral in spherical coordinates.

45. Convert the integral $\int_{0}^{4} \int_{0}^{\sqrt{16-x^2}} \int_{-\sqrt{16-x^2-y^2}}^{\sqrt{16-x^2-y^2}} (x^2 + y^2 + z^2)^2\ dz\ dy\ dx$ into an integral in spherical coordinates.

Answer

$\int_{0}^{\pi/2} \int_{0}^{\pi} \int_{0}^{4} \rho^6 \sin\varphi\ d\rho\ d\phi\ d\theta$

46. Convert the integral $\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{16-x^2-y^2}} dz\ dy\ dx$ into an integral in spherical coordinates and evaluate it.

47. [T] Use a CAS to graph the solid whose volume is given by the iterated integral in spherical coordinates $\int_{\pi/2}^{\pi} \int_{5\pi}^{\pi/6} \int_{0}^{2} \rho^2 \sin\varphi\ d\rho\ d\varphi\ d\theta$. Find the volume V of the solid. Round your answer to three decimal places.

Answer

$V = \frac{4\pi\sqrt{3}}{3} \approx 7.255$

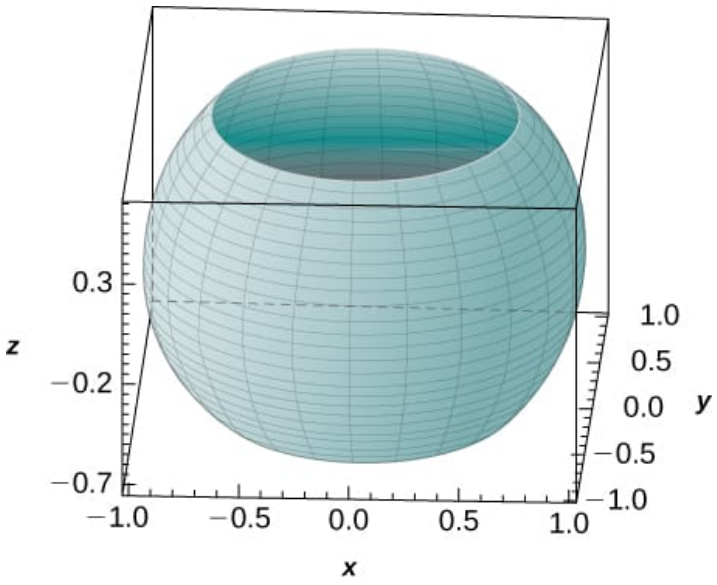


Figure $2.6E.6$: The graph illustrates the solid E.

48. [T] Use a CAS to graph the solid whose volume is given by the iterated integral in spherical coordinates as $\int_{0}^{2\pi} \int_{3\pi/4}^{\pi/4} \int_{0}^{1} \rho^2 \sin\varphi\ d\rho\ d\varphi\ d\theta$. Find the volume V of the solid. Round your answer to three decimal places.

49. [T] Use a CAS to evaluate the integral $\iiint_E (x^2 + y^2)\ dV$ where E lies above the paraboloid $z = x^2 + y^2$ and below the plane $z = 3y$.

Answer

$\frac{343\pi}{32}$

50. [T]

a. Evaluate the integral $\iiint_E e^{\sqrt{x^2+y^2+z^2}}\ dV$, where E is bounded by spheres $4x^2+4y^2+4z^2=1$ and $x^2+y^2+z^2=1$.

b. Use a CAS to find an approximation of the previous integral. Round your answer to two decimal places.

51. Express the volume of the solid inside the sphere $x^2+y^2+z^2=16$ and outside the cylinder $x^2+y^2=4$ as triple integrals in cylindrical coordinates and spherical coordinates, respectively.

Answer

$\int_0^{2\pi}\int_2^4\int_{-\sqrt{16-r^2}}^{\sqrt{16-r^2}} r\ dz\ dr\ d\theta$ and $\int_{\pi/6}^{5\pi/6}\int_0^{2\pi}\int_{2\csc\phi}^4 \rho^2\sin\rho\ d\rho\ d\theta\ d\phi$

52. Express the volume of the solid inside the sphere $x^2+y^2+z^2=16$ and outside the cylinder $x^2+y^2=4$ that is located in the first octant as triple integrals in cylindrical coordinates and spherical coordinates, respectively.

53. The power emitted by an antenna has a power density per unit volume given in spherical coordinates by $p(\rho,\theta,\varphi)=\frac{P_0}{\rho^2}\cos^2\theta\ \sin^4\varphi$, where P_0 is a constant with units in watts. The total power within a sphere B of radius r meters is defined as $P=\iiint_B p(\rho,\theta,\varphi)\ dV$. Find the total power P.

Answer

$P=\frac{32P_0\pi}{3}$ watts

54. Use the preceding exercise to find the total power within a sphere B of radius 5 meters when the power density per unit volume is given by $p(\rho,\theta,\varphi)=\frac{30}{\rho^2}\cos^2\theta\sin^4\varphi$.

55. A charge cloud contained in a sphere B of radius r centimeters centered at the origin has its charge density given by $q(x,y,z)=k\sqrt{x^2+y^2+z^2}\frac{\mu C}{cm^3}$, where $k>0$. The total charge contained in B is given by $Q=\iiint_B q(x,y,z)\ dV$. Find the total charge Q.

Answer

$Q=kr^4\pi\mu C$

56. Use the preceding exercise to find the total charge cloud contained in the unit sphere if the charge density is $q(x,y,z)=20\sqrt{x^2+y^2+z^2}\frac{\mu C}{cm^3}$.

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2.7: Calculating Centers of Mass and Moments of Inertia

Learning Objectives

- Use double integrals to locate the center of mass of a two-dimensional object.
- Use double integrals to find the moment of inertia of a two-dimensional object.
- Use triple integrals to locate the center of mass of a three-dimensional object.

We have already discussed a few applications of multiple integrals, such as finding areas, volumes, and the average value of a function over a bounded region. In this section we develop computational techniques for finding the center of mass and moments of inertia of several types of physical objects, using double integrals for a lamina (flat plate) and triple integrals for a three-dimensional object with variable density. The density is usually considered to be a constant number when the lamina or the object is homogeneous; that is, the object has uniform density.

Center of Mass in Two Dimensions

The center of mass is also known as the center of gravity if the object is in a uniform gravitational field. If the object has uniform density, the center of mass is the geometric center of the object, which is called the centroid. Figure $2.7.1$ shows a point P as the center of mass of a lamina. The lamina is perfectly balanced about its center of mass.

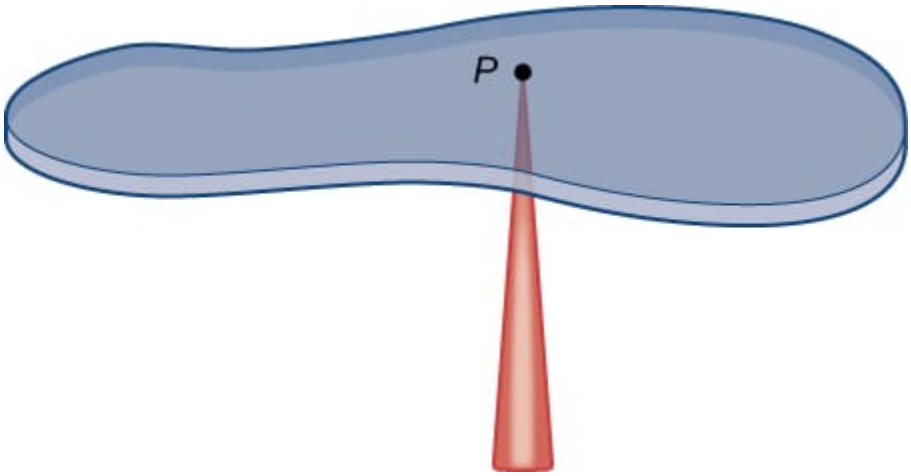


Figure $2.7.1$: A lamina is perfectly balanced on a spindle if the lamina's center of mass sits on the spindle. A point P is located at the center of mass of a lamina, and it is at the tip of the spindle.

To find the coordinates of the center of mass $P(\bar{x}, \bar{y})$ of a lamina, we need to find the moment M_x of the lamina about the x-axis and the moment M_y about the y-axis. We also need to find the mass m of the lamina. Then

$$\bar{x} = \frac{M_y}{m} \tag{2.7.1}$$

and

$$\bar{y} = \frac{M_x}{m}. \tag{2.7.2}$$

Refer to Moments and Centers of Mass for the definitions and the methods of single integration to find the center of mass of a one-dimensional object (for example, a thin rod). We are going to use a similar idea here except that the object is a two-dimensional lamina and we use a double integral.

If we allow a constant density function, then $\bar{x} = \frac{M_y}{m}$ and $\bar{y} = \frac{M_x}{m}$ give the *centroid* of the lamina.

Suppose that the lamina occupies a region R in the xy-plane and let $\rho(x,y)$ be its density (in units of mass per unit area) at any point (x,y). Hence,

$$\rho(x,y) = \lim_{\Delta A \to 0} \frac{\Delta m}{\Delta A} \tag{2.7.3}$$

where Δm and ΔA are the mass and area of a small rectangle containing the point (x,y) and the limit is taken as the dimensions of the rectangle go to 0 (see the Figure 2.7.2).

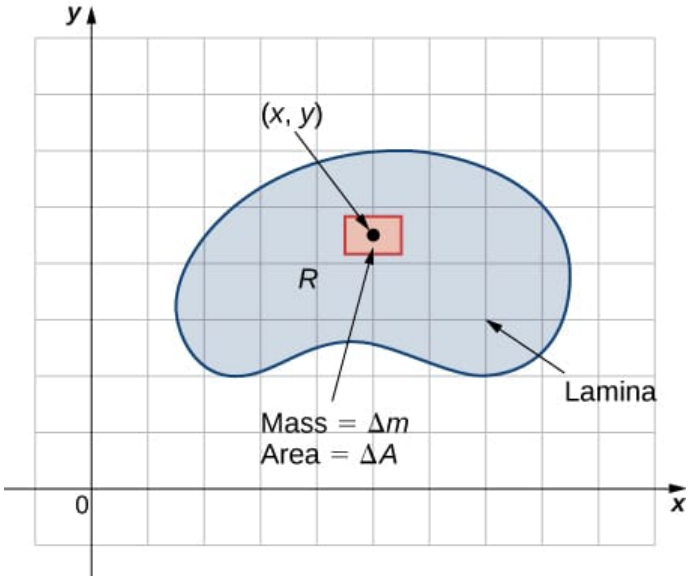


Figure 2.7.2: The density of a lamina at a point is the limit of its mass per area in a small rectangle about the point as the area goes to zero.

Just as before, we divide the region R into tiny rectangles R_{ij} with area ΔA and choose (x_{ij}^*, y_{ij}^*) as sample points. Then the mass m_{ij} of each R_{ij} is equal to $\rho(x_{ij}^*, y_{ij}^*)\Delta A$ (Figure 2.7.2). Let k and l be the number of subintervals in x and y respectively. Also, note that the shape might not always be rectangular but the limit works anyway, as seen in previous sections.

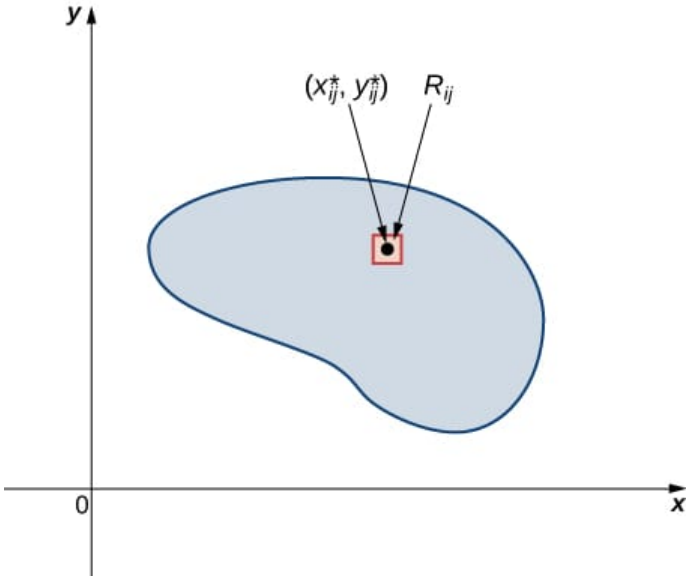


Figure 2.7.3: A sample rectangular region R_{ij} in the lamina. It contains a sample point (x_{ij}^*, y_{ij}^*).

Hence, the mass of the lamina is

$$m = \lim_{k,l \to \infty} \sum_{i=1}^{k} \sum_{j=1}^{l} m_{ij} = \lim_{k,l \to \infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_R \rho(x,y)\,dA. \tag{2.7.4}$$

Let's see an example now of finding the total mass of a triangular lamina.

Example 2.7.1: Finding the Total Mass of a Lamina

Consider a triangular lamina R with vertices $(0,0),\ (0,3),\ (3,0)$ and with density $\rho(x,y) = xy\ kg/m^2$. Find the total mass.

Solution

A sketch of the region R is always helpful, as shown in the figure 2.7.4.

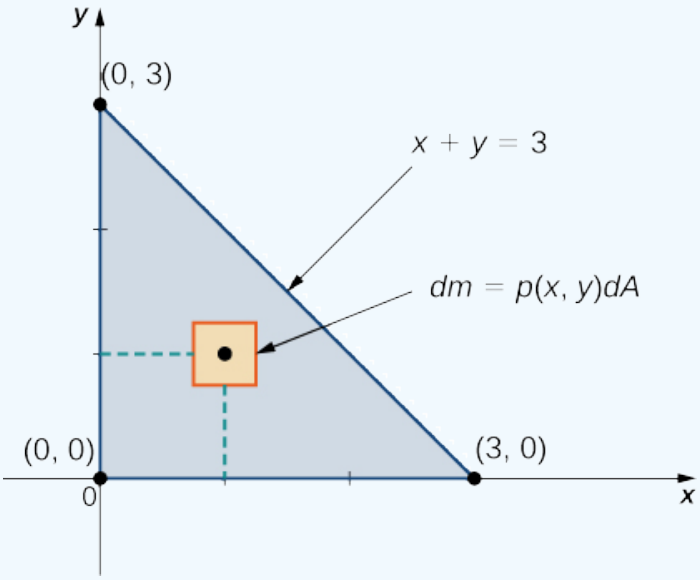


Figure 2.7.4: A triangular lamina in the xy-plane with vertices $(0,0),\ (0,3),\ (3,0)$ and with density $\rho(x,y) = xy$.

Using the expression developed for mass, we see that

$$m = \iint_R dm = \iint_R \rho(x,y)dA = \int_{x=0}^{x=3}\int_{y=0}^{y=3-x} xy\,dy\,dx = \int_{x=0}^{x=3}\left[x\frac{y^2}{2}\Big|_{y=0}^{y=3-x}\right]dx = \int_{x=0}^{x=3}\frac{1}{2}x(3-x)^2dx = \left[\frac{9x^2}{4} - x^3 + \frac{x^4}{8}\right.$$

The computation is straightforward, giving the answer $m = \dfrac{27}{8}\ kg$.

Exercise 2.7.1

Consider the same region R as in the previous example, and use the density function $\rho(x,y) = \sqrt{xy}$. Find the total mass.

Answer available at the end of this section.

Now that we have established the expression for mass, we have the tools we need for calculating moments and centers of mass. The moment M_z about the x-axis for R is the limit of the sums of moments of the regions R_{ij} about the x-axis. Hence

$$M_x = \lim_{k,l\to\infty}\sum_{i=1}^{k}\sum_{j=1}^{l}(y_{ij}^*)m_{ij} = \lim_{k,l\to\infty}\sum_{i=1}^{k}\sum_{j=1}^{l}(y_{ij}^*)\rho(x_{ij}^*,y_{ij}^*)\,\Delta A = \iint_R y\rho(x,y)\,dA \tag{2.7.6}$$

Similarly, the moment M_y about the y-axis for R is the limit of the sums of moments of the regions R_{ij} about the y-axis. Hence

$$M_y = \lim_{k,l\to\infty}\sum_{i=1}^{k}\sum_{j=1}^{l}(x_{ij}^*)m_{ij} = \lim_{k,l\to\infty}\sum_{i=1}^{k}\sum_{j=1}^{l}(x_{ij}^*)\rho(x_{ij}^*,y_{ij}^*)\,\Delta A = \iint_R x\rho(x,y)\,dA \tag{2.7.7}$$

Example 2.7.2: Finding Moments

Consider the same triangular lamina R with vertices $(0,0),\ (0,3),\ (3,0)$ and with density $\rho(x,y)=xy$. Find the moments M_x and M_y.

Solution

Use double integrals for each moment and compute their values:

$$M_x=\iint_R y\rho(x,y)\,dA=\int_{x=0}^{x=3}\int_{y=0}^{y=3-x} xy^2\,dy\,dx=\frac{81}{20}, \tag{2.7.8}$$

$$M_y=\iint_R x\rho(x,y)\,dA=\int_{x=0}^{x=3}\int_{y=0}^{y=3-x} x^2y\,dy\,dx=\frac{81}{20}, \tag{2.7.9}$$

The computation is quite straightforward.

Exercise 2.7.2

Consider the same lamina R as above and use the density function $\rho(x,y)=\sqrt{xy}$. Find the moments M_x and M_y.

Answer available at the end of this section.

Finally we are ready to restate the expressions for the center of mass in terms of integrals. We denote the *x*-coordinate of the center of mass by $\bar{x}$ and the *y*-coordinate by $\bar{y}$. Specifically,

$$\bar{x}=\frac{M_y}{m}=\frac{\iint_R x\rho(x,y)\,dA}{\iint_R \rho(x,y)\,dA} \tag{2.7.10}$$

and

$$\bar{y}=\frac{M_x}{m}=\frac{\iint_R y\rho(x,y)\,dA}{\iint_R \rho(x,y)\,dA} \tag{2.7.11}$$

Example 2.7.3: Center of Mass

Again consider the same triangular region R with vertices $(0,0),\ (0,3),\ (3,0)$ and with density function $\rho(x,y)=xy$. Find the center of mass.

Solution

Using the formulas we developed, we have

$$\bar{x}=\frac{M_y}{m}=\frac{\iint_R x\rho(x,y)\,dA}{\iint_R \rho(x,y)\,dA}=\frac{81/20}{27/8}=\frac{6}{5}, \tag{2.7.12}$$

$$\bar{y}=\frac{M_x}{m}=\frac{\iint_R y\rho(x,y)\,dA}{\iint_R \rho(x,y)\,dA}=\frac{81/20}{27/8}=\frac{6}{5}. \tag{2.7.13}$$

Therefore, the center of mass is the point $\left(\frac{6}{5},\frac{6}{5}\right)$.

Analysis

If we choose the density $\rho(x,y)$ instead to be uniform throughout the region (i.e., constant), such as the value 1 (any constant will do), then we can compute the centroid,

$$x_c = \frac{M_y}{m} = \frac{\iint_R x\,dA}{\iint_R dA} = \frac{9/2}{9/2} = 1, \tag{2.7.14}$$

$$y_c = \frac{M_x}{m} = \frac{\iint_R y\,dA}{\iint_R dA} = \frac{9/2}{9/2} = 1. \tag{2.7.15}$$

Notice that the center of mass $\left(\frac{6}{5}, \frac{6}{5}\right)$ is not exactly the same as the centroid $(1, 1)$ of the triangular region. This is due to the variable density of R. If the density is constant, then we just use $\rho(x, y) = c$ (constant). This value cancels out from the formulas, so for a constant density, the center of mass coincides with the centroid of the lamina.

Exercise 2.7.3

Again use the same region R as above and use the density function $\rho(x, y) = \sqrt{xy}$. Find the center of mass.

Answer available at the end of this section.

Once again, based on the comments at the end of Example 2.7.3, we have expressions for the centroid of a region on the plane:

$$x_c = \frac{M_y}{m} = \frac{\iint_R x\,dA}{\iint_R dA} \text{ and } y_c = \frac{M_x}{m} = \frac{\iint_R y\,dA}{\iint_R dA}. \tag{2.7.16}$$

We should use these formulas and verify the centroid of the triangular region referred to in the last three examples.

Example 2.7.4: Finding Mass, Moments, and Center of Mass

Find the mass, moments, and the center of mass of the lamina of density $\rho(x, y) = x + y$ occupying the region R under the curve $y = x^2$ in the interval $0 \le x \le 2$ (see the figure 2.7.5).

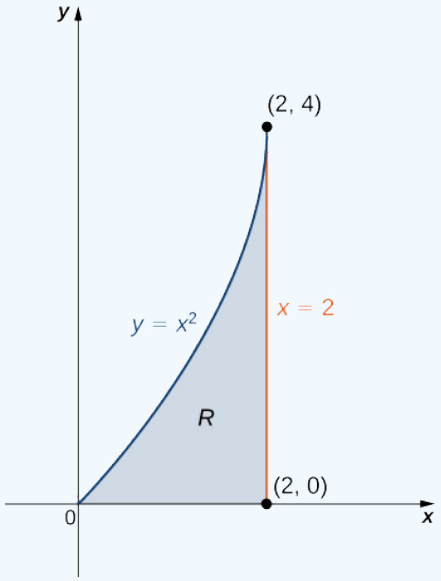


Figure 2.7.5: A lamina R with density $\rho(x, y) = x + y$ occupying the region R under the curve $y = x^2$ in the interval $0 \le x \le 2$.

Solution

First we compute the mass m. We need to describe the region between the graph of $y = x^2$ and the vertical lines $x = 0$ and $x = 2$:

$$m = \iint_R dm = \iint_R \rho(x,y)\,dA = \int_{x=0}^{x=2}\int_{y=0}^{y=x^2} (x+y)dy\,dx = \int_{x=0}^{x=2}\left[xy + \frac{y^2}{2}\Big|_{y=0}^{y=x^2}\right] dx \tag{2.7.17}$$

$$= \int_{x=0}^{x=2}\left[x^3 + \frac{x^4}{2}\right] dx = \left[\frac{x^4}{4} + \frac{x^5}{10}\right]\Big|_{x=0}^{x=2} = \frac{36}{5}. \tag{2.7.18}$$

Now compute the moments M_x and M_y:

$$M_x = \iint_R y\rho(x,y)\,dA = \int_{x=0}^{x=2}\int_{y=0}^{y=x^2} y(x+y)\,dy\,dx = \frac{80}{7}, \tag{2.7.19}$$

$$M_y = \iint_R x\rho(x,y)\,dA = \int_{x=0}^{x=2}\int_{y=0}^{y=x^2} x(x+y)\,dy\,dx = \frac{176}{15}. \tag{2.7.20}$$

Finally, evaluate the center of mass,

$$\bar{x} = \frac{M_y}{m} = \frac{\iint_R x\rho(x,y)\,dA}{\iint_R \rho(x,y)\,dA} = \frac{176/15}{36/5} = \frac{44}{27}, \tag{2.7.21}$$

$$\bar{y} = \frac{M_x}{m} = \frac{\iint_R y\rho(x,y)\,dA}{\iint_R \rho(x,y)\,dA} = \frac{80/7}{36/5} = \frac{100}{63}. \tag{2.7.22}$$

Hence the center of mass is $(\bar{x},\bar{y}) = \left(\frac{44}{27}, \frac{100}{63}\right)$.

Exercise 2.7.4

Calculate the mass, moments, and the center of mass of the region between the curves $y = x$ and $y = x^2$ with the density function $\rho(x,y) = x$ in the interval $0 \le x \le 1$.

Answer available at the end of this section.

Example 2.7.5: Finding a Centroid

Find the centroid of the region under the curve $y = e^x$ over the interval $1 \le x \le 3$ (Figure 2.7.6).

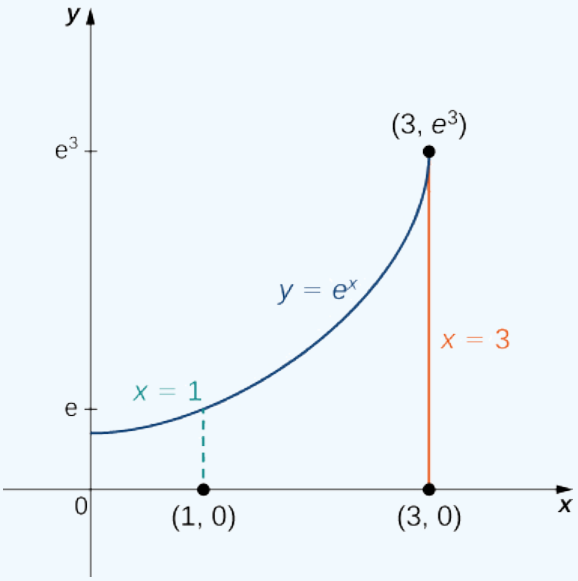


Figure 2.7.6: A region bounded below the curve $y = e^x$ and above the x-axis over the interval $1 \le x \le 3$.

Solution

To compute the centroid, we assume that the density function is constant and hence it cancels out:

$$x_c = \frac{M_y}{m} = \frac{\iint_R x\,dA}{\iint_R dA} \; and \; y_c = \frac{M_x}{m} = \frac{\iint_R y\,dA}{\iint_R dA}, \tag{2.7.23}$$

$$x_c = \frac{M_y}{m} = \frac{\iint_R x\,dA}{\iint_R dA} = \frac{\int_{x=1}^{x=3}\int_{y=0}^{y=e^x} x\,dy\,dx}{\int_{x=1}^{x=3}\int_{y=0}^{y=e^x} dy\,dx} = \frac{\int_{x=1}^{x=3} xe^x dx}{\int_{x=1}^{x=3} e^x dx} = \frac{2e^3}{e^3-e} = \frac{2e^2}{e^2-1}, \tag{2.7.24}$$

$$y_c = \frac{M_x}{m} = \frac{\iint_R y\,dA}{\iint_R dA} = \frac{\int_{x=1}^{x=3}\int_{y=0}^{y=e^x} y\,dy\,dx}{\int_{x=1}^{x=3}\int_{y=0}^{y=e^x} dy\,dx} = \frac{\int_{x=1}^{x=3} \frac{e^{2x}}{2} dx}{\int_{x=1}^{x=3} e^x dx} = \frac{\frac{1}{4}e^2(e^4-1)}{e(e^2-1)} = \frac{1}{4}e(e^2+1). \tag{2.7.25}$$

Thus the centroid of the region is

$$(x_c, y_c) = \left(\frac{2e^2}{e^2-1}, \frac{1}{4}e(e^2+1)\right). \tag{2.7.26}$$

Watch the accompanying video for a visual demonstration of the concept.

- **Video length**: 3 minutes 38 seconds.
- **Context**: This video demonstrates how to use a double integral to find the center of mass.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Exercise 2.7.5

Calculate the centroid of the region between the curves $y = x$ and $y = \sqrt{x}$ with uniform density in the interval $0 \le x \le 1$.

Answer available at the end of this section.

Moments of Inertia

For a clear understanding of how to calculate moments of inertia using double integrals, we need to go back to the general definition in Section 6.6. The moment of inertia of a particle of mass m about an axis is mr^2 where r is the distance of the particle from the axis. We can see from Figure $PageIndex3$ that the moment of inertia of the subrectangle R_{ij} about the x-axis is $(y_{ij}^*)^2\rho(x_{ij}^*, y_{ij}^*)\Delta A$. Similarly, the moment of inertia of the subrectangle R_{ij} about the y-axis is $(x_{ij}^*)^2\rho(x_{ij}^*, y_{ij}^*)\Delta A$. The moment of inertia is related to the rotation of the mass; specifically, it measures the tendency of the mass to resist a change in rotational motion about an axis.

The moment of inertia I_x about the x-axis for the region R is the limit of the sum of moments of inertia of the regions R_{ij} about the x-axis. Hence

$$I_x = \lim_{k,l\to\infty}\sum_{i=1}^{k}\sum_{j=1}^{l}(y_{ij}^*)^2 m_{ij} = \lim_{k,l\to\infty}\sum_{i=1}^{k}\sum_{j=1}^{l}(y_{ij}^*)^2\rho(x_{ij}^*, y_{ij}^*)\,\Delta A = \iint_R y^2\rho(x,y)\,dA. \tag{2.7.27}$$

Similarly, the moment of inertia I_y about the y-axis for R is the limit of the sum of moments of inertia of the regions R_{ij} about the y-axis. Hence

$$I_y = \lim_{k,l\to\infty} \sum_{i=1}^{k} \sum_{j=1}^{l} (x_{ij}^*)^2 m_{ij} = \lim_{k,l\to\infty} \sum_{i=1}^{k} \sum_{j=1}^{l} (x_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*)\, \Delta A = \iint_R x^2 \rho(x,y)\, dA. \tag{2.7.28}$$

Sometimes, we need to find the moment of inertia of an object about the origin, which is known as the polar moment of inertia. We denote this by I_0 and obtain it by adding the moments of inertia I_x and I_y. Hence

$$I_0 = I_x + I_y = \iint_R (x^2 + y^2)\rho(x,y)\, dA. \tag{2.7.29}$$

All these expressions can be written in polar coordinates by substituting $x = r\ \cos\ \theta,\ y = r\ \sin\ \theta$, and $dA = r\ dr\ d\theta$. For example, $I_0 = \iint_R r^2 \rho(r\ \cos\ \theta,\ r\ \sin\ \theta)\ dA$.

Example 2.7.6: Finding Moments of Inertia for a Triangular Lamina

Use the triangular region R with vertices $(0,0),\ (2,2)$, and $(2,0)$ and with density $\rho(x,y) = xy$ as in previous examples. Find the moments of inertia.

Solution

Using the expressions established above for the moments of inertia, we have

$$I_x = \iint_R y^2 \rho(x,y)\, dA = \int_{x=0}^{x=2} \int_{y=0}^{y=x} xy^3\, dy\, dx = \frac{8}{3}, \tag{2.7.30}$$

$$I_y = \iint_R x^2 \rho(x,y)\, dA = \int_{x=0}^{x=2} \int_{y=0}^{y=x} x^3 y\, dy\, dx = \frac{16}{3}, \tag{2.7.31}$$

$$I_0 = \iint_R (x^2 + y^2)\rho(x,y)\, dA = \int_0^2 \int_0^x (x^2 + y^2) xy\, dy\, dx = I_x + I_y = 8 \tag{2.7.32}$$

Exercise 2.7.6

Again use the same region R as above and the density function $\rho(x,y) = \sqrt{xy}$. Find the moments of inertia.

Answer available at the end of this section.

As mentioned earlier, the moment of inertia of a particle of mass m about an axis is mr^2 where r is the distance of the particle from the axis, also known as the **radius of gyration**.

Hence the radii of gyration with respect to the x-axis, the y-axis and the origin are

$$R_x = \sqrt{\frac{I_x}{m}},\ R_y = \sqrt{\frac{I_y}{m}},\ and\ R_0 = \sqrt{\frac{I_0}{m}}, \tag{2.7.33}$$

respectively. In each case, the radius of gyration tells us how far (perpendicular distance) from the axis of rotation the entire mass of an object might be concentrated. The moments of an object are useful for finding information on the balance and torque of the object about an axis, but radii of gyration are used to describe the distribution of mass around its centroidal axis. There are many applications in engineering and physics. Sometimes it is necessary to find the radius of gyration, as in the next example.

Example 2.7.7: Finding the Radius of Gyration for a Triangular Lamina

Consider the same triangular lamina R with vertices $(0,0),\ (2,2)$, and $(2,0)$ and with density $\rho(x,y) = xy$ as in previous examples. Find the radii of gyration with respect to the x-axis the y-axis and the origin.

Solution

If we compute the mass of this region we find that $m = 2$. We found the moments of inertia of this lamina in Example 2.7.4. From these data, the radii of gyration with respect to the x-axis, y-axis and the origin are, respectively,

$$R_x = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{8/3}{2}} = \sqrt{\frac{8}{6}} = \frac{2\sqrt{3}}{3}, \tag{2.7.34}$$

$$R_y = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{16/3}{2}} = \sqrt{\frac{8}{3}} = \frac{2\sqrt{6}}{3}, \tag{2.7.35}$$

$$R_0 = \sqrt{\frac{I_0}{m}} = \sqrt{\frac{8}{2}} = \sqrt{4} = 2. \tag{2.7.36}$$

Exercise 2.7.7

Use the same region R from Example 2.7.7 and the density function $\rho(x, y) = \sqrt{xy}$. Find the radii of gyration with respect to the x-axis, the y-axis, and the origin.

Hint available at the end of this section.

Answer available at the end of this section.

Center of Mass and Moments of Inertia in Three Dimensions

All the expressions of double integrals discussed so far can be modified to become triple integrals.

Definitions: Moment of Masses and Center of Mass

If we have a solid object Q with a density function $\rho(x, y, z)$ at any point (x, y, z) in space, then its mass is

$$m = \iiint_Q \rho(x, y, z)\, dV. \tag{2.7.37}$$

Its moments about the xy-plane the xz-plane and the yz-plane are

$$M_{xy} = \iiint_Q z\rho(x, y, z)\, dV,\ M_{xz} = \iiint_Q y\rho(x, y, z)\, dV,\ M_{yz} = \iiint_Q x\rho(x, y, z)\, dV. \tag{2.7.38}$$

If the center of mass of the object is the point $(\bar{x}, \bar{y}, \bar{z})$, then

$$\bar{x} = \frac{M_{yz}}{m},\ \bar{y} = \frac{M_{xz}}{m},\ \bar{z} = \frac{M_{xy}}{m}. \tag{2.7.39}$$

Also, if the solid object is homogeneous (with constant density), then the center of mass becomes the centroid of the solid. Finally, the moments of inertia about the yz-plane, xz-plane, and the xy-plane are

$$I_x = \iiint_Q (y^2 + z^2)\, \rho(x, y, z)\, dV, \tag{2.7.40}$$

$$I_y = \iiint_Q (x^2 + z^2)\, \rho(x, y, z)\, dV, \tag{2.7.41}$$

$$I_z = \iiint_Q (x^2 + y^2)\, \rho(x, y, z)\, dV. \tag{2.7.42}$$

Example 2.7.8: Finding the Mass of a Solid

Suppose that Q is a solid region bounded by $x + 2y + 3z = 6$ and the coordinate planes and has density $\rho(x, y, z) = x^2yz$. Find the total mass.

Solution

The region Q is a tetrahedron (Figure 2.7.7) meeting the axes at the points $(6, 0, 0)$, $(0, 3, 0)$, and $(0, 0, 2)$. To find the limits of integration, let $z = 0$ in the slanted plane $z = \frac{1}{3}(6 - x - 2y)$. Then for x and y find the projection of Q onto the xy-plane, which is bounded by the axes and the line $x + 2y = 6$. Hence the mass is

$$m = \iiint_Q \rho(x, y, z)\, dV = \int_{x=0}^{x=6} \int_{y=0}^{y=1/2(6-x)} \int_{z=0}^{z=1/3(6-x-2y)} x^2yz\, dz\, dy\, dx = \frac{108}{35} \tag{2.7.43}$$

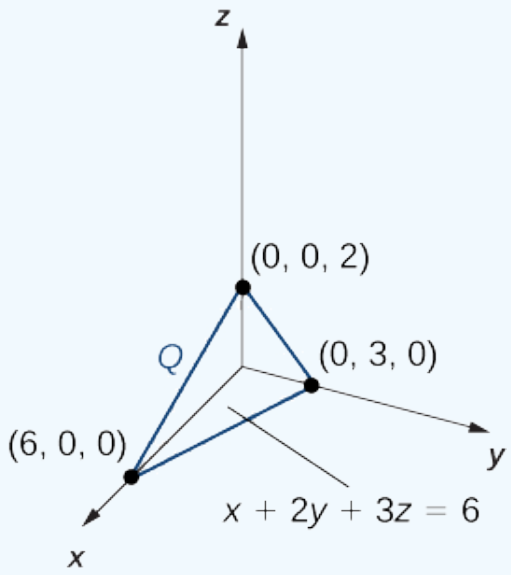


Figure 2.7.7: A three-dimensional solid Q bounded by $x + 2y + 3z = 6$ and the coordinate planes and has density $\rho(x, y, z) = x^2yz$.

Exercise 2.7.8

Consider the same region Q (Figure 2.7.7), and use the density function $\rho(x, y, z) = xy^2z$. Find the mass.

Hint available at the end of this section.

Answer available at the end of this section.

Example 2.7.9: Finding the Center of Mass of a Solid

Suppose Q is a solid region bounded by the plane $x + 2y + 3z = 6$ and the coordinate planes with density $\rho(x, y, z) = x^2yz$ (see Figure 2.7.7). Find the center of mass using decimal approximation.

Solution

We have used this tetrahedron before and know the limits of integration, so we can proceed to the computations right away. First, we need to find the moments about the xy-plane, the xz-plane, and the yz-plane:

$$M_{xy} = \iiint_Q z\rho(x, y, z)\, dV = \int_{x=0}^{x=6} \int_{y=0}^{y=\frac{1}{2}(6-x)} \int_{z=0}^{z=\frac{1}{3}(6-x-2y)} x^2yz^2\, dz\, dy\, dx = \frac{54}{35} \approx 1.543, \tag{2.7.44}$$

$$M_{xz} = \iiint_Q y\rho(x,y,z)\,dV = \int_{x=0}^{x=6}\int_{y=0}^{y=\frac{1}{2}(6-x)}\int_{z=0}^{z=\frac{1}{3}(6-x-2y)} x^2y^2z\,dz\,dy\,dx = \frac{81}{35} \approx 2.314, \tag{2.7.45}$$

$$M_{yz} = \iiint_Q x\rho(x,y,z)\,dV = \int_{x=0}^{x=6}\int_{y=0}^{y=\frac{1}{2}(6-x)}\int_{z=0}^{z=\frac{1}{3}(6-x-2y)} x^3yz\,dz\,dy\,dx = \frac{243}{35} \approx 6.943. \tag{2.7.46}$$

Hence the center of mass is

$$\bar{x} = \frac{M_{yz}}{m},\ \bar{y} = \frac{M_{xz}}{m},\ \bar{z} = \frac{M_{xy}}{m}, \tag{2.7.47}$$

$$\bar{x} = \frac{M_{yz}}{m} = \frac{243/35}{108/35} = \frac{243}{108} = 2.25, \tag{2.7.48}$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{81/35}{108/35} = \frac{81}{108} = 0.75, \tag{2.7.49}$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{54/35}{108/35} = \frac{54}{108} = 0.5 \tag{2.7.50}$$

The center of mass for the tetrahedron Q is the point $(2.25, 0.75, 0.5)$.

Exercise 2.7.9

Consider the same region Q (Figure 2.7.7) and use the density function $\rho(x,y,z) = xy^2z$. Find the center of mass.

Hint available at the end of this section.

Answer available at the end of this section.

We conclude this section with an example of finding moments of inertia $I_x,\ I_y$, and I_z.

Example 2.7.10: Finding the Moments of Inertia of a Solid

Suppose that Q is a solid region and is bounded by $x + 2y + 3z = 6$ and the coordinate planes with density $\rho(x,y,z) = x^2yz$ (see Figure 2.7.7). Find the moments of inertia of the tetrahedron Q about the yz-plane, the xz-plane, and the xy-plane.

Solution

Once again, we can almost immediately write the limits of integration and hence we can quickly proceed to evaluating the moments of inertia. Using the formula stated before, the moments of inertia of the tetrahedron Q about the yz-plane, the xz-plane, and the xy-plane are

$$I_x = \iiint_Q (y^2+z^2)\rho(x,y,z)\,dV, \tag{2.7.51}$$

$$I_y = \iiint_Q (x^2+z^2)\rho(x,y,z)\,dV, \tag{2.7.52}$$

and

$$I_z = \iiint_Q (x^2+y^2)\rho(x,y,z)\,dV\ with\ \rho(x,y,z) = x^2yz. \tag{2.7.53}$$

Proceeding with the computations, we have

$$I_x = \iiint_Q (y^2+z^2)x^2\rho(x,y,z)\,dV$$

$$= \int_{x=0}^{x=6}\int_{y=0}^{y=\frac{1}{2}(6-x)}\int_{z=0}^{z=\frac{1}{3}(6-x-2y)} (y^2+z^2)x^2yz\,dz\,dy\,dx = \frac{117}{35} \approx 3.343,$$

$$I_y = \iiint_Q (x^2+z^2)x^2\rho(x,y,z)\,dV$$

$$= \int_{x=0}^{x=6}\int_{y=0}^{y=\frac{1}{2}(6-x)}\int_{z=0}^{z=\frac{1}{3}(6-x-2y)} (x^2+z^2)x^2yz\,dz\,dy\,dx = \frac{684}{35} \approx 19.543,$$

$$I_z = \iiint_Q (x^2+y^2)x^2\rho(x,y,z)\,dV$$

$$= \int_{x=0}^{x=6}\int_{y=0}^{y=\frac{1}{2}(6-x)}\int_{z=0}^{z=\frac{1}{3}(6-x-2y)} (x^2+y^2)x^2yz\,dz\,dy\,dx = \frac{729}{35} \approx 20.829.$$

Thus, the moments of inertia of the tetrahedron Q about the yz-plane, the xz-plane, and the xy-plane are $117/35,\ 684/35$, and $729/35$, respectively.

Exercise 2.7.10

Consider the same region Q (Figure 2.7.7), and use the density function $\rho(x,y,z) = xy^2z$. Find the moments of inertia about the three coordinate planes.

Answer available at the end of this section.

Key Concepts

Finding the mass, center of mass, moments, and moments of inertia in double integrals:

- For a lamina R with a density function $\rho(x,y)$ at any point (x,y) in the plane, the mass is
$$m = \iint_R \rho(x,y)\,dA. \tag{2.7.54}$$
- The moments about the x-axis and y-axis are
$$M_x = \iint_R y\rho(x,y)\,dA \text{ and } M_y = \iint_R x\rho(x,y)\,dA. \tag{2.7.55}$$
- The center of mass is given by $\bar{x} = \dfrac{M_y}{m},\ \bar{y} = \dfrac{M_x}{m}$.
- The center of mass becomes the centroid of the plane when the density is constant.
- The moments of inertia about the x-axis, y-axis, and the origin are
$$I_x = \iint_R y^2\rho(x,y)\,dA,\ I_y = \iint_R x^2\rho(x,y)\,dA, \text{ and } I_0 = I_x + I_y = \iint_R (x^2+y^2)\rho(x,y)\,dA. \tag{2.7.56}$$

Finding the mass, center of mass, moments, and moments of inertia in triple integrals:

- For a solid object Q with a density function $\rho(x,y,z)$ at any point (x,y,z) in space, the mass is
$$m = \iiint_Q \rho(x,y,z)\,dV. \tag{2.7.57}$$
- The moments about the xy-plane, the xz-plane, and the yz-plane are
$$M_{xy} = \iiint_Q z\rho(x,y,z)\,dV,\ M_{xz} = \iiint_Q y\rho(x,y,z)\,dV,\ M_{yz} = \iiint_Q x\rho(x,y,z)\,dV \tag{2.7.58}$$
- The center of mass is given by $\bar{x} = \dfrac{M_{yz}}{m},\ \bar{y} = \dfrac{M_{xz}}{m},\ \bar{z} = \dfrac{M_{xy}}{m}$.

- The center of mass becomes the centroid of the solid when the density is constant.
- The moments of inertia about the yz-plane, the xz-plane, and the xy-plane are

$$I_x = \iiint_Q (y^2 + z^2)\,\rho(x,y,z)\,dV,\; I_y = \iiint_Q (x^2 + z^2)\,\rho(x,y,z)\,dV,\; I_z = \iiint_Q (x^2 + y^2)\,\rho(x,y,z)\,dV. \tag{2.7.59}$$

Key Equations

- **Mass of a lamina**

$$m = \lim_{k,l\to\infty} \sum_{i=1}^{k}\sum_{j=1}^{l} m_{ij} = \lim_{k,l\to\infty} \sum_{i=1}^{k}\sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*)\,\Delta A = \iint_R \rho(x,y)\,dA \tag{2.7.60}$$

- **Moment about the *x*-axis**

$$M_x = \lim_{k,l\to\infty} \sum_{i=1}^{k}\sum_{j=1}^{l} (y_{ij}^*) m_{ij} = \lim_{k,l\to\infty} \sum_{i=1}^{k}\sum_{j=1}^{l} (y_{ij}^*)\rho(x_{ij}^*, y_{ij}^*)\,\Delta A = \iint_R y\rho(x,y)\,dA \tag{2.7.61}$$

- **Moment about the *y*-axis**

$$M_y = \lim_{k,l\to\infty} \sum_{i=1}^{k}\sum_{j=1}^{l} (x_{ij}^*) m_{ij} = \lim_{k,l\to\infty} \sum_{i=1}^{k}\sum_{j=1}^{l} (x_{ij}^*)\rho(x_{ij}^*, y_{ij}^*)\,\Delta A = \iint_R x\rho(x,y)\,dA \tag{2.7.62}$$

- **Center of mass of a lamina**

$$\bar{x} = \frac{M_y}{m} = \frac{\iint_R x\rho(x,y)\,dA}{\iint_R \rho(x,y)\,dA} \; and \; \bar{y} = \frac{M_x}{m} = \frac{\iint_R y\rho(x,y)\,dA}{\iint_R \rho(x,y)\,dA} \tag{2.7.63}$$

Glossary

Radius of gyration
: the distance from an object's center of mass to its axis of rotation

Contributors and Attributions

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Answers

Exercise 2.7.1

$\frac{9\pi}{8}\ kg$

Exercise 2.7.2

$M_x = \frac{81\pi}{64}$ and $M_y = \frac{81\pi}{64}$

Exercise 2.7.3

$\bar{x} = \frac{M_y}{m} = \frac{81\pi/64}{9\pi/8} = \frac{9}{8}$ and $\bar{y} = \frac{M_x}{m} = \frac{81\pi/64}{9\pi/8} = \frac{9}{8}$.

Exercise 2.7.4

$\bar{x} = \frac{M_y}{m} = \frac{1/20}{1/12} = \frac{3}{5}$ and $\bar{y} = \frac{M_x}{m} = \frac{1/24}{1/12} = \frac{1}{2}$

Exercise 2.7.5

$x_c = \frac{M_y}{m} = \frac{1/15}{1/6} = \frac{2}{5}$ and $y_c = \frac{M_x}{m} = \frac{1/12}{1/6} = \frac{1}{2}$

Exercise 2.7.6

$$I_x = \int_{x=0}^{x=2}\int_{y=0}^{y=x} y^2\sqrt{xy}\,dy\,dx = \frac{64}{35} \tag{2.7.64}$$

and

$$I_y = \int_{x=0}^{x=2}\int_{y=0}^{y=x} x^2\sqrt{xy}\,dy\,dx = \frac{64}{35}. \tag{2.7.65}$$

Also,

$$I_0 = \int_{x=0}^{x=2} \int_{y=0}^{y=x} (x^2 + y^2)\sqrt{xy}\, dy\, dx = \frac{128}{21} \tag{2.7.66}$$

Exercise 2.7.7

Follow the steps shown in the previous example.

Exercise 2.7.7

$R_x = \frac{6\sqrt{35}}{35}$, $R_y = \frac{6\sqrt{15}}{15}$, and $R_0 = \frac{4\sqrt{42}}{7}$.

Exercise 2.7.8

Follow the steps in the previous example.

Exercise 2.7.8

$\frac{54}{35} = 1.543$

Exercise 2.7.9

Check that $M_{xy} = \frac{27}{35}$, $M_{xz} = \frac{243}{140}$, and $M_{yz} = \frac{81}{35}$. Then use m from a previous checkpoint question.

Exercise 2.7.9

$\left(\frac{3}{2}, \frac{9}{8}, \frac{1}{2}\right)$

Exercise 2.7.10

The moments of inertia of the tetrahedron Q about the yz-plane, the xz-plane, and the xy-plane are $99/35$, $36/7$ and $243/35$, respectively.

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2.7E: Exercises

Exercises 1 - 12

The region R occupied by a lamina is shown in a graph. Find the mass of R with the density function ρ.

1. R is the triangular region with vertices $(0,0),\ (0,3)$, and $(6,0);\ \rho(x,y)=xy$.

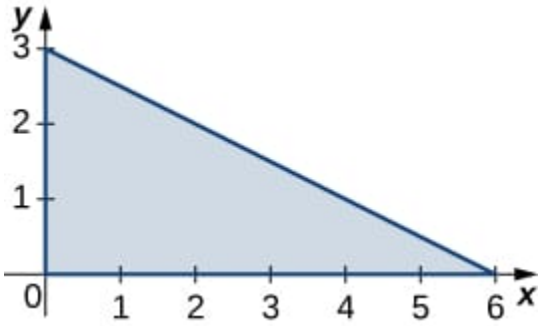


Figure $2.7E.1$: The graph illustrates the triangular region with vertices $(0,0)$, $(0,3)$, and $(6,0)$.

Answer

$\frac{27}{2}$

2. R is the triangular region with vertices $(0,0),\ (1,1)$, and $(0,5);\ \rho(x,y)=x+y$.

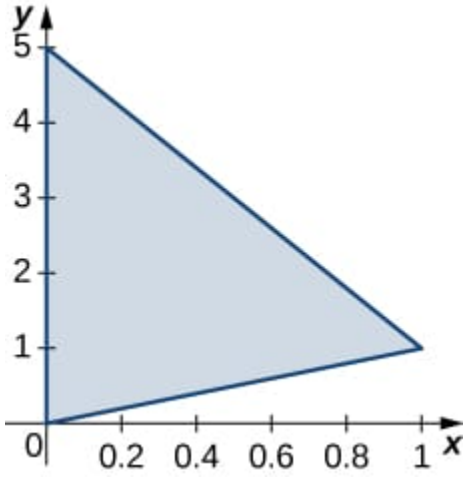


Figure $2.7E.2$: The graph illustrates the triangular region with vertices $(0,0)$, $(1,1)$, and $(0,5)$.

3. R is the rectangular region with vertices $(0,0),\ (0,3),\ (6,3)$ and $(6,0);\ \rho(x,y)=\sqrt{xy}$.

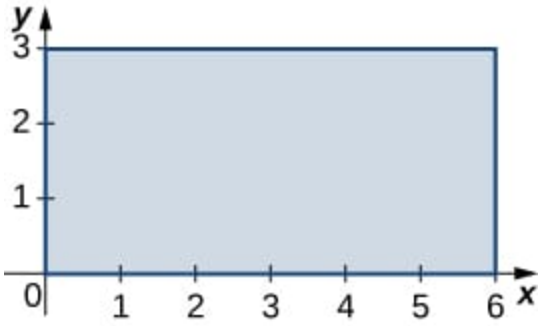


Figure $2.7E.3$: The graph illustrates the rectangular region with vertices $(0,0)$, $(0,3)$, $(6,3)$, and $(6,0)$.

Answer

$24\sqrt{2}$

4. R is the rectangular region with vertices $(0,1),\ (0,3),\ (3,3)$ and $(3,1);\ \rho(x,y)=x^2y$.

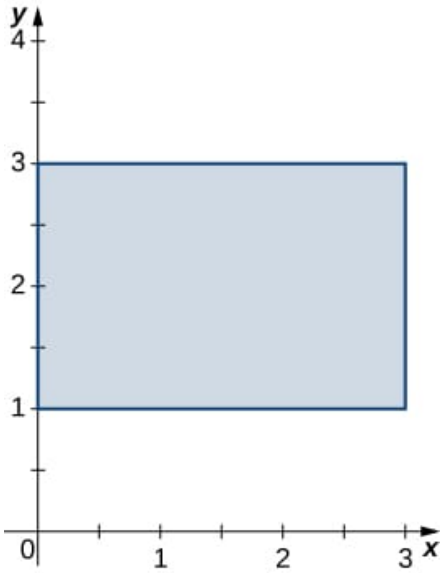


Figure $2.7E.4$: The graph illustrates the rectangular region with vertices $(0,1)$, $(0,3)$, $(3,3)$, and $(3,1)$.

5. R is the trapezoidal region determined by the lines $y=-\frac{1}{4}x+\frac{5}{2},\ y=0,\ y=2$, and $x=0;\ \rho(x,y)=3xy$.

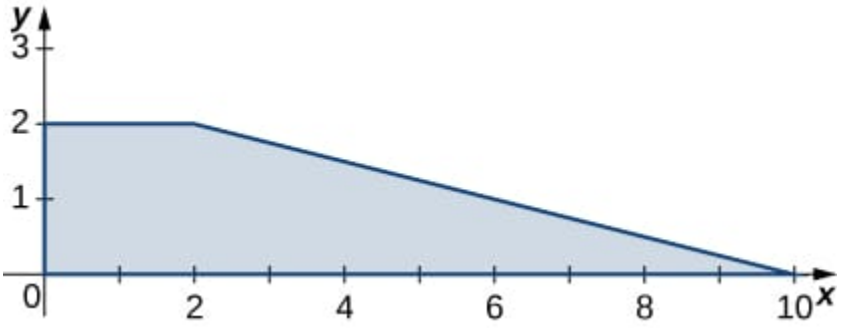


Figure $2.7E.5$: The graph illustrates the trapezoidal region determined by the lines $y=-\frac{1}{4}x+\frac{5}{2}$, $y=0$, $y=2$, and $x=0$.

Answer

76

6. R is the trapezoidal region determined by the lines $y=0,\ y=1,\ y=x$ and $y=-x+3;\ \rho(x,y)=2x+y$.

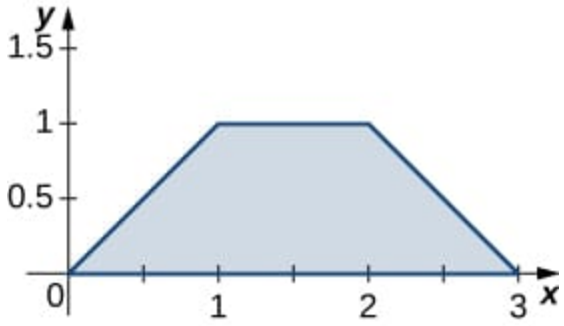


Figure $2.7E.6$: The graph illustrates the trapezoidal region determined by the lines $y = 0, \quad y = 1, \quad y = x, \quad \text{and} \quad y = -x + 3$.

7. R is the disk of radius 2 centered at $(1, 2)$; $\rho(x, y) = x^2 + y^2 - 2x - 4y + 5$.

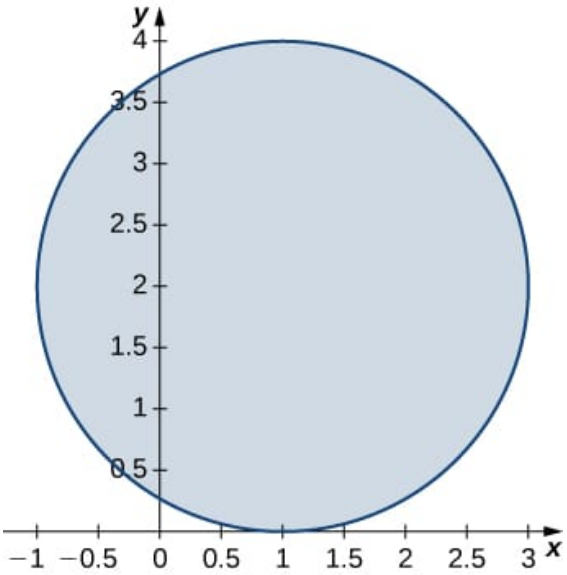


Figure $2.7E.7$: The graph illustrates the disk of radius 2 centered at $(1, 2)$.

Answer

8π

8. R is the unit disk; $\rho(x, y) = 3x^4 + 6x^2y^2 + 3y^4$.

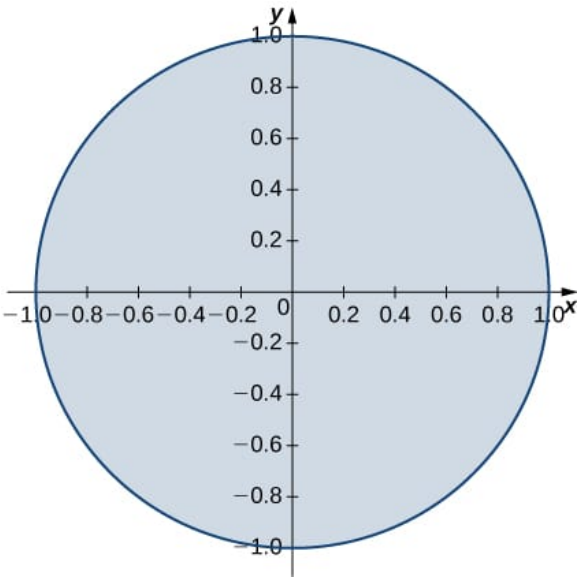


Figure $2.7E.8$: The graph illustrates the unit disk region of radius 1 centered at $(0, 0)$.

9. R is the region enclosed by the ellipse $x^2 + 4y^2 = 1;\ \rho(x, y) = 1$.

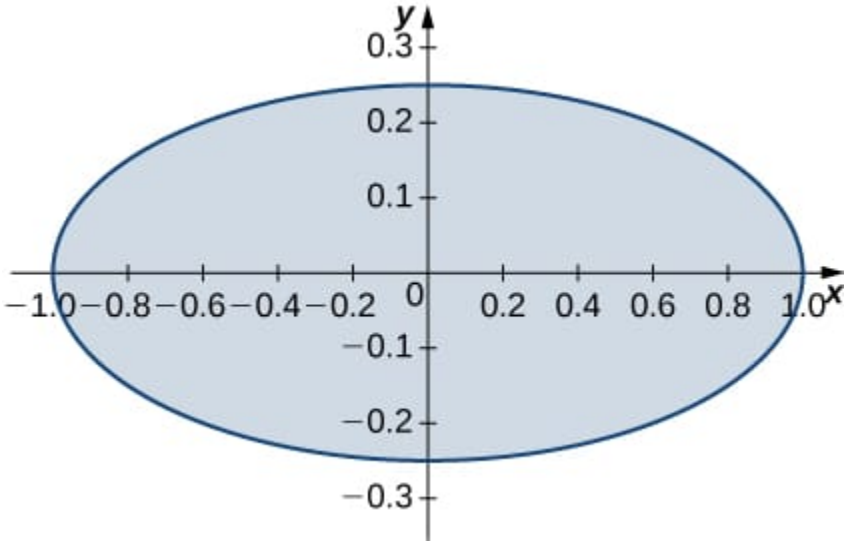


Figure $2.7E.9$: The graph illustrates the ellipse with equation $x^2 + 4y^2 = 1$.

Answer

$\frac{\pi}{2}$

10. $R = \{(x, y) \mid 9x^2 + y^2 \le 1,\ x \ge 0,\ y \ge 0\};\ \rho(x, y) = \sqrt{9x^2 + y^2}$.

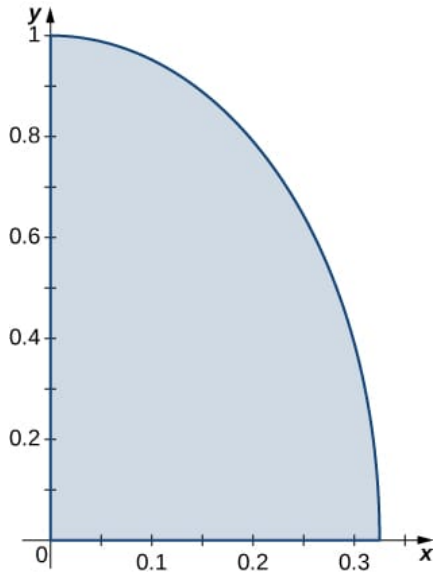


Figure $2.7E.10$: The graph illustrates the region R defined by $9x^2 + y^2 \leq 1, \quad x \geq 0, \quad y \geq 0$.

11. R is the region bounded by $y = x,\ y = -x,\ y = x + 2,\ y = -x + 2;\ \rho(x, y) = 1$.

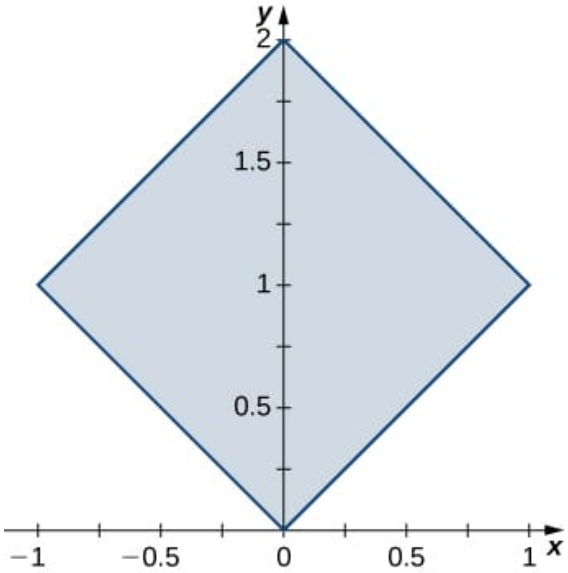


Figure $2.7E.11$: The graph illustrates the region R bounded by $y = x,\ y = -x,\ y = x + 2,\ y = -x + 2$.

Answer

2

12. R is the region bounded by $y = \frac{1}{x},\ y = \frac{2}{x},\ y = 1$, and $y = 2;\ \rho(x, y) = 4(x + y)$.

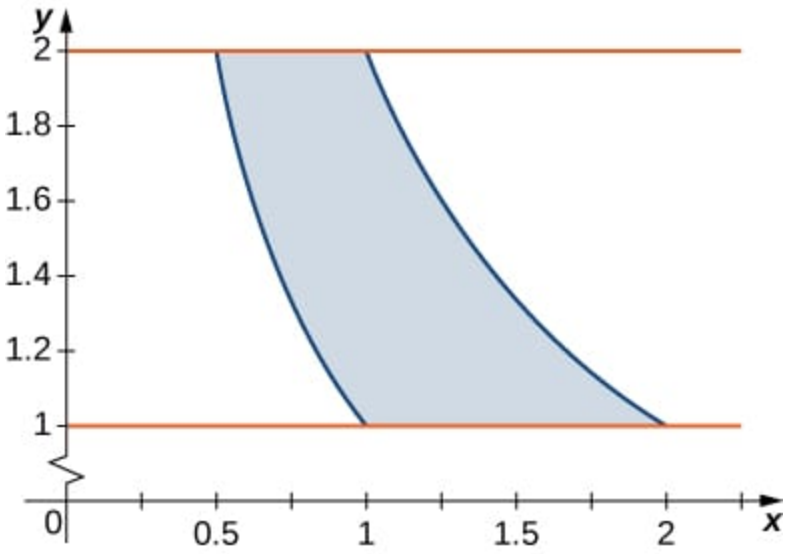


Figure $2.7E.12$: The graph illustrates the region R bounded by $y = \frac{1}{x},\ y = \frac{2}{x},\ y = 1$, and $y = 2$.

Exercises 13 - 24

Consider a lamina occupying the region R and having the density function ρ given in the preceding group of exercises. Use a computer algebra system (CAS) to answer the following questions.

a. **Find the moments M_x and M_y about the x-axis and y-axis, respectively.**
b. **Calculate and plot the center of mass of the lamina.**
c. **[T] Use a CAS to locate the center of mass on the graph of R.**

13. [T] R is the triangular region with vertices $(0,0),\ (0,3)$, and $(6,0);\ \rho(x,y) = xy$.

Answer

a. $M_x = \frac{81}{5},\ M_y = \frac{162}{5}$;
b. $\bar{x} = \frac{12}{5},\ \bar{y} = \frac{6}{5}$;
c.

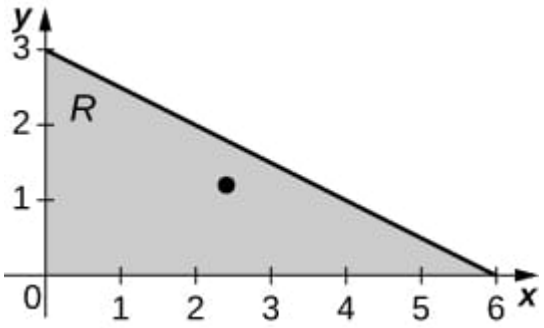


Figure $2.7E.13$: The graph illustrates the region R.

14. [T] R is the triangular region with vertices $(0,0),\ (1,1)$, and $(0,5);\ \rho(x,y) = x + y$.

15. [T] R is the rectangular region with vertices $(0,0),\ (0,3),\ (6,3)$, and $(6,0);\ \rho(x,y) = \sqrt{xy}$.

Answer

a. $M_x = \frac{216\sqrt{2}}{5},\ M_y = \frac{432\sqrt{2}}{5}$;

b. $\bar{x} = \frac{18}{5},\ \bar{y} = \frac{9}{5}$;
c.

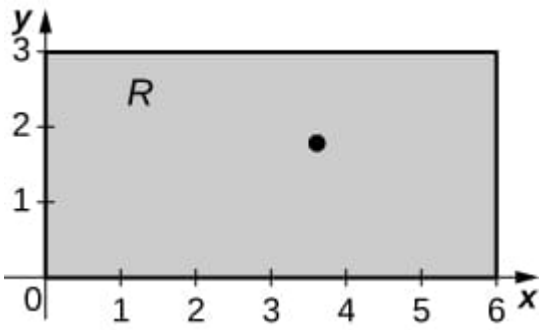


Figure $2.7E.14$: The graph illustrates the region R.

16. [T] R is the rectangular region with vertices $(0,1),\ (0,3),\ (3,3)$, and $(3,1);\ \rho(x,y) = x^2y$.

17. [T] R is the trapezoidal region determined by the lines $y = -\frac{1}{4}x + \frac{5}{2},\ y = 0,\ y = 2$, and $x = 0;\ \rho(x,y) = 3xy$.

Answer

a. $M_x = \frac{368}{5},\ M_y = \frac{1552}{5}$;
b. $\bar{x} = \frac{92}{95},\ \bar{y} = \frac{388}{95}$;
c.

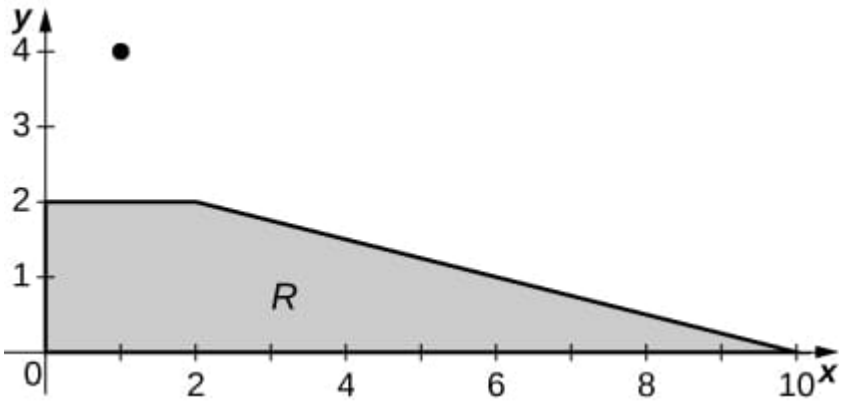


Figure $2.7E.15$: The graph illustrates the region R.

18. [T] R is the trapezoidal region determined by the lines $y = 0,\ y = 1,\ y = x$, and $y = -x + 3;\ \rho(x,y) = 2x + y$.

19. [T] R is the disk of radius 2 centered at $(1,2);\ \rho(x,y) = x^2 + y^2 - 2x - 4y + 5$.

Answer

a. $M_x = 16\pi,\ M_y = 8\pi$;
b. $\bar{x} = 1,\ \bar{y} = 2$;
c.

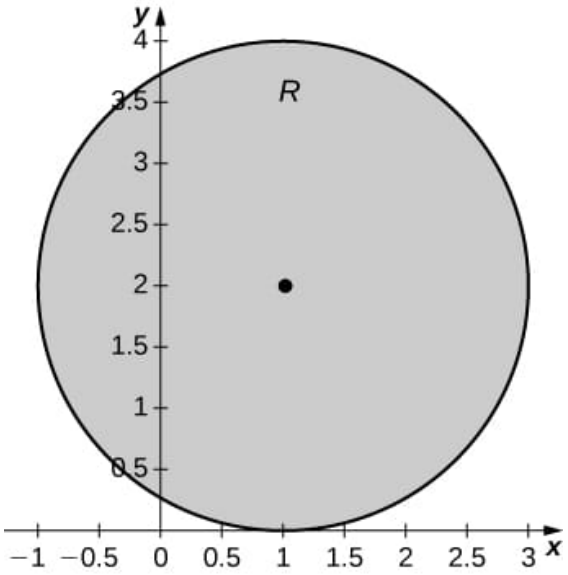


Figure $2.7E.16$: The graph illustrates the region R.

20. [T] R is the unit disk; $\rho(x,y) = 3x^4 + 6x^2y^2 + 3y^4$.

21. [T] R is the region enclosed by the ellipse $x^2 + 4y^2 = 1;\ \rho(x,y) = 1$.

Answer

a. $M_x = 0,\ M_y = 0)$;
b. $\bar{x} = 0,\ \bar{y} = 0$;
c.

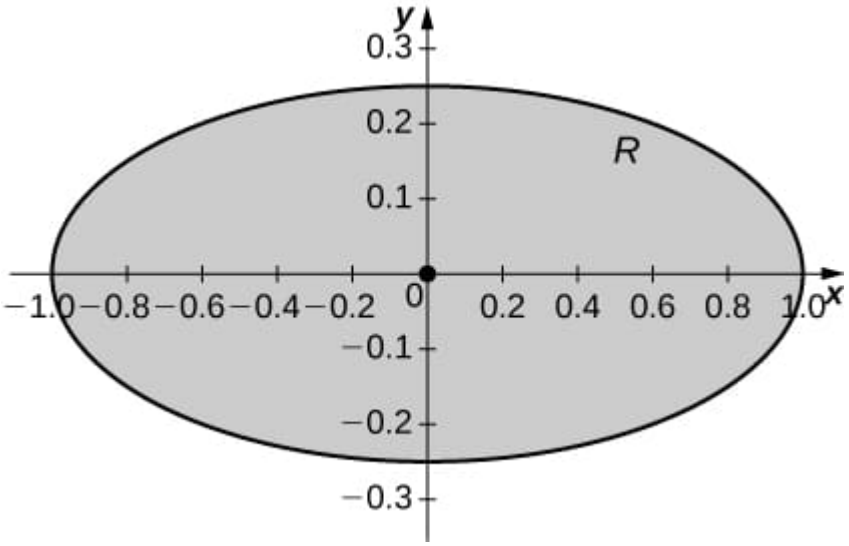


Figure $2.7E.17$: The graph illustrates the region R.

22. [T] $R = \{(x,y) \mid 9x^2 + y^2 \le 1,\ x \ge 0,\ y \ge 0\};\ \rho(x,y) = \sqrt{9x^2 + y^2}$.

23. [T] R is the region bounded by $y = x,\ y = -x,\ y = x + 2$, and $y = -x + 2;\ \rho(x,y) = 1$.

Answer

a. $M_x = 2,\ M_y = 0)$;
b. $\bar{x} = 0,\ \bar{y} = 1$;
c.

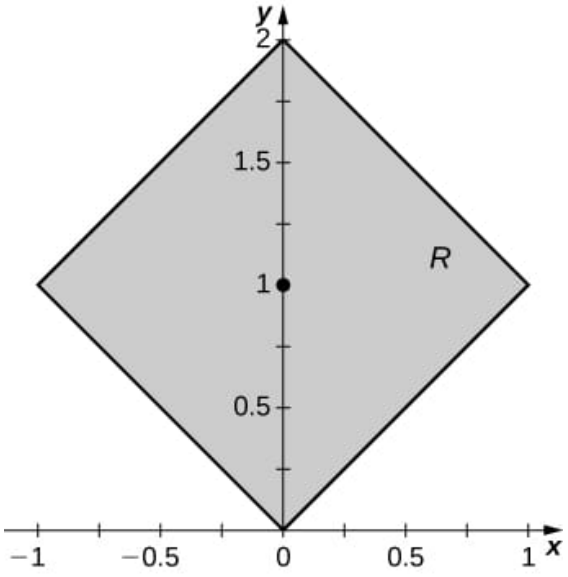


Figure $2.7E.18$: The graph illustrates the region R.

24. [T] R is the region bounded by $y = \frac{1}{x},\ y = \frac{2}{x},\ y = 1$, and $y = 2;\ \rho(x, y) = 4(x + y)$.

Exercises 25 - 36

Consider a lamina occupying the region R and having the density function ρ given in the first two groups of Exercises.

a. Find the moments of inertia I_x, I_y and I_0 about the x-axis, y-axis, and origin, respectively.

b. Find the radii of gyration with respect to the x-axis, y-axis, and origin, respectively.

25. R is the triangular region with vertices $(0, 0),\ (0, 3)$, and $(6, 0);\ \rho(x, y) = xy$.

Answer

a. $I_x = \frac{243}{10},\ I_y = \frac{486}{5}$, and $I_0 = \frac{243}{2}$;
b. $R_x = \frac{3\sqrt{5}}{5},\ R_y = \frac{6\sqrt{5}}{5}$, and $R_0 = 3$

26. R is the triangular region with vertices $(0, 0),\ (1, 1)$, and $(0, 5);\ \rho(x, y) = x + y$.

27. R is the rectangular region with vertices $(0, 0),\ (0, 3),\ (6, 3)$, and $(6, 0);\ \rho(x, y) = \sqrt{xy}$.

Answer

a. $I_x = \frac{2592\sqrt{2}}{7},\ I_y = \frac{648\sqrt{2}}{7}$, and $I_0 = \frac{3240\sqrt{2}}{7}$;
b. $R_x = \frac{6\sqrt{21}}{7},\ R_y = \frac{3\sqrt{21}}{7}$, and $R_0 = \frac{3\sqrt{106}}{7}$

28. R is the rectangular region with vertices $(0, 1),\ (0, 3),\ (3, 3)$, and $(3, 1);\ \rho(x, y) = x^2y$.

29. R is the trapezoidal region determined by the lines $y = -\frac{1}{4}x + \frac{5}{2},\ y = 0,\ y = 2$, and x = 0; \space \rho (x,y) = 3xy\).

Answer

a. $I_x = 88,\ I_y = 1560$, and $I_0 = 1648$;

b. $R_x = \frac{\sqrt{418}}{19},\ R_y = \frac{\sqrt{7410}}{10}$, and $R_0 = \frac{2\sqrt{1957}}{19}$

30. R is the trapezoidal region determined by the lines $y = 0,\ y = 1,\ y = x$, and y = -x + 3; \space \rho (x,y) = 2x + y\).

31. R is the disk of radius 2 centered at $(1, 2);\ \rho(x, y) = x^2 + y^2 - 2x - 4y + 5$.

Answer

a. $I_x = \frac{128\pi}{3},\ I_y = \frac{56\pi}{3}$, and $I_0 = \frac{184\pi}{3}$;
b. $R_x = \frac{4\sqrt{3}}{3},\ R_y = \frac{\sqrt{21}}{2}$, and $R_0 = \frac{\sqrt{69}}{3}$

32. R is the unit disk; $\rho(x, y) = 3x^4 + 6x^2y^2 + 3y^4$.

33. R is the region enclosed by the ellipse $x^2 + 4y^2 = 1;\ \rho(x, y) = 1$.

Answer

a. $I_x = \frac{\pi}{32},\ I_y = \frac{\pi}{8}$, and $I_0 = \frac{5\pi}{32}$;
b. $R_x = \frac{1}{4},\ R_y = \frac{1}{2}$, and $R_0 = \frac{\sqrt{5}}{4}$

34. $R = \{(x, y) \mid 9x^2 + y^2 \le 1,\ x \ge 0,\ y \ge 0\};\ \rho(x, y) = \sqrt{9x^2 + y^2}$.

35. R is the region bounded by $y = x,\ y = -x,\ y = x + 2$, and $y = -x + 2;\ \rho(x, y) = 1$.

Answer

a. $I_x = \frac{7}{3},\ I_y = \frac{1}{3}$, and $I_0 = \frac{8}{3}$;
b. $R_x = \frac{\sqrt{42}}{6},\ R_y = \frac{\sqrt{6}}{6}$, and $R_0 = \frac{2\sqrt{3}}{3}$

36. R is the region bounded by $y = \frac{1}{x},\ y = \frac{2}{x},\ y = 1$, and $y = 2;\ \rho(x, y) = 4(x + y)$.

37. Let Q be the solid unit cube. Find the mass of the solid if its density ρ is equal to the square of the distance of an arbitrary point of Q to the xy-plane.

Answer

$m = \frac{1}{3}$

38. Let Q be the solid unit hemisphere. Find the mass of the solid if its density ρ is proportional to the distance of an arbitrary point of Q to the origin.

39. The solid Q of constant density 1 is situated inside the sphere $x^2 + y^2 + z^2 = 16$ and outside the sphere $x^2 + y^2 + z^2 = 1$. Show that the center of mass of the solid is not located within the solid.

40. Find the mass of the solid $Q = \{(x, y, z) \mid 1 \le x^2 + z^2 \le 25,\ y \le 1 - x^2 - z^2\}$ whose density is $\rho(x, y, z) = k$, where $k > 0$.

41. [T] The solid $Q = \{(x, y, z) \mid x^2 + y^2 \le 9,\ 0 \le z \le 1,\ x \ge 0,\ y \ge 0\}$ has density equal to the distance to the xy-plane. Use a CAS to answer the following questions.

a. Find the mass of Q.

b. Find the moments $M_{xy},\ M_{xz}$ and M_{yz} about the xy-plane, xz-plane, and yz-plane, respectively.

c. Find the center of mass of Q.

d. Graph Q and locate its center of mass.

Answer

a. $m = \frac{9\pi}{4}$;
b. $M_{xy} = \frac{3\pi}{2},\ M_{xz} = \frac{81}{8},\ M_{yz} = \frac{81}{8}$;
c. $\bar{x} = \frac{9}{2\pi},\ \bar{y} = \frac{9}{2\pi},\ \bar{z} = \frac{2}{3}$;
d.

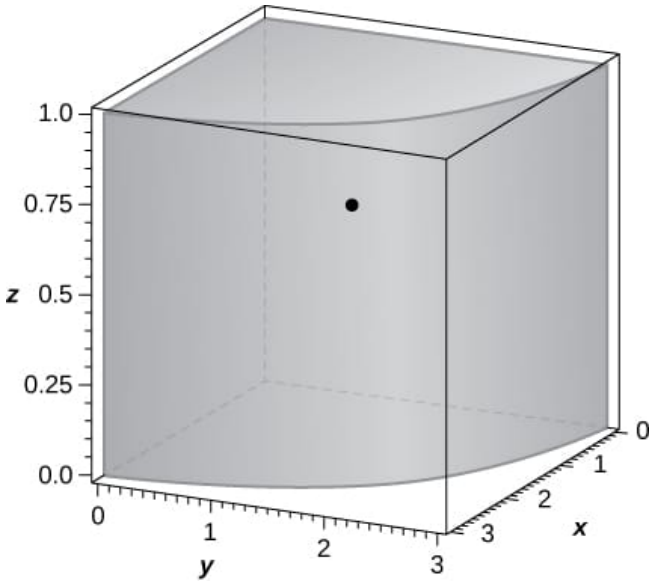


Figure $2.7E.19$: The graph illustrates the region R.

42. Consider the solid $Q = \{(x, y, z) \mid 0 \leq x \leq 1,\ 0 \leq y \leq 2,\ 0 \leq z \leq 3\}$ with the density function $\rho(x, y, z) = x + y + 1$.

    a. Find the mass of Q.

    b. Find the moments $M_{xy},\ M_{xz}$ and M_{yz} about the xy-plane, xz-plane, and yz-plane, respectively.

    c. Find the center of mass of Q.

43. [T] The solid Q has the mass given by the triple integral $\int_{-1}^{1} \int_{0}^{\pi/4} \int_{0}^{1} r^2\ dr\ d\theta\ dz.$

    Use a CAS to answer the following questions.

    - Show that the center of mass of Q is located in the xy-plane.
    - Graph Q and locate its center of mass.

    Answer

    $\bar{x} = \frac{3\sqrt{2}}{2\pi}$, $\bar{y} = \frac{3(2-\sqrt{2})}{2\pi}$, $\bar{z} = 0$; 2. the solid Q and its center of mass are shown in the figure $2.7E.20$.

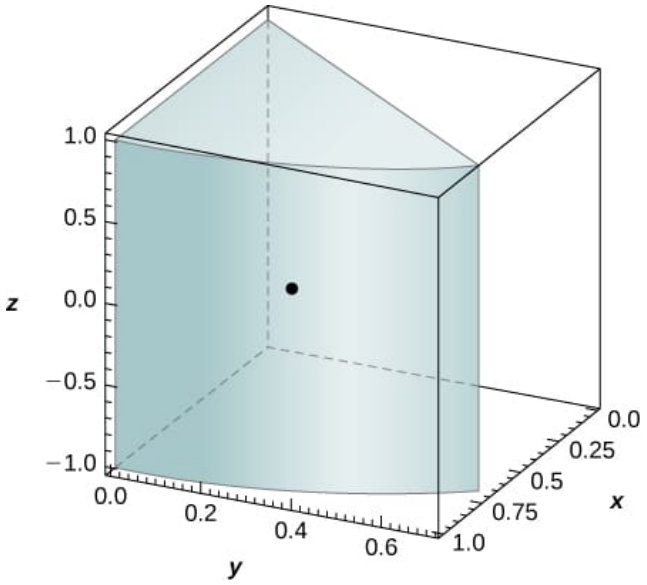


Figure $2.7E.20$: The graph illustrates the region R.

44. The solid Q is bounded by the planes $x + 4y + z = 8,\ x = 0,\ y = 0$, and $z = 0$. Its density at any point is equal to the distance to the xz-plane. Find the moments of inertia of the solid about the xz-plane.

45. The solid Q is bounded by the planes $x + y + z = 3,\ x = 0,\ y = 0$, and $z = 0$. Its density is $\rho(x, y, z) = x + ay$, where $a > 0$. Show that the center of mass of the solid is located in the plane $z = \frac{3}{5}$ for any value of a.

46. Let Q be the solid situated outside the sphere $x^2 + y^2 + z^2 = z$ and inside the upper hemisphere $x^2 + y^2 + z^2 = R^2$, where $R > 1$. If the density of the solid is $\rho(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$, find R such that the mass of the solid is $\frac{7\pi}{2}$.

47. The mass of a solid Q is given by $\displaystyle\int_0^2 \int_0^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{16-x^2-y^2}} (x^2 + y^2 + z^2)^n \, dz \, dy \, dx$, where n is an integer. Determine n such the mass of the solid is $(2 - \sqrt{2})\pi$.

Answer

$n = -2$

48. Let Q be the solid bounded above the cone $x^2 + y^2 = z^2$ and below the sphere $x^2 + y^2 + z^2 - 4z = 0$. Its density is a constant $k > 0$. Find k such that the center of mass of the solid is situated 7 units from the origin.

49. The solid $Q = \{(x, y, z) \mid 0 \le x^2 + y^2 \le 16,\ x \ge 0,\ y \ge 0,\ 0 \le z \le x\}$ has the density $\rho(x, y, z) = k$. Show that the moment M_{xy} about the xy-plane is half of the moment M_{yz} about the yz-plane.

50. The solid Q is bounded by the cylinder $x^2 + y^2 = a^2$, the paraboloid $b^2 - z = x^2 + y^2$, and the xy-plane, where $0 < a < b$. Find the mass of the solid if its density is given by $\rho(x, y, z) = \sqrt{x^2 + y^2}$.

51. Let Q be a solid of constant density k, where $k > 0$, that is located in the first octant, inside the circular cone $x^2 + y^2 = 9(z - 1)^2$, and above the plane $z = 0$. Show that the moment M_{xy} about the xy-plane is the same as the moment M_{yz} about the xz-plane.

52. The solid Q has the mass given by the triple integral $\displaystyle\int_0^1 \int_0^{\pi/2} \int_0^{r^3} (r^4 + r) \, dz \, d\theta \, dr$.

a. Find the density of the solid in rectangular coordinates.

b. Find the moment M_{xy} about the xy-plane.

53. The solid Q has the moment of inertia I_x about the yz-plane given by the triple integral $\displaystyle\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{\frac{1}{2}(x^2+y^2)}^{\sqrt{x^2+y^2}} (y^2 + z^2)(x^2 + y^2)$

$dz\ dx\ dy$.

a. Find the density of Q.

b. Find the moment of inertia I_z about the xy-plane.

Answer

a. $\rho(x, y, z) = x^2 + y^2$;
b. $\frac{16\pi}{7}$

54. The solid Q has the mass given by the triple integral $\int_0^{\pi/4} \int_0^{2\ sec\ \theta} \int_0^1 (r^3 \cos\theta \sin\theta + 2r)\ dz\ dr\ d\theta$.

a. Find the density of the solid in rectangular coordinates.

b. Find the moment M_{xz} about the xz-plane.

55. Let Q be the solid bounded by the xy-plane, the cylinder $x^2 + y^2 = a^2$, and the plane $z = 1$, where $a > 1$ is a real number. Find the moment M_{xy} of the solid about the xy-plane if its density given in cylindrical coordinates is $\rho(x, y, z) = \frac{d^2 f}{dr^2}(r)$, where f is a differentiable function with the first and second derivatives continuous and differentiable on $(0, a)$.

Answer

$M_{xy} = \pi(f(0) - f(a) + af'(a))$

56. A solid Q has a volume given by $\iint_D \int_a^b dA\ dz$, where D is the projection of the solid onto the xy-plane and $a < b$ are real numbers, and its density does not depend on the variable z. Show that its center of mass lies in the plane $z = \frac{a+b}{2}$.

57. Consider the solid enclosed by the cylinder $x^2 + z^2 = a^2$ and the planes $y = b$ and $y = c$, where $a > 0$ and $b < c$ are real numbers. The density of Q is given by $\rho(x, y, z) = f'(y)$, where f is a differential function whose derivative is continuous on (b, c). Show that if $f(b) = f(c)$, then the moment of inertia about the xz-plane of Q is null.

58. [T] The average density of a solid Q is defined as $\rho_{ave} = \frac{1}{V(Q)} \iiint_Q \rho(x, y, z)\ dV = \frac{m}{V(Q)}$, where $V(Q)$ and m are the volume and the mass of Q, respectively. If the density of the unit ball centered at the origin is $\rho(x, y, z) = e^{-x^2-y^2-z^2}$, use a CAS to find its average density. Round your answer to three decimal places.

59. Show that the moments of inertia I_x, I_y, and I_z about the yz-plane, xz-plane, and xy-plane, respectively, of the unit ball centered at the origin whose density is $\rho(x, y, z) = e^{-x^2-y^2-z^2}$ are the same. Round your answer to two decimal places.

Answer

$I_x = I_y = I_z \approx 0.84$

Contributors

Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

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2.8: Change of Variables in Multiple Integrals

Learning Objectives

- Determine the image of a region under a given transformation of variables.
- Compute the Jacobian of a given transformation.
- Evaluate a double integral using a change of variables.
- Evaluate a triple integral using a change of variables.

Recall from Substitution Rule the method of integration by substitution. When evaluating an integral such as

$$\int_2^3 x(x^2-4)^5 dx, \tag{2.8.1}$$

we substitute $u = g(x) = x^2 - 4$. Then $du = 2x\ dx$ or $x\ dx = \frac{1}{2}du$ and the limits change to $u = g(2) = 2^2 - 4 = 0$ and $u = g(3) = 9 - 4 = 5$. Thus the integral becomes

$$\int_0^5 \frac{1}{2}u^5 du \tag{2.8.2}$$

and this integral is much simpler to evaluate. In other words, when solving integration problems, we make appropriate substitutions to obtain an integral that becomes much simpler than the original integral.

We also used this idea when we transformed double integrals in rectangular coordinates to polar coordinates and transformed triple integrals in rectangular coordinates to cylindrical or spherical coordinates to make the computations simpler. More generally,

$$\int_a^b f(x)dx = \int_c^d f(g(u))g'(u)du, \tag{2.8.3}$$

Where $x = g(u),\ dx = g'(u)du$, and $u = c$ and $u = d$ satisfy $c = g(a)$ and $d = g(b)$.

A similar result occurs in double integrals when we substitute

- $x = f(r,\theta) = r\ \cos\ \theta$
- $y = g(r,\theta) = r\ \sin\ \theta$, and
- $dA = dx\ dy = r\ dr\ d\theta$.

Then we get

$$\iint_R f(x,y)dA = \iint_S (r\cos\theta,\ r\sin\theta)r\,dr\,d\theta \tag{2.8.4}$$

where the domain R is replaced by the domain S in polar coordinates. Generally, the function that we use to change the variables to make the integration simpler is called a transformation or mapping.

Planar Transformations

A planar transformation T is a function that transforms a region G in one plane into a region R in another plane by a change of variables. Both G and R are subsets of R^2. For example, Figure 2.8.1 shows a region G in the uv-plane transformed into a region R in the xy-plane by the change of variables $x = g(u,v)$ and $y = h(u,v)$, or sometimes we write $x = x(u,v)$ and $y = y(u,v)$. We shall typically assume that each of these functions has continuous first partial derivatives, which means $g_u,\ g_v,\ h_u$, and h_v exist and are also continuous. The need for this requirement will become clear soon.

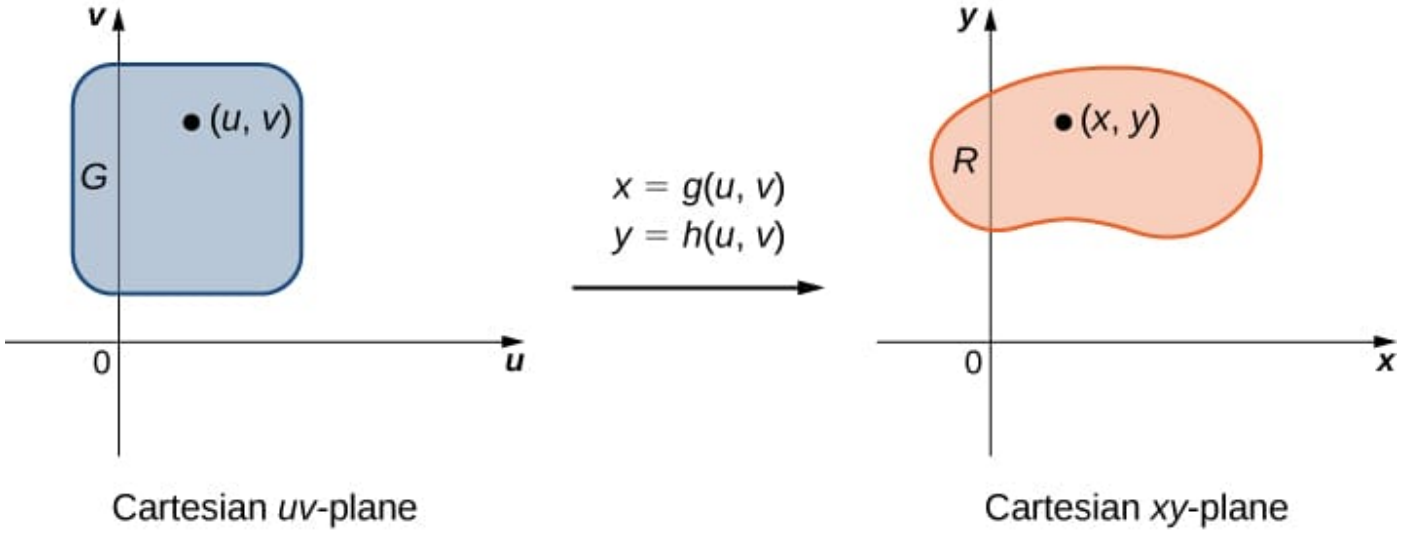


Figure 2.8.1: The transformation of a region G in the uv-plane into a region R in the xy-plane.

Definition: One-to-One Transformation

A transformation $T: \ G \to R$, defined as $T(u, v) = (x, y)$, is said to be a one-to-one transformation if no two points map to the same image point.

To show that T is a one-to-one transformation, we assume $T(u_1, v_1) = T(u_2, v_2)$ and show that as a consequence we obtain$(u_1, v_1) = (u_2, v_2)$. If the transformation T is one-to-one in the domain G, then the inverse T^{-1} exists with the domain R such that $T^{-1} \circ T$ and $T \circ T^{-1}$ are identity functions.

Figure 2.8.2 shows the mapping $T(u, v) = (x, y)$ where x and y are related to u and v by the equations $x = g(u, v)$ and $y = h(u, v)$. The region G is the domain of T and the region R is the range of T, also known as the *image* of G under the transformation T.

Example 2.8.1A: Determining How the Transformation Works

Suppose a transformation T is defined as $T(r, \theta) = (x, y)$ where $x = r \ \cos \ \theta, \ y = r \ \sin \ \theta$. Find the image of the polar rectangle $G = \{(r, \theta) | 0 \leq r \leq 1, \ 0 \leq \theta \leq \pi/2\}$ in the $r\theta$-plane to a region R in the xy-plane. Show that T is a one-to-one transformation in G and find $T^{-1}(x, y)$.

Solution

Since r varies from 0 to 1 in the $r\theta$-plane, we have a circular disc of radius 0 to 1 in the xy-plane. Because θ varies from 0 to $\pi/2$ in the $r\theta$-plane, we end up getting a quarter circle of radius 1 in the first quadrant of the xy-plane (Figure 2.8.2). Hence R is a quarter circle bounded by $x^2 + y^2 = 1$ in the first quadrant.

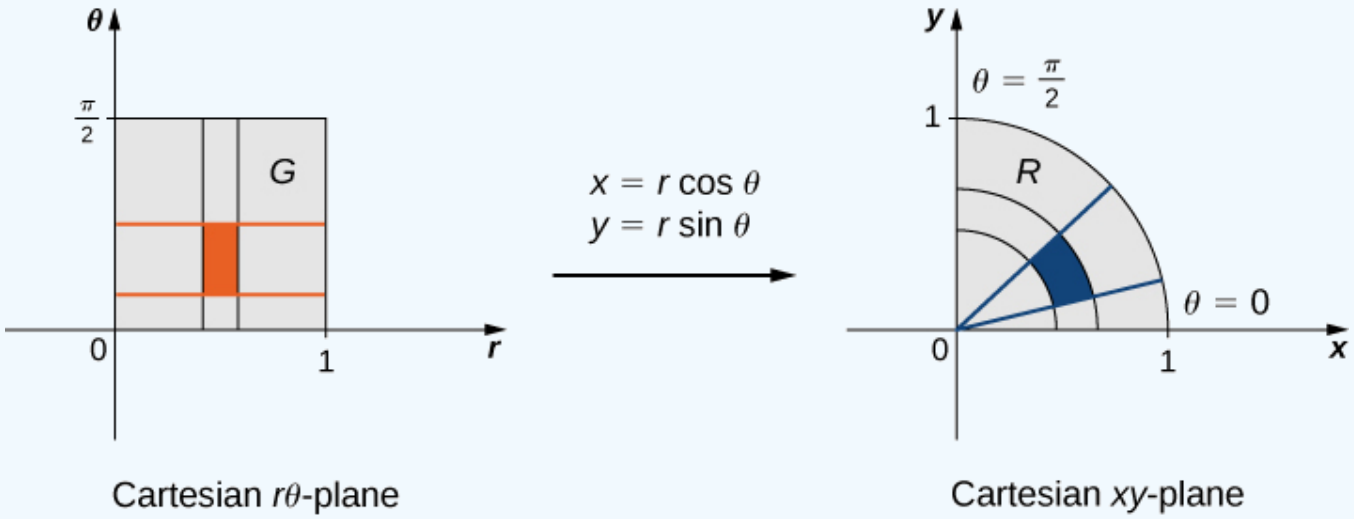


Figure 2.8.2: A rectangle in the $r\theta$-plane is mapped into a quarter circle in the xy-plane.

In order to show that T is a one-to-one transformation, assume $T(r_1, \theta_1) = T(r_2, \theta_2)$ and show as a consequence that $(r_1, \theta_1) = (r_2, \theta_2)$. In this case, we have

$$T(r_1, \theta_1) = T(r_2, \theta_2), \tag{2.8.5}$$

$$(x_1, y_1) = (x_2, y_2), \tag{2.8.6}$$

$$(r_1 \cos\, \theta_1, r_1 \sin\, \theta_1) = (r_2 \cos\, \theta_2, r_2 \sin\, \theta_2), \tag{2.8.7}$$

$$r_1 \cos\, \theta_1 = r_2 \cos\, \theta_2,\ r_1 \sin\, \theta_1 = r_2 \sin\, \theta_2. \tag{2.8.8}$$

Dividing, we obtain

$$\frac{r_1 \cos\, \theta_1}{r_1 \sin\, \theta_1} = \frac{r_2 \cos\, \theta_2}{r_2 \sin\, \theta_2} \tag{2.8.9}$$

$$\frac{\cos\, \theta_1}{\sin\, \theta_1} = \frac{\cos\, \theta_2}{\sin\, \theta_2} \tag{2.8.10}$$

$$\tan\, \theta_1 = \tan\, \theta_2 \tag{2.8.11}$$

$$\theta_1 = \theta_2 \tag{2.8.12}$$

since the tangent function is one-one function in the interval $0 \le \theta \le \pi/2$. Also, since $0 \le r \le 1$, we have $r_1 = r_2,\ \ \theta_1 = \theta_2$. Therefore, $(r_1, \theta_1) = (r_2, \theta_2)$ and T is a one-to-one transformation from G to R.

To find $T^{-1}(x, y)$ solve for r, θ in terms of x, y. We already know that $r^2 = x^2 + y^2$ and $\tan\ \ \theta = \frac{y}{x}$. Thus $T^{-1}(x, y) = (r, \theta)$ is defined as $r = \sqrt{x^2 + y^2}$ and $\tan^{-1}\left(\frac{y}{x}\right)$.

Example 2.8.1B: Finding the Image under T

Let the transformation T be defined by $T(u, v) = (x, y)$ where $x = u^2 - v^2$ and $y = uv$. Find the image of the triangle in the uv -plane with vertices $(0, 0),\ \ (0, 1)$, and $(1, 1)$.

Solution

The triangle and its image are shown in Figure 2.8.3. To understand how the sides of the triangle transform, call the side that joins $(0, 0)$ and $(0, 1)$ side A, the side that joins $(0, 0)$ and $(1, 1)$ side B, and the side that joins $(1, 1)$ and $(0, 1)$ side C.

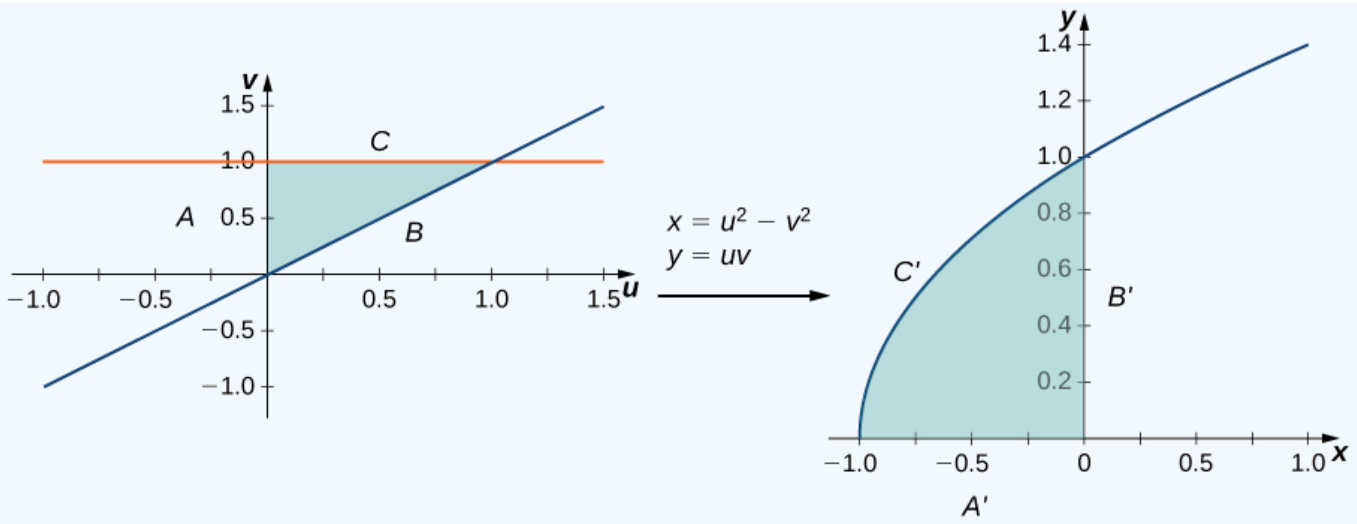


Figure 2.8.3: A triangular region with vertices $(0,0),\ (0,1)$, and $(1,1)$ in the uv-plane is transformed into an image in the xy-plane.

- For the side $A:\ u=0,\ 0\le v\le 1$ transforms to $x=-v^2,\ y=0$ so this is the side A' that joins $(-1,0)$ and $(0,0)$.
- For the side $B:\ u=v,\ 0\le u\le 1$ transforms to $x=0,\ y=u^2$ so this is the side B' that joins $(0,0)$ and $(0,1)$.
- For the side $C:\ 0\le u\le 1,\ v=1$ transforms to $x=u^2-1,\ y=u$ (hence $x=y^2-1$ so this is the side C' that makes the upper half of the parabolic arc joining $(-1,0)$ and $(0,1)$.

All the points in the entire region of the triangle in the uv-plane are mapped inside the parabolic region in the xy-plane.

Exercise 2.8.1

Let a transformation T be defined as $T(u,v)=(x,y)$ where $x=u+v,\ y=3v$. Find the image of the rectangle $G=\{(u,v):\ 0\le u\le 1,\ 0\le v\le 2\}$ from the uv-plane after the transformation into a region R in the xy-plane. Show that T is a one-to-one transformation and find $T^{-1}(x,y)$.

Hint available at the end of this section.

Answer available at the end of this section.

Jacobians

Recall that we mentioned near the beginning of this section that each of the component functions must have continuous first partial derivatives, which means that g_u, g_v, h_u and h_v exist and are also continuous. A transformation that has this property is called a C^1 transformation (here C denotes continuous). Let $T(u,v)=(g(u,v),\ h(u,v))$, where $x=g(u,v)$ and $y=h(u,v)$ be a one-to-one C^1 transformation. We want to see how it transforms a small rectangular region $S,\ \Delta u$ units by Δv units, in the uv-plane (Figure 2.8.4).

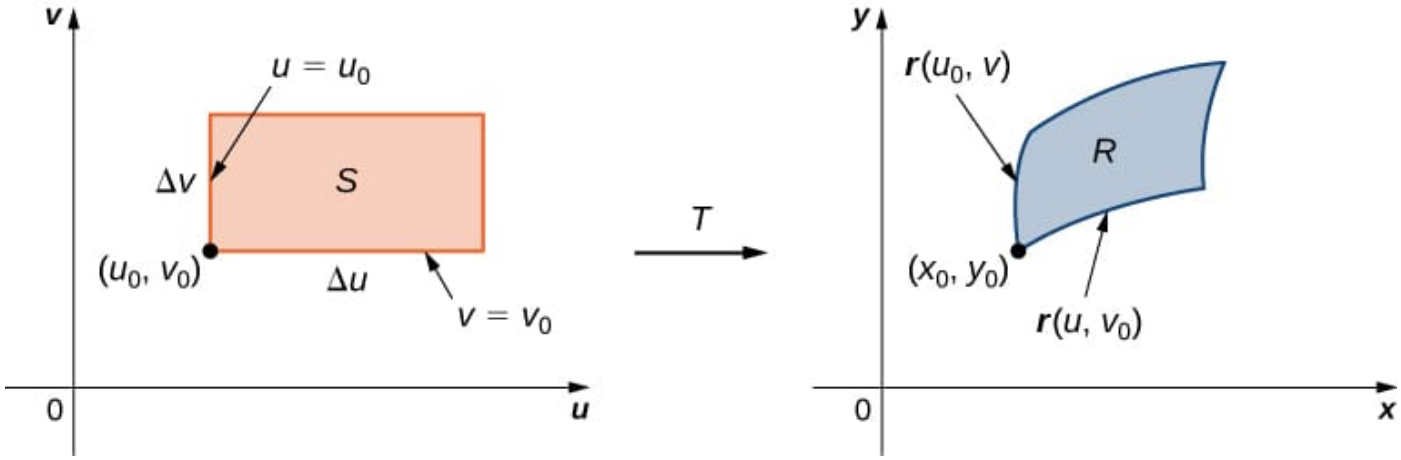


Figure 2.8.4: A small rectangle S in the uv-plane is transformed into a non-rectangular region R in the xy-plane.

Since $x = g(u, v)$ and $y = h(u, v)$, we have the position vector $\mathbf{r}(u, v) = g(u, v)\mathbf{i} + h(u, v)\mathbf{j}$ of the image of the point (u, v). Suppose that (u_0, v_0) is the coordinate of the point at the lower left corner that mapped to $(x_0, y_0) = T(u_0, v_0)$ The line $v = v_0$ maps to the image curve with vector function $\mathbf{r}(u, v_0)$, and the tangent vector at (x_0, y_0) to the image curve is

$$\mathbf{r}_u = g_u(u_0, v_0)\mathbf{i} + h_v(u_0, v_0)\mathbf{j} = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j}. \tag{2.8.13}$$

Similarly, the line $u = u_0$ maps to the image curve with vector function $\mathbf{r}(u_0, v)$, and the tangent vector at (x_0, y_0) to the image curve is

$$\mathbf{r}_v = g_v(u_0, v_0)\mathbf{i} + h_u(u_0, v_0)\mathbf{j} = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j}. \tag{2.8.14}$$

Now, note that

$$\mathbf{r}_u = \lim_{\Delta u \to 0} \frac{r(u_0 + \Delta u, v_0) - r(u_0, v_0)}{\Delta u} \; so \; r(u_0 + \Delta u, v_0) - r(u_0, v_0) \approx \Delta u \mathbf{r}_u. \tag{2.8.15}$$

Similarly,

$$\mathbf{r}_v = \lim_{\Delta v \to 0} \frac{r(u_0, v_0 + \Delta v) - r(u_0, v_0)}{\Delta v} \; so \; r(u_0, v_0 + \Delta v) - r(u_0, v_0) \approx \Delta v \mathbf{r}_v. \tag{2.8.16}$$

This allows us to estimate the area ΔA of the image R by finding the area of the parallelogram formed by the sides $\Delta v\mathbf{r}_v$ and $\Delta u\mathbf{r}_u$. By using the cross product of these two vectors by adding the $\mathbf{k}$th component as 0, the area ΔA of the image R (refer to The Cross Product) is approximately $|\Delta u\mathbf{r}_u \times \Delta v\mathbf{r}_v| = |\mathbf{r}_u \times \mathbf{r}_v|\Delta u\Delta v$. In determinant form, the cross product is

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} i & j & k \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \mathbf{k} = \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right) \mathbf{k} \tag{2.8.17}$$

Since $|k| = 1$, we have

$$\Delta A \approx |\mathbf{r}_u \times \mathbf{r}_v| \, \Delta u\Delta v = \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right) \Delta u\Delta v. \tag{2.8.18}$$

Definition: Jacobian

The Jacobian of the C^1 transformation $T(u, v) = (g(u, v),\ h(u, v))$ is denoted by $J(u, v)$ and is defined by the 2×2 determinant

$$J(u,v)=\left|\frac{\partial(x,y)}{\partial(u,v)}\right|=\begin{vmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u}\\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v}\end{vmatrix}=\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right). \tag{2.8.19}$$

Using the definition, we have

$$\Delta A\approx J(u,v)\Delta u\Delta v=\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\Delta u\Delta v. \tag{2.8.20}$$

Note that the Jacobian is frequently denoted simply by

$$J(u,v)=\frac{\partial(x,y)}{\partial(u,v)}. \tag{2.8.21}$$

Note also that

$$\begin{vmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u}\\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v}\end{vmatrix}=\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)=\begin{vmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v}\\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{vmatrix}.$$

Hence the notation $J(u,v)=\frac{\partial(x,y)}{\partial(u,v)}$ suggests that we can write the Jacobian determinant with partials of x in the first row and partials of y in the second row.

Example $2.8.2A$: Finding the Jacobian

Find the Jacobian of the transformation given in Example $2.8.1A$.

Solution

The transformation in the example is $T(r,\theta)=(r\ \cos\ \theta,\ r\ \sin\ \theta)$ where $x=r\ \cos\ \theta$ and $y=r\ \sin\ \theta$. Thus the Jacobian is

$$J(r,\theta)=\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix}\dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta}\\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta}\end{vmatrix}=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}=r\ \cos^2\theta+r\ \sin^2\theta=r(\cos^2\theta+\sin^2\theta)=r.$$

Example $2.8.2B$: Finding the Jacobian

Find the Jacobian of the transformation given in Example $2.8.1B$.

Solution

The transformation in the example is $T(u,v)=(u^2-v^2,uv)$ where $x=u^2-v^2$ and $y=uv$. Thus the Jacobian is

$$J(u,v)=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v}\\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}2u & -2v\\ v & u\end{vmatrix}=2u^2+2v^2.$$

Exercise $2.8.2$

Find the Jacobian of the transformation given in the previous checkpoint: $T(u,v)=(u+v,2v)$.

Hint available at the end of this section.

Answer available at the end of this section.

Change of Variables for Double Integrals

We have already seen that, under the change of variables $T(u,v)=(x,y)$ where $x=g(u,v)$ and $y=h(u,v)$, a small region ΔA in the xy-plane is related to the area formed by the product $\Delta u \Delta v$ in the uv-plane by the approximation

$$\Delta A \approx J(u,v)\Delta u,\ \Delta v. \tag{2.8.22}$$

Now let's go back to the definition of double integral for a minute:

$$\iint_R f(x,y)dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij},y_{ij})\Delta A. \tag{2.8.23}$$

Referring to Figure 2.8.5, observe that we divided the region S in the uv-plane into small subrectangles S_{ij} and we let the subrectangles R_{ij} in the xy-plane be the images of S_{ij} under the transformation $T(u,v)=(x,y)$.

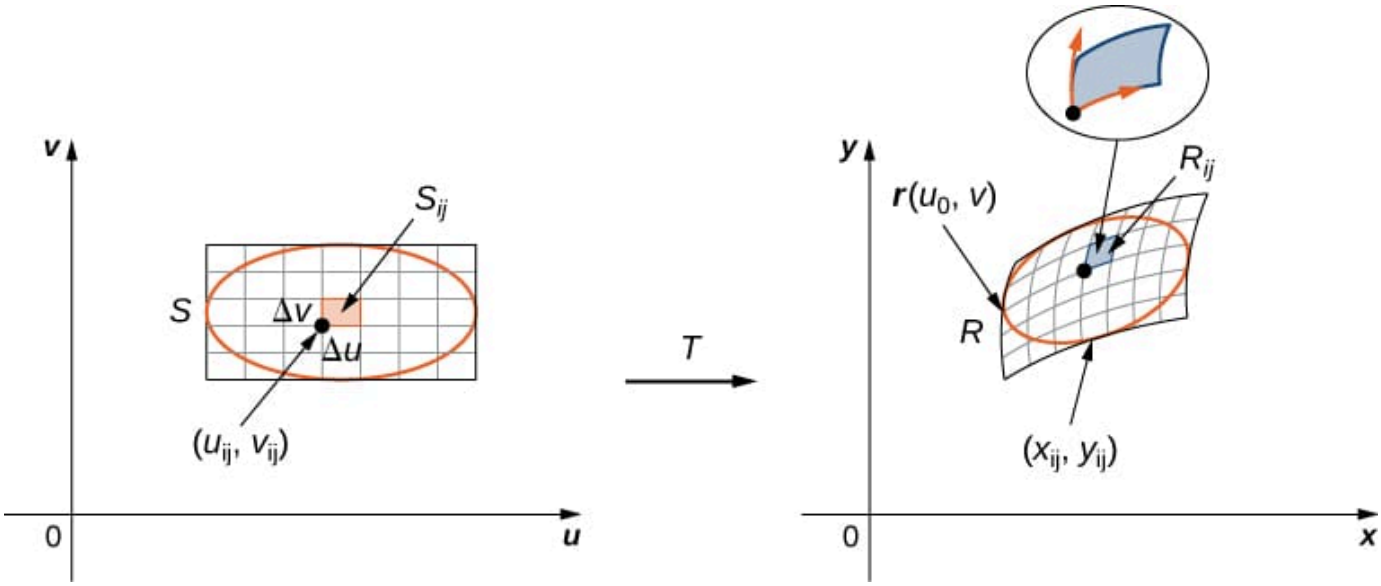


Figure 2.8.5: The region S in the uv-plane is divided into small subrectangles S_{ij}. The subrectangles S_{ij} in the uv-plane transform into subrectangles R_{ij} in the xy-plane.

Then the double integral becomes

$$\iint_R = f(x,y)dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij},y_{ij})\Delta A = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(g(u_{ij},v_{ij}),\ h(u_{ij},v_{ij}))|J(u_{ij},v_{ij})|\Delta u\Delta v. \tag{2.8.24}$$

Notice this is exactly the double Riemann sum for the integral

$$\iint_S f(g(u,v),\ h(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv. \tag{2.8.25}$$

Change of Variables for Double Integrals

Let $T(u,v)=(x,y)$ where $x=g(u,v)$ and $y=h(u,v)$ be a one-to-one C^1 transformation, with a nonzero Jacobian on the interior of the region S in the uv-plane it maps S into the region R in the xy-plane. If f is continuous on R, then

$$\iint_R f(x,y)dA = \iint_S f(g(u,v),\ h(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv. \tag{2.8.26}$$

With this theorem for double integrals, we can change the variables from (x, y) to (u, v) in a double integral simply by replacing

$$dA = dx\, dy = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\, dv \tag{2.8.27}$$

when we use the substitutions $x = g(u, v)$ and $y = h(u, v)$ and then change the limits of integration accordingly. This change of variables often makes any computations much simpler.

Example 2.8.3: Changing Variables from Rectangular to Polar Coordinates

Consider the integral

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2 + y^2} dy\, dx. \tag{2.8.28}$$

Use the change of variables $x = r\ \cos\ \theta$ and $y = r\ \sin\ \theta$, and find the resulting integral.

Solution

First we need to find the region of integration. This region is bounded below by $y = 0$ and above by $y = \sqrt{2x - x^2}$ (Figure 2.8.6).

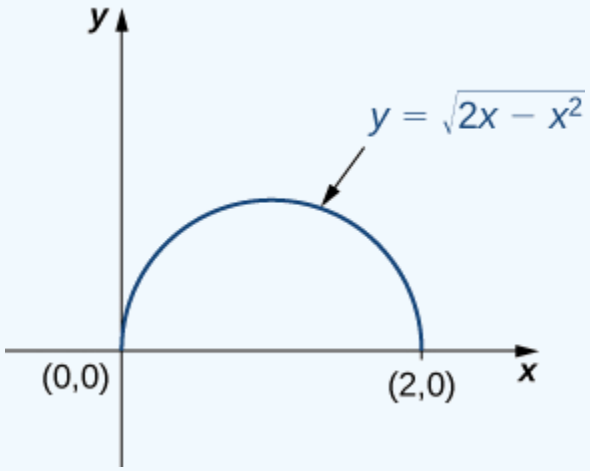


Figure 2.8.6: Changing the region bounded below by $y = 0$ and above by $y = \sqrt{2x - x^2}$ from rectangular coordinates to polar coordinates.

Squaring and collecting terms, we find that the region is the upper half of the circle $x^2 + y^2 - 2x = 0$, that is $y^2 + (x - 1)^2 = 1$. In polar coordinates, the circle is $r = 2\ cos\ \theta$ so the region of integration in polar coordinates is bounded by $0 \le r \le 2 \cos\ \theta$ and $0 \le \theta \le \frac{\pi}{2}$.

The Jacobian is $J(r, \theta) = r$, as shown in Example 2.8.2A. Since $r \ge 0$, we have $|J(r, \theta)| = r$.

The integrand $\sqrt{x^2 + y^2}$ changes to r in polar coordinates, so the double iterated integral is

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2 + y^2} dy\, dx = \int_0^{\pi/2} \int_0^{2\ cos\ \theta} r|j(r, \theta)| dr\, d\theta = \int_0^{\pi/2} \int_0^{2\ cos\ \theta} r^2 dr\, d\theta. \tag{2.8.29}$$

Watch the accompanying video for a visual demonstration of the concept.

- **Video length**: 3 minutes 25 seconds.
- **Context**: This video demonstrates how to evaluate double integration using a Jacobian.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Exercise 2.8.3

Considering the integral $\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2) dy\ dx$, use the change of variables $x = r\ cos\ \theta$ and $y = r\ sin\ \theta$ and find the resulting integral.

Hint available at the end of this section.

Answer available at the end of this section.

Notice in the next example that the region over which we are to integrate may suggest a suitable transformation for the integration. This is a common and important situation.

Example 2.8.4: Changing Variables

Consider the integral

$$\iint_R (x - y) dy\, dx, \tag{2.8.30}$$

where R is the parallelogram joining the points $(1, 2),\ (3, 4),\ (4, 3)$, and $(6, 5)$ (Figure 2.8.7). Make appropriate changes of variables, and write the resulting integral.

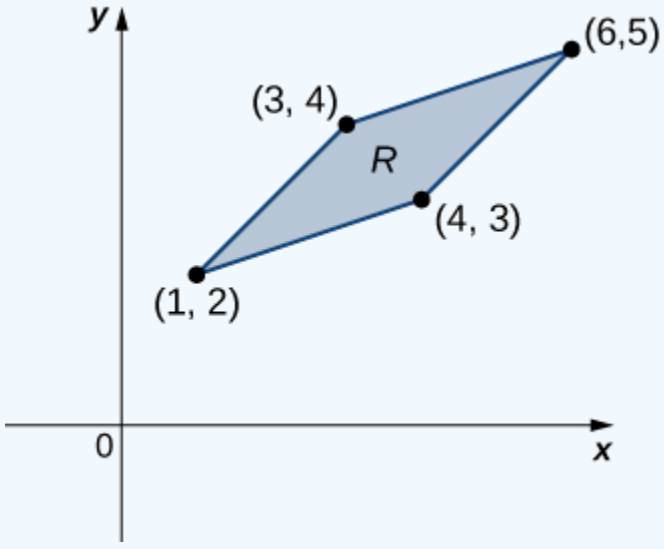


Figure 2.8.7: The region R is the parallelogram joining the points $(1, 2),\ (3, 4),\ (4, 3)$, and $(6, 5)$.

Solution

First, we need to understand the region over which we are to integrate. The sides of the parallelogram are $x - y + 1,\ x - y - 1 = 0,\ x - 3y + 5 = 0$ and $x - 3y + 9 = 0$ (Figure 2.8.8). Another way to look at them is $x - y = -1,\ x - y = 1,\ x - 3y = -5$, and $x - 3y = 9$.

Clearly the parallelogram is bounded by the lines $y = x + 1,\ y = x - 1,\ y = \frac{1}{3}(x + 5)$, and $y = \frac{1}{3}(x + 9)$.

Notice that if we were to make $u = x - y$ and $v = x - 3y$, then the limits on the integral would be $-1 \leq u \leq 1$ and $-9 \leq v \leq -5$.

To solve for x and y, we multiply the first equation by 3 and subtract the second equation, $3u - v = (3x - 3y) - (x - 3y) = 2x$. Then we have $x = \frac{3u-v}{2}$. Moreover, if we simply subtract the second equation from the first, we get $u - v = (x - y) - (x - 3y) = 2y$ and $y = \frac{u-v}{2}$.

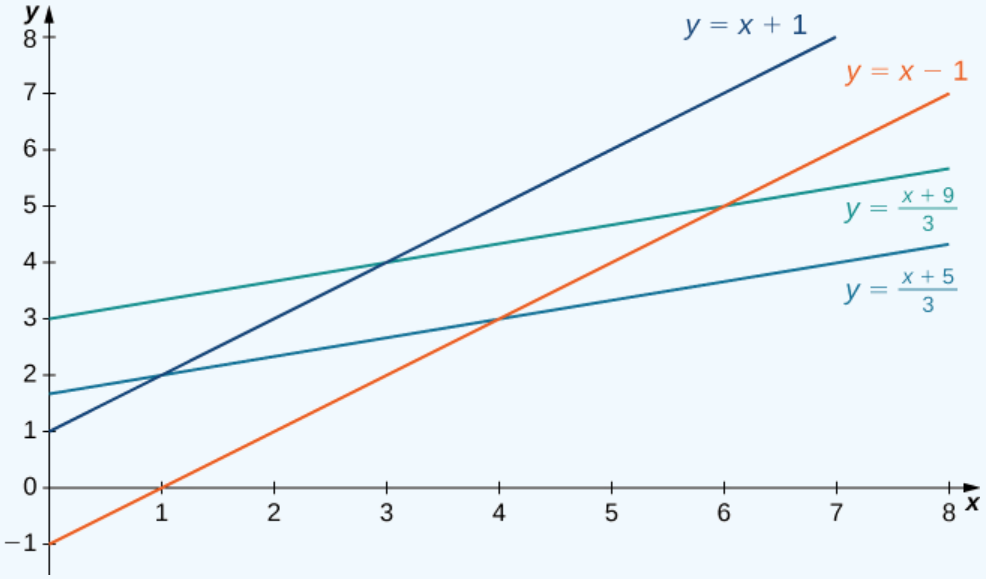


Figure 2.8.8: The sides of the parallelogram by the intersections of curves $x - y + 1,\ x - y - 1 = 0,\ x - 3y + 5 = 0$ and $x - 3y + 9 = 0$.

Thus, we can choose the transformation

$$T(u, v) = \left(\frac{3u - v}{2}, \frac{u - v}{2}\right) \tag{2.8.31}$$

and compute the Jacobian $J(u, v)$. We have

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 3/2 & -1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{3}{4} + \frac{1}{4} = -\frac{1}{2}$$

Therefore, $|J(u, v)| = \frac{1}{2}$. Also, the original integrand becomes

$$x - y = \frac{1}{2}[3u - v - u + v] = \frac{1}{2}[3u - u] = \frac{1}{2}[2u] = u.$$

Therefore, by the use of the transformation T, the integral changes to

$$\iint_R (x - y)dy\,dx = \int_{-9}^{-5}\int_{-1}^{1} J(u, v)u\,du\,dv = \int_{-9}^{-5}\int_{-1}^{1} \left(\frac{1}{2}\right) u\,du\,dv,$$

which is much simpler to compute.

Exercise 2.8.4

Make appropriate changes of variables in the integral

$$\iint_R \frac{4}{(x - y)^2} dy\,dx,$$

where R is the trapezoid bounded by the lines $x - y = 2,\ x - y = 4,\ x = 0$, and $y = 0$. Write the resulting integral.

Hint available at the end of this section.

Answer available at the end of this section.

We are ready to give a problem-solving strategy for change of variables.

Problem-Solving Strategy: Change of Variables

1. Sketch the region given by the problem in the xy-plane and then write the equations of the curves that form the boundary.
2. Depending on the region or the integrand, choose the transformations $x = g(u, v)$ and $y = h(u, v)$.
3. Determine the new limits of integration in the uv-plane.
4. Find the Jacobian $J(u, v)$.
5. In the integrand, replace the variables to obtain the new integrand.
6. Replace $dy\ dx$ or $dx\ dy$, whichever occurs, by $J(u, v)du\ dv$.

In the next example, we find a substitution that makes the integrand much simpler to compute.

Example 2.8.5: Evaluating an Integral

Using the change of variables $u = x - y$ and $v = x + y$, evaluate the integral

$$\iint_R (x - y)e^{x^2-y^2} dA, \tag{2.8.32}$$

where R is the region bounded by the lines $x + y = 1$ and $x + y = 3$ and the curves $x^2 - y^2 = -1$ and $x^2 - y^2 = 1$ (see the first region in Figure 2.8.9).

Solution

As before, first find the region R and picture the transformation so it becomes easier to obtain the limits of integration after the transformations are made (Figure 2.8.9).

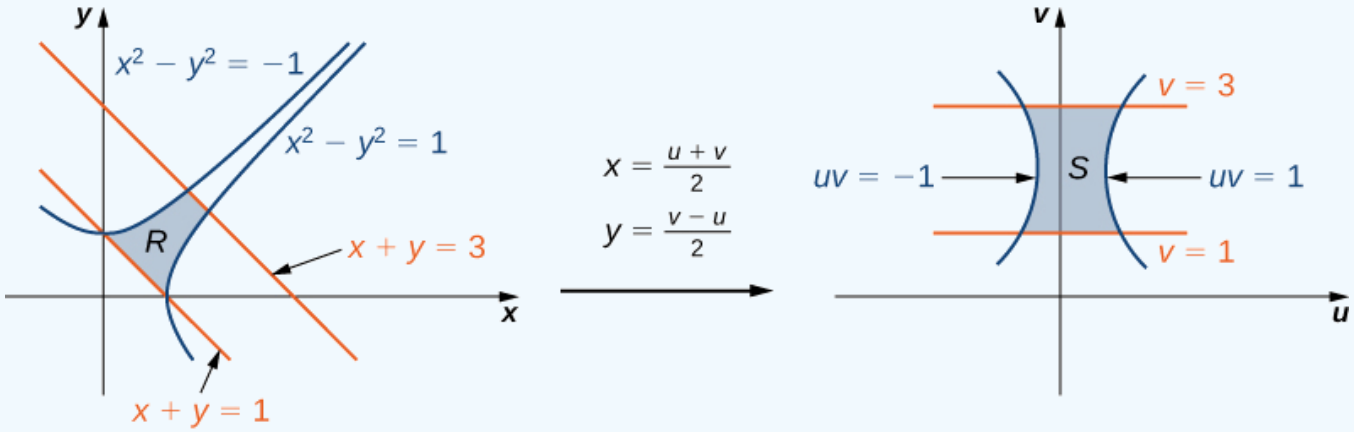


Figure 2.8.9: Transforming the region R which is bounded by the lines $x + y = 1$ and $x + y = 3$ and the curves $x^2 - y^2 = -1$ and $x^2 - y^2 = 1$ into the region S to simplify the computation of an integral.

Given $u = x - y$ and $v = x + y$, we have $x = \frac{u+v}{2}$ and $y = \frac{v-u}{2}$ and hence the transformation to use is $T(u, v) = (\frac{u+v}{2}, \frac{v-u}{2})$. The lines $x + y = 1$ and $x + y = 3$ become $v = 1$ and $v = 3$, respectively. The curves $x^2 - y^2 = 1$ and $x^2 - y^2 = -1$ become $uv = 1$ and $uv = -1$, respectively.

Thus we can describe the region S (see the second region Figure 2.8.9) as

$$S = \left\{(u,v) | 1 \le v \le 3,\ \frac{-1}{v} \le u \le \frac{1}{v}\right\}.$$

The Jacobian for this transformation is

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{vmatrix} = \frac{1}{2}.$$

Therefore, by using the transformation T, the integral changes to

$$\iint_R (x-y)e^{x^2-y^2}\,dA = \frac{1}{2}\int_1^3 \int_{-1/v}^{1/v} ue^{uv}\,du\,dv.$$

Doing the evaluation, we have

$$\frac{1}{2}\int_1^3 \int_{-1/v}^{1/v} ue^{uv}\,du\,dv = \frac{2}{3e} \approx 0.245.$$

Exercise 2.8.5

Using the substitutions $x = v$ and $y = \sqrt{u+v}$, evaluate the integral $\iint_R y\ \sin(y^2 - x)\ dA$, where R is the region bounded by the lines $y = \sqrt{x},\ x = 2$ and $y = 0$.

Hint available at the end of this section.

Answer available at the end of this section.

Change of Variables for Triple Integrals

Changing variables in triple integrals works in exactly the same way. Cylindrical and spherical coordinate substitutions are special cases of this method, which we demonstrate here.

Suppose that G is a region in uvw-space and is mapped to D in xyz-space (Figure 2.8.10) by a one-to-one C^1 transformation $T(u,v,w) = (x,y,z)$ where $x = g(u,v,w),\ y = h(u,v,w)$, and $z = k(u,v,w)$.

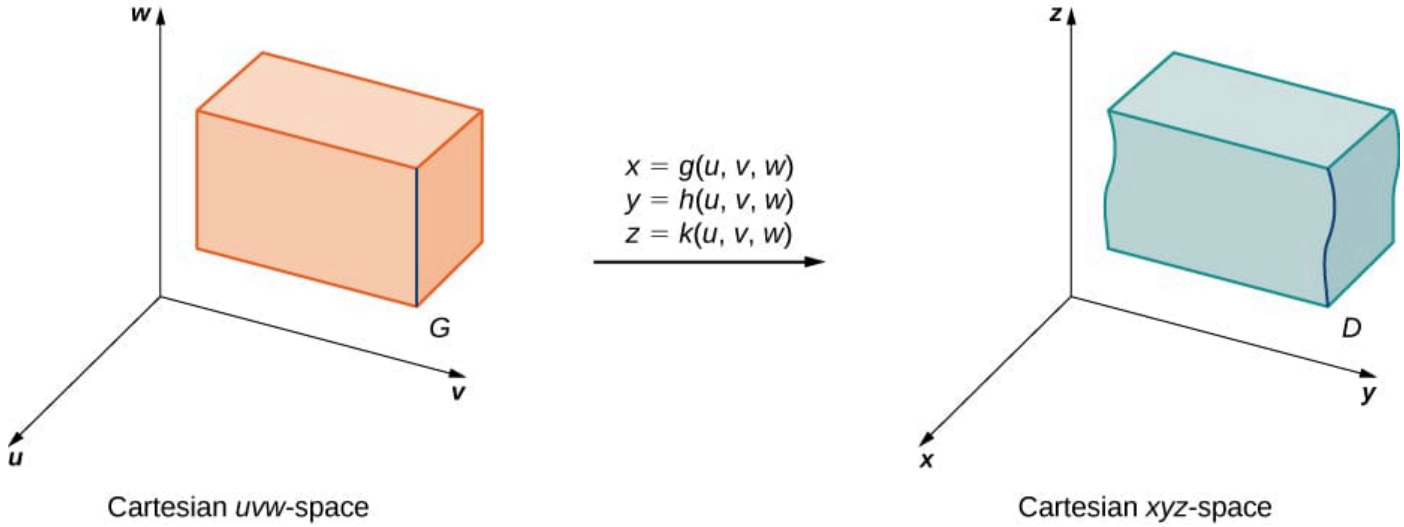


Figure 2.8.10: A region G in uvw-space mapped to a region D in xyz-space by a one-to-one C^1 transformation $T(u,v,w) = (x,y,z)$ where $x = g(u,v,w),\ y = h(u,v,w)$, and $z = k(u,v,w)$.

Then any function $F(x, y, z)$ defined on D can be thought of as another function $H(u, v, w)$ that is defined on G:

$$F(x, y, z) = F(g(u, v, w),\ h(u, v, w),\ k(u, v, w)) = H(u, v, w). \tag{2.8.33}$$

Now we need to define the Jacobian for three variables.

Definition: Jacobian Determinant

The Jacobian determinant $J(u, v, w)$ in three variables is defined as follows:

$$J(u, v, w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}. \tag{2.8.34}$$

This is also the same as

$$J(u, v, w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}. \tag{2.8.35}$$

The Jacobian can also be simply denoted as $\frac{\partial(x,y,z)}{\partial(u,v,w)}$.

With the transformations and the Jacobian for three variables, we are ready to establish the theorem that describes change of variables for triple integrals.

Change of Variables for Triple Integrals

Let $T(u, v, w) = (x, y, z)$ where $x = g(u, v, w),\ \ y = h(u, v, w)$, and $z = k(u, v, w)$, be a one-to-one C^1 transformation, with a nonzero Jacobian, that maps the region G in the uvw-space into the region D in the xyz-space. As in the two-dimensional case, if F is continuous on D, then

$$\iiint_D F(x, y, z)dV = \iiint_G f(g(u, v, w),\ h(u, v, w),\ k(u, v, w)) \left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| du\, dv\, dw \tag{2.8.36}$$

$$= \iiint_G H(u, v, w)|J(u, v, w)|du\, dv\, dw. \tag{2.8.37}$$

Let us now see how changes in triple integrals for cylindrical and spherical coordinates are affected by this theorem. We expect to obtain the same formulas as in Triple Integrals in Cylindrical and Spherical Coordinates.

Example 2.8.6A: Obtaining Formulas in Triple Integrals for Cylindrical and Spherical Coordinates

Derive the formula in triple integrals for

a. cylindrical and
b. spherical coordinates.

Solution

a.

For cylindrical coordinates, the transformation is $T(r,\theta,z)=(x,y,z)$ from the Cartesian $r\theta z$-space to the Cartesian xyz-space (Figure 2.8.11). Here $x=r\ \cos\ \theta,\ y=r\ \sin\theta$ and $z=z$. The Jacobian for the transformation is

$$J(r,\theta,z)=\frac{\partial(x,y,z)}{\partial(r,\theta,z)}=\begin{vmatrix}\frac{\partial x}{\partial r}&\frac{\partial x}{\partial\theta}&\frac{\partial x}{\partial z}\\ \frac{\partial y}{\partial r}&\frac{\partial y}{\partial\theta}&\frac{\partial y}{\partial z}\\ \frac{\partial z}{\partial r}&\frac{\partial z}{\partial\theta}&\frac{\partial z}{\partial z}\end{vmatrix} \tag{2.8.38}$$

$$\begin{vmatrix}\cos\theta&-r\sin\theta&0\\ \sin\theta&r\cos\theta&0\\ 0&0&1\end{vmatrix}=r\ \cos^2\theta+r\ \sin^2\theta=r. \tag{2.8.39}$$

We know that $r\geq 0$, so $|J(r,\theta,z)|=r$. Then the triple integral is

$$\iiint_D f(x,y,z)dV=\iiint_G f(r\ \cos\theta,\ r\ \sin\theta,\ z)r\,dr\,d\theta\,dz. \tag{2.8.40}$$

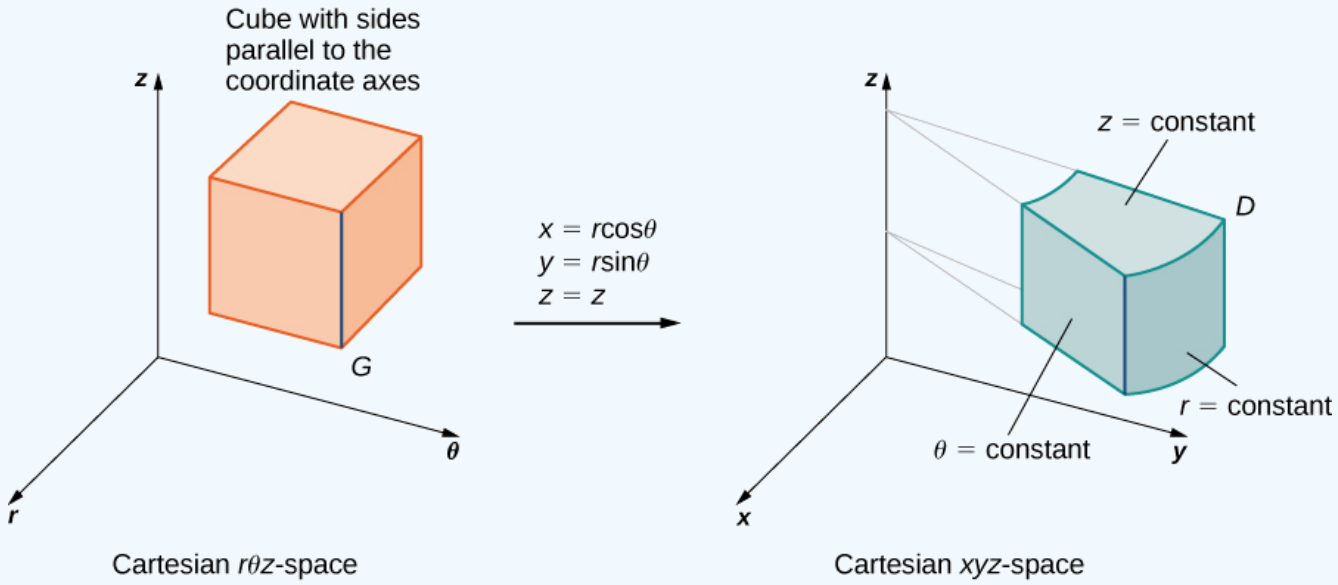


Figure 2.8.11: The transformation from rectangular coordinates to cylindrical coordinates can be treated as a change of variables from region G in $r\theta z$-space to region D in xyz-space.

b.

For spherical coordinates, the transformation is $T(\rho,\theta,\varphi)$ from the Cartesian $\rho\theta\varphi$-space to the Cartesian xyz-space (Figure 2.8.12). Here $x=\rho\ \sin\varphi\ \cos\theta,\ y=\rho\ \sin\varphi\ \sin\theta$, and $z=\rho\ \cos\varphi$. The Jacobian for the transformation is

$$J(\rho,\theta,\varphi)=\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)}=\begin{vmatrix}\frac{\partial x}{\partial\rho}&\frac{\partial x}{\partial\theta}&\frac{\partial x}{\partial\varphi}\\ \frac{\partial y}{\partial\rho}&\frac{\partial y}{\partial\theta}&\frac{\partial y}{\partial\varphi}\\ \frac{\partial z}{\partial\rho}&\frac{\partial z}{\partial\theta}&\frac{\partial z}{\partial\varphi}\end{vmatrix}=\begin{vmatrix}\sin\varphi\cos\theta&-\rho\sin\varphi\sin\theta&\rho\cos\varphi\cos\theta\\ \sin\varphi\sin\theta&\rho\sin\varphi\cos\theta&\rho\cos\varphi\sin\theta\\ \cos\varphi&0&-\rho\sin\varphi\end{vmatrix}. \tag{2.8.41}$$

Expanding the determinant with respect to the third row:

$$\begin{aligned}&=\cos\varphi\begin{vmatrix}-\rho\sin\varphi\sin\theta&\rho\cos\varphi\cos\theta\\ \rho\sin\varphi\cos\theta&\rho\cos\varphi\sin\theta\end{vmatrix}-\rho\sin\varphi\begin{vmatrix}\sin\varphi\cos\theta&-\rho\sin\varphi\sin\theta\\ \sin\varphi\sin\theta&\rho\sin\varphi\cos\theta\end{vmatrix}\\ &=\cos\varphi(-\rho^2\sin\varphi\ \cos\varphi\ \sin^2\theta-\rho^2\ \sin\varphi\ \cos\varphi\ \cos^2\theta)\\ &\quad-\rho\sin\varphi(\rho\sin^2\varphi\cos^2\theta+\rho\sin^2\varphi\sin^2\theta)\\ &=-\rho^2\sin\varphi\cos^2\varphi(\sin^2\theta+\cos^2\theta)-\rho^2\sin\varphi\sin^2\varphi(\sin^2\theta+\cos^2\theta)\\ &=-\rho^2\sin\varphi\cos^2\varphi-\rho^2\sin\varphi\sin^2\varphi\\ &=-\rho\sin\varphi(\cos^2\varphi+\sin^2\varphi)=-\rho^2\sin\varphi.\end{aligned}$$

Since $0 \le \varphi \le \pi$, we must have $\sin\varphi \ge 0$. Thus $|J(\rho,\theta,\varphi)| = |-\rho^2 \sin\varphi| = \rho^2 \sin\varphi$.

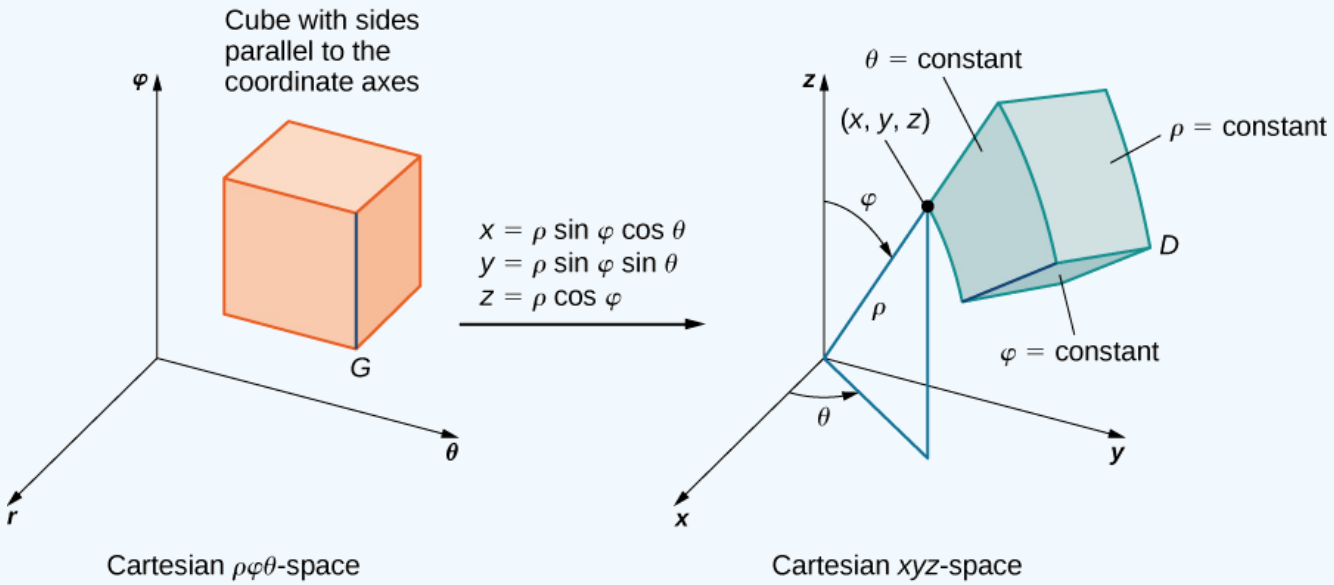


Figure 2.8.12: The transformation from rectangular coordinates to spherical coordinates can be treated as a change of variables from region G in $\rho\theta\varphi$-space to region D in xyz-space.

Then the triple integral becomes

$$\iiint_D f(x,y,z)dV = \iiint_G f(\rho\,\sin\varphi\,\cos\theta,\ \rho\,\sin\varphi\,\sin\theta, \rho\,\cos\varphi)\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta. \tag{2.8.42}$$

Let's try another example with a different substitution.

Example 2.8.6B: Evaluating a Triple Integral with a Change of Variables

Evaluate the triple integral

$$\int_0^3 \int_0^4 \int_{y/2}^{(y/2)+1} \left(x + \frac{z}{3}\right) dx\,dy\,dz \tag{2.8.43}$$

In xyz-space by using the transformation

$u = (2x - y)/2,\ \ v = y/2$, and $w = z/3$.

Then integrate over an appropriate region in uvw-space.

Solution

As before, some kind of sketch of the region G in xyz-space over which we have to perform the integration can help identify the region D in uvw-space (Figure 2.8.13). Clearly G in xyz-space is bounded by the planes $x = y/2,\ \ x = (y/2) + 1,\ \ y = 0,\ \ y = 4,\ \ z = 0$, and $z = 3$. We also know that we have to use $u = (2x - y)/2,\ \ v = y/2$, and $w = z/3$ for the transformations. We need to solve for x, y and z. Here we find that $x = u + v,\ \ y = 2v$, and $z = 3w$.

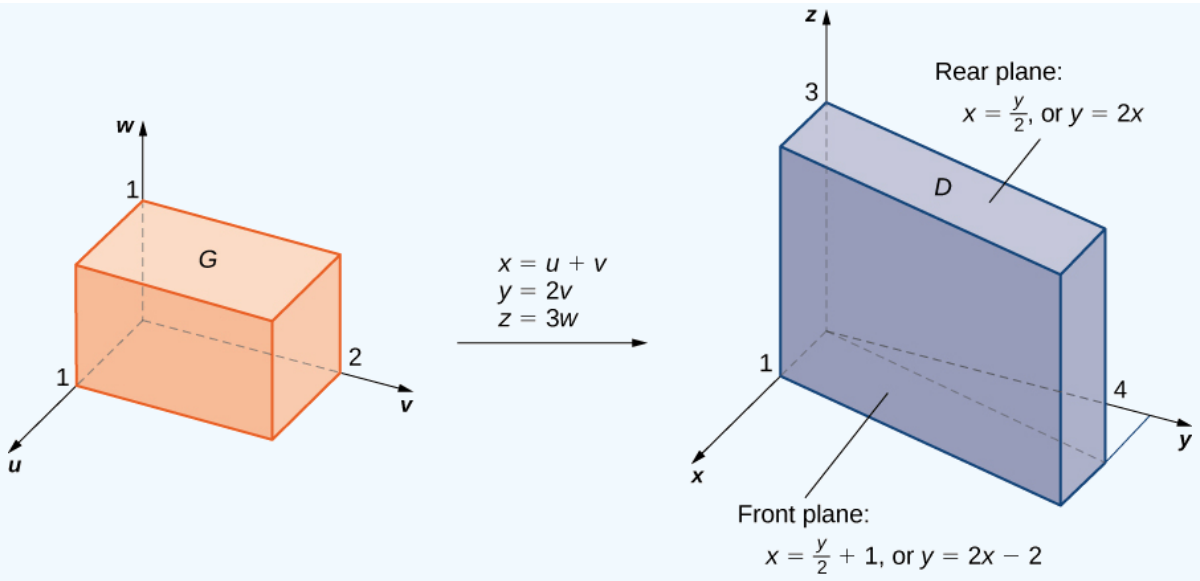


Figure 2.8.13: The region G in uvw-space is transformed to region D in xyz-space.

Using elementary algebra, we can find the corresponding surfaces for the region G and the limits of integration in uvw-space. It is convenient to list these equations in a table.

Table 2.8.1: The table shows the solution to a triple integral problem using a change of variables, including the transformation equations, Jacobian determinant, and final evaluation.

Equations in xyz for the region D	Corresponding equations in uvw for the region G	Limits for the integration in uvw
$x = y/2$	$u + v = 2v/2 = v$	$u = 0$
$x = y/2$	$u + v = (2v/2) + 1 = v + 1$	$u = 1$
$y = 0$	$2v = 0$	$v = 0$
$y = 4$	$2v = 4$	$v = 2$
$z = 0$	$3w = 0$	$w = 0$
$z = 3$	$3w = 3$	$w = 1$

Now we can calculate the Jacobian for the transformation:

$$J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 6.$$

The function to be integrated becomes

$$f(x,y,z) = x + \frac{z}{3} = u + v + \frac{3w}{3} = u + v + w. \tag{2.8.44}$$

We are now ready to put everything together and complete the problem.

$$\int_0^3\int_0^4\int_{y/2}^{(y/2)+1}\left(x+\frac{z}{3}\right)dx\,dy\,dz = \int_0^1\int_0^2\int_0^1 (u+v+w)|J(u,v,w)|du\,dv\,dw$$
$$= \int_0^1\int_0^2\int_0^1 (u+v+w)|6|du\,dv\,dw$$
$$= 6\int_0^1\int_0^2\int_0^1 (u+v+w)\,du\,dv\,dw$$
$$= 6\int_0^1\int_0^2\left[\frac{u^2}{2}+vu+wu\right]_0^1 dv\,dw$$
$$= 6\int_0^1\int_0^2\left(\frac{1}{2}+v+u\right)dv\,dw$$
$$= 6\int_0^1\left[\frac{1}{2}v+\frac{v^2}{2}+wv\right]_0^2 dw$$
$$= 6\int_0^1(3+2w)\,dw = 6\Big[3w+w^2\Big]_0^1 = 24.$$

Exercise 2.8.6

Let D be the region in xyz-space defined by $1 \le x \le 2,\ \ 0 \le xy \le 2$, and $0 \le z \le 1$.

Evaluate $\iiint_D (x^2y + 3xyz)\ dx\ dy\ dz$ by using the transformation $u = x,\ \ v = xy$, and $w = 3z$.

Hint available at the end of this section.

Answer available at the end of this section.

Glossary

Jacobian

the Jacobian $J(u,v)$ in two variables is a 2×2 determinant:

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix};$$

the Jacobian $J(u,v,w)$ in three variables is a 3×3 determinant:

$$J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}$$

One-to-one transformation
a transformation $T : G \to R$ defined as $T(u,v) = (x,y)$ is said to be one-to-one if no two points map to the same image point

Planar transformation
a function T that transforms a region G in one plane into a region R in another plane by a change of variables

Transformation
a function that transforms a region GG in one plane into a region RR in another plane by a change of variables

Contributors and Attributions

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Answers

Exercise 2.8.1

Follow the steps of Example 2.8.1B.

Exercise 2.8.1

$T^{-1}(x,y)=(u,v)$ where $u=\frac{3x-y}{3}$ and $v=\frac{y}{3}$

Exercise 2.8.2

Follow the steps in the previous two examples.

Exercise 2.8.2

$$J(u,v)=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}1 & 1\\ 0 & 2\end{vmatrix}=2$$

Exercise 2.8.3

Follow the steps in the previous example.

Exercise 2.8.3

$$\int_0^{\pi/2}\int_0^1 r^3 dr\,d\theta \tag{2.8.45}$$

Exercise 2.8.4

Follow the steps in the previous example.

Exercise 2.8.4

$x=\frac{1}{2}(v+u)$ and $y=\frac{1}{2}(v-u)$

and

$$\int_2^4\int_{-u}^{u}\left(\frac{1}{2}\right)\cdot\frac{4}{u^2}\,dv\,du.$$

Exercise 2.8.5

Sketch a picture and find the limits of integration.

Exercise 2.8.5

$\frac{1}{2}(\sin 2-2)$

Exercise 2.8.6

Make a table for each surface of the regions and decide on the limits, as shown in the example.

Exercise 2.8.6

$$\int_0^3\int_0^2\int_1^2\left(\frac{v}{3}+\frac{vw}{3u}\right)du\,dv\,dw=2+\ln 8 \tag{2.8.46}$$

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2.8E: Exercises

Exercises 1 - 6

The function $T : S \to R,\ T(u, v) = (x, y)$ on the region $S = \{(u, v) \mid 0 \le u \le 1,\ 0 \le v \le 1\}$ bounded by the unit square is given, where $R \in R^2$ is the image of S under T.

a. Justify that the function T is a C^1 transformation.

b. Find the images of the vertices of the unit square S through the function T.

c. Determine the image R of the unit square S and graph it.

1. $x = 2u,\ y = 3v$

2. $x = \frac{u}{2},\ y = \frac{v}{3}$

Answer

a. $T(u, v) = (g(u, v),\ h(u, v),\ x = g(u, v) = \frac{u}{2}$ and $y = h(u, v) = \frac{v}{3}$. The functions g and h are continuous and differentiable, and the partial derivatives $g_u(u, v) = \frac{1}{2},\ g_v(u, v) = 0,\ h_u(u, v) = 0$ and $h_v(u, v) = \frac{1}{3}$ are continuous on S;

b. $T(0, 0) = (0, 0),\ T(1, 0) = \left(\frac{1}{2}, 0\right),\ T(0, 1) = \left(0, \frac{1}{3}\right)$, and $T(1, 1) = \left(\frac{1}{2}, \frac{1}{3}\right)$;

c. R is the rectangle of vertices $(0, 0),\ \left(0, \frac{1}{3}\right),\ \left(\frac{1}{2}, \frac{1}{3}\right)$, and $\left(0, \frac{1}{3}\right)$ in the xy-plane; the figure $2.8E.1$.

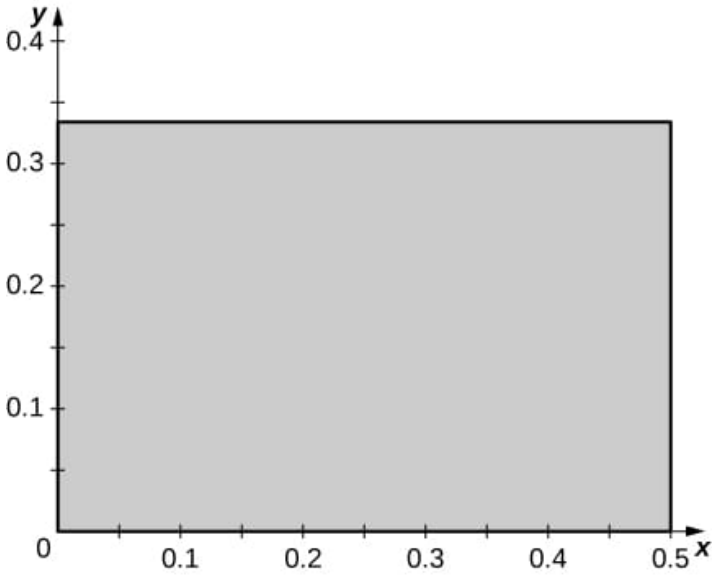


Figure $2.8E.1$: The graph illustrates the solid S.

3. $x = u - v,\ y = u + v$

4. $x = 2u - v,\ y = u + 2v$

Answer

a. $T(u, v) = (g(u, v),\ h(u, v),\ x = g(u, v) = 2u - v$ and $y = h(u, v) = u + 2v$. The functions g and h are continuous

and differentiable, and the partial derivatives $g_u(u,v) = 2,\ g_v(u,v) = -1,\ h_u(u,v) = 1$ and $h_v(u,v) = 2$ are continuous on S;

b. $T(0,0) = (0,0),\ T(1,0) = (2,1),\ T(0,1) = (-1,2)$, and $T(1,1) = (1,3)$;

c. R is the parallelogram of vertices $(0,0),\ (2,1)\ (1,3)$, and $(-1,2)$ in the xy-plane; the figure $2.8E.2$.

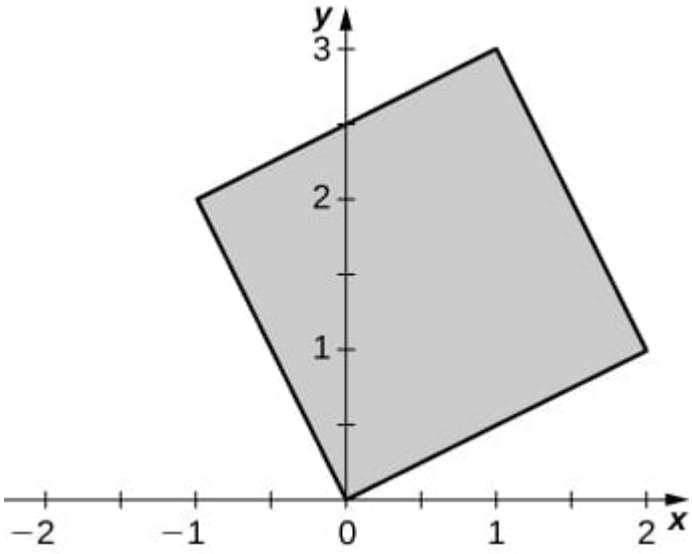


Figure $2.8E.2$: The graph illustrates the solid S.

5. $x = u^2,\ y = v^2$

6. $x = u^3,\ y = v^3$

Answer

a. $T(u,v) = (g(u,v),\ h(u,v),\ x = g(u,v) = u^3$ and $y = h(u,v) = v^3$. The functions g and h are continuous and differentiable, and the partial derivatives $g_u(u,v) = 3u^2,\ g_v(u,v) = 0,\ h_u(u,v) = 0$ and $h_v(u,v) = 3v^2$ are continuous on S;

b. $T(0,0) = (0,0),\ T(1,0) = (1,0),\ T(0,1) = (0,1)$, and $T(1,1) = (1,1)$;

c. R is the unit square in the xy-plane see the figure in the answer to the previous exercise.

Exercises 7 - 12

Determine whether the transformations $T : S \to R$ are one-to-one or not.

7. $x = u^2,\ y = v^2$, where S is the rectangle of vertices $(-1,0),\ (1,0),\ (1,1)$, and $(-1,1)$.

8. $x = u^4,\ y = u^2 + v$, where S is the triangle of vertices $(-2,0),\ (2,0)$, and $(0,2)$.

Answer

T is not one-to-one: two points of S have the same image. Indeed, $T(-2,0) = T(2,0) = (16,4)$.

9. $x = 2u,\ y = 3v$, where S is the square of vertices $(-1,1),\ (-1,-1),\ (1,-1)$, and $(1,1)$.

10. $T(u,v) = (2u - v, u)$, where S is the triangle with vertices $(-1,1),\ (-1,-1)$, and $(1,-1)$.

Answer

T is one-to-one: We argue by contradiction. $T(u_1,v_1) = T(u_2,v_2)$ implies $2u_1 - v_1 = 2u_2 - v_2$ and $u_1 = u_2$.

Thus, $u_1 = u + 2$ and $v_1 = v_2$.

11. $x = u + v + w,\ y = u + v,\ z = w$, where $S = R = R^3$.

12. $x = u^2 + v + w,\ y = u^2 + v,\ z = w$, where $S = R = R^3$.

Answer

T is not one-to-one: $T(1, v, w) = (-1, v, w)$

Exercises 13 - 18

The transformations $T : R \to S$ are one-to-one. Find their related inverse transformations $T^{-1} : R \to S$.

13. $x = 4u,\ y = 5v$, where $S = R = R^2$.

14. $x = u + 2v,\ y = -u + v$, where $S = R = R^2$.

Answer

$u = \frac{x-2y}{3},\ v = \frac{x+y}{3}$

15. $x = e^{2u+v},\ y = e^{u-v}$, where $S = R^2$ and $R = \{(x, y) \mid x > 0,\ y > 0\}$

16. $x = \ln u,\ y = \ln(uv)$, where $S = \{(u, v) \mid u > 0,\ v > 0\}$ and $R = R^2$.

Answer

$u = e^x,\ v = e^{-x+y}$

17. $x = u + v + w,\ y = 3v,\ z = 2w$, where $S = R = R^3$.

18. $x = u + v,\ y = v + w,\ z = u + w$, where $S = R = R^3$.

Answer

$u = \frac{x-y+z}{2},\ v = \frac{x+y-z}{2},\ w = \frac{-x+y+z}{2}$

Exercises 19 - 22

The transformation $T : S \to R,\ T(u, v) = (x, y)$ and the region $R \subset R^2$ are given. Find the region $S \subset R^2$.

19. $x = au,\ y = bv,\ R = \{(x, y) \mid x^2 + y^2 \le a^2b^2\}$ where $a, b > 0$

20. $x = au,\ y = bc,\ R = \{(x, y) \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1\}$, where $a, b > 0$

Answer

$S = \{(u, v) \mid u^2 + v^2 \le 1\}$

21. $x = \frac{u}{a},\ y = \frac{v}{b},\ z = \frac{w}{c},\ R = \{(x, y) \mid x^2 + y^2 + z^2 \le 1\}$, where $a, b, c > 0$

22. $x = au,\ y = bv,\ z = cw,\ R = \{(x, y) \mid \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} \le 1,\ z > 0\}$, where $a, b, c > 0$

Answer

$R = \{(u, v, w) \mid u^2 - v^2 - w^2 \le 1,\ w > 0\}$

Exercises 23 - 32

Find the Jacobian J of the transformation.

23. $x = u + 2v,\ y = -u + v$

24. $x = \frac{u^3}{2},\ y = \frac{v}{u^2}$

Answer

$\frac{3}{2}$

25. $x = e^{2u-v},\ y = e^{u+v}$

26. $x = ue^v,\ y = e^{-v}$

Answer

-1

27. $x = u\ \cos(e^v),\ y = u\ \sin(e^v)$

28. $x = v\ \sin(u^2),\ y = v\ \cos(u^2)$

Answer

$2uv$

29. $x = u\ \cosh v,\ y = u\ \sinh v,\ z = w$

30. $x = v\ \cosh\left(\frac{1}{u}\right),\ y = v\ \sinh\left(\frac{1}{u}\right),\ z = u + w^2$

Answer

$\frac{v}{u^2}$

31. $x = u + v,\ y = v + w,\ z = u$

32. $x = u - v,\ y = u + v,\ z = u + v + w$

Answer

2

33. The triangular region R with the vertices $(0,0),\ (1,1)$, and $(1,2)$ is shown in the figure $2.8E.3$.

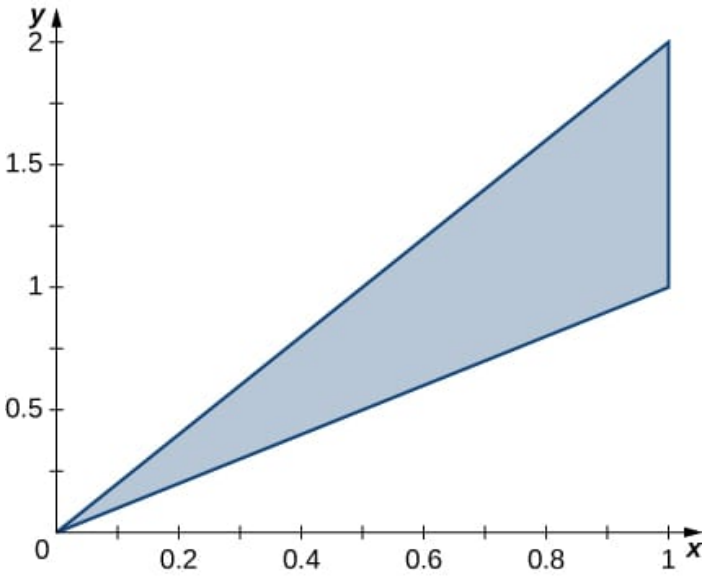


Figure $2.8E.3$: The graph illustrates the triangular region R with the vertices $(0,0),\ (1,1)$, and $(1,2)$.

a. Find a transformation $T : S \to R,\ T(u,v) = (x,y) = (au + bv + dv)$, where a, b, c, and d are real numbers with $ad - bc \neq 0$ such that $T^{-1}(0,0) = (0,0),\ T^{-1}(1,1) = (1,0)$, and $T^{-1}(1,2) = (0,1)$.

b. Use the transformation T to find the area $A(R)$ of the region R.

34. The triangular region R with the vertices $(0,0),\ (2,0)$, and $(1,3)$ is shown in the figure $2.8E.4$.

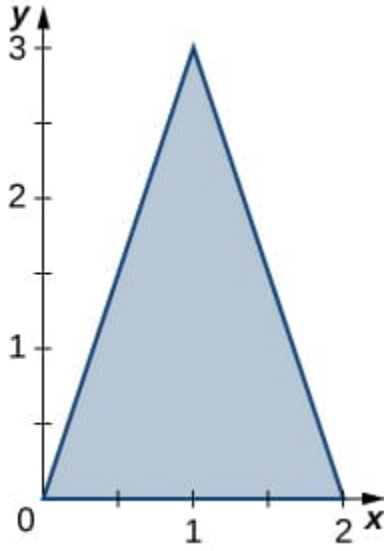


Figure $2.8E.4$: The graph illustrates the triangular region R with the vertices $(0,0),\ (2,0)$, and $(1,3)$.

a. Find a transformation $T : S \to R,\ T(u,v) = (x,y) = (au + bv + dv)$, where a, b, c, and d are real numbers with $ad - bc \neq 0$ such that $T^{-1}(0,0) = (0,0),\ T^{-1}(2,0) = (1,0)$, and $T^{-1}(1,3) = (0,1)$.

b. Use the transformation T to find the area $A(R)$ of the region R.

Answer

a. $T(u,v) = (2u + v,\ 3v)$

b. The area of R is $A(R) = \int_0^3 \int_{y/3}^{(6-y)/3} dx\ dy = \int_0^1 \int_0^{1-u} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dv\ du = \int_0^1 \int_0^{1-u} 6\ dv\ du = 3.$

Exercises 35 - 36

Use the transformation $u = y - x$, $v = y$, to evaluate the integrals on the parallelogram R of vertices $(0,0)$, $(1,0)$, $(2,1)$, and $(1,1)$ shown in the figure $2.8E.5$.

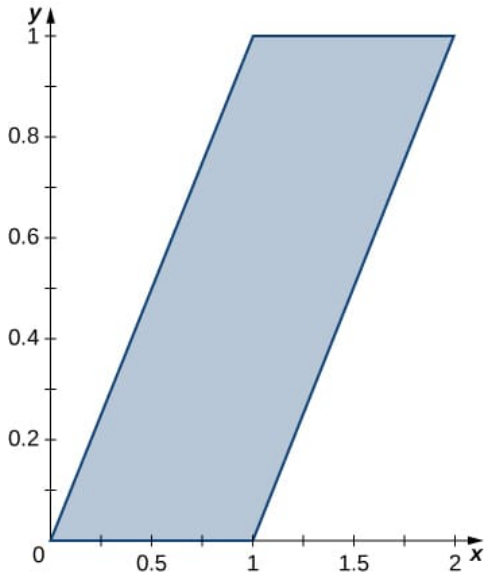


Figure $2.8E.5$: The graph illustrates the region R with vertices $(0,0)$, $(1,0)$, $(2,1)$, and $(1,1)$.

35. $\iint_R (y-x)\ dA$

36. $\iint_R (y^2-xy)\ dA$

Answer

$-\frac{1}{4}$

Exercises 37 - 38

Use the transformation $y-x=u$, $x+y=v$ to evaluate the integrals on the square R determined by the lines $y=x$, $y=-x+2$, $y=x+2$, and $y=-x$ shown in the figure $2.8E.6$.

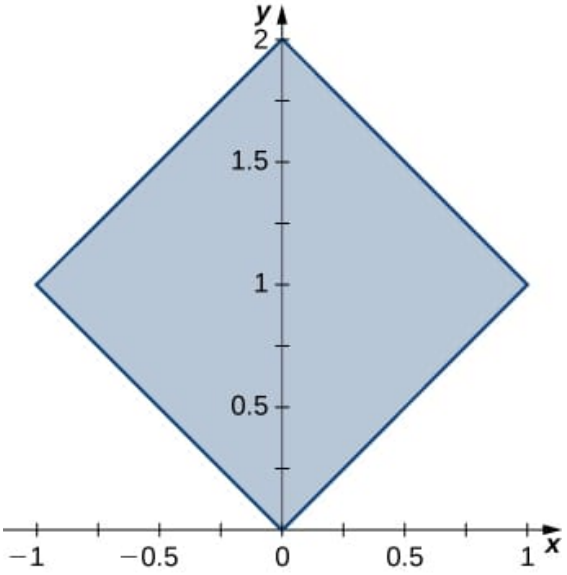


Figure $2.8E.6$: The graph illustrates the region R determined by the lines $y=x$, $y=-x+2$, $y=x+2$, and $y=-x$.

37. $\iint_R e^{x+y}\ dA$

38. $\iint_R \sin(x-y)\ dA$

Answer

$-1+\cos 2$

Exercises 39 - 40

Use the transformation $x=u$, $5y=v$ to evaluate the integrals on the region R bounded by the ellipse $x^2+25y^2=1$ shown in the figure $2.8E.7$.

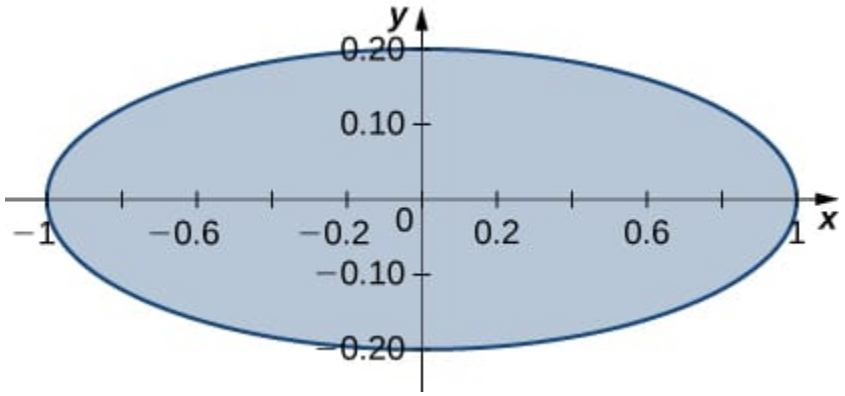


Figure $2.8E.7$: The graph illustrates the region R bounded by the ellipse $x^2+25y^2=1$.

39. $\iint_R \sqrt{x^2+25y^2}\ dA$

40. $\iint_R (x^2+25y^2)^2\ dA$

Answer

$\frac{\pi}{15}$

Exercises 41 - 42

Use the transformation $u=x+y$, $v=x-y$ to evaluate the integrals on the trapezoidal region R determined by the points $(1,0)$, $(2,0)$, $(0,2)$, and $(0,1)$ shown in the figure $2.8E.8$.

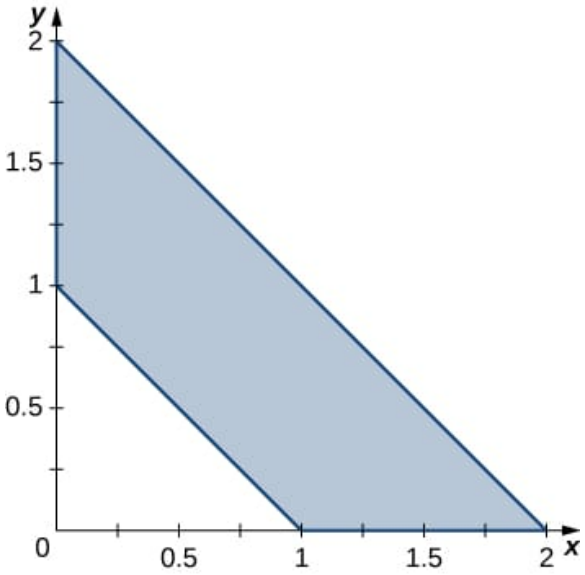


Figure $2.8E.8$: The graph illustrates the region R determined by the points $(1,0)$, $(2,0)$, $(0,2)$, and $(0,1)$.

41. $\iint_R (x^2 - 2xy + y^2)\ e^{x+y}\ dA$

42. $\iint_R (x^3 + 3x^2y + 3xy^2 + y^3)\ dA$

Answer

$\frac{31}{5}$

43. The circular annulus sector R bounded by the circles $4x^2 + 4y^2 = 1$ and $9x^2 + 9y^2 = 64$, the line $x = y\sqrt{3}$, and the y-axis is shown in the figure $2.8E.9$. Find a transformation T from a rectangular region S in the $r\theta$-plane to the region R in the xy-plane. Graph S.

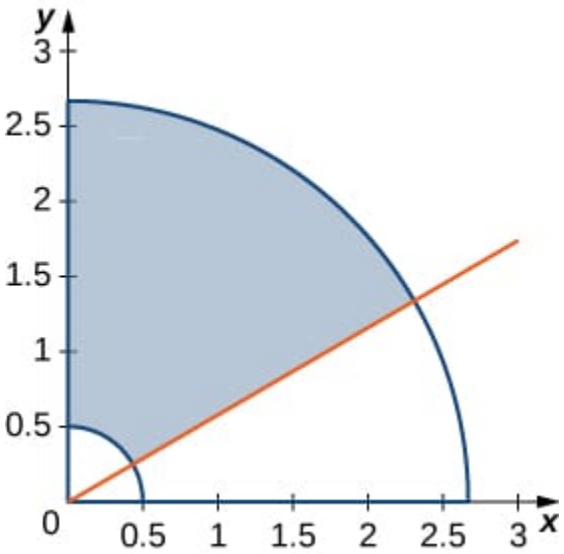


Figure $2.8E.9$: The graph illustrates the region R bounded by the circles $4x^2 + 4y^2 = 1$ and $9x^2 + 9y^2 = 64$, the line $x = y\sqrt{3}$, and the y-axis.

44. The solid R bounded by the circular cylinder $x^2 + y^2 = 9$ and the planes $z = 0,\ z = 1,\ x = 0$, and $y = 0$ is shown in the figure $2.8E.10$. Find a transformation T from a cylindrical box S in $r\theta z$-space to the solid R in xyz-space.

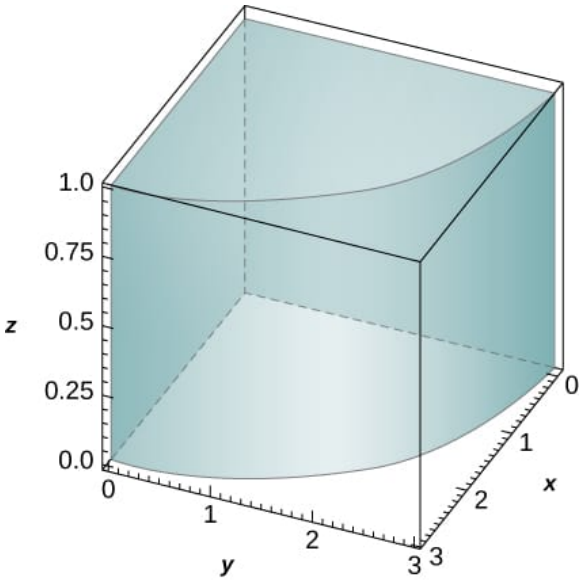


Figure $2.8E.10$: The graph illustrates the solid R bounded by the circular cylinder $x^2 + y^2 = 9$ and the planes $z = 0,\ z = 1,\ x = 0$, and $y = 0$.

Answer

$T(r, \theta, z) = (r\ \cos\theta,\ r\ \sin\theta,\ z);\ S = [0, 3] \times [0, \frac{\pi}{2}] \times [0, 1]$ in the $r\theta z$-space

45. Show that

$$\iint_R f\left(\sqrt{\frac{x^2}{3} + \frac{y^2}{3}}\right) dA = 2\pi\sqrt{15} \int_0^1 f(\rho)\rho\ d\rho,$$

where f is a continuous function on $[0, 1]$ and R is the region bounded by the ellipse $5x^2 + 3y^2 = 15$.

46. Show that

$$\iiint_R f\left(\sqrt{16x^2 + 4y^2 + z^2}\right) dV = \frac{\pi}{2} \int_0^1 f(\rho)\rho^2 d\rho,$$

where f is a continuous function on $[0, 1]$ and R is the region bounded by the ellipsoid $16x^2 + 4y^2 + z^2 = 1$.

47. [T] Find the area of the region bounded by the curves $xy = 1,\ xy = 3,\ y = 2x$, and $y = 3x$ by using the transformation $u = xy$ and $v = \frac{y}{x}$. Use a computer algebra system (CAS) to graph the boundary curves of the region R.

48. [T] Find the area of the region bounded by the curves $x^2y = 2,\ x^2y = 3,\ y = x$, and $y = 2x$ by using the transformation $u = x^2y$ and $v = \frac{y}{x}$. Use a CAS to graph the boundary curves of the region R.

Answer

The area of R is $10 - 4\sqrt{6}$; the boundary curves of R are graphed in the figure $2.8E.11$.

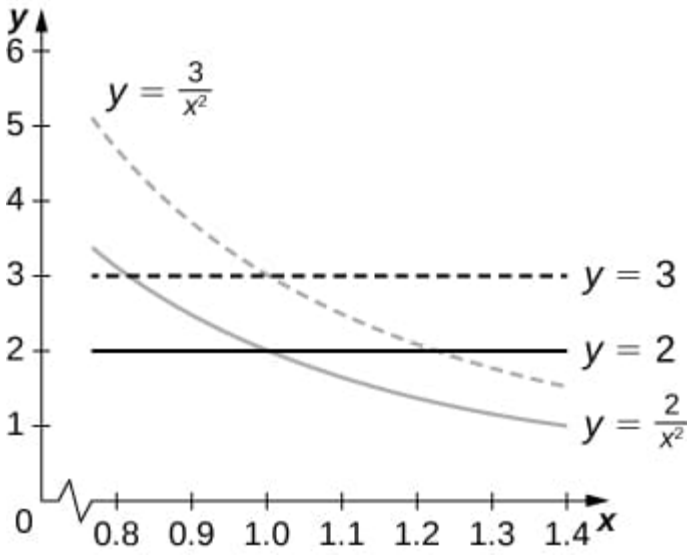


Figure $2.8E.11$: The graph illustrates the boundary curves of R.

49. Evaluate the triple integral

$$\int_0^1 \int_1^2 \int_z^{z+1} (y+1)\ dx\ dy\ dz$$

by using the transformation $u = x - z,\ v = 3y$, and $w = \frac{z}{2}$.

50. Evaluate the triple integral

$$\int_0^2 \int_4^6 \int_{3z}^{3z+2} (5-4y)\ dx\ dy\ dz$$

by using the transformation $u = x - 3z,\ v = 4y$, and $w = z$.

Answer

8

51. A transformation $T : R^2 \to R^2,\ T(u,v) = (x,y)$ of the form $x = au + bv,\ y = cu + dv$, where a, b, c, and d are real numbers, is called linear. Show that a linear transformation for which $ad - bc \neq 0$ maps parallelograms to parallelograms.

52. A transformation $T_\theta : R^2 \to R^2,\ T_\theta(u,v) = (x,y)$ of the form $x = u\ \cos\theta - v\ \sin\theta,\ y = u\ \sin\theta + v\ \cos\theta$, is called a rotation angle θ. Show that the inverse transformation of T_θ satisfies $T_\theta^{-1} = T_{-\theta}$ where $T_{-\theta}$ is the rotation of angle $-\theta$.

53. [T] Find the region S in the uv-plane whose image through a rotation of angle $\frac{\pi}{4}$ is the region R enclosed by the ellipse $x^2 + 4y^2 = 1$. Use a CAS to answer the following questions.

a. Graph the region S.

b. Evaluate the integral $\iint_S e^{-2uv}\ du\ dv$. Round your answer to two decimal places.

54. [T] The transformations $T_i : \mathbb{R}^2 \to \mathbb{R}^2,\ i = 1, ..., 4$, defined by $T_1(u,v) = (u,-v),\ T_2(u,v) = (-u,v),\ T_3(u,v) = (-u,-v)$, and $T_4(u,v) = (v,u)$ are called reflections about the x-axis, y-axis origin, and the line $y = x$, respectively.

1. Find the image of the region $S = \{(u,v) \mid u^2 + v^2 - 2u - 4v + 1 \leq 0\}$ in the xy-plane through the transformation $T_1 \circ T_2 \circ T_3 \circ T_4$.

2. Use a CAS to graph R.

3. Evaluate the integral $\iint_S \sin(u^2)\ du\ dv$ by using a CAS. Round your answer to two decimal places.

Answer

a. $R = \{(x, y) \mid y^2 + x^2 - 2y - 4x + 1 \leq 0\}$;
b. R is graphed in the figure $2.8E.12$;

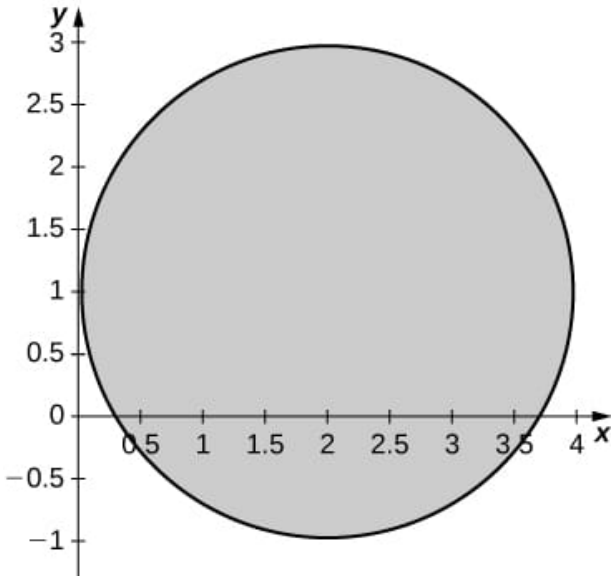


Figure $2.8E.12$: The graph illustrates the region R.

c. 3.16

55. [T] The transformations $T_{k,1,1} : \mathbb{R}^3 \to \mathbb{R}^3,\ T_{k,1,1}(u, v, w) = (x, y, z)$ of the form $x = ku,\ y = v,\ z = w$, where $k \neq 1$ is a positive real number, is called a stretch if $k > 1$ and a compression if $0 < k < 1$ in the x-direction. Use a CAS to evaluate the integral $\iiint_S e^{-(4x^2+9y^2+25z^2)}\ dx\ dy\ dz$ on the solid $S = \{(x, y, z) \mid 4x^2 + 9y^2 + 25z^2 \leq 1\}$ by considering the compression $T_{2,3,5}(u, v, w) = (x, y, z)$ defined by $x = \frac{u}{2},\ y = \frac{v}{3}$, and $z = \frac{w}{5}$. Round your answer to four decimal places.

56. [T] The transformation $T_{a,0} : \mathbb{R}^2 \to \mathbb{R}^2,\ T_{a,0}(u, v) = (u + av, v)$, where $a \neq 0$ is a real number, is called a shear in the x-direction. The transformation, $T_{b,0} : R^2 \to R^2,\ T_{o,b}(u, v) = (u, bu + v)$, where $b \neq 0$ is a real number, is called a shear in the y-direction.

a. Find transformations $T_{0,2} \circ T_{3,0}$.

b. Find the image R of the trapezoidal region S bounded by $u = 0,\ v = 0,\ v = 1$, and $v = 2 - u$ through the transformation $T_{0,2} \circ T_{3,0}$.

c. Use a CAS to graph the image R in the xy-plane.

d. Find the area of the region R by using the area of region S.

Answer

a. $T_{0,2} \circ T_{3,0}(u, v) = (u + 3v, 2u + 7v)$;

b. The image S is the quadrilateral of vertices $(0, 0),\ (3, 7),\ (2, 4)$, and $(4, 9)$;

c. S is graphed in the figure $2.8E.13$;

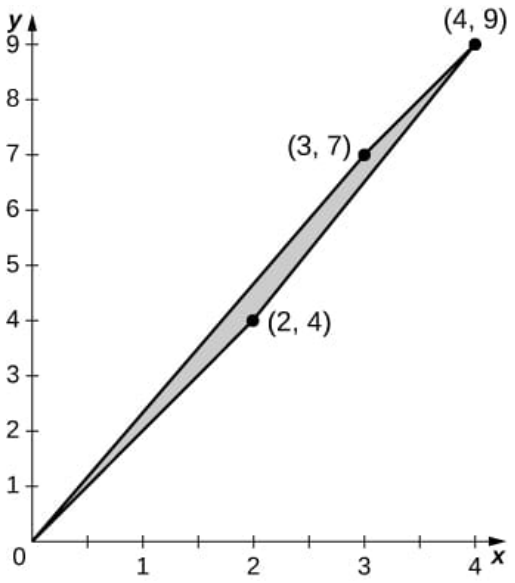


Figure $2.8E.13$: The graph illustrates the region S.

4. $\frac{3}{2}$

57. Use the transformation, $x = au,\ y = av,\ z = cw$ and spherical coordinates to show that the volume of a region bounded by the spheroid $\frac{x^2+y^2}{a^2} + \frac{z^2}{c^2} = 1$ is $\frac{4\pi a^2 c}{3}$.

58. Find the volume of a football whose shape is a spheroid $\frac{x^2+y^2}{a^2} + \frac{z^2}{c^2} = 1$ whose length from tip to tip is 11 inches and circumference at the center is 22 inches. Round your answer to two decimal places.

Answer

$\frac{2662}{3\pi} \approx 282.45\ in^3$

Contributors

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2.9: Chapter 2 Review Exercises

Exercises 1 - 4

***True or False?* Justify your answer with a proof or a counterexample.**

1. $\int_a^b \int_c^d f(x,y)\ dy\ dx = \int_c^d \int_a^b f(x,y)\ dy\ dx$

2. Fubini's theorem can be extended to three dimensions, as long as f is continuous in all variables.

Answer

True

3. The integral $\int_0^{2\pi} \int_0^1 \int_r^1 dz\ dr\ d\theta$ represents the volume of a right cone.

4. The Jacobian of the transformation for $x = u^2 - 2v,\ y = 3v - 2uv$ is given by $-4u^2 + 6u + 4v$.

Answer

False

Exercises 5 - 13

Evaluate the following integrals.

5. $\iint_R (5x^3y^2 - y^2)\ dA$, where $R = \{(x,y) \mid 0 \le x \le 2,\ 1 \le y \le 4\}$

6. $\iint_D \frac{y}{3x^2+1}\ dA$, where $D = \{(x,y) \mid 0 \le x \le 1,\ -x \le y \le x\}$

Answer

0

7. $\iint_D \sin(x^2+y^2)\ dA$ where D is a disk of radius 2 centered at the origin.

8. $\int_0^1 \int_y^1 xye^{x^2}\ dx\ dy$

Answer

$\frac{1}{4}$

9. $\int_{-1}^1 \int_0^z \int_0^{x-z} 6\ dy\ dx\ dz$

10. $\iiint_R 3y\ dV$, where $R = \{(x,y,z) \mid 0 \le x \le 1,\ 0 \le y \le x,\ 0 \le z \le 9 - y^2\}$

Answer

1.475

11. $\int_0^2 \int_0^{2\pi} \int_r^1 r\ dz\ d\theta\ dr$

12. $\int_0^{2\pi}\int_0^{\pi/2}\int_1^3 \rho^2\sin(\varphi)\ d\rho\ d\varphi\ d\theta$

Answer

$\frac{52\pi}{3}$

13. $\int_0^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dz\ dy\ dx$

Exercises 14 - 17

For the following problems, find the specified area or volume.

14. The area of region enclosed by one petal of $r=\cos(4\theta)$.

Answer

$\frac{\pi}{16}\ \text{units}^3$

15. The volume of the solid that lies between the paraboloid $z=2x^2+2y^2$ and the plane $z=8$.

16. The volume of the solid bounded by the cylinder $x^2+y^2=16$ and from $z=1$ to $z+x=2$.

Answer

$93.291\ \text{units}^3$

17. The volume of the intersection between two spheres of radius 1, the top whose center is $(0,\ 0,\ 0.25)$ and the bottom, which is centered at $(0,\ 0,\ 0)$.

Exercises 18 - 21

For the following problems, find the center of mass of the region.

18. $\rho(x,y)=xy$ on the circle with radius 1 in the first quadrant only.

Answer

$\left(\frac{8}{15},\ \frac{8}{15}\right)$

19. $\rho(x,y)=(y+1)\sqrt{x}$ in the region bounded by $y=e^x,\ y=0$, and $x=1$.

20. $\rho(x,y,z)=z$ on the inverted cone with radius 2 and height 2.

Answer

$\left(0,\ 0,\ \frac{8}{5}\right)$

21. The volume an ice cream cone that is given by the solid above $z=\sqrt{x^2+y^2}$ and below $z^2+x^2+y^2=z$.

Exercises 22 - 23

The following problems examine Mount Holly in the state of Michigan. Mount Holly is a landfill that was converted into a ski resort. The shape of Mount Holly can be approximated by a right circular cone of height 1100 ft and radius 6000 ft.

22. If the compacted trash used to build Mount Holly on average has a density $400\ \text{lb}/\text{ft}^3$, find the amount of work required to build the mountain.

Answer

$1.452\pi\times10^{15}$ ft-lb

23. In reality, it is very likely that the trash at the bottom of Mount Holly has become more compacted with all the weight of the above trash. Consider a density function with respect to height: the density at the top of the mountain is still density $400\ \text{lb/ft}^3$, and the density increases. Every 100 feet deeper, the density doubles. What is the total weight of Mount Holly?

Exercises 24 - 25

The following problems consider the temperature and density of Earth's layers.

24. The temperature of Earth's layers is exhibited in the table below. Use your calculator to fit a polynomial of degree 3 to the temperature along the radius of the Earth. Then find the average temperature of Earth. (*Hint*: begin at 0 in the inner core and increase outward toward the surface)

Table 2.9.1: The temperature of Earth's layers is exhibited in the table.

Layer	Depth from center (km)	Temperature °C
Rocky Crust	0 to 40	0
Upper Mantle	40 to 150	870
Mantle	400 to 650	870
Inner Mantel	650 to 2700	870
Molten Outer Core	2890 to 5150	4300
Inner Core	5150 to 6378	7200

Answer

$y = -1.238 \times 10^{-7}x^3 + 0.001196x^2 - 3.666x + 7208$;
The average temperature is approximately 2800 °C.

25. The density of Earth's layers is displayed in the table below. Using your calculator or a computer program, find the best-fit quadratic equation to the density. Using this equation, find the total mass of Earth.

Table 2.9.2: The table includes density of Earth's layers.

Layer	Depth from center (km)	Density (g/cm^3)
Inner Core	0	12.95
Outer Core	1228	11.05
Mantle	3488	5.00
Upper Mantle	6338	3.90
Crust	6378	2.55

Exercises 26 - 27

The following problems concern the Theorem of Pappus (see Moments and Centers of Mass for a refresher), a method for calculating volume using centroids. Assuming a region R, when you revolve around the x-axis the volume is given by $V_x = 2\pi A\bar{y}$, and when you revolve around the y-axis the volume is given by $V_y = 2\pi A\bar{x}$, where A is the area of R. Consider the region bounded by $x^2 + y^2 = 1$ and above $y = x + 1$.

26. Find the volume when you revolve the region around the x-axis.

Answer

$\frac{\pi}{3}\ \text{units}^3$

27. Find the volume when you revolve the region around the y-axis.

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3: VECTOR CALCULUS

In this chapter, we learn to model new kinds of integrals over fields such as magnetic fields, gravitational fields, or velocity fields. We also learn how to calculate the work done on a charged particle traveling through a magnetic field, the work done on a particle with mass traveling through a gravitational field, and the volume per unit time of water flowing through a net dropped in a river. All these applications are based on the concept of a vector field. The content in this Textmap's chapter is complemented by Vector Calculus Modules in the Core and the Vector Calculus (UCD Mat 21D) Libretext.

Hurricanes are huge storms that can produce tremendous amounts of damage to life and property, especially when they reach land. Predicting where and when they will strike and how strong the winds will be is of great importance for preparing for protection or evacuation. Scientists rely on studies of rotational vector fields for their forecasts.



Figure 3.1: Hurricanes form from rotating winds driven by warm temperatures over the ocean. Meteorologists forecast the motion of hurricanes by studying the rotating vector fields of their wind velocity. Shown is Cyclone Catarina in the South Atlantic Ocean in 2004, as seen from the International Space Station. (CC-BY-2.0 modification of work by NASA)

In this chapter, we learn to model new kinds of integrals over fields such as magnetic fields, gravitational fields, or velocity fields. We also learn how to calculate the work done on a charged particle traveling through a magnetic field, the work done on a particle with mass traveling through a gravitational field, and the volume per unit time of water flowing through a net dropped in a river.

All these applications are based on the concept of a vector field, which we explore in this chapter. Vector fields have many applications because they can be used to model real fields such as electromagnetic or gravitational fields. A deep understanding of physics or engineering is impossible without an understanding of vector fields. Furthermore, vector fields have mathematical properties that are worthy of study in their own right. In particular, vector fields can be used to develop several higher-dimensional versions of the Fundamental Theorem of Calculus.

3.1: Vector Fields

3.1E: Exercises

3.2: Line Integrals

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3.1: Vector Fields

Learning Objectives

- Recognize a vector field in a plane or in space.
- Sketch a vector field from a given equation.
- Identify a conservative field and its associated potential function.

Vector fields are an important tool for describing many physical concepts, such as gravitation and electromagnetism, which affect the behavior of objects over a large region of a plane or of space. They are also useful for dealing with large-scale behavior such as atmospheric storms or deep-sea ocean currents. In this section, we examine the basic definitions and graphs of vector fields so we can study them in more detail in the rest of this chapter.

Examples of Vector Fields

How can we model the gravitational force exerted by multiple astronomical objects? How can we model the velocity of water particles on the surface of a river? Figure $3.1.1$ gives visual representations of such phenomena.

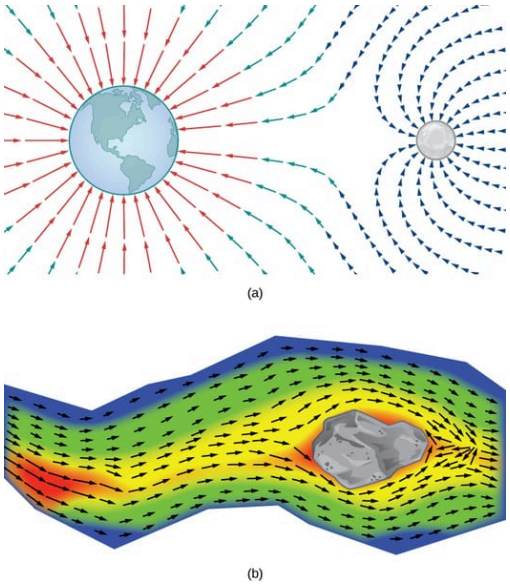


Figure $3.1.1$ (a) The gravitational field exerted by two astronomical bodies on a small object. (b) The vector velocity field of water on the surface of a river shows the varied speeds of water. Red indicates that the magnitude of the vector is greater, so the water flows more quickly; blue indicates a lesser magnitude and a slower speed of water flow.

Figure $3.1.1$ (a) shows a gravitational field exerted by two astronomical objects, such as a star and a planet or a planet and a moon. At any point in the figure, the vector associated with a point gives the net gravitational force exerted by the two objects on an object of unit mass. The vectors of largest magnitude in the figure are the vectors closest to the larger object. The larger object has greater mass, so it exerts a gravitational force of greater magnitude than the smaller object.

Figure $3.1.1$ (b) shows the velocity of a river at points on its surface. The vector associated with a given point on the river's surface gives the velocity of the water at that point. Since the vectors to the left of the figure are small in magnitude, the water is flowing slowly on that part of the surface. As the water moves from left to right, it encounters some rapids around a rock. The speed of the water increases, and a whirlpool occurs in part of the rapids.

Each figure illustrates an example of a vector field. Intuitively, a vector field is a map of vectors. In this section, we study vector fields in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Definition: Vector Field

- A vector field $\mathbf{F}$ in $\mathbb{R}^2$ is an assignment of a two-dimensional vector $\mathbf{F}(x, y)$ to each point (x, y) of a subset D of $\mathbb{R}^2$. The subset D is the domain of the vector field.
- A vector field $\mathbf{F}$ in $\mathbb{R}^3$ is an assignment of a three-dimensional vector $\mathbf{F}(x, y, z)$ to each point (x, y, z) of a subset D of $\mathbb{R}^3$. The subset D is the domain of the vector field.

Vector Fields in $\mathbb{R}^2$

A vector field in $\mathbb{R}^2$ can be represented in either of two equivalent ways. The first way is to use a vector with components that are two-variable functions:

$$\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle \tag{3.1.1}$$

The second way is to use the standard unit vectors:

$$\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}. \tag{3.1.2}$$

A vector field is said to be *continuous* if its component functions are continuous.

Example 3.1.1: Finding a Vector Associated with a Given Point

Let $\mathbf{F}(x, y) = (2y^2 + x - 4)\ \mathbf{i} + \cos(x)\ \mathbf{j}$ be a vector field in $\mathbb{R}^2$. Note that this is an example of a continuous vector field since both component functions are continuous. What vector is associated with point $(0, -1)$?

Solution

Substitute the point values for x and y:

$$\begin{aligned} \mathbf{F}(0, -1) &= (2(-1)^2 + 0 - 4)\,\mathbf{i} + \cos(0)\,\mathbf{j} \\ &= -2\,\mathbf{i} + \mathbf{j}. \end{aligned}$$

Exercise 3.1.1

Let $\mathbf{G}(x, y) = x^2 y\ \mathbf{i} - (x + y)\ \mathbf{j}$ be a vector field in $\mathbb{R}^2$. What vector is associated with the point $(-2, 3)$?

Hint available at the end of this section.

Drawing a Vector Field

We can now represent a vector field in terms of its components of functions or unit vectors, but representing it visually by sketching it is more complex because the domain of a vector field is in $\mathbb{R}^2$, as is the range. Therefore the "graph" of a vector field in $\mathbb{R}^2$ lives in four-dimensional space. Since we cannot represent four-dimensional space visually, we instead draw vector fields in $\mathbb{R}^2$ in a plane itself. To do this, draw the vector associated with a given point at the point in a plane. For example, suppose the vector associated with point $(4, -1)$ is $\langle 3, 1 \rangle$. Then, we would draw vector $\langle 3, 1 \rangle$ at point $(4, -1)$.

We should plot enough vectors to see the general shape, but not so many that the sketch becomes a jumbled mess. If we were to plot the image vector at each point in the region, it would fill the region completely and would be useless. Instead, we can choose points at the intersections of grid lines and plot a sample of several vectors from each quadrant of a rectangular coordinate system in $\mathbb{R}^2$.

There are two types of vector fields in $\mathbb{R}^2$ on which this chapter focuses: radial fields and rotational fields. Radial fields model certain gravitational fields and energy source fields, and rotational fields model the movement of a fluid in a vortex. In a radial

field, all vectors either point directly toward or directly away from the origin. Furthermore, the magnitude of any vector depends only on its distance from the origin. In a radial field, the vector located at point (x, y) is perpendicular to the circle centered at the origin that contains point (x, y), and all other vectors on this circle have the same magnitude.

Example 3.1.2: Drawing a Radial Vector Field

Sketch the vector field $\mathbf{F}(x, y) = \frac{x}{2}\mathbf{i} + \frac{y}{2}\mathbf{j}$.

Solution

To sketch this vector field, choose a sample of points from each quadrant and compute the corresponding vector. Table 3.1.1 gives a representative sample of points in a plane and the corresponding vectors.

Table 3.1.1: Shows a sample of points in the plane and their corresponding vectors for the vector field $\mathbf{F}(x, y) = \frac{x}{2}\mathbf{i} + \frac{y}{2}\mathbf{j}$. Each row contains a specific point (x, y) and the computed vector $\mathbf{F}(x, y)$ at that point. The table includes points from different quadrants to illustrate the behavior of the vector field across the plane.

(x, y)	$\mathbf{F}(x, y)$	(x, y)	$\mathbf{F}(x, y)$	(x, y)	$\mathbf{F}(x, y)$
$(1, 0)$	$\langle \frac{1}{2}, 0 \rangle$	$(2, 0)$	$\langle 1, 0 \rangle$	$(1, 1)$	$\langle \frac{1}{2}, \frac{1}{2} \rangle$
$(0, 1)$	$\langle 0, \frac{1}{2} \rangle$	$(0, 2)$	$\langle 0, 1 \rangle$	$(-1, 1)$	$\langle -\frac{1}{2}, \frac{1}{2} \rangle$
$(-1, 0)$	$\langle -\frac{1}{2}, 0 \rangle$	$(-2, 0)$	$\langle -1, 0 \rangle$	$(-1, -1)$	$\langle -\frac{1}{2}, -\frac{1}{2} \rangle$
$(0, -1)$	$\langle 0, -\frac{1}{2} \rangle$	$(0, -2)$	$\langle 0, -1 \rangle$	$(1, -1)$	$\langle \frac{1}{2}, -\frac{1}{2} \rangle$

Figure 3.1.2(a) shows the vector field. To see that each vector is perpendicular to the corresponding circle, Figure 3.1.2(b) shows circles overlain on the vector field.

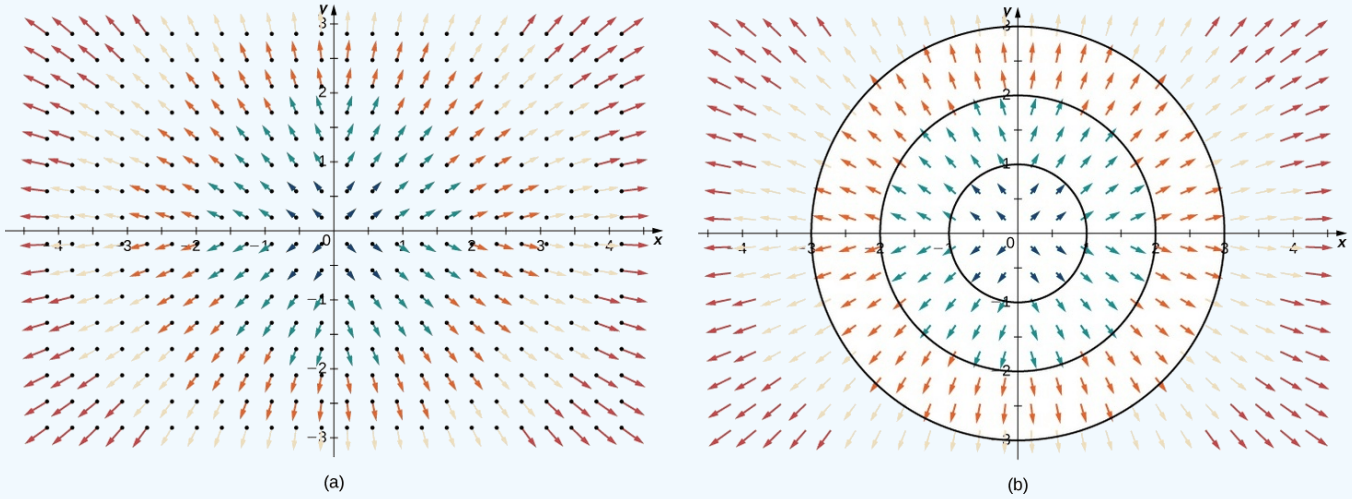


Figure 3.1.2: (a) A visual representation of the radial vector field $\mathbf{F}(x, y) = \frac{x}{2}\mathbf{i} + \frac{y}{2}\mathbf{j}$. (b) The radial vector field $\mathbf{F}(x, y) = \frac{x}{2}\mathbf{i} + \frac{y}{2}\mathbf{j}$ with overlaid circles. Notice that each vector is perpendicular to the circle on which it is located.

Exercise 3.1.2

Draw the radial field $\mathbf{F}(x, y) = -\frac{x}{3}\mathbf{i} - \frac{y}{3}\mathbf{j}$.

Hint available at the end of this section.

In contrast to radial fields, in a *rotational field*, the vector at point (x, y) is tangent (not perpendicular) to a circle with radius $r = \sqrt{x^2 + y^2}$. In a standard rotational field, all vectors point either in a clockwise direction or in a counterclockwise direction, and the magnitude of a vector depends only on its distance from the origin. Both of the following examples are clockwise rotational fields, and we see from their visual representations that the vectors appear to rotate around the origin.

Example 3.1.3: Drawing a Rotational Vector Field

Sketch the vector field $\mathbf{F}(x, y) = \langle y, \ -x \rangle$.

Solution

Create a table using a representative sample of points in a plane and their corresponding vectors. Figure 3.1.4 shows the resulting vector field.

Table 3.1.2: Shows the vector field for $\mathbf{F}(x, y) = \langle y, \ -x \rangle$.

(x, y)	$\mathbf{F}(x, y)$	(x, y)	$\mathbf{F}(x, y)$	(x, y)	$\mathbf{F}(x, y)$
$(1, 0)$	$\langle 0, -1 \rangle$	$(2, 0)$	$\langle 0, -2 \rangle$	$(1, 1)$	$\langle 1, -1 \rangle$
$(0, 1)$	$\langle 1, 0 \rangle$	$(0, 2)$	$\langle 2, 0 \rangle$	$(-1, 1)$	$\langle 1, 1 \rangle$
$(-1, 0)$	$\langle 0, 1 \rangle$	$(-2, 0)$	$\langle 0, 2 \rangle$	$(-1, -1)$	$\langle -1, 1 \rangle$
$(0, -1)$	$\langle -1, 0 \rangle$	$(0, -2)$	$\langle -2, 0 \rangle$	$(1, -1)$	$\langle -1, -1 \rangle$

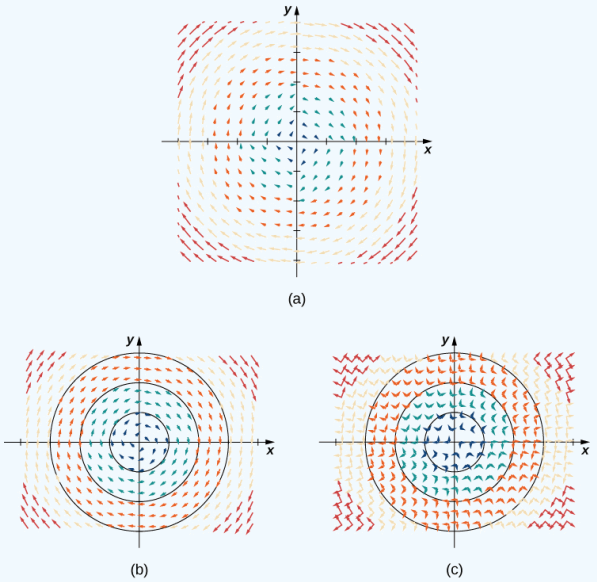


Figure 3.1.4: (a) A visual representation of vector field $\mathbf{F}(x, y) = \langle y, -x \rangle$. (b) Vector field $\mathbf{F}(x, y) = \langle y, -x \rangle$ with circles centered at the origin. (c) Vector $\mathbf{F}(a, b)$ is perpendicular to radial vector $\langle a, b \rangle$ at point (a, b).

Analysis

Note that vector $\mathbf{F}(a, b) = \langle b, -a \rangle$ points clockwise and is perpendicular to radial vector $\langle a, b \rangle$. (We can verify this assertion by computing the dot product of the two vectors: $\langle a, b \rangle \cdot \langle -b, a \rangle = -ab + ab = 0$.) Furthermore, vector $\langle b, -a \rangle$ has length $r = \sqrt{a^2 + b^2}$. Thus, we have a complete description of this rotational vector field: the vector associated with point (a, b) is the vector with length *r* tangent to the circle with radius *r*, and it points in the clockwise direction.

Sketches such as that in Figure 3.1.4 are often used to analyze major storm systems, including hurricanes and cyclones. In the northern hemisphere, storms rotate counterclockwise; in the southern hemisphere, storms rotate clockwise. (This is an effect caused by Earth's rotation about its axis and is called the Coriolis Effect.)



Figure 3.1.5: Hurricanes form from rotating winds driven by warm temperatures over the ocean. Meteorologists forecast the motion of hurricanes by studying the rotating vector fields of their wind velocity. Shown is Cyclone Catarina in the South Atlantic Ocean in 2004, as seen from the International Space Station. (CC-BY-2.0 modification of work by NASA)

Example 3.1.4: Sketching a Vector Field

Sketch vector field $\mathbf{F}(x,y) = \dfrac{y}{x^2+y^2}\mathbf{i} - \dfrac{x}{x^2+y^2}\mathbf{j}$.

Solution

To visualize this vector field, first note that the dot product $\mathbf{F}(a,b) \cdot (a\ \mathbf{i} + b\ \mathbf{j})$ is zero for any point (a,b). Therefore, each vector is tangent to the circle on which it is located. Also, as $(a,b) \to (0,0)$, the magnitude of $\mathbf{F}(a,b)$ goes to infinity. To see this, note that

$$||\mathbf{F}(a,b)|| = \sqrt{\frac{a^2+b^2}{(a^2+b^2)^2}} = \sqrt{\frac{1}{a^2+b^2}}.$$

Since $\dfrac{1}{a^2+b^2} \to \infty$ as $(a,b) \to (0,0)$, then $||\mathbf{F}(a,b)|| \to \infty$ as $(a,b) \to (0,0)$. This vector field looks similar to the vector field in Example 3.1.3, but in this case the magnitudes of the vectors close to the origin are large. Table 3.1.3 shows a sample of points and the corresponding vectors, and Figure 3.1.6 shows the vector field. Note that this vector field models the whirlpool motion of the river in Figure 3.1.6. The domain of this vector field is all of $\mathbb{R}^2$ except for point $(0,0)$.

Table 3.1.3: Shows the vector field for $\mathbf{F}(x,y) = \dfrac{y}{x^2+y^2}\mathbf{i} - \dfrac{x}{x^2+y^2}\mathbf{j}$.

(x,y)	$\mathbf{F}(x,y)$	(x,y)	$\mathbf{F}(x,y)$	(x,y)	$\mathbf{F}(x,y)$
$(1,0)$	$\langle 0,-1\rangle$	$(2,0)$	$\langle 0,-\frac{1}{2}\rangle$	$(1,1)$	$\langle \frac{1}{2},-\frac{1}{2}\rangle$
$(0,1)$	$\langle 1,0\rangle$	$(0,2)$	$\langle \frac{1}{2},0\rangle$	$(-1,1)$	$\langle \frac{1}{2},\frac{1}{2}\rangle$
$(-1,0)$	$\langle 0,1\rangle$	$(-2,0)$	$\langle 0,\frac{1}{2}\rangle$	$(-1,-1)$	$\langle -\frac{1}{2},\frac{1}{2}\rangle$
$(0,-1)$	$\langle -1,0\rangle$	$(0,-2)$	$\langle -\frac{1}{2},0\rangle$	$(1,-1)$	$\langle -\frac{1}{2},-\frac{1}{2}\rangle$

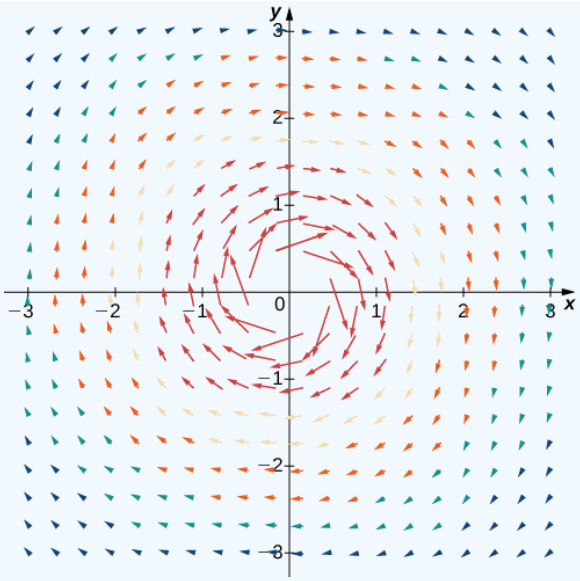


Figure 3.1.6: A visual representation of vector field $\mathbf{F}(x,y) = \frac{y}{x^2+y^2}\mathbf{i} - \frac{x}{x^2+y^2}\mathbf{j}$. *This vector field could be used to model whirlpool motion of a fluid.*

Exercise 3.1.4

Sketch vector field $\mathbf{F}(x,y) = \langle -2y,\ 2x\rangle$. Is the vector field radial, rotational, or neither?

Hint available at the end of this section.

Example 3.1.5: Velocity Field of a Fluid

Suppose that $\mathbf{v}(x,y) = -\frac{2y}{x^2+y^2}\mathbf{i} + \frac{2x}{x^2+y^2}\mathbf{j}$ is the velocity field of a fluid. How fast is the fluid moving at point $(1,-1)$? (Assume the units of speed are meters per second.)

Solution

To find the velocity of the fluid at point $(1,-1)$, substitute the point into $\mathbf{v}$:

$$\mathbf{v}(1,-1) = \frac{-2(-1)}{1+1}\mathbf{i} + \frac{2(1)}{1+1}\mathbf{j} = \mathbf{i} + \mathbf{j}.$$

The speed of the fluid at $(1,-1)$ is the magnitude of this vector. Therefore, the speed is $||\mathbf{i}+\mathbf{j}|| = \sqrt{2}$ m/sec.

Exercise 3.1.5

Vector field $\mathbf{v}(x,y) = \langle 4|x|,\ 1\rangle$ models the velocity of water on the surface of a river. What is the speed of the water at point $(2,3)$? Use meters per second as the units.

Hint available at the end of this section.

We have examined vector fields that contain vectors of various magnitudes, but just as we have unit vectors, we can also have a unit vector field. A vector field $\mathbf{F}$ is a **unit vector field** if the magnitude of each vector in the field is 1. In a unit vector field, the only relevant information is the direction of each vector.

Example 3.1.6: A Unit Vector Field

Show that vector field $\mathbf{F}(x,y) = \left\langle \frac{y}{\sqrt{x^2+y^2}}, -\frac{x}{\sqrt{x^2+y^2}} \right\rangle$ is a unit vector field.

Solution

To show that $\mathbf{F}$ is a unit field, we must show that the magnitude of each vector is 1. Note that

$$\begin{aligned}\sqrt{\left(\frac{y}{\sqrt{x^2+y^2}}\right)^2 + \left(-\frac{x}{\sqrt{x^2+y^2}}\right)^2} &= \sqrt{\frac{y^2}{x^2+y^2} + \frac{x^2}{x^2+y^2}} \\ &= \sqrt{\frac{x^2+y^2}{x^2+y^2}} \\ &= 1\end{aligned}$$

Therefore, $\mathbf{F}$ is a unit vector field.

Exercise 3.1.6

Is vector field $\mathbf{F}(x,y) = \langle -y,\ x\rangle$ a unit vector field?

Hint available at the end of this section.

Why are unit vector fields important? Suppose we are studying the flow of a fluid, and we care only about the direction in which the fluid is flowing at a given point. In this case, the speed of the fluid (which is the magnitude of the corresponding velocity vector) is irrelevant, because all we care about is the direction of each vector. Therefore, the unit vector field associated with velocity is the field we would study.

If $\mathbf{F} = \langle P, Q, R\rangle$ is a vector field, then the corresponding unit vector field is $\left\langle \frac{P}{||\mathbf{F}||}, \frac{Q}{||\mathbf{F}||}, \frac{R}{||\mathbf{F}||} \right\rangle$. Notice that if $\mathbf{F}(x,y) = \langle y,\ -x\rangle$ is the vector field from Example $3.1.6$, then the magnitude of $\mathbf{F}$ is $\sqrt{x^2+y^2}$, and therefore the corresponding unit vector field is the field $\mathbf{G}$ from the previous example.

If $\mathbf{F}$ is a vector field, then the process of dividing $\mathbf{F}$ by its magnitude to form unit vector field $\mathbf{F}/||\mathbf{F}||$ is called normalizing the field $\mathbf{F}$.

Vector Fields in $\mathbb{R}^3$

We have seen several examples of vector fields in $\mathbb{R}^2$; let's now turn our attention to vector fields in $\mathbb{R}^3$. These vector fields can be used to model gravitational or electromagnetic fields, and they can also be used to model fluid flow or heat flow in three dimensions. A two-dimensional vector field can really only model the movement of water on a two-dimensional slice of a river (such as the river's surface). Since a river flows through three spatial dimensions, to model the flow of the entire depth of the river, we need a vector field in three dimensions.

The extra dimension of a three-dimensional field can make vector fields in $\mathbb{R}^3$ more difficult to visualize, but the idea is the same. To visualize a vector field in $\mathbb{R}^3$, plot enough vectors to show the overall shape. We can use a similar method to visualizing a vector field in $\mathbb{R}^2$ by choosing points in each octant.

Just as with vector fields in $\mathbb{R}^2$, we can represent vector fields in $\mathbb{R}^3$ with component functions. We simply need an extra component function for the extra dimension. We write either

$$\mathbf{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle \tag{3.1.3}$$

or

$$\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}. \tag{3.1.4}$$

Example 3.1.7: Sketching a Vector Field in Three Dimensions

Describe vector field $\mathbf{F}(x,y,z) = \langle 1,\ 1,\ z\rangle$.

Solution

For this vector field, the x- and y-components are constant, so every point in $\mathbb{R}^3$ has an associated vector with x- and y -components equal to one. To visualize $\mathbf{F}$, we first consider what the field looks like in the xy-plane. In the xy-plane, $z = 0$. Hence, each point of the form $(a,b,0)$ has vector $\langle 1,1,0\rangle$ associated with it. For points not in the xy-plane but slightly above it, the associated vector has a small but positive z-component, and therefore the associated vector points slightly upward. For points that are far above the xy-plane, the z-component is large, so the vector is almost vertical. Figure 3.1.8 shows this vector field.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this Interactive Content online

Figure 3.1.8: A visual representation of vector field $\mathbf{F}(x,y,z) = \langle 1,\ 1,\ z\rangle$.

Exercise 3.1.7

Sketch vector field $\mathbf{G}(x,y,z) = \langle 2,\ \frac{z}{2},\ 1\rangle$.

Hint available at the end of this section.

Answer available at the end of this section.

In the next example, we explore one of the classic cases of a three-dimensional vector field: a gravitational field.

Example 3.1.8: Describing a Gravitational Vector Field

Newton's law of gravitation states that $\mathbf{F} = -G\frac{m_1 m_2}{r^2}\mathbf{r}$, where G is the universal gravitational constant. It describes the gravitational field exerted by an object (object 1) of mass m_1 located at the origin on another object (object 2) of mass m_2 located at point (x,y,z). Field $\mathbf{F}$ denotes the gravitational force that object 1 exerts on object 2, r is the distance between the two objects, and $\mathbf{r}$ indicates the unit vector from the first object to the second. The minus sign shows that the gravitational force attracts toward the origin; that is, the force of object 1 is attractive. Sketch the vector field associated with this equation.

Solution

Since object 1 is located at the origin, the distance between the objects is given by $r = \sqrt{x^2+y^2+z^2}$. The unit vector from object 1 to object 2 is $\mathbf{r} = \frac{\langle x,y,z\rangle}{||\langle x,y,z\rangle||}$, and hence $\mathbf{r} = \langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r}\rangle$. Therefore, gravitational vector field $\mathbf{F}$ exerted by object 1 on object 2 is

$$\mathbf{F} = -Gm_1m_2\langle \frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3}\rangle.$$

This is an example of a radial vector field in $\mathbb{R}^3$.

Figure 3.1.10 shows what this gravitational field looks like for a large mass at the origin. Note that the magnitudes of the vectors increase as the vectors get closer to the origin.

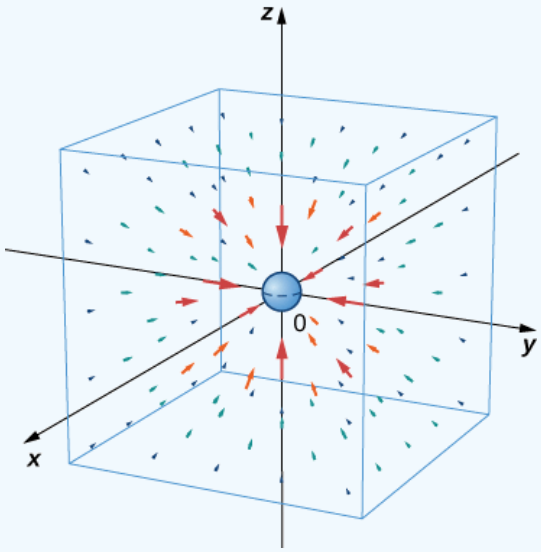


Figure 3.1.10: A visual representation of gravitational vector field $\mathbf{F} = -Gm_1m_2\langle \frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3} \rangle$ for a large mass at the origin.

Exercise 3.1.8

The mass of asteroid 1 is 750,000 kg and the mass of asteroid 2 is 130,000 kg. Assume asteroid 1 is located at the origin, and asteroid 2 is located at $(15, -5, 10)$, measured in units of 10 to the eighth power kilometers. Given that the universal gravitational constant is $G = 6.67384 \times 10^{-11} m^3 kg^{-1} s^{-2}$, find the gravitational force vector that asteroid 1 exerts on asteroid 2.

Hint available at the end of this section.

Gradient Fields (Conservative Fields)

In this section, we study a special kind of vector field called a gradient field or a **conservative field**. These vector fields are extremely important in physics because they can be used to model physical systems in which energy is conserved. Gravitational fields and electric fields associated with a static charge are examples of gradient fields.

Recall that if f is a (scalar) function of x and y, then the gradient of f is

$$\text{grad}\, f = \nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}. \tag{3.1.5}$$

We can see from the form in which the gradient is written that ∇f is a vector field in $\mathbb{R}^2$. Similarly, if f is a function of x, y, and z, then the gradient of f is

$$\text{grad}\, f = \nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}. \tag{3.1.6}$$

The gradient of a three-variable function is a vector field in $\mathbb{R}^3$. A gradient field is a vector field that can be written as the gradient of a function, and we have the following definition.

Definition: Gradient Field

A vector field $\mathbf{F}$ in $\mathbb{R}^2$ or in $\mathbb{R}^3$ is a gradient field if there exists a scalar function f such that $\nabla f = \mathbf{F}$.

Example 3.1.9: Sketching a Gradient Vector Field

Use technology to plot the gradient vector field of $f(x, y) = x^2y^2$.

Solution

The gradient of f is $\nabla f(x, y) = \langle 2xy^2,\ 2x^2y \rangle$. To sketch the vector field, use a computer algebra system such as Mathematica. Figure 3.1.11 shows ∇f.

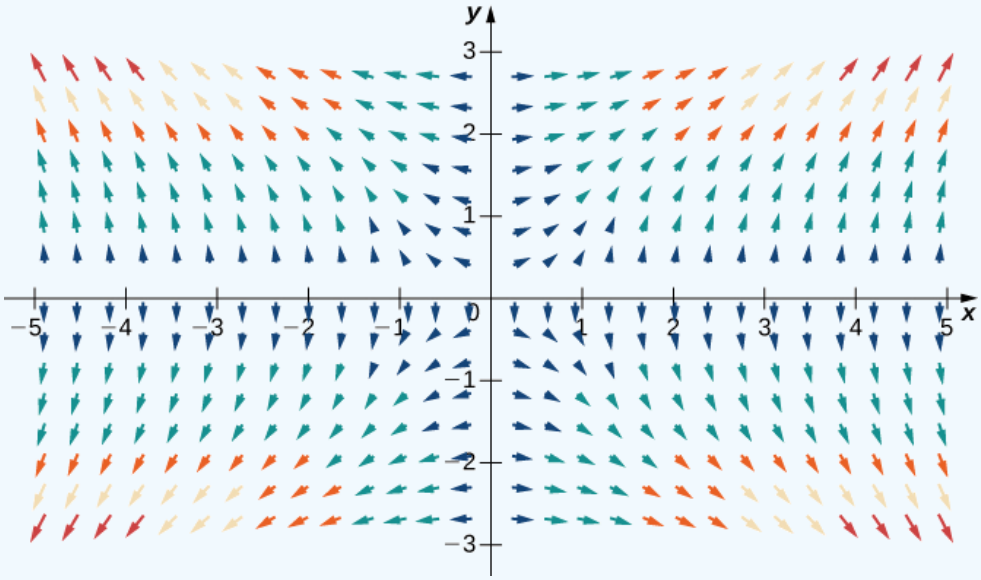


Figure 3.1.11: The gradient vector field is ∇f, where $f(x, y) = x^2y^2$.

Exercise 3.1.9

Use technology to plot the gradient vector field of $f(x, y) = \sin x \cos y$.

Hint available at the end of this section.

Consider the function $f(x, y) = x^2y^2$ from Example 3.1.13. Figure 3.1.13 shows the level curves of this function overlaid on the function's gradient vector field. The gradient vectors are perpendicular to the level curves, and the magnitudes of the vectors get larger as the level curves get closer together, because closely grouped level curves indicate the graph is steep, and the magnitude of the gradient vector is the largest value of the directional derivative. Therefore, you can see the local steepness of a graph by investigating the corresponding function's gradient field.

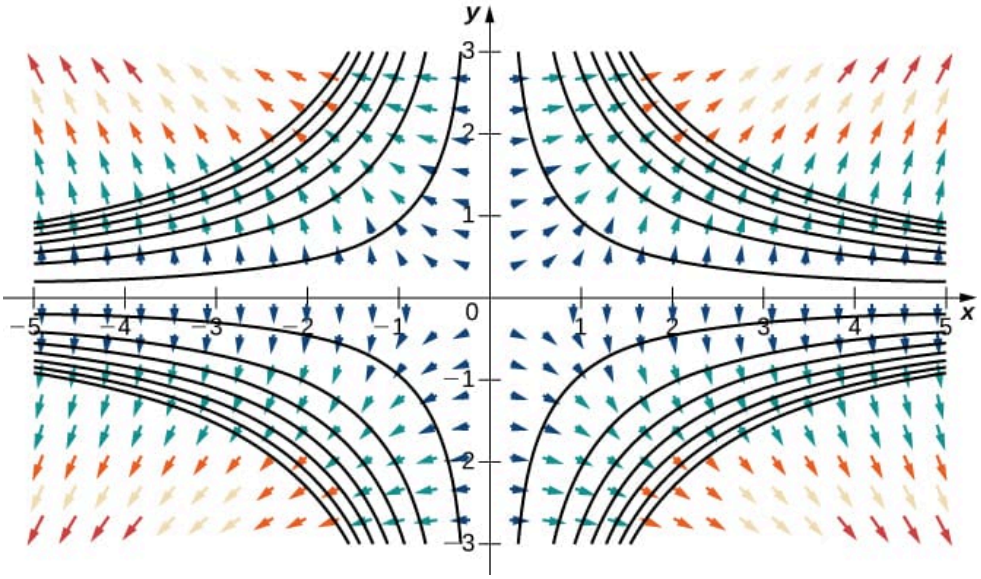


Figure 3.1.13: The gradient field of $f(x,y) = x^2y^2$ and several level curves of f. Notice that as the level curves get closer together, the magnitude of the gradient vectors increases.

As we learned earlier, a vector field $\mathbf{F}$ is a conservative vector field, or a gradient field if there exists a scalar function f such that $\nabla f = \mathbf{F}$. In this situation, f is called a **potential function** for $\mathbf{F}$. Conservative vector fields arise in many applications, particularly in physics. The reason such fields are called *conservative* is that they model forces of physical systems in which energy is conserved. We study conservative vector fields in more detail later in this chapter.

You might notice that, in some applications, a potential function f for $\mathbf{F}$ is defined instead as a function such that $-\nabla f = \mathbf{F}$. This is the case for certain contexts in physics, for example.

Example 3.1.10: Verifying a Potential Function

Is $f(x,y,z) = x^2yz - \sin(xy)$ a potential function for vector field

$$\mathbf{F}(x,y,z) = \langle 2xyz - y\cos(xy), x^2z - x\cos(xy), x^2y\rangle?$$

Solution

We need to confirm whether $\nabla f = \mathbf{F}$. We have

$$\begin{aligned} f_x(x,y) &= 2xyz - y\cos(xy) \\ f_y(x,y) &= x^2z - x\cos(xy). \\ f_z(x,y) &= x^2y \end{aligned}$$

Therefore, $\nabla f = \mathbf{F}$ and f is a potential function for $\mathbf{F}$.

Exercise 3.1.10

Is $f(x,y,z) = x^2\cos(yz) + y^2z^2$ a potential function for $\mathbf{F}(x,y,z) = \langle 2x\cos(yz), -x^2z\sin(yz) + 2yz^2, y^2\rangle$?

Hint available at the end of this section.

Example 3.1.11: Verifying a Potential Function

The velocity of a fluid is modeled by field $\mathbf{v}(x,y) = \langle xy, \frac{x^2}{2} - y\rangle$. Verify that $f(x,y) = \dfrac{x^2y}{2} - \dfrac{y^2}{2}$ is a potential function for $\mathbf{v}$.

Solution

To show that f is a potential function, we must show that $\nabla f = \mathbf{v}$. Note that $f_x(x,y) = xy$ and $f_y(x,y) = \dfrac{x^2}{2} - y$. Therefore, $\nabla f(x,y) = \langle xy, \frac{x^2}{2} - y\rangle$ and f is a potential function for $\mathbf{v}$ (Figure 3.1.14).

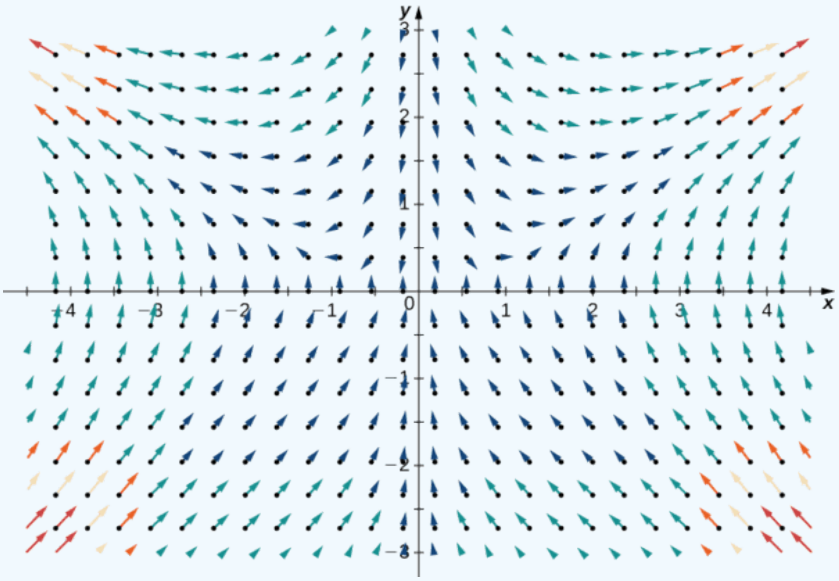


Figure 3.1.14: Velocity field $\mathbf{v}(x,y)$ has a potential function and is a conservative field.

Exercise 3.1.11

Verify that $f(x,y) = x^2y^2 + x$ is a potential function for velocity field $\mathbf{v}(x,y) = \langle 3x^2y^2 + 1, 2x^3y\rangle$.

Hint available at the end of this section.

If $\mathbf{F}$ is a conservative vector field, then there is at least one potential function f such that $\nabla f = \mathbf{F}$. But, could there be more than one potential function? If so, is there any relationship between two potential functions for the same vector field? Before answering these questions, let's recall some facts from single-variable calculus to guide our intuition. Recall that if $k(x)$ is an integrable function, then k has infinitely many antiderivatives. Furthermore, if $\mathbf{F}$ and $\mathbf{G}$ are both antiderivatives of k, then $\mathbf{F}$ and $\mathbf{G}$ **differ only by a constant**. That is, there is some number C such that $\mathbf{F}(x) = \mathbf{G}(x) + C$.

Now let $\mathbf{F}$ be a conservative vector field and let f and g be potential functions for $\mathbf{F}$. Since the gradient is like a derivative, $\mathbf{F}$ being conservative means that $\mathbf{F}$ is "integrable" with "antiderivatives" f and g. Therefore, if the analogy with single-variable calculus is valid, we expect there is some constant C such that $f(x) = g(x) + C$. The next theorem says that this is indeed the case.

To state the next theorem with precision, we need to assume the domain of the vector field is connected and open. To be connected means if P_1 and P_2 are any two points in the domain, then you can walk from P_1 to P_2 along a path that stays entirely inside the domain.

Uniqueness of Functions

Let $\mathbf{F}$ be a conservative vector field on an open and connected domain and let f and g be functions such that $\nabla f = \mathbf{F}$ and $\nabla g = \mathbf{F}$. Then, there is a constant C such that $f = g + C$.

Proof

Since f and g are both potential functions for $\mathbf{F}$, then $\nabla(f-g) = \nabla f - \nabla g = \mathbf{F} - \mathbf{F} = \mathbf{0}$. Let $h = f - g$, then we have $\nabla h = \mathbf{0}$ .We would like to show that h is a constant function.

Assume h is a function of x and y (the logic of this proof extends to any number of independent variables). Since $\nabla h = \mathbf{0}$, we have $h_x(x,y) = 0$ and $h_y(x,y) = 0$. The expression $h_x(x,y) = 0$ implies that h is a constant function with respect to x —that is, $h(x,y) = k_1(y)$ for some function k_1. Similarly, $h_y(x,y) = 0$ implies $h(x,y) = k_2(x)$ for some function k_2. Therefore, function h depends only on y and also depends only on x. Thus, $h(x,y) = C$ for some constant C on the connected domain of $\mathbf{F}$. Note that we really do need connectedness at this point; if the domain of $\mathbf{F}$ came in two separate pieces, then k could be a constant C_1 on one piece but could be a different constant C_2 on the other piece. Since $f - g = h = C$, we have that $f = g + C$, as desired.

□

Conservative vector fields also have a special property called the ***cross-partial property***. This property helps test whether a given vector field is conservative.

The Cross-Partial Property of Conservative Vector Field

Let $\mathbf{F}$ be a vector field in two or three dimensions such that the component functions of $\mathbf{F}$ have continuous second-order mixed-partial derivatives on the domain of $\mathbf{F}$.

If $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ is a conservative vector field in $\mathbb{R}^2$, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}. \tag{3.1.7}$$

If $\mathbf{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$ is a conservative vector field in $\mathbb{R}^3$, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$
$$\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$
$$\frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}.$$

Proof

Since $\mathbf{F}$ is conservative, there is a function $f(x,y)$ such that $\nabla f = \mathbf{F}$. Therefore, by the definition of the gradient, $f_x = P$ and $f_y = Q$. By Clairaut's theorem, $f_{xy} = f_{yx}$, But, $f_{xy} = P_y$ and $f_{yx} = Q_x$, and thus $P_y = Q_x$.

□

Clairaut's theorem gives a fast proof of the cross-partial property of conservative vector fields in $\mathbb{R}^3$, just as it did for vector fields in $\mathbb{R}^2$.

The Cross-Partial Property of Conservative Vector Fields shows that most vector fields are not conservative. The cross-partial property is difficult to satisfy in general, so most vector fields won't have equal cross-partials.

Example 3.1.12

Show that rotational vector field $\mathbf{F}(x,y) = \langle y,\ -x\rangle$ is not conservative.

Solution

Let $P(x,y)=y$ and $Q(x,y)=-x$. If $\mathbf{F}$ is conservative, then the cross-partials would be equal—that is, P_y would equal Q_x .Therefore, to show that $\mathbf{F}$ is not conservative, check that $P_y \neq Q_x$. Since $P_y=1$ and $Q_x=-1$, the vector field is not conservative.

Exercise 3.1.12

Show that vector field $\mathbf{F}(x,y)=xy\ \mathbf{i}-x^2y\ \mathbf{j}$ is not conservative.

Hint available at the end of this section.

Example 3.1.13: Showing a Vector Field Is Not Conservative

Is vector field $\mathbf{F}(x,y,z)=\langle 7,-2,x^3\rangle$ conservative?

Solution

Let $P(x,y,z)=7$, $Q(x,y,z)=-2$, and $R(x,y,z)=x^3$. If $\mathbf{F}$ is conservative, then all three cross-partial equations will be satisfied—that is, if $\mathbf{F}$ is conservative, then P_y would equal Q_x, Q_z would equal R_y, and R_x would equal P_z. Note that

$$P_y=Q_x=R_y=Q_z=0$$

so the first two necessary equalities hold. However, $R_x(x,y,z)=3x^2$ and $P_z(x,y,z)=0$ so $R_x \neq P_z$. Therefore, $\mathbf{F}$ is not conservative.

Exercise 3.1.13

Is vector field $\mathbf{G}(x,y,z)=\langle y,\ x,\ xyz\rangle$ conservative?

Hint available at the end of this section.

We conclude this section with a word of warning: The Cross-Partial Property of Conservative Vector Fields says that if $\mathbf{F}$ is conservative, then $\mathbf{F}$ has the cross-partial property. The theorem does *not* say that, if $\mathbf{F}$ has the cross-partial property, then $\mathbf{F}$ is conservative (the converse of an implication is not logically equivalent to the original implication). In other words, The Cross-Partial Property of Conservative Vector Fields can only help determine that a field is not conservative; it does not let you conclude that a vector field is conservative.

For example, consider vector field $\mathbf{F}(x,y)=\langle x^2y,\frac{x^3}{3}\rangle$. This field has the cross-partial property, so it is natural to try to use The Cross-Partial Property of Conservative Vector Fields to conclude this vector field is conservative. However, this is a misapplication of the theorem. We learn later how to conclude that $\mathbf{F}$ is conservative.

Key Concepts

- A vector field assigns a vector $\mathbf{F}(x,y)$ to each point (x,y) in a subset D of $\mathbb{R}^2$ or $\mathbb{R}^3$. $\mathbf{F}(x,y,z)$ to each point (x,y,z) in a subset D of $\mathbb{R}^3$.
- Vector fields can describe the distribution of vector quantities such as forces or velocities over a region of the plane or of space. They are in common use in such areas as physics, engineering, meteorology, oceanography.
- We can sketch a vector field by examining its defining equation to determine relative magnitudes in various locations and then drawing enough vectors to determine a pattern.
- A vector field $\mathbf{F}$ is called conservative if there exists a scalar function f such that $\nabla f=\mathbf{F}$.

Key Equations

- **Vector field in $\mathbb{R}^2$**
  $\mathbf{F}(x,y) = \langle P(x,y),\ Q(x,y)\rangle$
  or
  $\mathbf{F}(x,y) = P(x,y)\ \mathbf{i} + Q(x,y)\ \mathbf{j}$
- **Vector field in $\mathbb{R}^3$**
  $\mathbf{F}(x,y,z) = \langle P(x,y,z),\ Q(x,y,z),\ R(x,y,z)\rangle$
  or
  $\mathbf{F}(x,y,z) = P(x,y,z)\ \mathbf{i} + Q(x,y,z)\ \mathbf{j} + R(x,y,z)\ \mathbf{k}$

Glossary

Conservative field
: a vector field for which there exists a scalar function f such that $\nabla f = \mathbf{F}$

Gradient field
: a vector field $\mathbf{F}$ for which there exists a scalar function f such that $\nabla f = \mathbf{F}$; in other words, a vector field that is the gradient of a function; such vector fields are also called *conservative*

Potential function
: a scalar function f such that $\nabla f = \mathbf{F}$

Radial field
: a vector field in which all vectors either point directly toward or directly away from the origin; the magnitude of any vector depends only on its distance from the origin

Rotational field
: a vector field in which the vector at point (x,y) is tangent to a circle with radius $r = \sqrt{x^2+y^2}$; in a rotational field, all vectors flow either clockwise or counterclockwise, and the magnitude of a vector depends only on its distance from the origin

Unit vector field
: a vector field in which the magnitude of every vector is 1

Vector field
: measured in $\mathbb{R}^2$, an assignment of a vector $\mathbf{F}(x,y)$ to each point (x,y) of a subset D of $\mathbb{R}^2$; in $\mathbb{R}^3$, an assignment of a vector $\mathbf{F}(x,y,z)$ to each point (x,y,z) of a subset D of $\mathbb{R}^3$

Contributors and Attributions

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Answers

Exercise 3.1.1

Substitute the point values into the vector function.

$\mathbf{G}(-2,3) = 12\mathbf{i} - \mathbf{j}$

Exercise 3.1.2

Sketch enough vectors to get an idea of the shape.

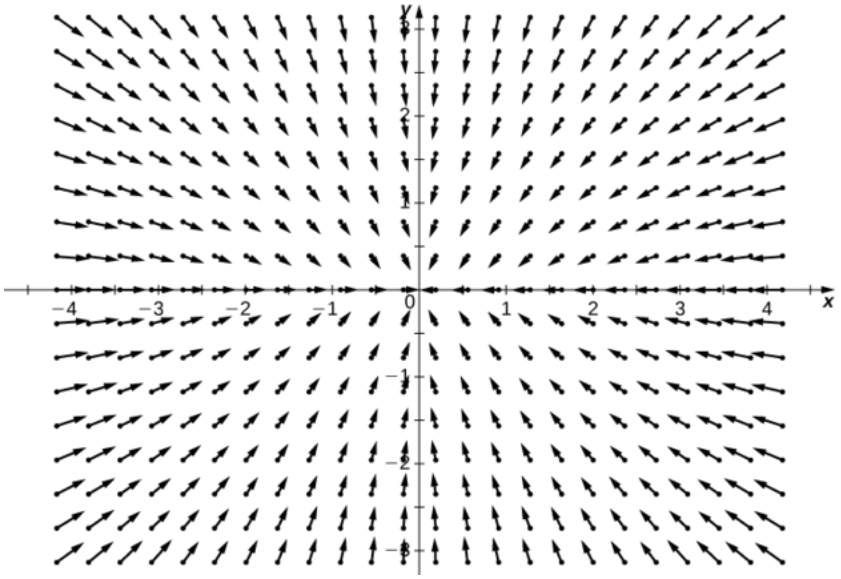


Figure 3.1.3: A visual representation of the vector field $\mathbf{F}(x,y) = -\frac{x}{3}\mathbf{i} - \frac{y}{3}\mathbf{j}$. The arrows point toward the origin.

Exercise 3.1.4

Substitute enough points into $\mathbf{F}$ to get an idea of the shape.

Rotational

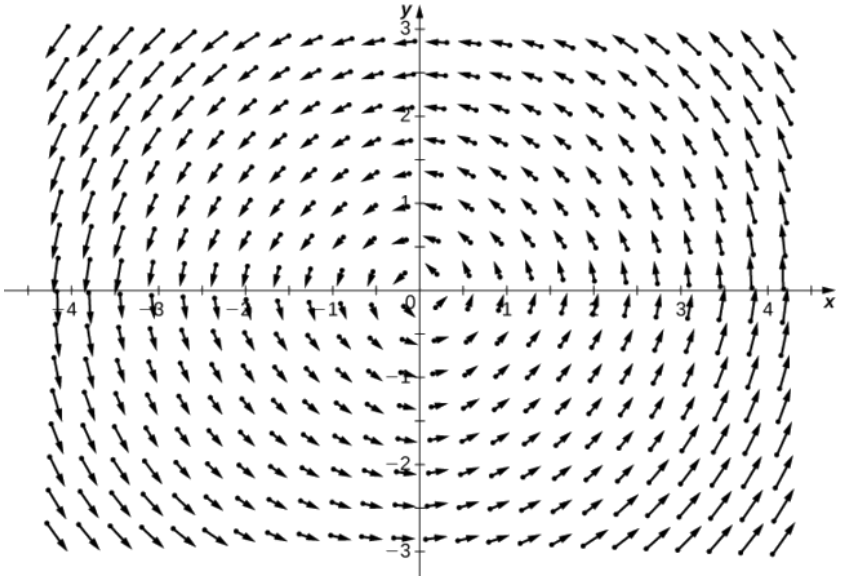


Figure 3.1.7: A visual representation of vector field $\mathbf{F}(x,y) = \langle -2y,\ 2x \rangle$. The arrows move in counterclockwise direction creating a rotational vector field.

Exercise 3.1.5

Remember, speed is the magnitude of velocity.

$\sqrt{65}$ m/sec

Exercise 3.1.6

Calculate the magnitude of $\mathbf{F}$ at an arbitrary point (x,y).

No.

Exercise 3.1.7

Substitute enough points into the vector field to get an idea of the general shape.

Exercise 3.1.7

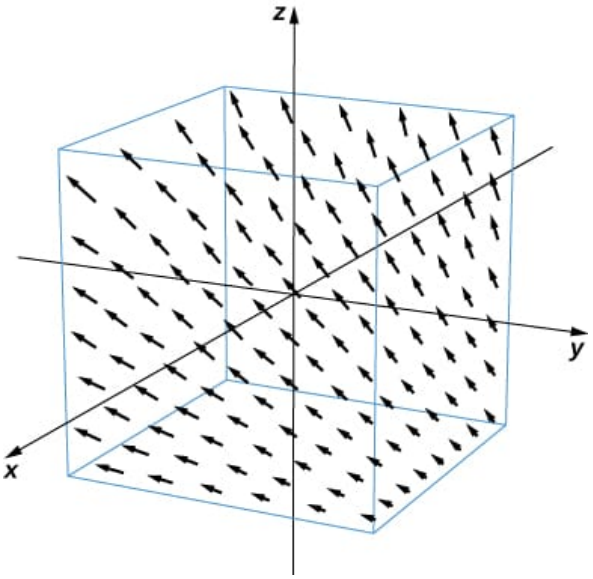


Figure 3.1.9: A visual representation of vector field $\mathbf{G}(x, y, z) = \langle 2,\ \frac{z}{2},\ 1\rangle$. The arrows points upwards to the left. Arrows appear to be flatter near the bottom of the box, and the arrows appear to be steeper near the top of the box.

Exercise 3.1.8

Follow Example 3.1.8 and first compute the distance between the asteroids.

1.49063×10^{-18}, 4.96876×10^{-19}, 9.93752×10^{-19} N

Exercise 3.1.9

Find the gradient of f.

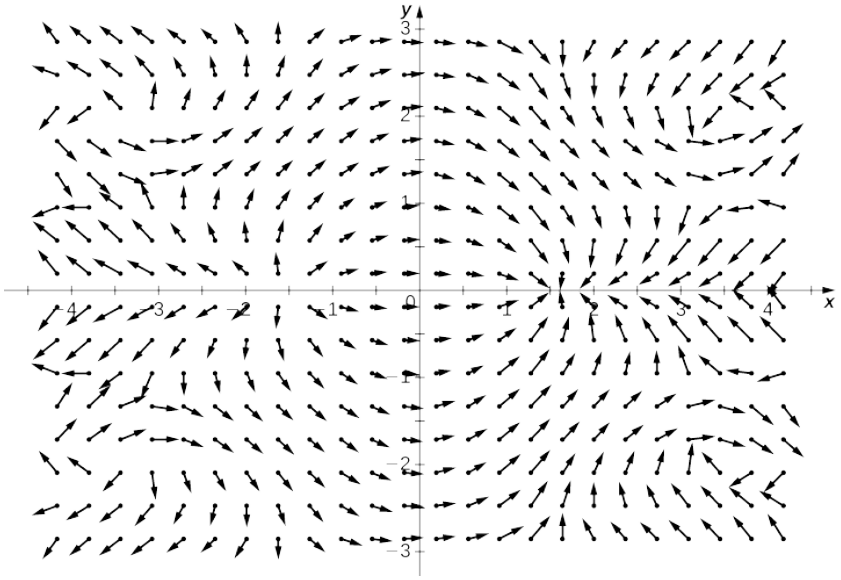


Figure 3.1.12: A visual representation of the gradient vector field of the scalar function $f(x, y) = \sin x \cos y$. The arrows move in non-uniform patterns creating four unique flows in each quadrant.

Exercise 3.1.10

Compute the gradient of f.

No

Exercise 3.1.11

Calculate the gradient.

$\nabla f(x, y) = \mathbf{v}(x, y)$

Exercise 3.1.12

Check the cross-partials.

$P_y(x, y) = x$ and $Q_x(x, y) = -2xy$. Since $P_y(x, y) \neq Q_x(x, y)$, $\mathbf{F}$ is not conservative.

Exercise 3.1.13

Check the cross-partials.

No

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3.1E: Exercises

1. The domain of vector field $\mathbf{F} = \mathbf{F}(x, y)$ is a set of points (x, y) in a plane, and the range of $\mathbf{F}$ is a set of *what* in the plane?

Answer

Vectors

Exercises 2 - 4

Determine whether the statement is *true or false*.

2. Vector field $\mathbf{F} = \langle 3x^2, 1 \rangle$ is a gradient field for both $\phi_1(x, y) = x^3 + y$ and $\phi_2(x, y) = y + x^3 + 100$.

3. Vector field $\mathbf{F} = \dfrac{\langle y, x \rangle}{\sqrt{x^2 + y^2}}$ is constant in direction and magnitude on a unit circle.

Answer

False

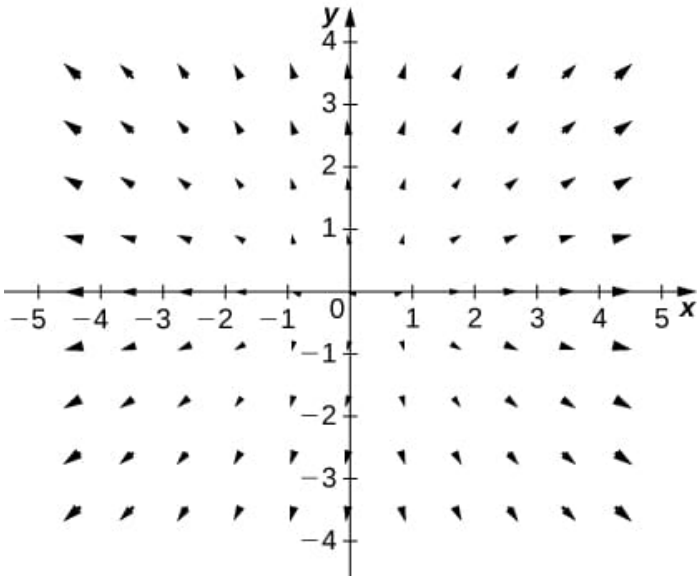
4. Vector field $\mathbf{F} = \dfrac{\langle y, x \rangle}{\sqrt{x^2 + y^2}}$ is neither a radial field nor a rotation field.

Exercises 5 - 14

Describe each vector field by drawing some of its vectors.

5. $\mathbf{F}(x, y) = x\ \mathbf{i} + y\ \mathbf{j}$

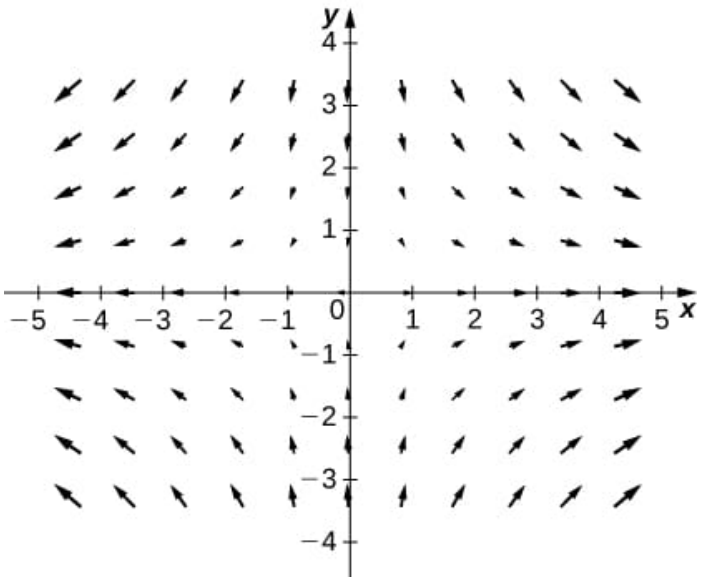
Answer



6. $\mathbf{F}(x, y) = -y\ \mathbf{i} + x\ \mathbf{j}$

7. $\mathbf{F}(x, y) = x\ \mathbf{i} - y\ \mathbf{j}$

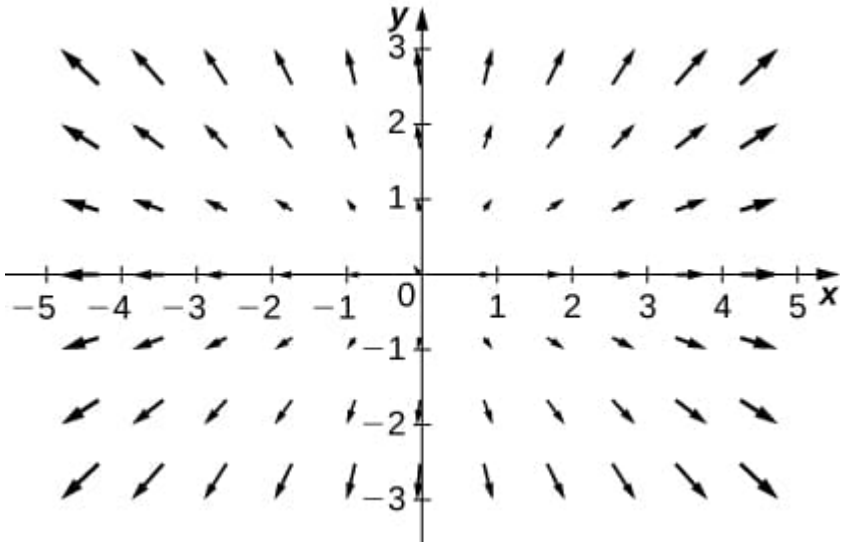
Answer



8. $\mathbf{F}(x, y) = \mathbf{i} + \mathbf{j}$

9. $\mathbf{F}(x, y) = 2x\ \mathbf{i} + 3y\ \mathbf{j}$

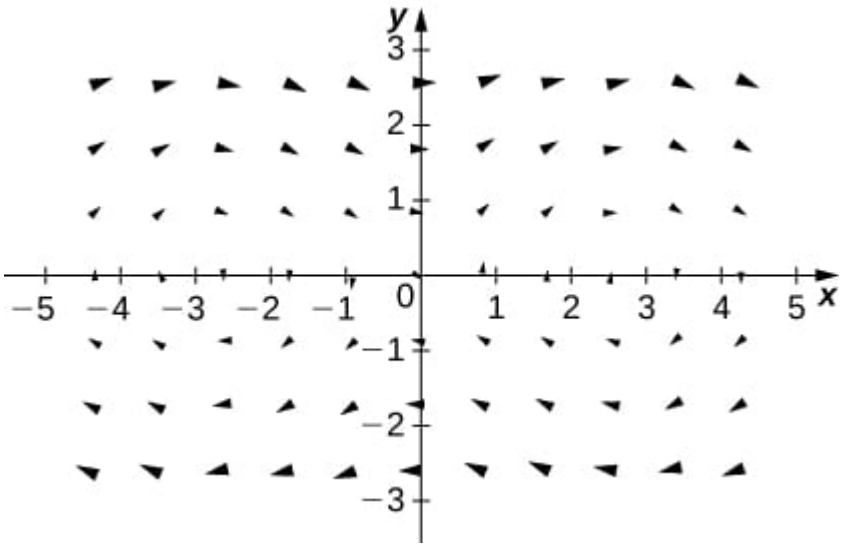
Answer



10. $\mathbf{F}(x, y) = 3\ \mathbf{i} + x\ \mathbf{j}$

11. $\mathbf{F}(x, y) = y\ \mathbf{i} + \sin x\ \mathbf{j}$

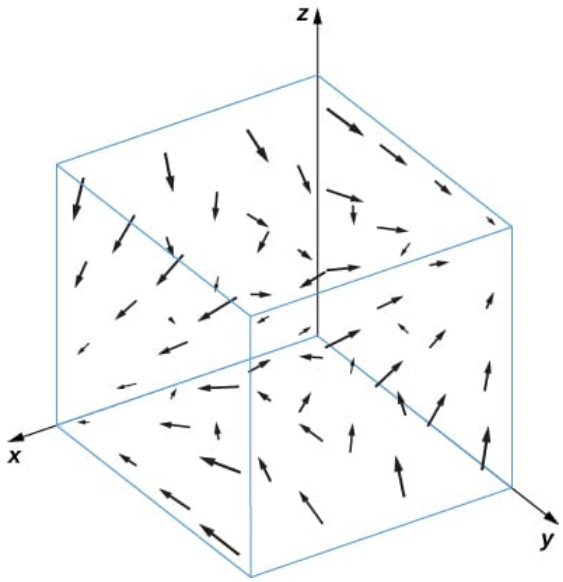
Answer



12. $\mathbf{F}(x, y, z) = x\ \mathbf{i} + y\ \mathbf{j} + z\ \mathbf{k}$

13. $\mathbf{F}(x, y, z) = 2x\ \mathbf{i} - 2y\ \mathbf{j} - 2z\ \mathbf{k}$

Answer



14. $\mathbf{F}(x, y, z) = yz\ \mathbf{i} - xz\ \mathbf{j}$

Exercises 15 - 21

Find the gradient vector field of each function f.

15. $f(x, y) = x \sin y + \cos y$

Answer

$$\mathbf{F}(x, y) = \sin(y)\ \mathbf{i} + (x \cos y - \sin y)\ \mathbf{j}$$

16. $f(x,y,z) = ze^{-xy}$

17. $f(x,y,z) = x^2y + xy + y^2z$

Answer

$$\mathbf{F}(x,y,z) = (2xy + y)\ \mathbf{i} + (x^2 + x + 2yz)\ \mathbf{j} + y^2\ \mathbf{k}$$

18. $f(x,y) = x^2 \sin(5y)$

19. $f(x,y) = \ln(1 + x^2 + 2y^2)$

Answer

$$\mathbf{F}(x,y) = \frac{2x}{1 + x^2 + 2y^2}\ \mathbf{i} + \frac{4y}{1 + x^2 + 2y^2}\ \mathbf{j}$$

20. $f(x,y,z) = x\cos\left(\frac{y}{z}\right)$

21. What is vector field $\mathbf{F}(x,y)$ with a value at (x,y) that is of unit length and points toward $(1,0)$?

Answer

$$\mathbf{F}(x,y) = \frac{(1-x)\,\mathbf{i} - y\mathbf{j}}{\sqrt{(1-x)^2 + y^2}}$$

Exercises 22 - 27

Write formulas for the vector fields with the given properties.

22. All vectors are parallel to the x-axis and all vectors on a vertical line have the same magnitude.

23. All vectors point toward the origin and have constant length.

Answer

$$\mathbf{F}(x,y) = \frac{(y\,\mathbf{i} - x\,\mathbf{j})}{\sqrt{x^2 + y^2}}$$

24. All vectors are of unit length and are perpendicular to the position vector at that point.

25. Give a formula $\mathbf{F}(x,y) = M(x,y)\ \mathbf{i} + N(x,y)\ \mathbf{j}$ for the vector field in a plane that has the properties that $\mathbf{F} = \mathbf{0}$ at $(0,0)$ and that at any other point (a,b), $\mathbf{F}$ is tangent to circle $x^2 + y^2 = a^2 + b^2$ and points in the clockwise direction with magnitude $\|\mathbf{F}\| = \sqrt{a^2 + b^2}$.

Answer

$$\mathbf{F}(x,y) = y\ \mathbf{i} - x\ \mathbf{j}$$

26. Is vector field $\mathbf{F}(x,y) = (P(x,y), Q(x,y)) = (\sin x + y)\ \mathbf{i} + (\cos y + x)\ \mathbf{j}$ a gradient field?

27. Find a formula for vector field $\mathbf{F}(x,y) = M(x,y)\ \mathbf{i} + N(x,y)\ \mathbf{j}$ given the fact that for all points (x,y), $\mathbf{F}$ points toward the origin and $\|\mathbf{F}\| = \dfrac{10}{x^2 + y^2}$.

Answer

$$\mathbf{F}(x,y) = \frac{-10}{(x^2 + y^2)^{3/2}}(x\ \mathbf{i} + y\ \mathbf{j})$$

Exercises 28 - 29

Assume that an electric field in the xy-plane caused by an infinite line of charge along the x-axis is a gradient field with potential function $V(x,y) = c\ln\left(\frac{r_0}{\sqrt{x^2+y^2}}\right)$, where $c > 0$ is a constant and r_0 is a reference distance at which the potential

is assumed to be zero.

28. Find the components of the electric field in the x- and y-directions, where $\mathbf{E}(x,y) = -\nabla V(x,y)$.

29. Show that the electric field at a point in the xy-plane is directed outward from the origin and has magnitude $\|\mathbf{E}\| = \dfrac{c}{r}$, where $r = \sqrt{x^2+y^2}$.

Answer

$$\|\mathbf{E}\| = \frac{c}{|r|^2} r = \frac{c}{|r|}\frac{r}{|r|}$$

A ***flow line*** (or *streamline*) of a vector field $\mathbf{F}$ is a curve $\mathbf{r}(t)$ such that $d\mathbf{r}/dt = \mathbf{F}(\mathbf{r}(t))$. If $\mathbf{F}$ represents the velocity field of a moving particle, then the flow lines are paths taken by the particle. Therefore, flow lines are tangent to the vector field.

Exercises 30 and 31

Show that the given curve $\mathbf{c}(t)$ is a flow line of the given velocity vector field $\mathbf{F}(x,y,z)$.

30. $\mathbf{c}(t) = \langle e^{2t}, \ln|t|, \frac{1}{t}\rangle, \ t \neq 0; \quad \mathbf{F}(x,y,z) = \langle 2x, z, -z^2\rangle$

31. $\mathbf{c}(t) = \langle \sin t, \cos t, e^t\rangle; \quad \mathbf{F}(x,y,z) = \langle y, -x, z\rangle$

Answer

$$\mathbf{c}\,'(t) = \langle \cos t, -\sin t, e^{-t}\rangle = \mathbf{F}(\mathbf{c}(t))$$

Exercises 32 - 34

Let $\mathbf{F} = x\ \mathbf{i} + y\ \mathbf{j}$, $\mathbf{G} = -y\ \mathbf{i} + x\ \mathbf{j}$, and $\mathbf{H} = x\ \mathbf{i} - y\ \mathbf{j}$. Match $\mathbf{F}$, $\mathbf{G}$, and $\mathbf{H}$ with their graphs.

32.

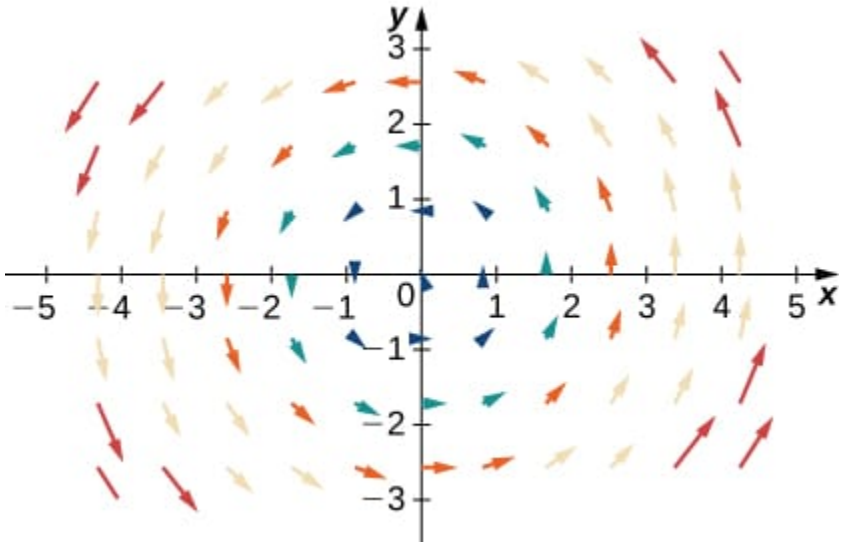


Figure $3.1E.1$: A visual representation of a counterclockwise rotational vector field with increasing magnitude as the distance from the origin grows.

33.

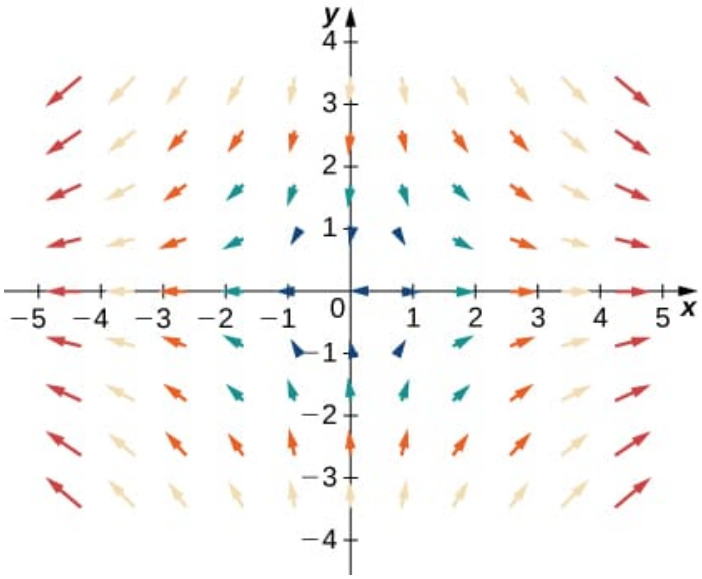


Figure $3.1E.2$: A visual representation of arrows point outward in a radial pattern from the origin, with the magnitude of the arrows increasing as the distance from the origin grows.

Answer

$\mathbf{H}$

34.

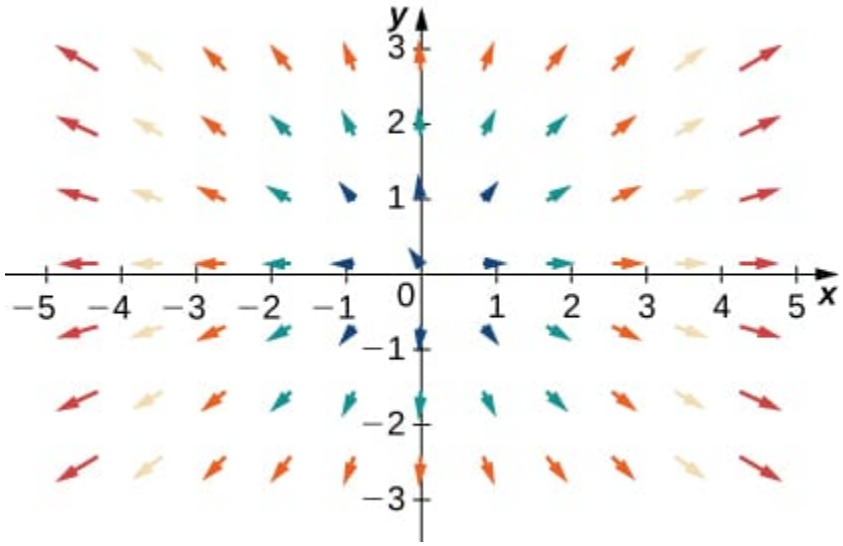


Figure $3.1E.3$: A visual representation of arrows point inward toward the origin in a radial pattern, with the magnitude increasing as the distance from the origin grows.

Exercises 35 - 38

Let $\mathbf{F} = x\ \mathbf{i} + y\ \mathbf{j}$, $\mathbf{G} = -y\ \mathbf{i} + x\ \mathbf{j}$, and $\mathbf{H} = -x\ \mathbf{i} + y\ \mathbf{j}$. Match the vector fields with their graphs in (I)–(IV).

1. $\mathbf{F} + \mathbf{G}$
2. $\mathbf{F} + \mathbf{H}$
3. $\mathbf{G} + \mathbf{H}$
4. $-\mathbf{F} + \mathbf{G}$

35.

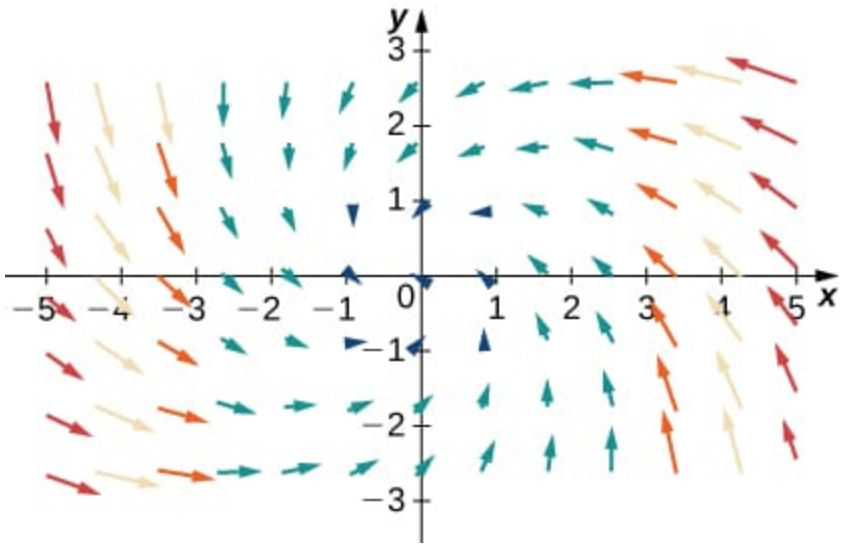


Figure $3.1E.4$: A visual representation of arrows display a combination of rotational and outward radial patterns.

Answer

4. $-\mathbf{F}+\mathbf{G}$

36.

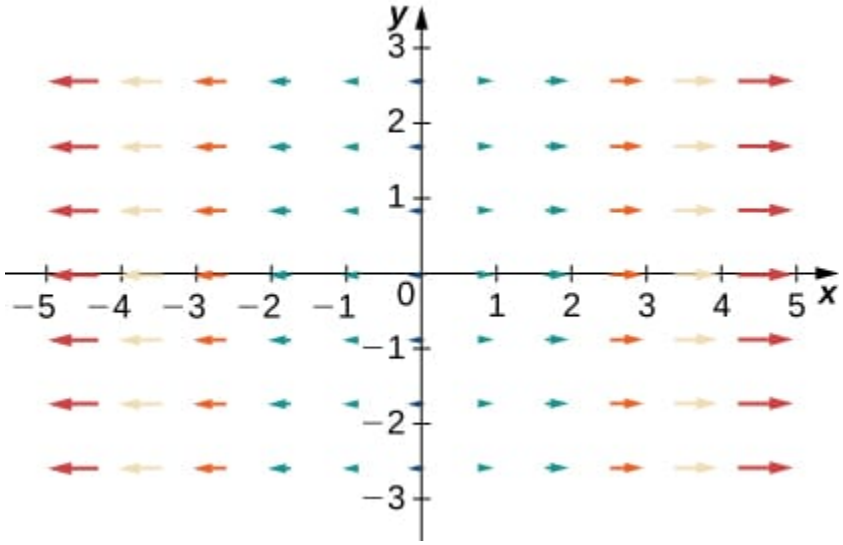


Figure $3.1E.5$: A visual representation of arrows are horizontal and uniformly spaced, all pointing to the right, indicating a uniform field with constant magnitude and direction.

37.

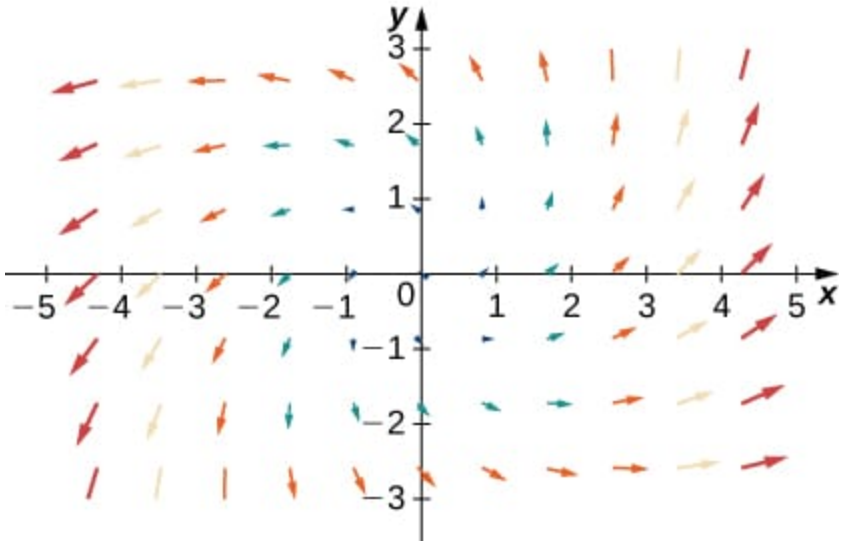


Figure $3.1E.6$: A visual representation of arrows indicate a counterclockwise rotational pattern combined with inward radial flow, suggesting a combination of rotation and contraction toward the origin.

Answer

1. $\mathbf{F} + \mathbf{G}$

38.

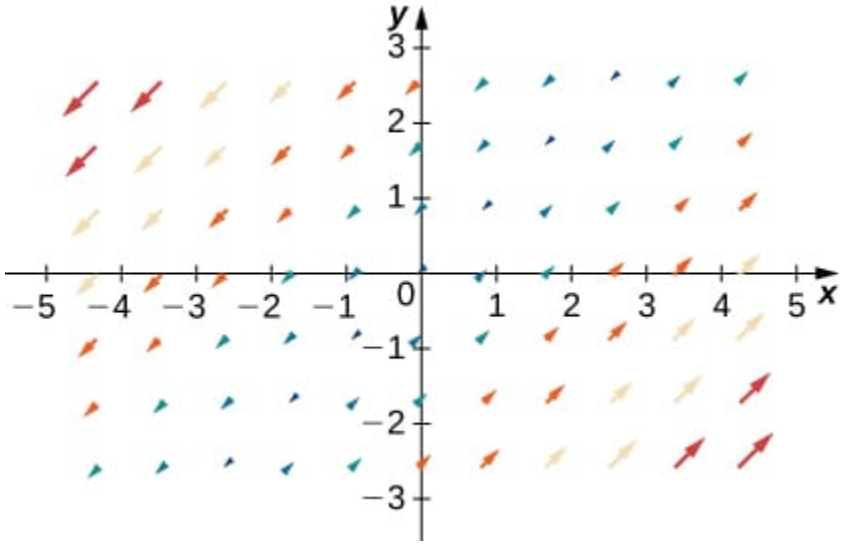


Figure $3.1E.7$: A visual representation of arrows display a counterclockwise rotational pattern with outward radial flow, representing a combination of rotation and expansion away from the origin.

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3.2: Line Integrals

Intro

Learning Objectives

- Calculate a scalar line integral along a curve.
- Calculate a vector line integral along an oriented curve in space.
- Use a line integral to compute the work done in moving an object along a curve in a vector field.
- Describe the flux and circulation of a vector field.

We are familiar with single-variable integrals of the form $\int_a^b f(x)\ dx$, where the domain of integration is an interval $[a, b]$. Such an interval can be thought of as a curve in the xy-plane, since the interval defines a line segment with endpoints $(a, 0)$ and $(b, 0)$—in other words, a line segment located on the x-axis. Suppose we want to integrate over *any* curve in the plane, not just over a line segment on the x-axis. Such a task requires a new kind of integral, called a *line integral*.

Line integrals have many applications to engineering and physics. They also allow us to make several useful generalizations of the Fundamental Theorem of Calculus. And, they are closely connected to the properties of vector fields, as we shall see.

Scalar Line Integrals

A line integral gives us the ability to integrate multivariable functions and vector fields over arbitrary curves in a plane or in space. There are two types of line integrals: scalar line integrals and vector line integrals. Scalar line integrals are integrals of a scalar function over a curve in a plane or in space. Vector line integrals are integrals of a vector field over a curve in a plane or in space. Let's look at scalar line integrals first.

A scalar line integral is defined just as a single-variable integral is defined, except that for a scalar line integral, the integrand is a function of more than one variable and the domain of integration is a curve in a plane or in space, as opposed to a curve on the x-axis.

For a scalar line integral, we let C be a smooth curve in a plane or in space and let f be a function with a domain that includes C. We chop the curve into small pieces. For each piece, we choose point P in that piece and evaluate f at P. (We can do this because all the points in the curve are in the domain of f.) We multiply $f(P)$ by the arc length of the piece Δs, add the product $f(P)\Delta s$ over all the pieces, and then let the arc length of the pieces shrink to zero by taking a limit. The result is the scalar line integral of the function over the curve.

For a formal description of a scalar line integral, let C be a smooth curve in space given by the parameterization $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, $a \le t \le b$. Let $f(x, y, z)$ be a function with a domain that includes curve C. To define the line integral of the function f over C, we begin as most definitions of an integral begin: we chop the curve into small pieces. Partition the parameter interval $[a, b]$ into n subintervals $[t_{i-l}, t_i]$ of equal width for $1 \le i \le n$, where $t_0 = a$ and $t_n = b$ (Figure 3.2.1). Let t_i^* be a value in the ith interval $[t_{i-l}, t_i]$. Denote the endpoints of $\mathbf{r}(t_0)$, $\mathbf{r}(t_1)$,..., $\mathbf{r}(t_n)$ by P_0,..., P_n. Points P_i divide curve C into n pieces C_1, C_2,..., C_n ,with lengths Δs_1, Δs_2,..., Δs_n, respectively. Let P_i^* denote the endpoint of $\mathbf{r}(t_i^*)$ for $1 \le i \le n$. Now, we evaluate the function f at point P_i^* for $1 \le i \le n$. Note that P_i^* is in piece C_1, and therefore P_i^* is in the domain of f. Multiply $f(P_i^*)$ by the length Δs_1 of C_1, which gives the area of the "sheet" with base C_1, and height $f(P_i^*)$. This is analogous to using rectangles to approximate area in a single-variable integral. Now, we form the sum $\sum_{i=1}^{n} f(P_i^*)\ \Delta s_i$.

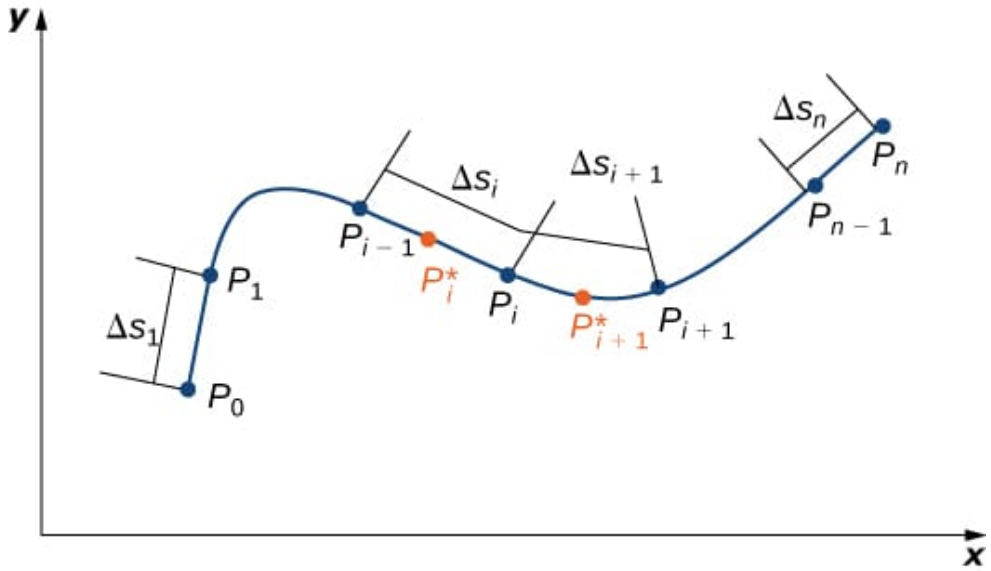


Figure 3.2.1: Curve C has been divided into n pieces, and a point inside each piece has been chosen.

Note the similarity of this sum versus a Riemann sum; in fact, this definition is a generalization of a Riemann sum to arbitrary curves in space. Just as with Riemann sums and integrals of form $\int_a^b g(x)\ dx$, we define an integral by letting the width of the pieces of the curve shrink to zero by taking a limit. The result is the scalar line integral of f along C.

You may have noticed a difference between this definition of a scalar line integral and a single-variable integral. In this definition, the arc lengths Δs_1, Δs_2,..., Δs_n aren't necessarily the same; in the definition of a single-variable integral, the curve in the x-axis is partitioned into pieces of equal length. This difference does not have any effect in the limit. As we shrink the arc lengths to zero, their values become close enough that any small difference becomes irrelevant.

Definition: Scalar Line Integral

Let f be a function with a domain that includes the smooth curve C that is parameterized by $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, $a \le t \le b$. The *scalar line integral* of f along C is

$$\int_C f(x,y,z)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(P_i^*)\,\Delta s_i \tag{3.2.1}$$

if this limit exists (t_i^* and Δs_i are defined as in the previous paragraphs). If C is a planar curve, then C can be represented by the parametric equations $x = x(t)$, $y = y(t)$, and $a \le t \le b$. If C is smooth and $f(x,y)$ is a function of two variables, then the scalar line integral of f along C is defined similarly as

$$\int_C f(x,y)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(P_i^*)\,\Delta s_i, \tag{3.2.2}$$

if this limit exists.

If f is a continuous function on a smooth curve C, then $\int_C f\ ds$ always exists. Since $\int_C f\ ds$ is defined as a limit of Riemann sums, the continuity of f is enough to guarantee the existence of the limit, just as the integral $\int_a^b g(x)\ dx$ exists if g is continuous over $[a,b]$.

Before looking at how to compute a line integral, we need to examine the geometry captured by these integrals. Suppose that $f(x,y) \ge 0$ for all points (x,y) on a smooth planar curve C. Imagine taking curve C and projecting it "up" to the surface defined by $f(x,y)$, thereby creating a new curve C' that lies in the graph of $f(x,y)$ (Figure 3.2.2). Now we drop a "sheet" from C' down to

the xy-plane. The area of this sheet is $\int_C f(x,y)ds$. If $f(x,y) \leq 0$ for some points in C, then the value of $\int_C f(x,y)\ ds$ is the area above the xy-plane less the area below the xy-plane. (Note the similarity with integrals of the form $\int_a^b g(x)\ dx$.)

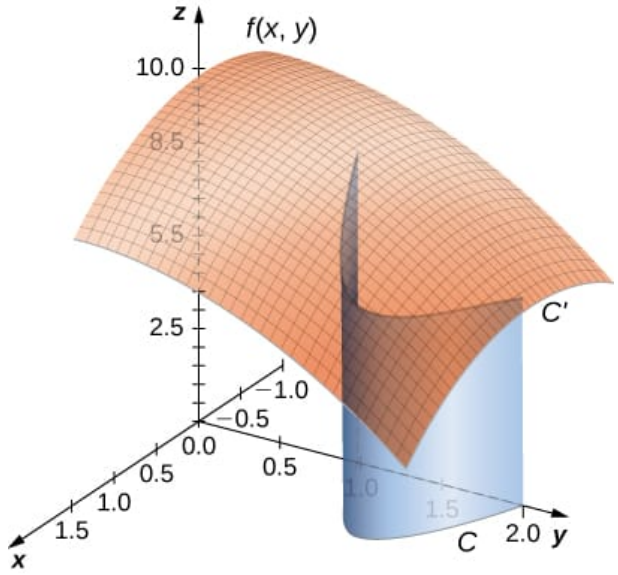


Figure 3.2.2: A three-dimensional surface $f(x,y)$ and a smooth curve C lying in the xy-plane. The curve C is projected upward onto the surface $f(x,y)$, forming a new curve C' that resides on the graph of $f(x,y)$. A translucent vertical "sheet" connects C' to C, representing the integral $\int_C f(x,y)\ ds$. This integral corresponds to the net area above the xy-plane (if $f(x,y) \geq 0$) or the area above minus the area below the xy-plane (if $f(x,y) \leq 0$ at some points).

From this geometry, we can see that line integral $\int_C f(x,y)\ ds$ does not depend on the parameterization $\mathbf{r}(t)$ of C. As long as the curve is traversed exactly once by the parameterization, the area of the sheet formed by the function and the curve is the same. This same kind of geometric argument can be extended to show that the line integral of a three-variable function over a curve in space does not depend on the parameterization of the curve.

Example 3.2.1: Finding the Value of a Line Integral

Find the value of integral $\int_C 2\ ds$, where C is the upper half of the unit circle.

Solution

The integrand is $f(x,y) = 2$. Figure 3.2.3 shows the graph of $f(x,y) = 2$, curve C, and the sheet formed by them. Notice that this sheet has the same area as a rectangle with width π and length 2. Therefore, $\int_C 2\ ds = 2\pi\ \text{units}^2$.

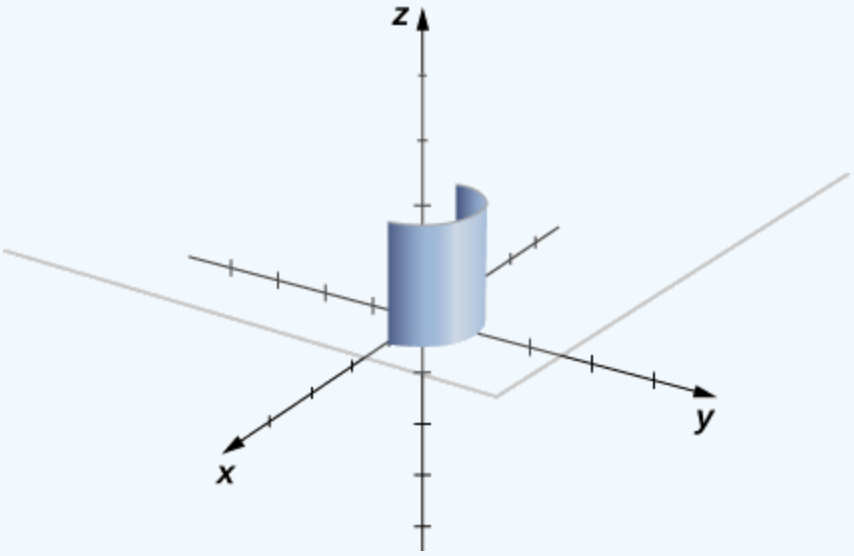


Figure 3.2.3: A sheet is formed by the upper half of the unit circle in a plane and the graph of $f(x,y)=2$.

To see that $\int_C 2\ ds = 2\pi$ using the definition of line integral, we let $\mathbf{r}(t)$ be a parameterization of C. Then, $f(\mathbf{r}(t_i)) = 2$ for any number t_i in the domain of $\mathbf{r}$. Therefore,

$$\begin{aligned}\int_C f\,ds &= \lim_{n\to\infty}\sum_{i=1}^{n} f(\mathbf{r}(t_i^*))\,\Delta s_i \\ &= \lim_{n\to\infty}\sum_{i=1}^{n} 2\,\Delta s_i \\ &= 2\lim_{n\to\infty}\sum_{i=1}^{n} \Delta s_i \\ &= 2(\text{length of } C) \\ &= 2\pi \text{ units}^2.\end{aligned}$$

Exercise 3.2.1

Find the value of $\int_C (x+y)\ ds$, where C is the curve parameterized by $x=t$, $y=t$, $0\le t\le 1$.

Hint available at the end of this section.

Note that in a scalar line integral, the integration is done with respect to arc length s, which can make a scalar line integral difficult to calculate. To make the calculations easier, we can translate $\int_C f\ ds$ to an integral with a variable of integration that is t.

Let $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ for $a\le t\le b$ be a parameterization of C. Since we are assuming that C is smooth, $\mathbf{r}\,'(t) = \langle x\,'(t), y\,'(t), z\,'(t)\rangle$ is continuous for all t in $[a,b]$. In particular, $x\,'(t)$, $y\,'(t)$, and $z\,'(t)$ exist for all t in $[a,b]$. According to the arc length formula, we have

$$\text{length}(C_i) = \Delta s_i = \int_{t_{i-1}}^{t_i} \|\mathbf{r}'(t)\|\,dt. \tag{3.2.3}$$

If width $\Delta t_i = t_i - t_{i-1}$ is small, then function $\int_{t_{i-1}}^{t_i} \|\mathbf{r}\,'(t)\|\ dt \ \approx\ \|\mathbf{r}\,'(t_i^*)\|\ \Delta t_i$, $\|\mathbf{r}\,'(t)\|$ is almost constant over the interval $[t_{i-1}, t_i]$ .Therefore,

$$\int_{t_{i-1}}^{t_i} \|\mathbf{r}'(t)\|\,dt \approx \|\mathbf{r}'(t_i^*)\|\,\Delta t_i, \tag{3.2.4}$$

and we have

$$\sum_{i=1}^{n} f(\mathbf{r}(t_i^*))\,\Delta s_i \approx \sum_{i=1}^{n} f(\mathbf{r}(t_i^*))\|\mathbf{r}'(t_i^*)\|\,\Delta t_i. \tag{3.2.5}$$

See Figure 3.2.4.

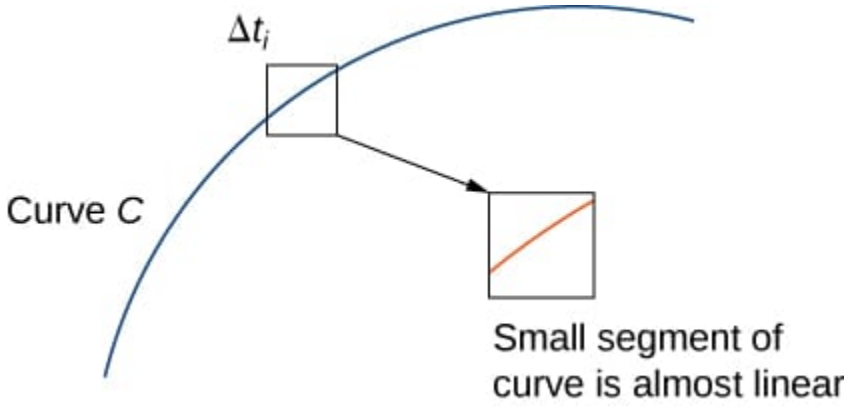


Figure 3.2.4: If we zoom in on the curve enough by making Δt_i very small, then the corresponding piece of the curve is approximately linear.

Note that

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(\mathbf{r}(t_i^*))\|\mathbf{r}'(t_i^*)\|\,\Delta t_i = \int_a^b f(\mathbf{r}(t))\|\mathbf{r}'(t)\|\,dt. \tag{3.2.6}$$

In other words, as the widths of intervals $[t_{i-1}, t_i]$ shrink to zero, the sum $\sum_{i=1}^{n} f(\mathbf{r}(t_i^*))\|\mathbf{r}\,'(t_i^*)\|\,\Delta t_i$ converges to the integral $\int_a^b f(\mathbf{r}(t))\|\mathbf{r}\,'(t)\|\,dt$. Therefore, we have the following theorem.

Theorem: Evaluating a Scalar Line Integral

Let f be a continuous function with a domain that includes the smooth curve C with parameterization $\mathbf{r}(t)$, $a \le t \le b$. Then

$$\int_C f\,ds = \int_a^b f(\mathbf{r}(t))\|\mathbf{r}'(t)\|\,dt. \tag{3.2.7}$$

Although we have labeled Equation 3.2.4 as an equation, it is more accurately considered an approximation because we can show that the left-hand side of Equation 3.2.4 approaches the right-hand side as $n \to \infty$. In other words, letting the widths of the pieces shrink to zero makes the right-hand sum arbitrarily close to the left-hand sum. Since

$$\|\mathbf{r}'(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}, \tag{3.2.8}$$

we obtain the following theorem, which we use to compute scalar line integrals.

Theorem: Scalar Line Integral Calculation

Let f be a continuous function with a domain that includes the smooth curve C with parameterization $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, $a \le t \le b$. Then

$$\int_C f(x,y,z)\,ds = \int_a^b f(\mathbf{r}(t))\sqrt{(x\prime(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt. \tag{3.2.9}$$

Similarly,

$$\int_C f(x,y)\,ds = \int_a^b f(\mathbf{r}(t))\sqrt{(x'(t))^2 + (y'(t))^2}\,dt \tag{3.2.10}$$

if C is a planar curve and f is a function of two variables.

Note that a consequence of this theorem is the equation $ds = \|\mathbf{r}\,'(t)\|\,dt$. In other words, the change in arc length can be viewed as a change in the t-domain, scaled by the magnitude of vector $\mathbf{r}\,'(t)$.

Example 3.2.2: Evaluating a Line Integral

Find the value of integral $\int_C (x^2 + y^2 + z)\ ds$, where C is part of the helix parameterized by $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$, $0 \le t \le 2\pi$.

Solution

To compute a scalar line integral, we start by converting the variable of integration from arc length s to t. Then, we can use Equation 3.2.1 to compute the integral with respect to t. Note that

$$f(\mathbf{r}(t)) = \cos^2 t + \sin^2 t + t = 1 + t$$

and

$$\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} = \sqrt{(-\sin(t))^2 + \cos^2(t) + 1} = \sqrt{2}.$$

Therefore,

$$\int_C (x^2 + y^2 + z)\,ds = \int_0^{2\pi} (1 + t)\sqrt{2}\,dt.$$

Notice that Equation 3.2.1 translated the original difficult line integral into a manageable single-variable integral. Since

$$\begin{aligned}\int_0^{2\pi} (1 + t)\sqrt{2}\,dt &= \left[\sqrt{2}t + \frac{\sqrt{2}t^2}{2}\right]_0^{2\pi} \\ &= 2\sqrt{2}\pi + 2\sqrt{2}\pi^2,\end{aligned}$$

we have

$$\int_C (x^2 + y^2 + z)\,ds = 2\sqrt{2}\pi + 2\sqrt{2}\pi^2.$$

Exercise 3.2.2

Evaluate $\int_C (x^2 + y^2 + z)ds$, where C is the curve with parameterization $\mathbf{r}(t) = \langle \sin(3t), \cos(3t)\rangle$, $0 \le t \le \frac{\pi}{4}$.

Hint available at the end of this section.

Example 3.2.3: Independence of Parameterization

Find the value of integral $\int_C (x^2+y^2+z)\ ds$, where C is part of the helix parameterized by $\mathbf{r}(t)=\langle\cos(2t),\sin(2t),2t\rangle$, $0\le t\le\pi$. Notice that this function and curve are the same as in the previous example; the only difference is that the curve has been reparameterized so that time runs twice as fast.

Solution

As with the previous example, we use Equation 3.2.1 to compute the integral with respect to t. Note that $f(\mathbf{r}(t))=\cos^2(2t)+\sin^2(2t)+2t=2t+1$ and

$$\begin{aligned}\sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}&=\sqrt{(-2\sin(2t))^2+(2\cos(2t))^2+(2)^2)}\\&=2\sqrt{2}\end{aligned}$$

so we have

$$\begin{aligned}\int_C (x^2+y^2+z)ds&=2\sqrt{2}\int_0^\pi (1+2t)dt\\&=2\sqrt{2}\Big[t+t^2\Big]_0^\pi\\&=2\sqrt{2}(\pi+\pi^2).\end{aligned}$$

Notice that this agrees with the answer in the previous example. Changing the parameterization did not change the value of the line integral. Scalar line integrals are independent of parameterization, as long as the curve is traversed exactly once by the parameterization.

Exercise 3.2.3

Evaluate line integral $\int_C (x^2+yz)\ ds$, where C is the line with parameterization $\mathbf{r}(t)=\langle 2t,5t,-t\rangle$, $0\le t\le 10$. Reparameterize C with parameterization $s(t)=\langle 4t,10t,-2t\rangle$, $0\le t\le 5$, recalculate line integral $\int_C (x^2+yz)\ ds$, and notice that the change of parameterization had no effect on the value of the integral.

Hint available at the end of this section.

Now that we can evaluate line integrals, we can use them to calculate arc length. If $f(x,y,z)=1$, then

$$\begin{aligned}\int_C f(x,y,z)\,ds&=\lim_{n\to\infty}\sum_{i=1}^n f(t_i^*)\,\Delta s_i\\&=\lim_{n\to\infty}\sum_{i=1}^n \Delta s_i\\&=\lim_{n\to\infty}\text{length}(C)\\&=\text{length}(C).\end{aligned}$$

Therefore, $\int_C 1\ ds$ is the arc length of C.

Example 3.2.4: Calculating Arc Length

A wire has a shape that can be modeled with the parameterization $\mathbf{r}(t) = \langle \cos t, \sin t, \frac{2}{3}t^{3/2} \rangle$, $0 \le t \le 4\pi$. Find the length of the wire.

Solution

The length of the wire is given by $\int_C 1 \; ds$, where C is the curve with parameterization $\mathbf{r}$. Therefore,

$$\begin{aligned} \text{The length of the wire} &= \int_C 1 \, ds \\ &= \int_0^{4\pi} ||\mathbf{r}'(t)|| \, dt \\ &= \int_0^{4\pi} \sqrt{(-\sin t)^2 + \cos^2 t + t} dt \\ &= \int_0^{4\pi} \sqrt{1+t} dt \\ &= \frac{2(1+t)^{\frac{3}{2}}}{3} \Bigg|_0^{4\pi} \\ &= \frac{2}{3}\left((1+4\pi)^{3/2} - 1\right). \end{aligned}$$

Exercise 3.2.4

Find the length of a wire with parameterization $\mathbf{r}(t) = \langle 3t+1, 4-2t, 5+2t \rangle$, $0 \le t \le 4$.

Hint available at the end of this section.

Vector Line Integrals

The second type of line integrals are vector line integrals, in which we integrate along a curve through a vector field. For example, let

$$\mathbf{F}(x,y,z) = P(x,y,z)\,\mathbf{i} + Q(x,y,z)\,\mathbf{j} + R(x,y,z)\,\mathbf{k} \tag{3.2.11}$$

be a continuous vector field in $\mathbb{R}^3$ that represents a force on a particle, and let C be a smooth curve in $\mathbb{R}^3$ contained in the domain of $\mathbf{F}$. How would we compute the work done by $\mathbf{F}$ in moving a particle along C?

To answer this question, first note that a particle could travel in two directions along a curve: a forward direction and a backward direction. The work done by the vector field depends on the direction in which the particle is moving. Therefore, we must specify a direction along curve C; such a specified direction is called an orientation of a curve. The specified direction is the positive direction along C; the opposite direction is the negative direction along C. When C has been given an orientation, C is called an oriented curve (Figure 3.2.5). The work done on the particle depends on the direction along the curve in which the particle is moving.

A closed curve is one for which there exists a parameterization $\mathbf{r}(t)$, $a \le t \le b$, such that $\mathbf{r}(a) = \mathbf{r}(b)$, and the curve is traversed exactly once. In other words, the parameterization is one-to-one on the domain (a, b).

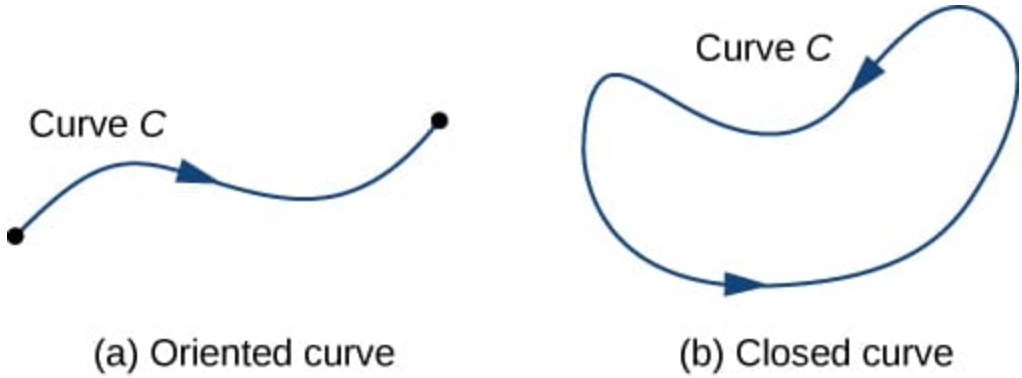


Figure 3.2.5: (a) An oriented curve between two points. (b) A closed oriented curve.

Let $\mathbf{r}(t)$ be a parameterization of C for $a \le t \le b$ such that the curve is traversed exactly once by the particle and the particle moves in the positive direction along C. Divide the parameter interval $[a, b]$ into n subintervals $[t_{i-1}, t_i]$, $0 \le i \le n$, of equal width. Denote the endpoints of $r(t_0)$, $r(t_1)$,..., $r(t_n)$ by P_0,...,P_n. Points P_i divide C into n pieces. Denote the length of the piece from P_{i-1} to P_i by Δs_i. For each i, choose a value t_i^* in the subinterval $[t_{i-1}, t_i]$. Then, the endpoint of $\mathbf{r}(t_i^*)$ is a point in the piece of C between P_{i-1} and P_i (Figure 3.2.6). If Δs_i is small, then as the particle moves from P_{i-1} to P_i along C, it moves approximately in the direction of $\mathbf{T}(P_i)$, the unit tangent vector at the endpoint of $\mathbf{r}(t_i^*)$. Let P_i^* denote the endpoint of $\mathbf{r}(t_i^*)$. Then, the work done by the force vector field in moving the particle from P_{i-1} to P_i is $\mathbf{F}(P_i^*) \cdot (\Delta s_i \mathbf{T}(P_i^*))$, so the total work done along C is

$$\sum_{i=1}^{n} \mathbf{F}(P_i^*) \cdot (\Delta s_i \mathbf{T}(P_i^*)) = \sum_{i=1}^{n} \mathbf{F}(P_i^*) \cdot \mathbf{T}(P_i^*)\, \Delta s_i. \tag{3.2.12}$$

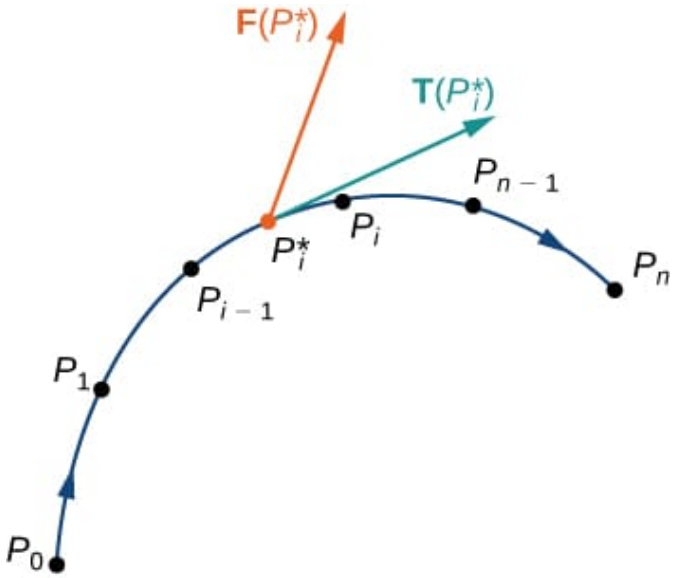


Figure 3.2.6: Curve C is divided into n pieces, and a point inside each piece is chosen. The dot product of any tangent vector in the ith piece with the corresponding vector $\mathbf{F}$ is approximated by $\mathbf{F}(P_i^*) \cdot \mathbf{T}(P_i^*)$.

Letting the arc length of the pieces of C get arbitrarily small by taking a limit as $n \to \infty$ gives us the work done by the field in moving the particle along C. Therefore, the work done by $\mathbf{F}$ in moving the particle in the positive direction along C is defined as

$$W = \int_C \mathbf{F} \cdot \mathbf{T}\, ds, \tag{3.2.13}$$

which gives us the concept of a vector line integral.

Definition: Line Integral of a Vector Field

The vector line integral of vector field $\mathbf{F}$ along oriented smooth curve C is

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \lim_{n\to\infty} \sum_{i=1}^{n} \mathbf{F}(P_i^*) \cdot \mathbf{T}(P_i^*) \Delta s_i \tag{3.2.14}$$

if that limit exists.

With scalar line integrals, neither the orientation nor the parameterization of the curve matters. As long as the curve is traversed exactly once by the parameterization, the value of the line integral is unchanged. With vector line integrals, the orientation of the curve does matter. If we think of the line integral as computing work, then this makes sense: if you hike up a mountain, then the gravitational force of Earth does negative work on you. If you walk down the mountain by the exact same path, then Earth's gravitational force does positive work on you. In other words, reversing the path changes the work value from negative to positive in this case. Note that if C is an oriented curve, then we let $-C$ represent the same curve but with opposite orientation.

As with scalar line integrals, it is easier to compute a vector line integral if we express it in terms of the parameterization function $\mathbf{r}$ and the variable t. To translate the integral $\int_C \mathbf{F} \cdot \mathbf{T} ds$ in terms of t, note that unit tangent vector $\mathbf{T}$ along C is given by $\mathbf{T} = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$ (assuming $\|\mathbf{r}\,'(t)\| \neq 0$). Since $ds = \|\mathbf{r}\,'(t)\|\ dt$, as we saw when discussing scalar line integrals, we have

$$\mathbf{F} \cdot \mathbf{T}\, ds = \mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \|\mathbf{r}'(t)\| dt = \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt. \tag{3.2.15}$$

Thus, we have the following formula for computing vector line integrals:

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt. \tag{3.2.16}$$

Because of Equation 3.2.16, we often use the notation $\int_C \mathbf{F} \cdot d\mathbf{r}$ for the line integral $\int_C \mathbf{F} \cdot \mathbf{T}\ ds$.

If $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, then $\dfrac{d\mathbf{r}}{dt}$ denotes vector $\langle x\,'(t), y\,'(t), z\,'(t)\rangle$, and $d\mathbf{r} = \mathbf{r}'(t)\ dt$.

Example 3.2.5: Evaluating a Vector Line Integral

Find the value of integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the semicircle parameterized by $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$, $0 \leq t \leq \pi$ and $\mathbf{F} = \langle -y, x\rangle$ in Figure 3.2.7.

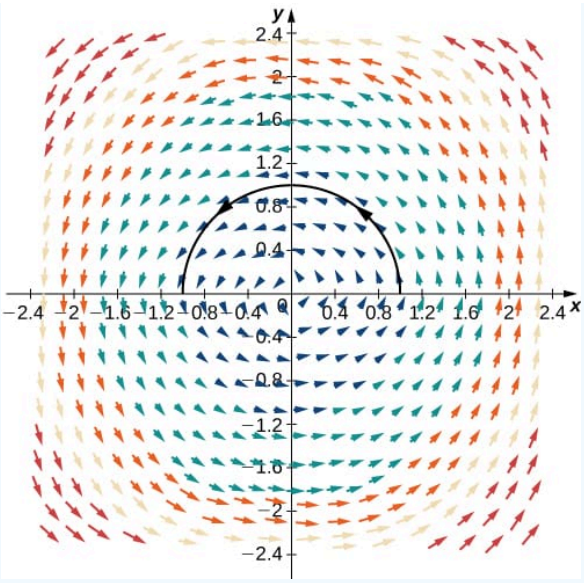


Figure 3.2.7: Curve $\mathbf{r}(t) = \langle \cos t,\ \sin t \rangle$, $0 \le t \le \pi$ in vector field $\mathbf{F} = \langle -y,\ x \rangle$.

Solution

We can use Equation 3.2.16 to convert the variable of integration from s to t. We then have

$$\mathbf{F}(\mathbf{r}(t)) = \langle -\sin t, \cos t \rangle \text{ and } \mathbf{r}'(t) = \langle -\sin t, \cos t \rangle.$$

Therefore,

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^\pi \langle -\sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle \, dt \\ &= \int_0^\pi \sin^2 t + \cos^2 t \, dt \\ &= \int_0^\pi 1 \, dt = \pi. \end{aligned}$$

Example 3.2.6: Reversing Orientation

Find the value of integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the semicircle parameterized by $\mathbf{r}(t) = \langle \cos(t + \pi), \sin t \rangle$, $0 \le t \le \pi$ and $\mathbf{F} = \langle -y, x \rangle$.

Solution

Notice that this is the same problem as Example 3.2.5, except the orientation of the curve has been reversed. In this example, the parameterization starts at $\mathbf{r}(0) = \langle -1, 0 \rangle$ and ends at $\mathbf{r}(\pi) = \langle 1, 0 \rangle$. By Equation 3.2.16,

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^\pi \langle -\sin t, \cos(t + \pi) \rangle \cdot \langle -\sin(t + \pi), \cos t \rangle dt \\ &= \int_0^\pi \langle -\sin t, -\cos t \rangle \cdot \langle \sin t, \cos t \rangle dt \\ &= \int_0^\pi (-\sin^2 t - \cos^2 t) dt \\ &= \int_0^\pi -1 dt \\ &= -\pi. \end{aligned}$$

Notice that this is the negative of the answer in Example 3.2.5. It makes sense that this answer is negative because the orientation of the curve goes against the "flow" of the vector field.

Let C be an oriented curve and let $-C$ denote the same curve but with the orientation reversed. Then, the previous two examples illustrate the following fact:

$$\int_{-C} \mathbf{F}\cdot d\mathbf{r} = -\int_{C} \mathbf{F}\cdot d\mathbf{r}. \tag{3.2.17}$$

That is, reversing the orientation of a curve changes the sign of a line integral.

Exercise 3.2.6

Let $\mathbf{F} = x\ \mathbf{i} + y\ \mathbf{j}$ be a vector field and let C be the curve with parameterization $\langle t, t^2\rangle$ for $0 \le t \le 2$. Which is greater: $\int_C \mathbf{F}\cdot\mathbf{T}\ ds$ or $\int_{-C} \mathbf{F}\cdot\mathbf{T}\ ds$?

Hint available at the end of this section.

Another standard notation for integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ is $\int_C P\ dx + Q\ dy + R\ dz$. In this notation, P, Q, and R are functions, and we think of $d\mathbf{r}$ as vector $\langle dx, dy, dz\rangle$. To justify this convention, recall that $d\mathbf{r} = \mathbf{T}\ ds = \mathbf{r}\,'(t)\ dt = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right\rangle\ dt$. Therefore,

$$\mathbf{F}\cdot d\mathbf{r} = \langle P, Q, R\rangle \cdot \langle dx, dy, dz\rangle = P\,dx + Q\,dy + R\,dz. \tag{3.2.18}$$

If $d\mathbf{r} = \langle dx, dy, dz\rangle$, then $\frac{dr}{dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right\rangle$, which implies that $d\mathbf{r} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right\rangle\ dt$. Therefore

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C P\,dx + Q\,dy + R\,dz \tag{3.2.19}$$

$$= \int_a^b \left(P\big(\mathbf{r}(t)\big)\frac{dx}{dt} + Q\big(\mathbf{r}(t)\big)\frac{dy}{dt} + R\big(\mathbf{r}(t)\big)\frac{dz}{dt}\right)\ dt. \tag{3.2.20}$$

Example 3.2.7: Finding the Value of an Integral of the Form $\int_C P\ dx + Q\ dy + R\ dz$

Find the value of integral $\int_C z\ dx + x\ dy + y\ dz$, where C is the curve parameterized by $\mathbf{r}(t) = \langle t^2, \sqrt{t}, t\rangle$, $1 \le t \le 4$.

Solution

As with our previous examples, to compute this line integral we should perform a change of variables to write everything in terms of t. In this case, Equation 3.2.20 allows us to make this change:

$$\begin{aligned}\int_C z\,dx + x\,dy + y\,dz &= \int_1^4 \left(t(2t) + t^2\left(\frac{1}{2\sqrt{t}}\right) + \sqrt{t}\right)\ dt \\ &= \int_1^4 \left(2t^2 + \frac{t^{3/2}}{2} + \sqrt{t}\right)\ dt \\ &= \left[\frac{2t^3}{3} + \frac{t^{5/2}}{5} + \frac{2t^{3/2}}{3}\right]_{t=1}^{t=4} \\ &= \frac{793}{15}.\end{aligned}$$

Exercise 3.2.7

Find the value of $\int_C 4x\ dx + z\ dy + 4y^2\ dz$, where C is the curve parameterized by $\mathbf{r}(t) = \langle 4\cos(2t), 2\sin(2t), 3\rangle$, $0 \le t \le \frac{\pi}{4}$.

Hint available at the end of this section.

We have learned how to integrate smooth oriented curves. Now, suppose that C is an oriented curve that is not smooth, but can be written as the union of finitely many smooth curves. In this case, we say that C is a piecewise smooth curve. To be precise, curve C is piecewise smooth if C can be written as a union of n smooth curves C_1, C_2,..., C_n such that the endpoint of C_i is the starting point of C_{i+1} (Figure 3.2.8). When curves C_i satisfy the condition that the endpoint of C_i is the starting point of C_{i+1}, we write their union as $C_1 + C_2 + \cdots + C_n$.

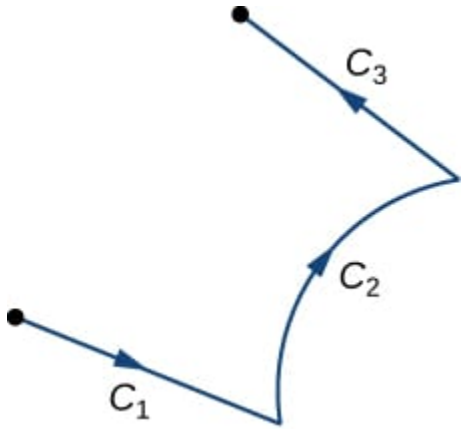


Figure 3.2.8: Curve C is piecewise smooth if C can be written as a union of n smooth curves C_1, C_2,..., C_n such that the endpoint of C_i is the starting point of C_{i+1}.

The next theorem summarizes several key properties of vector line integrals.

Theorem: Properties of Vector Line Integral

Let $\mathbf{F}$ and $\mathbf{G}$ be continuous vector fields with domains that include the oriented smooth curve C. Then

1. $\int_C (\mathbf{F} + \mathbf{G}) \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot d\mathbf{r} + \int_C \mathbf{G} \cdot d\mathbf{r}$
2. $\int_C k\mathbf{F} \cdot d\mathbf{r} = k\int_C \mathbf{F} \cdot d\mathbf{r}$, where k is a constant
3. $\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r}$
4. Suppose instead that C is a piecewise smooth curve in the domains of $\mathbf{F}$ and $\mathbf{G}$, where $C = C_1 + C_2 + \cdots + C_n$ and $C_1, C_2, \ldots, C_n$ are smooth curves such that the endpoint of C_i is the starting point of C_{i+1}. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \cdots + \int_{C_n} \mathbf{F} \cdot d\mathbf{r}. \tag{3.2.21}$$

Notice the similarities between these items and the properties of single-variable integrals. Properties i. and ii. say that line integrals are linear, which is true of single-variable integrals as well. Property iii. says that reversing the orientation of a curve changes the sign of the integral. If we think of the integral as computing the work done on a particle traveling along C, then this makes sense. If the particle moves backward rather than forward, then the value of the work done has the opposite sign. This is analogous to the equation $\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$. Finally, if $[a_1, a_2]$, $[a_2, a_3]$,..., $[a_{n-1}, a_n]$ are intervals, then

$$\int_{a_1}^{a_n} f(x)\,dx = \int_{a_1}^{a_2} f(x)\,dx + \int_{a_2}^{a_3} f(x)\,dx + \cdots + \int_{a_{n-1}}^{a_n} f(x)\,dx, \tag{3.2.22}$$

which is analogous to property iv.

Example 3.2.8: Using Properties to Compute a Vector Line Integral

Find the value of integral $\int_C \mathbf{F} \cdot \mathbf{T}\ ds$, where C is the rectangle (oriented counterclockwise) in a plane with vertices $(0,0)$, $(2,0)$, $(2,1)$, and $(0,1)$, and where $\mathbf{F} = \langle x - 2y, y - x \rangle$ (Figure 3.2.9).

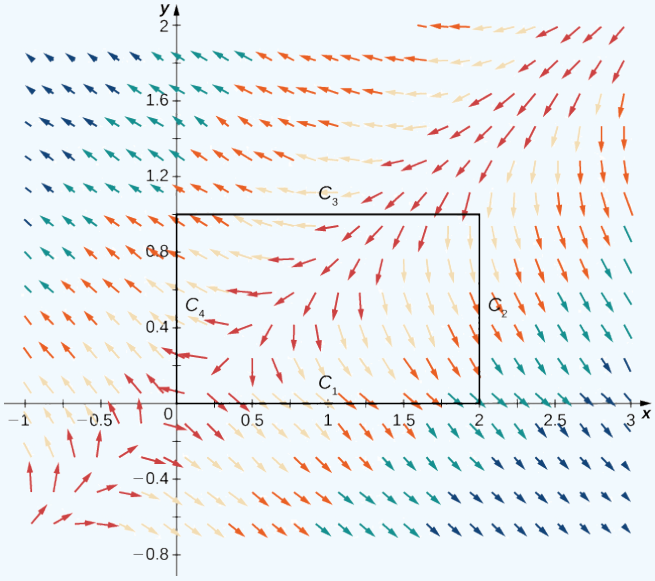


Figure 3.2.9: Curve C is the rectangle (oriented counterclockwise) in a plane with vertices $(0,0)$, $(2,0)$, $(2,1)$, and $(0,1)$, and where $\mathbf{F} = \langle x - 2y, y - x \rangle$.

Solution

Note that curve C is the union of its four sides, and each side is smooth. Therefore C is piecewise smooth. Let C_1 represent the side from $(0,0)$ to $(2,0)$, let C_2 represent the side from $(2,0)$ to $(2,1)$, let C_3 represent the side from $(2,1)$ to $(0,1)$, and let C_4 represent the side from $(0,1)$ to $(0,0)$ (Figure 3.2.9). Then,

$$\int_C \mathbf{F} \cdot \mathbf{T}\, dr = \int_{C_1} \mathbf{F} \cdot \mathbf{T}\, dr + \int_{C_2} \mathbf{F} \cdot \mathbf{T}\, dr + \int_{C_3} \mathbf{F} \cdot \mathbf{T}\, dr + \int_{C_4} \mathbf{F} \cdot \mathbf{T}\, dr. \tag{3.2.23}$$

We want to compute each of the four integrals on the right-hand side using Equation 3.2.1. Before doing this, we need a parameterization of each side of the rectangle. Here are four parameterizations (note that they traverse C counterclockwise):

$$\begin{aligned} C_1 &: \langle t, 0 \rangle, 0 \le t \le 2 \\ C_2 &: \langle 2, t \rangle, 0 \le t \le 1 \\ C_3 &: \langle 2 - t, 1 \rangle, 0 \le t \le 2 \\ C_4 &: \langle 0, 1 - t \rangle, 0 \le t \le 1. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{C_1} \mathbf{F} \cdot \mathbf{T}\, dr &= \int_0^2 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\ &= \int_0^2 \langle t - 2(0), 0 - t \rangle \cdot \langle 1, 0 \rangle\, dt = \int_0^2 t\, dt \\ &= \left[\tfrac{t^2}{2} \right]_0^2 = 2. \end{aligned}$$

Notice that the value of this integral is positive, which should not be surprising. As we move along curve C_1 from left to right, our movement flows in the general direction of the vector field itself. At any point along C_1, the tangent vector to the curve and the corresponding vector in the field form an angle that is less than 90°. Therefore, the tangent vector and the force vector have a positive dot product all along C_1, and the line integral will have positive value.

The calculations for the three other line integrals are done similarly:

$$\begin{aligned}\int_{C_2} \mathbf{F}\cdot d\mathbf{r} &= \int_0^1 \langle 2-2t, t-2\rangle \cdot \langle 0,1\rangle \, dt \\ &= \int_0^1 (t-2)\, dt \\ &= \left[\tfrac{t^2}{2} - 2t\right]_0^1 = -\frac{3}{2},\end{aligned}$$

$$\begin{aligned}\int_{C_3} \mathbf{F}\cdot\mathbf{T}\, ds &= \int_0^2 \langle (2-t)-2, 1-(2-t)\rangle \cdot \langle -1,0\rangle \, dt \\ &= \int_0^2 t\, dt = 2,\end{aligned}$$

and

$$\begin{aligned}\int_{C_4} \mathbf{F}\cdot d\mathbf{r} &= \int_0^1 \langle -2(1-t), 1-t\rangle \cdot \langle 0,-1\rangle \, dt \\ &= \int_0^1 (t-1)\, dt \\ &= \left[\tfrac{t^2}{2} - t\right]_0^1 = -\frac{1}{2}.\end{aligned}$$

Thus, we have $\int_C \mathbf{F}\cdot d\mathbf{r} = 2$.

Exercise 3.2.8

Calculate line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}$ is vector field $\langle y^2, 2xy+1\rangle$ and C is a triangle with vertices $(0,0)$, $(4,0)$, and $(0,5)$, oriented counterclockwise.

Hint available at the end of this section.

Applications of Line Integrals

Scalar line integrals have many applications. They can be used to calculate the length or mass of a wire, the surface area of a sheet of a given height, or the electric potential of a charged wire given a linear charge density. Vector line integrals are extremely useful in physics. They can be used to calculate the work done on a particle as it moves through a force field, or the flow rate of a fluid across a curve. Here, we calculate the mass of a wire using a scalar line integral and the work done by a force using a vector line integral.

Suppose that a piece of wire is modeled by curve C in space. The mass per unit length (the linear density) of the wire is a continuous function $\rho(x,y,z)$. We can calculate the total mass of the wire using the scalar line integral $\int_C \rho(x,y,z)\ ds$. The reason is that mass is density multiplied by length, and therefore the density of a small piece of the wire can be approximated by $\rho(x^*,y^*,z^*)\ \Delta s$ for some point (x^*,y^*,z^*) in the piece. Letting the length of the pieces shrink to zero with a limit yields the line integral $\int_C \rho(x,y,z)\ ds$.

Example 3.2.9: Calculating the Mass of a Wire

Calculate the mass of a spring in the shape of a curve parameterized by $\langle t, 2\cos t, 2\sin t\rangle$, $0 \le t \le \dfrac{\pi}{2}$, with a density function given by $\rho(x, y, z) = e^x + yz$ kg/m (Figure 3.2.10).

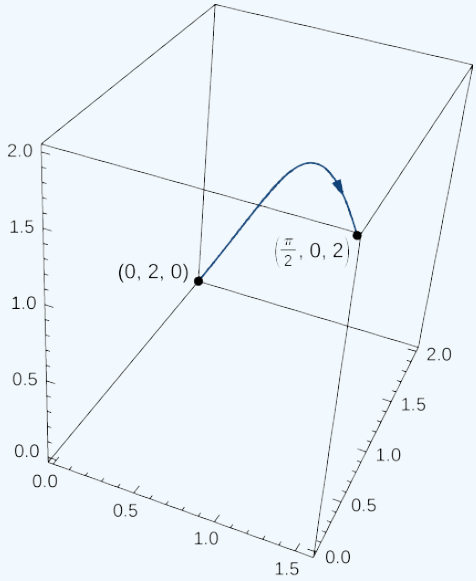


Figure 3.2.10: The wire is represented by a curve which is parameterized by $\langle t, 2\cos t, 2\sin t\rangle$, $0 \le t \le \dfrac{\pi}{2}$.

Solution

To calculate the mass of the spring, we must find the value of the scalar line integral $\displaystyle\int_C (e^x + yz)\ ds$, where C is the given helix. To calculate this integral, we write it in terms of t using Equation 3.2.1:

$$\begin{aligned}\int_C (e^x + yz)\ ds &= \int_0^{\frac{\pi}{2}} \left((e^t + 4\cos t \sin t)\sqrt{1 + (-2\cos t)^2 + (2\sin t)^2}\right)\ dt \\ &= \int_0^{\frac{\pi}{2}} \left((e^t + 4\cos t \sin t)\sqrt{5}\right)\ dt \\ &= \sqrt{5}\left[e^t + 2\sin^2 t\right]_{t=0}^{t=\pi/2} \\ &= \sqrt{5}(e^{\pi/2} + 1).\end{aligned}$$

Therefore, the mass is $\sqrt{5}(e^{\pi/2} + 1)$ kg.

Exercise 3.2.9

Calculate the mass of a spring in the shape of a helix parameterized by $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$, $0 \le t \le 6\pi$, with a density function given by $\rho(x, y, z) = x + y + z$ kg/m.

Hint available at the end of this section.

When we first defined vector line integrals, we used the concept of work to motivate the definition. Therefore, it is not surprising that calculating the **work done by a vector field** representing a force is a standard use of vector line integrals. Recall that if an object moves along curve C in force field $\mathbf{F}$, then the work required to move the object is given by $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$.

Example 3.2.10: Calculating Work

How much work is required to move an object in vector force field $\mathbf{F} = \langle yz, xy, xz \rangle$ along path $\mathbf{r}(t) = \langle t^2, t, t^4 \rangle,\ 0 \le t \le 1$ (Figure 3.2.11)?

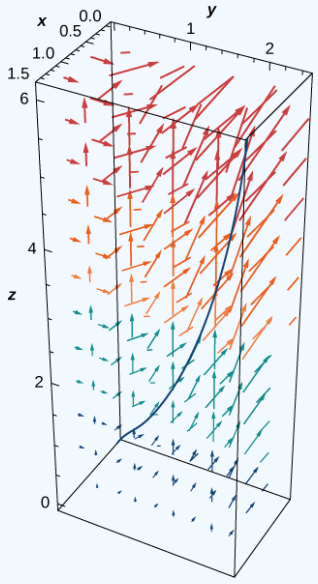


Figure 3.2.11: A three-dimensional vector field $\mathbf{F} = \langle yz, xy, xz \rangle$ and the parametric path $\mathbf{r}(t) = \langle t^2, t, t^4 \rangle$ for $0 \le t \le 1$. The vector field is represented by arrows whose direction and magnitude indicate the nature of the field. The path $\mathbf{r}(t)$ is shown as a smooth curve passing through the field, illustrating the trajectory along which the work integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is calculated. This visualization helps demonstrate the relationship between the vector field and the parametric curve.

Solution

Let C denote the given path. We need to find the value of $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$. To do this, we use Equation 3.2.16 :

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^1 (\langle t^5, t^3, t^6 \rangle \cdot \langle 2t, 1, 4t^3 \rangle)\, dt \\ &= \int_0^1 (2t^6 + t^3 + 4t^9)\, dt \\ &= \Big[\frac{2t^7}{7} + \frac{t^4}{4} + \frac{2t^{10}}{5}\Big]_{t=0}^{t=1} = \frac{131}{140} \text{ units of work.}\end{aligned}$$

Flux

We close this section by discussing two key concepts related to line integrals: flux across a plane curve and circulation along a plane curve. Flux is used in applications to calculate fluid flow across a curve, and the concept of circulation is important for characterizing conservative gradient fields in terms of line integrals. Both these concepts are used heavily throughout the rest of this chapter. The idea of flux is especially important for Green's theorem, and in higher dimensions for Stokes' theorem and the divergence theorem.

Let C be a plane curve and let $\mathbf{F}$ be a vector field in the plane. Imagine C is a membrane across which fluid flows, but C does not impede the flow of the fluid. In other words, C is an idealized membrane invisible to the fluid. Suppose $\mathbf{F}$ represents the velocity field of the fluid. How could we quantify the rate at which the fluid is crossing C?

Recall that the line integral of $\mathbf{F}$ along C is $\displaystyle\int_C \mathbf{F} \cdot \mathbf{T}\ ds$—in other words, the line integral is the dot product of the vector field with the unit tangential vector with respect to arc length. If we replace the unit tangential vector with unit normal vector $\mathbf{N}(t)$ and

instead compute integral $\int_C \mathbf{F} \cdot \mathbf{N}\ ds$, we determine the flux across C. To be precise, the definition of integral $\int_C \mathbf{F} \cdot \mathbf{N}\ ds$ is the same as integral $\int_C \mathbf{F} \cdot \mathbf{T}\ ds$, except the $\mathbf{T}$ in the Riemann sum is replaced with $\mathbf{N}$. Therefore, the flux across C is defined as

$$\int_C \mathbf{F} \cdot \mathbf{N}\, ds = \lim_{n\to\infty} \sum_{i=1}^{n} \mathbf{F}(P_i^*) \cdot \mathbf{N}(P_i^*)\, \Delta s_i, \tag{3.2.24}$$

where P_i^* and Δs_i are defined as they were for integral $\int_C \mathbf{F} \cdot \mathbf{T}\ ds$. Therefore, a flux integral is an integral that is *perpendicular* to a vector line integral, because $\mathbf{N}$ and $\mathbf{T}$ are perpendicular vectors.

If $\mathbf{F}$ is a velocity field of a fluid and C is a curve that represents a membrane, then the flux of $\mathbf{F}$ across C is the quantity of fluid flowing across C per unit time, or the rate of flow.

More formally, let C be a plane curve parameterized by $\mathbf{r}(t) = \langle x(t),\ y(t)\rangle$, $a \le t \le b$. Let $\mathbf{n}(t) = \langle y\,'(t),\ -x\,'(t)\rangle$ be the vector that is normal to C at the endpoint of $\mathbf{r}(t)$ and points to the right as we traverse C in the positive direction (Figure 3.2.12). Then, $\mathbf{N}(t) = \dfrac{\mathbf{n}(t)}{\|\mathbf{n}(t)\|}$ is the unit normal vector to C at the endpoint of $\mathbf{r}(t)$ that points to the right as we traverse C.

Definition: Flux

The *flux* of $\mathbf{F}$ across C is line integral

$$\int_C \mathbf{F} \cdot \frac{\mathbf{n}(t)}{\|\mathbf{n}(t)\|}\, ds. \tag{3.2.25}$$

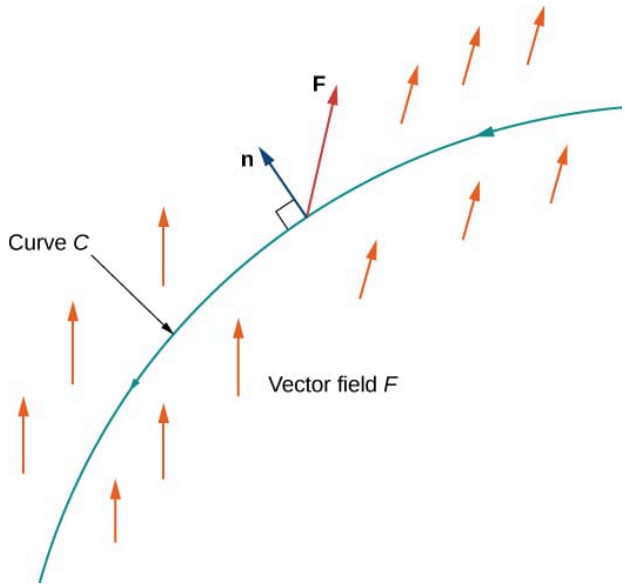


Figure 3.2.12: The concept of flux of a vector field $\mathbf{F}$ across a smooth curve C. The curve C is shown in the plane, with the vector field $\mathbf{F}$ represented by arrows indicating the direction and magnitude of the field. At a given point on C, a unit normal vector $\mathbf{n}$ is shown perpendicular to the tangent of C. The flux of $\mathbf{F}$ across C is computed using the integral: $\int_C \mathbf{F} \cdot \frac{\mathbf{n}(t)}{\|\mathbf{n}(t)\|}\ ds$. This formula calculates the flow of the vector field $\mathbf{F}$ through the curve C, analogous to a vector line integral.

We now give a formula for calculating the flux across a curve. This formula is analogous to the formula used to calculate a vector line integral (see Equation 3.2.16).

Theorem: Calculating Flux Across a Curve

Let $\mathbf{F}$ be a vector field and let C be a smooth curve with parameterization $r(t) = \langle x(t), y(t) \rangle$, $a \leq t \leq b$.Let $\mathbf{n}(t) = \langle y'(t), -x'(t) \rangle$. The flux of $\mathbf{F}$ across C is

$$\int_C \mathbf{F} \cdot \mathbf{N}\, ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{n}(t)\, dt. \tag{3.2.26}$$

Proof

Before deriving the formula, note that

$$\|\mathbf{n}(t)\| = \|\langle y'(t), -x'(t) \rangle\| = \sqrt{(y'(t))^2 + (x'(t))^2} = \|\mathbf{r}'(t)\|. \tag{3.2.27}$$

Therefore,

$$\begin{aligned} \int_C \mathbf{F} \cdot \mathbf{N}\, ds &= \int_C \mathbf{F} \cdot \frac{\mathbf{n}(t)}{\|\mathbf{n}(t)\|}\, ds \\ &= \int_a^b \mathbf{F} \cdot \frac{\mathbf{n}(t)}{\|\mathbf{n}(t)\|} \|\mathbf{r}'(t)\|\, dt \\ &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{n}(t)\, dt. \end{aligned}$$

□

Example 3.2.11: Flux across a Curve

Calculate the flux of $\mathbf{F} = \langle 2x, 2y \rangle$ across a unit circle oriented counterclockwise (Figure 3.2.13).

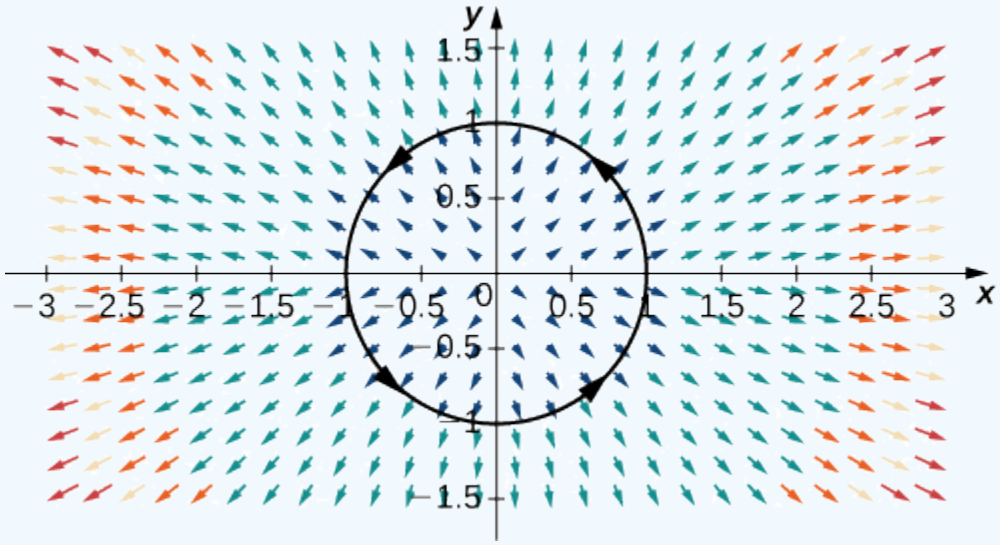


Figure 3.2.13: A two-dimensional vector field $\mathbf{F} = \langle 2x, 2y \rangle$ represented by arrows indicating the direction and magnitude of the field.

Solution

To compute the flux, we first need a parameterization of the unit circle. We can use the standard parameterization $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$, $0 \leq t \leq 2\pi$. The normal vector to a unit circle is $\langle \cos t, \sin t \rangle$. Therefore, the flux is

$$\begin{aligned}\int_C \mathbf{F}\cdot\mathbf{N}\,ds &= \int_0^{2\pi}\langle 2\cos t, 2\sin t\rangle\cdot\langle\cos t,\sin t\rangle\,dt\\ &= \int_0^{2\pi}(2\cos^2 t + 2\sin^2 t)\,dt\\ &= 2\int_0^{2\pi}(\cos^2 t+\sin^2 t)\,dt\\ &= 2\int_0^{2\pi} dt = 4\pi.\end{aligned}$$

Exercise 3.2.11

Calculate the flux of $\mathbf{F}=\langle x+y, 2y\rangle$ across the line segment from $(0,0)$ to $(2,3)$, where the curve is oriented from left to right.

Hint available at the end of this section.

Let $\mathbf{F}(x,y)=\langle P(x,y),Q(x,y)\rangle$ be a two-dimensional vector field. Recall that integral $\int_C \mathbf{F}\cdot\mathbf{T}\ ds$ is sometimes written as $\int_C P\ dx + Q\ dy$. Analogously, flux $\int_C \mathbf{F}\cdot\mathbf{N}\ ds$ is sometimes written in the notation $\int_C -Q\ dx + P\ dy$, because the unit normal vector $\mathbf{N}$ is perpendicular to the unit tangent $\mathbf{T}$. Rotating the vector $d\mathbf{r}=\langle dx, dy\rangle$ by 90° results in vector $\langle dy, -dx\rangle$. Therefore, the line integral in Example 3.2.8 can be written as $\int_C -2y\ dx + 2x\ dy$.

Circulation

Now that we have defined flux, we can turn our attention to circulation. The line integral of vector field $\mathbf{F}$ along an oriented closed curve is called the circulation of $\mathbf{F}$ along C. Circulation line integrals have their own notation: $\oint_C \mathbf{F}\cdot\mathbf{T}\ ds$. The circle on the integral symbol denotes that C is "circular" in that it has no endpoints. Example 3.2.5 shows a calculation of circulation.

To see where the term *circulation* comes from and what it measures, let $\mathbf{v}$ represent the velocity field of a fluid and let C be an oriented closed curve. At a particular point P, the closer the direction of $\mathbf{v}(P)$ is to the direction of $\mathbf{T}(P)$, the larger the value of the dot product $\mathbf{v}(P)\cdot\mathbf{T}(P)$. The maximum value of $\mathbf{v}(P)\cdot\mathbf{T}(P)$ occurs when the two vectors are pointing in the exact same direction; the minimum value of $\mathbf{v}(P)\cdot\mathbf{T}(P)$ occurs when the two vectors are pointing in opposite directions. Thus, the value of the circulation $\oint_C \mathbf{v}\cdot\mathbf{T}\ ds$ measures the tendency of the fluid to move in the direction of C.

Example 3.2.12: Calculating Circulation

Let $\mathbf{F}=\langle -y,\ x\rangle$ be the vector field from Example 3.2.3 and let C represent the unit circle oriented counterclockwise. Calculate the circulation of $\mathbf{F}$ along C.

Solution

We use the standard parameterization of the unit circle: $\mathbf{r}(t)=\langle\cos t,\sin t\rangle$, $0\le t\le 2\pi$. Then, $\mathbf{F}(\mathbf{r}(t))=\langle -\sin t,\cos t\rangle$ and $\mathbf{r}\,'(t)=\langle -\sin t,\cos t\rangle$. Therefore, the circulation of $\mathbf{F}$ along C is

$$\begin{aligned}\oint_C \mathbf{F}\cdot\mathbf{T}\,ds &= \int_0^{2\pi}\langle -\sin t,\cos t\rangle\cdot\langle -\sin t,\cos t\rangle\,dt\\ &= \int_0^{2\pi}(\sin^2 t+\cos^2 t)\,dt\\ &= \int_0^{2\pi} dt = 2\pi \text{ units of work.}\end{aligned}$$

Notice that the circulation is positive. The reason for this is that the orientation of C "flows" with the direction of $\mathbf{F}$. At any point along the circle, the tangent vector and the vector from $\mathbf{F}$ form an angle of less than 90°, and therefore the corresponding dot product is positive.

In Example 3.2.12, what if we had oriented the unit circle clockwise? We denote the unit circle oriented clockwise by $-C$. Then

$$\oint_{-C} \mathbf{F} \cdot \mathbf{T}\, ds = -\oint_{C} \mathbf{F} \cdot \mathbf{T}\, ds = -2\pi \text{ units of work.} \tag{3.2.28}$$

Notice that the circulation is negative in this case. The reason for this is that the orientation of the curve flows against the direction of $\mathbf{F}$.

Exercise 3.2.12

Calculate the circulation of $\mathbf{F}(x,y) = \langle -\dfrac{y}{x^2+y^2},\ \dfrac{x}{x^2+y^2} \rangle$ along a unit circle oriented counterclockwise.

Hint available at the end of this section.

Example 3.2.13: Calculating Work

Calculate the work done on a particle that traverses circle C of radius 2 centered at the origin, oriented counterclockwise, by field $\mathbf{F}(x,y) = \langle -2,\ y \rangle$. Assume the particle starts its movement at $(1,\ 0)$.

Solution

The work done by $\mathbf{F}$ on the particle is the circulation of $\mathbf{F}$ along C: $\oint_C \mathbf{F} \cdot \mathbf{T}\ ds$. We use the parameterization $\mathbf{r}(t) = \langle 2\cos t,\ 2\sin t \rangle$, $0 \le t \le 2\pi$ for C. Then, $\mathbf{r}\,'(t) = \langle -2\sin t,\ 2\cos t \rangle$ and $\mathbf{F}(\mathbf{r}(t)) = \langle -2,\ 2\sin t \rangle$. Therefore, the circulation of $\mathbf{F}$ along C is

$$\begin{aligned}
\oint_C \mathbf{F} \cdot \mathbf{T}\, ds &= \int_0^{2\pi} \langle -2, 2\sin t \rangle \cdot \langle -2\sin t, 2\cos t \rangle\, dt \\
&= \int_0^{2\pi} (4\sin t + 4\sin t \cos t)\, dt \\
&= \Big[-4\cos t + 4\sin^2 t \Big]_0^{2\pi} \\
&= \big(-4\cos(2\pi) + 2\sin^2(2\pi) \big) - \big(-4\cos(0) + 4\sin^2(0) \big) \\
&= -4 + 4 = 0 \text{ units of work.}
\end{aligned}$$

The force field does zero work on the particle.

Notice that the circulation of $\mathbf{F}$ along C is zero. Furthermore, notice that since $\mathbf{F}$ is the gradient of $f(x,y) = -2x + \dfrac{y^2}{2}$, $\mathbf{F}$ is conservative. We prove in a later section that under certain broad conditions, the circulation of a conservative vector field along a closed curve is zero.

Exercise 3.2.14

Calculate the work done by field $\mathbf{F}(x,y) = \langle 2x,\ 3y \rangle$ on a particle that traverses the unit circle. Assume the particle begins its movement at $(-1,\ 0)$.

Hint available at the end of this section.

Key Concepts

- Line integrals generalize the notion of a single-variable integral to higher dimensions. The domain of integration in a single-variable integral is a line segment along the x-axis, but the domain of integration in a line integral is a curve in a plane or in space.
- If C is a curve, then the length of C is $\int_C ds$.
- There are two kinds of line integral: scalar line integrals and vector line integrals. Scalar line integrals can be used to calculate the mass of a wire; vector line integrals can be used to calculate the work done on a particle traveling through a field.
- Scalar line integrals can be calculated using Equation 3.2.1; vector line integrals can be calculated using Equation 3.2.16.
- Two key concepts expressed in terms of line integrals are flux and circulation. Flux measures the rate that a field crosses a given line; circulation measures the tendency of a field to move in the same direction as a given closed curve.

Key Equations

- **Calculating a scalar line integral**
  $$\int_C f(x,y,z)\ ds = \int_a^b f(\mathbf{r}(t))\sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}\ dt$$
- **Calculating a vector line integral**
  $$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}\ ds = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}\,'(t)\ dt$$
  or
  $$\int_C P\ dx + Q\ dy + R\ dz = \int_a^b \left(P\big(\mathbf{r}(t)\big)\frac{dx}{dt} + Q\big(\mathbf{r}(t)\big)\frac{dy}{dt} + R\big(\mathbf{r}(t)\big)\frac{dz}{dt}\right)\ dt$$
- **Calculating flux**
  $$\int_C \mathbf{F}\cdot\frac{\mathbf{n}(t)}{\|\mathbf{n}(t)\|}\ ds = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{n}(t)\ dt$$

Glossary

Circulation
: the tendency of a fluid to move in the direction of curve C. If C is a closed curve, then the circulation of $\mathbf{F}$ along C is line integral $\int_C \mathbf{F}\cdot\mathbf{T}\ ds$, which we also denote $\oint_C \mathbf{F}\cdot\mathbf{T}\ ds$.

Closed curve
: a curve for which there exists a parameterization $\mathbf{r}(t), a\le t\le b$, such that $\mathbf{r}(a)=\mathbf{r}(b)$, and the curve is traversed exactly once

Flux
: the rate of a fluid flowing across a curve in a vector field; the flux of vector field $\mathbf{F}$ across plane curve C is line integral $\int_C \mathbf{F}\cdot\frac{\mathbf{n}(t)}{\|\mathbf{n}(t)\|}\ ds$

Line integral
: the integral of a function along a curve in a plane or in space

Orientation of a curve
: the orientation of a curve C is a specified direction of C

Piecewise smooth curve
: an oriented curve that is not smooth, but can be written as the union of finitely many smooth curves

Scalar line integral
: the scalar line integral of a function f along a curve C with respect to arc length is the integral $\int_C f\ ds$, it is the integral of a scalar function f along a curve in a plane or in space; such an integral is defined in terms of a Riemann sum, as is a single-variable integral

Vector line integral
: the vector line integral of vector field $\mathbf{F}$ along curve C is the integral of the dot product of $\mathbf{F}$ with unit tangent vector

$\mathbf{T}$ of C with respect to arc length, $\int_C \mathbf{F} \cdot \mathbf{T}\ ds$; such an integral is defined in terms of a Riemann sum, similar to a single-variable integral

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Answers

Exercise 3.2.1

Find the shape formed by C and the graph of function $f(x, y) = x + y$.

$\sqrt{2}$

Exercise 3.2.2

Use the two-variable version of scalar line integral definition (Equation 3.2.2).

$$\frac{3\pi}{4} \tag{3.2.29}$$

Exercise 3.2.3

Use Equation 3.2.1.

Both line integrals equal $-\frac{1000\sqrt{30}}{3}$.

Exercise 3.2.4

Find the line integral of one over the corresponding curve.

$4\sqrt{17}$

Exercise 3.2.6

Imagine moving along the path and computing the dot product $\mathbf{F} \cdot \mathbf{T}$ as you go.

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds \tag{3.2.30}$$

Exercise 3.2.7

Write the integral in terms of t using Equation 3.2.20.

-26

Exercise 3.2.8

Write the triangle as a union of its three sides, then calculate three separate line integrals.

0

Exercise 3.2.9

Calculate the line integral of ρ over the curve with parameterization $\mathbf{r}$.

$18\sqrt{2}\pi^2$ kg

Exercise 3.2.11

Use Equation 3.2.26.

$3/2$

Exercise 3.2.12

Use Equation 3.2.26.

2π units of work

Exercise 3.2.14

Use Equation 3.2.26.

0 units of work

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3.2E: Exercises

1. *True or False?* Line integral $\int_C f(x,y)\ ds$ is equal to a definite integral if C is a smooth curve defined on $[a,b]$ and if function f is continuous on some region that contains curve C.

Answer

True

2. *True or False?* Vector functions $\mathbf{r}_1 = t\ \mathbf{i} + t^2\ \mathbf{j}, \quad 0 \le t \le 1$, and $\mathbf{r}_2 = (1-t)\ \mathbf{i} + (1-t)^2\ \mathbf{j}, \quad 0 \le t \le 1$, define the same oriented curve.

3. *True or False?* $\int_{-C}(P\ dx + Q\ dy) = \int_C (P\ dx - Q\ dy)$

Answer

False

4. *True or False?* A piecewise smooth curve C consists of a finite number of smooth curves that are joined together end to end.

5. *True or False?* If C is given by $x(t) = t, \quad y(t) = t, \quad 0 \le t \le 1$, then $\int_C xy\ ds = \int_0^1 t^2\ dt$.

Answer

False

Exercises 1 - 10

Use a computer algebra system (CAS) to evaluate the line integrals over the indicated path.

6. $\int_C (x+y)\ ds$

$C: x = t, y = (1-t), z = 0$ from $(0,1,0)$ to $(1,0,0)$

7. $\int_C (x-y)ds$

$C: \mathbf{r}(t) = 4t\ \mathbf{i} + 3t\ \mathbf{j}$ when $0 \le t \le 2$

Answer

$$\int_C (x-y)\ ds = 10$$

8. $\int_C (x^2+y^2+z^2)\ ds$

$C: \mathbf{r}(t) = \sin t\ \mathbf{i} + \cos t\ \mathbf{j} + 8t\ \mathbf{k}$ when $0 \le t \le \dfrac{\pi}{2}$

9. Evaluate $\int_C xy^4\ ds$, where C is the right half of circle $x^2+y^2=16$ and is traversed in the clockwise direction.

Answer

$$\int_C xy^4\ ds = \frac{8192}{5}$$

10. Evaluate $\int_C 4x^3 ds$, where C is the line segment from $(-2,-1)$ to $(1,2)$.

Exercises 11 - 16

Find the work done.

11. Find the work done by vector field $\mathbf{F}(x,y,z) = x\ \mathbf{i} + 3xy\ \mathbf{j} - (x+z)\ \mathbf{k}$ on a particle moving along a line segment that goes from $(1,4,2)$ to $(0,5,1)$.

Answer

$W = 8$ units of work

12. Find the work done by a person weighing 150 lb walking exactly one revolution up a circular, spiral staircase of radius 3 ft if the person rises 10 ft.

13. Find the work done by force field $\mathbf{F}(x,y,z) = -\frac{1}{2}x\ \mathbf{i} - \frac{1}{2}y\ \mathbf{j} + \frac{1}{4}\ \mathbf{k}$ on a particle as it moves along the helix $\mathbf{r}(t) = \cos t\ \mathbf{i} + \sin t\ \mathbf{j} + t\ \mathbf{k}$ from point $(1,0,0)$ to point $(-1,0,3\pi)$.

Answer

$W = \frac{3\pi}{4}$ units of work

14. Find the work done by vector field $\mathbf{F}(x,y) = y\ \mathbf{i} + 2x\ \mathbf{j}$ in moving an object along path C, which joins points $(1,0)$ and $(0,1)$.

15. Find the work done by force $\mathbf{F}(x,y) = 2y\ \mathbf{i} + 3x\ \mathbf{j} + (x+y)\ \mathbf{k}$ in moving an object along curve $\mathbf{r}(t) = \cos(t)\ \mathbf{i} + \sin(t)\ \mathbf{j} + 16\ \mathbf{k}$, where $0 \le t \le 2\pi$.

Answer

$W = \pi$ units of work

16. Find the mass of a wire in the shape of a circle of radius 2 centered at (3, 4) with linear mass density $\rho(x,y) = y^2$.

Exercises 17 - 27

Evaluate the line integrals.

17. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x,y) = -1\ \mathbf{j}$, and C is the part of the graph of $y = \frac{1}{2}x^3 - x$ from $(2,2)$ to $(-2,-2)$.

Answer

$\int_C \mathbf{F} \cdot d\mathbf{r} = 4$ units of work

18. Evaluate $\int_\gamma (x^2+y^2+z^2)^{-1} ds$, where γ is the helix $x = \cos t, y = \sin t, z = t$, with $0 \le t \le T$.

19. Evaluate $\int_C yz\ dx + xz\ dy + xy\ dz$ over the line segment from $(1,1,1)$ to $(3,2,0)$.

Answer

$\int_C yz\ dx + xz\ dy + xy\ dz = -1$

20. Let C be the line segment from point (0, 1, 1) to point (2, 2, 3). Evaluate line integral $\int_C y\ ds$.

21. Use a computer algebra system to evaluate the line integral $\int_C y^2\ dx + x\ dy$, where C is the arc of the parabola $x = 4$

$-y^2$ from $(-5,-3)$ to $(0,2)$.

Answer

$$\int_C (y^2)\ dx + (x)\ dy = \frac{245}{6}$$

22. Use a computer algebra system to evaluate the line integral $\int_C (x+3y^2)\ dy$ over the path C given by $x = 2t, y = 10t$, where $0 \le t \le 1$.

23. Use a CAS to evaluate line integral $\int_C xy\ dx + y\ dy$ over path C given by $x = 2t, y = 10t$, where $0 \le t \le 1$.

Answer

$$\int_C xy\ dx + y\ dy = \frac{190}{3}$$

24. Evaluate line integral $\int_C (2x-y)\ dx + (x+3y)\ dy$, where C lies along the x-axis from $x = 0$ to $x = 5$.

26. Use a CAS to evaluate $\int_C \frac{y}{2x^2 - y^2}\ ds$, where C is defined by the parametric equations $x = t, y = t$, for $1 \le t \le 5$.

Answer

$$\int_C \frac{y}{2x^2 - y^2}\ ds = \sqrt{2}\ln 5$$

27. Use a CAS to evaluate $\int_C xy\ ds$, where C is defined by the parametric equations $x = t^2, y = 4t$, for $0 \le t \le 1$.

Exercises 28 - 40

Find the work done by force field $\mathbf{F}$ on an object moving along the indicated path.

28. $\mathbf{F}(x,y) = -x\ \mathbf{i} - 2y\ \mathbf{j}$

$C: y = x^3$ from $(0,0)$ to $(2,8)$

Answer

$W = -66$ units of work

29. $\mathbf{F}(x,y) = 2x\ \mathbf{i} + y\ \mathbf{j}$

C: counterclockwise around the triangle with vertices $(0,0), (1,0)$, and $(1,1)$

30. $\mathbf{F}(x,y,z) = x\ \mathbf{i} + y\ \mathbf{j} - 5z\ \mathbf{k}$

$C: \mathbf{r}(t) = 2\cos t\ \mathbf{i} + 2\sin t\ \mathbf{j} + t\ \mathbf{k}, \quad 0 \le t \le 2\pi$

Answer

$W = -10\pi^2$ units of work

31. Let $\mathbf{F}$ be vector field $\mathbf{F}(x,y) = (y^2 + 2xe^y + 1)\ \mathbf{i} + (2xy + x^2e^y + 2y)\ \mathbf{j}$. Compute the work of integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the path $\mathbf{r}(t) = \sin t\ \mathbf{i} + \cos t\ \mathbf{j}, \quad 0 \le t \le \frac{\pi}{2}$.

32. Compute the work done by force $\mathbf{F}(x,y,z) = 2x\ \mathbf{i} + 3y\ \mathbf{j} - z\ \mathbf{k}$ along path $\mathbf{r}(t) = t\ \mathbf{i} + t^2\ \mathbf{j} + t^3\ \mathbf{k}$, where $0 \le t \le 1$.

Answer

$W = 2$ units of work

33. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x,y) = \dfrac{1}{x+y}\ \mathbf{i} + \dfrac{1}{x+y}\ \mathbf{j}$ and C is the segment of the unit circle going counterclockwise from $(1,0)$ to $(0,1)$.

34. Force $\mathbf{F}(x,y,z) = zy\ \mathbf{i} + x\ \mathbf{j} + z^2x\ \mathbf{k}$ acts on a particle that travels from the origin to point $(1,2,3)$. Calculate the work done if the particle travels:

a. along the path $(0,0,0) \to (1,0,0) \to (1,2,0) \to (1,2,3)$ along straight-line segments joining each pair of endpoints;

b. along the straight line joining the initial and final points.

c. Is the work the same along the two paths?

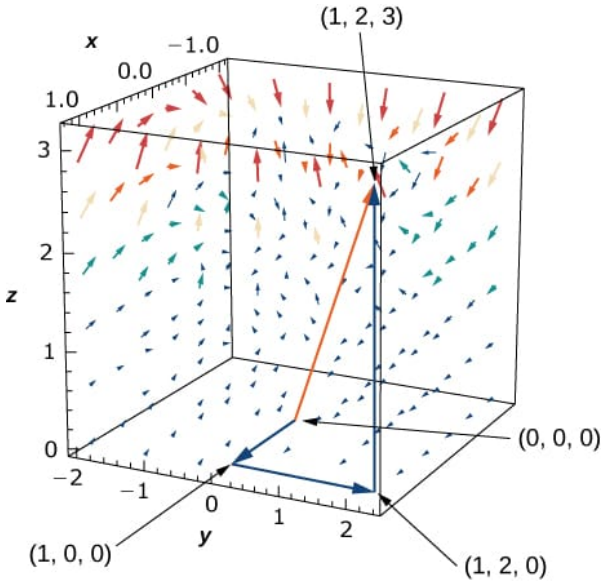


Figure $3.2E.1$: Force $\mathbf{F}(x,y,z) = zy\ \mathbf{i} + x\ \mathbf{j} + z^2x\ \mathbf{k}$ acts on a particle that travels from the origin to point $(1,2,3)$.

Answer

a. $W = 11$ units of work;
b. $W = \dfrac{39}{4} = 9\frac{3}{4}$ units of work;
c. No

35. Find the work done by vector field $\mathbf{F}(x,y,z) = x\ \mathbf{i} + 3xy\ \mathbf{j} - (x+z)\ \mathbf{k}$ on a particle moving along a line segment that goes from $(1,4,2)$ to $(0,5,1)$.

36. How much work is required to move an object in vector field $\mathbf{F}(x,y) = y\ \mathbf{i} + 3x\ \mathbf{j}$ along the upper part of ellipse $\dfrac{x^2}{4} + y^2 = 1$ from $(2,0)$ to $(-2,0)$?

Answer

$W = 2\pi$ units of work

37. A vector field is given by $\mathbf{F}(x,y) = (2x+3y)\ \mathbf{i} + (3x+2y)\ \mathbf{j}$. Evaluate the line integral of the field around a circle of unit radius traversed in a clockwise fashion.

38. Evaluate the line integral of scalar function xy along parabolic path $y = x^2$ connecting the origin to point $(1,1)$.

Answer

$$\int_C f\ ds = \frac{25\sqrt{5}+1}{120}$$

39. Find $\int_C y^2\ dx + (xy - x^2)\ dy$ along $C: y = 3x$ from $(0,0)$ to $(1,3)$.

40. Find $\int_C y^2\ dx + (xy - x^2)\ dy$ along $C : y^2 = 9x$ from $(0, 0)$ to $(1, 3)$.

Answer

$$\int_C y^2\ dx + (xy - x^2)\ dy = 6.15$$

Exercises 41 - 49

Use a CAS to evaluate the given line integrals.

41. Evaluate $\mathbf{F}(x, y, z) = x^2 z\ \mathbf{i} + 6y\ \mathbf{j} + yz^2\ \mathbf{k}$, where C is represented by $\mathbf{r}(t) = t\ \mathbf{i} + t^2\ \mathbf{j} + \ln t\ \mathbf{k}, 1 \le t \le 3$.

42. Evaluate line integral $\int_\gamma xe^y\ ds$ where, γ is the arc of curve $x = e^y$ from $(1, 0)$ to $(e, 1)$.

Answer

$$\int_\gamma xe^y\ ds \approx 7.157$$

43. Evaluate the integral $\int_\gamma xy^2\ ds$, where γ is a triangle with vertices $(0, 1, 2), (1, 0, 3)$, and $(0, -1, 0)$.

44. Evaluate line integral $\int_\gamma (y^2 - xy)\ dx$, where γ is curve $y = \ln x$ from $(1, 0)$ toward $(e, 1)$.

Answer

$$\int_\gamma (y^2 - xy)\ dx \approx -1.379$$

45. Evaluate line integral $\int_\gamma xy^4\ ds$, where γ is the right half of circle $x^2 + y^2 = 16$.

46. Evaluate $\int C\mathbf{F} \cdot d\mathbf{r}, \int C\mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = x^2 y\ \mathbf{i} + (x - z)\ \mathbf{j} + xyz\ \mathbf{k}$ and

$C : \mathbf{r}(t) = t\ \mathbf{i} + t^2\ \mathbf{j} + 2\ \mathbf{k}, 0 \le t \le 1.$

Answer

$$\int_C \mathbf{F} \cdot d\mathbf{r} \approx -1.133 \text{ units of work}$$

47. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = 2x \sin y\ \mathbf{i} + (x^2 \cos y - 3y^2)\ \mathbf{j}$ and

C is any path from $(-1, 0)$ to $(5, 1)$.

48. Find the line integral of $\mathbf{F}(x, y, z) = 12x^2\ \mathbf{i} - 5xy\ \mathbf{j} + xz\ \mathbf{k}$ over path C defined by $y = x^2, z = x^3$ from point $(0, 0, 0)$ to point $(2, 4, 8)$.

Answer

$$\int_C \mathbf{F} \cdot d\mathbf{r} \approx 22.857 \text{ units of work}$$

49. Find the line integral of $\int_C (1 + x^2 y)\ ds$, where C is ellipse $\mathbf{r}(t) = 2 \cos t\ \mathbf{i} + 3 \sin t\ \mathbf{j}$ from $0 \le t \le \pi$.

Exercises 50 - 43

Find the flux.

50. Compute the flux of $\mathbf{F} = x^2\ \mathbf{i} + y\ \mathbf{j}$ across a line segment from $(0, 0)$ to $(1, 2)$.

Answer

$\text{flux} = -\frac{1}{3}$

51. Let $\mathbf{F} = 5\ \mathbf{i}$ and let C be curve $y = 0$, with $0 \leq x \leq 4$. Find the flux across C.

52. Let $\mathbf{F} = 5\ \mathbf{j}$ and let C be curve $y = 0$, with $0 \leq x \leq 4$. Find the flux across C.

Answer

$\text{flux} = -20$

53. Let $\mathbf{F} = -y\ \mathbf{i} + x\ \mathbf{j}$ and let $C : \mathbf{r}(t) = \cos t\ \mathbf{i} + \sin t\ \mathbf{j}$ for $0 \leq t \leq 2\pi$. Calculate the flux across C.

54. Let $\mathbf{F} = (x^2 + y^3)\ \mathbf{i} + (2xy)\ \mathbf{j}$. Calculate flux $\mathbf{F}$ orientated counterclockwise across the curve $C : x^2 + y^2 = 9$.

Answer

$\text{flux} = 0$

55. Find the line integral of $\int_C z^2\ dx + y\ dy + 2y\ dz$, where C consists of two parts: C_1 and C_2. C_1 is the intersection of cylinder $x^2 + y^2 = 16$ and plane $z = 3$ from $(0, 4, 3)$ to $(-4, 0, 3)$. C_2 is a line segment from $(-4, 0, 3)$ to $(0, 1, 5)$.

56. A spring is made of a thin wire twisted into the shape of a circular helix $x = 2\cos t,\ \ y = 2\sin t,\ \ z = t$. Find the mass of two turns of the spring if the wire has a constant mass density of ρ grams per cm.

Answer

$m = 4\pi\rho\sqrt{5}$ grams

57. A thin wire is bent into the shape of a semicircle of radius a. If the linear mass density at point P is directly proportional to its distance from the line through the endpoints, find the mass of the wire.

58. An object moves in force field $\mathbf{F}(x, y, z) = y^2\ \mathbf{i} + 2(x + 1)y\ \mathbf{j}$ counterclockwise from point $(2, 0)$ along elliptical path $x^2 + 4y^2 = 4$ to $(-2, 0)$, and back to point $(2, 0)$ along the x-axis. How much work is done by the force field on the object?

Answer

$W = 0$ units of work

59. Find the work done when an object moves in force field $\mathbf{F}(x, y, z) = 2x\ \mathbf{i} - (x + z)\ \mathbf{j} + (y - x)\ \mathbf{k}$ along the path given by $\mathbf{r}(t) = t^2\ \mathbf{i} + (t^2 - t)\ \mathbf{j} + 3\ \mathbf{k},\ \ 0 \leq t \leq 1$.

60. If an inverse force field $\mathbf{F}$ is given by $\mathbf{F}(x, y, z) = \dfrac{k}{\|r\|^3} r$, where k is a constant, find the work done by $\mathbf{F}$ as its point of application moves along the x-axis from $A(1, 0, 0)$ to $B(2, 0, 0)$.

Answer

$W = \frac{k}{2}$ units of work

61. David and Sandra plan to evaluate line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ along a path in the xy-plane from $(0, 0)$ to $(1, 1)$. The force field is $\mathbf{F}(x, y) = (x + 2y)\ \mathbf{i} + (-x + y^2)\ \mathbf{j}$. David chooses the path that runs along the x-axis from $(0, 0)$ to $(1, 0)$ and then runs along the vertical line $x = 1$ from $(1, 0)$ to the final point $(1, 1)$. Sandra chooses the direct path along the diagonal line $y = x$ from $(0, 0)$ to $(1, 1)$. Whose line integral is larger and by how much?

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3.3: Conservative Vector Fields

Learning Objectives

- Describe simple and closed curves; define connected and simply connected regions.
- Explain how to find a potential function for a conservative vector field.
- Use the Fundamental Theorem for Line Integrals to evaluate a line integral in a vector field.
- Explain how to test a vector field to determine whether it is conservative.

In this section, we continue the study of conservative vector fields. We examine the Fundamental Theorem for Line Integrals, which is a useful generalization of the Fundamental Theorem of Calculus to line integrals of conservative vector fields. We also demonstrate how to determine if a vector field is conservative and construct its corresponding potential function.

Curves and Regions

Before continuing our study of conservative vector fields, we need some geometric definitions. The theorems in the subsequent sections all rely on integrating over certain kinds of curves and regions, so we develop the definitions of those curves and regions here. We first define two special kinds of curves: closed curves and simple curves. As we have learned, a closed curve is one that begins and ends at the same point. A simple curve is one that does not cross itself. A curve that is both closed and simple is a simple closed curve (Figure 3.3.1).

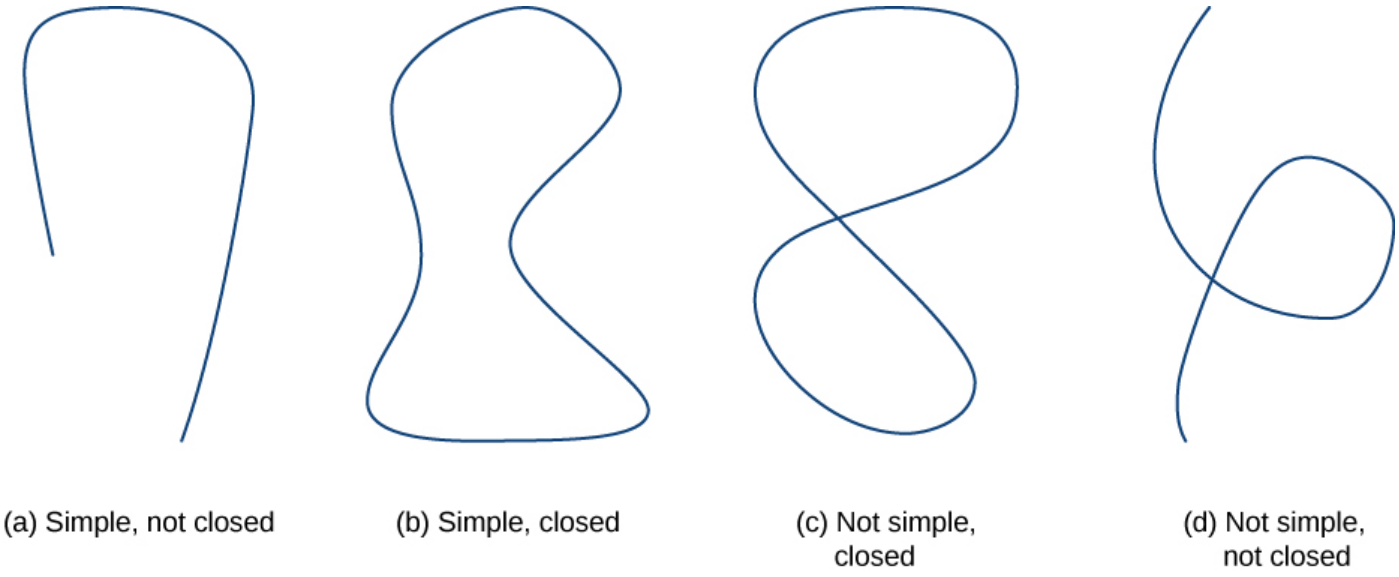


Figure 3.3.1: Different types of curves to define the concepts of simple and closed curves: (a) A simple, not closed curve: This curve does not intersect itself but does not end at its starting point. (b) A simple, closed curve: This curve does not intersect itself and begins and ends at the same point. (c) A not simple, closed curve: This curve intersects itself but starts and ends at the same point. (d) A not simple, not closed curve: This curve intersects itself and does not end at its starting point.

Figure 3.3.1 visually categorizes curves based on whether they are simple (non-intersecting) and/or closed (starting and ending at the same point).

Definition: Closed Curves

Curve C is a *closed curve* if there is a parameterization $\mathbf{r}(t)$, $a \leq t \leq b$ of C such that the parameterization traverses the curve exactly once and $\mathbf{r}(a) = \mathbf{r}(b)$. Curve C is a simple curve if C does not cross itself. That is, C is simple if there exists a parameterization $\mathbf{r}(t)$, $a \leq t \leq b$ of C such that $\mathbf{r}$ is one-to-one over (a, b). It is possible for $\mathbf{r}(a) = \mathbf{r}(b)$, meaning that the simple curve is also closed.

Example 3.3.1: Determining Whether a Curve Is Simple and Closed

Is the curve with parameterization $\mathbf{r}(t) = \left\langle \cos t, \frac{\sin(2t)}{2} \right\rangle$, $0 \leq t \leq 2\pi$ a simple closed curve?

Solution

Note that $\mathbf{r}(0) = \langle 1, 0 \rangle = \mathbf{r}(2\pi)$; therefore, the curve is closed. The curve is not simple, however. To see this, note that $\mathbf{r}\left(\frac{\pi}{2}\right) = \langle 0, 0 \rangle = \mathbf{r}\left(\frac{3\pi}{2}\right)$, and therefore the curve crosses itself at the origin (Figure 3.3.2).

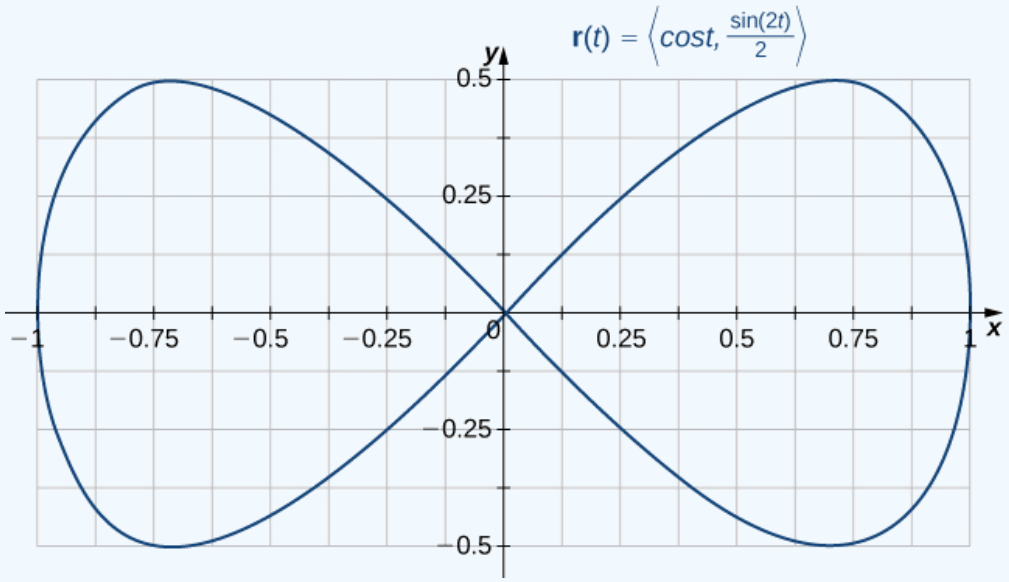


Figure 3.3.2: The curve with parameterization $\mathbf{r}(t) = \left\langle \cos t, \frac{\sin(2t)}{2} \right\rangle$, $0 \leq t \leq 2\pi$. The curve forms a closed figure as $\mathbf{r}(0) = \mathbf{r}(2\pi)$, meaning it starts and ends at the same point. However, the curve is not simple because it intersects itself at the origin $(0, 0)$, as $\mathbf{r}(\pi/2) = \mathbf{r}(3\pi/2) = (0, 0)$. The plot visually shows this intersection, highlighting the distinction between a closed and a simple curve.

Exercise 3.3.1

Is the curve given by parameterization $\mathbf{r}(t) = \langle 2\cos t, 3\sin t \rangle$, $0 \leq t \leq 6\pi$, a simple closed curve?

Hint available at the end of this section.

Many of the theorems in this chapter relate an integral over a region to an integral over the boundary of the region, where the region's boundary is a simple closed curve or a union of simple closed curves. To develop these theorems, we need two geometric definitions for regions: that of a connected region and that of a simply connected region. A connected region is one in which there is a path in the region that connects any two points that lie within that region. A simply connected region is a connected region that does not have any holes in it. These two notions, along with the notion of a simple closed curve, allow us to state several generalizations of the Fundamental Theorem of Calculus later in the chapter. These two definitions are valid for regions in any number of dimensions, but we are only concerned with regions in two or three dimensions.

Definition: Connected Regions

A region *D* is a connected *region* if, for any two points P_1 and P_2, there is a path from P_1 to P_2 with a trace contained entirely inside *D*. A region *D* is a simply connected region if *D* is connected for any simple closed curve *C* that lies inside *D*, and curve *C* can be shrunk continuously to a point while staying entirely inside *D*. In two dimensions, a region is simply connected if it is connected and has no holes.

All simply connected regions are connected, but not all connected regions are simply connected (Figure 3.3.3).

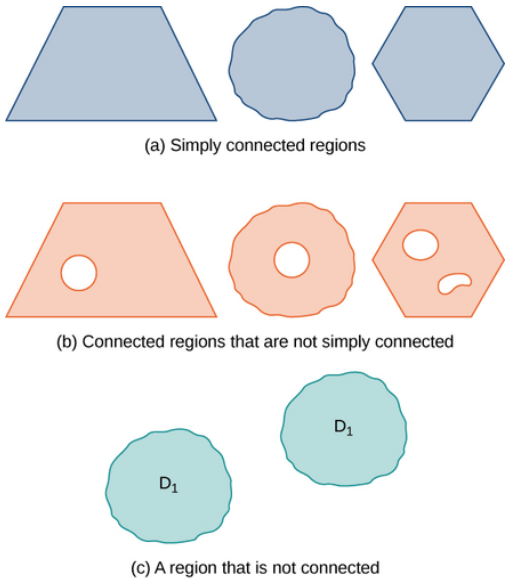


Figure 3.3.3: Not all connected regions are simply connected. (a) Simply connected regions have no holes. (b) Connected regions that are not simply connected may have holes but you can still find a path in the region between any two points. (c) A region that is not connected has some points that cannot be connected by a path in the region.

Fundamental Theorem for Line Integrals

Now that we understand some basic curves and regions, let's generalize the Fundamental Theorem of Calculus to line integrals. Recall that the Fundamental Theorem of Calculus says that if a function f has an antiderivative F, then the integral of f from a to b depends only on the values of F at a and at b—that is,

$$\int_a^b f(x)\,dx = F(b) - F(a). \tag{3.3.1}$$

If we think of the gradient as a derivative, then the same theorem holds for vector line integrals. We show how this works using a motivational example.

Example 3.3.2: Evaluating a Line Integral and the Antiderivatives of the Endpoints

Let $\mathbf{F}(x, y) = \langle 2x, 4y\rangle$. Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where *C* is the line segment from $(0, 0)$ to $(2, 2)$ (Figure 3.3.4).

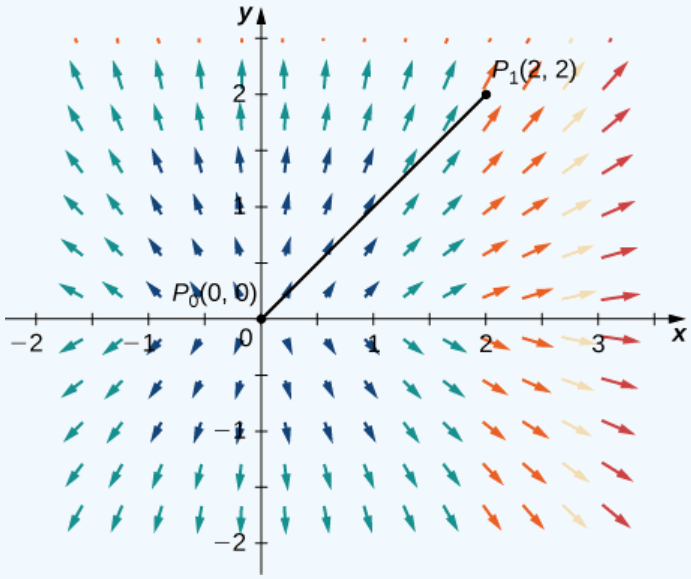


Figure 3.3.4: A two-dimensional vector field $\mathbf{F}(x, y) = \langle 2x, 4y \rangle$ represented by arrows, with their direction and magnitude illustrating the nature of the field. A line segment C is shown from the point $P(0, 0)$ to $Q(2, 2)$, indicating the path along which the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is to be evaluated.

Solution

We use the method from the previous section to calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$. Curve *C* can be parameterized by $\mathbf{r}(t) = \langle 2t, 2t \rangle$, $0 \leq t \leq 1$. Then, $\mathbf{F}(\mathbf{r}(t)) = \langle 4t, 8t \rangle$ and $\mathbf{r}\,'(t) = \langle 2, 2 \rangle$, which implies that

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \langle 4t, 8t \rangle \cdot \langle 2, 2 \rangle dt \\ &= \int_0^1 (8t + 16t) dt = \int_0^1 24t dt \\ &= \left[12t^2 \right]_0^1 = 12. \end{aligned}$$

Notice that $\mathbf{F} = \nabla f$, where $f(x, y) = x^2 + 2y^2$. If we think of the gradient as a derivative, then f is an "antiderivative" of $\mathbf{F}$. In the case of single-variable integrals, the integral of derivative $g\,'(x)$ is $g(b) - g(a)$, where *a* is the start point of the interval of integration and *b* is the endpoint. If vector line integrals work like single-variable integrals, then we would expect integral $\mathbf{F}$ to be $f(P_1) - f(P_0)$, where P_1 is the endpoint of the curve of integration and P_0 is the start point. Notice that this is the case for this example:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = 12$$

and

$$f(2, 2) - f(0, 0) = 4 + 8 - 0 = 12.$$

In other words, the integral of a "derivative" can be calculated by evaluating an "antiderivative" at the endpoints of the curve and subtracting, just as for single-variable integrals.

The following theorem says that, under certain conditions, what happened in the previous example holds for any gradient field. The same theorem holds for vector line integrals, which we call the **Fundamental Theorem for Line Integrals**.

Theorem: The Fundamental Theorem For Line Integrals

Let *C* be a piecewise smooth curve with parameterization $\mathbf{r}(t)$, $a \leq t \leq b$. Let f be a function of two or three variables with first-order partial derivatives that exist and are continuous on *C*. Then,

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)). \tag{3.3.2}$$

Proof

First,

$$\int_C \nabla f \cdot d\mathbf{r} = \int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt.$$

By the chain rule,

$$\frac{d}{dt}(f(\mathbf{r}(t)) = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$$

Therefore, by the Fundamental Theorem of Calculus,

$$\begin{aligned} \int_C \nabla f \cdot d\mathbf{r} &= \int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt \\ &= \int_a^b \frac{d}{dt}(f(\mathbf{r}(t)) dt \\ &= \left[f(\mathbf{r}(t))\right]_{t=a}^{t=b} \\ &= f(\mathbf{r}(b)) - f(\mathbf{r}(a)). \end{aligned}$$

□

We know that if $\mathbf{F}$ is a conservative vector field, there is a potential function f such that $\nabla f = \mathbf{F}$. Therefore

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)). \tag{3.3.3}$$

In other words, just as with the Fundamental Theorem of Calculus, computing the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}$ is conservative, is a two-step process:

1. Find a potential function ("antiderivative") f for $\mathbf{F}$ and
2. Compute the value of f at the endpoints of C and calculate their difference $f(\mathbf{r}(b)) - f(\mathbf{r}(a))$.

Keep in mind, however, there is one major difference between the Fundamental Theorem of Calculus and the Fundamental Theorem for Line Integrals:
A function of one variable that is continuous must have an antiderivative. However, a vector field, even if it is continuous, does not need to have a potential function.

Example 3.3.3: Applying the Fundamental Theorem

Calculate integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = \langle 2x \ln y, \frac{x^2}{y} + z^2, 2yz \rangle$ and C is a curve with parameterization $\mathbf{r}(t) = \langle t^2, t, t \rangle$, $1 \leq t \leq e$

a. without using the Fundamental Theorem of Line Integrals and
b. using the Fundamental Theorem of Line Integrals.

Solution

1. First, let's calculate the integral without the Fundamental Theorem for Line Integrals and instead use the method we learned in the previous section:

$$\begin{aligned}\int_C \mathbf{F}\cdot dr &= \int_1^e \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)\,dt \\ &= \int_1^e \langle 2t^2\ln t, \frac{t^4}{t}+t^2, 2t^2\rangle \cdot \langle 2t,1,1\rangle\,dt \\ &= \int_1^e (4t^3\ln t + t^3 + 3t^2)\,dt \\ &= \int_1^e 4t^3\ln t\,dt + \int_1^e (t^3+3t^2)\,dt \\ &= \int_1^e 4t^3\ln t\,dt + \left[\frac{t^4}{4}+t^3\right]_1^e \\ &= \int_1^e 4t^3\ln t\,dt + \frac{e^4}{4}+e^3-\frac{1}{4}-1 \\ &= \int_1^e 4t^3\ln t\,dt + \frac{e^4}{4}+e^3-\frac{5}{4}\end{aligned}$$

Integral $\int_1^e t^3\ln t\ dt$ requires integration by parts. Let $u=\ln t$ and $dv=t^3$. Then $u=\ln t$, $dv=t^3$

and

$$du=\frac{1}{t}\,dt,\quad v=\frac{t^4}{4}.$$

Therefore,

$$\begin{aligned}\int_1^e t^3\ln t\,dt &= \left[\frac{t^4}{4}\ln t\right]_1^e - \frac{1}{4}\int_1^e t^3\,dt \\ &= \frac{e^4}{4}-\frac{1}{4}\left(\frac{e^4}{4}-\frac{1}{4}\right).\end{aligned}$$

Thus,

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= 4\int_1^e t^3\ln t\,dt \quad + \quad \frac{e^4}{4}+e^3-\frac{5}{4} \\ &= 4\left(\frac{e^4}{4}-\frac{1}{4}\left(\frac{e^4}{4}-\frac{1}{4}\right)\right)+\frac{e^4}{4}+e^3-\frac{5}{4} \\ &= e^4-\frac{e^4}{4}+\frac{1}{4}+\frac{e^4}{4}+e^3-\frac{5}{4} \\ &= e^4+e^3-1.\end{aligned}$$

2. Given that $f(x,y,z)=x^2\ln y+yz^2$ is a potential function for $\mathbf{F}$, let's use the Fundamental Theorem for Line Integrals to calculate the integral. Note that

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_C \nabla f\cdot d\mathbf{r} \\ &= f(\mathbf{r}(e))-f(\mathbf{r}(1)) \\ &= f(e^2,e,e)-f(1,1,1) \\ &= e^4+e^3-1.\end{aligned}$$

This calculation is much more straightforward than the calculation we did in (a). As long as we have a potential function, calculating a line integral using the Fundamental Theorem for Line Integrals is much easier than calculating without the theorem.

Exercise 3.3.2

Given that $f(x,y) = (x-1)^2 y + (y+1)^2 x$ is a potential function for $\mathbf{F}(x,y) = \langle 2xy - 2y + (y+1)^2, (x-1)^2 + 2yx + 2x \rangle$, calculate integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the lower half of the unit circle oriented counterclockwise.

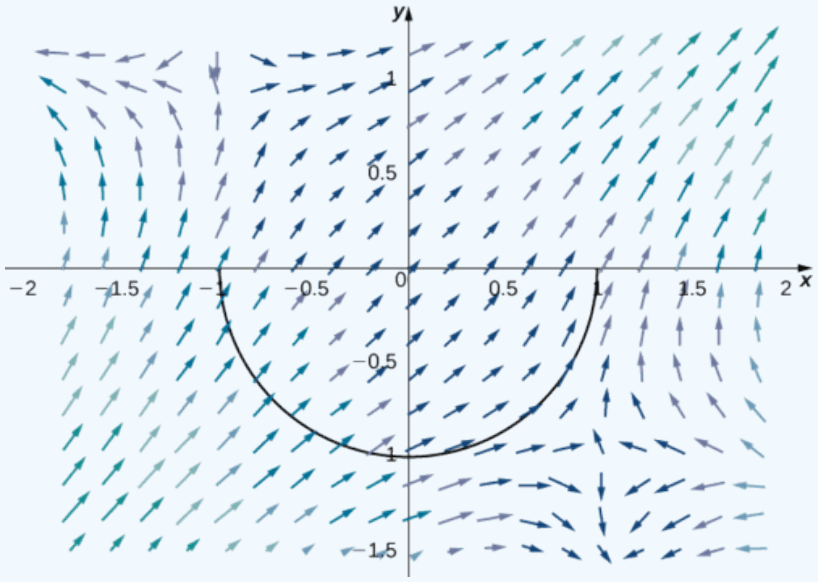


Figure 3.3.5: Curve C is the lower half of the unit circle oriented counterclockwise.

Hint available at the end of this section.

The Fundamental Theorem for Line Integrals has two important consequences. The first consequence is that if $\mathbf{F}$ is conservative and C is a closed curve, then the circulation of $\mathbf{F}$ along C is zero—that is, $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$. To see why this is true, let f be a potential function for $\mathbf{F}$. Since C is a closed curve, the terminal point $\mathbf{r}(b)$ of C is the same as the initial $\mathbf{r}(a)$ of C—that is, $\mathbf{r}(a) = \mathbf{r}(b)$. Therefore, by the Fundamental Theorem for Line Integrals,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C \nabla f \cdot d\mathbf{r} \tag{3.3.4}$$

$$= f(\mathbf{r}(b)) - f(\mathbf{r}(a)) \tag{3.3.5}$$

$$= f(\mathbf{r}(b)) - f(\mathbf{r}(b)) \tag{3.3.6}$$

$$= 0. \tag{3.3.7}$$

Recall that the reason a conservative vector field $\mathbf{F}$ is called "conservative" is because such vector fields model forces in which energy is conserved. We have shown gravity to be an example of such a force. If we think of vector field $\mathbf{F}$ in integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ as a gravitational field, then the equation $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ follows. If a particle travels along a path that starts and ends at the same place, then the work done by gravity on the particle is zero.

The second important consequence of the Fundamental Theorem for Line Integrals (Equation 3.3.2) is that line integrals of conservative vector fields are independent of path—meaning, they depend only on the endpoints of the given curve, and do not depend on the path between the endpoints.

Definition: Path Independence

Let $\mathbf{F}$ be a vector field with domain D; it is independent of path (or path independent) if

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r} \tag{3.3.8}$$

for any paths C_1 and C_2 in D with the same initial and terminal points.

The second consequence is stated formally in the following theorem.

Theorem: Conservative Fields

If $\mathbf{F}$ is a *conservative vector field*, then $\mathbf{F}$ is independent of path.

Proof

Let D denote the domain of $\mathbf{F}$ and let C_1 and C_2 be two paths in D with the same initial and terminal points (Figure 3.3.5). Call the initial point P_1 and the terminal point P_2. Since $\mathbf{F}$ is conservative, there is a potential function f for $\mathbf{F}$. By the Fundamental Theorem for Line Integrals,

$$\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = f(P_2) - f(P_1) = \int_{C_2} \mathbf{F}\cdot d\mathbf{r}.$$

Therefore, $\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{C_2} \mathbf{F}\cdot d\mathbf{r}$ and $\mathbf{F}$ is independent of path.

□

To visualize what independence of path means, imagine three hikers climbing from base camp to the top of a mountain. Hiker 1 takes a steep route directly from camp to the top. Hiker 2 takes a winding route that is not steep from camp to the top. Hiker 3 starts by taking the steep route but halfway to the top decides it is too difficult for him. Therefore he returns to camp and takes the non-steep path to the top. All three hikers are traveling along paths in a gravitational field. Since gravity is a force in which energy is conserved, the gravitational field is conservative. By independence of path, the total amount of work done by gravity on each of the hikers is the same because they all started in the same place and ended in the same place. The work done by the hikers includes other factors such as friction and muscle movement, so the total amount of energy each one expended is not the same, but the net energy expended against gravity is the same for all three hikers.

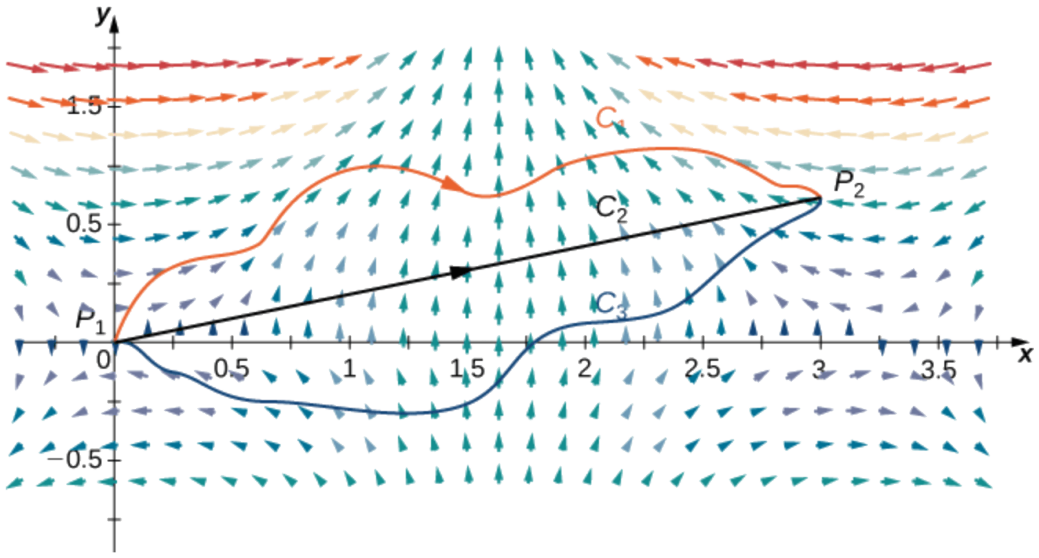


Figure 3.3.6: A demonstration of the concept of path independence in a conservative vector field. The vector field is represented by arrows showing its direction and magnitude. Three distinct paths (C_1, C_2, and C_3) are illustrated, all starting from the same initial point P_1 and ending at the same final point P_2. Each path represents a different route taken through the vector field.

We have shown that if $\mathbf{F}$ is conservative, then $\mathbf{F}$ is independent of path. It turns out that if the domain of $\mathbf{F}$ is open and connected, then the converse is also true. That is, if $\mathbf{F}$ is independent of path and the domain of $\mathbf{F}$ is open and connected, then $\mathbf{F}$ is

conservative. Therefore, the set of conservative vector fields on open and connected domains is precisely the set of vector fields independent of path.

Theorem: The Path Independence Test For Conservative Fields

If $\mathbf{F}$ is a continuous vector field that is independent of path and the domain D of $\mathbf{F}$ is open and connected, then $\mathbf{F}$ is conservative.

Proof

We prove the theorem for vector fields in $\mathbb{R}^2$. The proof for vector fields in $\mathbb{R}^3$ is similar. To show that $\mathbf{F} = \langle P, Q \rangle$ is conservative, we must find a potential function f for $\mathbf{F}$. To that end, let X be a fixed point in D. For any point (x, y) in D, let C be a path from X to (x, y). Define $f(x, y)$ by $f(x, y) = \int_C \mathbf{F} \cdot d\mathbf{r}$. (Note that this definition of f makes sense only because $\mathbf{F}$ is independent of path. If $\mathbf{F}$ was not independent of path, then it might be possible to find another path C' from X to (x, y) such that $\int_C \mathbf{F} \cdot d\mathbf{r} \neq \int_C \mathbf{F} \cdot d\mathbf{r}$, and in such a case $f(x, y)$ would not be a function.) We want to show that f has the property $\nabla f = \mathbf{F}$.

Since domain D is open, it is possible to find a disk centered at (x, y) such that the disk is contained entirely inside D. Let (a, y) with \(a

$$f(x, y) = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

The first integral does not depend on x, so

$$f_x(x, y) = \frac{\partial}{\partial x} \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

If we parameterize C_2 by $\mathbf{r}(t) = \langle t, y \rangle$, $a \leq t \leq x$, then

$$\begin{aligned} f_x(x, y) &= \frac{\partial}{\partial x} \int_{C_2} \mathbf{F} \cdot d\mathbf{r} \\ &= \frac{\partial}{\partial x} \int_a^x \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\ &= \frac{\partial}{\partial x} \int_a^x \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d}{dt}(\langle t, y \rangle)\, dt \\ &= \frac{\partial}{\partial x} \int_a^x \mathbf{F}(\mathbf{r}(t)) \cdot \langle 1, 0 \rangle\, dt \\ &= \frac{\partial}{\partial x} \int_a^x P(t, y)\, dt. \end{aligned}$$

By the Fundamental Theorem of Calculus (part 1),

$$f_x(x, y) = \frac{\partial}{\partial x} \int_a^x P(t, y)\, dt = P(x, y).$$

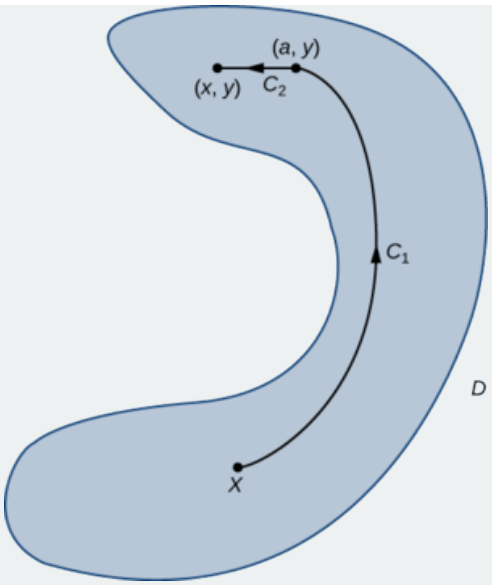


Figure 3.3.7: C_1 is any path from C to (a, y) that stays inside D, and C_2 is the horizontal line segment from (a, y) to (x, y).

A similar argument using a vertical line segment rather than a horizontal line segment shows that $f_y(x, y) = Q(x, y)$.

Therefore $\nabla f = \mathbf{F}$ and $\mathbf{F}$ is conservative.

□

We have spent a lot of time discussing and proving the theorems above, but we can summarize them simply: a vector field $\mathbf{F}$ on an open and connected domain is conservative if and only if it is independent of path. This is important to know because conservative vector fields are extremely important in applications, and these theorems give us a different way of viewing what it means to be conservative using path independence.

Example 3.3.4: Showing That a Vector Field Is Not Conservative

Use path independence to show that vector field $\mathbf{F}(x, y) = \langle x^2 y, y + 5\rangle$ is not conservative.

Solution

We can indicate that $\mathbf{F}$ is not conservative by showing that $\mathbf{F}$ is not path independent. We do so by giving two different paths, C_1 and C_2, that both start at $(0, 0)$ and end at $(1, 1)$, and yet $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} \neq \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$.

Let C_1 be the curve with parameterization $\mathbf{r}_1(t) = \langle t,\ t\rangle$, $0 \leq t \leq 1$ and let C_2 be the curve with parameterization $\mathbf{r}_2(t) = \langle t,\ t^2\rangle$, $0 \leq t \leq 1$ (Figure 3.3.7.). Then

$$\begin{aligned}\int_{C_1} \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(\mathbf{r}_1(t)) \cdot \mathbf{r}_1{}'(t)\, dt \\ &= \int_0^1 \langle t^3, t + 5\rangle \cdot \langle 1, 1\rangle\, dt = \int_0^1 (t^3 + t + 5)\, dt \\ &= \left[\frac{t^4}{4} + \frac{t^2}{2} + 5t\right]_0^1 = \frac{23}{4}\end{aligned}$$

and

$$\begin{aligned}\int_{C_2} \mathbf{F}\cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(\mathbf{r}_2(t))\cdot \mathbf{r}_2{}'(t)\,dt \\ &= \int_0^1 \langle t^4, t^2+5\rangle \cdot \langle 1, 2t\rangle\,dt = \int_0^1 (t^4+2t^3+10t)\,dt \\ &= \left[\frac{t^5}{5}+\frac{t^4}{2}+5t^2\right]_0^1 = \frac{57}{10}.\end{aligned}$$

Since $\int_{C_1} \mathbf{F}\cdot d\mathbf{r} \neq \int_{C_2} \mathbf{F}\cdot d\mathbf{r}$, the value of a line integral of $\mathbf{F}$ depends on the path between two given points. Therefore, $\mathbf{F}$ is not independent of path, and $\mathbf{F}$ is not conservative.

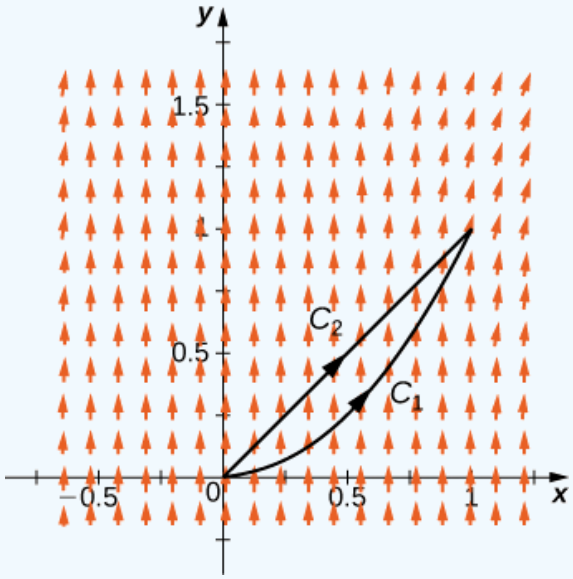


Figure 3.3.8: Two different paths, C_1 and C_2, that both start at $(0,0)$ and end at $(1,1)$, and yet $\int_{C_1} \mathbf{F}\cdot d\mathbf{r} \neq \int_{C_2} \mathbf{F}\cdot d\mathbf{r}$.

Exercise 3.3.3

Show that $\mathbf{F}(x,y) = \langle xy,\ x^2y^2\rangle$ is not path independent by considering the line segment from $(0,0)$ to $(0,2)$ and the piece of the graph of $y = \dfrac{x^2}{2}$ that goes from $(0,0)$ to $(0,2)$.

Hint available at the end of this section.

Conservative Vector Fields and Potential Functions

As we have learned, the Fundamental Theorem for Line Integrals says that if $\mathbf{F}$ is conservative, then calculating $\int_C \mathbf{F}\cdot d\mathbf{r}$ has two steps: first, find a potential function f for $\mathbf{F}$ and, second, calculate $f(P_1) - f(P_0)$, where P_1 is the endpoint of C and P_0 is the starting point. To use this theorem for a conservative field $\mathbf{F}$, we must be able to find a potential function f for $\mathbf{F}$. Therefore, we must answer the following question: Given a conservative vector field $\mathbf{F}$, how do we find a function f such that $\nabla f = \mathbf{F}$? Before giving a general method for finding a potential function, let's motivate the method with an example.

Example 3.3.5: Finding a Potential Function

Find a potential function for $\mathbf{F}(x,y) = \langle 2xy^3, 3x^2y^2 + \cos(y)\rangle$, thereby showing that $\mathbf{F}$ is conservative.

Solution

Suppose that $f(x,y)$ is a potential function for $\mathbf{F}$. Then, $\nabla f = \mathbf{F}$, and therefore

$$f_x(x,y) = 2xy^3 \text{ and } f_y(x,y) = 3x^2y^2 + \cos y.$$

Integrating the equation $f_x(x,y) = 2xy^3$ with respect to x yields the equation

$$f(x,y) = x^2y^3 + h(y).$$

Notice that since we are integrating a two-variable function with respect to x, we must add a constant of integration that is a constant with respect to x, but may still be a function of y. The equation $f(x,y) = x^2y^3 + h(y)$ can be confirmed by taking the partial derivative with respect to x:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2y^3) + \frac{\partial}{\partial x}(h(y)) = 2xy^3 + 0 = 2xy^3.$$

Since f is a potential function for $\mathbf{F}$,

$$f_y(x,y) = 3x^2y^2 + \cos(y),$$

and therefore

$$3x^2y^2 + g'(y) = 3x^2y^2 + \cos(y).$$

This implies that $h'(y) = \cos y$, so $h(y) = \sin y + C$. Therefore, *any* function of the form $f(x,y) = x^2y^3 + \sin(y) + C$ is a potential function. Taking, in particular, $C = 0$ gives the potential function $f(x,y) = x^2y^3 + \sin(y)$.

To verify that f is a potential function, note that $\nabla f(x,y) = \langle 2xy^3, 3x^2y^2 + \cos y\rangle = \mathbf{F}$.

Exercise 3.3.4

Find a potential function for $\mathbf{F}(x,y) = \langle e^x y^3 + y, 3e^x y^2 + x\rangle$.

Hint available at the end of this section.

The logic of the previous example extends to finding the potential function for any conservative vector field in $\mathbb{R}^2$. Thus, we have the following problem-solving strategy for finding potential functions:

Problem-Solving Strategy: Finding A Potential Function For A Conservative Vector Field $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$

1. Integrate P with respect to x. This results in a function of the form $g(x,y) + h(y)$, where $h(y)$ is unknown.
2. Take the partial derivative of $g(x,y) + h(y)$ with respect to y, which results in the function $gy(x,y) + h'(y)$.
3. Use the equation $gy(x,y) + h'(y) = Q(x,y)$ to find $h'(y)$.
4. Integrate $h'(y)$ to find $h(y)$.
5. Any function of the form $f(x,y) = g(x,y) + h(y) + C$, where C is a constant, is a potential function for $\mathbf{F}$.

We can adapt this strategy to find potential functions for vector fields in $\mathbb{R}^3$, as shown in the next example.

Example 3.3.6: Finding a Potential Function in $\mathbb{R}^3$

Find a potential function for $F(x,y,z) = \langle 2xy, x^2 + 2yz^3, 3y^2z^2 + 2z\rangle$, thereby showing that $\mathbf{F}$ is conservative.

Solution

Suppose that f is a potential function. Then, $\nabla f = \mathbf{F}$ and therefore $f_x(x,y,z) = 2xy$. Integrating this equation with respect to x yields the equation $f(x,y,z) = x^2y + g(y,z)$ for some function g. Notice that, in this case, the constant of integration with respect to x is a function of y and z.

Since f is a potential function,

$$x^2 + 2yz^3 = f_y(x, y, z) = x^2 + g_y(y, z).$$

Therefore,

$$g_y(y, z) = 2yz^3.$$

Integrating this function with respect to y yields

$$g(y, z) = y^2z^3 + h(z)$$

for some function $h(z)$ of z alone. (Notice that, because we know that g is a function of only y and z, we do not need to write $g(y, z) = y^2z^3 + h(x, z)$.) Therefore,

$$f(x, y, z) = x^2y + g(y, z) = x^2y + y^2z^3 + h(z).$$

To find f, we now must only find h. Since f is a potential function,

$$3y^2z^2 + 2z = g_z(y, z) = 3y^2z^2 + h'(z).$$

This implies that $h\,'(z) = 2z$, so $h(z) = z^2 + C$. Letting $C = 0$ gives the potential function

$$f(x, y, z) = x^2y + y^2z^3 + z^2.$$

To verify that f is a potential function, note that $\nabla f(x, y, z) = \langle 2xy, x^2 + 2yz^3, 3y^2z^2 + 2z\rangle = \mathbf{F}(x, y, z)$.

Exercise 3.3.5

Find a potential function for $\mathbf{F}(x, y, z) = \langle 12x^2, \cos y \cos z, 1 - \sin y \sin z\rangle$.

Hint available at the end of this section.

We can apply the process of finding a potential function to a gravitational force. Recall that, if an object has unit mass and is located at the origin, then the gravitational force in $\mathbb{R}^2$ that the object exerts on another object of unit mass at the point (x, y) is given by vector field

$$\mathbf{F}(x, y) = -G\left\langle \frac{x}{(x^2 + y^2)^{3/2}}, \frac{y}{(x^2 + y^2)^{3/2}} \right\rangle,$$

where G is the universal gravitational constant. In the next example, we build a potential function for $\mathbf{F}$, thus confirming what we already know: that gravity is conservative.

Example 3.3.7: Finding a Potential Function

Find a potential function f for $\mathbf{F}(x, y) = -G\left\langle \frac{x}{(x^2 + y^2)^{3/2}}, \frac{y}{(x^2 + y^2)^{3/2}} \right\rangle$.

Solution

Suppose that f is a potential function. Then, $\nabla f = \mathbf{F}$ and therefore

$$f_x(x, y) = \frac{-Gx}{(x^2 + y^2)^{3/2}}.$$

To integrate this function with respect to x, we can use u-substitution. If $u = x^2 + y^2$, then $\frac{du}{2} = x\ dx$, so

$$\int \frac{-Gx}{(x^2+y^2)^{3/2}}\,dx = \int \frac{-G}{2u^{3/2}}\,du$$
$$= \frac{G}{\sqrt{u}} + h(y)$$
$$= \frac{G}{\sqrt{x^2+y^2}} + h(y)$$

for some function $h(y)$. Therefore,

$$f(x,y) = \frac{G}{\sqrt{x^2+y^2}} + h(y).$$

Since f is a potential function for $\mathbf{F}$,

$$f_y(x,y) = \frac{-Gy}{(x^2+y^2)^{3/2}}$$

.

Since $f(x,y) = \frac{G}{\sqrt{x^2+y^2}} + h(y)$, $f_y(x,y)$ also equals $\frac{-Gy}{(x^2+y^2)^{3/2}} + h\,'(y)$.

Therefore,

$$\frac{-Gy}{(x^2+y^2)^{3/2}} + h'(y) = \frac{-Gy}{(x^2+y^2)^{3/2}},$$

which implies that $h\,'(y) = 0$. Thus, we can take $h(y)$ to be any constant; in particular, we can let $h(y) = 0$. The function

$$f(x,y) = \frac{G}{\sqrt{x^2+y^2}}$$

is a potential function for the gravitational field $\mathbf{F}$. To confirm that f is a potential function, note that

$$\nabla f(x,y) = \langle -\frac{1}{2}\frac{G}{(x^2+y^2)^{3/2}}(2x), -\frac{1}{2}\frac{G}{(x^2+y^2)^{3/2}}(2y)\rangle$$
$$= \langle \frac{-Gx}{(x^2+y^2)^{3/2}}, \frac{-Gy}{(x^2+y^2)^{3/2}}\rangle$$
$$= \mathbf{F}(x,y).$$

Exercise 3.3.6

Find a potential function f for the three-dimensional gravitational force $\mathbf{F}(x,y,z) = \left\langle \frac{-Gx}{(x^2+y^2+z^2)^{3/2}}, \frac{-Gy}{(x^2+y^2+z^2)^{3/2}}, \frac{-Gz}{(x^2+y^2+z^2)^{3/2}} \right\rangle$.

Hint available at the end of this section.

Testing a Vector Field

Until now, we have worked with vector fields that we know are conservative, but if we are not told that a vector field is conservative, we need to be able to test whether it is conservative. Recall that, if $\mathbf{F}$ is conservative, then $\mathbf{F}$ has the cross-partial property (see The Cross-Partial Property of Conservative Vector Fields). That is, if $\mathbf{F} = \langle P, Q, R\rangle$ is conservative, then $P_y = Q_x$, $P_z = R_x$, and $Q_z = R_y$. So, if $\mathbf{F}$ has the cross-partial property, then is $\mathbf{F}$ conservative? If the domain of $\mathbf{F}$ is open and simply connected, then the

answer is yes.

Theorem: The Cross-Partial Test For Conservative Fields

If $\mathbf{F} = \langle P, Q, R \rangle$ is a vector field on an open, simply connected region D and $P_y = Q_x$, $P_z = R_x$, and $Q_z = R_y$ throughout D, then $\mathbf{F}$ is conservative.

Although a proof of this theorem is beyond the scope of the text, we can discover its power with some examples. Later, we see why it is necessary for the region to be simply connected.

Combining this theorem with the cross-partial property, we can determine whether a given vector field is conservative:

Theorem: Cross-Partial Property Of Conservative Fields

Let $\mathbf{F} = \langle P, Q, R \rangle$ be a vector field on an open, simply connected region D. Then $P_y = Q_x$, $P_z = R_x$, and $Q_z = R_y$ throughout D if and only if $\mathbf{F}$ is conservative.

The version of this theorem in $\mathbb{R}^2$ is also true. If $\mathbf{F}(x, y) = \langle P, Q \rangle$ is a vector field on an open, simply connected domain in $\mathbb{R}^2$, then $\mathbf{F}$ is conservative if and only if $P_y = Q_x$.

Example 3.3.8: Determining Whether a Vector Field Is Conservative

Determine whether vector field $\mathbf{F}(x, y, z) = \langle xy^2z, x^2yz, z^2 \rangle$ is conservative.

Solution

Note that the domain of $\mathbf{F}$ is all of $\mathbb{R}^2$ and $\mathbb{R}^3$ is simply connected. Therefore, we can use The Cross-Partial Property of Conservative Vector Fields to determine whether $\mathbf{F}$ is conservative. Let

$$P(x, y, z) = xy^2z$$

$$Q(x, y, z) = x^2yz$$

and

$$R(x, y, z) = z^2.$$

Since $Q_z(x, y, z) = x^2y$ and $R_y(x, y, z) = 0$, the vector field is not conservative.

Example 3.3.9: Determining Whether a Vector Field Is Conservative

Determine vector field $\mathbf{F}(x, y) = \langle x \ln(y), \ \frac{x^2}{2y} \rangle$ is conservative.

Solution

Note that the domain of $\mathbf{F}$ is the part of $\mathbb{R}^2$ in which $y > 0$. Thus, the domain of $\mathbf{F}$ is part of a plane above the x-axis, and this domain is simply connected (there are no holes in this region and this region is connected). Therefore, we can use The Cross-Partial Property of Conservative Vector Fields to determine whether $\mathbf{F}$ is conservative. Let

$$P(x, y) = x \ln(y) \ \text{ and } \ Q(x, y) = \frac{x^2}{2y}.$$

Then $P_y(x, y) = \frac{x}{y} = Q_x(x, y)$ and thus $\mathbf{F}$ is conservative.

Exercise 3.3.7

Determine whether $\mathbf{F}(x,y) = \langle \sin x \cos y,\ \cos x \sin y \rangle$ is conservative.

Hint available at the end of this section.

When using The Cross-Partial Property of Conservative Vector Fields, it is important to remember that a theorem is a tool, and like any tool, it can be applied only under the right conditions. In the case of The Cross-Partial Property of Conservative Vector Fields, the theorem can be applied only if the domain of the vector field is simply connected.

To see what can go wrong when misapplying the theorem, consider the vector field from Example 3.3.4:

$$\mathbf{F}(x,y) = \frac{y}{x^2+y^2}\,\mathbf{i} + \frac{-x}{x^2+y^2}\,\mathbf{j}. \tag{3.3.9}$$

This vector field satisfies the cross-partial property, since

$$\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) = \frac{(x^2+y^2)-y(2y)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2} \tag{3.3.10}$$

and

$$\frac{\partial}{\partial x}\left(\frac{-x}{x^2+y^2}\right) = \frac{-(x^2+y^2)+x(2x)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}. \tag{3.3.11}$$

Since $\mathbf{F}$ satisfies the cross-partial property, we might be tempted to conclude that $\mathbf{F}$ is conservative. However, $\mathbf{F}$ is not conservative. To see this, let

$$\mathbf{r}(t) = \langle \cos t, \sin t \rangle,\quad 0 \le t \le \pi \tag{3.3.12}$$

be a parameterization of the upper half of a unit circle oriented counterclockwise (denote this C_1) and let

$$\mathbf{s}(t) = \langle \cos t, -\sin t \rangle,\quad 0 \le t \le \pi \tag{3.3.13}$$

be a parameterization of the lower half of a unit circle oriented clockwise (denote this C_2). Notice that C_1 and C_2 have the same starting point and endpoint. Since $\sin^2 t + \cos^2 t = 1$,

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = \langle \sin(t), -\cos(t) \rangle \cdot \langle -\sin(t), \cos(t) \rangle = -1 \tag{3.3.14}$$

and

$$\mathbf{F}(\mathbf{s}(t)) \cdot \mathbf{s}'(t) = \langle -\sin t, -\cos t \rangle \cdot \langle -\sin t, -\cos t \rangle = \sin^2 t + \cos^2 t = 1. \tag{3.3.15}$$

Therefore,

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi} -1\, dt = -\pi \tag{3.3.16}$$

and

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi} 1\, dt = \pi. \tag{3.3.17}$$

Thus, C_1 and C_2 have the same starting point and endpoint, but $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} \neq \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$. Therefore, $\mathbf{F}$ is not independent of path and $\mathbf{F}$ is not conservative.

To summarize: $\mathbf{F}$ satisfies the cross-partial property and yet $\mathbf{F}$ is not conservative. What went wrong? Does this contradict The Cross-Partial Property of Conservative Vector Fields? The issue is that the domain of $\mathbf{F}$ is all of $\mathbb{R}^2$ except for the origin. In other words, the domain of $\mathbf{F}$ has a hole at the origin, and therefore the domain is not simply connected. Since the domain is not simply

connected, The Cross-Partial Property of Conservative Vector Fields does not apply to $\mathbf{F}$.

We close this section by looking at an example of the usefulness of the Fundamental Theorem for Line Integrals. Now that we can test whether a vector field is conservative, we can always decide whether the Fundamental Theorem for Line Integrals can be used to calculate a vector line integral. If we are asked to calculate an integral of the form $\int_C \mathbf{F} \cdot d\mathbf{r}$, then our first question should be: Is $\mathbf{F}$ conservative? If the answer is yes, then we should find a potential function and use the Fundamental Theorem for Line Integrals to calculate the integral. If the answer is no, then the Fundamental Theorem for Line Integrals cannot help us and we have to use other methods, such as using the method from the previous section (using $\mathbf{F}(\mathbf{r}(t))$ and $\mathbf{r}'(t)$).

Example 3.3.10: Using the Fundamental Theorem for Line Integrals

Calculate line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = \langle 2xe^y z + e^x z,\ x^2 e^y z,\ x^2 e^y + e^x \rangle$ and C is any smooth curve that goes from the origin to $(1, 1, 1)$.

Solution

Before trying to compute the integral, we need to determine whether $\mathbf{F}$ is conservative and whether the domain of $\mathbf{F}$ is simply connected. The domain of $\mathbf{F}$ is all of $\mathbb{R}^3$, which is connected and has no holes. Therefore, the domain of $\mathbf{F}$ is simply connected. Let

$$P(x, y, z) = 2xe^y z + e^x z, \quad Q(x, y, z) = x^2 e^y z, \quad \text{and} \quad R(x, y, z) = x^2 e^y + e^x$$

so that $\mathbf{F}(x, y, z) = \langle P, Q, R \rangle$. Since the domain of $\mathbf{F}$ is simply connected, we can check the cross partials to determine whether $\mathbf{F}$ is conservative. Note that

$$\begin{aligned} P_y(x, y, z) &= 2xe^y z = Q_x(x, y, z) \\ P_z(x, y, z) &= 2xe^y + e^x = R_x(x, y, z) \\ Q_z(x, y, z) &= x^2 e^y = R_y(x, y, z). \end{aligned}$$

Therefore, $\mathbf{F}$ is conservative.

To evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ using the Fundamental Theorem for Line Integrals, we need to find a potential function f for $\mathbf{F}$. Let f be a potential function for $\mathbf{F}$. Then, $\nabla f = \mathbf{F}$, and therefore $f_x(x, y, z) = 2xe^y z + e^x z$. Integrating this equation with respect to x gives $f(x, y, z) = x^2 e^y z + e^x z + h(y, z)$ for some function h. Differentiating this equation with respect to y gives $x^2 e^y z + h_y(y, z) = Q(x, y, z) = x^2 e^y z$, which implies that $h_y(y, z) = 0$. Therefore, h is a function of z only, and $f(x, y, z) = x^2 e^y z + e^x z + h(z)$. To find h, note that $f_z = x^2 e^y + e^x + h'(z) = R = x^2 e^y + e^x$. Therefore, $h'(z) = 0$ and we can take $h(z) = 0$. A potential function for $\mathbf{F}$ is $f(x, y, z) = x^2 e^y z + e^x z$.

Now that we have a potential function, we can use the Fundamental Theorem for Line Integrals to evaluate the integral. By the theorem,

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C \nabla f \cdot d\mathbf{r} \\ &= f(1, 1, 1) - f(0, 0, 0) \\ &= 2e. \end{aligned}$$

Analysis

Notice that if we hadn't recognized that $\mathbf{F}$ is conservative, we would have had to parameterize C and use the method from the previous section. Since curve C is unknown, using the Fundamental Theorem for Line Integrals is much simpler.

Exercise 3.3.8

Calculate integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \langle \sin x \sin y, 5 - \cos x \cos y \rangle$ and C is a semicircle with starting point $(0, \pi)$ and endpoint $(0, -\pi)$.

Hint available at the end of this section.

Example 3.3.11: Work Done on a Particle

Let $\mathbf{F}(x,y) = \langle 2xy^2, 2x^2y\rangle$ be a force field. Suppose that a particle begins its motion at the origin and ends its movement at any point in a plane that is not on the x-axis or the y-axis. Furthermore, the particle's motion can be modeled with a smooth parameterization. Show that $\mathbf{F}$ does positive work on the particle.

Solution

We show that $\mathbf{F}$ does positive work on the particle by showing that $\mathbf{F}$ is conservative and then by using the Fundamental Theorem for Line Integrals.

To show that $\mathbf{F}$ is conservative, suppose $f(x,y)$ were a potential function for $\mathbf{F}$. Then, $\nabla f(x,y) = \mathbf{F}(x,y) = \langle 2xy^2, 2x^2y\rangle$ and therefore $f_x(x,y) = 2xy^2$ and $f_y(x,y) = 2x^2y$. The equation $fx(x,y) = 2xy^2$ implies that $f(x,y) = x^2y^2 + h(y)$. Deriving both sides with respect to y yields $f_y(x,y) = 2x^2y + h\,'(y)$. Therefore, $h\,'(y) = 0$ and we can take $h(y) = 0$.

If $f(x,y) = x^2y^2$, then note that $\nabla f(x,y) = \langle 2xy^2, 2x^2y\rangle = \mathbf{F}$, and therefore f is a potential function for $\mathbf{F}$.

Let (a,b) be the point at which the particle stops its motion, and let C denote the curve that models the particle's motion. The work done by $\mathbf{F}$ on the particle is $\int_C \mathbf{F}\cdot d\mathbf{r}$. By the Fundamental Theorem for Line Integrals,

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_C \nabla f\cdot d\mathbf{r}\\ &= f(a,b) - f(0,0)\\ &= a^2b^2.\end{aligned}$$

Since $a \neq 0$ and $b \neq 0$, by assumption, $a^2b^2 > 0$. Therefore, $\int_C \mathbf{F}\cdot d\mathbf{r} > 0$, and $\mathbf{F}$ does positive work on the particle.

Analysis

Notice that this problem would be much more difficult without using the Fundamental Theorem for Line Integrals. To apply the tools we have learned, we would need to give a curve parameterization and use the method from the previous section. Since the path of motion C can be as exotic as we wish (as long as it is smooth), it can be very difficult to parameterize the motion of the particle.

Exercise 3.3.9

Let $\mathbf{F}(x,y) = \langle 4x^3y^4, 4x^4y^3\rangle$, and suppose that a particle moves from point $(4,4)$ to $(1,1)$ along any smooth curve. Is the work done by $\mathbf{F}$ on the particle positive, negative, or zero?

Hint available at the end of this section.

Key Concepts

- The theorems in this section require curves that are closed, simple, or both, and regions that are connected or simply connected.
- The line integral of a conservative vector field can be calculated using the Fundamental Theorem for Line Integrals. This theorem is a generalization of the Fundamental Theorem of Calculus in higher dimensions. Using this theorem usually makes the calculation of the line integral easier.
- Conservative fields are independent of path. The line integral of a conservative field depends only on the value of the potential function at the endpoints of the domain curve.
- Given vector field $\mathbf{F}$, we can test whether $\mathbf{F}$ is conservative by using the cross-partial property. If $\mathbf{F}$ has the cross-partial property and the domain is simply connected, then $\mathbf{F}$ is conservative (and thus has a potential function). If $\mathbf{F}$ is conservative,

we can find a potential function by using the Problem-Solving Strategy.
- The circulation of a conservative vector field on a simply connected domain over a closed curve is zero.

Key Equations

- **Fundamental Theorem for Line Integrals**
  $\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$
- **Circulation of a conservative field over curve *C* that encloses a simply connected region**
  $\oint_C \nabla f \cdot d\mathbf{r} = 0$

Glossary

Closed curve
a curve that begins and ends at the same point

Connected region
a region in which any two points can be connected by a path with a trace contained entirely inside the region

Fundamental Theorem for Line Integrals
the value of line integral $\int_C \nabla f \cdot d\mathbf{r}$ depends only on the value of f at the endpoints of C : $\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$

Independence of path
a vector field $\mathbf{F}$ has path independence if $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ for any curves C_1 and C_2 in the domain of $\mathbf{F}$ with the same initial points and terminal points

Simple curve
a curve that does not cross itself

Simply connected region
a region that is connected and has the property that any closed curve that lies entirely inside the region encompasses points that are entirely inside the region

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Answers

Exercise 3.3.1
Sketch the curve.
Yes

Exercise 3.3.2
The Fundamental Theorem for Line Intervals says this integral depends only on the value of f at the endpoints of C.
2

Exercise 3.3.3
Calculate the corresponding line integrals.
If C_1 and C_2 represent the two curves, then

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} \neq \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

Exercise 3.3.4
Follow the steps in Example 3.3.5.
$f(x, y) = e^x y^3 + xy$

Exercise 3.3.5

Following Example 3.3.6, begin by integrating with respect to x.

$f(x, y, z) = 4x^3 + \sin y \cos z + z$

Exercise 3.3.6

Follow the Problem-Solving Strategy.

$f(x, y, z) = \dfrac{G}{\sqrt{x^2 + y^2 + z^2}}$

Exercise 3.3.7

Use The Cross-Partial Property of Conservative Vector Fields from the previous section.

It is conservative.

Exercise 3.3.8

Use the Fundamental Theorem for Line Integrals.

-10π

Exercise 3.3.9

Use the Fundamental Theorem for Line Integrals.

Negative

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3.3E: Exercises

1. *True* or *False?* If vector field $\mathbf{F}$ is conservative on the open and connected region D, then line integrals of $\mathbf{F}$ are path independent on D, regardless of the shape of D.

Answer

True

2. *True* or *False?* Function $\mathbf{r}(t) = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$, where $0 \leq t \leq 1$, parameterizes the straight-line segment from $\mathbf{a}$ to $\mathbf{b}$.

Answer

True

3. *True* or *False?* Vector field $\mathbf{F}(x, y, z) = (y \sin z)\ \mathbf{i} + (x \sin z)\ \mathbf{j} + (xy \cos z)\ \mathbf{k}$ is conservative.

Answer

True

4. *True* or *False?* Vector field $\mathbf{F}(x, y, z) = y\ \mathbf{i} + (x + z)\ \mathbf{j} - y\ \mathbf{k}$ is conservative.

5. Justify the Fundamental Theorem of Line Integrals for $\int_C \mathbf{F} \cdot d\mathbf{r}$ in the case when $\mathbf{F}(x, y) = (2x + 2y)\ \mathbf{i} + (2x + 2y)\ \mathbf{j}$ and C is a portion of the positively oriented circle $x^2 + y^2 = 25$ from $(5, 0)$ to $(3, 4)$.

Answer

$\int_C \mathbf{F} \cdot d\mathbf{r} = 24$ units of work

6. Find $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = (ye^{xy} + \cos x)\ \mathbf{i} + \left(xe^{xy} + \frac{1}{y^2+1}\right)\ \mathbf{j}$ and C is a portion of curve $y = \sin x$ from $x = 0$ to $x = \frac{\pi}{2}$.

7. Evaluate line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = (e^x \sin y - y)\ \mathbf{i} + (e^x \cos y - x - 2)\ \mathbf{j}$, and C is the path given by $\mathbf{r}(t) = (t^3 \sin \frac{\pi t}{2})\ \mathbf{i} - (\frac{\pi}{2} \cos(\frac{\pi t}{2} + \frac{\pi}{2}))\ \mathbf{j}$ for $0 \leq t \leq 1$.

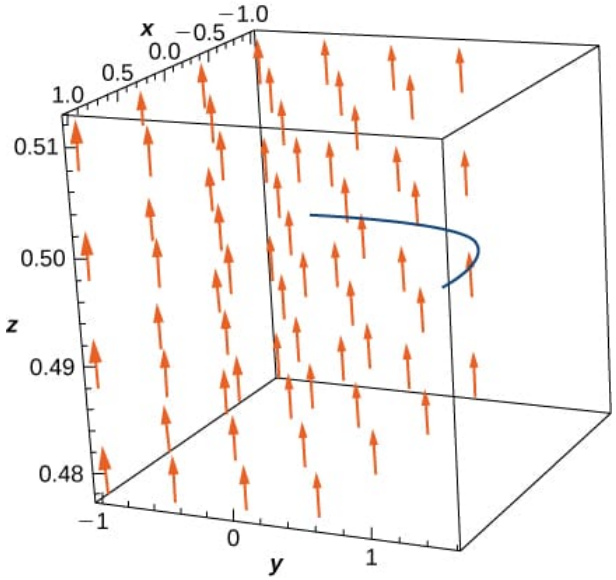


Figure $3.3E.1$: Curve C is the path given by $\mathbf{r}(t) = (t^3 \sin \frac{\pi t}{2})\ \mathbf{i} - (\frac{\pi}{2} \cos(\frac{\pi t}{2} + \frac{\pi}{2}))\ \mathbf{j}$ for $0 \leq t \leq 1$.

Answer

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \left(e - \frac{3\pi}{2}\right) \text{ units of work}$$

Exercises 8 - 13

Determine whether the vector field is conservative and, if it is, find the potential function.

8. $\mathbf{F}(x, y) = 2xy^3\ \mathbf{i} + 3y^2x^2\ \mathbf{j}$

9. $\mathbf{F}(x, y) = (-y + e^x \sin y)\ \mathbf{i} + ((x + 2)e^x \cos y)\ \mathbf{j}$

Answer

Not conservative

10. $\mathbf{F}(x, y) = (e^{2x} \sin y)\ \mathbf{i} + (e^{2x} \cos y)\ \mathbf{j}$

11. $\mathbf{F}(x, y) = (6x + 5y)\ \mathbf{i} + (5x + 4y)\ \mathbf{j}$

Answer

Conservative, $f(x, y) = 3x^2 + 5xy + 2y^2 + k$

12. $\mathbf{F}(x, y) = (2x\cos(y) - y\cos(x))\ \mathbf{i} + (-x^2\sin(y) - \sin(x))\ \mathbf{j}$

13. $\mathbf{F}(x, y) = (ye^x + \sin(y))\ \mathbf{i} + (e^x + x\cos(y))\ \mathbf{j}$

Answer

Conservative, $f(x, y) = ye^x + x\sin(y) + k$

Exercises 14 - 17

Evaluate the line integrals using the Fundamental Theorem of Line Integrals.

14. $\oint_C (y\ \mathbf{i} + x\ \mathbf{j}) \cdot d\mathbf{r}$, where C is any path from $(0, 0)$ to $(2, 4)$

15. $\oint_C (2y\ dx + 2x\ dy)$, where C is the line segment from $(0, 0)$ to $(4, 4)$

Answer

$$\oint_C (2y\ dx + 2x\ dy) = 32 \text{ units of work}$$

16. [T] $\oint_C \left[\arctan\frac{y}{x} - \frac{xy}{x^2 + y^2}\right]\ dx + \left[\frac{x^2}{x^2 + y^2} + e^{-y}(1 - y)\right]\ dy$, where C is any smooth curve from $(1, 1)$ to $(-1, 2)$.

17. Find the conservative vector field for the potential function $f(x, y) = 5x^2 + 3xy + 10y^2$.

Answer

$$\mathbf{F}(x, y) = (10x + 3y)\ \mathbf{i} + (3x + 20y)\ \mathbf{j}$$

Exercises 18 - 25

Determine whether the vector field is conservative and, if so, find a potential function.

18. $\mathbf{F}(x, y) = (12xy)\ \mathbf{i} + 6(x^2 + y^2)\ \mathbf{j}$

19. $\mathbf{F}(x, y) = (e^x \cos y)\ \mathbf{i} + 6(e^x \sin y)\ \mathbf{j}$

Answer

$\mathbf{F}$ is not conservative.

20. $\mathbf{F}(x,y) = (2xye^{x^2y})\ \mathbf{i} + 6(x^2e^{x^2y})\ \mathbf{j}$

21. $\mathbf{F}(x,y,z) = (ye^z)\ \mathbf{i} + (xe^z)\ \mathbf{j} + (xye^z)\ \mathbf{k}$

Answer

$\mathbf{F}$ is conservative and a potential function is $f(x,y,z) = xye^z + k$.

22. $\mathbf{F}(x,y,z) = (\sin y)\ \mathbf{i} - (x\cos y)\ \mathbf{j} + \ \mathbf{k}$

23. $\mathbf{F}(x,y,z) = \dfrac{1}{y}\ \mathbf{i} - \dfrac{x}{y^2}\ \mathbf{j} + (2z-1)\ \mathbf{k}$

Answer

$\mathbf{F}$ is conservative and a potential function is $f(x,y,z) = \dfrac{x}{y} + z^2 - z + k$.

24. $\mathbf{F}(x,y,z) = 3z^2\ \mathbf{i} - \cos y\ \mathbf{j} + 2xz\ \mathbf{k}$

25. $\mathbf{F}(x,y,z) = (2xy)\ \mathbf{i} + (x^2 + 2yz)\ \mathbf{j} + y^2\ \mathbf{k}$

Answer

$\mathbf{F}$ is conservative and a potential function is $f(x,y,z) = x^2y + y^2z + k$.

Exercises 26 - 27

Determine whether the given vector field is conservative and find a potential function.

26. $\mathbf{F}(x,y) = (e^x\cos y)\ \mathbf{i} + 6(e^x\sin y)\ \mathbf{j}$

27. $\mathbf{F}(x,y) = (2xye^{x^2y})\ \mathbf{i} + (x^2e^{x^2y})\ \mathbf{j}$

Answer

$\mathbf{F}$ is conservative and a potential function is $f(x,y) = e^{x^2y} + k$

Exercise 28 - 31

Evaluate the integral using the Fundamental Theorem of Line Integrals.

28. Evaluate $\displaystyle\int_C \nabla f \cdot d\mathbf{r}$, where $f(x,y,z) = \cos(\pi x) + \sin(\pi y) - xyz$ and C is any path that starts at $(1,12,2)$ and ends at $(2,1,-1)$.

29. [T] Evaluate $\displaystyle\int_C \nabla f \cdot d\mathbf{r}$, where $f(x,y) = xy + e^x$ and C is a straight line from $(0,0)$ to $(2,1)$.

Answer

$\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = \left(e^2 + 1\right)$ units of work

30. [T] Evaluate $\displaystyle\int_C \nabla f \cdot d\mathbf{r}$, where $f(x,y) = x^2y - x$ and C is any path in a plane from (1, 2) to (3, 2).

31. Evaluate $\displaystyle\int_C \nabla f \cdot d\mathbf{r}$, where $f(x,y,z) = xyz^2 - yz$ and C has initial point $(1,2,3)$ and terminal point $(3,5,2)$.

Answer

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 38 \text{ units of work}$$

Exercises 32 - 37

Let $\mathbf{F}(x, y) = 2xy^2\ \mathbf{i} + (2yx^2 + 2y)\ \mathbf{j}$ and $G(x, y) = (y + x)\ \mathbf{i} + (y - x)\ \mathbf{j}$, and let C_1 be the curve consisting of the circle of radius 2, centered at the origin and oriented counterclockwise, and C_2 be the curve consisting of a line segment from $(0, 0)$ to $(1, 1)$ followed by a line segment from $(1, 1)$ to $(3, 1)$.

32. Calculate the line integral of $\mathbf{F}$ over C_1.

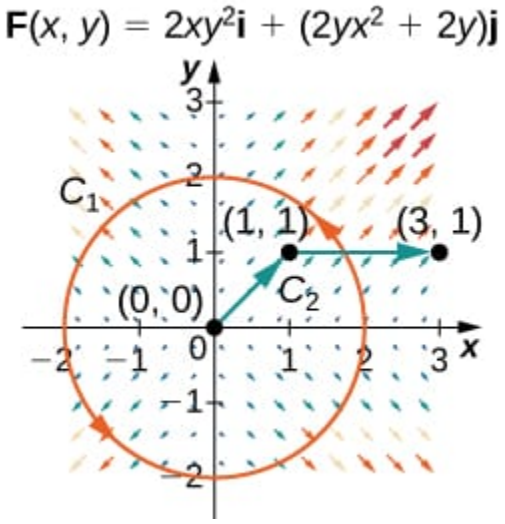


Figure $3.3E.2$: Curve C_1 be the curve consisting of the circle of radius 2, centered at the origin and oriented counterclockwise.

33. Calculate the line integral of $\mathbf{G}$ over C_1.

Answer

$$\oint_{C_1} \mathbf{G} \cdot d\mathbf{r} = -8\pi \text{ units of work}$$

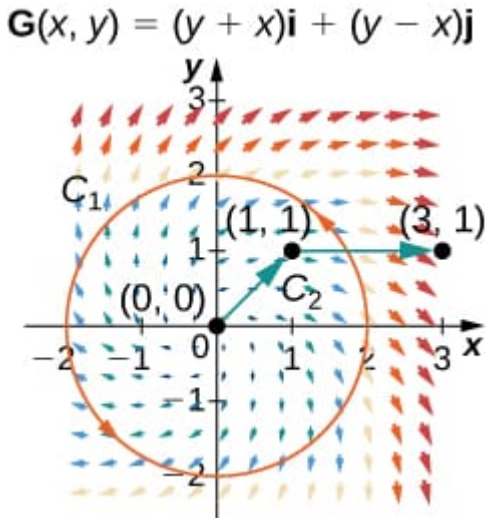


Figure $3.3E.3$: Curve C_1 be the curve consisting of the circle of radius 2, centered at the origin and oriented counterclockwise.

34. Calculate the line integral of $\mathbf{F}$ over C_2.

35. Calculate the line integral of $\mathbf{G}$ over C_2.

Answer

$\oint_{C_2} \mathbf{F} \cdot d\mathbf{r} = 7$ units of work

36. Let $\mathbf{F}(x, y, z) = x^2\ \mathbf{i} + z\sin(yz)\ \mathbf{j} + y\sin(yz)\ \mathbf{k}$. Calculate $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where C is a path from $A = (0, 0, 1)$ to $B = (3, 1, 2)$.

37. Find line integral $\oint_C \mathbf{F} \cdot dr$ of vector field $\mathbf{F}(x, y, z) = 3x^2z\ \mathbf{i} + z^2\ \mathbf{j} + (x^3 + 2yz)\ \mathbf{k}$ along curve C parameterized by $\mathbf{r}(t) = (\frac{\ln t}{\ln 2})\ \mathbf{i} + t^{3/2}\ \mathbf{j} + t\cos(\pi t), 1 \le t \le 4$.

Answer

$\int_C \mathbf{F} \cdot d\mathbf{r} = 159$ units of work

Exercises 38 - 40

Show that the following vector fields are conservative. Then calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ for the given curve.

38. $\mathbf{F}(x, y) = (xy^2 + 3x^2y)\ \mathbf{i} + (x + y)x^2\ \mathbf{j}$; C is the curve consisting of line segments from $(1, 1)$ to $(0, 2)$ to $(3, 0)$.

39. $\mathbf{F}(x, y) = \dfrac{2x}{y^2 + 1}\ \mathbf{i} - \dfrac{2y(x^2 + 1)}{(y^2 + 1)^2}\ \mathbf{j}$; C is parameterized by $x = t^3 - 1,\ \ y = t^6 - t$, for $0 \le t \le 1$.

Answer

$\int_C \mathbf{F} \cdot d\mathbf{r} = -1$ units of work

40. $\mathbf{F}(x, y) = [\cos(xy^2) - xy^2\sin(xy^2)]\ \mathbf{i} - 2x^2y\sin(xy^2)\ \mathbf{j}$; C is the curve $\langle e^t, e^{t+1}\rangle$, for $-1 \le t \le 0$.

41. The mass of Earth is approximately $6 \times 10^{27}g$ and that of the Sun is 330,000 times as much. The gravitational constant is $6.7 \times 10^{-8}cm^3/s^2 \cdot g$. The distance of Earth from the Sun is about $1.5 \times 10^{12}cm$. Compute, approximately, the work necessary to increase the distance of Earth from the Sun by $1\ \ cm$.

Answer

4×10^{31} erg

42. Let $\mathbf{F}(x, y, z) = (e^x \sin y)\ \mathbf{i} + (e^x \cos y)\ \mathbf{j} + z^2\ \mathbf{k}$. Evaluate the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{r}(t) = \langle\sqrt{t}, t^3, e^{\sqrt{t}}\rangle$, for $0 \le t \le 1$.

43. Let $C : [1, 2] \to \mathbb{R}^2$ be given by $x = e^{t-1}, y = \sin(\frac{\pi}{t})$. Use a computer to compute the integral $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C 2x\cos y\ \ dx - x^2\sin y\ \ dy$, where $\mathbf{F}(x, y) = (2x\cos y)\ \mathbf{i} - (x^2\sin y)\ \mathbf{j}$.

Answer

$\int_C \mathbf{F} \cdot d\mathbf{s} = 0.4687$ units of work

44. Use a computer algebra system to find the mass of a wire that lies along the curve $\mathbf{r}(t) = (t^2 - 1)\ \mathbf{j} + 2t\ \mathbf{k}$, where $0 \le t \le 1$, if the density is given by $d(t) = \dfrac{3}{2}t$.

45. Find the circulation and flux of field $\mathbf{F}(x, y) = -y\ \mathbf{i} + x\ \mathbf{j}$ around and across the closed semicircular path that consists of semicircular arch $\mathbf{r}_1(t) = (a\cos t)\ \mathbf{i} + (a\sin t)\ \mathbf{j}, \quad 0 \le t \le \pi$, followed by line segment $\mathbf{r}_2(t) = t\ \mathbf{i}, \quad -a \le t \le a$.

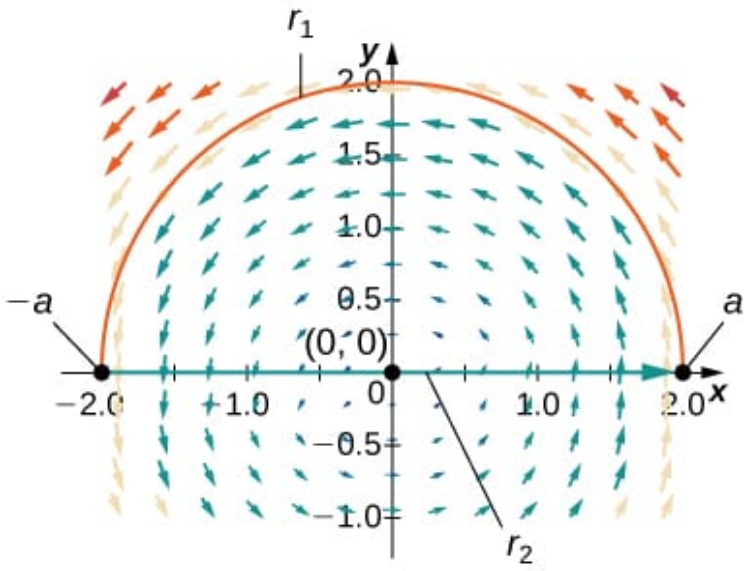


Figure $3.3E.4$: The image illustrates closed semicircular path consists of semicircular arch $\mathbf{r}_1(t) = (a\cos t)\ \mathbf{i} + (a\sin t)\ \mathbf{j}, \quad 0 \le t \le \pi$, followed by line segment $\mathbf{r}_2(t) = t\ \mathbf{i}, \quad -a \le t \le a$.

Answer

$\text{circulation} = \pi a^2$ and $\text{flux} = 0$

46. Compute $\int_C \cos x \cos y\ dx - \sin x \sin y\ dy$, where $\mathbf{r}(t) = \langle t, t^2 \rangle, \quad 0 \le t \le 1.$

47. Complete the proof of the theorem titled THE PATH INDEPENDENCE TEST FOR CONSERVATIVE FIELDS by showing that $f_y = Q(x, y)$.

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3.4: Green's Theorem

Learning Objectives

- Apply the circulation form of Green's theorem.
- Apply the flux form of Green's theorem.
- Calculate circulation and flux on more general regions.

In this section, we examine Green's theorem, which is an extension of the Fundamental Theorem of Calculus to two dimensions. Green's theorem has two forms: a circulation form and a flux form, both of which require region D in the double integral to be simply connected. However, we will extend Green's theorem to regions that are not simply connected.

Put simply, Green's theorem relates a line integral around a simply closed plane curve C and a double integral over the region enclosed by C. The theorem is useful because it allows us to translate difficult line integrals into simplier double integrals, or difficult double integrals into more simple line integrals.

Extending the Fundamental Theorem of Calculus

Recall that the Fundamental Theorem of Calculus says that

$$\int_a^b F'(x)\,dx = F(b) - F(a). \tag{3.4.1}$$

As a geometric statement, this equation says that the integral over the region below the graph of $F'(x)$ and above the line segment $[a, b]$ depends only on the value of F at the endpoints a and b of that segment. Since the numbers a and b are the boundary of the line segment $[a, b]$, the theorem says we can calculate integral $\int_a^b F'(x)\ dx$ based on information about the boundary of line segment $[a, b]$ (Figure 3.4.1). The same idea is true of the Fundamental Theorem for Line Integrals:

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)). \tag{3.4.2}$$

When we have a potential function (an "antiderivative"), we can calculate the line integral based solely on information about the boundary of curve C.

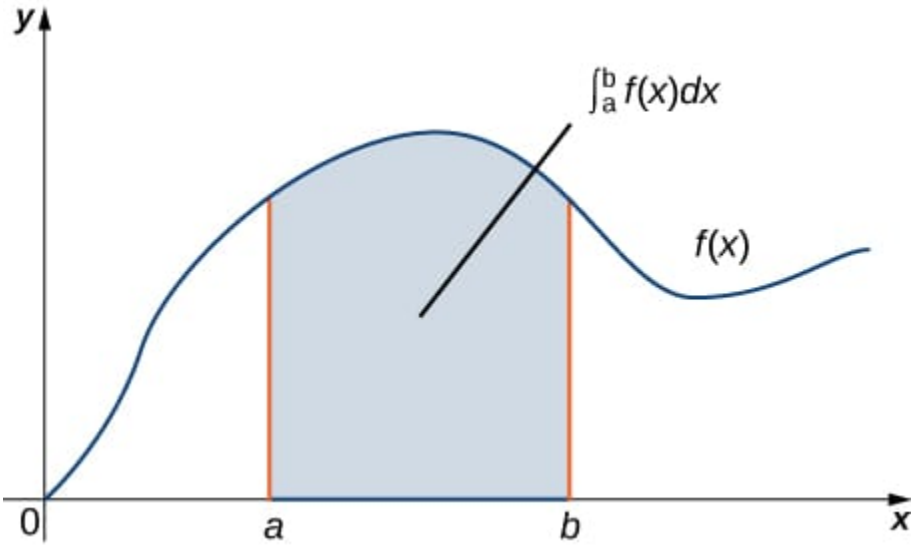


Figure 3.4.1: The Fundamental Theorem of Calculus says that the integral over line segment $[a, b]$ depends only on the values of the antiderivative at the endpoints of $[a, b]$.

Green's theorem takes this idea and extends it to calculating double integrals. Green's theorem says that we can calculate a double integral over region D based solely on information about the boundary of D. Green's theorem also says we can calculate a line integral over a simple closed curve C based solely on information about the region that C encloses. In particular, Green's theorem connects a double integral over region D to a line integral around the boundary of D.

Circulation Form of Green's Theorem

The first form of Green's theorem that we examine is the circulation form. This form of the theorem relates the vector line integral over a simple, closed plane curve C to a double integral over the region enclosed by C. Therefore, the circulation of a vector field along a simple closed curve can be transformed into a double integral and vice versa.

Green's Theorem (Circulation Form)

Let D be an open, simply connected region with a boundary curve C that is a piecewise smooth, simple closed curve oriented counterclockwise (Figure 3.4.2). Let $\mathbf{F} = \langle P, Q \rangle$ be a vector field with component functions that have continuous partial derivatives on D. Then,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P\,dx + Q\,dy \tag{3.4.3}$$

$$= \iint_D (Q_x - P_y)\,dA. \tag{3.4.4}$$

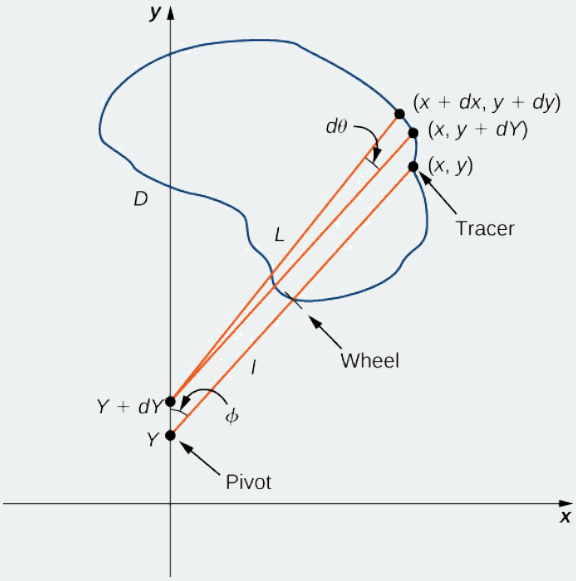


Figure 3.4.2: The image illustrates an open, simply connected region D with a boundary curve C that is piecewise smooth, simple, and closed, oriented counterclockwise.

Notice that Green's theorem can be used only for a two-dimensional vector field $\mathbf{F}$. If $\mathbf{F}$ is a three-dimensional field, then Green's theorem does not apply. Since

$$\int_C P\,dx + Q\,dy = \int_C \mathbf{F} \cdot \mathbf{T}\,ds \tag{3.4.5}$$

this version of Green's theorem is sometimes referred to as the *tangential form* of Green's theorem.

The proof of Green's theorem is rather technical, and beyond the scope of this text. Here we examine a proof of the theorem in the special case that D is a rectangle. For now, notice that we can quickly confirm that the theorem is true for the special case in which $\mathbf{F} = \langle P, Q \rangle$ is conservative. In this case,

$$\oint_C P\,dx + Q\,dy = 0 \tag{3.4.6}$$

because the circulation is zero in conservative vector fields. $\mathbf{F}$ satisfies the cross-partial condition, so $P_y = Q_x$. Therefore,

$$\iint_D (Q_x - P_y)\,dA = \int_D 0\,dA = 0 = \oint_C P\,dx + Q\,dy \tag{3.4.7}$$

which confirms Green's theorem in the case of conservative vector fields.

Proof

Let's now prove that the circulation form of Green's theorem is true when the region D is a rectangle. Let D be the rectangle $[a, b] \times [c, d]$ oriented counterclockwise. Then, the boundary C of D consists of four piecewise smooth pieces C_1, C_2, C_3, and C_4 (Figure 3.4.3). We parameterize each side of D as follows:

$$C_1 : \mathbf{r}_1(t) = \langle t, c \rangle,\ a \le t \le b$$

$$C_2 : \mathbf{r}_2(t) = \langle b, t \rangle,\ c \le t \le d$$

$$-C_3 : \mathbf{r}_3(t) = \langle t, d \rangle,\ a \le t \le b$$

$$-C_4 : \mathbf{r}_4(t) = \langle a, t \rangle,\ c \le t \le d.$$

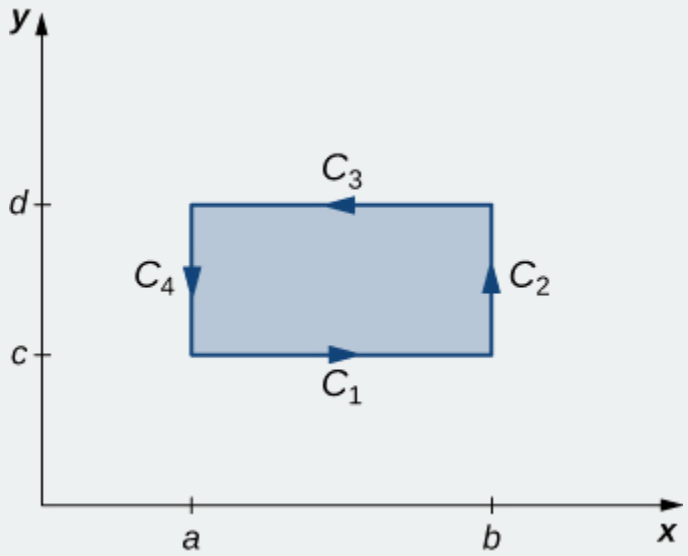


Figure 3.4.3: A rectangle $D = [a, b] \times [c, d]$ is oriented counterclockwise, and its boundary C consists of four piecewise smooth curves C_1, C_2, C_3, and C_4.

Then,

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_{C_1} \mathbf{F}\cdot d\mathbf{r} + \int_{C_2} \mathbf{F}\cdot d\mathbf{r} + \int_{C_3} \mathbf{F}\cdot d\mathbf{r} + \int_{C_4} \mathbf{F}\cdot d\mathbf{r} \\ &= \int_{C_1} \mathbf{F}\cdot d\mathbf{r} + \int_{C_2} \mathbf{F}\cdot d\mathbf{r} - \int_{-C_3} \mathbf{F}\cdot d\mathbf{r} - \int_{-C_4} \mathbf{F}\cdot d\mathbf{r} \\ &= \int_a^b \mathbf{F}(\mathbf{r}_1(t))\cdot \mathbf{r}_1'(t)\,dt + \int_c^d \mathbf{F}(\mathbf{r}_2(t))\cdot \mathbf{r}_2'(t)\,dt - \int_a^b \mathbf{F}(\mathbf{r}_3(t))\cdot \mathbf{r}_3'(t)\,dt - \int_c^d \mathbf{F}(\mathbf{r}_4(t))\cdot \mathbf{r}_4'(t)\,dt \\ &= \int_a^b P(t,c)\,dt + \int_c^d Q(b,t)\,dt - \int_a^b P(t,d)\,dt - \int_c^d Q(a,t)\,dt \\ &= \int_a^b (P(t,c)-P(t,d))\,dt + \int_c^d (Q(b,t)-Q(a,t))\,dt \\ &= -\int_a^b (P(t,d)-P(t,c))\,dt + \int_c^d (Q(b,t)-Q(a,t))\,dt.\end{aligned}$$

By the Fundamental Theorem of Calculus,

$$P(t,d)-P(t,c) = \int_c^d \frac{\partial}{\partial y}P(t,y)dy$$

and

$$Q(b,t)-Q(a,t) = \int_a^b \frac{\partial}{\partial x}Q(x,t)\,dx.$$

Therefore,

$$-\int_a^b (P(t,d)-P(t,c))\,dt + \int_c^d (Q(b,t)-Q(a,t))\,dt = -\int_a^b\int_c^d \frac{\partial}{\partial y}P(t,y)\,dy\,dt + \int_c^d\int_a^b \frac{\partial}{\partial x}Q(x,t)\,dx\,dt.$$

But,

$$\begin{aligned}-\int_a^b\int_c^d \frac{\partial}{\partial y}P(t,y)\,dy\,dt + \int_c^d\int_a^b \frac{\partial}{\partial x}Q(x,t)\,dx\,dt &= -\int_a^b\int_c^d \frac{\partial}{\partial y}P(x,y)\,dy\,dx + \int_c^d\int_a^b \frac{\partial}{\partial x}Q(x,y)\,dx\,dy \\ &= \int_a^b\int_c^d (Q_x - P_y)\,dy\,dx \\ &= \iint_D (Q_x - P_y)\,dA.\end{aligned}$$

Therefore, $\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_D (Q_x - P_y)\ dA$ and we have proved Green's theorem in the case of a rectangle.

□

To prove Green's theorem over a general region D, we can decompose D into many tiny rectangles and use the proof that the theorem works over rectangles. The details are technical, however, and beyond the scope of this text.

Example 3.4.1: Applying Green's Theorem over a Rectangle

Calculate the line integral

$$\oint_C x^2 y dx + (y-3)dy,$$

where C is a rectangle with vertices $(1,1)$, $(4,1)$, $(4,5)$, and $(1,5)$ oriented counterclockwise.

Solution

Let $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y)\rangle = \langle x^2 y, y-3\rangle$. Then, $Q_x(x,y) = 0$ and $P_y(x,y) = x^2$. Therefore, $Q_x - P_y = -x^2$.

Let D be the rectangular region enclosed by C (Figure 3.4.4). By Green's theorem,

$$\begin{aligned}\oint_C x^2 y dx + (y-3)\,dy &= \iint_D (Q_x - P_y)\,dA \\ &= \iint_D -x^2\,dA = \int_1^5 \int_1^4 -x^2\,dx\,dy \\ &= \int_1^5 -21\,dy = -84.\end{aligned}$$

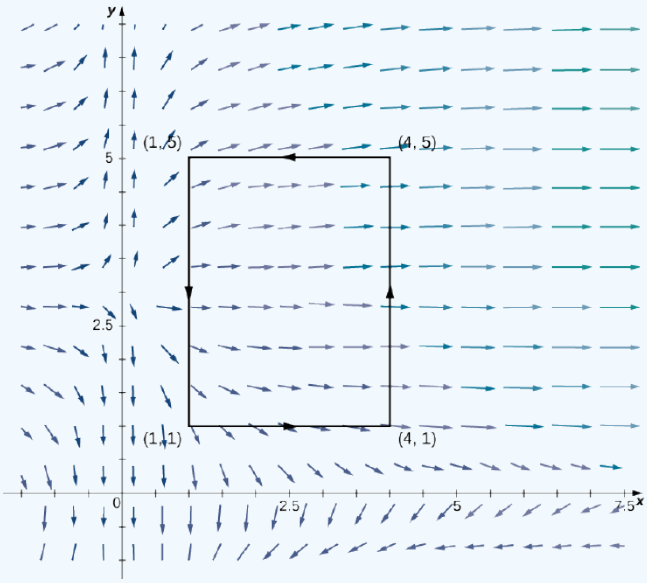


Figure 3.4.4: The vector field with arrows representing its magnitude and direction, and a rectangular region D enclosed by the boundary C. Key points of the rectangle, such as $(1,1)$, $(4,1)$, $(4,5)$, and $(1,5)$, are labeled. Green's Theorem simplifies the computation of the line integral by transforming it into a double integral, bypassing the need to evaluate individual line integrals along each side of the rectangle.

Analysis

If we were to evaluate this line integral without using Green's theorem, we would need to parameterize each side of the rectangle, break the line integral into four separate line integrals, and use the methods from the section titled **Line Integrals** to evaluate each integral. Furthermore, since the vector field here is not conservative, we cannot apply the Fundamental Theorem for Line Integrals. Green's theorem makes the calculation much simpler.

Example 3.4.2: Applying Green's Theorem to Calculate Work

Calculate the work done on a particle by force field

$$\mathbf{F}(x,y) = \langle y + \sin x, e^y - x\rangle$$

as the particle traverses circle $x^2 + y^2 = 4$ exactly once in the counterclockwise direction, starting and ending at point $(2,0)$.

Solution

Let C denote the circle and let D be the disk enclosed by C. The work done on the particle is

$$W = \oint_C (y + \sin x)\,dx + (e^y - x)\,dy.$$

As with Example 3.4.1, this integral can be calculated using tools we have learned, but it is easier to use the double integral given by Green's theorem (Figure 3.4.5).

Let $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y)\rangle = \langle y + \sin x, e^y - x\rangle$. Then, $Q_x = -1$ and $P_y = 1$. Therefore, $Q_x - P_y = -2$.

By Green's theorem,

$$\begin{aligned} W &= \oint_C (y + \sin(x))dx + (e^y - x)\,dy \\ &= \iint_D (Q_x - P_y)\,dA \\ &= \iint_D -2\,dA \\ &= -2(area(D)) = -2\pi(2^2) = -8\pi. \end{aligned}$$

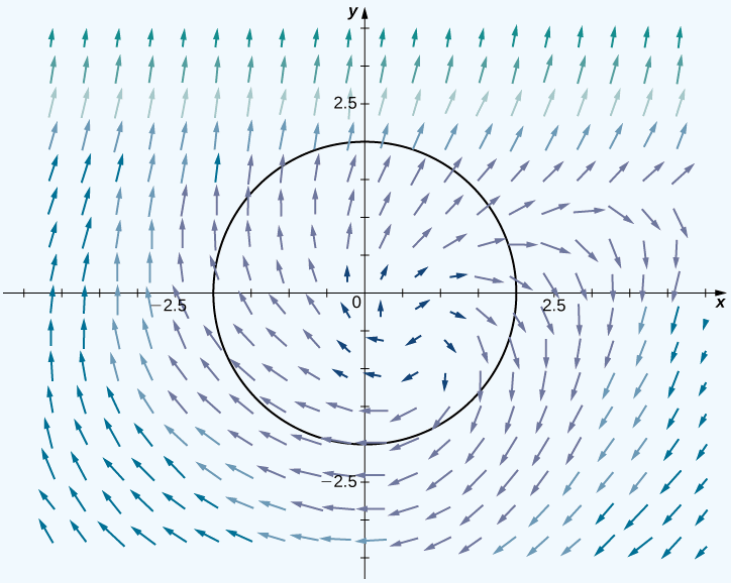


Figure 3.4.5: The particle traverses circle $x^2 + y^2 = 4$ exactly once in the counterclockwise direction, starting and ending at point $(2,0)$.

Exercise 3.4.2

Use Green's theorem to calculate line integral

$$\oint_C \sin(x^2)\,dx + (3x - y)\,dy.$$

where C is a right triangle with vertices $(-1,2)$, $(4,2)$, and $(4,5)$ oriented counterclockwise.

Hint available at the end of this section.

In the preceding two examples, the double integral in Green's theorem was easier to calculate than the line integral, so we used the theorem to calculate the line integral. In the next example, the double integral is more difficult to calculate than the line integral, so we use Green's theorem to translate a double integral into a line integral.

Example 3.4.3: Applying Green's Theorem over an Ellipse

Calculate the area enclosed by ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ (Figure 3.4.6).

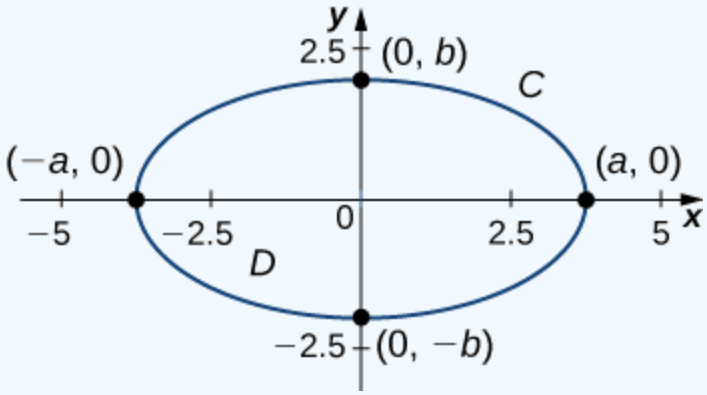


Figure 3.4.6: An ellipse defined by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. The boundary of the ellipse is denoted by C, and the region enclosed by the ellipse is labeled as D.

Solution

Let C denote the ellipse and let D be the region enclosed by C. Recall that ellipse C can be parameterized by

- $x = a\cos t$,
- $y = b\sin t$,
- $0 \le t \le 2\pi$.

Calculating the area of D is equivalent to computing double integral $\iint_D \, dA$. To calculate this integral without Green's theorem, we would need to divide D into two regions: the region above the *x*-axis and the region below. The area of the ellipse is

$$\int_{-a}^{a}\int_{0}^{\sqrt{b^2-(bx/a)^2}} \, dy\,dx + \int_{-a}^{a}\int_{-\sqrt{b^2-(bx/a)^2}}^{0} \, dy\,dx.$$

These two integrals are not straightforward to calculate (although when we know the value of the first integral, we know the value of the second by symmetry). Instead of trying to calculate them, we use Green's theorem to transform $\iint_D \, dA$ into a line integral around the boundary C.

Consider vector field

$$F(x,y) = \langle P, Q\rangle = \langle -\frac{y}{2}, \frac{x}{2}\rangle.$$

Then, $Q_x = \frac{1}{2}$ and $P_y = -\frac{1}{2}$, and therefore $Q_x - P_y = 1$. Notice that $\mathbf{F}$ was chosen to have the property that $Q_x - P_y = 1$. Since this is the case, Green's theorem transforms the line integral of $\mathbf{F}$ over C into the double integral of 1 over D.

By Green's theorem,

$$\begin{aligned}
\iint_D dA &= \iint_D (Q_x - P_y)\,dA \\
&= \int_C \mathbf{F}\cdot d\mathbf{r} = \frac{1}{2}\int_C -y\,dx + x\,dy \\
&= \frac{1}{2}\int_0^{2\pi} -b\sin t(-a\sin t) + a(\cos t)b\cos t\,dt \\
&= \frac{1}{2}\int_0^{2\pi} ab\cos^2 t + ab\sin^2 t\,dt \\
&= \frac{1}{2}\int_0^{2\pi} ab\,dt = \pi ab.
\end{aligned}$$

Therefore, the area of the ellipse is $\pi ab \ \text{units}^2$.

In Example 3.4.3, we used vector field $\mathbf{F}(x,y) = \langle P, Q\rangle = \langle -\frac{y}{2}, \frac{x}{2}\rangle$ to find the area of any ellipse. The logic of the previous example can be extended to derive a formula for the area of any region D. Let D be any region with a boundary that is a simple closed curve C oriented counterclockwise. If $F(x,y) = \langle P, Q\rangle = \langle -\frac{y}{2}, \frac{x}{2}\rangle$, then $Q_x - P_y = 1$. Therefore, by the same logic as in Example 3.4.3,

$$\text{area of } D = \iint_D dA = \frac{1}{2}\oint_C -y\,dx + x\,dy. \tag{3.4.8}$$

It's worth noting that if $F = \langle P, Q\rangle$ is any vector field with $Q_x - P_y = 1$, then the logic of the previous paragraph works. So. Equation 3.4.8 is not the only equation that uses a vector field's mixed partials to get the area of a region.

Exercise 3.4.3

Find the area of the region enclosed by the curve with parameterization $r(t) = \langle \sin t \cos t, \sin t\rangle$, $0 \le t \le \pi$.

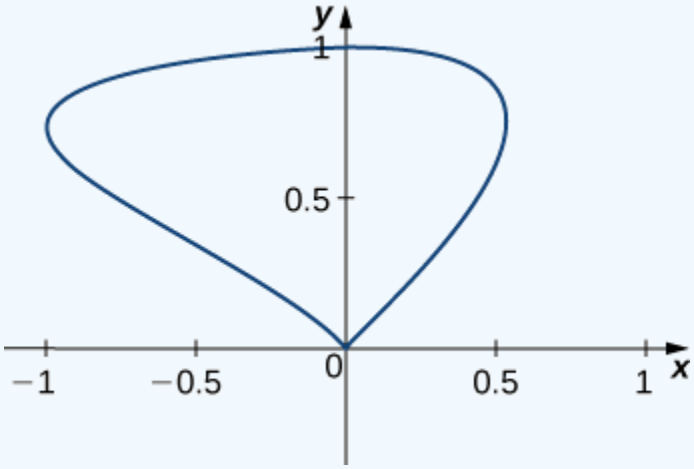


Figure 3.4.7: A region enclosed by a curve parameterized as $\mathbf{r}(t) = \langle \sin t \cos t, \sin t\rangle$, where $0 \le t \le \pi$. The curve forms a closed loop.

Hint available at the end of this section.

Flux Form of Green's Theorem

The circulation form of Green's theorem relates a double integral over region D to line integral $\oint_C \mathbf{F}\cdot\mathbf{T}ds$, where C is the boundary of D. The flux form of Green's theorem relates a double integral over region D to the flux across boundary C. The flux of a fluid across a curve can be difficult to calculate using the flux line integral. This form of Green's theorem allows us to translate a difficult flux integral into a double integral that is often easier to calculate.

Green's Theorem (Flux Form)

Let D be an open, simply connected region with a boundary curve C that is a piecewise smooth, simple closed curve that is oriented counterclockwise (Figure 3.4.8). Let $\mathbf{F} = \langle P, Q\rangle$ be a vector field with component functions that have continuous partial derivatives on an open region containing D. Then,

$$\oint_C \mathbf{F}\cdot\mathbf{N}\,ds = \iint_D P_x + Q_y\,dA. \tag{3.4.9}$$

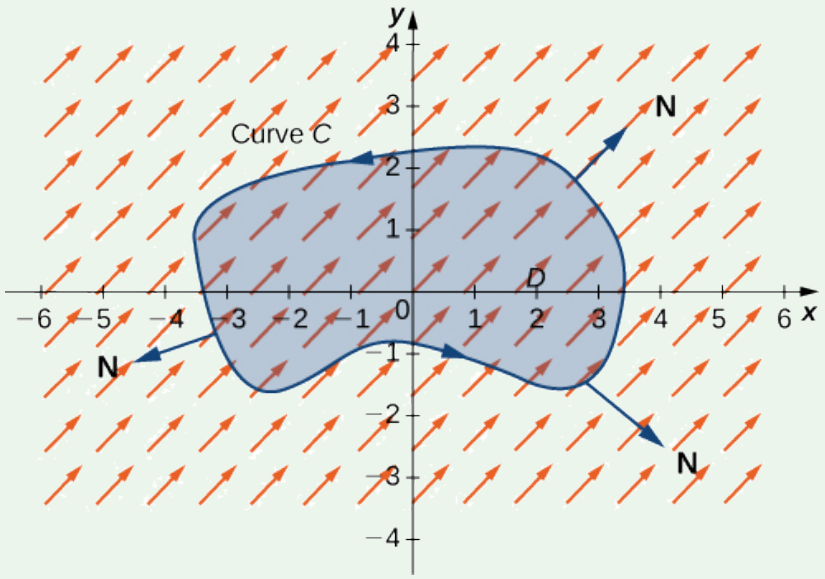


Figure 3.4.8: An open, simply connected region D bounded by a curve C. The curve C is a piecewise smooth, simple closed curve oriented counterclockwise, represented as a blue loop enclosing the shaded region D. Arrows labeled N indicate outward-pointing unit normal vectors at various points along C, consistent with the counterclockwise orientation.

Because this form of Green's theorem contains unit normal vector $\mathbf{N}$, it is sometimes referred to as the *normal form* of Green's theorem.

Proof

Recall that $\oint_C \mathbf{F} \cdot \mathbf{N}\ ds = \oint_C -Q\ dx + P\ dy$. Let $M = -Q$ and $N = P$. By the circulation form of Green's theorem,

$$\begin{aligned} \oint_C -Q\,dx + P\,dy &= \oint_C M\,dx + N\,dy \\ &= \iint_D N_x - M_y\,dA \\ &= \iint_D P_x - (-Q)_y\,dA \\ &= \iint_D P_x + Q_y\,dA. \end{aligned}$$

□

Example 3.4.4A: Applying Green's Theorem for Flux across a Circle

Let C be a circle of radius r centered at the origin (Figure 3.4.9) and let $\mathbf{F}(x, y) = \langle x, y \rangle$. Calculate the flux across C.

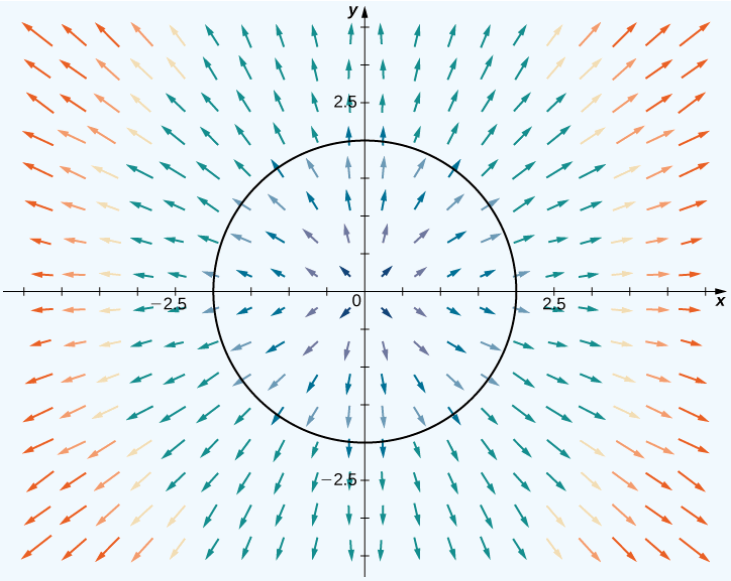


Figure 3.4.9: A circle C of radius r centered at the origin in a two-dimensional vector field. The vector field is represented by arrows radiating outward from the origin, with their direction and magnitude indicating the components of the field at various points. The curve C is shown as a black circle.

Solution

Let D be the disk enclosed by C. The flux across C is $\oint_C \mathbf{F} \cdot \mathbf{N}\, ds$. We could evaluate this integral using tools we have learned, but Green's theorem makes the calculation much more simple. Let $P(x, y) = x$ and $Q(x, y) = y$ so that $\mathbf{F} = \langle P, Q \rangle$. Note that $P_x = 1 = Q_y$, and therefore $P_x + Q_y = 2$. By Green's theorem,

$$\int_C \mathbf{F} \cdot \mathbf{N}\, ds = \iint_D 2\, dA = 2 \iint_D dA.$$

Since $\iint_D dA$ is the area of the circle, $\iint_D dA = \pi r^2$. Therefore, the flux across C is $2\pi r^2$.

Example 3.4.4B: Applying Green's Theorem for Flux across a Triangle

Let S be the triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 3)$ oriented clockwise (Figure 3.4.10). Calculate the flux of $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle = \langle x^2 + e^y, x + y \rangle$ across S.

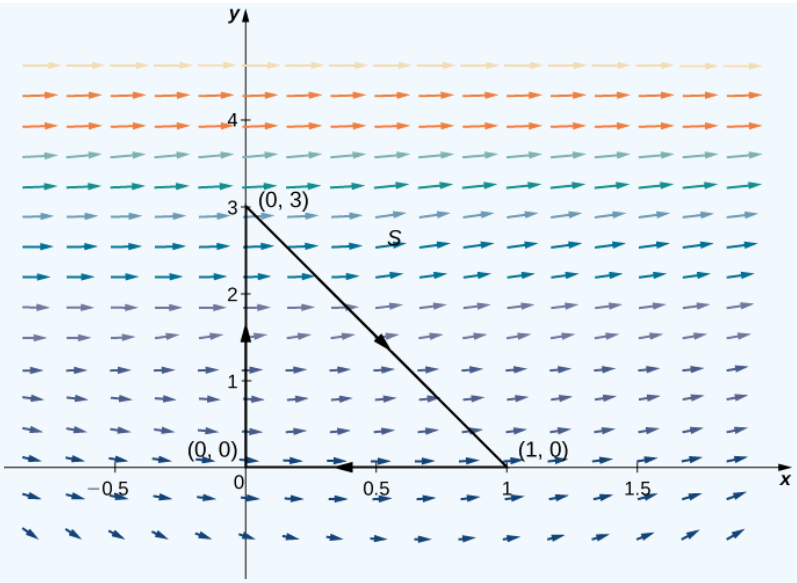


Figure 3.4.10: Curve S is a triangle with vertices $(0,0)$, $(1,0)$, and $(0,3)$ oriented clockwise in a two-dimensional vector field.

Solution

To calculate the flux without Green's theorem, we would need to break the flux integral into three line integrals, one integral for each side of the triangle. Using Green's theorem to translate the flux line integral into a single double integral is much more simple.

Let D be the region enclosed by S. Note that $P_x = 2x$ and $Q_y = 1$; therefore, $P_x + Q_y = 2x + 1$. Green's theorem applies only to simple closed curves oriented counterclockwise, but we can still apply the theorem because $\oint_C \mathbf{F} \cdot \mathbf{N}\, ds = -\oint_{-S} \mathbf{F} \cdot \mathbf{N}\, ds$ and $-S$ is oriented counterclockwise. By Green's theorem, the flux is

$$\begin{aligned}\oint_C \mathbf{F} \cdot \mathbf{N}\, ds &= \oint_{-S} \mathbf{F} \cdot \mathbf{N}\, ds \\ &= -\iint_D (P_x + Q_y)\, dA \\ &= -\iint_D (2x+1)\, dA.\end{aligned}$$

Notice that the top edge of the triangle is the line $y = -3x + 3$. Therefore, in the iterated double integral, the y-values run from $y = 0$ to $y = -3x + 3$, and we have

$$\begin{aligned}-\iint_D (2x+1)\, dA &= -\int_0^1 \int_0^{-3x+3} (2x+1)\, dy\, dx \\ &= -\int_0^1 (2x+1)(-3x+3)\, dx \\ &= -\int_0^1 (-6x^2 + 3x + 3)\, dx \\ &= -\left[-2x^3 + \frac{3x^2}{2} + 3x\right]_0^1 \\ &= -\frac{5}{2}.\end{aligned}$$

Exercise 3.4.4

Calculate the flux of $\mathbf{F}(x,y) = \langle x^3, y^3 \rangle$ across a unit circle oriented counterclockwise.

Hint available at the end of this section.

Example 3.4.5: Applying Green's Theorem for Water Flow across a Rectangle

Water flows from a spring located at the origin. The velocity of the water is modeled by vector field $\mathbf{v}(x,y) = \langle 5x + y, x + 3y \rangle$ m/sec. Find the amount of water per second that flows across the rectangle with vertices $(-1,-2)$, $(1,-2)$, $(1,3)$,and $(-1,3)$, oriented counterclockwise (Figure 3.4.11).

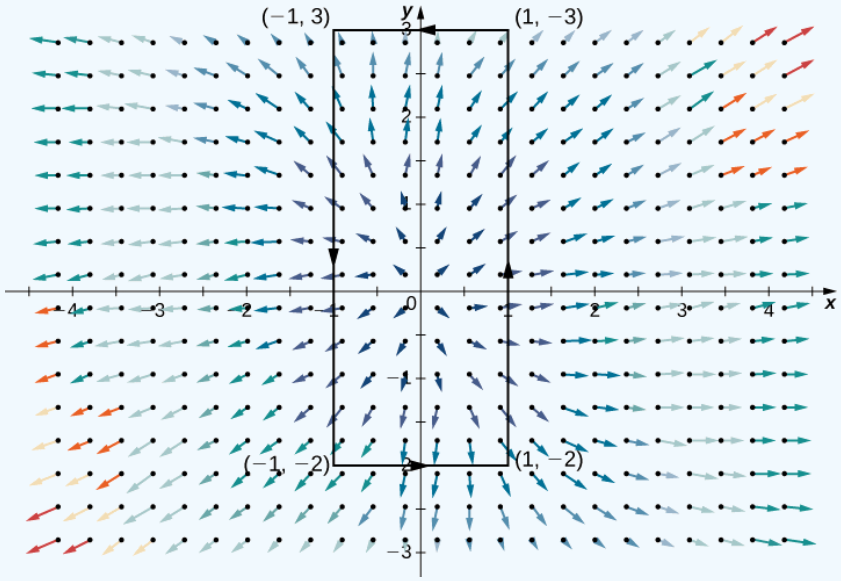


Figure 3.4.11: The rectangle with vertices $(-1,-2)$, $(1,-2)$, $(1,3)$,and $(-1,3)$ oriented counterclockwise.

Solution

Let C represent the given rectangle and let D be the rectangular region enclosed by C. To find the amount of water flowing across C, we calculate flux $\int_C \mathbf{v} \cdot d\mathbf{r}$. Let $P(x,y) = 5x + y$ and $Q(x,y) = x + 3y$ so that $\mathbf{v} = \langle P, Q \rangle$. Then, $P_x = 5$ and $Q_y = 3$. By Green's theorem,

$$\begin{aligned} \int_C \mathbf{v} \cdot d\mathbf{r} &= \iint_D (P_x + Q_y)\, dA \\ &= \iint_D 8\, dA \\ &= 8(area\ of\ D) = 80. \end{aligned}$$

Therefore, the water flux is 80 m^2/sec.

Recall that if vector field $\mathbf{F}$ is conservative, then $\mathbf{F}$ does no work around closed curves—that is, the circulation of $\mathbf{F}$ around a closed curve is zero. In fact, if the domain of $\mathbf{F}$ is simply connected, then $\mathbf{F}$ is conservative if and only if the circulation of $\mathbf{F}$ around any closed curve is zero. If we replace "circulation of $\mathbf{F}$" with "flux of $\mathbf{F}$," then we get a definition of a source-free vector field. The following statements are all equivalent ways of defining a source-free field $\mathbf{F} = \langle P, Q \rangle$ on a simply connected domain (note the similarities with properties of conservative vector fields):

1. The flux $\oint_C \mathbf{F} \cdot \mathbf{N}\ ds$ across any closed curve C is zero.
2. If C_1 and C_2 are curves in the domain of $\mathbf{F}$ with the same starting points and endpoints, then $\int_{C_1} \mathbf{F} \cdot \mathbf{N}\ ds = \int_{C_2} \mathbf{F} \cdot \mathbf{N}\ ds$. In

other words, flux is independent of path.

3. There is a stream function $g(x,y)$ for $\mathbf{F}$. A stream function for $\mathbf{F}=\langle P,Q\rangle$ is a function g such that $P=g_y$ and $Q=-g_x$ .Geometrically, $\mathbf{F}=\langle a,b\rangle$ is tangential to the level curve of g at (a,b). Since the gradient of g is perpendicular to the level curve of g at (a,b), stream function g has the property $\mathbf{F}(a,b)\cdot\nabla g(a,b)=0$ for any point (a,b) in the domain of g. (Stream functions play the same role for source-free fields that potential functions play for conservative fields.)
4. $P_x+Q_y=0$

Example 3.4.6: Finding a Stream Function

Verify that rotation vector field $\mathbf{F}(x,y)=\langle y,-x\rangle$ is source free, and find a stream function for $\mathbf{F}$.

Solution

Note that the domain of $\mathbf{F}$ is all of $\mathbb{R}^2$, which is simply connected. Therefore, to show that $\mathbf{F}$ is source free, we can show any of items 1 through 4 from the previous list to be true. In this example, we show that item 4 is true. Let $P(x,y)=y$ and $Q(x,y)=-x$. Then $P_x+0=Q_y$, and therefore $P_x+Q_y=0$. Thus, $\mathbf{F}$ is source free.

To find a stream function for $\mathbf{F}$, proceed in the same manner as finding a potential function for a conservative field. Let g be a stream function for $\mathbf{F}$. Then $g_y=y$, which implies that

$$g(x,y)=\frac{y^2}{2}+h(x).$$

Since $-g_x=Q=-x$, we have $h\,'(x)=x$. Therefore,

$$h(x)=\frac{x^2}{2}+C.$$

Letting $C=0$ gives stream function

$$g(x,y)=\frac{x^2}{2}+\frac{y^2}{2}.$$

To confirm that g is a stream function for $\mathbf{F}$, note that $g_y=y=P$ and $-g_x=-x=Q$.

Notice that source-free rotation vector field $\mathbf{F}(x,y)=\langle y,-x\rangle$ is perpendicular to conservative radial vector field $\nabla g=\langle x,y\rangle$ (Figure 3.4.12).

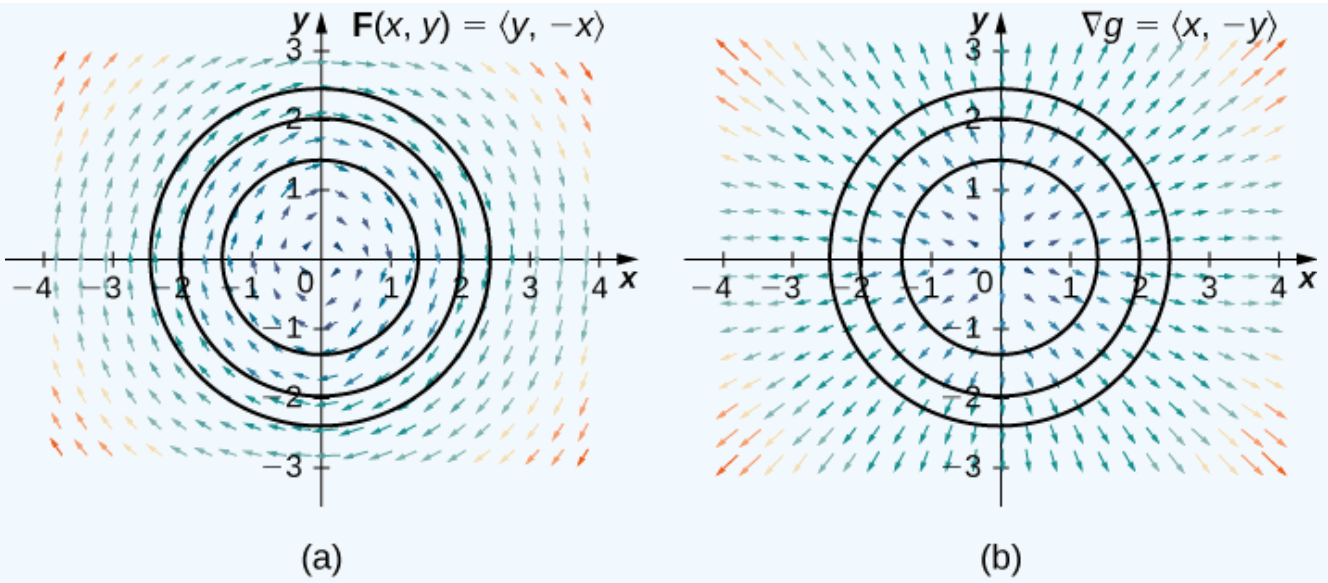


Figure 3.4.12: (a) The three-level curves of g and vector field $\mathbf{F}$. Note that the $\mathbf{F}$ vectors on a given level curve are tangent to the level curve. (b) The three-level curves of g and vector field ∇g. The gradient vectors are perpendicular to the corresponding level curve. Therefore, $\mathbf{F}(a,b) \cdot \nabla g(a,b) = 0$ for any point in the domain of g.

Exercise 3.4.6

Find a stream function for vector field $\mathbf{F}(x,y) = \langle x \sin y, \cos y \rangle$.

Hint available at the end of this section.

Vector fields that are both conservative and source free are important vector fields. One important feature of conservative and source-free vector fields on a simply connected domain is that any potential function f of such a field satisfies Laplace's equation $f_{xx} + f_{yy} = 0$. Laplace's equation is foundational in the field of partial differential equations because it models such phenomena as gravitational and magnetic potentials in space, and the velocity potential of an ideal fluid. A function that satisfies Laplace's equation is called a *harmonic* function. Therefore any potential function of a conservative and source-free vector field is harmonic.

To see that any potential function of a conservative and source-free vector field on a simply connected domain is harmonic, let f be such a potential function of vector field $\mathbf{F} = \langle P, Q \rangle$. Then, $f_x = P$ and $f_x = Q$ because $\nabla f = \mathbf{F}$. Therefore, $f_{xx} = P_x$ and $f_{yy} = Q_y$. Since $\mathbf{F}$ is source free, $f_{xx} + f_{yy} = P_x + Q_y = 0$, and we have that f is harmonic.

Example 3.4.7: Satisfying Laplace's Equation

For vector field $\mathbf{F}(x,y) = \langle e^x \sin y, e^x \cos y \rangle$, verify that the field is both conservative and source free, find a potential function for $\mathbf{F}$, and verify that the potential function is harmonic.

Solution

Let $P(x,y) = e^x \sin y$ and $Q(x,y) = e^x \cos y$. Notice that the domain of $\mathbf{F}$ is all of two-space, which is simply connected. Therefore, we can check the cross-partials of $\mathbf{F}$ to determine whether $\mathbf{F}$ is conservative. Note that $P_y = e^x \cos y = Q_x$, so $\mathbf{F}$ is conservative. Since $P_x = e^x \sin y$ and $Q_y = -e^x \sin y$, $P_x + Q_y = 0$ and the field is source free.

To find a potential function for $\mathbf{F}$, let f be a potential function. Then, $\nabla f = \mathbf{F}$, so $f_x(x,y) = e^x \sin y$. Integrating this equation with respect to x gives $f(x,y) = e^x \sin y + h(y)$. Since $f_y(x,y) = e^x \cos y$, differentiating f with respect to y gives $e^x \cos y = e^x \cos y + h\,'(y)$. Therefore, we can take $h(y) = 0$, and $f(x,y) = e^x \sin y$ is a potential function for f.

To verify that f is a harmonic function, note that $f_{xx}(x,y) = \dfrac{\partial}{\partial x}(e^x \sin y) = e^x \sin y$ and

$f_{yy}(x,y) = \dfrac{\partial}{\partial x}(e^x \cos y) = -e^x \sin y$. Therefore, $f_{xx} + f_{yy} = 0$, and f satisfies Laplace's equation.

Exercise 3.4.7

Is the function $f(x,y) = e^{x+5y}$ harmonic?

Hint available at the end of this section.

Green's Theorem on General Regions

Green's theorem, as stated, applies only to regions that are simply connected—that is, Green's theorem as stated so far cannot handle regions with holes. Here, we extend Green's theorem so that it does work on regions with finitely many holes (Figure 3.4.13).

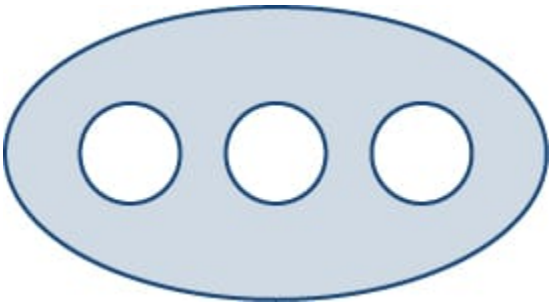


Figure 3.4.13: A nonsimply connected region with three holes. Green's theorem, as stated, does not apply to a nonsimply connected region with three holes like this one.

Before discussing extensions of Green's theorem, we need to go over some terminology regarding the boundary of a region. Let D be a region and let C be a component of the boundary of D. We say that C is *positively oriented* if, as we walk along C in the direction of orientation, region D is always on our left. Therefore, the counterclockwise orientation of the boundary of a disk is a positive orientation, for example. Curve C is *negatively oriented* if, as we walk along C in the direction of orientation, region D is always on our right. The clockwise orientation of the boundary of a disk is a negative orientation, for example.

Let D be a region with finitely many holes (so that D has finitely many boundary curves), and denote the boundary of D by ∂D (Figure 3.4.14). To extend Green's theorem so it can handle D, we divide region D into two regions, D_1 and D_2 (with respective boundaries ∂D_1 and ∂D_2), in such a way that $D = D_1 \cup D_2$ and neither D_1 nor D_2 has any holes (Figure 3.4.14).

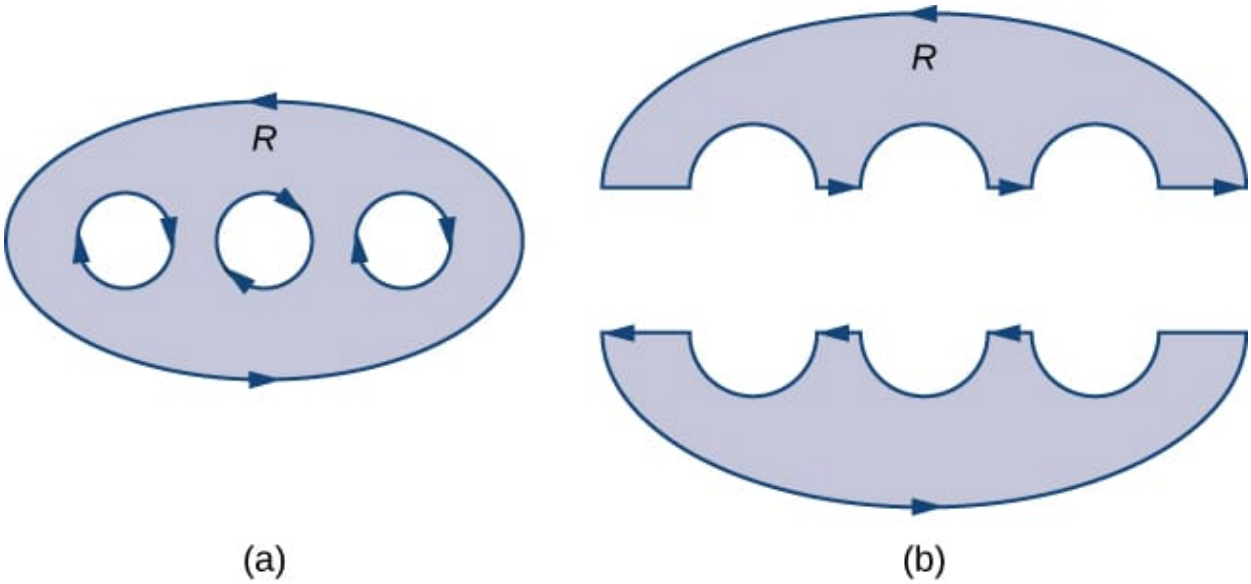


Figure 3.4.14: In part (a), the region D is shown as a shaded area with an outer boundary labeled R and three internal holes, each with their own oriented boundary. In part (b), the same region is divided into two subregions, which are shaded separately. These subregions are constructed such that neither contains any holes, making them simply connected regions.

Assume the boundary of D is oriented as in the figure, with the inner holes given a negative orientation and the outer boundary given a positive orientation. The boundary of each simply connected region D_1 and D_2 is positively oriented. If $\mathbf{F}$ is a vector field defined on D, then Green's theorem says that

$$\oint_{\partial D} \mathbf{F}\cdot d\mathbf{r} = \oint_{\partial D_1} \mathbf{F}\cdot d\mathbf{r} + \oint_{\partial D_2} \mathbf{F}\cdot d\mathbf{r} \tag{3.4.10}$$

$$= \iint_{D_1} Q_x - P_y \, dA + \iint_{D_2} Q_x - P_y \, dA \tag{3.4.11}$$

$$= \iint_{D} (Q_x - P_y) \, dA. \tag{3.4.12}$$

Therefore, Green's theorem still works on a region with holes.

To see how this works in practice, consider annulus D in Figure 3.4.15 and suppose that $F = \langle P, Q \rangle$ is a vector field defined on this annulus. Region D has a hole, so it is not simply connected. Orient the outer circle of the annulus counterclockwise and the inner circle clockwise (Figure 3.4.15) so that, when we divide the region into D_1 and D_2, we are able to keep the region on our left as we walk along a path that traverses the boundary. Let D_1 be the upper half of the annulus and D_2 be the lower half. Neither of these regions has holes, so we have divided D into two simply connected regions.

We label each piece of these new boundaries as P_i for some i, as in Figure 3.4.15. If we begin at P and travel along the oriented boundary, the first segment is P_1, then P_2, P_3, and P_4. Now we have traversed D_1 and returned to P. Next, we start at P again and traverse D_2. Since the first piece of the boundary is the same as P_4 in D_1, but oriented in the opposite direction, the first piece of D_2 is $-P_4$. Next, we have P_5, then $-P_2$, and finally P_6.

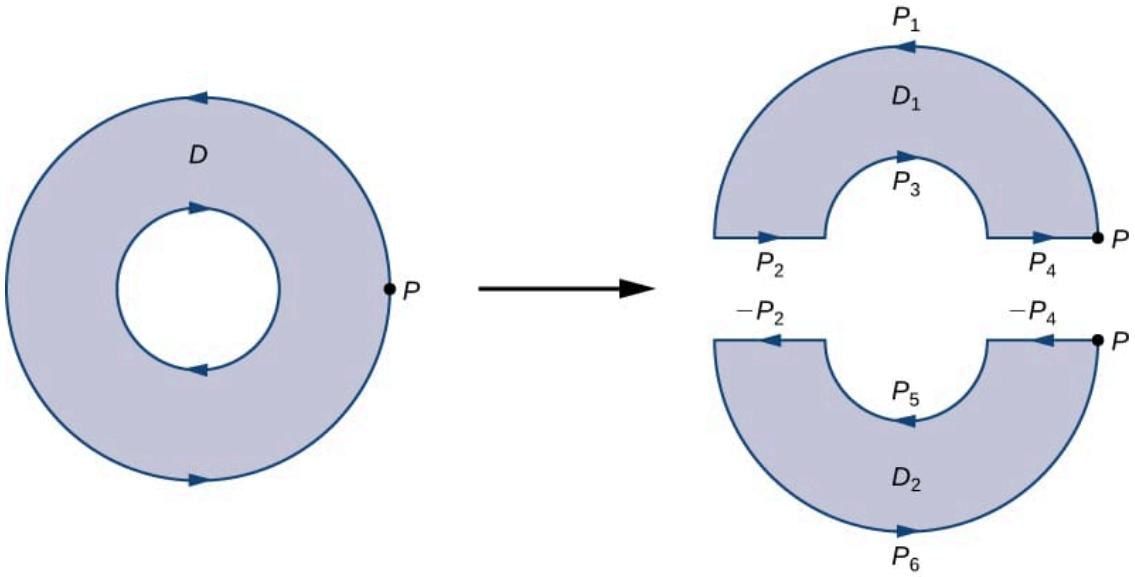


Figure 3.4.15: On the left side is the region D is shown as an annulus with a single hole and an oriented boundary. The outer boundary is oriented counterclockwise, while the inner boundary is oriented clockwise. The arrow pointing from the left image to the right image show how an annular region D with a hole can be split into two simply connected regions D_1 and D_2 for the application of Green's Theorem. On the right side is the annular region D is divided into two subregions, D_1 and D_2, with their respective boundaries labeled P_1, P_2, P_3, P_4, P_5, and P_6. The orientation of the boundaries is preserved, and the common boundaries between D_1 and D_2 have opposite orientations.

Figure 3.4.15 shows a path that traverses the boundary of D. Notice that this path traverses the boundary of region D_1, returns to the starting point, and then traverses the boundary of region D_2. Furthermore, as we walk along the path, the region is always on our left. Notice that this traversal of the P_i paths covers the entire boundary of region D. If we had only traversed one portion of the boundary of D, then we cannot apply Green's theorem to D.

The boundary of the upper half of the annulus, therefore, is $P_1 \cup P_2 \cup P_3 \cup P_4$ and the boundary of the lower half of the annulus is $-P_4 \cup P_5 \cup -P_2 \cup P_6$. Then, Green's theorem implies

$$\oint_{\partial D} \mathbf{F}\cdot d\mathbf{r} = \int_{P_1} \mathbf{F}\cdot d\mathbf{r} + \int_{P_2} \mathbf{F}\cdot d\mathbf{r} + \int_{P_3} \mathbf{F}\cdot d\mathbf{r} + \int_{P_4} \mathbf{F}\cdot d\mathbf{r} + \int_{-P_4} \mathbf{F}\cdot d\mathbf{r} + \int_{P_5} \mathbf{F}\cdot d\mathbf{r} + \int_{-P_2} \mathbf{F}\cdot d\mathbf{r} + \int_{P_6} \mathbf{F}\cdot d\mathbf{r} \tag{3.4.13}$$

$$= \int_{P_1} \mathbf{F}\cdot d\mathbf{r} + \int_{P_2} \mathbf{F}\cdot d\mathbf{r} + \int_{P_3} \mathbf{F}\cdot d\mathbf{r} + \int_{P_4} \mathbf{F}\cdot d\mathbf{r} + \int_{P_4} \mathbf{F}\cdot d\mathbf{r} + \int_{P_5} \mathbf{F}\cdot d\mathbf{r} + \int_{-P_2} \mathbf{F}\cdot d\mathbf{r} + \int_{P_6} \mathbf{F}\cdot d\mathbf{r} \tag{3.4.14}$$

$$= \int_{P_1} \mathbf{F}\cdot d\mathbf{r} + \int_{P_3} \mathbf{F}\cdot d\mathbf{r} + \int_{P_5} \mathbf{F}\cdot d\mathbf{r} + \int_{P_6} \mathbf{F}\cdot d\mathbf{r} \tag{3.4.15}$$

$$= \oint_{\partial D_1} \mathbf{F}\cdot d\mathbf{r} + \oint_{\partial D_2} \mathbf{F}\cdot d\mathbf{r} \tag{3.4.16}$$

$$= \iint_{D_1} (Q_x - P_y)\, dA + \iint_{D_2} (Q_x - P_y)\, dA \tag{3.4.17}$$

$$= \iint_{D} (Q_x - P_y)\, dA. \tag{3.4.18}$$

Therefore, we arrive at the equation found in Green's theorem—namely,

$$\oint_{\partial D} \mathbf{F}\cdot d\mathbf{r} = \iint_{D} (Q_x - P_y)\, dA. \tag{3.4.19}$$

The same logic implies that the flux form of Green's theorem can also be extended to a region with finitely many holes:

$$\oint_{C} F \cdot N\, ds = \iint_{D} (P_x + Q_y)\, dA. \tag{3.4.20}$$

Example 3.4.8A: Using Green's Theorem on a Region with Holes

Calculate the integral

$$\oint_{\partial D}(\sin x - \frac{y^3}{3})dx + (\frac{y^3}{3} + \sin y)dy, \tag{3.4.21}$$

where D is the annulus given by the polar inequalities $1 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$.

Solution

Although D is not simply connected, we can use the extended form of Green's theorem to calculate the integral. Since the integration occurs over an annulus, we convert to polar coordinates:

$$\begin{aligned}\oint_{\partial D}(\sin x - \frac{y^3}{3})\,dx + (\frac{x^3}{3} + \sin y)\,dy &= \iint_D (Q_x - P_y)\,dA \\ &= \iint_D (x^2 + y^2)\,dA \\ &= \int_0^{2\pi}\int_1^2 r^3\,dr d\theta = \int_0^{2\pi}\frac{15}{4}\,d\theta \\ &= \frac{15\pi}{2}.\end{aligned}$$

Example 3.4.8B: Using the Extended Form of Green's Theorem

Let $\mathbf{F} = \langle P, Q\rangle = \langle \frac{y}{x^2+y^2}, -\frac{x}{x^2+y^2}\rangle$ and let C be any simple closed curve in a plane oriented counterclockwise. What are the possible values of $\oint_C \mathbf{F}\cdot d\mathbf{r}$?

Solution

We use the extended form of Green's theorem to show that $\oint_C \mathbf{F}\cdot d\mathbf{r}$ is either 0 or -2π—that is, no matter how crazy curve C is, the line integral of $\mathbf{F}$ along C can have only one of two possible values. We consider two cases: the case when C encompasses the origin and the case when C does not encompass the origin.

Case 1: *C* Does Not Encompass the Origin

In this case, the region enclosed by C is simply connected because the only hole in the domain of $\mathbf{F}$ is at the origin. We showed in our discussion of cross-partials that $\mathbf{F}$ satisfies the cross-partial condition. If we restrict the domain of $\mathbf{F}$ just to C and the region it encloses, then $\mathbf{F}$ with this restricted domain is now defined on a simply connected domain. Since $\mathbf{F}$ satisfies the cross-partial property on its restricted domain, the field $\mathbf{F}$ is conservative on this simply connected region and hence the circulation $\oint_C \mathbf{F}\cdot d\mathbf{r}$ is zero.

Case 2: *C* Does Encompass the Origin

In this case, the region enclosed by C is not simply connected because this region contains a hole at the origin. Let C_1 be a circle of radius *a* centered at the origin so that C_1 is entirely inside the region enclosed by C (Figure 3.4.16). Give C_1 a clockwise orientation.

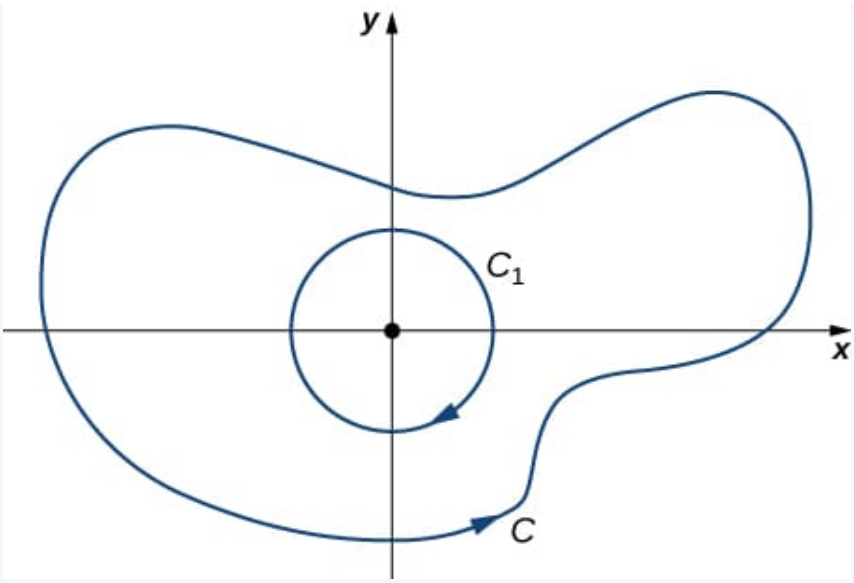


Figure 3.4.16: A region enclosed by a closed curve C that is not simply connected because it contains a hole at the origin. The outer boundary, C, is a smooth, simple closed curve with a counterclockwise orientation. Inside C, there is a smaller circular boundary C_1, which is centered at the origin and oriented clockwise.

Let D be the region between C_1 and C, and C is orientated counterclockwise. By the extended version of Green's theorem,

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} + \int_{C_1} \mathbf{F}\cdot d\mathbf{r} &= \iint_D Qx\_P_y\, dA \\ &= \iint_D -\frac{y^2-x^2}{(x^2+y^2)^2} + \frac{y^2-x^2}{(x^2+y^2)^2} dA \\ &= 0,\end{aligned}$$

and therefore

$$\int_C \mathbf{F}\cdot d\mathbf{r} = -\int_{C_1} \mathbf{F}\cdot d\mathbf{r}.$$

Since C_1 is a specific curve, we can evaluate $\int_{C_1} \mathbf{F}\cdot d\mathbf{r}$. Let

$$x = a\cos t, \quad y = a\sin t, \quad 0 \le t \le 2\pi$$

be a parameterization of C_1. Then,

$$\begin{aligned}\int_{C_1} \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} F(r(t))\cdot r'(t)dt \\ &= \int_0^{2\pi} \langle -\frac{\sin(t)}{a}, -\frac{\cos(t)}{a}\rangle \cdot \langle -a\sin(t), -a\cos(t)\rangle dt \\ &= \int_0^{2\pi} \sin^2(t) + \cos^2(t) dt \\ &= \int_0^{2\pi} dt = 2\pi.\end{aligned}$$

Therefore, $\int_C F\cdot ds = -2\pi$.

Exercise 3.4.8

Calculate integral $\oint_{\partial D} \mathbf{F} \cdot d\mathbf{r}$, where D is the annulus given by the polar inequalities $2 \leq r \leq 5$, $0 \leq \theta \leq 2\pi$, and $F(x, y) = \langle x^3, 5x + e^y \sin y \rangle$.

Hint available at the end of this section.

Measuring Area From A Boundary: The Planimeter

Imagine you are a doctor who has just received a magnetic resonance image of your patient's brain. The brain has a tumor (Figure 3.4.17). How large is the tumor? To be precise, what is the area of the red region? The red cross-section of the tumor has an irregular shape, and therefore it is unlikely that you would be able to find a set of equations or inequalities for the region and then be able to calculate its area by conventional means. You could approximate the area by chopping the region into tiny squares (a Riemann sum approach), but this method always gives an answer with some error.

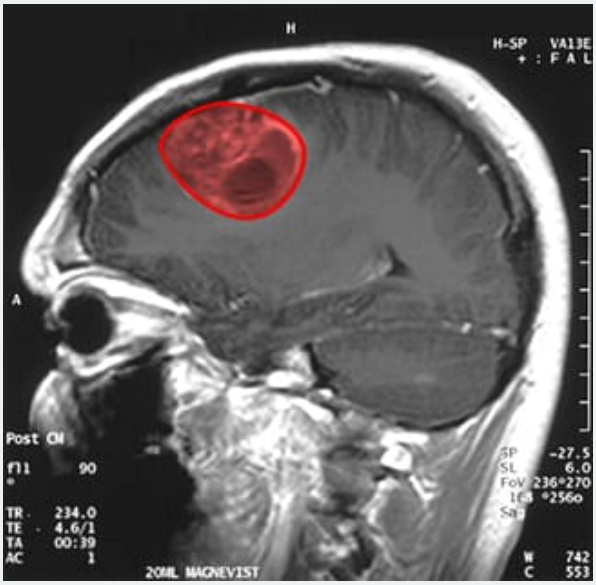


Figure 3.4.17: This magnetic resonance image of a patient's brain shows a tumor, which is highlighted in red. (CC-BY-2.0 modification of work by oatsy40, Wikipedia)

Instead of trying to measure the area of the region directly, we can use a device called a *rolling planimeter* to calculate the area of the region exactly, simply by measuring its boundary. In this project you investigate how a planimeter works, and you use Green's theorem to show the device calculates area correctly.

A rolling planimeter is a device that measures the area of a planar region by tracing out the boundary of that region (Figure 3.4.18). To measure the area of a region, we simply run the tracer of the planimeter around the boundary of the region. The planimeter measures the number of turns through which the wheel rotates as we trace the boundary; the area of the shape is proportional to this number of wheel turns. We can derive the precise proportionality equation using Green's theorem. As the tracer moves around the boundary of the region, the tracer arm rotates and the roller moves back and forth (but does not rotate).

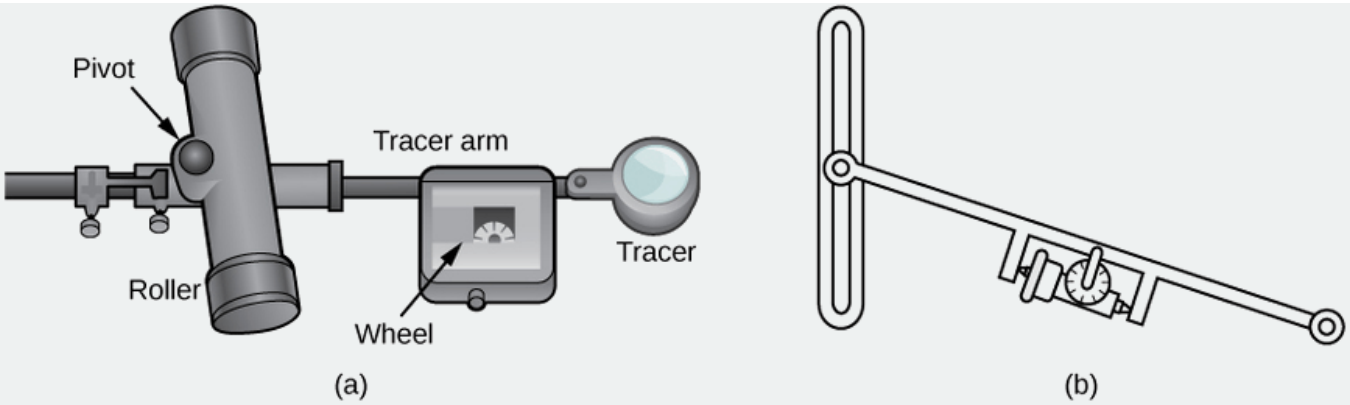


Figure 3.4.18: (a) A rolling planimeter. The pivot allows the tracer arm to rotate. The roller itself does not rotate; it only moves back and forth. (b) An interior view of a rolling planimeter. Notice that the wheel cannot turn if the planimeter is moving back and forth with the tracer arm perpendicular to the roller.

Let C denote the boundary of region D, the area to be calculated. As the tracer traverses curve C, assume the roller moves along the *y*-axis (since the roller does not rotate, one can assume it moves along a straight line). Use the coordinates (x, y) to represent points on boundary C, and coordinates $(0, Y)$ to represent the position of the pivot. As the planimeter traces C, the pivot moves along the *y*-axis while the tracer arm rotates on the pivot.

Begin the analysis by considering the motion of the tracer as it moves from point (x, y) counterclockwise to point $(x + dx, y + dy)$ that is close to (x, y) (Figure 3.4.19). The pivot also moves, from point $(0, Y)$ to nearby point $(0, Y + dY)$. How much does the wheel turn as a result of this motion? To answer this question, break the motion into two parts. First, roll the pivot along the *y*-axis from $(0, Y)$ to $(0, Y + dY)$ without rotating the tracer arm. The tracer arm then ends up at point $(x, y + dY)$ while maintaining a constant angle ϕ with the *x*-axis. Second, rotate the tracer arm by an angle $d\theta$ without moving the roller. Now the tracer is at point $(x + dx, y + dy)$. Let ll be the distance from the pivot to the wheel and let *L* be the distance from the pivot to the tracer (the length of the tracer arm).

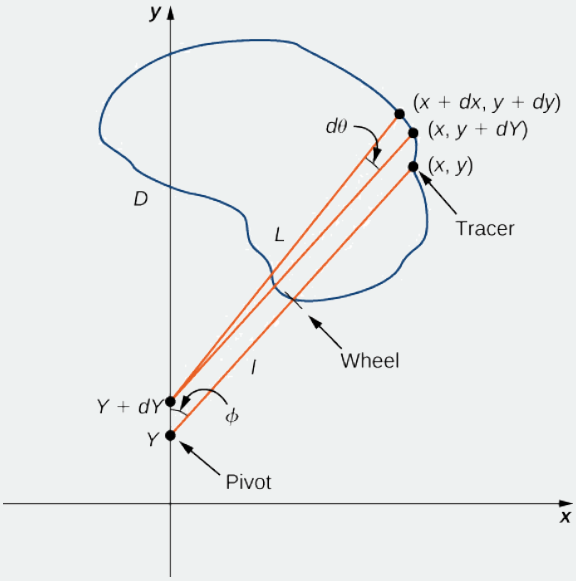


Figure 3.4.19: A mathematical analysis of the motion of a planimeter. The region D is bounded by a closed curve C, shown in blue. The planimeter's components are labeled as the "Tracer," "Wheel," and "Pivot," with the tracer point moving along the curve C. At any given moment, the tracer is at position (x, y), and its displacement is represented as $(x + dx, y + dy)$.

1. Explain why the total distance through which the wheel rolls the small motion just described is $\sin\phi dY + ld\theta = \frac{x}{L}dY + ld\theta$.
2. Show that $\oint_C d\theta = 0$.

3. Use step 2 to show that the total rolling distance of the wheel as the tracer traverses curve C is
   Total wheel roll $= \frac{1}{L}\oint_C x\,dY$.
   Now that you have an equation for the total rolling distance of the wheel, connect this equation to Green's theorem to calculate area D enclosed by C.
4. Show that $x^2 + (y - Y)^2 = L^2$.
5. Assume the orientation of the planimeter is as shown in Figure 3.4.18. Explain why $Y \leq y$, and use this inequality to show there is a unique value of Y for each point (x, y): $Y = y = \sqrt{L^2 - x^2}$.
6. Use step 5 to show that $dY = dy + \frac{x}{L^2 - x^2}dx$.
7. Use Green's theorem to show that $\oint_C \frac{x}{L^2 - x^2}dx = 0$.
8. Use step 7 to show that the total wheel roll is

$$\text{Total wheel roll} \quad = \quad 1L\oint_C x\,dy. \tag{3.4.22}$$

   It took a bit of work, but this equation says that the variable of integration *Y* in step 3 can be replaced with *y*.

9. Use Green's theorem to show that the area of D is $\oint_C x dy$. The logic is similar to the logic used to show that the area of $D = 12\oint_C -y\ dx + x\ dy$.
10. Conclude that the area of D equals the length of the tracer arm multiplied by the total rolling distance of the wheel.

You now know how a planimeter works and you have used Green's theorem to justify that it works. To calculate the area of a planar region D, use a planimeter to trace the boundary of the region. The area of the region is the length of the tracer arm multiplied by the distance the wheel rolled.

Key Concepts

- Green's theorem relates the integral over a connected region to an integral over the boundary of the region. Green's theorem is a version of the Fundamental Theorem of Calculus in one higher dimension.
- Green's Theorem comes in two forms: a circulation form and a flux form. In the circulation form, the integrand is $\mathbf{F} \cdot \mathbf{T}$. In the flux form, the integrand is $\mathbf{F} \cdot \mathbf{N}$.
- Green's theorem can be used to transform a difficult line integral into an easier double integral, or to transform a difficult double integral into an easier line integral.
- A vector field is source free if it has a stream function. The flux of a source-free vector field across a closed curve is zero, just as the circulation of a conservative vector field across a closed curve is zero.

Key Equations

- **Green's theorem, circulation form**
  $\oint_C P\ dx + Q\ dy = \iint_D Q_x - P_y\ dA$, where C is the boundary of D
- **Green's theorem, flux form**
  $\oint_C \mathbf{F} \cdot \mathbf{N}\ ds = \iint_D P_x + Q_y\ dA$, where C is the boundary of D
- **Green's theorem, extended version**
  $\oint_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint_D Q_x - P_y\ dA$

Glossary

Green's theorem
relates the integral over a connected region to an integral over the boundary of the region

Stream function
if $\mathbf{F} = \langle P, Q\rangle$ is a source-free vector field, then stream function g is a function such that $P = g_y$ and $Q = -g_x$

Contributors and Attributions

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Answers

Exercise 3.4.2

Transform the line integral into a double integral.

$\frac{45}{2}$

Exercise 3.4.3

Use Equation 3.4.8.

$\frac{4}{3}$

Exercise 3.4.4

Apply Green’s theorem and use polar coordinates.

$\frac{3\pi}{2}$

Exercise 3.4.6

Follow the outline provided in the previous example.

$g(x, y) = -x \cos y$

Exercise 3.4.7

Determine whether the function satisfies Laplace’s equation.

No

Exercise 3.4.8

Use the extended version of Green’s theorem.

105π

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3.4E: Exercises

Exercises 1 - 6

Evaluate the line integrals by applying Green's theorem.

1. $\int_C 2xy\ dx + (x+y)\ dy$, where C is the path from $(0,0)$ to $(1,1)$ along the graph of $y = x^3$ and from $(1,1)$ to $(0,0)$ along the graph of $y = x$ oriented in the counterclockwise direction

2. $\int_C 2xy\ dx + (x+y)\ dy$, where C is the boundary of the region lying between the graphs of $y = 0$ and $y = 4 - x^2$ oriented in the counterclockwise direction

Answer

$\int_C 2xy\ dx + (x+y)\ dy = \frac{32}{3}$ units of work

3. $\int_C 2\arctan\left(\frac{y}{x}\right)\ dx + \ln(x^2+y^2)\ dy$, where C is defined by $x = 4 + 2\cos\theta,\ \ y = 4\sin\theta$ oriented in the counterclockwise direction

4. $\int_C \sin x\cos y\ dx + (xy + \cos x\sin y)\ dy$, where C is the boundary of the region lying between the graphs of $y = x$ and $y = \sqrt{x}$ oriented in the counterclockwise direction

Answer

$\int_C \sin x\cos y\ dx + (xy + \cos x\sin y)\ dy = \frac{1}{12}$ units of work

5. $\int_C xy\ dx + (x+y)\ dy$, where C is the boundary of the region lying between the graphs of $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$ oriented in the counterclockwise direction

6. $\oint_C (-y\ dx + x\ dy)$, where C consists of line segment C_1 from $(-1,0)$ to $(1,0)$, followed by the semicircular arc C_2 from $(1,0)$ back to $(-1,0)$

Answer

$\oint_C (-y\ \ dx + x\ \ dy) = \pi$ units of work

Exercises 7 - 15

Use Green's theorem.

7. Let C be the curve consisting of line segments from $(0,0)$ to $(1,1)$ to $(0,1)$ and back to $(0,0)$. Find the value of $\int_C xy\ dx + \sqrt{y^2+1}\ dy$.

8. Evaluate line integral $\int_C xe^{-2x}\ dx + (x^4 + 2x^2y^2)\ dy$, where C is the boundary of the region between circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$, and is a positively oriented curve.

Answer

$\int_C xe^{-2x}\ dx + (x^4 + 2x^2y^2)\ dy = 0$ units of work

9. Find the counterclockwise circulation of field $\mathbf{F}(x,y) = xy\ \mathbf{i} + y^2\ \mathbf{j}$ around and over the boundary of the region enclosed by curves $y = x^2$ and $y = x$ in the first quadrant and oriented in the counterclockwise direction.

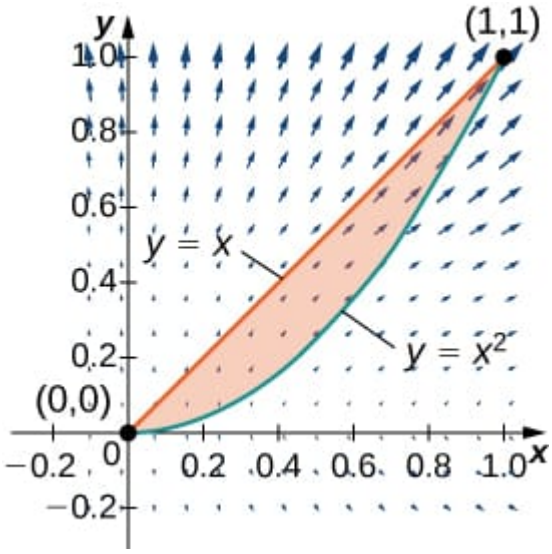


Figure $3.4E.1$: The region enclosed by curves $y = x^2$ and $y = x$ in the first quadrant in the counterclockwise direction.

10. Evaluate $\oint_C y^3\ dx - x^3y^2\ dy$, where C is the positively oriented circle of radius 2 centered at the origin.

Answer

$\oint_C y^3\ dx - x^3y^2\ dy = -20\pi$ units of work

11. Evaluate $\oint_C y^3\ dx - x^3\ dy$, where C includes the two circles of radius 2 and radius 1 centered at the origin, both with positive orientation.

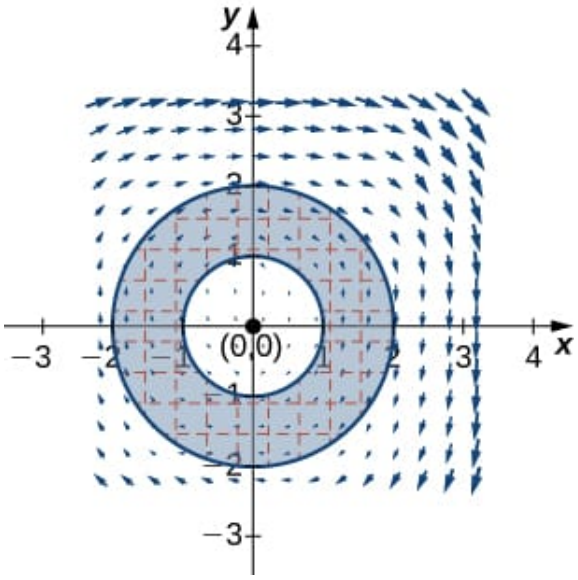


Figure $3.4E.2$: C includes the two circles of radius 2 and radius 1 centered at the origin.

12. Calculate $\oint_C -x^2y\ dx + xy^2\ dy$, where C is a circle of radius 2 centered at the origin and oriented in the counterclockwise direction.

Answer

$\oint_C -x^2y\ dx + xy^2\ dy = 8\pi$ units of work

13. Calculate integral $\oint_C 2[y + x\sin(y)]\ dx + [x^2\cos(y) - 3y^2]\ dy$ along triangle C with vertices $(0,0),\ (1,0)$ and $(1,1)$, oriented counterclockwise, using Green's theorem.

14. Evaluate integral $\oint_C (x^2 + y^2)\ dx + 2xy\ dy$, where C is the curve that follows parabola $y = x^2$ from $(0,0),\ (2,4)$, then the line from $(2,4)$ to $(2,0)$, and finally the line from $(2,0)$ to $(0,0)$.

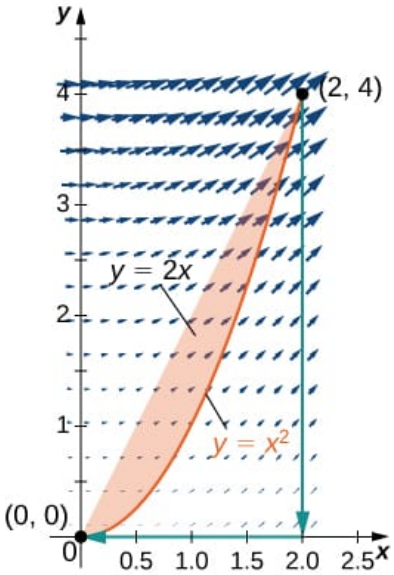


Figure $3.4E.3$: C is the curve that follows parabola $y = x^2$ from $(0,0),\ (2,4)$, then the line from $(2,4)$ to $(2,0)$, and finally the line from $(2,0)$ to $(0,0)$.

Answer

$\oint_C (x^2 + y^2)\ dx + 2xy\ dy = 0$ units of work

15. Evaluate line integral $\oint_C (y - \sin(y)\cos(y))\ dx + 2x\sin^2(y)\ dy$, where C is oriented in a counterclockwise path around the region bounded by $x = -1,\ x = 2,\ y = 4 - x^2$, and $y = x - 2$.

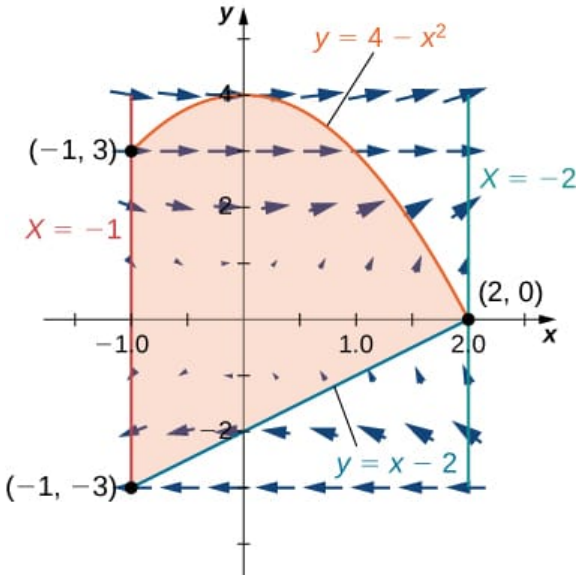


Figure $3.4E.4$: C is oriented in a counterclockwise path around the region bounded by $x = -1,\ x = 2,\ y = 4 - x^2$, and $y = x - 2$.

Exercises 16 - 28

Use Green's theorem to find the area.

16. Find the area between ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and circle $x^2 + y^2 = 25$.

Answer

$A = 19\pi \text{ units}^2$

17. Find the area of the region enclosed by parametric equation $\mathbf{p}(\theta) = (\cos(\theta) - \cos^2(\theta))\ \mathbf{i} + (\sin(\theta) - \cos(\theta)\sin(\theta))\ \mathbf{j}$ for $0 \leq \theta \leq 2\pi$.

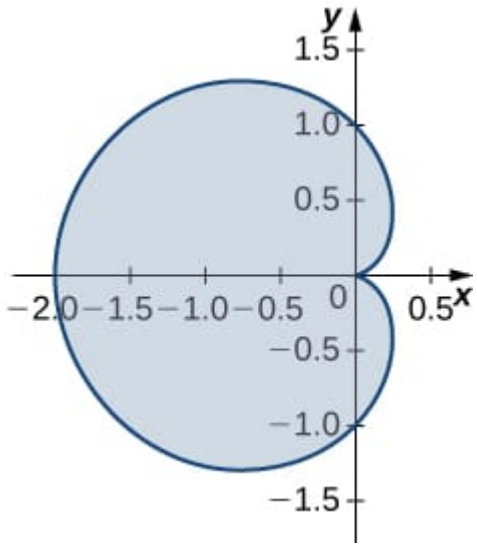


Figure $3.4E.5$: Region enclosed by parametric equation $\mathbf{p}(\theta) = (\cos(\theta) - \cos^2(\theta))\ \mathbf{i} + (\sin(\theta) - \cos(\theta)\sin(\theta))\ \mathbf{j}$ for $0 \leq \theta \leq 2\pi$.

18. Find the area of the region bounded by hypocycloid $\mathbf{r}(t) = \cos^3(t)\ \mathbf{i} + \sin^3(t)\ \mathbf{j}$. The curve is parameterized by $t \in [0, 2\pi]$.

Answer

$A = \frac{3}{8}\pi \text{ units}^2$

19. Find the area of a pentagon with vertices $(0,4),\ (4,1),\ (3,0),\ (-1,-1)$, and $(-2,2)$.

20. Use Green's theorem to evaluate $\int_{C^+} (y^2+x^3)\ dx + x^4\ dy$, where C^+ is the perimeter of square $[0,1]\times[0,1]$ oriented counterclockwise.

Answer

$$\int_{C^+} (y^2+x^3)\ dx + x^4\ dy = 0$$

21. Use Green's theorem to prove the area of a disk with radius a is $A = \pi a^2\ \text{units}^2$.

22. Use Green's theorem to find the area of one loop of a four-leaf rose $r = 3\sin 2\theta$. (*Hint*: $x\ dy - y\ dx = r^2\ d\theta$).

Answer

$$A = \tfrac{9\pi}{8}\ \text{units}^2$$

23. Use Green's theorem to find the area under one arch of the cycloid given by the parametric equations: $x = t - \sin t,\ \ y = 1-\cos t,\ \ t \geq 0$.

24. Use Green's theorem to find the area of the region enclosed by curve

$$\mathbf{r}(t) = t^2\ \mathbf{i} + \left(\tfrac{t^3}{3} - t\right)\ \mathbf{j}, \text{ for } -\sqrt{3} \leq t \leq \sqrt{3}.$$

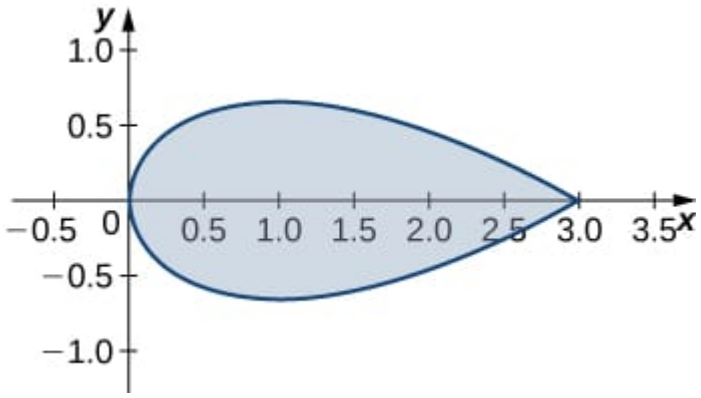


Figure $3.4E.6$: Region enclosed by curve $\mathbf{r}(t) = t^2\ \mathbf{i} + \left(\tfrac{t^3}{3} - t\right)\ \mathbf{j}$, for $-\sqrt{3} \leq t \leq \sqrt{3}$.

Answer

$$A = \tfrac{8\sqrt{3}}{5}\ \text{units}^2$$

25. Evaluate Green's theorem using a computer algebra system to evaluate the integral $\int_C xe^y\ dx + e^x\ dy$, where C is the circle given by $x^2+y^2=4$ and is oriented in the counterclockwise direction.

26. Evaluate $\int_C (x^2y - 2xy + y^2)\ ds$, where C is the boundary of the unit square $0 \leq x \leq 1,\ \ 0 \leq y \leq 1$, traversed counterclockwise.

Answer

$$\int_C (x^2y - 2xy + y^2)\ ds = 3$$

27. Evaluate $\int_C \dfrac{-(y+2)\,dx + (x-1)\,dy}{(x-1)^2+(y+2)^2}$, where C is any simple closed curve with an interior that does not contain point $(1,-2)$ traversed counterclockwise.

28. Evaluate $\int_C \dfrac{x\,dx + y\,dy}{x^2+y^2}$, where C is any piecewise, smooth simple closed curve enclosing the origin, traversed counterclockwise.

Answer

$$\int_C \frac{x\,dx + y\,dy}{x^2 + y^2} = 2\pi$$

Exercises 29 - 60

Use Green's theorem to calculate the work done by force $\mathbf{F}$ on a particle that is moving counterclockwise around closed path C.

29. $\mathbf{F}(x, y) = xy\ \mathbf{i} + (x + y)\ \mathbf{j}, \quad C : x^2 + y^2 = 4$

30. $\mathbf{F}(x, y) = (x^{3/2} - 3y)\ \mathbf{i} + (6x + 5\sqrt{y})\ \mathbf{j}, \quad C$: boundary of a triangle with vertices $(0, 0),\ (5, 0)$, and $(0, 5)$

Answer

$W = \frac{225}{2}$ units of work

31. Evaluate $\int_C (2x^3 - y^3)\ dx + (x^3 + y^3)\ dy$, where C is a unit circle oriented in the counterclockwise direction.

32. A particle starts at point $(-2, 0)$, moves along the x-axis to $(2, 0)$, and then travels along semicircle $y = \sqrt{4 - x^2}$ to the starting point. Use Green's theorem to find the work done on this particle by force field $\mathbf{F}(x, y) = x\ \mathbf{i} + (x^3 + 3xy^2)\ \mathbf{j}$.

Answer

$W = 12\pi$ units of work

33. David and Sandra are skating on a frictionless pond in the wind. David skates on the inside, going along a circle of radius 2 in a counterclockwise direction. Sandra skates once around a circle of radius 3, also in the counterclockwise direction. Suppose the force of the wind at point (x, y) is $\mathbf{F}(x, y) = (x^2y + 10y)\ \mathbf{i} + (x^3 + 2xy^2)\ \mathbf{j}$. Use Green's theorem to determine who does more work.

34. Use Green's theorem to find the work done by force field $\mathbf{F}(x, y) = (3y - 4x)\ \mathbf{i} + (4x - y)\ \mathbf{j}$ when an object moves once counterclockwise around ellipse $4x^2 + y^2 = 4$.

Answer

$W = 2\pi$ units of work

35. Use Green's theorem to evaluate line integral $\oint_C e^{2x} \sin 2y\ dx + e^{2x} \cos 2y\ dy$, where C is ellipse $9(x - 1)^2 + 4(y - 3)^2 = 36$ oriented counterclockwise.

36. Evaluate line integral $\oint_C y^2\ dx + x^2\ dy$, where C is the boundary of a triangle with vertices $(0, 0),\ (1, 1)$, and $(1, 0)$, with the counterclockwise orientation.

Answer

$\oint_C y^2\ dx + x^2\ dy = \frac{1}{3}$ units of work

37. Use Green's theorem to evaluate line integral $\int_C \mathbf{h} \cdot d\mathbf{r}$ if $\mathbf{h}(x, y) = e^y\ \mathbf{i} - \sin \pi x\ \mathbf{j}$, where C is a triangle with vertices $(1, 0)$, $(0, 1)$, and $(-1, 0)$, traversed counterclockwise.

38. Use Green's theorem to evaluate line integral $\int_C \sqrt{1 + x^3}\ dx + 2xy\ dy$ where C is a triangle with vertices $(0, 0),\ (1, 0)$, and $(1, 3)$ oriented clockwise.

Answer

$\int_C \sqrt{1 + x^3}\ dx + 2xy\ dy = 3$ units of work

39. Use Green's theorem to evaluate line integral $\int_C x^2 y \ dx - xy^2 \ dy$ where C is a circle $x^2 + y^2 = 4$ oriented counterclockwise.

40. Use Green's theorem to evaluate line integral $\int_C (3y - e^{\sin x}) \ dx + \left(7x + \sqrt{y^4 + 1}\right) \ dy$ where C is circle $x^2 + y^2 = 9$ oriented in the counterclockwise direction.

Answer

$\int_C (3y - e^{\sin x}) \ dx + \left(7x + \sqrt{y^4 + 1}\right) \ dy = 36\pi$ units of work

41. Use Green's theorem to evaluate line integral $\int_C (3x - 5y) \ dx + (x - 6y) \ dy$, where C is ellipse $\frac{x^2}{4} + y^2 = 1$ and is oriented in the counterclockwise direction.

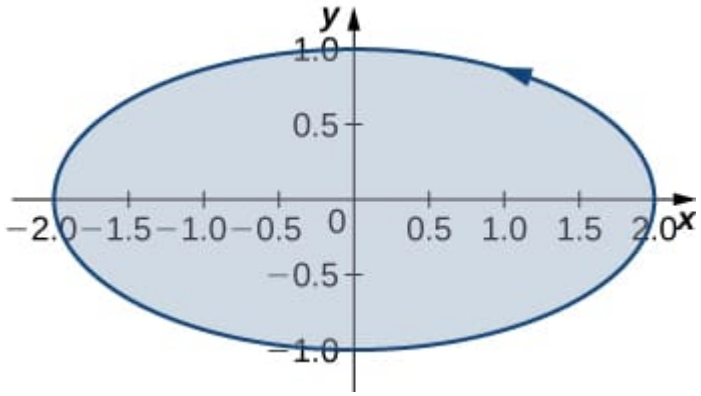


Figure 3.4E.7: C is ellipse $\frac{x^2}{4} + y^2 = 1$ and is oriented in the counterclockwise direction.

42. Let C be a triangular closed curve from $(0, 0)$ to $(1, 0)$ to $(1, 1)$ and finally back to $(0, 0)$. Let $\mathbf{F}(x, y) = 4y \ \mathbf{i} + 6x^2 \ \mathbf{j}$. Use Green's theorem to evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$.

Answer

$\oint_C \mathbf{F} \cdot d\mathbf{r} = 2$ units of work

43. Use Green's theorem to evaluate line integral $\oint_C y \ dx - x \ dy$, where C is circle $x^2 + y^2 = a^2$ oriented in the clockwise direction.

44. Use Green's theorem to evaluate line integral $\oint_C (y + x) \ dx + (x + \sin y) \ dy$, where C is any smooth simple closed curve joining the origin to itself oriented in the counterclockwise direction.

Answer

$\oint_C (y + x) \ dx + (x + \sin y) \ dy = 0$ units of work

45. Use Green's theorem to evaluate line integral $\oint_C \left(y - \ln(x^2 + y^2)\right) \ dx + \left(2 \arctan \frac{y}{x}\right) \ dy$, where C is the positively oriented circle $(x - 2)^2 + (y - 3)^2 = 1$.

46. Use Green's theorem to evaluate $\oint_C xy \ dx + x^3 y^3 \ dy$, where C is a triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 2)$ with positive orientation.

Answer

$\oint_C xy \ dx + x^3 y^3 \ dy = 2221$ units of work

47. Use Green's theorem to evaluate line integral $\int_C \sin y \ dx + x \cos y \ dy$, where C is ellipse $x^2 + xy + y^2 = 1$ oriented in the counterclockwise direction.

48. Let $\mathbf{F}(x, y) = (\cos(x^5) - 13y^3) \ \mathbf{i} + 13x^3 \ \mathbf{j}$. Find the counterclockwise circulation $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where C is a curve consisting of the line segment joining $(-2, 0)$ and $(-1, 0)$, half circle $y = \sqrt{1 - x^2}$, the line segment joining $(1, 0)$ and $(2, 0)$, and half circle $y = \sqrt{4 - x^2}$.

Answer

$\oint_C \mathbf{F} \cdot d\mathbf{r} = 15\pi^4$ units of work

49. Use Green's theorem to evaluate line integral $\int_C \sin(x^3) \ dx + 2ye^{x^2} \ dy$, where C is a triangular closed curve that connects the points $(0, 0), \ (2, 2)$, and $(0, 2)$ counterclockwise.

50. Let C be the boundary of square $0 \le x \le \pi, \ \ 0 \le y \le \pi$, traversed counterclockwise. Use Green's theorem to find $\int_C \sin(x + y) \ dx + \cos(x + y) \ dy$.

Answer

$\int_C \sin(x + y) \ dx + \cos(x + y) \ dy = 4$ units of work

51. Use Green's theorem to evaluate line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = (y^2 - x^2) \ \mathbf{i} + (x^2 + y^2) \ \mathbf{j}$, and C is a triangle bounded by $y = 0, \ \ x = 3$, and $y = x$, oriented counterclockwise.

52. Use Green's Theorem to evaluate integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = (xy^2) \ \mathbf{i} + x \ \mathbf{j}$, and C is a unit circle oriented in the counterclockwise direction.

Answer

$\int_C \mathbf{F} \cdot d\mathbf{r} = \pi$ units of work

53. Use Green's theorem in a plane to evaluate line integral $\oint_C (xy + y^2) \ dx + x^2 \ dy$, where C is a closed curve of a region bounded by $y = x$ and $y = x^2$ oriented in the counterclockwise direction.

54. Calculate the outward flux of $\mathbf{F}(x, y) = -x \ \mathbf{i} + 2y \ \mathbf{j}$ over a square with corners $(\pm 1, \ \pm 1)$, where the unit normal is outward pointing and oriented in the counterclockwise direction.

Answer

$\oint_C \mathbf{F} \cdot \mathbf{N} \ ds = 4$

55. Let C be circle $x^2 + y^2 = 4$ oriented in the counterclockwise direction. Evaluate $\oint_C \left[(3y - e^{\arctan x}) \, dx + (7x + \sqrt{y^4 + 1}) \ dy \right]$ using a computer algebra system.

56. Find the flux of field $\mathbf{F}(x, y) = -x \ \mathbf{i} + y \ \mathbf{j}$ across $x^2 + y^2 = 16$ oriented in the counterclockwise direction.

Answer

$\oint_C \mathbf{F} \cdot \mathbf{N} \ ds = 32\pi$

57. Let $\mathbf{F} = (y^2 - x^2) \ \mathbf{i} + (x^2 + y^2) \ \mathbf{j}$, and let C be a triangle bounded by $y = 0, \ \ x = 3$, and $y = x$ oriented in the counterclockwise direction. Find the outward flux of $\mathbf{F}$ through C.

58. Let C be unit circle $x^2 + y^2 = 1$ traversed once counterclockwise. Evaluate $\int_C \left[-y^3 + \sin(xy) + xy \cos(xy) \right] \ \ dx$

$+\left[x^3+x^2\cos(xy)\right]\ dy$ by using a computer algebra system.

Answer

$$\int_C \left[-y^3+\sin(xy)+xy\cos(xy)\right]\ dx+\left[x^3+x^2\cos(xy)\right]\ dy=4.7124 \text{ units of work}$$

59. Find the outward flux of vector field $\mathbf{F}(x,y)=xy^2\ \mathbf{i}+x^2y\ \mathbf{j}$ across the boundary of annulus $R=\left\{(x,y):1\le x^2+y^2\le 4\right\}=\left\{(r,\theta):1\le r\le 2,\ \ 0\le\theta\le 2\pi\right\}$ using a computer algebra system.

60. Consider region R bounded by parabolas $y=x^2$ and $x=y^2$. Let C be the boundary of R oriented counterclockwise. Use Green's theorem to evaluate $\oint_C \left(y+e^{\sqrt{x}}\right)\ dx+\left(2x+\cos(y^2)\right)\ dy$.

Answer

$$\oint_C \left(y+e^{\sqrt{x}}\right)\ dx+\left(2x+\cos(y^2)\right)\ dy=13 \text{ units of work}$$

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3.5: Divergence and Curl

Learning Objectives

- Determine divergence from the formula for a given vector field.
- Determine curl from the formula for a given vector field.
- Use the properties of curl and divergence to determine whether a vector field is conservative.

In this section, we examine two important operations on a vector field: divergence and curl. They are important to the field of calculus for several reasons, including the use of curl and divergence to develop some higher-dimensional versions of the Fundamental Theorem of Calculus. In addition, curl and divergence appear in mathematical descriptions of fluid mechanics, electromagnetism, and elasticity theory, which are important concepts in physics and engineering. We can also apply curl and divergence to other concepts we already explored. For example, under certain conditions, a vector field is conservative if and only if its curl is zero.

In addition to defining curl and divergence, we look at some physical interpretations of them, and show their relationship to conservative and source-free vector fields.

Divergence

Divergence is an operation on a vector field that tells us how the field behaves toward or away from a point. Locally, the divergence of a vector field $\mathbf{F}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ at a particular point P is a measure of the "outflowing-ness" of the vector field at P. If $\mathbf{F}$ represents the velocity of a fluid, then the divergence of $\mathbf{F}$ at P measures the net rate of change with respect to time of the amount of fluid flowing away from P (the tendency of the fluid to flow "out of" P). In particular, if the amount of fluid flowing into P is the same as the amount flowing out, then the divergence at P is zero.

Definition: Divergence in $\mathbb{R}^3$

If $\mathbf{F} = \langle P, Q, R \rangle$ is a vector field in $\mathbb{R}^3$ and $P_x,\ Q_y$, and R_z all exist, then the divergence of $\mathbf{F}$ is defined by

$$\begin{aligned} \operatorname{div} F &= P_x + Q_y + R_z \qquad (3.5.1) \\ &= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}. \qquad (3.5.2) \end{aligned}$$

Note the divergence of a vector field is not a vector field, but a scalar function. In terms of the gradient operator

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \qquad (3.5.3)$$

divergence can be written symbolically as the dot product

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F}. \qquad (3.5.4)$$

Note this is merely helpful notation, because the dot product of a vector of operators and a vector of functions is not meaningfully defined given our current definition of dot product.

If $\mathbf{F} = \langle P, Q \rangle$ is a vector field in $\mathbb{R}^2$, and P_x and Q_y both exist, then the divergence of $\mathbf{F}$ is defined similarly as

$$\begin{aligned} \operatorname{div} \mathbf{F} &= P_x + Q_y \\ &= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \\ &= \nabla \cdot \mathbf{F}. \end{aligned}$$

To illustrate this point, consider the two vector fields in Figure 3.5.1. At any particular point, the amount flowing in is the same as the amount flowing out, so at every point the "outflowing-ness" of the field is zero. Therefore, we expect the divergence of both fields to be zero, and this is indeed the case, as

$$\operatorname{div}(\langle 1, 2\rangle) = \frac{\partial}{\partial x}(1) + \frac{\partial}{\partial y}(2) = 0 \tag{3.5.5}$$

and

$$\operatorname{div}(\langle -y, x\rangle) = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) = 0. \tag{3.5.6}$$

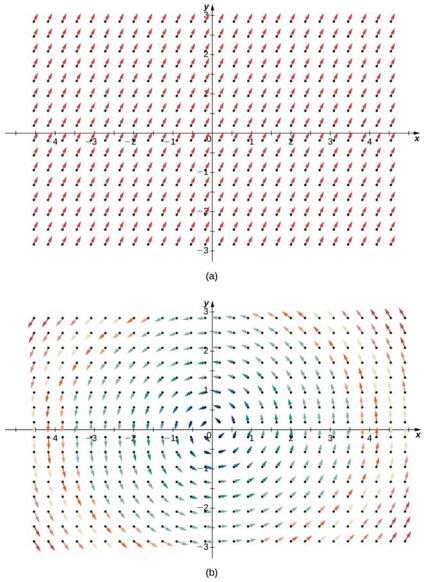


Figure 3.5.1: Two vector fields and their divergence properties: (a) The first vector field is represented by uniformly spaced and parallel arrows pointing in the same direction throughout the plane. This vector field has zero divergence as it represents a constant field with no sources or sinks. (b) The second vector field forms a circular pattern around the origin with arrows pointing tangentially to concentric circles. This field also has zero divergence as the field exhibits a rotational behavior with no net outward or inward flow at any point. Both subfigures emphasize that vector fields can exhibit different behaviors such as uniform flow or rotational flow while still having zero divergence.

By contrast, consider a radial vector field $\mathbf{R}(x, y) = \langle -x, -y\rangle$ in Figure 3.5.2. At any given point, more fluid is flowing in than is flowing out, and therefore the "outgoingness" of the field is negative. We expect the divergence of this field to be negative, and this is indeed the case, as

$$\operatorname{div}(\mathbf{R}) = \frac{\partial}{\partial x}(-x) + \frac{\partial}{\partial y}(-y) = -2. \tag{3.5.7}$$

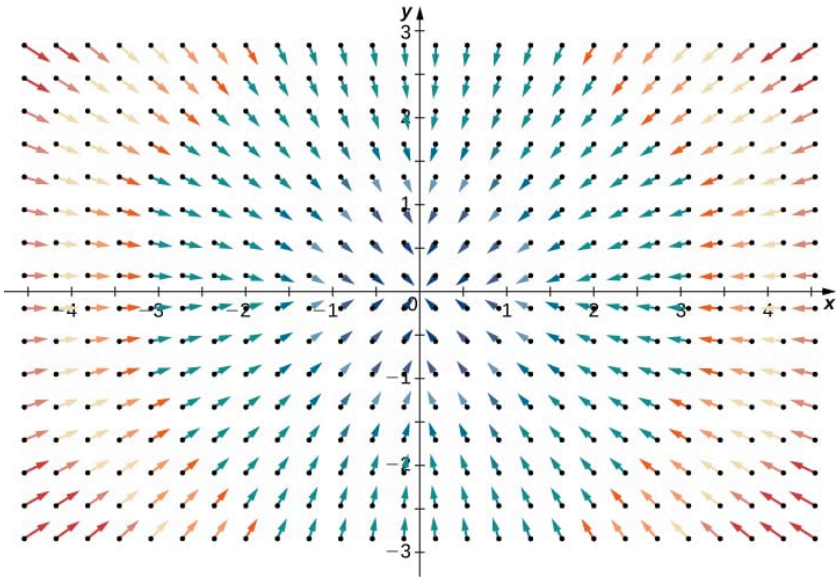


Figure 3.5.2: A radial vector field $\mathbf{R}(x, y) = \langle -x, -y \rangle$ with arrows pointing inward toward the origin from all directions. The magnitude of the arrows increases as their distance increases.

To get a global sense of what divergence is telling us, suppose that a vector field in $\mathbb{R}^2$ represents the velocity of a fluid. Imagine taking an elastic circle (a circle with a shape that can be changed by the vector field) and dropping it into a fluid. If the circle maintains its exact area as it flows through the fluid, then the divergence is zero. This would occur for both vector fields in Figure 3.5.1. On the other hand, if the circle's shape is distorted so that its area shrinks or expands, then the divergence is not zero. Imagine dropping such an elastic circle into the radial vector field in Figure 3.5.2 so that the center of the circle lands at point $(3, 3)$. The circle would flow toward the origin, and as it did so the front of the circle would travel more slowly than the back, causing the circle to "scrunch" and lose area. This is how you can see a negative divergence.

Example 3.5.1: Calculating Divergence at a Point

If $\mathbf{F}(x, y, z) = e^x i + yzj - yz^2 k$, then find the divergence of $\mathbf{F}$ at $(0, 2, -1)$.

Solution

The divergence of $\mathbf{F}$ is

$$\frac{\partial}{\partial x}(e^x) + \frac{\partial}{\partial y}(yz) - \frac{\partial}{\partial z}(yz^2) = e^x + z - 2yz.$$

Therefore, the divergence at $(0, 2, -1)$ is $e^0 - 1 + 4 = 4$. If $\mathbf{F}$ represents the velocity of a fluid, then more fluid is flowing out than flowing in at point $(0, 2, -1)$.

Exercise 3.5.1

Find $\operatorname{div}\ \mathbf{F}$ for

$$\mathbf{F}(x, y, z) = \langle xy,\ 5 - z^2,\ x^2 + y^2 \rangle.$$

Hint available at the end of this section.

One application for divergence occurs in physics, when working with magnetic fields. A magnetic field is a vector field that models the influence of electric currents and magnetic materials. Physicists use divergence in Gauss's law for magnetism, which states that if $\mathbf{B}$ is a magnetic field, then $\nabla \cdot \mathbf{B} = 0$; in other words, the divergence of a magnetic field is zero.

Example 3.5.2: Determining Whether a Field Is Magnetic

Is it possible for $\mathbf{F}(x,y) = \langle x^2 y,\ y - xy^2 \rangle$ to be a magnetic field?

Solution

If $\mathbf{F}$ were magnetic, then its divergence would be zero. The divergence of $\mathbf{F}$ is

$$\frac{\partial}{\partial x}(x^2 y) + \frac{\partial}{\partial y}(y - xy^2) = 2xy + 1 - 2xy = 1$$

and therefore $\mathbf{F}$ cannot model a magnetic field (Figure 3.5.3).

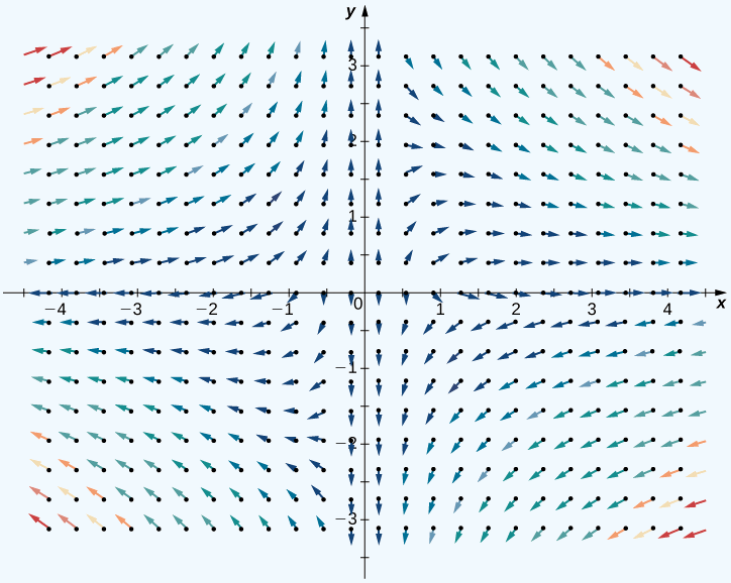


Figure 3.5.3: The vector field $\mathbf{F}(x,y) = \langle x^2 y,\ y - xy^2 \rangle$. The arrows represent the vector field's magnitude and direction at various points in the plane. The length and orientation of the arrows change depending on their location, reflecting the behavior of the vector field. The divergence of this vector field is calculated to be 1 everywhere, indicating that at each point in the field, there is a uniform source of fluid or energy being added. Because the divergence is nonzero, this vector field cannot represent a magnetic field, as magnetic fields are source-free and have zero divergence. This visualization highlights the uniform "outgoing" behavior of the field due to its constant divergence.

Another application for divergence is detecting whether a field is source free. Recall that a source-free field is a vector field that has a stream function; equivalently, a source-free field is a field with a flux that is zero along any closed curve. The next two theorems say that, under certain conditions, source-free vector fields are precisely the vector fields with zero divergence.

Theorem: Divergence of a Source-Free Vector Field

If $\mathbf{F} = \langle P, Q \rangle$ is a source-free continuous vector field with differentiable component functions, then $\text{div } \mathbf{F} = 0$.

Proof

Since $\mathbf{F}$ is source free, there is a function $g(x,y)$ with $g_y = P$ and $-g_x = Q$. Therefore, $\mathbf{F} = \langle g_y, -g_x \rangle$ and $\text{div } \mathbf{F} = g_{yx} - g_{xy} = 0$ by Clairaut's theorem.

□

The converse of Divergence of a Source-Free Vector Field is true on simply connected regions, but the proof is too technical to include here. Thus, we have the following theorem, which can test whether a vector field in $\mathbb{R}^2$ is source free.

Theorem: Divergence Test for Source-Free Vector Fields

Let $\mathbf{F} = \langle P, Q \rangle$ be a continuous vector field with differentiable component functions with a domain that is simply connected. Then, $\text{div } \mathbf{F} = 0$ if and only if $\mathbf{F}$ is source free.

Example 3.5.3: Determining Whether a Field Is Source Free

Is field $\mathbf{F}(x, y) = \langle x^2 y,\ 5 - xy^2 \rangle$ source free?

Solution

Note the domain of $\mathbf{F}$ is $\mathbb{R}^2$ which is simply connected. Furthermore, $\mathbf{F}$ is continuous with differentiable component functions. Therefore, we can use the Divergence Test for Source-Free Vector Fields to analyze $\mathbf{F}$. The divergence of $\mathbf{F}$ is

$$\frac{\partial}{\partial x}(x^2 y) + \frac{\partial}{\partial y}(5 - xy^2) = 2xy - 2xy = 0.$$

Therefore, $\mathbf{F}$ is source free by the Divergence Test for Source-Free Vector Fields.

Exercise 3.5.2

Let $\mathbf{F}(x, y) = \langle -ay, bx \rangle$ be a rotational field where a and b are positive constants. Is $\mathbf{F}$ source free?

Hint available at the end of this section.

Recall that the flux form of Green's theorem says that

$$\oint_C \mathbf{F} \cdot \mathbf{N}\, ds = \iint_D P_x + Q_y\, dA, \tag{3.5.8}$$

where C is a simple closed curve and D is the region enclosed by C. Since $P_x + Q_y = \text{div } \mathbf{F}$, Green's theorem is sometimes written as

$$\oint_C \mathbf{F} \cdot \mathbf{N}\, ds = \iint_D \text{div}\, \mathbf{F}\, dA. \tag{3.5.9}$$

Therefore, Green's theorem can be written in terms of divergence. If we think of divergence as a derivative of sorts, then Green's theorem says the "derivative" of $\mathbf{F}$ on a region can be translated into a line integral of $\mathbf{F}$ along the boundary of the region. This is analogous to the Fundamental Theorem of Calculus, in which the derivative of a function f on a line segment $[a, b]$ can be translated into a statement about f on the boundary of $[a, b]$. Using divergence, we can see that Green's theorem is a higher-dimensional analog of the Fundamental Theorem of Calculus.

We can use all of what we have learned in the application of divergence. Let $\mathbf{v}$ be a vector field modeling the velocity of a fluid. Since the divergence of $\mathbf{v}$ at point P measures the "outflowing-ness" of the fluid at P, $\text{div } v(P) > 0$ implies that more fluid is flowing out of P than flowing in. Similarly, $\text{div } v(P) < 0$ implies the more fluid is flowing in to P than is flowing out, and $\text{div } \mathbf{v}(P) = 0$ implies the same amount of fluid is flowing in as flowing out.

Example 3.5.4: Determining Flow of a Fluid

Suppose $\mathbf{v}(x, y) = \langle -xy, y \rangle,\ y > 0$ models the flow of a fluid. Is more fluid flowing into point $(1, 4)$ than flowing out?

Solution

To determine whether more fluid is flowing into $(1, 4)$ than is flowing out, we calculate the divergence of $\mathbf{v}$ at $(1, 4)$:

$$div(\mathbf{v}) = \frac{\partial}{\partial x}(-xy) + \frac{\partial}{\partial y}(y) = -y + 1.$$

To find the divergence at $(1, 4)$ substitute the point into the divergence: $-4 + 1 = -3$. Since the divergence of $\mathbf{v}$ at $(1, 4)$ is negative, more fluid is flowing in than flowing out (Figure 3.5.4).

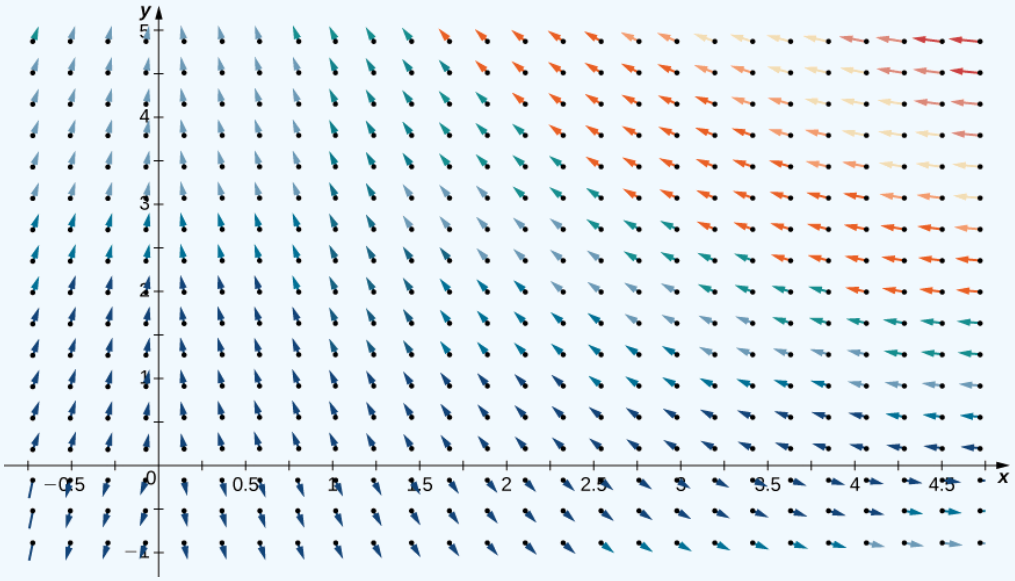


Figure 3.5.4: The vector field $\mathbf{v}(x, y) = \langle -xy, y \rangle$, which models the flow of a fluid. The field is visualized with arrows indicating the direction and magnitude of the flow at different points in the plane. The arrows' length and orientation vary, reflecting the changes in the flow's intensity and direction.

Exercise 3.5.3

For vector field $\mathbf{v}(x, y) = \langle -xy, y \rangle$, $y > 0$, find all points P such that the amount of fluid flowing in to P equals the amount of fluid flowing out of P.

Hint available at the end of this section.

Curl

The second operation on a vector field that we examine is the curl, which measures the extent of rotation of the field about a point. Suppose that $\mathbf{F}$ represents the velocity field of a fluid. Then, the curl of $\mathbf{F}$ at point P is a vector that measures the tendency of particles near P to rotate about the axis that points in the direction of this vector. The magnitude of the curl vector at P measures how quickly the particles rotate around this axis. In other words, the curl at a point is a measure of the vector field's "spin" at that point. Visually, imagine placing a paddlewheel into a fluid at P, with the axis of the paddlewheel aligned with the curl vector (Figure 3.5.5). The curl measures the tendency of the paddlewheel to rotate.

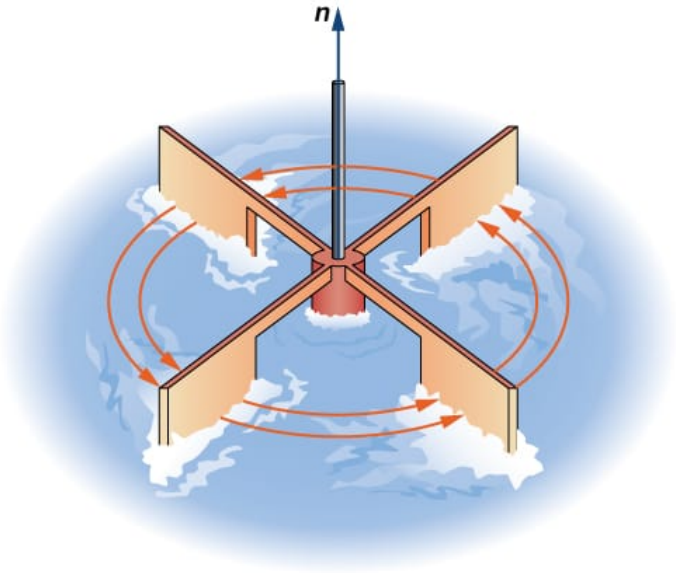


Figure 3.5.5: The concept of curl in a vector field, represented by a paddlewheel placed in the field. The paddlewheel rotates around an axis $\mathbf{n}$, which is perpendicular to the plane of rotation. The arrows around the paddlewheel show the direction of rotation, while the vertical axis represents the alignment of the curl vector with the field's rotation axis.

Consider the vector fields in Figure 3.5.1. In part (a), the vector field is constant and there is no spin at any point. Therefore, we expect the curl of the field to be zero, and this is indeed the case. Part (b) shows a rotational field, so the field has spin. In particular, if you place a paddlewheel into a field at any point so that the axis of the wheel is perpendicular to a plane, the wheel rotates counterclockwise. Therefore, we expect the curl of the field to be nonzero, and this is indeed the case (the curl is $2\ \mathbf{k}$).

To see what curl is measuring globally, imagine dropping a leaf into the fluid. As the leaf moves along with the fluid flow, the curl measures the tendency of the leaf to rotate. If the curl is zero, then the leaf doesn't rotate as it moves through the fluid.

Definition: Curl

If $\mathbf{F} = \langle P, Q, R \rangle$ is a vector field in $\mathbb{R}^3$, and $P_x,\ Q_y$, and R_z all exist, then the curl of $\mathbf{F}$ is defined by

$$\operatorname{curl} \mathbf{F} = (R_y - Q_z)\,\mathbf{i} + (P_z - R_x)\,\mathbf{j} + (Q_x - P_y)\,\mathbf{k} \tag{3.5.10}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}. \tag{3.5.11}$$

Note that the curl of a vector field is a vector field, in contrast to divergence.

The definition of curl can be difficult to remember. To help with remembering, we use the notation $\nabla \times \mathbf{F}$ to stand for a "determinant" that gives the curl formula:

$$\begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}. \tag{3.5.12}$$

The determinant of this matrix is

$$(R_y - Q_z)\,\mathbf{i} - (R_x - P_z)\,\mathbf{j} + (Q_x - P_y)\,\mathbf{k} = (R_y - Q_z)\,\mathbf{i} + (P_z - R_x)\,\mathbf{j} + (Q_x - P_y)\,\mathbf{k} = \operatorname{curl} \mathbf{F}. \tag{3.5.13}$$

Thus, this matrix is a way to help remember the formula for curl. Keep in mind, though, that the word *determinant* is used very loosely. A determinant is not really defined on a matrix with entries that are three vectors, three operators, and three functions.

If $\mathbf{F} = \langle P, Q \rangle$ is a vector field in $\mathbb{R}^2$, then the curl of $\mathbf{F}$, by definition, is

$$\text{curl}\,\mathbf{F} = (Q_x - P_y)\,\mathbf{k} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}. \tag{3.5.14}$$

Example 3.5.5: Finding the Curl of a Three-Dimensional Vector Field

Find the curl of $\mathbf{F}(P, Q, R) = \langle x^2 z, e^y + xz, xyz\rangle$.

Solution

The curl is

$$\begin{aligned}\text{curl}\, f &= \nabla \times \mathbf{F} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} \\ &= (R_y - Q_z)\,\mathbf{i} + (P_z - R_x)\,\mathbf{j} + (Q_x - P_y)\,\mathbf{k} \\ &= (xz - x)\,\mathbf{i} + (x^2 - yz)\,\mathbf{j} + z\,\mathbf{k}.\end{aligned}$$

Exercise 3.5.4

Find the curl of $\mathbf{F} = \langle \sin x \;\; \cos z, \;\; \sin y \;\; \sin z, \;\; \cos x \;\; \cos y\rangle$ at point $\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right)$.

Hint available at the end of this section.

Example 3.5.6: Finding the Curl of a Two-Dimensional Vector Field

Find the curl of $\mathbf{F} = \langle P, Q\rangle = \langle y, 0\rangle$.

Solution

Notice that this vector field consists of vectors that are all parallel. In fact, each vector in the field is parallel to the *x*-axis. This fact might lead us to the conclusion that the field has no spin and that the curl is zero. To test this theory, note that

$$\text{curl}\,\mathbf{F} = (Q_x - P_y)\,\mathbf{k} = -\mathbf{k} \neq \mathbf{0}. \tag{3.5.15}$$

Therefore, this vector field does have spin. To see why, imagine placing a paddlewheel at any point in the first quadrant (Figure 3.5.6). The larger magnitudes of the vectors at the top of the wheel cause the wheel to rotate. The wheel rotates in the clockwise (negative) direction, causing the coefficient of the curl to be negative.

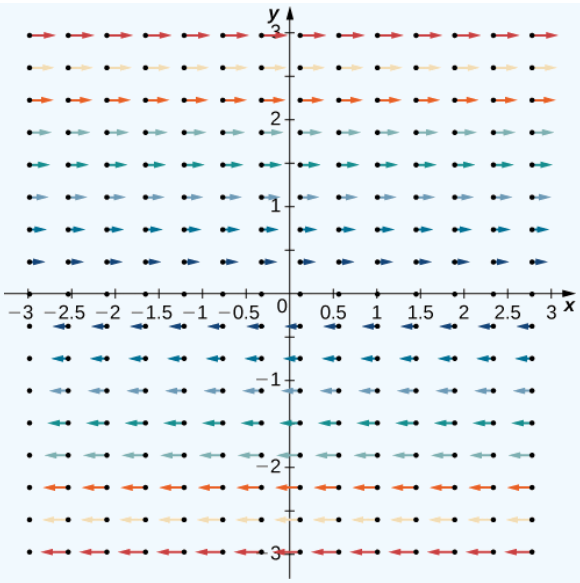


Figure 3.5.6: A vector field $\mathbf{F} = \langle P, Q \rangle = \langle y, 0 \rangle$ in the plane, where all vectors are parallel to the x-axis.

Note that if $\mathbf{F} = \langle P, Q \rangle$ is a vector field in a plane, then $\text{curl } \mathbf{F} \cdot \mathbf{k} = (Q_x - P_y)\ \mathbf{k} \cdot \mathbf{k} = Q_x - P_y$. Therefore, the circulation form of Green's theorem is sometimes written as

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \text{curl}\, \mathbf{F} \cdot\ \mathbf{k}\, dA, \tag{3.5.16}$$

where C is a simple closed curve and D is the region enclosed by C. Therefore, the circulation form of Green's theorem can be written in terms of the curl. If we think of curl as a derivative of sorts, then Green's theorem says that the "derivative" of $\mathbf{F}$ on a region can be translated into a line integral of $\mathbf{F}$ along the boundary of the region. This is analogous to the Fundamental Theorem of Calculus, in which the derivative of a function f on line segment $[a, b]$ can be translated into a statement about f on the boundary of $[a, b]$. Using curl, we can see the circulation form of Green's theorem is a higher-dimensional analog of the Fundamental Theorem of Calculus.

We can now use what we have learned about curl to show that gravitational fields have no "spin." Suppose there is an object at the origin with mass m_1 at the origin and an object with mass m_2. Recall that the gravitational force that object 1 exerts on object 2 is given by field

$$\mathbf{F}(x, y, z) = -Gm_1m_2 \left\langle \frac{x}{(x^2 + y^2 + z^2)^{3/2}}, \frac{y}{(x^2 + y^2 + z^2)^{3/2}}, \frac{z}{(x^2 + y^2 + z^2)^{3/2}} \right\rangle. \tag{3.5.17}$$

Example 3.5.7: Determining the Spin of a Gravitational Field

Show that a gravitational field has no spin.

Solution

To show that $\mathbf{F}$ has no spin, we calculate its curl. Let

- $P(x, y, z) = \dfrac{x}{(x^2 + y^2 + z^2)^{3/2}}$,
- $Q(x, y, z) = \dfrac{y}{(x^2 + y^2 + z^2)^{3/2}}$, and
- $R(x, y, z) = \dfrac{z}{(x^2 + y^2 + z^2)^{3/2}}$.

Then,

$$\begin{aligned}\operatorname{curl}\mathbf{F} &= -Gm_1m_2[(R_y-Q_z)\mathbf{i}+(P_z-R_x)\mathbf{j}+(Q_x-P_y)\mathbf{k}]\\ &= -Gm_1m_2\begin{pmatrix}\left(\frac{-3yz}{(x^2+y^2+z^2)^{5/2}}-\left(\frac{-3yz}{(x^2+y^2+z^2)^{5/2}}\right)\right)\mathbf{i}\\ +\left(\frac{-3xz}{(x^2+y^2+z^2)^{5/2}}-\left(\frac{-3xz}{(x^2+y^2+z^2)^{5/2}}\right)\right)\mathbf{j}\\ +\left(\frac{-3xy}{(x^2+y^2+z^2)^{5/2}}-\left(\frac{-3xy}{(x^2+y^2+z^2)^{5/2}}\right)\right)\mathbf{k}\end{pmatrix}\\ &= \mathbf{0}.\end{aligned}$$

Since the curl of the gravitational field is zero, the field has no spin.

Exercise 3.5.7

Field $\mathbf{v}(x,y)=\langle -\frac{y}{x^2+y^2},\frac{x}{x^2+y^2}\rangle$ models the flow of a fluid. Show that if you drop a leaf into this fluid, as the leaf moves over time, the leaf does not rotate.

Hint available at the end of this section.

Using Divergence and Curl

Now that we understand the basic concepts of divergence and curl, we can discuss their properties and establish relationships between them and conservative vector fields.

If $\mathbf{F}$ is a vector field in $\mathbb{R}^3$ then the curl of $\mathbf{F}$ is also a vector field in $\mathbb{R}^3$. Therefore, we can take the divergence of a curl. The next theorem says that the result is always zero. This result is useful because it gives us a way to show that some vector fields are not the curl of any other field. To give this result a physical interpretation, recall that divergence of a velocity field $\mathbf{v}$ at point P measures the tendency of the corresponding fluid to flow out of P. Since $\operatorname{div}(\operatorname{curl}\ \mathbf{v})=0$, the net rate of flow in vector field $\operatorname{curl}\ \mathbf{v}$ \) at any point is zero. Taking the curl of vector field $\mathbf{F}$ eliminates whatever divergence was present in $\mathbf{F}$.

Theorem: Divergence of the Curl

Let $\mathbf{F}=\langle P,Q,R\rangle$ be a vector field in $\mathbb{R}^3$ such that the component functions all have continuous second-order partial derivatives. Then,

$$\operatorname{div}(\operatorname{curl}\mathbf{F})=\nabla\cdot(\nabla\times\mathbf{F})=0. \tag{3.5.18}$$

Proof

By the definitions of divergence and curl, and by Clairaut's theorem,

$$\begin{aligned}\operatorname{div}(\operatorname{curl}\mathbf{F}) &= \operatorname{div}[(R_y-Q_z)\,\mathbf{i}+(P_z-R_x)\,\mathbf{j}+(Q_x-P_y)\,\mathbf{k}]\\ &= R_{yx}-Q_{xz}+P_{yz}-R_{yx}+Q_{zx}-P_{zy}\\ &= 0.\end{aligned}$$

□

Example 3.5.8: Showing That a Vector Field Is Not the Curl of Another

Show that $\mathbf{F}(x,y,z)=e^x\ \mathbf{i}+yz\ \mathbf{j}+xz^2\ \mathbf{k}$ is not the curl of another vector field. That is, show that there is no other vector $\mathbf{G}$ with $\operatorname{curl}\ \mathbf{G}=\mathbf{F}$.

Solution

Notice that the domain of $\mathbf{F}$ is all of $\mathbb{R}^3$ and the second-order partials of $\mathbf{F}$ are all continuous. Therefore, we can apply the previous theorem to $\mathbf{F}$.

The divergence of $\mathbf{F}$ is $e^x + z + 2xz$. If $\mathbf{F}$ were the curl of vector field $\mathbf{G}$, then $\text{div } \mathbf{F} = \text{div}(\text{curl } \mathbf{G}) = 0$. But, the divergence of $\mathbf{F}$ is not zero, and therefore $\mathbf{F}$ is not the curl of any other vector field.

Exercise 3.5.8

Is it possible for $\mathbf{G}(x, y, z) = \langle \sin x,\ \cos y,\ \sin(xyz) \rangle$ to be the curl of a vector field?

Hint available at the end of this section.

With the next two theorems, we show that if $\mathbf{F}$ is a conservative vector field then its curl is zero, and if the domain of $\mathbf{F}$ is simply connected then the converse is also true. This gives us another way to test whether a vector field is conservative.

Theorem: Curl of a Conservative Vector Field

If $\mathbf{F} = \langle P, Q, R \rangle$ is conservative, then $\text{curl } \mathbf{F} = \mathbf{0}$.

Proof

Since conservative vector fields satisfy the cross-partials property, all the cross-partials of $\mathbf{F}$ are equal. Therefore,

$$\begin{aligned} \text{curl}\, \mathbf{F} &= (R_y - Q_z)\, \mathbf{i} + (P_z - R_x)\, \mathbf{j} + (Q_x - P_y)\, \mathbf{k} \\ &= \mathbf{0}. \end{aligned}$$

□

The same theorem is true for vector fields in a plane.

Since a conservative vector field is the gradient of a scalar function, the previous theorem says that $\text{curl } (\nabla f) = \mathbf{0}$ for any scalar function f. In terms of our curl notation, $\nabla \times \nabla(f) = \mathbf{0}$. This equation makes sense because the cross product of a vector with itself is always the zero vector. Sometimes equation $\nabla \times \nabla(f) = \mathbf{0}$ is simplified as $\nabla \times \nabla = \mathbf{0}$.

Theorem: Curl Test for a Conservative Field

Let $\mathbf{F} = \langle P, Q, R \rangle$ be a vector field in space on a simply connected domain. If $\text{curl } \mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is conservative.

Proof

Since $\text{curl } \mathbf{F} = \mathbf{0}$, we have that $R_y = Q_z,\ P_z = R_x$, and $Q_x = P_y$. Therefore, $\mathbf{F}$ satisfies the cross-partials property on a simply connected domain, and the Cross-Partial Property of Conservative Fields implies that $\mathbf{F}$ is conservative.

□

The same theorem is also true in a plane. Therefore, if $\mathbf{F}$ is a vector field in a plane or in space and the domain is simply connected, then $\mathbf{F}$ is conservative if and only if $\text{curl } \mathbf{F} = \mathbf{0}$.

Example 3.5.9: Testing Whether a Vector Field Is Conservative

Use the curl to determine whether $\mathbf{F}(x, y, z) = \langle yz, xz, xy \rangle$ is conservative.

Solution

Note that the domain of $\mathbf{F}$ is all of $\mathbb{R}^3$ which is simply connected (Figure 3.5.7). Therefore, we can test whether $\mathbf{F}$ is conservative by calculating its curl.

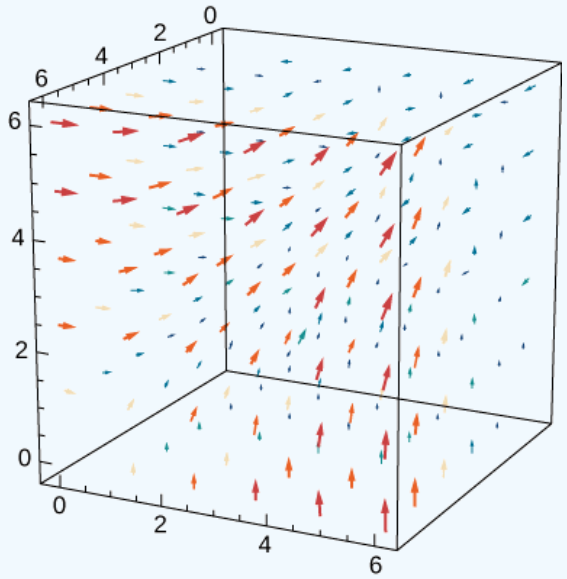


Figure 3.5.7: The vector field $\mathbf{F}(x, y, z) = \langle yz, xz, xy \rangle$ represented within a 3D cube. The field shows vectors emanating from different points, varying in direction and magnitude. The calculation of the curl of $\mathbf{F}$ confirms that it is zero, indicating that the vector field is conservative.

The curl of $\mathbf{F}$ is

$$\left(\frac{\partial}{\partial y}xy - \frac{\partial}{\partial z}xz\right)\mathbf{i} + \left(\frac{\partial}{\partial y}yz - \frac{\partial}{\partial z}xy\right)\mathbf{j} + \left(\frac{\partial}{\partial y}xz - \frac{\partial}{\partial z}yz\right)\mathbf{k} = (x - x)\,\mathbf{i} + (y - y)\,\mathbf{j} + (z - z)\,\mathbf{k} = \mathbf{0}.$$

Thus, $\mathbf{F}$ is conservative.

We have seen that the curl of a gradient is zero. What is the divergence of a gradient? If f is a function of two variables, then $\text{div}(\nabla f) = \nabla \cdot (\nabla f) = f_{xx} + f_{yy}$. We abbreviate this "double dot product" as ∇^2. This operator is called the *Laplace operator,* and in this notation Laplace's equation becomes $\nabla^2 f = 0$. Therefore, a harmonic function is a function that becomes zero after taking the divergence of a gradient.

Similarly, if f is a function of three variables then

$$\text{div}(\nabla f) = \nabla \cdot (\nabla f) = f_{xx} + f_{yy} + f_{zz}. \tag{3.5.19}$$

Using this notation we get Laplace's equation for harmonic functions of three variables:

$$\nabla^2 f = 0. \tag{3.5.20}$$

Harmonic functions arise in many applications. For example, the potential function of an electrostatic field in a region of space that has no static charge is harmonic.

Example 3.5.10: Finding a Potential Function

Is it possible for $f(x,y) = x^2 + x - y$ to be the potential function of an electrostatic field that is located in a region of $\mathbb{R}^2$ free of static charge?

Solution

If f were such a potential function, then f would be harmonic. Note that $f_{xx} = 2$ and $f_{yy} = 0$, and so $f_{xx} + f_{yy} \neq 0$. Therefore, f is not harmonic and f cannot represent an electrostatic potential.

Exercise 3.5.10

Is it possible for function $f(x,y) = x^2 - y^2 + x$ to be the potential function of an electrostatic field located in a region of $\mathbb{R}^2$ free of static charge?

Hint available at the end of this section.

Key Concepts

- The divergence of a vector field is a scalar function. Divergence measures the "outflowing-ness" of a vector field. If $\mathbf{v}$ is the velocity field of a fluid, then the divergence of $\mathbf{v}$ at a point is the outflow of the fluid less the inflow at the point.
- The curl of a vector field is a vector field. The curl of a vector field at point P measures the tendency of particles at P to rotate about the axis that points in the direction of the curl at P.
- A vector field with a simply connected domain is conservative if and only if its curl is zero.

Key Equations

- **Curl**

$$\nabla \times \mathbf{F} = (R_y - Q_z)\,\mathbf{i} + (P_z - R_x)\,\mathbf{j} + (Q_x - P_y)\,\mathbf{k}$$

- **Divergence**

$$\nabla \cdot \mathbf{F} = P_x + Q_y + R_z$$

- **Divergence of curl is zero**

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0$$

- **Curl of a gradient is the zero vector**

$$\nabla \times (\nabla f) = 0$$

Glossary

Curl

the curl of vector field $\mathbf{F} = \langle P, Q, R \rangle$, denoted $\boldsymbol{\nabla} \times \mathbf{F}$ is the "determinant" of the matrix

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}.$$

and is given by the expression $(R_y - Q_z)\ \mathbf{i} + (P_z - R_x)\ \mathbf{j} + (Q_x - P_y)\ \mathbf{k}$; it measures the tendency of particles at a point to rotate about the axis that points in the direction of the curl at the point

Divergence
: the divergence of a vector field $\mathbf{F} = \langle P, Q, R \rangle$, denoted $\boldsymbol{\nabla} \times \mathbf{F}$, is $P_x + Q_y + R_z$; it measures the "outflowing-ness" of a vector field

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Answers

Exercise 3.5.1
Follow Example 3.5.1.
$\text{div } \mathbf{F} = y$

Exercise 3.5.2
Calculate the divergence.
Yes

Exercise 3.5.3
Find where the divergence is zero.
All points on line $y = 1$.

Exercise 3.5.4
Find the determinant of matrix $\nabla \times \mathbf{F}$.
$-i$

Exercise 3.5.7
Calculate the curl.
$\text{curl } \mathbf{v} = \mathbf{0}$

Exercise 3.5.8
Find the divergence of $\mathbf{G}$.
No.

Exercise 3.5.10
Determine whether the function is harmonic.
Yes.

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3.5E: Exercises

Exercises 1 - 6

Determine whether the statement is *True* or *False*.

1. If the coordinate functions of $\mathbf{F}:\mathbb{R}^3\to\mathbb{R}^3$ have continuous second partial derivatives, then $\text{curl }(\text{div }\mathbf{F})$ equals zero.

2. $\nabla\cdot(x\ \mathbf{i}+y\ \mathbf{j}+z\ \mathbf{k})=1$.

Answer

False

3. All vector fields of the form $\mathbf{F}(x,y,z)=f(x)\ \mathbf{i}+g(y)\ \mathbf{j}+h(z)\ \mathbf{k}$ are conservative.

4. If $\text{curl }\mathbf{F}=\mathbf{0}$, then $\mathbf{F}$ is conservative.

Answer

True

5. If $\mathbf{F}$ is a constant vector field then $\text{div }\mathbf{F}=0$.

6. If $\mathbf{F}$ is a constant vector field then $\text{curl }\mathbf{F}=\mathbf{0}$.

Answer

True

Exercises 7 - 16

Find the curl of $\mathbf{F}$.

7. $\mathbf{F}(x,y,z)=xy^2z^4\ \mathbf{i}+(2x^2y+z)\ \mathbf{j}+y^3z^2\ \mathbf{k}$

8. $\mathbf{F}(x,y,z)=x^2z\ \mathbf{i}+y^2x\ \mathbf{j}+(y+2z)\ \mathbf{k}$

Answer

$\text{curl }\mathbf{F}=\ \mathbf{i}+x^2\ \mathbf{j}+y^2\ \mathbf{k}$

9. $\mathbf{F}(x,y,z)=3xyz^2\ \mathbf{i}+y^2\sin z\ \mathbf{j}+xe^{2z}\ \mathbf{k}$

10. $\mathbf{F}(x,y,z)=x^2yz\ \mathbf{i}+xy^2z\ \mathbf{j}+xyz^2\ \mathbf{k}$

Answer

$\text{curl }\mathbf{F}=(xz^2-xy^2)\ \mathbf{i}+(x^2y-yz^2)\ \mathbf{j}+(y^2z-x^2z)\ \mathbf{k}$

11. $\mathbf{F}(x,y,z)=(x\ \cos y)\ \mathbf{i}+xy^2\ \mathbf{j}$

12. $\mathbf{F}(x,y,z)=(x-y)\ \mathbf{i}+(y-z)\ \mathbf{j}+(z-x)\ \mathbf{k}$

Answer

$\text{curl }\mathbf{F}=\ \mathbf{i}+\ \mathbf{j}+\ \mathbf{k}$

13. $\mathbf{F}(x,y,z)=xyz\ \mathbf{i}+x^2y^2z^2\ \mathbf{j}+y^2z^3\ \mathbf{k}$

14. $\mathbf{F}(x,y,z)=xy\ \mathbf{i}+yz\ \mathbf{j}+xz\ \mathbf{k}$

Answer

$$\text{curl } \mathbf{F} = -y\ \mathbf{i} - z\ \mathbf{j} - x\ \mathbf{k}$$

15. $\mathbf{F}(x, y, z) = x^2\ \mathbf{i} + y^2\ \mathbf{j} + z^2\ \mathbf{k}$

16. $\mathbf{F}(x, y, z) = ax\ \mathbf{i} + by\ \mathbf{j} + c\ \mathbf{k}$ for constants $a,\ b,\ c$.

Answer

$$\text{curl } \mathbf{F} = \mathbf{0}$$

Exercises 17 - 26

Find the divergence of $\mathbf{F}$.

17. $\mathbf{F}(x, y, z) = x^2 z\ \mathbf{i} + y^2 x\ \mathbf{j} + (y + 2z)\ \mathbf{k}$

18. $\mathbf{F}(x, y, z) = 3xyz^2\ \mathbf{i} + y^2 \sin z\ \mathbf{j} + xe^{2z}\ \mathbf{k}$

Answer

$$\text{div } \mathbf{F} = 3yz^2 + 2y\ \sin z + 2xe^{2z}$$

19. $\mathbf{F}(x, y) = (\sin x)\ \mathbf{i} + (\cos y)\ \mathbf{j}$

20. $\mathbf{F}(x, y, z) = x^2\ \mathbf{i} + y^2\ \mathbf{j} + z^2\ \mathbf{k}$

Answer

$$\text{div } \mathbf{F} = 2(x + y + z)$$

21. $\mathbf{F}(x, y, z) = (x - y)\ \mathbf{i} + (y - z)\ \mathbf{j} + (z - x)\ \mathbf{k}$

22. $\mathbf{F}(x, y) = \dfrac{x}{\sqrt{x^2 + y^2}}\ \mathbf{i} + \dfrac{y}{\sqrt{x^2 + y^2}}\ \mathbf{j}$

Answer

$$\text{div } \mathbf{F} = \frac{1}{\sqrt{x^2 + y^2}}$$

23. $\mathbf{F}(x, y) = x\ \mathbf{i} - y\ \mathbf{j}$

24. $\mathbf{F}(x, y, z) = ax\ \mathbf{i} + by\ \mathbf{j} + c\ \mathbf{k}$ for constants $a,\ b,\ c$.

Answer

$$\text{div } \mathbf{F} = a + b$$

25. $\mathbf{F}(x, y, z) = xyz\ \mathbf{i} + x^2 y^2 z^2\ \mathbf{j} + y^2 z^3\ \mathbf{k}$

26. $\mathbf{F}(x, y, z) = xy\ \mathbf{i} + yz\ \mathbf{j} + xz\ \mathbf{k}$

Answer

$$\text{div } \mathbf{F} = x + y + z$$

Exercises 27 - 28

Determine whether each of the given scalar functions is harmonic.

27. $u(x, y, z) = e^{-x}(\cos y - \sin y)$

28. $w(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$

Answer

Harmonic

29. If $\mathbf{F}(x,y,z) = 2\ \mathbf{i} + 2xj + 3yk$ and $\mathbf{G}(x,y,z) = x\ \mathbf{i} - y\ \mathbf{j} + z\ \mathbf{k}$, find $\text{curl}\ (\mathbf{F} \times \mathbf{G})$.

30. If $\mathbf{F}(x,y,z) = 2\ \mathbf{i} + 2xj + 3yk$ and $\mathbf{G}(x,y,z) = x\ \mathbf{i} - y\ \mathbf{j} + z\ \mathbf{k}$, find $\text{div}\ (\mathbf{F} \times \mathbf{G})$.

Answer

$\text{div}\ (\mathbf{F} \times \mathbf{G}) = 2z + 3x$

31. Find $\text{div}\ \mathbf{F}$, given that $\mathbf{F} = \nabla f$, where $f(x,y,z) = xy^3z^2$.

32. Find the divergence of $\mathbf{F}$ for vector field $\mathbf{F}(x,y,z) = (y^2+z^2)(x+y)\ \mathbf{i} + (z^2+x^2)(y+z)\ \mathbf{j} + (x^2+y^2)(z+x)\ \mathbf{k}$.

Answer

$\text{div}\ \mathbf{F} = 2r^2$

33. Find the divergence of $\mathbf{F}$ for vector field $\mathbf{F}(x,y,z) = f_1(y,z)\ \mathbf{i} + f_2(x,z)\ \mathbf{j} + f_3(x,y)\ \mathbf{k}$.

For exercises 34 - 36, use $r = |\mathbf{r}|$ and $\mathbf{r}(x,y,z) = \langle x,y,z \rangle$.

34. Find the $\text{curl}\ \mathbf{r}$

Answer

$\text{curl}\ \mathbf{r} = \mathbf{0}$

35. Find the $\text{curl}\ \dfrac{\mathbf{r}}{r}$.

36. Find the $\text{curl}\ \dfrac{\mathbf{r}}{r^3}$.

Answer

$\text{curl}\ \dfrac{\mathbf{r}}{r^3} = \mathbf{0}$

37. Let $\mathbf{F}(x,y) = \dfrac{-y\,\mathbf{i} + x\,\mathbf{j}}{x^2+y^2}$, where $\mathbf{F}$ is defined on $\{(x,y) \in \mathbb{R} | (x,y) \neq (0,0)\}$. Find $\text{curl}\ \mathbf{F}$.

Exercises 38 - 39

Use a computer algebra system to find the curl of the given vector fields.

38. $\mathbf{F}(x,y,z) = \arctan\left(\dfrac{x}{y}\right)\ \mathbf{i} + \ln\sqrt{x^2+y^2}\ \mathbf{j} + \ \mathbf{k}$

Answer

$\text{curl}\ \ \mathbf{F} = \dfrac{2x}{x^2+y^2}\ \mathbf{k}$

39. $\mathbf{F}(x,y,z) = \sin(x-y)\ \mathbf{i} + \sin(y-z)\ \mathbf{j} + \sin(z-x)\ \mathbf{k}$

Exercises 40 - 44

Find the divergence of $\mathbf{F}$ at the given point.

40. $\mathbf{F}(x,y,z) = \ \mathbf{i} + \ \mathbf{j} + \ \mathbf{k}$ at $(2,-1,3)$

Answer

$\text{div}\ \mathbf{F} = 0$

41. $\mathbf{F}(x,y,z) = xyz\ \mathbf{i} + y\ \mathbf{j} + z\ \mathbf{k}$ at $(1,2,3)$

42. $\mathbf{F}(x,y,z) = e^{-xy}\ \mathbf{i} + e^{xz}\ \mathbf{j} + e^{yz}\ \mathbf{k}$ at $(3,2,0)$

Answer

$$\text{div}\ \mathbf{F} = 2 - 2e^{-6}$$

43. $\mathbf{F}(x,y,z) = xyz\ \mathbf{i} + y\ \mathbf{j} + z\ \mathbf{k}$ at $(1,2,1)$

44. $\mathbf{F}(x,y,z) = e^x \sin y\ \mathbf{i} - e^x \cos y\ \mathbf{j}$ at $(0,0,3)$

Answer

$$\text{div}\ \mathbf{F} = 0$$

Exercises 45- 49

Find the curl of $\mathbf{F}$ at the given point.

45. $\mathbf{F}(x,y,z) = \mathbf{i} + \mathbf{j} + \mathbf{k}$ at $(2,-1,3)$

46. $\mathbf{F}(x,y,z) = xyz\ \mathbf{i} + y\ \mathbf{j} + x\ \mathbf{k}$ at $(1,2,3)$

Answer

$$\text{curl}\ \ \mathbf{F} = \mathbf{j} - 3\ \mathbf{k}$$

47. $\mathbf{F}(x,y,z) = e^{-xy}\ \mathbf{i} + e^{xz}\ \mathbf{j} + e^{yz}\ \mathbf{k}$ at $(3,2,0)$

48. $\mathbf{F}(x,y,z) = xyz\ \mathbf{i} + y\ \mathbf{j} + z\ \mathbf{k}$ at $(1,2,1)$

Answer

$$\text{curl}\ \ \mathbf{F} = 2\ \mathbf{j} - \mathbf{k}$$

49. $\mathbf{F}(x,y,z) = e^x \sin y\ \mathbf{i} - e^x \cos y\ \mathbf{j}$ at $(0,0,3)$

50. Let $\mathbf{F}(x,y,z) = (3x^2y + az)\ \mathbf{i} + x^3\ \mathbf{j} + (3x + 3z^2)\ \mathbf{k}$. For what value of a is $\mathbf{F}$ conservative?

Answer

$$a = 3$$

51. Given vector field $\mathbf{F}(x,y) = \dfrac{1}{x^2+y^2}\langle -y, x\rangle$ on domain $D = \dfrac{\mathbb{R}^2}{\{(0,0)\}} = \{(x,y) \in \mathbb{R}^2 | (x,y) \neq (0,0)\}$, is $\mathbf{F}$ conservative?

52. Given vector field $\mathbf{F}(x,y) = \dfrac{1}{x^2+y^2}\langle x, y\rangle$ on domain $D = \dfrac{\mathbb{R}^2}{\{(0,0)\}}$, is $\mathbf{F}$ conservative?

Answer

$\mathbf{F}$ is conservative.

53. Find the work done by force field $\mathbf{F}(x,y) = e^{-y}\ \mathbf{i} - xe^{-y}\ \mathbf{j}$ in moving an object from$P(0,1)$ to $Q(2,0)$. Is the force field conservative?

54. Compute divergence $\mathbf{F}(x,y,z) = (\sinh x)\ \mathbf{i} + (\cosh y)\ \mathbf{j} - xyz\ \mathbf{k}$.

Answer

$$\text{div}\ \mathbf{F} = \cosh x + \sinh y - xy$$

55. Compute $\text{curl}\ \ \mathbf{F} = (\sinh x)\ \mathbf{i} + (\cosh y)\ \mathbf{j} - xyz\ \mathbf{k}$.

Exercises 56 - 58

Consider a rigid body that is rotating about the x-axis counterclockwise with constant angular velocity $\omega = \langle a, b, c \rangle$. If P is a point in the body located at $\mathrm{r} = x\ \mathbf{i} + y\ \mathbf{j} + z\ \mathbf{k}$, the velocity at P is given by vector field $\mathbf{F} = \omega \times \mathbf{r}$.

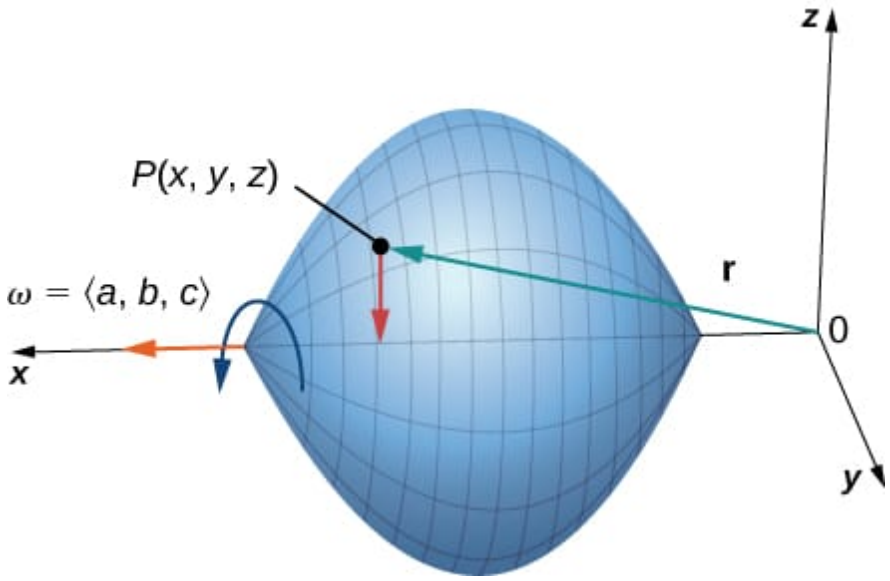


Figure $3.5E.1$: The rigid body that is rotating about the x-axis counterclockwise

56. Express $\mathbf{F}$ in terms of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ vectors.

Answer

$$\mathbf{F} = (bz - cy)\ \mathbf{i} + (cx - az)\ \mathbf{j} + (ay - bx)\ \mathbf{k}$$

57. Find $\mathrm{div}\ F$.

58. Find $\mathrm{curl}\ F$

Answer

$$\mathrm{curl}\ \ \mathbf{F} = 2\omega$$

Exercises 59 - 60

Suppose that $\nabla \cdot \mathbf{F} = 0$ and $\nabla \cdot \mathbf{G} = 0$.

59. Does $\mathbf{F} + \mathbf{G}$ necessarily have zero divergence?

60. Does $\mathbf{F} \times \mathbf{G}$ necessarily have zero divergence?

Answer

$\mathbf{F} \times \mathbf{G}$ does not have zero divergence.

Exercises 61 - 62

Suppose a solid object in $\mathbb{R}^3$ has a temperature distribution given by $T(x, y, z)$. The heat flow vector field in the object is $\mathbf{F} = -k\nabla T$, where $k > 0$ is a property of the material. The heat flow vector points in the direction opposite to that of the gradient, which is the direction of greatest temperature decrease. The divergence of the heat flow vector is $\nabla \cdot \mathbf{F} = -k\nabla \cdot \nabla T = -k\nabla^2 T$.

61. Compute the heat flow vector field.

62. Compute the divergence.

Answer

$$\nabla \cdot \mathbf{F} = -200k[1 + 2(x^2 + y^2 + z^2)]e^{-x^2+y^2+z^2}$$

63. Consider rotational velocity field $\mathbf{v} = \langle 0, 10z, -10y \rangle$. If a paddlewheel is placed in plane $x + y + z = 1$ with its axis normal to this plane, using a computer algebra system, calculate how fast the paddlewheel spins in revolutions per unit time.

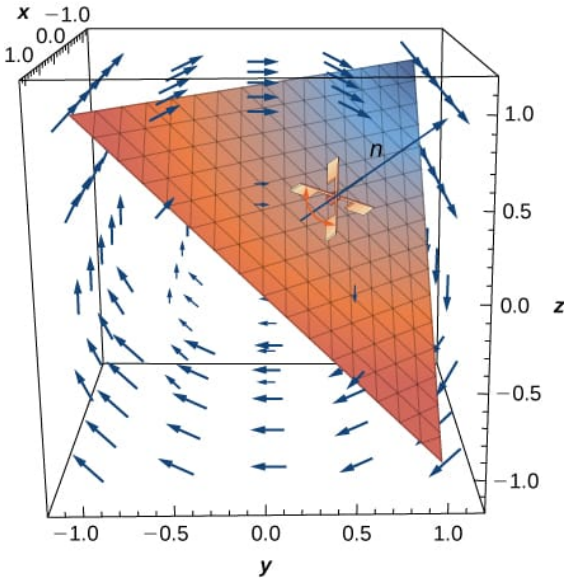


Figure $3.5E.2$: The image depicts the rotational velocity field $\mathbf{v} = \langle 0, 10z, -10y \rangle$. It illustrates the flow of vectors in 3D space with a plane $x + y + z = 1$ shown as a triangular slice.

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3.6: Surface Area

Learning Objectives

- Find the parametric representations of a cylinder, a cone, and a sphere.
- Describe the surface area of a parametric surface.

We have seen that a line integral is an integral over a path in a plane or in space. However, if we wish to integrate over a surface (a two-dimensional object) rather than a path (a one-dimensional object) in space, then we need a new kind of integral that can handle integration over objects in higher dimensions. We can extend the concept of a line integral to a surface integral to allow us to perform this integration.

Surface integrals are important for the same reasons that line integrals are important. They have many applications to physics and engineering, and they allow us to develop higher dimensional versions of the Fundamental Theorem of Calculus. In particular, surface integrals allow us to generalize Green's theorem to higher dimensions, and they appear in some important theorems we discuss in later sections.

Parametric Surfaces

A surface integral is similar to a line integral, except the integration is done over a surface rather than a path. In this sense, surface integrals expand on our study of line integrals. Just as with line integrals, there are two kinds of surface integrals: a surface integral of a scalar-valued function and a surface integral of a vector field.

However, before we can integrate over a surface, we need to consider the surface itself. Recall that to calculate a scalar or vector line integral over curve C, we first need to parameterize C. In a similar way, to calculate a surface integral over surface S, we need to parameterize S. That is, we need a working concept of a *parameterized surface* (or a *parametric surface*), in the same way that we already have a concept of a parameterized curve.

A parameterized surface is given by a description of the form

$$\mathbf{r}(u,v) = \langle x(u,v),\, y(u,v),\, z(u,v)\rangle. \tag{3.6.1}$$

Notice that this parameterization involves two parameters, u and v, because a surface is two-dimensional, and therefore two variables are needed to trace out the surface. The parameters u and v vary over a region called the parameter domain, or parameter space—the set of points in the uv-plane that can be substituted into $\mathbf{r}$. Each choice of u and v in the parameter domain gives a point on the surface, just as each choice of a parameter t gives a point on a parameterized curve. The entire surface is created by making all possible choices of u and v over the parameter domain.

Definition: Parameter Domain

Given a parameterization of surface

$$\mathbf{r}(u,v) = \langle x(u,v),\, y(u,v),\, z(u,v)\rangle. \tag{3.6.2}$$

the parameter domain of the parameterization is the set of points in the uv-plane that can be substituted into $\mathbf{r}$.

Example 3.6.1: Parameterizing a Cylinder

Describe surface S parameterized by

$$\mathbf{r}(u,v) = \langle \cos u,\, \sin u,\, v\rangle,\ -\infty < u < \infty,\ -\infty < v < \infty.$$

Solution

To get an idea of the shape of the surface, we first plot some points. Since the parameter domain is all of $\mathbb{R}^2$, we can choose any value for *u* and *v* and plot the corresponding point. If $u = v = 0$, then $\mathbf{r}(0,0) = \langle 1,0,0\rangle$, so point (1, 0, 0) is on S. Similarly, points $\mathbf{r}(\pi,2) = (-1,0,2)$ and $\mathbf{r}\left(\frac{\pi}{2},4\right) = (0,1,4)$ are on S.

Although plotting points may give us an idea of the shape of the surface, we usually need quite a few points to see the shape. Since it is time-consuming to plot dozens or hundreds of points, we use another strategy. To visualize S, we visualize two families of curves that lie on S. In the first family of curves we hold u constant; in the second family of curves we hold v constant. This allows us to build a "skeleton" of the surface, thereby getting an idea of its shape.

- Suppose that u is a constant K. Then the curve traced out by the parameterization is $\langle \cos K,\ \sin K,\ v\rangle$, which gives a vertical line that goes through point $(\cos K, \sin K, v\rangle$ in the xy-plane.
- Suppose that v is a constant K. Then the curve traced out by the parameterization is $\langle \cos u,\ \sin u,\ K\rangle$, which gives a circle in plane $z = K$ with radius 1 and center $(0,0,K)$.

If u is held constant, then we get vertical lines; if v is held constant, then we get circles of radius 1 centered around the vertical line that goes through the origin. Therefore the surface traced out by the parameterization is cylinder $x^2 + y^2 = 1$ (Figure 3.6.1).

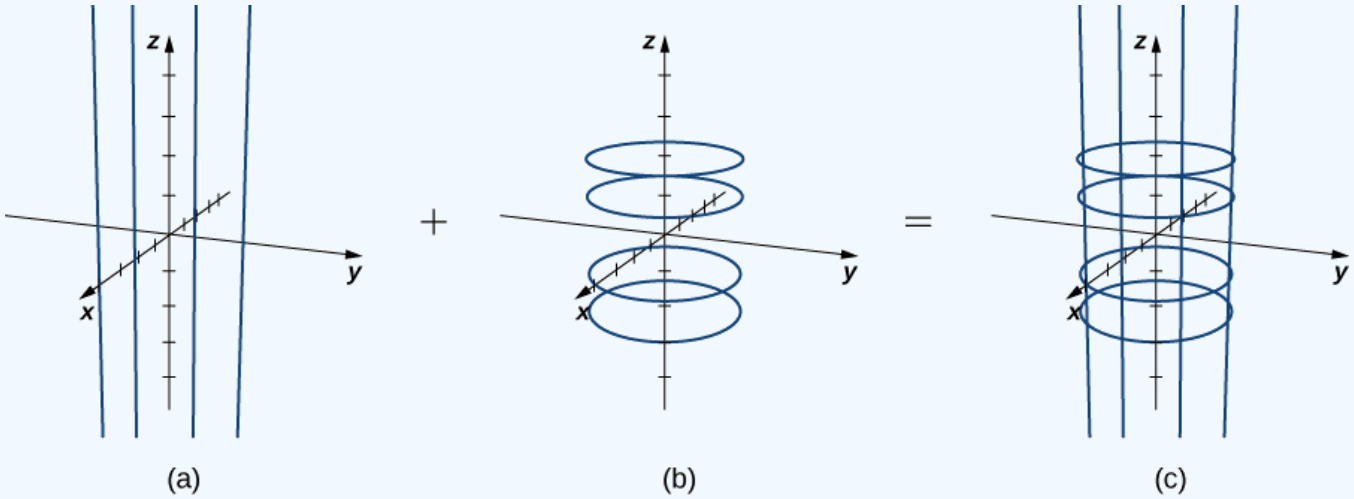


Figure 3.6.1: (a) Lines $\langle \cos K,\ \sin K, v\rangle$ for $K = 0,\ \frac{\pi}{2},\ \pi$, and $\frac{3\pi}{2}$. (b) Circles $\langle \cos u,\ \sin u,\ K\rangle$ for $K = -2, -1, 1$, and 2 (c) The lines and circles together. As u and v vary, they describe a cylinder.

Notice that if $x = \cos u$ and $y = \sin u$, then $x^2 + y^2 = 1$, so points from S do indeed lie on the cylinder. Conversely, each point on the cylinder is contained in some circle $\langle \cos u,\ \sin u,\ k\rangle$ for some k, and therefore each point on the cylinder is contained in the parameterized surface (Figure 3.6.2).

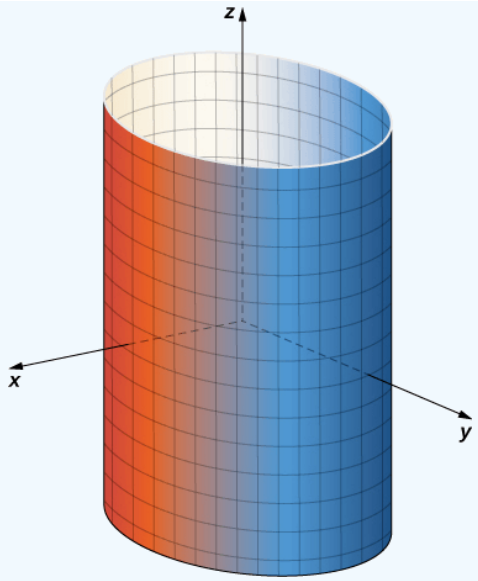


Figure 3.6.2: Cylinder $x^2 + y^2 = r^2$ has parameterization $\mathbf{r}(u, v) = \langle r\ \cos u,\ r\ \sin u,\ v\rangle,\ 0 \le u \le 2\pi,\ -\infty < v < \infty$.

Analysis

Notice that if we change the parameter domain, we could get a different surface. For example, if we restricted the domain to $0 \le u \le \pi,\ -\infty < v < 6$, then the surface would be a half-cylinder of height 6.

Exercise 3.6.1

Describe the surface with parameterization

$$\mathbf{r}(u, v) = \langle 2\ \cos u,\ 2\ \sin u,\ v\rangle,\ 0 \le u \le 2\pi,\ -\infty < v < \infty$$

Hint available at the end of this section.

Answer available at the end of this section.

It follows from Example 3.6.1 that we can parameterize all cylinders of the form $x^2 + y^2 = R^2$. If S is a cylinder given by equation $x^2 + y^2 = R^2$, then a parameterization of S is $\mathbf{r}(u, v) = \langle R\ \cos u,\ R\ \sin u,\ v\rangle,\ 0 \le u \le 2\pi,\ -\infty < v < \infty$.

We can also find different types of surfaces given their parameterization, or we can find a parameterization when we are given a surface.

Example 3.6.2: Describing a Surface

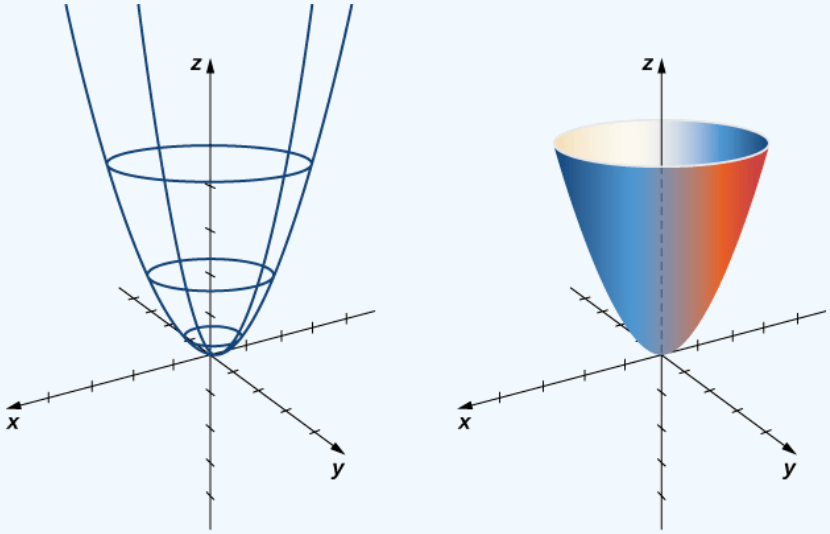
Describe surface S parameterized by $\mathbf{r}(u, v) = \langle u\ \cos v,\ u\ \sin v,\ u^2\rangle,\ 0 \le u < \infty,\ 0 \le v < 2\pi$.

Solution

Notice that if u is held constant, then the resulting curve is a circle of radius u in plane $z = u$. Therefore, as u increases, the radius of the resulting circle increases. If v is held constant, then the resulting curve is a vertical parabola. Therefore, we expect the surface to be an elliptic paraboloid. To confirm this, notice that

$$\begin{aligned} x^2 + y^2 &= (u\ \cos v)^2 + (u\ \sin v)^2 \\ &= u^2 \cos^2 v + u^2 sin^2 v \\ &= u^2 \\ &= z \end{aligned}$$

Therefore, the surface is the elliptic paraboloid $x^2 + y^2 = z$ (Figure 3.6.3).



(a) (b)

Figure 3.6.3: (a) Circles arise from holding u constant; the vertical parabolas arise from holding v constant. (b) An elliptic paraboloid results from all choices of u and v in the parameter domain.

Exercise 3.6.2

Describe the surface parameterized by $\mathbf{r}(u,v) = \langle u\ \cos v,\ u\ \sin v,\ u\rangle,\ -\infty < u < \infty,\ 0 \le v < 2\pi$.

Hint available at the end of this section.

Answer available at the end of this section.

Example 3.6.3: Finding a Parameterization

Give a parameterization of the cone $x^2 + y^2 = z^2$ lying on or above the plane $z = -2$.

Solution

The horizontal cross-section of the cone at height $z = u$ is circle $x^2 + y^2 = u^2$. Therefore, a point on the cone at height u has coordinates $(u\ \cos v,\ u\ \sin v,\ u)$ for angle v. Hence, a parameterization of the cone is $\mathbf{r}(u,v) = \langle u\ \cos v,\ u\ \sin v,\ u\rangle$. Since we are not interested in the entire cone, only the portion on or above plane $z = -2$, the parameter domain is given by $-2 < u < \infty,\ 0 \le v < 2\pi$ (Figure 3.6.4).

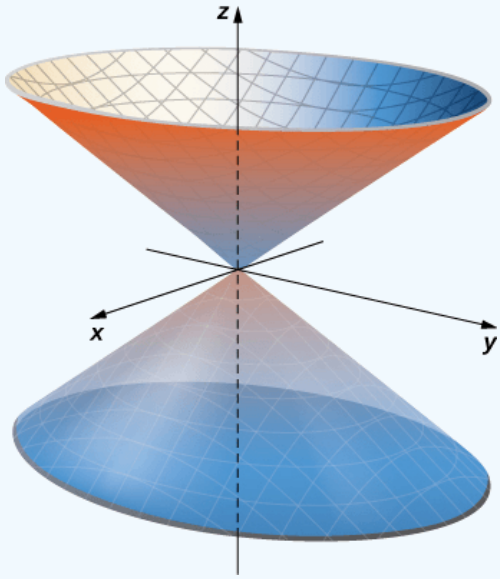


Figure 3.6.4: Cone $x^2 + y^2 = z^2$ has parametrization $\mathbf{r}(u, v) = \langle u\ \cos v,\ u\ \sin v, u\rangle, -\infty < u < \infty,\ 0 \le v \le 2\pi$.

Exercise 3.6.3

Give a parameterization for the portion of cone $x^2 + y^2 = z^2$ lying in the first octant.

Hint available at the end of this section.

Answer available at the end of this section.

We have discussed parameterizations of various surfaces, but two important types of surfaces need a separate discussion: spheres and graphs of two-variable functions. To parameterize a sphere, it is easiest to use spherical coordinates. The sphere of radius ρ centered at the origin is given by the parameterization

$$\mathbf{r}(\phi, \theta) = \langle \rho\ \cos\theta\ \sin\phi,\ \rho\ \sin\theta\ \sin\phi,\ \rho\ \cos\phi\rangle,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi.$$

The idea of this parameterization is that as ϕ sweeps downward from the positive z-axis, a circle of radius $\rho\ \sin\phi$ is traced out by letting θ run from 0 to 2π. To see this, let ϕ be fixed. Then

$$\begin{aligned} x^2 + y^2 &= (\rho\ \cos\theta\ \sin\phi)^2 + (\rho\ \sin\theta\ \sin\phi)^2 \\ &= \rho^2\ \sin^2\phi(\cos^2\theta + \sin^2\theta) \\ &= \rho^2\ \sin^2\phi \\ &= (\rho\ \sin\phi)^2. \end{aligned}$$

This results in the desired circle (Figure 3.6.5).

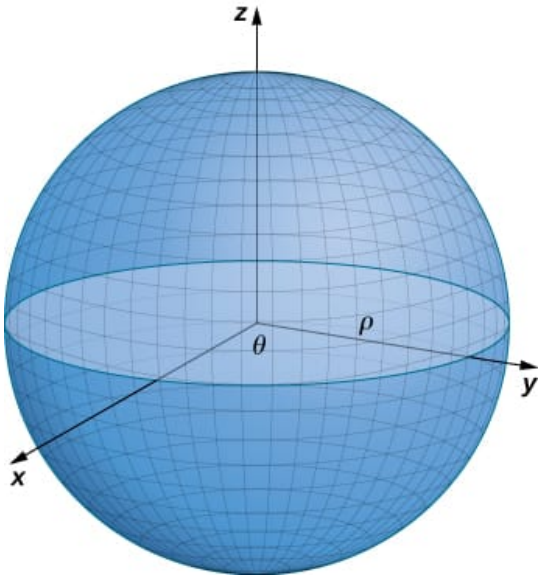


Figure 3.6.5: The sphere of radius ρ has parameterization $\mathbf{r}(\phi,\theta) = \langle \rho\ \cos\theta\ \sin\phi,\ \rho\ \sin\theta\ \sin\phi,\ \rho\ \cos\phi\rangle,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi.$

Finally, to parameterize the graph of a two-variable function, we first let $z = f(x,y)$ be a function of two variables. The simplest parameterization of the graph of f is $\mathbf{r}(x,y) = \langle x, y, f(x,y)\rangle$, where x and y vary over the domain of f (Figure 3.6.6). For example, the graph of $f(x,y) = x^2y$ can be parameterized by $\mathbf{r}(x,y) = \langle x, y, x^2y\rangle$, where the parameters x and y vary over the domain of f. If we only care about a piece of the graph of f - say, the piece of the graph over rectangle $[1,3] \times [2,5]$ - then we can restrict the parameter domain to give this piece of the surface:

$$\mathbf{r}(x,y) = \langle x, y, x^2y\rangle,\ 1 \le x \le 3,\ 2 \le y \le 5. \tag{3.6.3}$$

Similarly, if S is a surface given by equation $x = g(y,z)$ or equation $y = h(x,z)$, then a parameterization of S is $\mathbf{r}(y,z) = \langle g(y,z),\ y, z\rangle$ or $\mathbf{r}(x,z) = \langle x, h(x,z), z\rangle$, respectively. For example, the graph of paraboloid $2y = x^2 + z^2$ can be parameterized by $\mathbf{r}(x,y) = \left\langle x, \dfrac{x^2+z^2}{2}, z\right\rangle,\ 0 \le x < \infty,\ 0 \le z < \infty$. Notice that we do not need to vary over the entire domain of y because x and z are squared.

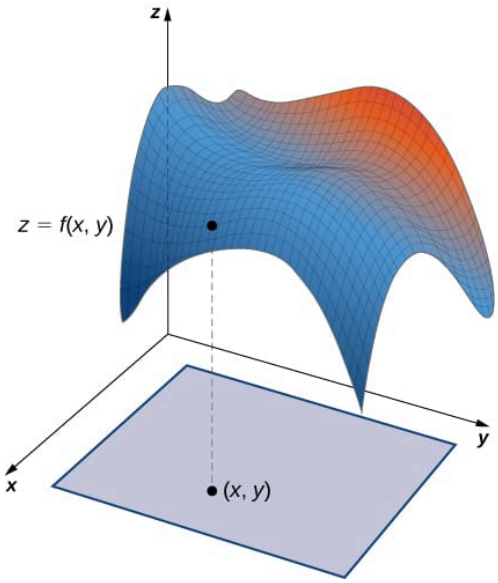


Figure 3.6.6: The graph of a surface $z = f(x,y)$, with height determined by $f(x,y)$ over the x-y plane.

Let's now generalize the notions of smoothness and regularity to a parametric surface. Recall that curve parameterization $\mathbf{r}(t),\ a \le t \le b$ is regular (or **smooth**) if $\mathbf{r}'(t) \neq \mathbf{0}$ for all t in $[a,b]$. For a curve, this condition ensures that the image of $\mathbf{r}$ really is a curve, and not just a point. For example, consider curve parameterization $\mathbf{r}(t) = \langle 1, 2\rangle,\ 0 \le t \le 5$. The image of this parameterization is

simply point $(1, 2)$, which is not a curve. Notice also that $\mathbf{r}'(t) = \mathbf{0}$. The fact that the derivative is the zero vector indicates we are not actually looking at a curve.

Analogously, we would like a notion of regularity (or smoothness) for surfaces so that a surface parameterization really does trace out a surface. To motivate the definition of regularity of a surface parameterization, consider the parameterization

$$\mathbf{r}(u, v) = \langle 0,\ \cos v,\ 1\rangle,\ 0 \leq u \leq 1,\ 0 \leq v \leq \pi. \tag{3.6.4}$$

Although this parameterization appears to be the parameterization of a surface, notice that the image is actually a line (Figure 3.6.7). How could we avoid parameterizations such as this? Parameterizations that do not give an actual surface? Notice that $\mathbf{r}_u = \langle 0, 0, 0\rangle$ and $\mathbf{r}_v = \langle 0, -\sin v, 0\rangle$, and the corresponding cross product is zero. The analog of the condition $\mathbf{r}'(t) = \mathbf{0}$ is that $\mathbf{r}_u \times \mathbf{r}_v$ is not zero for point (u, v) in the parameter domain, which is a regular parameterization.

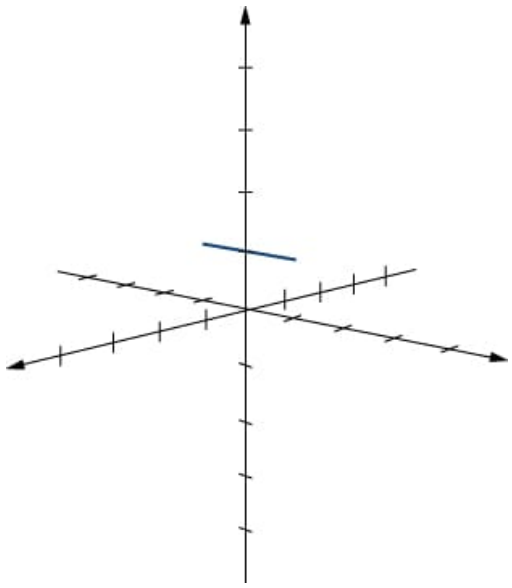


Figure 3.6.7: The parameterization $\mathbf{r}(u, v) = \langle 0, \cos v, 1\rangle$, where $0 \leq u \leq 1$ and $0 \leq v \leq \pi$. The result is straight line segment parallel to the z-axis.

Definition: Regular Parameterization

Parameterization $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$ is a *regular parameterization* if $\mathbf{r}_u \times \mathbf{r}_v$ is not zero for point (u, v) in the parameter domain.

If parameterization $\vec{r}$ is regular, then the image of $\vec{r}$ is a two-dimensional object, as a surface should be. Throughout this chapter, parameterizations $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$are assumed to be regular.

Recall that curve parameterization $\mathbf{r}(t),\ \ a \leq t \leq b$ is smooth if $\mathbf{r}'(t)$ is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$ for all t in $[a, b]$. Informally, a curve parameterization is smooth if the resulting curve has no sharp corners. The definition of a smooth surface parameterization is similar. Informally, a surface parameterization is *smooth* if the resulting surface has no sharp corners.

Definition: Smooth Surfaces

A surface parameterization $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$ is *smooth* if vector $\mathbf{r}_u \times \mathbf{r}_v$ is not zero for any choice of u and v in the parameter domain.

Exercise 3.6.4

Is the surface parameterization $\mathbf{r}(u, v) = \langle u^{2v}, v + 1,\ \sin u\rangle,\ \ 0 \leq u \leq 2,\ \ 0 \leq v \leq 3$ smooth?

Hint available at the end of this section.

Answer available at the end of this section.

Surface Area of a Parametric Surface

Our goal is to define a surface integral, and as a first step we have examined how to parameterize a surface. The second step is to define the surface area of a parametric surface. The notation needed to develop this definition is used throughout the rest of this chapter.

Let S be a surface with parameterization $\mathbf{r}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v)\rangle$ over some parameter domain D. We assume here and throughout that the surface parameterization $\mathbf{r}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v)\rangle$ is continuously differentiable—meaning, each component function has continuous partial derivatives. Assume for the sake of simplicity that D is a rectangle (although the following material can be extended to handle nonrectangular parameter domains). Divide rectangle D into subrectangles D_{ij} with horizontal width Δu and vertical length Δv. Suppose that *i* ranges from 1 to *m* and *j* ranges from 1 to *n* so that D is subdivided into *mn* rectangles. This division of D into subrectangles gives a corresponding division of surface S into pieces S_{ij}. Choose point P_{ij} in each piece S_{ij}. Point P_{ij} corresponds to point (u_i, v_j) in the parameter domain.

Note that we can form a grid with lines that are parallel to the u-axis and the v-axis in the uv-plane. These grid lines correspond to a set of grid curves on surface S that is parameterized by $\mathbf{r}(u,v)$. Without loss of generality, we assume that P_{ij} is located at the corner of two grid curves, as in Figure 3.6.9. If we think of $\mathbf{r}$ as a mapping from the uv-plane to $\mathbb{R}^3$, the grid curves are the image of the grid lines under $\mathbf{r}$. To be precise, consider the grid lines that go through point (u_i, v_j). One line is given by $x = u_i$, $y = v$; the other is given by $x = u,\ y = v_j$. In the first grid line, the horizontal component is held constant, yielding a vertical line through (u_i, v_j). In the second grid line, the vertical component is held constant, yielding a horizontal line through (u_i, v_j). The corresponding grid curves are $\mathbf{r}(u_i, v)$ and (u, v_j) and these curves intersect at point P_{ij}.

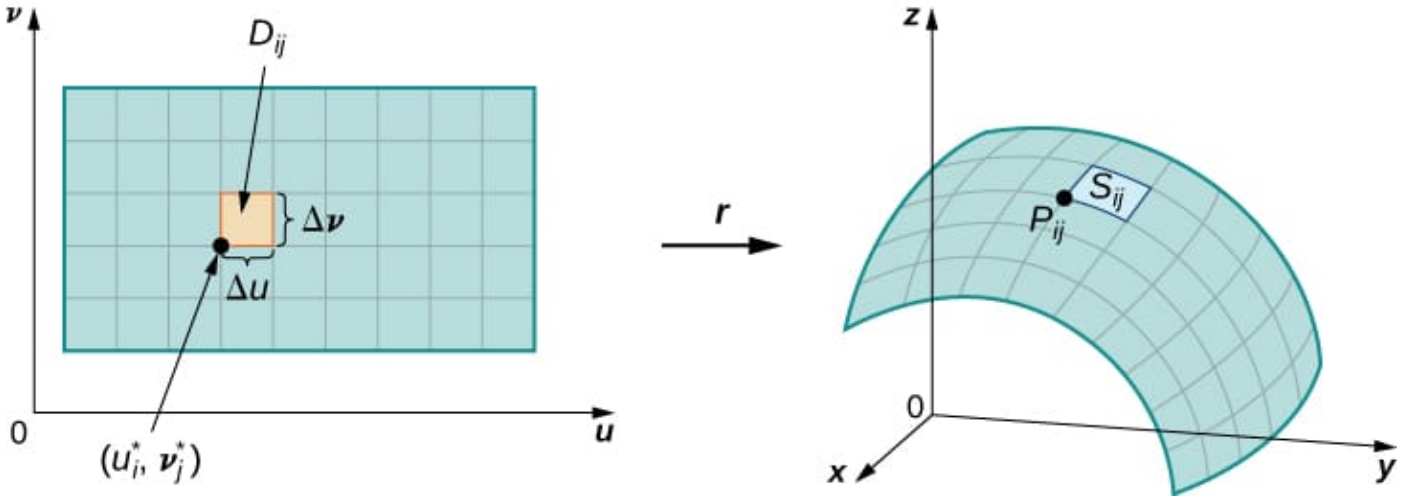


Figure 3.6.9: Grid lines on a parameter domain correspond to grid curves on a surface.

Now consider the vectors that are tangent to these grid curves. For grid curve $\mathbf{r}(u_i, v)$, the tangent vector at P_{ij} is

$$\mathbf{t}_v(P_{ij}) = \mathbf{r}_v(u_i, v_j) = \langle x_v(u_i, v_j),\ y_v(u_i, v_j),\ z_v(u_i, v_j)\rangle. \tag{3.6.5}$$

For grid curve $\mathbf{r}(u, v_j)$, the tangent vector at P_{ij} is

$$\mathbf{t}_u(P_{ij}) = \mathbf{r}_u(u_i, v_j) = \langle x_u(u_i, v_j),\ y_u(u_i, v_j),\ z_u(u_i, v_j)\rangle. \tag{3.6.6}$$

If vector $\mathbf{N} = \mathbf{t}_u(P_{ij}) \times \mathbf{t}_v(P_{ij})$ exists and is not zero, then the tangent plane at P_{ij} exists (Figure 3.6.10). If piece S_{ij} is small enough, then the tangent plane at point P_{ij} is a good approximation of piece S_{ij}.

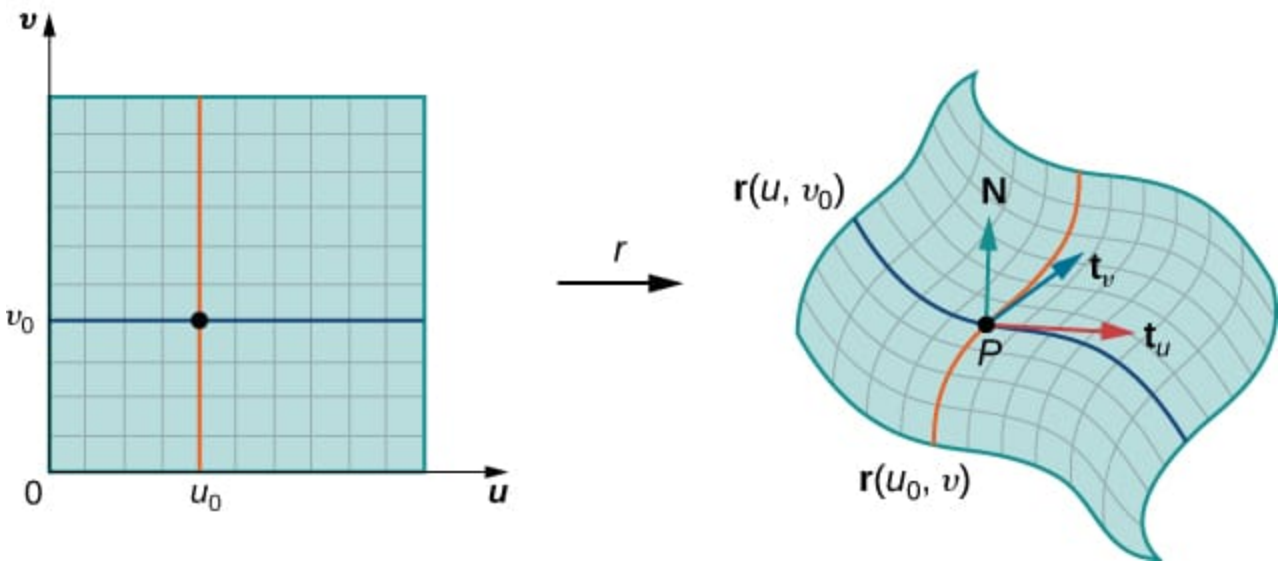


Figure 3.6.10: If the cross product of vectors $\mathbf{t}_u$ and $\mathbf{t}_v$ exists, then there is a tangent plane. On the left is a rectangular region divided into many subrectangles with the point (u_0, v_0). On the right is the same point transformed on a surface using the vector function $\mathbf{r}$.

The tangent plane at P_{ij} contains vectors $\mathbf{t}_u(P_{ij})$ and $\mathbf{t}_v(P_{ij})$ and therefore the parallelogram spanned by $\mathbf{t}_u(P_{ij})$ and $\mathbf{t}_v(P_{ij})$ is in the tangent plane. Since the original rectangle in the uv-plane corresponding to S_{ij} has width Δu and length Δv, the parallelogram that we use to approximate S_{ij} is the parallelogram spanned by $\Delta u\mathbf{t}_u(P_{ij})$ and $\Delta v\mathbf{t}_v(P_{ij})$. In other words, we scale the tangent vectors by the constants Δu and Δv to match the scale of the original division of rectangles in the parameter domain. Therefore, the area of the parallelogram used to approximate the area of S_{ij} is

$$\Delta S_{ij} \approx ||(\Delta u\mathbf{t}_u(P_{ij})) \times (\Delta v\mathbf{t}_v(P_{ij}))|| = ||\mathbf{t}_u(P_{ij}) \times \mathbf{t}_v(P_{ij})||\Delta u\,\Delta v. \tag{3.6.7}$$

Varying point P_{ij} over all pieces S_{ij} and the previous approximation leads to the following definition of surface area of a parametric surface (Figure 3.6.11).

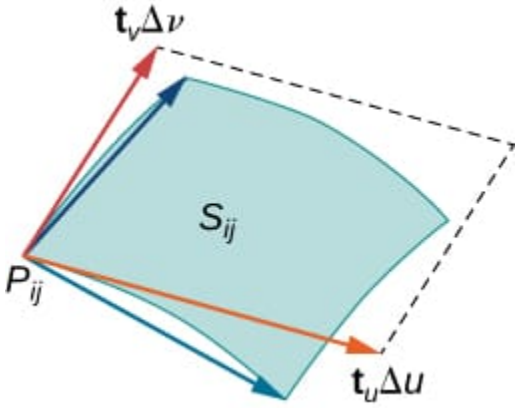


Figure 3.6.11: The parallelogram spanned by $\mathbf{t}_u$ *and* $\mathbf{t}_v$ *approximates the piece of surface* S_{ij}.

Definition: Smooth Parameterization of a Surface

Let $\mathbf{r}(u, v) = \langle x(u, v),\ y(u, v),\ z(u, v)\rangle$ with parameter domain D be a smooth parameterization of surface S. Furthermore, assume that S is traced out only once as (u, v) varies over D. The surface area of S is

$$\iint_D ||\mathbf{t}_u \times \mathbf{t}_v||\, dA, \tag{3.6.8}$$

where $\mathbf{t}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle$

and

$$\mathbf{t}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle. \tag{3.6.9}$$

Example 3.6.4: Calculating Surface Area

Calculate the lateral surface area (the area of the "side," not including the base) of the right circular cone with height h and radius r.

Solution

Before calculating the surface area of this cone using Equation 3.6.8, we need a parameterization. We assume this cone is in $\mathbb{R}^3$ with its vertex at the origin (Figure 3.6.12). To obtain a parameterization, let α be the angle that is swept out by starting at the positive z-axis and ending at the cone, and let $k = \tan\alpha$. For a height value v with $0 \le v \le h$, the radius of the circle formed by intersecting the cone with plane $z = v$ is kv. Therefore, a parameterization of this cone is

$$\mathbf{s}(u, v) = \langle kv\cos u,\ kv\sin u,\ v\rangle,\ 0 \le u < 2\pi,\ 0 \le v \le h.$$

The idea behind this parameterization is that for a fixed v-value, the circle swept out by letting u vary is the circle at height v and radius kv. As v increases, the parameterization sweeps out a "stack" of circles, resulting in the desired cone.

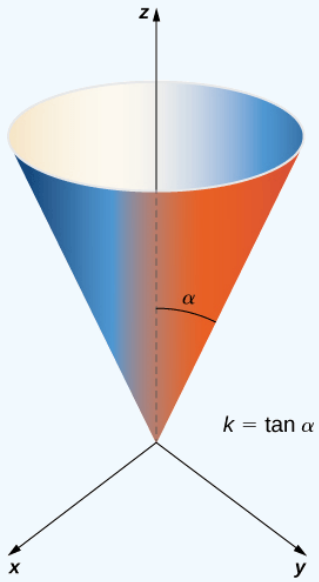


Figure 3.6.12: The right circular cone with radius $r = kh$ and height h has parameterization

$$\mathbf{s}(u, v) = \langle kv\cos u,\ kv\sin u,\ v\rangle,\ 0 \le u < 2\pi,\ 0 \le v \le h. \tag{3.6.10}$$

With a parameterization in hand, we can calculate the surface area of the cone using Equation 3.6.8. The tangent vectors are $\mathbf{t}_u = \langle -kv\ \sin u,\ kv\ \cos u,\ 0\rangle$ and $\mathbf{t}_v = \langle k\ \cos u,\ k\ \sin u,\ 1\rangle$. Therefore,

$$\begin{aligned} \vec{t}_u \times \vec{t}_v &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -kv\sin u & kv\cos u & 0 \\ k\cos u & k\sin u & 1 \end{vmatrix} \\ &= \langle kv\cos u,\ kv\sin u,\ -k^2v\sin^2 u - k^2v\cos^2 u\rangle \\ &= \langle kv\cos u,\ kv\sin u,\ -k^2v\rangle. \end{aligned}$$

The magnitude of this vector is

$$||\langle kv\cos u,\ kv\sin u,\ -k^2v\rangle|| = \sqrt{k^2v^2\cos^2 u + k^2v^2\sin^2 u + k^4v^2}$$
$$= \sqrt{k^2v^2 + k^4v^2}$$
$$= kv\sqrt{1+k^2}.$$

By Equation 3.6.8, the surface area of the cone is

$$\iint_D ||\mathbf{t}_u \times \mathbf{t}_v||\, dA = \int_0^h \int_0^{2\pi} kv\sqrt{1+k^2}\, du\, dv$$
$$= 2\pi k\sqrt{1+k^2}\int_0^h v\, dv$$
$$= 2\pi k\sqrt{1+k^2}\left[\frac{v^2}{2}\right]_0^h$$
$$= \pi k h^2\sqrt{1+k^2}.$$

Since $k = \tan\alpha = r/h$,

$$\pi k h^2\sqrt{1+k^2} = \pi\frac{r}{h}h^2\sqrt{1+\frac{r^2}{h^2}}$$
$$= \pi r h\sqrt{1+\frac{r^2}{h^2}}$$
$$= \pi r\sqrt{h^2 + h^2\left(\frac{r^2}{h^2}\right)}$$
$$= \pi r\sqrt{h^2+r^2}.$$

Therefore, the lateral surface area of the cone is $\pi r\sqrt{h^2+r^2}$.

Analysis

The surface area of a right circular cone with radius r and height h is usually given as $\pi r^2 + \pi r\sqrt{h^2+r^2}$. The reason for this is that the circular base is included as part of the cone, and therefore the area of the base πr^2 is added to the lateral surface area $\pi r\sqrt{h^2+r^2}$ that we found.

Exercise 3.6.5

Find the surface area of the surface with parameterization $\mathbf{r}(u,v) = \langle u+v,\ u^2,\ 2v\rangle,\ 0\le u\le 3,\ 0\le v\le 2$.

Hint available at the end of this section.

Answer available at the end of this section.

Example 3.6.5: Calculating Surface Area

Show that the surface area of the sphere $x^2+y^2+z^2 = r^2$ is $4\pi r^2$.

Solution

The sphere has parameterization

$$r\ \cos\theta\ \sin\phi,\ \ r\ \sin\theta\ \sin\phi,\ \ r\ \cos\phi\rangle,\ \ 0\le\theta<2\pi,\ \ 0\le\phi\le\pi.$$

The tangent vectors are

$$\mathbf{t}_\theta = \langle -r\ \sin\theta\ \sin\phi,\ r\ \cos\theta\ \sin\phi,\ 0\rangle$$

and

$$\mathbf{t}_\phi = \langle r\ \cos\theta\ \cos\phi,\ r\ \sin\theta\ \cos\phi,\ -r\ \sin\phi\rangle.$$

Therefore,

$$\begin{aligned}\mathbf{t}_\phi \times \mathbf{t}_\theta &= \langle r^2\cos\theta\,\sin^2\phi,\, r^2\sin\theta\,\sin^2\phi,\, r^2\sin^2\theta\,\sin\phi\,\cos\phi + r^2\cos^2\theta\,\sin\phi\,\cos\phi\rangle \\ &= \langle r^2\cos\theta\,\sin^2\phi,\, r^2\sin\theta\,\sin^2\phi,\, r^2\sin\phi\,\cos\phi\rangle.\end{aligned}$$

Now,

$$\begin{aligned}||\mathbf{t}_\phi \times \mathbf{t}_\theta|| &= \sqrt{r^4\sin^4\phi\,\cos^2\theta + r^4\sin^4\phi\,\sin^2\theta + r^4\sin^2\phi\,\cos^2\phi} \\ &= \sqrt{r^4\sin^4\phi + r^4\sin^2\phi\,\cos^2\phi} \\ &= r^2\sqrt{\sin^2\phi} \\ &= r\,\sin\phi.\end{aligned}$$

Notice that $\sin\phi \geq 0$ on the parameter domain because $0 \leq \phi < \pi$, and this justifies equation $\sqrt{\sin^2\phi} = \sin\phi$. The surface area of the sphere is

$$\int_0^{2\pi}\int_0^{\pi} r^2\sin\phi\,d\phi\,d\theta = r^2\int_0^{2\pi} 2\,d\theta = 4\pi r^2.$$

We have derived the familiar formula for the surface area of a sphere using surface integrals.

Exercise 3.6.6

Show that the surface area of cylinder $x^2 + y^2 = r^2,\ 0 \leq z \leq h$ is $2\pi rh$. Notice that this cylinder does not include the top and bottom circles.

Hint available at the end of this section.

Answer available at the end of this section.

In addition to parameterizing surfaces given by equations or standard geometric shapes such as cones and spheres, we can also parameterize surfaces of revolution. Therefore, we can calculate the surface area of a surface of revolution by using the same techniques. Let $y = f(x) \geq 0$ be a positive single-variable function on the domain $a \leq x \leq b$ and let S be the surface obtained by rotating f about the x-axis (Figure 3.6.13). Let θ be the angle of rotation. Then, S can be parameterized with parameters x and θ by

$$\mathbf{r}(x,\theta) = \langle x, f(x)\ \cos\theta,\ f(x)\sin\theta\rangle,\ a \leq x \leq b,\ 0 \leq \theta \leq 2\pi. \tag{3.6.11}$$

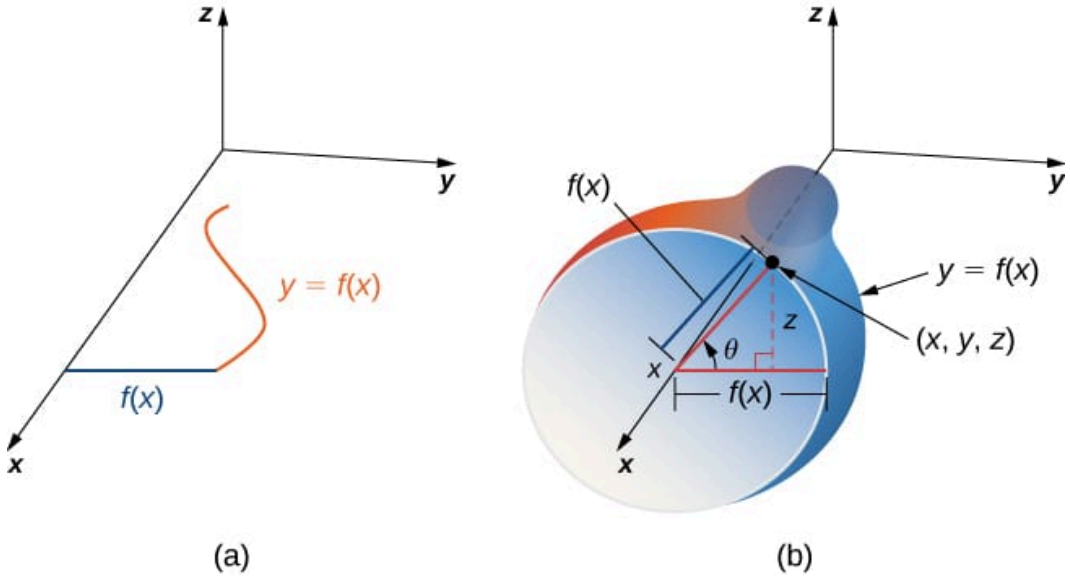


Figure 3.6.13: Parametrization of a surface of revolution by $\mathbf{r}(x,\theta) = \langle x, f(x)\cos\theta, f(x)\sin\theta\rangle,\ a \le x \le b,\ 0 \le x < 2\pi$.

Example 3.6.6: Calculating Surface Area

Find the area of the surface of revolution obtained by rotating $y = x^2,\ 0 \le x \le b$ about the *x*-axis (Figure 3.6.14).

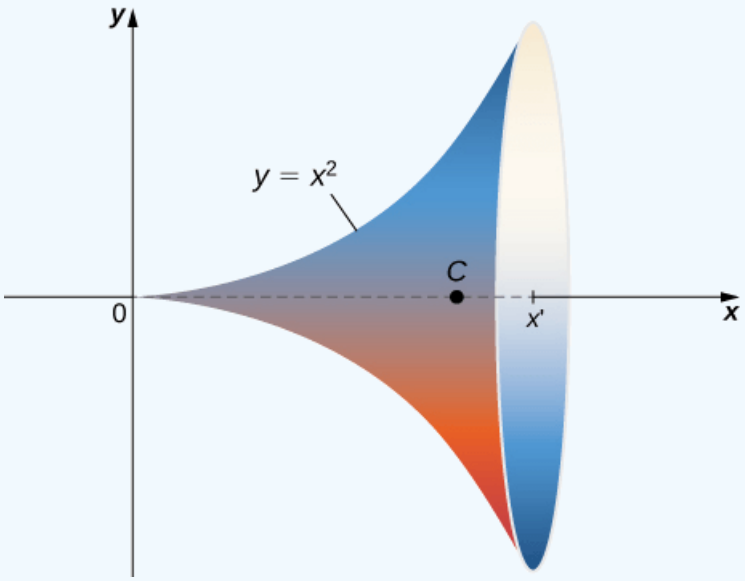


Figure 3.6.14: The surface of revolution obtained by rotating $y = x^2,\ 0 \le x \le b$ about the *x*-axis.

Solution

This surface has parameterization $\mathbf{r}(x,\theta) = \langle x,\ x^2\cos\theta,\ x^2\sin\theta\rangle,\ 0 \le x \le b,\ 0 \le x < 2\pi.$

The tangent vectors are $\mathbf{t}_x = \langle 1,\ 2x\ \cos\theta,\ 2x\ \sin\theta\rangle$ and $\mathbf{t}_\theta = \langle 0,\ -x^2\sin\theta,\ x^2\cos\theta\rangle$.

Therefore,

$$\begin{aligned}\mathbf{t}_x \times \mathbf{t}_\theta &= \langle 2x^3\cos^2\theta + 2x^3\sin^2\theta,\ -x^2\cos\theta,\ -x^2\sin\theta\rangle \\ &= \langle 2x^3,\ -x^2\cos\theta,\ -x^2\sin\theta\rangle\end{aligned}$$

and

$$\begin{aligned}\mathbf{t}_x \times \mathbf{t}_\theta &= \sqrt{4x^6 + x^4\cos^2\theta + x^4\sin^2\theta} \\ &= \sqrt{4x^6 + x^4} \\ &= x^2\sqrt{4x^2+1}\end{aligned}$$

The area of the surface of revolution is

$$\begin{aligned}\int_0^b \int_0^{2\pi} x^2\sqrt{4x^2+1}\,d\theta\,dx &= 2\pi\int_0^b x^2\sqrt{4x^2+1}\,dx \\ &= 2\pi\left[\frac{1}{64}\left(2\sqrt{4x^2+1}(8x^3+x)\,\sinh^{-1}(2x)\right)\right]_0^b \\ &= 2\pi\left[\frac{1}{64}\left(2\sqrt{4b^2+1}(8b^3+b)\,\sinh^{-1}(2b)\right)\right].\end{aligned}$$

Exercise 3.6.7

Use Equation 3.6.8 to find the area of the surface of revolution obtained by rotating curve $y = \sin x,\ 0 \le x \le \pi$ about the x-axis.

Hint available at the end of this section.

Answer available at the end of this section.

Key Concepts

- Surfaces can be parameterized, just as curves can be parameterized. In general, surfaces must be parameterized with two parameters.
- Surfaces can sometimes be oriented, just as curves can be oriented.
- If S is a surface, then the area of S is

$$\iint_S dS. \tag{3.6.12}$$

Glossary

Grid curves
curves on a surface that are parallel to grid lines in a coordinate plane

Parameter domain (parameter space)
the region of the uv-plane over which the parameters u and v vary for parameterization $\mathbf{r}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v)\rangle$

Parameterized surface (parametric surface)
a surface given by a description of the form $\mathbf{r}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v)\rangle$, where the parameters u and v vary over a parameter domain in the uv-plane

Regular parameterization
parameterization $\mathbf{r}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v)\rangle$ such that $r_u \times r_v$ is not zero for point (u,v) in the parameter domain

Surface area
the area of surface S given by the surface integral

$$\iint_S dS \tag{3.6.13}$$

Contributors and Attributions

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Answers

Exercise 3.6.1

Hold u and v constant, and see what kind of curves result.

Exercise 3.6.1

Cylinder $x^2 + y^2 = 4$

Exercise 3.6.2

Hold u constant and see what kind of curves result. Imagine what happens as u increases or decreases.

Exercise 3.6.2

Cone $x^2 + y^2 = z^2$

Exercise 3.6.3

Consider the parameter domain for this surface.

Exercise 3.6.3

$\mathbf{r}(u, v) = \langle u\ \cos v,\ u\ \sin v,\ u\rangle,\ 0 < u < \infty,\ 0 \leq v < \dfrac{\pi}{2}$

Exercise 3.6.4

Investigate the cross product $\mathbf{r}_u \times \mathbf{r}_v$.

Exercise 3.6.4

Yes

Exercise 3.6.5

Use Equation 3.6.8.

Exercise 3.6.5

$\backslash\approx 43.02$

Exercise 3.6.6

Use the standard parameterization of a cylinder and follow the previous example.

Exercise 3.6.6

With the standard parameterization of a cylinder, Equation 3.6.8 shows that the surface area is $2\pi rh$.

Exercise 3.6.7

Use the parameterization of surfaces of revolution given before Example 3.6.7.

Exercise 3.6.7

$2\pi(\sqrt{2} + \sinh^{-1}(1))$

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3.6E: Exercises

Exercises 1 - 4

Determine whether the statements are *true or false*.

1. If surface S is given by $\{(x,y,z):\ 0\le x\le 1,\ 0\le y\le 1,\ z=10\}$, then $\iint_S f(x,y,z)\ dS=\int_0^1\int_0^1 f(x,y,10)\ dx\ dy.$

Answer

True

2. If surface S is given by $\{(x,y,z):\ 0\le x\le 1,\ 0\le y\le 1,\ z=x\}$, then $\iint_S f(x,y,z)\ dS=\int_0^1\int_0^1 f(x,y,x)\ dx\ dy.$

3. Surface $\mathbf{r}=\langle v\ \cos u,\ v\ \sin u,\ v^2\rangle$, for $0\le u\le\pi,\ 0\le v\le 2$ is the same surface $\mathbf{r}=\langle\sqrt{v}\ \cos 2u,\ \sqrt{v}\ \sin 2u,\ v\rangle$, for $0\le u\le\dfrac{\pi}{2},\ 0\le v\le 4$.

Answer

True

4. Given the standard parameterization of a sphere, normal vectors $t_u\times t_v$ are outward normal vectors.

Exercises 5 - 10

Find parametric descriptions for the following surfaces.

5. Plane $3x-2y+z=2$

Answer

$\mathbf{r}(u,v)=\langle u,\ v,\ 2-3u+2v\rangle$ for $-\infty<u<\infty$ and $-\infty<v<\infty$.

6. Paraboloid $z=x^2+y^2$, for $0\le z\le 9$.

7. Plane $2x-4y+3z=16$

Answer

$\mathbf{r}(u,v)=\langle u,\ v,\ \dfrac{1}{3}(16-2u+4v)\rangle$ for $|u|<\infty$ and $|v|<\infty$.

8. The frustum of cone $z^2=x^2+y^2$, for $2\le z\le 8$

9. The portion of cylinder $x^2+y^2=9$ in the first octant, for $0\le z\le 3$

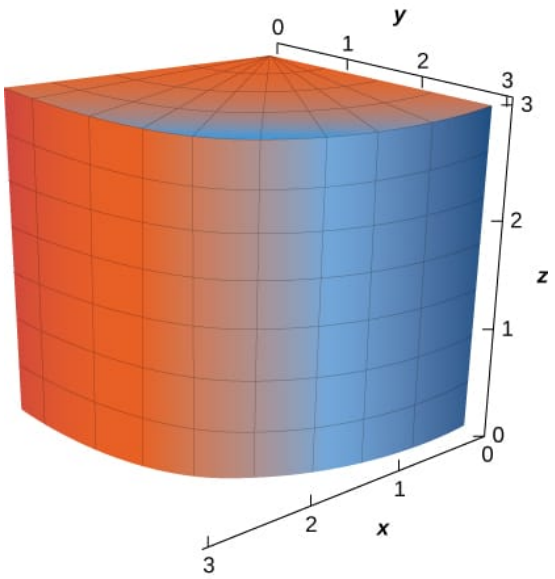


Figure $3.6E.1$: Portion of cylinder $x^2 + y^2 = 9$ in the first octant.

Answer

$$\mathbf{r}(u, v) = \langle 3\ \cos u,\ 3\ \sin u,\ v \rangle \text{ for } 0 \le u \le \frac{\pi}{2},\ 0 \le v \le 3$$

10. A cone with base radius r and height h, where r and h are positive constants.

Exercises 11 - 12

Use a computer algebra system to approximate the area of the following surfaces using a parametric description of the surface.

11. Half cylinder $\{(r, \theta, z) :\ r = 4,\ 0 \le \theta \le \pi,\ 0 \le z \le 7\}$

Answer

$$A = 87.9646$$

12. Plane $z = 10 - z - y$ above square $|x| \le 2,\ |y| \le 2$

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3.7: Surface Integral

Learning Objectives

- Describe the surface integral of a scalar-valued function over a parametric surface.
- Use a surface integral to calculate the area of a given surface.
- Explain the meaning of an oriented surface, giving an example.
- Describe the surface integral of a vector field.
- Use surface integrals to solve applied problems.

Surface Integral of a Scalar-Valued Function

Now that we can parameterize surfaces and we can calculate their surface areas, we are able to define surface integrals. First, let's look at the surface integral of a scalar-valued function. Informally, the surface integral of a scalar-valued function is an analog of a scalar line integral in one higher dimension. The domain of integration of a scalar line integral is a parameterized curve (a one-dimensional object); the domain of integration of a scalar surface integral is a parameterized surface (a two-dimensional object). Therefore, the definition of a surface integral follows the definition of a line integral quite closely. For scalar line integrals, we chopped the domain curve into tiny pieces, chose a point in each piece, computed the function at that point, and took a limit of the corresponding Riemann sum. For scalar surface integrals, we chop the domain *region* (no longer a curve) into tiny pieces and proceed in the same fashion.

Let S be a piecewise smooth surface with parameterization $\mathbf{r}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v)\rangle$ with parameter domain D and let $f(x,y,z)$ be a function with a domain that contains S. For now, assume the parameter domain D is a rectangle, but we can extend the basic logic of how we proceed to any parameter domain (the choice of a rectangle is simply to make the notation more manageable). Divide rectangle D into subrectangles D_{ij} with horizontal width Δu and vertical length Δv. Suppose that i ranges from 1 to m and j ranges from 1 to n so that D is subdivided into mn rectangles. This division of D into subrectangles gives a corresponding division of S into pieces S_{ij}. Choose point P_{ij} in each piece S_{ij} evaluate f at P_{ij}, and multiply by area S_{ij} to form the Riemann sum

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij})\,\Delta S_{ij}.$$

To define a surface integral of a scalar-valued function, we let the areas of the pieces of S shrink to zero by taking a limit.

Definition: Surface Integral of a Scalar-Valued Function

The surface integral of a scalar-valued function of f over a piecewise smooth surface S is

$$\iint_S f(x,y,z)dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij})\Delta S_{ij}. \tag{3.7.1}$$

Again, notice the similarities between this definition and the definition of a scalar line integral. In the definition of a line integral we chop a curve into pieces, evaluate a function at a point in each piece, and let the length of the pieces shrink to zero by taking the limit of the corresponding Riemann sum. In the definition of a surface integral, we chop a surface into pieces, evaluate a function at a point in each piece, and let the area of the pieces shrink to zero by taking the limit of the corresponding Riemann sum. Thus, a surface integral is similar to a line integral but in one higher dimension.

The definition of a scalar line integral can be extended to parameter domains that are not rectangles by using the same logic used earlier. The basic idea is to chop the parameter domain into small pieces, choose a sample point in each piece, and so on. The exact shape of each piece in the sample domain becomes irrelevant as the areas of the pieces shrink to zero.

Scalar surface integrals are difficult to compute from the definition, just as scalar line integrals are. To develop a method that

makes surface integrals easier to compute, we approximate surface areas ΔS_{ij} with small pieces of a tangent plane, just as we did in the previous subsection. Recall the definition of vectors $\mathbf{t}_u$ and $\mathbf{t}_v$:

$$\mathbf{t}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle \text{ and } \mathbf{t}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle.$$

From the material we have already studied, we know that

$$\Delta S_{ij} \approx ||\mathbf{t}_u(P_{ij}) \times \mathbf{t}_v(P_{ij})|| \, \Delta u \, \Delta v.$$

Therefore,

$$\iint_S f(x,y,z)\, dS \approx \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}) ||\mathbf{t}_u(P_{ij}) \times \mathbf{t}_v(P_{ij})|| \, \Delta u \, \Delta v.$$

This approximation becomes arbitrarily close to $\lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}) \Delta S_{ij}$ as we increase the number of pieces S_{ij} by letting m and n go to infinity. Therefore, we have the following equation to calculate **scalar surface integrals**:

$$\iint_S f(x,y,z)\, dS = \iint_D f(\mathbf{r}(u,v)) ||\mathbf{t}_u \times \mathbf{t}_v|| \, dA. \tag{3.7.2}$$

Equation 3.7.2 allows us to calculate a surface integral by transforming it into a double integral. This equation for surface integrals is analogous to the equation for line integrals:

$$\iint_C f(x,y,z)\, ds = \int_a^b f(\mathbf{r}(t)) ||\mathbf{r}'(t)|| \, dt.$$

In this case, vector $\mathbf{t}_u \times \mathbf{t}_v$ is perpendicular to the surface, whereas vector $\mathbf{r}'(t)$ is tangent to the curve.

Example 3.7.1: Calculating a Surface Integral

Calculate surface integral

$$\iint_S 5\, dS,$$

where S is the surface with parameterization $\mathbf{r}(u,v) = \langle u,\ u^2,\ v \rangle$ for $0 \le u \le 2$ and $0 \le v \le u$.

Solution

Notice that this parameter domain$D$ is a triangle, and therefore the parameter domain is not rectangular. This is not an issue though, because Equation 3.7.2 does not place any restrictions on the shape of the parameter domain.

To use Equation 3.7.2 to calculate the surface integral, we first find vectors $\mathbf{t}_u$ and $\mathbf{t}_v$. Note that $\mathbf{t}_u = \langle 1, 2u, 0 \rangle$ and $\mathbf{t}_v = \langle 0, 0, 1 \rangle$. Therefore,

$$\mathbf{t}_u \times \mathbf{t}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2u & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle 2u,\ -1,\ 0 \rangle$$

and

$$||\mathbf{t}_u \times \mathbf{t}_v|| = \sqrt{1 + 4u^2}. \tag{3.7.3}$$

By Equation 3.7.2,

$$\begin{aligned}\iint_S 5\,dS &= 5\iint_D u\sqrt{1+4u^2}\,dA \\ &= 5\int_0^2\int_0^u \sqrt{1+4u^2}\,dv\,du = 5\int_0^2 u\sqrt{1+4u^2}\,du \\ &= 5\left[\frac{(1+4u^2)^{3/2}}{3}\right]_0^2 \\ &= \frac{5(17^{3/2}-1)}{3} \approx 115.15.\end{aligned}$$

Example 3.7.2: Calculating the Surface Integral of a Cylinder

Calculate surface integral

$$\iint_S (x+y^2)\,dS, \tag{3.7.4}$$

where S is cylinder $x^2+y^2=1,\ 0\le z\le 3$ (Figure 3.7.1).

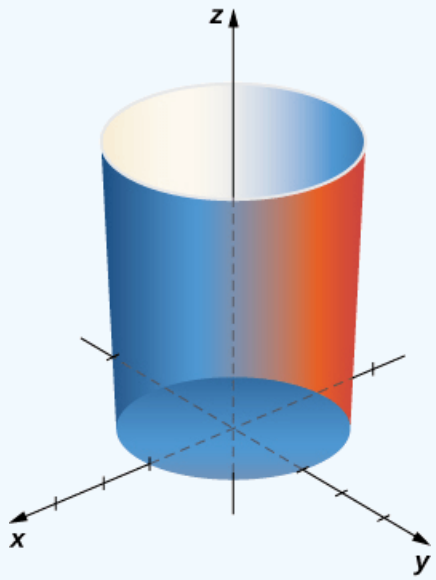


Figure 3.7.1: S is cylinder $x^2+y^2=1,\ 0\le z\le 3$

Solution

To calculate the surface integral, we first need a parameterization of the cylinder. A parameterization is $\mathbf{r}(u,v)=\langle\cos u,\ \sin u,\ v\rangle, 0\le u\le 2\pi,\ 0\le v\le 3$.

The tangent vectors are $\mathbf{t}_u=\langle-\sin u,\ \cos u,\ 0\rangle$ and $\mathbf{t}_v=\langle 0,0,1\rangle$. Then,

$$\mathbf{t}_u\times\mathbf{t}_v=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -\sin u & \cos u & 0\\ 0 & 0 & 1\end{vmatrix}=\langle\cos u,\ \sin u,\ 0\rangle \tag{3.7.5}$$

and $||\mathbf{t}_u\times\mathbf{t}_v||=\sqrt{\cos^2 u+\sin^2 u}=1$. By Equation 3.7.2,

$$\begin{aligned}\iint_S f(x,y,z)dS &= \iint_D f(\mathbf{r}(u,v))||\mathbf{t}_u \times \mathbf{t}_v||\, dA \\ &= \int_0^3 \int_0^{2\pi} (\cos u + \sin^2 u)\, du\, dv \\ &= \int_0^3 \left[\sin u + \frac{u}{2} - \frac{\sin(2u)}{4}\right]_0^{2\pi} dv \\ &= \int_0^3 \pi\, dv = 3\pi.\end{aligned}$$

Exercise 3.7.1

Calculate

$$\iint_S (x^2 - z)\, dS, \tag{3.7.6}$$

where S is the surface with parameterization $\mathbf{r}(u,v) = \langle v,\ u^2 + v^2,\ 1\rangle,\ 0 \le u \le 2,\ 0 \le v \le 3.$

Hint available at the end of this section.

Answer available at the end of this section.

Example 3.7.3: Calculating the Surface Integral of a Piece of a Sphere

Calculate surface integral

$$\iint_S f(x,y,z)\, dS, \tag{3.7.7}$$

where $f(x,y,z) = z^2$ and S is the surface that consists of the piece of sphere $x^2 + y^2 + z^2 = 4$ that lies on or above plane $z = 1$ and the disk that is enclosed by intersection plane $z = 1$ and the given sphere (Figure 3.7.2).

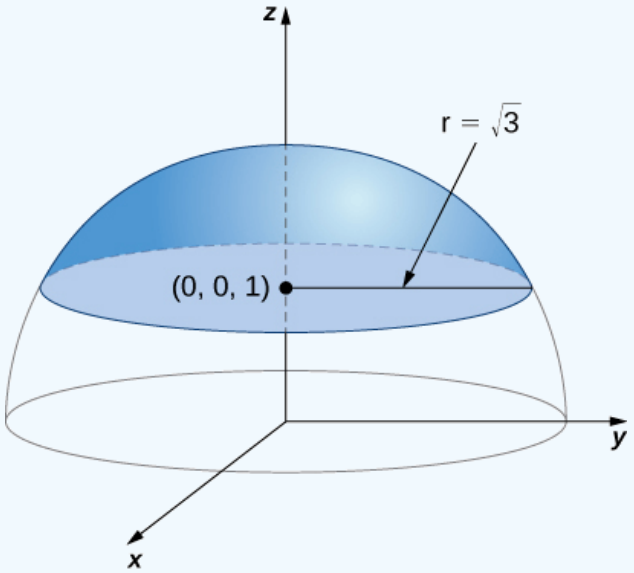


Figure 3.7.2: S is the surface that consists of the piece of sphere $x^2 + y^2 + z^2 = 4$ that lies on or above plane $z = 1$ and the disk that is enclosed by intersection plane $z = 1$ and the given sphere.

Solution

Notice that S is not smooth but is piecewise smooth; S can be written as the union of its base S_1 and its spherical top S_2, and both S_1 and S_2 are smooth. Therefore, to calculate

$$\iint_S z^2 dS,$$

we write this integral as

$$\iint_{S_1} z^2\, dS + \iint_{S_2} z^2\, dS$$

and we calculate integrals

$$\iint_{S_1} z^2\, dS$$

and

$$\iint_{S_2} z^2\, dS.$$

First, we calculate $\iint_{S_1} z^2\ dS$. To calculate this integral we need a parameterization of S_1. This surface is a disk in plane $z = 1$ centered at $(0, 0, 1)$. To parameterize this disk, we need to know its radius. Since the disk is formed where plane $z = 1$ intersects sphere $x^2 + y^2 + z^2 = 4$, we can substitute $z = 1$ into equation $x^2 + y^2 + z^2 = 4$:

$$x^2 + y^2 + 1 = 4 \Rightarrow x^2 + y^2 = 3.$$

Therefore, the radius of the disk is $\sqrt{3}$ and a parameterization of S_1 is $\mathbf{r}(u, v) = \langle u\ \cos v,\ u\ \sin v,\ 1\rangle,\ 0 \le u \le \sqrt{3},\ 0 \le v \le 2\pi$. The tangent vectors are $\mathbf{t}_u = \langle \cos v,\ \sin v,\ 0\rangle$ and $\mathbf{t}_v = \langle -u\ \sin v,\ u\ \cos v,\ 0\rangle$, and thus

$$\mathbf{t}_u \times \mathbf{t}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 0 \end{vmatrix} = \langle 0,\ 0,\ u\ \cos^2 v + u\ \sin^2 v\rangle = \langle 0, 0, u\rangle. \tag{3.7.8}$$

The magnitude of this vector is u. Therefore,

$$\begin{aligned} \iint_{S_1} z^2\, dS &= \int_0^{\sqrt{3}} \int_0^{2\pi} f(r(u,v))||t_u \times t_v||\, dv\, du \\ &= \int_0^{\sqrt{3}} \int_0^{2\pi} u\, dv\, du \\ &= 2\pi \int_0^{\sqrt{3}} u\, du \\ &= 2\pi\sqrt{3}. \end{aligned}$$

Now we calculate

$$\iint_{S_2} dS.$$

To calculate this integral, we need a parameterization of S_2. The parameterization of full sphere $x^2 + y^2 + z^2 = 4$ is

$$\mathbf{r}(\phi, \theta) = \langle 2\ \cos\theta\ \sin\phi,\ 2\ \sin\theta\ \sin\phi,\ 2\ \cos\phi\rangle,\ 0 \le \theta \le 2\pi, 0 \le \phi \le \pi. \tag{3.7.9}$$

Since we are only taking the piece of the sphere on or above plane $z = 1$, we have to restrict the domain of ϕ. To see how far this angle sweeps, notice that the angle can be located in a right triangle, as shown in Figure 3.7.17 (the $\sqrt{3}$ comes from the fact that the base of S is a disk with radius $\sqrt{3}$). Therefore, the tangent of ϕ is $\sqrt{3}$, which implies that ϕ is $\pi/3$. We now have a parameterization of S_2:

$\mathbf{r}(\phi, \theta) = \langle 2\ \cos\theta\ \sin\phi,\ 2\ \sin\theta\ \sin\phi,\ 2\ \cos\phi\rangle,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi/3.$

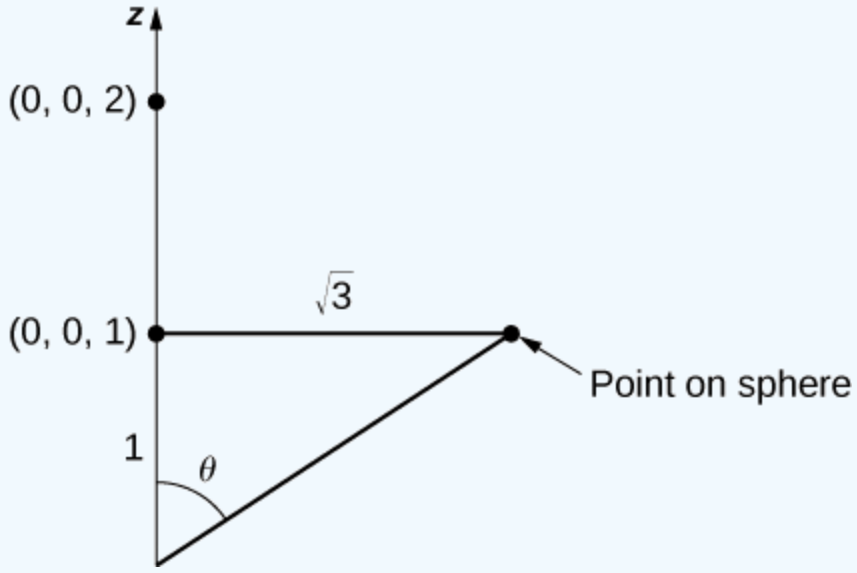


Figure 3.7.3: The maximum value of ϕ has a tangent value of $\sqrt{3}$.

The tangent vectors are $\mathbf{t}_\phi = \langle 2\ \cos\theta\ \cos\phi,\ 2\ \sin\theta\ \cos\phi,\ -2\ \sin\phi\rangle$ and $\mathbf{t}_\theta = \langle -2\sin\theta\sin\phi,\ u\cos\theta\sin\phi,\ 0\rangle$, and thus

$$\begin{aligned}\mathbf{t}_\phi \times \mathbf{t}_\theta &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\cos\theta\cos\phi & 2\sin\theta\cos\phi & -2\sin\phi \\ -2\sin\theta\sin\phi & 2\cos\theta\sin\phi & 0 \end{vmatrix} \\ &= \langle 4\ \cos\theta\ \sin^2\phi,\ 4\ \sin\theta\ \sin^2\phi,\ 4\ \cos^2\theta\ \cos\phi\ \sin\phi + 4\ \sin^2\theta\ \cos\phi\ \sin\phi\rangle \\ &= \langle 4\ \cos\theta\ \sin^2\phi,\ 4\ \sin\theta\ \sin^2\phi,\ 4\ \cos\phi\ \sin\phi\rangle.\end{aligned}$$

The magnitude of this vector is

$$\begin{aligned}\mathbf{t}_\phi \times \mathbf{t}_\theta &= \sqrt{16\ \cos^2\theta\ \sin^4\phi + 16\ \sin^2\theta\ \sin^4\phi + 16\ \cos^2\phi\ \sin^2\phi} \\ &= 4\sqrt{\sin^4\phi + \cos^2\phi\ \sin^2\phi}.\end{aligned}$$

Therefore,

$$\begin{aligned}\iint_{S_2} z\,dS &= \int_0^{\pi/6}\int_0^{2\pi} f(\mathbf{r}(\phi,\theta))||\mathbf{t}_\phi \times \mathbf{t}_\theta||\,d\theta\,d\phi \\ &= \int_0^{\pi/6}\int_0^{2\pi} 16\ \cos^2\phi\sqrt{\sin^4\phi + \cos^2\phi\ \sin^2\phi}\,d\theta\,d\phi \\ &= 32\pi\int_0^{\pi/6} \cos^2\phi\sqrt{\sin^4\phi + \cos^2\phi\ \sin^2\phi}\,d\phi \\ &= 32\pi\int_0^{\pi/6} \cos^2\phi\ \sin\phi\sqrt{\sin^2\phi + \cos^2\phi}\,d\phi \\ &= 32\pi\int_0^{\pi/6} \cos^2\phi\ \sin\phi\,d\phi \\ &= 32\pi\left[-\frac{\cos^3\phi}{3}\right]_0^{\pi/6} \\ &= 32\pi\left[\frac{1}{3} - \frac{\sqrt{3}}{8}\right] = \frac{32\pi}{3} - 4\sqrt{3}\pi.\end{aligned}$$

Since

$$\iint_S z^2\,dS = \iint_{S_1} z^2\,dS + \iint_{S_2} z^2\,dS,$$

we have

$$\iint_S z^2\,dS = (2\pi - 4)\sqrt{3} + \frac{32\pi}{3}.$$

Analysis

In this example we broke a surface integral over a piecewise surface into the addition of surface integrals over smooth subsurfaces. There were only two smooth subsurfaces in this example, but this technique extends to finitely many smooth subsurfaces.

Exercise 3.7.2

Calculate line integral $\iint_S (x-y)\ dS$, where S is cylinder $x^2+y^2=1,\ 0\le z\le 2$, including the circular top and bottom.

Hint available at the end of this section.

Answer available at the end of this section.

Scalar surface integrals have several real-world applications. Recall that scalar line integrals can be used to compute the mass of a wire given its density function. In a similar fashion, we can use scalar surface integrals to compute the mass of a sheet given its density function. If a thin sheet of metal has the shape of surface S and the density of the sheet at point (x,y,z) is $\rho(x,y,z)$ then mass m of the sheet is

$$m = \iint_S \rho(x,y,z)\,dS. \tag{3.7.10}$$

Example 3.7.4: Calculating the Mass of a Sheet

A flat sheet of metal has the shape of surface $z = 1 + x + 2y$ that lies above rectangle $0 \le x \le 4$ and $0 \le y \le 2$. If the density of the sheet is given by $\rho(x,y,z) = x^2yz$, what is the mass of the sheet?

Solution

Let S be the surface that describes the sheet. Then, the mass of the sheet is given by $m = \iint_S x^2yx\ dS$. To compute this surface integral, we first need a parameterization of S. Since S is given by the function $f(x,y) = 1 + x + 2y$, a parameterization of S is $\mathbf{r}(x,y) = \langle x,\ y,\ 1+x+2y\rangle,\ 0\le x\le 4,\ 0\le y\le 2$.

The tangent vectors are $\mathbf{t}_x = \langle 1,0,1\rangle$ and $\mathbf{t}_y = \langle 1,0,2\rangle$. Therefore, $\mathbf{t}_x + \mathbf{t}_y = \langle -1,-2,1\rangle$ and $||\mathbf{t}_x \times \mathbf{t}_y|| = \sqrt{6}$.

By the definition of the line integral (Section 16.2),

$$\begin{aligned} m &= \iint_S x^2yz\,dS \\ &= \sqrt{6}\int_0^4 \int_0^2 x^2y(1+x+2y)\,dy\,dx \\ &= \sqrt{6}\int_0^4 \frac{22x^2}{3} + 2x^3\,dx \\ &= \frac{2560\sqrt{6}}{9} \approx 696.74. \end{aligned}$$

Exercise 3.7.3

A piece of metal has a shape that is modeled by paraboloid $z = x^2 + y^2,\ 0 \le z \le 4$, and the density of the metal is given by $\rho(x, y, z) = z + 1$. Find the mass of the piece of metal.

Answer available at the end of this section.

Orientation of a Surface

Recall that when we defined a scalar line integral, we did not need to worry about an orientation of the curve of integration. The same was true for scalar surface integrals: we did not need to worry about an "orientation" of the surface of integration.

On the other hand, when we defined vector line integrals, the curve of integration needed an orientation. That is, we needed the notion of an oriented curve to define a vector line integral without ambiguity. Similarly, when we define a surface integral of a vector field, we need the notion of an oriented surface. An oriented surface is given an "upward" or "downward" orientation or, in the case of surfaces such as a sphere or cylinder, an "outward" or "inward" orientation.

Let S be a smooth surface. For any point (x, y, z) on S, we can identify two unit normal vectors $\mathbf{N}$ and $-\mathbf{N}$. If it is possible to choose a unit normal vector $\mathbf{N}$ at every point (x, y, z) on S so that $\mathbf{N}$ varies continuously over S, then S is "*orientable*." Such a choice of unit normal vector at each point gives the **orientation of a surface** S. If you think of the normal field as describing water flow, then the side of the surface that water flows toward is the "negative" side and the side of the surface at which the water flows away is the "positive" side. Informally, a choice of orientation gives S an "outer" side and an "inner" side (or an "upward" side and a "downward" side), just as a choice of orientation of a curve gives the curve "forward" and "backward" directions.

Closed surfaces such as spheres are orientable: if we choose the outward normal vector at each point on the surface of the sphere, then the unit normal vectors vary continuously. This is called the *positive orientation of the closed surface* (Figure 3.7.18). We also could choose the inward normal vector at each point to give an "inward" orientation, which is the negative orientation of the surface.

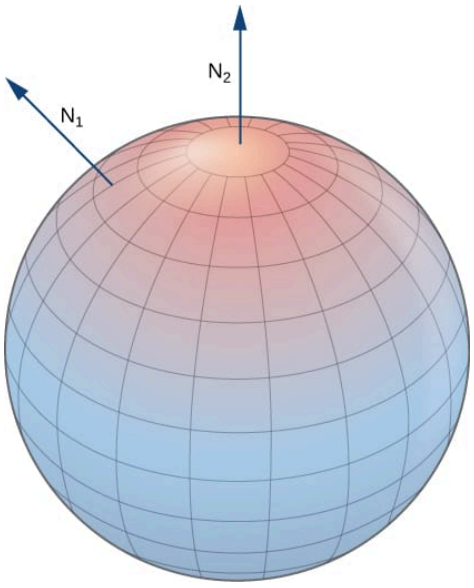


Figure 3.7.4: An oriented sphere with positive orientation. Two normal vectors $\mathbf{N}_1$ and $\mathbf{N}_2$ point outward.

A portion of the graph of any smooth function $z = f(x, y)$ is also orientable. If we choose the unit normal vector that points "above" the surface at each point, then the unit normal vectors vary continuously over the surface. We could also choose the unit normal vector that points "below" the surface at each point. To get such an orientation, we parameterize the graph of f in the standard way: $\mathbf{r}(x, y) = \langle x,\ y,\ f(x, y)\rangle$, where x and y vary over the domain of f. Then, $\mathbf{t}_x = \langle 1, 0, f_x\rangle$ and $\mathbf{t}_y = \langle 0, 1, f_y\rangle$, and therefore the cross product $\mathbf{t}_x \times \mathbf{t}_y$ (which is normal to the surface at any point on the surface) is $\langle -f_x,\ -f_y,\ 1\rangle$Since the z-component of this vector is one, the corresponding unit normal vector points "upward," and the upward side of the surface is chosen to be the

"positive" side.

Let S be a smooth orientable surface with parameterization $\mathbf{r}(u,v)$. For each point $\mathbf{r}(a,b)$ on the surface, vectors $\mathbf{t}_u$ and $\mathbf{t}_v$ lie in the tangent plane at that point. Vector $\mathbf{t}_u \times \mathbf{t}_v$ is normal to the tangent plane at $\mathbf{r}(a,b)$ and is therefore normal to S at that point. Therefore, the choice of unit normal vector

$$\mathbf{N} = \frac{\mathbf{t}_u \times \mathbf{t}_v}{||\mathbf{t}_u \times \mathbf{t}_v||} \tag{3.7.11}$$

gives an orientation of surface S.

Example 3.7.5:Choosing an Orientation

Give an orientation of cylinder $x^2 + y^2 = r^2,\ 0 \le z \le h$.

Solution

This surface has parameterization $\mathbf{r}(u,v) = \langle r\ \cos u,\ r\ \sin u,\ v\rangle,\ 0 \le u < 2\pi,\ 0 \le v \le h$.

The tangent vectors are $\mathbf{t}_u = \langle -r\ \sin u,\ r\ \cos u,\ 0\rangle$ and $\mathbf{t}_v = \langle 0,0,1\rangle$. To get an orientation of the surface, we compute the unit normal vector

$$\mathbf{N} = \frac{\mathbf{t}_u \times \mathbf{t}_v}{||\mathbf{t}_u \times \mathbf{t}_v||}$$

In this case, $\mathbf{t}_u \times \mathbf{t}_v = \langle r\ \cos u,\ r\ \sin u,\ 0\rangle$ and therefore

$$||\mathbf{t}_u \times \mathbf{t}_v|| = \sqrt{r^2\cos^2 u + r^2 \sin^2 u} = r.$$

An orientation of the cylinder is

$$\mathbf{N}(u,v) = \frac{\langle r\ \cos u,\ r\ \sin u,\ 0\rangle}{r} = \langle \cos u,\ \sin u,\ 0\rangle.$$

Notice that all vectors are parallel to the xy-plane, which should be the case with vectors that are normal to the cylinder. Furthermore, all the vectors point outward, and therefore this is an outward orientation of the cylinder (Figure 3.7.5).



Figure 3.7.5: A normal vector pointing outward at the middle the cylinder.

Exercise 3.7.4

Give the "upward" orientation of the graph of $f(x,y) = xy$.

Hint available at the end of this section.

Answer available at the end of this section.

Since every curve has a "forward" and "backward" direction (or, in the case of a closed curve, a clockwise and counterclockwise direction), it is possible to give an orientation to any curve. Hence, it is possible to think of every curve as an oriented curve. This is not the case with surfaces, however. Some surfaces cannot be oriented; such surfaces are called *nonorientable*. Essentially, a

surface can be oriented if the surface has an “inner” side and an “outer” side, or an “upward” side and a “downward” side. Some surfaces are twisted in such a fashion that there is no well-defined notion of an “inner” or “outer” side.

The classic example of a nonorientable surface is the Möbius strip. To create a Möbius strip, take a rectangular strip of paper, give the piece of paper a half-twist, and the glue the ends together (Figure 3.7.20). Because of the half-twist in the strip, the surface has no “outer” side or “inner” side. If you imagine placing a normal vector at a point on the strip and having the vector travel all the way around the band, then (because of the half-twist) the vector points in the opposite direction when it gets back to its original position. Therefore, the strip really only has one side.



Figure 3.7.6: A Möbius strip by taking a rectangular strip of paper, give the piece of paper a half-twist, and the glue the ends together.

Since some surfaces are *nonorientable*, it is not possible to define a vector surface integral on all piecewise smooth surfaces. This is in contrast to vector line integrals, which can be defined on any piecewise smooth curve.

Surface Integral of a Vector Field

With the idea of orientable surfaces in place, we are now ready to define a surface integral of a vector field. The definition is analogous to the definition of the flux of a vector field along a plane curve. Recall that if $\mathbf{F}$ is a two-dimensional vector field and C is a plane curve, then the definition of the flux of $\mathbf{F}$ along C involved chopping C into small pieces, choosing a point inside each piece, and calculating $\mathbf{F} \cdot \mathbf{N}$ at the point (where $\mathbf{N}$ is the unit normal vector at the point). The definition of a surface integral of a vector field proceeds in the same fashion, except now we chop surface S into small pieces, choose a point in the small (two-dimensional) piece, and calculate $\mathbf{F} \cdot \mathbf{N}$ at the point.

To place this definition in a real-world setting, let S be an oriented surface with unit normal vector $\mathbf{N}$. Let $\mathbf{v}$ be a velocity field of a fluid flowing through S, and suppose the fluid has density $\rho(x, y, z)$ Imagine the fluid flows through S, but S is completely permeable so that it does not impede the fluid flow (Figure 3.7.7). The mass flux of the fluid is the rate of mass flow per unit area. The mass flux is measured in mass per unit time per unit area. How could we calculate the mass flux of the fluid across S?

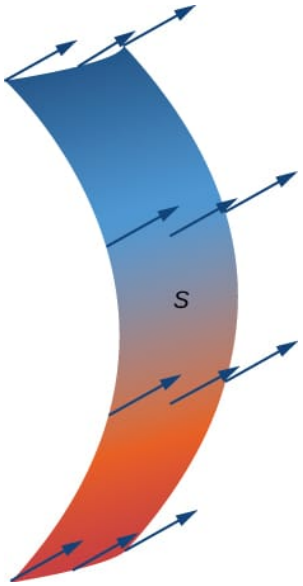


Figure 3.7.7: Fluid flows across a completely permeable surface S.

The rate of flow, measured in mass per unit time per unit area, is $\rho\mathbf{N}$. To calculate the mass flux across S, chop S into small pieces S_{ij}. If S_{ij} is small enough, then it can be approximated by a tangent plane at some point P in S_{ij}. Therefore, the unit normal vector at P can be used to approximate $\mathbf{N}(x, y, z)$ across the entire piece S_{ij} because the normal vector to a plane does not change as

we move across the plane. The component of the vector ρv at P in the direction of $\mathbf{N}$ is $\rho\mathbf{v} \cdot \mathbf{N}$ at P. Since S_{ij} is small, the dot product $\rho v \cdot N$ changes very little as we vary across S_{ij} and therefore $\rho\mathbf{v} \cdot \mathbf{N}$ can be taken as approximately constant across S_{ij}. To approximate the mass of fluid per unit time flowing across S_{ij} (and not just locally at point P), we need to multiply $(\rho\mathbf{v} \cdot \mathbf{N})(P)$ by the area of S_{ij}. Therefore, the mass of fluid per unit time flowing across S_{ij} in the direction of $\mathbf{N}$ can be approximated by $(\rho\mathbf{v} \cdot \mathbf{N})\Delta S_{ij}$ where $\mathbf{N}$, ρ and $\mathbf{v}$ are all evaluated at P (Figure 3.7.22). This is analogous to the flux of two-dimensional vector field $\mathbf{F}$ across plane curve C, in which we approximated flux across a small piece of C with the expression $(\mathbf{F} \cdot \mathbf{N})\ \Delta s$. To approximate the mass flux across S, form the sum

$$\sum_{i=1} m \sum_{j=1}^{n} (\rho\mathbf{v} \cdot \mathbf{N})\Delta S_{ij}.$$

As pieces S_{ij} get smaller, the sum

$$\sum_{i=1} m \sum_{j=1}^{n} (\rho\mathbf{v} \cdot \mathbf{N})\Delta S_{ij}$$

gets arbitrarily close to the mass flux. Therefore, the mass flux is

$$\iint_s \rho\mathbf{v} \cdot \mathbf{N}\, dS = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} (\rho\mathbf{v} \cdot \mathbf{N})\Delta S_{ij}. \tag{3.7.12}$$

This is a surface integral of a vector field. Letting the vector field $\rho\mathbf{v}$ be an arbitrary vector field $\mathbf{F}$ leads to the following definition.

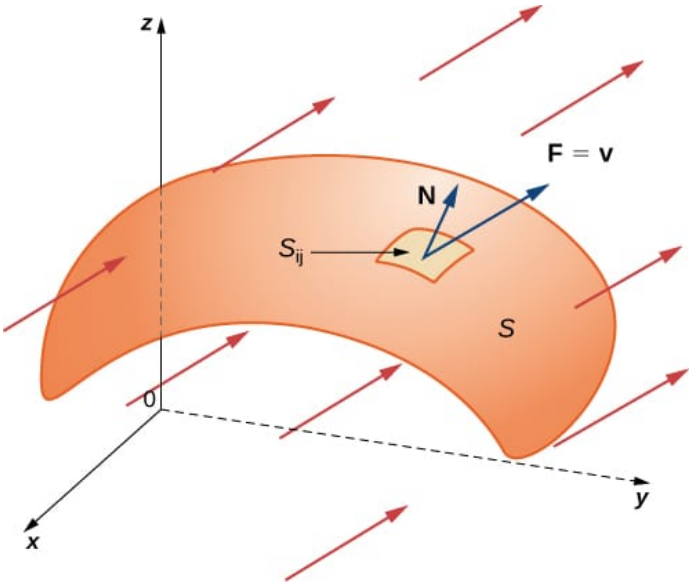


Figure 3.7.8: The mass of fluid per unit time flowing across S_{ij} in the direction of $\mathbf{N}$ can be approximated by $(\rho\mathbf{v} \cdot \mathbf{N})\Delta S_{ij}$.

Definition: Surface Integrals

Let $\mathbf{F}$ be a continuous vector field with a domain that contains oriented surface S with unit normal vector $\mathbf{N}$. The surface integral of $\mathbf{F}$ over S is

$$\iint_S \mathbf{F} \cdot \mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{N}\, dS. \tag{3.7.13}$$

Notice the parallel between this definition and the definition of vector line integral $\int_C \mathbf{F} \cdot \mathbf{N}\ dS$. A surface integral of a vector field is defined in a similar way to a flux line integral across a curve, except the domain of integration is a surface (a two-dimensional object) rather than a curve (a one-dimensional object). Integral $\iint_S \mathbf{F} \cdot \mathbf{N}\ dS$ is called the *flux of* $\mathbf{F}$ *across* S, just as integral $\int_C \mathbf{F} \cdot \mathbf{N}\ dS$ is the flux of $\mathbf{F}$ across curve C. A surface integral over a vector field is also called a **flux integral**.

Just as with vector line integrals, surface integral $\iint_S \mathbf{F} \cdot \mathbf{N}\ dS$ is easier to compute after surface S has been parameterized. Let $\mathbf{r}(u, v)$ be a parameterization of S with parameter domain D. Then, the unit normal vector is given by $\mathbf{N} = \dfrac{\mathbf{t}_u \times \mathbf{t}_v}{||\mathbf{t}_u \times \mathbf{t}_v||}$ and we have

$$\begin{aligned}\int_C \mathbf{F} \cdot \mathbf{N}\, dS &= \iint_S \mathbf{F} \cdot \frac{\mathbf{t}_u \times \mathbf{t}_v}{||\mathbf{t}_u \times \mathbf{t}_v||}\, dS \\ &= \iint_D \left(\mathbf{F}(\mathbf{r}(u,v)) \cdot \frac{\mathbf{t}_u \times \mathbf{t}_v}{||\mathbf{t}_u \times \mathbf{t}_v||}\right) ||\mathbf{t}_u \times \mathbf{t}_v||\, dA \\ &= \iint_D (\mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{t}_u \times \mathbf{t}_v))\, dA.\end{aligned}$$

Therefore, to compute a surface integral over a vector field we can use the equation

$$\iint_S \mathbf{F} \cdot \mathbf{N}\, dS = \iint_D (\mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{t}_u \times \mathbf{t}_v))\, dA. \tag{3.7.14}$$

Example 3.7.6: Calculating a Surface Integral

Calculate the surface integral

$$\iint_S \mathbf{F} \cdot \mathbf{N}\, dS,$$

where $\mathbf{F} = \langle -y, x, 0\rangle$ and S is the surface with parameterization

$$\mathbf{r}(u, v) = \langle u, v^2 - u,\ u + v\rangle,\ 0 \le u \le 3,\ 0 \le v \le 4.$$

Solution

The tangent vectors are $\mathbf{t}_u = \langle 1, -1, 1\rangle$ and $\mathbf{t}_v = \langle 0, 2v, 1\rangle$. Therefore,

$$\mathbf{t}_u \times \mathbf{t}_v = \langle -1 - 2v, -1, 2v\rangle.$$

By Equation 3.7.14,

$$\begin{aligned}\iint_S \mathbf{F} \cdot dS &= \int_0^4 \int_0^3 F(\mathbf{r}(u,v)) \cdot (\mathbf{t}_u \times \mathbf{t}_v)\, du\, dv \\ &= \int_0^4 \int_0^3 \langle u - v^2,\ u,\ 0\rangle \cdot \langle -1 - 2v,\ -1,\ 2v\rangle\, du\, dv \\ &= \int_0^4 \int_0^3 [(u - v^2)(-1 - 2v) - u]\, du\, dv \\ &= \int_0^4 \int_0^3 (2v^3 + v^2 - 2uv - 2u)\, du\, dv \\ &= \int_0^4 [2v^3 u + v^2 u - v u^2 - u^2]\Big|_0^3\, dv \\ &= \int_0^4 (6v^3 + 3v^2 - 9v - 9)\, dv \\ &= \left[\frac{3v^4}{2} + v^3 - \frac{9v^2}{2} - 9v\right]_0^4 \\ &= 340.\end{aligned}$$

Therefore, the flux of $\mathbf{F}$ across S is 340.

Exercise 3.7.5

Calculate surface integral

$$\iint_S \mathbf{F} \cdot \mathbf{N}\, dS,$$

where $\mathbf{F} = \langle 0, -z, y \rangle$ and S is the portion of the unit sphere in the first octant with outward orientation.

Answer available at the end of this section.

Example 3.7.7: Calculating Mass Flow Rate

Let $\mathbf{v}(x, y, z) = \langle 2x,\ 2y,\ z \rangle$ represent a velocity field (with units of meters per second) of a fluid with constant density 80 kg/m^3. Let S be hemisphere $x^2 + y^2 + z^2 = 9$ with $z \geq 0$ such that S is oriented outward. Find the mass flow rate of the fluid across S.

Solution

A parameterization of the surface is

$$\mathbf{r}(\phi, \theta) = \langle 3\cos\theta \sin\phi,\ 3\sin\theta\sin\phi,\ 3\cos\phi \rangle,\ 0 \leq \theta \leq 2\pi,\ 0 \leq \phi \leq \pi/2. \tag{3.7.15}$$

As in Example, the tangent vectors are $\mathbf{t}_\theta = \langle -3\ \sin\theta\ \sin\phi,\ 3\ \cos\theta\ \sin\phi,\ 0 \rangle$ and $\mathbf{t}_\phi = \langle 3\ \cos\theta\ \cos\phi,\ 3\ \sin\theta\ \cos\phi,\ -3\ \sin\phi \rangle$, and their cross product is

$$\mathbf{t}_\phi \times \mathbf{t}_\theta = \langle 9\cos\theta\sin^2\phi,\ 9\sin\theta\sin^2\phi,\ 9\sin\phi\cos\phi \rangle. \tag{3.7.16}$$

Notice that each component of the cross product is positive, and therefore this vector gives the outward orientation. Therefore we use the orientation

$\mathbf{N} = \langle 9\ \cos\theta\ \sin^2\phi,\ 9\ \sin\theta\ \sin^2\phi,\ 9\ \sin\phi\ \cos\phi \rangle$

for the sphere.

By Equation 3.7.14,

$$\begin{aligned}
\iint_S \rho v \cdot dS &= 80 \int_0^{2\pi} \int_0^{\pi/2} v(r(\phi,\theta)) \cdot (t_\phi \times t_\theta)\, d\phi\, d\theta \\
&= 80 \int_0^{2\pi} \int_0^{\pi/2} \langle 6\cos\theta\sin\phi,\ 6\sin\theta\sin\phi,\ 3\cos\phi \rangle \cdot \langle 9\cos\theta\sin^2\phi,\ 9\sin\theta\sin^2\phi,\ 9\sin\phi\cos\phi \rangle\, d\phi\, d\theta \\
&= 80 \int_0^{2\pi} \int_0^{\pi/2} 54\sin^3\phi + 27\cos^2\phi\sin\phi\, d\phi\, d\theta \\
&= 80 \int_0^{2\pi} \int_0^{\pi/2} 54(1-\cos^2\phi)\sin\phi + 27\cos^2\phi\sin\phi\, d\phi\, d\theta \\
&= 80 \int_0^{2\pi} \int_0^{\pi/2} 54\sin\phi - 27\cos^2\phi\sin\phi\, d\phi\, d\theta \\
&= 80 \int_0^{2\pi} \Big[-54\cos\phi + 9\cos^3\phi \Big]_{\phi=0}^{\phi=2\pi} d\theta \\
&= 80 \int_0^{2\pi} 45\, d\theta \\
&= 7200\pi.
\end{aligned}$$

Therefore, the mass flow rate is 7200π kg/sec/m^2.

Exercise 3.7.6

Let $\mathbf{v}(x,y,z) = \langle x^2+y^2,\ z,\ 4y\rangle$ m/sec represent a velocity field of a fluid with constant density 100 kg/m^3. Let S be the half-cylinder $\mathbf{r}(u,v) = \langle \cos u,\ \sin u,\ v\rangle,\ 0 \le u \le \pi,\ 0 \le v \le 2$ oriented outward. Calculate the mass flux of the fluid across S.

Hint available at the end of this section.

Answer available at the end of this section.

If we want to find the flow rate (measured in volume per time), we can use flux integral

$$\iint_S \mathbf{v} \cdot \mathbf{N}\, dS, \tag{3.7.17}$$

which leaves out the density. Since the flow rate of a fluid is measured in volume per unit time, flow rate does not take mass into account. Therefore, we have the following characterization of the flow rate of a fluid with velocity $\mathbf{v}$ across a surface S:

$$\text{Flow rate of fluid across S} = \iint_S \mathbf{v}\cdot dS.$$

To compute the flow rate of the fluid in Example, we simply remove the density constant, which gives a flow rate of $90\pi\ m^3/sec$.

Both mass flux and flow rate are important in physics and engineering. Mass flux measures how much mass is flowing across a surface; flow rate measures how much volume of fluid is flowing across a surface.

In addition to modeling fluid flow, surface integrals can be used to model heat flow. Suppose that the temperature at point (x,y,z) in an object is $T(x,y,z)$. Then the heat flow is a vector field proportional to the negative temperature gradient in the object. To be precise, the heat flow is defined as vector field $F = -k\nabla T$, where the constant *k* is the *thermal conductivity* of the substance from which the object is made (this constant is determined experimentally). The rate of heat flow across surface S in the object is given by the flux integral

$$\iint_S \mathbf{F}\cdot dS = \iint_S -k\nabla T\cdot dS. \tag{3.7.18}$$

Example 3.7.8: Calculating Heat Flow

A cast-iron solid cylinder is given by inequalities $x^2+y^2 \le 1,\ 1 \le z \le 4$. The temperature at point (x,y,z) in a region containing the cylinder is $T(x,y,z) = (x^2+y^2)z$. Given that the thermal conductivity of cast iron is 55, find the heat flow across the boundary of the solid if this boundary is oriented outward.

Solution

Let S denote the boundary of the object. To find the heat flow, we need to calculate flux integral

$$\iint_S -k\nabla T\cdot dS. \tag{3.7.19}$$

Notice that S is not a smooth surface but is piecewise smooth, since S is the union of three smooth surfaces (the circular top and bottom, and the cylindrical side). Therefore, we calculate three separate integrals, one for each smooth piece of S. Before calculating any integrals, note that the gradient of the temperature is $\nabla T = \langle 2xz,\ 2yz,\ x^2+y^2\rangle$.

First we consider the circular bottom of the object, which we denote S_1. We can see that S_1 is a circle of radius 1 centered at point $(0,0,1)$ sitting in plane $z=1$. This surface has parameterization $\mathbf{r}(u,v) = \langle v\ \cos u,\ v\ \sin u,\ 1\rangle,\ 0 \le u < 2\pi,\ 0 \le v \le 1.$

Therefore,

$\mathbf{t}_u = \langle -v\ \sin u,\ v\ \cos u,\ 0\rangle$ and $\mathbf{t}_v = \langle \cos u,\ v\ \sin u,\ 0\rangle$, and $\mathbf{t}_u \times \mathbf{t}_v = \langle 0,\ 0, -v\ \sin^2 u - v\ \cos^2 u\rangle = \langle 0,\ 0, -v\rangle$.

Since the surface is oriented outward and S_1 is the bottom of the object, it makes sense that this vector points downward. By Equation, the heat flow across S_1 is

$$\begin{aligned}\iint_{S_1} -k\nabla T \cdot dS &= -55\int_0^{2\pi}\int_0^1 \nabla T(u,v)\cdot(\mathbf{t}_u\times\mathbf{t}_v)\,dv\,du\\ &= -55\int_0^{2\pi}\int_0^1 \langle 2v\,\cos u,\,2v\,\sin u,\,v^2\cos^2 u+v^2\sin^2 u\rangle\cdot\langle 0,0,-v\rangle\,dv\,du\\ &= -55\int_0^{2\pi}\int_0^1 \langle 2v\,\cos u,\,2v\,\sin u,\,v^2\rangle\cdot\langle 0,0,-v\rangle\,dv\,du\\ &= -55\int_0^{2\pi}\int_0^1 -v^3\,dv\,du\\ &= -55\int_0^{2\pi} -\frac{1}{4}du\\ &= \frac{55\pi}{2}.\end{aligned}$$

Now let's consider the circular top of the object, which we denote S_2. We see that S_2 is a circle of radius 1 centered at point $(0,0,4)$, sitting in plane $z=4$. This surface has parameterization $\mathbf{r}(u,v)=\langle v\ \cos u,\ v\ \sin u,\ 4\rangle,\ 0\le u<2\pi,\ 0\le v\le 1$.

Therefore, $\mathbf{t}_u=\langle -v\ \sin u,\ v\ \cos u,\ 0\rangle$ and $\mathbf{t}_v=\langle\cos u,\ v\ \sin u,\ 0\rangle$, and $\mathbf{t}_u\times\mathbf{t}_v=\langle 0,\ 0,-v\ \sin^2 u-v\ \cos^2 u\rangle=\langle 0,0,-v\rangle$.

Since the surface is oriented outward and S_1 is the top of the object, we instead take vector $\mathbf{t}_v\times\mathbf{t}_u=\langle 0,0,v\rangle$. By Equation, the heat flow across S_1 is

$$\begin{aligned}\iint_{S_2} -k\nabla T \cdot dS &= -55\int_0^{2\pi}\int_0^1 \nabla T(u,v)\cdot\ (\mathbf{t}_u\times\mathbf{t}_v)\,dv\,du\\ &= -55\int_0^{2\pi}\int_0^1 \langle 8v\,\cos u,\,8v\,\sin u,\,v^2\cos^2 u+v^2\sin^2 u\rangle\cdot\langle 0,0,-v\rangle\,dv\,du\\ &= -55\int_0^{2\pi}\int_0^1 \langle 8v\,\cos u,\,8v\,\sin u,\,v^2\rangle\cdot\langle 0,0,-v\rangle\ dv\,du\\ &= -55\int_0^{2\pi}\int_0^1 -v^3\,dv\,du = -55\int_0^{2\pi} -\frac{1}{4}\,du = -\frac{55\pi}{2}.\end{aligned}$$

Last, let's consider the cylindrical side of the object. This surface has parameterization $\mathbf{r}(u,v)=\langle\cos u,\ \sin u,\ v\rangle,\ 0\le u<2\pi,\ 1\le v\le 4$. By Example, we know that $\mathbf{t}_u\times\mathbf{t}_v=\langle\cos u,\ \sin u,\ 0\rangle$. By Equation,

$$\begin{aligned}\iint_{S_3} -k\nabla T \cdot dS &= -55\int_0^{2\pi}\int_1^4 \nabla T(u,v)\cdot(\mathbf{t}_u\times\mathbf{t}_v)\,dv\,du\\ &= -55\int_0^{2\pi}\int_1^4 \langle 2v\,\cos u,\,2v\,\sin u,\,\cos^2 u+\sin^2 u\rangle\cdot\langle\cos u,\,\sin u,\,0\rangle\,dv\,du\\ &= -55\int_0^{2\pi}\int_0^1 \langle 2v\,\cos^2 u,\,2v\,\sin u,\,1\rangle\cdot\langle\cos u,\,\sin u,\,0\rangle\,dv\ du\\ &= -55\int_0^{2\pi}\int_0^1 (2v\,\cos^2 u+2v\,\sin^2 u)\,dv\,du\\ &= -55\int_0^{2\pi}\int_0^1 2v\,dv\,du\\ &= -55\int_0^{2\pi} du\\ &= -110\pi.\end{aligned}$$

Therefore, the rate of heat flow across S is

$$\frac{55\pi}{2} - \frac{55\pi}{2} - 110\pi = -110\pi.$$

Exercise 3.7.7

A cast-iron solid ball is given by inequality $x^2 + y^2 + z^2 \leq 1$. The temperature at a point in a region containing the ball is $T(x, y, z) = \frac{1}{3}(x^2 + y^2 + z^2)$. Find the heat flow across the boundary of the solid if this boundary is oriented outward.

Answer available at the end of this section.

Key Concepts

- Surfaces can sometimes be oriented, just as curves can be oriented. Some surfaces, such as a Möbius strip, cannot be oriented.
- A surface integral is like a line integral in one higher dimension. The domain of integration of a surface integral is a surface in a plane or space, rather than a curve in a plane or space.
- The integrand of a surface integral can be a scalar function or a vector field. To calculate a surface integral with an integrand that is a function, use Equation. To calculate a surface integral with an integrand that is a vector field, use Equation.

Key Equations

- **Scalar surface integral**

$$\iint_S f(x, y, z)\, dS = \iint_D f(\mathbf{r}(u, v))||\mathbf{t}_u \times \mathbf{t}_v||\, dA$$

- **Flux integral**

$$\iint_S \mathbf{F} \cdot \mathbf{N}\, dS = \iint_S \mathbf{F} \cdot dS = \iint_D \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{t}_u \times \mathbf{t}_v)\, dA$$

Glossary

Flux integral
another name for a surface integral of a vector field; the preferred term in physics and engineering

Grid curves
curves on a surface that are parallel to grid lines in a coordinate plane

Heat flow
a vector field proportional to the negative temperature gradient in an object

Mass flux
the rate of mass flow of a fluid per unit area, measured in mass per unit time per unit area

Orientation of a surface
if a surface has an “inner” side and an “outer” side, then an orientation is a choice of the inner or the outer side; the surface could also have “upward” and “downward” orientations

Surface integral
an integral of a function over a surface

Surface integral of a scalar-valued function
a surface integral in which the integrand is a scalar function

Surface integral of a vector field
a surface integral in which the integrand is a vector field

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Answers

Exercise 3.7.1
Use Equation 3.7.2.

Exercise 3.7.1
24

Exercise 3.7.2
Break the integral into three separate surface integrals.

Exercise 3.7.2
0

Exercise 3.7.3
$38.401\pi \approx 120.640$

Exercise 3.7.4
Parameterize the surface and use the fact that the surface is the graph of a function.

Exercise 3.7.4

$$\mathbf{N}(x,y) = \left\langle \frac{-y}{\sqrt{1+x^2+y^2}}, \frac{-x}{\sqrt{1+x^2+y^2}}, \frac{1}{\sqrt{1+x^2+y^2}} \right\rangle$$

Exercise 3.7.5
0

Exercise 3.7.6
Use Equation 3.7.14.

Exercise 3.7.6
400 kg/sec/m

Exercise 3.7.7
$-\frac{440\pi}{3}$

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3.7E: Exercises

Exercises 1 - 3

Let S be the hemisphere $x^2 + y^2 + z^2 = 4$, with $z \geq 0$, and evaluate each surface integral, in the counterclockwise direction.

1. $\iint_S z \; dS$

Answer

$$\iint_S z \; dS = 8\pi$$

2. $\iint_S (x - 2y) \; dS$

3. $\iint_S (x^2 + y^2) \; dS$

Answer

$$\iint_S (x^2 + y^2) \; dS = \frac{64}{3\pi}$$

Exercises 4 - 6

Evaluate $\int \int_S \mathbf{F} \cdot \mathbf{N} \; dS$ for vector field $\mathbf{F}$ where $\mathbf{N}$ is an outward normal vector to surface S.

4. $\mathbf{F}(x, y, z) = x \; \mathbf{i} + 2y \; \mathbf{j} = 3z \; \mathbf{k}$, and S is that part of plane $15x - 12y + 3z = 6$ that lies above unit square $0 \leq x \leq 1, \;\; 0 \leq y \leq 1$.

5. $\mathbf{F}(x, y) = x \; \mathbf{i} + y \; \mathbf{j}$, and S is hemisphere $z = \sqrt{1 - x^2 - y^2}$.

Answer

$$\iint_S \mathbf{F} \cdot \mathbf{N} \; dS = \frac{4\pi}{3}$$

6. $\mathbf{F}(x, y, z) = x^2 \; \mathbf{i} + y^2 \; \mathbf{j} + z^2 \; \mathbf{k}$, and S is the portion of plane $z = y + 1$ that lies inside cylinder $x^2 + y^2 = 1$.

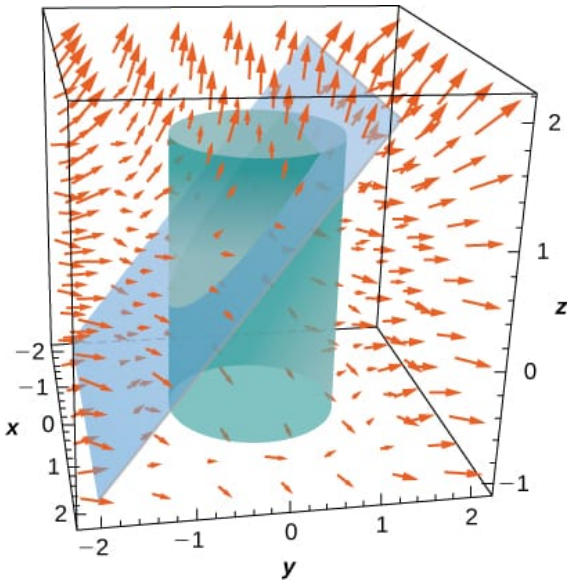


Figure $3.7E.1$: S is the portion of plane $z = y + 1$ that lies inside cylinder $x^2 + y^2 = 1$.

Exercises 7 - 9

Approximate the mass of the homogeneous lamina that has the shape of given surface S. Round to four decimal places.

7. S is surface $z = 4 - x - 2y$, with $z \geq 0,\ \ x \geq 0,\ \ y \geq 0;\ \ \xi = x.$

Answer

$m \approx 13.0639$

8. S is surface $z = x^2 + y^2$, with $z \leq 1;\ \ \xi = z.$

9. S is surface $x^2 + y^2 + x^2 = 5$, with $z \geq 1;\ \ \xi = \theta^2.$

Answer

$m \approx 228.5313$

10. Evaluate surface integral $\iint_S g\ dS$, where $g(x, y, z) = xz + 2x^2 - 3xy$ and S is the portion of plane $2x - 3y + z = 6$ that lies over unit square $R : 0 \leq x \leq 1,\ \ 0 \leq y \leq 1.$

Answer

$\iint_S g\ dS = 3\sqrt{14}$

11. Evaluate $\iint_S (x + y + z)\ dS$, where S is the surface defined parametrically by $\mathbf{R}(u, v) = (2u + v)\ \mathbf{i} + (u - 2v)\ \mathbf{j} + (u + 3v)\ \mathbf{k}$ for $0 \leq u \leq 1$, and $0 \leq v \leq 2.$

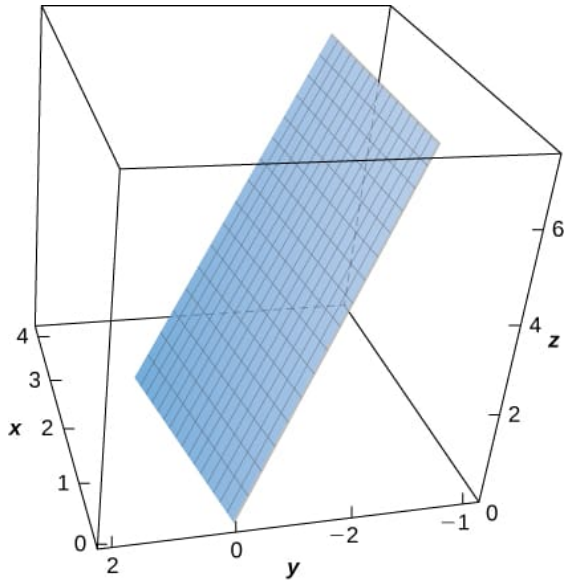


Figure $3.7E.2$: S is the surface defined parametrically by $\mathbf{R}(u,v)$.

12. Evaluate $\iint_S (x - y^2 + z)\ dS$, where S is the surface defined parametrically by $\mathbf{R}(u,v) = u^2\ \mathbf{i} + v\ \mathbf{j} + u\ \mathbf{k}$ for $0 \le u \le 1,\ 0 \le v \le 1$.

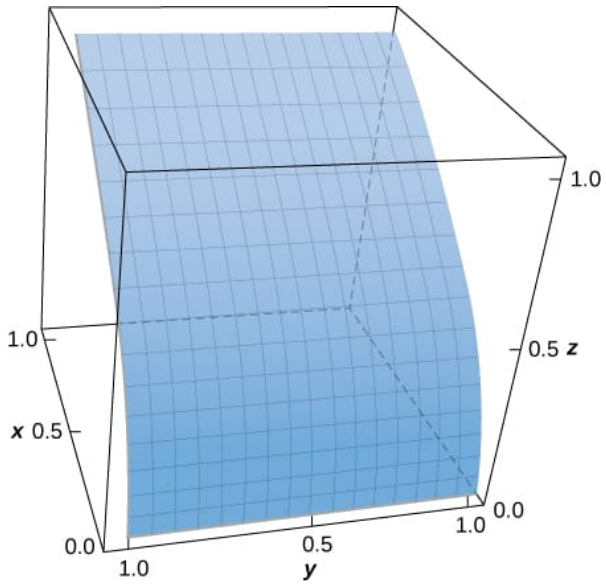


Figure $3.7E.3$: S is the surface defined parametrically by $\mathbf{R}(u,v)$.

Answer

$$\iint_S (x^2 + y - z)\ dS \approx 0.9617$$

13. Evaluate where S is the surface defined by $\mathbf{R}(u,v) = u\ \mathbf{i} - u^2\ \mathbf{j} + v\ \mathbf{k},\ 0 \le u \le 2,\ 0 \le v \le 1$ for $0 \le u \le 1,\ 0 \le v \le 2$.

14. Evaluate $\iint_S (x^2 + y^2)\ dS$, where S is the surface bounded above hemisphere $z = \sqrt{1 - x^2 - y^2}$, and below by plane $z = 0$.

Answer

$$\iint_S (x^2 + y^2)\ dS = \frac{4\pi}{3}$$

15. Evaluate $\iint_S (x^2 + y^2 + z^2)\ dS$, where S is the portion of plane that lies inside cylinder $x^2 + y^2 = 1$.

16. Evaluate $\iint_S x^2 z \, dS$, where S is the portion of cone $z^2 = x^2 + y^2$ that lies between planes $z = 1$ and $z = 4$.

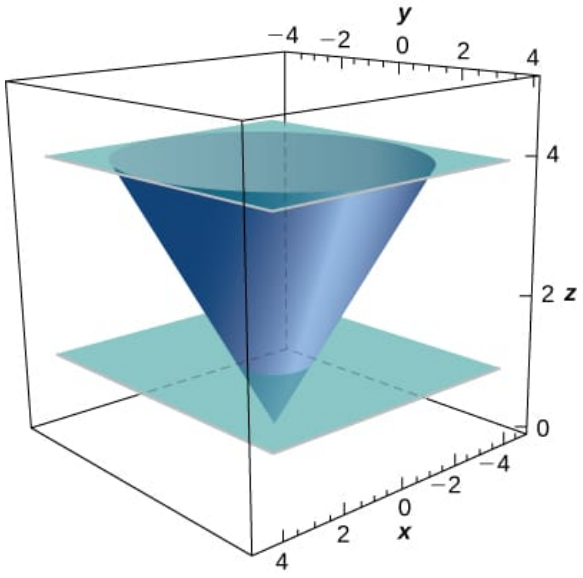


Figure $3.7E.4$: S is the portion of cone $z^2 = x^2 + y^2$ that lies between planes $z = 1$ and $z = 4$.

Answer

$$\iint_S x^2 z dS = \frac{1023\sqrt{2}\pi}{5}$$

17. Evaluate $\iint_S \frac{xz}{y} \, dS$, where S is the portion of cylinder $x = y^2$ that lies in the first octant between planes $z = 0, \ z = 5$, and $y = 4$.

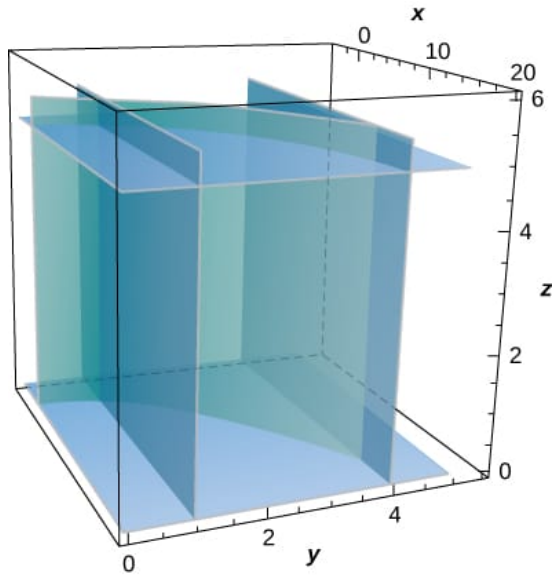


Figure $3.7E.5$: S is the portion of cylinder $x = y^2$ that lies in the first octant between planes $z = 0, \ z = 5$, and $y = 4$.

18. Evaluate $\iint_S (z + y) \, dS$, where S is the part of the graph of $z = \sqrt{1 - x^2}$ in the first octant between the xy-plane and plane $y = 3$.

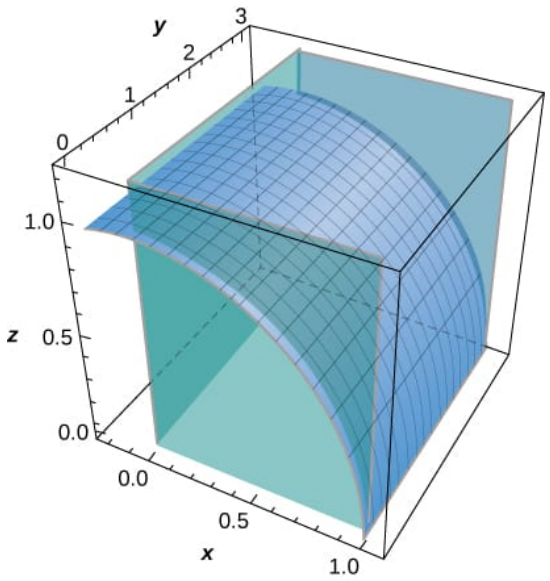


Figure $3.7E.6$: S is the part of the graph of $z = \sqrt{1 - x^2}$ in the first octant between the xy-plane and plane $y = 3$.

Answer

$$\iint_S (z + y)\ dS \approx 10.1$$

19. Evaluate $\iint_S xyz\ dS$ if S is the part of plane $z = x + y$ that lies over the triangular region in the xy-plane with vertices (0, 0, 0), (1, 0, 0), and (0, 2, 0).

20. Find the mass of a lamina of density $\xi(x, y, z) = z$ in the shape of hemisphere $z = (a^2 - x^2 - y^2)^{1/2}$.

Answer

$$m = \pi a^3$$

21. Compute $\int\int_S \mathbf{F} \cdot \mathbf{N}\ dS$, where $\mathbf{F}(x, y, z) = x\ \mathbf{i} - 5y\ \mathbf{j} + 4z\ \mathbf{k}$ and $\mathbf{N}$ is an outward normal vector S, where S is the union of two squares $S_1 : x = 0,\ 0 \le y \le 1,\ 0 \le z \le 1$ and $S_2\ :\ z = 1,\ 0 \le x \le 1,\ 0 \le y \le 1$.

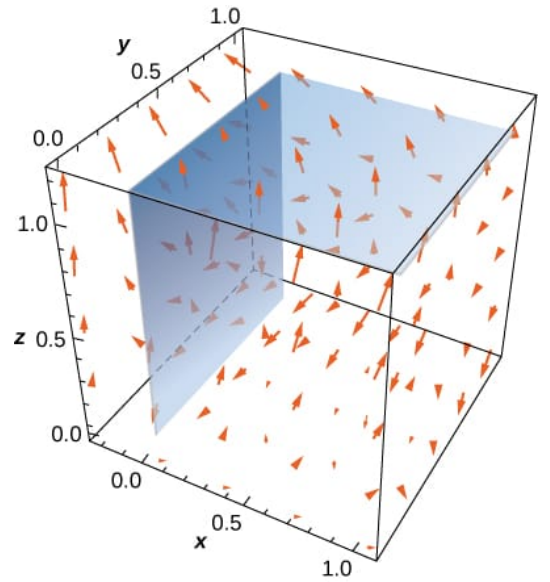


Figure $3.7E.7$: S is the union of two squares $S_1 : x = 0,\ 0 \le y \le 1,\ 0 \le z \le 1$ and $S_2\ :\ z = 1,\ 0 \le x \le 1,\ 0 \le y \le 1$.

22. Compute $\int\int_S \mathbf{F} \cdot \mathbf{N}\ dS$, where $\mathbf{F}(x, y, z) = xy\ \mathbf{i} + z\ \mathbf{j} + (x + y)\ \mathbf{k}$ and $\mathbf{N}$ is an outward normal vector S, where S is the triangular

region cut off from plane $x + y + z = 1$ by the positive coordinate axes.

Answer

$$\iint_S \mathbf{F} \cdot \mathbf{N}\ dS = \frac{13}{24}$$

23. Compute $\int\int_S \mathbf{F} \cdot \mathbf{N}\ dS$, where $\mathbf{F}(x, y, z) = 2yz\ \mathbf{i} + (\tan^{-1} xz)\ \mathbf{j} + e^{xy}\ \mathbf{k}$ and $\mathbf{N}$ is an outward normal vector S, where S is the surface of sphere $x^2 + y^2 + z^2 = 1$.

24. Compute $\int\int_S \mathbf{F} \cdot \mathbf{N}\ dS$, where $\mathbf{F}(x, y, z) = xyz\ \mathbf{i} + xyz\ \mathbf{j} + xyz\ \mathbf{k}$ and $\mathbf{N}$ is an outward normal vector S, where S is the surface of the five faces of the unit cube $0 \le x \le 1,\ \ 0 \le y \le 1,\ \ 0 \le z \le 1$ missing $z = 0$.

Answer

$$\iint_S \mathbf{F} \cdot \mathbf{N}\ dS = \frac{3}{4}$$

Exercises 25 - 26

Express the surface integral as an iterated double integral by using a projection on S on the yz-plane.

25. $\iint_S xy^2z^3\ dS$; S is the first-octant portion of plane $2x + 3y + 4z = 12$.

26. $\iint_S (x^2 - 2y + z)\ dS$; S is the portion of the graph of $4x + y = 8$ bounded by the coordinate planes and plane $z = 6$.

Answer

$$\int_0^8 \int_0^6 \left(4 - 3y + \frac{1}{16}y^2 + z\right)\left(\frac{1}{4}\sqrt{17}\right)\ dz\ dy$$

Exercises 27 - 28

Express the surface integral as an iterated double integral by using a projection on S on the xz-plane.

27. $\iint_S xy^2z^3\ dS$; S is the first-octant portion of plane $2x + 3y + 4z = 12$.

28. $\iint_S (x^2 - 2y + z)\ dS$; is the portion of the graph of $4x + y = 8$ bounded by the coordinate planes and plane $z = 6$.

Answer

$$\int_0^2 \int_0^6 \left[x^2 - 2(8 - 4x) + z\right]\sqrt{17}\ dz\ dx$$

29. Evaluate surface integral $\iint_S yz\ dS$, where S is the first-octant part of plane $x + y + z = \lambda$, where λ is a positive constant.

30. Evaluate surface integral $\iint_S (x^2z + y^2z)\ dS$, where S is hemisphere $x^2 + y^2 + z^2 = a^2,\ \ z \ge 0$.

Answer

$$\iint_S (x^2z + y^2z)\ dS = \frac{\pi a^5}{2}$$

31. Evaluate surface integral $\iint_S z\ dA$, where S is surface $z = \sqrt{x^2 + y^2},\ \ 0 \le z \le 2$.

32. Evaluate surface integral $\iint_S x^2yz\ dS$, where S is the part of plane $z = 1 + 2x + 3y$ that lies above rectangle $0 \le x \le 3$ and 0

$\leq y \leq 2$.

Answer

$$\iint_S x^2yz \; dS = 171\sqrt{14}$$

33. Evaluate surface integral $\iint_S yz \; dS$, where S is plane $x + y + z = 1$ that lies in the first octant.

34. Evaluate surface integral $\iint_S yz \; dS$, where S is the part of plane $z = y + 3$ that lies inside cylinder $x^2 + y^2 = 1$.

Answer

$$\iint_S yz \; dS = \frac{\sqrt{2}\pi}{4}$$

Exercises 35 - 37

Use geometric reasoning to evaluate the given surface integrals.

35. $\iint_S \sqrt{x^2 + y^2 + z^2} \; dS$, where S is surface $x^2 + y^2 + z^2 = 4, \;\; z \geq 0$

36. $\iint_S (x \; \mathbf{i} + y \; \mathbf{j}) \cdot dS$, where S is surface $x^2 + y^2 = 4, \;\; 1 \leq z \leq 3$, oriented with unit normal vectors pointing outward

Answer

$$\iint_S (x \; \mathbf{i} + y \; \mathbf{j}) \cdot dS = 16\pi$$

37. $\iint_S (z \; \mathbf{k}) \cdot dS$, where S is disc $x^2 + y^2 \leq 9$ on plane $z = 4$ oriented with unit normal vectors pointing upward

38. A lamina has the shape of a portion of sphere $x^2 + y^2 + z^2 = a^2$ that lies within cone $z = \sqrt{x^2 + y^2}$. Let S be the spherical shell centered at the origin with radius *a*, and let C be the right circular cone with a vertex at the origin and an axis of symmetry that coincides with the z-axis. Determine the mass of the lamina if $\delta(x, y, z) = x^2y^2z$.

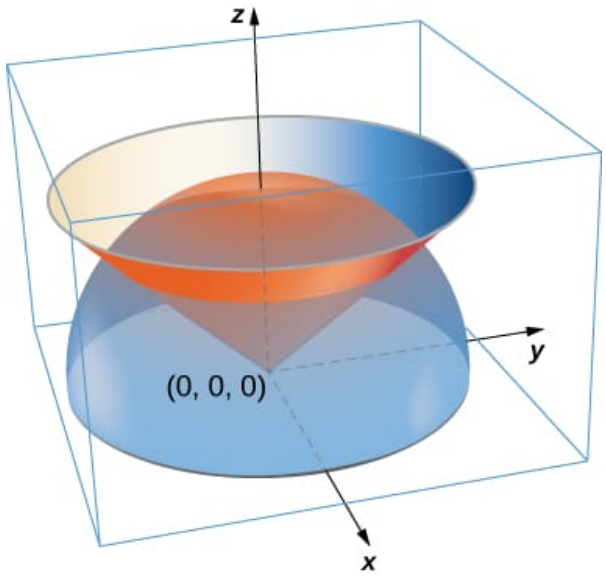


Figure $3.7E.8$: The image shows a sphere and a cone intersecting.

Answer

$$m = \frac{\pi a^7}{192}$$

39. A lamina has the shape of a portion of sphere $x^2 + y^2 + z^2 = a^2$ that lies within cone $z = \sqrt{x^2 + y^2}$. Let S be the spherical shell centered at the origin with radius *a*, and let C be the right circular cone with a vertex at the origin and an axis of symmetry that coincides with the *z*-axis. Suppose the vertex angle of the cone is ϕ_0, with $0 \leq \phi_0 < \frac{\pi}{2}$. Determine the mass of that portion of the shape enclosed in the intersection of S and C. Assume $\delta(x, y, z) = x^2 y^2 z$.

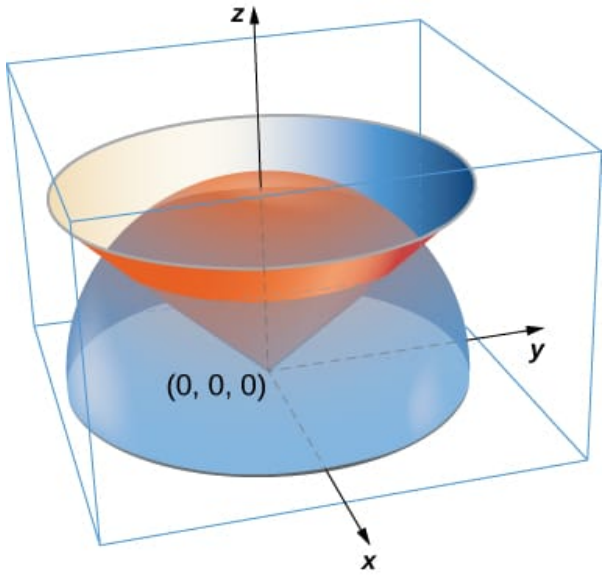


Figure $3.7E.9$: The image shows a sphere and a cone intersecting.

40. A paper cup has the shape of an inverted right circular cone of height 6 in. and radius of top 3 in. If the cup is full of water weighing $62.5\ lb/ft^3$, find the magnitude of the total force exerted by the water on the inside surface of the cup.

Answer

$F \approx 4.57\ lb$

Exercises 41 - 42

The heat flow vector field for conducting objects is $\mathbf{F} = -k\nabla T$, where $T(x, y, z)$ is the temperature in the object and $k > 0$ is a constant that depends on the material. Find the outward flux of $\mathbf{F}$ across the following surfaces S for the given temperature distributions and assume $k = 1$.

41. $T(x, y, z) = 100e^{-x-y}$; S consists of the faces of cube $|x| \leq 1,\ |y| \leq 1,\ |z| \leq 1$.

42. $T(x, y, z) = -\ln(x^2 + y^2 + z^2)$; S is sphere $x^2 + y^2 + z^2 = a^2$.

Answer

$8\pi a$

Exercises 43 - 44

Consider the radial fields $\mathbf{F} = \dfrac{\langle x, y, z\rangle}{(x^2 + y^2 + z^2)^{\frac{p}{2}}} = \dfrac{\mathbf{r}}{|\mathbf{r}|^p}$, where p is a real number. Let S consist of spheres A and B centered at the origin with radii $0 < a < b$. The total outward flux across S is given by the outward flux across the outer sphere B minus the flux into S across the inner sphere A. Mathematically, this is expressed as: $\text{Flux}(S) = \text{Flux}(B) - \text{Flux}(A)$, where the flux across each sphere is computed as: $\text{Flux}(R) = \int_{S_R} \mathbf{F} \cdot \mathbf{n}\ dS$, with S_R representing the surface of a sphere of radius R and $\mathbf{n}$ being the outward normal vector at each point on S_R.

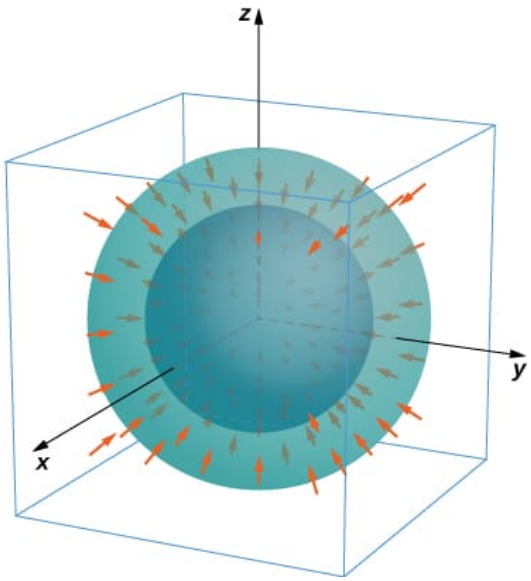


Figure $3.7E.10$: S consist of spheres A and B centered at the origin with radii $0 < a < b$.

43. Find the total flux across S with $p = 0$.

44. Show that for $p = 3$ the flux across S is independent of a and b.

Answer

The net flux is zero.

Contributors

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3.8: Stokes' Theorem

Learning Objectives

- Explain the meaning of Stokes' theorem.
- Use Stokes' theorem to evaluate a line integral.
- Use Stokes' theorem to calculate a surface integral.
- Use Stokes' theorem to calculate a curl.

In this section, we study Stokes' theorem, a higher-dimensional generalization of Green's theorem. This theorem, like the Fundamental Theorem for Line Integrals and Green's theorem, is a generalization of the Fundamental Theorem of Calculus to higher dimensions. Stokes' theorem relates a vector surface integral over the surface S in space to a line integral around the boundary of S. Therefore, just as the theorems before it, Stokes' theorem can be used to reduce an integral over a geometric object S to an integral over the boundary of S. In addition to allowing us to translate between line integrals and surface integrals, Stokes' theorem connects the concepts of curl and circulation. Furthermore, the theorem has applications in fluid mechanics and electromagnetism. We use Stokes' theorem to derive Faraday's law, an important result involving electric fields.

Stokes' Theorem

Stokes' theorem says we can calculate the flux of $curl\ \mathbf{F}$ across surface S by knowing information only about the values of $\mathbf{F}$ along the boundary of S. Conversely, we can calculate the line integral of vector field $\mathbf{F}$ along the boundary of surface S by translating to a double integral of the curl of $\mathbf{F}$ over S.

Let S be an oriented smooth surface with unit normal vector $\mathbf{N}$. Furthermore, suppose the boundary of S is a simple closed curve C. The orientation of S induces the positive orientation of C if, as you walk in the positive direction around C with your head pointing in the direction of $\mathbf{N}$, the surface is always on your left. With this definition in place, we can state *Stokes' theorem*.

Theorem 3.8.1: Stokes' Theorem

Let S be a piecewise smooth oriented surface with a boundary that is a simple closed curve C with positive orientation (Figure 3.8.1). If $\mathbf{F}$ is a vector field with component functions that have continuous partial derivatives on an open region containing S, then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S curl\,\mathbf{F}\cdot d\mathbf{S}. \tag{3.8.1}$$

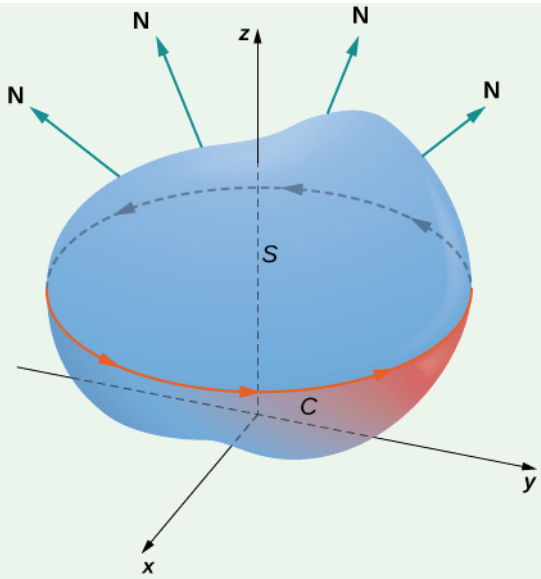


Figure 3.8.1: S is a piecewise smooth oriented surface with a boundary that is a simple closed curve C with positive orientation.

Suppose surface S is a flat region in the xy-plane with upward orientation. Then the unit normal vector is $\mathbf{k}$ and surface integral

$$\iint_S curl\,\mathbf{F}\cdot d\mathbf{S} \tag{3.8.2}$$

is actually the double integral

$$\iint_S curl\,\mathbf{F}\cdot\mathbf{k}\,dA. \tag{3.8.3}$$

In this special case, Stokes' theorem gives

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S curl\,\mathbf{F}\cdot\mathbf{k}\,dA. \tag{3.8.4}$$

However, this is the flux form of Green's theorem, which shows us that Green's theorem is a special case of Stokes' theorem. Green's theorem can only handle surfaces in a plane, but Stokes' theorem can handle surfaces in a plane or in space.

The complete proof of Stokes' theorem is beyond the scope of this text. We look at an intuitive explanation for the truth of the theorem and then see proof of the theorem in the special case that surface S is a portion of a graph of a function, and S, the boundary of S, and $\mathbf{F}$ are all fairly tame.

Proof

First, we look at an informal proof of the theorem. This proof is not rigorous, but it is meant to give a general feeling for why the theorem is true. Let S be a surface and let D be a small piece of the surface so that D does not share any points with the boundary of S. We choose D to be small enough so that it can be approximated by an oriented square E. Let D inherit its orientation from S, and give E the same orientation. This square has four sides; denote them E_l, E_r, E_u, and E_d for the left, right, up, and down sides, respectively. On the square, we can use the flux form of Green's theorem:

$$\int_{E_l+E_d+E_r+E_u} \mathbf{F}\cdot d\mathbf{r} = \iint_E curl\,\mathbf{F}\cdot\mathbf{N}\,d\mathbf{S} = \iint_E curl\,\mathbf{F}\cdot d\mathbf{S}. \tag{3.8.5}$$

To approximate the flux over the entire surface, we add the values of the flux on the small squares approximating small pieces of the surface (Figure 3.8.2).

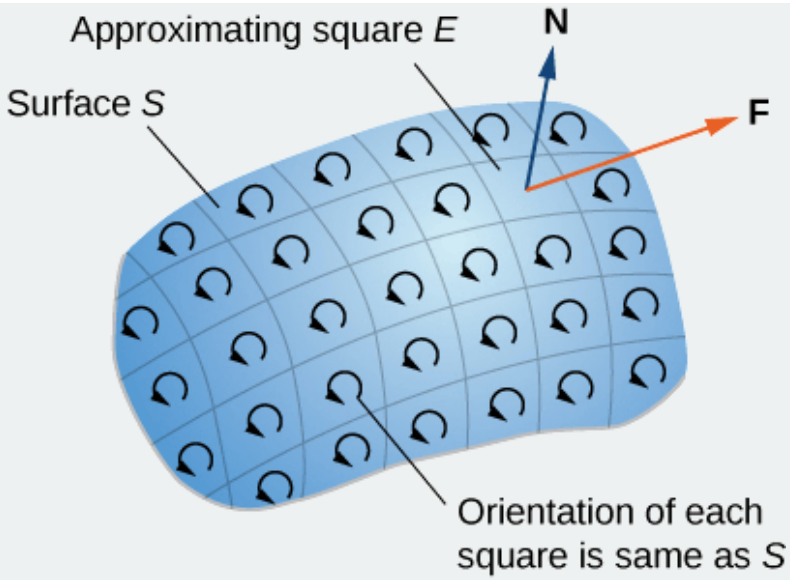


Figure 3.8.2: Chop the surface into small pieces. The pieces should be small enough that they can be approximated by a square.

By Green's theorem, the flux across each approximating square is a line integral over its boundary. Let F be an approximating square with an orientation inherited from S and with a right side E_l (so F is to the left of E). Let F_r denote the right side of F; then, $E_l = -F_r$. In other words, the right side of F is the same curve as the left side of E, just oriented in the opposite direction. Therefore,

$$\int_{E_l} \mathbf{F} \cdot d\mathbf{r} = -\int_{F_r} \mathbf{F} \cdot d\mathbf{r}.$$

As we add up all the fluxes over all the squares approximating surface S, line integrals

$$\int_{E_l} \mathbf{F} \cdot d\mathbf{r} \tag{3.8.6}$$

and

$$\int_{F_r} \mathbf{F} \cdot d\mathbf{r} \tag{3.8.7}$$

cancel each other out. The same goes for the line integrals over the other three sides of E. These three line integrals cancel out with the line integral of the lower side of the square above E, the line integral over the left side of the square to the right of E, and the line integral over the upper side of the square below E (Figure 3.8.3). After all this cancelation occurs over all the approximating squares, the only line integrals that survive are the line integrals over sides approximating the boundary of S. Therefore, the sum of all the fluxes (which, by Green's theorem, is the sum of all the line integrals around the boundaries of approximating squares) can be approximated by a line integral over the boundary of S. In the limit, as the areas of the approximating squares go to zero, this approximation gets arbitrarily close to the flux.

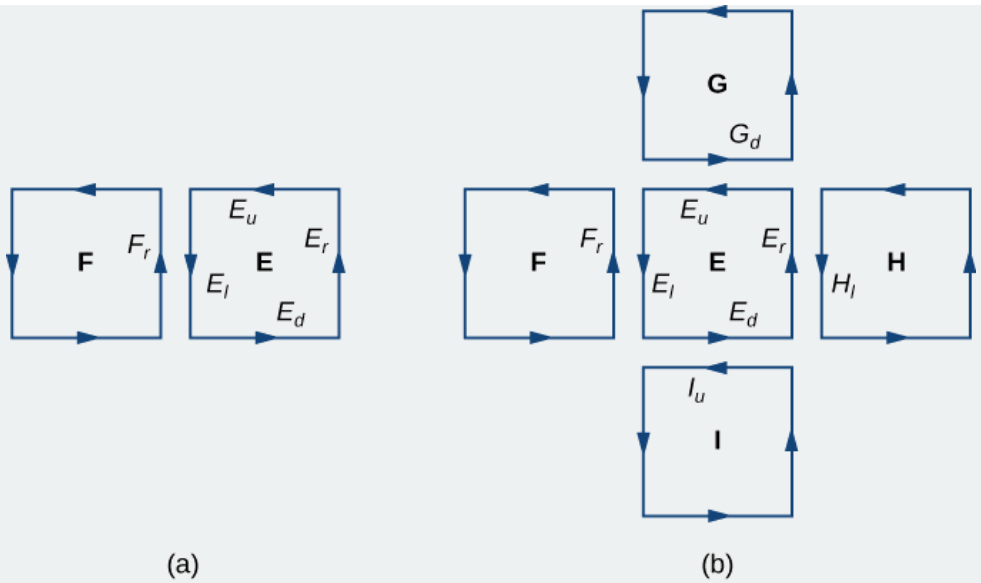


Figure 3.8.3: (a) The line integral along E_l cancels out the line integral along F_r because $E_l = -F_r$. (b) The line integral along any of the sides of E cancels out with the line integral along a side of an adjacent approximating square.

Let's now look at a rigorous proof of the theorem in the special case that S is the graph of function $z = f(x, y)$, where x and y vary over a bounded, simply connected region D of finite area (Figure 3.8.4). Furthermore, assume that f has continuous second-order partial derivatives. Let C denote the boundary of S and let C' denote the boundary of D. Then, D is the "shadow" of S in the plane and C' is the "shadow" of C. Suppose that S is oriented upward. The counterclockwise orientation of C is positive, as is the counterclockwise orientation of C'. Let $\mathbf{F}(x, y, z) = \langle P, Q, R\rangle$ be a vector field with component functions that have continuous partial derivatives.

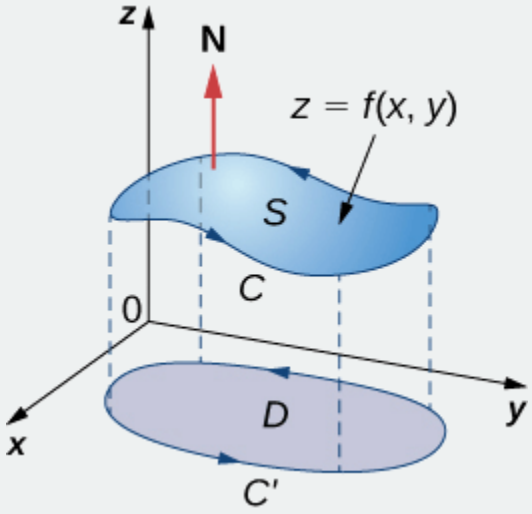


Figure 3.8.4: D is the "shadow," or projection, of S in the plane and C' is the projection of C.

We take the standard parameterization of S : $x = x,\ y = y,\ z = g(x, y)$. The tangent vectors are $\mathbf{t}_x = \langle 1, 0, g_x\rangle$ and $\mathbf{t}_y = \langle 0, 1, g_y\rangle$, and therefore $\mathbf{t}_x \times \mathbf{t}_y = \langle -g_x,\ -g_y,\ 1\rangle$.

$$\iint_S curl\,\mathbf{F}\cdot d\mathbf{S} = \iint_D [-(R_y - Q_z)z_x - (P_z - R_x)z_y + (Q_x - P_y)]\,dA,$$

where the partial derivatives are all evaluated at $(x, y, g(x, y))$, making the integrand depend on x and y only. Suppose $\langle x(t),\ y(t)\rangle,\ a \le t \le b$ is a parameterization of C'. Then, a parameterization of C is $\langle x(t),\ y(t),\ g(x(t),\ y(t))\rangle,\ a \le t \le b$. Armed with these parameterizations, the Chain rule, and Green's theorem, and keeping in mind that P, Q and R are all functions of x and y, we can evaluate line integral

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_a^b (Px'(t)+Qy'(t)+Rz'(t))\,dt \\ &= \int_a^b \left[Px'(t)+Qy'(t)+R\left(\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}\right)\right]dt \\ &= \int_a^b \left[\left(P+R\frac{\partial z}{\partial x}\right)x'(t)+\left(Q+R\frac{\partial z}{\partial y}\right)y'(t)\right]dt \\ &= \int_{C'}\left(P+R\frac{\partial z}{\partial x}\right)dx+\left(Q+R\frac{\partial z}{\partial y}\right)dy \\ &= \iint_D\left[\frac{\partial}{\partial x}\left(Q+R\frac{\partial z}{\partial y}\right)-\frac{\partial}{\partial y}\left(P+R\frac{\partial z}{\partial x}\right)\right]dA \\ &= \iint_D\left(\frac{\partial Q}{\partial x}+\frac{\partial Q}{\partial z}\frac{\partial z}{\partial x}+\frac{\partial R}{\partial x}\frac{\partial z}{\partial y}+\frac{\partial R}{\partial z}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}+R\frac{\partial^2 z}{\partial x\partial y}\right)-\left(\frac{\partial P}{\partial y}+\frac{\partial P}{\partial z}\frac{\partial z}{\partial y}+\frac{\partial R}{\partial z}\frac{\partial z}{\partial y}\frac{\partial z}{\partial x}+R\frac{\partial^2 z}{\partial y\partial x}\right)\end{aligned}$$

By Clairaut's theorem,

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x}$$

Therefore, four of the terms disappear from this double integral, and we are left with

$$\iint_D\left[-(R_y-Q_z)Z_x-(P_z-R_x)z_y+(Q_x-P_y)\right]dA,$$

which equals

$$\iint_S curl\,\mathbf{F}\cdot d\mathbf{S}.$$

□

We have shown that Stokes' theorem is true in the case of a function with a domain that is a simply connected region of finite area. We can quickly confirm this theorem for another important case: when vector field $\mathbf{F}$ is a conservative field. If $\mathbf{F}$ is conservative, the curl of $\mathbf{F}$ is zero, so

$$\iint_S curl\,\mathbf{F}\cdot d\mathbf{S}=0. \tag{3.8.8}$$

Since the boundary of S is a closed curve, the integral

$$\int_C \mathbf{F}\cdot d\mathbf{r}. \tag{3.8.9}$$

is also zero.

Example 3.8.1: Verifying Stokes' Theorem for a Specific Case

Verify that Stokes' theorem is true for vector field $\mathbf{F}(x,y)=\langle -z,x,0\rangle$ and surface S, where S is the hemisphere, oriented outward, with parameterization $\mathbf{r}(\phi,\theta)=\langle \sin\phi\ \cos\theta,\ \sin\phi\ \sin\theta,\ \cos\phi\rangle,\ 0\le\theta\le\pi,\ 0\le\phi\le\pi$ as shown in Figure 3.8.5.

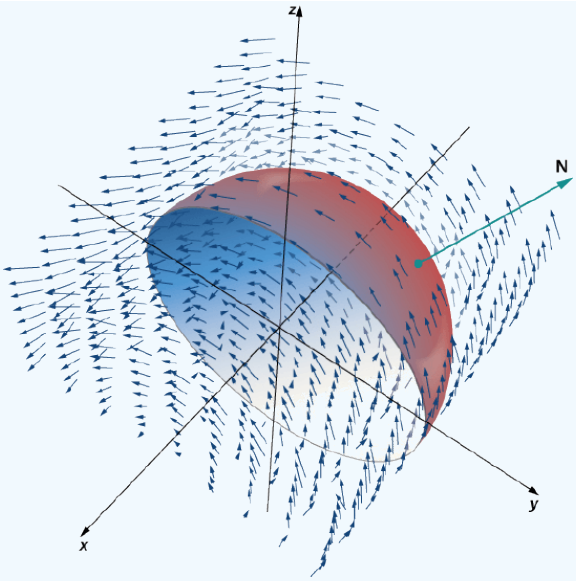


Figure 3.8.5: S is the hemisphere, oriented outward, with parameterization $\mathbf{r}(\phi,\theta) = \langle \sin\phi\ \cos\theta,\ \sin\phi\ \sin\theta,\ \cos\phi\rangle,\ 0 \le \theta \le \pi,\ 0 \le \phi \le \pi$.

Solution

Let C be the boundary of S. Note that C is a circle of radius 1, centered at the origin, sitting in plane $y = 0$. This circle has parameterization $\langle \cos t,\ 0,\ \sin t\rangle,\ 0 \le t \le 2\pi$. the equation for scalar surface integrals

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} \langle -\sin t,\ \cos t,\ 0\rangle \cdot \langle -\sin t,\ 0,\ \cos t\rangle\, dt \\ &= \int_0^{2\pi} \sin^2 t\, dt \\ &= \pi.\end{aligned}$$

By the equation for vector line integrals,

$$\begin{aligned}\iint_S curl\,\mathbf{F}\cdot d\mathbf{S} &= \iint_D curl\,\mathbf{F}(\mathbf{r}(\phi,\theta))\cdot(\mathbf{t}_\phi \times \mathbf{t}_\theta)\, dA \\ &= \iint_D \langle 0,-1,1\rangle \cdot \langle \cos\theta\,\sin^2\phi,\ \sin\theta\,\sin^2\phi,\ \sin\phi\,\cos\phi\rangle\, dA \\ &= \int_0^{\pi}\int_0^{\pi} (\sin\phi\,\cos\phi - \sin\theta\,\sin^2\phi)\, d\phi d\theta \\ &= \frac{\pi}{2}\int_0^{\pi} \sin\theta\, d\theta \\ &= \pi.\end{aligned}$$

Therefore, we have verified Stokes' theorem for this example.

Exercise 3.8.1

Verify that Stokes' theorem is true for vector field $\mathbf{F}(x,y,z) = \langle y, x, -z\rangle$ and surface S, where S is the upwardly oriented portion of the graph of $f(x,y) = x^2 y$ over a triangle in the xy-plane with vertices $(0,0),\ (2,0)$, and $(0,2)$.

Hint available at the end of this section.

Answer available at the end of this section.

Applying Stokes' Theorem

Stokes' theorem translates between the flux integral of surface S to a line integral around the boundary of S. Therefore, the theorem allows us to compute surface integrals or line integrals that would ordinarily be quite difficult by translating the line integral into a surface integral or vice versa. We now study some examples of each kind of translation.

Example 3.8.2: Calculating a Surface Integral

Calculate surface integral

$$\iint_S curl\,\mathbf{F}\cdot d\mathbf{S},$$

where S is the surface, oriented outward, in Figure 3.8.6 and $\mathbf{F} = \langle z,\ 2xy,\ x+y\rangle$.

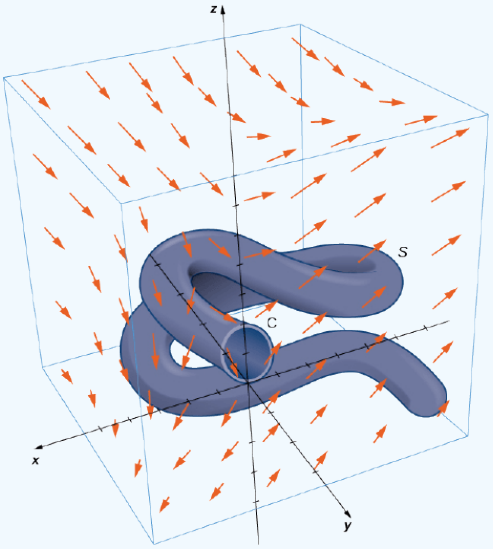


Figure 3.8.6: S is a complicated surface in a vector field.

Solution

Note that to calculate

$$\iint_S curl\,\mathbf{F}\cdot d\mathbf{S}$$

without using Stokes' theorem, we would need the equation for scalar surface integrals. Use of this equation requires a parameterization of S. Surface S is complicated enough that it would be extremely difficult to find a parameterization. Therefore, the methods we have learned in previous sections are not useful for this problem. Instead, we use Stokes' theorem, noting that the boundary C of the surface is merely a single circle with radius 1.

The curl of $\mathbf{F}$ is $\langle 1, 1, 2y\rangle$. By Stokes' theorem,

$$\iint_S curl\,\mathbf{F}\cdot d\mathbf{S} = \int_C \mathbf{F}\cdot d\mathbf{r},$$

where C has parameterization $\langle \cos t,\ \sin t,\ 1\rangle, 0 \le t \le 2\pi$. By the equation for vector line integrals,

$$\begin{aligned}\iint_S curl\, F \cdot d\mathbf{S} &= \int_C \mathbf{F}\cdot d\mathbf{r} \\ &= \int_0^2 \langle 1,\, \sin t\, \cos t,\, \cos t + \sin t\rangle \cdot \langle -\sin t,\, \cos t,\, 0\rangle\, dt \\ &= \int_0^{2\pi} (-\sin t + 2\, \sin t\, \cos^2 t)\, dt \\ &= \left[\cos t - \frac{2\, \cos^3 t}{3}\right]_0^{2\pi} \\ &= \cos(2\pi) - \frac{2\, \cos^3(2\pi)}{3} - \left(\cos(0) - \frac{2\, \cos^3(0)}{3}\right) \\ &= 0.\end{aligned}$$

An amazing consequence of Stokes' theorem is that if S' is any other smooth surface with boundary C and the same orientation as S, then

$$\iint_S curl\, \mathbf{F}\cdot d\mathbf{S} = \int_C \mathbf{F}\cdot d\mathbf{r} = 0 \tag{3.8.10}$$

because Stokes' theorem says the surface integral depends on the line integral around the boundary only.

In Example 3.8.2, we calculated a surface integral simply by using information about the boundary of the surface. In general, let S_1 and S_2 be smooth surfaces with the same boundary C and the same orientation. By Stokes' theorem,

$$\iint_{S_1} curl\, \mathbf{F}\cdot d\mathbf{S} = \int_C \mathbf{F}\cdot d\mathbf{r} = \iint_{S_2} curl\, \mathbf{F}\cdot d\mathbf{S}. \tag{3.8.11}$$

Therefore, if

$$\iint_{S_1} curl\, \mathbf{F}\cdot d\mathbf{S} \tag{3.8.12}$$

is difficult to calculate but

$$\iint_{S_2} curl\, \mathbf{F}\cdot d\mathbf{S} \tag{3.8.13}$$

is easy to calculate, Stokes' theorem allows us to calculate the easier surface integral. In Example 3.8.2, we could have calculated

$$\iint_S curl\, \mathbf{F}\cdot d\mathbf{S} \tag{3.8.14}$$

by calculating

$$\iint_{S'} curl\, \mathbf{F}\cdot d\mathbf{S}, \tag{3.8.15}$$

where $\mathbf{S}'$ is the disk enclosed by boundary curve C (a much more simple surface with which to work).

Equation 3.8.11 shows that flux integrals of curl vector fields are surface independent in the same way that line integrals of gradient fields are path independent. Recall that if $\mathbf{F}$ is a two-dimensional conservative vector field defined on a simply connected domain, f is a *potential function* for $\mathbf{F}$, and C is a curve in the domain of $\mathbf{F}$, then

$$\int_C \mathbf{F}\cdot d\mathbf{r} \tag{3.8.16}$$

depends only on the endpoints of C. Therefore if C' is any other curve with the same starting point and endpoint as C (that is, C' has the same orientation as C), then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{C'} \mathbf{F}\cdot d\mathbf{r} \tag{3.8.17}$$

In other words, the value of the integral depends on the boundary of the path only; it does not really depend on the path itself.

Analogously, suppose that S and S' are surfaces with the same boundary and same orientation, and suppose that $\mathbf{G}$ is a three-dimensional vector field that can be written as the curl of another vector field $\mathbf{F}$ (so that $\mathbf{F}$ is like a "potential field" of $\mathbf{G}$). By Equation 3.8.11,

$$\iint_S \mathbf{G}\cdot d\mathbf{S} = \iint_S curl\,\mathbf{F}\cdot d\mathbf{S} = \int_C \mathbf{F}\cdot d\mathbf{r} = \iint_{S'} curl\,\mathbf{F}\cdot d\mathbf{S} = \iint_{S'} \mathbf{G}\cdot d\mathbf{S}.$$

Therefore, the flux integral of $\mathbf{G}$ does not depend on the surface, only on the boundary of the surface. Flux integrals of vector fields that can be written as the curl of a vector field are surface independent in the same way that line integrals of vector fields that can be written as the gradient of a scalar function are path independent.

Example 3.8.3: Calculating a Line Integral

Calculate the line integral

$$\int_C \mathbf{F}\cdot d\mathbf{r},$$

where $\mathbf{F} = \langle xy,\ x^2+y^2+z^2,\ yz\rangle$ and C is the boundary of the parallelogram with vertices $(0,0,1),\ (0,1,0),\ (2,0,-1)$, and $(2,1,-2)$.

Solution

To calculate the line integral directly, we need to parameterize each side of the parallelogram separately, calculate four separate line integrals, and add the result. This is not overly complicated, but it is time-consuming.

By contrast, let's calculate the line integral using Stokes' theorem. Let S denote the surface of the parallelogram. Note that S is the portion of the graph of $z = 1-x-y$ for (x,y) varying over the rectangular region with vertices $(0,0),\ (0,1),\ (2,0)$, and $(2,1)$ in the xy-plane. Therefore, a parameterization of S is $\langle x, y,\ 1-x-y\rangle,\ 0\le x\le 2,\ 0\le y\le 1$. The curl of $\mathbf{F}$ is $\langle -z,\ 0,\ x\rangle$,and Stokes' theorem and the equation for scalar surface integrals

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= \iint_S curl\,\mathbf{F}\cdot d\mathbf{S} \\
&= \int_0^2\int_0^1 curl\,\mathbf{F}(x,y)\cdot(\mathbf{t}_x\times\mathbf{t}_y)\,dy\,dx \\
&= \int_0^2\int_0^1 \langle -(1-x-y),\ 0,\ x\rangle\cdot(\langle 1,\ 0,-1\rangle\times\langle 0,\ 1,\ -1\rangle)\,dy\,dx \\
&= \int_0^2\int_0^1 \langle x+y-1,\ 0,\ x\rangle\cdot\langle 1,1,1\rangle\,dy\,dx \\
&= \int_0^2\int_0^1 2x+y-1\,dy\,dx \\
&= 3.
\end{aligned}$$

Exercise 3.8.2

Use Stokes' theorem to calculate line integral

$$\int_C \mathbf{F}\cdot d\mathbf{r},$$

where $\mathbf{F} = \langle z, x, y\rangle$ and C is the boundary of a triangle with vertices $(0,0,1),\ (3,0,-2)$, and $(0,1,2)$.

Hint available at the end of this section.

Answer available at the end of this section.

Interpretation of Curl

In addition to translating between line integrals and flux integrals, Stokes' theorem can be used to justify the physical interpretation of curl that we have learned. Here we investigate the relationship between curl and circulation, and we use Stokes' theorem to state Faraday's law—an important law in electricity and magnetism that relates the curl of an electric field to the rate of change of a magnetic field.

Recall that if C is a closed curve and $\mathbf{F}$ is a vector field defined on C, then the circulation of $\mathbf{F}$ around C is line integral

$$\int_C \mathbf{F}\cdot d\mathbf{r}. \tag{3.8.18}$$

If $\mathbf{F}$ represents the velocity field of a fluid in space, then the circulation measures the tendency of the fluid to move in the direction of C.

Let $\mathbf{F}$ be a continuous vector field and let D_τ be a small disk of radius r with center P_0 (Figure 3.8.7). If D_τ is small enough, then $(curl\ \mathbf{F})(P) \approx (curl\ \mathbf{F})(P_0)$ for all points P in D_τ because the curl is continuous. Let C_τ be the boundary circle of D_τ: By Stokes' theorem,

$$\int_{C_\tau} \mathbf{F}\cdot d\mathbf{r} = \iint_{D_\tau} curl\,\mathbf{F}\cdot\mathbf{N}\,d\mathbf{S} \approx \iint_{D_\tau} (curl\,\mathbf{F})(P_0)\cdot\mathbf{N}(P_0)\,d\mathbf{S}. \tag{3.8.19}$$

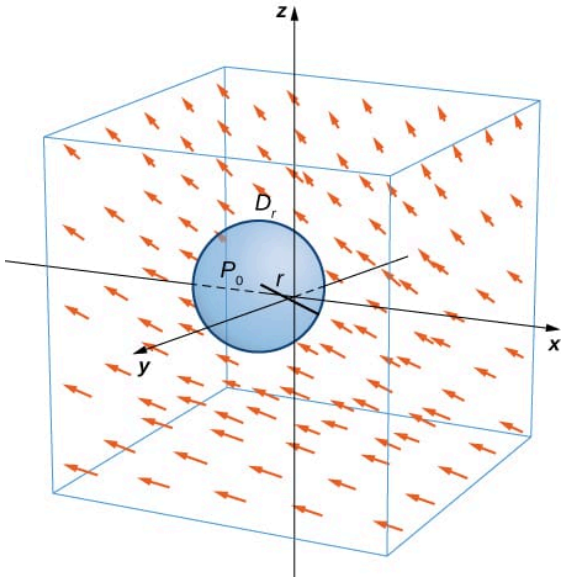


Figure 3.8.7: Disk D_τ is a small disk in a continuous vector field.

The quantity $(curl\ \mathbf{F})(P_0)\cdot\mathbf{N}(P_0)$ is constant, and therefore

$$\iint_{D_\tau} (curl\,\mathbf{F})(P_0)\cdot\mathbf{N}(P_0)\,d\mathbf{S} = \pi r^2[(curl\,\mathbf{F})(P_0)\cdot\mathbf{N}(P_0)].$$

Thus

$$\int_{C_\tau} \mathbf{F}\cdot d\mathbf{r} \approx \pi r^2[(curl\,\mathbf{F})(P_0)\cdot\mathbf{N}(P_0)],$$

and the approximation gets arbitrarily close as the radius shrinks to zero. Therefore Stokes' theorem implies that

$$(curl\,\mathbf{F})(P_0)\cdot\mathbf{N}(P_0)=\lim_{r\to 0^+}\frac{1}{\pi r^2}\int_{C_\tau}\mathbf{F}\cdot d\mathbf{r}.$$

This equation relates the curl of a vector field to the circulation. Since the area of the disk is πr^2, this equation says we can view the curl (in the limit) as the circulation per unit area. Recall that if $\mathbf{F}$ is the velocity field of a fluid, then circulation

$$\oint_{C_\tau}\mathbf{F}\cdot d\mathbf{r}=\oint_{C_\tau}\mathbf{F}\cdot\mathbf{T}\,ds \tag{3.8.20}$$

is a measure of the tendency of the fluid to move around C_τ: The reason for this is that $\mathbf{F}\cdot\mathbf{T}$ is a component of $\mathbf{F}$ in the direction of $\mathbf{T}$, and the closer the direction of $\mathbf{F}$ is to $\mathbf{T}$, the larger the value of $\mathbf{F}\cdot\mathbf{T}$ (remember that if $\mathbf{a}$ and $\mathbf{b}$ are vectors and $\mathbf{b}$ is fixed, then the dot product $\mathbf{a}\cdot\mathbf{b}$ is maximal when $\mathbf{a}$ points in the same direction as $\mathbf{b}$). Therefore, if $\mathbf{F}$ is the velocity field of a fluid, then $curl\ \mathbf{F}\cdot\mathbf{N}$ is a measure of how the fluid rotates about axis $\mathbf{N}$. The effect of the curl is largest about the axis that points in the direction of $\mathbf{N}$, because in this case $curl\ \mathbf{F}\cdot\mathbf{N}$ is as large as possible.

To see this effect in a more concrete fashion, imagine placing a tiny paddlewheel at point P_0 (Figure 3.8.8). The paddlewheel achieves its maximum speed when the axis of the wheel points in the direction of curl $\mathbf{F}$. This justifies the interpretation of the curl we have learned: curl is a measure of the rotation in the vector field about the axis that points in the direction of the normal vector $\mathbf{N}$, and Stokes' theorem justifies this interpretation.

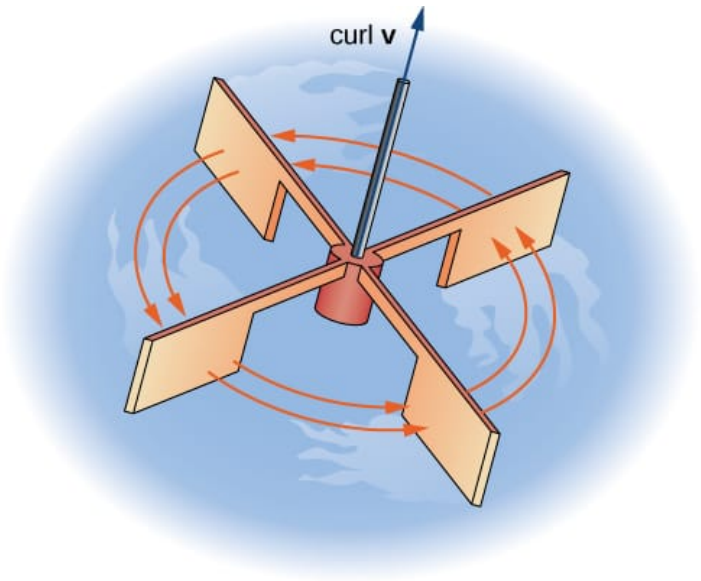


Figure 3.8.8: The image illustrates a tiny paddlewheel at a point in the vector field.

Now that we have learned about Stokes' theorem, we can discuss applications in the area of electromagnetism. In particular, we examine how we can use Stokes' theorem to translate between two equivalent forms of Faraday's law. Before stating the two forms of Faraday's law, we need some background terminology.

Let C be a closed curve that models a thin wire. In the context of electric fields, the wire may be moving over time, so we write $C(t)$ to represent the wire. At a given time t, curve $C(t)$ may be different from original curve C because of the movement of the wire, but we assume that $C(t)$ is a closed curve for all times t. Let $D(t)$ be a surface with $C(t)$ as its boundary, and orient $C(t)$ so that $D(t)$ has positive orientation. Suppose that $C(t)$is in a magnetic field $\mathbf{B}(t)$ that can also change over time. In other words, $\mathbf{B}$ has the form

$$\mathbf{B}(x,y,z)=\langle P(x,y,z),\,Q(x,y,z),\,R(x,y,z)\rangle, \tag{3.8.21}$$

where P, Q, and R can all vary continuously over time. We can produce current along the wire by changing field $\mathbf{B}(t)$ (this is a consequence of Ampere's law). Flux $\phi(t)=\iint_{D(t)}\mathbf{B}(t)\cdot d\mathbf{S}$ creates electric field $\mathbf{E}(t)$ that does work. The integral form of Faraday's law states that

$$Work=\int_{C(t)}\mathbf{E}(t)\cdot d\mathbf{r}=-\frac{\partial\phi}{\partial t}. \tag{3.8.22}$$

In other words, the work done by $\mathbf{E}$ is the line integral around the boundary, which is also equal to the rate of change of the flux with respect to time. The differential form of Faraday's law states that

$$curl\,\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{3.8.23}$$

Using Stokes' theorem, we can show that the differential form of Faraday's law is a consequence of the integral form. By Stokes' theorem, we can convert the line integral in the integral form into surface integral

$$-\frac{\partial \phi}{\partial t} = \int_{C(t)} \mathbf{E}(t)\cdot d\mathbf{r} = \iint_{D(t)} curl\,\mathbf{E}(t)\cdot d\mathbf{S}. \tag{3.8.24}$$

Since

$$\phi(t) = \iint_{D(t)} B(t)\cdot d\mathbf{S}, \tag{3.8.25}$$

then as long as the integration of the surface does not vary with time we also have

$$-\frac{\partial \phi}{\partial t} = \iint_{D(t)} -\frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{S}. \tag{3.8.26}$$

Therefore,

$$\iint_{D(t)} -\frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{S} = \iint_{D(t)} curl\,\mathbf{E}\cdot d\mathbf{S}. \tag{3.8.27}$$

To derive the differential form of Faraday's law, we would like to conclude that $curl\ \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$: In general, the equation

$$\iint_{D(t)} -\frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{S} = \iint_{D(t)} curl\,\mathbf{E}\cdot d\mathbf{S} \tag{3.8.28}$$

is not enough to conclude that $curl\ \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$: The integral symbols do not simply "cancel out," leaving equality of the integrands. To see why the integral symbol does not just cancel out in general, consider the two single-variable integrals $\int_0^1 x\ dx$ and $\int_0^1 f(x)\ dx$, where

$$f(x) = \begin{cases} 1, & \text{if } 0 \le x \le 1/2 \\ 0, & \text{if } 1/2 \le x \le 1. \end{cases} \tag{3.8.29}$$

Both of these integrals equal $\frac{1}{2}$, so $\int_0^1 x\ dx = \int_0^1 f(x)\ dx$.

However, $x \neq f(x)$. Analogously, with our equation

$$\iint_{D(t)} -\frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{S} = \iint_{D(t)} curl\,\mathbf{E}\cdot d\mathbf{S}, \tag{3.8.30}$$

we cannot simply conclude that $curl\ \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ just because their integrals are equal. However, in our context, equation

$$\iint_{D(t)} -\frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{S} = \iint_{D(t)} curl\,\mathbf{E}\cdot d\mathbf{S} \tag{3.8.31}$$

is true for *any* region, however small (this is in contrast to the single-variable integrals just discussed). If $\mathbf{F}$ and $\mathbf{G}$ are three-dimensional vector fields such that

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{G}\cdot d\mathbf{S} \tag{3.8.32}$$

for any surface S, then it is possible to show that $\mathbf{F} = \mathbf{G}$ by shrinking the area of S to zero by taking a limit (the smaller the area of S, the closer the value of $\iint_S \mathbf{F}\cdot d\mathbf{S}$ to the value of $\mathbf{F}$ at a point inside S). Therefore, we can let area $D(t)$ shrink to zero by taking

a limit and obtain the differential form of Faraday's law:

$$curl\,\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{3.8.33}$$

In the context of electric fields, the curl of the electric field can be interpreted as the negative of the rate of change of the corresponding magnetic field with respect to time.

Example 3.8.4: Using Faraday's Law

Calculate the curl of electric field $\mathbf{E}$ if the corresponding magnetic field is constant field $\mathbf{B}(t) = \langle 1, -4, 2\rangle$.

Solution

Since the magnetic field does not change with respect to time, $-\frac{\partial \mathbf{B}}{\partial t} = \mathbf{0}$. By Faraday's law, the curl of the electric field is therefore also zero.

A consequence of Faraday's law is that the curl of the electric field corresponding to a constant magnetic field is always zero.

Exercise 3.8.3

Calculate the curl of electric field $\mathbf{E}$ if the corresponding magnetic field is $\mathbf{B}(t) = \langle tx,\ ty,\ -2tz\rangle,\ 0 \le t < \infty$.

Hint available at the end of this section.

Answer available at the end of this section.

Key Concepts

- Stokes' theorem relates a flux integral over a surface to a line integral around the boundary of the surface. Stokes' theorem is a higher dimensional version of Green's theorem, and therefore is another version of the Fundamental Theorem of Calculus in higher dimensions.
- Stokes' theorem can be used to transform a difficult surface integral into an easier line integral, or a difficult line integral into an easier surface integral.
- Through Stokes' theorem, line integrals can be evaluated using the simplest surface with boundary C.
- Faraday's law relates the curl of an electric field to the rate of change of the corresponding magnetic field. Stokes' theorem can be used to derive Faraday's law.

Key Equations

- **Stokes' theorem**

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S curl\,\mathbf{F}\cdot d\mathbf{S}$$

Glossary

Stokes' theorem
: relates the flux integral over a surface S to a line integral around the boundary C of the surface S

Surface independent
: flux integrals of curl vector fields are surface independent if their evaluation does not depend on the surface but only on the boundary of the surface

Contributors and Attributions

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Answers

Exercise 3.8.1

Calculate the double integral and line integral separately.

Exercise 3.8.1

Both integrals give $-\frac{136}{45}$:

Exercise 3.8.2

This triangle lies in plane $z = 1 - x + y$.

Exercise 3.8.2

$\frac{3}{2}$

Exercise 3.8.3

- Use the differential form of Faraday’s law.
- Notice that the curl of the electric field does not change over time, although the magnetic field does change over time.

Exercise 3.8.3

$curl\ \mathbf{E} = \langle -x,\ -y,\ 2z \rangle$

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3.8E: Exercises

Exercises 1 - 6

Without using Stokes' theorem, calculate directly both the flux of $curl\ \mathbf{F} \cdot \mathbf{N}$ over the given surface and the circulation integral around its boundary, assuming all are oriented clockwise.

1. $\mathbf{F}(x, y, z) = y^2\ \mathbf{i} + z^2\ \mathbf{j} + x^2\ \mathbf{k}$; S is the first-octant portion of plane $x + y + z = 1$.

2. $\mathbf{F}(x, y, z) = z\ \mathbf{i} + x\ \mathbf{j} + y\ \mathbf{k}$; S is hemisphere $z = (a^2 - x^2 - y^2)^{1/2}$.

Answer

$$\iint_S (curl\ \mathbf{F} \cdot \mathbf{N})\ dS = \pi a^2$$

3. $\mathbf{F}(x, y, z) = y^2\ \mathbf{i} + 2x\ \mathbf{j} + 5\ \mathbf{k}$; S is hemisphere $z = (4 - x^2 - y^2)^{1/2}$.

4. $\mathbf{F}(x, y, z) = z\ \mathbf{i} + 2x\ \mathbf{j} + 3y\ \mathbf{k}$; S is upper hemisphere $z = \sqrt{9 - x^2 - y^2}$.

Answer

$$\iint_S (curl\ \mathbf{F} \cdot \mathbf{N})\ dS = 18\pi$$

5. $\mathbf{F}(x, y, z) = (x + 2z)\ \mathbf{i} + (y - x)\ \mathbf{j} + (z - y)\ \mathbf{k}$; S is a triangular region with vertices $(3, 0, 0),\ (0, 3/2, 0)$, and $(0, 0, 3)$.

6. $\mathbf{F}(x, y, z) = 2y\ \mathbf{i} + 6z\ \mathbf{j} + 3x\ \mathbf{k}$; S is a portion of paraboloid $z = 4 - x^2 - y^2$ and is above the xy-plane.

Answer

$$\iint_S (curl\ \mathbf{F} \cdot \mathbf{N})\ dS = -8\pi$$

Exercises 7 - 9

Use Stokes' theorem to evaluate $\iint_S (curl\ \mathbf{F} \cdot \mathbf{N})\ dS$ for the vector fields and surface.

7. $\mathbf{F}(x, y, z) = xy\ \mathbf{i} - z\ \mathbf{j}$ and S is the surface of the cube $0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1$, except for the face where $z = 0$ and using the outward unit normal vector.

8. $\mathbf{F}(x, y, z) = xy\ \mathbf{i} + x^2\ \mathbf{j} + z^2\ \mathbf{k}$; and C is the intersection of paraboloid $z = x^2 + y^2$ and plane $z = y$, and using the outward normal vector.

Answer

$$\iint_S (curl\ \mathbf{F} \cdot \mathbf{N})\ dS = 0$$

9. $\mathbf{F}(x, y, z) = 4y\ \mathbf{i} + z\ \mathbf{j} + 2y\ \mathbf{k}$; and C is the intersection of sphere $x^2 + y^2 + z^2 = 4$ with plane $z = 0$, and using the outward normal vector.

10. Use Stokes' theorem to evaluate $\int_C \left[2xy^2z\ dx + 2x^2yz\ dy + (x^2y^2 - 2z)\ dz\right]$, where C is the curve given by $x = \cos t,\ y = \sin t$, $0 \le t \le 2\pi$, traversed in the direction of increasing t.

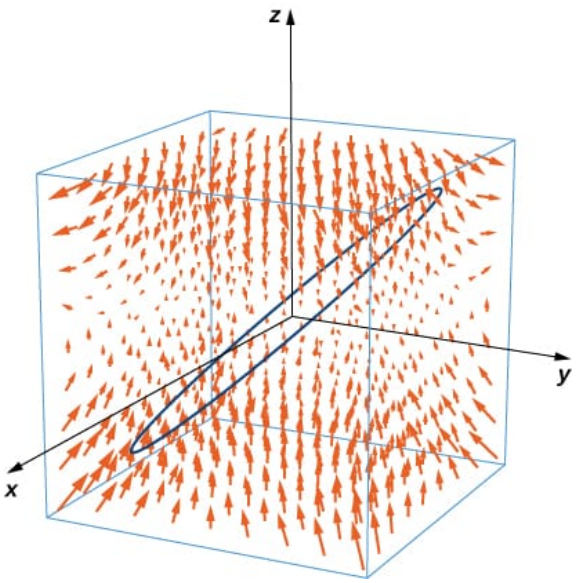


Figure $3.8E.1$: C is the curve given by $x = \cos t,\ \ y = \sin t,\ \ 0 \le t \le 2\pi$, traversed in the direction of increasing t.

Answer

$$\int_C \mathbf{F} \cdot dS = 0$$

11. Use a computer algebraic system (CAS) and Stokes' theorem to approximate line integral $\int_C (y\ dx + z\ dy + x\ dz)$, where C is the intersection of plane $x + y = 2$ and surface $x^2 + y^2 + z^2 = 2(x + y)$, traversed counterclockwise viewed from the origin.

12. Use a CAS and Stokes' theorem to approximate line integral $\int_C (3y\ dx + 2z\ dy - 5x\ dz)$, where C is the intersection of the xy -plane and hemisphere $z = \sqrt{1 - x^2 - y^2}$, traversed counterclockwise viewed from the top—that is, from the positive z-axis toward the xy-plane.

Answer

$$\int_C \mathbf{F} \cdot dS = -9.4248$$

13. Use a CAS and Stokes' theorem to approximate line integral $\int_C [(1 + y)\ z\ dx + (1 + z)x\ dy + (1 + x)y\ dz]$, where C is a triangle with vertices $(1, 0, 0),\ \ (0, 1, 0)$, and $(0, 0, 1)$ oriented counterclockwise.

14. Use Stokes' theorem to evaluate $\iint_S curl\ \mathbf{F} \cdot dS$, where $\mathbf{F}(x, y, z) = e^{xy} cos\ \ z\ \mathbf{i} + x^2 z\ \mathbf{j} + xy\ \mathbf{k}$, and S is half of sphere $x = \sqrt{1 - y^2 - z^2}$, oriented out toward the positive x-axis.

Answer

$$\iint_S \mathbf{F} \cdot dS = 0$$

15. Use a CAS and Stokes' theorem to evaluate $\iint_S (curl\ \mathbf{F} \cdot \mathbf{N})\ dS$, where $\mathbf{F}(x, y, z) = x^2 y\ \mathbf{i} + xy^2\ \mathbf{j} + z^3\ \mathbf{k}$ and C is the curve of the intersection of plane $3x + 2y + z = 6$ and cylinder $x^2 + y^2 = 4$, oriented clockwise when viewed from above.

16. Use a CAS and Stokes' theorem to evaluate $\iint_S curl\ \mathbf{F} \cdot dS$, where $\mathbf{F}(x, y, z) = \left(\sin(y + z) - yx^2 - \frac{y^3}{3}\right)\ \mathbf{i} + x\ \ \cos(y + z)\ \mathbf{j} + \cos(2y)\ \mathbf{k}$ and S consists of the top and the four sides but not the bottom of the cube with vertices $(\pm 1,\ \pm 1,\ \pm 1)$, oriented outward.

Answer

$$\iint_S curl\ \mathbf{F} \cdot dS = 2.6667$$

17. Use a CAS and Stokes' theorem to evaluate $\iint_S curl\ \mathbf{F} \cdot dS$, where $\mathbf{F}(x,y,z) = z^2\ \mathbf{i} + 3xy\ \mathbf{j} + x^3y^3\ \mathbf{k}$ and S is the top part of $z = 5 - x^2 - y^2$ above plane $z = 1$ and S is oriented upward.

18. Use Stokes' theorem to evaluate $\iint_S (curl\ \mathbf{F} \cdot \mathbf{N})dS$, where $\mathbf{F}(x,y,z) = z^2\ \mathbf{i} + y^2\ \mathbf{j} + x\ \mathbf{k}$ and S is a triangle with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ with counterclockwise orientation.

Answer

$$\iint_S (curl\ \mathbf{F} \cdot \mathbf{N})dS = -\frac{1}{6}$$

19. Use Stokes' theorem to evaluate line integral $\int_C (z\ dx + x\ dy + y\ dz)$, where C is a triangle with vertices $(3,0,0)$, $(0,0,2)$, and $(0,6,0)$ traversed in the given order.

20. Use Stokes' theorem to evaluate $\int_C \left(\frac{1}{2}y^2\,dx + z\,dy + x\,dz\right)$, where C is the curve of intersection of plane $x + z = 1$ and ellipsoid $x^2 + 2y^2 + z^2 = 1$, oriented clockwise from the origin.

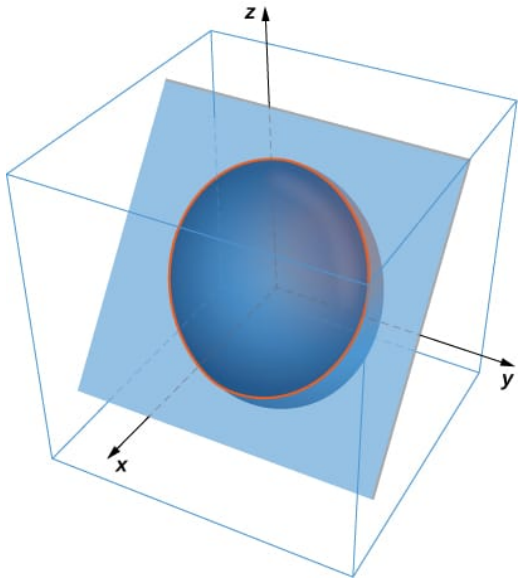


Figure 3.8E.2: C is the curve of intersection of plane $x + z = 1$ and ellipsoid $x^2 + 2y^2 + z^2 = 1$, oriented clockwise from the origin.

Answer

$$\int_C \left(\frac{1}{2}y^2\,dx + z\,dy + x\,dz\right) = -\frac{\pi}{4}$$

21. Use Stokes' theorem to evaluate $\iint_S (curl\ \mathbf{F} \cdot \mathbf{N})dS$, where $\mathbf{F}(x,y,z) = x\ \mathbf{i} + y^2\ \mathbf{j} + ze^{xy}\ \mathbf{k}$ and S is the part of surface $z = 1 - x^2 - 2y^2$ with $z \geq 0$, oriented counterclockwise.

22. Use Stokes' theorem for vector field $\mathbf{F}(x,y,z) = z\ \mathbf{i} + 3x\ \mathbf{j} + 2z\ \mathbf{k}$ where S is surface $z = 1 - x^2 - 2y^2,\ z \geq 0$, C is boundary circle $x^2 + y^2 = 1$, and S is oriented in the positive z-direction.

Answer

$$\iint_S (curl\ \mathbf{F} \cdot \mathbf{N})dS = -3\pi$$

23. Use Stokes' theorem for vector field $\mathbf{F}(x,y,z) = -\frac{3}{2}y^2\ \mathbf{i} - 2xy\ \mathbf{j} + yz\ \mathbf{k}$, where S is that part of the surface of plane $x + y + z = 1$ contained within triangle C with vertices $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, traversed counterclockwise as viewed from above.

24. A certain closed path C in plane $2x + 2y + z = 1$ is known to project onto unit circle $x^2 + y^2 = 1$ in the xy-plane. Let C be a constant and let $\mathbf{R}(x,y,z) = x\ \mathbf{i} + y\ \mathbf{j} + z\ \mathbf{k}$. Use Stokes' theorem to evaluate $\int_C (c\ \mathbf{k} \times \mathbf{R}) \cdot dS$.

Answer

$$\int_C (c\ \mathbf{k} \times \mathbf{R}) \cdot dS = 2\pi c$$

25. Use Stokes' theorem and let C be the boundary of surface $z = x^2 + y^2$ with $0 \le x \le 2$ and $0 \le y \le 1$ oriented with upward facing normal. Define $\mathbf{F}(x, y, z) = \left(\sin(x^3) + xz\right)\ \mathbf{i} + (x - yz)\ \mathbf{j} + \cos(z^4)\ \mathbf{k}$ and evaluate $\int_C \mathbf{F} \cdot dS$.

26. Let S be hemisphere $x^2 + y^2 + z^2 = 4$ with $z \ge 0$, oriented upward. Let $\mathbf{F}(x, y, z) = x^2 e^{yz}\ \mathbf{i} + y^2 e^{xz}\ \mathbf{j} + z^2 e^{xy}\ \mathbf{k}$ be a vector field. Use Stokes' theorem to evaluate $\iint_S curl\ \mathbf{F} \cdot dS$.

Answer

$$\iint_S curl\ \mathbf{F} \cdot dS = 0$$

27. Let $\mathbf{F}(x, y, z) = xy\ \mathbf{i} + (e^{z^2} + y)\ \mathbf{j} + (x + y)\ \mathbf{k}$ and let S be the graph of function $y = \dfrac{x^2}{9} + \dfrac{z^2}{9} - 1$ with $z \le 0$ oriented so that the normal vector S has a positive *y* component. Use Stokes' theorem to compute integral $\iint_S curl\ \mathbf{F} \cdot dS$.

28. Use Stokes' theorem to evaluate $\oint \mathbf{F} \cdot dS$, where $\mathbf{F}(x, y, z) = y\ \mathbf{i} + z\ \mathbf{j} + x\ \mathbf{k}$ and C is a triangle with vertices $(0, 0, 0),\ (2, 0, 0)$ and $0, -2, 2)$ oriented counterclockwise when viewed from above.

Answer

$$\oint \mathbf{F} \cdot dS = -4$$

29. Use the surface integral in Stokes' theorem to calculate the circulation of field $\mathbf{F}$, $\mathbf{F}(x, y, z) = x^2 y^3\ \mathbf{i} +\ \mathbf{j} + z\ \mathbf{k}$ around C, which is the intersection of cylinder $x^2 + y^2 = 4$ and hemisphere $x^2 + y^2 + z^2 = 16,\ z \ge 0$, oriented counterclockwise when viewed from above.

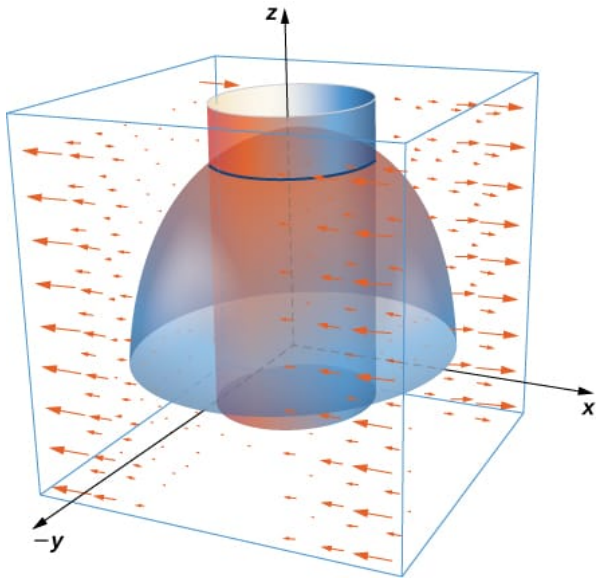


Figure 3.8E.3: C is the intersection of cylinder $x^2 + y^2 = 4$ and hemisphere $x^2 + y^2 + z^2 = 16,\ z \ge 0$, oriented counterclockwise when viewed from above

30. Use Stokes' theorem to compute $\iint_S curl\ \mathbf{F} \cdot dS$. where $\mathbf{F}(x, y, z) =\ \mathbf{i} + xy^2\ \mathbf{j} + xy^2\ \mathbf{k}$ and S is a part of plane $y + z = 2$ inside cylinder $x^2 + y^2 = 1$ and oriented counterclockwise.

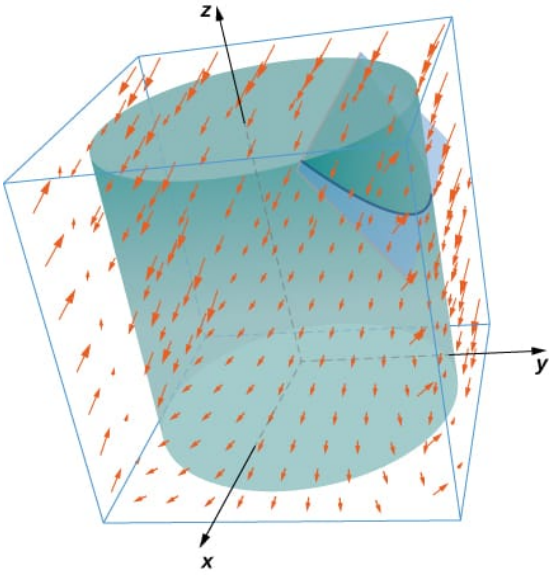


Figure 3.8E.4: S is a part of plane $y + z = 2$ inside cylinder $x^2 + y^2 = 1$ and oriented counterclockwise.

Answer

$$\iint_S curl\ \mathbf{F} \cdot dS = 0$$

31. Use Stokes' theorem to evaluate $\iint_S curl\ \mathbf{F} \cdot dS$, where $\mathbf{F}(x, y, z) = -y^2\ \mathbf{i} + x\ \mathbf{j} + z^2\ \mathbf{k}$ and S is the part of plane $x + y + z = 1$ in the positive octant and oriented counterclockwise $x \geq 0,\ y \geq 0,\ z \geq 0$.

32. Let $\mathbf{F}(x, y, z) = xy\ \mathbf{i} + 2z\ \mathbf{j} - 2y\ \mathbf{k}$ and let C be the intersection of plane $x + z = 5$ and cylinder $x^2 + y^2 = 9$, which is oriented counterclockwise when viewed from the top. Compute the line integral of $\mathbf{F}$ over C using Stokes' theorem.

Answer

$$\iint_S curl\ \mathbf{F} \cdot dS = -36\pi$$

33. [T] Use a CAS and let $\mathbf{F}(x, y, z) = xy^2\ \mathbf{i} + (yz - x)\ \mathbf{j} + e^{yxz}\ \mathbf{k}$. Use Stokes' theorem to compute the surface integral of curl $\mathbf{F}$ over surface S with inward orientation consisting of cube $[0, 1] \times [0, 1] \times [0, 1]$ with the right side missing.

34. Let S be ellipsoid $\frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1$ oriented counterclockwise and let $\mathbf{F}$ be a vector field with component functions that have continuous partial derivatives.

Answer

$$\iint_S curl\ \mathbf{F} \cdot \mathbf{N} = 0$$

35. Let S be the part of paraboloid $z = 9 - x^2 - y^2$ with $z \geq 0$. Verify Stokes' theorem for vector field $\mathbf{F}(x, y, z) = 3z\ \mathbf{i} + 4x\ \mathbf{j} + 2y\ \mathbf{k}$.

36. [T] Use a CAS and Stokes' theorem to evaluate $\oint \mathbf{F} \cdot dS$, if $\mathbf{F}(x, y, z) = (3z - \sin x)\ \mathbf{i} + (x^2 + e^y)\ \mathbf{j} + (y^3 - \cos z)\ \mathbf{k}$, where C is the curve given by $x = \cos t,\ y = \sin t,\ z = 1;\ 0 \leq t \leq 2\pi$.

Answer

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$$

37. [T] Use a CAS and Stokes' theorem to evaluate $\mathbf{F}(x, y, z) = 2y\ \mathbf{i} + e^z\ \mathbf{j} - \arctan x\ \mathbf{k}$ with S as a portion of paraboloid $z = 4 - x^2 - y^2$ cut off by the xy-plane oriented counterclockwise.

38. [T] Use a CAS to evaluate $\iint_S curl(F) \cdot dS$, where $\mathbf{F}(x, y, z) = 2z\ \mathbf{i} + 3x\ \mathbf{j} + 5y\ \mathbf{k}$ and S is the surface parametrically by $\mathbf{r}(r, \theta)$

$= r\ \cos\theta\ \mathbf{i} + r\ \sin\theta\ \mathbf{j} + (4 - r^2)\ \mathbf{k}\ (0 \le \theta \le 2\pi,\ 0 \le r \le 3)$.

Answer

$$\iint_S curl(F) \cdot dS = 84.8230$$

39. Let S be paraboloid $z = a(1 - x^2 - y^2)$, for $z \ge 0$, where $a > 0$ is a real number. Let $\mathbf{F}(x, y, z) = \langle x - y,\ y + z,\ z - x \rangle$. For what value(s) of a (if any) does $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\ dS$ have its maximum value?

Exercises 40 - 41

The goal is to evaluate $A = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\ dS$, where $\mathbf{F} = \langle xz,\ -xz,\ xy \rangle$ and S is the upper half of ellipsoid $x^2 + y^2 + 8z^2 = 1$, where $z \ge 0$.

40. Evaluate a surface integral over a more convenient surface to find the value of A.

Answer

$$A = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\ dS = 0$$

41. Evaluate A using a line integral.

42. Take paraboloid $z = x^2 + y^2$, for $0 \le z \le 4$, and slice it with plane $y = 0$. Let S be the surface that remains for $y \ge 0$, including the planar surface in the xz-plane. Let C be the semicircle and line segment that bounded the cap of S in plane $z = 4$ with counterclockwise orientation. Let $\mathbf{F} = \langle 2z + y,\ 2x + z,\ 2y + x \rangle$. Evaluate $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\ dS$.

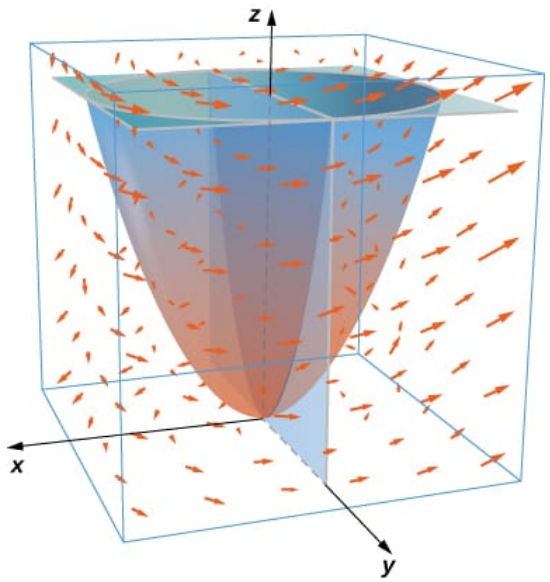


Figure $3.8E.5$: S is the surface that remains for $y \ge 0$, including the planar surface in the xz-plane.

Answer

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\ dS = 2\pi$$

Exercises 43 - 45

Let S be the disk enclosed by curve $C\ :\ \mathbf{r}(t) = \langle \cos\varphi\ \cos t,\ \sin t,\ \sin\varphi\ \cos t \rangle$, for $0 \le t \le 2\pi$, where $0 \le \varphi \le \frac{\pi}{2}$ is a fixed angle.

43. What is the length of C in terms of φ?

44. What is the circulation of C of vector field $\mathbf{F} = \langle -y,\ -z,\ x \rangle$ as a function of φ?

Answer

$$C = \pi(\cos\varphi - \sin\varphi)$$

45. For what value of φ is the circulation a maximum?

46. Circle C in plane $x + y + z = 8$ has radius 4 and center $(2, 3, 3)$. Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ for $\mathbf{F} = \langle 0,\ -z,\ 2y \rangle$, where C has a counterclockwise orientation when viewed from above.

Answer

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 48\pi$$

47. Velocity field $v = \langle 0,\ 1 - x^2,\ 0 \rangle$, for $|x| \leq 1$ and $|z| \leq 1$, represents a horizontal flow in the y-direction. Compute the curl of $\mathbf{v}$ in a clockwise rotation.

48. Evaluate integral $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\ dS$, where $\mathbf{F} = -xz\ \mathbf{i} + yz\ \mathbf{j} + xye^z\ \mathbf{k}$ and S is the cap of paraboloid $z = 5 - x^2 - y^2$ above plane $z = 3$, and $\mathbf{n}$ points in the positive z-direction on S.

Answer

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} = 0$$

In exercises 49 - 50

Use Stokes' theorem to find the circulation of the following vector fields around any smooth, simple closed curve C.

49. $\mathbf{F} = \nabla(x\ \sin ye^z)$

50. $\mathbf{F} = \langle y^2z^3,\ z2xyz^3, 3xy^2z^2 \rangle$

Answer

$$0$$

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3.9: The Divergence Theorem

Learning Objectives

- Explain the meaning of the divergence theorem.
- Use the divergence theorem to calculate the flux of a vector field.
- Apply the divergence theorem to an electrostatic field.

We have examined several versions of the Fundamental Theorem of Calculus in higher dimensions that relate the integral around an oriented boundary of a domain to a "derivative" of that entity on the oriented domain. In this section, we state the divergence theorem, which is the final theorem of this type that we will study. The divergence theorem has many uses in physics; in particular, the divergence theorem is used in the field of partial differential equations to derive equations modeling heat flow and conservation of mass. We use the theorem to calculate flux integrals and apply it to electrostatic fields.

Overview of Theorems

Before examining the divergence theorem, it is helpful to begin with an overview of the versions of the Fundamental Theorem of Calculus we have discussed:

1. The Fundamental Theorem of Calculus:
$$\int_a^b f'(x)\,dx = f(b) - f(a). \tag{3.9.1}$$
This theorem relates the integral of derivative f' over line segment $[a, b]$ along the x-axis to a difference of f evaluated on the boundary.
2. The Fundamental Theorem for Line Integrals:
$$\int_C \nabla f \cdot d\mathbf{r} = f(P_1) - f(P_0), \tag{3.9.2}$$
where P_0 is the initial point of C and P_1 is the terminal point of C. The Fundamental Theorem for Line Integrals allows path C to be a path in a plane or in space, not just a line segment on the x-axis. If we think of the gradient as a derivative, then this theorem relates an integral of derivative ∇f over path C to a difference of f evaluated on the boundary of C.
3. Green's theorem, circulation form:
$$\iint_D (Q_x - P_y)\,dA = \int_C \mathbf{F} \cdot d\mathbf{r}. \tag{3.9.3}$$
Since $Q_x - P_y = \text{curl } \mathbf{F} \cdot \mathbf{k}$ and curl is a derivative of sorts, Green's theorem relates the integral of derivative curl $\mathbf{F}$ over planar region D to an integral of $\mathbf{F}$ over the boundary of D.
4. Green's theorem, flux form:
$$\iint_D (P_x + Q_y)\,dA = \int_C \mathbf{F} \cdot \mathbf{N}\,dS. \tag{3.9.4}$$
Since $P_x + Q_y = \text{div } \mathbf{F}$ and divergence is a derivative of sorts, the flux form of Green's theorem relates the integral of derivative div $\mathbf{F}$ over planar region D to an integral of $\mathbf{F}$ over the boundary of D.
5. Stokes' theorem:
$$\iint_S curl\, \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}. \tag{3.9.5}$$
If we think of the curl as a derivative of sorts, then Stokes' theorem relates the integral of derivative curl $\mathbf{F}$ over surface S (not necessarily planar) to an integral of $\mathbf{F}$ over the boundary of S.

Stating the Divergence Theorem

The divergence theorem follows the general pattern of these other theorems. If we think of divergence as a derivative of sorts, then the divergence theorem relates a triple integral of derivative div $\mathbf{F}$ over a solid to a flux integral of $\mathbf{F}$ over the boundary of the solid. More specifically, the divergence theorem relates a flux integral of vector field $\mathbf{F}$ over a closed surface S to a triple integral of the divergence of $\mathbf{F}$ over the solid enclosed by S.

The Divergence Theorem

Let S be a piecewise, smooth closed surface that encloses solid E in space. Assume that S is oriented outward, and let $\mathbf{F}$ be a vector field with continuous partial derivatives on an open region containing E (Figure 3.9.1). Then

$$\iiint_E \text{div } \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot d\mathbf{S}. \tag{3.9.6}$$

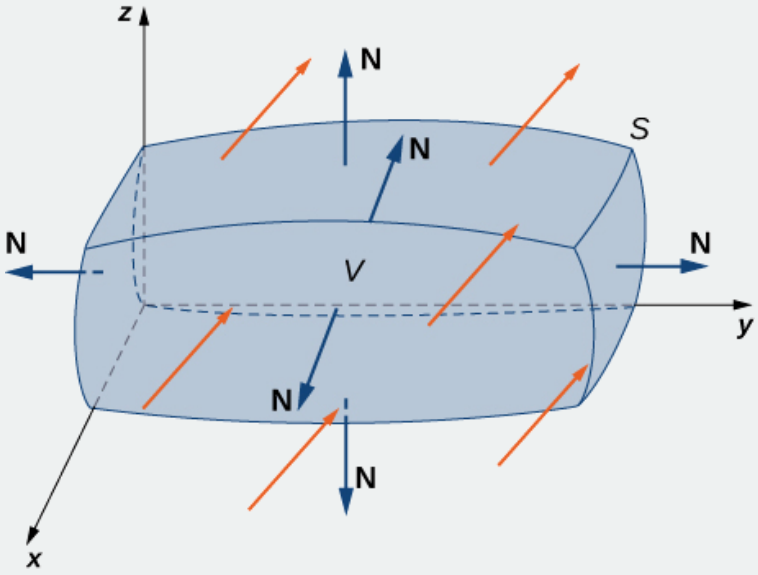


Figure 3.9.1: S is a piecewise, smooth closed surface which is oriented outward that encloses solid E in space.

Recall that the flux form of Green's theorem states that

$$\iint_D \text{div } \mathbf{F}\, dA = \int_C \mathbf{F} \cdot \mathbf{N}\, dS. \tag{3.9.7}$$

Therefore, the divergence theorem is a version of Green's theorem in one higher dimension.

The proof of the divergence theorem is beyond the scope of this text. However, we look at an informal proof that gives a general feel for why the theorem is true, but does not prove the theorem with full rigor. This explanation follows the informal explanation given for why Stokes' theorem is true.

Proof

Let B be a small box with sides parallel to the coordinate planes inside E (Figure 3.9.2a). Let the center of B have coordinates (x, y, z) and suppose the edge lengths are Δx, Δy, and Δz. (Figure 3.9.1b). The normal vector out of the top of the box is $\mathbf{k}$ and the normal vector out of the bottom of the box is $-\mathbf{k}$. The dot product of $\mathbf{F} = \langle P, Q, R \rangle$ with $\mathbf{k}$ is R and the dot product with $-\mathbf{k}$ is $-R$. The area of the top of the box (and the bottom of the box) ΔS is $\Delta x \Delta y$.

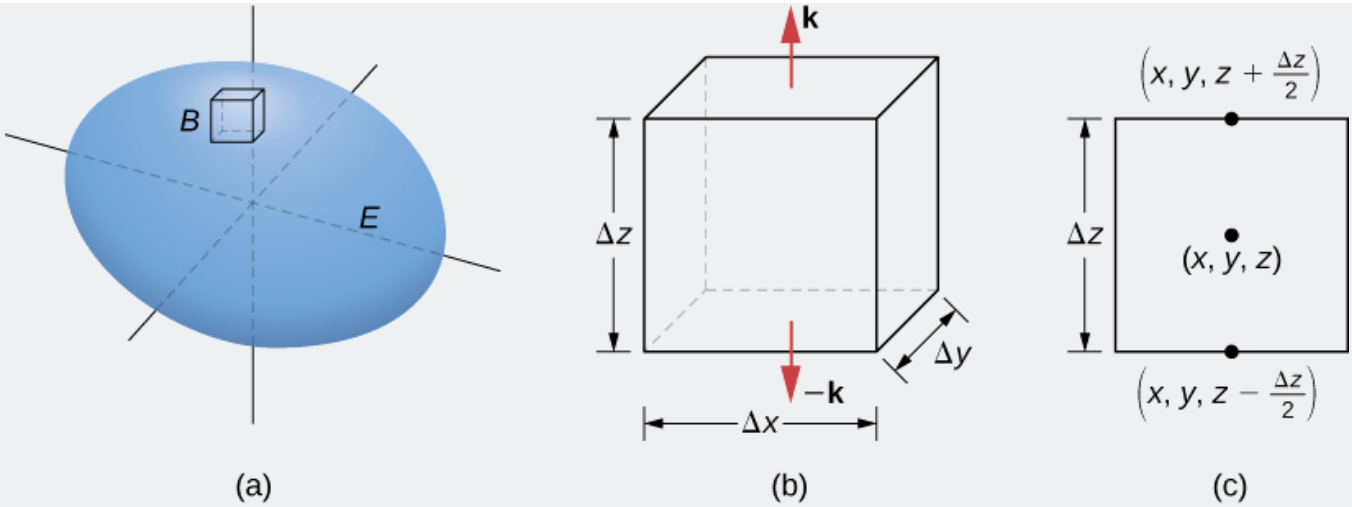


Figure 3.9.2: (a) A small box B inside surface E has sides parallel to the coordinate planes. (b) Box B has side lengths$\Delta x,\ \Delta y$, and Δz (c) If we look at the side view of B, we see that, since (x, y, z) is the center of the box, to get to the top of the box we must travel a vertical distance of $\frac{\Delta z}{2}$ up from (x, y, z). Similarly, to get to the bottom of the box we must travel a distance $\frac{\Delta z}{2}$ down from (x, y, z).

The flux out of the top of the box can be approximated by $R\left(x,\, y,\, z+\frac{\Delta z}{2}\right)\ \Delta x\ \Delta y$ (Figure 3.9.2c) and the flux out of the bottom of the box is $-R\left(x,\, y,\, z-\frac{\Delta z}{2}\right)\ \Delta x\ \Delta y$. If we denote the difference between these values as ΔR, then the net flux in the vertical direction can be approximated by $\Delta R\ \Delta x\ \Delta y$. However,

$$\Delta R\,\Delta x\,\Delta y = \left(\frac{\Delta R}{\Delta z}\right)\Delta x\,\Delta y\Delta z \approx \left(\frac{\partial R}{\partial z}\right)\Delta V.$$

Therefore, the net flux in the vertical direction can be approximated by $\left(\frac{\partial R}{\partial z}\right)\Delta V$. Similarly, the net flux in the x-direction can be approximated by $\left(\frac{\partial P}{\partial x}\right)\ \Delta V$ and the net flux in the y-direction can be approximated by $\left(\frac{\partial Q}{\partial y}\right)\ \Delta V$. Adding the fluxes in all three directions gives an approximation of the total flux out of the box:

$$\text{Total flux } \approx \left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)\Delta V = \operatorname{div}\mathbf{F}\,\Delta V.$$

This approximation becomes arbitrarily close to the value of the total flux as the volume of the box shrinks to zero.

The sum of $\operatorname{div}\mathbf{F}\ \Delta V$ over all the small boxes approximating E is approximately $\iiint_E \operatorname{div}\mathbf{F}\ dV$. On the other hand, the sum of $\operatorname{div}\mathbf{F}\ \Delta V$ over all the small boxes approximating E is the sum of the fluxes over all these boxes. Just as in the informal proof of Stokes' theorem, adding these fluxes over all the boxes results in the cancelation of a lot of the terms. If an approximating box shares a face with another approximating box, then the flux over one face is the negative of the flux over the shared face of the adjacent box. These two integrals cancel out. When adding up all the fluxes, the only flux integrals that survive are the integrals over the faces approximating the boundary of E. As the volumes of the approximating boxes shrink to zero, this approximation becomes arbitrarily close to the flux over S.

□

Example 3.9.1: Verifying the Divergence Theorem

Verify the divergence theorem for vector field $\mathbf{F} = \langle x-y,\ x+z,\ z-y\rangle$ and surface S that consists of cone $x^2+y^2=z^2,\ 0 \le z \le 1$, and the circular top of the cone in Figure 3.9.3. Assume this surface is positively oriented.

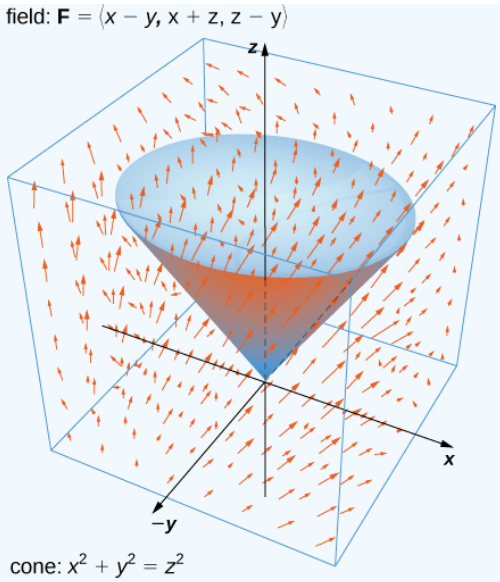


Figure 3.9.3: S consists of cone $x^2 + y^2 = z^2,\ 0 \le z \le 1$, and the circular top of the cone.

Solution

Let E be the solid cone enclosed by S. To verify the theorem for this example, we show that

$$\iiint_E \text{div } \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$$

by calculating each integral separately.

To compute the triple integral, note that $\text{div } \mathbf{F} = P_x + Q_y + R_z = 2$, and therefore the triple integral is

$$\begin{aligned} \iiint_E \text{div } \mathbf{F}\, dV &= 2 \iiint_E dV \\ &= 2\,(volume\ of\ E). \end{aligned}$$

The volume of a right circular cone is given by $\pi r^2 \frac{h}{3}$. In this case, $h = r = 1$. Therefore,

$$\iiint_E \text{div } \mathbf{F}\, dV = 2\,(volume\ of\ E) = \frac{2\pi}{3}.$$

To compute the flux integral, first note that S is piecewise smooth; S can be written as a union of smooth surfaces. Therefore, we break the flux integral into two pieces: one flux integral across the circular top of the cone and one flux integral across the remaining portion of the cone. Call the circular top S_1 and the portion under the top S_2. We start by calculating the flux across the circular top of the cone. Notice that S_1 has parameterization

$$\mathbf{r}(u, v) = \langle u\, \cos v,\ u\, \sin v,\ 1\rangle,\ 0 \le u \le 1,\ 0 \le v \le 2\pi.$$

Then, the tangent vectors are $\mathbf{t}_u = \langle \cos v,\ \sin v,\ 0\rangle$ and $\mathbf{t}_v = \langle -u\ \sin v,\ u\ \cos v, 0\rangle$. Therefore, the flux across S_1 is

$$\begin{aligned} \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} &= \int_0^1 \int_0^{2\pi} \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{t}_u \times \mathbf{t}_v)\, dA \\ &= \int_0^1 \int_0^{2\pi} \langle u\, \cos v - u\, \sin v,\ u\, \cos v + 1,\ 1 - u\, \sin v\rangle \cdot \langle 0, 0, u\rangle\, dv\, du \\ &= \int_0^1 \int_0^{2\pi} u - u^2 \sin v\, dv du \\ &= \pi. \end{aligned}$$

We now calculate the flux over S_2. A parameterization of this surface is

$$\mathbf{r}(u,v) = \langle u\,\cos v,\ u\,\sin v,\ u\rangle,\ 0 \le u \le 1,\ 0 \le v \le 2\pi.$$

The tangent vectors are $\mathbf{t}_u = \langle \cos v,\ \sin v,\ 1\rangle$ and $\mathbf{t}_v = \langle -u\ \sin v,\ u\ \cos v, 0\rangle$, so the cross product is

$$\mathbf{t}_u \times \mathbf{t}_v = \langle -u\,\cos v,\ -u\,\sin v,\ u\rangle.$$

Notice that the negative signs on the x and y components induce the negative (or inward) orientation of the cone. Since the surface is positively oriented, we use vector $\mathbf{t}_v \times \mathbf{t}_u = \langle u\ \cos v,\ u\ \sin v,\ -u\rangle$ in the flux integral. The flux across S_2 is then

$$\begin{aligned}\iint_{S_2} \mathbf{F}\cdot d\mathbf{S} &= \int_0^1 \int_0^{2\pi} \mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{t}_u \times \mathbf{t}_v)\,dA \\ &= \int_0^1 \int_0^{2\pi} \langle u\,\cos v - u\,\sin v,\ u\,\cos v + u,\ u\ - u\sin v\rangle \cdot \langle u\,\cos v,\ u\,\sin v,\ -u\rangle\,dv\,du \\ &= \int_0^1 \int_0^{2\pi} u^2\cos^2 v + 2u^2\sin v - u^2\,dv\,du \\ &= -\frac{\pi}{3}\end{aligned}$$

The total flux across S is

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_{S_1} \mathbf{F}\cdot d\mathbf{S} + \iint_{S_2} \mathbf{F}\cdot d\mathbf{S} = \frac{2\pi}{3} = \iiint_E \text{div }\mathbf{F}\,dV,$$

and we have verified the divergence theorem for this example.

Exercise 3.9.1

Verify the divergence theorem for vector field $\mathbf{F}(x,y,z) = \langle x+y+z,\ y,\ 2x-y\rangle$ and surface S given by the cylinder $x^2+y^2 = 1,\ 0 \le z \le 3$ plus the circular top and bottom of the cylinder. Assume that S is positively oriented.

Hint available at the end of this section.

Answer available at the end of this section.

Recall that the divergence of continuous field $\mathbf{F}$ at point P is a measure of the "outflowing-ness" of the field at P. If $\mathbf{F}$ represents the velocity field of a fluid, then the divergence can be thought of as the rate per unit volume of the fluid flowing out less the rate per unit volume flowing in. The divergence theorem confirms this interpretation. To see this, let P be a point and let B_τ be a ball of small radius r centered at P (Figure 3.9.3). Let S_τ be the boundary sphere of B_τ. Since the radius is small and $\mathbf{F}$ is continuous, $\text{div }\mathbf{F}(Q) \approx \text{div }\mathbf{F}(P)$ for all other points Q in the ball. Therefore, the flux across S_τ can be approximated using the divergence theorem:

$$\iint_{S_\tau} \mathbf{F}\cdot d\mathbf{S} = \iiint_{B_\tau} \text{div }\mathbf{F}\,dV \approx \iiint_{B_\tau} \text{div }\mathbf{F}(P)\,dV.$$

Since $divF(P)$ is a constant,

$$\iiint_{B_\tau} \text{div }\mathbf{F}(P)\,dV = \text{div }\mathbf{F}(P)\,V(B_\tau).$$

Therefore, flux

$$\iint_{S_\tau} \mathbf{F}\cdot d\mathbf{S} \tag{3.9.8}$$

can be approximated by $\mathbf{F}(P)\ V(B_\tau)$. This approximation gets better as the radius shrinks to zero, and therefore

$$\text{div } \mathbf{F}(P) = \lim_{\tau \to 0} \frac{1}{V(B_\tau)} \iint_{S_\tau} \mathbf{F} \cdot d\mathbf{S}.$$

This equation says that the divergence at P is the net rate of outward flux of the fluid per unit volume.

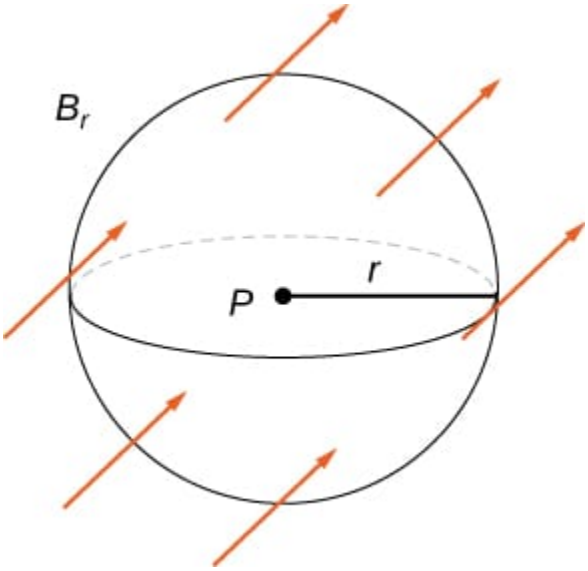


Figure 3.9.4: Ball B_τ of small radius r centered at P.

Using the Divergence Theorem

The divergence theorem translates between the flux integral of closed surface S and a triple integral over the solid enclosed by S. Therefore, the theorem allows us to compute flux integrals or triple integrals that would ordinarily be difficult to compute by translating the flux integral into a triple integral and vice versa.

Example 3.9.2: Applying the Divergence Theorem

Calculate the surface integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S},$$

where S is cylinder $x^2 + y^2 = 1,\ 0 \le z \le 2$, including the circular top and bottom, and $\mathbf{F} = \left\langle \frac{x^3}{3} + yz,\ \frac{y^3}{3} - \sin(xz),\ z - x - y \right\rangle$.

Solution

We could calculate this integral without the divergence theorem, but the calculation is not straightforward because we would have to break the flux integral into three separate integrals: one for the top of the cylinder, one for the bottom, and one for the side. Furthermore, each integral would require parameterizing the corresponding surface, calculating tangent vectors and their cross product..

By contrast, the divergence theorem allows us to calculate the single triple integral

$$\iiint_E \text{div } \mathbf{F}\, dV,$$

where E is the solid enclosed by the cylinder. Using the divergence theorem (Equation 3.9.6) and converting to cylindrical coordinates, we have

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iiint_E \operatorname{div}\,\mathbf{F}\,dV,\\ &= \iiint_E (x^2+y^2+1)\,dV\\ &= \int_0^{2\pi}\int_0^1\int_0^2 (r^2+1)\,r\,dz\,dr\,d\theta\\ &= \frac{3}{2}\int_0^{2\pi} d\theta\\ &= 3\pi.\end{aligned}$$

Example 3.9.3: Applying the Divergence Theorem

Let $\mathbf{v} = \langle -\frac{y}{z}, \frac{x}{z}, 0\rangle$ be the velocity field of a fluid. Let C be the solid cube given by $1 \le x \le 4,\ 2 \le y \le 5,\ 1 \le z \le 4$, and let S be the boundary of this cube (see the Figure 3.9.5). Find the flow rate of the fluid across S.

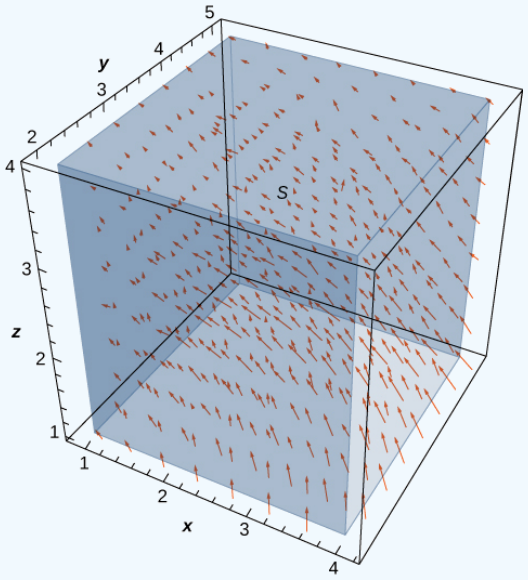


Figure 3.9.5: C is the solid cube given by $1 \le x \le 4,\ 2 \le y \le 5,\ 1 \le z \le 4$, and S is the boundary of this cube.

Solution

The flow rate of the fluid across S is $\iint_S \mathbf{v}\cdot d\mathbf{S}$. Before calculating this flux integral, let's discuss what the value of the integral should be. Based on Figure 3.9.4, we see that if we place this cube in the fluid (as long as the cube doesn't encompass the origin), then the rate of fluid entering the cube is the same as the rate of fluid exiting the cube. The field is rotational in nature and, for a given circle parallel to the xy-plane that has a center on the z-axis, the vectors along that circle are all the same magnitude. That is how we can see that the flow rate is the same entering and exiting the cube. The flow into the cube cancels with the flow out of the cube, and therefore the flow rate of the fluid across the cube should be zero.

To verify this intuition, we need to calculate the flux integral. Calculating the flux integral directly requires breaking the flux integral into six separate flux integrals, one for each face of the cube. We also need to find tangent vectors, compute their cross product. However, using the divergence theorem makes this calculation go much more quickly:

$$\begin{aligned}\iint_S \mathbf{v}\cdot d\mathbf{S} &= \iiint_C \operatorname{div}\,\mathbf{v}\,dV\\ &= \iiint_C 0\,dV = 0.\end{aligned}$$

Therefore the flux is zero, as expected.

Example illustrates a remarkable consequence of the divergence theorem. Let S be a piecewise, smooth closed surface and let $\mathbf{F}$ be a vector field defined on an open region containing the surface enclosed by S. If $\mathbf{F}$ has the form $F = \langle f(y,z),\ g(x,z),\ h(x,y)\rangle$, then the divergence of $\mathbf{F}$ is zero. By the divergence theorem, the flux of $\mathbf{F}$ across S is also zero. This makes certain flux integrals incredibly easy to calculate. For example, suppose we wanted to calculate the flux integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$ where S is a cube and

$$\mathbf{F} = \langle \sin(y)\, e^{yz},\ x^2 z^2,\ \cos(xy)\, e^{\sin x}\rangle. \tag{3.9.9}$$

Calculating the flux integral directly would be difficult, if not impossible, using techniques we studied previously. At the very least, we would have to break the flux integral into six integrals, one for each face of the cube. But, because the divergence of this field is zero, the divergence theorem immediately shows that the flux integral is zero.

We can now use the divergence theorem to justify the physical interpretation of divergence that we discussed earlier. Recall that if $\mathbf{F}$ is a continuous three-dimensional vector field and P is a point in the domain of $\mathbf{F}$, then the divergence of $\mathbf{F}$ at P is a measure of the "outflowing-ness" of $\mathbf{F}$ at P. If $\mathbf{F}$ represents the velocity field of a fluid, then the divergence of $\mathbf{F}$ at P is a measure of the net flow rate out of point P (the flow of fluid out of P less the flow of fluid in to P). To see how the divergence theorem justifies this interpretation, let B_τ be a ball of very small radius *r* with center P, and assume that B_τ is in the domain of $\mathbf{F}$. Furthermore, assume that B_τ has a positive, outward orientation. Since the radius of B_τ is small and $\mathbf{F}$ is continuous, the divergence of $\mathbf{F}$ is approximately constant on B_τ. That is, if P' is any point in B_τ, then $\text{div }\mathbf{F}(P) \approx \text{div }\mathbf{F}(P')$. Let S_τ denote the boundary sphere of B_τ. We can approximate the flux across S_τ using the divergence theorem as follows:

$$\begin{aligned}\iint_{S_\tau} \mathbf{F}\cdot d\mathbf{S} &= \iiint_{B_\tau} \text{div }\mathbf{F}\, dV \\ &\approx \iiint_{B_\tau} \text{div }\mathbf{F}(P)\, dV \\ &= \text{div }\mathbf{F}(P)\, V(B_\tau).\end{aligned}$$

As we shrink the radius r to zero via a limit, the quantity $\text{div }\mathbf{F}(P)\ V(B_\tau)$ gets arbitrarily close to the flux. Therefore,

$$\text{div }\mathbf{F}(P) = \lim_{\tau\to 0} \frac{1}{V(B_\tau)} \iint_{S_\tau} \mathbf{F}\cdot d\mathbf{S} \tag{3.9.10}$$

and we can consider the divergence at P as measuring the net rate of outward flux per unit volume at P. Since "outflowing-ness" is an informal term for the net rate of outward flux per unit volume, we have justified the physical interpretation of divergence we discussed earlier, and we have used the divergence theorem to give this justification.

Application to Electrostatic Fields

The divergence theorem has many applications in physics and engineering. It allows us to write many physical laws in both an integral form and a differential form (in much the same way that Stokes' theorem allowed us to translate between an integral and differential form of Faraday's law). Areas of study such as fluid dynamics, electromagnetism, and quantum mechanics have equations that describe the conservation of mass, momentum, or energy, and the divergence theorem allows us to give these equations in both integral and differential forms.

One of the most common applications of the divergence theorem is to **electrostatic fields**. An important result in this subject is **Gauss' law**. This law states that if S is a closed surface in electrostatic field $\mathbf{E}$, then the flux of $\mathbf{E}$ across S is the total charge enclosed by S (divided by an electric constant). We now use the divergence theorem to justify the special case of this law in which the electrostatic field is generated by a stationary point charge at the origin.

If (x,y,z) is a point in space, then the distance from the point to the origin is $r = \sqrt{x^2+y^2+z^2}$. Let $\mathbf{F}_\tau$ denote radial vector field $\mathbf{F}_\tau = \frac{1}{\tau^2}\left\langle \frac{x}{\tau}, \frac{y}{\tau}, \frac{z}{\tau}\right\rangle$.The vector at a given position in space points in the direction of unit radial vector $\left\langle \frac{x}{\tau}, \frac{y}{\tau}, \frac{z}{\tau}\right\rangle$ and is scaled by the quantity $1/\tau^2$. Therefore, the magnitude of a vector at a given point is inversely proportional to the square of the vector's distance from the origin. Suppose we have a stationary charge of q Coulombs at the origin, existing in a vacuum. The charge generates electrostatic field $\mathbf{E}$ given by

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\mathbf{F}_\tau, \tag{3.9.11}$$

where the approximation $\epsilon_0 = 8.854 \times 10^{-12}$ farad (F)/m is an electric constant. (The constant ϵ_0 is a measure of the resistance

encountered when forming an electric field in a vacuum.) Notice that $\mathbf{E}$ is a radial vector field similar to the gravitational field. The difference is that this field points outward whereas the gravitational field points inward. Because

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\mathbf{F}_\tau = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\tau^2}\left\langle\frac{x}{\tau}, \frac{y}{\tau}, \frac{z}{\tau}\right\rangle\right), \tag{3.9.12}$$

we say that electrostatic fields obey an inverse-square law. That is, the electrostatic force at a given point is inversely proportional to the square of the distance from the source of the charge (which in this case is at the origin). Given this vector field, we show that the flux across closed surface S is zero if the charge is outside of S, and that the flux is $q/epsilon_0$ if the charge is inside of S. In other words, the flux across S is the charge inside the surface divided by constant ϵ_0. This is a special case of Gauss' law, and here we use the divergence theorem to justify this special case.

To show that the flux across S is the charge inside the surface divided by constant ϵ_0, we need two intermediate steps. First we show that the divergence of $\mathbf{F}_\tau$ is zero and then we show that the flux of $\mathbf{F}_\tau$ across any smooth surface S is either zero or 4π. We can then justify this special case of Gauss' law.

Example 3.9.4: The Divergence of F_τ is Zero

Verify that the divergence of $\mathbf{F}_\tau$ is zero where $\mathbf{F}_\tau$ is defined (away from the origin).

Solution

Since $\tau = \sqrt{x^2+y^2+z^2}$, the quotient rule gives us

$$\begin{aligned}\frac{\partial}{\partial x}\left(\frac{x}{\tau^3}\right) &= \frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right)\\ &= \frac{(x^2+y^2+z^2)^{3/2} - x\left[\frac{3}{2}(x^2+y^2+z^2)^{1/2}2x\right]}{(x^2+y^2+z^2)^3}\\ &= \frac{\tau^3 - 3x^2\tau}{\tau^6} = \frac{\tau^2-3x^2}{\tau^5}.\end{aligned}$$

Similarly,

$$\frac{\partial}{\partial y}\left(\frac{y}{\tau^3}\right) = \frac{\tau^2-3y^2}{\tau^5}\, and\, \frac{\partial}{\partial z}\left(\frac{z}{\tau^3}\right) = \frac{\tau^2-3z^2}{\tau^5}.$$

Therefore,

$$\begin{aligned}\text{div }\mathbf{F}_\tau &= \frac{\tau^2-3x^2}{\tau^5} + \frac{\tau^2-3y^2}{\tau^5} + \frac{\tau^2-3z^2}{\tau^5}\\ &= \frac{3\tau^2 - 3(x^2+y^2+z^2)}{\tau^5}\\ &= \frac{3\tau^2-3\tau^2}{\tau^5} = 0.\end{aligned}$$

Notice that since the divergence of $\mathbf{F}_\tau$ is zero and $\mathbf{E}$ is $\mathbf{F}_\tau$ scaled by a constant, the divergence of electrostatic field $\mathbf{E}$ is also zero (except at the origin).

Flux Across a Smooth Surface

Let S be a connected, piecewise smooth closed surface and let $\mathbf{F}_\tau = \frac{1}{\tau^2}\left\langle\frac{x}{\tau}, \frac{y}{\tau}, \frac{z}{\tau}\right\rangle$. Then,

$$\iint_S \mathbf{F}_\tau \cdot d\mathbf{S} = \begin{cases}0, & \text{if } S \text{ does not encompass the origin}\\ 4\pi, & \text{if } S \text{ encompasses the origin.}\end{cases} \tag{3.9.13}$$

In other words, this theorem says that the flux of $\mathbf{F}_\tau$ across any piecewise smooth closed surface S depends only on whether the origin is inside of S.

Proof

We use the divergence theorem rather than Green's theorem.

First, suppose that S does not encompass the origin. In this case, the solid enclosed by S is in the domain of $\mathbf{F}_\tau$, and since the divergence of $\mathbf{F}_\tau$ is zero, we can immediately apply the divergence theorem and find that

$$\iint_S \mathbf{F}\cdot d\mathbf{S} \tag{3.9.14}$$

is zero.

Now suppose that S does encompass the origin. We cannot just use the divergence theorem to calculate the flux, because the field is not defined at the origin. Let S_a be a sphere of radius *a* inside of S centered at the origin. The outward normal vector field on the sphere, in spherical coordinates, is

$$\mathbf{t}_\phi \times \mathbf{t}_\theta = \langle a^2 \cos\theta \sin^2\phi,\ a^2 \sin\theta \sin^2\phi,\ a^2 \sin\phi \cos\phi\rangle \tag{3.9.15}$$

. Therefore, on the surface of the sphere, the dot product $\mathbf{F}_\tau \cdot \mathbf{N}$ (in spherical coordinates) is

$$\begin{aligned}\mathbf{F}_\tau \cdot \mathbf{N} &= \left\langle \frac{\sin\phi\cos\theta}{a^2}, \frac{\sin\phi\sin\theta}{a^2}, \frac{\cos\phi}{a^2}\right\rangle \cdot \langle a^2\cos\theta\sin^2\phi, a^2\sin\theta\sin^2\phi,\ a^2\sin\phi\cos\phi\rangle \\ &= \sin\phi(\langle\sin\phi\cos\theta,\ \sin\phi\sin\theta,\ \cos\phi\rangle \cdot \langle\sin\phi\cos\theta, \sin\phi\sin\theta,\ \cos\phi\rangle) \\ &= \sin\phi.\end{aligned}$$

The flux of $\mathbf{F}_\tau$ across S_a is

$$\iint_{S_a} \mathbf{F}_\tau \cdot \mathbf{N} dS = \int_0^{2\pi}\int_0^{\pi} \sin\phi\, d\phi\, d\theta = 4\pi. \tag{3.9.16}$$

Now, remember that we are interested in the flux across S, not necessarily the flux across S_a. To calculate the flux across S, let E be the solid between surfaces S_a and S. Then, the boundary of E consists of S_a and S. Denote this boundary by $S - S_a$ to indicate that S is oriented outward but now S_a is oriented inward. We would like to apply the divergence theorem to solid E. Notice that the divergence theorem, as stated, can't handle a solid such as E because E has a hole. However, the divergence theorem can be extended to handle solids with holes, just as Green's theorem can be extended to handle regions with holes. This allows us to use the divergence theorem in the following way. By the divergence theorem,

$$\begin{aligned}\iint_{S-S_a} \mathbf{F}_\tau\cdot d\mathbf{S} &= \iint_S \mathbf{F}_\tau\cdot d\mathbf{S} - \iint_{S_a} \mathbf{F}_\tau\cdot d\mathbf{S} \\ &= \iiint_E \operatorname{div}\mathbf{F}_\tau\, dV \\ &= \iiint_E 0\, dV = 0.\end{aligned}$$

Therefore,

$$\iint_S \mathbf{F}_\tau\cdot d\mathbf{S} = \iint_{S_a} \mathbf{F}_\tau\cdot d\mathbf{S} = 4\pi,$$

and we have our desired result.

□

Now we return to calculating the flux across a smooth surface in the context of electrostatic field $\mathbf{E} = \dfrac{q}{4\pi\epsilon_0}\mathbf{F}_\tau$ of a point charge at the origin. Let S be a piecewise smooth closed surface that encompasses the origin. Then

$$\begin{aligned}\iint_S \mathbf{E}\cdot d\mathbf{S} &= \iint_S \frac{q}{4\pi\epsilon_0}\mathbf{F}_\tau\cdot d\mathbf{S}\\ &= \frac{q}{4\pi\epsilon_0}\iint_S \mathbf{F}_\tau\cdot d\mathbf{S}\\ &= \frac{q}{\epsilon_0}.\end{aligned}$$

If S does not encompass the origin, then

$$\iint_S \mathbf{E}\cdot d\mathbf{S} = \frac{q}{4\pi\epsilon_0}\iint_S \mathbf{F}_\tau\cdot d\mathbf{S} = 0.$$

Therefore, we have justified the claim that we set out to justify: the flux across closed surface S is zero if the charge is outside of S, and the flux is q/ϵ_0 if the charge is inside of S.

This analysis works only if there is a single point charge at the origin. In this case, Gauss' law says that the flux of $\mathbf{E}$ across S is the total charge enclosed by S. Gauss' law can be extended to handle multiple charged solids in space, not just a single point charge at the origin. The logic is similar to the previous analysis, but beyond the scope of this text. In full generality, Gauss' law states that if S is a piecewise smooth closed surface and Q is the total amount of charge inside of S, then the flux of $\mathbf{E}$ across S is Q/ϵ_0.

Example 3.9.5: Using Gauss' law

Suppose we have four stationary point charges in space, all with a charge of 0.002 Coulombs (C). The charges are located at $(0,0,1)$, $(1,1,4)$, $(-1,0,0)$, and $(-2,-2,2)$. Let $\mathbf{E}$ denote the electrostatic field generated by these point charges. If S is the sphere of radius 2 oriented outward and centered at the origin, then find

$$\iint_S \mathbf{E}\cdot d\mathbf{S}.$$

Solution

According to Gauss' law, the flux of $\mathbf{E}$ across S is the total charge inside of S divided by the electric constant. Since S has radius 2, notice that only two of the charges are inside of S: the charge at $(0,1,1)$ and the charge at $(-1,0,0)$. Therefore, the total charge encompassed by S is 0.004 and, by Gauss' law,

$$\iint_S \mathbf{E}\cdot d\mathbf{S} = \frac{0.004}{8.854\times 10^{-12}} \approx 4.418\times 10^9\, V-m.$$

Exercise 3.9.2

Work the previous example for surface S that is a sphere of radius 4 centered at the origin, oriented outward.

Hint available at the end of this section.

Answer available at the end of this section.

Key Concepts

- The divergence theorem relates a surface integral across closed surface S to a triple integral over the solid enclosed by S. The divergence theorem is a higher dimensional version of the flux form of Green's theorem, and is therefore a higher dimensional version of the Fundamental Theorem of Calculus.
- The divergence theorem can be used to transform a difficult flux integral into an easier triple integral and vice versa.
- The divergence theorem can be used to derive Gauss' law, a fundamental law in electrostatics.

Key Equations

- **Divergence theorem**

$$\iiint_E \operatorname{div} \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$$

Glossary

Divergence theorem
: a theorem used to transform a difficult flux integral into an easier triple integral and vice versa

Gauss' law
: if S is a piecewise, smooth closed surface in a vacuum and Q is the total stationary charge inside of S, then the flux of electrostatic field $\mathbf{E}$ across S is Q/ϵ_0

Inverse-square law
: the electrostatic force at a given point is inversely proportional to the square of the distance from the source of the charge

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Answers

Exercise $3.9.1$

Calculate both the flux integral and the triple integral with the divergence theorem and verify they are equal.

Exercise $3.9.1$

Both integrals equal 6π.

Exercise $3.9.2$

Use Gauss' law.

Exercise $3.9.2$

$\approx 6.777 \times 10^9$

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3.9E: Exercises

Exercises 1 - 9

Use a computer algebraic system (CAS) and the divergence theorem to evaluate surface integral $\int_S \mathbf{F} \cdot \mathbf{n}\ ds$ for the given choice of $\mathbf{F}$ and the boundary surface S. For each closed surface, assume $\mathbf{N}$ is the outward unit normal vector.

1. $\mathbf{F}(x,y,z) = x\ \mathbf{i} + y\ \mathbf{j} + z\ \mathbf{k}$; S is the surface of cube $0 \le x \le 1,\ 0 \le y \le 1,\ 0 < z \le 1$.

2. $\mathbf{F}(x,y,z) = (\cos yz)\ \mathbf{i} + e^{xz}\ \mathbf{j} + 3z^2\ \mathbf{k}$; S is the surface of hemisphere $z = \sqrt{4 - x^2 - y^2}$ together with disk $x^2 + y^2 \le 4$ in the xy -plane.

Answer

$$\int_S \mathbf{F} \cdot \mathbf{n}\ ds = 75.3982$$

3. $\mathbf{F}(x,y,z) = (x^2 + y^2 - x^2)\ \mathbf{i} + x^2 y\ \mathbf{j} + 3z\ \mathbf{k}$; S is the surface of the five faces of unit cube $0 \le x \le 1,\ 0 \le y \le 1,\ 0 < z \le 1$.

4. $\mathbf{F}(x,y,z) = x\ \mathbf{i} + y\ \mathbf{j} + z\ \mathbf{k}$; S is the surface of paraboloid $z = x^2 + y^2$ for $0 \le z \le 9$.

Answer

$$\int_S \mathbf{F} \cdot \mathbf{n}\ ds = 127.2345$$

5. $\mathbf{F}(x,y,z) = x^2\ \mathbf{i} + y^2\ \mathbf{j} + z^2\ \mathbf{k}$; S is the surface of sphere $x^2 + y^2 + z^2 = 4$.

6. $\mathbf{F}(x,y,z) = x\ \mathbf{i} + y\ \mathbf{j} + (z^2 - 1)\ \mathbf{k}$; S is the surface of the solid bounded by cylinder $x^2 + y^2 = 4$ and planes $z = 0$ and $z = 1$.

Answer

$$\int_S \mathbf{F} \cdot \mathbf{n}\ ds = 37.699$$

7. $\mathbf{F}(x,y,z) = x^2\ \mathbf{i} + y^2\ \mathbf{j} + z^2\ \mathbf{k}$; S is the surface bounded above by sphere $\rho = 2$ and below by cone $\varphi = \dfrac{\pi}{4}$ in spherical coordinates. (Think of S as the surface of an “ice cream cone.”)

8. $\mathbf{F}(x,y,z) = x^3\ \mathbf{i} + y^3\ \mathbf{j} + 3a^2 z\ \mathbf{k}\ (constant\ a > 0)$; S is the surface bounded by cylinder $x^2 + y^2 = a^2$ and planes $z = 0$ and $z = 1$.

Answer

$$\int_S \mathbf{F} \cdot \mathbf{n}\ ds = \frac{9\pi a^4}{2}$$

9. Surface integral $\iint_S \mathbf{F} \cdot dS$, where S is the solid bounded by paraboloid $z = x^2 + y^2$ and plane $z = 4$, and $\mathbf{F}(x,y,z) = (x + y^2 z^2)\ \mathbf{i} + (y + z^2 x^2)\ \mathbf{j} + (z + x^2 y^2)\ \mathbf{k}$

10. Use the divergence theorem to calculate surface integral $\iint_S \mathbf{F} \cdot dS$, where $\mathbf{F}(x,y,z) = (e^{y^2}\ \mathbf{i} + (y + \sin(z^2))\ \mathbf{j} + (z - 1)\ \mathbf{k}$ and S is upper hemisphere $x^2 + y^2 + z^2 = 1,\ z \ge 0$, oriented upward.

Answer

$$\iint_S \mathbf{F} \cdot dS = \frac{\pi}{3}$$

11. Use the divergence theorem to calculate surface integral $\iint_S \mathbf{F} \cdot dS$, where $\mathbf{F}(x,y,z) = x^4\ \mathbf{i} - x^3 z^2\ \mathbf{j} + 4xy^2 z\ \mathbf{k}$ and S is the surface bounded by cylinder $x^2 + y^2 = 1$ and planes $z = x + 2$ and $z = 0$.

12. Use the divergence theorem to calculate surface integral $\iint_S \mathbf{F} \cdot dS$, when $\mathbf{F}(x, y, z) = x^2 z^3\ \mathbf{i} + 2xyz^3\ \mathbf{j} + xz^4\ \mathbf{k}$ and S is the surface of the box with vertices $(\pm 1,\ \pm 2,\ \pm 3)$.

Answer

$$\iint_S \mathbf{F} \cdot dS = 0$$

13. Use the divergence theorem to calculate surface integral $\iint_S \mathbf{F} \cdot dS$, when $\mathbf{F}(x, y, z) = z\ \tan^{-1}(y^2)\ \mathbf{i} + z^3 \ln(x^2 + 1)\ \mathbf{j} + z\ \mathbf{k}$ and S is a part of paraboloid $x^2 + y^2 + z = 2$ that lies above plane $z = 1$ and is oriented upward.

14. [T] Use a CAS and the divergence theorem to calculate flux $\iint_S \mathbf{F} \cdot dS$, where $\mathbf{F}(x, y, z) = (x^3 + y^3)\ \mathbf{i} + (y^3 + z^3)\ \mathbf{j} + (z^3 + x^3)\ \mathbf{k}$ and S is a sphere with center $(0, 0)$ and radius 2.

Answer

$$\iint_S \mathbf{F} \cdot dS = 241.2743$$

15. Use the divergence theorem to compute the value of flux integral $\iint_S \mathbf{F} \cdot dS$, where $\mathbf{F}(x, y, z) = (y^3 + 3x)\ \mathbf{i} + (xz + y)\ \mathbf{j} + (z + x^4 \cos(x^2 y))\ \mathbf{k}$ and S is the area of the region bounded by $x^2 + y^2 = 1,\ x \geq 0,\ y \geq 0$, and $0 \leq z \leq 1$.

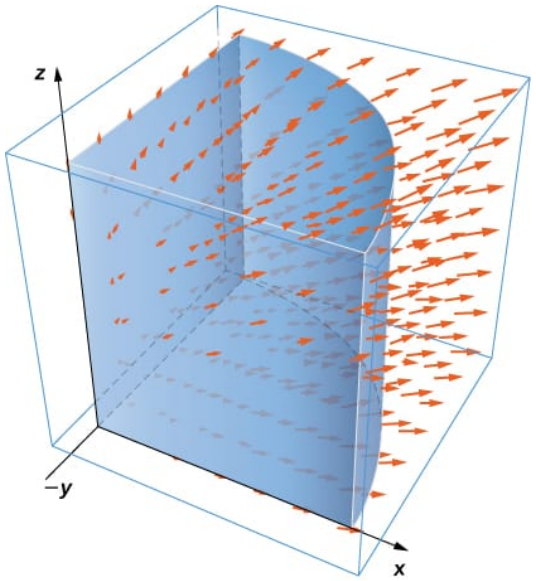


Figure $3.9E.1$: S is the area of the region bounded by $x^2 + y^2 = 1,\ x \geq 0,\ y \geq 0$, and $0 \leq z \leq 1$.

16. Use the divergence theorem to compute flux integral $\iint_S \mathbf{F} \cdot dS$, where $\mathbf{F}(x, y, z) = y\ \mathbf{j} - z\ \mathbf{k}$ and S consists of the union of paraboloid $y = x^2 + z^2,\ 0 \leq y \leq 1$, and disk $x^2 + z^2 \leq 1,\ y = 1$, oriented outward. What is the flux through just the paraboloid?

Answer

$$\iint_S \mathbf{F} \cdot dS = -\pi$$

17. Use the divergence theorem to compute flux integral $\iint_S \mathbf{F} \cdot dS$, where $\mathbf{F}(x, y, z) = x\ \mathbf{i} + y\ \mathbf{j} + z^4\ \mathbf{k}$ and S is a part of cone $z = \sqrt{x^2 + y^2}$ beneath top plane $z = 1$ oriented downward.

18. Use the divergence theorem to calculate surface integral $\iint_S \mathbf{F} \cdot dS$ for $\mathbf{F}(x, y, z) = x^4\ \mathbf{i} - x^3 z^2\ \mathbf{j} + 4xy^2 z\ \mathbf{k}$, where S is the surface bounded by cylinder $x^2 + y^2 = 1$ and planes $z = x + 2$ and $z = 0$.

Answer

$$\iint_S \mathbf{F} \cdot dS = \frac{2\pi}{3}$$

19. Consider $\mathbf{F}(x, y, z) = x^2\ \mathbf{i} + xy\ \mathbf{j} + (z+1)\ \mathbf{k}$. Let E be the solid enclosed by paraboloid $z = 4 - x^2 - y^2$ and plane $z = 0$ with normal vectors pointing outside E. Compute flux $\mathbf{F}$ across the boundary of E using the divergence theorem.

Exercises 20 - 23

Use a CAS along with the divergence theorem to compute the net outward flux for the fields across the given surfaces S.

20. [T] $\mathbf{F} = \langle x,\ -2y,\ 3z \rangle$; S is sphere $\{(x, y, z) : x^2 + y^2 + z^2 = 6\}$.

Answer

$$15\sqrt{6}\pi$$

21. [T] $\mathbf{F} = \langle x,\ 2y,\ z \rangle$; S is the boundary of the tetrahedron in the first octant formed by plane $x + y + z = 1$.

22. [T] $\mathbf{F} = \langle y - 2x,\ x^3 - y,\ y^2 - z \rangle$; S is sphere $\{(x, y, z)\ :\ x^2 + y^2 + z^2 = 4\}$.

Answer

$$-\frac{128}{3}\pi$$

23. [T] $\mathbf{F} = \langle x, y, z \rangle$; S is the surface of paraboloid $z = 4 - x^2 - y^2$, for $z \geq 0$, plus its base in the xy-plane.

Exercises 24 - 26

Use a CAS and the divergence theorem to compute the net outward flux for the vector fields across the boundary of the given regions D.

24. [T] $\mathbf{F} = \langle z - x,\ x - y,\ 2y - z \rangle$; D is the region between spheres of radius 2 and 4 centered at the origin.

Answer

$$-703.7168$$

25. [T] $\mathbf{F} = \dfrac{\mathbf{r}}{\|\mathbf{r}\|} = \dfrac{\langle x, y, z \rangle}{\sqrt{x^2 + y^2 + z^2}}$; D is the region between spheres of radius 1 and 2 centered at the origin.

26. [T] $\mathbf{F} = \langle x^2,\ -y^2,\ z^2 \rangle$; D is the region in the first octant between planes $z = 4 - x - y$ and $z = 2 - x - y$.

Answer

$$20$$

27. Let $\mathbf{F}(x, y, z) = 2x\ \mathbf{i} - 3xy\ \mathbf{j} + xz^2\ \mathbf{k}$. Use the divergence theorem to calculate $\iint_S \mathbf{F} \cdot dS$, where S is the surface of the cube with corners at $(0, 0, 0),\ (1, 0, 0),\ (0, 1, 0),\ (1, 1, 0),\ (0, 0, 1),\ (1, 0, 1),\ (0, 1, 1)$, and $(1, 1, 1)$, oriented outward.

28. Use the divergence theorem to find the outward flux of field $\mathbf{F}(x, y, z) = (x^3 - 3y)\ \mathbf{i} + (2yz + 1)\ \mathbf{j} + xyz\ \mathbf{k}$ through the cube bounded by planes $x = \pm 1,\ y = \pm 1$, and $z = \pm 1$.

Answer

$$\iint_S \mathbf{F} \cdot dS = 8$$

29. Let $\mathbf{F}(x, y, z) = 2x\ \mathbf{i} - 3y\ \mathbf{j} + 5z\ \mathbf{k}$ and let S be hemisphere $z = \sqrt{9 - x^2 - y^2}$ together with disk $x^2 + y^2 \leq 9$ in the xy-plane. Use the divergence theorem.

30. Evaluate $\iint_S \mathbf{F} \cdot \mathbf{n}\ dS$, where $\mathbf{F}(x,y,z) = x^2\ \mathbf{i} + xy\ \mathbf{j} + x^3y^3\ \mathbf{k}$ and S is the surface consisting of all faces except the tetrahedron bounded by plane $x + y + z = 1$ and the coordinate planes, with outward unit normal vector $\mathbf{N}$.

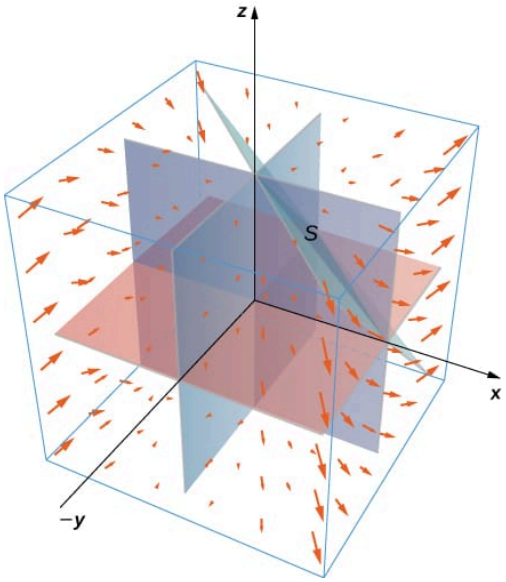


Figure $3.9E.2$: S is the surface consisting of all faces except the tetrahedron bounded by plane $x + y + z = 1$ and the coordinate planes, with outward unit normal vector $\mathbf{N}$.

Answer

$$\iint_S \mathbf{F} \cdot \mathbf{n}\ dS = \frac{1}{8}$$

31. Find the net outward flux of field $\mathbf{F} = \langle bz - cy,\ cx - az,\ ay - bx \rangle$ across any smooth closed surface in R^3 where $a,\ b$, and c are constants.

32. Use the divergence theorem to evaluate $\iint_S ||\mathbf{R}||\mathbf{R} \cdot \mathbf{n}\ ds$, where $\mathbf{R}(x,y,z) = x\ \mathbf{i} + y\ \mathbf{j} + z\ \mathbf{k}$ and S is sphere $x^2 + y^2 + z^2 = a^2$, with constant $a > 0$.

Answer

$$\iint_S ||\mathbf{R}||\mathbf{R} \cdot \mathbf{n}\ ds = 4\pi a^4$$

33. Use the divergence theorem to evaluate $\iint_S \mathbf{F} \cdot dS$, where $\mathbf{F}(x,y,z) = y^2z\ \mathbf{i} + y^3\ \mathbf{j} + xz\ \mathbf{k}$ and S is the boundary of the cube defined by $-1 \le x \le 1,\ -1 \le y \le 1$, and $0 \le z \le 2$.

34. Let R be the region defined by $x^2 + y^2 + z^2 \le 1$. Use the divergence theorem to find $\iiint_R z^2\ dV$.

Answer

$$\iiint_R z^2 dV = \frac{4\pi}{15}$$

35. Let E be the solid bounded by the xy-plane and paraboloid $z = 4 - x^2 - y^2$ so that S is the surface of the paraboloid piece together with the disk in the xy-plane that forms its bottom. If $\mathbf{F}(x,y,z) = (xz\ \sin(yz) + x^3)\ \mathbf{i} + \cos(yz)\ \mathbf{j} + (3zy^2 - e^{x^2+y^2})\ \mathbf{k}$, find $\iint_S \mathbf{F} \cdot dS$ using the divergence theorem.

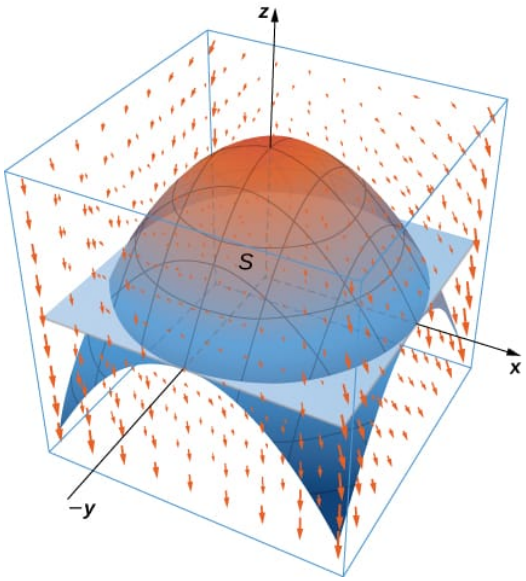


Figure $3.9E.3$: S is the surface of the paraboloid piece together with the disk in the xy-plane.

36. Let E be the solid unit cube with diagonally opposite corners at the origin and $(1, 1, 1)$, and faces parallel to the coordinate planes. Let S be the surface of E, oriented with the outward-pointing normal. Use a CAS to find $\iint_S \mathbf{F} \cdot dS$ using the divergence theorem if $\mathbf{F}(x, y, z) = 2xy\ \mathbf{i} + 3ye^z\ \mathbf{j} + x \sin z\ \mathbf{k}$.

Answer

$$\iint_S \mathbf{F} \cdot dS = 6.5759$$

37. Use the divergence theorem to calculate the flux of $\mathbf{F}(x, y, z) = x^3\ \mathbf{i} + y^3\ \mathbf{j} + z^3\ \mathbf{k}$ through sphere $x^2 + y^2 + z^2 = 1$.

38. Find $\iint_S \mathbf{F} \cdot dS$, where $\mathbf{F}(x, y, z) = x\ \mathbf{i} + y\ \mathbf{j} + z\ \mathbf{k}$ and S is the outwardly oriented surface obtained by removing cube $[1, 2] \times [1, 2] \times [1, 2]$ from cube $[0, 2] \times [0, 2] \times [0, 2]$.

Answer

$$\iint_S \mathbf{F} \cdot dS = 21$$

39. Consider radial vector field $\mathbf{F} = \dfrac{\mathbf{r}}{\|\mathbf{r}\|} = \dfrac{\langle x, y, z\rangle}{(x^2 + y^2 + z^2)^{1/2}}$. Compute the surface integral, where S is the surface of a sphere of radius *a* centered at the origin.

40. Compute the flux of water through parabolic cylinder $S\ :\ y = x^2$, from $0 \le x \le 2,\ \ 0 \le z \le 3$, if the velocity vector is $\mathbf{F}(x, y, z) = 3z^2\ \mathbf{i} + 6\ \mathbf{j} + 6xz\ \mathbf{k}$.

Answer

$$\iint_S \mathbf{F} \cdot dS = 72$$

41. [T] Use a CAS to find the flux of vector field $\mathbf{F}(x, y, z) = z\ \mathbf{i} + z\ \mathbf{j} + \sqrt{x^2 + y^2}\ \mathbf{k}$ across the portion of hyperboloid $x^2 + y^2 = z^2 + 1$ between planes $z = 0$ and $z = \dfrac{\sqrt{3}}{3}$, oriented so the unit normal vector points away from the z-axis.

42. Use a CAS to find the flux of vector field $\mathbf{F}(x, y, z) = (e^y + x)\ \mathbf{i} + (3\ \cos(xz) - y)\ \mathbf{j} + z\ \mathbf{k}$ through surface S, where S is given by $z^2 = 4x^2 + 4y^2$ from $0 \le z \le 4$, oriented so the unit normal vector points downward.

Answer

$$\iint_S \mathbf{F} \cdot dS = -33.5103$$

43. [T] Use a CAS to compute $\iint_S \mathbf{F} \cdot dS$, where $\mathbf{F}(x, y, z) = x\ \mathbf{i} + y\ \mathbf{j} + 2z\ \mathbf{k}$ and S is a part of sphere $x^2 + y^2 + z^2 = 2$ with $0 \leq z \leq 1$.

44. Evaluate $\iint_S \mathbf{F} \cdot dS$, where $\mathbf{F}(x, y, z) = bxy^2\ \mathbf{i} + bx^2y\ \mathbf{j} + (x^2 + y^2)z^2\ \mathbf{k}$ and S is a closed surface bounding the region and consisting of solid cylinder $x^2 + y^2 \leq a^2$ and $0 \leq z \leq b$.

Answer

$$\iint_S \mathbf{F} \cdot dS = \pi a^4 b^2$$

45. [T] Use a CAS to calculate the flux of $\mathbf{F}(x, y, z) = (x^3 + y\ \sin z)\ \mathbf{i} + (y^3 + z\ \sin x)\ \mathbf{j} + 3z\ \mathbf{k}$ across surface S, where S is the boundary of the solid bounded by hemispheres $z = \sqrt{4 - x^2 - y^2}$ and $z = \sqrt{1 - x^2 - y^2}$, and plane $z = 0$.

46. Use the divergence theorem to evaluate $\iint_S \mathbf{F} \cdot dS$, where $\mathbf{F}(x, y, z) = xy\ \mathbf{i} - \frac{1}{2}y^2\ \mathbf{j} + z\ \mathbf{k}$ and S is the surface consisting of three pieces: $z = 4 - 3x^2 - 3y^2,\ 1 \leq z \leq 4$ on the top; $x^2 + y^2 = 1,\ 0 \leq z \leq 1$ on the sides; and $z = 0$ on the bottom.

Answer

$$\iint_S \mathbf{F} \cdot dS = \frac{5}{2}\pi$$

47. [T] Use a CAS and the divergence theorem to evaluate $\iint_S \mathbf{F} \cdot dS$, where $\mathbf{F}(x, y, z) = (2x + y\ \cos z)\ \mathbf{i} + (x^2 - y)\ \mathbf{j} + y^2z\ \mathbf{k}$ and S is sphere $x^2 + y^2 + z^2 = 4$ orientated outward.

48. Use the divergence theorem to evaluate $\iint_S \mathbf{F} \cdot dS$, where $\mathbf{F}(x, y, z) = x\ \mathbf{i} + y\ \mathbf{j} + z\ \mathbf{k}$ and S is the boundary of the solid enclosed by paraboloid $y = x^2 + z^2 - 2$, cylinder $x^2 + z^2 = 1$, and plane $x + y = 2$, and S is oriented outward.

Answer

$$\iint_S \mathbf{F} \cdot dS = \frac{21\pi}{2}$$

For the following exercises, **Fourier's law of heat transfer** states that the heat flow vector $\mathbf{F}$ at a point is proportional to the negative gradient of the temperature; that is, $\mathbf{F} = -k\nabla T$, which means that heat energy flows hot regions to cold regions. The constant $k > 0$ is called the *conductivity*, which has metric units of joules per meter per second-kelvin or watts per meter-kelvin. A temperature function for region D s given. Use the divergence theorem to find net outward heat flux $\iint_S \mathbf{F} \cdot \mathbf{n}\ dS = -k \iint_S \nabla T \cdot N\ dS$ across the boundary S of D, where $k = 1$.

49. $T(x, y, z) = 100 + x + 2y + z$;

$D = \{(x, y, z) : 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ 0 \leq z \leq 1\}$

50. $T(x, y, z) = 100 + e^{-z}$;

$D = \{(x, y, z) : 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ 0 \leq z \leq 1\}$

Answer

$$-(1 - e^{-1})$$

51. $T(x, y, z) = 100e^{-x^2-y^2-z^2}$; D is the sphere of radius a centered at the origin.

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3.10: Chapter 3 Review Exercises

Exercises 1 - 4

***True or False?* Justify your answer with a proof or a counterexample.**

1. The vector field $\mathbf{F}(x,y) = x^2y\ \mathbf{i} + y^2x\ \mathbf{j}$ is conservative.

Answer

False

2. For vector field $\mathbf{F}(x,y) = P(x,y)\ \mathbf{i} + Q(x,y)\ \mathbf{j}$, if $P_y(x,y) = Q_z(x,y)$ in open region D, then $\int_{\partial D} P\ dx + Q\ dy = 0$.

3. The divergence of a vector field is a vector field.

Answer

False

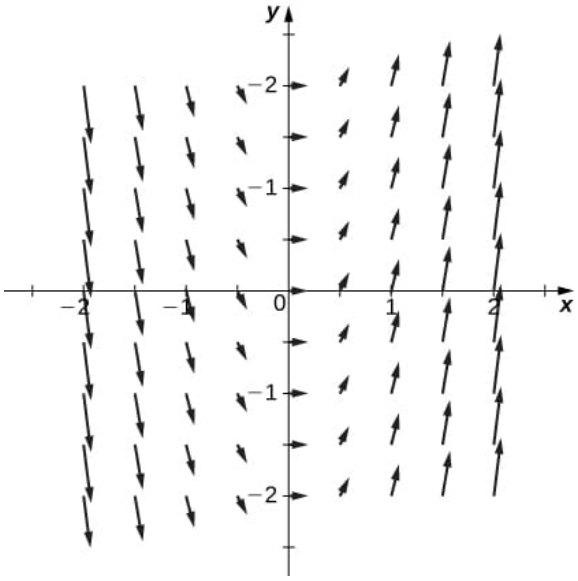
4. If $curl\ \mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is a conservative vector field.

Exercises 5 - 6

Draw the following vector fields.

5. $\mathbf{F}(x,y) = \frac{1}{2}\ \mathbf{i} + 2x\ \mathbf{j}$

Answer



6. $\mathbf{F}(x,y) = \sqrt{\frac{y\,\mathbf{i} + 3x\,\mathbf{j}}{x^2+y^2}}$

Exercises 7 - 10

Are the following the vector fields conservative? If so, find the potential function $\mathbf{F}$ such that $\mathbf{F} = \nabla f$.

7. $\mathbf{F}(x, y) = y\ \mathbf{i} + (x - 2e^y)\ \mathbf{j}$

Answer

Conservative, $f(x, y) = xy - 2e^y$

8. $\mathbf{F}(x, y) = (6xy)\ \mathbf{i} + (3x^2 - ye^y)\ \mathbf{j}$

9. $\mathbf{F}(x, y) = (2xy + z^2)\ \mathbf{i} + (x^2 + 2yz)\ \mathbf{j} + (2xz + y^2)\ \mathbf{k}$

Answer

Conservative, $f(x, y, z) = x^2y + y^2z + z^2x$

10. $\mathbf{F}(x, y, z) = (e^x y)\ \mathbf{i} + (e^x + z)\ \mathbf{j} + (e^x + y^2)\ \mathbf{k}$

Exercises 11 - 13

Evaluate the following integrals.

11. $\int_C x^2\ dy + (2x - 3xy)\ dx$, along $C : y = \frac{1}{2}x$ from $(0, 0)$ to $(4, 2)$

Answer

$-\frac{16}{3}$

12. $\int_C y\ dx + xy^2\ dy$, where $C : x = \sqrt{t},\ \ y = t - 1,\ \ 0 \le t \le 1$

13. $\iint_S xy^2\ dS$, where S is the surface $z = x^2 - y,\ \ 0 \le x \le 1,\ \ 0 \le y \le 4$

Answer

$\frac{32\sqrt{2}}{9}(3\sqrt{3} - 1)$

Exercises 14 - 15

Find the divergence and curl for the following vector fields.

14. $\mathbf{F}(x, y, z) = 3xyz\ \mathbf{i} + xye^x\ \mathbf{j} - 3xy\ \mathbf{k}$

15. $\mathbf{F}(x, y, z) = e^x\ \mathbf{i} + e^{xy}\ \mathbf{j} - e^{xyz}\ \mathbf{k}$

Answer

Divergence: $e^x + x\ e^{xy} + xy\ e^{xyz}$
Curl: $xze^{xyz}\ \mathbf{i} - yze^{xyz}\ \mathbf{j} + ye^{xy}\ \mathbf{k}$

Exercises 16 - 17

Use Green's theorem to evaluate the following integrals.

16. $\int_C 3xy\ dx + 2xy^2\ dy$, where C is a square with vertices $(0,0),\ (0,2),\ (2,2)$ and $(2,0)$.

17. $\oint_C 3y\ dx + (x + e^y)\ dy$, where C is a circle centered at the origin with radius 3.

Answer

-18π

Exercises 18 - 19

Use Stokes' theorem to evaluate $\iint_S curl\ \mathbf{F} \cdot dS$.

18. $\mathbf{F}(x,y,z) = y\ \mathbf{i} - x\ \mathbf{j} + z\ \mathbf{k}$, where S is the upper half of the unit sphere

19. $\mathbf{F}(x,y,z) = y\ \mathbf{i} + xyz\ \mathbf{j} - 2zx\ \mathbf{k}$, where S is the upward-facing paraboloid $z = x^2 + y^2$ lying in cylinder $x^2 + y^2 = 1$

Answer

$-\pi$

Exercises 20 - 25

Use the divergence theorem to evaluate $\iint_S \mathbf{F} \cdot dS$.

20. $\mathbf{F}(x,y,z) = (x^3y)\ \mathbf{i} + (3y - e^x)\ \mathbf{j} + (z + x)\ \mathbf{k}$, over cube S defined by $-1 \le x \le 1,\ 0 \le y \le 2,\ 0 \le z \le 2$

21. $\mathbf{F}(x,y,z) = (2xy)\ \mathbf{i} + (-y^2)\ \mathbf{j} + (2z^3)\ \mathbf{k}$, where S is bounded by paraboloid $z = x^2 + y^2$ and plane $z = 2$

Answer

$31\pi/2$

22. Find the amount of work performed by a 50-kg woman ascending a helical staircase with radius 2 m and height 100 m. The woman completes five revolutions during the climb.

23. Find the total mass of a thin wire in the shape of a semicircle with radius $\sqrt{2}$, and a density function of $\rho(x,y) = y + x^2$.

Answer

$\sqrt{2}(2 + \pi)$

24. Find the total mass of a thin sheet in the shape of a hemisphere with radius 2 for $z \ge 0$ with a density function $\rho(x,y,z) = x + y + z$.

25. Use the divergence theorem to compute the value of the flux integral over the unit sphere with $\mathbf{F}(x,y,z) = 3z\ \mathbf{i} + 2y\ \mathbf{j} + 2x\ \mathbf{k}$.

Answer

$2\pi/3$

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