Linear Algebra: A First Course

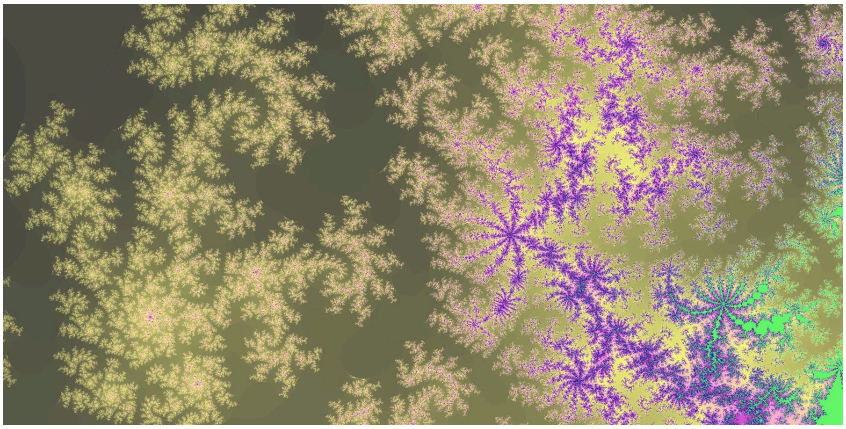
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De Anza College

Mathematics

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Linear Algebra: A First Course

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Detailed Licensing

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About the Book

Publication Date: August, 2025

This **Mathematics textbook** is part of the **De Anza College Zero Textbook Cost (ZTC) Pathways** initiative, supporting both students and instructors with free, adaptable Open Educational Resources (OER).

This book offers a comprehensive and accessible introduction to linear algebra. It begins with methods for solving linear systems, followed by an exploration of matrices and their properties, including a dedicated chapter on determinants. The text then introduces vectors in $\mathbb{R}^n$, covering essential concepts such as span, linear independence, subspaces, and linear transformations. From there, it develops spectral theory, including diagonalization and orthogonal diagonalization, along with their applications. The final chapter explores general vector spaces and revisits core linear algebra concepts in that broader context. With numerous examples and practical applications, the material is designed to engage students and support their development in applying abstract mathematical ideas to problem solving—making it a valuable resource for both individual study and classroom use.

This book builds upon the foundation established in A Course in Linear Algebra by Ken Kuttler, one of the most widely used open educational resources in the subject. Additional material was drawn from three other sources: the Gauss-Seidel method from Autar Kaw's Introduction to Matrix Algebra, determinant geometry from Dan Margalit and Joseph Rabinoff's Interactive Linear Algebra, and inner product spaces from W. Keith Nicholson's Linear Algebra with Applications. We also acknowledge Lake Tahoe Community College's derivative of Kuttler's textbook, which incorporates video resources and served as an important starting point for our own work. In creating this resource, we have aimed to uphold the principles of open education while enhancing accessibility for a diverse range of learners.

In addition to the core text, we provide ancillary materials to enhance teaching and learning.

Included Ancillary Materials

- **Instructor Notes**: Guidance for educators with teaching tips, common misconceptions, and activity ideas.

This ancillary can be found under **"assets"** on the De Anza Commons page for Linear Algebra: A First Course

Online Homework

Aligned online homework is available on **MyOpenMath**. After logging into an instructor account, select *Add New Course* → *Copy a template or promoted course* → choose *Calculus I: Differential Calculus (De Anza ZTC)*.

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About the Authors

Information about the authors

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	**Doli Bambhania** is a Mathematics Instructor at De Anza College in Cupertino, CA, with over two decades of teaching experience. She brings a vibrant energy and a student-centered approach to her work, striving to ensure success for students not just in her class but throughout their entire math journey. Doli has been involved in several programs aimed at removing barriers to math education, such as De Anza's Math Performance Success (MPS) and Learning in Communities (LinC) programs, and Statway. Additionally, she has developed open educational resources (OER) workbooks for algebra and statistics, reflecting her strong commitment to using OER materials in all her classes. Outside of teaching, Doli enjoys spending time with her family, playing games, hiking, dancing, and music. Doli received her Bachelors degree in Mathematics from University of California, San Diego, and Master degrees in Mathematics and Computer Science from University of California, Santa Barbara.
	**Bill Wilson** started his career as a mathematics professor. After teaching mathematics at a four-year college in Massachusetts for nine years, he undertook a career in cybersecurity. Starting as a researcher, he moved into management and eventually started a company, serving as its CEO for 10 years until its acquisition. In 2006 Bill was elected to the Board of Trustees of the Fremont Union High School District where he served sixteen years. He returned to his love of teaching mathematics as a part time instructor in the mathematics department at De Anza College. He has found renewed energy from engaging with the resilient students and dedicated faculty at De Anza. Bill received his bachelor's degree in mathematics from Princeton University and his master's and Ph.D. degrees in mathematics from Harvard University.

Author Images	Author Bios
	**Fatemeh Yarahmadi** is a dedicated math instructor at De Anza College, whose passion for mathematics extends far beyond the textbook. She believes that math is not just a subject, but a language, a tool for understanding the world, and she is committed to helping every student unlock its power. With a deep commitment to education, she strives to build a nurturing and stimulating learning environment, where students feel empowered to ask questions and explore complex concepts without fear. Her teaching philosophy is rooted in the belief that mathematics is accessible to everyone, regardless of their background or prior experiences. She employs innovative teaching methods, incorporating real-world examples and interactive exercises, to make abstract concepts tangible and relatable. She understands that each student learns differently, and she dedicates herself to providing personalized support, ensuring that no one is left behind.

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1: SYSTEMS OF EQUATIONS

1.1: Systems of Linear Equations

1.1E: Exercises for Section 1.1

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1.1: Systems of Linear Equations

Learning Objectives

- Relate the types of solution sets of a system of two (three) variables graphically to the intersections of lines in a plane (the intersection of planes in three space).
- Use elementary operations to find the solution to a linear system of equations algebraically.

Systems of Equations, Geometry

As you may remember, linear equations like $2x + 3y = 6$ can be graphed as straight lines in the coordinate plane. We say that this equation is in two variables, in this case x and y. Suppose you have two such equations, each of which can be graphed as a straight line, and consider the resulting graph of two lines. What would it mean if there exists a point of intersection between the two lines? This point, which lies on *both* graphs, gives x and y values for which both equations are true. In other words, this point gives the ordered pair (x, y) that satisfy both equations. If the point (x, y) is a point of intersection, we say that (x, y) is a **solution** to the two equations. In linear algebra, we often are concerned with finding the solution(s) to a system of equations, if such solutions exist. First, we consider graphical representations of solutions and later we will consider the algebraic methods for finding solutions.

When looking for the intersection of two lines in a graph, several situations may arise. The following picture (Figure 1.1.1) demonstrates the possible situations when considering two equations (two lines in the graph) involving two variables.

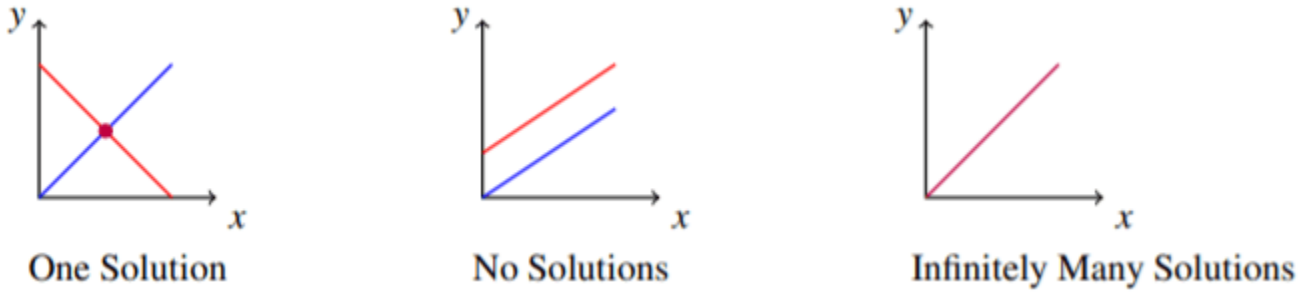


Figure 1.1.1: The possibilities for a system with 2 equations and 2 variables: two crossing lines for one solution, two parallel lines for no solution, and two identical lines for infinitely many solutions. (CC BY-NC-SA 4.0; Kuttler via A First Course in LINEAR ALGEBRA)

In the first diagram, there is a unique point of intersection, which means that there is only one (unique) solution to the two equations. In the second, there are no points of intersection and no solution. When no solution exists, this means that the two lines are parallel and they never intersect. The third situation which can occur, as demonstrated in diagram three, is that the two lines are really the same line. For example, $x + y = 1$ and $2x + 2y = 2$ are equations which when graphed yield the same line. In this case there are infinitely many points which are solutions of these two equations, as every ordered pair which is on the graph of the line satisfies both equations. When considering linear systems of equations, there are always three types of solutions possible; exactly one (unique) solution, infinitely many solutions, or no solution.

Example 1.1.1: A Graphical Solution

Use a graph to find the solution to the following system of equations

$$x + y = 3$$
$$y - x = 5$$

Solution

Through graphing the above equations and identifying the point of intersection, we can find the solution(s). Remember that we must have either one solution, infinitely many, or no solutions at all. The following graph (Figure $1.1.2$) shows the two equations, as well as the intersection. Remember, the point of intersection represents the solution of the two equations, or the (x, y) which satisfy both equations. In this case, there is one point of intersection at $(-1, 4)$ which means we have one unique solution, $x = -1, y = 4$.

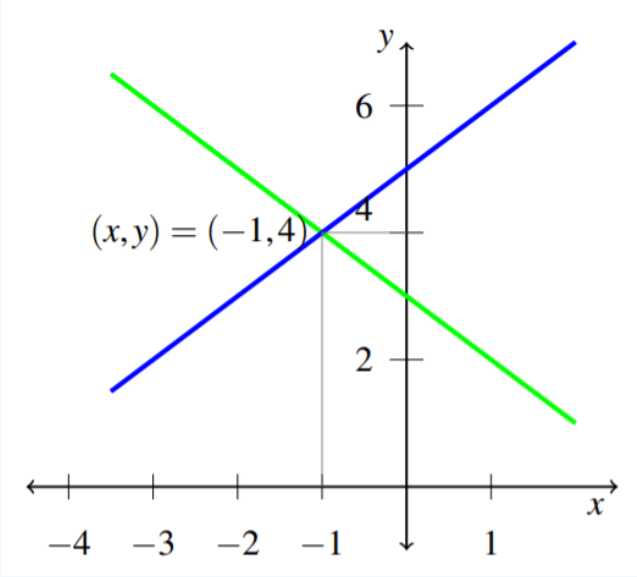


Figure $1.1.2$: The lines corresponding to the equations $x + y = 3$ and $y - x = 5$ intersect at $(-1, 4)$. (CC BY-NC-SA 4.0; Kuttler via A First Course in LINEAR ALGEBRA)

In the above example, we investigated the intersection point of two equations in two variables, x and y. Now we will consider the graphical solutions of three equations in two variables.

Consider a system of three equations in two variables. Again, these equations can be graphed as straight lines in the plane, so that the resulting graph contains three straight lines. Recall the three possible types of solutions; no solution, one solution, and infinitely many solutions. There are now more complex ways of achieving these situations, due to the presence of the third line. For example, you can imagine the case of three intersecting lines having no common point of intersection. Perhaps you can also imagine three intersecting lines which do intersect at a single point. These two situations are illustrated below (Figure $1.1.3$).

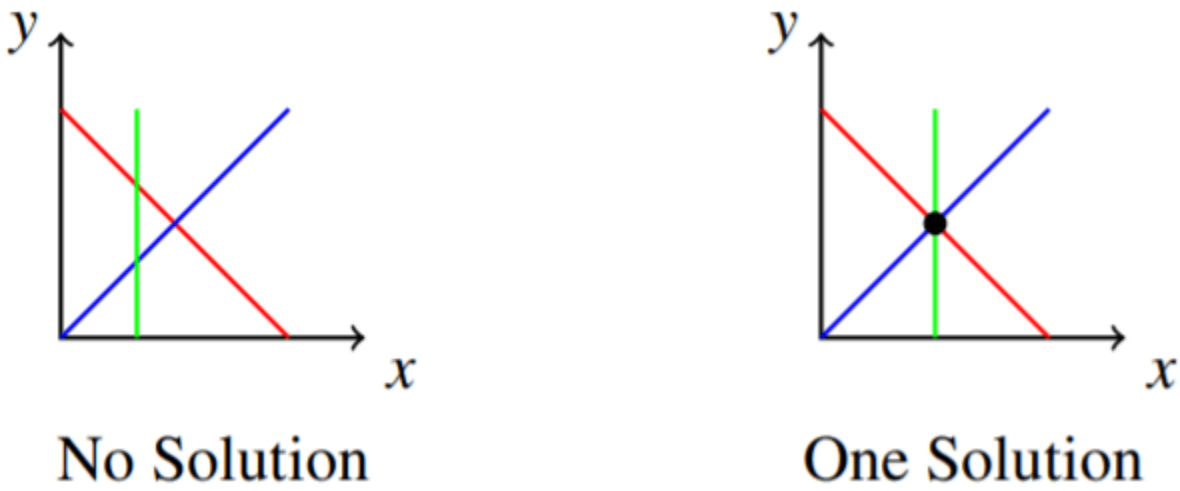


Figure $1.1.3$: The first diagram shows a linear system corresponding to three pairwise non-parallel lines with no common point of intersection has no solution. The second diagram shows linear system corresponding to three pairwise non-parallel lines with one common point of intersection has one solution. (CC BY-NC-SA 4.0; Kuttler via A First Course in LINEAR ALGEBRA)

Consider the first picture above. While all three lines intersect with one another, there is no common point of intersection where all three lines meet at one point. Hence, there is no solution to the three equations. Remember, a solution is a point (x, y) which satisfies **all** three equations. In the case of the second picture, the lines intersect at a common point. This means that there is one solution to the three equations whose graphs are the given lines. You should take a moment now to draw the graph of a system which results in three parallel lines. Next, try the graph of three identical lines. Which type of solution is represented in each of these graphs?

We have now considered the graphical solutions of systems of two equations in two variables, as well as three equations in two variables. However, there is no reason to limit our investigation to equations in two variables. We will now consider equations in three variables.

You may recall that equations in three variables, such as $2x + 4y - 5z = 8$, form a plane. Above, we were looking for intersections of lines in order to identify any possible solutions. When graphically solving systems of equations in three variables, we look for intersections of planes. These points of intersection give the (x, y, z) that satisfy all the equations in the system. What types of solutions are possible when working with three variables? Consider the following picture involving two planes, which are given by two equations in three variables (Figure $1.1.4$).

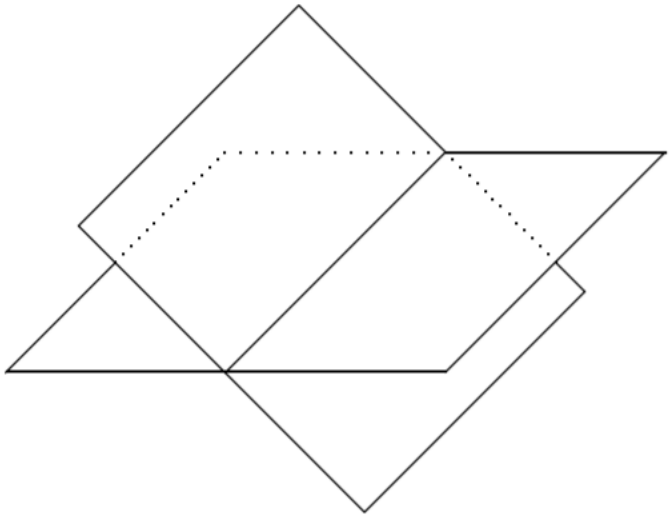


Figure $1.1.4$: Two planes intersecting in a line. (CC BY-NC-SA 4.0; Kuttler via A First Course in LINEAR ALGEBRA)

Notice how these two planes intersect in a line. This means that the points (x, y, z) on this line satisfy both equations in the system. Since the line contains infinitely many points, this system has infinitely many solutions.

It could also happen that the two planes fail to intersect. However, is it possible to have two planes intersect at a single point? Take a moment to attempt drawing this situation, and convince yourself that it is not possible! This means that when we have only two equations in three variables, there is no way to have a unique solution! Hence, the types of solutions possible for two equations in three variables are no solution or infinitely many solutions.

Now imagine adding a third plane. In other words, consider three equations in three variables. What types of solutions are now possible? Consider the following diagram (Figure $1.1.5$).

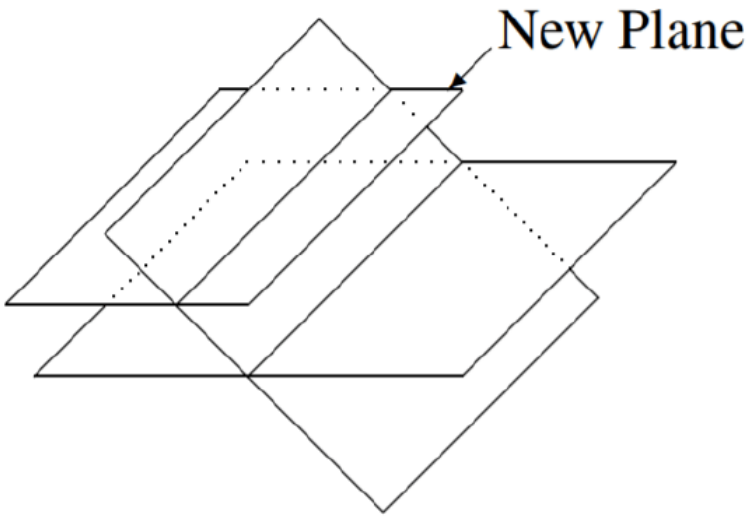


Figure 1.1.5: A third plane added to the system of two planes from Figure 1.1.4 that intersected in a line. The third plane is parallel to one of the original planes, which results in no common points of intersection. (CC BY-NC-SA 4.0; Kuttler via A First Course in LINEAR ALGEBRA)

In this diagram, there is no point which lies in all three planes. There is no intersection between **all** planes so there is no solution. The picture illustrates the situation in which the line of intersection of the new plane with one of the original planes forms a line parallel to the line of intersection of the first two planes. However, in three dimensions, it is possible for two lines to fail to intersect even though they are not parallel. Such lines are called **skew lines**.

Recall that when working with two equations in three variables, it was not possible to have a unique solution. Is it possible when considering three equations in three variables? In fact, it is possible, and we demonstrate this situation in the following picture (Figure 1.1.6).

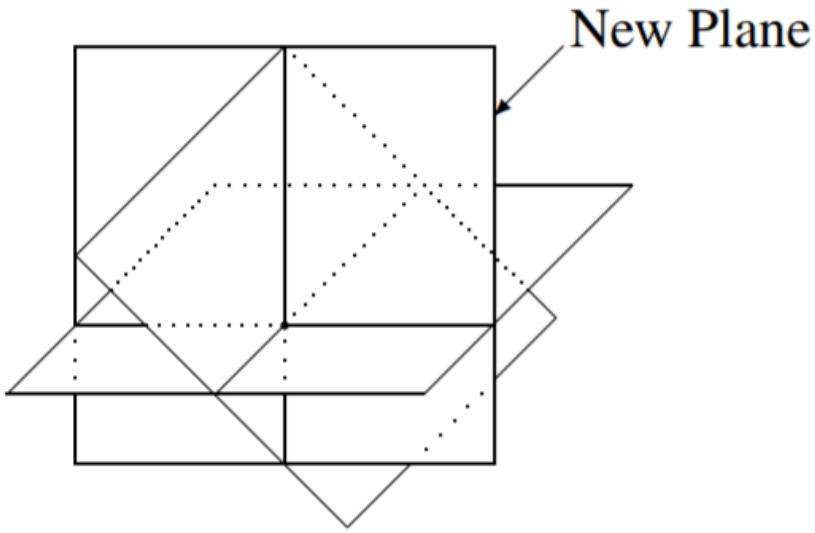


Figure 1.1.6: A third plane added to the system of two planes from Figure 1.1.4 that intersected in a line. The third plane is perpendicular to one of the original planes, which results in one point of intersection. (CC BY-NC-SA 4.0; Kuttler via A First Course in LINEAR ALGEBRA)

In this case, the three planes have a single point of intersection. Can you think of other types of solutions possible? Another is that the three planes could intersect in a line, resulting in infinitely many solutions, as in the following diagram (Figure 1.1.7).

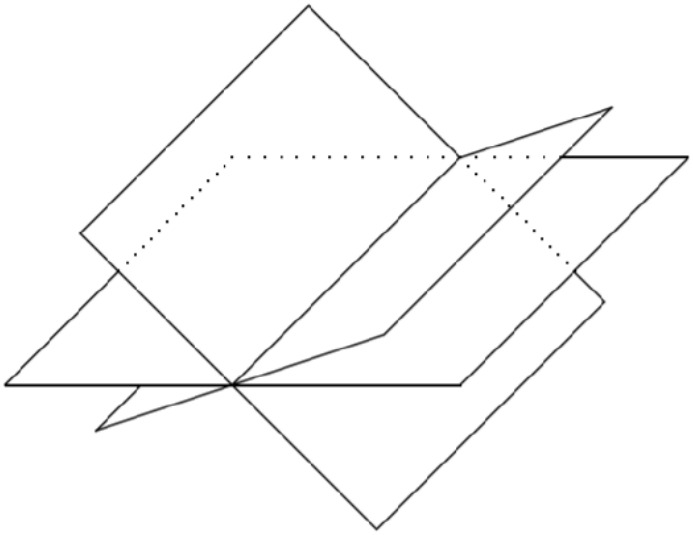


Figure 1.1.7: A third plane added to the system of two planes from Figure 1.1.4 that intersected in a line. The third plane passes through the line of intersection, which results all three planes intersecting in that line. (CC BY-NC-SA 4.0; Kuttler via A First Course in LINEAR ALGEBRA)

We have now seen how three equations in three variables can have no solution, a unique solution, or intersect in a line resulting in infinitely many solutions. It is also possible that the three equations graph the same plane, which also leads to infinitely many solutions.

You can see that when working with equations in three variables, there are many more ways to achieve the different types of solutions than when working with two variables. It may prove enlightening to spend time imagining (and drawing) many possible scenarios, and you should take some time to try a few.

You should also take some time to imagine (and draw) graphs of systems in more than three variables. Equations like $x + y - 2z + 4w = 8$ with more than three variables are often called **hyper-planes**. You may soon realize that it is tricky to draw the graphs of hyper-planes! Through the tools of linear algebra, we can algebraically examine these types of systems which are difficult to graph. In the following section, we will consider these algebraic tools.

Algebraic Procedures

We have taken an in depth look at graphical representations of systems of equations, as well as how to find possible solutions graphically. Our attention now turns to working with systems algebraically.

Definition 1.1.1: System of Linear Equations

A **system of linear equations** is a list of equations,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

where a_{ij} and b_j are real numbers. The above is a system of m equations in the n variables, $x_1, x_2 \cdots, x_n$. Written more simply in terms of summation notation, the above can be written in the form

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \ \ i = 1, 2, 3, \cdots, m$$

The relative size of m and n is not important here. Notice that we have allowed a_{ij} and b_j to be any real number. We can also call these numbers **scalars** . We will use this term throughout the text, so keep in mind that the term **scalar** just means that we are

working with real numbers.

Now, suppose we have a system where $b_i = 0$ for all i. In other words every equation equals 0. This is a special type of system.

Definition 1.1.2: Homogeneous System of Equations

A system of equations is called **homogeneous** if each equation in the system is equal to 0. A homogeneous system has the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

where a_{ij} are scalars and x_i are variables.

Recall from the previous section that our goal when working with systems of linear equations was to find the point of intersection of the equations when graphed. In other words, we looked for the solutions to the system. We now wish to find these solutions algebraically. We want to find values for $x_1, \cdots, x_n$ which solve all of the equations. If such a set of values exists, we call $(x_1, \cdots, x_n)$ the **solution set**.

Recall the above discussions about the types of solutions possible. We will see that systems of linear equations will have one unique solution, infinitely many solutions, or no solution. Consider the following definition.

Definition 1.1.3: Consistent and Inconsistent Systems

A system of linear equations is called **consistent** if there exists at least one solution. It is called **inconsistent** if there is no solution.

If you think of each equation as a condition which must be satisfied by the variables, consistent would mean there is some choice of variables which can satisfy **all** the conditions. Inconsistent would mean there is no choice of the variables which can satisfy all of the conditions.

The following sections provide methods for determining if a system is consistent or inconsistent, and finding solutions if they exist.

Elementary Operations

We begin with an example. Recall from Example 1.1.1 that the solution to the given system was $(x, y) = (-1, 4)$.

Example 1.1.2: Verifying an Ordered Pair is a Solution

Algebraically verify that $(x, y) = (-1, 4)$ is a solution to the following system of equations.

$$\begin{aligned} x + y &= 3 \\ y - x &= 5 \end{aligned}$$

Solution

By graphing these two equations and identifying the point of intersection, we previously found that $(x, y) = (-1, 4)$ is the unique solution.

We can verify algebraically by substituting these values into the original equations, and ensuring that the equations hold. First, we substitute the values into the first equation and check that it equals 3.

$$x + y = (-1) + (4) = 3$$

This equals 3 as needed, so we see that $(-1, 4)$ is a solution to the first equation. Substituting the values into the second equation yields

$$y - x = (4) - (-1) = 4 + 1 = 5$$

which is true. For $(x, y) = (-1, 4)$ each equation is true and therefore, this is a solution to the system.

Now, the interesting question is this: If you were not given these numbers to verify, how could you algebraically determine the solution? Linear algebra gives us the tools needed to answer this question. The following basic operations are important tools that we will utilize.

Definition 1.1.4: Elementary Operations

Elementary operations are those operations consisting of the following.

1. Interchange the order in which the equations are listed.
2. Multiply any equation by a nonzero number.
3. Replace any equation with itself added to a multiple of another equation.

It is important to note that none of these operations will change the set of solutions of the system of equations. In fact, elementary operations are the *key tool* we use in linear algebra to find solutions to systems of equations.

Consider the following example.

Example 1.1.3: Effects of an Elementary Operation

Show that the system

$$\begin{aligned} x + y &= 7 \\ 2x - y &= 8 \end{aligned}$$

has the same solution as the system

$$\begin{aligned} x + y &= 7 \\ -3y &= -6 \end{aligned}$$

Solution

Notice that the second system has been obtained by taking the second equation of the first system and adding -2 times the first equation, as follows:

$$2x - y + (-2)(x + y) = 8 + (-2)(7)$$

By simplifying, we obtain

$$-3y = -6$$

which is the second equation in the second system. Now, from here we can solve for y and see that $y = 2$. Next, we substitute this value into the first equation as follows

$$x + y = x + 2 = 7$$

Hence $x = 5$ and so $(x, y) = (5, 2)$ is a solution to the second system. We want to check if $(5, 2)$ is also a solution to the first system. We check this by substituting $(x, y) = (5, 2)$ into the system and ensuring the equations are true.

$$\begin{aligned} x + y &= (5) + (2) = 7 \\ 2x - y &= 2\,(5) - (2) = 8 \end{aligned}$$

Hence, $(5, 2)$ is also a solution to the first system.

This example illustrates how an elementary operation applied to a system of two equations in two variables does not affect the

solution set. However, a linear system may involve many equations and many variables and there is no reason to limit our study to small systems. For any size of system in any number of variables, the solution set is still the collection of solutions to the equations. In every case, the above operations of Definition 1.1.4 do not change the set of solutions to the system of linear equations.

In the following theorem, we use the notation E_i to represent an equation, while b_i denotes a constant.

Theorem 1.1.1: Elementary Operations and Solutions

Suppose you have a system of two linear equations

$$\begin{aligned} E_1 &= b_1 \\ E_2 &= b_2 \end{aligned} \tag{1.1.1}$$

Then the following systems have the same solution set:

1. $$\begin{aligned} E_2 &= b_2 \\ E_1 &= b_1 \end{aligned} \tag{1.1.2}$$

2. $$\begin{aligned} E_1 &= b_1 \\ kE_2 &= kb_2 \end{aligned} \tag{1.1.3}$$

   for any scalar k, provided $k \neq 0$.

3. $$\begin{aligned} E_1 &= b_1 \\ E_2 + kE_1 &= b_2 + kb_1 \end{aligned} \tag{1.1.4}$$

   for any scalar k (including $k = 0$).

Answer available at the end of this section.

Before we proceed with the proof of Theorem 1.1.1, let us consider this theorem in context of Example 1.1.3. Then,

$$\begin{aligned} E_1 &= x + y, \quad b_1 = 7 \\ E_2 &= 2x - y, \quad b_2 = 8 \end{aligned}$$

Recall the elementary operations that we used to modify the system in the solution to the example. First, we added (-2) times the first equation to the second equation. In terms of Theorem 1.1.1, this action is given by

$$E_2 + (-2)\, E_1 = b_2 + (-2)\, b_1$$

or

$$2x - y + (-2)\,(x + y) = 8 + (-2)\, 7$$

This gave us the second system in Example 1.1.3, given by

$$\begin{aligned} E_1 &= b_1 \\ E_2 + (-2)\, E_1 &= b_2 + (-2)\, b_1 \end{aligned}$$

From this point, we were able to find the solution to the system. Theorem 1.1.1 tells us that the solution we found is in fact a solution to the original system.

We will now prove Theorem 1.1.1.

Theorem 1.1.1

Proof available at the end of this section.

Stated simply, the above theorem shows that the elementary operations do not change the solution set of a system of equations.

We will now look at an example of a system of three equations and three variables. Similarly to the previous examples, the goal is to find values for x, y, z such that each of the given equations are satisfied when these values are substituted in.

Example 1.1.4: Solving a System of Equations with Elementary Operations

Find the solutions to the system,

$$\begin{aligned} x+3y+6z &= 25 \\ 2x+7y+14z &= 58 \\ 2y+5z &= 19 \end{aligned} \tag{1.1.5}$$

Solution

We can relate this system to Theorem 1.1.1 above. In this case, we have

$$\begin{aligned} E_1 &= x+3y+6z, & b_1 &= 25 \\ E_2 &= 2x+7y+14z, & b_2 &= 58 \\ E_3 &= 2y+5z, & b_3 &= 19 \end{aligned}$$

Theorem 1.1.1 claims that if we do elementary operations on this system, we will not change the solution set. Therefore, we can solve this system using the elementary operations given in Definition 1.1.4. First, replace the second equation by (-2) times the first equation added to the second. This yields the system

$$\begin{aligned} x+3y+6z &= 25 \\ y+2z &= 8 \\ 2y+5z &= 19 \end{aligned} \tag{1.1.6}$$

Now, replace the third equation with (-2) times the second added to the third. This yields the system

$$\begin{aligned} x+3y+6z &= 25 \\ y+2z &= 8 \\ z &= 3 \end{aligned} \tag{1.1.7}$$

At this point, we can easily find the solution. Simply take $z=3$ and substitute this back into the previous equation to solve for y, and similarly to solve for x.

$$\begin{aligned} x+3y+6\left(3\right) = x+3y+18 &= 25 \\ y+2\left(3\right) = y+6 &= 8 \\ z &= 3 \end{aligned}$$

The second equation is now

$$y+6=8$$

You can see from this equation that $y=2$. Therefore, we can substitute this value into the first equation as follows:

$$x+3\left(2\right)+18=25$$

By simplifying this equation, we find that $x=1$. Hence, the solution to this system is $(x,y,z)=(1,2,3)$. This process is called **back substitution**.

Alternatively, in (1.1.7) you could have continued as follows. Add (-2) times the third equation to the second and then add (-6) times the second to the first. This yields

$$\begin{aligned} x+3y &= 7 \\ y &= 2 \\ z &= 3 \end{aligned}$$

Now add (-3) times the second to the first. This yields

$$\begin{aligned} x &= 1 \\ y &= 2 \\ z &= 3 \end{aligned}$$

a system which has the same solution set as the original system. This avoided back substitution and led to the same solution set. It is your decision which you prefer to use, as both methods lead to the correct solution, $(x,y,z)=(1,2,3)$.

Answers

Theorem 1.1.1**: Elementary Operations and Solutions**

Theorem 1.1.1

1. The proof that the systems (1.1.1) and (1.1.2) have the same solution set is as follows. Suppose that $(x_1, \cdots, x_n)$ is a solution to $E_1 = b_1, E_2 = b_2$. We want to show that this is a solution to the system in (1.1.2) above. This is clear, because the system in (1.1.2) is the original system, but listed in a different order. Changing the order does not effect the solution set, so $(x_1, \cdots, x_n)$ is a solution to (1.1.2).
2. Next we want to prove that the systems (1.1.1) and (1.1.3) have the same solution set. That is $E_1 = b_1, E_2 = b_2$ has the same solution set as the system $E_1 = b_1, kE_2 = kb_2$ provided $k \neq 0$. Let $(x_1, \cdots, x_n)$ be a solution of $E_1 = b_1, E_2 = b_2$,. We want to show that it is a solution to $E_1 = b_1, kE_2 = kb_2$. Notice that the only difference between these two systems is that the second involves multiplying the equation, $E_2 = b_2$ by the scalar k. Recall that when you multiply both sides of an equation by the same number, the sides are still equal to each other. Hence if $(x_1, \cdots, x_n)$ is a solution to $E_2 = b_2$, then it will also be a solution to $kE_2 = kb_2$. Hence, $(x_1, \cdots, x_n)$ is also a solution to (1.1.3). Similarly, let $(x_1, \cdots, x_n)$ be a solution of $E_1 = b_1, kE_2 = kb_2$. Then we can multiply the equation $kE_2 = kb_2$ by the scalar $1/k$, which is possible only because we have required that $k \neq 0$. Just as above, this action preserves equality and we obtain the equation $E_2 = b_2$. Hence $(x_1, \cdots, x_n)$ is also a solution to $E_1 = b_1, E_2 = b_2$.
3. Finally, we will prove that the systems (1.1.1) and (1.1.4) have the same solution set. We will show that any solution of $E_1 = b_1, E_2 = b_2$ is also a solution of (1.1.4). Then, we will show that any solution of (1.1.4) is also a solution of $E_1 = b_1, E_2 = b_2$. Let $(x_1, \cdots, x_n)$ be a solution to $E_1 = b_1, E_2 = b_2$. Then in particular it solves $E_1 = b_1$. Hence, it solves the first equation in (1.1.4). Similarly, it also solves $E_2 = b_2$. By our proof of (1.1.3), it also solves $kE_1 = kb_1$. Notice that if we add E_2 and kE_1, this is equal to $b_2 + kb_1$. Therefore, if $(x_1, \cdots, x_n)$ solves $E_1 = b_1, E_2 = b_2$ it must also solve $E_2 + kE_1 = b_2 + kb_1$. Now suppose $(x_1, \cdots, x_n)$ solves the system $E_1 = b_1, E_2 + kE_1 = b_2 + kb_1$. Then in particular it is a solution of $E_1 = b_1$. Again by our proof of (1.1.3), it is also a solution to $kE_1 = kb_1$. Now if we subtract these equal quantities from both sides of $E_2 + kE_1 = b_2 + kb_1$ we obtain $E_2 = b_2$, which shows that the solution also satisfies \(E_{1}=b_{1},E_{2}=b_{2}.\

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1.1E: Exercises for Section 1.1

Exercise 1.1E.1

Graphically, find the point (x_1, y_1) which lies on both lines, $x + 3y = 1$ and $4x - y = 3$. That is, graph each line and see where they intersect.

Exercise 1.1E.2

Graphically, find the point of intersection of the two lines, $3x + y = 3$ and $x + 2y = 1$. That is, graph each line and see where they intersect

Exercise 1.1E.3

You have a system of k equations in two variables, $k \geq 2$. Explain the geometric significance of

a. No solution.
b. A unique solution.
c. An infinite number of solutions.

Exercise 1.1E.4

Graphically, find the point (x_1, y_1) which lies on both lines, $x + 3y = 1$ and $4x - y = 3$. That is, graph each line and see where they intersect.

Answer available at the end of this section.

Exercise 1.1E.5

Graphically, find the point of intersection of the two lines $3x + y = 3$ and $x + 2y = 1$. That is, graph each line and see where they intersect.

Answer available at the end of this section.

Exercise 1.1E.6

Do the three lines, $x + 2y = 1$, $2x - y = 1$, and $4x + 3y = 3$ have a common point of intersection? If so, find the point and if not, tell why they don't have such a common point of intersection.

Answer available at the end of this section.

Exercise 1.1E.7

Do the three planes, $x + y - 3z = 2$, $2x + y + z = 1$, and $3x + 2y - 2z = 0$ have a common point of intersection? If so, find one and if not, tell why there is no such point.

Answer available at the end of this section.

Exercise $1.1E.8$

Four times the weight of Gaston is 150 pounds more than the weight of Ichabod. Four times the weight of Ichabod is 660 pounds less than seventeen times the weight of Gaston. Four times the weight of Gaston plus the weight of Siegfried equals 290 pounds. Brunhilde would balance all three of the others. Find the weights of the four people.

Answer available at the end of this section.

Exercise $1.1E.9$

Find a linear system with 3 equations and 3 variables that has the solution $(-2, 1, 3)$. Describe the solution graphically.

Exercise $1.1E.10$

Find a linear system with 3 equations and 3 variables that has infinitely many solutions. Come up with a sytem in which none of the equations is a multiple of another equation. Describe the relationship between the corresponding planes graphically.

Exercise $1.1E.11$

Find a linear system with 3 equations and 3 variables that has no solutions. Come up with a sytem in which none of the planes correspoding to your equations is parallel to another one in the system. Describe the relationship between the 3 planes graphically.

Answers

Exercise $1.1E.4$

$\begin{matrix} x+3y=1 \\ 4x-y=3 \end{matrix}$, Solution is: $[x=\frac{10}{13}, y=\frac{1}{13}]$.

Exercise $1.1E.5$

$\begin{matrix} 3x+y=3 \\ x+2y=1 \end{matrix}$, Solution is: $[x=1, y=0]$

Exercise $1.1E.6$

$\begin{matrix} x+2y=1 \\ 2x-y=1 \\ 4x+3y=3 \end{matrix}$, Solution is: $[x=\frac{3}{5}, y=\frac{1}{5}]$

Exercise $1.1E.7$

No solution exists.

Exercise $1.1E.8$

$\begin{matrix} 4g-I=150 \\ 4I-17g=-660 \\ 4g+s=290 \\ g+I+s-b=0 \end{matrix}$, Solution is: $\{g=60, I=90, b=200, s=50\}$

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1.2: Gaussian Elimination

Learning Objectives

- Find the row-echelon form and reduced row-echelon form of a matrix.
- Determine whether a system of linear equations has no solution, a unique solution, or an infinite number of solutions from its row-echelon or reduced row-echelon form.
- Solve a system of equations using Gaussian Elimination and Gauss-Jordan Elimination.
- Establish the uniqueness of the reduced row-echelon form.

The work we did in the previous section will always find the solution to the system. In this section, we will explore a less cumbersome way to find the solutions. First, we will represent a linear system with an **augmented matrix.** A **matrix** is simply a rectangular array of numbers. The size or dimension of a matrix is defined as $m \times n$ where m is the number of rows and n is the number of columns. In order to construct an augmented matrix from a linear system, we create a **coefficient matrix** from the coefficients of the variables in the system, as well as a **constant matrix** from the constants. The coefficients from one equation of the system create one row of the augmented matrix.

For example, consider the linear system in Example 1.1.4

$$\begin{aligned} x + 3y + 6z &= 25 \\ 2x + 7y + 14z &= 58 \\ 2y + 5z &= 19 \end{aligned}$$

This system can be written as an augmented matrix, as follows

$$\left[\begin{array}{cccc} 1 & 3 & 6 & 25 \\ 2 & 7 & 14 & 58 \\ 0 & 2 & 5 & 19 \end{array}\right]$$

Notice that it has exactly the same information as the original system. Here it is understood that the first column contains the coefficients from x in each equation, in order, $\left[\begin{array}{c} 1 \\ 2 \\ 0 \end{array}\right]$. Similarly, we create a column from the coefficients on y in each equation, $\left[\begin{array}{c} 3 \\ 7 \\ 2 \end{array}\right]$ and a column from the coefficients on z in each equation, $\left[\begin{array}{c} 6 \\ 14 \\ 5 \end{array}\right]$. For a system of more than three variables, we would continue in this way constructing a column for each variable. Similarly, for a system of less than three variables, we simply construct a column for each variable.

Finally, we construct a column from the constants of the equations, $\left[\begin{array}{c} 25 \\ 58 \\ 19 \end{array}\right]$.

In this system, the coefficient matrix refers to $\left[\begin{array}{ccc} 1 & 3 & 6 \\ 2 & 7 & 14 \\ 0 & 2 & 5 \end{array}\right]$ and the constant matrix refers to $\left[\begin{array}{c} 25 \\ 58 \\ 19 \end{array}\right]$.

The rows of the augmented matrix correspond to the equations in the system. For example, the top row in the augmented matrix, $\left[\begin{array}{ccc|c} 1 & 3 & 6 & 25 \end{array}\right]$ corresponds to the equation

$$x + 3y + 6z = 25.$$

Consider the following definition.

Definition 1.2.1: Augmented Matrix of a Linear System

For a linear system of the form

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

where the x_i are variables and the a_{ij} and b_i are constants, the augmented matrix of this system is given by

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{bmatrix}$$

Now, consider elementary operations in the context of the augmented matrix. The elementary operations in Definition 1.1.4 can be used on the rows just as we used them on equations previously. Changes to a system of equations as a result of an elementary operation are equivalent to changes in the augmented matrix resulting from the corresponding row operation. Note that Theorem 1.1.1 implies that any elementary row operations used on an augmented matrix will *not* change the solution to the corresponding system of equations. We now formally define elementary row operations. These are the *key tool* we will use to find solutions to systems of equations.

Definition 1.2.2: Elementary Row Operations

The **elementary row operations** (also known as **row operations**) consist of the following

1. Switch two rows.
2. Multiply a row by a nonzero number.
3. Replace a row by any multiple of another row added to it.

Recall how we solved Example 1.1.4. We can do the exact same steps as above, except now in the context of an augmented matrix and using row operations. The augmented matrix of this system is

$$\begin{bmatrix} 1 & 3 & 6 & 25 \\ 2 & 7 & 14 & 58 \\ 0 & 2 & 5 & 19 \end{bmatrix}$$

Thus the first step in solving the system given by (1.1.5) in Example 1.1.4 would be to take (-2) times the first row of the augmented matrix and add it to the second row,

$$\begin{bmatrix} 1 & 3 & 6 & 25 \\ 0 & 1 & 2 & 8 \\ 0 & 2 & 5 & 19 \end{bmatrix}$$

Note how this corresponds to (1.1.6) in Example 1.14. Next take (-2) times the second row and add to the third,

$$\begin{bmatrix} 1 & 3 & 6 & 25 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

This augmented matrix corresponds to the system

$$\begin{aligned} x + 3y + 6z &= 25 \\ y + 2z &= 8 \\ z &= 3 \end{aligned}$$

which is the same as (1.1.7) in Example 1.1.4. By back substitution you obtain the solution $x = 1, y = 2$, and $z = 3$.

Through a systematic procedure of row operations, we can simplify an augmented matrix and carry it to **row-echelon form** or **reduced row-echelon form**, which we define next. These forms are used to find the solutions of the system of equations corresponding to the augmented matrix.

In the following definitions, the term **leading entry** refers to the first nonzero entry of a row when scanning the row from left to right.

Definition 1.2.3: Row-Echelon Form

An augmented matrix is in **row-echelon form** if

1. All nonzero rows are above any rows of zeros.
2. Each leading entry of a row is in a column to the right of the leading entries of any row above it.
3. Each leading entry of a row is equal to 1.

We also consider another reduced form of the augmented matrix which has one further condition.

Definition 1.2.4: Reduced Row-Echelon Form

An augmented matrix is in **reduced row-echelon form** if

1. All nonzero rows are above any rows of zeros.
2. Each leading entry of a row is in a column to the right of the leading entries of any rows above it.
3. Each leading entry of a row is equal to 1.
4. All entries in a column above and below a leading entry are zero.

Notice that the first three conditions on a reduced row-echelon form matrix are the same as those for row-echelon form.

Hence, every reduced row-echelon form matrix is also in row-echelon form. The converse is not necessarily true; we cannot assume that every matrix in row-echelon form is also in reduced row-echelon form. However, it often happens that the row-echelon form is sufficient to provide information about the solution of a system.

The following examples describe matrices in these various forms. As an exercise, take the time to carefully verify that they are in the specified form.

Example 1.2.1: Not in Row-Echelon Form

The following augmented matrices are not in row-echelon form (and therefore also not in reduced row-echelon form).

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & -6 \\ 4 & 0 & 7 \end{bmatrix}, \begin{bmatrix} 0 & 2 & 3 & 3 \\ 1 & 5 & 0 & 2 \\ 7 & 5 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Example 1.2.2: Matrices in Row-Echelon Form

The following augmented matrices are in row-echelon form, but not in reduced row-echelon form.

$$\begin{bmatrix} 1 & 0 & 6 & 5 & 8 & 2 \\ 0 & 0 & 1 & 2 & 7 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 3 & 5 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 6 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Notice that we could apply further row operations to these matrices to carry them to reduced row-echelon form. Take the time to try that on your own. Consider the following matrices, which are in reduced row-echelon form.

Example 1.2.3: Matrices in Reduced Row-Echelon Form

The following augmented matrices are in reduced row-echelon form.

$$\left[\begin{array}{cccccc} 1 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{cccc} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

One way in which the row-echelon form of a matrix is useful is in identifying the pivot positions and pivot columns of the matrix.

Definition 1.2.5: Pivot Position and Pivot Column

A **pivot position** in a matrix is the location of a leading entry in the row-echelon form of a matrix.

A **pivot column** is a column that contains a pivot position.

For example consider the following.

Example 1.2.4: Pivot Position

Let

$$A = \left[\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 6 \\ 4 & 4 & 4 & 10 \end{array}\right]$$

Where are the pivot positions and pivot columns of the augmented matrix A?

Solution

The row-echelon form of this matrix is

$$\left[\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This is all we need in this example, but note that this matrix is not in reduced row-echelon form.

In order to identify the pivot positions in the original matrix, we look for the leading entries in the row-echelon form of the matrix. Here, the entry in the first row and first column, as well as the entry in the second row and second column are the leading entries. Hence, these locations are the pivot positions. We identify the pivot positions in the original matrix, as in the following:

$$\left[\begin{array}{cccc} \boxed{1} & 2 & 3 & 4 \\ 3 & \boxed{2} & 1 & 6 \\ 4 & 4 & 4 & 10 \end{array}\right]$$

Thus the pivot columns in the matrix are the first two columns.

The following is an algorithm for carrying a matrix to row-echelon form and reduced row-echelon form. You may wish to use this algorithm to carry the above matrix to row-echelon form or reduced row-echelon form yourself for practice.

Algorithm 1.2.1: Reduced Row-Echelon Form Algorithm

This algorithm provides a method for using row operations to take a matrix to its reduced row-echelon form. We begin with the matrix in its original form.

1. Starting from the left, find the first nonzero column. This is the first pivot column, and the position at the top of this column is the first pivot position. Switch rows if necessary to place a nonzero number in the first pivot position.
2. Use row operations to make the entries below the first pivot position (in the first pivot column) equal to zero.
3. Ignoring the row containing the first pivot position, repeat steps 1 and 2 with the remaining rows. Repeat the process until there are no more rows to modify.
4. Divide each nonzero row by the value of the leading entry, so that the leading entry becomes 1. The matrix will then be in row-echelon form.

   The following step will carry the matrix from row-echelon form to reduced row-echelon form.

5. Moving from right to left, use row operations to create zeros in the entries of the pivot columns which are above the pivot positions. The result will be a matrix in reduced row-echelon form.

Most often we will apply this algorithm to an augmented matrix in order to find the solution to a system of linear equations. However, we can use this algorithm to compute the reduced row-echelon form of any matrix which could be useful in other applications.

Consider the following example of applying Algorithm $1.2.1$.

Example 1.2.5: Finding Row-Echelon Form and Reduced Row-Echelon Form of a Matrix

Let

$$A = \begin{bmatrix} 0 & -5 & -4 \\ 1 & 4 & 3 \\ 5 & 10 & 7 \end{bmatrix}$$

Find the row-echelon form of A. Then complete the process until A is in reduced row-echelon form.

Solution

In working through this example, we will use the steps outlined in Algorithm $1.2.1$.

1. The first pivot column is the first column of the matrix, as this is the first nonzero column from the left. Hence the first pivot position is the one in the first row and first column. Switch the first two rows to obtain a nonzero entry in the first pivot position, outlined in a box below.

$$\begin{bmatrix} \boxed{1} & 4 & 3 \\ 0 & -5 & -4 \\ 5 & 10 & 7 \end{bmatrix}$$

2. Step two involves creating zeros in the entries below the first pivot position. The first entry of the second row is already a zero. All we need to do is subtract 5 times the first row from the third row. The resulting matrix is

$$\begin{bmatrix} 1 & 4 & 3 \\ 0 & -5 & -4 \\ 0 & 10 & 8 \end{bmatrix}$$

3. Now ignore the top row. Apply steps 1 and 2 to the smaller matrix

$$\begin{bmatrix} -5 & -4 \\ 10 & 8 \end{bmatrix}$$

In this matrix, the first column is a pivot column, and -5 is in the first pivot position. Therefore, we need to create a zero below it. To do this, add 2 times the first row (of this matrix) to the second. The resulting matrix is

$$\begin{bmatrix} -5 & -4 \\ 0 & 0 \end{bmatrix}$$

Our original matrix now looks like

$$\begin{bmatrix} 1 & 4 & 3 \\ 0 & -5 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

We can see that there are no more rows to modify.

4. Now, we need to create leading 1s in each row. The first row already has a leading 1 so no work is needed here. Divide the second row by -5 to create a leading 1. The resulting matrix is

$$\begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & \frac{4}{5} \\ 0 & 0 & 0 \end{bmatrix}$$

This matrix is now in row-echelon form.

5. Now create zeros in the entries above pivot positions in each column, in order to carry this matrix all the way to reduced row-echelon form. Notice that there is no pivot position in the third column so we do not need to create any zeros in this column! The column in which we need to create zeros is the second. To do so, subtract 4 times the second row from the first row. The resulting matrix is

$$\begin{bmatrix} 1 & 0 & -\frac{1}{5} \\ 0 & 1 & \frac{4}{5} \\ 0 & 0 & 0 \end{bmatrix}$$

This matrix is now in reduced row-echelon form.

The above algorithm gives you a simple way to obtain the row-echelon form and reduced row-echelon form of a matrix. The main idea is to do row operations in such a way as to end up with a matrix in row-echelon form or reduced row-echelon form. This process is important because the resulting matrix will allow you to describe the solutions to the corresponding linear system of equations in a meaningful way.

In the next example, we look at how to solve a system of equations using the corresponding augmented matrix.

Example $1.2.6$: Finding the Solution to a System

Give the complete solution to the following system of equations

$$\begin{aligned} 2x + 4y - 3z &= -1 \\ 5x + 10y - 7z &= -2 \\ 3x + 6y + 5z &= 9 \end{aligned}$$

Solution

The augmented matrix for this system is

$$\begin{bmatrix} 2 & 4 & -3 & -1 \\ 5 & 10 & -7 & -2 \\ 3 & 6 & 5 & 9 \end{bmatrix}$$

In order to find the solution to this system, we wish to carry the augmented matrix to reduced row-echelon form. We will do so using Algorithm $1.2.1$. Notice that the first column is nonzero, so this is our first pivot column. The first entry in the first row, 2, is the first leading entry and it is in the first pivot position. We will use row operations to create zeros in the entries below the 2. First, replace the second row with -5 times the first row plus 2 times the second row. This yields

$$\begin{bmatrix} 2 & 4 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 3 & 6 & 5 & 9 \end{bmatrix}$$

Now, replace the third row with -3 times the first row plus 2 times the third row. This yields

$$\begin{bmatrix} 2 & 4 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 19 & 21 \end{bmatrix}$$

Now the entries in the first column below the pivot position are zeros. We now look for the second pivot column, which in this case is column three. Here, the 1 in the second row and third column is in the pivot position. We need to do just one row operation to create a zero below the 1.

Taking -19 times the second row and adding it to the third row yields

$$\begin{bmatrix} 2 & 4 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

We could proceed with the algorithm to carry this matrix to row-echelon form or reduced row-echelon form. However, remember that we are looking for the solutions to the system of equations. Take another look at the third row of the matrix. Notice that it corresponds to the equation

$$0x + 0y + 0z = 2$$

There is no solution to this equation because for all x, y, z, the left side will equal 0 and $0 \neq 2$. This shows there is no solution to the given system of equations. In other words, this system is inconsistent.

Watch the accompanying video for a visual demonstration of the this concept.

- **Video Length**: 9 minutes 41 seconds.
- **Context**: This video demonstrates how to solve a system of three equations using an augmented matrix and finding the reduced row echelon form for the linear system:

$$\begin{aligned} x + 2y - z &= -10 \\ 2x - 3y + 2z &= 2 \\ x + y + 3z &= 0 \end{aligned}$$

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

The following is another example of how to find the solution to a system of equations by carrying the corresponding augmented matrix to reduced row-echelon form.

Example 1.2.7: An Infinite Set of Solutions

Give the complete solution to the system of equations

$$\begin{aligned} 3x - y - 5z &= 9 \\ y - 10z &= 0 \\ -2x + y &= -6 \end{aligned} \tag{1.2.1}$$

Solution

The augmented matrix of this system is

$$\begin{bmatrix} 3 & -1 & -5 & 9 \\ 0 & 1 & -10 & 0 \\ -2 & 1 & 0 & -6 \end{bmatrix}$$

In order to find the solution to this system, we will carry the augmented matrix to reduced row-echelon form, using Algorithm $1.2.1$. The first column is the first pivot column. We want to use row operations to create zeros beneath the first entry in this column, which is in the first pivot position. Replace the third row with 2 times the first row added to 3 times the third row. This gives

$$\begin{bmatrix} 3 & -1 & -5 & 9 \\ 0 & 1 & -10 & 0 \\ 0 & 1 & -10 & 0 \end{bmatrix}$$

Now, we have created zeros beneath the 3 in the first column, so we move on to the second pivot column (which is the second column) and repeat the procedure. Take -1 times the second row and add to the third row.

$$\begin{bmatrix} 3 & -1 & -5 & 9 \\ 0 & 1 & -10 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The entry below the pivot position in the second column is now a zero. Notice that we have no more pivot columns because we have only two leading entries.

At this stage, we also want the leading entries to be equal to one. To do so, divide the first row by 3.

$$\begin{bmatrix} 1 & -\frac{1}{3} & -\frac{5}{3} & 3 \\ 0 & 1 & -10 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix is now in row-echelon form.

Let's continue with row operations until the matrix is in reduced row-echelon form. This involves creating zeros above the pivot positions in each pivot column. This requires only one step, which is to add $\frac{1}{3}$ times the second row to the first row.

$$\begin{bmatrix} 1 & 0 & -5 & 3 \\ 0 & 1 & -10 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is in reduced row-echelon form, which you should verify using Definition $1.2.4$. The equations corresponding to this reduced row-echelon form are

$$\begin{aligned} x - 5z &= 3 \\ y - 10z &= 0 \end{aligned}$$

or

$$\begin{aligned} x &= 3 + 5z \\ y &= 10z \end{aligned}$$

Observe that z is not restrained by any equation. In fact, z can equal any number. For example, we can let $z = t$, where we can choose t to be any number. In this context t is called a **parameter** . Therefore, the solution set of this system is

$$\begin{aligned} x &= 3 + 5t \\ y &= 10t \\ z &= t \end{aligned}$$

where t is arbitrary. The system has an infinite set of solutions which are given by these equations. For any value of t we select, $x, y,$ and z will be given by the above equations. For example, if we choose $t = 4$ then the corresponding solution would be

$$x = 3 + 5(4) = 23$$
$$y = 10(4) = 40$$
$$z = 4$$

In Example 1.2.7 the solution involved one parameter. It may happen that the solution to a system involves more than one parameter, as shown in the following example.

Example 1.2.8: A Two Parameter Set of Solutions

Find the solution to the system

$$x + 2y - z + w = 3$$
$$x + y - z + w = 1$$
$$x + 3y - z + w = 5$$

Solution

The augmented matrix is

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 3 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 3 & -1 & 1 & 5 \end{bmatrix}$$

We wish to carry this matrix to row-echelon form. Here, we will outline the row operations used. However, make sure that you understand the steps in terms of Algorithm 1.2.1.

Take -1 times the first row and add to the second. Then take -1 times the first row and add to the third. This yields

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 3 \\ 0 & -1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 2 \end{bmatrix}$$

Now add the second row to the third row and divide the second row by -1.

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 3 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.2.2}$$

This matrix is in row-echelon form and we can see that x and y correspond to pivot columns, while z and w do not. Therefore, we will assign parameters to the variables z and w. Assign the parameter s to z and the parameter t to w. Then the first row yields the equation $x + 2y - s + t = 3$, while the second row yields the equation $y = 2$. Since $y = 2$, the first equation becomes $x + 4 - s + t = 3$ showing that the solution is given by

$$x = -1 + s - t$$
$$y = 2$$
$$z = s$$
$$w = t$$

It is customary to write this solution in the form

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -1 + s - t \\ 2 \\ s \\ t \end{bmatrix} \tag{1.2.3}$$

This example shows a system of equations with an infinite solution set which depends on two parameters. It can be less confusing in the case of an infinite solution set to first place the augmented matrix in reduced row-echelon form rather than just row-echelon form before seeking to write down the description of the solution.

In the above steps, this means we don't stop with the row-echelon form in equation (). Instead we first place it in reduced row-echelon form as follows.

$$\begin{bmatrix} 1 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then the solution is $y = 2$ from the second row and $x = -1 + z - w$ from the first. Thus letting $z = s$ and $w = t$, the solution is given by (1.2.3).

You can see here that there are two paths to the correct answer, which both yield the same answer. Hence, either approach may be used. The process which we first used in the above solution is called **Gaussian Elimination** This process involves carrying the matrix to row-echelon form, converting back to equations, and using back substitution to find the solution. When you do row operations until you obtain reduced row-echelon form, the process is called **Gauss-Jordan Elimination**.

We have now found solutions for systems of equations with no solution and infinitely many solutions, with one parameter as well as two parameters. Recall the three types of solution sets which we discussed in the previous section; no solution, one solution, and infinitely many solutions. Each of these types of solutions could be identified from the graph of the system. It turns out that we can also identify the type of solution from the reduced row-echelon form of the augmented matrix.

- *No Solution:* In the case where the system of equations has no solution, the row-echelon form of the augmented matrix will have a row of the form

  $$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 1 \end{array}\right]$$

  This row indicates that the system is inconsistent and has no solution.
- *One Solution:* In the case where the system of equations has one solution, every column of the coefficient matrix is a pivot column. The following is an example of an augmented matrix in reduced row-echelon form for a system of equations with one solution.

  $$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

- *Infinitely Many Solutions:* In the case where the system of equations has infinitely many solutions, the solution contains parameters. There will be columns of the coefficient matrix which are not pivot columns. The following are examples of augmented matrices in reduced row-echelon form for systems of equations with infinitely many solutions.

  $$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

  or

  $$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -3 \end{bmatrix}$$

Uniqueness of Reduced Row Echelon Form

As we have seen, we know that every matrix can be brought into reduced row-echelon form by a sequence of elementary row operations. We will now prove that the resulting matrix is unique; in other words, the resulting matrix in reduced row-echelon does not depend upon the particular sequence of elementary row operations or the order in which they were performed.

Let A be the augmented matrix of a homogeneous system of linear equations in the variables $x_1, x_2, \cdots, x_n$ which is also in reduced row-echelon form. The matrix A divides the set of variables in two different types. We say that x_i is a **basic variable** whenever A has a leading 1 in column number i, in other words, when column i is a pivot column. Otherwise we say that x_i is a **free variable**.

Recall Example $1.2.8$ above. We continue with it in the following example.

Example 1.2.9: Basic and Free Variables

Find the basic and free variables in the system

$$\begin{aligned} x + 2y - z + w &= 3 \\ x + y - z + w &= 1 \\ x + 3y - z + w &= 5 \end{aligned}$$

Solution

Recall from the solution of Example 1.2.8 that the row-echelon form of the augmented matrix of this system is given by

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 3 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

You can see that columns 1 and 2 are pivot columns. These columns correspond to variables x and y, making these the basic variables. Columns 3 and 4 are not pivot columns, which means that z and w are free variables.

We can write the solution to this system as

$$\begin{aligned} x &= -1 + s - t \\ y &= 2 \\ z &= s \\ w &= t \end{aligned}$$

Here the free variables are written as parameters, and the basic variables are given by linear functions of these parameters.

In general, all solutions can be written in terms of the free variables. In such a description, the free variables can take any values (they become parameters), while the basic variables become simple linear functions of these parameters. Indeed, a basic variable x_i is a linear function of *only* those free variables x_j with $j > i$. This leads to the following observation.

Proposition 1.2.1: Basic and Free Variables

If x_i is a basic variable of a homogeneous system of linear equations, then any solution of the system with $x_j = 0$ for all those free variables x_j with $j > i$ must also have $x_i = 0$.

Using this proposition, we prove a lemma which will be used in the proof of the main result of this section below.

Lemma 1.2.1: Solutions and the Reduced Row-Echelon Form of a Matrix

Let A and B be two distinct augmented matrices for two homogeneous systems of m equations in n variables, such that A and B are each in reduced row-echelon. Then, the two systems do not have exactly the same solutions.

Proof available at the end of this section.

Now, we say that the matrix B is **equivalent** (or **row equivalent**) to the matrix A provided that B can be obtained from A by performing a sequence of elementary row operations beginning with A. The importance of this concept lies in the following result.

Theorem 1.2.1: Row Equivalent Matrices

The two linear systems of equations corresponding to two row equivalent augmented matrices have exactly the same solutions.

Proof available at the end of this section.

Now, we can use Lemma 1.2.1 and Theorem 1.2.1 to prove the main result of this section.

Theorem 1.2.2: Uniqueness of the Reduced Row-Echelon Form

Every matrix A is row equivalent to a unique matrix in reduced row-echelon form.

Proof available at the end of this section.

According to this theorem we can say that each matrix A has a unique reduced row-echelon form.

Answers

Lemma 1.2.1: Solutions and the Reduced Row-Echelon Form of a Matrix

With respect to the linear systems associated with the matrices A and B, there are two cases to consider:

- Case 1: the two systems have the same basic variables
- Case 2: the two systems do not have the same basic variables

In case 1, the two matrices will have exactly the same pivot positions. However, since A and B are not identical, there is some row of A which is different from the corresponding row of B and yet the rows each have a pivot in the same column position. Let i be the index of this column position. Since the matrices are in reduced row-echelon form, the two rows must differ at some entry in a column $j > i$. Let these entries be a in A and b in B, where $a \neq b$. Since A is in reduced row-echelon form, if x_j were a basic variable for its linear system, we would have $a = 0$. Similarly, if x_j were a basic variable for the linear system of the matrix B , we would have $b = 0$. Since a and b are unequal, they cannot both be equal to 0, and hence x_j cannot be a basic variable for both linear systems. However, since the systems have the same basic variables, x_j must then be a free variable for each system. We now look at the solutions of the systems in which x_j is set equal to 1 and all other free variables are set equal to 0. For this choice of parameters, the solution of the system for matrix A has $x_j = -a$, while the solution of the system for matrix B has $x_j = -b$, so that the two systems have different solutions.

In case 2, there is a variable x_i which is a basic variable for one matrix, let's say A, and a free variable for the other matrix B. The system for matrix B has a solution in which $x_i = 1$ and $x_j = 0$ for all other free variables x_j. However, by Proposition 1.2.1this cannot be a solution of the system for the matrix A. This completes the proof of case 2.

Theorem 1.2.1: Row Equivalent Matrices

The proof of this theorem is left as an exercise.

Theorem 1.2.2: Uniqueness of the Reduced Row-Echelon Form

Let A be an $m \times n$ matrix and let B and C be matrices in reduced row-echelon form, each equivalent to A. It suffices to show that $B = C$.

Let A^+ be the matrix A augmented with a new rightmost column consisting entirely of zeros. Similarly, augment matrices B and C each with a rightmost column of zeros to obtain B^+ and C^+. Note that B^+ and C^+ are matrices in reduced row-echelon form which are obtained from A^+ by respectively applying the same sequence of elementary row operations which were used to obtain B and C from A.

Now, A^+, B^+, and C^+ can all be considered as augmented matrices of homogeneous linear systems in the variables $x_1, x_2, \cdots, x_n$. Because B^+ and C^+ are each equivalent to A^+, Theorem 1.2.1 ensures that all three homogeneous linear systems have exactly the same solutions. By Lemma 1.2.1 we conclude that $B^+ = C^+$. By construction, we must also have $B = C$.

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1.2E: Exercises for Section 1.2

Exercise $1.2E.1$

Consider the following augmented matrix in which $*$ denotes an arbitrary number and $\blacksquare$ denotes a nonzero number. Determine whether the given augmented matrix is consistent. If consistent, is the solution unique?

$$\left[\begin{array}{cccccc} \blacksquare & * & * & * & * & * \\ 0 & \blacksquare & * & * & 0 & * \\ 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * \end{array}\right]$$

Answer available at the end of this section.

Exercise $1.2E.2$

Consider the following augmented matrix in which $*$ denotes an arbitrary number and $\blacksquare$ denotes a nonzero number. Determine whether the given augmented matrix is consistent. If consistent, is the solution unique?

$$\left[\begin{array}{cccc} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \end{array}\right]$$

Answer available at the end of this section.

Exercise $1.2E.3$

Consider the following augmented matrix in which $*$ denotes an arbitrary number and $\blacksquare$ denotes a nonzero number. Determine whether the given augmented matrix is consistent. If consistent, is the solution unique?

$$\left[\begin{array}{cccccc} \blacksquare & * & * & * & * & * \\ 0 & \blacksquare & 0 & * & 0 & * \\ 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * \end{array}\right]$$

Exercise $1.2E.4$

Consider the following augmented matrix in which $*$ denotes an arbitrary number and $\blacksquare$ denotes a nonzero number. Determine whether the given augmented matrix is consistent. If consistent, is the solution unique?

$$\left[\begin{array}{cccccc} \blacksquare & * & * & * & * & * \\ 0 & \blacksquare & * & * & 0 & * \\ 0 & 0 & 0 & 0 & \blacksquare & 0 \\ 0 & 0 & 0 & 0 & * & \blacksquare \end{array}\right]$$

Answer available at the end of this section.

Exercise $1.2E.5$

Suppose a system of equations has fewer equations than variables. Will such a system necessarily be consistent? If so, explain why and if not, give an example which is not consistent.

Answer available at the end of this section.

Exercise $1.2E.6$

If a system of equations has more equations than variables, can it have a solution? If so, give an example and if not, tell why not.

Answer available at the end of this section.

Exercise $1.2E.7$

Find h such that

$$\begin{bmatrix} 2 & h & 4 \\ 3 & 6 & 7 \end{bmatrix}$$

is the augmented matrix of an *inconsistent* system.

Answer available at the end of this section.

Exercise $1.2E.8$

Find h such that

$$\begin{bmatrix} 1 & h & 3 \\ 2 & 4 & 6 \end{bmatrix}$$

is the augmented matrix of a *consistent* system with a unique solution.

Answer available at the end of this section.

Exercise $1.2E.9$

Find h such that

$$\begin{bmatrix} 1 & 1 & 4 \\ 3 & h & 12 \end{bmatrix}$$

is the augmented matrix of a *consistent* system with infinitely many solutions.

Answer available at the end of this section.

Exercise $1.2E.10$

Choose h and k such that the augmented matrix shown has each of the following:

a. one solution
b. no solution
c. infinitely many solutions

$$\begin{bmatrix} 1 & h & 2 \\ 2 & 4 & k \end{bmatrix}$$

Answer available at the end of this section.

Exercise $1.2E.11$

Choose h and k such that the augmented matrix shown has each of the following:

a. one solution
b. no solution
c. infinitely many solutions

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & h & k \end{bmatrix}$$

Answer available at the end of this section.

Exercise $1.2E.12$

Determine if the system is consistent. If so, is the solution unique?

$$\begin{aligned} x + 2y + z - w &= 2 \\ x - y + z + w &= 1 \\ 2x + y - z &= 1 \\ 4x + 2y + z &= 5 \end{aligned}$$

Answer available at the end of this section.

Exercise $1.2E.13$

Determine if the system is consistent. If so, is the solution unique?

$$\begin{aligned} x + 2y + z - w &= 2 \\ x - y + z + w &= 0 \\ 2x + y - z &= 1 \\ 4x + 2y + z &= 3 \end{aligned}$$

Answer available at the end of this section.

Exercise $1.2E.14$

Determine which matrices are in reduced row-echelon form.

a. $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 7 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{bmatrix}$

Answer available at the end of this section.

Exercise $1.2E.15$

Row reduce the following matrix to obtain the row-echelon form. Then continue to obtain the reduced row-echelon form.

$$\begin{bmatrix} 2 & -1 & 3 & -1 \\ 1 & 0 & 2 & 1 \\ 1 & -1 & 1 & -2 \end{bmatrix}$$

Exercise $1.2E.16$

Row reduce the following matrix to obtain the row-echelon form. Then continue to obtain the reduced row-echelon form.

$$\begin{bmatrix} 0 & 0 & -1 & -1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & -1 \end{bmatrix}$$

Exercise $1.2E.17$

Row reduce the following matrix to obtain the row-echelon form. Then continue to obtain the reduced row-echelon form.

$$\begin{bmatrix} 3 & -6 & -7 & -8 \\ 1 & -2 & -2 & -2 \\ 1 & -2 & -3 & -4 \end{bmatrix}$$

Exercise $1.2E.18$

Row reduce the following matrix to obtain the row-echelon form. Then continue to obtain the reduced row-echelon form.

$$\begin{bmatrix} 2 & 4 & 5 & 15 \\ 1 & 2 & 3 & 9 \\ 1 & 2 & 2 & 6 \end{bmatrix}$$

Exercise $1.2E.19$

Row reduce the following matrix to obtain the row-echelon form. Then continue to obtain the reduced row-echelon form.

$$\begin{bmatrix} 4 & -1 & 7 & 10 \\ 1 & 0 & 3 & 3 \\ 1 & -1 & -2 & 1 \end{bmatrix}$$

Exercise $1.2E.20$

Row reduce the following matrix to obtain the row-echelon form. Then continue to obtain the reduced row-echelon form.

$$\begin{bmatrix} 3 & 5 & -4 & 2 \\ 1 & 2 & -1 & 1 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

Exercise $1.2E.21$

Row reduce the following matrix to obtain the row-echelon form. Then continue to obtain the reduced row-echelon form.

$$\begin{bmatrix} -2 & 3 & -8 & 7 \\ 1 & -2 & 5 & -5 \\ 1 & -3 & 7 & -8 \end{bmatrix}$$

Exercise $1.2E.22$

Find the solution of the system whose augmented matrix is

$$\begin{bmatrix} 1 & 2 & 0 & 2 \\ 1 & 3 & 4 & 2 \\ 1 & 0 & 2 & 1 \end{bmatrix}$$

Exercise $1.2E.23$

Find the solution of the system whose augmented matrix is

$$\begin{bmatrix} 1 & 2 & 0 & 2 \\ 2 & 0 & 1 & 1 \\ 3 & 2 & 1 & 3 \end{bmatrix}$$

Answer available at the end of this section.

Exercise $1.2E.24$

Find the solution of the system whose augmented matrix is

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 4 & 2 \end{bmatrix}$$

Answer available at the end of this section.

Exercise $1.2E.25$

Find the solution of the system whose augmented matrix is

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 1 & 2 & 0 & 0 & 1 & 3 \\ 1 & 0 & 1 & 0 & 2 & 2 \end{bmatrix}$$

Answer available at the end of this section.

Exercise $1.2E.26$

Find the solution of the system whose augmented matrix is

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 2 & 0 & 0 & 1 & 3 \\ 1 & -1 & 2 & 2 & 2 & 0 \end{bmatrix}$$

Answer available at the end of this section.

Exercise $1.2E.27$

Find the solution to the system of equations, $7x + 14y + 15z = 22$, $2x + 4y + 3z = 5$, and $3x + 6y + 10z = 13$.

Answer available at the end of this section.

Exercise $1.2E.28$

Find the solution to the system of equations, $3x - y + 4z = 6$, $y + 8z = 0$, and $-2x + y = -4$.

Answer available at the end of this section.

Exercise $1.2E.29$

Find the solution to the system of equations, $9x - 2y + 4z = -17$, $13x - 3y + 6z = -25$, and $-2x - z = 3$.

Answer available at the end of this section.

Exercise $1.2E.30$

Find the solution to the system of equations, $65x + 84y + 16z = 546$, $81x + 105y + 20z = 682$, and $84x + 110y + 21z = 713$.

Answer available at the end of this section.

Exercise $1.2E.31$

Find the solution to the system of equations, $8x + 2y + 3z = -3, 8x + 3y + 3z = -1$, and $4x + y + 3z = -9$.

Answer available at the end of this section.

Exercise $1.2E.32$

Find the solution to the system of equations, $-8x + 2y + 5z = 18, -8x + 3y + 5z = 13$, and $-4x + y + 5z = 19$.

Answer available at the end of this section.

Exercise $1.2E.33$

Find the solution to the system of equations, $3x - y - 2z = 3$, $y - 4z = 0$, and $-2x + y = -2$.

Answer available at the end of this section.

Exercise $1.2E.34$

Find the solution to the system of equations, $-9x + 15y = 66, -11x + 18y = 79$, $-x + y = 4$, and $z = 3$.

Answer available at the end of this section.

Exercise $1.2E.35$

Find the solution to the system of equations, $-19x + 8y = -108$, $-71x + 30y = -404$, $-2x + y = -12$, $4x + z = 14$.

Answer available at the end of this section.

Exercise $1.2E.36$

Suppose a system of equations has fewer equations than variables and you have found a solution to this system of equations. Is it possible that your solution is the only one? Explain.

Answer available at the end of this section.

Exercise $1.2E.37$

Suppose a system of linear equations has a 2×4 augmented matrix and the last column is a pivot column. Could the system of linear equations be consistent? Explain.

Answer available at the end of this section.

Exercise $1.2E.38$

Suppose the coefficient matrix of a system of n equations with n variables has the property that every column is a pivot column. Does it follow that the system of equations must have a solution? If so, must the solution be unique? Explain.

Answer available at the end of this section.

Exercise $1.2E.39$

Suppose there is a unique solution to a system of linear equations. What must be true of the pivot columns in the augmented matrix?

Answer available at the end of this section.

Exercise $1.2E.40$

The steady state temperature, u, of a plate solves Laplace's equation, $\Delta u = 0$. One way to approximate the solution is to divide the plate into a square mesh and require the temperature at each node to equal the average of the temperature at the four adjacent nodes. In the following picture, the numbers represent the observed temperature at the indicated nodes. Find the temperature at the interior nodes, indicated by x, y, z, and w. One of the equations is $z = \frac{1}{4}(10 + 0 + w + x)$.

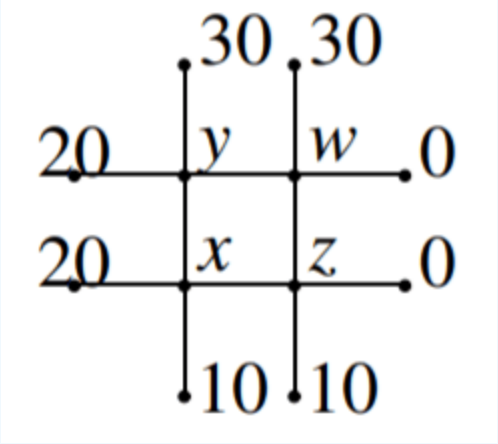


Figure $1.2E.1$ A grid with 2 horizontal and 2 vertical line segments, creating 12 vertices. The temperatures at the exterior vertices are known to be 30, 30, 20, 0, 20, 0, 10, 10 - scanning left to right, starting at top. The temperatures at the interior vertices are unknown and represented by the variables y, w, x, z - scanning left to right, starting at top.

Answer available at the end of this section.

Answers

Exercise $1.2E.1$

The solution exists but is not unique.

Exercise $1.2E.2$

A solution exists and is unique.

Exercise $1.2E.4$

There might be a solution. If so, there are infinitely many.

Exercise $1.2E.5$

No. Consider $x+y+z=2$ and $x+y+z=1$.

Exercise $1.2E.6$

These can have a solution. For example, $x+y=1, 2x+2y=2, 3x+3y=3$ even has an infinite set of solutions.

Exercise $1.2E.7$

$h=4$

Exercise $1.2E.8$

Any $h \neq 2$ will work.

Exercise $1.2E.9$

$h=3$

Exercise $1.2E.10$

If $h \neq 2$ there will be a unique solution for any k. If $h=2$ and $k \neq 4$, there are no solutions. If $h=2$ and $k=4$, then there are infinitely many solutions.

Exercise $1.2E.11$

If $h \neq 4$, then there is exactly one solution. If $h=4$ and $k \neq 4$, then there are no solutions. If $h=4$ and $k=4$, then there are infinitely many solutions.

Exercise $1.2E.12$

There is no solution. The system is inconsistent. You can see this from the augmented matrix. $\begin{bmatrix} 1 & 2 & 1 & -1 & 2 \\ 1 & -1 & 1 & 1 & 1 \\ 2 & 1 & -1 & 0 & 1 \\ 4 & 2 & 1 & 0 & 5 \end{bmatrix}$, reduced row-echelon form: $\begin{bmatrix} 1 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 1 & 0 & -\frac{2}{3} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

Exercise $1.2E.13$

Solution is: $\left[w=\frac{3}{2}y-1, x=\frac{2}{3}-\frac{1}{2}y, z=\frac{1}{3}\right]$

Exercise $1.2E.14$

a. This one is not.
b. This one is.
c. This one is.

Exercise $1.2E.23$

The reduced row-echelon form is $\begin{bmatrix} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -\frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Therefore, the solution is of the form $z = t, y = \frac{3}{4} + t\left(\frac{1}{4}\right), x = \frac{1}{2} - \frac{1}{2}t$ where $t \in \mathbb{R}$.

Exercise $1.2E.24$

The reduced row-echelon form is $\begin{bmatrix} 1 & 0 & 4 & 2 \\ 0 & 1 & -4 & -1 \end{bmatrix}$ and so the solution is $z = t, y = 4t, x = 2 - 4t$.

Exercise $1.2E.25$

The reduced row-echelon form is $\begin{bmatrix} 1 & 0 & 0 & 0 & 9 & 3 \\ 0 & 1 & 0 & 0 & -4 & 0 \\ 0 & 0 & 1 & 0 & -7 & -1 \\ 0 & 0 & 0 & 1 & 6 & 1 \end{bmatrix}$ and so $x_5 = t, x_4 = 1 - 6t, x_3 = -1 + 7t, x_2 = 4t, x_1 = 3 - 9t$.

Exercise $1.2E.26$

The reduced row-echelon form is $\begin{bmatrix} 1 & 0 & 2 & 0 & -\frac{1}{2} & \frac{5}{2} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 0 & 1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Therefore, let $x_5 = t, x_3 = s$. Then the other variables are given by $x_4 = -\frac{1}{2} - \frac{3}{2}t, x_2 = \frac{3}{2} - t\frac{1}{2}, , x_1 = \frac{5}{2} + \frac{1}{2}t - 2s$.

Exercise $1.2E.27$

Solution is: $[x = 1 - 2t, z = 1, y = t]$

Exercise $1.2E.28$

Solution is: $[x = 2 - 4t, y = -8t, z = t]$

Exercise $1.2E.29$

Solution is: $[x = -1, y = 2, z = -1]$

Exercise $1.2E.30$

Solution is: $[x = 2, y = 4, z = 5]$

Exercise $1.2E.31$

Solution is: $[x = 1, y = 2, z = -5]$

Exercise $1.2E.32$

Solution is: $[x = -1, y = -5, z = 4]$

Exercise $1.2E.33$

Solution is: $[x = 2t + 1, y = 4t, z = t]$

Exercise $1.2E.34$

Solution is: $[x = 1, y = 5, z = 3]$

Exercise $1.2E.35$

Solution is: $[x = 4, y = -4, z = -2]$

Exercise $1.2E.36$

No. Consider $x + y + z = 2$ and $x + y + z = 1$.

Exercise $1.2E.37$

No. This would lead to $0 = 1$.

Exercise $1.2E.38$

Yes. It has a unique solution.

Exercise $1.2E.39$

The last column must not be a pivot column. The remaining columns must each be pivot columns.

Exercise $1.2E.40$

You need $\begin{array}{l} \frac{1}{4}(20 + 30 + w + x) - y = 0 \\ \frac{1}{4}(y + 30 + 0 + z) - w = 0 \\ \frac{1}{4}(20 + y + z + 10) - x = 0 \\ \frac{1}{4}(x + w + 0 + 10) - z = 0 \end{array}$, Solution is: $[w = 15, x = 15, y = 20, z = 10]$.

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1.3: Rank and Homogeneous Systems

Learning Objectives

- Solve a homogeneous system of equations.
- Find and interpret the rank of a matrix.

There is a special type of system which requires additional study. This type of system is called a *homogeneous* system of equations, which we defined in Definition 1.1.2. Our focus in this section is to consider what types of solutions are possible for a homogeneous system of equations.

Consider the following definition.

Definition 1.3.1: Trivial Solution

Consider the homogeneous system of equations given by

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

Then, $x_1 = 0, x_2 = 0, \cdots, x_n = 0$ is always a solution to this system. We call this the **trivial solution** .

If the system has a solution in which not all of the $x_1, \cdots, x_n$ are equal to zero, then we call this solution **nontrivial** . The trivial solution does not tell us much about the system, as it says that $0 = 0$! Therefore, when working with homogeneous systems of equations, we want to know when the system has a nontrivial solution.

Suppose we have a homogeneous system of m equations, using n variables, and suppose that $n > m$. In other words, there are more variables than equations. Then, it turns out that this system always has a nontrivial solution. Not only will the system have a nontrivial solution, but it also will have infinitely many solutions. It is also possible, but not required, to have a nontrivial solution if $n = m$ and \(n.

Consider the following example.

Example 1.3.1: Solutions to a Homogeneous System of Equations

Find the nontrivial solutions to the following homogeneous system of equations

$$\begin{aligned} 2x + y - z &= 0 \\ x + 2y - 2z &= 0 \end{aligned}$$

Solution

Notice that this system has $m = 2$ equations and $n = 3$ variables, so $n > m$. Therefore by our previous discussion, we expect this system to have infinitely many solutions.

The process we use to find the solutions for a homogeneous system of equations is the same process we used in the previous section. First, we construct the augmented matrix, given by

$$\begin{bmatrix} 2 & 1 & -1 & 0 \\ 1 & 2 & -2 & 0 \end{bmatrix}$$

Then, we carry this matrix to its reduced row-echelon form, given below.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

The corresponding system of equations is

$$\begin{aligned} x &= 0 \\ y - z &= 0 \end{aligned}$$

Since z is not restrained by any equation, we know that this variable will become our parameter. Let $z = t$ where t is any number. Therefore, our solution has the form

$$\begin{aligned} x &= 0 \\ y &= z = t \\ z &= t \end{aligned}$$

Hence this system has infinitely many solutions, with one parameter t.

Suppose we were to write the solution to the previous example in another form. Specifically,

$$\begin{aligned} x &= 0 \\ y &= 0 + t \\ z &= 0 + t \end{aligned}$$

can be written as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Notice that we have constructed a column from the constants in the solution (all equal to 0), as well as a column corresponding to the coefficients on t in each equation. While we will discuss this form of solution more in further chapters, for now consider the column of coefficients of the parameter t. In this case, this is the column $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

There is a special name for this column, which is **basic solution**. The basic solutions of a system are columns constructed from the coefficients on parameters in the solution. We often denote basic solutions by X_1, X_2 etc., depending on how many solutions occur. Therefore, Example 1.3.1 has the basic solution $X_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

We explore this further in the following example.

Example 1.3.2: Basic Solutions of a Homogeneous System

Consider the following homogeneous system of equations.

$$\begin{aligned} x + 4y + 3z &= 0 \\ 3x + 12y + 9z &= 0 \end{aligned}$$

Find the basic solutions to this system.

Solution

The augmented matrix of this system and the resulting reduced row-echelon form are

$$\begin{bmatrix} 1 & 4 & 3 & 0 \\ 3 & 12 & 9 & 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

When written in equations, this system is given by

$$x + 4y + 3z = 0$$

Notice that only x corresponds to a pivot column. In this case, we will have two parameters, one for y and one for z. Let $y = s$ and $z = t$ for any numbers s and t. Then, our solution becomes

$$\begin{aligned} x &= -4s - 3t \\ y &= s \\ z &= t \end{aligned}$$

which can be written as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

You can see here that we have two columns of coefficients corresponding to parameters, specifically one for s and one for t. Therefore, this system has two basic solutions! These are

$$X_1 = \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix}, X_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

We now present a new definition.

Definition 1.3.2: Linear Combination

Let $X_1, \cdots, X_n, V$ be column matrices. Then V is said to be a **linear combination** of the columns $X_1, \cdots, X_n$ if there exist scalars, $a_1, \cdots, a_n$ such that

$$V = a_1 X_1 + \cdots + a_n X_n$$

A remarkable result of this section is that a linear combination of the basic solutions is again a solution to the system. Even more remarkable is that every solution can be written as a linear combination of these solutions. Therefore, if we take a linear combination of the two solutions to Example 1.3.2, this would also be a solution. For example, we could take the following linear combination

$$3 \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -18 \\ 3 \\ 2 \end{bmatrix}$$

You should take a moment to verify that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -18 \\ 3 \\ 2 \end{bmatrix}$$

is in fact a solution to the system in Example 1.3.2.

Below is a video example of solving a system of non-homogenous equations (as we did in Section 1.2.1). However, in this example, we see how the solution to a non-homogeneous system is related to the solution of the corresponding homogeneous system. The corresponding homogeneous system refers to the linear system in which the constant terms are all 0.

Example 1.3.3

Solve the system of equations. Determine the particular solution and the solution set to the homogeneous system.

$$\begin{aligned} 5x - 15y - z - 25w &= -27 \\ 12x - 36y - 2z - 58w &= -64 \\ 3x - 9y - 12w &= -15 \end{aligned}$$

- **Video Length**: 4 minutes 30 seconds

- **Context**: This video demonstrates how to solve a system of equations and give the particular solution and homogeneous solution.

Solution

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Another way in which we can find out more information about the solutions of a homogeneous system is to consider the **rank** of the associated coefficient matrix. We now define what is meant by the rank of a matrix.

Definition 1.3.3: Rank of a Matrix

Let A be a matrix and consider any row-echelon form of A. Then, the number r of leading entries of A does not depend on the row-echelon form you choose, and is called the **rank** of A. We denote it by rank(A).

Similarly, we could count the number of pivot positions (or pivot columns) to determine the rank of A.

Example 1.3.4: Finding the Rank of a Matrix

Consider the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 5 & 9 \\ 2 & 4 & 6 \end{bmatrix}$$

What is its rank?

Solution

First, we need to find the reduced row-echelon form of A. Through the usual algorithm, we find that this is

$$\begin{bmatrix} \boxed{1} & 0 & -1 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Here we have two leading entries, or two pivot positions, shown above in boxes.The rank of A is $r = 2$.

Notice that we would have achieved the same answer if we had found the row-echelon form of A instead of the reduced row-echelon form.

Suppose we have a homogeneous system of m equations in n variables, and suppose that $n > m$. From our above discussion, we know that this system will have infinitely many solutions. If we consider the rank of the coefficient matrix of this system, we can find out even more about the solution. Note that we are looking at just the coefficient matrix, not the entire augmented matrix.

Theorem 1.3.1: Rank and Solutions to a Homogeneous System

Let A be the $m \times n$ coefficient matrix corresponding to a homogeneous system of equations, and suppose A has rank r. Then, the solution to the corresponding system has $n - r$ parameters.

Consider our above Example 1.3.2 in the context of this theorem. The system in this example has $m = 2$ equations in $n = 3$

variables. First, because $n > m$, we know that the system has a nontrivial solution, and therefore infinitely many solutions. This tells us that the solution will contain at least one parameter. The rank of the coefficient matrix can tell us even more about the solution! The rank of the coefficient matrix of the system is 1, as it has one leading entry in row-echelon form. Theorem 1.3.1 tells us that the solution will have $n - r = 3 - 1 = 2$ parameters. You can check that this is true in the solution to Example 1.3.2.

Notice that if $n = m$ or \(n, it is possible to have either a unique solution (which will be the trivial solution) or infinitely many solutions.

Example 1.3.5

For each of the following matrices. a row-echelon form is provided Find the rank of the matrix, and use it to find the basic and free variables of the homogeneous system with the given matrix as its coefficient matrix.

a. $\begin{bmatrix} 1 & 4 & 2 \\ -5 & 25 & 15 \\ 2 & -8 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

b. $\begin{bmatrix} 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 7 \\ 1 & 0 & 0 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

- **Video length**: 3 minutes 25 seconds.
- **Context**: This video demonstrates the concepts of rank, basic variables, and free variables for a given coefficient matrix.

Solution

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

It turns out that the rank of a matrix can be used to learn about the solutions of any system of linear equations, not just homogeneous linear system. In the previous section, we discussed that a system of equations can have no solution, a unique solution, or infinitely many solutions. Suppose the system is consistent, whether it is homogeneous or not. The following theorem tells us how we can use the rank to learn about the type of solution we have.

Theorem 1.3.2: Rank and Solutions to a Consistent System of Equations

Let A be the $m \times (n+1)$ augmented matrix corresponding to a *consistent* linear system of m equations in n variables, and suppose A has rank r. Then

1. the system has a unique solution if $r = n$
2. the system has infinitely many solutions if $r < n$

We will not present a formal proof of this, but consider the following discussions.

1. *No Solution* The above theorem assumes that the system is consistent; that is, it has a solution. It turns out that it is possible for the augmented matrix of a system with no solution to have any rank r as long as $r > 1$. Therefore, we must know that the system is *consistent* in order to use this theorem!
2. *Unique Solution* Suppose $r = n$. Then, there is a pivot position in every column of the coefficient matrix of A. Hence, there is a unique solution.
3. *Infinitely Many Solutions* Suppose \(r. Then there are infinitely many solutions. There are less pivot positions (and hence less leading entries) than columns, meaning that not every column is a pivot column. The columns which are not pivot columns correspond to parameters. In fact, in this case we have $n - r$ parameters

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1.3E: Exercises for Section 1.3

Exercise $1.3E.1$

Find the rank of the following matrix.

$$\begin{bmatrix} 4 & -16 & -1 & -5 \\ 1 & -4 & 0 & -1 \\ 1 & -4 & -1 & -2 \end{bmatrix}$$

Answer available at the end of this section.

Exercise $1.3E.2$

Find the rank of the following matrix.

$$\begin{bmatrix} 3 & 6 & 5 & 12 \\ 1 & 2 & 2 & 5 \\ 1 & 2 & 1 & 2 \end{bmatrix}$$

Exercise $1.3E.3$

Find the rank of the following matrix.

$$\begin{bmatrix} 0 & 0 & -1 & 0 & 3 \\ 1 & 4 & 1 & 0 & -8 \\ 1 & 4 & 0 & 1 & 2 \\ -1 & -4 & 0 & -1 & -2 \end{bmatrix}$$

Answer available at the end of this section.

Exercise $1.3E.4$

Find the rank of the following matrix.

$$\begin{bmatrix} 4 & -4 & 3 & -9 \\ 1 & -1 & 1 & -2 \\ 1 & -1 & 0 & -3 \end{bmatrix}$$

Exercise $1.3E.5$

Find the rank of the following matrix.

$$\begin{bmatrix} 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 7 \\ 1 & 0 & 0 & 1 & 7 \end{bmatrix}$$

Answer available at the end of this section.

Exercise 1.3E.6

Find the rank of the following matrix.

$$\begin{bmatrix} 4 & 15 & 29 \\ 1 & 4 & 8 \\ 1 & 3 & 5 \\ 3 & 9 & 15 \end{bmatrix}$$

Exercise 1.3E.7

Find the rank of the following matrix.

$$\begin{bmatrix} 0 & 0 & -1 & 0 & 1 \\ 1 & 2 & 3 & -2 & -18 \\ 1 & 2 & 2 & -1 & -11 \\ -1 & -2 & -2 & 1 & 11 \end{bmatrix}$$

Answer available at the end of this section.

Exercise 1.3E.8

Find the rank of the following matrix.

$$\begin{bmatrix} 1 & -2 & 0 & 3 & 11 \\ 1 & -2 & 0 & 4 & 15 \\ 1 & -2 & 0 & 3 & 11 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Exercise 1.3E.9

Find the rank of the following matrix.

$$\begin{bmatrix} -2 & -3 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ -3 & 0 & -3 \end{bmatrix}$$

Answer available at the end of this section.

Exercise 1.3E.10

Find the rank of the following matrix.

$$\begin{bmatrix} 4 & 4 & 20 & -1 & 17 \\ 1 & 1 & 5 & 0 & 5 \\ 1 & 1 & 5 & -1 & 2 \\ 3 & 3 & 15 & -3 & 6 \end{bmatrix}$$

Exercise 1.3E.11

Find the rank of the following matrix.

$$\begin{bmatrix} -1 & 3 & 4 & -3 & 8 \\ 1 & -3 & -4 & 2 & -5 \\ 1 & -3 & -4 & 1 & -2 \\ -2 & 6 & 8 & -2 & 4 \end{bmatrix}$$

Exercise $1.3E.12$

Suppose A is an $m \times n$ matrix. Explain why the rank of A is always no larger than $\min(m, n)$.

Answer available at the end of this section.

Exercise $1.3E.13$

State whether each of the following sets of data are possible for the matrix equation $AX = B$. If possible, describe the solution set. That is, tell whether there exists a unique solution, no solution or infinitely many solutions. Here, $[A|B]$ denotes the augmented matrix.

a. A is a 5×6 matrix, $rank(A) = 4$ and $rank[A|B] = 4$.
b. A is a 3×4 matrix, $rank(A) = 3$ and $rank[A|B] = 2$.
c. A is a 4×2 matrix, $rank(A) = 4$ and $rank[A|B] = 4$.
d. A is a 5×5 matrix, $rank(A) = 4$ and $rank[A|B] = 5$.
e. A is a 4×2 matrix, $rank(A) = 2$ and $rank[A|B] = 2$.

Answer available at the end of this section.

Exercise $1.3E.14$

Consider the system $-5x + 2y - z = 0$ and $-5x - 2y - z = 0$. Both equations equal zero and so $-5x + 2y - z = -5x - 2y - z$ which is equivalent to $y = 0$. Does it follow that x and z can equal anything? Notice that when $x = 1$, $z = -4$, and $y = 0$ are plugged in to the equations, the equations do not equal 0. Why?

Answer available at the end of this section.

Answers

Exercise $1.3E.1$

Rank = 2

Exercise $1.3E.3$

Rank = 3

Exercise $1.3E.5$

Rank = 3

Exercise $1.3E.7$

Rank = 3

Exercise $1.3E.9$

Rank = 2

Exercise $1.3E.12$

It is because you cannot have more than $\min(m, n)$ nonzero rows in the reduced row-echelon form. Recall that the number of pivot columns is the same as the number of nonzero rows from the description of this reduced row-echelon form.

Exercise $1.3E.13$

a. This says that both A and $A|B$ have exactly one row of zeros at the bottom. This means a solution exists. Since A has 5 columns but only 4 pivots, the system will have at least one free variable, leading to an infinite solution set.
b. This surely can't happen. The rank cannot get smaller after augmenting another column since we can't *lose* a pivot position in doing this.
c. This is impossible, since A with only 2 columns cannot possibly have 4 pivots.
d. Since $A|B$ has one more pivot than A, in the reduced row echelon form, $A|B$'s last row must be $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$. In this case, there is no solution to the

system of equations represented by the augmented matrix.

e. In this case, there is a unique solution since the columns of A are independent. In the reduced row echoelon form of both A and $A|B$, the last two rows are rows of 0s.

Exercise $1.3E.14$

These are not legitimate row operations. They do not preserve the solution set of the system.

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1.4: Balancing Chemical Reactions

Learning Objectives

- Using augmented matrices to balance chemical reactions.

The tools of linear algebra can also be used in the subject area of Chemistry, specifically for balancing chemical reactions.

Consider the chemical reaction

$$SnO_2 + H_2 \to Sn + H_2O$$

Here the elements involved are tin (Sn), oxygen (O), and hydrogen (H). A chemical reaction occurs and the result is a combination of tin (Sn) and water (H_2O). When considering chemical reactions, we want to investigate how much of each element we began with and how much of each element is involved in the result.

An important theory we will use here is the **Mass Balance Theory**. It tells us that we cannot create or delete elements within a chemical reaction. For example, in the above expression, we must have the same number of oxygen, tin, and hydrogen on both sides of the reaction. Notice that this is not currently the case. For example, there are two oxygen atoms on the left and only one on the right. In order to fix this, we want to find numbers x, y, z, w such that

$$xSnO_2 + yH_2 \to zSn + wH_2O$$

where both sides of the reaction have the same number of atoms of the various elements.

This is a familiar problem. We can solve it by setting up a system of equations in the variables x, y, z, w. Thus you need

$$\begin{aligned} Sn: &\quad x = z \\ O: &\quad 2x = w \\ H: &\quad 2y = 2w \end{aligned}$$

We can rewrite these equations as

$$\begin{aligned} Sn: &\quad x - z = 0 \\ O: &\quad 2x - w = 0 \\ H: &\quad 2y - 2w = 0 \end{aligned}$$

The augmented matrix for this system of equations is given by

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 2 & 0 & 0 & -1 & 0 \\ 0 & 2 & 0 & -2 & 0 \end{bmatrix}$$

The reduced row-echelon form of this matrix is

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 \end{bmatrix}$$

The solution is given by

$$\begin{aligned} x - \tfrac{1}{2}w &= 0 \\ y - w &= 0 \\ z - \tfrac{1}{2}w &= 0 \end{aligned}$$

which we can write as

$$x = \frac{1}{2}t$$
$$y = t$$
$$z = \frac{1}{2}t$$
$$w = t$$

For example, let $w = 2$ and this would yield $x = 1, y = 2,$ and $z = 1$. We can put these values back into the expression for the reaction which yields

$$SnO_2 + 2H_2 \rightarrow Sn + 2H_2O$$

Observe that each side of the expression contains the same number of atoms of each element. This means that it preserves the total number of atoms, as required, and so the chemical reaction is balanced.

Consider another example.

Example 1.4.1: Balancing a Chemical Reaction

Potassium is denoted by K, oxygen by O, phosphorus by P and hydrogen by H. Consider the reaction given by

$$KOH + H_3PO_4 \rightarrow K_3PO_4 + H_2O$$

Balance this chemical reaction.

Solution

We will use the same procedure as above to solve this problem. We need to find values for x, y, z, w such that

$$xKOH + yH_3PO_4 \rightarrow zK_3PO_4 + wH_2O$$

preserves the total number of atoms of each element.

Finding these values can be done by finding the solution to the following system of equations.

$$\begin{aligned} K: &\quad x = 3z \\ O: &\quad x + 4y = 4z + w \\ H: &\quad x + 3y = 2w \\ P: &\quad y = z \end{aligned}$$

The augmented matrix for this system is

$$\begin{bmatrix} 1 & 0 & -3 & 0 & 0 \\ 1 & 4 & -4 & -1 & 0 \\ 1 & 3 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{bmatrix}$$

and the reduced row-echelon form is

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The solution is given by

$$x - w = 0$$
$$y - \frac{1}{3}w = 0$$
$$z - \frac{1}{3}w = 0$$

which can be written as

$$x = t$$
$$y = \frac{1}{3}t$$
$$z = \frac{1}{3}t$$
$$w = t$$

Choose a value for t, say 3. Then $w = 3$ and this yields $x = 3, y = 1, z = 1$. It follows that the balanced reaction is given by

$$3KOH + 1H_3PO_4 \rightarrow 1K_3PO_4 + 3H_2O$$

Note that this results in the same number of atoms on both sides.

Of course these numbers you are finding would typically be the number of moles of the molecules on each side. Thus three moles of KOH added to one mole of H_3PO_4 yields one mole of K_3PO_4 and three moles of H_2O.

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1.4E: Exercises for Section 1.4

Exercise 1.4E.1

Balance the following chemical reaction.

$C_3H_8 + O_2 \rightarrow CO_2 + H_2O$

Answer available at the end of this section.

Exercise 1.4E.2

Balance the following chemical reaction.

$KNO_3 + H_2CO_3 \rightarrow K_2CO_3 + HNO_3$

Exercise 1.4E.3

Balance the following chemical reaction.

$C_2H_5OH + O_2 \rightarrow CO_2 + H_2O$

Answer available at the end of this section.

Exercise 1.4E.4

Balance the following chemical reaction.

$AgI + Na_2S \rightarrow Ag_2S + NaI$

Exercise 1.4E.5

Balance the following chemical reaction.

$P_4O_{10} + H_2O \rightarrow H_3PO_4$

Answer available at the end of this section.

Exercise 1.4E.6

Balance the following chemical reaction.

$Ba_3N_2 + H_2O \rightarrow Ba\left(OH\right)_2 + NH_3$

Exercise 1.4E.7

Balance the following chemical reaction.

$Na_2CO_3 + H_2SO_4 \rightarrow Na_2SO_4 + CO_2 + H_2O$

Answer available at the end of this section.

Exercise $1.4E.8$

Balance the following chemical reaction.

$CaCl_2 + Na_3PO_4 \rightarrow Ca_3\left(PO_4\right)_2 + NaCl$

Answers

Exercise $1.4E.1$

$C_3H_8 + 5O_2 \rightarrow 3CO_2 + 4H_2O$

Exercise $1.4E.3$

$C_2H_5OH + 3O_2 \rightarrow 2CO_2 + 3H_2O$

Exercise $1.4E.5$

$P_4O_{10} + 6H_2O \rightarrow 4H_3PO_4$

Exercise $1.4E.7$

$Na_2CO_3 + H_2SO_4 \rightarrow Na_2SO_4 + CO_2 + H_2O$

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1.5: Applications to Physics

Learning Objectives

- Determine the current for circuit diagram by solving a linear system.
- Determine dimensionless variables by solving a linear system.

Resistor Networks

The tools of linear algebra can be used to study the application of resistor networks. An example of an electrical circuit is below.

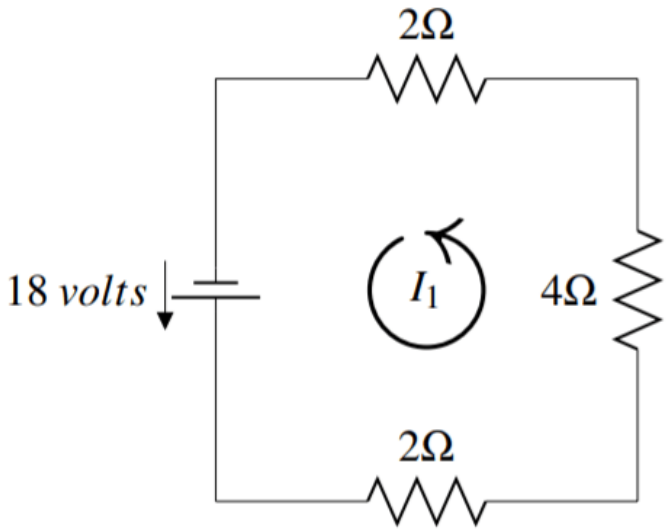


Figure 1.5.1 A circuit with a voltage source of 18 volts that forms a closed rectangular path with three resistors and an arrow indicating current flow. The resistors have resistance 2Ω, 4Ω, and 2Ω, as we move clockwise from the voltage source. The current flow, I_1, is counterclockwise.

The jagged lines () denote resistors and the numbers next to them give their resistance in ohms, written as Ω. The voltage source () causes the current to flow in the direction from the shorter of the two lines toward the longer (as indicated by the arrow). The current for a circuit is labeled I_k.

In the Figure 1.5.1, the current I_1 has been labeled with an arrow in the counter clockwise direction. This is an entirely arbitrary decision and we could have chosen to label the current in the clockwise direction. With our choice of direction here, we define a positive current to flow in the counter clockwise direction and a negative current to flow in the clockwise direction.

The goal of this section is to use the values of resistors and voltage sources in a circuit to determine the current. An essential theorem for this application is Kirchhoff's law.

Theorem 1.5.1: Kirchhoff's Law

The sum of the resistance (R) times the amps (I) in the counter clockwise direction around a loop equals the sum of the voltage sources (V) in the same direction around the loop.

Kirchhoff's law allows us to set up a system of linear equations and solve for any unknown variables. When setting up this system, it is important to trace the circuit in the counter clockwise direction. If a resistor or voltage source is crossed against this direction, the related term must be given a negative sign.

We will explore this in the next example where we determine the value of the current in the initial diagram.

Example 1.5.1: Solving for Current

Applying Kirchhoff's Law to the circuit in Figure 1.5.1, determine the value for I_1.

Solution

Begin in the bottom left corner, and trace the circuit in the counter clockwise direction. At the first resistor, multiplying resistance and current gives $2I_1$. Continuing in this way through all three resistors gives $2I_1 + 4I_1 + 2I_1$. This must equal the voltage source in the same direction. Notice that the direction of the voltage source matches the counter clockwise direction specified, so the voltage is positive.

Therefore the equation and solution are given by

$$\begin{aligned} 2I_1 + 4I_1 + 2I_1 &= 18 \\ 8I_1 &= 18 \\ I_1 &= \frac{9}{4}A \end{aligned} \tag{1.5.1}$$

Since the answer is positive, this confirms that the current flows counter clockwise.

Example 1.5.2 Solving for Current

Applying Kirchhoff's Law to the circuit diagram below, determine the value for I_1.

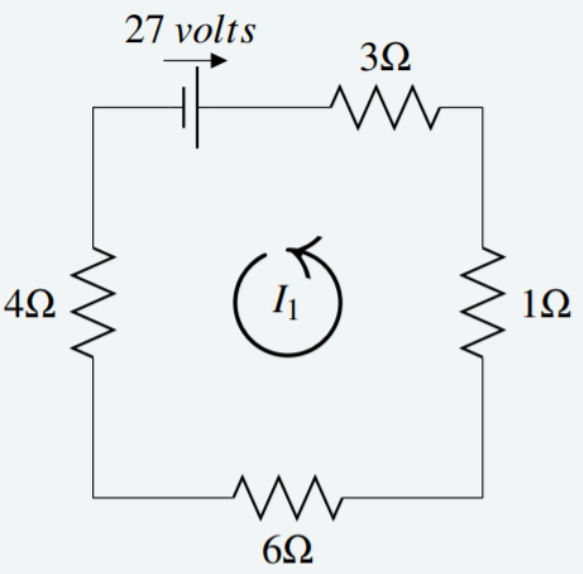


Figure 1.5.2 A circuit with a voltage source of 27 volts that forms a closed rectangular path with four resistors and an arrow indicating current flow. The resistors have resistance 3Ω, 1Ω, 6Ω, and 4Ω, as we move clockwise from the voltage source. The current flow, I_1, is counterclockwise.

Solution

Begin in the top left corner this time, and trace the circuit in the counter clockwise direction. At the first resistor, multiplying resistance and current gives $4I_1$. Continuing in this way through the four resistors gives $4I_1 + 6I_1 + 1I_1 + 3I_1$. This must equal the voltage source in the same direction. Notice that the direction of the voltage source is opposite to the counter clockwise direction, so the voltage is negative.

Therefore the equation and solution are given by

$$\begin{aligned} 4I_1 + 6I_1 + 1I_1 + 3I_1 &= -27 \\ 14I_1 &= -27 \\ I_1 &= -\frac{27}{14} A \end{aligned} \tag{1.5.2}$$

Since the answer is negative, this tells us that the current flows clockwise.

A more complicated example follows. Two of the circuits below may be familiar; they were examined in the examples above. However as they are now part of a larger system of circuits, the answers will differ.

Example 1.5.3: Unknown Currents

The diagram below consists of four circuits. The current (I_k) in the four circuits is denoted by I_1, I_2, I_3, I_4. Using Kirchhoff's Law, write an equation for each circuit and solve for each current.

Solution

The circuits are given in the following diagram.

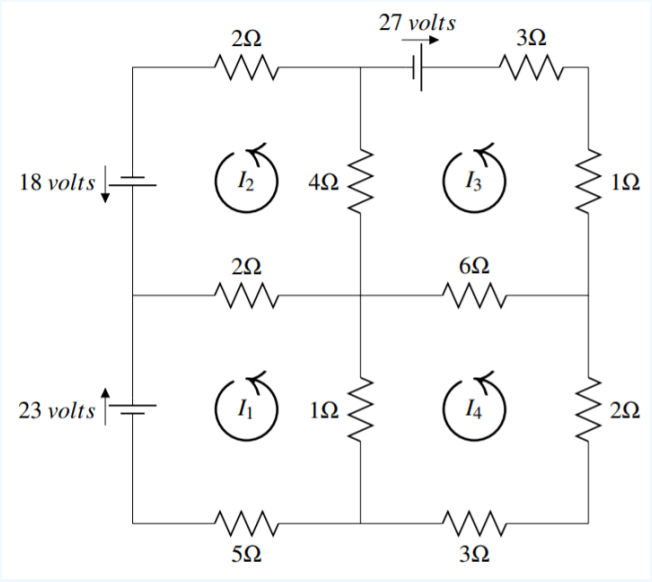


Figure 1.5.3 A circuit with multiple interconnected components, including resistors, voltage sources, and current loops. The circuit is arranged in a 2x2 grid, dividing the circuit into 4 squares: Square 1 on the bottom left, Square 2 on the top left, Square 3 on the top right and Square 4 on the bottom right. Square 1 has counterclockwise current flow I_1, with a 23V voltage source with arrow pointing up, and resistors 5Ω along the bottom, 1Ω on the right shared with Square 4, and 2Ω above shared with Square 2. Square 2 has counterclockwise current flow I_2, with an 18V voltage source with arrow pointing down, and resistors 2Ω on the bottom shared with Square 1, 4Ω on the right, shared iwth Square 3, and 2Ω above. Square 3 has counterclockwise current flow I_3, with resistors 4Ω on the left shared with Square 2, 6Ω on the bottom shared with Square 4, 1Ω on the right, 3Ω above, and a 27V voltage source with arrow pointing right. Square 4 has counterclockwise current flow I_4, with resistors 1Ω on the left shared with Square 1, 3Ω below, 2Ω on the right and 6Ω above shared with Square 3.

Starting with the top left circuit, multiply the resistance by the amps and sum the resulting products. Specifically, consider the resistor labeled 2Ω that is part of the circuits of I_1 and I_2. Notice that current I_2 runs through this in a positive (counter clockwise) direction, and I_1 runs through in the opposite (negative) direction. The product of resistance and amps is then $2(I_2 - I_1) = 2I_2 - 2I_1$. Continue in this way for each resistor, and set the sum of the products equal to the voltage source to write the equation:

$$2I_2 - 2I_1 + 4I_2 - 4I_3 + 2I_2 = 18$$

The above process is used on each of the other three circuits, and the resulting equations are:

Upper right circuit:

$$4I_3 - 4I_2 + 6I_3 - 6I_4 + I_3 + 3I_3 = -27$$

Lower right circuit:

$$3I_4 + 2I_4 + 6I_4 - 6I_3 + I_4 - I_1 = 0$$

Lower left circuit:

$$5I_1 + I_1 - I_4 + 2I_1 - 2I_2 = -23$$

Notice that the voltage for the upper right and lower left circuits are negative due to the clockwise direction they indicate.

The resulting system of four equations in four unknowns is

$$\begin{aligned} 2I_2 - 2I_1 + 4I_2 - 4I_3 + 2I_2 &= 18 \\ 4I_3 - 4I_2 + 6I_3 - 6I_4 + I_3 + I_3 &= -27 \\ 2I_4 + 3I_4 + 6I_4 - 6I_3 + I_4 - I_1 &= 0 \\ 5I_1 + I_1 - I_4 + 2I_1 - 2I_2 &= -23 \end{aligned} \tag{1.5.3}$$

Simplifying and rearranging with variables in order, we have:

$$\begin{aligned} -2I_1 + 8I_2 - 4I_3 &= 18 \\ -4I_2 + 14I_3 - 6I_4 &= -27 \\ -I_1 - 6I_3 + 12I_4 &= 0 \\ 8I_1 - 2I_2 - I_4 &= -23 \end{aligned} \tag{1.5.4}$$

The augmented matrix is

$$\begin{bmatrix} -2 & 8 & -4 & 0 & 18 \\ 0 & -4 & 14 & -6 & -27 \\ -1 & 0 & -6 & 12 & 0 \\ 8 & -2 & 0 & -1 & -23 \end{bmatrix}$$

The solution to this matrix is

$$\begin{aligned} I_1 &= -3A \\ I_2 &= \frac{1}{4}A \\ I_3 &= -\frac{5}{2}A \\ I_4 &= -\frac{3}{2}A \end{aligned} \tag{1.5.5}$$

This tells us that currents $I_1, I_3,$ and I_4 travel clockwise while I_2 travels counter clockwise.

Dimensionless Variables

This section shows how solving systems of equations can be used to determine appropriate dimensionless variables. It is only an introduction to this topic and considers a specific example of a simple airplane wing shown below. We assume for simplicity that it is a flat plane at an angle to the wind which is blowing against it with speed V as shown in Figure $1.5.4$.

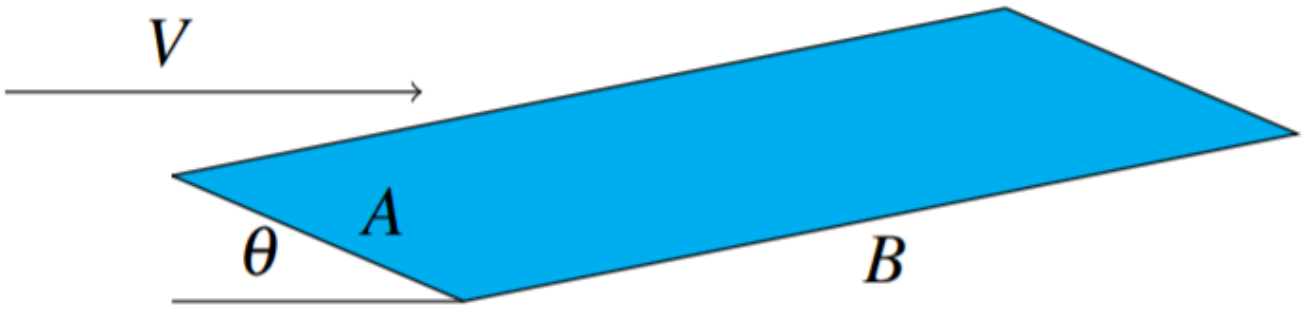


Figure 1.5.4 The airplane, represented as a slanted rectangle with sides A and B, is slanted to the right. The horizontal arrow on the left, labeled V and pointing towards the airplane, indicates wind speed. The slanted side of the rectangle, labeled A, makes an angle θ with the horizontal, whereas the other side of the rectangle, labeled B, is horizontal.

The angle θ is called the angle of incidence, B is the span of the wing and A is called the chord. Denote by l the lift. Then this should depend on various quantities like θ, V, B, A and so forth. Here is a table which indicates various quantities on which it is reasonable to expect l to depend.

Table 1.5.1 *Quantities with corresponding symbol and units*

Variable	Symbol	Units
chord	A	m
span	B	m
angle incidence	θ	$m^0 kg^0 sec^0$
speed of wind	V	$msec^{-1}$
speed of sound	V_0	$msec^{-1}$
density of air	ρ	kgm^{-3}
viscosity	μ	$kgsec^{-1}m^{-1}$
lift	l	$kgsec^{-2}m$

Here m denotes meters, $\sec$ refers to seconds and kg refers to kilograms. All of these are likely familiar except for μ, which we will discuss in further detail now.

Viscosity is a measure of how much internal friction is experienced when the fluid moves. It is roughly a measure of how "sticky" the fluid is. Consider a piece of area parallel to the direction of motion of the fluid. To say that the viscosity is large is to say that the tangential force applied to this area must be large in order to achieve a given change in speed of the fluid in a direction normal to the tangential force. Thus

$$\mu\,(\text{area})\,(\text{velocity gradient}) = \text{ tangential force}$$

Hence

$$(\text{units on } \mu)\, m^2 \left(\frac{m}{\sec m}\right) = kg \sec^{-2} m$$

Thus the units on μ are

$$kg \sec^{-1} m^{-1}$$

as claimed above.

Returning to our original discussion, you may think that we would want

$$l = f(A, B, \theta, V, V_0, \rho, \mu)$$

This is very cumbersome because it depends on seven variables. Also, it is likely that without much care, a change in the units such as going from meters to feet would result in an incorrect value for l. The way to get around this problem is to look for l as a function of dimensionless variables multiplied by something which has units of force. It is helpful because first of all, you will likely have fewer independent variables and secondly, you could expect the formula to hold independent of the way of specifying length, mass and so forth. One looks for

$$l = f\left(g_1, \cdots, g_k\right) \rho V^2 AB$$

where the units on $\rho V^2 AB$ are

$$\frac{kg}{m^3}\left(\frac{m}{\sec}\right)^2 m^2 = \frac{kg \times m}{\sec^2}$$

which are the units of force. Each of these g_i is of the form

$$A^{x_1} B^{x_2} \theta^{x_3} V^{x_4} V_0^{x_5} \rho^{x_6} \mu^{x_7} \tag{1.5.6}$$

and each g_i is independent of the dimensions. That is, this expression must not depend on meters, kilograms, seconds, etc. Thus, placing in the units for each of these quantities, one needs

$$m^{x_1} m^{x_2} \left(m^{x_4} \sec^{-x_4}\right)\left(m^{x_5} \sec^{-x_5}\right)\left(kgm^{-3}\right)^{x_6}\left(kg \sec^{-1} m^{-1}\right)^{x_7} = m^0 kg^0 \sec^0$$

Notice that there are no units on θ because it is just the radian measure of an angle. Hence its dimensions consist of length divided by length, thus it is dimensionless. Then this leads to the following equations for the x_i.

$$\begin{aligned} m: \quad & x_1 + x_2 + x_4 + x_5 - 3x_6 - x_7 = 0 \\ \sec: \quad & -x_4 - x_5 - x_7 = 0 \\ kg: \quad & x_6 + x_7 = 0 \end{aligned}$$

The augmented matrix for this system is

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & -3 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

The reduced row-echelon form is given by

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

and so the solutions are of the form

$$\begin{aligned} x_1 &= -x_2 - x_7 \\ x_3 &= x_3 \\ x_4 &= -x_5 - x_7 \\ x_6 &= -x_7 \end{aligned} \tag{1.5.7}$$

Thus, in terms of vectors, the solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} -x_2 - x_7 \\ x_2 \\ x_3 \\ -x_5 - x_7 \\ x_5 \\ -x_7 \\ x_7 \end{bmatrix}$$

Thus the free variables are x_2, x_3, x_5, x_7. By assigning values to these, we can obtain dimensionless variables by placing the values obtained for the x_i in the formula (1.5.6). For example, let $x_2 = 1$ and all the rest of the free variables are 0. This yields

$$x_1 = -1, x_2 = 1, x_3 = 0, x_4 = 0, x_5 = 0, x_6 = 0, x_7 = 0$$

The dimensionless variable is then $A^{-1}B^1$. This is the ratio between the span and the chord. It is called the aspect ratio, denoted as AR. Next let $x_3 = 1$ and all others equal zero. This gives for a dimensionless quantity the angle θ. Next let $x_5 = 1$ and all others equal zero. This gives

$$x_1 = 0, x_2 = 0, x_3 = 0, x_4 = -1, x_5 = 1, x_6 = 0, x_7 = 0$$

Then the dimensionless variable is $V^{-1}V_0^1$. However, it is written as V/V_0. This is called the Mach number $\mathcal{M}$.

Finally, let $x_7 = 1$ and all the other free variables equal 0. Then

$$x_1 = -1, x_2 = 0, x_3 = 0, x_4 = -1, x_5 = 0, x_6 = -1, x_7 = 1$$

then the dimensionless variable which results from this is $A^{-1}V^{-1}\rho^{-1}\mu$. It is customary to write it as $Re = (AV\rho)/\mu$. This one is called the Reynold's number. It is the one which involves viscosity. Thus we would look for

$$l = f(Re, AR, \theta, \mathcal{M})\, kg \times m/\sec^2$$

This is quite interesting because it is easy to vary Re by simply adjusting the velocity or A but it is hard to vary things like μ or ρ. Note that all the quantities are easy to adjust. Now this could be used, along with wind tunnel experiments to get a formula for the lift which would be reasonable. You could also consider more variables and more complicated situations in the same way.

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1.5E: Exercises for Section 1.5

Exercise $1.5E.1$

Consider the following diagram of three circuits.

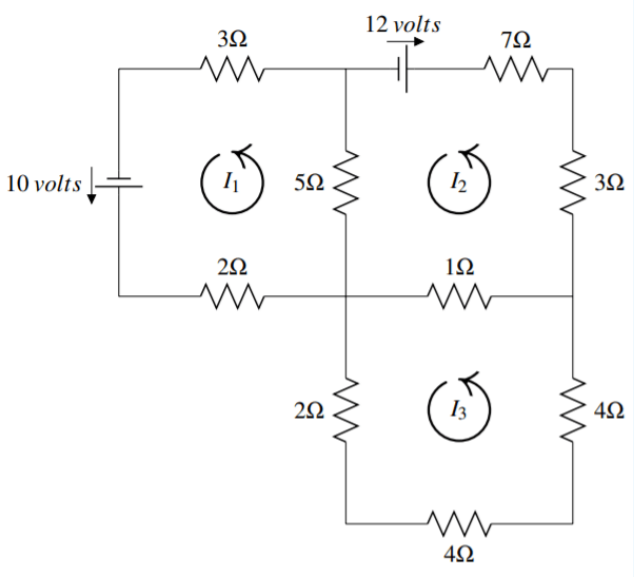


Figure $1.5E.2$: A circuit with multiple interconnected components, including resistors, voltage sources, and current loops. The circuit is composed of 3 squares: Square 1 on the top left, Square 2 on the top right and Square 3 on the bottom right. Square 1 has counterclockwise current flow I_1, with a 10V voltage source with arrow pointing down, and resistors 2Ω on the bottom, 5Ω on the right, shared with Square 2, and 3Ω above. Square 2 has counterclockwise current flow I_3, with resistors 5Ω on the left shared with Square 1, 1Ω on the bottom shared with Square 3, 3Ω on the right, 7Ω above, and a 12V voltage source with arrow pointing right. Square 3 has counterclockwise current flow I_3, with resistors 2Ω on the left, 4Ω below, 2Ω on the right and 1Ω above shared with Square 2.

The current in amps in the four circuits is denoted by I_1, I_2, I_3 and it is understood that the motion is in the counter clockwise direction. If I_k ends up being negative, then it just means the current flows in the clockwise direction.

Find I_1, I_2, I_3.

Answer available at the end of this section.

Example $1.5E.2$

Consider the following diagram of four circuits.

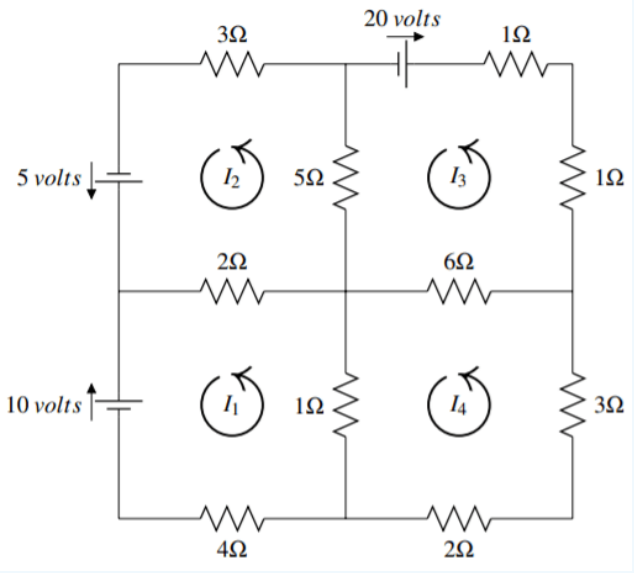


Figure $1.5E.1$: A circuit with multiple interconnected components, including resistors, voltage sources, and current loops. The circuit is arranged in a 2x2 grid, dividing the circuit into 4 squares: Square 1 on the bottom left, Square 2 on the top left, Square 3 on the top right and Square 4 on the bottom right. Square 1 has counterclockwise current flow I_1, with a 10V voltage source with arrow pointing up, and resistors 4Ω along the bottom, 1Ω on the right shared with Square 4, and 2Ω above shared with Square 2. Square 2 has counterclockwise current flow I_2, with a 5V voltage source with arrow pointing down, and resistors 2Ω on the bottom shared with Square 1, 5Ω on the right, shared with Square 3, and 3Ω above. Square 3 has counterclockwise current flow I_3, with resistors 5Ω on the left shared with Square 2, 6Ω on the bottom shared with Square 4, 1Ω on the right, 1Ω above, and a 20V voltage source with arrow pointing right. Square 4 has counterclockwise current flow I_4, with resistors 1Ω on the left shared with Square 1, 3Ω below, 2Ω on the right and 6Ω above shared with Square 3.

The current in amps in the four circuits is denoted by I_1, I_2, I_3, I_4 and it is understood that the motion is in the counter clockwise direction. If I_k ends up being negative, then it just means the current flows in the clockwise direction.

In the above diagram, the top left circuit should give the equation

$$2I_2 - 2I_1 + 5I_2 - 5I_3 + 3I_2 = 5$$

For the circuit on the lower left, you should have

$$4I_1 + I_1 - I_4 + 2I_1 - 2I_2 = -10$$

Write equations for each of the other two circuits and then give a solution to the resulting system of equations.

Answer available at the end of this section.

Exercise $1.5E.3$

Consider a circuit with the following description:

- Left Loop:
  - A 10 V battery
  - A 2 Ω resistor
  - A shared 3 Ω resistor
- Right Loop:
  - A 5 V battery
  - A 4 Ω resistor
  - The same shared 3 Ω resistor (which lies between the two loops)

Suppose that:

- i_1 is the clockwise loop current in the left loop.
- i_2 is the clockwise loop current in the right loop.
- The current through the shared resistor is given by the difference $(i_1 - i_2)$

Write the equations for each loop and then solve for i_1 and i_2.

Exercise $1.5E.4$

Consider a circuit with three loops connected through shared resistors with the following description:

- Loop 1:
  - A 12 V battery
  - A 2 Ω resistor
  - A shared 4 Ω resistor with Loop 2
- Loop 2:
  - A 9 V battery
  - A 3 Ω resistor
  - A shared 4 Ω resistor with Loop 1
  - A shared 5 Ω resistor with Loop 3
- Loop 3:
  - A 6 V battery
  - A 2 Ω resistor
  - A shared 5 Ω resistor with Loop 2

Currents:

- i_1 flows clockwise in Loop 1
- i_2 flows clockwise in Loop 2
- i_3 flows clockwise in Loop 3

The shared resistors carry the current differences between loops:

- 4 Ω resistor (Loops 1 & 2): current is $i_1 - i_2$
- 5 Ω resistor (Loops 2 & 3): current is $i_2 - i_3$

Write the equations for each loop and then solve for i_1, i_2 and i_3.

Exercise $1.5E.5$

In the section, it was observed that $\rho V^2 AB$ has the units of force. Describe a systematic way to obtain such combinations of the variables which will yield something which has the units of force.

Answers

Exercise $1.5E.1$

You have

$$\begin{aligned} 2I_1 + 5I_1 + 3I_1 - 5I_2 &= 10 \\ I_2 - I_3 + 3I_2 + 7I_2 + 5I_2 - 5I_1 &= -12 \\ 2I_3 + 4I_3 + 4I_3 + I_3 - I_2 &= 0 \end{aligned} \tag{1.5E.1}$$

Simplifying this yields

$$\begin{aligned} 10I_1 - 5I_2 &= 10 \\ -5I_1 + 16I_2 - I_3 &= -12 \\ -I_2 + 11I_3 &= 0 \end{aligned} \tag{1.5E.2}$$

The solution is given by

$$I_1 = \frac{218}{295}, I_2 = -\frac{154}{295}, I_3 = -\frac{14}{295}$$

Example $1.5E.2$

The other two equations are

$$\begin{aligned} 6I_3 - 6I_4 + I_3 + I_3 + 5I_3 - 5I_2 &= -20 \\ 2I_4 + 3I_4 + 6I_4 - 6I_3 + I_4 - I_1 &= 0 \end{aligned} \tag{1.5E.3}$$

Then the system is

$$\begin{aligned} 2I_2 - 2I_1 + 5I_2 - 5I_3 + 3I_2 &= 5 \\ 4I_1 + I_1 - I_4 + 2I_1 - 2I_2 &= -10 \\ 6I_3 - 6I_4 + I_3 + I_3 + 5I_3 - 5I_2 &= -20 \\ 2I_4 + 3I_4 + 6I_4 - 6I_3 + I_4 - I_1 &= 0 \end{aligned}$$

The solution is:

$$\begin{aligned} I_1 &= -\frac{750}{373} \\ I_2 &= -\frac{1421}{1119} \\ I_3 &= -\frac{3061}{1119} \\ I_4 &= -\frac{1718}{1119} \end{aligned} \tag{1.5E.4}$$

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1.6: Application to Polynomial Interpolation

Learning Objectives

- Given data points, find an appropriate interpolating polynomial and use it to estimate points.

Polynomial Interpolation

In studying a set of data that relates variables x and y, it may be the case that we can use a polynomial to "fit" to the data. If such a polynomial can be established, it can be used to estimate values of x and y which have not been provided.

Consider the following example.

Example 1.6.1: Polynomial Interpolation

Given data points $(1,4),(2,9),(3,12)$, find an interpolating polynomial $p(x)$ of degree at most 2 and then estimate the value corresponding to $x=\frac{1}{2}$.

Solution

We want to find a polynomial given by

$$p(x)=r_0+r_1x+r_2x^2$$

such that $p(1)=4, p(2)=9$ and $p(3)=12$. To find this polynomial, substitute the known values in for x and solve for r_0, r_1, and r_2.

$$\begin{aligned} p(1)&=r_0+r_1+r_2=4\\ p(2)&=r_0+2r_1+4r_2=9\\ p(3)&=r_0+3r_1+9r_2=12 \end{aligned} \tag{1.6.1}$$

Writing the augmented matrix, we have

$$\begin{bmatrix} 1 & 1 & 1 & 4\\ 1 & 2 & 4 & 9\\ 1 & 3 & 9 & 12 \end{bmatrix}$$

After row operations, the resulting matrix is

$$\begin{bmatrix} 1 & 0 & 0 & -3\\ 0 & 1 & 0 & 8\\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Therefore the solution to the system is $r_0=-3, r_1=8, r_2=-1$ and the required interpolating polynomial is

$$p(x)=-3+8x-x^2$$

To estimate the value for $x=\frac{1}{2}$, we calculate $p(\frac{1}{2})$:

$$\begin{aligned} p(\frac{1}{2})&=-3+8(\frac{1}{2})-(\frac{1}{2})^2\\ &=-3+4-\frac{1}{4}\\ &=\frac{3}{4} \end{aligned} \tag{1.6.2}$$

This procedure can be used for any number of data points, and any degree of polynomial. The steps are outlined below.

Procedure 1.6.1: Finding an Interpolation Polynomial

Suppose that values of x and corresponding values of y are given, such that the actual relationship between x and y is unknown. Then, values of y can be estimated using an **interpolating polynomial** $p(x)$. If given $x_1, ..., x_n$ and the corresponding $y_1, ..., y_n$, the procedure to find $p(x)$ is as follows:

1. The desired polynomial $p(x)$ is given by
$$p(x) = r_0 + r_1 x + r_2 x^2 + ... + r_{n-1} x^{n-1}$$
2. $p(x_i) = y_i$ for all $i = 1, 2, ..., n$ so that
$$\begin{aligned} r_0 + r_1 x_1 + r_2 x_1^2 + ... + r_{n-1} x_1^{n-1} &= y_1 \\ r_0 + r_1 x_2 + r_2 x_2^2 + ... + r_{n-1} x_2^{n-1} &= y_2 \\ &\vdots \\ r_0 + r_1 x_n + r_2 x_n^2 + ... + r_{n-1} x_n^{n-1} &= y_n \end{aligned}$$
3. Set up the augmented matrix of this system of equations
$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} & y_1 \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} & y_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} & y_n \end{bmatrix}$$
4. Solving this system will result in a unique solution $r_0, r_1, \cdots, r_{n-1}$. Use these values to construct $p(x)$, and estimate the value of $p(a)$ for any $x = a$.

This procedure motivates the following theorem.

Theorem 1.6.1: Polynomial Interpolation

Given n data points $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$ with the x_i distinct, there is a unique polynomial $p(x) = r_0 + r_1 x + r_2 x^2 + \cdots + r_{n-1} x^{n-1}$ such that $p(x_i) = y_i$ for $i = 1, 2, \cdots, n$. The resulting polynomial $p(x)$ is called the **interpolating polynomial** for the data points.

We conclude this section with another example.

Example 1.6.2: Polynomial Interpolation

Consider the data points $(0, 1), (1, 2), (3, 22), (5, 66)$. Find an interpolating polynomial $p(x)$ of degree at most three, and find the value of $p(2)$.

Solution

The desired polynomial $p(x)$ is given by:
$$p(x) = r_0 + r_1 x + r_2 x^2 + r_3 x^3$$

Using the given points, the system of equations is
$$\begin{aligned} p(0) &= r_0 = 1 \\ p(1) &= r_0 + r_1 + r_2 + r_3 = 2 \\ p(3) &= r_0 + 3r_1 + 9r_2 + 27r_3 = 22 \\ p(5) &= r_0 + 5r_1 + 25r_2 + 125r_3 = 66 \end{aligned} \tag{1.6.3}$$

The augmented matrix is given by:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 3 & 9 & 27 & 22 \\ 1 & 5 & 25 & 125 & 66 \end{bmatrix}$$

The resulting matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Therefore, $r_0 = 1, r_1 = -2, r_2 = 3, r_3 = 0$ and $p(x) = 1 - 2x + 3x^2$. To find the value of $p(2)$, we compute $p(2) = 1 - 2(2) + 3(2^2) = 1 - 4 + 12 = 9$.

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1.6E: Exercises for Section 1.7

Exercise $1.6E.1$

Given data points $(-1, 10), (1, 0), (3, 6)$ find an interpolating polynomial $p(x)$ of degree at most 2.

Answer available at the end of this section.

Exercise $1.6E.2$

Given data points $(-2, 25), (1, 1), (2, 1), (4, -23)$ find an interpolating polynomial $p(x)$ of degree at most 3.

Answer available at the end of this section.

Answer available at the end of this section.

Exercise $1.6E.3$

Given data points $(-6, 17), (2, 1), (4, 7)$ find an interpolating polynomial $p(x)$ of degree at most 2.

Exercise $1.6E.4$

Given data points $(-1, -8), (0, -1), (2, -11), (4, -13)$ find an interpolating polynomial $p(x)$ of degree at most 3.

Exercise $1.6E.5$

Suppose you are given three distinct collinear points and tasked with finding an interpolating polynomial $p(x)$ of degree at most 2. What must be true about your solution?

Answer available at the end of this section.

Exercise $1.6E.6$

Suppose you are given n distinct points, each with y component of 0 and tasked with finding interpolating polynomial $p(x)$ of degree at most $n - 1$. What must be true about your solution?

Answer available at the end of this section.

Exercise $1.6E.7$

Suppose you are given n distinct points, each with y component of 0 and tasked with finding interpolating polynomial $p(x)$ of degree at most n. What must be true about your solution?

Exercise $1.6E.8$

Suppose you are tasked with interpolating polynomial $p(x)$ corresponding to a set of data points. The data points contain two distinct points with the same x components. What must be true about your solution?

Answer available at the end of this section.

Answers

Exercise $1.6E.1$

$p(x) = 2x^2 - 5x + 3$

Exercise $1.6E.2$

Exercise $1.6E.2$

$p(x) = -x^3 + 3x^2 - 2x + 1$

Exercise $1.6E.5$

The coefficient of the quadratic term must be 0.

Exercise $1.6E.6$

$p(x)$ must be the 0 polynomial.

Exercise $1.6E.8$

There is no solution as the corresponding system will be an inconsistent system.

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1.7: Gauss-Seidel Method

Learning Objectives

- Solve linear systems using the Gauss-Seidel Method
- Recognize the advantages and pitfalls of the Gauss-Seidel Method

Why do we need another method to solve a set of simultaneous linear equations?

In certain cases, such as when a system of equations is large, iterative methods of solving equations are more advantageous. Elimination methods, such as Gaussian elimination, are prone to large round-off errors for a large set of equations. Iterative methods, such as the Gauss-Seidel method, give the user control of the round-off error. Also, if the physics of the problem are well known, initial guesses needed in iterative methods can be made more judiciously leading to faster convergence.

What is the algorithm for the Gauss-Seidel method? Given a general set of n equations and n unknowns, we have

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = c_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = c_2$$

$$\vdots \qquad\qquad \vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = c_n$$

If the diagonal elements are non-zero, each equation is rewritten for the corresponding unknown, that is, the first equation is rewritten with x_1 on the left hand side, the second equation is rewritten with x_2 on the left hand side and so on as follows

$$x_1 = \frac{c_1 - a_{12}x_2 - a_{13}x_3 \ldots\ldots - a_{1n}x_n}{a_{11}}$$

$$x_2 = \frac{c_2 - a_{21}x_1 - a_{23}x_3 \ldots\ldots - a_{2n}x_n}{a_{22}}$$

$$\vdots$$

$$\vdots$$

$$x_{n-1} = \frac{c_{n-1} - a_{n-1,1}x_1 - a_{n-1,2}x_2 \ldots\ldots - a_{n-1,n-2}x_{n-2} - a_{n-1,n}x_n}{a_{n-1,n-1}}$$

$$x_n = \frac{c_n - a_{n1}x_1 - a_{n2}x_2 - \ldots\ldots - a_{n,n-1}x_{n-1}}{a_{nn}}$$

These equations can be rewritten in a summation form as

$$x_1 = \frac{c_1 - \sum_{\substack{j=1 \\ j \neq 1}}^{n} a_{1j}x_j}{a_{11}}$$

$$x_2 = \frac{c_2 - \sum_{\substack{j=1 \\ j \neq 2}}^{n} a_{2j}x_j}{a_{22}}$$

$$\vdots$$

$$x_{n-1} = \frac{c_{n-1} - \sum_{j=1}^{n} a_{n-1,j} x_j}{j \neq n-1} \over a_{n-1,n-1}$$

$$x_n = \frac{c_n - \sum_{j=1}^{n} a_{nj} x_j}{j \neq n} \over a_{nn}$$

Hence for any row i,

$$x_i = \frac{c_i - \sum_{j=1}^{n} a_{ij} x_j}{j \neq i} \over a_{ii}, i = 1, 2, \ldots, n.$$

Now to find x_i's, one assumes an initial guess for the x_i's and then uses the rewritten equations to calculate the new estimates. Remember, one always uses the most recent estimates to calculate the next estimates, x_i. At the end of each iteration, one calculates the absolute relative approximate error for each x_i as

$$|\in_a|_i = \left| \frac{x_i^{\text{new}} - x_i^{\text{old}}}{x_i^{\text{new}}} \right| \times 100$$

where $x_i^{\text{new}}$is the recently obtained value of x_i, and x_i^{old} is the previous value of x_i.

When the absolute relative approximate error for each x_i is less than the pre-specified tolerance, the iterations are stopped.

Example 1.7.1: A Velocity Application

The upward velocity of a rocket is given at three different times in the following table.

Table 1.7.1: Velocity of the rocket at time t

Time, $t(s)$	**Velocity,** $v(m/s)$
5	106.8
8	177.2
12	279.2

The velocity data is approximated by a polynomial as

$$v(t) = a_1 t^2 + a_2 t + a_3, 5 \leq t \leq 12$$

Find the values of $a_1,\ a_2, and\ a_3$ using the Gauss-Seidel method. Assume an initial guess of the solution as

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

and conduct two iterations.

Solution

The polynomial is going through three data points $(t_1, v_1), (t_2, v_2), and\ (t_3, v_3)$ where from the above table

$$t_1 = 5, v_1 = 106.8$$

$$t_2 = 8, v_2 = 177.2$$

$$t_3 = 12, v_3 = 279.2$$

Requiring that $v(t) = a_1 t^2 + a_2 t + a_3$ passes through the three data points gives

$$v(t_1) = v_1 = a_1 t_1^2 + a_2 t_1 + a_3$$

$$v(t_2) = v_2 = a_1 t_2^2 + a_2 t_2 + a_3$$

$$v(t_3) = v_3 = a_1 t_3^2 + a_2 t_3 + a_3$$

Substituting the data $(t_1, v_1), (t_2, v_2), and (t_3, v_3)$ gives

$$a_1 (5^2) + a_2 (5) + a_3 = 106.8$$

$$a_1 (8^2) + a_2 (8) + a_3 = 177.2$$

$$a_1 (12^2) + a_2 (12) + a_3 = 279.2$$

or

$$25a_1 + 5a_2 + a_3 = 106.8$$

$$64a_1 + 8a_2 + a_3 = 177.2$$

$$144a_1 + 12a_2 + a_3 = 279.2$$

The coefficients $a_1, a_2, \text{and } a_3$ for the above expression are given by

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Rewriting the equations gives

$$a_1 = \frac{106.8 - 5a_2 - a_3}{25}$$

$$a_2 = \frac{177.2 - 64a_1 - a_3}{8}$$

$$a_3 = \frac{279.2 - 144a_1 - 12a_2}{1}$$

Iteration #1

Given the initial guess of the solution vector as

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

we get

$$a_1 = \frac{106.8 - 5(2) - (5)}{25} = 3.6720$$

$$a_2 = \frac{177.2 - 64(3.6720) - (5)}{8} = -7.8150$$

$$a_3 = \frac{279.2 - 144(3.6720) - 12(-7.8510)}{1} = -155.36$$

The absolute relative approximate error for each x_i then is

$$\begin{aligned}|\epsilon_a|_1 &= \left|\frac{3.6720-1}{3.6720}\right| \times 100 \\ &= 72.76\%\end{aligned}$$

$$\begin{aligned}|\epsilon_a|_2 &= \left|\frac{-7.8510-2}{-7.8510}\right| \times 100 \\ &= 125.47\%\end{aligned}$$

$$\begin{aligned}|\epsilon_a|_3 &= \left|\frac{-155.36-5}{-155.36}\right| \times 100 \\ &= 103.22\%\end{aligned}$$

At the end of the first iteration, the estimate of the solution vector is

$$\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix} = \begin{bmatrix}3.6720\\-7.8510\\-155.36\end{bmatrix}$$

and the maximum absolute relative approximate error is 125.47.

Iteration #2

The estimate of the solution vector at the end of Iteration #1 is

$$\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix} = \begin{bmatrix}3.6720\\-7.8510\\-155.36\end{bmatrix}$$

Now we get

$$\begin{aligned}a_1 &= \frac{106.8-5\left(-7.8510\right)-\left(-155.36\right)}{25} \\ &= 12.056\end{aligned}$$

$$\begin{aligned}a_2 &= \frac{177.2-64\left(12.056\right)-\left(-155.36\right)}{8} \\ &= -54.882\end{aligned}$$

$$\begin{aligned}a_3 &= \frac{279.2-144\left(12.056\right)-12\left(-54.882\right)}{1} \\ &= -798.34\end{aligned}$$

The absolute relative approximate error for each x_i then is

$$\begin{aligned}|\epsilon_a|_1 &= \left|\frac{12.056-3.6720}{12.056}\right| \times 100 \\ &= 69.543\%\end{aligned}$$

$$\begin{aligned}|\epsilon_a|_2 &= \left|\frac{-54.882-\left(-7.8510\right)}{-54.882}\right| \times 100 \\ &= 85.695\%\end{aligned}$$

$$\begin{aligned}|\epsilon_a|_3 &= \left|\frac{-798.34-\left(-155.36\right)}{-798.34}\right| \times 100 \\ &= 80.540\%\end{aligned}$$

At the end of the second iteration the estimate of the solution vector is

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 12.056 \\ -54.882 \\ -798.54 \end{bmatrix}$$

and the maximum absolute relative approximate error is 85.695%.

Conducting more iterations gives the following values for the solution vector and the corresponding absolute relative approximate errors.

Table 1.7.2: Table summarizing the results of 6 iterations of the Gauss-Seidel Method for Example 1.7.1

Iteration	a_1	$\lvert\epsilon_a\rvert_1\ \%$	a_2	$\lvert\epsilon_a\rvert_2\ \%$	a_3	$\lvert\epsilon_a\rvert_3\ \%$
1	3.672	72.767	-7.8510	125.47	-155.36	103.22
2	12.056	69.543	-54.882	85.695	-798.34	80.54
3	47.182	74.447	-255.51	78.521	-3448.9	76.852
4	193.33	75.595	-1093.4	76.632	-14440	76.116
5	800.53	75.85	-4577.2	76.112	-60072	75.963
6	3322.6	75.906	-19049	75.972	-249580	75.931

As seen in the above table (Table 1.7.2), the solution estimates are not converging to the true solution of

$$a_1 = 0.29048$$

$$a_2 = 19.690$$

$$a_3 = 1.0857$$

The above system of equations does not seem to converge. Why?

Well, a pitfall of most iterative methods is that they may or may not converge. However, the solution to a certain class of system of simultaneous equations does always converge using the Gauss-Seidel method.

Definition: Diagonally Dominant

The coefficient matrix *A* in *AX* = *C* is **diagonally dominant** if

$$|a_{ii}| \geq \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \text{ for all } i, \text{and}$$

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \text{ for at least one } i$$

If a system of equations has a coefficient matrix that is not diagonally dominant, it may or may not converge. Fortunately, many physical systems that result in simultaneous linear equations have a diagonally dominant coefficient matrix, which then assures convergence for iterative methods such as the Gauss-Seidel method of solving simultaneous linear equations.

Example 1.7.2: Using Gauss-Seidel Method

Find the solution to the following system of equations using the Gauss-Seidel method.

$$12x_1 + 3x_2 - 5x_3 = 1$$

$$x_1 + 5x_2 + 3x_3 = 28$$

$$3x_1 + 7x_2 + 13x_3 = 76$$

Use

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

as the initial guess and conduct two iterations.

Solution

The coefficient matrix

$$A = \begin{bmatrix} 12 & 3 & -5 \\ 1 & 5 & 3 \\ 3 & 7 & 13 \end{bmatrix}$$

is diagonally dominant as

$$|a_{11}| = |12| = 12 \geq |a_{12}| + |a_{13}| = |3| + |-5| = 8$$

$$|a_{22}| = |5| = 5 \geq |a_{21}| + |a_{23}| = |1| + |3| = 4$$

$$|a_{33}| = |13| = 13 \geq |a_{31}| + |a_{32}| = |3| + |7| = 10$$

and the inequality is strictly greater than for at least one row. Hence, the solution should converge using the Gauss-Seidel method.

Rewriting the equations, we get

$$x_1 = \frac{1 - 3x_2 + 5x_3}{12}$$

$$x_2 = \frac{28 - x_1 - 3x_3}{5}$$

$$x_3 = \frac{76 - 3x_1 - 7x_2}{13}$$

Assuming an initial guess of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Iteration #1

$$\begin{aligned} x_1 &= \frac{1 - 3\,(0) + 5\,(1)}{12} \\ &= 0.50000 \end{aligned}$$

$$\begin{aligned} x_2 &= \frac{28 - (0.50000) - 3\,(1)}{5} \\ &= 4.9000 \end{aligned}$$

$$\begin{aligned} x_3 &= \frac{76 - 3\,(0.50000) - 7\,(4.9000)}{13} \\ &= 3.0923 \end{aligned}$$

The absolute relative approximate error at the end of the first iteration is

$$\begin{aligned}|\epsilon_a|_1 &= \left|\frac{0.50000-1}{0.50000}\right| \times 100 \\ &= 100.00\%\end{aligned}$$

$$\begin{aligned}|\epsilon_a|_2 &= \left|\frac{4.9000-0}{4.9000}\right| \times 100 \\ &= 100.00\%\end{aligned}$$

$$\begin{aligned}|\epsilon_a|_3 &= \left|\frac{3.0923-1}{3.0923}\right| \times 100 \\ &= 67.662\%\end{aligned}$$

The maximum absolute relative approximate error is 100.00%

Iteration #2

$$\begin{aligned}x_1 &= \frac{1-3\left(4.9000\right)+5\left(3.0923\right)}{12} \\ &= 0.14679\end{aligned}$$

$$\begin{aligned}x_2 &= \frac{28-\left(0.14679\right)-3\left(3.0923\right)}{5} \\ &= 3.7153\end{aligned}$$

$$\begin{aligned}x_3 &= \frac{76-3\left(0.14679\right)-7\left(3.7153\right)}{13} \\ &= 3.8118\end{aligned}$$

At the end of second iteration, the absolute relative approximate error is

$$\begin{aligned}|\epsilon_a|_1 &= \left|\frac{0.14679-0.50000}{0.14679}\right| \times 100 \\ &= 240.61\%\end{aligned}$$

$$\begin{aligned}|\epsilon_a|_2 &= \left|\frac{3.7153-4.9000}{3.7153}\right| \times 100 \\ &= 31.889\%\end{aligned}$$

$$\begin{aligned}|\epsilon_a|_3 &= \left|\frac{3.8118-3.0923}{3.8118}\right| \times 100 \\ &= 18.874\%\end{aligned}$$

The maximum absolute relative approximate error is 240.61%. This is greater than the value of 100.00% we obtained in the first iteration. Is the solution diverging? No, as you conduct more iterations, the solution converges as follows.

Table 1.7.3: Table summarizing the results of 6 iterations of the Gauss-Seidel Method for Example 1.7.2

Iteration	a_1	$\|\epsilon_a\|_1 \%$	a_2	$\|\epsilon_a\|_2 \%$	a_3	$\|\epsilon_a\|_3 \%$
1	0.5	100	4.9	100	3.0923	67.662
2	0.14679	240.61	3.7153	31.889	3.8118	18.874
3	0.74275	80.236	3.1644	17.408	3.9708	4.0064
4	0.94675	21.546	3.0281	4.4996	3.9971	0.65772
5	0.99177	4.5391	3.0034	0.82499	4.0001	0.074383
6	0.99919	0.74307	3.0001	0.10856	4.0001	0.00101

This is close to the exact solution vector of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

Example 1.7.3: A Non-Converging Case

Given the system of equations

$$3x_1 + 7x_2 + 13x_3 = 76$$

$$x_1 + 5x_2 + 3x_3 = 28$$

$$12x_1 + 3x_2 - 5x_3 = 1$$

find the solution using the Gauss-Seidel method. Use

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

as the initial guess.

Solution

Rewriting the equations, we get

$$x_1 = \frac{76 - 7x_2 - 13x_3}{3}$$

$$x_2 = \frac{28 - x_1 - 3x_3}{5}$$

$$x_3 = \frac{1 - 12x_1 - 3x_2}{-5}$$

Assuming an initial guess of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

the next six iterative values are given in the table below.

Table 1.7.4: Table summarizing the results of 6 iterations of the Gauss-Seidel Method for Example 1.7.3

Iteration	a_1	$\lvert\epsilon_a\rvert_1\,\%$	a_2	$\lvert\epsilon_a\rvert_2\,\%$	a_3	$\lvert\epsilon_a\rvert_3\,\%$
1	21	95.238	0.8	100	50.68	98.027
2	-196.15	110.71	14.421	94.453	-462.30	110.96
3	1995	109.83	-116.02	112.43	4718.1	109.8
4	-20149	109.9	1204.6	109.63	-47636	109.9
5	2.0364×10^5	109.89	-12140	109.92	4.8144×10^5	109.89
6	-2.0579×10^6	109.89	1.2272×10^5	109.89	-4.8653×10^6	109.89

You can see that this solution is not converging and the coefficient matrix is not diagonally dominant. The coefficient matrix

$$A = \begin{bmatrix} 3 & 7 & 13 \\ 1 & 5 & 3 \\ 12 & 3 & -5 \end{bmatrix}$$

is not diagonally dominant as

$$|a_{11}| = |3| = 3 \leq |a_{12}| + |a_{13}| = |7| + |13| = 20$$

Hence, the Gauss-Seidel method may or may not converge.

However, it is the same set of equations as the previous example and that converged. The only difference is that we exchanged first and the third equation with each other and that made the coefficient matrix not diagonally dominant.

Therefore, it is possible that a system of equations can be made diagonally dominant if one exchanges the equations with each other. However, it is not possible for all cases. For example, the following set of equations

$$x_1 + x_2 + x_3 = 3$$

$$2x_1 + 3x_2 + 4x_3 = 9$$

$$x_1 + 7x_2 + x_3 = 9$$

cannot be rewritten to make the coefficient matrix diagonally dominant.

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1.7E: Exercises for Section 1.8

Exercise $1.7E.1$

In a system of equation $AX = C$, if A is diagonally dominant, then Gauss-Seidel method

A. always converges
B. may or may not converge
C. always diverges

Answer available at the end of this section.

Exercise $1.7E.2$

In a system of equations $AX = C$, if A is not diagonally dominant, then Gauss-Seidel method

A. Always converges
B. May or may not converge
C. Always diverges.

Answer available at the end of this section.

Exercise $1.7E.3$

In a system of equations $AX = C$, if A is not diagonally dominant, the system of equations can always be rewritten to make it diagonally dominant.

A. True
B. False

Answer available at the end of this section.

Exercise $1.7E.4$

Solve the following system of equations using Gauss-Seidel method

$$\begin{aligned} 12x_1 + 7x_2 + 3x_3 &= 2 \\ x_1 + 5x_2 + x_3 &= -5 \\ 2x_1 + 7x_2 - 11x_3 &= 6 \end{aligned}$$

Conduct 3 iterations, calculate the maximum absolute relative approximate error at the end of each iteration and choose $[x_1 \quad x_2 \quad x_3] = [1 \quad 3 \quad 5]$ as your initial guess.

Answer available at the end of this section.

Exercise $1.7E.5$

Solve the following system of equations using Gauss-Seidel method

$12x_1 + 7x_2 + 3x_3 = 2$
$x_1 + 5x_2 + x_3 = -5$
$2x_1 + 7x_2 - 11x_3 = 6$
Conduct 3 iterations, calculate the maximum absolute relative approximate error at the end of each iteration, and choose $[x_1 \quad x_2 \quad x_3] = [1 \quad 3 \quad 5]$ as your initial guess.

Answer available at the end of this section.

Exercise $1.7E.6$

Solve the following system of equations using Gauss-Seidel Method
$x_1 + 5x_2 + x_3 = 5$
$12x_1 + 7x_2 + 3x_3 = 2$
$2x_1 + 7x_2 - 11x_3 = 6$
Conduct 3 iterations, calculate the maximum absolute relative approximate error at the end of each iteration, and choose $[x_1 \quad x_2 \quad x_3] = [1 \quad 3 \quad 5]$ as your initial guess.

Answer available at the end of this section.

Answers

Exercise $1.7E.1$

A

Exercise $1.7E.2$

B

Exercise $1.7E.3$

B

Exercise $1.7E.4$

$[x_1 \quad x_2 \quad x_3] = [0.90666 \quad -1.0115 \quad -1.0243]$ $[|\epsilon_a|_1 \quad |\epsilon_a|_2 \quad |\epsilon_a|_3] = [65.001\% \quad 10.564\% \quad 17.099\%]$ $[|\epsilon_a|_1 \quad |\epsilon_a|_2 \quad |\epsilon_a|_3] = [65.001\% \quad 10.564\% \quad 17.099\%]$

Exercise $1.7E.5$

$[x_1 \quad x_2 \quad x_3] = [0.90666 \quad -1.0115 \quad -1.0243]$

Exercise $1.7E.6$

$[x_1 \quad x_2 \quad x_3] = [-1163.7 \quad 1947.6 \quad 1027.2]$
$[|\epsilon_a|_1 \quad |\epsilon_a|_2 \quad |\epsilon_a|_3] = [89.156\% \quad 89.139\% \quad 89.183\%]$

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2: MATRICES

2.1: Matrix Operations

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2.1: Matrix Operations

Learning Objectives

- Perform the matrix operations of matrix addition, scalar multiplication, transposition and matrix multiplication. Identify when these operations are not defined. Represent these operations in terms of the entries of a matrix.
- Prove algebraic properties for matrix addition, scalar multiplication, transposition, and matrix multiplication. Apply these properties to manipulate an algebraic expression involving matrices.

You have now solved systems of equations by writing them in terms of an augmented matrix and then doing row operations on this augmented matrix. It turns out that matrices are important not only for systems of equations but also in many applications.

Recall that a **matrix** is a rectangular array of numbers. Several of them are referred to as **matrices**. For example, here is a matrix.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 2 & 8 & 7 \\ 6 & -9 & 1 & 2 \end{bmatrix} \tag{2.1.1}$$

Recall that the size or dimension of a matrix is defined as $m \times n$ where m is the number of rows and n is the number of columns. The above matrix is a 3×4 matrix because there are three rows and four columns. You can remember the columns are like columns in a Greek temple. They stand upright while the rows lay flat like rows made by a tractor in a plowed field.

When specifying the size of a matrix, you always list the number of rows before the number of columns. You might remember that you always list the rows before the columns by using the phrase **Row**man **C**atholic.

Consider the following definition.

Definition 2.1.1: Square Matrix

A matrix A which has size $n \times n$ is called a **square matrix** . In other words, A is a square matrix if it has the same number of rows and columns.

There is some notation specific to matrices which we now introduce. We denote the columns of a matrix A by A_j as follows

$$A = \begin{bmatrix} A_1 & A_2 & \cdots & A_n \end{bmatrix}$$

Therefore, A_j is the jth column of A, when counted from left to right.

The individual elements of the matrix are called **entries** or **components** of A. Elements of the matrix are identified according to their position. The $(\mathbf{i}, \mathbf{j})$**-entry** of a matrix is the entry in the ith row and j^{th} column. For example, in the matrix (2.1.1) above, 8 is in position $(2,3)$ (and is called the $(2,3)$-entry) because it is in the second row and the third column.

In order to remember which matrix we are speaking of, we will denote the entry in the ith row and the jth column of matrix A by a_{ij}. Then, we can write A in terms of its entries, as $A = [a_{ij}]$. Using this notation on the matrix in (2.1.1), $a_{23} = 8, a_{32} = -9, a_{12} = 2$, etc.

There are various operations which are done on matrices of appropriate sizes. Matrices can be added to and subtracted from other matrices, multiplied by a scalar, and multiplied by other matrices. We will never divide a matrix by another matrix, but we will see later how matrix inverses play a similar role.

In doing arithmetic with matrices, we often define the action by what happens in terms of the entries (or components) of the matrices. Before looking at these operations in depth, consider a few general definitions.

Definition 2.1.2: The Zero Matrix

The $m \times n$ **zero matrix** is the $m \times n$ matrix having every entry equal to zero. It is denoted by 0.

One possible zero matrix is shown in the following example.

Example 2.1.1: The Zero Matrix

The 2×3 zero matrix is $0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Note there is a 2×3 zero matrix, a 3×4 zero matrix, etc. In fact there is a zero matrix for every size!

Definition 2.1.3: Equality of Matrices

Let A and B be two $m \times n$ matrices. Then $A = B$ means that for $A = [a_{ij}]$ and $B = [b_{ij}]$, $a_{ij} = b_{ij}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

In other words, two matrices are equal exactly when they are the same size and the corresponding entries are identical. Thus

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

because they are different sizes. Also,

$$\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$$

because, although they are the same size, their corresponding entries are not identical.

In the following section, we explore addition of matrices.

Addition of Matrices

When adding matrices, all matrices in the sum need have the same size. For example,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 2 \end{bmatrix}$$

and

$$\begin{bmatrix} -1 & 4 & 8 \\ 2 & 8 & 5 \end{bmatrix}$$

cannot be added, as one has size 3×2 while the other has size 2×3.

However, the addition

$$\begin{bmatrix} 4 & 6 & 3 \\ 5 & 0 & 4 \\ 11 & -2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 5 & 0 \\ 4 & -4 & 14 \\ 1 & 2 & 6 \end{bmatrix}$$

is possible.

The formal definition is as follows.

Definition 2.1.4: Addition of Matrices

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two $m \times n$ matrices. Then $A + B = C$ where C is the $m \times n$ matrix $C = [c_{ij}]$ defined by

$$c_{ij} = a_{ij} + b_{ij}$$

This definition tells us that when adding matrices, we simply add corresponding entries of the matrices. This is demonstrated in the next example.

Example 2.1.2: Addition of Matrices of Same Size

Add the following matrices, if possible.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 4 \end{bmatrix}, B = \begin{bmatrix} 5 & 2 & 3 \\ -6 & 2 & 1 \end{bmatrix}$$

Solution

Notice that both A and B are of size 2×3. Since A and B are of the same size, the addition is possible. Using Definition 2.1.4 , the addition is done as follows.

$$A + B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 2 & 3 \\ -6 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1+5 & 2+2 & 3+3 \\ 1+-6 & 0+2 & 4+1 \end{bmatrix} = \begin{bmatrix} 6 & 4 & 6 \\ -5 & 2 & 5 \end{bmatrix}$$

Addition of matrices obeys very much the same properties as normal addition with numbers. Note that when we write for example $A + B$ then we assume that both matrices are of equal size so that the operation is indeed possible.

Proposition 2.1.1: Properties of Matrix Addition

Let A, B and C be matrices. Then, the following properties hold.

- Commutative Law of Addition

$$A + B = B + A \tag{2.1.2}$$

- Associative Law of Addition

$$(A + B) + C = A + (B + C) \tag{2.1.3}$$

- Existence of an Additive Identity

$$\begin{gathered} \text{There exists a zero matrix } 0 \text{ such that} \\ A + 0 = A \end{gathered} \tag{2.1.4}$$

- Existence of an Additive Inverse

$$\begin{gathered} \text{There exists a matrix } -A \text{ such that} \\ A + (-A) = 0 \end{gathered} \tag{2.1.5}$$

Proof available at the end of this section.

We call the zero matrix in (2.1.4) the **additive identity**. Similarly, we call the matrix $-A$ in (2.1.5) the **additive inverse**. $-A$ is defined to equal $(-1)\, A = [-a_{ij}]$. In other words, every entry of A is multiplied by -1.

In the next section we will study scalar multiplication in more depth to understand what is meant by $(-1)\, A$.

Scalar Multiplication of Matrices

Recall that we use the word *scalar* when referring to numbers. Therefore, *scalar multiplication of a matrix* is the multiplication of

a matrix by a number. To illustrate this concept, consider the following example in which a matrix is multiplied by the scalar 3.

$$3\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 2 & 8 & 7 \\ 6 & -9 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 & 12 \\ 15 & 6 & 24 & 21 \\ 18 & -27 & 3 & 6 \end{bmatrix}$$

The new matrix is obtained by multiplying every entry of the original matrix by the given scalar.

The formal definition of scalar multiplication is as follows.

Definition 2.1.5: Scalar Multiplication of Matrices

If $A = [a_{ij}]$ and k is a scalar, then $kA = [ka_{ij}]$.

Consider the following example.

Example 2.1.3: Effect of Multiplication by a Scalar

Find the result of multiplying the following matrix A by 7.

$$A = \begin{bmatrix} 2 & 0 \\ 1 & -4 \end{bmatrix}$$

Solution

By Definition 2.1.5, we multiply each element of A by 7. Therefore,

$$7A = 7\begin{bmatrix} 2 & 0 \\ 1 & -4 \end{bmatrix} = \begin{bmatrix} 7(2) & 7(0) \\ 7(1) & 7(-4) \end{bmatrix} = \begin{bmatrix} 14 & 0 \\ 7 & -28 \end{bmatrix}$$

Similarly to addition of matrices, there are several properties of scalar multiplication which hold.

Proposition 2.1.2: Properties of Scalar Multiplication

Let A, B be matrices, and k, p be scalars. Then, the following properties hold.

- Distributive Law over Matrix Addition
$$k(A+B) = kA + kB$$
- Distributive Law over Scalar Addition
$$(k+p)A = kA + pA$$
- Associative Law for Scalar Multiplication
$$k(pA) = (kp)A$$
- Rule for Multiplication by 1
$$1A = A$$

Proof available at the end of this section.

Matrix Multiplication

The next important matrix operation we will explore is multiplication of matrices. The operation of matrix multiplication is one of the most important and useful of the matrix operations. Throughout this section, we will also demonstrate how matrix multiplication relates to linear systems of equations.

First, we provide a formal definition of row and column vectors.

Definition 2.1.6: Row and Column Vectors

Matrices of size $n \times 1$ or $1 \times n$ are called **vectors**. If X is such a matrix, then we write x_i to denote the entry of X in the ith row of a column matrix, or the ith column of a row matrix.

The $n \times 1$ matrix

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

is called a **column vector.** The $1 \times n$ matrix

$$X = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}$$

is called a **row vector**.

We may simply use the term **vector** throughout this text to refer to either a column or row vector. If we do so, the context will make it clear which we are referring to.

A linear combination of vectors is a sum consisting of those vectors multiplied by scalars. For example,

$$\begin{bmatrix} 50 \\ 122 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + 8 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + 9 \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

is a linear combination of three vectors.

It turns out that we can express any system of linear equations as a linear combination of vectors. In fact, the vectors that we will use are just the columns of the corresponding augmented matrix!

Definition 2.1.7: The Vector Form of a System of Linear Equations

Suppose we have a system of equations given by

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

We can express this system in **vector form** which is as follows:

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Notice that each vector used here is one column from the corresponding augmented matrix. There is one vector for each variable in the system, along with the constant vector.

The first important form of matrix multiplication is multiplying a matrix by a vector. Consider the product given by

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

We will soon see that this equals

$$7 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + 8 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + 9 \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 50 \\ 122 \end{bmatrix}$$

In general terms,

$$\begin{aligned}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= x_1 \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} + x_3 \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} \\ &= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{bmatrix}\end{aligned} \tag{2.1.6}$$

Thus you take x_1 times the first column, add to x_2 times the second column, and finally x_3 times the third column. The above sum is a linear combination of the columns of the matrix. When you multiply a matrix on the left by a vector on the right, the numbers making up the vector are just the scalars to be used in the linear combination of the columns as illustrated above.

Here is the formal definition of how to multiply an $m \times n$ matrix by an $n \times 1$ column vector.

Definition 2.1.8: Multiplication of Vector by Matrix

Let $A = [a_{ij}]$ be an $m \times n$ matrix and let X be an $n \times 1$ matrix given by

$$A = [A_1 \cdots A_n], X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Then the product AX is the $m \times 1$ column vector which equals the following linear combination of the columns of A:

$$x_1A_1 + x_2A_2 + \cdots + x_nA_n = \sum_{j=1}^{n} x_jA_j$$

If we write the columns of A in terms of their entries, they are of the form

$$A_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$$

Then, we can write the product AX as

$$AX = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Note that multiplication of an $m \times n$ matrix and an $n \times 1$ vector produces an $m \times 1$ vector.

Here is an example.

Example 2.1.4: A Vector Multiplied by a Matrix

Compute the product AX for

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 2 & 1 & -2 \\ 2 & 1 & 4 & 1 \end{bmatrix}, X = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

Solution

We will use Definition 2.1.3 to compute the product. Therefore, we compute the product AX as follows.

$$\begin{aligned} 1\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} + 1\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \\ = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \\ = \begin{bmatrix} 8 \\ 2 \\ 5 \end{bmatrix} \end{aligned} \tag{2.1.7}$$

Using the above operation, we can also write a system of linear equations in **matrix form**. In this form, we express the system as a matrix multiplied by a vector. Consider the following definition.

Definition 2.1.9: The Matrix Form of a System of Linear Equations

Suppose we have a system of equations given by

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Then we can express this system in **matrix form** as follows.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

The expression $AX = B$ is also known as the **Matrix Form** of the corresponding system of linear equations. The matrix A is simply the coefficient matrix of the system, the vector X is the column vector constructed from the variables of the system, and finally the vector B is the column vector constructed from the constants of the system. It is important to note that any system of linear equations can be written in this form.

Notice that if we write a homogeneous system of equations in matrix form, it would have the form $AX = 0$, for the zero vector 0.

You can see from this definition that a vector

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

will satisfy the equation $AX = B$ only when the entries $x_1, x_2, \cdots, x_n$ of the vector X are solutions to the original system.

Now that we have examined how to multiply a matrix by a vector, we wish to consider the case where we multiply two matrices of more general sizes, although these sizes still need to be appropriate as we will see. For example, in Example 2.1.1, we multiplied a 3×4 matrix by a 4×1 vector. We want to investigate how to multiply other sizes of matrices.

We have not yet given any conditions on when matrix multiplication is possible! For matrices A and B, in order to form the product AB, the number of columns of A must equal the number of rows of B. Consider a product AB where A has size $m \times n$ and B has size $n \times p$. Then, the product in terms of size of matrices is given by

$$(m \times \overbrace{n)\,(n}^{\text{these must match!}} \times p\ ) = m \times p$$

Note the two outside numbers give the size of the product. One of the most important rules regarding matrix multiplication is the following. If the two middle numbers don't match, you can't multiply the matrices!

When the number of columns of A equals the number of rows of B the two matrices are said to be **conformable** and the product AB is obtained as follows.

Definition 2.1.10: Multiplication of Two Matrices

Let A be an $m \times n$ matrix and let B be an $n \times p$ matrix of the form

$$B = [B_1 \cdots B_p]$$

where $B_1, ..., B_p$ are the $n \times 1$ columns of B. Then the $m \times p$ matrix AB is defined as follows:

$$AB = A[B_1 \cdots B_p] = [(AB)_1 \cdots (AB)_p]$$

where $(AB)_k$ is an $m \times 1$ matrix or column vector which gives the kth column of AB.

Consider the following example.

Example 2.1.5: Multiplying Two Matrices

Find AB if possible.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ -2 & 1 & 1 \end{bmatrix}$$

Solution

The first thing you need to verify when calculating a product is whether the multiplication is possible. The first matrix has size 2×3 and the second matrix has size 3×3. The inside numbers are equal, so A and B are conformable matrices. According to the above discussion AB will be a 2×3 matrix. Definition 2.1.5 gives us a way to calculate each column of AB, as follows.

$$\begin{bmatrix} \overbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}}^{\text{First column}}, & \overbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}}^{\text{Second column}}, & \overbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}^{\text{Third column}} \end{bmatrix}$$

You know how to multiply a matrix times a vector, using Definition 2.1.3 for each of the three columns. Thus

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ -2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 9 & 3 \\ -2 & 7 & 3 \end{bmatrix}$$

Since vectors are simply $n \times 1$ or $1 \times m$ matrices, we can also multiply a vector by another vector.

Example 2.1.6: Vector Times Vector Multiplication

Multiply if possible $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 & 0 \end{bmatrix}$.

Solution

In this case we are multiplying a matrix of size 3×1 by a matrix of size 1×4. The inside numbers match so the product is defined. Note that the product will be a matrix of size 3×4. Using Definition 2.1.5, we can compute this product as follows

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \overbrace{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix}}^{\text{First column}}, \overbrace{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \end{bmatrix}}^{\text{Second column}}, \overbrace{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix}}^{\text{Third column}}, \overbrace{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix}}^{\text{Fourth column}} \end{bmatrix}$$

You can use Definition 2.1.3 to verify that this product is

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 2 & 0 \\ 1 & 2 & 1 & 0 \end{bmatrix}$$

Example 2.1.7: A Multiplication Which is Not Defined

Find BA if possible.

$$B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ -2 & 1 & 1 \end{bmatrix}, A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

Solution

First check if it is possible. This product is of the form $(3 \times 3)(2 \times 3)$. The inside numbers do not match and so you can't do this multiplication.

In this case, we say that the multiplication is not defined. Notice that these are the same matrices which we used in Example 2.1.2. In this example, we tried to calculate BA instead of AB. This demonstrates another property of matrix multiplication. While the product AB maybe be defined, we cannot assume that the product BA will be possible. Therefore, it is important to always check that the product is defined before carrying out any calculations.

Earlier, we defined the zero matrix 0 to be the matrix (of appropriate size) containing zeros in all entries. Consider the following example for multiplication by the zero matrix.

Example 2.1.8: Multiplication by the Zero Matrix

Compute the product $A0$ for the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

and the 2×2 zero matrix given by

$$0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Solution

In this product, we compute

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence, $A0 = 0$.

Notice that we could also multiply A by the 2×1 zero vector given by $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The result would be the 2×1 zero vector. Therefore, it is always the case that $A0 = 0$, for an appropriately sized zero matrix or vector.

The ij-th Entry of a Matrix Product

Up to this point, we have used the entries of a matrix to describe the action of matrix addition and scalar multiplication. We can also study matrix multiplication using the entries of matrices.

What is the ij^{th} entry of AB? It is the entry in the ith row and the jth column of the product AB.

Now if A is $m \times n$ and B is $n \times p$, then we know that the product AB has the form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2p} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nj} & \cdots & b_{np} \end{bmatrix}$$

The jth column of AB is of the form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}$$

which is an $m \times 1$ column vector. It is calculated by

$$b_{1j} \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + b_{2j} \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + b_{nj} \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Therefore, the ij^{th} entry is the entry in row i of this vector. This is computed by

$$a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

The following is the formal definition for the ij^{th} entry of a product of matrices.

Definition 2.1.11: The ij^{th} Entry of a Product

Let $A = [a_{ij}]$ be an $m \times n$ matrix and let $B = [b_{ij}]$ be an $n \times p$ matrix. Then AB is an $m \times p$ matrix and the (i, j)-entry of AB is defined as

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

Another way to write this is

$$(AB)_{ij} = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

In other words, to find the (i, j)-entry of the product AB, or $(AB)_{ij}$, you multiply the ith row of A, on the left by the jth column of B. To express AB in terms of its entries, we write $AB = [(AB)_{ij}]$.

Consider the following example.

Example 2.1.9: The Entries of a Product

Compute AB if possible. If it is, find the $(3, 2)$-entry of AB using Definition 2.1.1.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 6 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 & 1 \\ 7 & 6 & 2 \end{bmatrix}$$

Solution

First check if the product is possible. It is of the form $(3 \times 2)(2 \times 3)$ and since the inside numbers match, it is possible to do the multiplication. The result should be a 3×3 matrix. We can first compute AB:

$$\left[\begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right]$$

where the commas separate the columns in the resulting product. Thus the above product equals

$$\begin{bmatrix} 16 & 15 & 5 \\ 13 & 15 & 5 \\ 46 & 42 & 14 \end{bmatrix}$$

which is a 3×3 matrix as desired. Thus, the $(3,2)$-entry equals 42.

Now using Definition 2.1.1, we can find that the $(3,2)$-entry equals

$$\begin{aligned} \sum_{k=1}^{2} a_{3k} b_{k2} &= a_{31} b_{12} + a_{32} b_{22} \\ &= 2 \times 3 + 6 \times 6 = 42 \end{aligned} \tag{2.1.8}$$

Consulting our result for AB above, this is correct!

You may wish to use this method to verify that the rest of the entries in AB are correct.

Here is another example.

Example 2.1.10: Finding the Entries of a Product

Determine if the product AB is defined. If it is, find the $(2,1)$-entry of the product.

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 7 & 6 & 2 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 6 \end{bmatrix}$$

Solution

This product is of the form $(3 \times 3)(3 \times 2)$. The middle numbers match so the matrices are conformable and it is possible to compute the product.

We want to find the $(2,1)$-entry of AB, that is, the entry in the second row and first column of the product. We will use Definition 2.1.1, which states

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

In this case, $n = 3$, $i = 2$ and $j = 1$. Hence the $(2,1)$-entry is found by computing

$$(AB)_{21} = \sum_{k=1}^{3} a_{2k} b_{k1} = \begin{bmatrix} a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix}$$

Substituting in the appropriate values, this product becomes

$$\begin{bmatrix} a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 7 & 6 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = 1 \times 7 + 3 \times 6 + 2 \times 2 = 29$$

Hence, $(AB)_{21} = 29$.

You should take a moment to find a few other entries of AB. You can multiply the matrices to check that your answers are correct. The product AB is given by

$$AB = \begin{bmatrix} 13 & 13 \\ 29 & 32 \\ 0 & 0 \end{bmatrix}$$

Properties of Matrix Multiplication

As pointed out above, it is sometimes possible to multiply matrices in one order but not in the other order. However, even if both AB and BA are defined, they may not be equal.

Example 2.1.11: Matrix Multiplication is Not Commutative

Compare the products AB and BA, for matrices $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Solution

First, notice that A and B are both of size 2×2. Therefore, both products AB and BA are defined. The first product, AB is

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$$

The second product, BA is

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

Therefore, $AB \neq BA$.

This example illustrates that you cannot assume $AB = BA$ even when multiplication is defined in both orders. If for some matrices A and B it is true that $AB = BA$, then we say that A and B **commute**. This is one important property of matrix multiplication.

The following are other important properties of matrix multiplication. Notice that these properties hold only when the size of matrices are such that the products are defined.

Proposition 2.1.3: Properties of Matrix Multiplication

The following hold for matrices A, B, and C and for scalars r and s,

$$A(rB + sC) = r(AB) + s(AC) \tag{2.1.9}$$

$$(B + C)A = BA + CA \tag{2.1.10}$$

$$A(BC) = (AB)C \tag{2.1.11}$$

Proof available at the end of this section.

The Transpose of a Matrix

Another important operation on matrices is that of taking the **transpose**. For a matrix A, we denote the **transpose** of A by A^T. Before formally defining the transpose, we explore this operation on the following matrix.

$$\begin{bmatrix} 1 & 4 \\ 3 & 1 \\ 2 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 1 & 6 \end{bmatrix}$$

What happened? The first column became the first row and the second column became the second row. Thus the 3×2 matrix became a 2×3 matrix. The number 4 was in the first row and the second column and it ended up in the second row and first column.

The definition of the transpose is as follows.

Definition 2.1.12: The Transpose of a Matrix

Let A be an $m \times n$ matrix. Then A^T, the **transpose** of A, denotes the $n \times m$ matrix given by

$$A^T = [a_{ij}]^T = [a_{ji}]$$

The (i, j)-entry of A becomes the (j, i)-entry of A^T.

Consider the following example.

Example 2.1.12: The Transpose of a Matrix

Calculate A^T for the following matrix

$$A = \begin{bmatrix} 1 & 2 & -6 \\ 3 & 5 & 4 \end{bmatrix}$$

Solution

By Definition 2.1.1, we know that for $A = [a_{ij}]$, $A^T = [a_{ji}]$. In other words, we switch the row and column location of each entry. The $(1, 2)$-entry becomes the $(2, 1)$-entry.

Thus,

$$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 5 \\ -6 & 4 \end{bmatrix}$$

Notice that A is a 2×3 matrix, while A^T is a 3×2 matrix.

The transpose of a matrix has the following important properties.

Proposition 2.1.4: Properties of the Transpose of a Matrix

Let A be an $m \times n$ matrix, B an $n \times p$ matrix, and r and s scalars. Then

1. $$\left(A^T\right)^T = A$$
2. $$(AB)^T = B^T A^T$$

3. $$(rA + sB)^T = rA^T + sB^T$$

Proof available at the end of this section.

The transpose of a matrix is related to other important topics. Consider the following definition.

Definition 2.1.13: Symmetric and Skew Symmetric Matrices

An $n \times n$ matrix A is said to be **symmetric** if $A = A^T$. It is said to be **skew symmetric** if $A = -A^T$.

We will explore these definitions in the following examples.

Example 2.1.13: Symmetric Matrices

Let

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 5 & -3 \\ 3 & -3 & 7 \end{bmatrix}$$

Use Definition 2.1.2 to show that A is symmetric.

Solution

By Definition 2.1.2, we need to show that $A = A^T$. Now, using Definition 2.1.1,

$$A^T = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 5 & -3 \\ 3 & -3 & 7 \end{bmatrix}$$

Hence, $A = A^T$, so A is symmetric.

Below is a video example that goes over the the product of a matrix with its transpose.

Example 2.1.14

Let $A = \begin{bmatrix} 6 & -1 \\ -2 & 4 \\ 1 & 3 \end{bmatrix}$. Find $A^T A$.

Solution

- **Video Length**: 4 minutes 46 seconds.
- **Context**: This video demonstrates using the transpose of a matrix in a calculation.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Proposition 2.1.5

Let A be an $m \times n$ matrix. Then

1. $A^T A$ is a symmetric matrix.
2. AA^T is a symmetric matrix.

Proof available at the end of this section.

Example 2.1.15: A Skew Symmetric Matrix

Let

$$A = \begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & 2 \\ -3 & -2 & 0 \end{bmatrix}$$

Show that A is skew symmetric.

Solution

By Definition 2.1.2,

$$A^T = \begin{bmatrix} 0 & -1 & -3 \\ 1 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix}$$

You can see that each entry of A^T is equal to -1 times the same entry of A. Hence, $A^T = -A$ and so by Definition 2.1.2, A is skew symmetric.

Answers

Proposition 2.1.1: Properties of Matrix Addition

Consider the Commutative Law of Addition given in (2.1.2). Let A, B, C, and D be matrices such that $A + B = C$ and $B + A = D$. We want to show that $D = C$. To do so, we will use the definition of matrix addition given in Definition 2.1.4. Now,

$$c_{ij} = a_{ij} + b_{ij} = b_{ij} + a_{ij} = d_{ij}$$

Therefore, $C = D$ because the ij^{th} entries are the same for all i and j. Note that the conclusion follows from the commutative law of addition of numbers, which says that if a and b are two numbers, then $a + b = b + a$. The proof of the other results are similar, and are left as an exercise.

Proposition 2.1.2: Properties of Scalar Multiplication

The proof of this proposition is similar to the proof of Proposition 2.1.1 and is left an exercise to the reader.

Proposition 2.1.3: Properties of Matrix Multiplication

First we will prove (2.1.9). We will use Definition 2.3.1 and prove this statement using the ij^{th} entries of a matrix. Therefore,

$$\begin{aligned} (A(rB + sC))_{ij} &= \sum_k a_{ik} (rB + sC)_{kj} \\ &= \sum_k a_{ik} (rb_{kj} + sc_{kj}) \\ &= r \sum_k a_{ik} b_{kj} + s \sum_k a_{ik} c_{kj} \\ &= r (AB)_{ij} + s (AC)_{ij} \\ &= (r(AB) + s(AC))_{ij} \end{aligned} \tag{2.1.12}$$

Thus $A(rB + sC) = r(AB) + s(AC)$ as claimed.

The proof of Equation (2.1.10) follows the same pattern and is left as an exercise.

Statement Equation (2.1.11) is the **associative law of multiplication**. Using Definition 2.3.1,

$$\begin{aligned}(A(BC))_{ij} &= \sum_k a_{ik}(BC)_{kj} \\ &= \sum_k a_{ik}\sum_l b_{kl}c_{lj} \\ &= \sum_l (AB)_{il}c_{lj} = ((AB)C)_{ij}.\end{aligned}$$

This proves (2.1.11).

Proposition 2.1.4**: Properties of the Transpose of a Matrix**

First we prove 2. From Definition 2.1.1,

$$\begin{aligned}(AB)^T = [(AB)_{ij}]^T = [(AB)_{ji}] = \sum_k a_{jk}b_{ki} = \sum_k b_{ki}a_{jk} \\ = \sum_k [b_{ik}]^T [a_{kj}]^T = [b_{ij}]^T [a_{ij}]^T = B^T A^T\end{aligned} \tag{2.1.13}$$

The proof of Formula 3 is left as an exercise.

Proposition 2.1.5

We prove statement 1. To show that A^TA is symmetric, we must show that it is equal to its transpose. Notice that

$$(A^TA)^T = A^T(A^T)^T = A^TA \tag{2.1.14}$$

Since $A^TA)^T = A^TA$, A^TA is symmetric.

Proof of statement 2 is similar.

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2.1E: Exercises for Section 2.1

Exercise $2.1E.1$

For the following pairs of matrices, determine if the sum $A+B$ is defined. If so, find the sum.

a. $A=\begin{bmatrix}1&0\\0&1\end{bmatrix}, B=\begin{bmatrix}0&1\\1&0\end{bmatrix}$

b. $A=\begin{bmatrix}2&1&2\\1&1&0\end{bmatrix}, B=\begin{bmatrix}-1&0&3\\0&1&4\end{bmatrix}$

c. $A=\begin{bmatrix}1&0\\-2&3\\4&2\end{bmatrix}, B=\begin{bmatrix}2&7&-1\\0&3&4\end{bmatrix}$

Exercise $2.1E.2$

For each matrix A, find the matrix $-A$ such that $A+(-A)=0$.

a. $A=\begin{bmatrix}1&2\\2&1\end{bmatrix}$

b. $A=\begin{bmatrix}-2&3\\0&2\end{bmatrix}$

c. $A=\begin{bmatrix}0&1&2\\1&-1&3\\4&2&0\end{bmatrix}$

Exercise $2.1E.3$

In the context of Proposition 2.1.1, describe $-A$ and 0.

Answer available at the end of this section.

Exercise $2.1E.4$

For each matrix A, find the product $(-2)A, 0A$, and $3A$.

a. $A=\begin{bmatrix}1&2\\2&1\end{bmatrix}$

b. $A=\begin{bmatrix}-2&3\\0&2\end{bmatrix}$

c. $A=\begin{bmatrix}0&1&2\\1&-1&3\\4&2&0\end{bmatrix}$

Exercise $2.1E.5$

Using only the properties given in Proposition 2.1.1 and Proposition 2.1.2, show $-A$ is unique.

Answer available at the end of this section.

Exercise $2.1E.6$

Using only the properties given in Proposition 2.1.1 and Proposition 2.1.2, show 0 is unique.

Answer available at the end of this section.

Exercise $2.1E.7$

Using only the properties given in Proposition 2.1.1 and Proposition 2.1.2 show $0A = 0$. Here the 0 on the left is the scalar 0 and the 0 on the right is the zero matrix of appropriate size.

Answer available at the end of this section.

Exercise $2.1E.8$

Using only the properties given in Proposition 2.1.1 and Proposition 2.1.2, as well as previous problems show $(-1)\,A = -A$.

Answer available at the end of this section.

Exercise $2.1E.9$

Consider the matrices $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 7 \end{bmatrix}, B = \begin{bmatrix} 3 & -1 & 2 \\ -3 & 2 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}, D = \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}, E = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Find the following if possible. If it is not possible explain why.

a. $-3A$
b. $3B - A$
c. AC
d. CB
e. AE
f. EA

Answer available at the end of this section.

Exercise $2.1E.10$

Consider the matrices $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 1 & -1 \end{bmatrix}, B = \begin{bmatrix} 2 & -5 & 2 \\ -3 & 2 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix}, D = \begin{bmatrix} -1 & 1 \\ 4 & -3 \end{bmatrix}, E = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Find the following if possible. If it is not possible explain why.

a. $-3A$
b. $3B - A$
c. AC
d. CA
e. AE
f. EA
g. BE
h. DE

Answer available at the end of this section.

Exercise 2.1E.11

Let $A = \begin{bmatrix} 1 & 1 \\ -2 & -1 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 & -2 \\ 2 & 1 & -2 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 1 & -3 \\ -1 & 2 & 0 \\ -3 & -1 & 0 \end{bmatrix}$. Find the following if possible.

a. AB
b. BA
c. AC
d. CA
e. CB
f. BC

Answer available at the end of this section.

Exercise 2.1E.12

Let $A = \begin{bmatrix} -1 & -1 \\ 3 & 3 \end{bmatrix}$. Find all 2×2 matrices, B such that $AB = 0$.

Answer available at the end of this section.

Exercise 2.1E.13

Let $X = \begin{bmatrix} -1 & -1 & 1 \end{bmatrix}$ and $Y = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}$. Find X^TY and XY^T if possible.

Answer available at the end of this section.

Exercise 2.1E.14

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 3 & k \end{bmatrix}$. Is it possible to choose k such that $AB = BA$? If so, what should k equal?

Answer available at the end of this section.

Exercise 2.1E.15

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 1 & k \end{bmatrix}$. Is it possible to choose k such that $AB = BA$? If so, what should k equal?

Answer available at the end of this section.

Exercise 2.1E.16

Find 2×2 matrices, A, B, and C such that $A \neq 0, C \neq B$, but $AC = AB$.

Answer available at the end of this section.

Exercise $2.1E.17$

Give an example of matrices (of any size), A, B, C such that $B \neq C$, $A \neq 0$, and yet $AB = AC$.

Exercise $2.1E.18$

Find 2×2 matrices A and B such that $A \neq 0$ and $B \neq 0$ but $AB = 0$.

Answer available at the end of this section.

Exercise $2.1E.19$

Give an example of matrices (of any size), A, B such that $A \neq 0$ and $B \neq 0$ but $AB = 0$.

Exercise $2.1E.20$

Find 2×2 matrices A and B such that $A \neq 0$ and $B \neq 0$ with $AB \neq BA$.

Answer available at the end of this section.

Exercise $2.1E.21$

Write the system

$$\begin{array}{c} x_1 - x_2 + 2x_3 \\ 2x_3 + x_1 \\ 3x_3 \\ 3x_4 + 3x_2 + x_1 \end{array}$$

in the form $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ where A is an appropriate matrix.

Answer available at the end of this section.

Exercise $2.1E.22$

Write the system

$$\begin{array}{c} x_1 + 3x_2 + 2x_3 \\ 2x_3 + x_1 \\ 6x_3 \\ x_4 + 3x_2 + x_1 \end{array}$$

in the form $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ where A is an appropriate matrix.

Answer available at the end of this section.

Exercise $2.1E.23$

Write the system

$$\begin{array}{c} x_1 + x_2 + x_3 \\ 2x_3 + x_1 + x_2 \\ x_3 - x_1 \\ 3x_4 + x_1 \end{array}$$

in the form $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ where A is an appropriate matrix.

Answer available at the end of this section.

Exercise $2.1E.24$

A matrix A is called *idempotent* if $A^2 = A$. Let

$$A = \begin{bmatrix} 2 & 0 & 2 \\ 1 & 1 & 2 \\ -1 & 0 & -1 \end{bmatrix}$$

and show that A is idempotent.

Exercise $2.1E.25$

For each pair of matrices, find the $(1,2)$-entry and $(2,3)$-entry of the product AB.

1. $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 4 & 0 \\ 2 & 5 & 1 \end{bmatrix}, B = \begin{bmatrix} 4 & 6 & -2 \\ 7 & 2 & 1 \\ -1 & 0 & 0 \end{bmatrix}$
2. $A = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 4 \\ 1 & 0 & 5 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 & 0 \\ -4 & 16 & 1 \\ 0 & 2 & 2 \end{bmatrix}$

Exercise $2.1E.26$

Suppose A and B are square matrices of the same size. Which of the following are necessarily true?

a. $(A-B)^2 = A^2 - 2AB + B^2$
b. $(AB)^2 = A^2B^2$
c. $(A+B)^2 = A^2 + 2AB + B^2$
d. $(A+B)^2 = A^2 + AB + BA + B^2$
e. $A^2B^2 = A(AB)B$
f. $(A+B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$
g. $(A+B)(A-B) = A^2 - B^2$

Answer available at the end of this section.

Exercise $2.1E.27$

Consider the matrices $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 1 & -1 \end{bmatrix}, B = \begin{bmatrix} 2 & -5 & 2 \\ -3 & 2 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix}, D = \begin{bmatrix} -1 & 1 \\ 4 & -3 \end{bmatrix}, E = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Find the following if possible. If it is not possible explain why.

a. $-3A^T$
b. $3B - A^T$
c. $E^T B$
d. EE^T
e. $B^T B$
f. CA^T
g. $D^T BE$

Answer available at the end of this section.

Exercise $2.1E.28$

Let A be an $n \times n$ matrix. Show A equals the sum of a symmetric and a skew symmetric matrix.

Hint available at the end of this section.

Exercise $2.1E.29$

Show that the main diagonal of every skew symmetric matrix consists of only zeros. Recall that the main diagonal consists of every entry of the matrix which is of the form a_{ii}.

Answer available at the end of this section.

Exercise $2.1E.30$

Prove 3 from Proposition 2.1.4. That is, show that for an $m \times n$ matrix A, an $m \times n$ matrix B, and scalars r, s, the following holds:

$$(rA + sB)^T = rA^T + sB^T$$

Answers

Exercise $2.1E.3$

To get $-A$, just replace every entry of A with its additive inverse. The $\mathbb{0}$ matrix is the one which has all zeros in it.

Exercise $2.1E.5$

Suppose B also works. Then

$$-A = -A + (A + B) = (-A + A) + B = 0 + B = B$$

Exercise $2.1E.6$

Suppose $0'$ also works. Then $0' = 0' + 0 = 0$.

Exercise $2.1E.7$

$0A = (0 + 0) A = 0A + 0A$. Now add $-(0A)$ to both sides. Then $0 = 0A$.

Exercise $2.1E.8$

$A + (-1) A = (1 + (-1)) A = 0A = 0$. Therefore, from the uniqueness of the additive inverse proved in the above Problem $2.1E.7$, it follows that $-A = (-1) A$.

Exercise $2.1E.9$

a. $\begin{bmatrix} -3 & -6 & -9 \\ -6 & -3 & -21 \end{bmatrix}$
b. $\begin{bmatrix} 8 & -5 & 3 \\ -11 & 5 & -4 \end{bmatrix}$
c. Not possible
d. $\begin{bmatrix} -3 & 3 & 4 \\ 6 & -1 & 7 \end{bmatrix}$
e. Not possible
f. Not possible

Exercise $2.1E.10$

a. $\begin{bmatrix} -3 & -6 \\ -9 & -6 \\ -3 & 3 \end{bmatrix}$
b. Not possible.
c. $\begin{bmatrix} 11 & 2 \\ 13 & 6 \\ -4 & 2 \end{bmatrix}$
d. Not possible.
e. $\begin{bmatrix} 7 \\ 9 \\ -2 \end{bmatrix}$
f. Not possible.
g. Not possible.
h. $\begin{bmatrix} 2 \\ -5 \end{bmatrix}$

Exercise $2.1E.11$

a. $\begin{bmatrix} 3 & 0 & -4 \\ -4 & 1 & 6 \\ 5 & 1 & -6 \end{bmatrix}$
b. $\begin{bmatrix} 1 & -2 \\ -2 & -3 \end{bmatrix}$
c. Not possible
d. $\begin{bmatrix} -4 & -6 \\ -5 & -3 \\ -1 & -2 \end{bmatrix}$
e. $\begin{bmatrix} 8 & 1 & -3 \\ 7 & 6 & -6 \end{bmatrix}$

Exercise $2.1E.12$

$$\begin{aligned} \begin{bmatrix} -1 & -1 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} &= \begin{bmatrix} -x-z & -w-y \\ 3x+3z & 3w+3y \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned} \tag{2.1E.1}$$

Solution is: $w=-y, x=-z$ so the matrices are of the form $\begin{bmatrix} x & y \\ -x & -y \end{bmatrix}$.

Exercise $2.1E.13$

$X^TY = \begin{bmatrix} 0 & -1 & -2 \\ 0 & -1 & -2 \\ 0 & 1 & 2 \end{bmatrix}, XY^T = 1$

Exercise $2.1E.14$

$$\begin{aligned} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & k \end{bmatrix} &= \begin{bmatrix} 7 & 2k+2 \\ 15 & 4k+6 \end{bmatrix} \\ \begin{bmatrix} 1 & 2 \\ 3 & k \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} &= \begin{bmatrix} 7 & 10 \\ 3k+3 & 4k+6 \end{bmatrix} \end{aligned} \tag{2.1E.2}$$

Thus you must have $\begin{matrix} 3k+3=15 \\ 2k+2=10 \end{matrix}$, Solution is: $[k=4]$

Exercise $2.1E.15$

$$\begin{aligned} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & k \end{bmatrix} &= \begin{bmatrix} 3 & 2k+2 \\ 7 & 4k+6 \end{bmatrix} \\ \begin{bmatrix} 1 & 2 \\ 1 & k \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} &= \begin{bmatrix} 7 & 10 \\ 3k+1 & 4k+2 \end{bmatrix} \end{aligned} \tag{2.1E.3}$$

However, $7 \neq 3$ and so there is no possible choice of k which will make these matrices commute.

Exercise $2.1E.16$

Let $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$.

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{2.1E.4}$$

Exercise $2.1E.18$

Let $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Exercise $2.1E.20$

Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \tag{2.1E.5}$$

Exercise $2.1E.21$

$$A = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 3 & 0 \\ 1 & 3 & 0 & 3 \end{bmatrix}$$

Exercise $2.1E.22$

$$A = \begin{bmatrix} 1 & 3 & 2 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 6 & 0 \\ 1 & 3 & 0 & 1 \end{bmatrix}$$

Exercise $2.1E.23$

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$$

Exercise $2.1E.26$

a. Not necessarily true.
b. Not necessarily true.
c. Not necessarily true.
d. Necessarily true.
e. Necessarily true.
f. Not necessarily true.
g. Not necessarily true.

Exercise $2.1E.27$

a. $\begin{bmatrix} -3 & -9 & -3 \\ -6 & -6 & 3 \end{bmatrix}$
b. $\begin{bmatrix} 5 & -18 & 5 \\ -11 & 4 & 4 \end{bmatrix}$
c. $\begin{bmatrix} -7 & 1 & 5 \end{bmatrix}$
d. $\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$
e. $\begin{bmatrix} 13 & -16 & 1 \\ -16 & 29 & -8 \\ 1 & -8 & 5 \end{bmatrix}$
f. $\begin{bmatrix} 5 & 7 & -1 \\ 5 & 15 & 5 \end{bmatrix}$
g. Not possible.

Exercise $2.1E.28$

Show that $\frac{1}{2}(A^T + A)$ is symmetric and then consider using this as one of the matrices.

Exercise $2.1E.29$

If A is symmetric then $A = -A^T$. It follows that $a_{ii} = -a_{ii}$ and so each $a_{ii} = 0$.

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2.2: The Inverse of a Matrix

Learning Objectives

- Compute the inverse of a matrix using row operations, and prove identities involving matrix inverses.
- Solve a linear system using matrix algebra.

There is a special matrix, denoted I, which is called the **identity matrix**. The identity matrix is always a square matrix, and it has the property that there are ones down the main diagonal and zeroes elsewhere. Here are some identity matrices of various sizes.

$$[1], \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The first is the 1×1 identity matrix, the second is the 2×2 identity matrix, and so on. By extension, you can likely see what the $n \times n$ identity matrix would be. When it is necessary to distinguish the size of identity matrix is being discussed, we will use the notation I_n for the $n \times n$ identity matrix.

The identity matrix is so important that there is a special symbol to denote the ij^{th} entry of the identity matrix. This symbol is given by $I_{ij} = \delta_{ij}$ where δ_{ij} is the **Kronecker delta** defined by

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

I_n is called the **identity matrix** because it is a **multiplicative identity** in the following sense.

Proposition 2.2.1: Multiplication by the Identity Matrix

Suppose A is an $m \times n$ matrix and I_n is the $n \times n$ identity matrix**.** Then $AI_n = A$. If I_m is the $m \times m$ identity matrix, it also follows that $I_m A = A$.

Proof available at the end of this section.

We now define the matrix operation which in some ways plays the role of division.

Definition 2.2.1: The Inverse of a Matrix

A square $n \times n$ matrix A is said to have an **inverse** A^{-1} if and only if

$$AA^{-1} = A^{-1}A = I_n$$

In this case, the matrix A is called **invertible**. An invertible matrix is also known as **non-singular** matrix.

If no such inverse exists, then A is said to be **non-invertible** (also known as **singular**).

Such a matrix A^{-1} will have the same size as the matrix A. It is very important to observe that the inverse of a matrix, if it exists, is unique. Another way to think of this is that if it acts like the inverse, then it is the inverse.

Theorem 2.2.1: Uniqueness of Inverse

Suppose A is an $n \times n$ matrix such that an inverse A^{-1} exists. Then there is only one such inverse matrix. That is, given any matrix B such that $AB = BA = I$, $B = A^{-1}$.

Proof available at the end of this section.

The next example demonstrates how to check the inverse of a matrix.

Example 2.2.1: Verifying the Inverse of a Matrix

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. Show $\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$ is the inverse of A.

Solution

To check this, multiply

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

and

$$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

showing that this matrix is indeed the inverse of A.

Unlike ordinary multiplication of numbers, it can happen that $A \neq 0$ but A may fail to have an inverse. This is illustrated in the following example.

Example 2.2.2: A Nonzero Matrix With No Inverse

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Show that A does not have an inverse.

Solution

One might think A would have an inverse because it does not equal zero. However, note that

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If A^{-1} existed, we would have the following

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= A^{-1}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) \\ &= A^{-1}\left(A \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) \\ &= (A^{-1}A) \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= I \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{aligned} \tag{2.2.1}$$

This says that

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

which is impossible! Therefore, A does not have an inverse.

Next, we will explore how to find the inverse of a matrix, if it exists.

In Example 2.2.1 we were given A^{-1} and asked to verify that this matrix was in fact the inverse of A. In this section, we explore how to find A^{-1}.

Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

as in Example 2.2.1. In order to find A^{-1}, we need to find a matrix $\begin{bmatrix} x & z \\ y & w \end{bmatrix}$ such that

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x & z \\ y & w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We can multiply these two matrices, and see that in order for this equation to be true, we must find the solution to the systems of equations,

$$\begin{aligned} x + y &= 1 \\ x + 2y &= 0 \end{aligned}$$

and

$$\begin{aligned} z + w &= 0 \\ z + 2w &= 1 \end{aligned}$$

Writing the augmented matrix for these two systems gives

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

for the first system and

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \tag{2.2.2}$$

for the second.

Let's solve the first system. Take -1 times the first row and add to the second to get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

Now take -1 times the second row and add to the first to get

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

Writing in terms of variables, this says $x = 2$ and $y = -1$.

Now solve the second system, (2.2.2) to find z and w. You will find that $z = -1$ and $w = 1$.

If we take the values found for x, y, z, and w and put them into our inverse matrix, we see that the inverse is

$$A^{-1} = \begin{bmatrix} x & z \\ y & w \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

After taking the time to solve the second system, you may have noticed that exactly the same row operations were used to solve both systems. In each case, the end result was something of the form $[I|X]$ where I is the identity and X gave a column of the inverse. In the above,

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

the first column of the inverse was obtained by solving the first system and then the second column

$$\begin{bmatrix} z \\ w \end{bmatrix}$$

To simplify this procedure, we could have solved both systems at once! To do so, we could have written

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

and row reduced until we obtained

$$\begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

and read off the inverse as the 2×2 matrix on the right side.

This exploration motivates the following important algorithm.

Algorithm 2.2.1: Matrix Inverse Algorithm

Suppose A is an $n \times n$ matrix. To find A^{-1} if it exists, form the augmented $n \times 2n$ matrix

$$[A|I]$$

If possible do row operations until you obtain an $n \times 2n$ matrix of the form

$$[I|B]$$

When this has been done, $B = A^{-1}$. In this case, we say that A is **invertible**. If it is impossible to row reduce to a matrix of the form $[I|B]$, then A has no inverse.

This algorithm shows how to find the inverse if it exists. It will also tell you if A does not have an inverse.

Consider the following example.

Example 2.2.3: Finding the Inverse

Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 3 & 1 & -1 \end{bmatrix}$. Find A^{-1} if it exists.

Solution

Set up the augmented matrix

$$[A|I] = \begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 3 & 1 & -1 & 0 & 0 & 1 \end{bmatrix}$$

Now we row reduce, with the goal of obtaining the 3×3 identity matrix on the left hand side. First, take -1 times the first row and add to the second followed by -3 times the first row added to the third row. This yields

$$\begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & -5 & -7 & -3 & 0 & 1 \end{bmatrix}$$

Then take 5 times the second row and add to -2 times the third row.

$$\begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -10 & 0 & -5 & 5 & 0 \\ 0 & 0 & 14 & 1 & 5 & -2 \end{bmatrix}$$

Next take the third row and add to -7 times the first row. This yields

$$\begin{bmatrix} -7 & -14 & 0 & -6 & 5 & -2 \\ 0 & -10 & 0 & -5 & 5 & 0 \\ 0 & 0 & 14 & 1 & 5 & -2 \end{bmatrix}$$

Now take $-\frac{7}{5}$ times the second row and add to the first row.

$$\begin{bmatrix} -7 & 0 & 0 & 1 & -2 & -2 \\ 0 & -10 & 0 & -5 & 5 & 0 \\ 0 & 0 & 14 & 1 & 5 & -2 \end{bmatrix}$$

Finally divide the first row by -7, the second row by -10 and the third row by 14 which yields

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{1}{7} & \frac{2}{7} & \frac{2}{7} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{14} & \frac{5}{14} & -\frac{1}{7} \end{bmatrix}$$

Notice that the left hand side of this matrix is now the 3×3 identity matrix I_3. Therefore, the inverse is the 3×3 matrix on the right hand side, given by

$$\begin{bmatrix} -\frac{1}{7} & \frac{2}{7} & \frac{2}{7} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{14} & \frac{5}{14} & -\frac{1}{7} \end{bmatrix}$$

It may happen that through this algorithm, you discover that the left hand side cannot be row reduced to the identity matrix. Consider the following example of this situation.

Example 2.2.4: A Matrix Which Has No Inverse

Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 2 & 2 & 4 \end{bmatrix}$. Find A^{-1} if it exists.

Solution

Write the augmented matrix $[A|I]$

$$\begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 2 & 2 & 4 & 0 & 0 & 1 \end{bmatrix}$$

and proceed to do row operations attempting to obtain $[I|A^{-1}]$. Take -1 times the first row and add to the second. Then take -2 times the first row and add to the third row.

$$\begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & -2 & 0 & -2 & 0 & 1 \end{bmatrix}$$

Next add -1 times the second row to the third row.

$$\begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{bmatrix}$$

At this point, you can see there will be no way to obtain I on the left side of this augmented matrix. Hence, there is no way to complete this algorithm, and therefore the inverse of A does not exist. In this case, we say that A is not invertible.

If the algorithm provides an inverse for the original matrix, it is always possible to check your answer. To do so, use the method demonstrated in Example 2.6.1. Check that the products AA^{-1} and $A^{-1}A$ both equal the identity matrix. Through this method, you can always be sure that you have calculated A^{-1} properly!

One way in which the inverse of a matrix is useful is to find the solution of a system of linear equations. Recall from Definition 2.1.9 that we can write a system of equations in matrix form, which is of the form $AX = B$. Suppose you find the inverse of the matrix A^{-1}. Then you could multiply both sides of this equation on the left by A^{-1} and simplify to obtain

$$\begin{aligned} \left(A^{-1}\right) AX &= A^{-1}B \\ \left(A^{-1}A\right) X &= A^{-1}B \\ IX &= A^{-1}B \\ X &= A^{-1}B \end{aligned}$$

Therefore we can find X, the solution to the system, by computing $X = A^{-1}B$. Note that once you have found A^{-1}, you can easily get the solution for different right hand sides (different B). It is always just $A^{-1}B$.

We will explore this method of finding the solution to a system in the following example.

Example 2.2.5: Using the Inverse to Solve a System of Equations

Consider the following system of equations. Use the inverse of a suitable matrix to give the solutions to this system.

$$\begin{aligned} x + z &= 1 \\ x - y + z &= 3 \\ x + y - z &= 2 \end{aligned}$$

Solution

First, we can write the system of equations in matrix form

$$AX = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = B \tag{2.2.3}$$

The inverse of the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

is

$$A^{-1} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & -1 & 0 \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

Verifying this inverse is left as an exercise.

From here, the solution to the given system (2.2.3) is found by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}B = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & -1 & 0 \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ -2 \\ -\frac{3}{2} \end{bmatrix}$$

What if the right side, B, of (2.2.3) had been $\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$? In other words, what would be the solution to

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}?$$

By the above discussion, the solution is given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}B = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & -1 & 0 \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$$

This illustrates that for a system $AX = B$ where A^{-1} exists, it is easy to find the solution when the vector B is changed.

We conclude this section with some important properties of the inverse.

Theorem 2.2.2: Inverses of Transposes and Products

Let A, B, and A_i for $i = 1, ..., k$ be $n \times n$ matrices.

1. If A is an invertible matrix, then $(A^T)^{-1} = (A^{-1})^T$
2. If A and B are invertible matrices, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$
3. If $A_1, A_2, ..., A_k$ are invertible, then the product $A_1A_2 \cdots A_k$ is invertible, and $(A_1A_2 \cdots A_k)^{-1} = A_k^{-1}A_{k-1}^{-1} \cdots A_2^{-1}A_1^{-1}$

Consider the following theorem.

Theorem 2.2.3: Properties of the Inverse

Let A be an $n \times n$ matrix and I the usual identity matrix.

1. I is invertible and $I^{-1} = I$
2. If A is invertible then so is A^{-1}, and $(A^{-1})^{-1} = A$
3. If A is invertible then so is A^k, and $(A^k)^{-1} = (A^{-1})^k$
4. If A is invertible and p is a nonzero real number, then pA is invertible and $(pA)^{-1} = \frac{1}{p}A^{-1}$

Answers

Proposition 2.2.1: Multiplication by the Identity Matrix

The (i, j)-entry of AI_n is given by:

$$\sum_k a_{ik}\delta_{kj} = a_{ij}$$

and so $AI_n = A$. The other case is left as an exercise for you.

Theorem 2.2.1: Uniqueness of Inverse

In this proof, it is assumed that I is the $n \times n$ identity matrix. Let A, B be $n \times n$ matrices such that A^{-1} exists and $AB = BA = I$. We want to show that $A^{-1} = B$. Now using properties we have seen, we get:

$$A^{-1} = A^{-1}I = A^{-1}(AB) = (A^{-1}A)B = IB = B$$

Hence, $A^{-1} = B$ which tells us that the inverse is unique.

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2.2E: Exercises for Section 2.2

Exercise 2.2E.1

Prove that $I_m A = A$ where A is an $m \times n$ matrix.

Answer available at the end of this section.

Exercise 2.2E.2

Suppose $AB = AC$ and A is an invertible $n \times n$ matrix. Does it follow that $B = C$? Explain why or why not.

Answer available at the end of this section.

Exercise 2.2E.3

Suppose $AB = AC$ and A is a non invertible $n \times n$ matrix. Does it follow that $B = C$? Explain why or why not.

Exercise 2.2E.4

Give an example of a matrix A such that $A^2 = I$ and yet $A \neq I$ and $A \neq -I$.

Answer available at the end of this section.

Exercise 2.2E.5

Let

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$$

Find A^{-1} if possible. If A^{-1} does not exist, explain why.

Answer available at the end of this section.

Exercise 2.2E.6

Let

$$A = \begin{bmatrix} 0 & 1 \\ 5 & 3 \end{bmatrix}$$

Find A^{-1} if possible. If A^{-1} does not exist, explain why.

Answer available at the end of this section.

Exercise 2.2E.7

Add exercises text here.Let

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$$

Find A^{-1} if possible. If A^{-1} does not exist, explain why.

Answer available at the end of this section.

Exercise $2.2E.8$

Let

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$$

Find A^{-1} if possible. If A^{-1} does not exist, explain why.

Answer available at the end of this section.

Exercise $2.2E.9$

Let A be a 2×2 invertible matrix, with $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Find a formula for A^{-1} in terms of a, b, c, d.

Answer available at the end of this section.

Exercise $2.2E.10$

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 1 & 0 & 2 \end{bmatrix}$$

Find A^{-1} if possible. If A^{-1} does not exist, explain why.

Answer available at the end of this section.

Exercise $2.2E.11$

Let

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix}$$

Find A^{-1} if possible. If A^{-1} does not exist, explain why.

Answer available at the end of this section.

Exercise $2.2E.12$

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 4 & 5 & 10 \end{bmatrix}$$

Find A^{-1} if possible. If A^{-1} does not exist, explain why.

Answer available at the end of this section.

Exercise $2.2E.13$

Let

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & -3 & 2 \\ 1 & 2 & 1 & 2 \end{bmatrix}$$

Find A^{-1} if possible. If A^{-1} does not exist, explain why.

Answer available at the end of this section.

Exercise $2.2E.14$

Using the inverse of the matrix, find the solution to the systems:

a. $$\begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

b. $$\begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Now give the solution in terms of a and b to

$$\begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

Exercise $2.2E.15$

Using the inverse of the matrix, find the solution to the systems:

a. $$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

b. $$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$$

Now give the solution in terms of $a, b,$ and c to the following:

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Answer available at the end of this section.

Exercise $2.2E.16$

Show that if A is an $n \times n$ invertible matrix and X is a $n \times 1$ matrix such that $AX = B$ for B an $n \times 1$ matrix, then $X = A^{-1}B$.

Answer available at the end of this section.

Exercise $2.2E.17$

Prove that if A^{-1} exists and $AX = 0$ then $X = 0$.

Answer available at the end of this section.

Exercise $2.2E.18$

Show that if A^{-1} exists for an $n \times n$ matrix, then it is unique. That is, if $BA = I$ and $AB = I$, then $B = A^{-1}$.

Answer available at the end of this section.

Exercise $2.2E.19$

Show that if A is an invertible $n \times n$ matrix, then so is A^T and $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$.

Answer available at the end of this section.

Exercise $2.2E.20$

Show $(AB)^{-1} = B^{-1}A^{-1}$ by verifying that

$$AB\left(B^{-1}A^{-1}\right) = I$$

and

$$B^{-1}A^{-1}(AB) = I$$

Hint: Use Problem $2.2E.18$.

Answer available at the end of this section.

Exercise $2.2E.21$

Show that $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$ by verifying that

$$(ABC)\left(C^{-1}B^{-1}A^{-1}\right) = I$$

and

$$\left(C^{-1}B^{-1}A^{-1}\right)(ABC) = I$$

Hint: Use Problem $2.2E.18$.

Answer available at the end of this section.

Exercise $2.2E.22$

If A is invertible, show $\left(A^2\right)^{-1} = \left(A^{-1}\right)^2$.

Answer available at the end of this section.

Exercise $2.2E.23$

If A is invertible, show $\left(A^{-1}\right)^{-1} = A$.

Answer available at the end of this section.

Answers

Exercise $2.2E.1$

$(I_m A)_{ij} \equiv \sum_k \delta_{ik} A_{kj} = A_{ij}$

Exercise $2.2E.2$

Yes $B = C$. Multiply $AB = AC$ on the left by A^{-1}.

Exercise $2.2E.4$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Exercise $2.2E.5$

$$\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{3}{7} & -\frac{1}{7} \\ \frac{1}{7} & \frac{2}{7} \end{bmatrix}$$

Exercise $2.2E.6$

$$\begin{bmatrix} 0 & 1 \\ 5 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{3}{5} & \frac{1}{5} \\ 1 & 0 \end{bmatrix}$$

Exercise $2.2E.7$

$$\begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & \frac{1}{3} \\ 1 & -\frac{2}{3} \end{bmatrix}$$

Exercise $2.2E.8$

$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}^{-1}$ does not exist. The of this matrix is $\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$

Exercise $2.2E.9$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

Exercise $2.2E.10$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 1 & 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 4 & -5 \\ 0 & 1 & -2 \\ 1 & -2 & 3 \end{bmatrix}$$

Exercise $2.2E.11$

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 0 & 3 \\ 0 & \frac{1}{3} & -\frac{2}{3} \\ 1 & 0 & -1 \end{bmatrix}$$

Exercise $2.2E.12$

The reduced row-echelon form is $\begin{bmatrix} 1 & 0 & \frac{5}{3} \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix}$. There is no inverse.

Exercise $2.2E.13$

$$\begin{bmatrix} 1 & 2 & 0 & 2 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & -3 & 2 \\ 1 & 2 & 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 3 & \frac{1}{2} & -\frac{1}{2} & -\frac{5}{2} \\ -1 & 0 & 0 & 1 \\ -2 & -\frac{3}{4} & \frac{1}{4} & \frac{9}{4} \end{bmatrix}$$

Exercise $2.2E.15$

a. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{2}{3} \\ 0 \end{bmatrix}$

b. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -12 \\ 1 \\ 5 \end{bmatrix}$

c. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3c-2a \\ \frac{1}{3}b-\frac{2}{3}c \\ a-c \end{bmatrix}$

Exercise $2.2E.16$

Multiply both sides of $AX = B$ on the left by A^{-1}.

Exercise $2.2E.17$

Multiply on both sides on the left by A^{-1}. Thus

$$0 = A^{-1}0 = A^{-1}(AX) = \left(A^{-1}A\right)X = IX = X$$

Exercise $2.2E.18$

$A^{-1} = A^{-1}I = A^{-1}(AB) = \left(A^{-1}A\right)B = IB = B.$

Exercise $2.2E.19$

You need to show that $\left(A^{-1}\right)^T$ acts like the inverse of A^T because from uniqueness in the above problem, this will imply it is the inverse. From properties of the transpose,

$$\begin{aligned} A^T\left(A^{-1}\right)^T &= \left(A^{-1}A\right)^T = I^T = I \\ \left(A^{-1}\right)^T A^T &= \left(AA^{-1}\right)^T = I^T = I \end{aligned} \tag{2.2E.1}$$

Hence $\left(A^{-1}\right)^T = \left(A^T\right)^{-1}$ and this last matrix exists.

Exercise $2.2E.20$

$(AB)B^{-1}A^{-1} = A\left(BB^{-1}\right)A^{-1} = AA^{-1} = I$ $B^{-1}A^{-1}(AB) = B^{-1}\left(A^{-1}A\right)B = B^{-1}IB = B^{-1}B = I$

Exercise $2.2E.21$

The proof of this exercise follows from the previous one.

Exercise $2.2E.22$

$A^2\left(A^{-1}\right)^2 = AAA^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I$ $\left(A^{-1}\right)^2A^2 = A^{-1}A^{-1}AA = A^{-1}IA = A^{-1}A = I$

Exercise $2.2E.23$

$A^{-1}A = AA^{-1} = I$ and so by uniqueness, $\left(A^{-1}\right)^{-1} = A$.

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2.3: Elementary Matrices

Learning Objectives

- Use multiplication by an elementary matrix to apply row operations.
- Write a matrix as a product of elementary matrices.

We now turn our attention to a special type of matrix called an **elementary matrix**. An elementary matrix is always a square matrix. Recall the row operations given in Definition 1.2.2. Any elementary matrix, which we often denote by E, is obtained from applying *one* row operation to the identity matrix of the same size.

For example, the matrix

$$E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is the elementary matrix obtained from switching the two rows. The matrix

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is the elementary matrix obtained from multiplying the second row of the 3×3 identity matrix by 3. The matrix

$$E = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

is the elementary matrix obtained from adding -3 times the first row to the third row.

You may construct an elementary matrix from any row operation, but remember that you can only apply one operation.

Consider the following definition.

Definition 2.3.1: Elementary Matrices and Row Operations

Let E be an $n \times n$ matrix. Then E is an **elementary matrix** if it is the result of applying one row operation to the $n \times n$ identity matrix I_n.

Those which involve switching rows of the identity matrix are called permutation matrices.

Therefore, E constructed above by switching the two rows of I_2 is called a permutation matrix.

Elementary matrices can be used in place of row operations and therefore are very useful. It turns out that multiplying (on the left hand side) by an elementary matrix E will have the same effect as doing the row operation used to obtain E.

The following theorem is an important result which we will use throughout this text.

Theorem 2.3.1: Multiplication by an Elementary Matrix and Row Operations

To perform any of the three row operations on a matrix A it suffices to take the product EA, where E is the elementary matrix obtained by using the desired row operation on the identity matrix.

Therefore, instead of performing row operations on a matrix A, we can row reduce through matrix multiplication with the appropriate elementary matrix. We will examine this theorem in detail for each of the three row operations given in Definition 1.3.2.

First, consider the following lemma.

Lemma 2.3.1: Action of Permutation Matrix

Let P^{ij} denote the elementary matrix which involves switching the ith and the jth rows. Then P^{ij} is a permutation matrix and

$$P^{ij}A = B$$

where B is obtained from A by switching the ith and the jth rows.

We will explore this idea more in the following example.

Example 2.3.1: Switching Rows with an Elementary Matrix

Let

$$P^{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} a & b \\ g & d \\ e & f \end{bmatrix}$$

Find B where $B = P^{12}A$.

Solution

You can see that the matrix P^{12} is obtained by switching the first and second rows of the 3×3 identity matrix I.

Using our usual procedure, compute the product $P^{12}A = B$. The result is given by

$$B = \begin{bmatrix} g & d \\ a & b \\ e & f \end{bmatrix}$$

Notice that B is the matrix obtained by switching rows 1 and 2 of A. Therefore by multiplying A by P^{12}, the row operation which was applied to I to obtain P^{12} is applied to A to obtain B.

Theorem 2.3.1 applies to all three row operations, and we now look at the row operation of multiplying a row by a scalar. Consider the following lemma.

Lemma 2.3.2: Multiplication by a Scalar and Elementary Matrices

Let $E(k,i)$ denote the elementary matrix corresponding to the row operation in which the ith row is multiplied by the nonzero scalar, k. Then

$$E(k,i)A = B$$

where B is obtained from A by multiplying the ith row of A by k.

We will explore this lemma further in the following example.

Example 2.3.2: Multiplication of a Row by 5 Using Elementary Matrix

Let

$$E(5,2)=\begin{bmatrix}1&0&0\\0&5&0\\0&0&1\end{bmatrix}, A=\begin{bmatrix}a&b\\c&d\\e&f\end{bmatrix}$$

Find the matrix B where $B=E(5,2)A$

Solution

You can see that $E(5,2)$ is obtained by multiplying the second row of the identity matrix by 5.

Using our usual procedure for multiplication of matrices, we can compute the product $E(5,2)A$. The resulting matrix is given by

$$B=\begin{bmatrix}a&b\\5c&5d\\e&f\end{bmatrix}$$

Notice that B is obtained by multiplying the second row of A by the scalar 5.

There is one last row operation to consider. The following lemma discusses the final operation of adding a multiple of a row to another row.

Lemma 2.3.3: Adding Multiples of Rows and Elementary Matrices

Let $E(k\times i+j)$ denote the elementary matrix obtained from I by adding k times the ith row to the jth. Then

$$E(k\times i+j)A=B$$

where B is obtained from A by adding k times the ith row to the jth row of A.

Consider the following example.

Example 2.3.3: Adding Two Times the First Row to the Last

Let

$$E(2\times 1+3)=\begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix}, A=\begin{bmatrix}a&b\\c&d\\e&f\end{bmatrix}$$

Find B where $B=E(2\times 1+3)A$.

Solution

You can see that the matrix $E(2\times 1+3)$ was obtained by adding 2 times the first row of I to the third row of I.

Using our usual procedure, we can compute the product $E(2\times 1+3)A$. The resulting matrix B is given by

$$B=\begin{bmatrix}a&b\\c&d\\2a+e&2b+f\end{bmatrix}$$

You can see that B is the matrix obtained by adding 2 times the first row of A to the third row.

Suppose we have applied a row operation to a matrix A. Consider the row operation required to return A to its original form, to undo the row operation. It turns out that this action is how we find the inverse of an elementary matrix E.

Consider the following theorem.

Theorem 2.3.2: Elementary Matrices and Inverses

Every elementary matrix is invertible and its inverse is also an elementary matrix.

In fact, the inverse of an elementary matrix is constructed by doing the *reverse* row operation on I. E^{-1} will be obtained by performing the row operation which would carry E back to I.

- If E is obtained by switching rows i and j, then E^{-1} is also obtained by switching rows i and j.
- If E is obtained by multiplying row i by the scalar k, then E^{-1} is obtained by multiplying row i by the scalar $\frac{1}{k}$.
- If E is obtained by adding k times row i to row j, then E^{-1} is obtained by subtracting k times row i from row j.

Consider the following example.

Example 2.3.4: Inverse of an Elementary Matrix

Let

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Find E^{-1}.

Solution

Consider the elementary matrix E given by

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Here, E is obtained from the 2×2 identity matrix by multiplying the second row by 2. In order to carry E back to the identity, we need to multiply the second row of E by $\frac{1}{2}$. Hence,

E^{-1} is given by

$$E^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

We can verify that $EE^{-1} = I$. Take the product EE^{-1}, given by

$$EE^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This equals I so we know that we have compute E^{-1} properly.

Suppose an $m \times n$ matrix A is row reduced to its reduced row-echelon form. By tracking each row operation completed, this row reduction can be completed through multiplication by elementary matrices.

Consider the following definition.

Definition 2.3.2: The Form $B = UA$

Let A be an $m \times n$ matrix and let B be the reduced row-echelon form of A. Then we can write $B = UA$ where U is the product of all elementary matrices representing the row operations done to A to obtain B.

Consider the following example.

Example 2.3.5: The Form $B = UA$

Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 0 \end{bmatrix}$. Find B, the reduced row-echelon form of A and write it in the form $B = UA$.

Solution

To find B, row reduce A. For each step, we will record the appropriate elementary matrix. First, switch rows 1 and 2.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 0 \end{bmatrix}$$

The resulting matrix is equivalent to finding the product of $P^{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and A.

Next, add (-2) times row 1 to row 3.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

This is equivalent to multiplying by the matrix $E(-2 \times 1 + 3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$. Notice that the resulting matrix is B, the required reduced row-echelon form of A.

We can then write

$$\begin{aligned} B &= E(-2 \times 1 + 2)\left(P^{12}A\right) \\ &= \left(E(-2 \times 1 + 2)P^{12}\right)A \\ &= UA \end{aligned} \tag{2.3.1}$$

It remains to find the matrix U.

$$\begin{aligned} U &= E(-2 \times 1 + 2)P^{12} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix} \end{aligned} \tag{2.3.2}$$

We can verify that $B = UA$ holds for this matrix U:

$$\begin{aligned} UA &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \\ &= B \end{aligned} \tag{2.3.3}$$

While the process used in the above example is reliable and simple when only a few row operations are used, it becomes cumbersome in a case where many row operations are needed to carry A to B. The following theorem provides an alternate way to find the matrix U.

Theorem 2.3.3: Finding the Matrix U

Let A be an $m \times n$ matrix and let B be its reduced row-echelon form. Then $B = UA$ where U is an invertible $m \times m$ matrix found by forming the matrix $[A|I_m]$ and row reducing to $[B|U]$.

Let's revisit the above example using the process outlined in Theorem 2.3.3.

Example 2.3.6: The Form $B = UA$, Revisited

Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 0 \end{bmatrix}$. Using the process outlined in Theorem 2.3.3, find U such that $B = UA$.

Solution

First, set up the matrix $[A|I_m]$.

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Now, row reduce this matrix until the left side equals the reduced row-echelon form of A.

$$\begin{aligned} \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \end{bmatrix} &\rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 \end{bmatrix} \\ &\rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 \end{bmatrix} \end{aligned} \tag{2.3.4}$$

The left side of this matrix is B, and the right side is U. Comparing this to the matrix U found above in Example 2.3.5, you can see that the same matrix is obtained regardless of which process is used.

Recall from Algorithm 2.2.1 that an $n \times n$ matrix A is invertible if and only if A can be carried to the $n \times n$ identity matrix using the usual row operations. This leads to an important consequence related to the above discussion.

Suppose A is an $n \times n$ invertible matrix. Then, set up the matrix $[A|I_n]$ as done above, and row reduce until it is of the form $[B|U]$. In this case, $B = I_n$ because A is invertible.

$$\begin{aligned} B &= UA \\ I_n &= UA \\ U^{-1} &= A \end{aligned} \tag{2.3.5}$$

Now suppose that $U = E_1 E_2 \cdots E_k$ where each E_i is an elementary matrix representing a row operation used to carry A to I. Then,

$$U^{-1} = (E_1 E_2 \cdots E_k)^{-1} = E_k^{-1} \cdots E_2^{-1} E_1^{-1}$$

Remember that if E_i is an elementary matrix, so too is E_i^{-1}. It follows that

$$\begin{aligned} A &= U^{-1} \\ &= E_k^{-1} \cdots E_2^{-1} E_1^{-1} \end{aligned} \tag{2.3.6}$$

and A can be written as a product of elementary matrices.

Theorem 2.3.4: Product of Elementary Matrices

Let A be an $n \times n$ matrix. Then A is invertible if and only if it can be written as a product of elementary matrices.

Consider the following example.

Example 2.3.7: Product of Elementary Matrices

Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$. Write A as a product of elementary matrices.

Solution

We will use the process outlined in Theorem 2.3.3 to write A as a product of elementary matrices. We will set up the matrix $[A|I]$ and row reduce, recording each row operation as an elementary matrix.

First:

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{bmatrix}$$

represented by the elementary matrix $E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Secondly:

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{bmatrix}$$

represented by the elementary matrix $E_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Finally:

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{bmatrix}$$

represented by the elementary matrix $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$.

Notice that the reduced row-echelon form of A is I. Hence $I = UA$ where U is the product of the above elementary matrices. It follows that $A = U^{-1}$. Since we want to write A as a product of elementary matrices, we wish to express U^{-1} as a product of elementary matrices.

$$\begin{aligned} U^{-1} &= (E_3E_2E_1)^{-1} \\ &= E_1^{-1}E_2^{-1}E_3^{-1} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \\ &= A \end{aligned} \tag{2.3.7}$$

This gives A written as a product of elementary matrices. By Theorem 2.3.4 it follows that A is invertible.

More on Matrix Inverses

In this section, we will prove three theorems which will clarify the concept of matrix inverses. In order to do this, first recall some important properties of elementary matrices.

Recall that an elementary matrix is a square matrix obtained by performing an elementary operation on an identity matrix. Each elementary matrix is invertible, and its inverse is also an elementary matrix. If E is an $m \times m$ elementary matrix and A is an $m \times n$ matrix, then the product EA is the result of applying to A the same elementary row operation that was applied to the $m \times m$ identity matrix in order to obtain E.

Let R be the reduced row-echelon form of an $m \times n$ matrix A. R is obtained by iteratively applying a sequence of elementary row operations to A. Denote by $E_1, E_2, \cdots, E_k$ the elementary matrices associated with the elementary row operations which were applied, in order, to the matrix A to obtain the resulting R. We then have that $R = (E_k \cdots (E_2 (E_1 A))) = E_k \cdots E_2 E_1 A$. Let E denote the product matrix $E_k \cdots E_2 E_1$ so that we can write $R = EA$ where E is an invertible matrix whose inverse is the product $(E_1)^{-1}(E_2)^{-1} \cdots (E_k)^{-1}$.

Now, we will consider some preliminary lemmas.

Lemma 2.3.4: Invertible Matrix and Zeros

Suppose that A and B are matrices such that the product AB is an identity matrix. Then the reduced row-echelon form of A does not have a row of zeros.

Proof available at the end of this section.

We now consider a second important lemma.

Lemma 2.3.5: Size of Invertible Matrix

Suppose that A and B are matrices such that the product AB is an identity matrix. Then A has at least as many columns as it has rows.

Proof available at the end of this section.

An important theorem follows from this lemma.

Theorem 2.3.5: Invertible Matrices are Square

Only square matrices can be invertible.

Proof available at the end of this section.

Of course, not all square matrices are invertible. In particular, zero matrices are not invertible, along with many other square matrices.

The following proposition will be useful in proving the next theorem.

Proposition 2.3.1: Reduced Row-Echelon Form of a Square Matrix

If R is the reduced row-echelon form of a square matrix, then either R has a row of zeros or R is an identity matrix.

The proof of this proposition is left as an exercise to the reader. We now consider the second important theorem of this section.

Theorem 2.3.6: Unique Inverse of a Matrix

Suppose A and B are square matrices such that $AB = I$ where I is an identity matrix. Then it follows that $BA = I$. Further, both A and B are invertible and $B = A^{-1}$ and $A = B^{-1}$.

Proof available at the end of this section.

This theorem is very useful, since with it we need only test one of the products AB or BA in order to check that B is the inverse of A. The hypothesis that A and B are square matrices is very important, and without this the theorem does not hold.

We will now consider an example.

Example 2.3.8: Non Square Matrices

Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

Show that $A^T A = I$ but $AA^T \neq 0$.

Solution

Consider the product $A^T A$ given by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{2.3.8}$$

Therefore, $A^T A = I_2$, where I_2 is the 2×2 identity matrix. However, the product AA^T is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.3.9}$$

Hence AA^T is not the 3×3 identity matrix. This shows that for Theorem 2.3.2, it is essential that both matrices be square and of the same size.

Is it possible to have matrices A and B such that $AB = I$, while $BA = 0$? This question is left to the reader to answer, and you should take a moment to consider the answer.

We conclude this section with an important theorem.

Theorem 2.3.7: The reduced row-echelon form of an Invertible Matrix

For any matrix A the following conditions are equivalent:

- A is invertible
- The reduced row-echelon form of A is an identity matrix

Proof available at the end of this section.

Theorem 2.3.3 corresponds to Algorithm 2.2.1, which claims that A^{-1} is found by row reducing the augmented matrix $[A|I]$ to the form $[I|A^{-1}]$. This will be a matrix product $E[A|I]$ where E is a product of elementary matrices. By the rules of matrix multiplication, we have that $E[A|I] = [EA|EI] = [EA|E]$.

It follows that the reduced row-echelon form of $[A|I]$ is $[EA|E]$, where EA gives the reduced row-echelon form of A. By Theorem 2.3.3, if $EA \neq I$, then A is not invertible, and if $EA = I$, A is invertible. If $EA = I$, then by Theorem 2.3.2, $E = A^{-1}$. This proves that Algorithm 2.2.1 does in fact find A^{-1}.

Answers

Lemma 2.3.4: Invertible Matrix and Zeros

Let R be the reduced row-echelon form of A. Then $R = EA$ for some invertible square matrix E as described above. By hypothesis $AB = I$ where I is an identity matrix, so we have a chain of equalities

$$R(BE^{-1}) = (EA)(BE^{-1}) = E(AB)E^{-1} = EIE^{-1} = EE^{-1} = I$$

If R would have a row of zeros, then so would the product $R(BE^{-1})$. But since the identity matrix I does not have a row of zeros, neither can R have one.

Lemma 2.3.5: Size of Invertible Matrix

Let R be the reduced row-echelon form of A. By Lemma 2.3.1, we know that R does not have a row of zeros, and therefore each row of R has a leading 1. Since each column of R contains at most one of these leading 1s, R must have at least as many columns as it has rows.

Theorem 2.3.5: Invertible Matrices are Square

Suppose that A and B are matrices such that both products AB and BA are identity matrices. We will show that A and B must be square matrices of the same size. Let the matrix A have m rows and n columns, so that A is an $m \times n$ matrix. Since the product AB exists, B must have n rows, and since the product BA exists, B must have m columns so that B is an $n \times m$ matrix. To finish the proof, we need only verify that $m = n$.

We first apply Lemma 2.3.2 with A and B, to obtain the inequality $m \leq n$. We then apply Lemma 2.3.2 again (switching the order of the matrices), to obtain the inequality $n \leq m$. It follows that $m = n$, as we wanted.

Theorem 2.3.6: Unique Inverse of a Matrix

Let R be the reduced row-echelon form of a square matrix A. Then, $R = EA$ where E is an invertible matrix. Since $AB = I$, Lemma 2.3.1 gives us that R does not have a row of zeros. By noting that R is a square matrix and applying Proposition 2.3.1, we see that $R = I$. Hence, $EA = I$.

Using both that $EA = I$ and $AB = I$, we can finish the proof with a chain of equalities as given by

$$\begin{aligned} BA = IBIA &= (EA)B(E^{-1}E)A \\ &= E(AB)E^{-1}(EA) \\ &= EIE^{-1}I \\ &= EE^{-1} = I \end{aligned} \tag{2.3.10}$$

It follows from the definition of the inverse of a matrix that $B = A^{-1}$ and $A = B^{-1}$.

Theorem 2.3.7: The reduced row-echelon form of an Invertible Matrix

In order to prove this, we show that for any given matrix A, each condition implies the other. We first show that if A is invertible, then its reduced row-echelon form is an identity matrix, then we show that if the reduced row-echelon form of A is an identity matrix, then A is invertible.

If A is invertible, there is some matrix B such that $AB = I$. By Lemma 2.3.1, we get that the of A does not have a row of zeros. Then by Theorem 2.3.1, it follows that A and the reduced row-echelon form of A are square matrices. Finally, by Proposition 2.3.1, this reduced row-echelon form of A must be an identity matrix. This proves the first implication.

Now suppose the reduced row-echelon form of A is an identity matrix I. Then $I = EA$ for some product E of elementary matrices. By Theorem 2.3.2, we can conclude that A is invertible.

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2.3E: Exercises for Section 2.3

Exercise 2.3E.1

Let $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$. Suppose a row operation is applied to A and the result is $B = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$. Find the elementary matrix E that represents this row operation.

Exercise 2.3E.2

Let $A = \begin{bmatrix} 4 & 0 \\ 2 & 1 \end{bmatrix}$. Suppose a row operation is applied to A and the result is $B = \begin{bmatrix} 8 & 0 \\ 2 & 1 \end{bmatrix}$. Find the elementary matrix E that represents this row operation.

Exercise 2.3E.3

Let $A = \begin{bmatrix} 1 & -3 \\ 0 & 5 \end{bmatrix}$. Suppose a row operation is applied to A and the result is $B = \begin{bmatrix} 1 & -3 \\ 2 & -1 \end{bmatrix}$. Find the elementary matrix E that represents this row operation.

Exercise 2.3E.4

Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 1 \\ 2 & -1 & 4 \end{bmatrix}$. Suppose a row operation is applied to A and the result is $B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 4 \\ 0 & 5 & 1 \end{bmatrix}$.

a. Find the elementary matrix E such that $EA = B$.
b. Find the inverse of E, E^{-1}, such that $E^{-1}B = A$.

Exercise 2.3E.5

Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 1 \\ 2 & -1 & 4 \end{bmatrix}$. Suppose a row operation is applied to A and the result is $B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 10 & 2 \\ 2 & -1 & 4 \end{bmatrix}$.

a. Find the elementary matrix E such that $EA = B$.
b. Find the inverse of E, E^{-1}, such that $E^{-1}B = A$.

Exercise 2.3E.6

Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 1 \\ 2 & -1 & 4 \end{bmatrix}$. Suppose a row operation is applied to A and the result is $B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 1 \\ 1 & -\frac{1}{2} & 2 \end{bmatrix}$.

a. Find the elementary matrix E such that $EA = B$.
b. Find the inverse of E, E^{-1}, such that $E^{-1}B = A$.

Exercise $2.3E.7$

Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 1 \\ 2 & -1 & 4 \end{bmatrix}$. Suppose a row operation is applied to A and the result is $B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 5 \\ 2 & -1 & 4 \end{bmatrix}$.

a. Find the elementary matrix E such that $EA = B$.
b. Find the inverse of E, E^{-1}, such that $E^{-1}B = A$.

Exercise $2.3E.8$

Prove Theorem 2.3.2: Every elementary matrix is invertible and its inverse is also an elementary matrix.

Exercise $2.3E.9$

Suppose $E_1, E_2, \dots, E_k$ are elementary matrices. Prove that the product $E_1E_2 \dots E_k$ is invertible.

Exercise $2.3E.10$

Suppose $EA = B$ for some elementary matrix E. Show that the system $Ax = b$ has the same solution set as the system $Bx = Eb$.

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2.4: LU Factorization

Learning Objectives

- Find an LU factorization of a matrix A, if possible
- Use LU Factorization of A to solve a linear system $AX = B$

LU Factorization

Also known as LU decomposition, an LU factorization of a matrix involves writing the given matrix as the product of a lower triangular matrix L which has the main diagonal consisting entirely of ones, and an upper triangular matrix U in the indicated order. This is the version discussed here but it is sometimes the case that the L has numbers other than 1 down the main diagonal. It is still a useful concept. The L goes with "lower" and the U with "upper".

It turns out many matrices can be written in this way and when this is possible, people get excited about slick ways of solving the system of equations, $AX = B$. It is for this reason that you want to study the LU factorization. It allows you to work only with triangular matrices. It turns out that it takes about half as many operations to obtain an LU factorization as it does to find the row reduced echelon form.

First it should be noted not all matrices have an LU factorization and so we will emphasize the techniques for achieving it rather than formal proofs.

Example 2.4.1: A Matrix with NO LU Factorization

Can you write $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ in the form LU as just described?

Solution

To do so you would need

$$\begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} a & b \\ xa & xb+c \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Therefore, $b = 1$ and $a = 0$. Also, from the bottom rows, $xa = 1$ which can't happen and have $a = 0$. Therefore, you can't write this matrix in the form It has no LU factorization. This is what we mean above by saying the method lacks generality.

Nevertheless the method is often extremely useful, and we will describe below one the many methods used to produce an LU factorization when possible.

Finding An LU Factorization By Inspection

Which matrices have an LU factorization? It turns out it is those whose row-echelon form can be achieved without switching rows. In other words matrices which only involve using row operations of type 2 or 3 to obtain the row-echelon form.

Example 2.4.2: An LU factorization

Find an LU factorization of $A = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 1 & 3 & 2 & 1 \\ 2 & 3 & 4 & 0 \end{bmatrix}$.

Solution

One way to find the LU factorization is to simply look for it directly. You need

$$\begin{bmatrix} 1 & 2 & 0 & 2 \\ 1 & 3 & 2 & 1 \\ 2 & 3 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & z & 1 \end{bmatrix} \begin{bmatrix} a & d & h & j \\ 0 & b & e & i \\ 0 & 0 & c & f \end{bmatrix}.$$

Then multiplying these you get

$$\begin{bmatrix} a & d & h & j \\ xa & xd+b & xh+e & xj+i \\ ya & yd+zb & yh+ze+c & yj+iz+f \end{bmatrix}$$

and so you can now tell what the various quantities equal. From the first column, you need $a = 1, x = 1, y = 2$. Now go to the second column. You need $d = 2, xd + b = 3$ so $b = 1, yd + zb = 3$ so $z = -1$. From the third column, $h = 0, e = 2, c = 6$. Now from the fourth column, $j = 2, i = -1, f = -5$. Therefore, an LU factorization is

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 6 & -5 \end{bmatrix}.$$

You can check whether you got it right by simply multiplying these two.

LU Factorization, Multiplier Method

Remember that for a matrix A to be written in the form $A = LU$, you must be able to reduce it to its row-echelon form without interchanging rows. The following method gives a process for calculating the LU factorization of such a matrix A.

Example 2.4.3: LU factorization

Find an LU factorization for

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ -2 & 3 & -2 \end{bmatrix}$$

Solution

Write the matrix as the following product.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ -2 & 3 & -2 \end{bmatrix}$$

In the matrix on the right, begin with the left row and zero out the entries below the top using the row operation which involves adding a multiple of a row to another row. You do this and also update the matrix on the left so that the product will be unchanged. Here is the first step. Take -2 times the top row and add to the second. Then take 2 times the top row and add to the second in the matrix on the left.

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ -2 & 3 & -2 \end{bmatrix}$$

The next step is to take 2 times the top row and add to the bottom in the matrix on the right. To ensure that the product is unchanged, you place a in the bottom left in the matrix on the left. Thus the next step yields

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & 7 & 4 \end{bmatrix}$$

Next take 7 times the middle row on right and add to bottom row. Updating the matrix on the left in a similar manner to what was done earlier,

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & -7 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & 0 & -31 \end{bmatrix}$$

At this point, stop. You are done.

The method just described is called the **Multiplier Method**.

Below is a video on the LU factorization using the Multiplier Method, invoking elementary matrices corresponding to the elementary row operations.

Example 2.4.4

Determine the LU decomposition for

$$A = \begin{bmatrix} 4 & -16 \\ 3 & -8 \end{bmatrix} \tag{2.4.1}$$

Solution

- **Video Length**: 10 minutes 22 seconds.
- **Context**: This video demonstrates performing the LU decomposition using elementary matrices.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

There is a shortcut to finding an LU decomposition. Below is a video that demonstrates this method.

Example 2.4.4

Determine the LU decomposition for

$$A = \begin{bmatrix} 1 & 4 & -3 \\ -2 & 8 & 5 \\ 3 & 4 & 7 \end{bmatrix} \tag{2.4.2}$$

Solution

- **Video Length**: 9 minutes 34 seconds
- **Context**: This video demonstrates performing the LU decomposition using a "shortcut" method.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Solving Systems using LU Factorization

One reason people care about the LU factorization is it allows the quick solution of systems of equations. Here is an example.

Example 2.4.4: LU factorization to Solve Equations

Suppose you want to find the solutions to

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 4 & 3 & 1 & 1 \\ 1 & 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Solution

Of course one way is to write the augmented matrix and grind away. However, this involves more row operations than the computation of the LU factorization and it turns out that the LU factorization can give the solution quickly. Here is how. The following is an LU factorization for the matrix.

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 4 & 3 & 1 & 1 \\ 1 & 2 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -5 & -11 & -7 \\ 0 & 0 & 0 & -2 \end{bmatrix}.$$

Let $UX = Y$ and consider $LY = B$ where in this case, $B = [1, 2, 3]^T$. Thus

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

which yields very quickly that $Y = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$.

Now you can find X by solving $UX = Y$. Thus in this case,

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -5 & -11 & -7 \\ 0 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

which yields

$$X = \begin{bmatrix} -\frac{3}{5} + \frac{7}{5}t \\ \frac{9}{5} - \frac{11}{5}t \\ t \\ -1 \end{bmatrix}, \quad t \in \mathbb{R}.$$

Justification for the Multiplier Method

Why does the multiplier method work for finding the LU factorization? Suppose A is a matrix which has the property that the row-echelon form for A may be achieved without switching rows. Thus every row which is replaced using this row operation in obtaining the row-echelon form may be modified by using a row which is above it.

Lemma 2.4.1: Multiplier Method and Triangular Matrices

Let L be a lower (upper) triangular matrix $m \times m$ which has ones down the main diagonal. Then L^{-1} also is a lower (upper) triangular matrix which has ones down the main diagonal. In the case that L is of the form

$$L = \begin{bmatrix} 1 & & & \\ a_1 & 1 & & \\ \vdots & & \ddots & \\ a_n & & & 1 \end{bmatrix} \tag{2.4.3}$$

where all entries are zero except for the left column and main diagonal, it is also the case that L^{-1} is obtained from L by simply multiplying each entry below the main diagonal in L with -1. The same is true if the single nonzero column is in another position.

Proof available at the end of this section.

For a simple illustration of the last claim,

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & a & 1 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -a & 1 \end{bmatrix}$$

Now let A be an $m \times n$ matrix, say

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

and assume A can be row reduced to an upper triangular form using only row operation 3. Thus, in particular, $a_{11} \neq 0$. Multiply on the left by $E_1 =$

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ -\frac{a_{21}}{a_{11}} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{a_{m1}}{a_{11}} & 0 & \cdots & 1 \end{bmatrix}$$

This is the product of elementary matrices which make modifications in the first column only. It is equivalent to taking $-a_{21}/a_{11}$ times the first row and adding to the second. Then taking $-a_{31}/a_{11}$ times the first row and adding to the third and so forth. The quotients in the first column of the above matrix are the multipliers. Thus the result is of the form

$$E_1 A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a'_{1n} \\ 0 & a'_{22} & \cdots & a'_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & a'_{m2} & \cdots & a'_{mn} \end{bmatrix}$$

By assumption, $a'_{22} \neq 0$ and so it is possible to use this entry to zero out all the entries below it in the matrix on the right by multiplication by a matrix of the form $E_2 = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & E \end{bmatrix}$ where E is an $[m-1] \times [m-1]$ matrix of the form

$$E = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -\frac{a'_{32}}{a'_{22}} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{a'_{m2}}{a'_{22}} & 0 & \cdots & 1 \end{bmatrix}$$

Again, the entries in the first column below the 1 are the multipliers. Continuing this way, zeroing out the entries below the diagonal entries, finally leads to

$$E_{m-1}E_{n-2}\cdots E_1 A = U$$

where U is upper triangular. Each E_j has all ones down the main diagonal and is lower triangular. Now multiply both sides by the inverses of the E_j in the reverse order. This yields

$$A = E_1^{-1}E_2^{-1}\cdots E_{m-1}^{-1}U$$

By Lemma 2.4.1, this implies that the product of those E_j^{-1} is a lower triangular matrix having all ones down the main diagonal.

The above discussion and lemma gives the justification for the multiplier method. The expressions

$$\frac{-a_{21}}{a_{11}}, \frac{-a_{31}}{a_{11}}, \cdots, \frac{-a_{m1}}{a_{11}}$$

denoted respectively by $M_{21}, \cdots, M_{m1}$ to save notation which were obtained in building E_1 are the multipliers. Then according to the lemma, to find E_1^{-1} you simply write

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ -M_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -M_{m1} & 0 & \cdots & 1 \end{bmatrix}$$

Similar considerations apply to the other E_j^{-1}. Thus L is a product of the form

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ -M_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -M_{m1} & 0 & \cdots & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & -M_{m[m-1]} & 1 \end{bmatrix}$$

each factor having at most one nonzero column, the position of which moves from left to right in scanning the above product of matrices from left to right. It follows from what we know about the effect of multiplying on the left by an elementary matrix that the above product is of the form

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ -M_{21} & 1 & \cdots & 0 & 0 \\ \vdots & -M_{32} & \ddots & \vdots & \vdots \\ -M_{[m-1]1} & \vdots & \cdots & 1 & 0 \\ -M_{m1} & -M_{m2} & \cdots & -M_{m[m-1]} & 1 \end{bmatrix}$$

In words, beginning at the left column and moving toward the right, you simply insert, into the corresponding position in the identity matrix, -1 times the multiplier which was used to zero out an entry in that position below the main diagonal in A, while retaining the main diagonal which consists entirely of ones. This is L.

Answers

Lemma 2.4.1: Multiplier Method and Triangular Matrices

Consider the usual setup for finding the inverse $\begin{bmatrix} L & I \end{bmatrix}$. Then each row operation done to L to reduce to row reduced echelon form results in changing only the entries in I below the main diagonal. In the special case of L given in (2.4.3) or the single nonzero column is in another position, multiplication by -1 as described in the lemma clearly results in L^{-1}.

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2.4E: Exercises for Section 2.4

Exercise 2.4E.1

Find an LU factorization of $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 3 \\ 1 & 2 & 3 \end{bmatrix}$.

Answer available at the end of this section.

Exercise 2.4E.2

Find an LU factorization of $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 1 & 3 & 2 & 1 \\ 5 & 0 & 1 & 3 \end{bmatrix}$.

Answer available at the end of this section.

Exercise 2.4E.3

Find an LU factorization of the matrix $\begin{bmatrix} 1 & -2 & -5 & 0 \\ -2 & 5 & 11 & 3 \\ 3 & -6 & -15 & 1 \end{bmatrix}$.

Answer available at the end of this section.

Exercise 2.4E.4

Find an LU factorization of the matrix $\begin{bmatrix} 1 & -1 & -3 & -1 \\ -1 & 2 & 4 & 3 \\ 2 & -3 & -7 & -3 \end{bmatrix}$.

Answer available at the end of this section.

Exercise 2.4E.5

Find an LU factorization of the matrix $\begin{bmatrix} 1 & -3 & -4 & -3 \\ -3 & 10 & 10 & 10 \\ 1 & -6 & 2 & -5 \end{bmatrix}$.

Answer available at the end of this section.

Exercise 2.4E.6

Find an LU factorization of the matrix $\begin{bmatrix} 1 & 3 & 1 & -1 \\ 3 & 10 & 8 & -1 \\ 2 & 5 & -3 & -3 \end{bmatrix}$.

Answer available at the end of this section.

Exercise 2.4E.7

Find an LU factorization of the matrix $\begin{bmatrix} 3 & -2 & 1 \\ 9 & -8 & 6 \\ -6 & 2 & 2 \\ 3 & 2 & -7 \end{bmatrix}$.

Answer available at the end of this section.

Exercise 2.4E.8

Find an LU factorization of the matrix $\begin{bmatrix} -3 & -1 & 3 \\ 9 & 9 & -12 \\ 3 & 19 & -16 \\ 12 & 40 & -26 \end{bmatrix}$.

Exercise 2.4E.9

Find an LU factorization of the matrix $\begin{bmatrix} -1 & -3 & -1 \\ 1 & 3 & 0 \\ 3 & 9 & 0 \\ 4 & 12 & 16 \end{bmatrix}$.

Answer available at the end of this section.

Exercise 2.4E.10

Find the LU factorization of the coefficient matrix and use it to solve the system of equations.

$$x + 2y = 5$$
$$2x + 3y = 6$$

Answer available at the end of this section.

Exercise 2.4E.11

Find the LU factorization of the coefficient matrix and use it to solve the system of equations.

$$x + 2y + z = 1$$
$$y + 3z = 2$$
$$2x + 3y = 6$$

Answer available at the end of this section.

Exercise 2.4E.12

Find the LU factorization of the coefficient matrix and use it to solve the system of equations.

$$x + 2y + 3z = 5$$
$$2x + 3y + z = 6$$
$$x - y + z = 2$$

Exercise $2.4E.13$

Find the LU factorization of the coefficient matrix and use it to solve the system of equations.

$$x + 2y + 3z = 5$$
$$2x + 3y + z = 6$$
$$3x + 5y + 4z = 11$$

Answer available at the end of this section.

Exercise $2.4E.14$

Find the LU factorization of the coefficient matrix and use it to solve the system of equations.

$$x + y + z = 6$$
$$2x + 3y + 5z = 15$$
$$4x + 5y + 6z = 24$$

Answer available at the end of this section.

Answer available at the end of this section.

Exercise $2.4E.15$

Not all linear systems can be solved using LU factorization. What happens when you try to use LU factorization of the coefficient matrix and use it to solve the following system of equations? \[

$$\begin{aligned} 2x + 4y + z &= 7 \\ 4x + 8y + 2z &= 14 \\ -6x + -9y + 3z &= -3 \end{aligned} \tag{2.4E.1}$$

\nonumber \

Exercise $2.4E.16$

Is there only one LU factorization for a given matrix? **Hint:** Consider the equation

$$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Look for all possible LU factorizations.

Answer available at the end of this section.

Exercise $2.4E.17$

Let A be an $n \times n$ matrix that admits an LU factorization, meaning there exist a lower triangular matrix L and an upper triangular matrix U such that $A = LU$. Prove that if L is **unit lower triangular** (i.e., all its diagonal entries are 1), then the LU factorization of A is unique.

Hint available at the end of this section.

Answers

Exercise $2.4E.1$

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 3 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -3 & 3 \\ 0 & 0 & 3 \end{bmatrix}$$

Exercise $2.4E.2$

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 1 & 3 & 2 & 1 \\ 5 & 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 5 & -10 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -24 & -17 \end{bmatrix}$$

Exercise $2.4E.3$

$$\begin{bmatrix} 1 & -2 & -5 & 0 \\ -2 & 5 & 11 & 3 \\ 3 & -6 & -15 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -5 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Exercise $2.4E.4$

$$\begin{bmatrix} 1 & -1 & -3 & -1 \\ -1 & 2 & 4 & 3 \\ 2 & -3 & -7 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -3 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Exercise $2.4E.5$

$$\begin{bmatrix} 1 & -3 & -4 & -3 \\ -3 & 10 & 10 & 10 \\ 1 & -6 & 2 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 1 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & -4 & -3 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Exercise $2.4E.6$

$$\begin{bmatrix} 1 & 3 & 1 & -1 \\ 3 & 10 & 8 & -1 \\ 2 & 5 & -3 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 & -1 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Exercise $2.4E.7$

$$\begin{bmatrix} 3 & -2 & 1 \\ 9 & -8 & 6 \\ -6 & 2 & 2 \\ 3 & 2 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ 1 & -2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 & 1 \\ 0 & -2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Exercise $2.4E.9$

$$\begin{bmatrix} -1 & -3 & -1 \\ 1 & 3 & 0 \\ 3 & 9 & 0 \\ 4 & 12 & 16 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ -4 & 0 & -4 & 1 \end{bmatrix} \begin{bmatrix} -1 & -3 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

Exercise $2.4E.10$

An LU factorization of the coefficient matrix is

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$$

First solve

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

which gives $\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$. Then solve

$$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$$

which says that $y = 4$ and $x = -3$.

Exercise $2.4E.11$

An LU factorization of the coefficient matrix is

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 2 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

First solve

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$$

which yields $u = 1, v = 2, w = 6$. Next solve

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$$

This yields $z = 6, y = -16, x = 27$.

Exercise $2.4E.13$

An LU factorization of the coefficient matrix is

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 5 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

First solve

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 11 \end{bmatrix}$$

Solution is: $\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ 0 \end{bmatrix}$. Next solve

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ 0 \end{bmatrix}$$

Solution is: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7t - 3 \\ 4 - 5t \\ t \end{bmatrix}, t \in \mathbb{R}$.

Exercise $2.4E.14$

Exercise $2.4E.14$

Add texts here. Do not delete this text first.

Exercise $2.4E.16$

Sometimes there is more than one LU factorization as is the case in this example. The given equation clearly gives an LU factorization. However, it appears that the following equation gives another LU factorization.

$$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Exercise $2.4E.17$

Begin by assuming that A has two LU factorizations: L_1U_1 and L_2U_2, where L_1 and L_2 are unit lower triangular. Then, through using inverses, arrive at a conclusion that $L_1 = L_2$ and $U_1 = U_2$.

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3: DETERMINANTS

3.1: Basic Techniques

3.1E: Exercises for Section 3.1

3.2: Properties of Determinants

3.2E: Exercises for Section 3.2

3.3: Application of the Determinant to Inverses; Cramer's Rule

3.3E: Exercises for Section 3.3

3.4: Determinants and Geometry

3.4E: Exercises for Section 3.4

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3.1: Basic Techniques

Learning Objectives

- Evaluate the determinant of a square matrix using Laplace Expansion (cofactor expansion)

Cofactors and 2 x 2 Determinants

Let A be an $n \times n$ matrix. That is, let A be a square matrix. The **determinant** of A, denoted by $\det(A)$ is a very important number which we will explore throughout this section.

If A is a 2×2 matrix, the determinant is given by the following formula.

Definition 3.1.1: Determinant of a Two By Two Matrix

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then

$$\det(A) = ad - cb$$

The determinant is also often denoted by enclosing the matrix with two vertical lines. Thus

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The following is an example of finding the determinant of a 2×2 matrix.

Example 3.1.1: A Two by Two Determinant

Find $\det(A)$ for the matrix $A = \begin{bmatrix} 2 & 4 \\ -1 & 6 \end{bmatrix}$.

Solution

From Definition 3.1.1,

$$\det(A) = (2)(6) - (-1)(4) = 12 + 4 = 16$$

The 2×2 determinant can be used to find the determinant of larger matrices. We will now explore how to find the determinant of a 3×3 matrix, using several tools including the 2×2 determinant.

We begin with the following definition.

Definition 3.1.2: The ij^{th} Minor of a Matrix

Let A be a 3×3 matrix. The ij^{th} **minor** of A, denoted as $\text{minor}(A)_{ij}$, is the determinant of the 2×2 matrix which results from deleting the ith row and the jth column of A.

In general, if A is an $n \times n$ matrix, then the ij^{th} minor of A is the determinant of the $n-1 \times n-1$ matrix which results from deleting the ith row and the jth column of A.

Hence, there is a minor associated with each entry of A. Consider the following example which demonstrates this definition.

Example 3.1.2: Finding Minors of a Matrix

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

Find $\text{minor}\,(A)_{12}$ and $\text{minor}\,(A)_{23}$.

Solution

First we will find $\text{minor}\,(A)_{12}$. By Definition 3.1.2, this is the determinant of the 2×2 matrix which results when you delete the first row and the second column. This minor is given by

$$\text{minor}\,(A)_{12} = \det \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$$

Using Definition 3.1.1, we see that

$$\det \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix} = (4)(1) - (3)(2) = 4 - 6 = -2$$

Therefore $\text{minor}\,(A)_{12} = -2$.

Similarly, $\text{minor}\,(A)_{23}$ is the determinant of the 2×2 matrix which results when you delete the second row and the third column. This minor is therefore

$$\text{minor}\,(A)_{23} = \det \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} = -4$$

Finding the other minors of A is left as an exercise.

The ij^{th} minor of a matrix A is used in another important definition, given next.

Definition 3.1.3: The ij^{th} Cofactor of a Matrix

Suppose A is an $n \times n$ matrix. The ij^{th} **cofactor**, denoted by $\text{cof}\,(A)_{ij}$ is defined to be

$$\text{cof}\,(A)_{ij} = (-1)^{i+j}\,\text{minor}\,(A)_{ij}$$

It is also convenient to refer to the cofactor of an entry of a matrix as follows. If a_{ij} is the ij^{th} entry of the matrix, then its cofactor is just $\text{cof}\,(A)_{ij}$.

Example 3.1.3: Finding Cofactors of a Matrix

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

Find $\text{cof}\,(A)_{12}$ and $\text{cof}\,(A)_{23}$.

Solution

We will use Definition 3.1.3 to compute these cofactors.

First, we will compute $\text{cof}\,(A)_{12}$. Therefore, we need to find $\text{minor}\,(A)_{12}$. This is the determinant of the 2×2 matrix which results when you delete the first row and the second column. Thus $\text{minor}\,(A)_{12}$ is given by

$$\det \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix} = -2$$

Then,

$$\text{cof}\,(A)_{12} = (-1)^{1+2}\,\text{minor}\,(A)_{12} = (-1)^{1+2}\,(-2) = 2$$

Hence, $\text{cof}\,(A)_{12} = 2$.

Similarly, we can find $\text{cof}\,(A)_{23}$. First, find $\text{minor}\,(A)_{23}$, which is the determinant of the 2×2 matrix which results when you delete the second row and the third column. This minor is therefore

$$\det \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} = -4$$

Hence,

$$\text{cof}\,(A)_{23} = (-1)^{2+3}\,\text{minor}\,(A)_{23} = (-1)^{2+3}\,(-4) = 4$$

You may wish to find the remaining cofactors for the above matrix. Remember that there is a cofactor for every entry in the matrix.

Watch the accompanying video for a visual demonstration of the this concept.

- **Video Length**: 6 minutes 44 seconds
- **Context**: This video demonstrates how to find the minors and cofactors of a 3×3 matrix: $\begin{bmatrix} 1 & 3 & 2 \\ -2 & 4 & 5 \\ 0 & -6 & 9 \end{bmatrix}$

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

We have now established the tools we need to find the determinant of a 3×3 matrix.

Definition 3.1.4: The Determinant of a Three By Three Matrix

Let A be a 3×3 matrix. Then, $\det(A)$ is calculated by picking a row (or column) and taking the product of each entry in that row (column) with its cofactor and adding these products together.

This process when applied to the ith row (column) is known as **expanding along the ith row (column)** as is given by

$$\det(A) = a_{i1}\text{cof}(A)_{i1} + a_{i2}\text{cof}(A)_{i2} + a_{i3}\text{cof}(A)_{i3}$$

When calculating the determinant, you can choose to expand any row or any column. Regardless of your choice, you will always get the same number which is the determinant of the matrix A. This method of evaluating a determinant by expanding along a row or a column is called **Laplace Expansion** or **Cofactor Expansion**.

Consider the following example.

Example 3.1.4: Finding the Determinant of a Three by Three Matrix

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

Find $\det(A)$ using the method of Laplace Expansion.

Solution

First, we will calculate $\det(A)$ by expanding along the first column. Using Definition 3.1.4, we take the 1 in the first column and multiply it by its cofactor,

$$1(-1)^{1+1}\begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} = (1)(1)(-1) = -1$$

Similarly, we take the 4 in the first column and multiply it by its cofactor, as well as with the 3 in the first column. Finally, we add these numbers together, as given in the following equation.

$$\det(A) = \overbrace{1(-1)^{1+1}\begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix}}^{\mathrm{cof}(A)_{11}} + \overbrace{4(-1)^{2+1}\begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix}}^{\mathrm{cof}(A)_{21}} + \overbrace{3(-1)^{3+1}\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix}}^{\mathrm{cof}(A)_{31}}$$

Calculating each of these, we obtain

$$\det(A) = 1(1)(-1) + 4(-1)(-4) + 3(1)(-5) = -1 + 16 + -15 = 0$$

Hence, $\det(A) = 0$.

As mentioned in Definition 3.1.4, we can choose to expand along any row or column. Let's try now by expanding along the second row. Here, we take the 4 in the second row and multiply it to its cofactor, then add this to the 3 in the second row multiplied by its cofactor, and the 2 in the second row multiplied by its cofactor. The calculation is as follows.

$$\det(A) = \overbrace{4(-1)^{2+1}\begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix}}^{\mathrm{cof}(A)_{21}} + \overbrace{3(-1)^{2+2}\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix}}^{\mathrm{cof}(A)_{22}} + \overbrace{2(-1)^{2+3}\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix}}^{\mathrm{cof}(A)_{23}}$$

Calculating each of these products, we obtain

$$\det(A) = 4(-1)(-2) + 3(1)(-8) + 2(-1)(-4) = 0$$

You can see that for both methods, we obtained $\det(A) = 0$.

Watch the accompanying video for a visual demonstration of the this concept.

- **Video Length**: 5 minutes 25 seconds
- **Context**: This video demonstrates how to find a 3×3 determinant using cofactor expansion: $A = \begin{bmatrix} 5 & 15 & 10 \\ -3 & -8 & -6 \\ -2 & -5 & -3 \end{bmatrix}$

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

As mentioned above, we will always come up with the same value for $\det(A)$ regardless of the row or column we choose to expand along. You should try to compute the above determinant by expanding along other rows and columns. This is a good way to check your work, because you should come up with the same number each time!

We present this idea formally in the following theorem.

Theorem 3.1.1: The Determinant is Well Defined

Expanding the $n \times n$ matrix along any row or column always gives the same answer, which is the determinant.

We have now looked at the determinant of 2×2 and 3×3 matrices. It turns out that the method used to calculate the determinant of a 3×3 matrix can be used to calculate the determinant of any sized matrix. Notice that Definition 3.1.2, Definition 3.1.3 and Definition 3.1.4 can all be applied to a matrix of any size.

For example, the ij^{th} minor of a 4×4 matrix is the determinant of the 3×3 matrix you obtain when you delete the ith row and the jth column. Just as with the 3×3 determinant, we can compute the determinant of a 4×4 matrix by Laplace Expansion, along any row or column

Consider the following example.

Example 3.1.5: Determinant of a Four by Four Matrix

Find $\det(A)$ where

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 4 & 2 & 3 \\ 1 & 3 & 4 & 5 \\ 3 & 4 & 3 & 2 \end{bmatrix}$$

Solution

As in the case of a 3×3 matrix, you can expand this along any row or column. Lets pick the third column. Then, using Laplace Expansion,

$$\det(A) = 3(-1)^{1+3}\begin{vmatrix} 5 & 4 & 3 \\ 1 & 3 & 5 \\ 3 & 4 & 2 \end{vmatrix} + 2(-1)^{2+3}\begin{vmatrix} 1 & 2 & 4 \\ 1 & 3 & 5 \\ 3 & 4 & 2 \end{vmatrix} +$$
$$4(-1)^{3+3}\begin{vmatrix} 1 & 2 & 4 \\ 5 & 4 & 3 \\ 3 & 4 & 2 \end{vmatrix} + 3(-1)^{4+3}\begin{vmatrix} 1 & 2 & 4 \\ 5 & 4 & 3 \\ 1 & 3 & 5 \end{vmatrix}$$

Now, you can calculate each 3×3 determinant using Laplace Expansion, as we did above. You should complete these as an exercise and verify that $\det(A) = -12$.

The following provides a formal definition for the determinant of an $n \times n$ matrix. You may wish to take a moment and consider the above definitions for 2×2 and 3×3 determinants in context of this definition.

Definition 3.1.5: The Determinant of an $n \times n$ Matrix

Let A be an $n \times n$ matrix where $n \geq 2$ and suppose the determinant of an $(n-1) \times (n-1)$ has been defined. Then

$$\det(A) = \sum_{j=1}^{n} a_{ij}\text{cof}(A)_{ij} = \sum_{i=1}^{n} a_{ij}\text{cof}(A)_{ij}$$

The first formula consists of expanding the determinant along the ith row and the second expands the determinant along the jth column.

In the following sections, we will explore some important properties and characteristics of the determinant.

The Determinant of a Triangular Matrix

There is a certain type of matrix for which finding the determinant is a very simple procedure. Consider the following definition.

Definition 3.1.6: Triangular Matrices

A matrix A is upper triangular if $a_{ij} = 0$ whenever $i > j$. Thus the entries of such a matrix below the main diagonal equal 0, as shown. Here, $*$ refers to any nonzero number.

$$\begin{bmatrix} * & * & \cdots & * \\ 0 & * & \cdots & \vdots \\ \vdots & \vdots & \ddots & * \\ 0 & \cdots & 0 & * \end{bmatrix}$$

A lower triangular matrix is defined similarly as a matrix for which all entries above the main diagonal are equal to zero.

The following theorem provides a useful way to calculate the determinant of a triangular matrix.

Theorem 3.1.2: Determinant of a Triangular Matrix

Let A be an upper or lower triangular matrix. Then $\det(A)$ is obtained by taking the product of the entries on the main diagonal.

The verification of this Theorem can be done by computing the determinant using Laplace Expansion along the first row or column.

Consider the following example.

Example 3.1.6: Determinant of a Triangular Matrix

Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 77 \\ 0 & 2 & 6 & 7 \\ 0 & 0 & 3 & 33.7 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Find $\det(A)$.

Solution

From Theorem 3.1.2, it suffices to take the product of the elements on the main diagonal. Thus $\det(A) = 1 \times 2 \times 3 \times (-1) = -6$.

Without using Theorem 3.1.2, you could use Laplace Expansion. We will expand along the first column. This gives

$$\begin{aligned} \det(A) =& 1 \begin{vmatrix} 2 & 6 & 7 \\ 0 & 3 & 33.7 \\ 0 & 0 & -1 \end{vmatrix} + 0(-1)^{2+1} \begin{vmatrix} 2 & 3 & 77 \\ 0 & 3 & 33.7 \\ 0 & 0 & -1 \end{vmatrix} + \\ & 0(-1)^{3+1} \begin{vmatrix} 2 & 3 & 77 \\ 2 & 6 & 7 \\ 0 & 0 & -1 \end{vmatrix} + 0(-1)^{4+1} \begin{vmatrix} 2 & 3 & 77 \\ 2 & 6 & 7 \\ 0 & 3 & 33.7 \end{vmatrix} \end{aligned} \tag{3.1.1}$$

and the only nonzero term in the expansion is

$$1\begin{vmatrix} 2 & 6 & 7 \\ 0 & 3 & 33.7 \\ 0 & 0 & -1 \end{vmatrix}$$

Now find the determinant of this 3×3 matrix, by expanding along the first column to obtain

$$\det(A) = 1 \times \left(2 \times \begin{vmatrix} 3 & 33.7 \\ 0 & -1 \end{vmatrix} + 0(-1)^{2+1} \begin{vmatrix} 6 & 7 \\ 0 & -1 \end{vmatrix} + 0(-1)^{3+1} \begin{vmatrix} 6 & 7 \\ 3 & 33.7 \end{vmatrix}\right)$$

$$= 1 \times 2 \times \begin{vmatrix} 3 & 33.7 \\ 0 & -1 \end{vmatrix}$$

Next use Definition $3.1.1$ to find the determinant of this 2×2 matrix, which is just $3 \times -1 - 0 \times 33.7 = -3$. Putting all these steps together, we have

$$\det(A) = 1 \times 2 \times 3 \times (-1) = -6$$

which is just the product of the entries down the main diagonal of the original matrix!

You can see that while both methods result in the same answer, Theorem $3.1.2$ provides a much quicker method.

In the next section, we explore some important properties of determinants.

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3.1E: Exercises for Section 3.1

Exercise $3.1E.1$

Find the determinants of the following matrices.

a. $\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$

b. $\begin{bmatrix} 0 & 3 \\ 0 & 2 \end{bmatrix}$

c. $\begin{bmatrix} 4 & 3 \\ 6 & 2 \end{bmatrix}$

Answer available at the end of this section.

Exercise $3.1E.2$

Let $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ -2 & 5 & 1 \end{bmatrix}$. Find the following.

a. $\text{minor}(A)_{11}$
b. $\text{minor}(A)_{21}$
c. $\text{minor}(A)_{32}$
d. $\text{cof}(A)_{11}$
e. $\text{cof}(A)_{21}$
f. $\text{cof}(A)_{32}$

Answer available at the end of this section.

Exercise $3.1E.3$

Find the determinants of the following matrices.

a. $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \\ 0 & 9 & 8 \end{bmatrix}$

b. $\begin{bmatrix} 4 & 3 & 2 \\ 1 & 7 & 8 \\ 3 & -9 & 3 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 1 & 3 & 2 & 3 \\ 4 & 1 & 5 & 0 \\ 1 & 2 & 1 & 2 \end{bmatrix}$

Answer available at the end of this section.

Exercise $3.1E.4$

Find the following determinant by expanding along the first row and second column.

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 2 & 1 & 1 \end{vmatrix}$$

Answer available at the end of this section.

Exercise $3.1E.5$

Find the following determinant by expanding along the first column and third row.

$$\begin{vmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{vmatrix}$$

Answer available at the end of this section.

Exercise $3.1E.6$

Find the following determinant by expanding along the second row and first column.

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 2 & 1 & 1 \end{vmatrix}$$

Answer available at the end of this section.

Exercise $3.1E.7$

Compute the determinant by cofactor expansion. Pick the easiest row or column to use.

$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 2 & 1 & 3 & 1 \end{vmatrix}$$

Answer available at the end of this section.

Exercise $3.1E.8$

Find the determinant of the following matrices.

a. $A = \begin{bmatrix} 1 & -34 \\ 0 & 2 \end{bmatrix}$

b. $A = \begin{bmatrix} 4 & 3 & 14 \\ 0 & -2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

c. $A = \begin{bmatrix} 2 & 3 & 15 & 0 \\ 0 & 4 & 1 & 7 \\ 0 & 0 & -3 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Exercise $3.1E.9$

Prove that for any 2×2 or 3×3 matrix A, if one row is a multiple of another row, then the $\det(A) = 0$

Exercise $3.1E.10$

Find the value of k such that $\det(A) = 0$ for the following matrix.

$$\begin{vmatrix} k & 4 & 7 \\ -2 & 0 & 8 \\ 0 & -6 & 9 \end{vmatrix}$$

Answer available at the end of this section.

Exercise $3.1E.11$

Suppose A is a 2×2 matrix. What effect will the following have on the determinant of the resulting matrix?

a. Multiplying the first row by 2.
b. Multiplying the second column by -3.
c. Multiplying the entire matrix by 5.

Answer available at the end of this section.

Exercise $3.1E.12$

Find the value of c such that $\det(A) \neq 0$ for the following matrix.

$$\begin{vmatrix} 3 & 2 & 0 & 6 \\ -1 & 5 & 6 & c \\ 0 & -2 & -2 & 0 \\ 4 & -3 & 1 & 8 \end{vmatrix}$$

Answer available at the end of this section.

Answers

Exercise $3.1E.1$

Exercise $3.1E.2$

Exercise $3.1E.3$

a. The answer is 31.
b. The answer is 375.
c. The answer is -2.

Exercise $3.1E.4$

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 2 & 1 & 1 \end{vmatrix} = 6$$

Exercise $3.1E.5$

$$\begin{vmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{vmatrix} = 2$$

Exercise $3.1E.6$

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 2 & 1 & 1 \end{vmatrix} = 6$$

Exercise $3.1E.7$

$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 2 & 1 & 3 & 1 \end{vmatrix} = -4$$

Exercise $3.1E.10$

$k = -\frac{13}{4}$

Exercise $3.1E.11$

a. The determinant will be multiplied by a factor of 2.
b. The determinant will be multiplied by a factor of -3.
c. The determinant will be multiplied by a factor of 5^2.

Exercise $3.1E.12$

$c \neq -2$

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3.2: Properties of Determinants

Learning Objectives

- Demonstrate the effects that elementary row operations have on determinants.
- Verify the following:
  - The determinant of a product of matrices is the product of the determinants
  - The determinant of a matrix is equal to the determinant of its transpose

Properties of Determinants I: Examples

There are many important properties of determinants.

Recall **elementary row operations** (Definition 1.2.2) from Chapter 1. In Chapter 1, we performed them on an augmented matrix to put it into row echelon form or reduced row echelon form. Elementary row operations are:

1. Switching two rows.
2. Multiplying a row by a nonzero number.
3. Replacing a row by a multiple of another row added to itself.

We will now consider the effect of row operations on the determinant of a matrix. In future sections, we will see that using the following properties can greatly assist in finding determinants. This section will initially use the theorems as motivation to provide various examples of the usefulness of the properties. We prove all of the results at the end of the section.

The first theorem explains the effect on the determinant of a matrix when two rows are switched.

Theorem 3.2.1: Switching Two Rows

Let A be an $n \times n$ matrix and let B be a matrix which results from switching two rows of A. Then $\det(B) = -\det(A)$.

When we switch two rows of a matrix, the determinant is multiplied by -1. Consider the following example.

Example 3.2.1: Switching Two Rows

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and let $B = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$. Knowing that $\det(A) = -2$, find $\det(B)$.

Solution

By Definition 3.1.1, $\det(A) = 1 \times 4 - 3 \times 2 = -2$. Notice that the rows of B are the rows of A but switched. By Theorem 3.2.1 since two rows of A have been switched, $\det(B) = -\det(A) = -(-2) = 2$. You can verify this using Definition 3.1.1.

The next theorem demonstrates the effect on the determinant of a matrix when we multiply a row by a scalar.

Theorem 3.2.2: Multiplying a Row by a Scalar

Let A be an $n \times n$ matrix and let B be a matrix which results from multiplying some row of A by a scalar k. Then $\det(B) = k\det(A)$.

Notice that this theorem is true when we multiply *one* row of the matrix by k. If we were to multiply *two* rows of A by k to obtain B, we would have $\det(B) = k^2 \det(A)$. Suppose we were to multiply all n rows of A by k to obtain the matrix B, so that $B = kA$. Then, $\det(B) = k^n \det(A)$. This gives the next theorem.

Theorem 3.2.3: Scalar Multiplication

Let A and B be $n \times n$ matrices and k a scalar, such that $B = kA$. Then $\det(B) = k^n \det(A)$.

Consider the following example.

Example 3.2.2: Multiplying a Row by a Scalar

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 10 \\ 3 & 4 \end{bmatrix}$. Knowing that $\det(A) = -2$, find $\det(B)$.

Solution

By Definition 3.1.1, $\det(A) = -2$. We can also compute $\det(B)$ using Definition 3.1.1, and we see that $\det(B) = -10$.

Now, let's compute $\det(B)$ using Theorem 3.2.2 and see if we obtain the same answer. Notice that the first row of B is 5 times the first row of A, while the second row of B is equal to the second row of A. By Theorem 3.2.2, $\det(B) = 5 \times \det(A) = 5 \times -2 = -10$.

You can see that this matches our answer above.

Finally, consider the next theorem for the last row operation, that of adding a multiple of a row to another row.

Theorem 3.2.4: Adding a Multiple of a Row to Another Row

Let A be an $n \times n$ matrix and let B be a matrix which results from adding a multiple of a row to another row. Then $\det(A) = \det(B)$.

Therefore, when we add a multiple of a row to another row, the determinant of the matrix is unchanged. Note that if a matrix A contains a row which is a multiple of another row, $\det(A)$ will equal 0. To see this, suppose the first row of A is equal to -1 times the second row. By Theorem 3.2.4, we can add the first row to the second row, and the determinant will be unchanged. However, this row operation will result in a row of zeros. Using Laplace Expansion along the row of zeros, we find that the determinant is 0.

Consider the following example.

Example 3.2.3: Adding a Multiple of a Row to Another Row

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and let $B = \begin{bmatrix} 1 & 2 \\ 5 & 8 \end{bmatrix}$. Find $\det(B)$.

Solution

By Definition 3.1.1, $\det(A) = -2$. Notice that the second row of B is two times the first row of A added to the second row. By Theorem 3.2.1, $\det(B) = \det(A) = -2$. As usual, you can verify this answer using Definition 3.1.1.

Example 3.2.4: Multiple of a Row

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. Show that $\det(A) = 0$.

Solution

Using Definition 3.1.1, the determinant is given by

$$\det(A) = 1 \times 4 - 2 \times 2 = 0$$

However notice that the second row is equal to 2 times the first row. Then by the discussion above following Theorem $3.2.4$ the determinant will equal 0.

Until now, our focus has primarily been on row operations. However, we can carry out the same operations with columns, rather than rows. The three operations outlined in Definition $3.2.1$ can be done with columns instead of rows. In this case, in Theorems $3.2.1$, $3.2.2$, and $3.2.4$ you can replace the word "row" with the word "column".

There are several other major properties of determinants which do not involve row (or column) operations. The first is the determinant of a product of matrices. This is a major result whose proof appears at the end of the section.

Theorem 3.2.5: Determinant of a Product

Let A and B be two $n \times n$ matrices. Then

$$\det(AB) = \det(A)\det(B)$$

In order to find the determinant of a product of matrices, we can simply take the product of the determinants.

Consider the following example.

Example 3.2.5: The Determinant of a Product

Compare $\det(AB)$ and $\det(A)\det(B)$ for

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 2 \end{bmatrix}, B = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$$

Solution

First compute AB, which is given by

$$AB = \begin{bmatrix} 1 & 2 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 4 \\ -1 & -4 \end{bmatrix}$$

and so by Definition 3.1.1

$$\det(AB) = \det \begin{bmatrix} 11 & 4 \\ -1 & -4 \end{bmatrix} = -40$$

Now

$$\det(A) = \det \begin{bmatrix} 1 & 2 \\ -3 & 2 \end{bmatrix} = 8$$

and

$$\det(B) = \det \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} = -5$$

Computing $\det(A) \times \det(B)$ we have $8 \times -5 = -40$. This is the same answer as above and you can see that $\det(A)\det(B) = 8 \times (-5) = -40 = \det(AB)$.

Consider the next important property.

Theorem 3.2.6: Determinant of the Transpose

Let A be a matrix where A^T is the transpose of A. Then,

$$\det\left(A^T\right) = \det(A)$$

This theorem is illustrated in the following example.

Example 3.2.6: Determinant of the Transpose

Let

$$A = \begin{bmatrix} 2 & 5 \\ 4 & 3 \end{bmatrix}$$

Find $\det(A^T)$.

Solution

First, note that

$$A^T = \begin{bmatrix} 2 & 4 \\ 5 & 3 \end{bmatrix}$$

Using Definition 3.1.1, we can compute $\det(A)$ and $\det(A^T)$. It follows that $\det(A) = 2 \times 3 - 4 \times 5 = -14$ and $\det(A^T) = 2 \times 3 - 5 \times 4 = -14$. Hence, $\det(A) = \det(A^T)$.

The following provides an essential property of the determinant, as well as a useful way to determine if a matrix is invertible.

Theorem 3.2.7: Determinant of the Inverse

Let A be an $n \times n$ matrix. Then A is invertible if and only if $\det(A) \neq 0$. If this is true, it follows that

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Consider the following example.

Example 3.2.7: Determinant of an Invertible Matrix

Let $A = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}$. For each matrix, determine if it is invertible. If so, find the determinant of the inverse.

Solution

Consider the matrix A first. Using Definition 3.1.1 we can find the determinant as follows:

$$\det(A) = 3 \times 4 - 2 \times 6 = 12 - 12 = 0$$

By Theorem 3.2.7 A is not invertible.

Now consider the matrix B. Again by Definition 3.1.1 we have

$$\det(B) = 2 \times 1 - 5 \times 3 = 2 - 15 = -13$$

By Theorem 3.2.7 B is invertible and the determinant of the inverse is given by

$$\begin{aligned} \det(A^{-1}) &= \frac{1}{\det(A)} \\ &= \frac{1}{-13} \\ &= -\frac{1}{13} \end{aligned} \tag{3.2.1}$$

Below is a video example combining the various properties of determinants.

Example 3.2.8:

Let A and B be 3×3 matrix with $\det(A) = 4$ and $\det(B) = 8$. Compute the following determinants:

1. $\det(AB)$
2. $\det(B^{-1})$
3. $\det(A^T A)$
4. $\det(5B)$

Solution

- **Video Length**: 2 minutes 20 seconds.
- **Context**: This video demonstrates the applying the various properties of the determinant.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Properties of Determinants II: Some Important Proofs

This section includes some important proofs on determinants and cofactors.

First we recall the definition of a determinant. If $A = [a_{ij}]$ is an $n \times n$ matrix, then $\det A$ is defined by computing the expansion along the first row:

$$\det A = \sum_{i=1}^{n} a_{1,i} \text{cof}(A)_{1,i}. \tag{3.2.2}$$

If $n = 1$ then $\det A = a_{1,1}$.

The following example is straightforward and strongly recommended as a means for getting used to definitions.

Example 3.2.9:

(1) Let E_{ij} be the elementary matrix obtained by interchanging ith and jth rows of I. Then $\det E_{ij} = -1$.

(2) Let E_{ik} be the elementary matrix obtained by multiplying the ith row of I by k. Then $\det E_{ik} = k$.

(3) Let E_{ijk} be the elementary matrix obtained by multiplying ith row of I by k and adding it to its jth row. Then $\det E_{ijk} = 1$.

(4) If C and B are such that CB is defined and the ith row of C consists of zeros, then the ith row of CB consists of zeros.

(5) If E is an elementary matrix, then $\det E = \det E^T$.

Many of the proofs in section use the Principle of Mathematical Induction. This concept is discussed in Appendix A.2 and is reviewed here for convenience. First we check that the assertion is true for $n = 2$ (the case $n = 1$ is either completely trivial or meaningless).

Next, we assume that the assertion is true for $n - 1$ (where $n \geq 3$) and prove it for n. Once this is accomplished, by the Principle of Mathematical Induction we can conclude that the statement is true for all $n \times n$ matrices for every $n \geq 2$.

If A is an $n \times n$ matrix and $1 \leq j \leq n$, then the matrix obtained by removing 1st column and jth row from A is an $n - 1 \times n - 1$ matrix (we shall denote this matrix by $A(j)$ below). Since these matrices are used in computation of cofactors $\operatorname{cof}(A)_{1,i}$, for $1 \leq i \neq n$, the inductive assumption applies to these matrices.

Consider the following lemma.

Lemma 3.2.1

If A is an $n \times n$ matrix such that one of its rows consists of zeros, then $\det A = 0$.

Proof available at the end of this section.

Lemma 3.2.2:

Assume A, B and C are $n \times n$ matrices that for some $1 \leq i \leq n$ satisfy the following.

1. jth rows of all three matrices are identical, for $j \neq i$.
2. Each entry in the jth row of A is the sum of the corresponding entries in jth rows of B and C.

Then $\det A = \det B + \det C$.

Proof available at the end of this section.

Next, we prove Theorems 3.2.1, 3.2.2, 3.2.4 and one other statement.

Theorem 3.2.1, 2, 4:

Let A and B be $n \times n$ matrices.

1. If A is obtained by interchanging ith and jth rows of B (with $i \neq j$), then $\det A = -\det B$.
2. If A is obtained by multiplying ith row of B by k then $\det A = k \det B$.
3. If two rows of A are identical then $\det A = 0$.
4. If A is obtained by multiplying ith row of B by k and adding it to jth row of B ($i \neq j$) then $\det A = \det B$.

Proof available at the end of this section.

Next, we prove the previously stated important theorem about the determinant of a product of matrices.

Theorem 3.2.5:

Let A and B be two $n \times n$ matrices. Then

$$\det(AB) = \det(A)\det(B)$$

Proof available at the end of this section.

The same 'machine' used in the previous proof will be used again to prove the previously stated theorem about the determinant of the transpose.

Theorem 3.2.6:

Let A be a matrix where A^T is the transpose of A. Then,

$$\det(A^T) = \det(A)$$

Proof available at the end of this section.

The above discussions allow us to now prove Theorem 3.1.1. It is restated below.

Theorem 3.2.8:

Expanding an $n \times n$ matrix along any row or column always gives the same result, which is the determinant.

Proof available at the end of this section.

Finding Determinant Using Row Operations

Theorems 3.2.1, 3.2.2 and 3.2.4 illustrate how row operations affect the determinant of a matrix. Now we look at two examples where row operations are used to find the determinant of a large matrix. Recall that when working with large matrices, Laplace Expansion is effective but time consuming, as there are many steps involved. This section provides useful tools for an alternative method. By first applying row operations, we can obtain a simpler matrix to which we apply Laplace Expansion.

While working through questions such as these, it is useful to record your row operations as you go along. Keep this in mind as you read through the next example.

Example 3.2.10: Finding a Determinant

Find the determinant of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 1 & 2 & 3 \\ 4 & 5 & 4 & 3 \\ 2 & 2 & -4 & 5 \end{bmatrix}$$

Solution

We will use the properties of determinants outlined above to find $\det(A)$. First, add -5 times the first row to the second row. Then add -4 times the first row to the third row, and -2 times the first row to the fourth row. This yields the matrix

$$B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -9 & -13 & -17 \\ 0 & -3 & -8 & -13 \\ 0 & -2 & -10 & -3 \end{bmatrix}$$

Notice that the only row operation we have done so far is adding a multiple of a row to another row. Therefore, by Theorem 3.2.4, $\det(B) = \det(A)$.

At this stage, you could use Laplace Expansion to find $\det(B)$. However, we will continue with row operations to find an even simpler matrix to work with.

Add -3 times the third row to the second row. By Theorem 3.2.4 this does not change the value of the determinant. Then, multiply the fourth row by -3. This results in the matrix

$$C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 11 & 22 \\ 0 & -3 & -8 & -13 \\ 0 & 6 & 30 & 9 \end{bmatrix}$$

Here, $\det(C) = -3\det(B)$, which means that $\det(B) = \left(-\frac{1}{3}\right)\det(C)$

Since $\det(A) = \det(B)$, we now have that $\det(A) = \left(-\frac{1}{3}\right)\det(C)$. Again, you could use Laplace Expansion here to find $\det(C)$. However, we will continue with row operations.

Now add 2 times the third row to the fourth row. This does not change the value of the determinant by Theorem 3.2.4. Finally switch the third and second rows. This causes the determinant to be multiplied by -1. Thus $\det(C) = -\det(D)$ where

$$D = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -8 & -13 \\ 0 & 0 & 11 & 22 \\ 0 & 0 & 14 & -17 \end{bmatrix}$$

Hence, $\det(A) = \left(-\frac{1}{3}\right)\det(C) = \left(\frac{1}{3}\right)\det(D)$

You could do more row operations or you could note that this can be easily expanded along the first column. Then, expand the resulting 3×3 matrix also along the first column. This results in

$$\det(D) = 1(-3)\begin{vmatrix} 11 & 22 \\ 14 & -17 \end{vmatrix} = 1485$$

and so $\det(A) = \left(\frac{1}{3}\right)(1485) = 495$.

You can see that by using row operations, we can simplify a matrix to the point where Laplace Expansion involves only a few steps. In Example 3.2.10, we also could have continued until the matrix was in upper triangular form, and taken the product of the entries on the main diagonal. Whenever computing the determinant, it is useful to consider all the possible methods and tools.

Consider the next example.

Example 3.2.11: Find the Determinant

Find the determinant of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 1 & -3 & 2 & 1 \\ 2 & 1 & 2 & 5 \\ 3 & -4 & 1 & 2 \end{bmatrix}$$

Solution

Once again, we will simplify the matrix through row operations. Add -1 times the first row to the second row. Next add -2 times the first row to the third and finally take -3 times the first row and add to the fourth row. This yields

$$B = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -5 & -1 & -1 \\ 0 & -3 & -4 & 1 \\ 0 & -10 & -8 & -4 \end{bmatrix}$$

By Theorem 3.2.4, $\det(A) = \det(B)$.

Remember you can work with the columns also. Take -5 times the fourth column and add to the second column. This yields

$$C = \begin{bmatrix} 1 & -8 & 3 & 2 \\ 0 & 0 & -1 & -1 \\ 0 & -8 & -4 & 1 \\ 0 & 10 & -8 & -4 \end{bmatrix}$$

By Theorem 3.2.4 $\det(A) = \det(C)$.

Now take -1 times the third row and add to the top row. This gives.

$$D = \begin{bmatrix} 1 & 0 & 7 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & -8 & -4 & 1 \\ 0 & 10 & -8 & -4 \end{bmatrix}$$

which by Theorem 3.2.4 has the same determinant as A.

Now, we can find $\det(D)$ by expanding along the first column as follows. You can see that there will be only one non zero term.

$$\det(D) = 1 \det \begin{bmatrix} 0 & -1 & -1 \\ -8 & -4 & 1 \\ 10 & -8 & -4 \end{bmatrix} + 0 + 0 + 0$$

Expanding again along the first column, we have

$$\det(D) = 1 \left(0 + 8 \det \begin{bmatrix} -1 & -1 \\ -8 & -4 \end{bmatrix} + 10 \det \begin{bmatrix} -1 & -1 \\ -4 & 1 \end{bmatrix} \right) = -82$$

Now since $\det(A) = \det(D)$, it follows that $\det(A) = -82$.

Remember that you can verify these answers by using Laplace Expansion on A. Similarly, if you first compute the determinant using Laplace Expansion, you can use the row operation method to verify.

Answers

Lemma 3.2.1

We will prove this lemma using Mathematical Induction.

If $n = 2$ this is easy (check!).

Let $n \geq 3$ be such that every matrix of size $n-1 \times n-1$ with a row consisting of zeros has determinant equal to zero. Let i be such that the ith row of A consists of zeros. Then we have $a_{ij} = 0$ for $1 \leq j \leq n$.

Fix $j \in \{1, 2, \ldots, n\}$ such that $j \neq i$. Then matrix $A(j)$ used in computation of $\mathrm{cof}(A)_{1,j}$ has a row consisting of zeros, and by our inductive assumption $\mathrm{cof}(A)_{1,j} = 0$.

On the other hand, if $j = i$ then $a_{1,j} = 0$. Therefore $a_{1,j}\mathrm{cof}(A)_{1,j} = 0$ for all j and by (3.2.2) we have

$$\det A = \sum_{j=1}^{n} a_{1,j}\mathrm{cof}(A)_{1,j} = 0$$

as each of the summands is equal to 0.

Lemma 3.2.2**:**

This is not difficult to check for $n = 2$ (do check it!).

Now assume that the statement of Lemma is true for $n-1 \times n-1$ matrices and fix A, B and C as in the statement. The assumptions state that we have $a_{l,j} = b_{l,j} = c_{l,j}$ for $j \neq i$ and for $1 \leq l \leq n$ and $a_{l,i} = b_{l,i} + c_{l,i}$ for all $1 \leq l \leq n$. Therefore $A(i) = B(i) = C(i)$, and $A(j)$ has the property that its ith row is the sum of ith rows of $B(j)$ and $C(j)$ for $j \neq i$ while the other rows of all three matrices are identical. Therefore by our inductive assumption we have $\mathrm{cof}(A)_{1j} = \mathrm{cof}(B)_{1j} + \mathrm{cof}(C)_{1j}$ for j

$\neq i$.

By (3.2.2) we have (using all equalities established above)

$$\begin{aligned}\det A &= \sum_{l=1}^{n} a_{1,l}\text{cof}(A)_{1,l} \\ &= \sum_{l\neq i} a_{1,l}(\text{cof}(B)_{1,l} + \text{cof}(C)_{1,l}) + (b_{1,i} + c_{1,i})\text{cof}(A)_{1,i} \\ &= \det B + \det C\end{aligned} \tag{3.2.3}$$

This proves that the assertion is true for all n and completes the proof.

Theorem 3.2.1, 2, 4**:**

We prove all statements by induction. The case $n = 2$ is easily checked directly (and it is strongly suggested that you do check it).

We assume $n \geq 3$ and (1)–(4) are true for all matrices of size $n-1 \times n-1$.

(1) We prove the case when $j = i+1$, i.e., we are interchanging two consecutive rows.

Let $l \in \{1, \dots, n\} \setminus \{i, j\}$. Then $A(l)$ is obtained from $B(l)$ by interchanging two of its rows (draw a picture) and by our assumption

$$\text{cof}(A)_{1,l} = -\text{cof}(B)_{1,l}. \tag{3.2.4}$$

Now consider $a_{1,i}\text{cof}(A)_{1,l}$. We have that $a_{1,i} = b_{1,j}$ and also that $A(i) = B(j)$. Since $j = i+1$, we have

$$(-1)^{1+j} = (-1)^{1+i+1} = -(-1)^{1+i}$$

and therefore $a_{1i}\text{cof}(A)_{1i} = -b_{1j}\text{cof}(B)_{1j}$ and $a_{1j}\text{cof}(A)_{1j} = -b_{1i}\text{cof}(B)_{1i}$. Putting this together with (3.2.4) into (3.2.2) we see that if in the formula for $\det A$ we change the sign of each of the summands we obtain the formula for $\det B$.

$$\det A = \sum_{l=1}^{n} a_{1l}\text{cof}(A)_{1l} = -\sum_{l=1}^{n} b_{1l}B_{1l} = \det B.$$

We have therefore proved the case of (1) when $j = i+1$. In order to prove the general case, one needs the following fact. If \(i, then in order to interchange ith and j th row one can proceed by interchanging two adjacent rows $2(j-i)+1$ times: First swap ith and $i+1$st, then $i+1$st and $i+2$nd, and so on. After one interchanges $j-1$st and jth row, we have ith row in position of jth and lth row in position of $l-1$st for $i+1 \leq l \leq j$. Then proceed backwards swapping adjacent rows until everything is in place.

Since $2(j-i)+1$ is an odd number $(-1)^{2(j-i)+1} = -1$ and we have that $\det A = -\det B$.

(2) This is like (1)... but much easier. Assume that (2) is true for all $n-1 \times n-1$ matrices. We have that $a_{ji} = kb_{ji}$ for $1 \leq j \leq n$. In particular $a_{1i} = kb_{1i}$, and for $l \neq i$ matrix $A(l)$ is obtained from $B(l)$ by multiplying one of its rows by k. Therefore $\text{cof}(A)_{1l} = k\text{cof}(B)_{1l}$ for $l \neq i$, and for all l we have $a_{1l}\text{cof}(A)_{1l} = kb_{1l}\text{cof}(B)_{1l}$. By (3.2.2) , we have $\det A = k \det B$.

(3) This is a consequence of (1). If two rows of A are identical, then A is equal to the matrix obtained by interchanging those two rows and therefore by (1) $\det A = -\det A$. This implies $\det A = 0$.

(4) Assume (4) is true for all $n-1 \times n-1$ matrices and fix A and B such that A is obtained by multiplying ith row of B by k and adding it to jth row of B ($i \neq j$) then $\det A = \det B$. If $k = 0$ then $A = B$ and there is nothing to prove, so we may assume $k \neq 0$.

Let C be the matrix obtained by replacing the jth row of B by the ith row of B multiplied by k. By Lemma 3.2.2, we have that

$$\det A = \det B + \det C$$

and we 'only' need to show that $\det C = 0$. But ith and jth rows of C are proportional. If D is obtained by multiplying the jth row of C by $\frac{1}{k}$ then by (2) we have $\det C = \frac{1}{k}\det D$ (recall that $k \neq 0$!). But ith and jth rows of D are identical, hence by (3) we have $\det D = 0$ and therefore $\det C = 0$.

Theorem 3.2.5**:**

If A is an elementary matrix of either type, then multiplying by A on the left has the same effect as performing the corresponding elementary row operation. Therefore the equality $\det(AB) = \det A \det B$ in this case follows by Example 3.2.8 and Theorem 3.2.8.

If C is the reduced row-echelon form of A then we can write $A = E_1 \cdot E_2 \cdot \cdots \cdot E_m \cdot C$ for some elementary matrices $E_1, \dots, E_m$.

Now we consider two cases.

Assume first that $C = I$. Then $A = E_1 \cdot E_2 \cdot \cdots \cdot E_m$ and $AB = E_1 \cdot E_2 \cdot \cdots \cdot E_m B$. By applying the above equality m times, and then $m-1$ times, we have that

$$\det(AB) = \det E_1 \det E_2 \cdot \cdots \cdot \det E_m \cdot \det B = \det(E_1 \cdot E_2 \cdot \cdots \cdot E_m) \det B = \det A \det B.$$

Now assume $C \neq I$. Since it is in reduced row-echelon form, its last row consists of zeros and by (4) of Example 3.2.8 the last row of CB consists of zeros. By Lemma 3.2.1 we have $\det C = \det(CB) = 0$ and therefore

$$\det A = \det(E_1 \cdot \cdots \cdot E_2 \cdot E_m) \cdot \det(C) = \det(E_1 \cdot E_2 \cdot \cdots \cdot E_m) \cdot 0 = 0$$

and also

$$\det(AB) = \det(E_1 \cdot E_2 \cdot \cdots \cdot E_m) \cdot \det(CB) = \det(E_1 \cdot E_2 \cdot \cdots \cdot E_m)0 = 0$$

hence $\det(AB) = 0 = \det A \det B$.

Theorem 3.2.6**:**

Note first that the conclusion is true if A is elementary by (5) of Example 3.2.8.

Let C be the reduced row-echelon form of A. Then we can write $A = E_1 \cdot E_2 \cdot \cdots \cdot E_m C$. Then $A^T = C^T \cdot E_m^T \cdot \cdots \cdot E_2^T \cdot E_1$. By Theorem 3.2.9 we have

$$\det(A^T) = \det(C^T) \cdot \det(E_m^T) \cdot \cdots \cdot \det(E_2^T) \cdot \det(E_1).$$

By (5) of Example 3.2.8 we have that $\det E_j = \det E_j^T$ for all j. Also, $\det C$ is either 0 or 1 (depending on whether $C = I$ or not) and in either case $\det C = \det C^T$. Therefore $\det A = \det A^T$.

Theorem 3.2.8**:**

We first show that the determinant can be computed along any row. The case $n = 1$ does not apply and thus let $n \geq 2$.

Let A be an $n \times n$ matrix and fix $j > 1$. We need to prove that

$$\det A = \sum_{i=1}^{n} a_{j,i}\text{cof}(A)_{j,i}.$$

Let us prove the case when $j = 2$.

Let B be the matrix obtained from A by interchanging its 1st and 2nd rows. Then by Theorem 3.2.8 we have

$$\det A = -\det B.$$

Now we have

$$\det B = \sum_{i=1}^{n} b_{1,i}\text{cof}(B)_{1,i}.$$

Since B is obtained by interchanging the 1st and 2nd rows of A we have that $b_{1,i} = a_{2,i}$ for all i and one can see that $\text{minor}(B)_{1,i} = \text{minor}(A)_{2,i}$.

Further,

$$\text{cof}(B)_{1,i} = (-1)^{1+i}\text{minor}B_{1,i} = -(-1)^{2+i}\text{minor}(A)_{2,i} = -\text{cof}(A)_{2,i}$$

hence $\det B = -\sum_{i=1}^{n} a_{2,i}\text{cof}(A)_{2,i}$, and therefore $\det A = -\det B = \sum_{i=1}^{n} a_{2,i}\text{cof}(A)_{2,i}$ as desired.

The case when $j > 2$ is very similar; we still have $\text{minor}(B)_{1,i} = \text{minor}(A)_{j,i}$ but checking that $\det B = -\sum_{i=1}^{n} a_{j,i}\text{cof}(A)_{j,i}$ is slightly more involved.

Now the cofactor expansion along column j of A is equal to the cofactor expansion along row j of A^T, which is by the above result just proved equal to the cofactor expansion along row 1 of A^T, which is equal to the cofactor expansion along column 1 of A. Thus the cofactor cofactor along any column yields the same result.

Finally, since $\det A = \det A^T$ by Theorem 3.2.10, we conclude that the cofactor expansion along row 1 of A is equal to the cofactor expansion along row 1 of A^T, which is equal to the cofactor expansion along column 1 of A. Thus the proof is complete.

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3.2E: Exercises for Section 3.2

Exercise 3.2E.1

An operation is done to get from the first matrix to the second. Identify what was done and tell how it will affect the value of the determinant.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \to \cdots \to \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Answer available at the end of this section.

Exercise 3.2E.2

An operation is done to get from the first matrix to the second. Identify what was done and tell how it will affect the value of the determinant.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \to \cdots \to \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

Answer available at the end of this section.

Exercise 3.2E.3

An operation is done to get from the first matrix to the second. Identify what was done and tell how it will affect the value of the determinant.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \to \cdots \to \begin{bmatrix} a & b \\ a+c & b+d \end{bmatrix}$$

Answer available at the end of this section.

Exercise 3.2E.4

An operation is done to get from the first matrix to the second. Identify what was done and tell how it will affect the value of the determinant.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \to \cdots \to \begin{bmatrix} a & b \\ 2c & 2d \end{bmatrix}$$

Answer available at the end of this section.

Exercise 3.2E.5

An operation is done to get from the first matrix to the second. Identify what was done and tell how it will affect the value of the determinant.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \to \cdots \to \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

Answer available at the end of this section.

Exercise $3.2E.6$

Let A be an $r \times r$ matrix and suppose there are $r-1$ rows (columns) such that all rows (columns) are linear combinations of these $r-1$ rows (columns). Show $\det(A) = 0$.

Answer available at the end of this section.

Exercise $3.2E.7$

Show $\det(aA) = a^n \det(A)$ for an $n \times n$ matrix A and scalar a.

Answer available at the end of this section.

Exercise $3.2E.8$

Construct 2×2 matrices A and B to show that the $\det A \det B = \det(AB)$.

Answer available at the end of this section.

Exercise $3.2E.9$

Is it true that $\det(A+B) = \det(A) + \det(B)$? If this is so, explain why. If it is not so, give a counter example.

Answer available at the end of this section.

Exercise $3.2E.10$

An $n \times n$ matrix is called **nilpotent** if for some positive integer, k it follows $A^k = 0$. If A is a nilpotent matrix and k is the smallest possible integer such that $A^k = 0$, what are the possible values of $\det(A)$?

Answer available at the end of this section.

Exercise $3.2E.11$

A matrix is said to be **orthogonal** if $A^T A = I$. Thus the inverse of an orthogonal matrix is just its transpose. What are the possible values of $\det(A)$ if A is an orthogonal matrix?

Answer available at the end of this section.

Exercise $3.2E.12$

Let A and B be two $n \times n$ matrices. $A \sim B$ (A is **similar** to B) means there exists an invertible matrix P such that $A = P^{-1}BP$. Show that if $A \sim B$, then $\det(A) = \det(B)$.

Answer available at the end of this section.

Exercise $3.2E.13$

Tell whether each statement is true or false. If true, provide a proof. If false, provide a counter example.

a. If A is a 3×3 matrix with a zero determinant, then one column must be a multiple of some other column.
b. If any two columns of a square matrix are equal, then the determinant of the matrix equals zero.
c. For two $n \times n$ matrices A and B, $\det(A+B) = \det(A) + \det(B)$.
d. For an $n \times n$ matrix A, $\det(3A) = 3\det(A)$
e. If A^{-1} exists then $\det(A^{-1}) = \det(A)^{-1}$.
f. If B is obtained by multiplying a single row of A by 4 then $\det(B) = 4\det(A)$.
g. For A an $n \times n$ matrix, $\det(-A) = (-1)^n \det(A)$.
h. If A is a real $n \times n$ matrix, then $\det(A^T A) \geq 0$.
i. If $A^k = 0$ for some positive integer k, then $\det(A) = 0$.
j. If $AX = 0$ for some $X \neq 0$, then $\det(A) = 0$.

Answer available at the end of this section.

Exercise $3.2E.14$

Find the determinant using row operations to first simplify.

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ -4 & 1 & 2 \end{vmatrix}$$

Answer available at the end of this section.

Exercise $3.2E.15$

Find the determinant using row operations to first simplify.

$$\begin{vmatrix} 2 & 1 & 3 \\ 2 & 4 & 2 \\ 1 & 4 & -5 \end{vmatrix}$$

Answer available at the end of this section.

Exercise $3.2E.16$

Find the determinant using row operations to first simplify.

$$\begin{vmatrix} 1 & 2 & 1 & 2 \\ 3 & 1 & -2 & 3 \\ -1 & 0 & 3 & 1 \\ 2 & 3 & 2 & -2 \end{vmatrix}$$

Answer available at the end of this section.

Exercise $3.2E.17$

Find the determinant using row operations to first simplify.

$$\begin{vmatrix} 1 & 4 & 1 & 2 \\ 3 & 2 & -2 & 3 \\ -1 & 0 & 3 & 3 \\ 2 & 1 & 2 & -2 \end{vmatrix}$$

Answer available at the end of this section.

Answers

Exercise $3.2E.1$

It does not change the determinant. This was just taking the transpose.

Exercise $3.2E.2$

In this case two rows were switched and so the resulting determinant is -1 times the first

Exercise $3.2E.3$

The determinant is unchanged. It was just the first row added to the second.

Exercise $3.2E.4$

The second row was multiplied by 2 so the determinant of the result is 2 times the original determinant.

Exercise $3.2E.5$

In this case the two columns were switched so the determinant of the second is -1 times the determinant of the first.

Exercise $3.2E.6$

If the determinant is nonzero, then it will remain nonzero with row operations applied to the matrix. However, by assumption, you can obtain a row of zeros by doing row operations. Thus the determinant must have been zero after all.

Exercise $3.2E.7$

$\det(aA) = \det(aIA) = \det(aI)\det(A) = a^n \det(A)$. The matrix which has a down the main diagonal has determinant equal to a^n.

Exercise $3.2E.8$

$$\det\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ -5 & 6 \end{bmatrix}\right) = -8$$

$$\det\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\det\begin{bmatrix} -1 & 2 \\ -5 & 6 \end{bmatrix} = -2 \times 4 = -8$$

Exercise $3.2E.9$

This is not true at all. Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

Exercise $3.2E.10$

It must be 0 because $0 = \det(0) = \det(A^k) = (\det(A))^k$.

Exercise $3.2E.11$

You would need $\det(AA^T) = \det(A)\det(A^T) = \det(A)^2 = 1$ and so $\det(A) = 1$, or -1.

Exercise $3.2E.12$

$\det(A) = \det(S^{-1}BS) = \det(S^{-1})\det(B)\det(S) = \det(B)\det(S^{-1}S) = \det(B)$.

Exercise $3.2E.13$

a. False. Consider $\begin{bmatrix} 1 & 1 & 2 \\ -1 & 5 & 4 \\ 0 & 3 & 3 \end{bmatrix}$

b. True.

c. False.

d. False.

e. True.

f. False.

g. True.

h. True.

i. True.

j. True.

Exercise $3.2E.14$

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ -4 & 1 & 2 \end{vmatrix} = -6$$

Exercise $3.2E.15$

$$\begin{vmatrix} 2 & 1 & 3 \\ 2 & 4 & 2 \\ 1 & 4 & -5 \end{vmatrix} = -32$$

Exercise $3.2E.16$

One can row reduce this using only row operation 3 to

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -5 & -5 & -3 \\ 0 & 0 & 2 & \frac{9}{5} \\ 0 & 0 & 0 & -\frac{63}{10} \end{bmatrix}$$

and therefore, the determinant is -63.

$$\begin{vmatrix} 1 & 2 & 1 & 2 \\ 3 & 1 & -2 & 3 \\ -1 & 0 & 3 & 1 \\ 2 & 3 & 2 & -2 \end{vmatrix} = 63$$

Exercise $3.2E.17$

One can row reduce this using only row operation 3 to

$$\begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & -10 & -5 & -3 \\ 0 & 0 & 2 & \frac{19}{5} \\ 0 & 0 & 0 & -\frac{211}{20} \end{bmatrix}$$

Thus the determinant is given by

$$\begin{vmatrix} 1 & 4 & 1 & 2 \\ 3 & 2 & -2 & 3 \\ -1 & 0 & 3 & 3 \\ 2 & 1 & 2 & -2 \end{vmatrix} = 211$$

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3.3: Application of the Determinant to Inverses; Cramer's Rule

Learning Objectives

- Use determinants to determine whether a matrix has an inverse, and evaluate the inverse using cofactors.
- Apply Cramer's Rule to solve a 2×2 or a 3×3 linear system.

A Formula for the Inverse

The determinant of a matrix also provides a way to find the inverse of a matrix. Recall the definition of the inverse of a matrix in Definition 2.2.1. We say that A^{-1}, an $n \times n$ matrix, is the inverse of A, also $n \times n$, if $AA^{-1} = I$ and $A^{-1}A = I$.

We now define a new matrix called the **cofactor matrix** of A. The cofactor matrix of A is the matrix whose ij^{th} entry is the ij^{th} cofactor of A. The formal definition is as follows.

Definition 3.3.1: The Cofactor Matrix

Let $A = [a_{ij}]$ be an $n \times n$ matrix. Then the **cofactor matrix of** A, denoted $\mathrm{cof}(A)$, is defined by $\mathrm{cof}(A) = \left[\mathrm{cof}(A)_{ij}\right]$ where $\mathrm{cof}(A)_{ij}$ is the ij^{th} cofactor of A.

Note that $\mathrm{cof}(A)_{ij}$ denotes the ij^{th} entry of the cofactor matrix.

We will use the cofactor matrix to create a formula for the inverse of A. First, we define the **adjugate** of A to be the transpose of the cofactor matrix. We can also call this matrix the **classical adjoint** of A, and we denote it by $\mathrm{adj}(A)$.

In the specific case where A is a 2×2 matrix given by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then $\mathrm{adj}(A)$ is given by

$$\mathrm{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

In general, $\mathrm{adj}(A)$ can always be found by taking the transpose of the cofactor matrix of A. The following theorem provides a formula for A^{-1} using the determinant and adjugate of A.

Theorem 3.3.1: The Inverse and the Determinant

Let A be an $n \times n$ matrix. Then

$$A\,\mathrm{adj}(A) = \mathrm{adj}(A)\,A = \det(A) I$$

Moreover A is invertible if and only if $\det(A) \neq 0$. In this case we have:

$$A^{-1} = \frac{1}{\det(A)} \mathrm{adj}(A)$$

Notice that the first formula holds for any $n \times n$ matrix A, and in the case A is invertible we actually have a formula for A^{-1}.

Before proving the theorem, let's consider the following example.

Example 3.3.1: Find Inverse Using the Determinant

Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

using the formula in Theorem 3.3.1.

Solution

According to Theorem 3.3.1,

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

First we will find the determinant of this matrix. Using Theorems 3.2.1, 3.2.2, and 3.2.4, we can first simplify the matrix through row operations. First, add -3 times the first row to the second row. Then add -1 times the first row to the third row to obtain

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -6 & -8 \\ 0 & 0 & -2 \end{bmatrix}$$

By Theorem 3.2.4, $\det(A) = \det(B)$. By Theorem 3.1.2, $\det(B) = 1 \times -6 \times -2 = 12$. Hence, $\det(A) = 12$.

Now, we need to find $\operatorname{adj}(A)$. To do so, first we will find the cofactor matrix of A. This is given by

$$\operatorname{cof}(A) = \begin{bmatrix} -2 & -2 & 6 \\ 4 & -2 & 0 \\ 2 & 8 & -6 \end{bmatrix}$$

Here, the ij^{th} entry is the ij^{th} cofactor of the original matrix A which you can verify. Therefore, from Theorem 3.3.1, the inverse of A is given by

$$A^{-1} = \frac{1}{12} \begin{bmatrix} -2 & -2 & 6 \\ 4 & -2 & 0 \\ 2 & 8 & -6 \end{bmatrix}^T = \begin{bmatrix} -\frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{6} & \frac{2}{3} \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}$$

Remember that we can always verify our answer for A^{-1}. Compute the product AA^{-1} and $A^{-1}A$ and make sure each product is equal to I.

Compute $A^{-1}A$ as follows

$$A^{-1}A = \begin{bmatrix} -\frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{6} & \frac{2}{3} \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

You can verify that $AA^{-1} = I$ and hence our answer is correct.

We will look at another example of how to use this formula to find A^{-1}.

Example 3.3.2: Find the Inverse From a Formula

Find the inverse of the matrix

$$A = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{6} & \frac{1}{3} & -\frac{1}{2} \\ -\frac{5}{6} & \frac{2}{3} & -\frac{1}{2} \end{bmatrix}$$

using the formula given in Theorem 3.3.1.

Solution

First we need to find $\det(A)$. This step is left as an exercise and you should verify that $\det(A) = \frac{1}{6}$. The inverse is therefore equal to

$$A^{-1} = \frac{1}{(1/6)} \operatorname{adj}(A) = 6 \operatorname{adj}(A)$$

We continue to calculate as follows. Here we show the 2×2 determinants needed to find the cofactors.

$$A^{-1} = 6 \begin{bmatrix} \begin{vmatrix} \frac{1}{3} & -\frac{1}{2} \\ \frac{2}{3} & -\frac{1}{2} \end{vmatrix} & -\begin{vmatrix} -\frac{1}{6} & -\frac{1}{2} \\ -\frac{5}{6} & -\frac{1}{2} \end{vmatrix} & \begin{vmatrix} -\frac{1}{6} & \frac{1}{3} \\ -\frac{5}{6} & \frac{2}{3} \end{vmatrix} \\ -\begin{vmatrix} 0 & \frac{1}{2} \\ \frac{2}{3} & -\frac{1}{2} \end{vmatrix} & \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{5}{6} & -\frac{1}{2} \end{vmatrix} & -\begin{vmatrix} \frac{1}{2} & 0 \\ -\frac{5}{6} & \frac{2}{3} \end{vmatrix} \\ \begin{vmatrix} 0 & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{2} \end{vmatrix} & -\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{6} & -\frac{1}{2} \end{vmatrix} & \begin{vmatrix} \frac{1}{2} & 0 \\ -\frac{1}{6} & \frac{1}{3} \end{vmatrix} \end{bmatrix}^T$$

Expanding all the 2×2 determinants, this yields

$$A^{-1} = 6 \begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & -2 & 1 \end{bmatrix}$$

Again, you can always check your work by multiplying $A^{-1}A$ and AA^{-1} and ensuring these products equal I.

$$A^{-1}A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{6} & \frac{1}{3} & -\frac{1}{2} \\ -\frac{5}{6} & \frac{2}{3} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This tells us that our calculation for A^{-1} is correct. It is left to the reader to verify that $AA^{-1} = I$.

The verification step is very important, as it is a simple way to check your work! If you multiply $A^{-1}A$ and AA^{-1} and these products are not both equal to I, be sure to go back and double check each step. One common error is to forget to take the transpose of the cofactor matrix, so be sure to complete this step.

We will now restate and prove Theorem 3.3.1.

Theorem 3.3.1

Let A be an $n \times n$ matrix. Then

$$A \operatorname{adj}(A) = \operatorname{adj}(A) A = \det(A) I$$

Moreover A is invertible if and only if $\det(A) \neq 0$. In this case we have:

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

Proof available at the end of this section.

This method for finding the inverse of A is useful in many contexts. In particular, it is useful with complicated matrices where the

entries are functions, rather than numbers.

Consider the following example.

Example 3.3.3: Inverse for Non-Constant Matrix

Suppose

$$A(t) = \begin{bmatrix} e^t & 0 & 0 \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{bmatrix}$$

Show that $A(t)^{-1}$ exists and then find it.

Solution

First note $\det(A(t)) = e^t(\cos^2 t + \sin^2 t) = e^t \neq 0$ so $A(t)^{-1}$ exists.

The cofactor matrix is

$$C(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^t \cos t & e^t \sin t \\ 0 & -e^t \sin t & e^t \cos t \end{bmatrix}$$

and so the inverse is

$$\frac{1}{e^t}\begin{bmatrix} 1 & 0 & 0 \\ 0 & e^t \cos t & e^t \sin t \\ 0 & -e^t \sin t & e^t \cos t \end{bmatrix}^T = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{bmatrix}$$

Cramer's Rule

Another context in which the formula given in Theorem 3.3.1 is important is **Cramer's Rule**. Recall that we can represent a system of linear equations in the form $AX = B$, where the solutions to this system are given by X. Cramer's Rule gives a formula for the solutions X in the special case that A is a square invertible matrix. Note this rule does not apply if you have a system of equations in which there is a different number of equations than variables (in other words, when A is not square), or when A is not invertible.

Suppose we have a system of equations given by $AX = B$, and we want to find solutions X which satisfy this system. Then recall that if A^{-1} exists,

$$\begin{aligned} AX &= B \\ A^{-1}(AX) &= A^{-1}B \\ (A^{-1}A)X &= A^{-1}B \\ IX &= A^{-1}B \\ X &= A^{-1}B \end{aligned} \tag{3.3.1}$$

Hence, the solutions X to the system are given by $X = A^{-1}B$. Since we assume that A^{-1} exists, we can use the formula for A^{-1} given above. Substituting this formula into the equation for X, we have

$$X = A^{-1}B = \frac{1}{\det(A)}\operatorname{adj}(A)B$$

Let x_i be the ith entry of X and b_j be the jth entry of B. Then this equation becomes

$$x_i = \sum_{j=1}^{n} [a_{ij}]^{-1} b_j = \sum_{j=1}^{n} \frac{1}{\det(A)}\operatorname{adj}(A)_{ij} b_j$$

where $\operatorname{adj}(A)_{ij}$ is the ij^{th} entry of $\operatorname{adj}(A)$.

By the formula for the expansion of a determinant along a column,

$$x_i = \frac{1}{\det(A)} \det \begin{bmatrix} * & \cdots & b_1 & \cdots & * \\ \vdots & & \vdots & & \vdots \\ * & \cdots & b_n & \cdots & * \end{bmatrix}$$

where here the ith column of A is replaced with the column vector $[b_1 \cdots, b_n]^T$. The determinant of this modified matrix is taken and divided by $\det(A)$. This formula is known as Cramer's rule.

We formally define this method now.

Procedure 3.3.1: Using Cramer's Rule

Suppose A is an $n \times n$ invertible matrix and we wish to solve the system $AX = B$ for $X = [x_1, \cdots, x_n]^T$. Then Cramer's rule says

$$x_i = \frac{\det(A_i)}{\det(A)}$$

where A_i is the matrix obtained by replacing the ith column of A with the column matrix

$$B = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

We illustrate this procedure in the following example.

Example 3.3.4: Using Cramer's Rule

Find x, y, z if

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 2 & -3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Solution

We will use method outlined in Procedure 3.3.1 to find the values for x, y, z which give the solution to this system. Let

$$B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

In order to find x, we calculate

$$x = \frac{\det(A_1)}{\det(A)}$$

where A_1 is the matrix obtained from replacing the first column of A, the column corresponding to x, with B.

Hence, A_1 is given by

$$A_1 = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 3 & -3 & 2 \end{bmatrix}$$

Therefore,

$$x = \frac{\det(A_1)}{\det(A)} = \frac{\begin{vmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 3 & -3 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 2 & -3 & 2 \end{vmatrix}} = \frac{1}{2}$$

Similarly, to find y we construct A_2 by replacing the second column of A, the column corresponding to y, with B. Hence, A_2 is given by

$$A_2 = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 3 & 2 \end{bmatrix}$$

Therefore,

$$y = \frac{\det(A_2)}{\det(A)} = \frac{\begin{vmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 3 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 2 & -3 & 2 \end{vmatrix}} = -\frac{1}{7}$$

Similarly, A_3 is constructed by replacing the third column of A, the column corresponding to z, with B. Then, A_3 is given by

$$A_3 = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 2 \\ 2 & -3 & 3 \end{bmatrix}$$

Therefore, z is calculated as follows.

$$z = \frac{\det(A_3)}{\det(A)} = \frac{\begin{vmatrix} 1 & 2 & 1 \\ 3 & 2 & 2 \\ 2 & -3 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 2 & -3 & 2 \end{vmatrix}} = \frac{11}{14}$$

Cramer's Rule gives you another tool to consider when solving a system of linear equations.

We can also use Cramer's Rule for systems of non linear equations. Consider the following system where the matrix A has functions rather than numbers for entries.

Example 3.3.5: Use Cramer's Rule for Non-Constant Matrix

Solve for z if

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & e^t \cos t & e^t \sin t \\ 0 & -e^t \sin t & e^t \cos t \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ t \\ t^2 \end{bmatrix}$$

Solution

We are asked to find the value of z in the solution. We will solve using Cramer's rule. Thus

$$z = \frac{\begin{vmatrix} 1 & 0 & 1 \\ 0 & e^t \cos t & t \\ 0 & -e^t \sin t & t^2 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 0 \\ 0 & e^t \cos t & e^t \sin t \\ 0 & -e^t \sin t & e^t \cos t \end{vmatrix}} = t\left(\left(\cos t\right) t + \sin t\right) e^{-t}$$

Answers

Theorem 3.3.1

Recall that the (i,j)-entry of $\mathrm{adj}(A)$ is equal to $\mathrm{cof}(A)_{ji}$. Thus the (i,j)-entry of $B = A \cdot \mathrm{adj}(A)$ is :

$$B_{ij} = \sum_{k=1}^{n} a_{ik}\mathrm{adj}(A)_{kj} = \sum_{k=1}^{n} a_{ik}\mathrm{cof}(A)_{jk}$$

By the cofactor expansion theorem, we see that this expression for B_{ij} is equal to the determinant of the matrix obtained from A by replacing its jth row by $a_{i1}, a_{i2}, \ldots a_{in}$ — i.e., its ith row.

If $i = j$ then this matrix is A itself and therefore $B_{ii} = \det A$. If on the other hand $i \neq j$, then this matrix has its ith row equal to its jth row, and therefore $B_{ij} = 0$ in his case. Thus we obtain:

$$A \,\mathrm{adj}\,(A) = \det(A) I$$

Similarly we can verify that:

$$\mathrm{adj}\,(A)\, A = \det(A) I$$

And this proves the first part of the theorem.

Further if A is invertible, then by Theorem 3.2.5 we have:

$$1 = \det(I) = \det\left(AA^{-1}\right) = \det(A)\det\left(A^{-1}\right)$$

and thus $\det(A) \neq 0$. Equivalently, if $\det(A) = 0$, then A is not invertible.

Finally if $\det(A) \neq 0$, then the above formula shows that A is invertible and that:

$$A^{-1} = \frac{1}{\det(A)}\mathrm{adj}\,(A)$$

This completes the proof.

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3.3E: Exercises for Section 3.3

Exercise 3.3E.1

Determine whether the following matrices are invertible by computing their determinants.

a. $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$

b. $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Answer available at the end of this section.

Exercise 3.3E.2

Find the adjugate of the matrix:

$$A = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$$

Answer available at the end of this section.

Exercise 3.3E.3

Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

using the determinant and the adjugate matrix.

Answer available at the end of this section.

Exercise 3.3E.4

Find the inverse of the matrix

$$A = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$$

using the determinant and the adjugate matrix

Exercise 3.3E.5

Find the cofactor matrix for:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}$$

Exercise 3.3E.6

Find the inverse of

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

by computing the cofactor matrix and adjugate.

Exercise 3.3E.7

Solve the system

$$\begin{cases} 2x + 3y = 5 \\ 4x + y = 6 \end{cases}$$

using Cramer's Rule.

Answer available at the end of this section.

Exercise 3.3E.8

Solve the system

$$\begin{cases} x + 2y + 3z = 6 \\ 2x - y + z = 3 \\ 3x + y - 2z = 4 \end{cases}$$

using Cramer's Rule

Exercise 3.3E.9

Find a 3×3 system where Cramer's Rule does not apply, and explain why.

Hint available at the end of this section.

Exercise 3.3E.10

Show that for any invertible 2×2 matrix A,

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A).$$

Exercise 3.3E.11

If A is an upper triangular 3×3 matrix, explain how Cramer's Rule simplifies solving $Ax = b$.

Exercise 3.3E.12

In an electrical circuit, Kirchhoff's Laws lead to the system:

$$\begin{cases} 3I_1 + I_2 - I_3 = 5 \\ 2I_1 + 4I_2 + I_3 = 6 \\ -I_1 + 2I_2 + 5I_3 = 2 \end{cases}$$

Solve for the currents I_1, I_2, I_3 using Cramer's Rule.

Answer available at the end of this section.

Exercise $3.3E.13$

If the determinant of a matrix is very small but nonzero, what does this tell us about the matrix?

Answer available at the end of this section.

Exercise $3.3E.14$

Solve the system of equations using Cramer's Rule:

$$\begin{cases} x + y + z + w = 4 \\ 2x - y + 3z + w = 5 \\ x + 2y - z - w = 2 \\ 3x + y + 2z + 4w = 8 \end{cases}$$

Answer available at the end of this section.

Exercise $3.3E.15$

Consider the system:

$$\begin{cases} ax + y = 3 \\ 2x + (a+1)y = 5 \end{cases}$$

a. For what values of a does the system have no unique solution?
b. For what values of a does the system have a unique solution, and what is it in terms of a?

Answers

Exercise $3.3E.1$

a. $\det(A) = 5$. Invertible.
b. $\det(B) = 0$. Not invertible.

Exercise $3.3E.2$

The cofactor matrix is:

$$\text{cof}(A) = \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}$$

The adjugate (adjoint) matrix is the transpose of the cofactor matrix:

$$\text{adj}(A) = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}$$

Exercise $3.3E.3$

First, compute the determinant:

$$\det(A) = (1)(4) - (2)(3) = 4 - 6 = -2.$$

Next, compute the cofactor matrix:

$$\text{cof}(A) = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

Now, compute the adjugate (transpose of the cofactor matrix):

$$\text{adj}(A) = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$$

Using the formula for the inverse:

$$A^{-1} = \frac{1}{\det(A)}\text{adj}(A),$$

we substitute the values:

$$A^{-1} = \frac{1}{-2}\begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$$

Thus, the inverse is:

$$A^{-1} = \begin{bmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{bmatrix}.$$

Exercise $3.3E.7$

$x = \frac{\det(A_x)}{det(A)}, y = \frac{\det(A_y)}{det(A)}$, yielding $x = 1, y = 1$

Exercise $3.3E.9$

What must be true about determinant of the coefficient matrix for Cramer's Rule to work?

Exercise $3.3E.12$

Using determinants, $I_1 = 1, I_2 = 2, I_3 = -1$.

Exercise $3.3E.13$

There are at least two reasons:

1. While the matrix technically has an inverse, numerical computations involving the inverse may be unstable due to round-off errors.
2. Even small changes in input (such as rounding errors) can cause large errors in the output when solving systems of equations.

Exercise $3.3E.14$

$x = 2, y = -1, z = 3, w = 4$

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3.4: Determinants and Geometry

Learning Objectives

- Understand the relationship between the determinant of a matrix and the volume of a parallelepiped.
- Learn to use determinants to compute areas of parallelograms and triangles.

In this section we give a geometric interpretation of determinants, in terms of *areas* and *volumes*.

Recall the three properties of the determinants related to the elementary row operations. For an $n \times n$ matrix A:

1. If B is a matrix that results from switching two rows of A, then $\det(B) = -\det(A)$.
2. If B is a matrix that results from multiplying some row of A by a scalar k, then $\det(B) = k\det(A)$.
3. If B is a matrix that results from adding a multiple of a row of A to another row of A, then $\det(B) = \det(A)$.

Additionally, note that the determinant of an $n \times n$ identify matrix I_n is 1.

Parallelograms and Parallelepipeds

The determinant computes the volume of the following kind of geometric object.

Definition 3.4.1: Parallelepiped

The ***parallelepiped*** determined by n vectors $\vec{v}_1, \vec{v}_, \dots, \vec{v}_n$ in $\mathbb{R}^n$ is the subset

$$P = \left\{ a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n \;\middle|\; 0 \leq a_1, a_2, \dots, a_n \leq 1 \right\}.$$

In other words, a parallelepiped is the set of all linear combinations of n vectors with coefficients in $[0, 1]$. We can draw parallelepipeds using the parallelogram law for vector addition.

Example 3.4.1: The Unit Cube

The parallelepiped determined by the standard coordinate vectors $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$ is the unit n-dimensional cube.

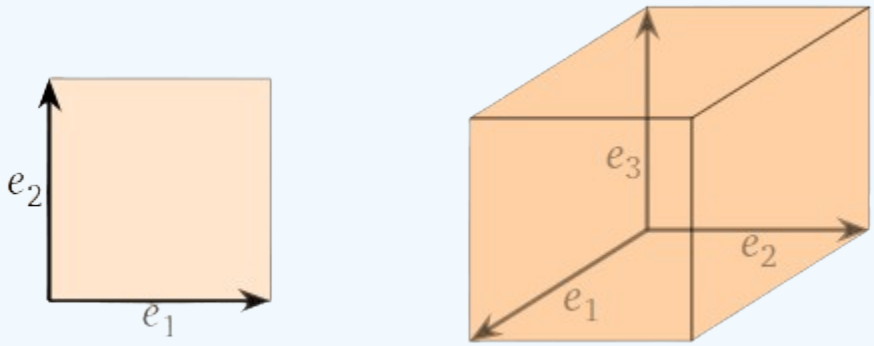


Figure 3.4.1: A 2D square with axes $\vec{e}_1$ and $\vec{e}_2$ on the left, and a 3D cube with axes $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ on the right, representing coordinate systems. (CC-BY-SA; Margalit and Rabinoff via Interactive Linear Algebra)

Example 3.4.2: Parallelograms

When $n = 2$, a parallelepiped is just a parallelogram in $\mathbb{R}^2$. Note that the edges come in parallel pairs.

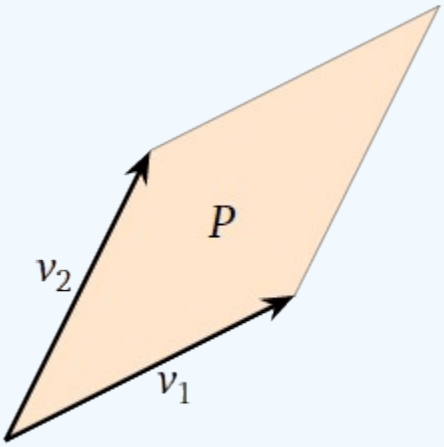


Figure 3.4.2: A parallelogram P formed by vectors $\vec{v}_1$ and $\vec{v}_2$, represented as black arrows from a common point. (CC-BY-SA; Margalit and Rabinoff via Interactive Linear Algebra)

Example 3.4.3

When $n = 3$, a parallelepiped is a kind of a skewed cube. Note that the faces come in parallel pairs.

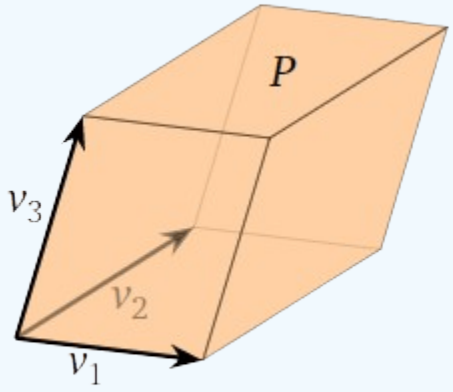


Figure 3.4.3: A parallelepiped P formed by vectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$, represented as black arrows from a common point. (CC-BY-SA; Margalit and Rabinoff via Interactive Linear Algebra)

When does a parallelepiped have zero volume? This can happen only if the parallelepiped is flat, i.e., it is squashed into a lower dimension.

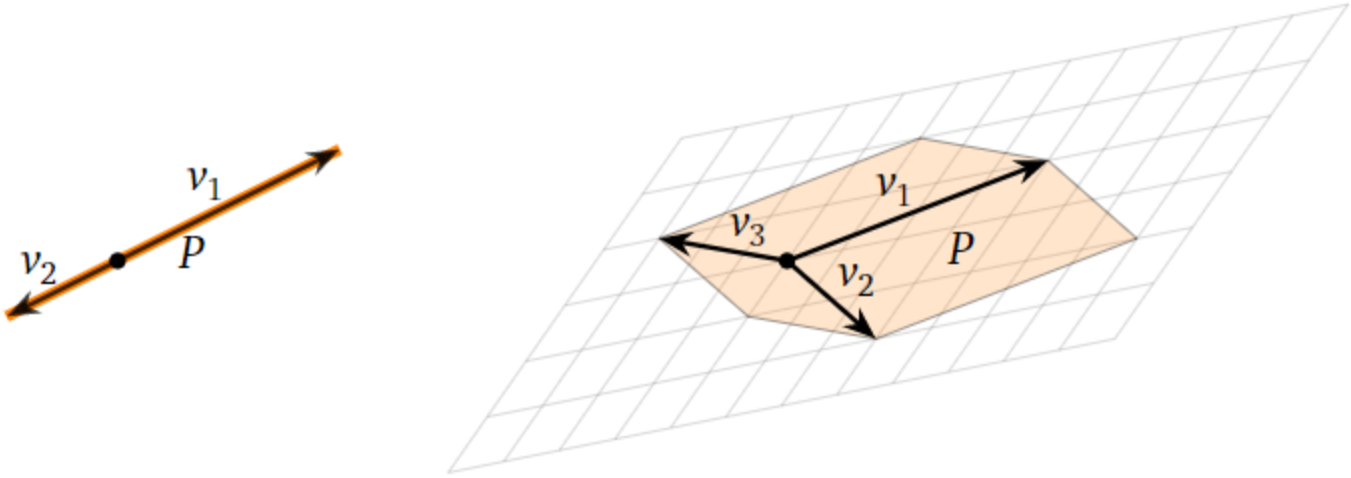


Figure 3.4.4: An illustration comparing geometric objects in one and two dimensions. On the left, a one-dimensional space is represented by a line segment with two vectors, $\vec{v}_1$ and $\vec{v}_2$, extending in opposite directions from a point P. On the right, a two-dimensional parallelogram is shown lying in a tilted plane on a grid. The parallelogram is defined by three vectors, $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$, originating from a common point P and forming a shaded area. (CC-BY-SA; Margalit and Rabinoff via Interactive Linear Algebra)

This means exactly that the set $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ has some dependence, implying that the matrix with rows $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ has determinant zero. To summarize:

Key Observation 3.4.1

The parallelepiped defined by $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ has zero volume if and only if the matrix with rows $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ has zero determinant.

Determinants and Volumes

The key observation above is only the beginning of the story: the volume of a parallelepiped is *always* a determinant.

Theorem 3.4.1: Determinants and Volumes

Let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ be vectors in $\mathbb{R}^n$, let P be the parallelepiped determined by these vectors, and let A be the matrix with rows $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$. Then the absolute value of the determinant of A is the volume of P:

$$|\det(A)| = \text{vol}(P).$$

Proof available at the end of this section.

Note

Since $\det(A) = \det(A^T)$, the absolute value of $\det(A)$ is also equal to the volume of the parallelepiped determined by the *columns* of A as well.

Example 3.4.4: Length

A 1×1 matrix A is just a number $\begin{pmatrix} a \end{pmatrix}$. In this case, the parallelepiped P determined by its one row is just the interval $[0, a]$ (or $[a, 0]$ if $a < 0$). The "volume" of a region in $\mathbb{R}^1 = \mathbb{R}$ is just its length, so it is clear in this case that $\text{vol}(P) = |a|$.

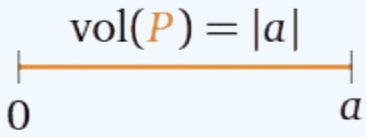


Figure 3.4.8: A diagram illustrating the volume (length) of a one-dimensional interval. A horizontal orange line segment extends from 0 to a, with vertical bars at both endpoints. Above the segment, the equation $\text{vol}(P) = |a|$ is displayed, indicating that the length of the interval is given by the absolute value of a. The endpoints are labeled 0 and a, representing the interval's boundaries. (CC-BY-SA; Margalit and Rabinoff via Interactive Linear Algebra)

Example 3.4.5: Area

When A is a 2×2 matrix, its rows determine a parallelogram in $\mathbb{R}^2$. The "volume" of a region in $\mathbb{R}^2$ is its area, so we obtain a formula for the area of a parallelogram: it is the determinant of the matrix whose rows are the vectors forming two adjacent sides of the parallelogram.

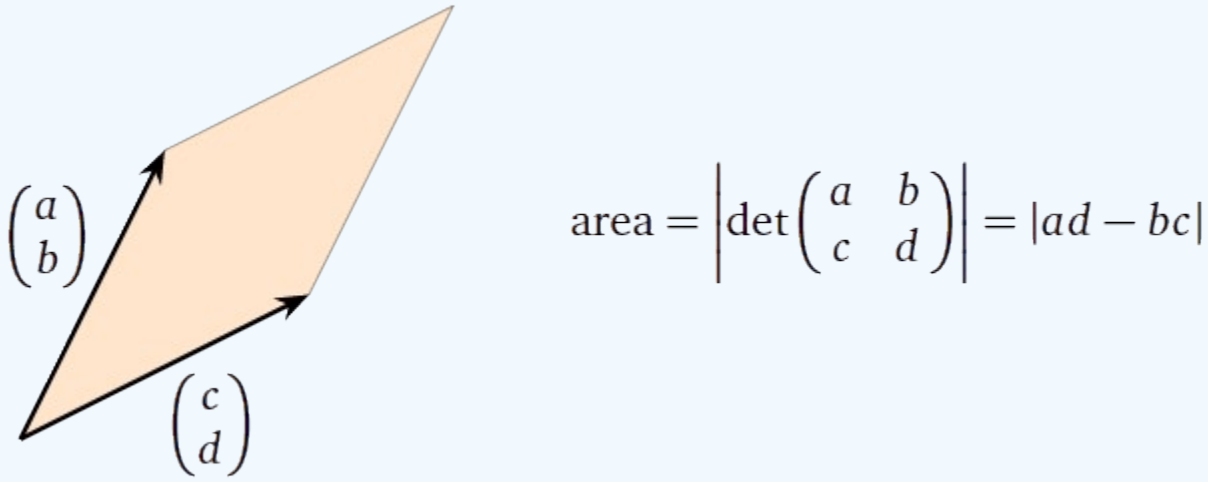


Figure 3.4.9: A diagram illustrating the area of a parallelogram using the determinant. On the left, a parallelogram is shown in a two-dimensional plane, defined by two black vectors originating from a common point. These vectors are labeled as column vectors $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} c \\ d \end{pmatrix}$. The parallelogram is shaded in light orange. On the right, the formula for the area of the parallelogram is displayed: $\text{area} = \left|\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right| = |ad - bc|$ which shows that the absolute value of the determinant of the matrix formed by the two vectors gives the area of the parallelogram. This diagram visually demonstrates how the determinant provides a measure of area in two dimensions. (CC-BY-SA; Margalit and Rabinoff via Interactive Linear Algebra)

It is perhaps surprising that it is possible to compute the area of a parallelogram without trigonometry. It is a fun geometry problem to prove this formula by hand. [Hint: first think about the case when the first row of A lies on the x-axis.]

Example 3.4.6

Find the area of the parallelogram with sides $(1, 3)$ and $(2, -3)$.

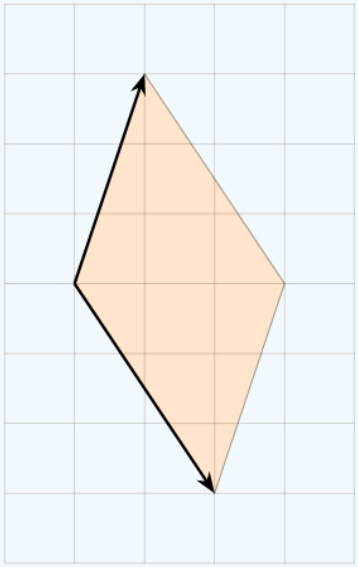


Figure 3.4.10: A parallelogram plotted on a coordinate grid, defined by two black arrows representing vectors originating from a common point. The parallelogram is shaded in light orange, illustrating the region whose area is determined by the determinant of the matrix formed by the vectors. The grid provides a reference for the positioning of the parallelogram in two-dimensional space. (CC-BY-SA; Margalit and Rabinoff via Interactive Linear Algebra)

Solution

The area is

$$\left|\det\begin{pmatrix}1 & 3\\ 2 & -3\end{pmatrix}\right| = |-3-6| = 9.$$

Example 3.4.7

Find the area of the parallelogram in the picture.

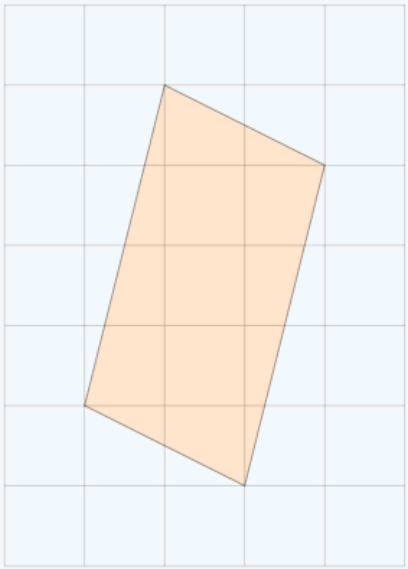


Figure 3.4.11: A parallelogram plotted on a coordinate grid. The points are not labeled. Starting with the top-left point as the first point and moving clockwise, the second point is 2 units right and 1 unit down from the first point. The third point is 1 unit left and 4 units down from the second point. The fourth point is 2 units left and 1 unit up from the third point. The first point is 1 unit right and 4 units up from the fourth point. (CC-BY-SA; Margalit and Rabinoff via Interactive Linear Algebra)

Solution

We choose two adjacent sides to be the rows of a matrix. We choose the top two:

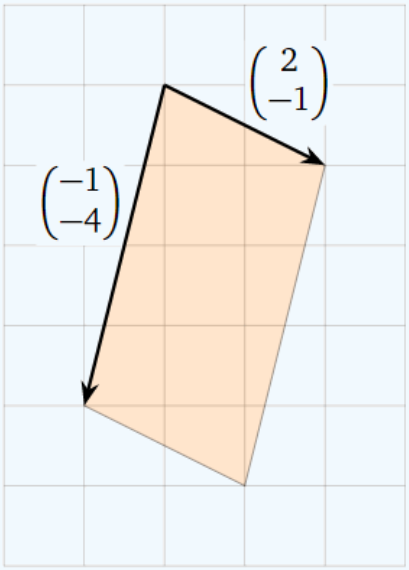


Figure 3.4.12: A parallelogram plotted on a coordinate grid, defined by two vectors originating from a common point (the top-left point of the parallelogram). The vectors are labeled with their components: $\begin{pmatrix}-1\\-4\end{pmatrix}$ and $\begin{pmatrix}2\\-1\end{pmatrix}$. (CC-BY-SA; Margalit and Rabinoff via Interactive Linear Algebra)

Note that we do not need to know where the origin is in the picture: vectors are determined by their length and direction, not where they start. The area is

$$\left|\det\begin{pmatrix}-1 & -4\\ 2 & -1\end{pmatrix}\right| = |1+8| = 9.$$

Example 3.4.8: Area of a triangle

Find the area of the triangle with vertices $(-1,-2),\ (2,-1),\ (1,3)$.

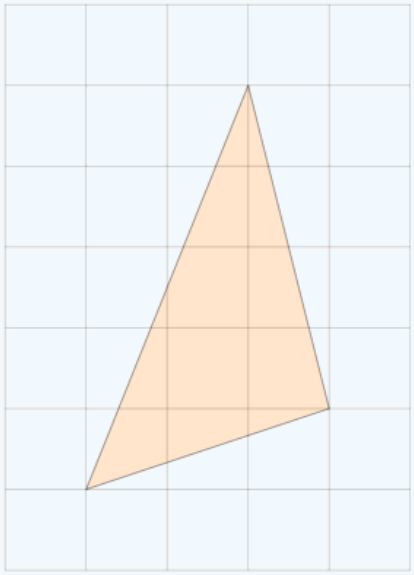


Figure 3.4.13: A triangle plotted on a coordinate grid. The triangle is formed by three points, creating a geometric region in two-dimensional space. The points are not labeled. Starting with the bottom-left point as the first point and moving clockwise, the second point is 2 units right and 5 units up. The third point is 1 unit right and 4 units down from the second point. The first point is 3 units left and 1 unit down from the third point. (CC-BY-SA; Margalit and Rabinoff via Interactive Linear Algebra)

Solution

Doubling a triangle makes a parallelogram. We choose two of its sides to be the rows of a matrix.

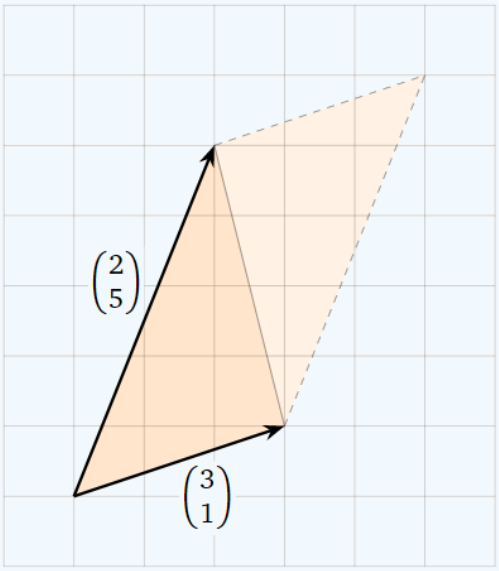


Figure 3.4.14: A triangle given in the example is extended to a parallelogram defined by two vectors originating from a common point. The vectors are labeled with their components: $\begin{pmatrix}2\\5\end{pmatrix}$ and $\begin{pmatrix}3\\1\end{pmatrix}$. The original triangle is shaded darker than the triangle added to form the parallelogram. (CC-BY-SA; Margalit and Rabinoff via Interactive Linear Algebra)

The area of the parallelogram is

$$\left|\det\begin{pmatrix}2&5\\3&1\end{pmatrix}\right| = |2-15| = 13,$$

so the area of the triangle is $13/2$.

You might be wondering: if the absolute value of the determinant is a volume, what is the geometric meaning of the determinant without the absolute value? The next remark explains that we can think of the determinant as a *signed* volume. If you have taken an integral calculus course, you probably computed negative areas under curves; the idea here is similar.

Remark: Signed volumes

Theorem 3.4.1 on determinants and volumes tells us that the *absolute value* of the determinant is the volume of a paralellepiped. This raises the question of whether the sign of the determinant has any geometric meaning.

A 1×1 matrix A is just a number $\begin{pmatrix} a \end{pmatrix}$. In this case, the parallelepiped P determined by its one row is just the interval $[0, a]$ if $a \geq 0$, and it is $[a, 0]$ if $a < 0$. In this case, the sign of the determinant determines whether the interval is to the left or the right of the origin.

For a 2×2 matrix with rows v_1, v_2, the sign of the determinant determines whether v_2 is counterclockwise or clockwise from v_1. That is, if the counterclockwise angle from v_1 to v_2 is less than 180°, then the determinant is positive; otherwise it is negative (or zero).

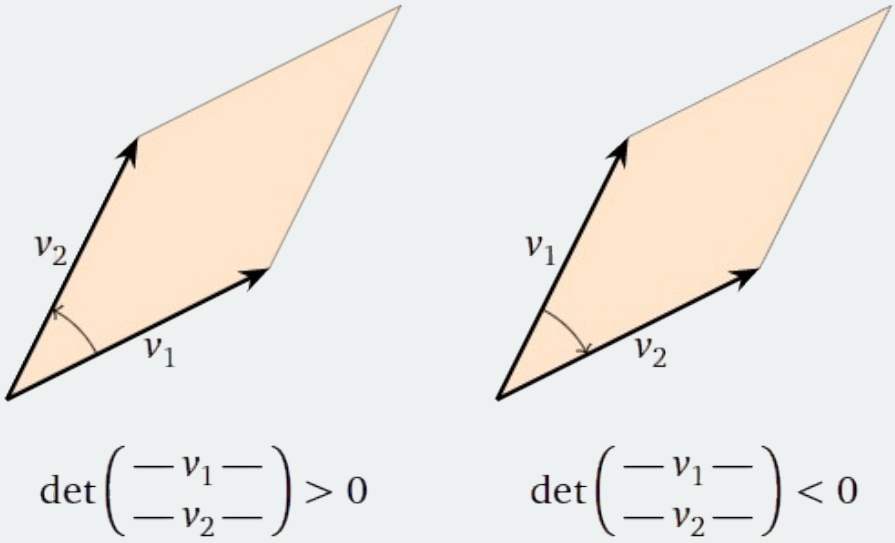


Figure 3.4.15: A comparison of two parallelograms, each defined by two vectors, $\vec{v}_1$ and $\vec{v}_2$, originating from a common point. On the left, the parallelogram is oriented such that the determinant of the matrix formed by $\vec{v}_1$ and $\vec{v}_2$ is positive, indicated by the expression: $\det(\vec{v}_1 \quad \vec{v}_2) > 0$. A signed angle between the vectors is shown and is positive, emphasizing their orientation. On the right, the parallelogram is oriented in the opposite way, resulting in a negative determinant: $\det(\vec{v}_1 \quad \vec{v}_2) < 0$. The reversed orientation demonstrates the effect of swapping the vectors on the determinant's sign. (CC-BY-SA; Margalit and Rabinoff via Interactive Linear Algebra)

For example, if $v_1 = \begin{pmatrix} a \\ b \end{pmatrix}$, then the counterclockwise rotation of v_1 by 90° is $v_2 = \begin{pmatrix} -b \\ a \end{pmatrix}$ by Example 3.3.8 in Section 3.3, and

$$\det\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = a^2 + b^2 > 0.$$

On the other hand, the *clockwise* rotation of v_1 by 90° is $\begin{pmatrix} b \\ -a, \end{pmatrix}$ and

$$\det\begin{pmatrix} a & b \\ b & -a \end{pmatrix} = -a^2 - b^2 < 0.$$

For a 3×3 matrix with rows v_1, v_2, v_3, the *right-hand rule* determines the sign of the determinant. If you point the index finger of your right hand in the direction of v_1 and your middle finger in the direction of v_2, then the determinant is positive if your thumb points roughly in the direction of v_3, and it is negative otherwise.

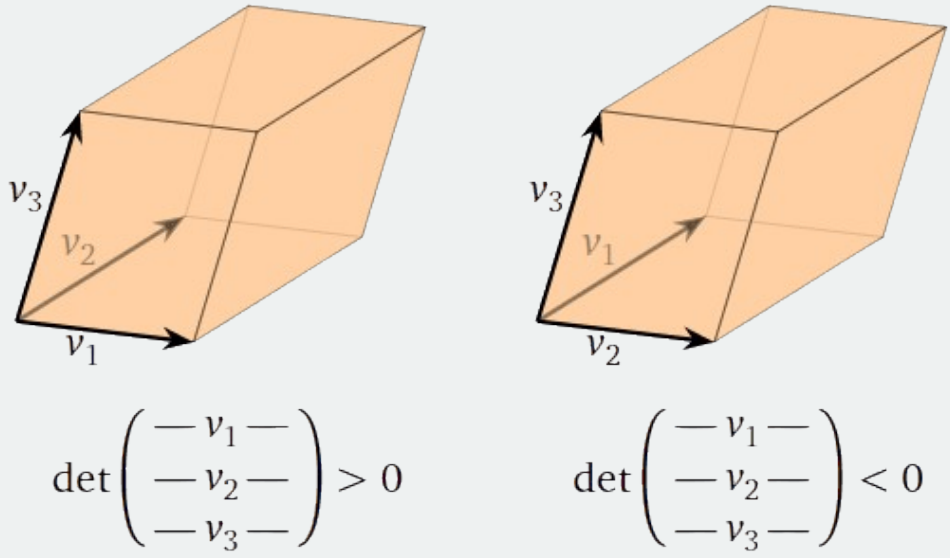


Figure 3.4.16: A comparison of two three-dimensional parallelepipeds, each defined by three vectors, $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$, originating from a common point. On the left, the parallelepiped is oriented such that the determinant of the matrix formed by $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ is positive, indicated by the expression: $\det(\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3) > 0$. The vectors are labeled and shown extending from a common vertex. On the right, the parallelepiped is oriented differently, resulting in a negative determinant: $\det(\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3) < 0$. The reversed orientation demonstrates the effect of changing the order of the vectors on the determinant's sign. (CC-BY-SA; Margalit and Rabinoff via Interactive Linear Algebra)

In higher dimensions, the notion of signed volume is still important, but it is usually *defined* in terms of the sign of a determinant.

Answers

Theorem 3.4.1: Determinants and Volumes

Case 1: There is a dependence in the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$. In this case, we established that the volume of the parallelopiped is 0 in Key Observation 3.4.1.

Case 2: There is no dependence in the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$. Hence the matrix A must row reduce to I_n. Suppose the required row operations correspond to left multiplying the row of A by elementary matrices $E_1, E_2, \ldots, E_k$. That is

$$E_m \ldots E_2 E_1 A = I_n$$

Since all elementary matrices are invertible and are elementary matrices themselves, we can express the above relationship as

$$A = E_1^{-1} E_2^{-1} \ldots E_m^{-1} I_n$$

Let's begin with the $n \times n$ identity matrix I_n. This matrix consists of row vectors $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$. The volume of the parallelepiped formed by these vectors is 1.

Now let us apply the opposite of the elementary row operations in reverse used to row reduce A to I_n. This would give us the row operations needed to transform I_n to A using elementary row operations. What effect do these row operations have on the absolute value of the determinant?

We begin with I_n, and note that $|\det(I_n)| = 1$.

Suppose the next row operation consists of switching two rows. Switching two rows of a matrix just reorders the vectors $v_1, v_2, \ldots, v_n$, hence has no effect on the parallelepiped determined by those vectors. Thus, the volume is unchanged by row swaps. The determinant after swapping rows will be negative, but the *absolute value* of the determinant is unchanged.

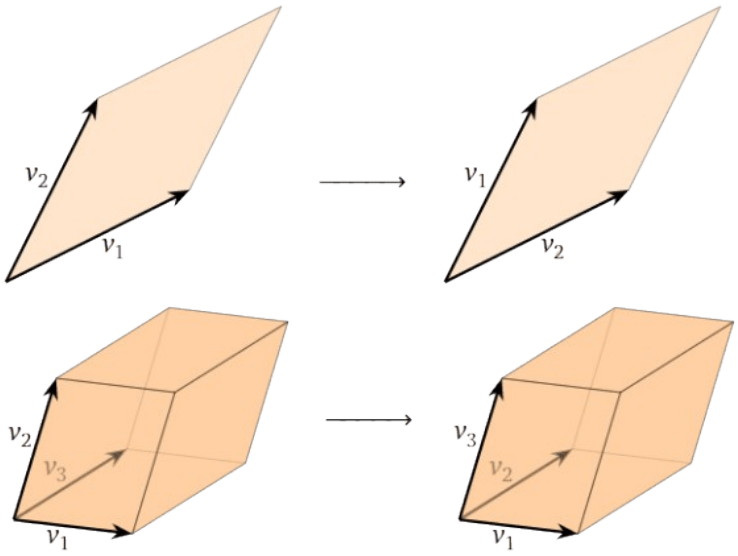


Figure 3.4.5: A diagram illustrating the effect of row swapping on area and volume in linear algebra. In the top row, two parallelograms are shown. On the left, the parallelogram is defined by vectors $\vec{v}_1$ and $\vec{v}_2$, with $\vec{v}_1$ positioned as the first vector and $\vec{v}_2$ as the second. On the right, the parallelogram remains the same in shape and size, but the roles of $\vec{v}_1$ and $\vec{v}_2$ are swapped. This transformation demonstrates that swapping two rows negates the determinant. In the bottom row, two parallelepipeds are shown. On the left, the parallelepiped is defined by vectors $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$, with the volume determined by their orientation. On the right, $\vec{v}_2$ and $\vec{v}_3$ have swapped places, maintaining the shape but flipping the determinant's sign. (CC-BY-SA; Margalit and Rabinoff via Interactive Linear Algebra)

Now , suppose the next row operation consists of multiplying a row by a constant k. For simplicity, we consider a row scaling of the form $kR_n -> R_n$. This scales the length of $\vec{v}_n$ by a factor of $|k|$, which also scales the perpendicular distance of $\vec{v}_n$ from the base by a factor of $|k|$. Thus, the volume of the resulting parallelepiped is scaled by $|k|$, as is the determinant of the matrix made with the tranformed vectors as rows. >

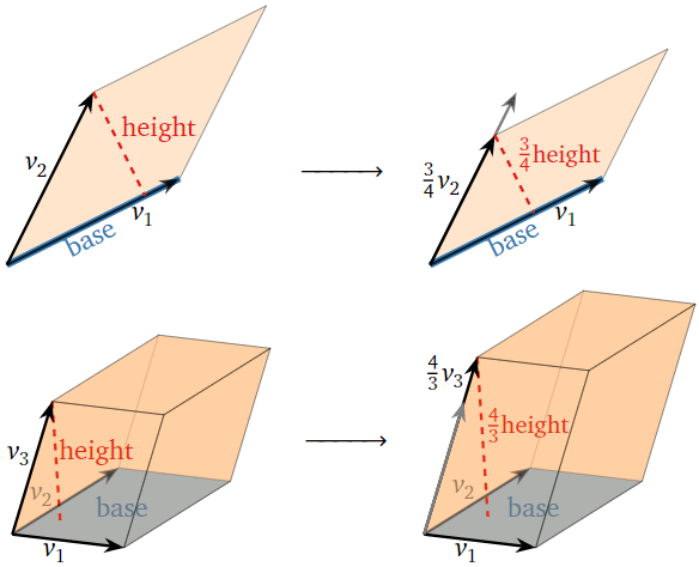


Figure 3.4.6: A diagram illustrating the effect of row scaling on area and volume in linear algebra. In the top row, two parallelograms are shown. On the left, a parallelogram is spanned by vectors $\vec{v}_1$ and $\vec{v}_2$, where $\vec{v}_1$ is labeled as the "base" in blue, and a red dashed line represents the "height" perpendicular to it. On the right, the vector v_2 is scaled by $\frac{3}{4}$, reducing the height proportionally to $\frac{3}{4}$ of its original value. This transformation illustrates that scaling a vector changes the determinant by the same factor. In the bottom row, two parallelepipeds are shown. On the left, the parallelepiped is defined by vectors $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$, where $\vec{v}_1$ and $\vec{v}_2$ form the "base" (shaded in gray), and a red dashed line represents the "height" along $\vec{v}_3$. On the right, the vector $\vec{v}_3$ is scaled by $\frac{4}{3}$, increasing the height proportionally to $\frac{4}{3}$ of its original value. This transformation illustrates that scaling a vector changes the volume by the same factor. (CC-BY-SA; Margalit and Rabinoff via Interactive Linear Algebra)

Now, suppose the next row operation consists of adding a multiple of one row to another. For simplicity, we consider a row replacement of the form $R_n + cR_i -> R_n$. The volume of a parallelepiped is the volume of its base, times its height: Here the “base” is the parallelepiped determined by $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{n-1}$, and the “height” is the perpendicular distance of $\vec{v}_n$ from the base. Translating $\vec{v}_n$ by a multiple of $\vec{v}_i$ moves $\vec{v}_n$ in a direction parallel to the base. This changes neither the base nor the height! Thus, the volume of the parallelepiped is unchanged by row replacements. And as we know from before, this row operation keeps the determinant of the matrix made of the updated rows unchanged.

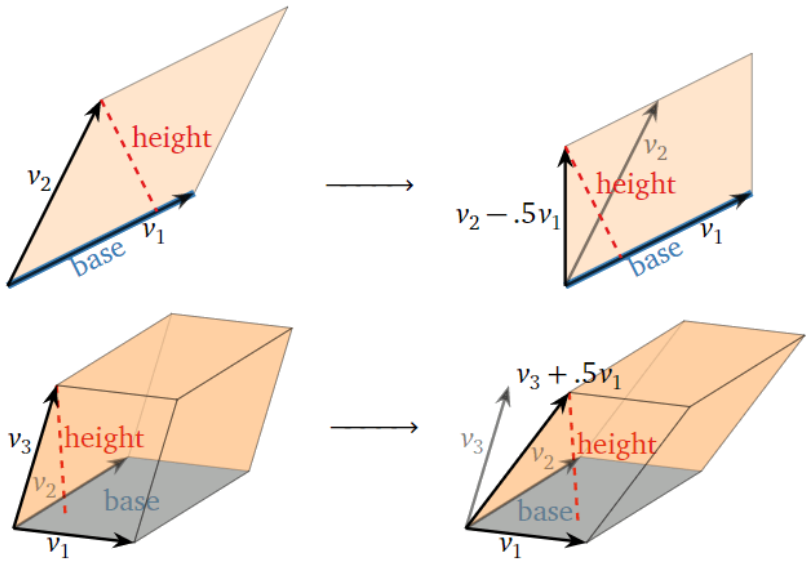


Figure 3.4.7: A diagram demonstrating the invariance of area and volume under row operations. In the top row, two parallelograms are shown. On the left, a parallelogram is spanned by vectors $\vec{v}_1$ and $\vec{v}_2$, where $\vec{v}_1$ is labeled as the "base" in blue, and a red dashed line represents the "height" perpendicular to it. On the right, the parallelogram undergoes a row operation where $\vec{v}_2$ is replaced by $\vec{v}_2 - 0.5\vec{v}_1$, but the height remains unchanged, preserving the area. In the bottom row, two parallelepipeds are shown. On the left, the parallelepiped is defined by vectors $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$, where $\vec{v}_1$ and $\vec{v}_2$ form the "base" (shaded in gray), and a red dashed line represents the "height" along $\vec{v}_3$. On the right, after the row operation $\vec{v}_3 \to \vec{v}_3 + 0.5\vec{v}_1$, the height remains the same, illustrating that the volume is preserved. (CC-BY-SA; Margalit and Rabinoff via Interactive Linear Algebra)

As we can see, the effect of performing elementary row operations on I_n is identical on the volume of the parallelepiped and the absolute value of the determinant. Hence,

$$|\det(A)| = \text{vol}(P).$$

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3.4E: Exercises for Section 3.4

Exercise 3.4E.1

Find the area of the parallelogram formed by the vectors

$\vec{v}_1 = \langle 3, 1\rangle, \quad \vec{v}_2 = \langle 2, -2\rangle.$

Answer available at the end of this section.

Exercise 3.4E.2

For what values of k do the vectors $\vec{v}_1 = \langle 2, -3\rangle, \quad \vec{v}_2 = \langle -4, k\rangle$ form a parallelogram with zero area?

Answer available at the end of this section.

Exercise 3.4E.3

Find the volume of the parallelepiped formed by

$$\vec{v}_1 = \langle 1, 2, 3\rangle, \quad \vec{v}_2 = \langle 2, 3, 1\rangle, \quad \vec{v}_3 = \langle 3, 1, 2\rangle. \tag{3.4E.1}$$

Answer available at the end of this section.

Exercise 3.4E.4

For what values of k do the vectors

$$\vec{v}_1 = \langle 1, 2, 3\rangle, \quad \vec{v}_2 = \langle 4, k, 6\rangle, \quad \vec{v}_3 = \langle 7, 8, 9\rangle \tag{3.4E.2}$$

form a parallelepiped with zero volume?

Answer available at the end of this section.

Exercise 3.4E.5

Find the area of the triangle with vertices $A(0, 0)$, $B(2, 3)$, and $C(4, 1)$.

Answer available at the end of this section.

Exercise 3.4E.6

Find the area of the triangle with vertices $(1, 2)$, $(4, 6)$, and $(5, 3)$.

Exercise 3.4E.7

For what values of k do the vectors

$$(1, 2, 3), \quad (2, 4, 6), \quad (3, 6, k) \tag{3.4E.3}$$

do not determine a 3 dimensional parallelepiped?

Exercise 3.4E.8

Suppose vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are the rows of a skew symmetric matrix A. Prove that the volume of the n-dimensional parallelepiped formed by $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ has zero volume.

Hint available at the end of this section.

Exercise 3.4E.9

a. Show that if three vectors in $\mathbb{R}^3$ lie in the same plane, the determinant of their matrix representation is zero.
b. Why does this happen geometrically?
c. Provide an example of three linearly dependent vectors and compute their determinant.

Exercise 3.4E.10

a. Given three points $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$, show that they are collinear if and only if

$$\det \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = 0. \tag{3.4E.4}$$

b. Interpret this result in terms of area.
c. What happens when four points in $\mathbb{R}^3$ satisfy a similar determinant condition?

Exercise 3.4E.11

Suppose we start with the identity matrix I_3. Perform the following operations to obtain matrix A:

- $2R_1 + R_2 -> R_2$
- $3R_2 + R_3 -> R_3$
- $R_2 <-> R_3$
- $-4R_2 + R_1 -> R_1$

a. What is A?
b. What is $\det(A)$?

Answers

Exercise 3.4E.1
Area = 8

Exercise 3.4E.2
$k = 6$

Exercise 3.4E.3
Volume = 18

Exercise 3.4E.4
$k = 6$.

Exercise 3.4E.5
Area = 5.

Exercise 3.4E.8
Since A is skew symmetric, $A^T = -A$.

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4: R^n

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4.2E: Exercises for Section 4.2

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4.1: Review of Vectors

Learning Objectives

- Find the position vector of a point in $\mathbb{R}^n$.
- Understand vector addition and scalar multiplication, algebraically.
- Understand vector addition, geometrically.
- Find the length of a vector and the distance between two points in $\mathbb{R}^n$.
- Find the corresponding unit vector to a vector in $\mathbb{R}^n$.
- Understand scalar multiplication, geometrically.
- Introduce the notion of linear combination of vectors.

In this section we review the concept and operations between vectors.

The notation $\mathbb{R}^n$ refers to the collection of ordered lists of n real numbers, that is

$$\mathbb{R}^n = \{(x_1, \cdots, x_n) : x_j \in \mathbb{R} \text{ for } j = 1, \cdots, n\}$$

In this chapter, we take a closer look at vectors in $\mathbb{R}^n$. First, we will consider what $\mathbb{R}^n$ looks like in more detail. Recall that the point given by $0 = (0, \cdots, 0)$ is called the **origin**.

Now, consider the case of $\mathbb{R}^n$ for $n = 1$. Then from the definition we can identify $\mathbb{R}$ with points in $\mathbb{R}^1$ as follows:

$$\mathbb{R} = \mathbb{R}^1 = \{(x_1) : x_1 \in \mathbb{R}\}$$

Hence, $\mathbb{R}$ is defined as the set of all real numbers and geometrically, we can describe this as all the points on a line.

Now suppose $n = 2$. Then, from the definition,

$$\mathbb{R}^2 = \{(x_1, x_2) : x_j \in \mathbb{R} \text{ for } j = 1, 2\}$$

Consider the familiar coordinate plane, with an x axis and a y axis. Any point within this coordinate plane is identified by where it is located along the x axis, and also where it is located along the y axis. Consider as an example the following diagram.

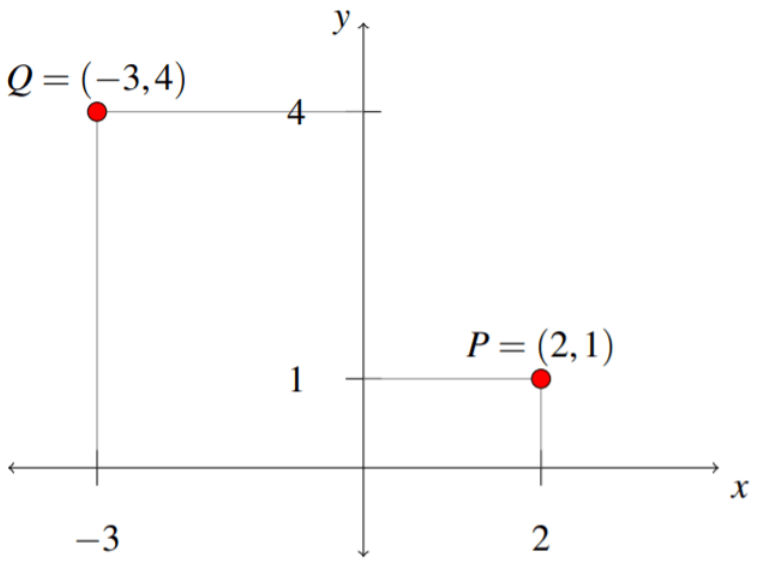


Figure 4.1.1: Diagram illustrates points $Q = (-3, 4)$ and $P = (2, 1)$ in $\mathbb{R}^2$. (CC BY-NC-SA 4.0; Kuttler via A First Course in LINEAR ALGEBRA)

Hence, every element in $\mathbb{R}^2$ is identified by two components, x and y, in the usual manner. The coordinates x, y (or x_1,x_2) uniquely determine a point in the plan. Note that while the definition uses x_1 and x_2 to label the coordinates and you may be used to x and y, these notations are equivalent.

Now suppose $n = 3$. You may have previously encountered the 3-dimensional coordinate system, given by

$$\mathbb{R}^3 = \{(x_1, x_2, x_3) : x_j \in \mathbb{R} \text{ for } j = 1, 2, 3\}$$

Points in $\mathbb{R}^3$ will be determined by three coordinates, often written (x, y, z) which correspond to the x, y, and z axes. We can think as above that the first two coordinates determine a point in a plane. The third component determines the height above or below the plane, depending on whether this number is positive or negative, and all together this determines a point in space. You see that the ordered triples correspond to points in space just as the ordered pairs correspond to points in a plane and single real numbers correspond to points on a line.

The idea behind the more general $\mathbb{R}^n$ is that we can extend these ideas beyond $n = 3$. This discussion regarding points in $\mathbb{R}^n$ leads into a study of vectors in $\mathbb{R}^n$. While we consider $\mathbb{R}^n$ for all n, we will largely focus on $n = 2, 3$ in this section.

Consider the following definition.

Definition 4.1.1 The Position Vector

Let $P = (p_1, \cdots, p_n)$ be the coordinates of a point in $\mathbb{R}^n$. Then the vector $\overrightarrow{0P}$ with its tail at $0 = (0, \cdots, 0)$ and its tip at P is called the **position vector** of the point P. We write

$$\overrightarrow{0P} = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix}$$

For this reason we may write both $P = (p_1, \cdots, p_n) \in \mathbb{R}^n$ and $\overrightarrow{0P} = [p_1 \cdots p_n]^T \in \mathbb{R}^n$.

This definition is illustrated in the following picture for the special case of $\mathbb{R}^3$.

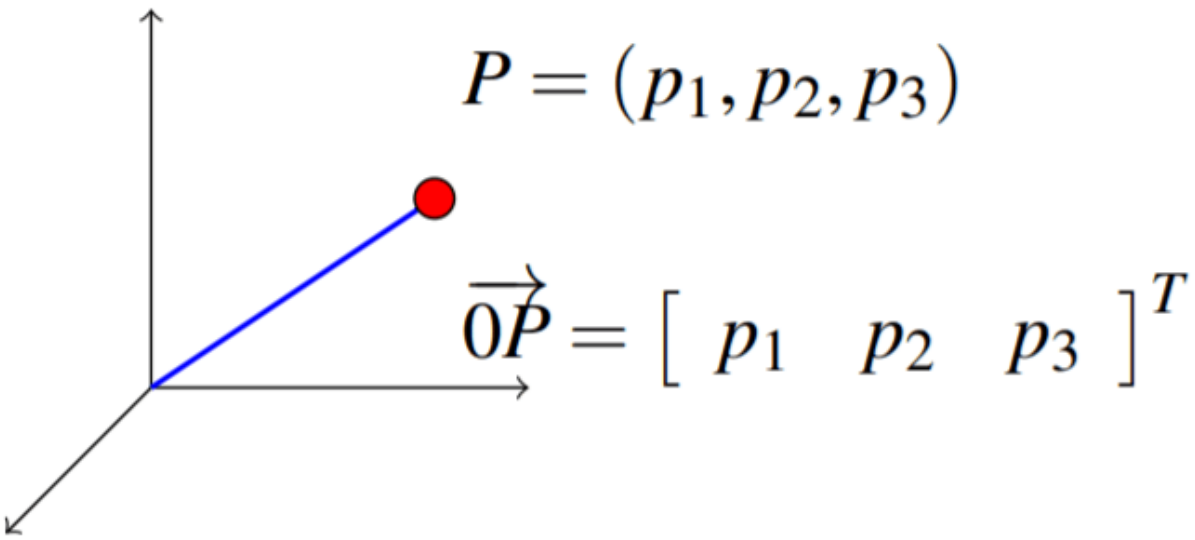


Figure 4.1.2: Diagram illustrates a 3D coordinate system where $P(p_1, p_2, p_3)$ is a point in space. The position vector $\vec{OP}$ extends from the origin O to the point P. (CC BY-NC-SA 4.0; Kuttler via A First Course in LINEAR ALGEBRA)

Thus every point P in $\mathbb{R}^n$ determines its position vector $\overrightarrow{0P}$. Conversely, every such position vector $\overrightarrow{0P}$ which has its tail at 0 and point at P determines the point P of $\mathbb{R}^n$.

Now suppose we are given two points, P, Q whose coordinates are $(p_1, \cdots, p_n)$ and $(q_1, \cdots, q_n)$ respectively. We can also determine the **position vector from** P **to** Q (also called the **vector from** P **to** Q) defined as follows.

$$\overrightarrow{PQ} = \begin{bmatrix} q_1 - p_1 \\ \vdots \\ q_n - p_n \end{bmatrix} = \overrightarrow{0Q} - \overrightarrow{0P}$$

Now, imagine taking a vector in $\mathbb{R}^n$ and moving it around, always keeping it pointing in the same direction as shown in the following picture.

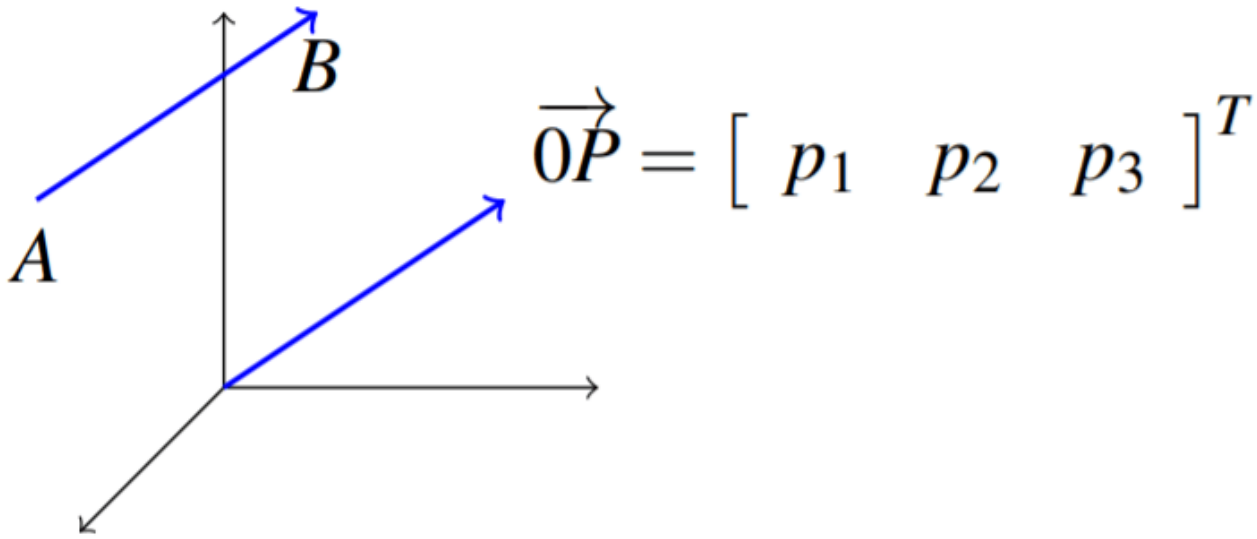


Figure 4.1.3: Diagram illustrates vectors in $\mathbb{R}^3$. $\vec{OP}$ is the position vector from O to point P. $\vec{AB}$ is a vector parallel to $\vec{OP}$. (CC BY-NC-SA 4.0; Kuttler via A First Course in LINEAR ALGEBRA)

After moving it around, it is regarded as the same vector. Each vector, $\overrightarrow{0P}$ and $\overrightarrow{AB}$ has the same length (or magnitude) and direction. Therefore, they are equal.

Consider now the general definition for a vector in $\mathbb{R}^n$.

Definition 4.1.2 Vectors in $\mathbb{R}^n$

Let $\mathbb{R}^n = \{(x_1, \cdots, x_n) : x_j \in \mathbb{R} \text{ for } j = 1, \cdots, n\}$. Then,

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

is called a **vector**. Vectors have both size (magnitude) and direction. The numbers x_j are called the **components** of $\vec{x}$.

Using this notation, we may use $\vec{p}$ to denote the position vector of point P. Notice that in this context, $\vec{p} = \overrightarrow{0P}$. These notations may be used interchangeably.

You can think of the components of a vector as directions for obtaining the vector. Consider $n = 3$. Draw a vector with its tail at the point $(0, 0, 0)$ and its tip at the point (a, b, c). This vector it is obtained by starting at $(0, 0, 0)$, moving parallel to the x axis to $(a, 0, 0)$ and then from here, moving parallel to the y axis to $(a, b, 0)$ and finally parallel to the z axis to (a, b, c). Observe that the same vector would result if you began at the point (d, e, f), moved parallel to the x axis to $(d + a, e, f)$, then parallel to the y axis to $(d + a, e + b, f)$, and finally parallel to the z axis to $(d + a, e + b, f + c)$. Here, the vector would have its tail sitting at the point determined by $A = (d, e, f)$ and its point at $B = (d + a, e + b, f + c)$. It is the **same vector** because it will point in the same direction and have the same length. It is like you took an actual arrow, and moved it from one location to another keeping it pointing the same direction.

We conclude this section with a brief discussion regarding notation. In previous sections, we have written vectors as columns, or

$n \times 1$ matrices. For convenience in this chapter we may write vectors as the transpose of row vectors, or $1 \times n$ matrices. These are of course equivalent and we may move between both notations. Therefore, recognize that

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 & 3 \end{bmatrix}^T$$

Notice that two vectors $\vec{u} = [u_1 \cdots u_n]^T$ and $\vec{v} = [v_1 \cdots v_n]^T$ are equal if and only if all corresponding components are equal. Precisely,

$$\vec{u} = \vec{v} \text{ if and only if}$$
$$u_j = v_j \text{ for all } j = 1, \cdots, n$$

Thus $\begin{bmatrix} 1 & 2 & 4 \end{bmatrix}^T \in \mathbb{R}^3$ and $\begin{bmatrix} 2 & 1 & 4 \end{bmatrix}^T \in \mathbb{R}^3$ but $\begin{bmatrix} 1 & 2 & 4 \end{bmatrix}^T \neq \begin{bmatrix} 2 & 1 & 4 \end{bmatrix}^T$ because, even though the same numbers are involved, the order of the numbers is different.

For the specific case of $\mathbb{R}^3$, there are three special vectors which we often use. They are given by

$$\vec{i} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$$
$$\vec{j} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$$
$$\vec{k} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$$

We can write any vector $\vec{u} = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^T$ as a linear combination of these vectors, written as $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$. This notation will be used throughout this chapter.

Addition and scalar multiplication are two important algebraic operations done with vectors. Notice that these operations apply to vectors in $\mathbb{R}^n$, for any value of n. We will explore these operations in more detail in the following sections.

Recall that an element of $\mathbb{R}^n$ is an ordered list of numbers. For the specific case of $n = 2, 3$ this can be used to determine a point in two or three dimensional space. This point is specified relative to some coordinate axes.

Consider the case $n = 3$. Recall that taking a vector and moving it around without changing its length or direction does not change the vector. This is important in the geometric representation of vector addition.

Suppose we have two vectors, $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$. Each of these can be drawn geometrically by placing the tail of each vector at 0 and its point at (u_1, u_2, u_3) and (v_1, v_2, v_3) respectively. Suppose we slide the vector $\vec{v}$ so that its tail sits at the point of $\vec{u}$. We know that this does not change the vector $\vec{v}$. Now, draw a new vector from the tail of $\vec{u}$ to the point of $\vec{v}$. This vector is $\vec{u} + \vec{v}$.

The geometric significance of vector addition in $\mathbb{R}^n$ for any n is given in the following definition.

Definition 4.1.3: Geometry of Vector Addition

Let $\vec{u}$ and $\vec{v}$ be two vectors. Slide $\vec{v}$ so that the tail of $\vec{v}$ is on the point of $\vec{u}$. Then draw the arrow which goes from the tail of $\vec{u}$ to the point of $\vec{v}$. This arrow represents the vector $\vec{u} + \vec{v}$.

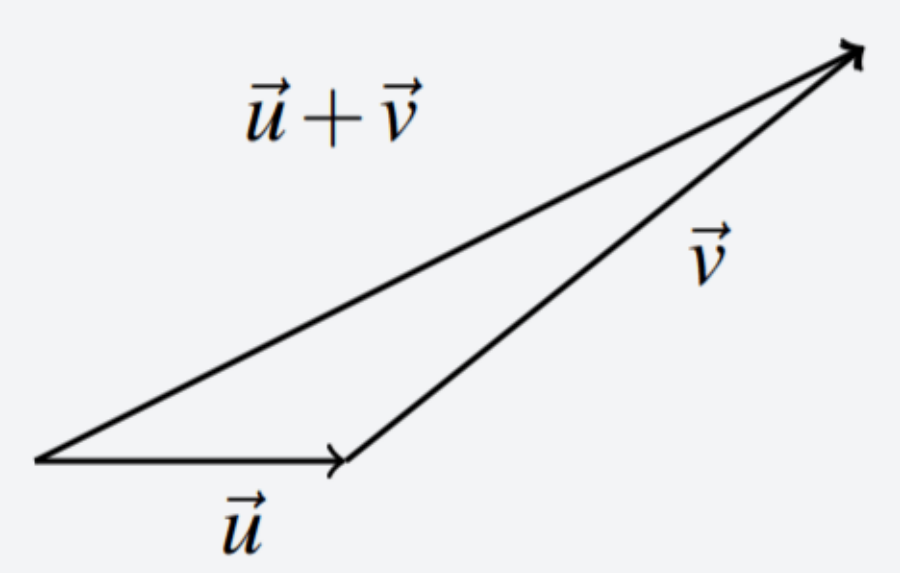


Figure 4.1.4: Diagram illustrates vector addition in $\mathbb{R}^2$. The vector $\vec{u}$ starts at the origin, the vector $\vec{v}$ is placed at the tip of $\vec{u}$, and their sum $\vec{u}$ + $\vec{v}$ is the resulting vector from the origin to the tip of $\vec{v}$. (CC BY-NC-SA 4.0; Kuttler via A First Course in LINEAR ALGEBRA)

This definition is illustrated in the following picture in which $\vec{u}+\vec{v}$ is shown for the special case $n=3$.

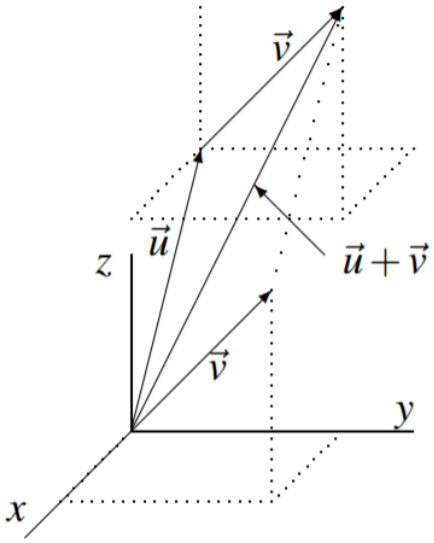


Figure 4.1.5: Diagram illustrating vector addition in $\mathbb{R}^3$, showing two vectors $\vec{u}$ and $\vec{v}$ in three-dimensional space. The image demonstrates how to add two vectors geometrically by placing the tail of one vector at the tip of the other. (CC BY-NC-SA 4.0; Kuttler via A First Course in LINEAR ALGEBRA)

Notice the parallelogram created by $\vec{u}$ and $\vec{v}$ in the above diagram. Then $\vec{u}+\vec{v}$ is the directed diagonal of the parallelogram determined by the two vectors $\vec{u}$ and $\vec{v}$.

When you have a vector $\vec{v}$, its additive inverse $-\vec{v}$ will be the vector which has the same magnitude as $\vec{v}$ but the opposite direction. When one writes $\vec{u}-\vec{v}$, the meaning is $\vec{u}+(-\vec{v})$ as with real numbers. The following example illustrates these definitions and conventions.

Example 4.1.1: Graphing Vector Addition

Consider the following picture of vectors $\vec{u}$ and $\vec{v}$.



Figure 4.1.6: Diagram illustrates two vectors in $\mathbb{R}^2$. (CC BY-NC-SA 4.0; Kuttler via A First Course in LINEAR ALGEBRA)

Sketch a picture of $\vec{u}+\vec{v}, \vec{u}-\vec{v}$.

Solution

We will first sketch $\vec{u}+\vec{v}$. Begin by drawing $\vec{u}$ and then at the point of $\vec{u}$, place the tail of $\vec{v}$ as shown. Then $\vec{u}+\vec{v}$ is the vector which results from drawing a vector from the tail of $\vec{u}$ to the tip of $\vec{v}$.

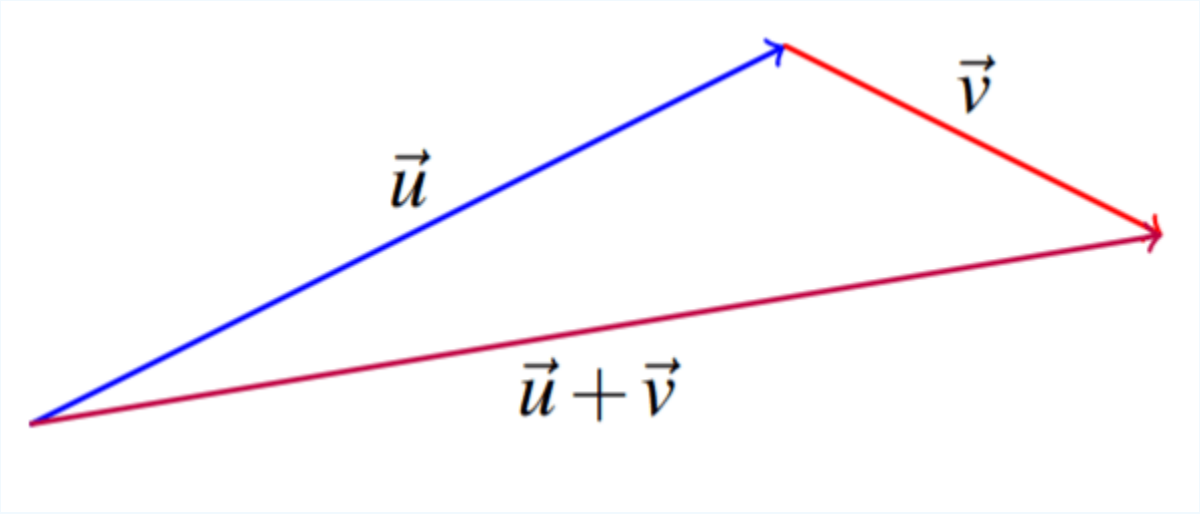


Figure 4.1.7: Diagram illustrates vector addition in $\mathbb{R}^2$, showing two vectors $\vec{u}$ and $\vec{v}$. The image demonstrates how to add two vectors geometrically by placing the tail of one vector at the tip of the other. (CC BY-NC-SA 4.0; Kuttler via A First Course in LINEAR ALGEBRA)

Next consider $\vec{u}-\vec{v}$. This means $\vec{u}+\left(-\vec{v}\right)$. From the above geometric description of vector addition, $-\vec{v}$ is the vector which has the same length but which points in the opposite direction to $\vec{v}$. Here is a picture.

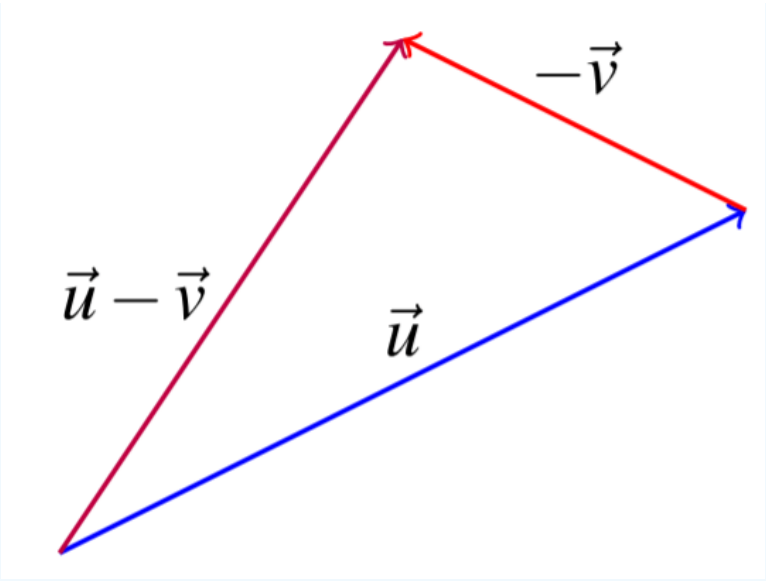


Figure 4.1.8: Diagram illustrates vector subtraction in $\mathbb{R}^2$, showing two vectors $\vec{u}$ and $\vec{v}$. The image demonstrates how to subtract two vectors geometrically by placing the tail of $-\vec{v}$ at the tip of $\vec{u}$. (CC BY-NC-SA 4.0; Kuttler via A First Course in LINEAR ALGEBRA)

Addition of Vectors in $\mathbb{R}^n$

Addition of vectors in $\mathbb{R}^n$ is defined as follows.

Definition 4.1.4: Addition of Vectors in $\mathbb{R}^n$

If $\vec{u}=\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$, $\vec{v}=\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$ then $\vec{u}+\vec{v} \in \mathbb{R}^n$ and is defined by

$$\begin{aligned} \vec{u}+\vec{v} &= \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \\ &= \begin{bmatrix} u_1+v_1 \\ \vdots \\ u_n+v_n \end{bmatrix} \end{aligned} \tag{4.1.1}$$

To add vectors, we simply add corresponding components. Therefore, in order to add vectors, they must be the same size.

Addition of vectors satisfies some important properties which are outlined in the following theorem.

Theorem 4.1.1: Properties of Vector Addition

The following properties hold for vectors $\vec{u},\vec{v},\vec{w} \in \mathbb{R}^n$.

- The Commutative Law of Addition

$$\vec{u}+\vec{v}=\vec{v}+\vec{u}$$

- The Associative Law of Addition

$$(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w})$$

- The Existence of an Additive Identity

$$\vec{u} + \vec{0} = \vec{u} \tag{4.1.2}$$

- The Existence of an Additive Inverse

$$\vec{u} + (-\vec{u}) = \vec{0}$$

The additive identity shown in Equation (4.1.2) is also called the **zero vector**, the $n \times 1$ vector in which all components are equal to 0. Further, $-\vec{u}$ is simply the vector with all components having same value as those of $\vec{u}$ but opposite sign; this is just $(-1)\vec{u}$. This will be made more explicit in the next section when we explore scalar multiplication of vectors. Note that subtraction is defined as $\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$.

We now explore what is meant by the length of a vector in $\mathbb{R}^n$. We develop this concept by first looking at the distance between two points in $\mathbb{R}^n$.

First, we will consider the concept of distance for $\mathbb{R}$, that is, for points in $\mathbb{R}^1$. Here, the distance between two points P and Q is given by the absolute value of their difference. We denote the distance between P and Q by $d(P,Q)$ which is defined as

$$d(P,Q) = \sqrt{(P-Q)^2} \tag{4.1.3}$$

Consider now the case for $n = 2$, demonstrated by the following picture.

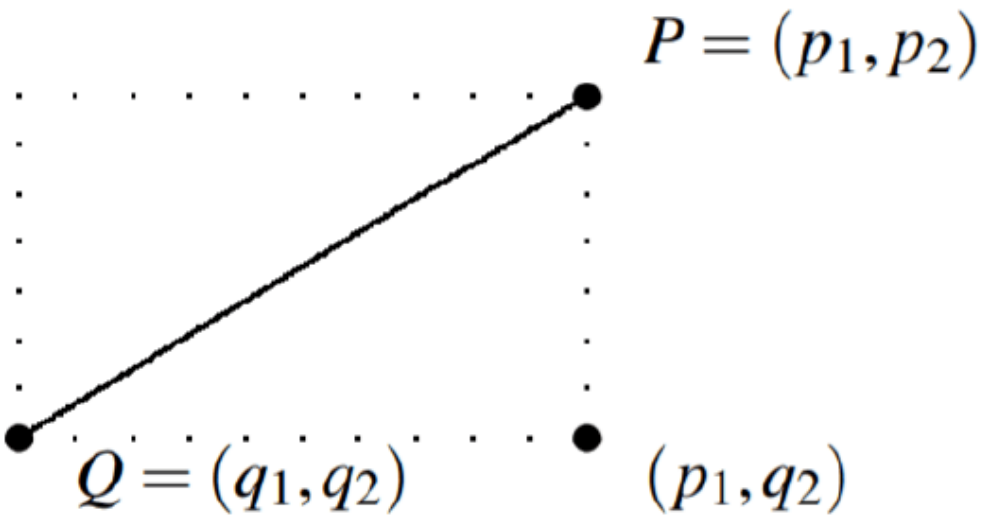


Figure 4.1.9: Diagram illustrates the distance between two points P and Q in $\mathbb{R}^2$. The image demonstrates how to calculate the distance using the Pythagorean theorem. (CC BY-NC-SA 4.0; Kuttler via A First Course in LINEAR ALGEBRA)

There are two points $P = (p_1, p_2)$ and $Q = (q_1, q_2)$ in the plane. The distance between these points is shown in the picture as a solid line. Notice that this line is the hypotenuse of a right triangle which is half of the rectangle shown in dotted lines. We want to find the length of this hypotenuse which will give the distance between the two points. Note the lengths of the sides of this triangle are $|p_1 - q_1|$ and $|p_2 - q_2|$, the absolute value of the difference in these values. Therefore, the Pythagorean Theorem implies the length of the hypotenuse (and thus the distance between P and Q) equals

$$\left(|p_1 - q_1|^2 + |p_2 - q_2|^2\right)^{1/2} = \left((p_1 - q_1)^2 + (p_2 - q_2)^2\right)^{1/2} \tag{4.1.4}$$

Now suppose $n = 3$ and let $P = (p_1, p_2, p_3)$ and $Q = (q_1, q_2, q_3)$ be two points in $\mathbb{R}^3$. Consider the following picture in which the solid line joins the two points and a dotted line joins the points (q_1, q_2, q_3) and (p_1, p_2, q_3) .

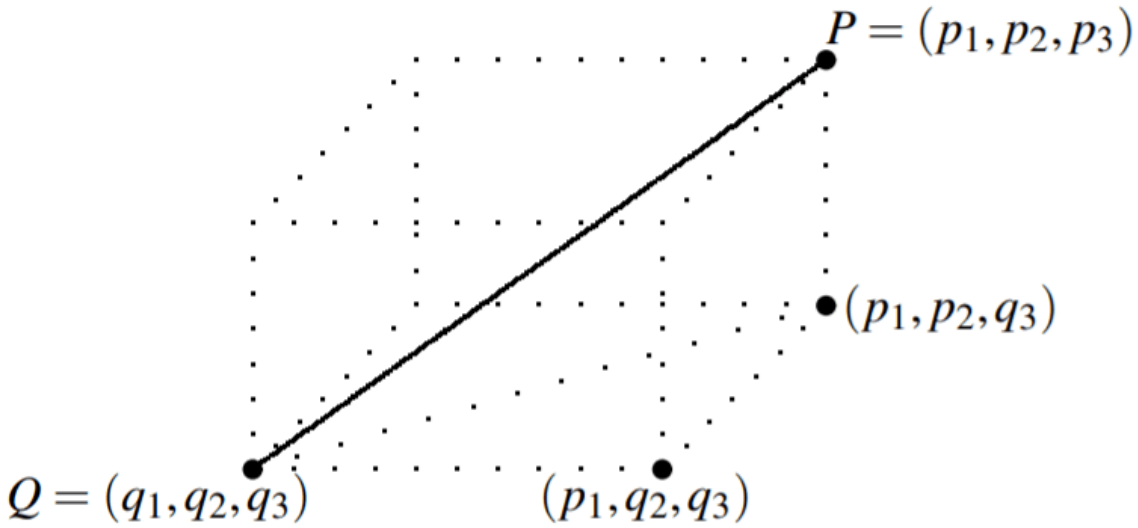


Figure 4.1.10: Diagram illustrates the distance between two points P and Q in $\mathbb{R}^3$. The image demonstrates how to calculate the distance using the Pythagorean theorem in three-dimensional space. (CC BY-NC-SA 4.0; Kuttler via A First Course in LINEAR ALGEBRA)

Here, we need to use Pythagorean Theorem twice in order to find the length of the solid line. First, by the Pythagorean Theorem, the length of the dotted line joining (q_1, q_2, q_3) and (p_1, p_2, q_3) equals

$$\left((p_1 - q_1)^2 + (p_2 - q_2)^2\right)^{1/2}$$

while the length of the line joining (p_1, p_2, q_3) to (p_1, p_2, p_3) is just $|p_3 - q_3|$. Therefore, by the Pythagorean Theorem again, the length of the line joining the points $P = (p_1, p_2, p_3)$ and $Q = (q_1, q_2, q_3)$ equals

$$\begin{aligned} &\left(\left(\left((p_1 - q_1)^2 + (p_2 - q_2)^2\right)^{1/2}\right)^2 + (p_3 - q_3)^2\right)^{1/2} \\ &= \left((p_1 - q_1)^2 + (p_2 - q_2)^2 + (p_3 - q_3)^2\right)^{1/2} \end{aligned} \tag{4.1.5}$$

This discussion motivates the following definition for the distance between points in $\mathbb{R}^n$.

Definition 4.1.5: Distance Between Points

Let $P = (p_1, \cdots, p_n)$ and $Q = (q_1, \cdots, q_n)$ be two points in $\mathbb{R}^n$. Then the distance between these points is defined as

$$\text{distance between } P \text{ and } Q \ = d(P, Q) = \left(\sum_{k=1}^{n} |p_k - q_k|^2\right)^{1/2}$$

This is called the **distance formula**. We may also write $|P - Q|$ as the distance between P and Q.

From the above discussion, you can see that Definition 4.1.1 holds for the special cases $n = 1, 2, 3$, as in Equations (4.1.3), (4.1.4), (4.1.5). In the following example, we use Definition 4.1.1 to find the distance between two points in $\mathbb{R}^4$.

Example 4.1.2: Distance Between Points

Find the distance between the points P and Q in $\mathbb{R}^4$, where P and Q are given by

$$P = (1, 2, -4, 6)$$

and

$$Q = (2, 3, -1, 0)$$

Solution

We will use the formula given in Definition 4.1.1 to find the distance between P and Q. Use the distance formula and write

$$d(P,Q) = \left((1-2)^2 + (2-3)^2 + (-4-(-1))^2 + (6-0)^2\right)^{\frac{1}{2}} = 47$$

Therefore, $d(P,Q) = \sqrt{47}$.

There are certain properties of the distance between points which are important in our study. These are outlined in the following theorem.

Theorem 4.1.2: Properties of Distance

Let P and Q be points in $\mathbb{R}^n$, and let the distance between them, $d(P,Q)$, be given as in Definition 4.1.1. Then, the following properties hold.

- $d(P,Q) = d(Q,P)$
- $d(P,Q) \geq 0$, and equals 0 exactly when $P = Q$.

There are many applications of the concept of distance. For instance, given two points, we can ask what collection of points are all the same distance between the given points. This is explored in the following example.

Example 4.1.3: The Plane Between Two Points

Describe the points in $\mathbb{R}^3$ which are at the same distance between $(1,2,3)$ and $(0,1,2)$.

Solution

Let $P = (p_1, p_2, p_3)$ be such a point. Therefore, P is the same distance from $(1,2,3)$ and $(0,1,2)$. Then byDefinition 4.1.1,

$$\sqrt{(p_1-1)^2 + (p_2-2)^2 + (p_3-3)^2} = \sqrt{(p_1-0)^2 + (p_2-1)^2 + (p_3-2)^2}$$

Squaring both sides we obtain

$$(p_1-1)^2 + (p_2-2)^2 + (p_3-3)^2 = p_1^2 + (p_2-1)^2 + (p_3-2)^2$$

and so

$$p_1^2 - 2p_1 + 14 + p_2^2 - 4p_2 + p_3^2 - 6p_3 = p_1^2 + p_2^2 - 2p_2 + 5 + p_3^2 - 4p_3$$

Simplifying, this becomes

$$-2p_1 + 14 - 4p_2 - 6p_3 = -2p_2 + 5 - 4p_3$$

which can be written as

$$2p_1 + 2p_2 + 2p_3 = -9 \tag{4.1.6}$$

Therefore, the points $P = (p_1, p_2, p_3)$ which are the same distance from each of the given points form a plane whose equation is given by (4.1.6).

We can now use our understanding of the distance between two points to define what is meant by the length of a vector. Consider the following definition.

Definition 4.1.6: Length of a Vector

Let $\vec{u} = [u_1 \cdots u_n]^T$ be a vector in $\mathbb{R}^n$. Then, the length of $\vec{u}$, written $\|\vec{u}\|$ is given by

$$\|\vec{u}\| = \sqrt{u_1^2 + \cdots + u_n^2}$$

This definition corresponds to Definition 4.1.1, if you consider the vector $\vec{u}$ to have its tail at the point $0 = (0, \cdots, 0)$ and its tip at

the point $U = (u_1, \cdots, u_n)$. Then the length of $\vec{u}$ is equal to the distance between 0 and U, $d(0, U)$. In general, $d(P, Q) = ||\vec{PQ}||$.

ConsiderExample $4.1.1$. ByDefinition $4.1.2$, we could also find the distance between P and Q as the length of the vector connecting them. Hence, if we were to draw a vector $\overrightarrow{PQ}$ with its tail at P and its point at Q, this vector would have length equal to $\sqrt{47}$.

We conclude this section with a new definition for the special case of vectors of length 1.

Definition $4.1.7$: Unit Vector

Let $\vec{u}$ be a vector in $\mathbb{R}^n$. Then, we call $\vec{u}$ a **unit vector** if it has length 1, that is if

$$\|\vec{u}\| = 1$$

Let $\vec{v}$ be a vector in $\mathbb{R}^n$. Then, the vector $\vec{u}$ which has the same direction as $\vec{v}$ but length equal to 1 is the corresponding unit vector of $\vec{v}$. This vector is given by

$$\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v}$$

We often use the term **normalize** to refer to this process. When we **normalize** a vector, we find the corresponding unit vector of length 1. Consider the following example.

Example $4.1.4$: Finding a Unit Vector

Let $\vec{v}$ be given by

$$\vec{v} = \begin{bmatrix} 1 & -3 & 4 \end{bmatrix}^T$$

Find the unit vector $\vec{u}$ which has the same direction as $\vec{v}$.

Solution

We will use Definition $4.1.3$ to solve this. Therefore, we need to find the length of $\vec{v}$ which, by Definition $4.1.2$ is given by

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

Using the corresponding values we find that

$$\begin{aligned} \|\vec{v}\| &= \sqrt{1^2 + (-3)^2 + 4^2} \\ &= \sqrt{1 + 9 + 16} \\ &= \sqrt{26} \end{aligned} \tag{4.1.7}$$

In order to find $\vec{u}$, we divide $\vec{v}$ by $\sqrt{26}$. The result is

$$\begin{aligned} \vec{u} &= \frac{1}{\|\vec{v}\|}\vec{v} \\ &= \frac{1}{\sqrt{26}}\begin{bmatrix} 1 & -3 & 4 \end{bmatrix}^T \\ &= \begin{bmatrix} \frac{1}{\sqrt{26}} & -\frac{3}{\sqrt{26}} & \frac{4}{\sqrt{26}} \end{bmatrix}^T \end{aligned} \tag{4.1.8}$$

You can verify using the Definition $4.1.1$ that $\|\vec{u}\| = 1$.

Now suppose we have a vector $\vec{u} = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^T$ and we multiply $\vec{u}$ by a scalar k. By Definition 4.2.2, $k\vec{u} = \begin{bmatrix} ku_1 & ku_2 & ku_3 \end{bmatrix}^T$. Then, by using Definition 4.4.1, the length of this vector is given by

$$\sqrt{\left((ku_1)^2 + (ku_2)^2 + (ku_3)^2\right)} = |k|\sqrt{u_1^2 + u_2^2 + u_3^2}$$

Thus the following holds.

$$\|k\vec{u}\| = |k| \, \|\vec{u}\|$$

In other words, multiplication by a scalar magnifies or shrinks the length of the vector by a factor of $|k|$. If $|k| > 1$, the length of the resulting vector will be magnified. If $|k| < 1$, the length of the resulting vector will shrink. Remember that by the definition of the absolute value, $|k| > 0$.

What about the direction? Draw a picture of $\vec{u}$ and $k\vec{u}$ where k is negative. Notice that this causes the resulting vector to point in the opposite direction while if $k > 0$ it preserves the direction the vector points. Therefore the direction can either reverse, if $k < 0$, or remain preserved, if $k > 0$.

Consider the following example.

Example 4.1.5: Graphing Scalar Multiplication

Consider the vectors $\vec{u}$ and $\vec{v}$ drawn below.



Figure 4.1.11: Diagram illustrates two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^2$. (CC BY-NC-SA 4.0; Kuttler via A First Course in LINEAR ALGEBRA)

Draw $-\vec{u}$, $2\vec{v}$, and $-\frac{1}{2}\vec{v}$.

Solution

In order to find $-\vec{u}$, we preserve the length of $\vec{u}$ and simply reverse the direction. For $2\vec{v}$, we double the length of $\vec{v}$, while preserving the direction. Finally $-\frac{1}{2}\vec{v}$ is found by taking half the length of $\vec{v}$ and reversing the direction. These vectors are shown in the following diagram.

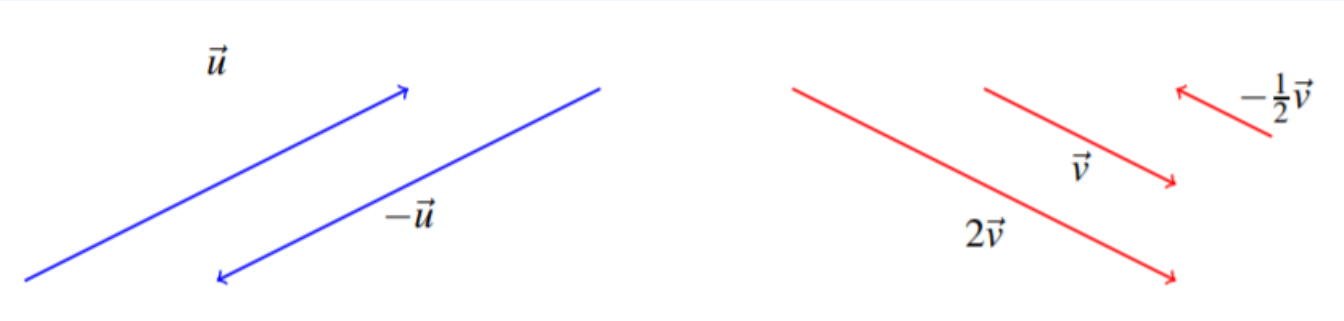


Figure 4.1.12: Diagram illustrating scalar multiplication of vectors in $\mathbb{R}^2$. The image demonstrates how to scale a vector by multiplying it by a scalar. (CC BY-NC-SA 4.0; Kuttler via A First Course in LINEAR ALGEBRA)

In the following example, we examine the geometric meaning of this concept.

Example 4.1.6: Graphing a Linear Combination of Vectors

Consider the following picture of the vectors $\vec{u}$ and $\vec{v}$:



Figure 4.1.13: Diagram illustrates two vectors (\vec{u}\) and $\vec{v}$ in $\mathbb{R}^2$. (CC BY-NC-SA 4.0; Kuttler via A First Course in LINEAR ALGEBRA)

Sketch a picture of $\vec{u}+2\vec{v}, \vec{u}-\frac{1}{2}\vec{v}$.

Solution

The two vectors are shown below.

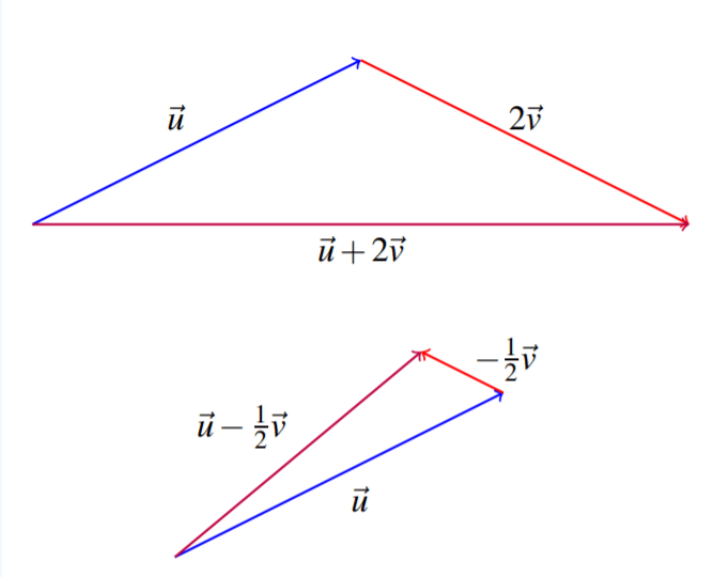


Figure 4.1.14: Diagram illustrates vector addition and scalar multiplication in $\mathbb{R}^2$. The image demonstrates how to add and scale vectors geometrically. (CC BY-NC-SA 4.0; Kuttler via A First Course in LINEAR ALGEBRA)

Scalar Multiplication of Vectors in $\mathbb{R}^n$

Scalar multiplication of vectors in $\mathbb{R}^n$ is defined as follows.

Definition 4.1.8: Scalar Multiplication of Vectors in $\mathbb{R}^n$

If $\vec{u} \in \mathbb{R}^n$ and $k \in \mathbb{R}$ is a scalar, then $k\vec{u} \in \mathbb{R}^n$ is defined by

$$k\vec{u} = k\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} ku_1 \\ \vdots \\ ku_n \end{bmatrix}$$

Just as with addition, scalar multiplication of vectors satisfies several important properties. These are outlined in the following theorem.

Theorem 4.1.3: Properties of Scalar Multiplication

The following properties hold for vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ and k, p scalars.

- The Distributive Law over Vector Addition
$$k\left(\vec{u}+\vec{v}\right) = k\vec{u} + k\vec{v}$$
- The Distributive Law over Scalar Addition
$$\left(k+p\right)\vec{u} = k\vec{u} + p\vec{u}$$
- The Associative Law for Scalar Multiplication
$$k\left(p\vec{u}\right) = \left(kp\right)\vec{u}$$
- Rule for Multiplication by 1
$$1\vec{u} = \vec{u}$$

Proof available at the end of this section.

We now present a useful notion you may have seen earlier combining vector addition and scalar multiplication

Definition 4.1.9: Linear Combination

A vector $\vec{v}$ is said to be a **linear combination** of the vectors $\vec{u}_1, \cdots, \vec{u}_n$ if there exist scalars, $a_1, \cdots, a_n$ such that
$$\vec{v} = a_1\vec{u}_1 + \cdots + a_n\vec{u}_n$$

For example,
$$3\begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -18 \\ 3 \\ 2 \end{bmatrix}.$$

Thus we can say that
$$\vec{v} = \begin{bmatrix} -18 \\ 3 \\ 2 \end{bmatrix}$$

is a linear combination of the vectors
$$\vec{u}_1 = \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} \text{ and } \vec{u}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

Answers

Theorem 4.1.3: Properties of Scalar Multiplication

We will show the proof of:
$$k\left(\vec{u}+\vec{v}\right) = k\vec{u} + k\vec{v}$$

Note that:
$$\begin{aligned} k\left(\vec{u}+\vec{v}\right) &= k\left[u_1 + v_1 \cdots u_n + v_n\right]^T \\ &= \left[k\left(u_1 + v_1\right) \cdots k\left(u_n + v_n\right)\right]^T \\ &= \left[ku_1 + kv_1 \cdots ku_n + kv_n\right]^T \\ &= \left[ku_1 \cdots ku_n\right]^T + \left[kv_1 \cdots kv_n\right]^T \\ &= k\vec{u} + k\vec{v} \end{aligned}$$

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4.1E: Exercises for Section 4.1

Exercise $4.1E.1$

Find $-3\begin{bmatrix} 5 \\ -1 \\ 2 \\ -3 \end{bmatrix} + 5\begin{bmatrix} -8 \\ 2 \\ -3 \\ 6 \end{bmatrix}$.

Answer available at the end of this section.

Exercise $4.1E.2$

Find $-7\begin{bmatrix} 6 \\ 0 \\ 4 \\ -1 \end{bmatrix} + 6\begin{bmatrix} -13 \\ -1 \\ 1 \\ 6 \end{bmatrix}$.

Exercise $4.1E.3$

Let

$$\vec{a} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} -4 \\ 5 \\ 0 \end{bmatrix}.$$

Compute:

a. $\vec{a} + \vec{b}$
b. $3\vec{a}$
c. $-2\vec{b}$
d. $\vec{a} - \vec{b}$

Exercise $4.1E.4$

Let $\vec{v}$ and $\vec{w}$ be two vectors in $\mathbb{R}^2$. How is the vector sum $\vec{v} + \vec{w}$ related to the diagonal of the parallelogram formed by $\vec{v}$ and $\vec{w}$?

Exercise $4.1E.5$

Let

$$\vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

a. What happens to the magnitude of $\vec{v}$ when it is multiplied by 2?
b. What happens to the direction of $\vec{v}$ when it is multiplied by -1?

Answer available at the end of this section.

Exercise 4.1E.6

Let $\vec{v}_1$ and $\vec{v}_2$ be vectors in $\mathbb{R}^2$. What geometric shape does the set $\{a\vec{v}_1 + b\vec{v}_2 \mid 0 \leq a \leq 1, 0 \leq b \leq 1\}$

Exercise 4.1E.7

Decide whether

$$\vec{v} = \begin{bmatrix} 4 \\ 4 \\ -3 \end{bmatrix}$$

is a linear combination of the vectors

$$\vec{u}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \vec{u}_2 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

Answer available at the end of this section.

Exercise 4.1E.8

Decide whether

$$\vec{v} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$$

is a linear combination of the vectors

$$\vec{u}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \vec{u}_2 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

Answer available at the end of this section.

Exercise 4.1E.9

Decide whether

$$\vec{v} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

is a linear combination of the vectors

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 5 \\ -4 \\ 0 \end{bmatrix} \text{ and } \quad \vec{u}_3 = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}.$$

Exercise 4.1E.10

a. Prove that there exists a unique zero vector $\vec{0}$ such that for every vector $\vec{a} \in \mathbb{R}^n$,

$$\vec{a} + \vec{0} = \vec{a}.$$

b. Prove that for every vector $\vec{a} \in \mathbb{R}^n$, there exists a unique vector $-\vec{a}$ such that \[\vec{a} + (-\vec{a}) = \vec{0}. \nonumber\

Exercise $4.1E.11$

Let $\vec{a}$ and $\vec{b}$ be two vectors in $\mathbb{R}^n$. Show geometrically that

$$\|\vec{a}+\vec{b}\|^2 + \|\vec{a}-\vec{b}\|^2 = 2\|\vec{a}\|^2 + 2\|\vec{b}\|^2.$$

Hint available at the end of this section.

Answers

Exercise $4.1E.1$

$$\begin{bmatrix} -55 \\ 13 \\ -21 \\ 39 \end{bmatrix}$$

Exercise $4.1E.5$

Exercise $4.1E.7$

$$\begin{bmatrix} 4 \\ 4 \\ -3 \end{bmatrix} = 2\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

Exercise $4.1E.8$

The system

$$\begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} = a_1\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} + a_2\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

has no solution.

Exercise $4.1E.11$

Sketch the parallelogram formed by the vectors $\vec{a}$ and $\vec{b}$. Notice that the two diagonal vectors of this parallelogram are given by:

$$\text{One diagonal:} \quad \vec{a}+\vec{b}$$

$$\text{Other diagonal:} \quad \vec{a}-\vec{b}$$

Then use the Law of Cosines.

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4.2: Dot and Cross Product

Learning Objectives

- Compute the dot product of vectors, and use it to compute vector projections.
- Compute the cross product and the box product of vectors in $\mathbb{R}^3$.

This section explores two fundamental operations for vectors: the dot product and the cross product. These operations are widely used in applications involving vectors in $\mathbb{R}^n$.

The Dot Product

There are two ways of multiplying vectors which are of great importance in applications. The first of these is called the **dot product**. When we take the dot product of vectors, the result is a scalar. For this reason, the dot product is also called the **scalar product** and sometimes the **inner product**. The definition is as follows.

Definition 4.2.1: Dot Product

Let $\vec{u}, \vec{v}$ be two vectors in $\mathbb{R}^n$. Then we define the **dot product** $\vec{u} \bullet \vec{v}$ as

$$\vec{u} \bullet \vec{v} = \sum_{k=1}^{n} u_k v_k$$

The dot product $\vec{u} \bullet \vec{v}$ is sometimes denoted as $(\vec{u}, \vec{v})$. It can also be written as $\langle \vec{u}, \vec{v} \rangle$. If we write the vectors as column or row matrices, it is equal to the matrix product $\vec{v}\vec{w}^T$.

With this definition, there are several important properties satisfied by the dot product.

Proposition 4.2.1: Properties of the Dot Product

Let k and p denote scalars and $\vec{u}, \vec{v}, \vec{w}$ denote vectors. Then the dot product $\vec{u} \bullet \vec{v}$ satisfies the following properties.

- $\vec{u} \bullet \vec{v} = \vec{v} \bullet \vec{u}$
- $\vec{u} \bullet \vec{u} \geq 0$ and equals zero if and only if $\vec{u} = \vec{0}$
- $(k\vec{u} + p\vec{v}) \bullet \vec{w} = k(\vec{u} \bullet \vec{w}) + p(\vec{v} \bullet \vec{w})$
- $\vec{u} \bullet (k\vec{v} + p\vec{w}) = k(\vec{u} \bullet \vec{v}) + p(\vec{u} \bullet \vec{w})$
- $\|\vec{u}\|^2 = \vec{u} \bullet \vec{u}$

Proof available at the end of this section.

This proposition tells us that we can also use the dot product to find the length of a vector.

The **Cauchy Schwarz inequality** is a fundamental inequality satisfied by the dot product. It is given in the following theorem.

Theorem 4.2.1: Cauchy Schwarz Inequality

The dot product satisfies the inequality

$$\left|\vec{u} \bullet \vec{v}\right| \leq \|\vec{u}\|\|\vec{v}\| \tag{4.2.1}$$

Furthermore equality is obtained if and only if one of $\vec{u}$ or $\vec{v}$ is a scalar multiple of the other.

Proof available at the end of this section.

Notice that this proof was based only on the properties of the dot product listed in Proposition $4.2.1$. This means that whenever an operation satisfies these properties, the Cauchy Schwarz inequality holds. There are many other instances of these properties besides vectors in $\mathbb{R}^n$.

The Cauchy Schwarz inequality provides another proof of the **triangle inequality** for distances in $\mathbb{R}^n$.

Theorem $4.2.2$: Triangle Inequality

For $\vec{u}, \vec{v} \in \mathbb{R}^n$

$$\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\| \tag{4.2.2}$$

and equality holds if and only if one of the vectors is a non-negative scalar multiple of the other.

Also

$$\|\|\vec{u}\| - \|\vec{v}\|\| \leq \|\vec{u} - \vec{v}\| \tag{4.2.3}$$

Proof available at the end of this section.

The Geometric Significance of the Dot Product

Given two vectors, $\vec{u}$ and $\vec{v}$, the **included angle** is the angle between these two vectors which is given by θ such that $0 \leq \theta \leq \pi$. The dot product can be used to determine the included angle between two vectors. Consider the following picture where θ gives the included angle.

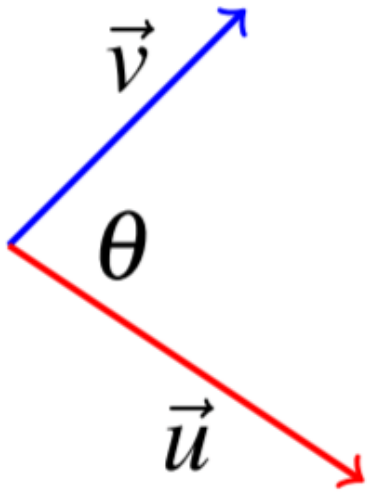


Figure $4.2.1$: Diagram illustrates the included angle θ between two vectors $\vec{u}$ and $\vec{v}$. (CC BY-NC-SA 4.0; Kuttler via A First Course in LINEAR ALGEBRA)

Proposition $4.2.2$: The Dot Product and the Included Angle

Let $\vec{u}$ and $\vec{v}$ be two vectors in $\mathbb{R}^n$, and let θ be the included angle. Then the following equation holds.

$$\vec{u} \bullet \vec{v} = \|\vec{u}\|\|\vec{v}\| \cos\theta$$

In words, the dot product of two vectors equals the product of the magnitude (or length) of the two vectors multiplied by the cosine of the included angle. Note this gives a geometric description of the dot product which does not depend explicitly on the coordinates of the vectors.

Another application of the geometric description of the dot product is in finding the angle between two lines. Typically one would assume that the lines intersect. In some situations, however, it may make sense to ask this question when the lines do not intersect, such as the angle between two object trajectories. In any case we understand it to mean the smallest angle between (any of) their direction vectors. The only subtlety here is that if $\vec{u}$ is a direction vector for a line, then so is any multiple $k\vec{u}$, and thus we will find complementary angles among all angles between direction vectors for two lines, and we simply take the smaller of the two.

Two nonzero vectors are said to be **perpendicular**, sometimes also called **orthogonal**, if the included angle is $\pi/2$ radians (90°).

Consider the following proposition.

Proposition 4.2.3: Perpendicular Vectors

Let $\vec{u}$ and $\vec{v}$ be nonzero vectors in $\mathbb{R}^n$. Then, $\vec{u}$ and $\vec{v}$ are said to be **perpendicular** exactly when

$$\vec{u} \bullet \vec{v} = 0$$

Proof available at the end of this section.

Projections

In some applications, we wish to write a vector as a sum of two related vectors. Through the concept of projections, we can find these two vectors. First, we explore an important theorem. The result of this theorem will provide our definition of a vector projection.

Theorem 4.2.3: Vector Projections

Let $\vec{v}$ and $\vec{u}$ be nonzero vectors. Then there exist unique vectors $\vec{v}_{||}$ and $\vec{v}_{\perp}$ such that

$$\vec{v} = \vec{v}_{||} + \vec{v}_{\perp} \tag{4.2.4}$$

where $\vec{v}_{||}$ is a scalar multiple of $\vec{u}$, and $\vec{v}_{\perp}$ is perpendicular to $\vec{u}$.

Proof available at the end of this section.

The vector $\vec{v}_{||}$ in Theorem 4.2.3 is called the **projection** of $\vec{v}$ onto $\vec{u}$ and is denoted by

$$\vec{v}_{||} = \text{proj}_{\vec{u}}\left(\vec{v}\right)$$

We now make a formal definition of the vector projection.

Definition 4.2.2: Vector Projection

Let $\vec{u}$ and $\vec{v}$ be vectors. Then, the **projection of $\vec{v}$ onto $\vec{u}$** is given by

$$\text{proj}_{\vec{u}}\left(\vec{v}\right) = \left(\frac{\vec{v} \bullet \vec{u}}{\vec{u} \bullet \vec{u}}\right)\vec{u} = \frac{\vec{v} \bullet \vec{u}}{\|\vec{u}\|^2}\vec{u}$$

Consider the following example of a projection.

Example 4.2.1: Find the Projection of One Vector Onto Another

Find $\text{proj}_{\vec{u}}(\vec{v})$ if

$$\vec{u} = \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Solution

We can use the formula provided in Definition 4.2.2 to find $\text{proj}_{\vec{u}}(\vec{v})$. First, compute $\vec{v} \bullet \vec{u}$. This is given by

$$\begin{aligned} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} &= (2)(1) + (3)(-2) + (-4)(1) \\ &= 2 - 6 - 4 \\ &= -8 \end{aligned} \tag{4.2.5}$$

Similarly, $\vec{u} \bullet \vec{u}$ is given by

$$\begin{aligned} \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} &= (2)(2) + (3)(3) + (-4)(-4) \\ &= 4 + 9 + 16 \\ &= 29 \end{aligned} \tag{4.2.6}$$

Therefore, the projection is equal to

$$\begin{aligned} \text{proj}_{\vec{u}}(\vec{v}) &= -\frac{8}{29} \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{16}{29} \\ -\frac{24}{29} \\ \frac{32}{29} \end{bmatrix} \end{aligned} \tag{4.2.7}$$

We will conclude the dot product discussion with an important application of projections. Suppose a line L and a point P are given such that P is not contained in L. Through the use of projections, we can determine the shortest distance from P to L.

Example 4.2.3: Shortest Distance from a Point to a Line

Let $P = (1, 3, 5)$ be a point in $\mathbb{R}^3$, and let L be the line which goes through point $P_0 = (0, 4, -2)$ with direction vector $\vec{d} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$. Find the shortest distance from P to the line L, and find the point Q on L that is closest to P.

Solution

In order to determine the shortest distance from P to L, we will first find the vector $\overrightarrow{P_0P}$ and then find the projection of this vector onto L. The vector $\overrightarrow{P_0P}$ is given by

$$\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} - \begin{bmatrix} 0 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix}$$

Then, if Q is the point on L closest to P, it follows that

$$\begin{aligned}\overrightarrow{P_0Q} &= \text{proj}_{\vec{d}}\overrightarrow{P_0P} \\ &= \left(\frac{\overrightarrow{P_0P} \bullet \vec{d}}{\|\vec{d}\|^2}\right)\vec{d} \\ &= \frac{15}{9}\begin{bmatrix}2\\1\\2\end{bmatrix} \\ &= \frac{5}{3}\begin{bmatrix}2\\1\\2\end{bmatrix}\end{aligned} \tag{4.2.8}$$

Now, the distance from P to L is given by

$$\|\overrightarrow{QP}\| = \|\overrightarrow{P_0P} - \overrightarrow{P_0Q}\| = \sqrt{26}$$

The point Q is found by adding the vector $\overrightarrow{P_0Q}$ to the position vector $\overrightarrow{0P_0}$ for P_0 as follows

$$\begin{bmatrix}0\\4\\-2\end{bmatrix} + \frac{5}{3}\begin{bmatrix}2\\1\\2\end{bmatrix} = \begin{bmatrix}\frac{10}{3}\\\frac{17}{3}\\\frac{4}{3}\end{bmatrix} \tag{4.2.9}$$

Therefore, $Q = (\frac{10}{3}, \frac{17}{3}, \frac{4}{3})$.

Recall that the dot product is one of two important products for vectors. The second type of product for vectors is called the cross product. It is important to note that the cross product is only defined in $\mathbb{R}^3$. First we discuss the geometric meaning and then a description in terms of coordinates is given, both of which are important. The geometric description is essential in order to understand the applications to physics and geometry while the coordinate description is necessary to compute the cross product.

The Cross Product

Consider the following definition.

Definition 4.2.3: Right Hand System of Vectors

Three vectors, $\vec{u}, \vec{v}, \vec{w}$ form a right hand system if when you extend the fingers of your right hand along the direction of vector $\vec{u}$ and close them in the direction of $\vec{v}$, the thumb points roughly in the direction of $\vec{w}$.

For an example of a right handed system of vectors, see the following picture.

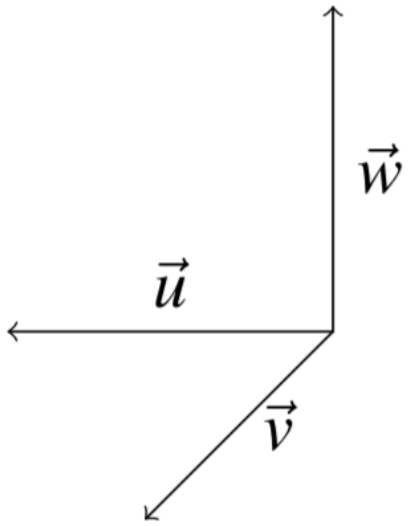


Figure 4.2.2: Diagram illustrates a right-handed coordinate system. The image shows how to determine the direction of the cross product of two vectors using the right-hand rule. (CC BY-NC-SA 4.0; Kuttler via A First Course in LINEAR ALGEBRA)

In this picture the vector $\vec{w}$ points upwards from the plane determined by the other two vectors. Point the fingers of your right hand along $\vec{u}$, and close them in the direction of $\vec{v}$. Notice that if you extend the thumb on your right hand, it points in the direction of $\vec{w}$.

You should consider how a right hand system would differ from a left hand system. Try using your left hand and you will see that the vector $\vec{w}$ would need to point in the opposite direction.

Notice that the special vectors, $\vec{i}, \vec{j}, \vec{k}$ will always form a right handed system. If you extend the fingers of your right hand along $\vec{i}$ and close them in the direction $\vec{j}$, the thumb points in the direction of $\vec{k}$.

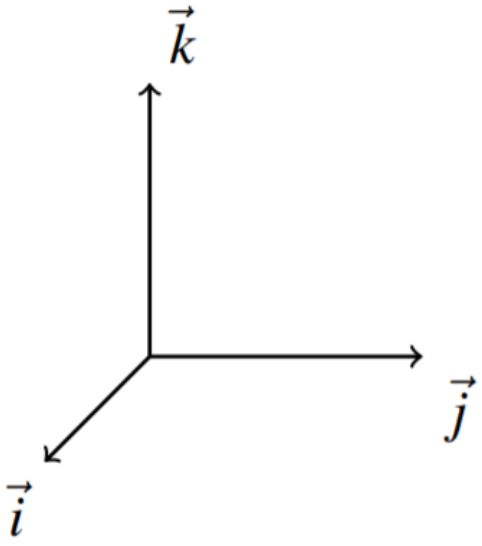


Figure 4.2.3: Diagram illustrates a coordinate system in $\mathbb{R}^3$. The image shows the standard unit vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$ along the x, y, and z axes, respectively. (CC BY-NC-SA 4.0; Kuttler via A First Course in LINEAR ALGEBRA)

The following is the geometric description of the cross product. Recall that the dot product of two vectors results in a scalar. In contrast, the cross product results in a vector, as the product gives a direction as well as magnitude.

Definition 4.2.4: Geometric Definition of Cross Product

Let $\vec{u}$ and $\vec{v}$ be two vectors in $\mathbb{R}^3$. Then the **cross product**, written $\vec{u} \times \vec{v}$, is defined by the following two rules.

1. Its length is
$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{v}\| \sin\theta,$$
where θ is the included angle between $\vec{u}$ and $\vec{v}$.
2. It is perpendicular to both $\vec{u}$ and $\vec{v}$, that is
$$\begin{aligned} (\vec{u} \times \vec{v}) \cdot \vec{u} &= 0, \\ (\vec{u} \times \vec{v}) \cdot \vec{v} &= 0, \end{aligned} \tag{4.2.10}$$
and
$$\vec{u}, \vec{v}, \vec{u} \times \vec{v}$$
form a right hand system.

The cross product of the special vectors $\vec{i}, \vec{j}, \vec{k}$ is as follows.

$$\begin{aligned} \vec{i} \times \vec{j} &= \vec{k} \quad \vec{j} \times \vec{i} = -\vec{k} \\ \vec{k} \times \vec{i} &= \vec{j} \quad \vec{i} \times \vec{k} = -\vec{j} \\ \vec{j} \times \vec{k} &= \vec{i} \quad \vec{k} \times \vec{j} = -\vec{i} \end{aligned}$$

With this information, the following gives the coordinate description of the cross product.

Recall that the vector $\vec{u} = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^T$ can be written in terms of $\vec{i}, \vec{j}, \vec{k}$ as $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$.

Theorem 4.2.4: Coordinate Description of Cross Product

Let $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$ and $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ be two vectors. Then

$$\vec{u} \times \vec{v} = (u_2v_3 - u_3v_2)\,\vec{i} - (u_1v_3 - u_3v_1)\,\vec{j} + (u_1v_2 - u_2v_1)\,\vec{k} \tag{4.2.11}$$

Writing $\vec{u} \times \vec{v}$ in the usual way, it is given by

$$\vec{u} \times \vec{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ -(u_1v_3 - u_3v_1) \\ u_1v_2 - u_2v_1 \end{bmatrix}$$

We now prove this proposition.

Proof available at the end of this section.

There is another version of (4.2.11) which may be easier to remember. We can express the cross product as the determinant of a matrix, as follows.

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \tag{4.2.12}$$

Expanding the determinant along the top row yields

$$\vec{i}\,(-1)^{1+1}\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} + \vec{j}\,(-1)^{2+1}\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \vec{k}\,(-1)^{3+1}\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

\

$$= \vec{i} \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \vec{j} \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \vec{k} \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

Expanding these determinants leads to

$$(u_2v_3 - u_3v_2)\,\vec{i} - (u_1v_3 - u_3v_1)\,\vec{j} + (u_1v_2 - u_2v_1)\,\vec{k}$$

which is the same as (4.2.20).

The cross product satisfies the following properties.

Proposition 4.2.4: Properties of the Cross Product

Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in $\mathbb{R}^3$, and k a scalar. Then, the following properties of the cross product hold.

1. $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$, and $\vec{u} \times \vec{u} = \vec{0}$
2. $(k\vec{u}) \times \vec{v} = k(\vec{u} \times \vec{v}) = \vec{u} \times (k\vec{v})$
3. $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
4. $(\vec{v} + \vec{w}) \times \vec{u} = \vec{v} \times \vec{u} + \vec{w} \times \vec{u}$

Proof available at the end of this section.

An important geometrical application of the cross product is as follows. The size of the cross product, $\|\vec{u} \times \vec{v}\|$, is the area of the parallelogram determined by $\vec{u}$ and $\vec{v}$, as shown in the following picture.

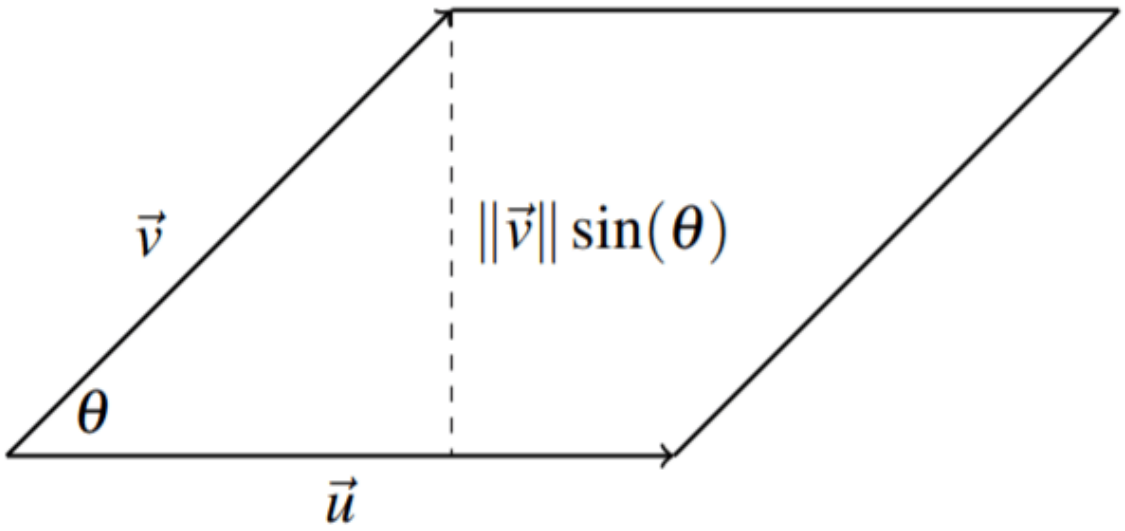


Figure 4.2.4: Diagram illustrating the geometric interpretation of the cross product of two vectors $\vec{u}$ and $\vec{v}$. The image shows the parallelogram formed by the two vectors, and the magnitude of the cross product is equal to the area of the parallelogram. (CC BY-NC-SA 4.0; Kuttler via A First Course in LINEAR ALGEBRA)

We examine this concept in the following example.

Example 4.2.3: Area of a Parallelogram

Find the area of the parallelogram determined by the vectors $\vec{u}$ and $\vec{v}$ given by

$$\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \vec{v} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

Solution

Notice that these vectors are the same as the ones given in Example 4.2.1. Recall from the geometric description of the cross product, that the area of the parallelogram is simply the magnitude of $\vec{u} \times \vec{v}$. From Example 4.2.1, $\vec{u} \times \vec{v} = 3\vec{i} + 5\vec{j} + \vec{k}$. We can also write this as

$$\vec{u} \times \vec{v} = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}$$

Thus the area of the parallelogram is

$$\|\vec{u} \times \vec{v}\| = \sqrt{(3)(3) + (5)(5) + (1)(1)} = \sqrt{9 + 25 + 1} = \sqrt{35}$$

We can also use this concept to find the area of a triangle. Consider the following example.

Example 4.2.4: Area of Triangle

Find the area of the triangle determined by the points $(1, 2, 3), (0, 2, 5), (5, 1, 2)$

Solution

This triangle is obtained by connecting the three points with lines. Picking $(1, 2, 3)$ as a starting point, there are two displacement vectors, $\begin{bmatrix} -1 & 0 & 2 \end{bmatrix}^T$ and $\begin{bmatrix} 4 & -1 & -1 \end{bmatrix}^T$. Notice that if we add either of these vectors to the position vector of the starting point, the result is the position vectors of the other two points. Now, the area of the triangle is half the area of the parallelogram determined by $\begin{bmatrix} -1 & 0 & 2 \end{bmatrix}^T$ and $\begin{bmatrix} 4 & -1 & -1 \end{bmatrix}^T$. The required cross product is given by

$$\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \times \begin{bmatrix} 4 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & 1 \end{bmatrix}$$

Taking the size of this vector gives the area of the parallelogram, given by

$$\sqrt{(2)(2) + (7)(7) + (1)(1)} = \sqrt{4 + 49 + 1} = \sqrt{54}$$

Hence the area of the triangle is $\frac{1}{2}\sqrt{54} = \frac{3}{2}\sqrt{6}$.

In general, if you have three points in $\mathbb{R}^3$, P, Q, R, the area of the triangle is given by

$$\frac{1}{2}\|\vec{PQ} \times \vec{PR}\|$$

Recall that $\vec{PQ}$ is the vector running from point P to point Q.

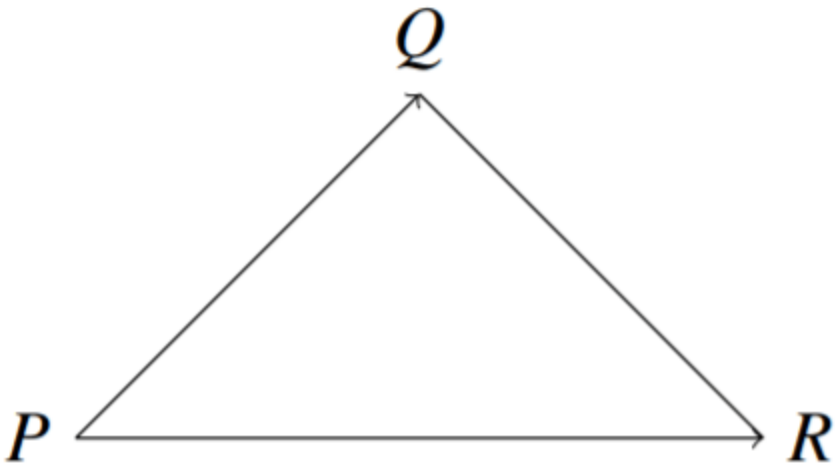


Figure 4.2.5: Diagram illustrates a triangle formed by three points P, Q, and R in $\mathbb{R}^3$. (CC BY-NC-SA 4.0; Kuttler via A First Course in LINEAR ALGEBRA)

In the next section, we explore another application of the cross product.

The Box Product

Recall that we can use the cross product to find the the area of a parallelogram. It follows that we can use the cross product together with the dot product to find the volume of a parallelepiped. We begin with a definition.

Definition 4.2.5: Parallelepiped

A **parallelepiped** determined by the three vectors, $\vec{u}, \vec{v}$, and $\vec{w}$ consists of

$$\{r\vec{u} + s\vec{v} + t\vec{w} : r, s, t \in [0, 1]\}$$

That is, if you pick three numbers, r, s, and t each in $[0, 1]$ and form $r\vec{u} + s\vec{v} + t\vec{w}$ then the collection of all such points makes up the parallelepiped determined by these three vectors.

The following is an example of a parallelepiped.

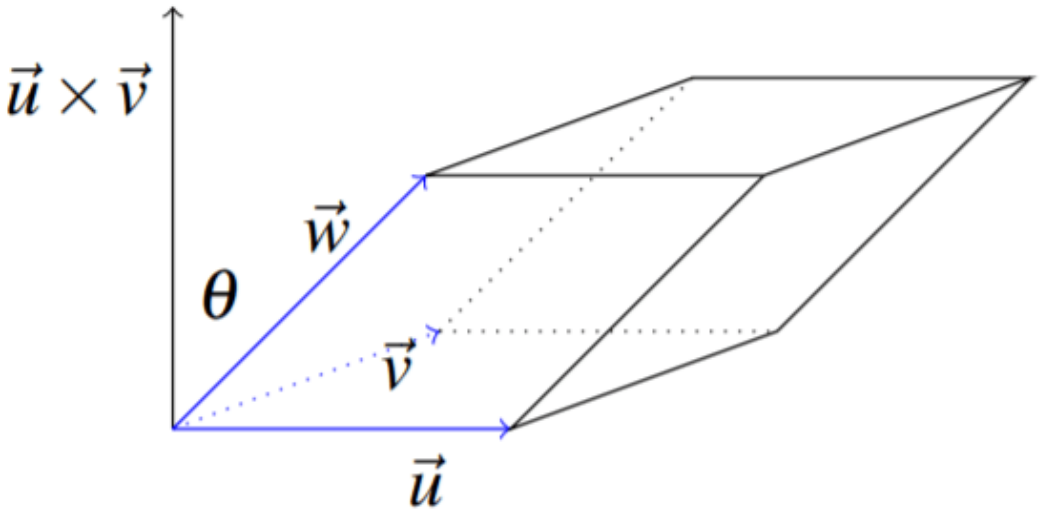


Figure 4.2.6: Diagram illustrates a parallelepiped determined by three vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$. The image shows the three-dimensional shape formed by the vectors. (CC BY-NC-SA 4.0; Kuttler via A First Course in LINEAR ALGEBRA)

Notice that the base of the parallelepiped is the parallelogram determined by the vectors $\vec{u}$ and $\vec{v}$. Therefore, its area is equal to $\|\vec{u} \times \vec{v}\|$. The height of the parallelepiped is $\|\vec{w}\| \cos\theta$ where θ is the angle shown in the picture between $\vec{w}$ and $\vec{u} \times \vec{v}$. The volume of this parallelepiped is the area of the base times the height which is just

$$\|\vec{u} \times \vec{v}\| \|\vec{w}\| \cos\theta = (\vec{u} \times \vec{v}) \cdot \vec{w}$$

This expression is known as the **box product** and is sometimes written as $[\vec{u}, \vec{v}, \vec{w}]$. You should consider what happens if you interchange the $\vec{v}$ with the $\vec{w}$ or the $\vec{u}$ with the $\vec{w}$. You can see geometrically from drawing pictures that this merely introduces a minus sign. In any case the box product of three vectors always equals either the volume of the parallelepiped determined by the three vectors or else -1 times this volume.

Note: Connection to Determinants

Recall from Section 3.4 that the determinant is useful in computing the area of a parallelogram formed by two vectors and the volume of the parallelepiped formed by three vectors.

- In $\mathbb{R}^2$: Suppose $\vec{v}_1 = \langle a, b \rangle$ and $\vec{v}_2 = \langle c, d \rangle$. Then, the area of the parallelogram formed by $\vec{v}_1$ and $\vec{v}_2$ is simply the magnitude of the cross product of the $\mathbb{R}^3$ vectors $\langle a, b, 0 \rangle$ and $\langle c, d, 0 \rangle$. This is simply $|ad - bc|$, which is the absolute value of the determinant of the 2×2 matrix formed by the row vectors $\vec{v}_1$ and $\vec{v}_2$.
- In $\mathbb{R}^3$: Suppose $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ are vectors in $\mathbb{R}^3$. Then, the volume of the parallelepiped formed by $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ is simply the absolute value of the box product of the three vectors. We leave it to the reader to check that indeed this is also the absolute value of the determinant of the 3×3 matrix formed by the row vectors $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$.

Answers

Proposition 4.2.1: Properties of the Dot Product

The proof is left as an exercise.

Theorem 4.2.1: Cauchy Schwarz Inequality

First note that if $\vec{v} = \vec{0}$ both sides of (4.2.1) equal zero and so the inequality holds in this case. Therefore, it will be assumed in what follows that $\vec{v} \neq \vec{0}$.

Define a function of $t \in \mathbb{R}$ by

$$f(t) = (\vec{u} + t\vec{v}) \bullet (\vec{u} + t\vec{v})$$

Then by Proposition 4.2.1, $f(t) \geq 0$ for all $t \in \mathbb{R}$. Also from Proposition 4.2.1

$$\begin{aligned} f(t) &= \vec{u} \bullet (\vec{u} + t\vec{v}) + t\vec{v} \bullet (\vec{u} + t\vec{v}) \\ &= \vec{u} \bullet \vec{u} + t(\vec{u} \bullet \vec{v}) + t\vec{v} \bullet \vec{u} + t^2 \vec{v} \bullet \vec{v} \\ &= \|\vec{u}\|^2 + 2t(\vec{u} \bullet \vec{v}) + \|\vec{v}\|^2 t^2 \end{aligned} \tag{4.2.13}$$

Now this means the graph of $y = f(t)$ is a parabola which opens up and either its vertex touches the t axis or else the entire graph is above the t axis. In the first

case, there exists some t where $f(t)=0$ and this requires $\vec{u}+t\vec{v}=\vec{0}$ so one vector is a multiple of the other. Then clearly equality holds in (4.2.1). In the case where $\vec{v}$ is not a multiple of $\vec{u}$, it follows $f(t)>0$ for all t which says $f(t)$ has no real zeros and so from the quadratic formula,

$$\left(2\left(\vec{u}\bullet\vec{v}\right)\right)^2-4\|\vec{u}\|^2\|\vec{v}\|^2<0$$

which is equivalent to $\left|\vec{u}\bullet\vec{v}\right|<\|\vec{u}\|\|\vec{v}\|$.

Theorem 4.2.2**: Triangle Inequality**

By properties of the dot product and the Cauchy Schwarz inequality,

$$\begin{aligned}\|\vec{u}+\vec{v}\|^2&=\left(\vec{u}+\vec{v}\right)\bullet\left(\vec{u}+\vec{v}\right)\\&=\left(\vec{u}\bullet\vec{u}\right)+\left(\vec{u}\bullet\vec{v}\right)+\left(\vec{v}\bullet\vec{u}\right)+\left(\vec{v}\bullet\vec{v}\right)\\&=\|\vec{u}\|^2+2\left(\vec{u}\bullet\vec{v}\right)+\|\vec{v}\|^2\\&\leq\|\vec{u}\|^2+2\left|\vec{u}\bullet\vec{v}\right|+\|\vec{v}\|^2\\&\leq\|\vec{u}\|^2+2\|\vec{u}\|\|\vec{v}\|+\|\vec{v}\|^2=\left(\|\vec{u}\|+\|\vec{v}\|\right)^2\end{aligned}\tag{4.2.14}$$

Hence,

$$\|\vec{u}+\vec{v}\|^2\leq\left(\|\vec{u}\|+\|\vec{v}\|\right)^2$$

Taking square roots of both sides you obtain (4.2.2).

It remains to consider when equality occurs. Suppose $\vec{u}=\vec{0}$. Then, $\vec{u}=0\vec{v}$ and the claim about when equality occurs is verified. The same argument holds if $\vec{v}=\vec{0}$. Therefore, it can be assumed both vectors are nonzero. To get equality in (4.2.2) above, Theorem 4.2.1 implies one of the vectors must be a multiple of the other. Say $\vec{v}=k\vec{u}$. If $k<0$ then equality cannot occur in (4.2.2) because in this case

$$\vec{u}\bullet\vec{v}=k\|\vec{u}\|^2<0<|k|\,\|\vec{u}\|^2=\left|\vec{u}\bullet\vec{v}\right|$$

Therefore, $k\geq0$.

To get the other form of the triangle inequality write

$$\vec{u}=\vec{u}-\vec{v}+\vec{v}$$

so

$$\begin{aligned}\|\vec{u}\|&=\|\vec{u}-\vec{v}+\vec{v}\|\\&\leq\|\vec{u}-\vec{v}\|+\|\vec{v}\|\end{aligned}\tag{4.2.15}$$

Therefore,

$$\|\vec{u}\|-\|\vec{v}\|\leq\|\vec{u}-\vec{v}\|\tag{4.2.16}$$

Similarly,

$$\|\vec{v}\|-\|\vec{u}\|\leq\|\vec{v}-\vec{u}\|=\|\vec{u}-\vec{v}\|\tag{4.2.17}$$

It follows from (4.2.16) and (4.2.17) that (4.2.3) holds. This is because $\left|\|\vec{u}\|-\|\vec{v}\|\right|$ equals the left side of either (4.2.16) or (4.2.17) and either way, $\left|\|\vec{u}\|-\|\vec{v}\|\right|\leq\|\vec{u}-\vec{v}\|$.

Proposition 4.2.3**: Perpendicular Vectors**

This follows directly from Proposition 4.2.2. First if the dot product of two nonzero vectors is equal to 0, this tells us that $\cos\theta=0$ (this is where we need nonzero vectors). Thus $\theta=\pi/2$ and the vectors are perpendicular.

If on the other hand $\vec{v}$ is perpendicular to $\vec{u}$, then the included angle is $\pi/2$ radians. Hence $\cos\theta=0$ and $\vec{u}\bullet\vec{v}=0$.

Theorem 4.2.3**: Vector Projections**

Suppose (4.2.4) holds and $\vec{v}_{||}=k\vec{u}$. Taking the dot product of both sides of (4.2.4) with $\vec{u}$ and using $\vec{v}_{\perp}\bullet\vec{u}=0$, this yields

$$\begin{aligned}\vec{v}\bullet\vec{u}&=\left(\vec{v}_{||}+\vec{v}_{\perp}\right)\bullet\vec{u}\\&=k\vec{u}\bullet\vec{u}+\vec{v}_{\perp}\bullet\vec{u}\\&=k\|\vec{u}\|^2\end{aligned}$$

which requires $k=\vec{v}\bullet\vec{u}/\|\vec{u}\|^2$. Thus there can be no more than one vector $\vec{v}_{||}$. It follows $\vec{v}_{\perp}$ must equal $\vec{v}-\vec{v}_{||}$. This verifies there can be no more than one choice for both $\vec{v}_{||}$ and $\vec{v}_{\perp}$ and proves their uniqueness.

Now let

$$\vec{v}_{||}=\frac{\vec{v}\bullet\vec{u}}{\|\vec{u}\|^2}\vec{u}$$

and let

$$\vec{v}_{\perp}=\vec{v}-\vec{v}_{||}=\vec{v}-\frac{\vec{v}\bullet\vec{u}}{\|\vec{u}\|^2}\vec{u}$$

Then $\vec{v}_{||}=k\vec{u}$ where $k=\frac{\vec{v}\bullet\vec{u}}{\|\vec{u}\|^2}$. It only remains to verify $\vec{v}_{\perp}\bullet\vec{u}=0$. But

$$\begin{aligned}\vec{v}_{\perp}\bullet\vec{u}&=\vec{v}\bullet\vec{u}-\frac{\vec{v}\bullet\vec{u}}{\|\vec{u}\|^2}\vec{u}\bullet\vec{u}\\&=\vec{v}\bullet\vec{u}-\vec{v}\bullet\vec{u}\\&=0\end{aligned}\tag{4.2.18}$$

Theorem 4.2.4**: Coordinate Description of Cross Product**

From the above table and the properties of the cross product listed,

$$\begin{aligned}\vec{u}\times\vec{v} &= \left(u_1\vec{i}+u_2\vec{j}+u_3\vec{k}\right)\times\left(v_1\vec{i}+v_2\vec{j}+v_3\vec{k}\right)\\ &= u_1v_2\vec{i}\times\vec{j}+u_1v_3\vec{i}\times\vec{k}+u_2v_1\vec{j}\times\vec{i}+u_2v_3\vec{j}\times\vec{k}++u_3v_1\vec{k}\times\vec{i}+u_3v_2\vec{k}\times\vec{j}\\ &= u_1v_2\vec{k}-u_1v_3\vec{j}-u_2v_1\vec{k}+u_2v_3\vec{i}+u_3v_1\vec{j}-u_3v_2\vec{i}\\ &= \left(u_2v_3-u_3v_2\right)\vec{i}+\left(u_3v_1-u_1v_3\right)\vec{j}+\left(u_1v_2-u_2v_1\right)\vec{k}\end{aligned} \tag{4.2.19}$$

(4.2.20)

Proposition 4.2.4**: Properties of the Cross Product**

Formula 1. follows immediately from the definition. The vectors $\vec{u}\times\vec{v}$ and $\vec{v}\times\vec{u}$ have the same magnitude, $|\vec{u}|\,|\vec{v}|\sin\theta$, and an application of the right hand rule shows they have opposite direction.

Formula 2. is proven as follows. If k is a non-negative scalar, the direction of $(k\vec{u})\times\vec{v}$ is the same as the direction of $\vec{u}\times\vec{v}, k\left(\vec{u}\times\vec{v}\right)$ and $\vec{u}\times(k\vec{v})$. The magnitude is k times the magnitude of $\vec{u}\times\vec{v}$ which is the same as the magnitude of $k\left(\vec{u}\times\vec{v}\right)$ and $\vec{u}\times(k\vec{v})$. Using this yields equality in 2. In the case where $k<0$, everything works the same way except the vectors are all pointing in the opposite direction and you must multiply by $|k|$ when comparing their magnitudes.

The distributive laws, 3. and 4., are much harder to establish. For now, it suffices to notice that if we know that 3. is true, 4. follows. Thus, assuming 3., and using 1.,

$$\begin{aligned}\left(\vec{v}+\vec{w}\right)\times\vec{u} &= -\vec{u}\times\left(\vec{v}+\vec{w}\right)\\ &= -\left(\vec{u}\times\vec{v}+\vec{u}\times\vec{w}\right)\\ &= \vec{v}\times\vec{u}+\vec{w}\times\vec{u}\end{aligned} \tag{4.2.21}$$

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4.2E: Exercises for Section 4.2

Exercise $4.2E.1$

Find $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 0 \\ 1 \\ 3 \end{bmatrix}$.

Answer available at the end of this section.

Exercise $4.2E.2$

Use the formula given in Proposition 4.2.1 to verify the Cauchy Schwarz inequality and to show that equality occurs if and only if one of the vectors is a scalar multiple of the other

Answer available at the end of this section.

Exercise $4.2E.3$

Let $\vec{a}$, $\vec{b}$ be vectors. Show that $\left(\vec{a} \bullet \vec{b}\right) = \frac{1}{4}\left(||\vec{a}+\vec{b}||^2 - ||\vec{a}-\vec{b}||^2\right)$.

Exercise $4.2E.4$

Using the axioms of the dot product, prove the parallelogram identity:

$$||\vec{a}+\vec{b}||^2 + ||\vec{a}-\vec{b}||^2 = 2||\vec{a}||^2 + 2||\vec{b}||^2$$

Exercise $4.2E.5$

Let A be a real $m \times n$ matrix and let $\vec{u} \in \mathbb{R}^n$ and $\vec{v} \in \mathbb{R}^m$. Show $A\vec{u} \bullet \vec{v} = \vec{u} \bullet A^T\vec{v}$. Hint: Use the definition of matrix multiplication to do this.

Answer available at the end of this section.

Exercise $4.2E.6$

Use the result of Problem $4.2E.5$ to verify directly that $(AB)^T = B^TA^T$ without making any reference to subscripts.

Answer available at the end of this section.

Exercise $4.2E.7$

Find the angle between the vectors

$$\vec{u} = \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$$

Answer available at the end of this section.

Exercise $4.2E.8$

Find the angle between the vectors

$$\vec{u} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 2 \\ -7 \end{bmatrix}$$

Answer available at the end of this section.

Exercise $4.2E.9$

Find $\text{proj}_{\vec{v}}(\vec{w})$ where $\vec{w} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Answer available at the end of this section.

Exercise $4.2E.10$

Find $\text{proj}_{\vec{v}}(\vec{w})$ where $\vec{w} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$.

Answer available at the end of this section.

Exercise $4.2E.11$

Find $\text{proj}_{\vec{v}}(\vec{w})$ where $\vec{w} = \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$.

Answer available at the end of this section.

Exercise $4.2E.12$

Let $P = (1, 2, 3)$ be a point in $\mathbb{R}^3$. Let L be the line through the point $P_0 = (1, 4, 5)$ with direction vector $\vec{d} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. Find the shortest distance from P to L, and find the point Q on L that is closest to P.

Exercise $4.2E.13$

Let $P = (0, 2, 1)$ be a point in $\mathbb{R}^3$. Let L be the line through the point $P_0 = (1, 1, 1)$ with direction vector $\vec{d} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$. Find the shortest distance from P to L, and find the point Q on L that is closest to P.

Exercise $4.2E.14$

Does it make sense to speak of $\text{proj}_{\vec{0}}(\vec{w})$?

Answer available at the end of this section.

Exercise $4.2E.15$

The Cauchy Schwarz inequality in $\mathbb{R}^n$ is:

For vectors $\vec{u}$, $\vec{v}$, $(\vec{u} - \text{proj}_{\vec{v}}\vec{u}) \bullet (\vec{u} - \text{proj}_{\vec{v}}\vec{u}) \geq 0$.

a. Prove the Cauchy Schwarz Inequality. HINT: Begin by simplifying using the axioms of the dot product and then putting in the formula for the projection. Notice that this expression equals 0 and you get equality in the Cauchy Schwarz inequality if and only if $\vec{u} = \text{proj}_{\vec{v}}\vec{u}$.
b. What is the geometric meaning of $\vec{w} = \text{proj}_{\vec{v}}\vec{w}$?

Answer available at the end of this section.

Exercise $4.2E.16$

Let $\vec{v}$, $\vec{w}$, $\vec{u}$ be vectors. Show that $(\vec{w} + \vec{u})_\perp = \vec{w}_\perp + \vec{u}_\perp$ where $\vec{w}_\perp = \vec{w} - \text{proj}_{\vec{v}}(\vec{w})$.

Answer available at the end of this section.

Exercise $4.2E.17$

Show that

$$(\vec{v} - \text{proj}_{\vec{u}}(\vec{v}), \vec{u}) = (\vec{v} - \text{proj}_{\vec{u}}(\vec{v})) \bullet \vec{u} = 0$$

and conclude every vector in $\mathbb{R}^n$ can be written as the sum of two vectors, one which is perpendicular and one which is parallel to the given vector.

Answer available at the end of this section.

Exercise $4.2E.18$

Show that if $\vec{a} \times \vec{u} = \vec{0}$ for any unit vector $\vec{u}$, then $\vec{a} = \vec{0}$.

Answer available at the end of this section.

Exercise $4.2E.19$

Find the area of the triangle determined by the three points $(1,2,3),\ (4,2,0)$ and $(-3,2,1)$.

Answer available at the end of this section.

Exercise $4.2E.20$

Find the area of the triangle determined by the three points $(1,0,3),\ (4,1,0)$ and $(-3,1,1)$.

Answer available at the end of this section.

Exercise $4.2E.21$

Find the area of the triangle determined by the three points, $(1,2,3),\ (2,3,4)$ and $(3,4,5)$. Did something interesting happen here? What does it mean geometrically?

Answer available at the end of this section.

Exercise $4.2E.22$

Find the area of the parallelogram determined by the vectors $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$.

Answer available at the end of this section.

Exercise $4.2E.23$

Find the area of the parallelogram determined by the vectors $\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$.

Answer available at the end of this section.

Exercise $4.2E.24$

Is $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \times \vec{w}$? What is the meaning of $\vec{u} \times \vec{v} \times \vec{w}$? Explain. Hint: Try $\left(\vec{i} \times \vec{j}\right) \times \vec{k}$.

Answer available at the end of this section.

Exercise $4.2E.25$

Verify directly that the coordinate description of the cross product, $\vec{u} \times \vec{v}$ has the property that it is perpendicular to both $\vec{u}$ and $\vec{v}$. Then show by direct computation that this coordinate description satisfies

$$\begin{aligned} ||\vec{u} \times \vec{v}||^2 &= ||\vec{u}||^2||\vec{v}||^2 - (\vec{u} \bullet \vec{v})^2 \\ &= ||\vec{u}||^2||\vec{v}||^2(1 - \cos^2(\theta)) \end{aligned} \tag{4.2E.1}$$

where θ is the angle included between the two vectors. Explain why $||\vec{u} \times \vec{v}||$ has the correct magnitude.

Answer available at the end of this section.

Exercise $4.2E.26$

Suppose A is a 3×3 skew symmetric matrix such that $A^T = -A$. Show there exists a vector $\vec{\Omega}$ such that for all $\vec{u} \in \mathbb{R}^3$

$$A\vec{u} = \vec{\Omega} \times \vec{u}$$

Hint: Explain why since A is skew symmetric it is of the form

$$A = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

where the ω_i are numbers. Then consider $\omega_1\vec{i} + \omega_2\vec{j} + \omega_3\vec{k}$.

Exercise $4.2E.27$

Find the volume of the parallelepiped determined by the vectors $\begin{bmatrix} 1 \\ -7 \\ -5 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -2 \\ -6 \end{bmatrix}$, and $\begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}$.

Answer available at the end of this section.

Exercise $4.2E.28$

Suppose $\vec{u}$, $\vec{v}$, and $\vec{w}$ are three vectors whose components are all integers. Can you conclude the volume of the parallelepiped determined from these three vectors will always be an integer?

Answer available at the end of this section.

Exercise $4.2E.29$

What does it mean geometrically if the box product of three vectors gives zero?

Answer available at the end of this section.

Answers

Exercise $4.2E.1$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 0 \\ 1 \\ 3 \end{bmatrix} = 17$$

Exercise $4.2E.2$

This formula says that $\vec{u} \bullet \vec{v} = ||\vec{u}||\ ||\vec{v}|| \cos\theta$ where θ is the included angle between the two vectors. Thus

$$||\vec{u} \bullet \vec{v}|| = ||\vec{u}||\ ||\vec{v}||\ |\cos\theta|| \leq ||\vec{u}||\ ||\vec{v}||$$

and equality holds if and only if $\theta = 0$ or π. This means that the two vectors either point in the same direction or opposite directions. Hence one is a multiple of the other.

Exercise $4.2E.5$

$A\vec{x} \bullet \vec{y} = \sum_k (A\vec{x})_k y_k = \sum_k \sum_i A_{ki} x_i y_k = \sum_i \sum_k A^T_{ik} x_i y_k = \vec{x} \bullet A^T\vec{y}$

Exercise $4.2E.6$

$$\begin{aligned} AB\vec{x} \bullet \vec{y} &= B\vec{x} \bullet A^T\vec{y} \\ &= \vec{x} \bullet B^T A^T \vec{y} \\ &= \vec{x} \bullet (AB)^T \vec{y} \end{aligned} \tag{4.2E.2}$$

Since this is true for all $\vec{x}$, it follows that, in particular, it holds for

$$\vec{x} = B^T A^T \vec{y} - (AB)^T \vec{y}$$

and so from the axioms of the dot product,

$$\left(B^T A^T \vec{y} - (AB)^T \vec{y}\right) \bullet \left(B^T A^T \vec{y} - (AB)^T \vec{y}\right) = 0$$

and so $B^T A^T \vec{y} - (AB)^T \vec{y} = \vec{0}$. However, this is true for all $\vec{y}$ and so $B^T A^T - (AB)^T = 0$.

Exercise $4.2E.7$

$\frac{\begin{bmatrix} 3 & -1 & -1 \end{bmatrix}^T \bullet \begin{bmatrix} 1 & 4 & 2 \end{bmatrix}^T}{\sqrt{9+1+1}\sqrt{1+16+4}} = \frac{-3}{\sqrt{11}\sqrt{21}} = -0.19739 = \cos\theta$ Therefore we need to solve

$$-0.19739 = \cos\theta$$

Thus $\theta = 1.7695$ radians.

Exercise $4.2E.8$

$\frac{-10}{\sqrt{1+4+1}\sqrt{1+4+49}} = -0.55555 = \cos\theta$ Therefore we need to solve $-0.55555 = \cos\theta$, which gives $\theta = 2.0313$ radians.

Exercise $4.2E.9$

$$\frac{\vec{u}\bullet\vec{v}}{\vec{u}\bullet\vec{u}}\vec{u} = \frac{-5}{14}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -\frac{5}{14} \\ -\frac{5}{7} \\ -\frac{15}{14} \end{bmatrix}$$

Exercise $4.2E.10$

$$\frac{\vec{u}\bullet\vec{v}}{\vec{u}\bullet\vec{u}}\vec{u} = \frac{-5}{10}\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ -\frac{3}{2} \end{bmatrix}$$

Exercise $4.2E.11$

$$\frac{\vec{u}\bullet\vec{v}}{\vec{u}\bullet\vec{u}}\vec{u} = \frac{\begin{bmatrix} 1 & 2 & -2 & 1 \end{bmatrix}^T \bullet \begin{bmatrix} 1 & 2 & 3 & 0 \end{bmatrix}^T}{1+4+9}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{14} \\ -\frac{1}{7} \\ -\frac{3}{14} \\ 0 \end{bmatrix}$$

Exercise $4.2E.14$

No, it does not. The 0 vector has no direction. The formula for $\text{proj}_{\vec{0}}(\vec{w})$ doesn't make sense either.

Exercise $4.2E.15$

a. $$\left(\vec{u} - \frac{\vec{u}\bullet\vec{v}}{||\vec{v}||^2}\vec{v}\right) \bullet \left(\vec{u} - \frac{\vec{u}\bullet\vec{v}}{||\vec{v}||^2}\vec{v}\right) = ||\vec{u}||^2 - 2(\vec{u}\bullet\vec{v})^2\frac{1}{||\vec{v}||^2} + (\vec{u}\bullet\vec{v})^2\frac{1}{||\vec{v}||^2} \geq 0$$

And so

$$||\vec{u}||^2||\vec{v}||^2 \geq (\vec{u}\bullet\vec{v})^2$$

You get equality exactly when $\vec{u} = \text{proj}_{\vec{v}}\vec{u} = \frac{\vec{u}\bullet\vec{v}}{||\vec{v}||^2}\vec{v}$.

b. Equality indicates that $\vec{u}$ is a multiple of $\vec{v}$.

Exercise $4.2E.16$

$$\begin{aligned} \vec{w} - \text{proj}_{\vec{v}}(\vec{w}) + \vec{u} - \text{proj}_{\vec{v}}(\vec{u}) &= \vec{w} + \vec{u} - (\text{proj}_{\vec{v}}(\vec{w}) + \text{proj}_{\vec{v}}(\vec{u})) \\ &= \vec{w} + \vec{u} - \text{proj}_{\vec{v}}(\vec{w} + \vec{u}) \end{aligned} \tag{4.2E.3}$$

This follows because

$$\begin{aligned} \text{proj}_{\vec{v}}(\vec{w}) + \text{proj}_{\vec{v}}(\vec{u}) &= \frac{\vec{u}\bullet\vec{v}}{||\vec{v}||^2}\vec{v} + \frac{\vec{w}\bullet\vec{v}}{||\vec{v}||^2}\vec{v} \\ &= \frac{(\vec{u}+\vec{w})\bullet\vec{v}}{||\vec{v}||^2}\vec{v} \\ &= \text{proj}_{\vec{v}}(\vec{w}+\vec{u}) \end{aligned} \tag{4.2E.4}$$

Exercise $4.2E.17$

$(\vec{v} - \text{proj}_{\vec{u}}(\vec{v})) \bullet \vec{u} = \vec{v}\bullet\vec{u} - \left(\frac{(\vec{v}\cdot\vec{u}}{||\vec{u}||^2}\vec{u}\right) \bullet \vec{u} = \vec{v}\bullet\vec{u} - \vec{v}\bullet\vec{u} = 0$. Therefore, $\vec{v} = \vec{v} - \text{proj}_{\vec{u}}(\vec{v}) + \text{proj}_{\vec{u}}(\vec{v})$. The first is perpendicular to $\vec{u}$ and the second is a multiple of $\vec{u}$ so it is parallel to $\vec{u}$.

Exercise $4.2E.18$

If $\vec{a} \neq \vec{0}$, then the condition says that $||\vec{a}\times\vec{u}|| = ||\vec{a}||\sin\theta = 0$ for all angles θ. Hence $\vec{a} = \vec{0}$ after all.

Exercise $4.2E.19$

$\begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix} \times \begin{bmatrix} -4 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 18 \\ 0 \end{bmatrix}$. So the area is 9.

Exercise $4.2E.20$

$\begin{bmatrix} 3 \\ 1 \\ -3 \end{bmatrix} \times \begin{bmatrix} -4 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 18 \\ 7 \end{bmatrix}$. The area is given by

$$\frac{1}{2}\sqrt{1+(18)^2+49} = \frac{1}{2}\sqrt{374}$$

Exercise $4.2E.21$

$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$. The area is 0. It means the three points are on the same line.

Exercise $4.2E.22$

$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \times \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 8 \\ -8 \end{bmatrix}$. The area is $8\sqrt{3}$.

Exercise $4.2E.23$

$\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \times \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \\ -2 \end{bmatrix}$. The area is $\sqrt{36+121+4} = \sqrt{161}$.

Exercise $4.2E.24$

$\left(\vec{i} \times \vec{j}\right) \times \vec{j} = \vec{k} \times \vec{j} = i\vec{i}$. However, $\vec{i} \times \left(\vec{j} \times \vec{j}\right) = \vec{0}$ and so the cross product is not associative.

Exercise $4.2E.25$

Verify directly from the coordinate description of the cross product that the right hand rule applies to the vectors $\vec{i}, \vec{j}, \vec{k}$. Next verify that the distributive law holds for the coordinate description of the cross product. This gives another way to approach the cross product. First define it in terms of coordinates and then get the geometric properties from this. However, this approach does not yield the right hand rule property very easily. From the coordinate description,

$$\vec{a} \times \vec{b} \cdot \vec{a} = \epsilon_{ijk} a_j b_k a_i = -\epsilon_{jik} a_k b_k a_i = -\epsilon_{jik} b_k a_i a_j = -\vec{a} \times \vec{b} \cdot \vec{a}$$

and so $\vec{a} \times \vec{b}$ is perpendicular to $\vec{a}$. Similarly, $\vec{a} \times \vec{b}$ is perpendicular to $\vec{b}$. Now we need that

$$||\vec{a} \times \vec{b}||^2 = ||\vec{a}||^2 ||\vec{b}||^2 (1 - \cos^2\theta) = ||\vec{a}||^2 ||\vec{b}||^2 \sin^2\theta$$

and so $||\vec{a} \times \vec{b}|| = ||\vec{a}||\ ||\vec{b}|| \sin\theta$, the area of the parallelogram determined by $\vec{a}$, $\vec{b}$. Only the right hand rule is a little problematic. However, you can see right away from the component definition that the right hand rule holds for each of the standard unit vectors. Thus $\vec{i} \times \vec{j} = \vec{k}$ etc.

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \vec{k}$$

Exercise $4.2E.27$

$$\begin{vmatrix} 1 & -7 & -5 \\ 1 & -2 & -6 \\ 3 & 2 & 3 \end{vmatrix} = 113$$

Exercise $4.2E.28$

Yes. It will involve the sum of product of integers and so it will be an integer.

Exercise $4.2E.29$

It means that if you place them so that they all have their tails at the same point, the three will lie in the same plane.

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4.3: Lines and Planes

Learning Objectives

- Find the vector and parametric equations of a line.
- Find the vector and scalar equations of a plane.

We can use the concept of vectors and points to find equations for arbitrary lines in $\mathbb{R}^n$, although in this section the focus will be on lines in $\mathbb{R}^3$.

To begin, consider the case $n = 1$ so we have $\mathbb{R}^1 = \mathbb{R}$. There is only one line here which is the familiar number line, that is $\mathbb{R}$ itself. Therefore it is not necessary to explore the case of $n = 1$ further.

Now consider the case where $n = 2$, in other words $\mathbb{R}^2$. Let P and P_0 be two different points in $\mathbb{R}^2$ which are contained in a line L. Let $\vec{p}$ and $\vec{p_0}$ be the position vectors for the points P and P_0 respectively. Suppose that Q is an arbitrary point on L. Consider the following diagram.

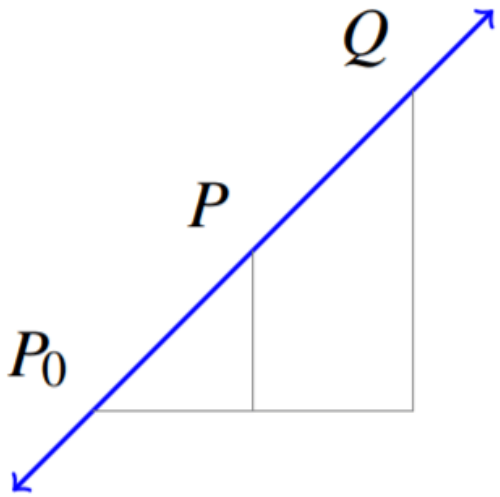


Figure 4.3.1 This diagram illustrates the concept of a line in two dimensions. (CC BY-NC-SA 4.0; Kuttler via A First Course in LINEAR ALGEBRA)

Our goal is to be able to define Q in terms of P and P_0. Consider the vector $\overrightarrow{P_0P} = \vec{p} - \vec{p_0}$ which has its tail at P_0 and point at P. If we add $\vec{p} - \vec{p_0}$ to the position vector $\vec{p_0}$ for P_0, the sum would be a vector with its point at P. In other words,

$$\vec{p} = \vec{p_0} + (\vec{p} - \vec{p_0})$$

Now suppose we were to add $t(\vec{p} - \vec{p_0})$ to $\vec{p}$ where t is some scalar. You can see that by doing so, we could find a vector with its point at Q. In other words, we can find t such that

$$\vec{q} = \vec{p_0} + t\,(\vec{p} - \vec{p_0})$$

This equation determines the line L in $\mathbb{R}^2$. In fact, it determines a line L in $\mathbb{R}^n$. Consider the following definition.

Definition 4.3.1: Vector Equation of a Line

Suppose a line L in $\mathbb{R}^n$ contains the two different points P and P_0. Let $\vec{p}$ and $\vec{p_0}$ be the position vectors of these two points, respectively. Then, L is the collection of points Q which have the position vector $\vec{q}$ given by

$$\vec{q} = \vec{p_0} + t\left(\vec{p} - \vec{p_0}\right)$$

where $t \in \mathbb{R}$.

Let $\vec{d} = \vec{p} - \vec{p_0}$. Then $\vec{d}$ is the **direction vector for** L and the **vector equation** for L is given by

$$\vec{p} = \vec{p_0} + t\vec{d}, t \in \mathbb{R}$$

Note that this definition agrees with the usual notion of a line in two dimensions and so this is consistent with earlier concepts. Consider now points in $\mathbb{R}^3$. If a point $P \in \mathbb{R}^3$ is given by $P = (x, y, z)$, $P_0 \in \mathbb{R}^3$ by $P_0 = (x_0, y_0, z_0)$, then we can write

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

where $\vec{d} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. This is the vector equation of L written in **component form** .

The following theorem claims that such an equation is in fact a line.

Proposition 4.3.1: Algebraic Description of a Straight Line

Let $\vec{a}, \vec{b} \in \mathbb{R}^n$ with $\vec{b} \neq \vec{0}$. Then $\vec{x} = \vec{a} + t\vec{b}, \ \ t \in \mathbb{R}$, is a line.

Proof available at the end of this section.

We can use the above discussion to find the equation of a line when given two distinct points. Consider the following example.

Example 4.3.1: A Line From Two Points

Find a vector equation for the line through the points $P_0 = (1, 2, 0)$ and $P = (2, -4, 6)$.

Solution

We will use the definition of a line given above in Definition 4.3.1 to write this line in the form

$$\vec{q} = \vec{p_0} + t\left(\vec{p} - \vec{p_0}\right)$$

Let $\vec{q} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Then, we can find $\vec{p}$ and $\vec{p_0}$ by taking the position vectors of points P and P_0 respectively. Then,

$$\vec{q} = \vec{p_0} + t\left(\vec{p} - \vec{p_0}\right)$$

can be written as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -6 \\ 6 \end{bmatrix}, \ t \in \mathbb{R}$$

Here, the direction vector $\begin{bmatrix} 1 \\ -6 \\ 6 \end{bmatrix}$ is obtained by $\vec{p} - \vec{p_0} = \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ as indicated above in Definition 4.3.1.

Notice that in the above example we said that we found "a" vector equation for the line, not "the" equation. The reason for this terminology is that there are infinitely many different vector equations for the same line. To see this, replace t with another parameter, say $3s$. Then you obtain a different vector equation for the same line because the same set of points is obtained.

In Example 4.3.1, the vector given by $\begin{bmatrix} 1 \\ -6 \\ 6 \end{bmatrix}$ is the direction vector defined in Definition 4.3.1. If we know the direction vector of a line, as well as a point on the line, we can find the vector equation.

Consider the following example.

Example 4.3.2: A Line From a Point and a Direction Vector

Find a vector equation for the line which contains the point $P_0 = (1, 2, 0)$ and has direction vector $\vec{d} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Solution

We will use Definition 4.3.1 to write this line in the form $\vec{p} = \vec{p_0} + t\vec{d}, \ \ t \in \mathbb{R}$. We are given the direction vector $\vec{d}$. In order to find $\vec{p_0}$, we can use the position vector of the point P_0. This is given by $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$. Letting $\vec{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, the equation for the line is given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \ t \in \mathbb{R} \tag{4.3.1}$$

We sometimes elect to write a line such as the one given in (4.3.1) in the form

$$\left.\begin{array}{l} x = 1 + t \\ y = 2 + 2t \\ z = t \end{array}\right\} \text{ where } t \in \mathbb{R} \tag{4.3.2}$$

This set of equations give the same information as (4.3.1), and is called the **parametric equation of the line**.

Consider the following definition.

Definition 4.3.2: Parametric Equation of a Line

Let L be a line in $\mathbb{R}^3$ which has direction vector $\vec{d} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and goes through the point $P_0 = (x_0, y_0, z_0)$. Then, letting t be a parameter, we can write L as

$$\left.\begin{array}{l} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{array}\right\} \text{ where } t \in \mathbb{R}$$

This is called a **parametric equation** of the line L.

You can verify that the form discussed following Example 4.3.2 in equation (4.3.2) is of the form given in Definition 4.3.2.

There is one other form for a line which is useful, which is the **symmetric form**. Consider the line given by (4.3.2). You can solve for the parameter t to write

$$\begin{aligned} t &= x-1 \\ t &= \tfrac{y-2}{2} \\ t &= z \end{aligned}$$

Therefore,

$$x-1 = \frac{y-2}{2} = z$$

This is the **symmetric form** of the line.

In the following example, we look at how to take the equation of a line from symmetric form to parametric form.

Example 4.3.3: Change Symmetric Form to Parametric Form

Suppose the **symmetric form of a line** is

$$\frac{x-2}{3} = \frac{y-1}{2} = z+3$$

Write the line in parametric form as well as vector form.

Solution

We want to write this line in the form given by Definition 4.3.2. This is of the form

$$\left.\begin{aligned} x &= x_0 + ta \\ y &= y_0 + tb \\ z &= z_0 + tc \end{aligned}\right\} \text{ where } t \in \mathbb{R}$$

Let $t = \frac{x-2}{3}, t = \frac{y-1}{2}$ and $t = z+3$, as given in the symmetric form of the line. Then solving for x, y, z, yields

$$\left.\begin{aligned} x &= 2 + 3t \\ y &= 1 + 2t \\ z &= -3 + t \end{aligned}\right\} \text{ with } t \in \mathbb{R}$$

This is the parametric equation for this line.

Now, we want to write this line in the form given by Definition 4.3.1. This is the form

$$\vec{p} = \vec{p_0} + t\vec{d}$$

where $t \in \mathbb{R}$. This equation becomes

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} + t \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \ t \in \mathbb{R}$$

Much like the above discussion with lines, vectors can be used to determine planes in $\mathbb{R}^n$. Given a vector $\vec{n}$ in $\mathbb{R}^n$ and a point P_0, it is possible to find a **unique** plane which contains P_0 and is perpendicular to the given vector.

Definition 4.3.3: Normal Vector

Let $\vec{n}$ be a nonzero vector in $\mathbb{R}^n$. Then $\vec{n}$ is called a **normal vector** to a plane if and only if

$$\vec{n} \bullet \vec{v} = 0$$

for every vector $\vec{v}$ in the plane.

In other words, we say that $\vec{n}$ is orthogonal (perpendicular) to every vector in the plane.

Consider now a plane with normal vector given by $\vec{n}$, and containing a point P_0. Notice that this plane is unique. If P is an arbitrary point on this plane, then by definition the normal vector is orthogonal to the vector between P_0 and P. Letting $\overrightarrow{0P}$ and $\overrightarrow{0P_0}$ be the position vectors of points P and P_0 respectively, it follows that

$$\vec{n} \bullet (\overrightarrow{0P} - \overrightarrow{0P_0}) = 0$$

or

$$\vec{n} \bullet \overrightarrow{P_0P} = 0$$

The first of these equations gives the vector equation of the plane.

Definition 4.3.4: Vector Equation of a Plane

Let $\vec{n}$ be the normal vector for a plane which contains a point P_0. If P is an arbitrary point on this plane, then the vector equation of the plane is given by

$$\vec{n} \bullet (\overrightarrow{0P} - \overrightarrow{0P_0}) = 0$$

Notice that this equation can be used to determine if a point P is contained in a certain plane.

Example 4.3.4: A Point in a Plane

Let $\vec{n} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ be the normal vector for a plane which contains the point $P_0 = (2, 1, 4)$. Determine if the point $P = (5, 4, 1)$ is contained in this plane.

Solution

By Definition 4.3.2, P is a point in the plane if it satisfies the equation

$$\vec{n} \bullet (\overrightarrow{0P} - \overrightarrow{0P_0}) = 0$$

Given the above $\vec{n}$, P_0, and P, this equation becomes

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \bullet \left(\begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \bullet \left(\begin{bmatrix} 3 \\ 3 \\ -3 \end{bmatrix} \right) \\ = 3 + 6 - 9 = 0 \tag{4.3.3}$$

Therefore $P = (5, 4, 1)$ is contained in the plane.

Suppose $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, $P = (x, y, z)$ and $P_0 = (x_0, y_0, z_0)$.

Then

$$\begin{aligned} \vec{n} \bullet (\overrightarrow{0P} - \overrightarrow{0P_0}) &= 0 \\ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \bullet \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \right) &= 0 \\ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \bullet \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} &= 0 \\ a(x - x_0) + b(y - y_0) + c(z - z_0) &= 0 \end{aligned} \tag{4.3.4}$$

We can also write this equation as

$$ax + by + cz = ax_0 + by_0 + cz_0$$

Notice that since P_0 is given, $ax_0 + by_0 + cz_0$ is a known scalar, which we can call d. This equation becomes

$$ax + by + cz = d$$

Definition 4.3.5: Scalar Equation of a Plane

Let $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be the normal vector for a plane which contains the point $P_0 = (x_0, y_0, z_0)$.Then if $P = (x, y, z)$ is an arbitrary point on the plane, the scalar equation of the plane is given by

$$ax + by + cz = d$$

where $a, b, c, d \in \mathbb{R}$ and $d = ax_0 + by_0 + cz_0$.

Consider the following equation.

Example 4.3.5: Finding the Equation of a Plane

Find an equation of the plane containing $P_0 = (3, -2, 5)$ and orthogonal to $\vec{n} = \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix}$.

Solution

The above vector $\vec{n}$ is the normal vector for this plane. Using Definition 4.3.2, we can determine the vector equation for this plane.

$$\begin{aligned} \vec{n} \bullet (\overrightarrow{0P} - \overrightarrow{0P_0}) &= 0 \\ \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix} \bullet \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} \right) &= 0 \\ \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} x - 3 \\ y + 2 \\ z - 5 \end{bmatrix} &= 0 \end{aligned} \tag{4.3.5}$$

Using Definition 4.3.3, we can determine the scalar equation of the plane.

$$-2x + 4y + 1z = -2(3) + 4(-2) + 1(5) = -9$$

Hence, the vector equation of the plane is

$$\begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} x-3 \\ y+2 \\ z-5 \end{bmatrix} = 0$$

and the scalar equation is

$$-2x + 4y + 1z = -9$$

Example 4.3.6: Shortest Distance From a Point to a Plane

Find the shortest distance from the point $P = (3, 2, 3)$ to the plane given by $2x + y + 2z = 2$, and find the point Q on the plane that is closest to P.

Solution

Pick an arbitrary point P_0 on the plane. Then, it follows that

$$\overrightarrow{QP} = proj_{\vec{n}} \overrightarrow{P_0P}$$

and $\|\overrightarrow{QP}\|$ is the shortest distance from P to the plane. Further, the vector $\overrightarrow{0Q} = \overrightarrow{0P} - \overrightarrow{QP}$ gives the necessary point Q.

From the above scalar equation, we have that $\vec{n} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$. Now, choose $P_0 = (1, 0, 0)$ so that $\vec{n} \bullet \overrightarrow{0P} = 2 = d$. Then, $\overrightarrow{P_0P} = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$.

Next, compute $\overrightarrow{QP} = proj_{\vec{n}} \overrightarrow{P_0P}$.

$$\begin{aligned} \overrightarrow{QP} &= proj_{\vec{n}} \overrightarrow{P_0P} \\ &= \left(\frac{\overrightarrow{P_0P} \bullet \vec{n}}{\|\vec{n}\|^2} \right) \vec{n} \\ &= \frac{12}{9} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \\ &= \frac{4}{3} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \end{aligned} \tag{4.3.6}$$

Then, $\|\overrightarrow{QP}\| = 4$ so the shortest distance from P to the plane is 4.

Next, to find the point Q on the plane which is closest to P we have

$$\begin{aligned} \overrightarrow{0Q} &= \overrightarrow{0P} - \overrightarrow{QP} \\ &= \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} - \frac{4}{3} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \end{aligned} \tag{4.3.7}$$

Therefore, $Q = (\frac{1}{3}, \frac{2}{3}, \frac{1}{3})$.

Answers

Proposition $4.3.1$**: Algebraic Description of a Straight Line**

Let $\vec{x}_1, \vec{x}_2 \in \mathbb{R}^n$. Define $\vec{x}_1 = \vec{a}$ and let $\vec{x}_2 - \vec{x}_1 = \vec{b}$. Since $\vec{b} \neq \vec{0}$, it follows that $\vec{x}_2 \neq \vec{x}_1$. Then $\vec{a} + t\vec{b} = \vec{x}_1 + t\left(\vec{x}_2 - \vec{x}_1\right)$. It follows that $\vec{x} = \vec{a} + t\vec{b}$ is a line containing the two different points X_1 and X_2 whose position vectors are given by $\vec{x}_1$ and $\vec{x}_2$ respectively.

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4.3E: Exercises for Section 4.3

Exercise $4.3E.1$

Find the vector equation for the line through $(-7, 6, 0)$ and $(-1, 1, 4)$. Then, find the parametric equations for this line.

Answer available at the end of this section.

Exercise $4.3E.2$

Find parametric equations for the line through the point $(7, 7, 1)$ with a direction vector $\vec{d} = \begin{bmatrix} 1 \\ 6 \\ 2 \end{bmatrix}$.

Exercise $4.3E.3$

Parametric equations of the line are

$$\begin{aligned} x &= t + 2 \\ y &= 6 - 3t \\ x &= -t = 6 \end{aligned} \tag{4.3E.1}$$

Find a direction vector for the line and a point on the line.

Exercise $4.3E.4$

Find the vector equation for the line through the two points $(-5, 5, 1),\ (2, 2, 4)$. Then, find the parametric equations.

Answer available at the end of this section.

Exercise $4.3E.5$

The equation of a line in two dimensions is written as $y = x - 5$. Find parametric equations for this line.

Exercise $4.3E.6$

Find parametric equations for the line through $(6, 5, -2)$ and $(5, 1, 2)$.

Exercise $4.3E.7$

Find the vector equation and parametric equations for the line through the point $(-7, 10, -6)$ with a direction vector $\vec{d} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$.

Exercise $4.3E.8$

Parametric equations of the line are

$$\begin{aligned} x &= 2t + 2 \\ y &= 5 - 4t \backslash \text{nonumver} \\ z &= -t - 3 \end{aligned} \tag{4.3E.2}$$

Find a direction vector for the line and a point on the line, and write the vector equation of the line.

Answer available at the end of this section.

Exercise $4.3E.9$

Find the vector equation and parametric equations for the line through the two points $(4, 10, 0),\ \ (1, -5, -6)$.

Exercise $4.3E.10$

Find the point on the line segment from $P = (-4, 7, 5)$ to $Q = (2, -2, -3)$ which is $\frac{1}{7}$ of the way from P to Q.

Answer available at the end of this section.

Exercise $4.3E.11$

Suppose a triangle in $\mathbb{R}^n$ has vertices at $P_1,\ \ P_2$, and P_3. Consider the lines which are drawn from a vertex to the mid point of the opposite side. Show these three lines intersect in a point and find the coordinates of this point.

Answers

Exercise $4.3E.1$

Vector Equation:

$$\vec{r}(t) = \begin{bmatrix} -7 \\ 6 \\ 0 \end{bmatrix} + t \begin{bmatrix} 6 \\ -5 \\ 4 \end{bmatrix}, \quad t \in \mathbb{R}.$$

Parametric Equations:

$$x = -7 + 6t, \quad y = 6 - 5t, \quad z = 4t, \quad t \in \mathbb{R}.$$

Exercise $4.3E.4$

Vector Equation:

$$\vec{r}(t) = \begin{bmatrix} -5 \\ 5 \\ 1 \end{bmatrix} + t \begin{bmatrix} 7 \\ -3 \\ 3 \end{bmatrix}, \quad t \in \mathbb{R}.$$

Parametric Equations:

$$x = -5 + 7t, \quad y = 5 - 3t, \quad z = 1 + 3t, \quad t \in \mathbb{R}.$$

Exercise $4.3E.8$

Direction Vector: The coefficients of t in the parametric equations give the direction vector:

$$\vec{d} = \begin{bmatrix} 2 \\ -4 \\ -1 \end{bmatrix}.$$

A Point on the Line: Setting $t = 0$ in the parametric equations gives a point on the line:

$$P = (2, 5, -3).$$

Vector Equation: The vector equation of the line is:

$$\vec{r}(t) = \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix} + t \begin{bmatrix} 2 \\ -4 \\ -1 \end{bmatrix}, \quad t \in \mathbb{R}.$$

Exercise $4.3E.10$

The point that is $\frac{1}{7}$ of the way from $P = (-4, 7, 5)$ to $Q = (2, -2, -3)$ is:

$$\left(\frac{-22}{7}, \frac{40}{7}, \frac{27}{7}\right).$$

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4.4: Spanning Sets in Rⁿ

Learning Objectives

- Determine the span of a set of vectors.
- Determine if a vector is contained in a specified span.

By generating all linear combinations of a set of vectors one can obtain various subsets of $\mathbb{R}^n$ which we call subspaces. For example what set of vectors in $\mathbb{R}^3$ generate the XY-plane? What is the smallest such set of vectors can you find? The tools of spanning, linear independence and basis are exactly what is needed to answer these and similar questions and are the focus of this section. The following definition is essential.

Definition 4.4.1: Subset

Let U and W be sets of vectors in $\mathbb{R}^n$. If all vectors in U are also in W, we say that U is a **subset** of W, denoted

$$U \subseteq W$$

We begin this section with a definition.

Definition 4.4.2: Span of a Set of Vectors

The collection of all linear combinations of a set of vectors $\{\vec{u}_1, \cdots, \vec{u}_k\}$ in $\mathbb{R}^n$ is known as the **span** of these vectors and is written as $\text{span}\{\vec{u}_1, \cdots, \vec{u}_k\}$.

Consider the following example.

Example 4.4.1: Span of Vectors

Describe the span of the vectors $\vec{u} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T$ and $\vec{v} = \begin{bmatrix} 3 & 2 & 0 \end{bmatrix}^T \in \mathbb{R}^3$.

Solution

You can see that any linear combination of the vectors $\vec{u}$ and $\vec{v}$ yields a vector of the form $\begin{bmatrix} x & y & 0 \end{bmatrix}^T$ in the XY-plane.

Moreover every vector in the XY-plane is in fact such a linear combination of the vectors $\vec{u}$ and $\vec{v}$. That's because

$$\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = (-2x + 3y) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + (x - y) \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$$

Thus $\text{span}\{\vec{u}, \vec{v}\}$ is precisely the XY-plane. This concept is explained below in a interactive video format. Feel free to click on it and see the theory in action.

You can convince yourself that no single vector can span the XY-plane. In fact, take a moment to consider what is meant by the span of a single vector.

However you can make the set larger if you wish. For example consider the larger set of vectors $\{\vec{u}, \vec{v}, \vec{w}\}$ where $\vec{w} = \begin{bmatrix} 4 & 5 & 0 \end{bmatrix}^T$. Since the first two vectors already span the entire XY-plane, the span is once again precisely the XY-plane and nothing has been gained. Of course if you add a new vector such as $\vec{w} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$ then it does span a different space. What is the span of $\vec{u}, \vec{v}, \vec{w}$ in this case?

The distinction between the sets $\{\vec{u},\vec{v}\}$ and $\{\vec{u},\vec{v},\vec{w}\}$ will be made using the concept of linear independence.

Consider the vectors $\vec{u},\vec{v}$, and $\vec{w}$ discussed above. In the next example, we will show how to formally demonstrate that $\vec{w}$ is in the span of $\vec{u}$ and $\vec{v}$.

Example 4.4.3: Vector in a Span

Let $\vec{u}=\begin{bmatrix}1 & 1 & 0\end{bmatrix}^T$ and $\vec{v}=\begin{bmatrix}3 & 2 & 0\end{bmatrix}^T\in\mathbb{R}^3$. Show that $\vec{w}=\begin{bmatrix}4 & 5 & 0\end{bmatrix}^T$ is in $\text{span}\{\vec{u},\vec{v}\}$.

Solution

For a vector to be in $\text{span}\{\vec{u},\vec{v}\}$, it must be a linear combination of these vectors. If $\vec{w}\in\text{span}\{\vec{u},\vec{v}\}$, we must be able to find scalars a,b such that

$$\vec{w}=a\vec{u}+b\vec{v}$$

We proceed as follows.

$$\begin{bmatrix}4\\5\\0\end{bmatrix}=a\begin{bmatrix}1\\1\\0\end{bmatrix}+b\begin{bmatrix}3\\2\\0\end{bmatrix}$$

This is equivalent to the following system of equations

$$\begin{aligned}a+3b&=4\\a+2b&=5\end{aligned}\tag{4.4.1}$$

We solving this system the usual way, constructing the augmented matrix and row reducing to find the reduced row-echelon form.

$$\begin{bmatrix}1&3&4\\1&2&5\end{bmatrix}\to\cdots\to\begin{bmatrix}1&0&7\\0&1&-1\end{bmatrix}$$

The solution is $a=7,b=-1$. This means that

$$\vec{w}=7\vec{u}-\vec{v}$$

Therefore we can say that $\vec{w}$ is in $\text{span}\{\vec{u},\vec{v}\}$.

Example 4.4.4

Let $V=\{\langle 5,-6,-1\rangle,\langle 1,4,5\rangle\}$. Which of the following vectors are in the space spanned by V ?

- $\langle 5,-5,-3\rangle$
- $\langle -1,-4,5\rangle$
- $\langle 2,1,-4\rangle$
- $\langle -5,-1,-1\rangle$
- $\langle -5,2,-3\rangle$

Solution

- **Video Length**: 7 minutes 41 seconds
- **Context**: This video demonstrates how to determine which given vectors lie in the span of a set of vectors by checking if they can be expressed as linear combinations of the spanning set.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

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4.4.E: Exercise for Section 4.4

Exercise 4.4.E.1

Here are some vectors.

$$\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ -4 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \\ -10 \end{bmatrix}, \begin{bmatrix} 12 \\ 17 \\ -24 \end{bmatrix}$$

Describe the span of these vectors as the span of as few vectors as possible.

Answer available at the end of this section.

Exercise 4.4.E.2

Here are some vectors.

$$\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 12 \\ 29 \\ -24 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 9 \\ -4 \end{bmatrix}, \begin{bmatrix} 5 \\ 12 \\ -10 \end{bmatrix}.$$

Describe the span of these vectors as the span of as few vectors as possible.

Exercise 4.4.E.3

Here are some vectors.

$$\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$$

Describe the span of these vectors as the span of as few vectors as possible.

Exercise 4.4.E.4

Here are some vectors.

$$\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

Now here is another vector:

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Is this vector in the span of the first four vectors? If it is, exhibit a linear combination of the first four vectors which equals this vector, using as few vectors as possible in the linear combination.

Answer available at the end of this section.

Exercise $4.4.E.5$

Here are some vectors.

$$\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

Now here is another vector:

$$\begin{bmatrix} 2 \\ -3 \\ -4 \end{bmatrix}$$

Is this vector in the span of the first four vectors? If it is, exhibit a linear combination of the first four vectors which equals this vector, using as few vectors as possible in the linear combination.

Exercise $4.4.E.6$

Here are some vectors.

$$\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Now here is another vector:

$$\begin{bmatrix} 1 \\ 9 \\ 1 \end{bmatrix}$$

Is this vector in the span of the first four vectors? If it is, exhibit a linear combination of the first four vectors which equals this vector, using as few vectors as possible in the linear combination.

Exercise $4.4.E.7$

Here are some vectors,

$$\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix}$$

Now here is another vector:

$$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Is this vector in the span of the first four vectors? If it is, exhibit a linear combination of the first four vectors which equals this vector, using as few vectors as possible in the linear combination.

Exercise $4.4.E.8$

Here are some vectors.

$$\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix}$$

Now here is another vector:

$$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Is this vector in the span of the first four vectors? If it is, exhibit a linear combination of the first four vectors which equals this vector, using as few vectors as possible in the linear combination.

Exercise $4.4.E.9$

Here are some vectors.

$$\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}$$

Now here is another vector:

$$\begin{bmatrix} -1 \\ -4 \\ 2 \end{bmatrix}$$

Is this vector in the span of the first four vectors? If it is, exhibit a linear combination of the first four vectors which equals this vector, using as few vectors as possible in the linear combination.

Exercise $4.4.E.10$

Suppose $\{\vec{x}_1, \cdots, \vec{x}_k\}$ is a set of vectors from $\mathbb{R}^n$. Show that $\vec{0}$ is in $\text{span}\{\vec{x}_1, \cdots, \vec{x}_k\}$.

Answer available at the end of this section.

Exercise $4.4.E.11$

Prove that the span of any set of vectors contains the zero vector.

Exercise $4.4.E.12$

Consider the vectors $\vec{v}_1 = \langle 2, 3, -1 \rangle$ and $\vec{v}_2 = \langle 0, 1, 2 \rangle$.

a. Describe $\text{span}(\vec{v}_1, \vec{v}_2)$ geometrically.
b. Express $\text{span}(\vec{v}_1, \vec{v}_2)$ in standard form for the type of object you concluded it is in part a.

Answer

a. Geometric Interpretation: The span of the two given vectors forms a plane through the origin in $\mathbb{R}^3$.
b. Standard Form of the Plane: The equation of the plane is:

$$7x - 4y + 2z = 0.$$

.

Exercise $4.4.E.13$

Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a set of vectors that spans a vector space V. Suppose one of the vectors, say $\vec{v}_i$, is a linear combination of the other vectors in S. Prove that the set $S' = S \setminus \{\vec{v}_i\}$ still spans V, i.e.,

$$\text{span}(S) = \text{span}(S').$$

Note: $S \setminus \{\vec{v}_i\}$ means the set S with the vector $\vec{v}_i\}$ removed.

Exercise $4.4.E.14$

Prove that if a set of vectors contains the zero vector, removing the zero vector does not change the span of the set

Answers

Exercise $4.4.E.1$

The given vectors span the same set as:

$$\text{span}\left\{\begin{bmatrix}1\\1\\-2\end{bmatrix}, \begin{bmatrix}1\\2\\-2\end{bmatrix}\right\}.$$

Exercise $4.4.E.4$

The vector

$$\begin{bmatrix}1\\2\\-1\end{bmatrix} \tag{4.4.E.1}$$

is in the span of the given four vectors. A minimal linear combination expressing this vector is:

$$\begin{bmatrix}1\\2\\-1\end{bmatrix} = \frac{5}{3}\begin{bmatrix}1\\1\\-2\end{bmatrix} - \frac{2}{3}\begin{bmatrix}1\\2\\-2\end{bmatrix}.$$

Exercise $4.4.E.10$

$\sum_{i=1}^{k} 0\vec{x}_k = \vec{0}$

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4.5: Linear Independence

Learning Objectives

- Determine if a set of vectors is linearly independent.
- Evaluate the linear independence of vectors through theoretical analysis and solving systems of equations.

We now turn our attention to the following question: what linear combinations of a given set of vectors $\{\vec{u}_1, \cdots, \vec{u}_k\}$ in $\mathbb{R}^n$ yields the zero vector? Clearly $0\vec{u}_1 + 0\vec{u}_2 + \cdots + 0\vec{u}_k = \vec{0}$, but is it possible to have $\sum_{i=1}^{k} a_i\vec{u}_i = \vec{0}$ without all coefficients being zero?

You can create examples where this easily happens. For example if $\vec{u}_1 = \vec{u}_2$, then $1\vec{u}_1 - \vec{u}_2 + 0\vec{u}_3 + \cdots + 0\vec{u}_k = \vec{0}$, no matter the vectors $\{\vec{u}_3, \cdots, \vec{u}_k\}$. But sometimes it can be more subtle.

Example 4.5.1: Linearly Dependent Set of Vectors

Consider the vectors

$$\vec{u}_1 = \begin{bmatrix} 0 & 1 & -2 \end{bmatrix}^T, \vec{u}_2 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T, \vec{u}_3 = \begin{bmatrix} -2 & 3 & 2 \end{bmatrix}^T, \text{ and } \vec{u}_4 = \begin{bmatrix} 1 & -2 & 0 \end{bmatrix}^T$$

in $\mathbb{R}^3$.

Then verify that

$$1\vec{u}_1 + 0\vec{u}_2 + -\vec{u}_3 - 2\vec{u}_4 = \vec{0}$$

You can see that the linear combination does yield the zero vector but has some non-zero coefficients. Thus we define a set of vectors to be *linearly dependent* if this happens.

Definition 4.5.1: Linearly Dependent Set of Vectors

A set of non-zero vectors $\{\vec{u}_1, \cdots, \vec{u}_k\}$ in $\mathbb{R}^n$ is said to be linearly dependent if a linear combination of these vectors without all coefficients being zero does yield the zero vector.

Note that if $\sum_{i=1}^{k} a_i\vec{u}_i = \vec{0}$ and some coefficient is non-zero, say $a_1 \neq 0$, then

$$\vec{u}_1 = \frac{-1}{a_1}\sum_{i=2}^{k} a_i\vec{u}_i$$

and thus $\vec{u}_1$ is in the span of the other vectors. And the converse clearly works as well, so we get that a set of vectors is linearly dependent precisely when one of its vector is in the span of the other vectors of that set.

In particular, you can show that the vector $\vec{u}_1$ in the above example is in the span of the vectors $\{\vec{u}_2, \vec{u}_3, \vec{u}_4\}$.

If a set of vectors is NOT linearly dependent, then it must be that any linear combination of these vectors which yields the zero vector must use all zero coefficients. This is a very important notion, and we give it its own name of *linear independence*.

Definition 4.5.2: Linearly Independent Set of Vectors

A set of non-zero vectors $\{\vec{u}_1, \cdots, \vec{u}_k\}$ in $\mathbb{R}^n$ is said to be linearly independent if whenever

$$\sum_{i=1}^{k} a_i\vec{u}_i = \vec{0}$$

it follows that each $a_i = 0$.

Note also that we require all vectors to be non-zero to form a linearly independent set.

To view this in a more familiar setting, form the $n \times k$ matrix A having these vectors as columns. Then all we are saying is that the set $\{\vec{u}_1, \cdots, \vec{u}_k\}$ is linearly independent precisely when $AX = 0$ has only the trivial solution.

Here is an example.

Example 4.5.2: Linearly Independent Vectors

Consider the vectors $\vec{u} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T$, $\vec{v} = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T$, and $\vec{w} = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T$ in $\mathbb{R}^3$. Verify whether the set $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent.

Solution

So suppose that we have a linear combinations $a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}$. Then you can see that this can only happen with $a = b = c = 0$.

As mentioned above, you can equivalently form the 3×3 matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, and show that $AX = 0$ has only the trivial solution.

Thus this means the set $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent.

Example 4.5.3

Consider the vectors: $\vec{v_1} = \begin{bmatrix} 5 & 2 & -3 \end{bmatrix}^T$, $\vec{v_2} = \begin{bmatrix} 14 & 13 & 5 \end{bmatrix}^T$, and $\vec{v_3} = \begin{bmatrix} -3 & 5 & 5 \end{bmatrix}^T$ in $\mathbb{R}^3$. Are these vectors linearly independent?

Solution

- **Video Length**: 8 minutes 8 seconds
- **Context**: Determine if the given set of vectors in $\mathbb{R}^3$ is linearly independent.

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In terms of spanning, a set of vectors is linearly independent if it does not contain unnecessary vectors; that is, not vector is in the span of the others.

Thus we put all this together in the following important theorem.

Theorem 4.5.1: Linear Independence as a Linear Combination

Let $\{\vec{u}_1, \cdots, \vec{u}_k\}$ be a collection of vectors in $\mathbb{R}^n$. Then the following are equivalent:

1. It is linearly independent, that is whenever

$$\sum_{i=1}^{k} a_i \vec{u}_i = \vec{0}$$

it follows that each coefficient $a_i = 0$.

2. No vector is in the span of the others.
3. The system of linear equations $AX = 0$ has only the trivial solution, where A is the $n \times k$ matrix having these vectors as columns.

The last sentence of this theorem is useful as it allows us to use the reduced row-echelon form of a matrix to determine if a set of vectors is linearly independent. Let the vectors be columns of a matrix A. Find the reduced row-echelon form of A. If each column has a leading one, then it follows that the vectors are linearly independent.

Sometimes we refer to the condition regarding sums as follows: The set of vectors, $\{\vec{u}_1, \cdots, \vec{u}_k\}$ is linearly independent if and only if there is no nontrivial linear combination which equals the zero vector. A nontrivial linear combination is one in which not all the scalars equal zero. Similarly, a trivial linear combination is one in which all scalars equal zero.

Here is a detailed example in $\mathbb{R}^4$.

Example 4.5.4: Linear Independence

Determine whether the set of vectors given by

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 2 \\ 0 \end{bmatrix} \right\}$$

is linearly independent. If it is linearly dependent, express one of the vectors as a linear combination of the others.

Solution

In this case the matrix of the corresponding homogeneous system of linear equations is

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 2 & 1 & 1 & 2 & 0 \\ 3 & 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 & 0 \end{bmatrix}$$

The reduced row-echelon form is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and so every column is a pivot column and the corresponding system $AX = 0$ only has the trivial solution. Therefore, these vectors are linearly independent and there is no way to obtain one of the vectors as a linear combination of the others.

Consider another example.

Example 4.5.5: Linear Independence

Determine whether the set of vectors given by

$$\left\{\begin{bmatrix}1\\2\\3\\0\end{bmatrix},\begin{bmatrix}2\\1\\0\\1\end{bmatrix},\begin{bmatrix}0\\1\\1\\2\end{bmatrix},\begin{bmatrix}3\\2\\2\\-1\end{bmatrix}\right\}$$

is linearly independent. If it is linearly dependent, express one of the vectors as a linear combination of the others.

Solution

Form the 4×4 matrix A having these vectors as columns:

$$A=\begin{bmatrix}1&2&0&3\\2&1&1&2\\3&0&1&2\\0&1&2&-1\end{bmatrix}$$

Then by Theorem $4.5.1$, the given set of vectors is linearly independent exactly if the system $AX=0$ has only the trivial solution.

The augmented matrix for this system and corresponding reduced row-echelon form are given by

$$\begin{bmatrix}1&2&0&3&0\\2&1&1&2&0\\3&0&1&2&0\\0&1&2&-1&0\end{bmatrix}\to\cdots\to\begin{bmatrix}1&0&0&1&0\\0&1&0&1&0\\0&0&1&-1&0\\0&0&0&0&0\end{bmatrix}$$

Not all the columns of the coefficient matrix are pivot columns and so the vectors are not linearly independent. In this case, we say the vectors are linearly dependent.

It follows that there are infinitely many solutions to $AX=0$, one of which is

$$\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}$$

Therefore we can write

$$1\begin{bmatrix}1\\2\\3\\0\end{bmatrix}+1\begin{bmatrix}2\\1\\0\\1\end{bmatrix}-1\begin{bmatrix}0\\1\\1\\2\end{bmatrix}-1\begin{bmatrix}3\\2\\2\\-1\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

This can be rearranged as follows

$$1\begin{bmatrix}1\\2\\3\\0\end{bmatrix}+1\begin{bmatrix}2\\1\\0\\1\end{bmatrix}-1\begin{bmatrix}0\\1\\1\\2\end{bmatrix}=\begin{bmatrix}3\\2\\2\\-1\end{bmatrix}$$

This gives the last vector as a linear combination of the first three vectors.

Notice that we could rearrange this equation to write any of the four vectors as a linear combination of the other three.

When given a linearly independent set of vectors, we can determine if related sets are linearly independent.

Example $4.5.6$: Related Sets of Vectors

Let $\{\vec{u},\vec{v},\vec{w}\}$ be an independent set of $\mathbb{R}^n$. Is $\{\vec{u}+\vec{v},2\vec{u}+\vec{w},\vec{v}-5\vec{w}\}$ linearly independent?

Solution

Suppose $a(\vec{u}+\vec{v})+b(2\vec{u}+\vec{w})+c(\vec{v}-5\vec{w})=\vec{0}_n$ for some $a,b,c\in\mathbb{R}$. Then

$$(a+2b)\vec{u}+(a+c)\vec{v}+(b-5c)\vec{w}=\vec{0}_n.$$

Since $\{\vec{u},\vec{v},\vec{w}\}$ is independent,

$$\begin{aligned} a+2b&=0\\ a+c&=0\\ b-5c&=0 \end{aligned} \tag{4.5.1}$$

This system of three equations in three variables has the unique solution $a=b=c=0$. Therefore, $\{\vec{u}+\vec{v},2\vec{u}+\vec{w},\vec{v}-5\vec{w}\}$ is independent.

The following corollary follows from the fact that if the augmented matrix of a homogeneous system of linear equations has more columns than rows, the system has infinitely many solutions.

Corollary 4.5.1: Linear Dependence in $\mathbb{R}^n$

Let $\{\vec{u}_1,\cdots,\vec{u}_k\}$ be a set of vectors in $\mathbb{R}^n$. If $k>n$, then the set is linearly dependent (i.e. NOT linearly independent).

Proof available at the end of this section.

Example 4.5.7: Linear Dependence

Consider the vectors

$$\left\{\begin{bmatrix}1\\4\end{bmatrix},\begin{bmatrix}2\\3\end{bmatrix},\begin{bmatrix}3\\2\end{bmatrix}\right\}$$

Are these vectors linearly independent?

Solution

This set contains three vectors in $\mathbb{R}^2$. By Corollary 4.5.1 these vectors are linearly dependent. In fact, we can write

$$(-1)\begin{bmatrix}1\\4\end{bmatrix}+(2)\begin{bmatrix}2\\3\end{bmatrix}=\begin{bmatrix}3\\2\end{bmatrix}$$

showing that this set is linearly dependent.

The third vector in the previous example is in the span of the first two vectors. We could find a way to write this vector as a linear combination of the other two vectors. It turns out that the linear combination which we found is the **only** one, provided that the set is linearly independent.

Theorem 4.5.2: Unique Linear Combination

Let $U\subseteq\mathbb{R}^n$ be an independent set. Then any vector $\vec{x}\in\text{span}(U)$ can be written uniquely as a linear combination of vectors of U.

Proof available at the end of this section.

Suppose that $\vec{u},\vec{v}$ and $\vec{w}$ are nonzero vectors in $\mathbb{R}^3$, and that $\{\vec{v},\vec{w}\}$ is independent. Consider the set $\{\vec{u},\vec{v},\vec{w}\}$. When can we know that this set is independent? It turns out that this follows exactly when $\vec{u}\notin\text{span}\{\vec{v},\vec{w}\}$.

Example 4.5.8

Suppose that $\vec{u}, \vec{v}$ and $\vec{w}$ are nonzero vectors in $\mathbb{R}^3$, and that $\{\vec{v}, \vec{w}\}$ is independent. Prove that $\{\vec{u}, \vec{v}, \vec{w}\}$ is independent if and only if $\vec{u} \notin \text{span}\{\vec{v}, \vec{w}\}$.

If $\vec{u} \in \text{span}\{\vec{v}, \vec{w}\}$, then there exist $a, b \in \mathbb{R}$ so that $\vec{u} = a\vec{v} + b\vec{w}$. This implies that $\vec{u} - a\vec{v} - b\vec{w} = \vec{0}_3$, so $\vec{u} - a\vec{v} - b\vec{w}$ is a nontrivial linear combination of $\{\vec{u}, \vec{v}, \vec{w}\}$ that vanishes, and thus $\{\vec{u}, \vec{v}, \vec{w}\}$ is dependent.

Now suppose that $\vec{u} \notin \text{span}\{\vec{v}, \vec{w}\}$, and suppose that there exist $a, b, c \in \mathbb{R}$ such that $a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}_3$. If $a \neq 0$, then $\vec{u} = -\frac{b}{a}\vec{v} - \frac{c}{a}\vec{w}$, and $\vec{u} \in \text{span}\{\vec{v}, \vec{w}\}$, a contradiction. Therefore, $a = 0$, implying that $b\vec{v} + c\vec{w} = \vec{0}_3$. Since $\{\vec{v}, \vec{w}\}$ is independent, $b = c = 0$, and thus $a = b = c = 0$, i.e., the only linear combination of $\vec{u}, \vec{v}$ and $\vec{w}$ that vanishes is the trivial one.

Therefore, $\{\vec{u}, \vec{v}, \vec{w}\}$ is independent.

Consider the following useful theorem.

Theorem 4.5.3: Invertible Matrices

Let A be an invertible $n \times n$ matrix. Then the columns of A are independent and span $\mathbb{R}^n$. Similarly, the rows of A are independent and span the set of all $1 \times n$ vectors.

This theorem also allows us to determine if a matrix is invertible. If an $n \times n$ matrix A has columns which are independent, or span $\mathbb{R}^n$, then it follows that A is invertible. If it has rows that are independent, or span the set of all $1 \times n$ vectors, then A is invertible.

A Short Application to Chemistry

The following section applies the concepts of spanning and linear independence to the subject of chemistry.

When working with chemical reactions, there are sometimes a large number of reactions and some are in a sense redundant. Suppose you have the following chemical reactions.

$$CO + \tfrac{1}{2}O_2 \to CO_2$$
$$H_2 + \tfrac{1}{2}O_2 \to H_2O$$
$$CH_4 + \tfrac{3}{2}O_2 \to CO + 2H_2O$$
$$CH_4 + 2O_2 \to CO_2 + 2H_2O$$

There are four chemical reactions here but they are not independent reactions. There is some redundancy. What are the independent reactions? Is there a way to consider a shorter list of reactions? To analyze this situation, we can write the reactions in a matrix as follows

$$\begin{bmatrix} CO & O_2 & CO_2 & H_2 & H_2O & CH_4 \\ 1 & 1/2 & -1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1 & -1 & 0 \\ -1 & 3/2 & 0 & 0 & -2 & 1 \\ 0 & 2 & -1 & 0 & -2 & 1 \end{bmatrix}$$

Each row contains the coefficients of the respective elements in each reaction. For example, the top row of numbers comes from $CO + \frac{1}{2}O_2 - CO_2 = 0$ which represents the first of the chemical reactions.

We can write these coefficients in the following matrix

$$\begin{bmatrix} 1 & 1/2 & -1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1 & -1 & 0 \\ -1 & 3/2 & 0 & 0 & -2 & 1 \\ 0 & 2 & -1 & 0 & -2 & 1 \end{bmatrix}$$

Rather than listing all of the reactions as above, it would be more efficient to only list those which are independent by throwing out that which is redundant. We can use the concepts of the previous section to accomplish this.

First, take the reduced row-echelon form of the above matrix.

$$\begin{bmatrix} 1 & 0 & 0 & 3 & -1 & -1 \\ 0 & 1 & 0 & 2 & -2 & 0 \\ 0 & 0 & 1 & 4 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The top three rows represent “independent" reactions which come from the original four reactions. One can obtain each of the original four rows of the matrix given above by taking a suitable linear combination of rows of this reduced row-echelon matrix.

With the redundant reaction removed, we can consider the simplified reactions as the following equations

$$CO + 3H_2 - 1H_2O - 1CH_4 = 0$$
$$O_2 + 2H_2 - 2H_2O = 0$$
$$CO_2 + 4H_2 - 2H_2O - 1CH_4 = 0$$

In terms of the original notation, these are the reactions

$$CO + 3H_2 \rightarrow H_2O + CH_4$$
$$O_2 + 2H_2 \rightarrow 2H_2O$$
$$CO_2 + 4H_2 \rightarrow 2H_2O + CH_4$$

These three reactions provide an equivalent system to the original four equations. The idea is that, in terms of what happens chemically, you obtain the same information with the shorter list of reactions. Such a simplification is especially useful when dealing with very large lists of reactions which may result from experimental evidence.

The main theorem about bases is not only they exist, but that they must be of the same size. To show this, we will need the the following fundamental result, called the Exchange Theorem.

Answers

Corollary $4.5.1$**: Linear Dependence in** $\mathbb{R}^{\prime\prime}$

Form the $n \times k$ matrix A having the vectors $\{\vec{u}_1, \cdots, \vec{u}_k\}$ as its columns and suppose $k > n$. Then A has rank \(r \leq n

Theorem $4.5.2$**: Unique Linear Combination**

To prove this theorem, we will show that two linear combinations of vectors in U that equal $\vec{x}$ must be the same. Let $U = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$. Suppose that there is a vector $\vec{x} \in \text{span}(U)$ such that

$$\begin{aligned} \vec{x} &= s_1\vec{u}_1 + s_2\vec{u}_2 + \cdots + s_k\vec{u}_k, \text{ for some } s_1, s_2, \ldots, s_k \in \mathbb{R}, \text{ and} \\ \vec{x} &= t_1\vec{u}_1 + t_2\vec{u}_2 + \cdots + t_k\vec{u}_k, \text{ for some } t_1, t_2, \ldots, t_k \in \mathbb{R}. \end{aligned} \tag{4.5.2}$$

Then $\vec{0}_n = \vec{x} - \vec{x} = (s_1 - t_1)\vec{u}_1 + (s_2 - t_2)\vec{u}_2 + \cdots + (s_k - t_k)\vec{u}_k$.

Since U is independent, the only linear combination that vanishes is the trivial one, so $s_i - t_i = 0$ for all i, $1 \leq i \leq k$.

Therefore, $s_i = t_i$ for all i, $1 \leq i \leq k$, and the representation is unique.Let $U \subseteq \mathbb{R}^n$ be an independent set. Then any vector $\vec{x} \in \text{span}(U)$ can be written uniquely as a linear combination of vectors of U.

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4.5.E: Exercise for Section 4.5

Exercise 4.5.E.1

Are the following vectors linearly independent? If they are, explain why and if they are not, exhibit one of them as a linear combination of the others. Also give a linearly independent set of vectors which has the same span as the given vectors.

$$\begin{bmatrix} 1 \\ 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 10 \\ 2 \\ 1 \end{bmatrix}$$

Answer available at the end of this section.

Exercise 4.5.E.2

Are the following vectors linearly independent? If they are, explain why and if they are not, exhibit one of them as a linear combination of the others. Also give a linearly independent set of vectors which has the same span as the given vectors.

$$\begin{bmatrix} -1 \\ -2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ -4 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 6 \\ 4 \end{bmatrix}$$

Exercise 4.5.E.3

Are the following vectors linearly independent? If they are, explain why and if they are not, exhibit one of them as a linear combination of the others. Also give a linearly independent set of vectors which has the same span as the given vectors.

$$\begin{bmatrix} 1 \\ 5 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 6 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 6 \\ -2 \\ 1 \end{bmatrix}$$

Exercise 4.5.E.4

Are the following vectors linearly independent? If they are, explain why and if they are not, exhibit one of them as a linear combination of the others. Also give a linearly independent set of vectors which has the same span as the given vectors.

$$\begin{bmatrix} 1 \\ -1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 6 \\ 34 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 7 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 8 \\ 1 \end{bmatrix}$$

Answer available at the end of this section.

Exercise 4.5.E.5

Are the following vectors linearly independent? If they are, explain why and if they are not, exhibit one of them as a linear combination of the others.

$$\begin{bmatrix} 1 \\ 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ -10 \\ 3 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 0 \\ 1 \end{bmatrix}$$

Exercise 4.5.E.6

Are the following vectors linearly independent? If they are, explain why and if they are not, exhibit one of them as a linear combination of the others. Also give a linearly independent set of vectors which has the same span as the given vectors.

$$\begin{bmatrix} 1 \\ 3 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 10 \\ -14 \\ 1 \end{bmatrix}$$

Exercise 4.5.E.7

Are the following vectors linearly independent? If they are, explain why and if they are not, exhibit one of them as a linear combination of the others. Also give a linearly independent set of vectors which has the same span as the given vectors.

$$\begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 8 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ 34 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 7 \\ 1 \end{bmatrix}$$

Exercise 4.5.E.8

Are the following vectors linearly independent? If they are, explain why and if they are not, exhibit one of them as a linear combination of the others. Also give a linearly independent set of vectors which has the same span as the given vectors.

$$\begin{bmatrix} 1 \\ 4 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -2 \\ 1 \end{bmatrix}$$

Exercise 4.5.E.9

Are the following vectors linearly independent? If they are, explain why and if they are not, exhibit one of them as a linear combination of the others.

$$\begin{bmatrix} 1 \\ 2 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 1 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -2 \\ 5 \end{bmatrix}$$

Answer available at the end of this section.

Exercise 4.5.E.10

Are the following vectors linearly independent? If they are, explain why and if they are not, exhibit one of them as a linear combination of the others. Also give a linearly independent set of vectors which has the same span as the given vectors.

$$\begin{bmatrix} 2 \\ 3 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -5 \\ -6 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 4 \end{bmatrix}$$

Exercise $4.5.E.11$

Suppose $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a linearly dependent set of vectors in $\mathbb{R}^n$. Prove that at least one of these vectors can be written as a linear combination of the others.

Answer available at the end of this section.

Exercise $4.5.E.12$

Let $B = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ be a linearly independent set. Prove that every vector in $\text{span}(B)$ can be written uniquely as a linear combination of the vectors in B.

Exercise $4.5.E.13$

Prove that any set containing the zero vector is a linearly dependent set.

Answer available at the end of this section.

Exercise $4.5.E.14$

Suppose $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent vectors in $\mathbb{R}^3$. Prove that $\vec{v}_1 + \vec{v}_2, \vec{v}_2 + \vec{v}_3, \vec{v}_3 + \vec{v}_1$ are also linearly independent

Exercise $4.5.E.15$

Let A be an $n \times n$ matrix. Prove that the column vectors of A are linearly independent if and only if $\det(A) \neq 0$.

Hint available at the end of this section.

Exercise $4.5.E.16$

Suppose the set $B = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ of vectors spans $\mathbb{R}^n$. Suppose $\vec{v}$ is any vector in $\mathbb{R}^n$. Prove that $B \cup \{\vec{v}\}$ is a linearly dependent set

Exercise $4.5.E.17$

Suppose $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is a linearly independent set of vectors. Prove that any subset of S is also linearly independent

Answers

Exercise $4.5.E.1$

The given vectors are linearly dependent. We express one of them as a linear combination of the others:

$$\vec{v}_4 = -2\vec{v}_1 + 3\vec{v}_2 + 1\vec{v}_3.$$

A linearly independent set that spans the same space is:

$$\left\{\begin{bmatrix}1\\3\\-1\\1\end{bmatrix},\begin{bmatrix}1\\4\\-1\\1\end{bmatrix},\begin{bmatrix}1\\4\\0\\1\end{bmatrix}\right\}.$$

Exercise $4.5.E.4$

The given vectors are linearly dependent. We express one of them as a linear combination of the others:

$$\vec{v}_4 = -\frac{13.43}{11.43}\vec{v}_3 + \text{other terms.}$$

A linearly independent set that spans the same space is:

$$\left\{\begin{bmatrix}1\\-1\\3\\1\end{bmatrix},\begin{bmatrix}1\\6\\34\\1\end{bmatrix},\begin{bmatrix}1\\0\\7\\1\end{bmatrix}\right\}.$$

Exercise $4.5.E.9$

The given vectors are linearly independent because the only solution to the homogeneous system

$$a\begin{bmatrix}1\\2\\2\\-4\end{bmatrix}+b\begin{bmatrix}3\\4\\1\\-4\end{bmatrix}+c\begin{bmatrix}0\\-1\\0\\4\end{bmatrix}+d\begin{bmatrix}0\\-1\\-2\\5\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

is the trivial solution $a=b=c=d=0$. Therefore, the vectors are linearly independent.

Exercise $4.5.E.11$

We are given that the set $S=\{\vec{v}_1,\vec{v}_2,\dots,\vec{v}_n\}$ is \textbf{linearly dependent} in $\mathbb{R}^n$. By definition, this means that there exist scalars $a_1,a_2,\dots,a_n$, not all zero, such that:

$$a_1\vec{v}_1+a_2\vec{v}_2+\cdots+a_n\vec{v}_n=\vec{0}.$$

Since not all of the a_i are zero, at least one of them is nonzero. Without loss of generality, assume that $a_k\neq 0$ for some k, meaning the term $a_k\vec{v}_k$ is present in the equation. We can solve for $\vec{v}_k$ in terms of the other vectors:

$$a_k\vec{v}_k=-(a_1\vec{v}_1+a_2\vec{v}_2+\cdots+a_{k-1}\vec{v}_{k-1}+a_{k+1}\vec{v}_{k+1}+\cdots+a_n\vec{v}_n).$$

Since $a_k\neq 0$, we can divide by a_k to express $\vec{v}_k$ as a linear combination of the other vectors:

$$\vec{v}_k=-\frac{a_1}{a_k}\vec{v}_1-\frac{a_2}{a_k}\vec{v}_2-\cdots-\frac{a_{k-1}}{a_k}\vec{v}_{k-1}-\frac{a_{k+1}}{a_k}\vec{v}_{k+1}-\cdots-\frac{a_n}{a_k}\vec{v}_n.$$

Thus, we have shown that at least one vector in S can be written as a linear combination of the others.

Exercise $4.5.E.13$

Let $S=\{\vec{v}_1,\vec{v}_2,\dots,\vec{v}_k,\vec{0}\}$ be a set of vectors in $\mathbb{R}^n$ that contains the zero vector $\vec{0}$. By the definition of linear dependence, we need to check if there exists a nontrivial linear combination of the vectors in S that equals the zero vector:

$$a_1\vec{v}_1+a_2\vec{v}_2+\cdots+a_k\vec{v}_k+a_{k+1}\vec{0}=\vec{0}$$

for some scalars $a_1,a_2,\dots,a_k,a_{k+1}$, not all zero. Consider setting $a_{k+1}=1$ and all other coefficients $a_1,a_2,\dots,a_k$ equal to zero:

$$0\cdot\vec{v}_1+0\cdot\vec{v}_2+\cdots+0\cdot\vec{v}_k+1\cdot\vec{0}=\vec{0}.$$

Since this is a nontrivial linear combination (not all coefficients are zero), the set S is linearly dependent.

Exercise $4.5.E.15$

Since this is an "if and only if" proof, you must prove each direction.

Let A be an $n\times n$ matrix with column vectors $\vec{a}_1,\vec{a}_2,\dots,\vec{a}_n$. We need to prove that the column vectors of A are linearly independent if and only if $\det(A)\neq 0$.

($\Leftarrow$) Suppose $\det(A)\neq 0$. [We must show that the column vectors of A are linearly independent.] Consider the homogeneous system of equations:

$$A\vec{x}=\vec{0}$$

where $\vec{x}=\langle x_1,x_2,\dots,x_n\rangle^T$ is an $n\times 1$ vector of unknowns. (What does that fact that $\det(A)=0$ say about the solution to $A\vec{x}=\vec{0}$? Finish the argument.)

($\Rightarrow$) Suppose the column vectors of A are linearly independent. [We must show that $\det(A)\neq 0$.] If the column vectors of A are linearly independent, then the only solution to the homogeneous system $A\vec{x}=\vec{0}$ is the trivial solution $\vec{x}=\vec{0}$ since each column of A is a pivot column. (Finish the proof from here.)

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4.6: Subspaces and Bases

Learning Objectives

- Define and identify subspaces within $\mathbb{R}^n$.
- Analyze and verify if a set of vectors forms a basis for a given subspace.
- Determine the dimension of a subspace and extend or shrink bases as needed.

The goal of this section is to develop an understanding of a subspace of $\mathbb{R}^n$.

A subspace is simply a set of vectors with the property that linear combinations of these vectors remain in the set. Geometrically in $\mathbb{R}^3$, it turns out that a subspace can be represented by either the origin as a single point, lines and planes which contain the origin, or the entire space $\mathbb{R}^3$.

Definition 4.6.1: Subspace

Let V be a nonempty collection of vectors in $\mathbb{R}^n$. Then V is called a **subspace** if whenever a and b are scalars and $\vec{u}$ and $\vec{v}$ are vectors in V, the linear combination $a\vec{u}+b\vec{v}$ is also in V.

Sometimes, instead of using the definition of subspace, it's easier to use the following test, especially the first part. Since a subspace must be closed under all possible linear combinations, it must contain the trivial linear combination, $\vec{0}$. Hence, if a set doesn't contain $\vec{0}$, it cannot possibly be a subspace.

Theorem 4.6.1: Subspace Test

A subset V of $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ if

1. the zero vector of $\mathbb{R}^n$, $\vec{0}_n$, is in V;
2. V is closed under addition, i.e., for all $\vec{u},\vec{w}\in V$, $\vec{u}+\vec{w}\in V$;
3. V is closed under scalar multiplication, i.e., for all $\vec{u}\in V$ and $k\in\mathbb{R}$, $k\vec{u}\in V$.

Consider the following example of a line in $\mathbb{R}^3$.

Example 4.6.1: Subspace of $\mathbb{R}^3$

In $\mathbb{R}^3$, the line L through the origin that is parallel to the vector $\vec{d}=\begin{bmatrix}-5\\1\\-4\end{bmatrix}$ has (vector) equation $\begin{bmatrix}x\\y\\z\end{bmatrix}=t\begin{bmatrix}-5\\1\\-4\end{bmatrix}, t\in\mathbb{R}$,

so

$$L=\left\{t\vec{d}\mid t\in\mathbb{R}\right\}.$$

Then L is a subspace of $\mathbb{R}^3$.

Solution

Using the subspace test given above we can verify that L is a subspace of $\mathbb{R}^3$.

- First: $\vec{0}_3\in L$ since $0\vec{d}=\vec{0}_3$.
- Suppose $\vec{u},\vec{v}\in L$. Then by definition, $\vec{u}=s\vec{d}$ and $\vec{v}=t\vec{d}$, for some $s,t\in\mathbb{R}$. Thus

$$\vec{u}+\vec{v}=s\vec{d}+t\vec{d}=(s+t)\vec{d}.$$

Since $s+t\in\mathbb{R}$, $\vec{u}+\vec{v}\in L$; i.e., L is closed under addition.

- Suppose $\vec{u}\in L$ and $k\in\mathbb{R}$ (k is a scalar). Then $\vec{u}=t\vec{d}$, for some $t\in\mathbb{R}$, so

$$k\vec{u}=k(t\vec{d})=(kt)\vec{d}.$$

Since $kt\in\mathbb{R}$, $k\vec{u}\in L$; i.e., L is closed under scalar multiplication.

Since L satisfies all conditions of the subspace test, it follows that L is a subspace.

Theorem 4.6.2: Subspaces are Spans

Let V be a nonempty collection of vectors in $\mathbb{R}^n$. Then V is a subspace of $\mathbb{R}^n$ if and only if there exist vectors $\{\vec{u}_1,\cdots,\vec{u}_k\}$ in V such that

$$V=\text{span}\{\vec{u}_1,\cdots,\vec{u}_k\}$$

Furthermore, let W be another subspace of $\mathbb{R}^n$ and suppose $\{\vec{u}_1,\cdots,\vec{u}_k\}\in W$. Then it follows that V is a subset of W.

Note that since W is arbitrary, the statement that $V\subseteq W$ means that any other subspace of $\mathbb{R}^n$ that contains these vectors will also contain V.

Proof available at the end of this section.

Since the vectors $\vec{u}_i$ we constructed in the proof above are not in the span of the previous vectors (by definition), they must be linearly independent and thus we obtain the following corollary.

Corollary 4.6.1: Subspaces are Spans of Independent Vectors

If V is a subspace of $\mathbb{R}^n$, then there exist linearly independent vectors $\{\vec{u}_1,\cdots,\vec{u}_k\}$ in V such that $V=\text{span}\{\vec{u}_1,\cdots,\vec{u}_k\}$.

In summary, subspaces of $\mathbb{R}^n$ consist of spans of finite, linearly independent collections of vectors of $\mathbb{R}^n$. Such a collection of vectors is called a basis.

Definition 4.6.2: Basis of a Subspace

Let V be a subspace of $\mathbb{R}^n$. Then $\{\vec{u}_1,\cdots,\vec{u}_k\}$ is a **basis** for V if the following two conditions hold.

1. $\text{span}\{\vec{u}_1,\cdots,\vec{u}_k\}=V$
2. $\{\vec{u}_1,\cdots,\vec{u}_k\}$ is linearly independent

Note the plural of basis is **bases**.

The following is a simple but very useful example of a basis, called the standard basis.

Definition 4.6.3: Standard Basis of $\mathbb{R}^n$

Let $\vec{e}_i$ be the vector in $\mathbb{R}^n$ which has a 1 in the ith entry and zeros elsewhere, that is the ith column of the identity matrix. Then the collection $\{\vec{e}_1,\vec{e}_2,\cdots,\vec{e}_n\}$ is a basis for $\mathbb{R}^n$ and is called the **standard basis** of $\mathbb{R}^n$.

The main theorem about bases is not only they exist, but that any two bases of a subspace must be of the same size. To show this, we will need the the following fundamental result, called the Exchange Theorem.

Theorem 4.6.3: Exchange Theorem

Suppose $\{\vec{u}_1, \cdots, \vec{u}_r\}$ is a linearly independent set of vectors in $\mathbb{R}^n$, and each $\vec{u}_k$ is contained in $\text{span}\{\vec{v}_1, \cdots, \vec{v}_s\}$ Then $s \geq r$. In words, spanning sets have at least as many vectors as linearly independent sets.

Proof available at the end of this section.

We are now ready to show that any two bases are of the same size.

Theorem 4.6.4: Bases of $\mathbb{R}^n$ are of the Same Size

Let V be a subspace of $\mathbb{R}^n$ with two bases B_1 and B_2. Suppose B_1 contains s vectors and B_2 contains r vectors. Then $s = r$.

Proof available at the end of this section.

The following definition can now be stated.

Definition 4.6.4: Dimension of a Subspace

Let V be a subspace of $\mathbb{R}^n$. Then the **dimension** of V, written $\dim(V)$ is defined to be the number of vectors in a basis.

The next result follows.

Corollary 4.6.2: Dimension of $\mathbb{R}^n$

The dimension of $\mathbb{R}^n$ is n.

Proof available at the end of this section.

Consider the following example.

Example 4.6.2: Basis of Subspace

Let

$$V = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4 \ : \ a - b = d - c \right\}.$$

Show that V is a subspace of $\mathbb{R}^4$, find a basis of V, and find $\dim(V)$.

Solution

The condition $a - b = d - c$ is equivalent to the condition $a = b - c + d$, so we may write

$$V = \left\{ \begin{bmatrix} b - c + d \\ b \\ c \\ d \end{bmatrix} \ : \ b, c, d \in \mathbb{R} \right\} = \left\{ b \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \ : \ b, c, d \in \mathbb{R} \right\}$$

This shows that V is a subspace of $\mathbb{R}^4$, since $V = \text{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ where

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Furthermore,

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is linearly independent, as can be seen by taking the reduced row-echelon form of the matrix whose columns are $\vec{u}_1, \vec{u}_2$ and $\vec{u}_3$.

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since every column of the reduced row-echelon form matrix has a leading one, the columns are linearly independent.

Therefore $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is linearly independent and spans V, so is a basis of V. Hence V has dimension three.

We continue by stating further properties of a set of vectors in $\mathbb{R}^n$.

Corollary 4.6.3: Linearly Independent and Spanning Sets in $\mathbb{R}^n$

The following properties hold in $\mathbb{R}^n$:

- Suppose $\{\vec{u}_1, \cdots, \vec{u}_n\}$ is linearly independent. Then $\{\vec{u}_1, \cdots, \vec{u}_n\}$ is a basis for $\mathbb{R}^n$.
- Suppose $\{\vec{u}_1, \cdots, \vec{u}_m\}$ spans $\mathbb{R}^n$. Then $m \geq n$.
- If $\{\vec{u}_1, \cdots, \vec{u}_n\}$ spans $\mathbb{R}^n$, then $\{\vec{u}_1, \cdots, \vec{u}_n\}$ is linearly independent.

Proof available at the end of this section.

The next theorem follows from the above claim.

Theorem 4.6.5: Existence of Basis

Let V be a subspace of $\mathbb{R}^n$. Then there exists a basis of V with $\dim(V) \leq n$.

Consider Corollary 4.6.4 together with Theorem 4.6.8. Let $\dim(V) = r$. Suppose there exists an independent set of vectors in V. If this set contains r vectors, then it is a basis for V. If it contains less than r vectors, then vectors can be added to the set to create a basis of V. Similarly, any spanning set of V which contains more than r vectors can have vectors removed to create a basis of V.

We illustrate this concept in the next example.

Example 4.6.3: Extending an Independent Set

Consider the set U given by

$$U = \left\{ \left[\begin{array}{c} a \\ b \\ c \\ d \end{array} \right] \in \mathbb{R}^4 \left| \begin{array}{c} \\ \\ \\ \\ \end{array} \right. a - b = d - c \right\}$$

Then U is a subspace of $\mathbb{R}^4$ and $\dim(U) = 3$.

Then

$$S = \left\{ \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array} \right], \left[\begin{array}{c} 2 \\ 3 \\ 3 \\ 2 \end{array} \right] \right\},$$

is an independent subset of U. Show that S can be extended to a basis of U.

Solution

To extend S to a basis of U, find a vector in U that is **not** in $\text{span}(S)$.

$$\left[\begin{array}{ccc} 1 & 2 & ? \\ 1 & 3 & ? \\ 1 & 3 & ? \\ 1 & 2 & ? \end{array} \right]$$

$$\left[\begin{array}{ccc} 1 & 2 & 1 \\ 1 & 3 & 0 \\ 1 & 3 & -1 \\ 1 & 2 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right]$$

Therefore, S can be extended to the following basis of U:

$$\left\{ \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array} \right], \left[\begin{array}{c} 2 \\ 3 \\ 3 \\ 2 \end{array} \right], \left[\begin{array}{c} 1 \\ 0 \\ -1 \\ 0 \end{array} \right] \right\},$$

Next we consider the case of removing vectors from a spanning set to result in a basis.

Theorem 4.6.6: Finding a Basis from a Span

Let W be a subspace. Also suppose that $W = span\left\{\vec{w}_1, \cdots, \vec{w}_m\right\}$. Then there exists a subset of $\left\{\vec{w}_1, \cdots, \vec{w}_m\right\}$ which is a basis for W.

Proof available at the end of this section.

The following example illustrates how to carry out this shrinking process which will obtain a subset of a span of vectors which is linearly independent.

Example 4.6.4: Subset of a Span

Let W be the subspace

$$span \left\{ \left[\begin{array}{r} 1 \\ 2 \\ -1 \\ 1 \end{array} \right], \left[\begin{array}{r} 1 \\ 3 \\ -1 \\ 1 \end{array} \right], \left[\begin{array}{r} 8 \\ 19 \\ -8 \\ 8 \end{array} \right], \left[\begin{array}{r} -6 \\ -15 \\ 6 \\ -6 \end{array} \right], \left[\begin{array}{r} 1 \\ 3 \\ 0 \\ 1 \end{array} \right], \left[\begin{array}{r} 1 \\ 5 \\ 0 \\ 1 \end{array} \right] \right\}$$

Find a basis for W which consists of a subset of the given vectors.

Solution

You can use the reduced row-echelon form to accomplish this reduction. Form the matrix which has the given vectors as columns.

$$\begin{bmatrix} 1 & 1 & 8 & -6 & 1 & 1 \\ 2 & 3 & 19 & -15 & 3 & 5 \\ -1 & -1 & -8 & 6 & 0 & 0 \\ 1 & 1 & 8 & -6 & 1 & 1 \end{bmatrix}$$

Then take the reduced row-echelon form

$$\begin{bmatrix} 1 & 0 & 5 & -3 & 0 & -2 \\ 0 & 1 & 3 & -3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

It follows that a basis for W is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Since the first, second, and fifth columns are obviously a basis for the column space of the , the same is true for the matrix having the given vectors as columns.

Consider the following theorems regarding a subspace contained in another subspace.

Theorem 4.6.7: Subset of a Subspace

Let V and W be subspaces of $\mathbb{R}^n$, and suppose that $W \subseteq V$. Then $\dim(W) \leq \dim(V)$ with equality when $W = V$.

Theorem 4.6.8: Extending a Basis

Let W be any non-zero subspace $\mathbb{R}^n$ and let $W \subseteq V$ where V is also a subspace of $\mathbb{R}^n$. Then every basis of W can be extended to a basis for V.

The proof is left as an exercise but proceeds as follows. Begin with a basis for $W, \{\vec{w}_1, \cdots, \vec{w}_s\}$ and add in vectors from V until you obtain a basis for V. Not that the process will stop because the dimension of V is no more than n.

Consider the following example.

Example 4.6.5: Extending a Basis

Let $V = \mathbb{R}^4$ and let

$$W = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Extend this basis of W to a basis of $\mathbb{R}^n$.

Solution

An easy way to do this is to take the reduced row-echelon form of the matrix

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.6.1}$$

Note how the given vectors were placed as the first two columns and then the matrix was extended in such a way that it is clear that the span of the columns of this matrix yield all of $\mathbb{R}^4$. Now determine the pivot columns. The reduced row-echelon form is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix} \tag{4.6.2}$$

Therefore the pivot columns are

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

and now this is an extension of the given basis for W to a basis for $\mathbb{R}^4$.

Why does this work? The columns of (4.6.1) obviously span $\mathbb{R}^4$. In fact the span of the first four is the same as the span of all six.

Consider another example.

Example 4.6.6: Extending a Basis

Let W be the span of $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ in $\mathbb{R}^4$. Let V consist of the span of the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ -6 \\ 1 \\ -6 \end{bmatrix}, \begin{bmatrix} -5 \\ 7 \\ 2 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Find a basis for V which extends the basis for W.

Solution

Note that the above vectors are not linearly independent, but their span, denoted as V is a subspace which does include the subspace W.

Using the process outlined in the previous example, form the following matrix

$$\begin{bmatrix} 1 & 0 & 7 & -5 & 0 \\ 0 & 1 & -6 & 7 & 0 \\ 1 & 1 & 1 & 2 & 0 \\ 0 & 1 & -6 & 7 & 1 \end{bmatrix}$$

Next find its reduced row-echelon form

$$\begin{bmatrix} 1 & 0 & 7 & -5 & 0 \\ 0 & 1 & -6 & 7 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

It follows that a basis for V consists of the first two vectors and the last.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Thus V is of dimension 3 and it has a basis which extends the basis for W.

Answers

Theorem 4.6.2**: Subspaces are Spans**

We first show that if V is a subspace, then it can be written as $V = \operatorname{span}\{\vec{u}_1, \cdots, \vec{u}_k\}$. Pick a vector $\vec{u}_1$ in V. If $V = \operatorname{span}\{\vec{u}_1\}$, then you have found your list of vectors and are done. If $V \neq \operatorname{span}\{\vec{u}_1\}$, then there exists $\vec{u}_2$ a vector of V which is not in $\operatorname{span}\{\vec{u}_1\}$. Consider $\operatorname{span}\{\vec{u}_1, \vec{u}_2\}$. If $V = \operatorname{span}\{\vec{u}_1, \vec{u}_2\}$, we are done. Otherwise, pick $\vec{u}_3$ not in $\operatorname{span}\{\vec{u}_1, \vec{u}_2\}$. Continue this way. Note that since V is a subspace, these spans are each contained in V. The process must stop with $\vec{u}_k$ for some $k \leq n$ by Corollary 4.6.1, and thus $V = \operatorname{span}\{\vec{u}_1, \cdots, \vec{u}_k\}$.

Now suppose $V = \operatorname{span}\{\vec{u}_1, \cdots, \vec{u}_k\}$, we must show this is a subspace. So let $\sum_{i=1}^{k} c_i \vec{u}_i$ and $\sum_{i=1}^{k} d_i \vec{u}_i$ be two vectors in V, and let a and b be two scalars. Then

$$a \sum_{i=1}^{k} c_i \vec{u}_i + b \sum_{i=1}^{k} d_i \vec{u}_i = \sum_{i=1}^{k} (ac_i + bd_i) \vec{u}_i$$

which is one of the vectors in $\operatorname{span}\{\vec{u}_1, \cdots, \vec{u}_k\}$ and is therefore contained in V. This shows that $\operatorname{span}\{\vec{u}_1, \cdots, \vec{u}_k\}$ has the properties of a subspace.

To prove that $V \subseteq W$, we prove that if $\vec{u}_i \in V$, then $\vec{u}_i \in W$.

Suppose $\vec{u} \in V$. Then $\vec{u} = a_1\vec{u}_1 + a_2\vec{u}_2 + \cdots + a_k\vec{u}_k$ for some $a_i \in \mathbb{R}$, $1 \leq i \leq k$. Since W contain each $\vec{u}_i$ and W is a vector space, it follows that $a_1\vec{u}_1 + a_2\vec{u}_2 + \cdots + a_k\vec{u}_k \in W$.

Theorem 4.6.3**: Exchange Theorem**

Since each $\vec{u}_j$ is in $\operatorname{span}\{\vec{v}_1, \cdots, \vec{v}_s\}$, there exist scalars a_{ij} such that

$$\vec{u}_j = \sum_{i=1}^{s} a_{ij} \vec{v}_i$$

Suppose for a contradiction that \(s

Theorem 4.6.4**: Bases of $\mathbb{R}^n$ are of the Same Size**

This follows from Theorem 4.6.3. Observe that $B_1 = \{\vec{u}_1, \cdots, \vec{u}_s\}$ is a spanning set for V while $B_2 = \{\vec{v}_1, \cdots, \vec{v}_r\}$ is linearly independent, so $s \geq r$. Similarly $B_2 = \{\vec{v}_1, \cdots, \vec{v}_r\}$ is a spanning set for V while $B_1 = \{\vec{u}_1, \cdots, \vec{u}_s\}$ is linearly independent, so $r \geq s$.

Corollary 4.6.2**: Dimension of $\mathbb{R}^n$**

You only need to exhibit a basis for $\mathbb{R}^n$ which has n vectors. Such a basis is the standard basis $\{\vec{e}_1, \cdots, \vec{e}_n\}$.

Corollary 4.6.3**: Linearly Independent and Spanning Sets in $\mathbb{R}^n$**

Assume first that $\{\vec{u}_1, \cdots, \vec{u}_n\}$ is linearly independent, and we need to show that this set spans $\mathbb{R}^n$. To do so, let $\vec{v}$ be a vector of $\mathbb{R}^n$, and we need to write $\vec{v}$ as a linear combination of $\vec{u}_i$'s. Consider the matrix A having the vectors $\vec{u}_i$ as columns:

$$A = \begin{bmatrix} \vec{u}_1 & \cdots & \vec{u}_n \end{bmatrix}$$

By linear independence of the $\vec{u}_i$'s, the reduced row-echelon form of A is the identity matrix. Therefore the system $A\vec{x} = \vec{v}$ has a (unique) solution, so $\vec{v}$ is a linear combination of the $\vec{u}_i$'s.

To establish the second claim, suppose that \(m

Finally consider the third claim. If $\{\vec{u}_1, \cdots, \vec{u}_n\}$ is not linearly independent, then replace this list with $\{\vec{u}_{i_1}, \cdots, \vec{u}_{i_k}\}$ where these are the pivot columns of the matrix

$$\begin{bmatrix} \vec{u}_1 & \cdots & \vec{u}_n \end{bmatrix}$$

Then $\{\vec{u}_{i_1}, \cdots, \vec{u}_{i_k}\}$ spans $\mathbb{R}^n$ and is linearly independent, so it is a basis having less than n vectors again contrary to Corollary 4.6.3.

Theorem 4.6.6**: Finding a Basis from a Span**

Let S denote the set of positive integers such that for $k \in S$, there exists a subset of $\{\vec{w}_1, \cdots, \vec{w}_m\}$ consisting of exactly k vectors which is a spanning set for W. Thus $m \in S$. Pick the smallest positive integer in S. Call it k. Then there exists $\{\vec{u}_1, \cdots, \vec{u}_k\} \subseteq \{\vec{w}_1, \cdots, \vec{w}_m\}$ such that $\operatorname{span}\{\vec{u}_1, \cdots, \vec{u}_k\} = W$. If

$$\sum_{i=1}^{k} c_i \vec{w}_i = \vec{0}$$

and not all of the $c_i = 0$, then you could pick $c_j \neq 0$, divide by it and solve for $\vec{u}_j$ in terms of the others,

$$\vec{w}_j = \sum_{i \neq j} \left(-\frac{c_i}{c_j} \right) \vec{w}_i$$

Then you could delete $\vec{w}_j$ from the list and have the same span. Any linear combination involving $\vec{w}_j$ would equal one in which $\vec{w}_j$ is replaced with the above sum, showing that it could have been obtained as a linear combination of $\vec{w}_i$ for $i \neq j$. Thus $k-1 \in S$ contrary to the choice of k. Hence each $c_i = 0$ and so $\{\vec{u}_1, \cdots, \vec{u}_k\}$ is a basis for W consisting of vectors of $\{\vec{w}_1, \cdots, \vec{w}_m\}$.

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4.6.E: Exercise for Section 4.6

Exercise 4.6.E.1

Let $H = \text{span}\left\{\begin{bmatrix}2\\1\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\0\\-1\\-1\end{bmatrix}, \begin{bmatrix}5\\2\\3\\3\end{bmatrix}, \begin{bmatrix}-1\\1\\-2\\-2\end{bmatrix}\right\}$. Find the dimension of H and determine a basis.

Exercise 4.6.E.2

Let $H = \text{span}\left\{\begin{bmatrix}0\\1\\1\\-1\end{bmatrix}, \begin{bmatrix}-1\\-1\\-2\\2\end{bmatrix}, \begin{bmatrix}2\\3\\5\\-5\end{bmatrix}, \begin{bmatrix}0\\1\\2\\-2\end{bmatrix}\right\}$. Find the dimension of H and determine a basis.

Exercise 4.6.E.3

Let $H = \text{span}\left\{\begin{bmatrix}-2\\1\\1\\-3\end{bmatrix}, \begin{bmatrix}-9\\4\\3\\-9\end{bmatrix}, \begin{bmatrix}-33\\15\\12\\-36\end{bmatrix}, \begin{bmatrix}-22\\10\\8\\-24\end{bmatrix}\right\}$. Find the dimension of H and determine a basis.

Exercise 4.6.E.4

Let $H = \text{span}\left\{\begin{bmatrix}-1\\1\\-1\\-2\end{bmatrix}, \begin{bmatrix}-4\\3\\-2\\-4\end{bmatrix}, \begin{bmatrix}-3\\2\\-1\\-2\end{bmatrix}, \begin{bmatrix}-1\\1\\-2\\-4\end{bmatrix}, \begin{bmatrix}-7\\5\\-3\\-6\end{bmatrix}\right\}$. Find the dimension of H and determine a basis.

Exercise 4.6.E.5

Let $H = \text{span}\left\{\begin{bmatrix}2\\3\\2\\1\end{bmatrix}, \begin{bmatrix}8\\15\\6\\3\end{bmatrix}, \begin{bmatrix}3\\6\\2\\1\end{bmatrix}, \begin{bmatrix}4\\6\\6\\3\end{bmatrix}, \begin{bmatrix}8\\15\\6\\3\end{bmatrix}\right\}$. Find the dimension of H and determine a basis.

Exercise 4.6.E.6

Let $H = \text{span}\left\{\begin{bmatrix}0\\2\\0\\-1\end{bmatrix}, \begin{bmatrix}-1\\6\\0\\-2\end{bmatrix}, \begin{bmatrix}-2\\16\\0\\-6\end{bmatrix}, \begin{bmatrix}-3\\22\\0\\-8\end{bmatrix}\right\}$. Find the dimension of H and determine a basis.

Exercise 4.6.E.7

Let $H = \text{span}\left\{\begin{bmatrix}5\\1\\1\\4\end{bmatrix}, \begin{bmatrix}14\\3\\2\\8\end{bmatrix}, \begin{bmatrix}38\\8\\6\\24\end{bmatrix}, \begin{bmatrix}47\\10\\7\\28\end{bmatrix}, \begin{bmatrix}10\\2\\3\\12\end{bmatrix}\right\}$. Find the dimension of H and determine a basis.

Exercise 4.6.E.8

Let $H = \text{span}\left\{\begin{bmatrix}6\\1\\1\\5\end{bmatrix}, \begin{bmatrix}17\\3\\2\\10\end{bmatrix}, \begin{bmatrix}52\\9\\7\\35\end{bmatrix}, \begin{bmatrix}18\\3\\4\\20\end{bmatrix}\right\}$. Find the dimension of H and determine a basis.

Exercise 4.6.E.9

Let $M = \left\{\vec{u} = \begin{bmatrix}u_1\\u_2\\u_3\\u_4\end{bmatrix} \in \mathbb{R}^4 : \sin(u_1) = 1\right\}$. Is M a subspace? Explain.

Answer available at the end of this section.

Exercise 4.6.E.10

Let $M = \left\{\vec{u} = \begin{bmatrix}u_1\\u_2\\u_3\\u_4\end{bmatrix} \in \mathbb{R}^4 : ||u_1|| \leq 4\right\}$. Is M a subspace? Explain.

Answer available at the end of this section.

Exercise 4.6.E.11

Let $M = \left\{\vec{u} = \begin{bmatrix}u_1\\u_2\\u_3\\u_4\end{bmatrix} \in \mathbb{R}^4 : u_1 \geq 0 \text{ for each } i = 1, 2, 3, 4\right\}$. Is M a subspace? Explain.

Answer available at the end of this section.

Exercise 4.6.E.12

Let $\vec{w}$, $\vec{w}_1$ be given vectors in $\mathbb{R}^4$ and define

$$M = \left\{\vec{u} = \begin{bmatrix}u_1\\u_2\\u_3\\u_4\end{bmatrix} \in \mathbb{R}^4 : \vec{w} \bullet \vec{u} = 0 \text{ and } \vec{w}_1 \bullet \vec{u} = 0\right\}.$$

Is M a subspace? Explain.

Answer available at the end of this section.

Exercise 4.6.E.13

Let $\vec{w} \in \mathbb{R}^4$ and let $M = \left\{ \vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \in \mathbb{R}^4 : \vec{w} \bullet \vec{u} = 0 \right\}$. Is M a subspace? Explain.

Answer available at the end of this section.

Exercise 4.6.E.14

Let $M = \left\{ \vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \in \mathbb{R}^4 : u_3 \geq u_1 \right\}$. Is M a subspace? Explain.

Answer available at the end of this section.

Exercise 4.6.E.15

Let $M = \left\{ \vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \in \mathbb{R}^4 : u_3 = u_1 = 0 \right\}$. Is M a subspace? Explain.

Answer available at the end of this section.

Exercise 4.6.E.16

Consider the set of vectors S given by

$$S = \left\{ \begin{bmatrix} 4u + v - 5w \\ 12u + 6v - 6w \\ 4u + 4v + 4w \end{bmatrix} : u, v, w \in \mathbb{R} \right\}.$$

Is S a subspace of $\mathbb{R}^3$? If so, explain why, give a basis for the subspace and find its dimension.

Exercise 4.6.E.17

Consider the set of vectors S given by

$$S = \left\{ \begin{bmatrix} 2u + 6v + 7w \\ -3u - 9v - 12w \\ 2u + 6v + 6w \\ u + 3v + 3w \end{bmatrix} : u, v, w \in \mathbb{R} \right\}.$$

Is S a subspace of $\mathbb{R}^4$? If so, explain why, give a basis for the subspace and find its dimension.

Exercise 4.6.E.18

Consider the set of vectors S given by

$$S = \left\{ \begin{bmatrix} 2u+v \\ 6v-3u+3w \\ 3v-6u+3w \end{bmatrix} : u,v,w \in \mathbb{R} \right\}.$$

Is this set of vectors a subspace of $\mathbb{R}^3$? If so, explain why, give a basis for the subspace and find its dimension.

Exercise 4.6.E.19

Consider the vectors of the form

$$\left\{ \begin{bmatrix} 2u+v+7w \\ u-2v+w \\ -6v-6w \end{bmatrix} : u,v,w \in \mathbb{R} \right\}.$$

Is this set of vectors a subspace of $\mathbb{R}^3$? If so, explain why, give a basis for the subspace and find its dimension.

Exercise 4.6.E.20

Consider the vectors of the form

$$\left\{ \begin{bmatrix} 3u+v+11w \\ 18u+6v+66w \\ 28u+8v+100w \end{bmatrix} : u,v,w \in \mathbb{R} \right\}.$$

Is this set of vectors a subspace of $\mathbb{R}^3$? If so, explain why, give a basis for the subspace and find its dimension.

Exercise 4.6.E.21

Consider the vectors of the form

$$\left\{ \begin{bmatrix} 3u+v \\ 2w-4u \\ 2w-2v-8u \end{bmatrix} : u,v,w \in \mathbb{R} \right\}.$$

Is this set of vectors a subspace of $\mathbb{R}^3$? If so, explain why, give a basis for the subspace and find its dimension.

Exercise 4.6.E.22

Consider the set of vectors S given by

$$\left\{ \begin{bmatrix} u+v+w \\ 2u+2v+4w \\ u+v+w \\ 0 \end{bmatrix} : u,v,w \in \mathbb{R} \right\}.$$

Is S is a subspace of $\mathbb{R}^4$? If so, explain why, give a basis for the subspace and find its dimension.

Exercise 4.6.E.23

Consider the set of vectors S given by

$$\left\{ \begin{bmatrix} v \\ -3u-3w \\ 8u-4v+4w \end{bmatrix} : u,v,w \in \mathbb{R} \right\}.$$

Is S is a subspace of $\mathbb{R}^4$? If so, explain why, give a basis for the subspace and find its dimension.

Exercise $4.6.E.24$

If you have 5 vectors in $\mathbb{R}^5$ and the vectors are linearly independent, can it always be concluded they span $\mathbb{R}^5$? Explain.

Answer available at the end of this section.

Exercise $4.6.E.25$

If you have 6 vectors in $\mathbb{R}^5$, is it possible they are linearly independent? Explain.

Answer available at the end of this section.

Exercise $4.6.E.26$

Suppose A is an $m \times n$ matrix and $\{\vec{w}_1, \cdots, \vec{w}_k\}$ is a linearly independent set of vectors in $A(\mathbb{R}^n) \subseteq \mathbb{R}^m$. Now suppose $A\vec{z}_i = \vec{w}_i$. Show $\{\vec{z}_1, \cdots, \vec{z}_k\}$ is also independent.

Answer available at the end of this section.

Exercise $4.6.E.27$

Let V and W be subspaces of $\mathbb{R}^n$. Show that the intersection $V \cap W$ is also a subspace of $\mathbb{R}^n$

Answers

Exercise $4.6.E.9$

No. Let $\vec{u} = \begin{bmatrix} \frac{\pi}{2} \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Then $2\vec{u} \notin M$ although $\vec{u} \in M$.

Exercise $4.6.E.10$

No. $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in M$ but $10 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \notin M$.

Exercise $4.6.E.11$

This is not a subspace. $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is in it. However, $(-1) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is not.

Exercise $4.6.E.12$

This is a subspace because it is closed with respect to vector addition and scalar multiplication.

Exercise $4.6.E.13$

Yes, this is a subspace because it is closed with respect to vector addition and scalar multiplication.

Exercise $4.6.E.14$

This is not a subspace. $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ is in it. However $(-1)\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$ is not.

Exercise $4.6.E.15$

This is a subspace. It is closed with respect to vector addition and scalar multiplication.

Exercise $4.6.E.24$

Yes. If not, there would exist a vector not in the span. But then you could add in this vector and obtain a linearly independent set of vectors with more vectors than a basis.

Exercise $4.6.E.25$

They can't be.

Exercise $4.6.E.26$

Say $\sum_{i=1}^{k} c_i \vec{z}_i = \vec{0}$. Then apply A to it as follows.

$$\sum_{i=1}^{k} c_a A\vec{z}_i = \sum_{i=1}^{k} c_i \vec{w}_i = \vec{0}$$

and so, by linear independence of the $\vec{w}_i$, it follows that each $c_i = 0$.

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4.7: Row, Column, and Null Spaces

Learning Objectives

- Determine the row space, column space, and null space of a matrix.
- Calculate the rank and nullity of a matrix and understand their relationship through the Rank-Nullity Theorem.
- Verify if a set of vectors forms a basis for the row or column space.

We begin this section with a new definition.

Definition 4.7.1: Row and Column Space

Let A be an $m \times n$ matrix. The **column space** of A, written $\mathrm{col}(A)$, is the span of the column vectors of A. The **row space** of A, written $\mathrm{row}(A)$, is the span of the row vectors of A.

Using the reduced row-echelon form, we can obtain an efficient description of the row and column space of a matrix. Consider the following lemma.

Lemma 4.7.1: Effect of Row Operations on Row Space

Let A and B be $m \times n$ matrices such that A can be carried to B by elementary row $[\mathrm{column}]$ operations. Then $\mathrm{row}(A) = \mathrm{row}(B)$ $[\mathrm{col}(A) = \mathrm{col}(B)]$.

Proof available at the end of this section.

Consider the following lemma.

Lemma 4.7.2: Row Space of a reduced row-echelon form Matrix

Let A be an $m \times n$ matrix and let R be its reduced row-echelon form. Then the nonzero rows of R form a basis of $\mathrm{row}(R)$, and consequently of $\mathrm{row}(A)$.

This lemma suggests that we can examine the reduced row-echelon form of a matrix in order to obtain the row space. Consider now the column space. The column space can be obtained by simply saying that it equals the span of all the columns. However, you can often get the column space as the span of fewer columns than this. A variation of the previous lemma provides a solution. Suppose A is row reduced to its reduced row-echelon form R. Identify the pivot columns of R (columns which have leading ones), and take the corresponding columns of A. It turns out that this forms a basis of $\mathrm{col}(A)$.

Before proceeding to an example of this concept, we revisit the definition of rank.

Definition 4.7.2: Rank of a Matrix

Previously, we defined $\mathrm{rank}(A)$ to be the number of leading entries in the row-echelon form of A. Using an understanding of dimension and row space, we can now define **rank** as follows:

$$\mathrm{rank}(A) = \dim(\mathrm{row}(A))$$

Consider the following example.

Example 4.7.1: Rank, Column and Row Space

Find the rank of the following matrix and describe the column and row spaces.

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 1 & 3 & 6 & 0 & 2 \\ 3 & 7 & 8 & 6 & 6 \end{bmatrix}$$

Solution

The reduced row-echelon form of A is

$$\begin{bmatrix} 1 & 0 & -9 & 9 & 2 \\ 0 & 1 & 5 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, the rank is 2.

Notice that the first two columns of R are pivot columns. By the discussion following Lemma $4.7.2$, we find the corresponding columns of A, in this case the first two columns. Therefore a basis for $\mathrm{col}(A)$ is given by

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix} \right\}$$

For example, consider the third column of the original matrix. It can be written as a linear combination of the first two columns of the original matrix as follows.

$$\begin{bmatrix} 1 \\ 6 \\ 8 \end{bmatrix} = -9 \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + 5 \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix}$$

What about an efficient description of the row space? By Lemma $4.7.2$ we know that the nonzero rows of R create a basis of $\mathrm{row}(A)$. For the above matrix, the row space equals

$$\mathrm{row}(A) = \mathrm{span}\left\{ \begin{bmatrix} 1 & 0 & -9 & 9 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 5 & -3 & 0 \end{bmatrix} \right\}$$

Notice that the column space of A is given as the span of columns of the original matrix, while the row space of A is the span of rows of the reduced row-echelon form of A.

Consider another example.

Example 4.7.2: Rank, Column and Row Space

Find the rank of the following matrix and describe the column and row spaces.

$$\begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 1 & 3 & 6 & 0 & 2 \\ 1 & 2 & 1 & 3 & 2 \\ 1 & 3 & 2 & 4 & 0 \end{bmatrix}$$

Solution

The reduced row-echelon form is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \frac{13}{2} \\ 0 & 1 & 0 & 2 & -\frac{5}{2} \\ 0 & 0 & 1 & -1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and so the rank is 3. The row space is given by

$$\text{row}(A) = \text{span}\left\{\begin{bmatrix} 1 & 0 & 0 & 0 & \frac{13}{2} \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 2 & -\frac{5}{2} \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & -1 & \frac{1}{2} \end{bmatrix}\right\}$$

Notice that the first three columns of the reduced row-echelon form are pivot columns. The column space is the span of the first three columns in the **original matrix**,

$$\text{col}(A) = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 6 \\ 1 \\ 2 \end{bmatrix}\right\}$$

Consider the solution given above for Example $4.7.2$, where the rank of A equals 3. Notice that the row space and the column space each had dimension equal to 3. It turns out that this is not a coincidence, and this essential result is referred to as the Rank Theorem and is given now. Recall that we defined $\text{rank}(A) = \dim(\text{row}(A))$.

Theorem 4.7.1: Rank Theorem

Let A be an $m \times n$ matrix. Then $\dim(\text{col}(A))$, the dimension of the column space, is equal to the dimension of the row space, $\dim(\text{row}(A))$.

The following statements all follow from the Rank Theorem.

Corollary 4.7.1: Results of the Rank Theorem

Let A be a matrix. Then the following are true:

1. $\text{rank}(A) = \text{rank}(A^T)$.
2. For A of size $m \times n$, $\text{rank}(A) \leq m$ and $\text{rank}(A) \leq n$.
3. For A of size $n \times n$, A is invertible if and only if $\text{rank}(A) = n$.
4. For invertible matrices B and C of appropriate size, $\text{rank}(A) = \text{rank}(BA) = \text{rank}(AC)$.

Consider the following example.

Example 4.7.3: Rank of the Transpose

Let

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$$

Find $\text{rank}(A)$ and $\text{rank}(A^T)$.

Solution

To find $\text{rank}(A)$ we first row reduce to find the reduced row-echelon form.

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore the rank of A is 2. Now consider A^T given by

$$A^T = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$$

Again we row reduce to find the reduced row-echelon form.

$$\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

You can see that $\text{rank}(A^T) = 2$, the same as $\text{rank}(A)$.

We now define what is meant by the null space of a general $m \times n$ matrix.

Definition 4.7.3: Null Space or Kernel of A

The **null space** of a matrix A, also referred to as the **kernel** of A, is defined as follows.

$$\text{null}\,(A) = \left\{\vec{x} : A\vec{x} = \vec{0}\right\}$$

It can also be referred to using the notation $\ker(A)$. Similarly, we can discuss the image of A, denoted by $\text{im}\,(A)$. The image of A consists of the vectors of $\mathbb{R}^m$ which "get hit" by A. The formal definition is as follows.

Definition 4.7.4: Image of A

The **image** of A, written $\text{im}\,(A)$ is given by

$$\text{im}\,(A) = \{A\vec{x} : \vec{x} \in \mathbb{R}^n\}$$

Consider A as a mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$ whose action is given by multiplication. The following diagram displays this scenario.

$$\begin{matrix} \text{null}(A) & & \text{im}(A) \\ \mathbb{R}^n & \overset{A}{\to} & \mathbb{R}^m \end{matrix}$$

As indicated, $\text{im}\,(A)$ is a subset of $\mathbb{R}^m$ while $\text{null}\,(A)$ is a subset of $\mathbb{R}^n$.

It turns out that the null space and image of A are both subspaces. Consider the following example.

Example 4.7.4: Null Space

Let A be an $m \times n$ matrix. Then the null space of A, $\text{null}(A)$ is a subspace of $\mathbb{R}^n$.

Solution

- Since $A\vec{0}_n = \vec{0}_m$, $\vec{0}_n \in \text{null}(A)$.
- Let $\vec{x}, \vec{y} \in \text{null}(A)$. Then $A\vec{x} = \vec{0}_m$ and $A\vec{y} = \vec{0}_m$, so

  $$A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = \vec{0}_m + \vec{0}_m = \vec{0}_m,$$

  and thus $\vec{x} + \vec{y} \in \text{null}(A)$.
- Let $\vec{x} \in \text{null}(A)$ and $k \in \mathbb{R}$. Then $A\vec{x} = \vec{0}_m$, so

  $$A(k\vec{x}) = k(A\vec{x}) = k\vec{0}_m = \vec{0}_m,$$

  and thus $k\vec{x} \in \text{null}(A)$.

Therefore by the subspace test, $\text{null}(A)$ is a subspace of $\mathbb{R}^n$.

The proof that $\text{im}(A)$ is a subspace of $\mathbb{R}^m$ is similar and is left as an exercise to the reader.

We now wish to find a way to describe $\text{null}(A)$ for a matrix A. However, finding $\text{null}\,(A)$ is not new! There is just some new terminology being used, as $\text{null}\,(A)$ is simply the solution to the system $A\vec{x} = \vec{0}$.

Theorem 4.7.2: Basis of null(A)

Let A be an $m \times n$ matrix such that $\text{rank}(A) = r$. Then the system $A\vec{x} = \vec{0}_m$ has $n - r$ basic solutions, providing a basis of $\text{null}(A)$ with $\dim(\text{null}(A)) = n - r$.

Consider the following example.

Example 4.7.5: Null Space of A

Let

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & 3 \end{bmatrix}$$

Find $\text{null}\,(A)$ and $\text{im}\,(A)$.

Solution

In order to find $\text{null}\,(A)$, we simply need to solve the equation $A\vec{x} = \vec{0}$. This is the usual procedure of writing the augmented matrix, finding the reduced row-echelon form and then the solution. The augmented matrix and corresponding reduced row-echelon form are

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 2 & 3 & 3 & 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The third column is not a pivot column, and therefore the solution will contain a parameter. The solution to the system $A\vec{x} = \vec{0}$ is given by

$$\begin{bmatrix} -3t \\ t \\ t \end{bmatrix} : t \in \mathbb{R}$$

which can be written as

$$t \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} : t \in \mathbb{R}$$

Therefore, the null space of A is all multiples of this vector, which we can write as

$$\text{null}(A) = \text{span} \left\{ \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Finally $\text{im}\,(A)$ is just $\{A\vec{x} : \vec{x} \in \mathbb{R}^n\}$ and hence consists of the span of all columns of A, that is $\text{im}\,(A) = \text{col}(A)$.

Notice from the above calculation that that the first two columns of the reduced row-echelon form are pivot columns. Thus the column space is the span of the first two columns in the *original matrix*, and we get

$$\text{im}\,(A) = \text{col}(A) = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \right\}$$

Here is a larger example, but the method is entirely similar.

Example 4.7.6: Null Space of A

Let

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 & 1 \\ 2 & -1 & 1 & 3 & 0 \\ 3 & 1 & 2 & 3 & 1 \\ 4 & -2 & 2 & 6 & 0 \end{bmatrix}$$

Find the null space of A.

Solution

To find the null space, we need to solve the equation $AX = 0$. The augmented matrix and corresponding reduced row-echelon form are given by

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & -1 & 1 & 3 & 0 & 0 \\ 3 & 1 & 2 & 3 & 1 & 0 \\ 4 & -2 & 2 & 6 & 0 & 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & \frac{3}{5} & \frac{6}{5} & \frac{1}{5} & 0 \\ 0 & 1 & \frac{1}{5} & -\frac{3}{5} & \frac{2}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

It follows that the first two columns are pivot columns, and the next three correspond to parameters. Therefore, $\text{null}\,(A)$ is given by

$$\begin{bmatrix} \left(-\frac{3}{5}\right)s + \left(-\frac{6}{5}\right)t + \left(\frac{1}{5}\right)r \\ \left(-\frac{1}{5}\right)s + \left(\frac{3}{5}\right)t + \left(-\frac{2}{5}\right)r \\ s \\ t \\ r \end{bmatrix} : s, t, r \in \mathbb{R}.$$

We write this in the form

$$s \begin{bmatrix} -\frac{3}{5} \\ -\frac{1}{5} \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -\frac{6}{5} \\ \frac{3}{5} \\ 0 \\ 1 \\ 0 \end{bmatrix} + r \begin{bmatrix} \frac{1}{5} \\ -\frac{2}{5} \\ 0 \\ 0 \\ 1 \end{bmatrix} : s, t, r \in \mathbb{R}.$$

In other words, the null space of this matrix equals the span of the three vectors above. Thus

$$\text{null}\,(A) = \text{span}\left\{ \begin{bmatrix} -\frac{3}{5} \\ -\frac{1}{5} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{6}{5} \\ \frac{3}{5} \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{5} \\ -\frac{2}{5} \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Notice also that the three vectors above are linearly independent and so the dimension of $\text{null}\,(A)$ is 3. The following is true in general, the number of parameters in the solution of $AX = 0$ equals the dimension of the null space. Recall also that the number of leading ones in the reduced row-echelon form equals the number of pivot columns, which is the rank of the matrix, which is the same as the dimension of either the column or row space.

Before we proceed to an important theorem, we first define what is meant by the nullity of a matrix.

Definition 4.7.5: Nullity

The dimension of the null space of a matrix is called the **nullity**, denoted $\dim(\text{null}\,(A))$.

From our observation above we can now state an important theorem, sometimes known as the **Rank-Nullity Theorem**.

Theorem 4.7.3: Rank and Nullity

Let A be an $m \times n$ matrix. Then $\text{rank}\,(A) + \dim(\text{null}\,(A)) = n$.

Example 4.7.7: Rank and Nullity

Let

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & 3 \end{bmatrix}$$

Find $\text{rank}\,(A)$ and $\dim(\text{null}\,(A))$.

Solution

In the above Example 4.7.5 we determined that the reduced row-echelon form of A is given by

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore the rank of A is 2. We also determined that the null space of A is given by

$$\text{null}(A) = \text{span}\left\{ \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Therefore the nullity of A is 1. It follows from Theorem 4.7.3 that $\text{rank}\,(A) + \dim(\text{null}\,(A)) = 2 + 1 = 3$, which is the number of columns of A.

We conclude this section with two similar, and important, theorems.

Theorem 4.7.4

Let A be an $m \times n$ matrix. The following are equivalent.

1. $\text{rank}(A) = n$.
2. $\text{row}(A) = \mathbb{R}^n$, i.e., the rows of A span $\mathbb{R}^n$.
3. The columns of A are independent in $\mathbb{R}^m$.
4. The $n \times n$ matrix $A^T A$ is invertible.
5. There exists an $n \times m$ matrix C so that $CA = I_n$.
6. If $A\vec{x} = \vec{0}_m$ for some $\vec{x} \in \mathbb{R}^n$, then $\vec{x} = \vec{0}_n$.

Theorem 4.7.5

Let A be an $m \times n$ matrix. The following are equivalent.

1. $\text{rank}(A) = m$.
2. $\text{col}(A) = \mathbb{R}^m$, i.e., the columns of A span $\mathbb{R}^m$.
3. The rows of A are independent in $\mathbb{R}^n$.
4. The $m \times m$ matrix AA^T is invertible.

5. There exists an $n \times m$ matrix C so that $AC = I_m$.
6. The system $A\vec{x} = \vec{b}$ is consistent for every $\vec{b} \in \mathbb{R}^m$.

Answers

Lemma $4.7.1$**: Effect of Row Operations on Row Space**

We will prove that the above is true for row operations, which can be easily applied to column operations.

Let $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_m$ denote the rows of A.

- If B is obtained from A by a interchanging two rows of A, then A and B have exactly the same rows, so $\text{row}(B) = \text{row}(A)$.
- Suppose $p \neq 0$, and suppose that for some j, $1 \leq j \leq m$, B is obtained from A by multiplying row j by p. Then

$$\text{row}(B) = \text{span}\{\vec{r}_1, \ldots, p\vec{r}_j, \ldots, \vec{r}_m\}.$$

Since

$$\{\vec{r}_1, \ldots, p\vec{r}_j, \ldots, \vec{r}_m\} \subseteq \text{row}(A),$$

it follows that $\text{row}(B) \subseteq \text{row}(A)$. Conversely, since

$$\{\vec{r}_1, \ldots, \vec{r}_m\} \subseteq \text{row}(B),$$

it follows that $\text{row}(A) \subseteq \text{row}(B)$. Therefore, $\text{row}(B) = \text{row}(A)$.

Suppose $p \neq 0$, and suppose that for some i and j, $1 \leq i, j \leq m$, B is obtained from A by adding p time row j to row i. Without loss of generality, we may assume \(i

Then

$$\text{row}(B) = \text{span}\{\vec{r}_1, \ldots, \vec{r}_{i-1}, \vec{r}_i + p\vec{r}_j, \ldots, \vec{r}_j, \ldots, \vec{r}_m\}.$$

Since

$$\{\vec{r}_1, \ldots, \vec{r}_{i-1}, \vec{r}_i + p\vec{r}_j, \ldots, \vec{r}_m\} \subseteq \text{row}(A),$$

it follows that $\text{row}(B) \subseteq \text{row}(A)$.

Conversely, since

$$\{\vec{r}_1, \ldots, \vec{r}_m\} \subseteq \text{row}(B),$$

it follows that $\text{row}(A) \subseteq \text{row}(B)$. Therefore, $\text{row}(B) = \text{row}(A)$.

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4.7.E: Exercise for Section 4.7

Exercise 4.7.E.1

Given the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 0 & 1 & -1 & 2 \end{bmatrix}$$

a. Find a basis for the row space of A.
b. Find a basis for the column space of A.
c. Find a basis for the null space of A.

Answer available at the end of this section.

Exercise 4.7.E.2

Let

$$B = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ 3 & 6 & -3 \end{bmatrix}$$

a. Compute the rank of B.
b. Compute the nullity of B.
c. Verify the Rank-Nullity Theorem for this matrix.

Answer available at the end of this section.

Exercise 4.7.E.3

Given the matrix

$$D = \begin{bmatrix} 2 & 1 & -3 & 4 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 2 & -2 \end{bmatrix}$$

a. Compute the rank of D.
b. Compute the nullity of D.
c. Find a basis for the null space of D.
d. Does the Rank-Nullity Theorem hold for D?

Answer available at the end of this section.

Exercise 4.7.E.4

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix}. \tag{4.7.E.1}$$

a. Find a basis for the row space of A.
b. Find a basis for the column space of A.
c. Find a basis for the null space of A.
d. Compute rank and nullity of A, and verify the Rank-Nullity Theorem.

Exercise $4.7.E.5$

Let

$$B = \begin{bmatrix} 1 & -1 & 3 & 2 & 4 \\ 2 & 0 & 5 & 6 & 8 \\ 3 & -1 & 7 & 8 & 12 \end{bmatrix}$$

a. Compute the rank of B.
b. Compute the nullity of B.

Answer available at the end of this section.

Exercise $4.7.E.6$

Consider the matrix

$$D = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 2 \end{bmatrix}.$$

Find a basis for the null space of D

Exercise $4.7.E.7$

Show that if A is an $m \times n$ matrix, then $\ker(A)$ is a subspace of $\mathbb{R}^n$.

Answer available at the end of this section.

Exercise $4.7.E.8$

Find the rank of the following matrix. Also find a basis for the row and column spaces.

$$\begin{bmatrix} 1 & 3 & 0 & -2 & 0 & 3 \\ 3 & 9 & 1 & -7 & 0 & 8 \\ 1 & 3 & 1 & -3 & 1 & -1 \\ 1 & 3 & -1 & -1 & -2 & 10 \end{bmatrix}$$

Answer available at the end of this section.

Exercise $4.7.E.9$

Find the rank of the following matrix. Also find a basis for the row and column spaces.

$$\begin{bmatrix} 1 & 3 & 0 & -2 & 7 & 3 \\ 3 & 9 & 1 & -7 & 23 & 8 \\ 1 & 3 & 1 & -3 & 9 & 2 \\ 1 & 3 & -1 & -1 & 5 & 4 \end{bmatrix}$$

Exercise 4.7.E.10

Find the rank of the following matrix. Also find a basis for the row and column spaces.

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 7 & 0 \\ 3 & 1 & 10 & 0 & 23 & 0 \\ 1 & 1 & 4 & 1 & 7 & 0 \\ 1 & -1 & 2 & -2 & 9 & 1 \end{bmatrix}$$

Exercise 4.7.E.11

Find the rank of the following matrix. Also find a basis for the row and column spaces.

$$\begin{bmatrix} 1 & 0 & 3 \\ 3 & 1 & 10 \\ 1 & 1 & 4 \\ 1 & -1 & 2 \end{bmatrix}$$

Exercise 4.7.E.12

Find the rank of the following matrix. Also find a basis for the row and column spaces.

$$\begin{bmatrix} 0 & 0 & -1 & 0 & 1 \\ 1 & 2 & 3 & -2 & -18 \\ 1 & 2 & 2 & -1 & -11 \\ -1 & -2 & -2 & 1 & 11 \end{bmatrix}$$

Exercise 4.7.E.13

Find the rank of the following matrix. Also find a basis for the row and column spaces.

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 3 & 1 & 10 & 0 \\ -1 & 1 & -2 & 1 \\ 1 & -1 & 2 & -2 \end{bmatrix}$$

Exercise 4.7.E.14

Find $\ker(A)$ and $\operatorname{im}(A)$ for the following matrices.

a. $A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$

b. $A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$

c. $A = \begin{bmatrix} 2 & 4 & 0 \\ 3 & 6 & -2 \\ 1 & 2 & -2 \end{bmatrix}$

d. $A = \begin{bmatrix} 2 & -1 & 3 & 5 \\ 2 & 0 & 1 & 2 \\ 6 & 4 & -5 & -6 \\ 0 & 2 & -4 & -6 \end{bmatrix}$

Exercise $4.7.E.15$

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Show that

$$\dim(\ker(AB)) \leq \dim(\ker(A)) + \dim(\ker(B)).$$

Consider the subspace, $B(\mathbb{R}^p) \cap \ker(A)$ and suppose a basis for this subspace is $\{\vec{w}_1, \cdots, \vec{w}_k\}$. Now suppose $\{\vec{u}_1, \cdots, \vec{u}_r\}$ is a basis for $\ker(B)$. Let $\{\vec{z}_1, \cdots, \vec{z}_k\}$ be such that $B\vec{z}_1 = \vec{w}_i$ and argue that

$$\ker(AB) \subseteq span\{\vec{u}_1, \cdots, \vec{u}_r, \vec{z}_1, \cdots, \vec{z}_k\}.$$

Answer available at the end of this section.

Exercise $4.7.E.16$

Suppose A is an m×nm \times nm×n matrix. Prove that every vector in $\mathbb{R}^n$ can be uniquely written as a sum of a vector in the row space of A and a vector in the null space of A.

Exercise $4.7.E.17$

Let A and B be two $m \times n$ matrices such that BBB is obtained from A by elementary row operations. Prove that A and B have the same row space.

Answers

Exercise $4.7.E.1$

Given the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 0 & 1 & -1 & 2 \end{bmatrix}$$

a. Basis for the row space: The row space is spanned by the linearly independent rows of A after row reduction:

$$\text{Row space basis: } \{[1 \quad 2 \quad 3 \quad 4], [0 \quad 1 \quad -1 \quad 2]\}.$$

b. Basis for the column space: The column space is spanned by the pivot columns of A:

$$\text{Column space basis: } \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} \right\}.$$

c. Basis for the null space: Solve $A\mathbf{x} = 0$ to find the null space basis:

$$\text{Null space basis: } \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Exercise $4.7.E.2$

Let

$$B = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ 3 & 6 & -3 \end{bmatrix}$$

a. Rank of B: The rank is the number of linearly independent rows:

$$\text{Rank}(B) = 1.$$

b. Nullity of B: The nullity is computed as:

$$\text{Nullity}(B) = 3 - 1 = 2.$$

c. Verification of Rank-Nullity Theorem:

$$\text{Rank}(B) + \text{Nullity}(B) = 1 + 2 = 3,$$

which matches the number of columns, so the theorem holds.

Exercise $4.7.E.3$

Given the matrix

$$D = \begin{bmatrix} 2 & 1 & -3 & 4 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 2 & -2 \end{bmatrix}$$

\textbf{Rank of D:} \ The rank is the number of linearly independent rows:

$$\text{Rank}(D) = 2.$$

\textbf{Nullity of D:} \

$$\text{Nullity}(D) = 4 - 2 = 2.$$

\textbf{Basis for the null space:} \ Solve $D\mathbf{x} = 0$:

$$\text{Null space basis: } \left\{ \begin{bmatrix} 1 \\ -5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

\textbf{Verification of Rank-Nullity Theorem:} \

$$\text{Rank}(D) + \text{Nullity}(D) = 2 + 2 = 4,$$

which matches the number of columns, so the theorem holds.

Exercise $4.7.E.5$

a. Row reduce B:

$$\begin{bmatrix} 1 & -1 & 3 & 2 & 4 \\ 2 & 0 & 5 & 6 & 8 \\ 3 & -1 & 7 & 8 & 12 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 3 & 2 & 4 \\ 0 & 2 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{Rank}(B) = 2$$

b. Using the Rank-Nullity Theorem:

$$\text{Nullity}(B) = 5 - 2 = 3$$

Exercise $4.7.E.7$

If $\vec{x}$, $\vec{y} \in \ker(A)$, then $A(\vec{x}) = \vec{0}$ and $A(\vec{y}) = \vec{0}$. Now let's look at $a\vec{x} + b\vec{y}$ for some $a, b \in \mathbb{R}$. Then,

$$A(a\vec{x} + b\vec{y}) = aA\vec{x} + bA\vec{y} = a\vec{0} + b\vec{0} = \vec{0}$$

and so $\ker(A)$ is closed under linear combinations. Hence it is a subspace.

Exercise $4.7.E.8$

The rank of the given matrix is 3. A basis for the row space is:

$$\{[1 \quad 3 \quad 0 \quad -2 \quad 0 \quad 3], [0 \quad 0 \quad 1 \quad -1 \quad 1 \quad -3], [0 \quad 0 \quad 0 \quad 0 \quad 1 \quad -5]\}$$

A basis for the column space is:

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

Exercise $4.7.E.15$

Here is how you do this. Suppose $AB\vec{x} = \vec{0}$. Then $B\vec{x} \in \ker(A) \cap B(\mathbb{R}^p)$ and so $B\vec{x} = \sum_{i=1}^{k} B\vec{z}_i$ showing that

$$\vec{x} - \sum_{i=1}^{k} \vec{z}_i \in \ker(B)$$

Consider $B(\mathbb{R}^p) \cap \ker(A)$ and let a basis be $\{\vec{w}_1, \cdots, \vec{w}_k\}$. Then each $\vec{w}_i$ is of the form $B\vec{z}_i = \vec{w}_i$. Therefore, $\{\vec{z}_1, \cdots, \vec{z}_k\}$ is linearly independent and $AB\vec{z}_i = 0$. Now let $\{\vec{u}_1, \cdots, \vec{u}_r\}$ be a basis for $\ker(B)$. If $AB\vec{x} = \vec{0}$, then $B\vec{x} \in \ker(A) \cap B(\mathbb{R}^p)$ and so $B\vec{x} = \sum_{i=1}^{k} c_i B\vec{z}_1$ which implies

$$\vec{x} - \sum_{i=1}^{k} c_i \vec{z}_i \in \ker(B)$$

and so it is of the form

$$\vec{x} - \sum_{i=1}^{k} c_i \vec{z}_i = \sum_{j=1}^{r} d_j \vec{u}_j$$

It follows that if $AB\vec{x} = \vec{0}$ so that $\vec{x} \in \ker(AB)$, then

$$\vec{x} \in span(\vec{z}_1, \cdots, \vec{z}_k, \vec{u}_1, \cdots, \vec{u}_r).$$

Therefore,

$$\begin{aligned} \dim(\ker(AB)) &\leq k + r = \dim(B(\mathbb{R}^p) \cap \ker(A)) + \dim(\ker(B)) \\ &\leq \dim(\ker(A)) + \dim(\ker(B)) \end{aligned} \tag{4.7.E.2}$$

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4.8: Orthogonal Vectors and Matrices

Learning Objectives

- Determine if a given set is orthogonal or orthonormal.
- Determine if a given matrix is orthogonal.

In this section, we examine what it means for vectors (and sets of vectors) to be orthogonal and orthonormal.

Recall from the properties of the dot product of vectors that two vectors $\vec{u}$ and $\vec{v}$ are **orthogonal** if $\vec{u} \cdot \vec{v} = 0$. Suppose a vector is orthogonal to a spanning set of $\mathbb{R}^n$. What can be said about such a vector? This is the discussion in the following example.

Example 4.8.1: Orthogonal Vector to a Spanning Set

Let $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\} \in \mathbb{R}^n$ and suppose $\mathbb{R}^n = \text{span}\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$. Furthermore, suppose that there exists a vector $\vec{u} \in \mathbb{R}^n$ for which $\vec{u} \cdot \vec{x}_j = 0$ for all j, $1 \leq j \leq k$. What type of vector is $\vec{u}$?

Solution

Write $\vec{u} = t_1\vec{x}_1 + t_2\vec{x}_2 + \cdots + t_k\vec{x}_k$ for some $t_1, t_2, \ldots, t_k \in \mathbb{R}$ (this is possible because $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k$ span $\mathbb{R}^n$).

Then

$$\begin{aligned} \|\vec{u}\|^2 &= \vec{u} \cdot \vec{u} \\ &= \vec{u} \cdot (t_1\vec{x}_1 + t_2\vec{x}_2 + \cdots + t_k\vec{x}_k) \\ &= \vec{u} \cdot (t_1\vec{x}_1) + \vec{u} \cdot (t_2\vec{x}_2) + \cdots + \vec{u} \cdot (t_k\vec{x}_k) \\ &= t_1(\vec{u} \cdot \vec{x}_1) + t_2(\vec{u} \cdot \vec{x}_2) + \cdots + t_k(\vec{u} \cdot \vec{x}_k) \\ &= t_1(0) + t_2(0) + \cdots + t_k(0) = 0. \end{aligned} \tag{4.8.1}$$

Since $\|\vec{u}\|^2 = 0$, $\|\vec{u}\| = 0$. We know that $\|\vec{u}\| = 0$ if and only if $\vec{u} = \vec{0}_n$. Therefore, $\vec{u} = \vec{0}_n$. In conclusion, the only vector orthogonal to every vector of a spanning set of $\mathbb{R}^n$ is the zero vector.

We can now discuss what is meant by an orthogonal set of vectors.

Definition 4.8.1: Orthogonal Set of Vectors

Let $\{\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_m\}$ be a set of vectors in $\mathbb{R}^n$. Then this set is called an **orthogonal set** if the following conditions hold:

1. $\vec{u}_i \cdot \vec{u}_j = 0$ for all $i \neq j$
2. $\vec{u}_i \neq \vec{0}$ for all i

If we have an orthogonal set of vectors and normalize each vector so they have length 1, the resulting set is called an **orthonormal set** of vectors. They can be described as follows.

Definition 4.8.2: Orthonormal Set of Vectors

A set of vectors, $\{\vec{w}_1, \cdots, \vec{w}_m\}$ is said to be an **orthonormal** set if

$$\vec{w}_i \cdot \vec{w}_j = \delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

Note that all orthonormal sets are orthogonal, but the reverse is not necessarily true since the vectors may not be normalized. In order to normalize the vectors, we simply need divide each one by its length.

Definition 4.8.3: Normalizing an Orthogonal Set

Normalizing an orthogonal set is the process of turning an orthogonal (but not orthonormal) set into an orthonormal set. If $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$ is an orthogonal subset of $\mathbb{R}^n$, then

$$\left\{ \frac{1}{\|\vec{u}_1\|}\vec{u}_1, \frac{1}{\|\vec{u}_2\|}\vec{u}_2, \ldots, \frac{1}{\|\vec{u}_k\|}\vec{u}_k \right\}$$

is an orthonormal set.

We illustrate this concept in the following example.

Example 4.8.2: Orthonormal Set

Consider the set of vectors given by

$$\{\vec{u}_1, \vec{u}_2\} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

Show that it is an orthogonal set of vectors but not an orthonormal one. Find the corresponding orthonormal set.

Solution

One easily verifies that $\vec{u}_1 \cdot \vec{u}_2 = 0$ and $\{\vec{u}_1, \vec{u}_2\}$ is an orthogonal set of vectors. On the other hand one can compute that $\|\vec{u}_1\| = \|\vec{u}_2\| = \sqrt{2} \neq 1$ and thus it is not an orthonormal set.

Thus to find a corresponding orthonormal set, we simply need to normalize each vector. We will write $\{\vec{w}_1, \vec{w}_2\}$ for the corresponding orthonormal set. Then,

$$\begin{aligned} \vec{w}_1 &= \frac{1}{\|\vec{u}_1\|}\vec{u}_1 \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \end{aligned} \tag{4.8.2}$$

Similarly,

$$\begin{aligned} \vec{w}_2 &= \frac{1}{\|\vec{u}_2\|}\vec{u}_2 \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \end{aligned} \tag{4.8.3}$$

Therefore the corresponding orthonormal set is

$$\{\vec{w}_1, \vec{w}_2\} = \left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\}$$

You can verify that this set is orthogonal.

Consider an orthogonal set of vectors in $\mathbb{R}^n$, written $\{\vec{w}_1, \cdots, \vec{w}_k\}$ with $k \leq n$. The span of these vectors is a subspace W of $\mathbb{R}^n$. If

we could show that this orthogonal set is also linearly independent, we would have a basis of W. We will show this in the next theorem.

Theorem 4.8.1: Orthogonal Basis of a Subspace

Let $\{\vec{w}_1, \vec{w}_2, \cdots, \vec{w}_k\}$ be an orthonormal set of vectors in $\mathbb{R}^n$. Then this set is linearly independent and forms a basis for the subspace $W = \text{span}\{\vec{w}_1, \vec{w}_2, \cdots, \vec{w}_k\}$.

Proof available at the end of this section.

If an orthogonal set is a basis for a subspace, we call this an orthogonal basis. Similarly, if an orthonormal set is a basis, we call this an orthonormal basis.

We conclude this section with a discussion of Fourier expansions. Given any orthogonal basis B of $\mathbb{R}^n$ and an arbitrary vector $\vec{x} \in \mathbb{R}^n$, how do we express $\vec{x}$ as a linear combination of vectors in B? The solution is Fourier expansion.

Theorem 4.8.2: Fourier Expansion

Let V be a subspace of $\mathbb{R}^n$ and suppose $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m\}$ is an orthogonal basis of V. Then for any $\vec{x} \in V$,

$$\vec{x} = \left(\frac{\vec{x} \cdot \vec{u}_1}{\|\vec{u}_1\|^2}\right)\vec{u}_1 + \left(\frac{\vec{x} \cdot \vec{u}_2}{\|\vec{u}_2\|^2}\right)\vec{u}_2 + \cdots + \left(\frac{\vec{x} \cdot \vec{u}_m}{\|\vec{u}_m\|^2}\right)\vec{u}_m$$

This expression is called the **Fourier expansion** of $\vec{x}$, and

$$\frac{\vec{x} \cdot \vec{u}_j}{\|\vec{u}_j\|^2},$$

$j = 1, 2, \ldots, m$ are the Fourier coefficients.

Consider the following example.

Example 4.8.3: Fourier Expansion

Let $\vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$, and $\vec{u}_3 = \begin{bmatrix} 5 \\ 1 \\ -2 \end{bmatrix}$, and let $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Then $B = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthogonal basis of $\mathbb{R}^3$.

Compute the Fourier expansion of $\vec{x}$, thus writing $\vec{x}$ as a linear combination of the vectors of B.

Solution

Since B is a basis (verify!) there is a unique way to express $\vec{x}$ as a linear combination of the vectors of B. Moreover since B is an orthogonal basis (verify!), then this can be done by computing the Fourier expansion of $\vec{x}$.

That is:

$$\vec{x} = \left(\frac{\vec{x} \cdot \vec{u}_1}{\|\vec{u}_1\|^2}\right)\vec{u}_1 + \left(\frac{\vec{x} \cdot \vec{u}_2}{\|\vec{u}_2\|^2}\right)\vec{u}_2 + \left(\frac{\vec{x} \cdot \vec{u}_3}{\|\vec{u}_3\|^2}\right)\vec{u}_3.$$

We readily compute:

$$\frac{\vec{x} \cdot \vec{u}_1}{\|\vec{u}_1\|^2} = \frac{2}{6}, \quad \frac{\vec{x} \cdot \vec{u}_2}{\|\vec{u}_2\|^2} = \frac{3}{5}, \quad \text{and} \quad \frac{\vec{x} \cdot \vec{u}_3}{\|\vec{u}_3\|^2} = \frac{4}{30}.$$

Therefore,

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + \frac{3}{5}\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + \frac{2}{15}\begin{bmatrix} 5 \\ 1 \\ -2 \end{bmatrix}.$$

Recall that the process to find the inverse of a matrix was often cumbersome. In contrast, it was very easy to take the transpose of a matrix. Luckily for some special matrices, the transpose equals the inverse. When an $n \times n$ matrix has all real entries and its transpose equals its inverse, the matrix is called an **orthogonal matrix**.

The precise definition is as follows.

Definition 4.8.4: Orthogonal Matrices

A real $n \times n$ matrix U is called an **orthogonal** matrix if

$$UU^T = U^T U = I.$$

Note since U is assumed to be a square matrix, it suffices to verify only one of these equalities $UU^T = I$ or $U^T U = I$ holds to guarantee that U^T is the inverse of U.

Consider the following example.

Example 4.8.4: Orthogonal Matrix

Orthogonal Matrix Show the matrix

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

is orthogonal.

Solution

All we need to do is verify (one of the equations from) the requirements of Definition 4.8.4.

$$UU^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Since $UU^T = I$, this matrix is orthogonal.

Here is another example.

Example 4.8.5: Orthogonal Matrix

Orthogonal Matrix Let $U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$. Is U orthogonal?

Solution

Again the answer is yes and this can be verified simply by showing that $U^T U = I$:

$$\begin{aligned} U^TU &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{4.8.4}$$

When we say that U is orthogonal, we are saying that $UU^T = I$, meaning that

$$\sum_j u_{ij}u_{jk}^T = \sum_j u_{ij}u_{kj} = \delta_{ik}$$

where δ_{ij} is the **Kronecker delta** defined by

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

In words, the product of the ith row of U with the kth row gives 1 if $i = k$ and 0 if $i \neq k$. The same is true of the columns because $U^TU = I$ also. Therefore,

$$\sum_j u_{ij}^T u_{jk} = \sum_j u_{ji}u_{jk} = \delta_{ik}$$

which says that the product of one column with another column gives 1 if the two columns are the same and 0 if the two columns are different.

More succinctly, this states that if $\vec{u}_1, \cdots, \vec{u}_n$ are the columns of U, an orthogonal matrix, then

$$\vec{u}_i \cdot \vec{u}_j = \delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

We will say that the columns form an orthonormal set of vectors, and similarly for the rows. Thus a matrix is **orthogonal** if its rows (or columns) form an **orthonormal** set of vectors. Notice that the convention is to call such a matrix orthogonal rather than orthonormal (although this may make more sense!).

Proposition 4.8.1: Orthonormal Basis

The rows of an $n \times n$ orthogonal matrix form an orthonormal basis of $\mathbb{R}^n$. Further, any orthonormal basis of $\mathbb{R}^n$ can be used to construct an $n \times n$ orthogonal matrix.

Proof available at the end of this section.

Consider the following proposition.

Proposition 4.8.2: Determinant of Orthogonal Matrices

Suppose U is an orthogonal matrix. Then $\det(U) = \pm 1$.

Proof available at the end of this section.

Orthogonal matrices are divided into two classes, proper and improper. The proper orthogonal matrices are those whose determinant equals 1 and the improper ones are those whose determinant equals -1. The reason for the distinction is that the improper orthogonal matrices are sometimes considered to have no physical significance. These matrices cause a change in orientation which would correspond to material passing through itself in a non physical manner. Thus in considering which

coordinate systems must be considered in certain applications, you only need to consider those which are related by a proper orthogonal transformation. Geometrically, the linear transformations determined by the proper orthogonal matrices correspond to the composition of rotations.

We conclude this section with two useful properties of orthogonal matrices.

Example 4.8.6: Product and Inverse of Orthogonal Matrices

Suppose A and B are orthogonal matrices. Then AB and A^{-1} both exist and are orthogonal.

Solution

First we examine the product AB.

$$(AB)(B^TA^T) = A(BB^T)A^T = AA^T = I$$

Since AB is square, $B^TA^T = (AB)^T$ is the inverse of AB, so AB is invertible, and $(AB)^{-1} = (AB)^T$ Therefore, AB is orthogonal.

Next we show that $A^{-1} = A^T$ is also orthogonal.

$$(A^{-1})^{-1} = A = (A^T)^T = (A^{-1})^T$$

Therefore A^{-1} is also orthogonal.

Answers

Theorem 4.8.1: Orthogonal Basis of a Subspace

To show it is a linearly independent set, suppose a linear combination of these vectors equals $\vec{0}$, such as:

$$a_1\vec{w}_1 + a_2\vec{w}_2 + \cdots + a_k\vec{w}_k = \vec{0}, a_i \in \mathbb{R}$$

We need to show that all $a_i = 0$. To do so, take the dot product of each side of the above equation with the vector $\vec{w}_i$ and obtain the following.

$$\begin{aligned} \vec{w}_i \cdot (a_1\vec{w}_1 + a_2\vec{w}_2 + \cdots + a_k\vec{w}_k) &= \vec{w}_i \cdot \vec{0} \\ a_1(\vec{w}_i \cdot \vec{w}_1) + a_2(\vec{w}_i \cdot \vec{w}_2) + \cdots + a_k(\vec{w}_i \cdot \vec{w}_k) &= 0 \end{aligned} \tag{4.8.5}$$

Now since the set is orthogonal, $\vec{w}_i \cdot \vec{w}_m = 0$ for all $m \neq i$, so we have:

$$a_1(0) + \cdots + a_i(\vec{w}_i \cdot \vec{w}_i) + \cdots + a_k(0) = 0$$

$$a_i\|\vec{w}_i\|^2 = 0$$

Since the set is orthogonal, we know that $\|\vec{w}_i\|^2 \neq 0$. It follows that $a_i = 0$. Since the a_i was chosen arbitrarily, the set $\{\vec{w}_1, \vec{w}_2, \cdots, \vec{w}_k\}$ is linearly independent.

Finally since $W = \text{span}\{\vec{w}_1, \vec{w}_2, \cdots, \vec{w}_k\}$, the set of vectors also spans W and therefore forms a basis of W.

Proposition 4.8.1: Orthonormal Basis

Recall from Theorem 4.8.1 that an orthonormal set is linearly independent and forms a basis for its span. Since the rows of an $n \times n$ orthogonal matrix form an orthonormal set, they must be linearly independent. Now we have n linearly independent vectors, and it follows that their span equals $\mathbb{R}^n$. Therefore these vectors form an orthonormal basis for $\mathbb{R}^n$.

Suppose now that we have an orthonormal basis for $\mathbb{R}^n$. Since the basis will contain n vectors, these can be used to construct an $n \times n$ matrix, with each vector becoming a row. Therefore the matrix is composed of orthonormal rows, which by our above discussion, means that the matrix is orthogonal. Note we could also have construct a matrix with each vector becoming a column instead, and this would again be an orthogonal matrix. In fact this is simply the transpose of the previous matrix.

Proposition 4.8.2: Determinant of Orthogonal Matrices

This result follows from the properties of determinants. Recall that for any matrix A, $\det(A)^T = \det(A)$. Now if U is orthogonal, then:

$$(\det(U))^2 = \det(U^T)\det(U) = \det(U^TU) = \det(I) = 1$$

Therefore $(\det(U))^2 = 1$ and it follows that $\det(U) = \pm 1$.

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4.8.E: Exercise for Section 4.8

Exercise 4.8.E.1

Determine whether the following set of vectors is orthogonal. If it is orthogonal, determine whether it is also orthonormal.

$$\begin{bmatrix} \frac{1}{6}\sqrt{2}\sqrt{3} \\ \frac{1}{3}\sqrt{2}\sqrt{3} \\ -\frac{1}{6}\sqrt{2}\sqrt{3} \end{bmatrix}, \begin{bmatrix} \frac{1}{2}\sqrt{2} \\ 0 \\ \frac{1}{2}\sqrt{2} \end{bmatrix}, \begin{bmatrix} -\frac{1}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{3} \end{bmatrix}$$

If the set of vectors is orthogonal but not orthonormal, give an orthonormal set of vectors which has the same span.

Exercise 4.8.E.2

Determine whether the following set of vectors is orthogonal. If it is orthogonal, determine whether it is also orthonormal.

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

If the set of vectors is orthogonal but not orthonormal, give an orthonormal set of vectors which has the same span.

Answer available at the end of this section.

Exercise 4.8.E.3

Determine whether the following set of vectors is orthogonal. If it is orthogonal, determine whether it is also orthonormal.

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

If the set of vectors is orthogonal but not orthonormal, give an orthonormal set of vectors which has the same span.

Exercise 4.8.E.4

Determine whether the following set of vectors is orthogonal. If it is orthogonal, determine whether it is also orthonormal.

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

If the set of vectors is orthogonal but not orthonormal, give an orthonormal set of vectors which has the same span.

Exercise 4.8.E.5

Determine whether the following set of vectors is orthogonal. If it is orthogonal, determine whether it is also orthonormal.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

If the set of vectors is orthogonal but not orthonormal, give an orthonormal set of vectors which has the same span.

Exercise 4.8.E.6

Here are some matrices. Label according to whether they are symmetric, skew symmetric, or orthogonal.

a. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

b. $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & 4 \\ -3 & 4 & 7 \end{bmatrix}$

c. $\begin{bmatrix} 0 & -2 & -3 \\ 2 & 0 & -4 \\ 3 & 4 & 0 \end{bmatrix}$

Answer available at the end of this section.

Exercise 4.8.E.7

In this exercise, you will show that orthogonal matrices are exactly those which preserve length.

a. For U an orthogonal matrix, explain why $||U\vec{x}|| = ||\vec{x}||$ for any vector $\vec{x}$.
b. Next explain why if U is an $n \times n$ matrix with the property that $||U\vec{x}|| = ||\vec{x}||$ for all vectors, $\vec{x}$, then U must be orthogonal.

Answer available at the end of this section.

Exercise 4.8.E.8

Suppose U is an orthogonal $n \times n$ matrix. Explain why $rank(U) = n$.

Answer available at the end of this section.

Exercise 4.8.E.9

Fill in the missing entries to make the matrix orthogonal.

$$\begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & - & - \\ - & \frac{\sqrt{6}}{3} & - \end{bmatrix}.$$

Answer available at the end of this section.

Exercise 4.8.E.10

Fill in the missing entries to make the matrix orthogonal.

$$\begin{bmatrix} \frac{2}{3} & \frac{\sqrt{2}}{2} & \frac{1}{6}\sqrt{2} \\ \frac{2}{3} & - & - \\ - & 0 & - \end{bmatrix}$$

Answer available at the end of this section.

Exercise 4.8.E.11

Fill in the missing entries to make the matrix orthogonal.

$$\begin{bmatrix} \frac{1}{3} & -\frac{2}{\sqrt{5}} & - \\ \frac{2}{3} & 0 & - \\ - & - & \frac{4}{15}\sqrt{5} \end{bmatrix}$$

Answer available at the end of this section.

Exercise 4.8.E.12

Consider the set of $\mathbb{R}^3$ vectors:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}.$$

a. Show that this set forms an $\mathbf{orthogonal\ basis}$ for $\mathbb{R}^3$.

b. Given the vector $\vec{w} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$, find the Fourier expansion of $\vec{w}$ in terms of this basis.

Exercise 4.8.E.13

Consider the set of $\mathbb{R}^4$ vectors:

$$\vec{u}_1 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{u}_2 = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \quad \vec{u}_3 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \quad \vec{u}_4 = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}.$$

a. Show that this set forms an $\mathbf{orthonormal\ basis}$ for $\mathbb{R}^4$.

b. Given the vector: $\vec{w} = \begin{bmatrix} 3 \\ 1 \\ -2 \\ 0 \end{bmatrix}$, find the Fourier expansion of $\vec{w}$ in terms of this basis.

Exercise 4.8.E.14

Show that if a matrix A has orthogonal columns, then $A^T A$ is a diagonal matrix.

Hint available at the end of this section.

Exercise $4.8.E.15$

Prove that if Q is an orthogonal matrix, then it preserves inner products, i.e., for any vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$,

$$\langle Q\vec{x}, Q\vec{y} \rangle = \langle \vec{x}, \vec{y} \rangle.$$

Hint available at the end of this section.

Answers

Exercise $4.8.E.2$

The given set of vectors:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

is orthogonal since their pairwise dot products are zero. To normalize them, we compute their norms:

$$\|\vec{v}_1\| = \sqrt{6}, \quad \|\vec{v}_2\| = \sqrt{2}, \quad \|\vec{v}_3\| = \sqrt{3}$$

Thus, an orthonormal basis with the same span is:

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{bmatrix}$$

$$\vec{u}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\vec{u}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|} = \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

Exercise $4.8.E.6$

a. Orthogonal
b. Symmetric
c. Skew Symmetric

Exercise $4.8.E.7$

a. Suppose U is an orthogonal matrix. $||U\vec{x}||^2 = U\vec{x} \bullet U\vec{x} = U^T U\vec{x} \bullet \vec{x} = I\vec{x} \bullet \vec{x} = ||\vec{x}||^2$.
b. Now, suppose distance is preserved by U. Then

$$\begin{aligned} (U(\vec{x}+\vec{y})) \bullet (U(\vec{x}+\vec{y})) &= ||Ux||^2 + ||Uy||^2 + 2(Ux \bullet Uy) \\ &= ||\vec{x}||^2 + ||\vec{y}||^2 + 2(U^T U\vec{x} \bullet \vec{y}) \end{aligned} \tag{4.8.E.1}$$

But since U preserves distances, it is also the case that

$$(U(\vec{x}+\vec{y}) \bullet U(\vec{x}+\vec{y})) = ||\vec{x}||^2 + ||\vec{y}||^2 + 2(\vec{x} \bullet \vec{y})$$

Hence

$$\vec{x} \bullet \vec{y} = U^T U\vec{x} \bullet \vec{y}$$

and so

$$((U^T U - I)\vec{x}) \bullet \vec{y} = 0$$

Since $\vec{y}$ is arbitrary, it follows that $U^T U - I = 0$. Thus U is orthogonal.

Exercise $4.8.E.8$

You could observe that $\det(UU^T) = (\det(U))^2 - 1$ so $\det(U) \neq 0$.

Exercise $4.8.E.9$

$$\begin{aligned} &\begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & a \\ 0 & \frac{\sqrt{6}}{3} & b \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & a \\ 0 & \frac{\sqrt{6}}{3} & b \end{bmatrix}^T \\ &= \begin{bmatrix} 1 & \frac{1}{3}\sqrt{3}a - \frac{1}{3} & \frac{1}{3}\sqrt{3}b - \frac{1}{3} \\ \frac{1}{3}\sqrt{3}a - \frac{1}{3} & a^2 + \frac{2}{3} & ab - \frac{1}{3} \\ \frac{1}{3}\sqrt{3}b - \frac{1}{3} & ab - \frac{1}{3} & b^2 + \frac{2}{3} \end{bmatrix} \end{aligned} \tag{4.8.E.2}$$

This requires, $a = 1/\sqrt{3}, b = 1/\sqrt{3}$.

$$\begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & 1/\sqrt{3} \\ 0 & \frac{\sqrt{6}}{3} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & 1/\sqrt{3} \\ 0 & \frac{\sqrt{6}}{3} & 1/\sqrt{3} \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Exercise 4.8.E.10

$$\begin{bmatrix} \frac{2}{3} & \frac{\sqrt{2}}{2} & \frac{1}{6}\sqrt{2} \\ \frac{2}{3} & \frac{-\sqrt{2}}{2} & a \\ -\frac{1}{3} & 0 & b \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{\sqrt{2}}{2} & \frac{1}{6}\sqrt{2} \\ \frac{2}{3} & \frac{-\sqrt{2}}{2} & a \\ -\frac{1}{3} & 0 & b \end{bmatrix}^T = \begin{bmatrix} 1 & \frac{1}{6}\sqrt{2}a - \frac{1}{18} & \frac{1}{6}\sqrt{2}b - \frac{2}{9} \\ \frac{1}{6}\sqrt{2}a - \frac{1}{18} & a^2 + \frac{17}{18} & ab - \frac{2}{9} \\ \frac{1}{6}\sqrt{2}b - \frac{2}{9} & ab - \frac{2}{9} & b^2 + \frac{1}{9} \end{bmatrix}$$

This requires $a = \frac{1}{3\sqrt{2}}$, $b = \frac{4}{3\sqrt{2}}$.

$$\begin{bmatrix} \frac{2}{3} & \frac{\sqrt{2}}{2} & \frac{1}{6}\sqrt{2} \\ \frac{2}{3} & \frac{-\sqrt{2}}{2} & \frac{1}{3\sqrt{2}} \\ -\frac{1}{3} & 0 & \frac{4}{3\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{\sqrt{2}}{2} & \frac{1}{6}\sqrt{2} \\ \frac{2}{3} & \frac{-\sqrt{2}}{2} & \frac{1}{3\sqrt{2}} \\ -\frac{1}{3} & 0 & \frac{4}{3\sqrt{2}} \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Exercise 4.8.E.11

Try

$$\begin{aligned} &\begin{bmatrix} \frac{1}{3} & -\frac{2}{\sqrt{5}} & c \\ \frac{2}{3} & 0 & d \\ \frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{15}\sqrt{5} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & -\frac{2}{\sqrt{5}} & c \\ \frac{2}{3} & 0 & d \\ \frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{15}\sqrt{5} \end{bmatrix}^T \\ &= \begin{bmatrix} c^2 + \frac{41}{45} & cd + \frac{2}{9} & \frac{4}{15}\sqrt{5}c - \frac{8}{45} \\ cd + \frac{2}{9} & d^2 + \frac{4}{9} & \frac{4}{15}\sqrt{5}d + \frac{4}{9} \\ \frac{4}{15}\sqrt{5}c - \frac{8}{45} & \frac{4}{15}\sqrt{5}d + \frac{4}{9} & 1 \end{bmatrix} \end{aligned} \tag{4.8.E.3}$$

This requires that $c = \frac{2}{3\sqrt{5}}, d = \frac{-5}{3\sqrt{5}}$.

$$\begin{bmatrix} \frac{1}{3} & -\frac{2}{\sqrt{5}} & \frac{2}{3\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{-5}{3\sqrt{5}} \\ \frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{15}\sqrt{5} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & -\frac{2}{\sqrt{5}} & \frac{2}{3\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{-5}{3\sqrt{5}} \\ \frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{15}\sqrt{5} \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Exercise 4.8.E.14

Compute the (i, j) entry of A^TA explicitly.

Exercise 4.8.E.15

Express the inner product using matrix multiplication and use the fact that $Q^TQ = I$.

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4.9: Gram-Schmidt Process

Learning Objectives

- Given a linearly independent set, use the Gram-Schmidt Process to find corresponding orthogonal and orthonormal sets.

The Gram-Schmidt process is an algorithm to transform a set of vectors into an orthonormal set spanning the same subspace, that is generating the same collection of linear combinations.

The goal of the Gram-Schmidt process is to take a linearly independent set of vectors and transform it into an orthonormal set with the same span. The first objective is to construct an orthogonal set of vectors with the same span, since from there an orthonormal set can be obtained by simply dividing each vector by its length.

Algorithm 4.9.1: Gram-Schmidt Process

Let $\{\vec{u}_1, \cdots, \vec{u}_n\}$ be a set of linearly independent vectors in $\mathbb{R}^n$.

I: Construct a new set of vectors $\{\vec{v}_1, \cdots, \vec{v}_n\}$ as follows:

$$\begin{aligned}
\vec{v}_1 &= \vec{u}_1 \\
\vec{v}_2 &= \vec{u}_2 - \left(\frac{\vec{u}_2 \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\right)\vec{v}_1 \\
\vec{v}_3 &= \vec{u}_3 - \left(\frac{\vec{u}_3 \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\right)\vec{v}_1 - \left(\frac{\vec{u}_3 \cdot \vec{v}_2}{\|\vec{v}_2\|^2}\right)\vec{v}_2 \\
&\vdots \\
\vec{v}_n &= \vec{u}_n - \left(\frac{\vec{u}_n \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\right)\vec{v}_1 - \left(\frac{\vec{u}_n \cdot \vec{v}_2}{\|\vec{v}_2\|^2}\right)\vec{v}_2 - \cdots - \left(\frac{\vec{u}_n \cdot \vec{v}_{n-1}}{\|\vec{v}_{n-1}\|^2}\right)\vec{v}_{n-1}
\end{aligned}$$

II: Now let $\vec{w}_i = \dfrac{\vec{v}_i}{\|\vec{v}_i\|}$ for $i = 1, \cdots, n$.

Then

1. $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is an orthogonal set.
2. $\{\vec{w}_1, \cdots, \vec{w}_n\}$ is an orthonormal set.
3. $\text{span}\{\vec{u}_1, \cdots, \vec{u}_n\} = \text{span}\{\vec{v}_1, \cdots, \vec{v}_n\} = \text{span}\{\vec{w}_1, \cdots, \vec{w}_n\}$.

Solution

The full proof of this algorithm is beyond this material, however here is an indication of the arguments.

To show that $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is an orthogonal set, let

$$a_2 = \frac{\vec{u}_2 \cdot \vec{v}_1}{\|\vec{v}_1\|^2}$$

then:

$$\begin{aligned}
\vec{v}_1 \cdot \vec{v}_2 &= \vec{v}_1 \cdot (\vec{u}_2 - a_2\vec{v}_1) \\
&= \vec{v}_1 \cdot \vec{u}_2 - a_2(\vec{v}_1 \cdot \vec{v}_1) \\
&= \vec{v}_1 \cdot \vec{u}_2 - \frac{\vec{u}_2 \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\|\vec{v}_1\|^2 \\
&= (\vec{v}_1 \cdot \vec{u}_2) - (\vec{u}_2 \cdot \vec{v}_1) = 0
\end{aligned}$$

Now that you have shown that $\{\vec{v}_1, \vec{v}_2\}$ is orthogonal, use the same method as above to show that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is also orthogonal, and so on.

Then in a similar fashion you show that $\text{span}\{\vec{u}_1, \cdots, \vec{u}_n\} = \text{span}\{\vec{v}_1, \cdots, \vec{v}_n\}$.

Finally defining $\vec{w}_i = \dfrac{\vec{v}_i}{\|\vec{v}_i\|}$ for $i = 1, \cdots, n$ does not affect orthogonality and yields vectors of length 1, hence an orthonormal set. You can also observe that it does not affect the span either and the proof would be complete.

Consider the following example.

Example 4.9.1: Find Orthonormal Set with Same Span

Consider the set of vectors $\{\vec{u}_1, \vec{u}_2\}$ given as in Example 4.9.1. That is

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \in \mathbb{R}^3$$

Use the Gram-Schmidt algorithm to find an orthonormal set of vectors $\{\vec{w}_1, \vec{w}_2\}$ having the same span.

Solution

We already remarked that the set of vectors in $\{\vec{u}_1, \vec{u}_2\}$ is linearly independent, so we can proceed with the Gram-Schmidt algorithm:

$$\begin{aligned} \vec{v}_1 &= \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\ \vec{v}_2 &= \vec{u}_2 - \left(\frac{\vec{u}_2 \cdot \vec{v}_1}{\|\vec{v}_1\|^2} \right) \vec{v}_1 \\ &= \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} - \frac{5}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix} \end{aligned} \tag{4.9.1}$$

Now to normalize simply let

$$\begin{aligned} \vec{w}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \\ \vec{w}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \end{aligned} \tag{4.9.2}$$

You can verify that $\{\vec{w}_1, \vec{w}_2\}$ is an orthonormal set of vectors having the same span as $\{\vec{u}_1, \vec{u}_2\}$, namely the XY-plane.

In this example, we began with a linearly independent set and found an orthonormal set of vectors which had the same span. It turns out that if we start with a basis of a subspace and apply the Gram-Schmidt algorithm, the result will be an orthogonal basis of the same subspace. We examine this in the following example.

Example 4.9.2: Find a Corresponding Orthogonal Basis

Let

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \text{ and } \vec{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

and let $U = \text{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$. Use the Gram-Schmidt Process to construct an orthogonal basis B of U.

Solution

First $\vec{f}_1 = \vec{x}_1$.

Next,

$$\vec{f}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Finally,

$$\vec{f}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{0}{1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \\ -1/2 \\ 0 \end{bmatrix}.$$

Therefore,

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1 \\ -1/2 \\ 0 \end{bmatrix} \right\}$$

is an orthogonal basis of U. However, it is sometimes more convenient to deal with vectors having integer entries, in which case we take

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} \right\}.$$

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4.9.E: Exercises for Section 4.9

Exercise $4.9.E.1$

Apply the Gram-Schmidt process to the following set of vectors in $\mathbb{R}^2$ to create an orthonormal basis:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Answer available at the end of this section.

Exercise $4.9.E.2$

Find an orthonormal basis for the span of each of the following sets of vectors.

a. $\begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ 1 \end{bmatrix}$

b. $\begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}, \begin{bmatrix} 11 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 7 \end{bmatrix}$

c. $\begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 10 \end{bmatrix}, \begin{bmatrix} -7 \\ 1 \\ 1 \end{bmatrix}$

Answer available at the end of this section.

Exercise $4.9.E.3$

Using the Gram Schmidt process find an orthonormal basis for the following span:

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Answer available at the end of this section.

Exercise $4.9.E.4$

Using the Gram Schmidt process find an orthonormal basis for the following span:

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Answer available at the end of this section.

Exercise $4.9.E.5$

Consider the following vectors:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} a \\ b \end{bmatrix} \tag{4.9.E.1}$$

a. Find an orthonormal basis.
b. What restriction did you have to make for your answer to part a?

Answer available at the end of this section.

Exercise $4.9.E.6$

The set $V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : 2x + 3y - z = 0 \right\}$ is a subspace of $\mathbb{R}^3$. Find an orthonormal basis for this subspace.

Answer available at the end of this section.

Exercise $4.9.E.7$

Use the Gram Schmidt process to transform the given basis for a subspace in $\mathbb{R}^n$ into an orthogonal basis for the subspace.

a. $\begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$

b. $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Answers

Exercise $4.9.E.1$

$$\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

Exercise $4.9.E.2$

a. $\begin{bmatrix} \frac{3}{5} \\ -\frac{4}{5} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{4}{5} \\ \frac{3}{5} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

b. $\begin{bmatrix} \frac{3}{5} \\ 0 \\ -\frac{4}{5} \end{bmatrix}, \begin{bmatrix} \frac{4}{5} \\ 0 \\ \frac{3}{5} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

c. $\begin{bmatrix} \frac{3}{5} \\ 0 \\ -\frac{4}{5} \end{bmatrix}, \begin{bmatrix} \frac{4}{5} \\ 0 \\ \frac{3}{5} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Exercise $4.9.E.3$

A solution is

$$\begin{bmatrix} \frac{1}{6}\sqrt{6} \\ \frac{1}{3}\sqrt{6} \\ \frac{1}{6}\sqrt{6} \end{bmatrix}, \begin{bmatrix} \frac{3}{10}\sqrt{2} \\ -\frac{2}{5}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{bmatrix}, \begin{bmatrix} \frac{7}{15}\sqrt{3} \\ -\frac{1}{15}\sqrt{3} \\ -\frac{1}{3}\sqrt{3} \end{bmatrix}$$

Exercise $4.9.E.4$

Then a solution is

$$\left[\begin{array}{c} \frac{1}{6}\sqrt{6} \\ \frac{1}{3}\sqrt{6} \\ \frac{1}{6}\sqrt{6} \\ 0 \end{array}\right], \left[\begin{array}{c} \frac{1}{6}\sqrt{2}\sqrt{3} \\ -\frac{2}{9}\sqrt{2}\sqrt{3} \\ \frac{5}{18}\sqrt{2}\sqrt{3} \\ \frac{1}{9}\sqrt{2}\sqrt{3} \end{array}\right], \left[\begin{array}{c} \frac{5}{111}\sqrt{3}\sqrt{37} \\ \frac{1}{133}\sqrt{3}\sqrt{37} \\ -\frac{17}{333}\sqrt{3}\sqrt{37} \\ \frac{22}{333}\sqrt{3}\sqrt{37} \end{array}\right]$$

Exercise $4.9.E.5$

a. $$\left\{\frac{1}{\sqrt{5}}\begin{bmatrix}1\\2\end{bmatrix}, \frac{1}{\sqrt{a^2+b^2-\frac{(a+2b)^2}{5}}}\begin{bmatrix}a-\frac{a+2b}{5}\\ b-\frac{2(a+2b)}{5}\end{bmatrix}\right\}$$

b. $b \neq 2a$.

Exercise $4.9.E.6$

The subspace is of the form

$$\left[\begin{array}{c} x \\ y \\ 2x+3y \end{array}\right]$$

and a basis is $\left[\begin{array}{c} 1 \\ 0 \\ 2 \end{array}\right], \left[\begin{array}{c} 0 \\ 1 \\ 3 \end{array}\right]$. Therefore, an orthonormal basis is

$$\left[\begin{array}{c} \frac{1}{5}\sqrt{5} \\ 0 \\ \frac{2}{5}\sqrt{5} \end{array}\right], \left[\begin{array}{c} -\frac{3}{35}\sqrt{5}\sqrt{14} \\ \frac{1}{14}\sqrt{5}\sqrt{14} \\ \frac{3}{70}\sqrt{5}\sqrt{14} \end{array}\right]$$

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4.10: Orthogonal Projections

Learning Objectives

- Find the orthogonal projection of a vector onto a subspace.
- Find the orthogonal complement of a subspace.

In this section, we compute orthogonal projections of vectors onto subspaces.

An important use of the Gram-Schmidt Process is in orthogonal projections, the focus of this section.

You may recall that a subspace of $\mathbb{R}^n$ is a set of vectors which contains the zero vector, and is closed under addition and scalar multiplication. Let's call such a subspace W. In particular, a plane in $\mathbb{R}^n$ which contains the origin, $(0, 0, \cdots, 0)$, is a subspace of $\mathbb{R}^n$.

Suppose a point Y in $\mathbb{R}^n$ is not contained in W, then what point Z in W is closest to Y? Using the Gram-Schmidt Process, we can find such a point. Let $\vec{y}, \vec{z}$ represent the position vectors of the points Y and Z respectively, with $\vec{y} - \vec{z}$ representing the vector connecting the two points Y and Z. It will follow that if Z is the point on W closest to Y, then $\vec{y} - \vec{z}$ will be perpendicular to W (can you see why?); in other words, $\vec{y} - \vec{z}$ is orthogonal to W (and to every vector contained in W) as in the following diagram.

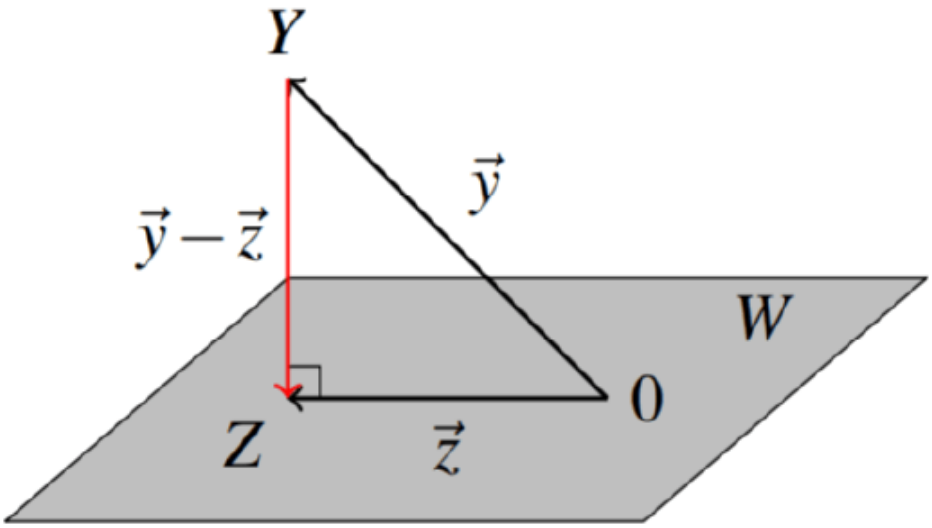


Figure 4.10.1 This diagram illustrates the concept of orthogonal projection of a vector onto a subspace. It shows a vector $\vec{y}$, a subspace W, and the orthogonal projection of $\vec{y}$ onto W, denoted as $\vec{z}$. The diagram helps visualize the relationship between the vector, the subspace, and the projection. (CC BY-NC-SA 4.0; Kuttler via A First Course in LINEAR ALGEBRA)

The vector $\vec{z}$ is called the **orthogonal projection** of $\vec{y}$ on W. The definition is given as follows.

Definition 4.10.1: Orthogonal Projection

Let W be a subspace of $\mathbb{R}^n$, and Y be any point in $\mathbb{R}^n$. Then the **orthogonal projection** of Y onto W is given by

$$\vec{z} = \text{proj}_W(\vec{y}) = \left(\frac{\vec{y} \cdot \vec{w}_1}{\|\vec{w}_1\|^2}\right) \vec{w}_1 + \left(\frac{\vec{y} \cdot \vec{w}_2}{\|\vec{w}_2\|^2}\right) \vec{w}_2 + \cdots + \left(\frac{\vec{y} \cdot \vec{w}_m}{\|\vec{w}_m\|^2}\right) \vec{w}_m$$

where $\{\vec{w}_1, \vec{w}_2, \cdots, \vec{w}_m\}$ is any orthogonal basis of W.

Therefore, in order to find the orthogonal projection, we must first find an orthogonal basis for the subspace. Note that one could use an orthonormal basis, but it is not necessary in this case since as you can see above the normalization of each vector is included in the formula for the projection.

Before we explore this further through an example, we show that the orthogonal projection does indeed yield a point Z (the point whose position vector is the vector $\vec{z}$ above) which is the point of W closest to Y.

Theorem 4.10.1: Approximation Theorem

Let W be a subspace of $\mathbb{R}^n$ and Y any point in $\mathbb{R}^n$. Let Z be the point whose position vector is the orthogonal projection of Y onto W.

Then, Z is the point in W closest to Y.

Proof available at the end of this section.

Consider the following example.

Example 4.10.1: Orthogonal Projection

Let W be the plane through the origin given by the equation $x - 2y + z = 0$.
Find the point in W closest to the point $Y = (1, 0, 3)$.

Solution

We must first find an orthogonal basis for W. Notice that W is characterized by all points (a, b, c) where $c = 2b - a$. In other words,

$$W = \begin{bmatrix} a \\ b \\ 2b - a \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \ a, b \in \mathbb{R}$$

We can thus write W as

$$\begin{aligned} W &= \text{span}\left\{\vec{u}_1, \vec{u}_2\right\} \\ &= \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}\right\} \end{aligned} \tag{4.10.1}$$

Notice that this span is a basis of W as it is linearly independent. We will use the Gram-Schmidt Process to convert this to an orthogonal basis, $\left\{\vec{w}_1, \vec{w}_2\right\}$. In this case, as we remarked it is only necessary to find an orthogonal basis, and it is not required that it be orthonormal.

$$\vec{w}_1 = \vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{aligned}\vec{w}_2 &= \vec{u}_2 - \left(\frac{\vec{u}_2 \cdot \vec{w}_1}{\|\vec{w}_1\|^2}\right)\vec{w}_1 \\ &= \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} - \left(\frac{-2}{2}\right)\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\end{aligned} \tag{4.10.2}$$

Therefore an orthogonal basis of W is

$$\{\vec{w}_1, \vec{w}_2\} = \left\{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$$

We can now use this basis to find the orthogonal projection of the point $Y = (1, 0, 3)$ on the subspace W. We will write the position vector $\vec{y}$ of Y as $\vec{y} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$. We compute the projection as follows:

$$\begin{aligned}\vec{z} &= \text{proj}_W(\vec{y}) \\ &= \left(\frac{\vec{y} \cdot \vec{w}_1}{\|\vec{w}_1\|^2}\right)\vec{w}_1 + \left(\frac{\vec{y} \cdot \vec{w}_2}{\|\vec{w}_2\|^2}\right)\vec{w}_2 \\ &= \left(\frac{-2}{2}\right)\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \left(\frac{4}{3}\right)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{3} \\ \frac{4}{3} \\ \frac{7}{3} \end{bmatrix}\end{aligned} \tag{4.10.3}$$

Therefore the point Z on W closest to the point $(1, 0, 3)$ is $(\frac{1}{3}, \frac{4}{3}, \frac{7}{3})$.

Recall that the vector $\vec{y} - \vec{z}$ is perpendicular (orthogonal) to all the vectors contained in the plane W. Using a basis for W, we can in fact find all such vectors which are perpendicular to W. We call this set of vectors the **orthogonal complement** of W and denote it $W^\perp$.

Definition 4.10.2: Orthogonal Complement

Let W be a subspace of $\mathbb{R}^n$. Then the **orthogonal complement** of W, written $W^\perp$, is the set of all vectors $\vec{x}$ such that $\vec{x} \cdot \vec{z} = 0$ for all vectors $\vec{z}$ in W.

$$W^\perp = \{\vec{x} \in \mathbb{R}^n \text{ such that } \vec{x} \cdot \vec{z} = 0 \text{ for all } \vec{z} \in W\}$$

The orthogonal complement is defined as the set of all vectors which are orthogonal to all vectors in the original subspace. It turns out that it is sufficient that the vectors in the orthogonal complement be orthogonal to a spanning set of the original space.

Proposition 4.10.1: Orthogonal to Spanning Set

Let W be a subspace of $\mathbb{R}^n$ such that $W = \text{span}\{\vec{w}_1, \vec{w}_2, \cdots, \vec{w}_m\}$. Then $W^\perp$ is the set of all vectors which are orthogonal to each $\vec{w}_i$ in the spanning set.

The following proposition demonstrates that the orthogonal complement of a subspace is itself a subspace.

Proposition 4.10.2: The Orthogonal Complement

Let W be a subspace of $\mathbb{R}^n$. Then the orthogonal complement $W^\perp$ is also a subspace of $\mathbb{R}^n$.

Consider the following proposition.

Proposition 4.10.3: Orthogonal Complement of $\mathbb{R}^n$

The complement of $\mathbb{R}^n$ is the set containing the zero vector:

$$(\mathbb{R}^n)^\perp = \left\{\vec{0}\right\}$$

Similarly,

$$\left\{\vec{0}\right\}^\perp = (\mathbb{R}^n).$$

Proof available at the end of this section.

In the next example, we will look at how to find $W^\perp$.

Example 4.10.2: Orthogonal Complement

Let W be the plane through the origin given by the equation $x - 2y + z = 0$. Find a basis for the orthogonal complement of W.

Solution

From Example 4.10.1 we know that we can write W as

$$W = \text{span}\left\{\vec{u}_1, \vec{u}_2\right\} = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}\right\}$$

In order to find $W^\perp$, we need to find all $\vec{x}$ which are orthogonal to every vector in this span.

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. In order to satisfy $\vec{x} \cdot \vec{u}_1 = 0$, the following equation must hold.

$$x_1 - x_3 = 0$$

In order to satisfy $\vec{x} \cdot \vec{u}_2 = 0$, the following equation must hold.

$$x_2 + 2x_3 = 0$$

Both of these equations must be satisfied, so we have the following system of equations.

$$\begin{aligned} x_1 - x_3 &= 0 \\ x_2 + 2x_3 &= 0 \end{aligned}$$

To solve, set up the augmented matrix.

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

Using Gaussian Elimination, we find that $W^\perp = \text{span}\left\{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right\}$, and hence $\left\{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right\}$ is a basis for $W^\perp$.

The following results summarize the important properties of the orthogonal projection.

Theorem 4.10.2: Orthogonal Projection

Let W be a subspace of $\mathbb{R}^n$, Y be any point in $\mathbb{R}^n$, and let Z be the point in W closest to Y. Then,

1. The position vector $\vec{z}$ of the point Z is given by $\vec{z} = \text{proj}_W(\vec{y})$
2. $\vec{z} \in W$ and $\vec{y} - \vec{z} \in W^\perp$
3. $|Y - Z| < |Y - Z_1|$ for all $Z_1 \neq Z \in W$

Consider the following example of this concept.

Example 4.10.3: Find a Vector Closest to a Given Vector

Let

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \vec{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \text{ and } \vec{v} = \begin{bmatrix} 4 \\ 3 \\ -2 \\ 5 \end{bmatrix}.$$

We want to find the vector in $W = \text{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ closest to $\vec{y}$.

Solution

We will first use the Gram-Schmidt Process to construct the orthogonal basis, B, of W:

$$B = \left\{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}\right\}.$$

By Theorem 4.10.2,

$$\text{proj}_U(\vec{v}) = \frac{2}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \frac{5}{1}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \frac{12}{6}\begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -1 \\ 5 \end{bmatrix}$$

is the vector in U closest to $\vec{y}$.

Consider the next example.

Example 4.10.4: Vector Written as a Sum of Two Vectors

Let W be a subspace given by $W = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}\right\}$, and $Y = (1, 2, 3, 4)$.

Find the point Z in W closest to Y, and moreover write $\vec{y}$ as the sum of a vector in W and a vector in $W^\perp$.

Solution

From Theorem 4.10.3 the point Z in W closest to Y is given by $\vec{z} = \text{proj}_W(\vec{y})$.

Notice that since the above vectors already give an orthogonal basis for W, we have:

$$\begin{aligned} \vec{z} &= \text{proj}_W(\vec{y}) \\ &= \left(\frac{\vec{y}\cdot\vec{w}_1}{\|\vec{w}_1\|^2}\right)\vec{w}_1 + \left(\frac{\vec{y}\cdot\vec{w}_2}{\|\vec{w}_2\|^2}\right)\vec{w}_2 \\ &= \left(\frac{4}{2}\right)\begin{bmatrix}1\\0\\1\\0\end{bmatrix} + \left(\frac{10}{5}\right)\begin{bmatrix}0\\1\\0\\2\end{bmatrix} \\ &= \begin{bmatrix}2\\2\\2\\4\end{bmatrix} \end{aligned} \tag{4.10.4}$$

Therefore the point in W closest to Y is $Z = (2,2,2,4)$.
Now, we need to write $\vec{y}$ as the sum of a vector in W and a vector in $W^\perp$. This can easily be done as follows:

$$\vec{y} = \vec{z} + (\vec{y} - \vec{z})$$

since $\vec{z}$ is in W and as we have seen $\vec{y} - \vec{z}$ is in $W^\perp$.
The vector $\vec{y} - \vec{z}$ is given by

$$\vec{y} - \vec{z} = \begin{bmatrix}1\\2\\3\\4\end{bmatrix} - \begin{bmatrix}2\\2\\2\\4\end{bmatrix} = \begin{bmatrix}-1\\0\\1\\0\end{bmatrix}$$

Therefore, we can write $\vec{y}$ as

$$\begin{bmatrix}1\\2\\3\\4\end{bmatrix} = \begin{bmatrix}2\\2\\2\\4\end{bmatrix} + \begin{bmatrix}-1\\0\\1\\0\end{bmatrix}$$

Example 4.10.5: Point in a Plane Closest to a Given Point

Find the point Z in the plane $3x + y - 2z = 0$ that is closest to the point $Y = (1,1,1)$.

Solution

The solution will proceed as follows.

1. Find a basis X of the subspace W of $\mathbb{R}^3$ defined by the equation $3x + y - 2z = 0$.
2. Orthogonalize the basis X to get an orthogonal basis B of W.
3. Find the projection on W of the position vector of the point Y.

We now begin the solution.

1. $3x + y - 2z = 0$ is a system of one equation in three variables. Putting the augmented matrix in reduced row-echelon form:

   $$\begin{bmatrix} 3 & 1 & -2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{3} & -\frac{2}{3} & 0 \end{bmatrix}$$

   gives general solution $x = \frac{1}{3}s + \frac{2}{3}t$, $y = s$, $z = t$ for any $s, t \in \mathbb{R}$. Then

$$W = \text{span}\left\{\begin{bmatrix} -\frac{1}{3} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{2}{3} \\ 0 \\ 1 \end{bmatrix}\right\}$$

Let $X = \left\{\begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}\right\}$. Then X is linearly independent and $\text{span}(X) = W$, so X is a basis of W.

2. Use the Gram-Schmidt Process to get an orthogonal basis of W:

$$\vec{f}_1 = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} \text{ and } \vec{f}_2 = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} - \frac{-2}{10}\begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 9 \\ 3 \\ 15 \end{bmatrix}.$$

Therefore $B = \left\{\begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}\right\}$ is an orthogonal basis of W.

3. To find the point Z on W closest to $Y = (1, 1, 1)$, compute

$$\begin{aligned} \text{proj}_W \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} &= \frac{2}{10}\begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} + \frac{9}{35}\begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} \\ &= \frac{1}{7}\begin{bmatrix} 4 \\ 6 \\ 9 \end{bmatrix}. \end{aligned} \tag{4.10.5}$$

Therefore, $Z = \left(\frac{4}{7}, \frac{6}{7}, \frac{9}{7}\right)$.

Answers

Theorem 4.10.1**: Approximation Theorem**

First Z is certainly a point in W since it is in the span of a basis of W.

To show that Z is the point in W closest to Y, we wish to show that $|\vec{y} - \vec{z}_1| > |\vec{y} - \vec{z}|$ for all $\vec{z}_1 \neq \vec{z} \in W$. We begin by writing $\vec{y} - \vec{z}_1 = (\vec{y} - \vec{z}) + (\vec{z} - \vec{z}_1)$. Now, the vector $\vec{y} - \vec{z}$ is orthogonal to W, and $\vec{z} - \vec{z}_1$ is contained in W. Therefore these vectors are orthogonal to each other. By the Pythagorean Theorem, we have that

$$\|\vec{y} - \vec{z}_1\|^2 = \|\vec{y} - \vec{z}\|^2 + \|\vec{z} - \vec{z}_1\|^2 > \|\vec{y} - \vec{z}\|^2$$

This follows because $\vec{z} \neq \vec{z}_1$ so $\|\vec{z} - \vec{z}_1\|^2 > 0$.

Hence, $\|\vec{y} - \vec{z}_1\|^2 > \|\vec{y} - \vec{z}\|^2$. Taking the square root of each side, we obtain the desired result.

Proposition 4.10.3**: Orthogonal Complement of** $\mathbb{R}^n$

Here, $\vec{0}$ is the zero vector of $\mathbb{R}^n$. Since $\vec{x} \cdot \vec{0} = 0$ for all $\vec{x} \in \mathbb{R}^n$, $\mathbb{R}^n \subseteq \{\vec{0}\}^\perp$. Since $\{\vec{0}\}^\perp \subseteq \mathbb{R}^n$, the equality follows, i.e., $\{\vec{0}\}^\perp = \mathbb{R}^n$.

Again, since $\vec{x} \cdot \vec{0} = 0$ for all $\vec{x} \in \mathbb{R}^n$, $\vec{0} \in (\mathbb{R}^n)^\perp$, so $\{\vec{0}\} \subseteq (\mathbb{R}^n)^\perp$. Suppose $\vec{x} \in \mathbb{R}^n$, $\vec{x} \neq \vec{0}$. Since $\vec{x} \cdot \vec{x} = ||\vec{x}||^2$ and $\vec{x} \neq \vec{0}$, $\vec{x} \cdot \vec{x} \neq 0$, so $\vec{x} \notin (\mathbb{R}^n)^\perp$. Therefore $(\mathbb{R}^n)^\perp \subseteq \{\vec{0}\}$, and thus $(\mathbb{R}^n)^\perp = \{\vec{0}\}$.

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4.10.E: Exercise for Section 4.10

Exercise 4.10.E.1

Let the subspace W of $\mathbb{R}^3$ be spanned by the vectors:

$$\vec{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Find the orthogonal projection of the vector:

$$\vec{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

onto the subspace W.

Answer available at the end of this section.

Exercise 4.10.E.2

Let W be the subspace of $\mathbb{R}^3$ given by the solution space of the equation:

$$x_1 + 2x_2 + 3x_3 = 0$$

Find a basis for $W^\perp$, the orthogonal complement of W.

Answer available at the end of this section.

Exercise 4.10.E.3

Let W be the subspace of $\mathbb{R}^4$ spanned by:

$$\vec{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Find a basis for $W^\perp$, the orthogonal complement.

Exercise 4.10.E.4

Let W be the subspace of $\mathbb{R}^4$ spanned by the vectors:

$$\vec{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

Find a basis for the orthogonal complement $W^\perp$.

Answer available at the end of this section.

Exercise $4.10.E.5$

Consider the following scalar equation of a plane.

$$2x - 3y + z = 0$$

Find the point on the plane which is closest to $Y = (3, 4, 1)$ using the steps outlined in Example 4.10.5.

a. Find a basis X of the subspace W of $\mathbb{R}^3$ defined by the equation $2x - 3y + z = 0$.
b. Orthogonalize the basis X to get an orthogonal basis B of W.
c. Find the projection on W of the position vector of the point Y.

Exercise $4.10.E.6$

Consider the following scalar equation of a plane.

$$x + 3y + z = 0$$

Find the point on the plane which is closest to $Y = (1, 2, 1)$.

Exercise $4.10.E.7$

Let $\vec{v}$ be a vector and let $\vec{n}$ be a normal vector for a plane through the origin. Find the equation of the line through the point determined by $\vec{v}$ which has direction vector $\vec{n}$. Show that it intersects the plane at the point determined by $\vec{v} - proj_{\vec{n}}\vec{v}$.

Hint available at the end of this section.

Exercise $4.10.E.8$

As shown in the above problem, one can find the closest point to $\vec{v}$ in a plane through the origin by finding the intersection of the line through $\vec{v}$ having direction vector equal to the normal vector to the plane with the plane. If the plane does not pass through the origin, this will still work to find the point on the plane closest to the point determined by $\vec{v}$. Here is a relation which defines a plane

$$2x + y + z = 11$$

and here is a point: $(1, 1, 2)$. Find the point on the plane which is closest to this point. Then determine the distance from the point to the plane by taking the distance between these two points.

Hint: available at the end of this section.

Exercise $4.10.E.9$

As shown in the above problem, one can find the closest point to~v in a plane through the origin by finding the intersection of the line through $\vec{v}$ having direction vector equal to the normal vector to the plane with the plane. If the plane does not pass through the origin, this will still work to find the point on the plane closest to the point determined by $\vec{v}$. Here is a relation which defines a plane

$$2x + y + z = 11$$

and here is a point: $(1, 1, 2)$. Find the point on the plane which is closest to this point. Then determine the distance from the point to the plane by taking the distance between these two points.

Hint: available at the end of this section.

Exercise $4.10.E.10$

Consider the plane

$$3x + 4y - 5z = 7$$

a. Find the point on this plane that is closest to the point $(2, -1, 3)$.
b. Compute the distance between $(2, -1, 3)$ and the plane.

Exercise $4.10.E.11$

Provide a proof for proposition 4.10.2: Let W be a subspace of $\mathbb{R}^n$. Then the orthogonal complement $W^\perp$ is also a subspace of $\mathbb{R}^n$.

Answers

Exercise $4.10.E.1$

$$\text{proj}_W \vec{v} = \begin{bmatrix} \frac{3}{2} \\ \frac{5}{2} \\ 1 \end{bmatrix}$$

Exercise $4.10.E.2$

$$W^\perp = \text{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix} \right\}$$

Exercise $4.10.E.4$

$$W^\perp = \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Exercise $4.10.E.7$

\[The line:$\vec{v} + t\vec{n}$. It is in the plane if $\vec{n} \bullet (\vec{v} + t\vec{n}) = 0$. Determine t. Then substitute in to the equation of the line.

Exercise $4.10.E.8$

Line: $(x, y, z) = (1, 1, 2) + t(2, 1, 1)$. Now require that it intersect the plane.

Exercise $4.10.E.9$

Line: $(x, y, z) = (1, 1, 2) + t(2, 1, 1)$. Now require that it intersect the plane.

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4.11: Least Squares Approximation

Learning Objectives

- Find the least squares solution for an inconsistent system.
- Apply the least squares approximate to find the regression line given a collection of points.

It should not be surprising to hear that many problems do not have a perfect solution, and in these cases the objective is always to try to do the best possible. For example what does one do if there are no solutions to a system of linear equations $A\vec{x} = \vec{b}$? It turns out that what we do is find $\vec{x}$ such that $A\vec{x}$ is as close to $\vec{b}$ as possible. A very important technique that follows from orthogonal projections is that of the least square approximation, and allows us to do exactly that.

We begin with a lemma.

Recall that we can form the image of an $m \times n$ matrix A by $\mathrm{im}\,(A) = \left\{A\vec{x} : \vec{x} \in \mathbb{R}^n\right\}$. Rephrasing Theorem $4.11.4$ using the subspace $W = \mathrm{im}\,(A)$ gives the equivalence of an orthogonality condition with a minimization condition. The following picture illustrates this orthogonality condition and geometric meaning of this theorem.

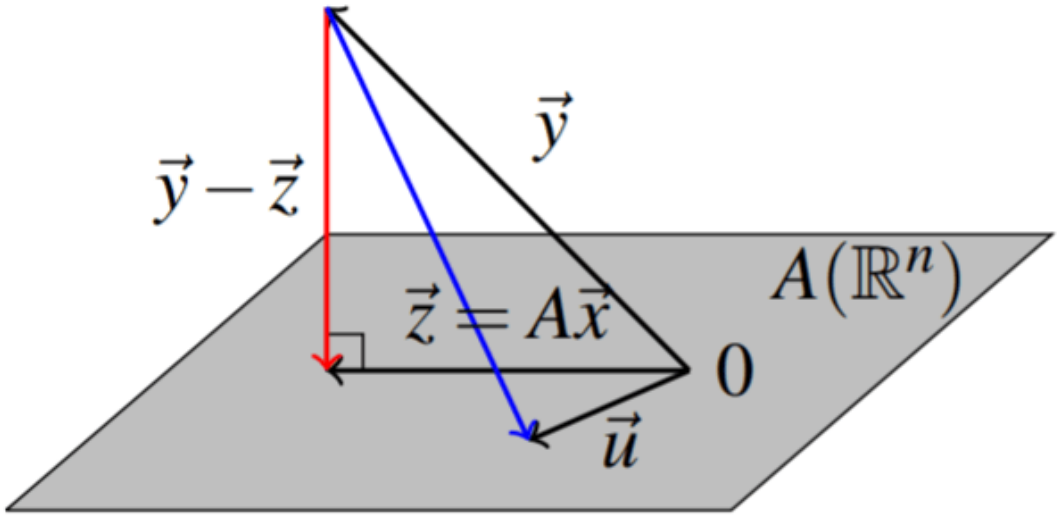


Figure $4.11.2$ A geometric representation of least squares regression. A gray plane represents the column space $A(\mathbb{R}^n)$ of matrix A, containing the origin 0. A black vector $\vec{y}$ extends from the origin to a point above the plane. The projection $\vec{z} = A\vec{x}$ is shown as a blue vector from the origin to the plane. A red vector $\vec{y} - \vec{z}$ is perpendicular to the plane, representing the residual. Another black vector $\vec{u}$ lies in the plane, completing the triangle. (CC BY-NC-SA 4.0; Kuttler via A First Course in LINEAR ALGEBRA)

Theorem $4.11.1$: Existence of Minimizers

Let $\vec{y} \in \mathbb{R}^m$ and let A be an $m \times n$ matrix.

Choose $\vec{z} \in W = \mathrm{im}\,(A)$ given by $\vec{z} = \mathrm{proj}_W(\vec{y})$, and let $\vec{x} \in \mathbb{R}^n$ such that $\vec{z} = A\vec{x}$.

Then

1. $\vec{y} - A\vec{x} \in W^\perp$
2. $\|\vec{y} - A\vec{x}\| < \|\vec{y} - \vec{u}\|$ for all $\vec{u} \neq \vec{z} \in W$

We note a simple but useful observation.

Lemma 4.11.1: Transpose and Dot Product

Let A be an $m \times n$ matrix. Then

$$A\vec{x} \cdot \vec{y} = \vec{x} \cdot A^T\vec{y}$$

Proof available at the end of this section.

The next corollary gives the technique of least squares.

Corollary 4.11.1: Least Squares and Normal Equation

A specific value of $\vec{x}$ which solves the problem of Theorem 4.11.5 is obtained by solving the equation

$$A^TA\vec{x} = A^T\vec{y}$$

Furthermore, there always exists a solution to this system of equations.

Proof available at the end of this section.

Note that $\vec{x}$ might not be unique but $A\vec{x}$, the closest point of $A(\mathbb{R}^n)$ to $\vec{y}$ is unique as was shown in the above argument.

Consider the following example.

Example 4.11.1: Least Squares Solution to a System

Find a least squares solution to the system

$$\begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

Solution

First, consider whether there exists a real solution. To do so, set up the augmnented matrix given by

$$\begin{bmatrix} 2 & 1 & 2 \\ -1 & 3 & 1 \\ 4 & 5 & 1 \end{bmatrix}$$

The reduced row-echelon form of this augmented matrix is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It follows that there is no real solution to this system. Therefore we wish to find the least squares solution. The normal equations are

$$A^TA\vec{x} = A^T\vec{y}$$

$$\begin{bmatrix} 2 & -1 & 4 \\ 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -1 & 4 \\ 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \tag{4.11.1}$$

and so we need to solve the system

$$\begin{bmatrix} 21 & 19 \\ 19 & 35 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \end{bmatrix}$$

This is a familiar exercise and the solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{5}{34} \\ \frac{7}{34} \end{bmatrix}$$

Consider another example.

Example 4.11.2: Least Squares Solution to a System

Find a least squares solution to the system

$$\begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 9 \end{bmatrix}$$

Solution

First, consider whether there exists a real solution. To do so, set up the augmnented matrix given by

$$\begin{bmatrix} 2 & 1 & 3 \\ -1 & 3 & 2 \\ 4 & 5 & 9 \end{bmatrix}$$

The reduced row-echelon form of this augmented matrix is

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

It follows that the system has a solution given by $x = y = 1$. However we can also use the normal equations and find the least squares solution.

$$\begin{bmatrix} 2 & -1 & 4 \\ 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -1 & 4 \\ 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 9 \end{bmatrix}$$

Then

$$\begin{bmatrix} 21 & 19 \\ 19 & 35 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 40 \\ 54 \end{bmatrix}$$

The least squares solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

which is the same as the solution found above.

An important application of Corollary 4.11.1 is the problem of finding the least squares regression line in statistics. Suppose you are given points in the xy plane

$$\{(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)\}$$

and you would like to find constants m and b such that the line $\vec{y} = m\vec{x} + b$ goes through all these points. Of course this will be impossible in general. Therefore, we try to find m, b such that the line will be as close as possible. The desired system is

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix}$$

which is of the form $\vec{y} = A\vec{x}$. It is desired to choose m and b to make

$$\left\| A \begin{bmatrix} m \\ b \end{bmatrix} - \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right\|^2$$

as small as possible. According to Theorem $4.11.5$ and Corollary $4.11.1$, the best values for m and b occur as the solution to

$$A^T A \begin{bmatrix} m \\ b \end{bmatrix} = A^T \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, \quad \text{where } A = \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix}$$

Thus, computing $A^T A$,

$$\begin{bmatrix} \sum_{i=1}^n x_i^2 & \sum_{i=1}^n x_i \\ \sum_{i=1}^n x_i & n \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n x_i y_i \\ \sum_{i=1}^n y_i \end{bmatrix}$$

Solving this system of equations for m and b (using Cramer's rule for example) yields:

$$m = \frac{-\left(\sum_{i=1}^n x_i\right)\left(\sum_{i=1}^n y_i\right) + \left(\sum_{i=1}^n x_i y_i\right) n}{\left(\sum_{i=1}^n x_i^2\right) n - \left(\sum_{i=1}^n x_i\right)^2}$$

and

$$b = \frac{-\left(\sum_{i=1}^n x_i\right) \sum_{i=1}^n x_i y_i + \left(\sum_{i=1}^n y_i\right) \sum_{i=1}^n x_i^2}{\left(\sum_{i=1}^n x_i^2\right) n - \left(\sum_{i=1}^n x_i\right)^2}.$$

Consider the following example.

Example $4.11.3$: Least Squares Regression

Find the least squares regression line $y = mx + b$ for the following set of data points:

$$\{(0,1),(1,2),(2,2),(3,4),(4,5)\}$$

Solution

In this case we have $n = 5$ data points and we obtain:

$$\sum_{i=1}^5 x_i = 10 \qquad \sum_{i=1}^5 y_i = 14$$

$$\sum_{i=1}^5 x_i y_i = 38 \quad \sum_{i=1}^5 x_i^2 = 30$$

and hence

$$\begin{aligned} m &= \frac{-10 * 14 + 5 * 38}{5 * 30 - 10^2} = 1.00 \\ b &= \frac{-10 * 38 + 14 * 30}{5 * 30 - 10^2} = 0.80 \end{aligned} \tag{4.11.2}$$

The least squares regression line for the set of data points is:

$$y = x + .8$$

One could use this line to approximate other values for the data. For example for $x = 6$ one could use $y(6) = 6 + .8 = 6.8$ as an approximate value for the data.

The following diagram shows the data points and the corresponding regression line.

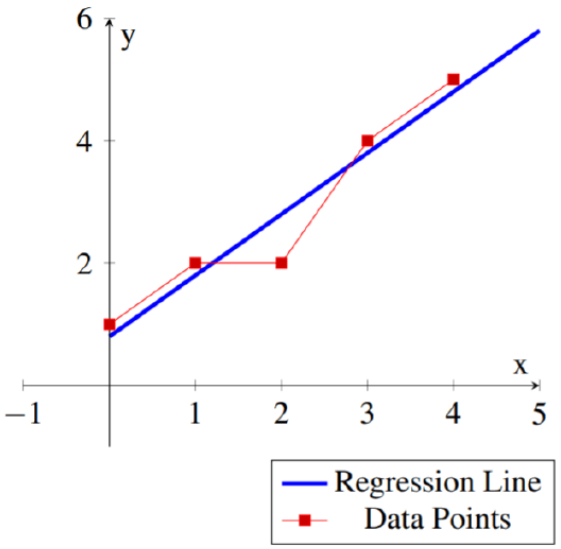


Figure 4.11.3 The diagram shows the data points $(0,1),(1,2),(2,2),(3,4),(4,5)$ and the corresponding regression line $y = x + .8$. (CC BY-NC-SA 4.0; Kuttler via A First Course in LINEAR ALGEBRA)

One could clearly do a least squares fit for curves of the form $y = ax^2 + bx + c$ in the same way. In this case you want to solve as well as possible for a, b, and c the system

$$\begin{bmatrix} x_1^2 & x_1 & 1 \\ \vdots & \vdots & \vdots \\ x_n^2 & x_n & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

and one would use the same technique as above. Many other similar problems are important, including many in higher dimensions and they are all solved the same way.

Answers

Lemma 4.11.1: Transpose and Dot Product

This follows from the definitions:

$$A\vec{x} \cdot \vec{y} = \sum_{i,j} a_{ij} x_j y_i = \sum_{i,j} x_j a_{ji} y_i = \vec{x} \cdot A^T \vec{y}$$

Corollary 4.11.1: Least Squares and Normal Equation

For $\vec{x}$ the minimizer of Theorem 4.11.5, $(\vec{y} - A\vec{x}) \cdot A\vec{u} = 0$ for all $\vec{u} \in \mathbb{R}^n$ and from Lemma 4.11.1, this is the same as saying

$$A^T (\vec{y} - A\vec{x}) \cdot \vec{u} = 0$$

for all $u \in \mathbb{R}^n$. This implies

$$A^T \vec{y} - A^T A\vec{x} = \vec{0}.$$

Therefore, there is a solution to the equation of this corollary, and it solves the minimization problem of Theorem 4.11.5.

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4.11.E: Exercises for Section 4.11

Exercise 4.11.E.1

Find the least squares solution to the following system.

$$\begin{aligned} x + 2y &= 1 \\ 2x + 3y &= 2 \\ 3x + 5y &= 4 \end{aligned} \tag{4.11.E.1}$$

Answer available at the end of this section.

Exercise 4.11.E.2

You are doing experiments and have obtained the ordered pairs,

$$(0,1), (1,2), (2,3.5), (3,4)$$

Find m and b such that $y = mx + b$ approximates these four points as well as possible.

Exercise 4.11.E.3

Given the ordered pairs $(2,1)$, $(1,2)$, $(0,1)$,$(-1,2)$, and $(-2,1)$, find m and b such that $y = mx + b$ approximates these five points as well as possible.

Answer available at the end of this section.

Exercise 4.11.E.4

Suppose you have several ordered triples, (x_i, y_i, z_i). Describe how to find a polynomial such as

$$z = a + bx + cy + dxy + ex^2 + fy^2$$

giving the best fit to the given ordered triples.

Exercise 4.11.E.5

Find the least squares solution for the system

$$\begin{aligned} x + y + z &= 2 \\ 2x + 3y + 4z &= 5 \\ 3x + 5y + 7z &= 8 \\ 4x + 7y + 10z &= 11 \end{aligned} \tag{4.11.E.2}$$

Exercise 4.11.E.6

In general, you have a point (x_0, y_0, z_0) and a scalar equation for a plane $ax + by + cz = d$ where $a^2 + b^2 + c^2 > 0$. Determine a formula for the closest point on the plane to the given point. Then use this point to get a formula for the distance from the given point to the plane.

Answer available at the end of this section.

Answers

Exercise $4.11.E.1$

$$\begin{aligned}\begin{bmatrix}1&2\\2&3\\3&5\end{bmatrix}^T\begin{bmatrix}1&2\\2&3\\3&5\end{bmatrix}&=\begin{bmatrix}14&23\\23&38\end{bmatrix}\begin{bmatrix}14&23\\23&38\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}\\&=\begin{bmatrix}1&2\\2&3\\3&5\end{bmatrix}^T\begin{bmatrix}1\\2\\4\end{bmatrix}=\begin{bmatrix}17\\28\end{bmatrix}\end{aligned}\tag{4.11.E.3}$$

$$\begin{aligned}\begin{bmatrix}14&23\\23&38\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}&=\begin{bmatrix}17\\28\end{bmatrix}\\\begin{bmatrix}14&23\\23&38\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}&=\begin{bmatrix}17\\28\end{bmatrix},\end{aligned}\tag{4.11.E.4}$$

Solution is: $\begin{bmatrix}\frac{2}{3}\\\frac{1}{3}\end{bmatrix}$

Exercise $4.11.E.3$

$y=1.4$

Exercise $4.11.E.6$

Find the line perpendicular to the plane which goes through the given point: $(x,y,z)=(x_0,y_0,z_0)+t(a,b,c)$. Now require that this point satisfy the equation for the plane to determine t.

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5: LINEAR TRANSFORMATIONS

5.1: Linear Transformations

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5.1: Linear Transformations

Learning Objectives

- Understand the definition of a linear transformation
- Understand how linear transformations are determined by matrix multiplication.

Recall that when we multiply an $m \times n$ matrix by an $n \times 1$ column vector, the result is an $m \times 1$ column vector. In this section we will discuss how, through matrix multiplication, an $m \times n$ matrix **transforms** an $n \times 1$ column vector into an $m \times 1$ column vector.

Recall that the $n \times 1$ vector given by

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

is said to belong to $\mathbb{R}^n$, which is the set of all $n \times 1$ vectors. In this section, we will discuss transformations of vectors in $\mathbb{R}^n$.

Consider the following example.

Example 5.1.1: A Function Which Transforms Vectors

Consider the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix}$. Show that by matrix multiplication A transforms vectors in $\mathbb{R}^3$ into vectors in $\mathbb{R}^2$.

Solution

First, recall that vectors in $\mathbb{R}^3$ are vectors of size 3×1, while vectors in $\mathbb{R}^2$ are of size 2×1. If we multiply A, which is a 2×3 matrix, by a 3×1 vector, the result will be a 2×1 vector. This what we mean when we say that A *transforms* vectors.

Now, for $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $\mathbb{R}^3$, multiply on the left by the given matrix to obtain the new vector. This product looks like

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 2y \\ 2x + y \end{bmatrix}$$

The resulting product is a 2×1 vector which is determined by the choice of x and y. Here are some numerical examples.

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

Here, the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in $\mathbb{R}^3$ was transformed by the matrix into the vector $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$ in $\mathbb{R}^2$.

Here is another example:

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 10 \\ 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 20 \\ 25 \end{bmatrix}$$

Watch the accompanying video for a visual demonstration of the this concept.

- **Video Length**: 6 minutes 47 seconds.
- **Context**: This video demonstrates how to transform vectors by matrix multiplication for matrices $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 4 \end{bmatrix}$.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

The notation $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ means that the function T transforms vectors in $\mathbb{R}^n$ into vectors in $\mathbb{R}^m$. The notation $T(\vec{x})$ means the transformation T applied to the vector $\vec{x}$. The above example demonstrated a transformation achieved by matrix multiplication. In this case, we often write

$$T_A(\vec{x}) = A\vec{x}$$

Therefore, T_A is the transformation determined by the matrix A. In this case we say that T is a matrix transformation.

Recall the property of matrix multiplication that states that for k and p scalars,

$$A(kB + pC) = kAB + pAC$$

In particular, for A an $m \times n$ matrix and B and C, $n \times 1$ vectors in $\mathbb{R}^n$, this formula holds.

In other words, this means that matrix multiplication gives an example of a linear transformation, which we will now define.

Definition 5.1.1: Linear Transformation

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a function, where for each $\vec{x} \in \mathbb{R}^n, T(\vec{x}) \in \mathbb{R}^m$. Then T is a **linear transformation** if whenever k, p are scalars and $\vec{x}_1$ and $\vec{x}_2$ are vectors in $\mathbb{R}^n$ ($n \times 1$ vectors),

$$T(k\vec{x}_1 + p\vec{x}_2) = kT(\vec{x}_1) + pT(\vec{x}_2)$$

Note

Sometimes, it's helpful to think of the condition required for a transformation $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ to be a linear transformation as two separate conditions:

1. **Linearity**: For vectors $\vec{x}_1$ and $\vec{x}_2$ in $\mathbb{R}^n$,
$$T(\vec{x}_1 + \vec{x}_2) = T(\vec{x}_1) + T(\vec{x}_2)$$
2. **Homogeneity**: For scalar k in $\mathbb{R}$ and $\vec{x}$ in $\mathbb{R}^n$,
$$T(k\vec{x}) = kT(\vec{x})$$

Consider the following example.

Example 5.1.2: Linear Transformation

Let T be a transformation defined by $T : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y \\ x-z \end{bmatrix} \text{ for all } \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$$

Show that T is a linear transformation.

Solution

By Definition 5.1.1 we need to show that $T\left(k\vec{x}_1 + p\vec{x}_2\right) = kT\left(\vec{x}_1\right) + pT\left(\vec{x}_2\right)$ for all scalars k, p and vectors $\vec{x}_1, \vec{x}_2$. Let

$$\vec{x}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$$

Then

$$\begin{aligned} T\left(k\vec{x}_1 + p\vec{x}_2\right) &= T\left(k\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + p\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) \\ &= T\left(\begin{bmatrix} kx_1 \\ ky_1 \\ kz_1 \end{bmatrix} + \begin{bmatrix} px_2 \\ py_2 \\ pz_2 \end{bmatrix}\right) \\ &= T\left(\begin{bmatrix} kx_1 + px_2 \\ ky_1 + py_2 \\ kz_1 + pz_2 \end{bmatrix}\right) \\ &= \begin{bmatrix} (kx_1 + px_2) + (ky_1 + py_2) \\ (kx_1 + px_2) - (kz_1 + pz_2) \end{bmatrix} \\ &= \begin{bmatrix} (kx_1 + ky_1) + (px_2 + py_2) \\ (kx_1 - kz_1) + (px_2 - pz_2) \end{bmatrix} \\ &= \begin{bmatrix} kx_1 + ky_1 \\ kx_1 - kz_1 \end{bmatrix} + \begin{bmatrix} px_2 + py_2 \\ px_2 - pz_2 \end{bmatrix} \\ &= k\begin{bmatrix} x_1 + y_1 \\ x_1 - z_1 \end{bmatrix} + p\begin{bmatrix} x_2 + y_2 \\ x_2 - z_2 \end{bmatrix} \\ &= kT(\vec{x}_1) + pT(\vec{x}_2) \end{aligned}$$

Therefore T is a linear transformation.

Two important examples of linear transformations are the zero transformation and identity transformation. The zero transformation defined by $T\left(\vec{x}\right) = \vec{0}$ for all $\vec{x}$ is an example of a linear transformation. Similarly the identity transformation defined by $T\left(\vec{x}\right) = \vec{x}$ is also linear. Take the time to prove these using the method demonstrated in Example 5.1.2.

We began this section by discussing matrix transformations, where multiplication by a matrix transforms vectors. These matrix transformations are in fact linear transformations.

Theorem 5.1.1: Matrix Transformations are Linear Transformations

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a transformation defined by $T(\vec{x}) = A\vec{x}$. Then T is a linear transformation.

It turns out that every linear transformation can be expressed as a matrix transformation, and thus linear transformations are exactly the same as matrix transformations.

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5.1E: Exercises for Section 5.1

Exercise $5.1E.1$

Show the map T: $\mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\vec{x}) = A\vec{x}$ where A is an $m \times n$ matrix and $\vec{x}$ is an $m \times 1$ column vector is a linear transformation.

Answer available at the end of this section.

Exercise $5.1E.2$

Show that the function $T_{\vec{u}}$ defined by $T_{\vec{u}}(\vec{v}) = \vec{v} - proj_{\vec{u}}(\vec{v})$ is also a linear transformation.

Answer available at the end of this section.

Exercise $5.1E.3$

Let $\vec{u}$ be a fixed vector. The function $T_{\vec{u}}$ defined by $T_{\vec{u}}\vec{v} = \vec{u} + \vec{v}$ has the effect of translating all vectors by adding $\vec{u} \neq \vec{0}$. Show this is not a linear transformation. Explain why it is not possible to represent $T_{\vec{u}}$ in $\mathbb{R}^3$ by multiplying by a 3×3 matrix.

Answer available at the end of this section.

Exercise $5.1E.4$

Consider the following functions $T : \mathbb{R}^3 \to \mathbb{R}^2$. Explain why each of these functions T is not linear.

a. $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + 2y + 3z + 1 \\ 2y - 3x + z \end{bmatrix}$

b. $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + 2y^2 + 3z \\ 2y + 3x + z \end{bmatrix}$

c. $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} \sin x + 2y + 3z \\ 2y + 3x + z \end{bmatrix}$

d. $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + 2y + 3z \\ 2y + 3x - \ln z \end{bmatrix}$

Exercise $5.1E.5$

Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation and $T\left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$, $T\left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, $T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$. Find $T\left(\begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}\right)$.

Exercise $5.1E.6$

Determine whether the following functions are linear transformations:

a. $T : \mathbb{R}^2 \to \mathbb{R}^2$, where $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -x+2 \\ y-3 \end{bmatrix}$

b. $T : \mathbb{R}^2 \to \mathbb{R}^2$, where $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} xy \\ x-y \end{bmatrix}$

c. $T : \mathbb{R}^2 \to \mathbb{R}^3$, where $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x-y \\ 3x+4y \\ 5x \end{bmatrix}$

d. Justify your answers.

Answer available at the end of this section.

Exercise $5.1E.7$

Suppose that for a function$T : \mathbb{R}^n \to \mathbb{R}^m$, we have that $T(\vec{0}_n) = \vec{0}_m$, where $\vec{0}_n$ is the zero vector in $\mathbb{R}^n$ and $\vec{0}_m$ is the zero vector in $\mathbb{R}^m$. Must T be a linear transformation?

Exercise $5.1E.8$

Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and k is a scalar. Show that the function $W : \mathbb{R}^n \to \mathbb{R}^m$ defined by $W(\vec{x}) = kT(\vec{x})$ is a linear transformation.

Answers

Exercise $5.1E.1$

This result follows from the properties of matrix multiplication.

Exercise $5.1E.2$

$$\begin{aligned} T_{\vec{u}}(a\vec{v}+b\vec{w}) &= a\vec{v}+b\vec{w}-\frac{(a\vec{v}+b\vec{w}\bullet\vec{u})}{||\vec{u}||^2}\vec{u} \\ &= a\vec{v}-a\frac{(\vec{v}\bullet\vec{u})}{||\vec{u}||^2}\vec{u}+b\vec{w}-b\frac{(\vec{w}\bullet\vec{u})}{||\vec{u}||^2}\vec{u} \\ &= aT_{\vec{u}}(\vec{v})+bT_{\vec{u}}(\vec{w}) \end{aligned} \tag{5.1E.1}$$

Exercise $5.1E.3$

Linear transformations take $\vec{0}$ to $\vec{0}$ which T does not. Also $T_{\vec{a}}(\vec{u}+\vec{v}) \neq T_{\vec{a}}\vec{u}+T_{\vec{a}}\vec{v}$.

Exercise $5.1E.6$

a. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -x+2 \\ y-3 \end{bmatrix}$ is not a linear transformation because adding the constant terms 2 and -3 violates homogeneity (i.e., $T(0,0) \neq 0$).

b. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} xy \\ x-y \end{bmatrix}$ is \textbf{not} a linear transformation because it involves a product of inputs (xy), which violates both additivity and homogeneity.

c. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x-y \\ 3x+4y \\ 5x \end{bmatrix}$ is a linear transformation. Show that it satisfies linearity and homogeneity.

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5.2: The Matrix of a Linear Transformation I

Learning Objectives

- Find the matrix of a linear transformation with respect to the standard basis.
- Determine the action of a linear transformation on a vector in $\mathbb{R}^n$.

In the above examples, the action of the linear transformations was to multiply by a matrix. It turns out that this is always the case for linear transformations. If T is *any* linear transformation which maps $\mathbb{R}^n$ to $\mathbb{R}^m$, there is *always* an $m \times n$ matrix A with the property that

$$T\left(\vec{x}\right) = A\vec{x} \tag{5.2.1}$$

for all $\vec{x} \in \mathbb{R}^n$.

Theorem 5.2.1: Matrix of a Linear Transformation

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation. Then we can find a matrix A such that $T(\vec{x}) = A\vec{x}$. In this case, we say that T is **determined** or **induced** by the matrix A.

Here is why. Suppose $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ is a linear transformation and you want to find the matrix defined by this linear transformation as described in equation (5.2.1). Note that

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = \sum_{i=1}^{n} x_i \vec{e}_i$$

where $\vec{e}_i$ is the ith column of I_n, that is the $n \times 1$ vector which has zeros in every slot but the ith and a 1 in this slot.

Then since T is linear,

$$\begin{aligned} T\left(\vec{x}\right) &= \sum_{i=1}^{n} x_i T\left(\vec{e}_i\right) \\ &= \begin{bmatrix} | & & | \\ T\left(\vec{e}_1\right) & \cdots & T\left(\vec{e}_n\right) \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\ &= A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \end{aligned} \tag{5.2.2}$$

The desired matrix is obtained from constructing the ith column as $T\left(\vec{e}_i\right)$. Recall that the set $\left\{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n\right\}$ is called the standard basis of $\mathbb{R}^n$. Therefore the matrix of T is found by applying T to the standard basis. We state this formally as the following theorem.

Theorem 5.2.2: Matrix of a Linear Transformation

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation. Then the matrix A satisfying $T\left(\vec{x}\right) = A\vec{x}$ is given by

$$A = \begin{bmatrix} | & & | \\ T\left(\vec{e}_1\right) & \cdots & T\left(\vec{e}_n\right) \\ | & & | \end{bmatrix}$$

where $\vec{e}_i$ is the ith column of I_n, and then $T\left(\vec{e}_i\right)$ is the ith column of A.

The following Corollary is an essential result.

Corollary 5.2.1: Matrix and Linear Transformation

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if and only if it is a matrix transformation.

Consider the following example of how to find the matrix induced by a linear transformation.

Example 5.2.1:The Matrix of a Linear Transformation

Suppose T is a linear transformation, $T : \mathbb{R}^3 \to \mathbb{R}^2$ where

$$T\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}, \; T\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}9\\-3\end{bmatrix}, \; T\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$$

Find the matrix A of T such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x}$.

Solution

By Theorem 5.2.2 we construct A as follows:

$$A = \begin{bmatrix} | & & | \\ T(\vec{e}_1) & \cdots & T(\vec{e}_n) \\ | & & | \end{bmatrix}$$

In this case, A will be a 2×3 matrix, so we need to find $T(\vec{e}_1), T(\vec{e}_2)$, and $T(\vec{e}_3)$. Luckily, we have been given these values so we can fill in A as needed, using these vectors as the columns of A. Hence,

$$A = \begin{bmatrix}1 & 9 & 1\\2 & -3 & 1\end{bmatrix}$$

Watch the accompanying video for a visual demonstration of this concept.

- **Video Length**: 4 minutes 55 seconds.
- **Context**: This video demonstrates how to find the matrix of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given that $T\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}2x_1 + 3x_2\\-x_1 - x_2 + x_3\\x_2 + x_3\end{bmatrix}$

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In this example, we were given the resulting vectors of $T(\vec{e}_1), T(\vec{e}_2)$, and $T(\vec{e}_3)$. Constructing the matrix A was simple, as we could simply use these vectors as the columns of A. The next example shows how to find A when we are not given the $T(\vec{e}_i)$ so clearly.

Example 5.2.2: The Matrix of Linear Transformation: Inconveniently Defined

Suppose T is a linear transformation, $T : \mathbb{R}^2 \to \mathbb{R}^2$ and

$$T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \; T\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Find the matrix A of T such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x}$.

Solution

By Theorem 5.2.2 to find this matrix, we need to determine the action of T on $\vec{e}_1$ and $\vec{e}_2$. In Example 5.2.1, we were given these resulting vectors. However, in this example, we have been given T of two different vectors. How can we find out the action of T on $\vec{e}_1$ and $\vec{e}_2$? In particular for $\vec{e}_1$, suppose there exist x and y such that

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = x\begin{bmatrix} 1 \\ 1 \end{bmatrix} + y\begin{bmatrix} 0 \\ -1 \end{bmatrix} \tag{5.2.3}$$

Then, since T is linear,

$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = xT\begin{bmatrix} 1 \\ 1 \end{bmatrix} + yT\begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Substituting in values, this sum becomes

$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = x\begin{bmatrix} 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} 3 \\ 2 \end{bmatrix} \tag{5.2.4}$$

Therefore, if we know the values of x and y which satisfy equation (5.2.3), we can substitute these into equation (5.2.4). By doing so, we find $T(\vec{e}_1)$ which is the first column of the matrix A.

We proceed to find x and y. We do so by solving (5.2.3), which can be done by solving the system

$$\begin{aligned} x &= 1 \\ x - y &= 0 \end{aligned}$$

We see that $x = 1$ and $y = 1$ is the solution to this system. Substituting these values into equation (5.2.4), we have

$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

Therefore $\begin{bmatrix} 4 \\ 4 \end{bmatrix}$ is the first column of A.

Computing the second column is done in the same way, and is left as an exercise.

The resulting matrix A is given by

$$A = \begin{bmatrix} 4 & -3 \\ 4 & -2 \end{bmatrix}$$

This example illustrates a very long procedure for finding the matrix of A. While this method is reliable and will always result in the correct matrix A, the following procedure provides an alternative method.

Procedure 5.2.1: Finding the Matrix of Inconveniently Defined Linear Transformation

Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. Suppose there exist vectors $\{\vec{a}_1, \cdots, \vec{a}_n\}$ in $\mathbb{R}^n$ such that $\begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}^{-1}$ exists, and

$$T(\vec{a}_i) = \vec{b}_i$$

Then the matrix of T must be of the form

$$\begin{bmatrix} \vec{b}_1 & \cdots & \vec{b}_n \end{bmatrix} \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}^{-1}$$

We will illustrate this procedure in the following example. You may also find it useful to work through Example 5.2.2 using this procedure.

Example 5.2.3: Matrix of a Linear Transformation Given Inconveniently

Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation and

$$T\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, T\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, T\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Find the matrix of this linear transformation.

Solution

By Procedure 5.2.1, $A = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}$

Then, Procedure 5.2.1 claims that the matrix of T is

$$C = BA^{-1} = \begin{bmatrix} 2 & -2 & 4 \\ 0 & 0 & 1 \\ 4 & -3 & 6 \end{bmatrix}$$

Indeed you can first verify that $T(\vec{x}) = C\vec{x}$ for the 3 vectors above:

$$\begin{bmatrix} 2 & -2 & 4 \\ 0 & 0 & 1 \\ 4 & -3 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 & -2 & 4 \\ 0 & 0 & 1 \\ 4 & -3 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -2 & 4 \\ 0 & 0 & 1 \\ 4 & -3 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

But more generally $T(\vec{x}) = C\vec{x}$ for any $\vec{x}$. To see this, let $\vec{y} = A^{-1}\vec{x}$ and then using linearity of T:

$$T(\vec{x}) = T(A\vec{y}) = T\left(\sum_i \vec{y}_i \vec{a}_i\right) = \sum \vec{y}_i T(\vec{a}_i) = \sum \vec{y}_i \vec{b}_i = B\vec{y} = BA^{-1}\vec{x} = C\vec{x}$$

Watch the accompanying video for a visual demonstration of this concept.

- **Video Length**: 6 minutes 27 seconds
- **Context**: This video demonstrates how to find the matrix for a linear transformation for which the image of three specified vectors, not necessarily standard unit vectors: Find the transformation matrix given $T : \mathbb{R}^3 \to \mathbb{R}^3$ with

$$T\begin{bmatrix} 1 \\ 1 \\ 8 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, T\begin{bmatrix} -1 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}, T\begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ -1 \end{bmatrix}$$

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Matrix Transformations and the Determinant

Since a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation for some $m \times n$ matrix A, the properties of matrices considered in chapters 2 and 3 apply to linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. In particular, the effect of a linear transformation T on the volume of a region can be described in terms of the determinant of its induced matrix.

For now, let's look at A, an $n \times n$ square matrix with columns $v_1, v_2, \ldots, v_n$, and let $T: \mathbb{R}^n \to \mathbb{R}^n$ be the associated matrix transformation $T(\vec{x}) = A\vec{x}$. Then $T(\vec{e}_1) = \vec{v}_1$, $T(\vec{e}_2) = \vec{v}_2$ and so on.

If $n = 2$, we can think of T transforming the unit cube (a square in this case) C int a parallelepiped (parallelogram in this case) P determined by $\vec{v}_1 \text{and} \vec{v}_2$, as shown in the figure below (Figure $5.2.1$).

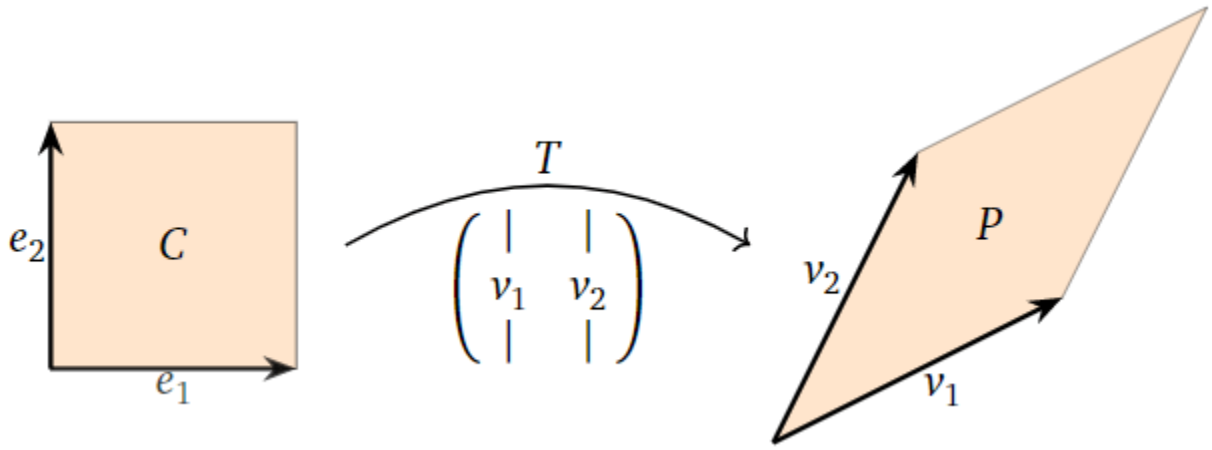


Figure $5.2.1$: A diagram illustrating the effect of a linear transformation on a unit square. On the left, a square labeled C is spanned by the standard basis vectors $\vec{e}_1$ and $\vec{e}_2$, representing the unit square in the standard coordinate system. In the center, an arrow labeled T represents the linear transformation applied to the unit square. Below the arrow, a transformation matrix is shown, composed of two column vectors $\vec{v}_1$ and $\vec{v}_2$, which define the new basis under the transformation. On the right, the transformed shape is a parallelogram labeled P, spanned by the new vectors $\vec{v}_1$ and $\vec{v}_2$. The parallelogram represents the image of the original unit square under the transformation T, and its area is given by the absolute value of the determinant of the transformation matrix. This diagram visually demonstrates how a linear transformation maps the unit square to a parallelogram, with the determinant controlling the scaling factor of the area. (CC-BY-SA; Margalit and Rabinoff via Interactive Linear Algebra)

Similarly, for $n = 3$, we can think of T transforming the unit cube C int a parallelepiped P determined by $\vec{v}_1, \vec{v}_2 \text{and} \vec{v}_3$.

In general, since the unit cube has volume 1 and its image has volume $|\det(A)|$, the transformation T scales the volume of the cube by a factor of $|\det(A)|$. Summarizing:

Note $5.2.1$

If A is an $n \times n$ matrix with corresponding matrix transformation $T: \mathbb{R}^n \to \mathbb{R}^n$, and if C is the unit cube in $\mathbb{R}^n$, then the volume of $T(C)$ is $|\det(A)|$.

The notation $T(S)$ means the image of the region S under the transformation T. That is,

$$T(S) = \{T(x) \mid x \text{ in } S\}.$$

In fact, T scales the volume of *any* region in $\mathbb{R}^n$ by the same factor, even for curvy regions.

The following theorem follows directly from Theorem 3.4.2 in section 3.4.

Theorem 5.2.3

Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with induced matrix A. If S is any region in $\mathbb{R}^n$, then

$$\text{vol}(T(S)) = |\det(A)| \cdot \text{vol}(S).$$

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- 4.3: Determinants and Volumes by Dan Margalit and Joseph Rabinoff is licensed GNU FDL. Original source: https://textbooks.math.gatech.edu/ila.

5.2E: Exercises for Section 5.2

Exercise $5.2E.1$

Consider the following functions which map $\mathbb{R}^n$ to $\mathbb{R}^n$.

a. T multiplies the jth component of $\vec{x}$ by a nonzero number b.
b. T replaces the ith component of $\vec{x}$ with b times the jth component added to the ith component.
c. T switches the ith and jth components.

Show these functions are linear transformations and describe their matrices A such that $T(\vec{x}) = A\vec{x}$.

Answer available at the end of this section.

Exercise $5.2E.2$

Suppose T is a linear transformation such that

$$\begin{aligned} T\left(\begin{bmatrix} 1 \\ 2 \\ -6 \end{bmatrix}\right) &= \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix} \\ T\left(\begin{bmatrix} -1 \\ -1 \\ 5 \end{bmatrix}\right) &= \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} \\ T\left(\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}\right) &= \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} \end{aligned} \tag{5.2E.1}$$

Find the matrix of T. That is find A such that $T(\vec{x}) = A\vec{x}$.

Answer available at the end of this section.

Exercise $5.2E.3$

Suppose T is a linear transformation such that

$$\begin{aligned} T\left(\begin{bmatrix} 1 \\ 1 \\ -8 \end{bmatrix}\right) &= \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \\ T\left(\begin{bmatrix} -1 \\ 0 \\ 6 \end{bmatrix}\right) &= \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} \\ T\left(\begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}\right) &= \begin{bmatrix} 6 \\ 1 \\ -1 \end{bmatrix} \end{aligned} \tag{5.2E.2}$$

Find the matrix of T. That is find A such that $T(\vec{x}) = A\vec{x}$.

Answer available at the end of this section.

Exercise $5.2E.4$

Suppose T is a linear transformation such that

$$\begin{aligned} T\left(\begin{bmatrix} 1 \\ 3 \\ -7 \end{bmatrix}\right) &= \begin{bmatrix} -3 \\ 1 \\ 3 \end{bmatrix} \\ T\left(\begin{bmatrix} -1 \\ -2 \\ 6 \end{bmatrix}\right) &= \begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix} \\ T\left(\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}\right) &= \begin{bmatrix} 5 \\ 3 \\ -3 \end{bmatrix} \end{aligned} \tag{5.2E.3}$$

Find the matrix of T. That is find A such that $T(\vec{x}) = A\vec{x}$.

Answer available at the end of this section.

Exercise $5.2E.5$

Suppose T is a linear transformation such that

$$\begin{aligned} T\left(\begin{bmatrix} 1 \\ 1 \\ -7 \end{bmatrix}\right) &= \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} \\ T\left(\begin{bmatrix} -1 \\ 0 \\ 6 \end{bmatrix}\right) &= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\ T\left(\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}\right) &= \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \end{aligned} \tag{5.2E.4}$$

Find the matrix of T. That is find A such that $T(\vec{x}) = A\vec{x}$.

Answer available at the end of this section.

Exercise $5.2E.6$

Suppose T is a linear transformation such that

$$\begin{aligned} T\left(\begin{bmatrix} 1 \\ 2 \\ -18 \end{bmatrix}\right) &= \begin{bmatrix} 5 \\ 2 \\ 5 \end{bmatrix} \\ T\left(\begin{bmatrix} -1 \\ -1 \\ 15 \end{bmatrix}\right) &= \begin{bmatrix} 3 \\ 3 \\ 5 \end{bmatrix} \\ T\left(\begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}\right) &= \begin{bmatrix} 2 \\ 5 \\ -2 \end{bmatrix} \end{aligned} \tag{5.2E.5}$$

Find the matrix of T. That is find A such that $T(\vec{x}) = A\vec{x}$.

Answer available at the end of this section.

Exercise $5.2E.7$

Consider the following functions $T : \mathbb{R}^3 \to \mathbb{R}^2$. Show that each is a linear transformation and determine for each the matrix A such that $T(\vec{x}) = A\vec{x}$.

a. $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + 2y + 3z \\ 2y - 3x + z \end{bmatrix}$

b. $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 7x + 2y + z \\ 3x - 11y + 2z \end{bmatrix}$

c. $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 3x + 2y + z \\ x + 2y + 6z \end{bmatrix}$

d. $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2y - 5x + z \\ x + y + z \end{bmatrix}$

Answer available at the end of this section.

Exercise $5.2E.8$

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by:

a. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + y \\ x - y \end{bmatrix}$

b. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x \\ 3y \end{bmatrix}$

c. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -y \\ x \end{bmatrix}$

Describe geometrically what each transformation does to a vector in $\mathbb{R}^2$. Think about how the transformation affects the standard unit vectors.

Answer available at the end of this section.

Exercise $5.2E.9$

Suppose

$$[A_1 \cdots A_n]^{-1}$$

exists where each $A_j \in \mathbb{R}^n$ and let vectors $\{B_1, \cdots, B_n\}$ in $\mathbb{R}^m$ be given. Show that there always exists a linear transformation T such that $T(A_i) = B_i$

Exercise $5.2E.10$

You are given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ and you know that

$$T(A_i) = B_i$$

where $[A_1 \cdots A_n]^{-1}$ exists. Show that the matrix of T is of the form

$$[B_1 \cdots B_n][A_1 \cdots A_n]^{-1}$$

Answer available at the end of this section.

Answers

Exercise $5.2E.1$

a. The matrix of T is the elementary matrix which multiplies the jth diagonal entry of the identity matrix by b.
b. The matrix of T is the elementary matrix which takes b times the jth row and adds to the ith row.
c. The matrix of T is the elementary matrix which switches the ith and the jth rows where the two components are in the ith and jth positions.

Exercise $5.2E.2$

$$\begin{bmatrix} 5 & 1 & 5 \\ 1 & 1 & 3 \\ 3 & 5 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 4 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 37 & 17 & 11 \\ 17 & 7 & 5 \\ 11 & 14 & 6 \end{bmatrix}$$

Exercise $5.2E.3$

$$\begin{bmatrix} 1 & 2 & 6 \\ 3 & 4 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 6 & 3 & 1 \\ 5 & 3 & 1 \\ 6 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 52 & 21 & 9 \\ 44 & 23 & 8 \\ 5 & 4 & 1 \end{bmatrix}$$

Exercise $5.2E.4$

$$\begin{bmatrix} -3 & 1 & 5 \\ 1 & 3 & 3 \\ 3 & -3 & -3 \end{bmatrix} \begin{bmatrix} 2 & 2 & 1 \\ 1 & 2 & 1 \\ 4 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 15 & 1 & 3 \\ 17 & 11 & 7 \\ -9 & -3 & -3 \end{bmatrix}$$

Exercise $5.2E.5$

$$\begin{bmatrix} 3 & 1 & 1 \\ 3 & 2 & 3 \\ 3 & 3 & -1 \end{bmatrix} \begin{bmatrix} 6 & 2 & 1 \\ 5 & 2 & 1 \\ 6 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 29 & 9 & 5 \\ 46 & 13 & 8 \\ 27 & 11 & 5 \end{bmatrix}$$

Exercise $5.2E.6$

$$\begin{bmatrix} 5 & 3 & 2 \\ 2 & 3 & 5 \\ 5 & 5 & -2 \end{bmatrix} \begin{bmatrix} 11 & 4 & 1 \\ 10 & 4 & 1 \\ 12 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 109 & 38 & 10 \\ 112 & 35 & 10 \\ 81 & 34 & 8 \end{bmatrix}$$

Exercise $5.2E.7$

Exercise $5.2E.8$

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x-y \end{bmatrix}$: This transformation rotates and scales the vector. It can be written as:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

This represents a shearing transformation that we will learn about later in the chapter.

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x \\ 3y \end{bmatrix}$: This transformation scales the x-coordinate by 2 and the y-coordinate by 3, stretching the vector accordingly.

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -y \\ x \end{bmatrix}$: This transformation rotates the vector counterclockwise by 90 degrees because the transformation matrix is:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Exercise $5.2E.10$

Suppose

$$\begin{bmatrix} \vec{c}_1^{\,T} \\ \vdots \\ \vec{c}_n^{\,T} \end{bmatrix} = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}^{-1}$$

Thus $\vec{c}_i^{\,T}\vec{a}_j = \delta_{ij}$. Therefore

$$\begin{aligned} \begin{bmatrix} \vec{b}_1 & \cdots & \vec{b}_n \end{bmatrix} \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}^{-1} \vec{a}_i &= \begin{bmatrix} \vec{b}_1 & \cdots & \vec{b}_n \end{bmatrix} \begin{bmatrix} \vec{c}_1^{\,T} \\ \vdots \\ \vec{c}_n^{\,T} \end{bmatrix} \vec{a}_i \\ &= \begin{bmatrix} \vec{b}_1 & \cdots & \vec{b}_n \end{bmatrix} \vec{e}_i \\ &= \vec{b}_i \end{aligned} \tag{5.2E.6}$$

Thus $T\vec{a}_i = \begin{bmatrix} \vec{b}_1 & \cdots & \vec{b}_n \end{bmatrix} \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}^{-1} \vec{a}_i = A\vec{a}_i$. If $\vec{x}$ is arbitrary, then since the matrix $\begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}$ is invertible, there exists a unique $\vec{y}$ such that $\begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix} \vec{y} = \vec{x}$ Hence

$$T\vec{x} = T\left(\sum_{i=1}^{n} y_i \vec{a}_i\right) = \sum_{i=1}^{n} y_i T\vec{a}_i = \sum_{i=1}^{n} y_i A\vec{a}_i = A\left(\sum_{i=1}^{n} y_i \vec{a}_i\right) = A\vec{x}$$

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5.3: Properties of Linear Transformations

Learning Objectives

- Use properties of linear transformations to solve problems.
- Find the composite of transformations and the inverse of a transformation.

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation. Then there are some important properties of T which will be examined in this section. Consider the following theorem.

Theorem 5.3.1: Properties of Linear Transformations

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation and let $\vec{x} \in \mathbb{R}^n$.

- T preserves the zero vector.

$$T(0\vec{x}) = 0T(\vec{x}). \text{ Hence } T(\vec{0}) = \vec{0}$$

- T preserves the negative of a vector:

$$T((-1)\vec{x}) = (-1)T(\vec{x}). \text{ Hence } T(-\vec{x}) = -T(\vec{x}).$$

- T preserves linear combinations:

$$\text{Let } \vec{x}_1, ..., \vec{x}_k \in \mathbb{R}^n \text{ and } a_1, ..., a_k \in \mathbb{R}.$$

$$\text{Then if } \vec{y} = a_1\vec{x}_1 + a_2\vec{x}_2 + ... + a_k\vec{x}_k, \text{ it follows that}$$

$$T(\vec{y}) = T(a_1\vec{x}_1 + a_2\vec{x}_2 + ... + a_k\vec{x}_k) = a_1T(\vec{x}_1) + a_2T(\vec{x}_2) + ... + a_kT(\vec{x}_k).$$

These properties are useful in determining the action of a transformation on a given vector. Consider the following example.

Example 5.3.1: Linear Combination

Let $T : \mathbb{R}^3 \mapsto \mathbb{R}^4$ be a linear transformation such that

$$T\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 0 \\ -2 \end{bmatrix}, T\begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ -1 \\ 5 \end{bmatrix}$$

Find $T\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix}$.

Solution

Using the third property in Theorem 5.3.1, we can find $T\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix}$ by writing $\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}$.

Therefore we want to find $a, b \in \mathbb{R}$ such that

$$\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix} = a \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + b \begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}$$

The necessary augmented matrix and resulting reduced row-echelon form are given by:

$$\begin{bmatrix} 1 & 4 & -7 \\ 3 & 0 & 3 \\ 1 & 5 & -9 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence $a = 1, b = -2$ and

$$\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + (-2) \begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}$$

Now, using the third property above, we have

$$\begin{aligned} T \begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix} &= T \left(1 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + (-2) \begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix} \right) \\ &= 1T \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} - 2T \begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ 4 \\ 0 \\ -2 \end{bmatrix} - 2 \begin{bmatrix} 4 \\ 5 \\ -1 \\ 5 \end{bmatrix} \\ &= \begin{bmatrix} -4 \\ -6 \\ 2 \\ -12 \end{bmatrix} \end{aligned} \tag{5.3.1}$$

Therefore, $T \begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix} = \begin{bmatrix} -4 \\ -6 \\ 2 \\ -12 \end{bmatrix}$.

Watch the accompanying video for a visual demonstration of this concept.

- **Video Length**: 2 minutes 17 seconds.
- **Context**: This video demonstrates how to use the properties of a linear transformation:

  Suppose T is a linear transformation with $T(\vec{u}) = \langle 5, 2 \rangle$ and $T(\vec{v}) = \langle -5, 2 \rangle$. Find the following:

  a. $T(-\vec{u})$
  b. $T(5\vec{u} - 6\vec{v})$

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Suppose two linear transformations act in the same way on $\vec{x}$ for all vectors. Then we say that these transformations are equal.

Definition 5.3.1: Equal Transformations

Let S and T be linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$. Then $S = T$ if and only if for every $\vec{x} \in \mathbb{R}^n$,

$$S(\vec{x}) = T(\vec{x})$$

Suppose two linear transformations act on the same vector $\vec{x}$, first the transformation T and then a second transformation given by S. We can find the **composite** transformation that results from applying both transformations.

Definition 5.3.2: Composition of Linear Transformations

Let $T : \mathbb{R}^k \mapsto \mathbb{R}^n$ and $S : \mathbb{R}^n \mapsto \mathbb{R}^m$ be linear transformations. Then the **composite** of S and T is

$$S \circ T : \mathbb{R}^k \mapsto \mathbb{R}^m$$

The action of $S \circ T$ is given by

$$(S \circ T)(\vec{x}) = S(T(\vec{x})) \text{ for all } \vec{x} \in \mathbb{R}^k$$

Notice that the resulting vector will be in $\mathbb{R}^m$. Be careful to observe the order of transformations. We write $S \circ T$ but apply the transformation T first, followed by S.

Theorem 5.3.2: Composition of Transformations

Let $T : \mathbb{R}^k \mapsto \mathbb{R}^n$ and $S : \mathbb{R}^n \mapsto \mathbb{R}^m$ be linear transformations such that T is induced by the matrix A and S is induced by the matrix B. Then $S \circ T$ is a linear transformation which is induced by the matrix BA.

Consider the following example.

Example 5.3.2: Composition of Transformations

Let T be a linear transformation induced by the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$$

and S a linear transformation induced by the matrix

$$B = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$$

Find the matrix of the composite transformation $S \circ T$. Then, find $(S \circ T)(\vec{x})$ for $\vec{x} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

Solution

By Theorem 5.3.2, the matrix of $S \circ T$ is given by BA.

$$BA = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ 2 & 0 \end{bmatrix}$$

To find $(S \circ T)(\vec{x})$, multiply $\vec{x}$ by BA as follows

$$\begin{bmatrix} 8 & 4 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 24 \\ 2 \end{bmatrix}$$

To check, first determine $T(\vec{x})$:

$$\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \end{bmatrix}$$

Then, compute $S(T(\vec{x}))$ as follows:

$$\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 9 \\ 2 \end{bmatrix} = \begin{bmatrix} 24 \\ 2 \end{bmatrix}$$

Watch the accompanying video for a visual demonstration of the this concept.

- **Video Length**: 7 minutes 41 seconds.
- **Context**: This video demonstrates how to find the standard matrices for the composition of transformations:

Let $T_1 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10x_1 - 3x_2 \\ 27x_1 - 8x_2 \end{bmatrix}$ and $T_2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1x_1 + 0x_2 \\ -3x_1 + 1x_2 \end{bmatrix}$. Write standard matrices for $T_1, T_2, \text{ and for } T_1 \circ T_2$.

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View this YouTube Video online

Consider a composite transformation $S \circ T$, and suppose that this transformation acted such that $(S \circ T)(\vec{x}) = \vec{x}$. That is, the transformation S took the vector $T(\vec{x})$ and returned it to $\vec{x}$. In this case, S and T are inverses of each other. Consider the following definition.

Definition 5.3.3: Inverse of a Transformation

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^n$ and $S : \mathbb{R}^n \mapsto \mathbb{R}^n$ be linear transformations. Suppose that for each $\vec{x} \in \mathbb{R}^n$,

$$(S \circ T)(\vec{x}) = \vec{x}$$

and

$$(T \circ S)(\vec{x}) = \vec{x}$$

Then, S is called an inverse of T and T is called an inverse of S. Geometrically, they reverse the action of each other.

The following theorem is crucial, as it claims that the above inverse transformations are unique.

Theorem 5.3.3: Inverse of a Transformation

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a linear transformation induced by the matrix A. Then T has an inverse transformation if and only if the matrix A is invertible. In this case, the inverse transformation is unique and denoted $T^{-1} : \mathbb{R}^n \mapsto \mathbb{R}^n$. T^{-1} is induced by the matrix A^{-1}.

Consider the following example.

Example 5.3.3: Inverse of a Transformation

Let $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be a linear transformation induced by the matrix

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$$

Show that T^{-1} exists and find the matrix B which it is induced by.

Solution

Since the matrix A is invertible, it follows that the transformation T is invertible. Therefore, T^{-1} exists.

You can verify that A^{-1} is given by:

$$A^{-1} = \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix}$$

Therefore the linear transformation T^{-1} is induced by the matrix A^{-1}.

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5.3E: Exercises for Section 5.3

Exercise 5.3E.1

Suppose that $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. Show that $T(\vec{0}) = \vec{0}$.

Exercise 5.3E.2

Let T be a linear transformation induced by the matrix $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ and S a linear transformation induced by $B = \begin{bmatrix} 0 & -2 \\ 4 & 2 \end{bmatrix}$. Find matrix of $S \circ T$ and find $(S \circ T)(\vec{x})$ for $\vec{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

Answer available at the end of this section.

Exercise 5.3E.3

Let T be a linear transformation and suppose $T\left(\begin{bmatrix} 1 \\ -4 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$. Suppose S is a linear transformation induced by the matrix $B = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$. Find $(S \circ T)(\vec{x})$ for $\vec{x} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$.

Answer available at the end of this section.

Exercise 5.3E.4

Let T be a linear transformation induced by the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$ and S a linear transformation induced by $B = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$. Find matrix of $S \circ T$ and find $(S \circ T)(\vec{x})$ for $\vec{x} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$.

Exercise 5.3E.5

Let T be a linear transformation induced by the matrix $A = \begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix}$. Find the matrix of T^{-1}.

Answer available at the end of this section.

Exercise 5.3E.6

Let T be a linear transformation induced by the matrix $A = \begin{bmatrix} 4 & -3 \\ 2 & -2 \end{bmatrix}$. Find the matrix of T^{-1}.

Exercise $5.3E.7$

Let T be a linear transformation and suppose $T\left(\begin{bmatrix}1\\2\end{bmatrix}\right)=\begin{bmatrix}9\\8\end{bmatrix}$, $T\left(\begin{bmatrix}0\\-1\end{bmatrix}\right)=\begin{bmatrix}-4\\-3\end{bmatrix}$. Find the matrix of T^{-1}.

Exercise $5.3E.8$

Let V and W be subspaces of $\mathbb{R}^n$ and $\mathbb{R}^n$ respectively, and let $T:V\to W$ be a linear transformation. Suppose that $\{T(\vec{v}_1),\cdots,T(\vec{v}_1)\}$ is a linearly independent set. Show that it must be the case that $\{\vec{v}_1,\cdots,\vec{v}_n\}$ is also linearly independent.

Hint available at the end of this section.

Answers

Exercise $5.3E.2$

The matrix of $S\circ T$ is given by BA.

$$\begin{bmatrix}0&-2\\4&2\end{bmatrix}\begin{bmatrix}3&1\\-1&2\end{bmatrix}=\begin{bmatrix}2&-4\\10&8\end{bmatrix}$$

Now, $(S\circ T)(\vec{x})=(BA)\vec{x}$.

$$\begin{bmatrix}2&-4\\10&8\end{bmatrix}\begin{bmatrix}2\\-1\end{bmatrix}=\begin{bmatrix}8\\12\end{bmatrix}$$

Exercise $5.3E.3$

To find $(S\circ T)(\vec{x})$ we compute $S(T(\vec{x}))$.

$$\begin{bmatrix}1&2\\-1&3\end{bmatrix}\begin{bmatrix}2\\-3\end{bmatrix}=\begin{bmatrix}-4\\-11\end{bmatrix}$$

Exercise $5.3E.5$

The matrix of T^{-1} is A^{-1}.

$$\begin{bmatrix}2&1\\5&2\end{bmatrix}^{-1}=\begin{bmatrix}-2&1\\5&-2\end{bmatrix}$$

Exercise $5.3E.8$

Start with $\sum_{i=1}^{r}a_i\vec{v}_i=0$. [We want to show that $a_1=\cdots=a_r=0$.] Use the properties of a linear transformation and the fact that $\{T(\vec{v}_1),\cdots,T(\vec{v}_1)\}$ is a linearly independent set to finish the argument.

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5.4: Special Linear Transformations in Two and Three Dimensions

Learning Objectives

- Find the matrix of rotations in $\mathbb{R}^2$ and $\mathbb{R}^3$ and determine the action of each on a vector in $\mathbb{R}^2$ and $\mathbb{R}^3$.
- Find the matrix of reflections in $\mathbb{R}^2$ and $\mathbb{R}^3$ and determine the action of each on a vector in $\mathbb{R}^2$ and $\mathbb{R}^3$.
- Find the matrix of projections in $\mathbb{R}^2$ and $\mathbb{R}^3$ and determine the action of each on a vector in $\mathbb{R}^2$ and $\mathbb{R}^3$.
- Find the matrix of other linear transformation in $\mathbb{R}^2$ and $\mathbb{R}^3$ and determine the action of each on a vector in $\mathbb{R}^2$ and $\mathbb{R}^3$, such as dilations, contractions, expansions, compressions, and shears.

In this section, we will examine some special examples of linear transformations in $\mathbb{R}^2$ and $\mathbb{R}^3$ including rotations, reflections and projections. We will use the geometric descriptions of vector addition and scalar multiplication discussed earlier to show these are examples of linear transformations.

Rotations

First, consider a transformation given by a rotation T. Why is such a transformation linear? Consider the following picture which illustrates a rotation in $\mathbb{R}^2$. Let $\vec{u}, \vec{v}$ denote vectors.

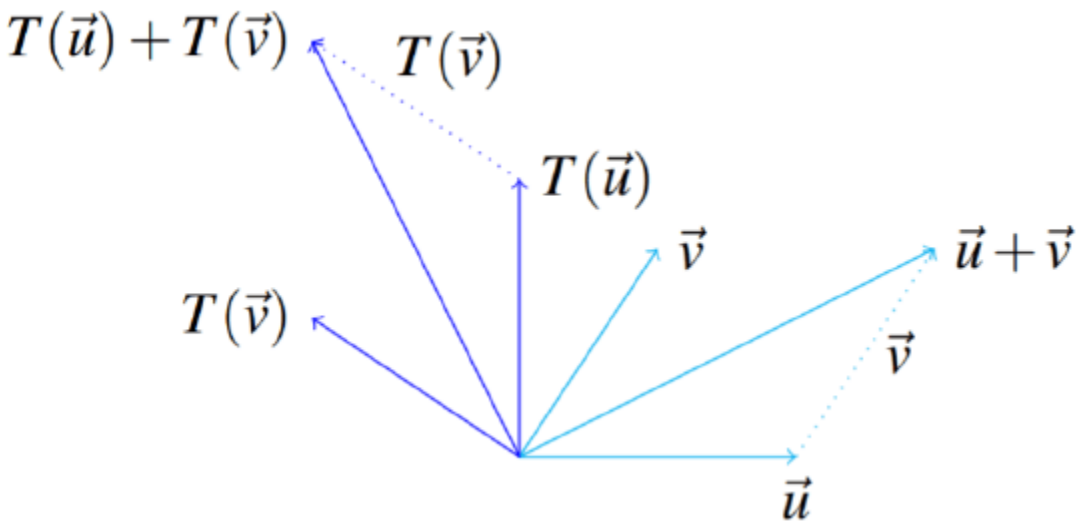


Figure 5.4.1 Vectors $\vec{u}, \vec{v}$, $\vec{u}$ + $\vec{v}$ and the result of rotating them by transformation T showing that $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ (CC BY-NC-SA 4.0; Kuttler via A First Course in LINEAR ALGEBRA)

Let's consider how to obtain $T(\vec{u}+\vec{v})$. Simply, you add $T(\vec{u})$ and $T(\vec{v})$. Here is why. If you add $T(\vec{u})$ to $T(\vec{v})$ you get the diagonal of the parallelogram determined by $T(\vec{u})$ and $T(\vec{v})$, as this action is our usual vector addition. Now, suppose we first add $\vec{u}$ and $\vec{v}$, and then apply the transformation T to $\vec{u}+\vec{v}$. Hence, we find $T(\vec{u}+\vec{v})$. As shown in the diagram, this will result in the same vector. In other words, $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$.

This is because the rotation preserves all angles between the vectors as well as their lengths. In particular, it preserves the shape of this parallelogram. Thus both $T(\vec{u}) + T(\vec{v})$ and $T(\vec{u}+\vec{v})$ give the same vector. It follows that T distributes across addition of the vectors of $\mathbb{R}^2$.

Similarly, if k is a scalar, it follows that $T(k\vec{u}) = kT(\vec{u})$. Thus rotations are an example of a linear transformation by Definition 5.1.1.

The following theorem gives the matrix of a linear transformation which rotates all vectors through an angle of θ.

Theorem 5.4.1: Rotation

Let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation given by rotating vectors through an angle of θ. Then the matrix A of R_θ is given by

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Proof available at the end of this section.

Consider the following example.

Example 5.4.1: Rotation in $\mathbb{R}^2$

Let $R_{\frac{\pi}{2}} : \mathbb{R}^2 \to \mathbb{R}^2$ denote rotation through $\frac{\pi}{2}$. Find the matrix of $R_{\frac{\pi}{2}}$. Then, find $R_{\frac{\pi}{2}}(\vec{x})$ where $\vec{x} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Solution

By Theorem 5.4.1, the matrix of $R_{\frac{\pi}{2}}$ is given by

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} = \begin{bmatrix} \cos(\pi/2) & -\sin(\pi/2) \\ \sin(\pi/2) & \cos(\pi/2) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

To find $R_{\frac{\pi}{2}}(\vec{x})$, we multiply the matrix of $R_{\frac{\pi}{2}}$ by $\vec{x}$ as follows

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Watch the accompanying video for a visual demonstration of the this concept.

- **Video Length**: 3 minutes 53 seconds.
- **Context**: This video demonstrates the transformation matrix for a $\mathbb{R}^2$ rotation: Write a standard matrix for a $\mathbb{R}^2 \to \mathbb{R}^2$ transformation that rotates points $\frac{2\pi}{3}$ radians about the origin.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

We now look at an example of a linear transformation involving two angles.

Example 5.4.2: The Rotation Matrix of the Sum of Two Angles

Find the matrix of the linear transformation which is obtained by first rotating all vectors through an angle of ϕ and then through an angle θ. Hence the linear transformation rotates all vectors through an angle of $\theta + \phi$.

Solution

Let $R_{\theta+\phi}$ denote the linear transformation which rotates every vector through an angle of $\theta + \phi$. Then to obtain $R_{\theta+\phi}$, we first apply R_ϕ and then R_θ where R_ϕ is the linear transformation which rotates through an angle of ϕ and R_θ is the linear transformation which rotates through an angle of θ. Denoting the corresponding matrices by $A_{\theta+\phi}$, A_ϕ, and A_θ, it follows that for every $\vec{u}$

$$R_{\theta+\phi}\left(\vec{u}\right) = A_{\theta+\phi}\vec{u} = A_\theta A_\phi \vec{u} = R_\theta R_\phi\left(\vec{u}\right)$$

Notice the order of the matrices here!

Consequently, you must have

$$\begin{aligned} A_{\theta+\phi} &= \begin{bmatrix} \cos\left(\theta+\phi\right) & -\sin\left(\theta+\phi\right) \\ \sin\left(\theta+\phi\right) & \cos\left(\theta+\phi\right) \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} = A_\theta A_\phi \end{aligned} \tag{5.4.1}$$

The usual matrix multiplication yields

$$\begin{aligned} A_{\theta+\phi} &= \begin{bmatrix} \cos\left(\theta+\phi\right) & -\sin\left(\theta+\phi\right) \\ \sin\left(\theta+\phi\right) & \cos\left(\theta+\phi\right) \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & -\cos\theta\sin\phi - \sin\theta\cos\phi \\ \sin\theta\cos\phi + \cos\theta\sin\phi & \cos\theta\cos\phi - \sin\theta\sin\phi \end{bmatrix} \\ &= A_\theta A_\phi \end{aligned} \tag{5.4.2}$$

Don't these look familiar? They are the usual trigonometric identities for the sum of two angles derived here using linear algebra concepts.

Here we have focused on rotations in two dimensions. However, you can consider rotations and other geometric concepts in any number of dimensions. This is one of the major advantages of linear algebra. You can break down a difficult geometrical procedure into small steps, each corresponding to multiplication by an appropriate matrix. Then by multiplying the matrices, you can obtain a single matrix which can give you numerical information on the results of applying the given sequence of simple procedures.

Reflections

Linear transformations which reflect vectors across a line are a second important type of transformations in $\mathbb{R}^2$. Consider the following theorem.

Theorem 5.4.2: Reflection

Let $Q_m : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation given by reflecting vectors over the line $y = mx$. Then the matrix of Q_m is given by

$$\frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}$$

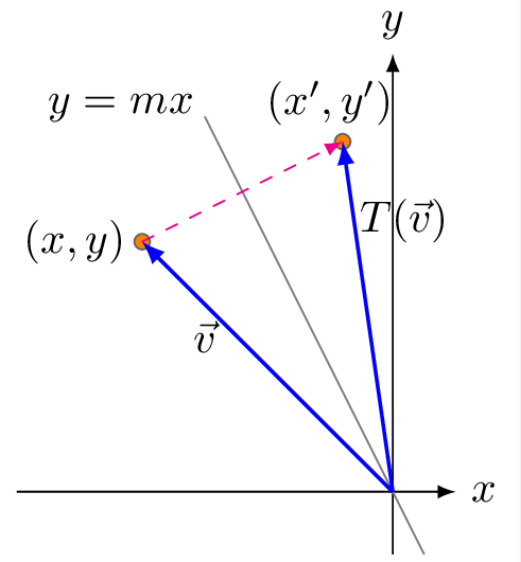


Figure 5.4.3: A 2D coordinate plane with x and y axes, depicting the reflection of a vector $\vec{v}$ across the line $y = mx$. The original vector $\vec{v}$ (in blue) extends from the origin to the point (x, y), and its reflected vector $T(\vec{v})$ extends to the point (x', y'). A dashed pink line connects (x, y) and (x', y'), indicating symmetry across the gray reflection line. (CC BY-NC-SA; Atomic Math via Studio Moraine LLC)

Consider the following example.

Example 5.4.3: Reflection in $\mathbb{R}^2$

Let $Q_2 : \mathbb{R}^2 \to \mathbb{R}^2$ denote reflection over the line $y = 2x$. Then Q_2 is a linear transformation. Find the matrix of Q_2. Then, find $Q_2(\vec{x})$ where $\vec{x} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Solution

By Theorem 5.4.2, the matrix of Q_2 is given by

$$\frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix} = \frac{1}{1+(2)^2}\begin{bmatrix} 1-(2)^2 & 2(2) \\ 2(2) & (2)^2-1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -3 & 8 \\ 8 & 3 \end{bmatrix}$$

To find $Q_2(\vec{x})$ we multiply $\vec{x}$ by the matrix of Q_2 as follows:

$$\frac{1}{5}\begin{bmatrix} -3 & 8 \\ 8 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -\frac{19}{5} \\ \frac{2}{5} \end{bmatrix}$$

We can also consider transformations that reflect a vector in $\mathbb{R}^3$ through a plane that passes through the origin. Consider, for example, reflection through the XY-plane (the plane $z = 0$). As shown in the illustration below, this reflection is given by the transformation $T(x, y, z) = (x, y, -z)$. The matrix of this transformation would be $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

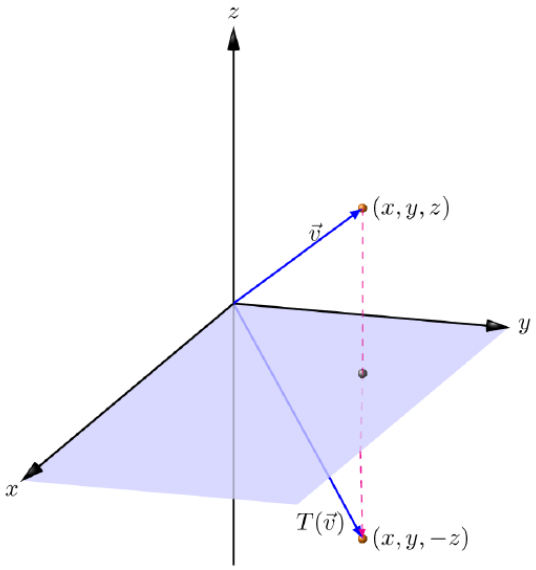


Figure 5.4.4: Reflection of a vector through the XY-plane. A 3D coordinate system with x, y, and z axes, illustrating the reflection of a vector $\vec{v}$ across the xy-plane. The original vector $\vec{v}$ (in blue) extends from the origin to the point (x, y, z). Its reflected counterpart, $T(\vec{v})$, extends to the point $(x, y, -z)$. (CC BY-NC-SA; Atomic Math via Studio Moraine LLC)

Consider the following example which incorporates a reflection as well as a rotation of vectors.

Example 5.4.4: Rotation Followed by a Reflection

Find the matrix of the linear transformation which is obtained by first rotating all vectors through an angle of $\pi/6$ and then reflecting through the x axis.

Solution

By Theorem 5.4.1, the matrix of the transformation which involves rotating through an angle of $\pi/6$ is

$$\begin{bmatrix} \cos(\pi/6) & -\sin(\pi/6) \\ \sin(\pi/6) & \cos(\pi/6) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} \end{bmatrix}$$

Reflecting across the x axis is the same action as reflecting vectors over the line $\vec{y} = m\vec{x}$ with $m = 0$. By Theorem 5.4.2, the matrix for the transformation which reflects all vectors through the x axis is

$$\frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix} = \frac{1}{1+(0)^2}\begin{bmatrix} 1-(0)^2 & 2(0) \\ 2(0) & (0)^2-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Therefore, the matrix of the linear transformation which first rotates through $\pi/6$ and then reflects through the x axis is given by

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2}\sqrt{3} \end{bmatrix}$$

Projections

Next consider the map $\vec{v} \mapsto \text{proj}_{\vec{u}}(\vec{v})$ which takes a vector and transforms it to its projection onto a given vector $\vec{u}$. It turns out that this map is linear, a result which follows from the properties of the dot product. This is shown as follows.

$$\begin{aligned}\text{proj}_{\vec{u}}\left(k\vec{v}+p\vec{w}\right) &= \left(\frac{\left(k\vec{v}+p\vec{w}\right)\bullet\vec{u}}{\vec{u}\bullet\vec{u}}\right)\vec{u} \\ &= k\left(\frac{\vec{v}\bullet\vec{u}}{\vec{u}\bullet\vec{u}}\right)\vec{u}+p\left(0.05in\frac{\vec{w}\bullet\vec{u}}{\vec{u}\bullet\vec{u}}\right)\vec{u} \\ &= k\,\text{proj}_{\vec{u}}\left(\vec{v}\right)+p\,\text{proj}_{\vec{u}}\left(\vec{w}\right)\end{aligned} \tag{5.4.3}$$

Example 5.4.5: Matrix of a Projection

Let $\vec{u}=\begin{bmatrix}1\\2\\3\end{bmatrix}$ and let T be the projection map $T:\mathbb{R}^3\mapsto\mathbb{R}^3$ defined by

$$T(\vec{v})=\text{proj}_{\vec{u}}\left(\vec{v}\right)$$

for any $\vec{v}\in\mathbb{R}^3$.

1. Does this transformation come from multiplication by a matrix?
2. If so, what is the matrix?

Solution

1. First, we have just seen that $T(\vec{v})=\text{proj}_{\vec{u}}\left(\vec{v}\right)$ is linear. Therefore, we can find a matrix A such that $T(\vec{x})=A\vec{x}$.
2. The columns of the matrix for T are defined above as $T(\vec{e}_i)$. It follows that $T(\vec{e}_i)=\text{proj}_{\vec{u}}\left(\vec{e}_i\right)$ gives the ith column of the desired matrix. Therefore, we need to find

$$\text{proj}_{\vec{u}}\left(\vec{e}_i\right)=\left(\frac{\vec{e}_i\bullet\vec{u}}{\vec{u}\bullet\vec{u}}\right)\vec{u}$$

For the given vector $\vec{u}$, this implies the columns of the desired matrix are

$$\frac{1}{14}\begin{bmatrix}1\\2\\3\end{bmatrix},\frac{2}{14}\begin{bmatrix}1\\2\\3\end{bmatrix},\frac{3}{14}\begin{bmatrix}1\\2\\3\end{bmatrix}$$

which you can verify. Hence the matrix of T is

$$\frac{1}{14}\begin{bmatrix}1&2&3\\2&4&6\\3&6&9\end{bmatrix}$$

Let's consider a linear transformation on $\mathbb{R}^2$ that projects a vector $vecv$ onto the line $y=mx$.

Theorem 5.4.1

Let $P_m:\mathbb{R}^2\to\mathbb{R}^2$ be a linear transformation given by projecting vectors on to the line $y=mx$. Then the matrix of P_m is given by

$$\frac{1}{1+m^2}\begin{bmatrix}1&m\\m&m^2\end{bmatrix}$$

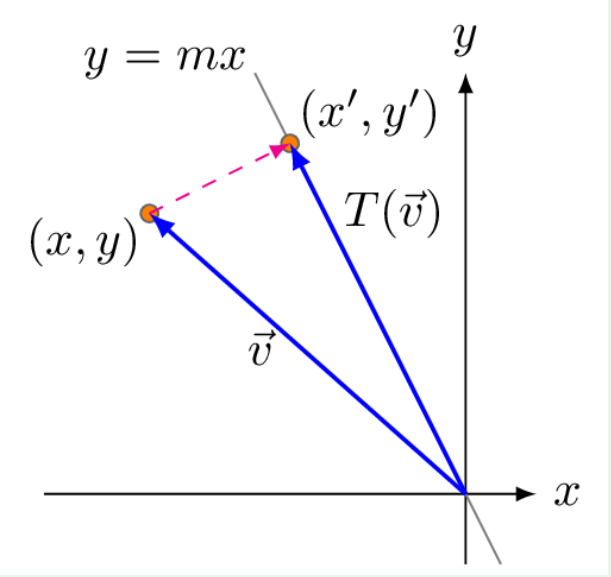


Figure 5.4.5: A 2D coordinate plane with x and y axes, illustrating the projection of a vector $\vec{v}$ onto the line $y = mx$. The original vector $\vec{v}$ (in blue) extends from the origin to the point (x, y). Its projection, $T(\vec{v})$, extends to the point (x', y'), which lies on the gray line $y = mx$. (CC BY-NC-SA; Atomic Math via Studio Moraine LLC)

Projections do not have to be onto a vector. In $\mathbb{R}^3$, we can project vectors onto a plane that passes through the origin also. Consider, for example, projection on to the XY-plane (the plane $z = 0$). As shown in the illustration below, this reflection is given by the transformation $T(x, y, z) = (x, y, 0)$. The matrix of this transformation would be $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

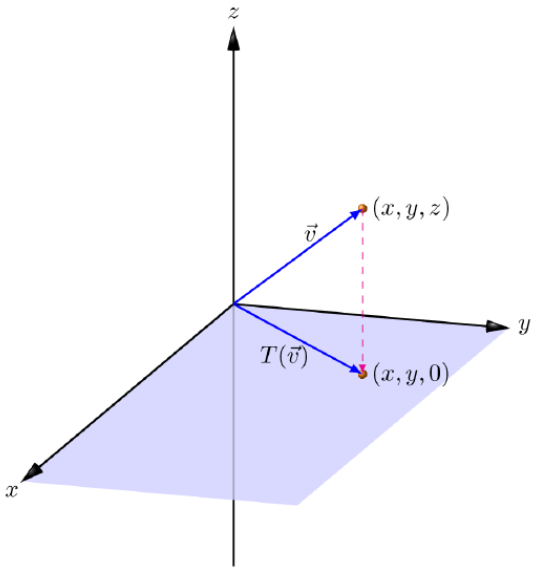


Figure 5.4.6: A 3D coordinate system with x, y, and z axes, illustrating the projection of a vector $\vec{v}$ onto the xy-plane. The original vector $\vec{v}$ (in blue) extends from the origin to the point (x, y, z). Its projection, $T(\vec{v})$, extends to the point $(x, y, 0)$, which lies on the shaded xy-plane. A dashed pink line connects (x, y, z) and $(x, y, 0)$, indicating the perpendicular distance from the point to the plane. (CC BY-NC-SA; Atomic Math via Studio Moraine LLC)

Dilations and Contractions

Definition: Dilation and Contraction

A **dilation** is a linear transformation that scales all vectors *away* from the origin by a factor k, where $k > 1$. This transformation enlarges objects in space while maintaining their relative proportions and the origin as a fixed point.

Mathematically, if T is a dilation transformation, then for any vector $\vec{v}$,

$$T(\vec{v}) = k\vec{v}, \quad \text{where } k > 1. \tag{5.4.4}$$

A **contraction** is a linear transformation that scales all vectors *toward* the origin by a factor k, where $0 < k < 1$. This transformation shrinks objects while keeping the origin fixed.

Mathematically, for a contraction transformation,

$$T(\vec{v}) = k\vec{v}, \quad \text{where } 0 < k < 1. \tag{5.4.5}$$

Below are visual representations of a dilation in $\mathbb{R}^2$ and a contraction in $\mathbb{R}^3$.

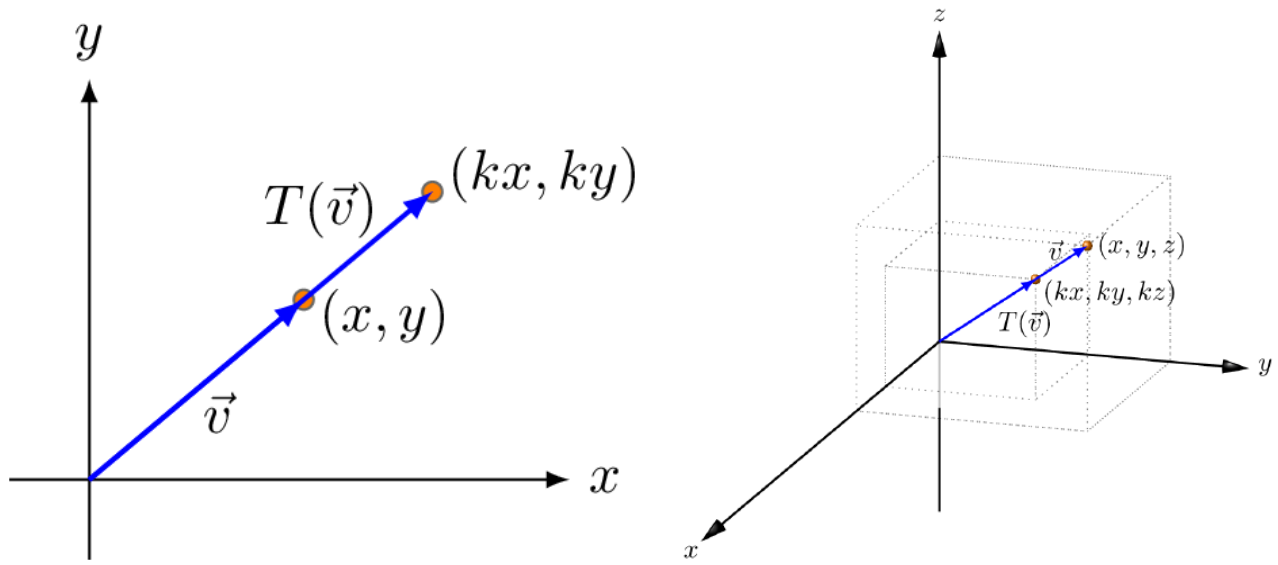


Figure 5.4.7: There are two images. In the first image, a 2D coordinate plane with x and y axes, illustrating the dilation of a vector $\vec{v}$ by a scalar factor k. The original vector $\vec{v}$ (in blue) extends from the origin to the point (x, y). Its transformed vector $T(\vec{v})$ extends to the point (kx, ky), indicating that the vector has been scaled by k. In the second image, a 3D coordinate system with x, y, and z axes, illustrating the contraction of a vector $\vec{v}$ by a scalar factor k, where $0 < k < 1$. The original vector $\vec{v}$ (in blue) extends from the origin to the point (x, y, z). Its transformed vector $T(\vec{v})$ extends to the point (kx, ky, kz). A dotted cube represents the relative scaling effect of the transformation. (CC BY-NC-SA; Atomic Math via Studio Moraine LLC)

Expansions and Compressions

Definition: Expansion and Compression

An **expansion** along a specific axis increases the magnitude of a vector's component along that axis by a factor k, where $k > 1$, while leaving the other coordinate(s) unchanged.

An **compression** along a specific axis decreases the magnitude of a vector's component along that axis by a factor k, where \(0

Below are visual representations of an expansion along the x-axis and a compression along the y-axis, both in $\mathbb{R}^2$.

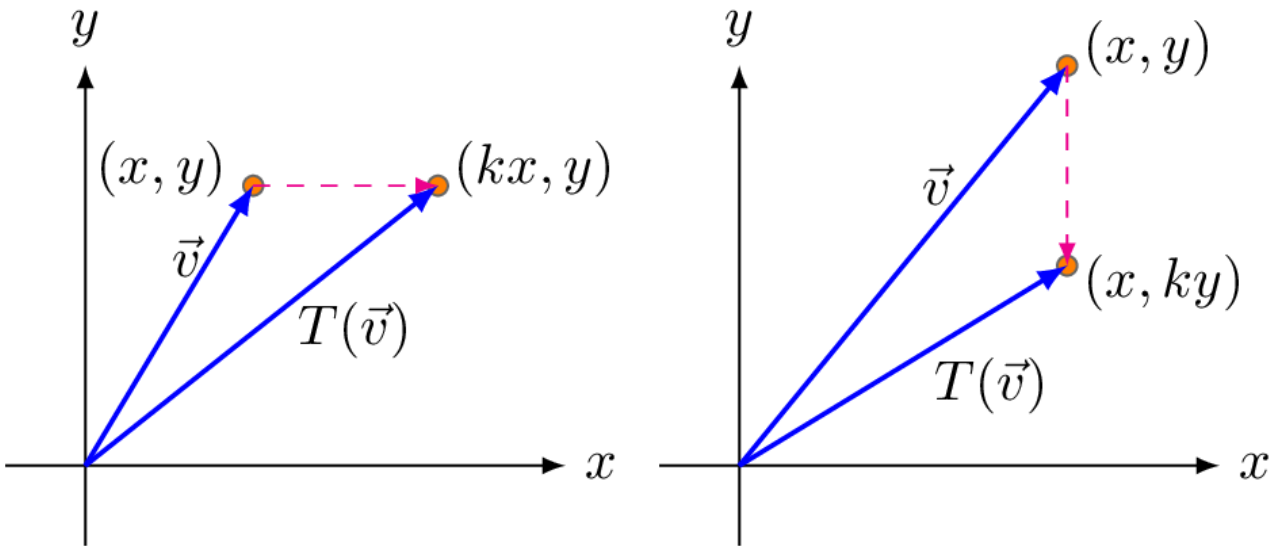


Figure 5.4.8: There are two images. In the first image, a 2D coordinate plane with x and y axes, illustrating the expansion of a vector $\vec{v}$ along the x-axis. The original vector $\vec{v}$ extends from the origin to the point (x, y), transforming into $T(\vec{v})$ given by (kx, y), showing that the x -coordinate has been scaled by a factor k, while the y-coordinate remains unchanged. In the second image, a 2D coordinate plane with x and y axes, illustrating the compression of a vector $\vec{v}$ along the y-axis. The original vector $\vec{v}$ extends from the origin to the point (x, y), transforming into $T(\vec{v})$ given by (x, ky), showing that the y-coordinate has been scaled by a factor k, while the x-coordinate remains unchanged. (CC BY-NC-SA; Atomic Math via Studio Moraine LLC)

Shears

Definition: Shear

A **shear transformation** in $\mathbb{R}^2$ is a linear transformation that shifts points in a fixed direction by an amount proportional to their perpendicular distance from a given axis. Shearing *preserves the area* of geometric shapes but distorts their angles, transforming rectangles into parallelograms.

Below are visual representations of a shear transformation in the x-direction with a positive shear factor, and a shear transformation in the y-direction with a negative shear factor.

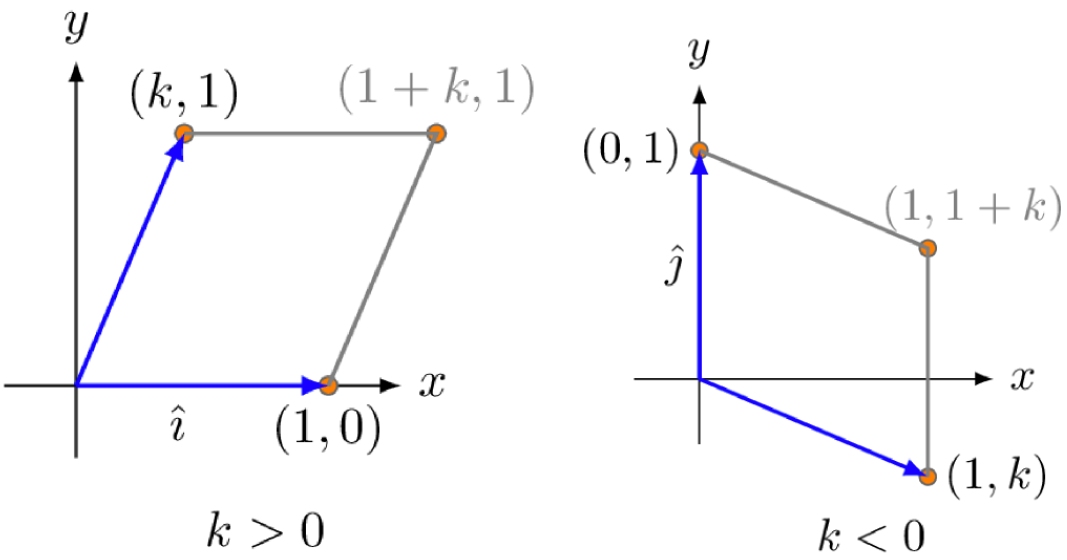


Figure 5.4.9: There are two images, each in 2D coordinate plane. In the first image, the standard unit vectors are shown, with $\hat{i}$ originally at $(1,0)$ and remaining unchanged, while the unit vector along the y-axis is transformed from $(0,1)$ to $(k,1)$, with $k>0$. In the second image, the standard unit vectors are shown, with $\hat{j}$ originally at $(0,1)$ and remaining unchanged, while the unit vector along the x-axis is transformed from $(1,0)$ to $(1,k)$, with $k<0$. (CC BY-NC-SA; Atomic Math via Studio Moraine LLC)

Answers

Theorem 5.4.1: Rotation

Let $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. These identify the geometric vectors which point along the positive x axis and positive y axis as shown.

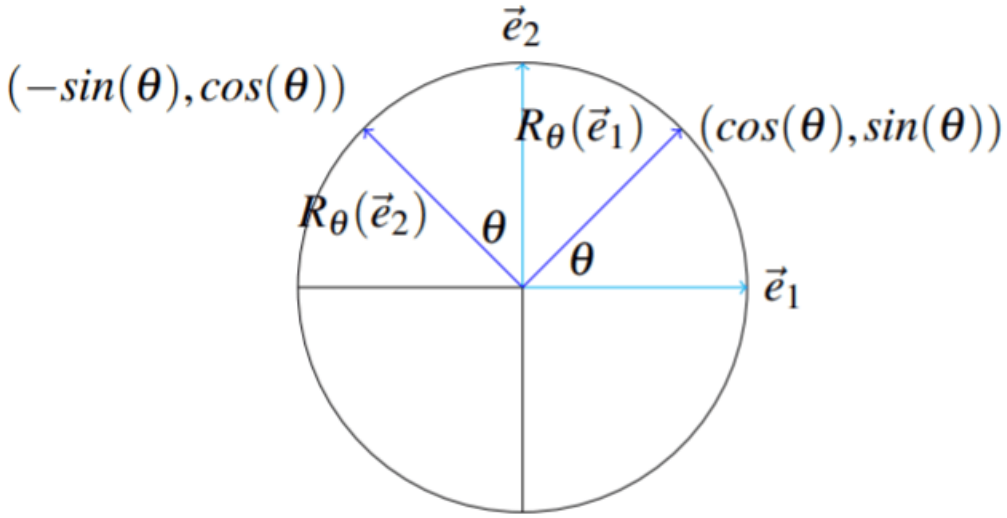


Figure 5.4.2 Standard basis vectors $\vec{e}_1, \vec{e}_2$ and the result of rotating them through angle θ (CC BY-NC-SA 4.0; Kuttler via A First Course in LINEAR ALGEBRA)

From Theorem 5.2.2, we need to find $R_\theta(\vec{e}_1)$ and $R_\theta(\vec{e}_2)$, and use these as the columns of the matrix A of T. We can use $\cos, \sin$ of the angle θ to find the coordinates of $R_\theta(\vec{e}_1)$ as shown in the above picture. The coordinates of $R_\theta(\vec{e}_2)$ also follow from trigonometry. Thus

$$R_\theta(\vec{e}_1) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, R_\theta(\vec{e}_2) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

Therefore, from Theorem 5.2.2,

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

We can also prove this algebraically without the use of the above picture. The definition of $[\cos[\theta], \sin[\theta]]$ is as the coordinates of the point of $R_\theta(\vec{e}_1)$. Now the point of the vector $\vec{e}_2$ is exactly $\pi/2$ further along the unit circle from the point of $\vec{e}_1$, and therefore after rotation through an angle of θ the coordinates x and y of the point of $R_\theta(\vec{e}_2)$ are given by

$$[x, y] = [\cos[\theta + \pi/2], \sin[\theta + \pi/2]] = [-\sin\theta, \cos\theta]$$

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5.4E: Exercises for Section 5.4

Exercise $5.4E.1$

Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $\frac{\pi}{3}$.

Answer available at the end of this section.

Exercise $5.4E.2$

Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $\frac{\pi}{4}$.

Answer available at the end of this section.

Exercise $5.4E.3$

Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $\frac{-\pi}{3}$.

Answer available at the end of this section.

Exercise $5.4E.4$

Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $\frac{2\pi}{3}$.

Answer available at the end of this section.

Exercise $5.4E.5$

Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $\frac{\pi}{12}$.

Hint: $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$

Answer available at the end of this section.

Exercise $5.4E.6$

Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $\frac{2\pi}{3}$ and then reflects across the x axis.

Answer available at the end of this section.

Exercise $5.4E.7$

Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $\frac{\pi}{3}$ and then reflects across the x axis.

Answer available at the end of this section.

Exercise $5.4E.8$

Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $\frac{\pi}{4}$ and then reflects across the x axis.

Answer available at the end of this section.

Exercise $5.4E.9$

Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $\frac{\pi}{6}$ and then reflects across the x axis followed by a reflection across the y axis.

Answer available at the end of this section.

Exercise $5.4E.10$

Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ across the x axis and then rotates every vector through an angle of $\frac{\pi}{4}$.

Answer available at the end of this section.

Exercise $5.4E.11$

Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ across the y axis and then rotates every vector through an angle of $\frac{\pi}{4}$.

Answer available at the end of this section.

Exercise $5.4E.12$

Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ across the x axis and then rotates every vector through an angle of $\frac{\pi}{6}$.

Answer available at the end of this section.

Exercise $5.4E.13$

Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ across the y axis and then rotates every vector through an angle of $\frac{\pi}{6}$.

Answer available at the end of this section.

Exercise $5.4E.14$

Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $\frac{5\pi}{12}$.

Answer

Answer available at the end of this section.

Exercise $5.4E.15$

Find the matrix of the linear transformation which rotates every vector in $\mathbb{R}^3$ counter clockwise about the z axis when viewed from the positive z axis through an angle of 30° and then reflects through the xy plane.

Answer available at the end of this section.

Exercise $5.4E.16$

Let $\vec{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ be a unit vector in $\mathbb{R}^2$. Find the matrix which reflects all vectors across this vector, as shown in the following picture.

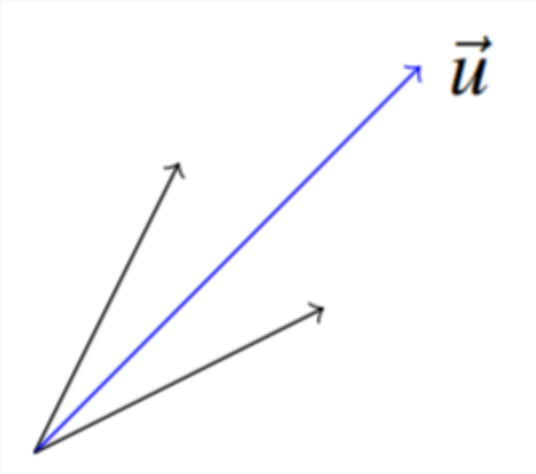


Figure $5.4E.1$ A vector u with an unnamed vector and its reflection through u all emananting from the same point.

Hint: Notice that $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ for some θ. First rotate through $-\theta$. Next reflect through the x axis. Finally rotate through θ.

Answer available at the end of this section.

Exercise $5.4E.17$

Find the matrix for $T(\vec{w}) = proj_{\vec{v}}(\vec{w})$ where $\vec{v} = \begin{bmatrix} 1 & -2 & 3 \end{bmatrix}^T$.

Answer available at the end of this section.

Exercise $5.4E.18$

Find the matrix for $T(\vec{w}) = proj_{\vec{v}}(\vec{w})$ where $\vec{v} = \begin{bmatrix} 1 & 5 & 3 \end{bmatrix}^T$.

Answer available at the end of this section.

Exercise $5.4E.19$

Find the matrix for $T(\vec{w}) = proj_{\vec{v}}(\vec{w})$ where $\vec{v} = \begin{bmatrix} 1 & 0 & 3 \end{bmatrix}^T$.

Answer available at the end of this section.

Exercise $5.4E.20$

Find the transformation matrix that reflects a vector across the x-axis, then applies a horizontal shear with factor k, and finally rotates the vector by 90° clockwise.

Answer available at the end of this section.

Exercise $5.4E.21$

Find the transformation matrix that first projects a vector onto the line $y = x$, then dilates the result by a factor of 2, and finally applies a vertical shear with factor m.

Answer available at the end of this section.

Exercise $5.4E.22$

Find the transformation matrix that rotates a vector by the 45°, then compresse it along the x-axis by a factor of k with \(0

Answer available at the end of this section.

Exercise $5.4E.23$

Find the transformation matrix that first contracts a vector by a factor c $(0 < c < 1)$, then projects the result onto the x-axis, and finally rotates it by 180°.

Answer available at the end of this section.

Answers

Exercise $5.4E.1$

$$\begin{bmatrix} \cos\left(\frac{\pi}{3}\right) & -\sin\left(\frac{\pi}{3}\right) \\ \sin\left(\frac{\pi}{3}\right) & \cos\left(\frac{\pi}{3}\right) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{bmatrix}$$

Exercise $5.4E.2$

$$\begin{bmatrix} \cos\left(\frac{\pi}{4}\right) & -\sin\left(\frac{\pi}{4}\right) \\ \sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix}$$

Exercise $5.4E.3$

$$\begin{bmatrix} \cos\left(-\frac{\pi}{3}\right) & -\sin\left(-\frac{\pi}{3}\right) \\ \sin\left(-\frac{\pi}{3}\right) & \cos\left(-\frac{\pi}{3}\right) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & \frac{1}{2} \end{bmatrix}$$

Exercise $5.4E.4$

$$\begin{bmatrix} \cos\left(\frac{2\pi}{3}\right) & -\sin\left(\frac{2\pi}{3}\right) \\ \sin\left(\frac{2\pi}{3}\right) & \cos\left(\frac{2\pi}{3}\right) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{bmatrix}$$

Exercise $5.4E.5$

$$\begin{aligned} &\begin{bmatrix} \cos\left(\frac{\pi}{3}\right) & -\sin\left(\frac{\pi}{3}\right) \\ \sin\left(\frac{\pi}{3}\right) & \cos\left(\frac{\pi}{3}\right) \end{bmatrix} \begin{bmatrix} \cos\left(-\frac{\pi}{4}\right) & -\sin\left(-\frac{\pi}{4}\right) \\ \sin\left(-\frac{\pi}{4}\right) & \cos\left(-\frac{\pi}{4}\right) \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{4}\sqrt{2}\sqrt{3} + \frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} - \frac{1}{4}\sqrt{2}\sqrt{3} \\ \frac{1}{4}\sqrt{2}\sqrt{3} - \frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2}\sqrt{3} + \frac{1}{4}\sqrt{2} \end{bmatrix} \end{aligned} \tag{5.4E.1}$$

Exercise $5.4E.6$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\left(\frac{2\pi}{3}\right) & -\sin\left(\frac{2\pi}{3}\right) \\ \sin\left(\frac{2\pi}{3}\right) & \cos\left(\frac{2\pi}{3}\right) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & \frac{1}{2} \end{bmatrix}$$

Exercise $5.4E.7$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\left(\frac{\pi}{3}\right) & -\sin\left(\frac{\pi}{3}\right) \\ \sin\left(\frac{\pi}{3}\right) & \cos\left(\frac{\pi}{3}\right) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{bmatrix}$$

Exercise $5.4E.8$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\left(\frac{\pi}{4}\right) & -\sin\left(\frac{\pi}{4}\right) \\ \sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ -\frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \end{bmatrix}$$

Exercise $5.4E.9$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\left(\frac{\pi}{6}\right) & -\sin\left(\frac{\pi}{6}\right) \\ \sin\left(\frac{\pi}{6}\right) & \cos\left(\frac{\pi}{6}\right) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\sqrt{3} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} \end{bmatrix}$$

Exercise $5.4E.10$

$$\begin{bmatrix} \cos\left(\frac{\pi}{4}\right) & -\sin\left(\frac{\pi}{4}\right) \\ \sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \end{bmatrix}$$

Exercise $5.4E.11$

$$\begin{bmatrix} \cos\left(\frac{\pi}{4}\right) & -\sin\left(\frac{\pi}{4}\right) \\ \sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right) \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix}$$

Exercise $5.4E.12$

$$\begin{bmatrix} \cos\left(\frac{\pi}{6}\right) & -\sin\left(\frac{\pi}{6}\right) \\ \sin\left(\frac{\pi}{6}\right) & \cos\left(\frac{\pi}{6}\right) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\sqrt{3} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2}\sqrt{3} \end{bmatrix}$$

Exercise $5.4E.13$

$$\begin{bmatrix} \cos\left(\frac{\pi}{6}\right) & -\sin\left(\frac{\pi}{6}\right) \\ \sin\left(\frac{\pi}{6}\right) & \cos\left(\frac{\pi}{6}\right) \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\sqrt{3} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2}\sqrt{3} \end{bmatrix}$$

Exercise $5.4E.14$

$$\begin{bmatrix} \cos\left(\frac{2\pi}{3}\right) & -\sin\left(\frac{2\pi}{3}\right) \\ \sin\left(\frac{2\pi}{3}\right) & \cos\left(\frac{2\pi}{3}\right) \end{bmatrix} \begin{bmatrix} \cos\left(-\frac{\pi}{4}\right) & -\sin\left(-\frac{\pi}{4}\right) \\ \sin\left(-\frac{\pi}{4}\right) & \cos\left(-\frac{\pi}{4}\right) \end{bmatrix} = \begin{bmatrix} \frac{1}{4}\sqrt{2}\sqrt{3} - \frac{1}{4}\sqrt{2} & -\frac{1}{4}\sqrt{2}\sqrt{3} - \frac{1}{4}\sqrt{2} \\ \frac{1}{4}\sqrt{2}\sqrt{3} + \frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2}\sqrt{3} - \frac{1}{4}\sqrt{2} \end{bmatrix}$$

Note that it doesn't matter about the order in this case.

Exercise $5.4E.15$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\left(\frac{\pi}{6}\right) & -\sin\left(\frac{\pi}{6}\right) & 0 \\ \sin\left(\frac{\pi}{6}\right) & \cos\left(\frac{\pi}{6}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Exercise $5.4E.16$

$$\begin{aligned} &\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos(-\theta) & -\sin(\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} \\ &= \begin{bmatrix} \cos^2\theta - \sin^2\theta & 2\cos\theta\sin\theta \\ 2\cos\theta\sin\theta & \sin^2\theta - \cos^2\theta \end{bmatrix} \end{aligned} \tag{5.4E.2}$$

Now to write in terms of (a, b), note that $a/\sqrt{a^2+b^2} = \cos\theta$, $b/\sqrt{a^2+b^2} = \sin\theta$. Now plug this in to the above. The result is

$$\begin{bmatrix} \frac{a^2-b^2}{a^2+b^2} & 2\frac{ab}{a^2+b^2} \\ 2\frac{ab}{a^2+b^2} & \frac{b^2-a^2}{a^2+b^2} \end{bmatrix} = \frac{1}{a^2+b^2} \begin{bmatrix} a^2-b^2 & 2ab \\ 2ab & b^2-a^2 \end{bmatrix}$$

Since this is a unit vector, $a^2 + b^2 = 1$ and so you get

$$\begin{bmatrix} a^2-b^2 & 2ab \\ 2ab & b^2-a^2 \end{bmatrix}$$

Exercise $5.4E.17$

Recall that $proj_{\vec{u}}(\vec{v}) = \frac{\vec{u}\bullet\vec{u}}{||\vec{u}||^2}\vec{u}$ and so the desired matrix has ith column equal to $proj_{\vec{u}}(\vec{e}_i)$. Therefore, the matrix desired is

$$\frac{1}{14}\begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & -6 \\ 3 & -6 & 9 \end{bmatrix}$$

Exercise $5.4E.18$

$$\frac{1}{35}\begin{bmatrix} 1 & 5 & 3 \\ 5 & 25 & 15 \\ 3 & 15 & 9 \end{bmatrix}$$

Exercise $5.4E.19$

$$\frac{1}{10}\begin{bmatrix} 1 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 9 \end{bmatrix}$$

Exercise $5.4E.20$

$M = \begin{bmatrix} 0 & 1 \\ 1 & k \end{bmatrix}$

Exercise $5.4E.21$

$M = \begin{bmatrix} 1 & 1 \\ m+1 & m+1 \end{bmatrix}$

Exercise $5.4E.22$

$M = \frac{1}{\sqrt{2}}\begin{bmatrix} -k & k \\ 1 & 1 \end{bmatrix}$

Exercise $5.4E.23$

$M = \begin{bmatrix} -c & 0 \\ 0 & 0 \end{bmatrix}$

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5.5: One-to-One and Onto Transformations

Learning Objectives

- Determine if a linear transformation is onto or one to one.

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation. We define the **range** or **image** of T as the set of vectors of $\mathbb{R}^m$ which are of the form $T(\vec{x})$ (equivalently, $A\vec{x}$) for some $\vec{x} \in \mathbb{R}^n$. It is common to write $T\mathbb{R}^n$, $T(\mathbb{R}^n)$, or $\mathrm{im}(T)$ to denote these vectors.

Lemma 5.5.1: Range of a Matrix Transformation

Let A be an $m \times n$ matrix where $A_1, \cdots, A_n$ denote the columns of A. Then, for a vector $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ in $\mathbb{R}^n$,

$$A\vec{x} = \sum_{k=1}^{n} x_k A_k$$

Therefore, $A(\mathbb{R}^n)$ is the collection of all linear combinations of these products.

Proof available at the end of this section.

This section is devoted to studying two important characterizations of linear transformations, called one to one and onto. We define them now.

Definition 5.5.1: One to One Transformation

Suppose $\vec{x}_1$ and $\vec{x}_2$ are vectors in $\mathbb{R}^n$. A linear transformation $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ is called **one to one** (often written as $1-1$) if whenever $\vec{x}_1 \neq \vec{x}_2$ it follows that :

$$T(\vec{x}_1) \neq T(\vec{x}_2)$$

Equivalently, if $T(\vec{x}_1) = T(\vec{x}_2)$, then $\vec{x}_1 = \vec{x}_2$. Thus, T is one to one if it never takes two different vectors to the same vector.

The second important characterization is called onto.

Definition 5.5.2: Onto Transformation

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation. Then T is called **onto** if whenever $\vec{y} \in \mathbb{R}^m$ there exists $\vec{x} \in \mathbb{R}^n$ such that $T(\vec{x}) = \vec{y}$.

We often call a linear transformation which is one-to-one an **injection**. Similarly, a linear transformation which is onto is often called a **surjection**.

The following proposition is an important result.

Proposition 5.5.1: One to One Transformation

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation. Then T is one to one if and only if $T(\vec{x}) = \vec{0}$ implies $\vec{x} = \vec{0}$.

Proof available at the end of this section.

Note that this proposition says that if $A = \begin{bmatrix} A_1 & \cdots & A_n \end{bmatrix}$ then A is one to one if and only if whenever

$$0 = \sum_{k=1}^{n} c_k A_k$$

it follows that each scalar $c_k = 0$.

We will now take a look at an example of a one to one and onto linear transformation.

Example 5.5.1: A One to One and Onto Linear Transformation

Suppose

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Then, $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation. Is T onto? Is it one to one?

Solution

Recall that because T can be expressed as matrix transformation, we know that T is a linear transformation. We will start by looking at onto. So suppose $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$. Does there exist $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$ such that $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$? If so, then since $\begin{bmatrix} a \\ b \end{bmatrix}$ is an arbitrary vector in $\mathbb{R}^2$, it will follow that T is onto.

This question is familiar to you. It is asking whether there is a solution to the equation

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

This is the same thing as asking for a solution to the following system of equations.

$$\begin{aligned} x + y &= a \\ x + 2y &= b \end{aligned}$$

Set up the augmented matrix and row reduce.

$$\begin{bmatrix} 1 & 1 & a \\ 1 & 2 & b \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2a - b \\ 0 & 1 & b - a \end{bmatrix} \tag{5.5.1}$$

You can see from this point that the system has a solution. Therefore, we have shown that for any a, b, there is a $\begin{bmatrix} x \\ y \end{bmatrix}$ such that $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$. Thus T is onto.

Now we want to know if T is one to one. By Proposition 5.5.1 it is enough to show that $A\vec{x} = 0$ implies $\vec{x} = 0$. Consider the system $A\vec{x} = 0$ given by:

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This is the same as the system given by

$$\begin{aligned} x + y &= 0 \\ x + 2y &= 0 \end{aligned}$$

We need to show that the solution to this system is $x = 0$ and $y = 0$. By setting up the augmented matrix and row reducing, we end up with

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

This tells us that $x = 0$ and $y = 0$. Returning to the original system, this says that if

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

then

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

In other words, $A\vec{x} = 0$ implies that $\vec{x} = 0$. By Proposition 5.5.1, A is one to one, and so T is also one to one.

We also could have seen that T is one to one from our above solution for onto. By looking at the matrix given by (5.5.1), you can see that there is a **unique** solution given by $x = 2a - b$ and $y = b - a$. Therefore, there is only one vector, specifically $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2a - b \\ b - a \end{bmatrix}$ such that $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$. Hence by Definition 5.5.1, T is one to one.

Example 5.5.2: An Onto Transformation

Let $T : \mathbb{R}^4 \mapsto \mathbb{R}^2$ be a linear transformation defined by

$$T \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a + d \\ b + c \end{bmatrix} \text{ for all } \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4$$

Prove that T is onto but not one to one.

Solution

You can prove that T is in fact linear.

To show that T is onto, let $\begin{bmatrix} x \\ y \end{bmatrix}$ be an arbitrary vector in $\mathbb{R}^2$. Taking the vector $\begin{bmatrix} x \\ y \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^4$ we have

$$T \begin{bmatrix} x \\ y \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x + 0 \\ y + 0 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

This shows that T is onto.

By Proposition 5.5.1 T is one to one if and only if $T(\vec{x}) = \vec{0}$ implies that $\vec{x} = \vec{0}$. Observe that

$$T \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 + -1 \\ 0 + 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

There exists a nonzero vector $\vec{x}$ in $\mathbb{R}^4$ such that $T(\vec{x}) = \vec{0}$. It follows that T is not one to one.

Example 5.5.3

Determine if the transformation given by the following is is one-to-one and/or onto:

a. $$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5x_1 + 3x_2 + 4x_3 \\ 20x_1 - 12x_2 - 16x_3 \end{bmatrix}$$

b. $$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_1 + 2x_2 + 3x_3 \\ 2x_1 - 2x_2 - 7x_3 \end{bmatrix}$$

- **Video Length**: 4 minutes 13 seconds.
- **Context**: This video demonstrates how to determine if a transformation is one-to-one and/or onto.

Solution

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

The above examples demonstrate a method to determine if a linear transformation T is one to one or onto. It turns out that the matrix A of T can provide this information.

Theorem 5.5.1: Matrix of a One to One or Onto Transformation

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation induced by the $m \times n$ matrix A. Then T is one to one if and only if the rank of A is n. T is onto if and only if the rank of A is m.

Consider Example 5.5.2. We showed that T was onto but not one to one. We can now use this theorem to determine this fact about T.

Example 5.5.4: An Onto Transformation

Let $T : \mathbb{R}^4 \mapsto \mathbb{R}^2$ be a linear transformation defined by

$$T\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a+d \\ b+c \end{bmatrix} \text{ for all } \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4$$

Prove that T is onto but not one to one.

Solution

Using Theorem 5.5.1 we can show that T is onto but not one to one from the matrix of T. Recall that to find the matrix A of T , we apply T to each of the standard basis vectors $\vec{e}_i$ of $\mathbb{R}^4$. The result is the 2×4 matrix A given by

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Fortunately, this matrix is already in reduced row-echelon form. The rank of A is 2. Therefore by the above theorem T is onto but not one to one.

Recall that if S and T are linear transformations, we can discuss their composite denoted $S \circ T$. The following examines what happens if both S and T are onto.

Example 5.5.4: Composite of Onto Transformations

Let $T : \mathbb{R}^k \mapsto \mathbb{R}^n$ and $S : \mathbb{R}^n \mapsto \mathbb{R}^m$ be linear transformations. If T and S are onto, then $S \circ T$ is onto.

Solution

Let $\vec{z} \in \mathbb{R}^m$. Since S is onto, there exists a vector $\vec{y} \in \mathbb{R}^n$ such that $S(\vec{y}) = \vec{z}$. Furthermore, since T is onto, there exists a vector $\vec{x} \in \mathbb{R}^k$ such that $T(\vec{x}) = \vec{y}$. Thus

$$\vec{z} = S(\vec{y}) = S(T(\vec{x})) = (ST)(\vec{x}),$$

showing that for each $\vec{z} \in \mathbb{R}^m$ there exists and $\vec{x} \in \mathbb{R}^k$ such that $(ST)(\vec{x}) = \vec{z}$. Therefore, $S \circ T$ is onto.

The next example shows the same concept with regards to one-to-one transformations.

Example 5.5.5: Composite of One to One Transformations

Let $T : \mathbb{R}^k \mapsto \mathbb{R}^n$ and $S : \mathbb{R}^n \mapsto \mathbb{R}^m$ be linear transformations. Prove that if T and S are one to one, then $S \circ T$ is one-to-one.

Solution

To prove that $S \circ T$ is one to one, we need to show that if $S(T(\vec{v})) = \vec{0}$ it follows that $\vec{v} = \vec{0}$. Suppose that $S(T(\vec{v})) = \vec{0}$. Since S is one to one, it follows that $T(\vec{v}) = \vec{0}$. Similarly, since T is one to one, it follows that $\vec{v} = \vec{0}$. Hence $S \circ T$ is one to one.

Answers

Lemma 5.5.1: Range of a Matrix Transformation

This follows from the definition of matrix multiplication.

Proposition 5.5.1: One to One Transformation

We need to prove two things here. First, we will prove that if T is one to one, then $T(\vec{x}) = \vec{0}$ implies that $\vec{x} = \vec{0}$. Second, we will show that if $T(\vec{x}) = \vec{0}$ implies that $\vec{x} = \vec{0}$, then it follows that T is one to one. Recall that a linear transformation has the property that $T(\vec{0}) = \vec{0}$.

Suppose first that T is one to one and consider $T(\vec{0})$.

$$T(\vec{0}) = T\left(\vec{0} + \vec{0}\right) = T(\vec{0}) + T(\vec{0})$$

and so, adding the additive inverse of $T(\vec{0})$ to both sides, one sees that $T(\vec{0}) = \vec{0}$. If $T(\vec{x}) = \vec{0}$ it must be the case that $\vec{x} = \vec{0}$ because it was just shown that $T(\vec{0}) = \vec{0}$ and T is assumed to be one to one.

Now assume that if $T(\vec{x}) = \vec{0}$, then it follows that $\vec{x} = \vec{0}$. If $T(\vec{v}) = T(\vec{u})$, then

$$T(\vec{v}) - T(\vec{u}) = T\left(\vec{v} - \vec{u}\right) = \vec{0}$$

which shows that $\vec{v} - \vec{u} = 0$. In other words, $\vec{v} = \vec{u}$, and T is one to one.

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5.5E: Exercises for Section 5.5

Exercise $5.5E.1$

Let T be a linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Is T one to one? Is T onto?

Exercise $5.5E.2$

Let T be a linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 2 \\ 2 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Is T one to one? Is T onto?

Exercise $5.5E.3$

Let T be a linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Is T one to one? Is T onto?

Exercise $5.5E.4$

Let T be a linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 3 & -5 \\ 2 & 0 & 2 \\ 2 & 4 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Is T one to one? Is T onto?

Exercise $5.5E.5$

Give an example of a 3×2 matrix with the property that the linear transformation determined by this matrix is one to one but not onto.

Answer available at the end of this section.

Exercise $5.5E.6$

Suppose A is an $m \times n$ matrix in which $m \leq n$. Suppose also that the rank of A equals m. Show that the transformation T determined by A maps $\mathbb{R}^n$ onto $\mathbb{R}^m$. Hint: The vectors $\vec{e}_1, \cdots, \vec{e}_m$ occur as columns in the reduced row-echelon form for A.

Answer available at the end of this section.

Exercise $5.5E.7$

Suppose A is an $m \times n$ matrix in which $m \geq n$. Suppose also that the rank of A equals n. Show that A is one to one. Hint: If not, there exists a vector, $\vec{x}$ such that $A\vec{x} = 0$, and this implies at least one column of A is a linear combination of the others. Show this would require the rank to be less than n.

Answer available at the end of this section.

Exercise $5.5E.8$

Explain why an $n \times n$ matrix A is both one to one and onto if and only if its rank is n.

Answer available at the end of this section.

Exercise $5.5E.9$

If $\{\vec{v}_1, \cdots, \vec{v}_r\}$ is linearly independent and T is a one to one linear transformation, show that $\{T(\vec{v}_1), \cdots, T(\vec{v}_r)\}$ is also linearly independent. Give an example which shows that if T is only linear, it can happen that, although $\{\vec{v}_1, \cdots, \vec{v}_r\}$ is linearly independent, $\{T(\vec{v}_1), \cdots, T(\vec{v}_r)\}$ is not. In fact, show that it can happen that each of the $T(\vec{v}_j)$ equals 0.

Exercise $5.5E.10$

Let V and W be subspaces of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively and let $T : V \to W$ be a linear transformation. Show that if T is onto W and if $\{\vec{v}_1, \cdots, \vec{v}_r\}$ is a basis for V, then $\text{span}\{T(\vec{v}_1), \cdots, T(\vec{v}_r)\} = W$.

Answers

Exercise $5.5E.5$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Exercise $5.5E.6$

This says that the columns of A have a subset of m vectors which are linearly independent. Therefore, this set of vectors is a basis for $\mathbb{R}^m$. It follows that the span of the columns is all of $\mathbb{R}^m$. Thus A is onto.

Exercise $5.5E.7$

The columns are independent. Therefore, A is one to one.

Exercise $5.5E.8$

The rank is n is the same as saying the columns are independent which is the same as saying A is one to one which is the same as saying the columns are a basis. Thus the span of the columns of A is all of $\mathbb{R}^n$ and so A is onto. If A is onto, then the columns must be linearly independent since otherwise the span of these columns would have dimension less than n and so the dimension of $\mathbb{R}^n$ would be less than n.

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5.6: Isomorphisms

Learning Objectives

- Determine if a linear transformation is an isomorphism.
- Determine if two subspaces of $\mathbb{R}^n$ are isomorphic.

Recall the definition of a linear transformation. Let V and W be two subspaces of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. A mapping $T : V \to W$ is called a **linear transformation** or **linear map** if it preserves the algebraic operations of addition and scalar multiplication. Specifically, if a, b are scalars and $\vec{x}, \vec{y}$ are vectors,

$$T\left(a\vec{x} + b\vec{y}\right) = aT(\vec{x}) + bT(\vec{y})$$

Consider the following important definition.

Definition 5.6.1: Isomorphism

A linear map T is called an **isomorphism** if the following two conditions are satisfied.

- T is one to one. That is, if $T(\vec{x}) = T(\vec{y})$, then $\vec{x} = \vec{y}$.
- T is onto. That is, if $\vec{w} \in W$, there exists $\vec{v} \in V$ such that $T(\vec{v}) = \vec{w}$.

Two such subspaces which have an isomorphism as described above are said to be **isomorphic.**

Since an isomorphism is both an injection (one to one) and onto (surjection), sometimes it is called a **bijection**.

Consider the following example of an isomorphism.

Example 5.6.1: Isomorphism

Let $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be defined by

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ x-y \end{bmatrix}$$

Show that T is an isomorphism.

Solution

To prove that T is an isomorphism we must show

1. T is a linear transformation;
2. T is one to one;
3. T is onto.

We proceed as follows.

1. T is a linear transformation:

   Let k, p be scalars.

$$\begin{aligned} T\left(k\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + p\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) &= T\left(\begin{bmatrix} kx_1 \\ ky_1 \end{bmatrix} + \begin{bmatrix} px_2 \\ py_2 \end{bmatrix}\right) \\ &= T\left(\begin{bmatrix} kx_1 + px_2 \\ ky_1 + py_2 \end{bmatrix}\right) \\ &= \begin{bmatrix} (kx_1 + px_2) + (ky_1 + py_2) \\ (kx_1 + px_2) - (ky_1 + py_2) \end{bmatrix} \\ &= \begin{bmatrix} (kx_1 + ky_1) + (px_2 + py_2) \\ (kx_1 - ky_1) + (px_2 - py_2) \end{bmatrix} \\ &= \begin{bmatrix} kx_1 + ky_1 \\ kx_1 - ky_1 \end{bmatrix} + \begin{bmatrix} px_2 + py_2 \\ px_2 - py_2 \end{bmatrix} \\ &= k\begin{bmatrix} x_1 + y_1 \\ x_1 - y_1 \end{bmatrix} + p\begin{bmatrix} x_2 + y_2 \\ x_2 - y_2 \end{bmatrix} \\ &= kT\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + pT\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) \end{aligned} \tag{5.6.1}$$

Therefore T is linear.

2. T is one to one:

   We need to show that if $T(\vec{x}) = \vec{0}$ for a vector $\vec{x} \in \mathbb{R}^2$, then it follows that $\vec{x} = \vec{0}$. Let $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$.

   $$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + y \\ x - y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

   This provides a system of equations given by

   $$\begin{aligned} x + y &= 0 \\ x - y &= 0 \end{aligned} \tag{5.6.2}$$

   You can verify that the solution to this system if $x = y = 0$. Therefore

   $$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

   and T is one to one.

3. T is onto:

   Let a, b be scalars. We want to check if there is always a solution to

   $$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + y \\ x - y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

   This can be represented as the system of equations

   $$\begin{aligned} x + y &= a \\ x - y &= b \end{aligned} \tag{5.6.3}$$

   Setting up the augmented matrix and row reducing gives

   $$\begin{bmatrix} 1 & 1 & a \\ 1 & -1 & b \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & \frac{a+b}{2} \\ 0 & 1 & \frac{a-b}{2} \end{bmatrix}$$

   This has a solution for all a, b and therefore T is onto.

Therefore T is an isomorphism.

An important property of isomorphisms is that its inverse is also an isomorphism.

Proposition 5.6.1: Inverse of an Isomorphism

Let $T : V \to W$ be an isomorphism and V, W be subspaces of $\mathbb{R}^n$. Then $T^{-1} : W \to V$ is also an isomorphism.

Proof available at the end of this section.

Another important result is that the composition of multiple isomorphisms is also an isomorphism.

Proposition 5.6.2: Composition of Isomorphisms

Let $T : V \to W$ and $S : W \to Z$ be isomorphisms where V, W, Z are subspaces of $\mathbb{R}^n$. Then $S \circ T$ defined by $(S \circ T)(\vec{v}) = S(T(\vec{v}))$ is also an isomorphism.

Proof available at the end of this section.

Consider two subspaces V and W, and suppose there exists an isomorphism mapping one to the other. In this way the two subspaces are related, which we can write as $V \sim W$. Then the previous two propositions together claim that $\sim$ is an equivalence relation. That is: $\sim$ satisfies the following conditions:

- $V \sim V$
- If $V \sim W$, it follows that $W \sim V$
- If $V \sim W$ and $W \sim Z$, then $V \sim Z$

We leave the verification of these conditions as an exercise.

Consider the following example.

Example 5.6.2: Matrix Isomorphism

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $T(\vec{x}) = A(\vec{x})$ where A is an invertible $n \times n$ matrix. Then T is an isomorphism.

Solution

The reason for this is that, since A is invertible, the only vector it sends to $\vec{0}$ is the zero vector. Hence if $A(\vec{x}) = A(\vec{y})$, then $A(\vec{x} - \vec{y}) = \vec{0}$ and so $\vec{x} = \vec{y}$. It is onto because if

$$\vec{y} \in \mathbb{R}^n, A\left(A^{-1}(\vec{y})\right) = \left(AA^{-1}\right)(\vec{y}) = \vec{y}.$$

In fact, all isomorphisms from $\mathbb{R}^n$ to $\mathbb{R}^n$ can be expressed as $T(\vec{x}) = A(\vec{x})$ where A is an invertible $n \times n$ matrix. One simply considers the matrix whose i^{th} column is $T\vec{e}_i$.

Recall that a basis of a subspace V is a set of linearly independent vectors which span V. The following fundamental lemma describes the relation between bases and isomorphisms.

Lemma 5.6.1: Mapping Bases

Let $T : V \to W$ be a linear transformation where V, W are subspaces of $\mathbb{R}^n$. If T is one to one, then it has the property that if $\{\vec{u}_1, \cdots, \vec{u}_k\}$ is linearly independent, so is $\{T(\vec{u}_1), \cdots, T(\vec{u}_k)\}$.

More generally, T is an isomorphism if and only if whenever $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a basis for V, it follows that $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ is a basis for W.

Proof available at the end of this section.

The following theorem illustrates a very useful idea for defining an isomorphism. Basically, if you know what it does to a basis, then you can construct the isomorphism.

Theorem 5.6.1: Isomorphic Subspaces

Suppose V and W are two subspaces of $\mathbb{R}^n$. Then the two subspaces are isomorphic if and only if they have the same dimension. In the case that the two subspaces have the same dimension, then for a linear map $T : V \to W$, the following are equivalent.

1. T is one to one.
2. T is onto.
3. T is an isomorphism.

Proof available at the end of this section.

Note the interesting way of defining a linear transformation in the first part of the argument by describing what it does to a basis and then "extending it linearly" to the entire subspace.

Example 5.6.4: Isomorphic Subspaces

Let $V = \mathbb{R}^3$ and let W denote

$$\text{span}\left\{\begin{bmatrix}1\\2\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\2\\0\end{bmatrix}\right\}$$

Show that V and W are isomorphic.

Solution

First observe that these subspaces are both of dimension 3 and so they are isomorphic by Theorem 5.6.1. The three vectors which span W are easily seen to be linearly independent by making them the columns of a matrix and row reducing to the reduced row-echelon form.

You can exhibit an isomorphism of these two spaces as follows.

$$T(\vec{e}_1) = \begin{bmatrix}1\\2\\1\\1\end{bmatrix}, T(\vec{e}_2) = \begin{bmatrix}0\\1\\0\\1\end{bmatrix}, T(\vec{e}_3) = \begin{bmatrix}1\\1\\2\\0\end{bmatrix}$$

and extend linearly. Recall that the matrix of this linear transformation is just the matrix having these vectors as columns. Thus the matrix of this isomorphism is

$$\begin{bmatrix}1&0&1\\2&1&1\\1&0&2\\1&1&0\end{bmatrix}$$

You should check that multiplication on the left by this matrix does reproduce the claimed effect resulting from an application by T.

Consider the following example.

Example 5.6.5: Finding the Matrix of an Isomorphism

Let $V = \mathbb{R}^3$ and let W denote

$$\text{span}\left\{\begin{bmatrix}1\\2\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\2\\0\end{bmatrix}\right\}$$

Let $T : V \mapsto W$ be defined as follows.

$$T\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}1\\2\\1\\1\end{bmatrix}, T\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}0\\1\\0\\1\end{bmatrix}, T\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1\\1\\2\\0\end{bmatrix}$$

Find the matrix of this isomorphism T.

Solution

First note that the vectors

$$\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}$$

are indeed a basis for $\mathbb{R}^3$ as can be seen by making them the columns of a matrix and using the reduced row-echelon form.

Now recall the matrix of T is a 4×3 matrix A which gives the same effect as T. Thus, from the way we multiply matrices,

$$A\begin{bmatrix}1&0&1\\1&1&1\\0&1&1\end{bmatrix} = \begin{bmatrix}1&0&1\\2&1&1\\1&0&2\\1&1&0\end{bmatrix}$$

Hence,

$$A = \begin{bmatrix}1&0&1\\2&1&1\\1&0&2\\1&1&0\end{bmatrix}\begin{bmatrix}1&0&1\\1&1&1\\0&1&1\end{bmatrix}^{-1} = \begin{bmatrix}1&0&0\\0&2&-1\\2&-1&1\\-1&2&-1\end{bmatrix}$$

Note how the span of the columns of this new matrix must be the same as the span of the vectors defining W.

This idea of defining a linear transformation by what it does on a basis works for linear maps which are not necessarily isomorphisms.

Example 5.6.6: Finding the Matrix of an Isomorphism

Let $V = \mathbb{R}^3$ and let W denote

$$\text{span}\left\{\begin{bmatrix}1\\0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\\2\end{bmatrix}\right\}$$

Let $T : V \mapsto W$ be defined as follows.

$$T\begin{bmatrix}1\\1\\0\end{bmatrix}=\begin{bmatrix}1\\0\\1\\1\end{bmatrix},T\begin{bmatrix}0\\1\\1\end{bmatrix}=\begin{bmatrix}0\\1\\0\\1\end{bmatrix},T\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}1\\1\\1\\2\end{bmatrix}$$

Find the matrix of this linear transformation.

Solution

Note that in this case, the three vectors which span W are not linearly independent. Nevertheless the above procedure will still work. The reasoning is the same as before. If A is this matrix, then

$$A\begin{bmatrix}1&0&1\\1&1&1\\0&1&1\end{bmatrix}=\begin{bmatrix}1&0&1\\0&1&1\\1&0&1\\1&1&2\end{bmatrix}$$

and so

$$A=\begin{bmatrix}1&0&1\\0&1&1\\1&0&1\\1&1&2\end{bmatrix}\begin{bmatrix}1&0&1\\1&1&1\\0&1&1\end{bmatrix}^{-1}=\begin{bmatrix}1&0&0\\0&0&1\\1&0&0\\1&0&1\end{bmatrix}$$

The columns of this last matrix are obviously not linearly independent.

Answers

Proposition 5.6.1**: Inverse of an Isomorphism**

Let T be an isomorphism. Since T is onto, a typical vector in W is of the form $T(\vec{v})$ where $\vec{v}\in V$. Consider then for a,b scalars,

$$T^{-1}\left(aT(\vec{v}_1)+bT(\vec{v}_2)\right)$$

where $\vec{v}_1,\vec{v}_2\in V$. Is this equal to

$$aT^{-1}\left(T(\vec{v}_1)\right)+bT^{-1}\left(T(\vec{v}_2)\right)=a\vec{v}_1+b\vec{v}_2?$$

Since T is one to one, this will be so if

$$T\left(a\vec{v}_1+b\vec{v}_2\right)=T\left(T^{-1}\left(aT(\vec{v}_1)+bT(\vec{v}_2)\right)\right)=aT(\vec{v}_1)+bT(\vec{v}_2).$$

However, the above statement is just the condition that T is a linear map. Thus T^{-1} is indeed a linear map. If $\vec{v}\in V$ is given, then $\vec{v}=T^{-1}\left(T(\vec{v})\right)$ and so T^{-1} is onto. If $T^{-1}(\vec{v})=0$, then

$$\vec{v}=T\left(T^{-1}(\vec{v})\right)=T(\vec{0})=\vec{0}$$

and so T^{-1} is one to one.

Proposition 5.6.2**: Composition of Isomorphisms**

Suppose $T:V\to W$ and $S:W\to Z$ are isomorphisms. Why is $S\circ T$ a linear map? For a,b scalars,

$$\begin{aligned}S\circ T\left(a\vec{v}_1+b(\vec{v}_2)\right)&=S\left(T\left(a\vec{v}_1+b\vec{v}_2\right)\right)=S\left(aT\vec{v}_1+bT\vec{v}_2\right)\\&=aS\left(T\vec{v}_1\right)+bS\left(T\vec{v}_2\right)=a\left(S\circ T\right)\left(\vec{v}_1\right)+b\left(S\circ T\right)\left(\vec{v}_2\right)\end{aligned}$$

Hence $S\circ T$ is a linear map. If $\left(S\circ T\right)\left(\vec{v}\right)=0$, then $S\left(T\left(\vec{v}\right)\right)=0$ and it follows that $T(\vec{v})=\vec{0}$ and hence, $\vec{v}=\vec{0}$. Thus $S\circ T$ is one to one. It remains to verify that it is onto. Let $\vec{z}\in Z$. Then since S is onto, there exists $\vec{w}\in W$ such that $S(\vec{w})=\vec{z}$. Also, since T is onto, there exists $\vec{v}\in V$ such that $T(\vec{v})=\vec{w}$. It follows that $S\left(T\left(\vec{v}\right)\right)=\vec{z}$ and so $S\circ T$ is also onto.

Lemma 5.6.1**: Mapping Bases**

First suppose that T is a linear transformation and is one to one and $\{\vec{u}_1,\cdots,\vec{u}_k\}$ is linearly independent. It is required to show that $\{T(\vec{u}_1),\cdots,T(\vec{u}_k)\}$ is also linearly independent. Suppose then that

$$\sum_{i=1}^{k}c_iT(\vec{u}_i)=\vec{0}$$

Then, since T is linear,

$$T\left(\sum_{i=1}^{n}c_i\vec{u}_i\right)=\vec{0}$$

Since T is one to one, it follows that

$$\sum_{i=1}^{n}c_i\vec{u}_i=0$$

Now the fact that $\{\vec{u}_1, \cdots, \vec{u}_n\}$ is linearly independent implies that each $c_i = 0$. Hence $\{T(\vec{u}_1), \cdots, T(\vec{u}_n)\}$ is linearly independent.

Now suppose that T is an isomorphism and $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a basis for V. It was just shown that $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ is linearly independent. It remains to verify that $\text{span}\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\} = W$. If $\vec{w} \in W$, then since T is onto there exists $\vec{v} \in V$ such that $T(\vec{v}) = \vec{w}$. Since $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a basis, it follows that there exists scalars $\{c_i\}_{i=1}^n$ such that

$$\sum_{i=1}^n c_i \vec{v}_i = \vec{v}.$$

Hence,

$$\vec{w} = T(\vec{v}) = T\left(\sum_{i=1}^n c_i \vec{v}_i\right) = \sum_{i=1}^n c_i T(\vec{v}_i)$$

It follows that $\text{span}\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\} = W$ showing that this set of vectors is a basis for W.

Next suppose that T is a linear transformation which takes a basis to a basis. This means that if $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a basis for V, it follows $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ is a basis for W. Then if $w \in W$, there exist scalars c_i such that $w = \sum_{i=1}^n c_i T(\vec{v}_i) = T\left(\sum_{i=1}^n c_i \vec{v}_i\right)$ showing that T is onto. If $T\left(\sum_{i=1}^n c_i \vec{v}_i\right) = \vec{0}$ then $\sum_{i=1}^n c_i T(\vec{v}_i) = \vec{0}$ and since the vectors $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ are linearly independent, it follows that each $c_i = 0$. Since $\sum_{i=1}^n c_i \vec{v}_i$ is a typical vector in V, this has shown that if $T(\vec{v}) = \vec{0}$ then $\vec{v} = \vec{0}$ and so T is also one to one. Thus T is an isomorphism.

Theorem $5.6.1$**: Isomorphic Subspaces**

Suppose first that these two subspaces have the same dimension. Let a basis for V be $\{\vec{v}_1, \cdots, \vec{v}_n\}$ and let a basis for W be $\{\vec{w}_1, \cdots, \vec{w}_n\}$. Now define T as follows.

$$T(\vec{v}_i) = \vec{w}_i$$

for $\sum_{i=1}^n c_i \vec{v}_i$ an arbitrary vector of V,

$$T\left(\sum_{i=1}^n c_i \vec{v}_i\right) = \sum_{i=1}^n c_i T\vec{v}_i = \sum_{i=1}^n c_i \vec{w}_i.$$

It is necessary to verify that this is well defined. Suppose then that

$$\sum_{i=1}^n c_i \vec{v}_i = \sum_{i=1}^n \hat{c}_i \vec{v}_i$$

Then

$$\sum_{i=1}^n \left(c_i - \hat{c}_i\right) \vec{v}_i = \vec{0}$$

and since $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a basis, $c_i = \hat{c}_i$ for each i. Hence

$$\sum_{i=1}^n c_i \vec{w}_i = \sum_{i=1}^n \hat{c}_i \vec{w}_i$$

and so the mapping is well defined. Also if a, b are scalars,

$$\begin{aligned} T\left(a\sum_{i=1}^n c_i \vec{v}_i + b\sum_{i=1}^n \hat{c}_i \vec{v}_i\right) &= T\left(\sum_{i=1}^n \left(ac_i + b\hat{c}_i\right) \vec{v}_i\right) = \sum_{i=1}^n \left(ac_i + b\hat{c}_i\right) \vec{w}_i \\ &= a\sum_{i=1}^n c_i \vec{w}_i + b\sum_{i=1}^n \hat{c}_i \vec{w}_i \\ &= aT\left(\sum_{i=1}^n c_i \vec{v}_i\right) + bT\left(\sum_{i=1}^n \hat{c}_i \vec{v}_i\right) \end{aligned} \tag{5.6.4}$$

Thus T is a linear transformation.

Now if

$$T\left(\sum_{i=1}^n c_i \vec{v}_i\right) = \sum_{i=1}^n c_i \vec{w}_i = \vec{0},$$

then since the $\{\vec{w}_1, \cdots, \vec{w}_n\}$ are independent, each $c_i = 0$ and so $\sum_{i=1}^n c_i \vec{v}_i = \vec{0}$ also. Hence T is one to one. If $\sum_{i=1}^n c_i \vec{w}_i$ is a vector in W, then it equals

$$\sum_{i=1}^n c_i T(\vec{v}_i) = T\left(\sum_{i=1}^n c_i \vec{v}_i\right)$$

showing that T is also onto. Hence T is an isomorphism and so V and W are isomorphic.

Next suppose $T : V \mapsto W$ is an isomorphism, so these two subspaces are isomorphic. Then for $\{\vec{v}_1, \cdots, \vec{v}_n\}$ a basis for V, it follows that a basis for W is $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ showing that the two subspaces have the same dimension.

Now suppose the two subspaces have the same dimension. Consider the three claimed equivalences.

First consider the claim that $1.) \Rightarrow 2.)$. If T is one to one and if $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a basis for V, then $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ is linearly independent. If it is not a basis, then it must fail to span W. But then there would exist $\vec{w} \notin \text{span}\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ and it follows that $\{T(\vec{v}_1), \cdots, T(\vec{v}_n), \vec{w}\}$ would be linearly independent which is impossible because there exists a basis for W of n vectors.

Hence $\text{span}\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\} = W$ and so $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ is a basis. If $\vec{w} \in W$, there exist scalars c_i such that

$$\vec{w} = \sum_{i=1}^n c_i T(\vec{v}_i) = T\left(\sum_{i=1}^n c_i \vec{v}_i\right)$$

showing that T is onto. This shows that $1.) \Rightarrow 2.)$.

Next consider the claim that $2.) \Rightarrow 3.)$. Since $2.)$ holds, it follows that T is onto. It remains to verify that T is one to one. Since T is onto, there exists a basis of the form $\{T(\vec{v}_i), \cdots, T(\vec{v}_n)\}$. Then it follows that $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is linearly independent. Suppose

$$\sum_{i=1}^n c_i \vec{v}_i = \vec{0}$$

Then

$$\sum_{i=1}^{n} c_i T(\vec{v}_i) = \vec{0}$$

Hence each $c_i = 0$ and so, $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a basis for V. Now it follows that a typical vector in V is of the form $\sum_{i=1}^{n} c_i \vec{v}_i$. If $T\left(\sum_{i=1}^{n} c_i \vec{v}_i\right) = \vec{0}$, it follows that

$$\sum_{i=1}^{n} c_i T(\vec{v}_i) = \vec{0}$$

and so, since $\{T(\vec{v}_i), \cdots, T(\vec{v}_n)\}$ is independent, it follows each $c_i = 0$ and hence $\sum_{i=1}^{n} c_i \vec{v}_i = \vec{0}$. Thus T is one to one as well as onto and so it is an isomorphism. If T is an isomorphism, it is both one to one and onto by definition so $3.)$ implies both $1.)$ and $2.)$.

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5.6E: Exercises for Section 5.6

Exercise $5.6E.1$

Define T: $\mathbb{R}^3 \to \mathbb{R}^3$ as follows.

$$T(\vec{x}) = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \vec{x}$$

where on the right, it is just matrix multiplication of the vector $\vec{x}$ which is meant. Explain why T is an isomorphism of $\mathbb{R}^3$ to $\mathbb{R}^3$.

Exercise $5.6E.2$

Suppose T: $\mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation given by

$$T(\vec{x}) = A\vec{x}$$

where A is a 3×3 matrix. Show that T is an isomorphism if and only if A is invertible.

Exercise $5.6E.3$

Suppose T: $\mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation given by

$$T(\vec{x}) = A\vec{x}$$

where A is an $m \times n$ matrix. Show that T is never an isomorphism if $m \neq n$. In particular, show that if $m > n$, T cannot be onto and if \(m

Exercise $5.6E.4$

Define $T : \mathbb{R}^2 \to \mathbb{R}^3$ as follows.

$$T(\vec{x}) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \vec{x}$$

a. Show that T is one to one.
b. Let $W = \text{im}(T)$. Show that T is an isomorphism of $\mathbb{R}^2$ and $\text{im}(T)$.

Exercise $5.6E.5$

Continuing with the previous exercise, find a 2×3 matrix A such that the restriction of A to $\text{im}(T)$ gives the same result as T^{-1} on $\text{im}(T)$.

Hint available at the end of this section.

Exercise $5.6E.6$

Continuing with the previous two exercises, now let $V = \text{span}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$ and let T: $V \to W$ be a linear transformation where

$$W = \text{span}\left\{\begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\\1\end{bmatrix}\right\}$$

and

$$T\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\0\\1\\0\end{bmatrix}, T\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right) = \begin{bmatrix}0\\1\\1\\1\end{bmatrix}$$

Explain why T is an isomorphism. Determine a matrix A which, when multiplied on the left gives the same result as T on V and a matrix B which delivers T^{-1} on W.

Hint available at the end of this section.

Exercise $5.6E.7$

Let $T : V \to W$ and $S : W \to Z$ be isomorphisms where V, W, Z are subspaces of $\mathbb{R}^n$. Show $(S \circ T)^{-1} = T^{-1} \circ S^{-1}$

Answers

Exercise $5.6E.5$

You might let A be such that $A\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$ and $A\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$.

Now find another vector $\vec{v} \in \mathbb{R}^3$ such that $\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \vec{v}\right\}$ is a basis. Then you could define $A\vec{v}$ to equal some vector in $\mathbb{R}^2$. Explain why there will be more than one such matrix A which will deliver the inverse isomorphism T^{-1} on $\text{im}(T)$.

Exercise $5.6E.6$

You need to have

$$A\begin{bmatrix}1&0\\0&1\\1&1\end{bmatrix} = \begin{bmatrix}1&0\\0&1\\1&1\\0&1\end{bmatrix}$$

Now enlarge $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}$ to obtain a basis for $\mathbb{R}^3$. You could add in $\begin{bmatrix}0\\0\\1\end{bmatrix}$ for example, and then pick another vector in $\mathbb{R}^4$ and let $A\begin{bmatrix}0\\0\\1\end{bmatrix}$ equal this other vector. Then you would have

$$A\begin{bmatrix}1&0&0\\0&1&0\\1&1&1\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\1&1&0\\0&1&1\end{bmatrix}$$

This would involve picking for the new vector in $\mathbb{R}^4$ the vector $\begin{bmatrix}0&0&0&1\end{bmatrix}^T$. Then you could find A. You can do something similar to find a matrix for T^{-1} denoted as B.

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5.7: The Kernel and Image of A Linear Map

Learning Objectives

- Describe the kernel and image of a linear transformation, and find a basis for each.

In this section we will consider the case where the linear transformation is not necessarily an isomorphism. First consider the following important definition.

Definition 5.7.1: Kernel and Image

Let V and W be subspaces of $\mathbb{R}^n$ and let $T : V \mapsto W$ be a linear transformation. Then the **image** of T denoted as $\mathrm{im}\,(T)$ is defined to be the set

$$\mathrm{im}\,(T) = \{T(\vec{v}) : \vec{v} \in V\}$$

In words, it consists of all vectors in W which equal $T(\vec{v})$ for some $\vec{v} \in V$.

The **kernel** of T, written $\ker(T)$, consists of all $\vec{v} \in V$ such that $T(\vec{v}) = \vec{0}$. That is,

$$\ker(T) = \left\{\vec{v} \in V : T(\vec{v}) = \vec{0}\right\}$$

It follows that $\mathrm{im}\,(T)$ and $\ker(T)$ are subspaces of W and V respectively.

Proposition 5.7.1: Kernel and Image as Subspaces

Let V, W be subspaces of $\mathbb{R}^n$ and let $T : V \to W$ be a linear transformation. Then $\ker(T)$ is a subspace of V and $\mathrm{im}\,(T)$ is a subspace of W.

Proof available at the end of this section.

We will now examine how to find the kernel and image of a linear transformation and describe the basis of each.

Example 5.7.1: Kernel and Image of a Linear Transformation

Let $T : \mathbb{R}^4 \mapsto \mathbb{R}^2$ be defined by

$$T\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a-b \\ c+d \end{bmatrix}$$

Then T is a linear transformation. Find a basis for $\ker(T)$ and $\mathrm{im}(T)$.

Solution

You can verify that T is a linear transformation.

First we will find a basis for $\ker(T)$. To do so, we want to find a way to describe all vectors $\vec{x} \in \mathbb{R}^4$ such that $T(\vec{x}) = \vec{0}$. Let $\vec{x} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ be such a vector. Then

$$T \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a-b \\ c+d \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The values of a, b, c, d that make this true are given by solutions to the system

$$\begin{aligned} a - b &= 0 \\ c + d &= 0 \end{aligned} \tag{5.7.1}$$

The solution to this system is $a = s, b = s, c = t, d = -t$ where s, t are scalars. We can describe $\ker(T)$ as follows.

$$\ker(T) = \left\{ \begin{bmatrix} s \\ s \\ t \\ -t \end{bmatrix} \right\} = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

Notice that this set is linearly independent and therefore forms a basis for $\ker(T)$.

We move on to finding a basis for $\text{im}(T)$. We can write the image of T as

$$\text{im}(T) = \left\{ \begin{bmatrix} a-b \\ c+d \end{bmatrix} \right\}$$

We can write this in the form

$$\text{span} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

This set is clearly not linearly independent. By removing unnecessary vectors from the set we can create a linearly independent set with the same span. This gives a basis for $\text{im}(T)$ as

$$\text{im}(T) = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

Recall that a linear transformation T is called one to one if and only if $T(\vec{x}) = \vec{0}$ implies $\vec{x} = \vec{0}$. Using the concept of kernel, we can state this theorem in another way.

Theorem 5.7.1: One to One and Kernel

Let T be a linear transformation where $\ker(T)$ is the kernel of T. Then T is one to one if and only if $\ker(T)$ consists of **only** the zero vector.

A major result is the relation between the dimension of the kernel and dimension of the image of a linear transformation. In the previous example $\ker(T)$ had dimension 2, and $\text{im}(T)$ also had dimension of 2. Is it a coincidence that the dimension of $\mathbb{R}^4$ is $4 = 2 + 2$? Consider the following theorem.

Theorem 5.7.2: Dimension of Kernel and Image

Let $T : V \to W$ be a linear transformation where V, W are subspaces of $\mathbb{R}^n$. Suppose the dimension of V is m. Then

$$m = \dim(\ker(T)) + \dim(\text{im}(T))$$

Proof available at the end of this section.

The above theorem leads to the next corollary.

Corollary 5.7.1

Let $T : V \to W$ be a linear transformation where V, W are subspaces of $\mathbb{R}^n$. Suppose the dimension of V is m. Then

$$\dim(\ker(T)) \leq m$$
$$\dim(\text{im}(T)) \leq m$$

This follows directly from the fact that $m = \dim(\ker(T)) + \dim(\text{im}(T))$.

Consider the following example.

Example 5.7.2

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by

$$T(\vec{x}) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{x}$$

Then $\text{im}(T) = V$ is a subspace of $\mathbb{R}^3$ and T is an isomorphism of $\mathbb{R}^2$ and V. Find a 2×3 matrix A such that the restriction of multiplication by A to $V = \text{im}(T)$ equals T^{-1}.

Solution

Since the two columns of the above matrix are linearly independent, we conclude that $\dim(\text{im}(T)) = 2$ and therefore $\dim(\ker(T)) = 2 - \dim(\text{im}(T)) = 2 - 2 = 0$ by Theorem 5.7.2. Then by Theorem 5.7.1 it follows that T is one to one.

Thus T is an isomorphism of $\mathbb{R}^2$ and the two dimensional subspace of $\mathbb{R}^3$ which is the span of the columns of the given matrix. Now in particular,

$$T(\vec{e}_1) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \; T(\vec{e}_2) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus

$$T^{-1} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \vec{e}_1, \; T^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \vec{e}_2$$

Extend T^{-1} to all of $\mathbb{R}^3$ by defining

$$T^{-1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \vec{e}_1$$

Notice that the vectors

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

are linearly independent so T^{-1} can be extended linearly to yield a linear transformation defined on $\mathbb{R}^3$. The matrix of T^{-1} denoted as A needs to satisfy

$$A\begin{bmatrix}1&0&0\\1&0&1\\0&1&0\end{bmatrix}=\begin{bmatrix}1&0&1\\0&1&0\end{bmatrix}$$

and so

$$A=\begin{bmatrix}1&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}1&0&0\\1&0&1\\0&1&0\end{bmatrix}^{-1}=\begin{bmatrix}0&1&0\\0&0&1\end{bmatrix}$$

Note that

$$\begin{bmatrix}0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$$

$$\begin{bmatrix}0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}$$

so the restriction to V of matrix multiplication by this matrix yields T^{-1}.

Answers

Proposition $5.7.1$**: Kernel and Image as Subspaces**

First consider $\ker(T)$. It is necessary to show that if $\vec{v}_1,\vec{v}_2$ are vectors in $\ker(T)$ and if a,b are scalars, then $a\vec{v}_1+b\vec{v}_2$ is also in $\ker(T)$. But

$$T(a\vec{v}_1+b\vec{v}_2)=aT(\vec{v}_1)+bT(\vec{v}_2)=a\vec{0}+b\vec{0}=\vec{0}$$

Thus $\ker(T)$ is a subspace of V.

Next suppose $T(\vec{v}_1),T(\vec{v}_2)$ are two vectors in $\mathrm{im}(T)$. Then if a,b are scalars,

$$aT(\vec{v}_1)+bT(\vec{v}_2)=T(a\vec{v}_1+b\vec{v}_2)$$

and this last vector is in $\mathrm{im}(T)$ by definition.

Theorem $5.7.2$**: Dimension of Kernel and Image**

From Proposition $5.7.1$, $\mathrm{im}(T)$ is a subspace of W. We know that there exists a basis for $\mathrm{im}(T)$, $\{T(\vec{v}_1),\cdots,T(\vec{v}_r)\}$. Similarly, there is a basis for $\ker(T)$, $\{\vec{u}_1,\cdots,\vec{u}_s\}$. Then if $\vec{v}\in V$, there exist scalars c_i such that

$$T(\vec{v})=\sum_{i=1}^{r}c_iT(\vec{v}_i)$$

Hence $T\left(\vec{v}-\sum_{i=1}^{r}c_i\vec{v}_i\right)=0$. It follows that $\vec{v}-\sum_{i=1}^{r}c_i\vec{v}_i$ is in $\ker(T)$. Hence there are scalars a_i such that

$$\vec{v}-\sum_{i=1}^{r}c_i\vec{v}_i=\sum_{j=1}^{s}a_j\vec{u}_j$$

Hence $\vec{v}=\sum_{i=1}^{r}c_i\vec{v}_i+\sum_{j=1}^{s}a_j\vec{u}_j$. Since $\vec{v}$ is arbitrary, it follows that

$$V=\mathrm{span}\{\vec{u}_1,\cdots,\vec{u}_s,\vec{v}_1,\cdots,\vec{v}_r\}$$

If the vectors $\{\vec{u}_1,\cdots,\vec{u}_s,\vec{v}_1,\cdots,\vec{v}_r\}$ are linearly independent, then it will follow that this set is a basis. Suppose then that

$$\sum_{i=1}^{r}c_i\vec{v}_i+\sum_{j=1}^{s}a_j\vec{u}_j=0$$

Apply T to both sides to obtain

$$\sum_{i=1}^{r}c_iT(\vec{v}_i)+\sum_{j=1}^{s}a_jT(\vec{u})_j=\sum_{i=1}^{r}c_iT(\vec{v}_i)=0$$

Since $\{T(\vec{v}_1),\cdots,T(\vec{v}_r)\}$ is linearly independent, it follows that each $c_i=0$. Hence $\sum_{j=1}^{s}a_j\vec{u}_j=0$ and so, since the $\{\vec{u}_1,\cdots,\vec{u}_s\}$ are linearly independent, it follows that each $a_j=0$ also. Therefore $\{\vec{u}_1,\cdots,\vec{u}_s,\vec{v}_1,\cdots,\vec{v}_r\}$ is a basis for V and so

$$n=s+r=\dim(\ker(T))+\dim(\mathrm{im}(T))$$

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5.7E: Exercises for Section 5.7

Exercise $5.7E.1$

Let $V = \mathbb{R}^3$ and let

$$W = \text{span}(S), \text{ where } S = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \right\}$$

Find a basis of W consisting of vectors in S.

Answer available at the end of this section.

Exercise $5.7E.2$

Let T be a linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Find a basis for $\ker(T)$ and $\text{im}(T)$.

Answer available at the end of this section.

Exercise $5.7E.3$

Let T be a linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Find a basis for $\ker(T)$ and $\text{im}(T)$.

Answer available at the end of this section.

Exercise $5.7E.4$

Let $V = \mathbb{R}^3$ and let

$$W = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \right\}$$

Extend this basis of W to a basis of V.

Answer available at the end of this section.

Exercise $5.7E.5$

Let T be a linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

What is $\dim(\ker(T))$?

Answer available at the end of this section.

Exercise $5.7E.6$

Define T: $\mathbb{R}^2 \to \mathbb{R}^3$ as follows.

$$T(\vec{x}) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \vec{x}$$

where on the right, it is just matrix multiplication of the vector $\vec{x}$ which is meant. Show that T is one to one. Next let $W = \text{im}(T)$. Show that T is an isomorphism of $\mathbb{R}^2$ and $\text{im}(T)$.

Exercise $5.7E.7$

In the above problem, find a 2×3 matrix A such that the restriction of A to $\text{im}(T)$ gives the same result as T^{-1} on $\text{im}(T)$.

Hint available at the end of this section.

Exercise $5.7E.8$

Let

$$V = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}\right\}$$

Define T: $V \to \mathbb{R}^4$ as follows:

Let $T(\vec{x}) = A\vec{x}$ where A is the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

Give a basis for $\text{im}(T)$.

Exercise $5.7E.9$

Let

$$V = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 4 \\ 1 \end{bmatrix}\right\}$$

Let $T(\vec{x}) = A\vec{x}$ where A is the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

Find a basis for $\mathrm{im}(T)$. In this case, the original vectors do not form an independent set.

Answer available at the end of this section.

Exercise $5.7E.10$

Define $T\colon \mathbb{R}^4 \to \mathbb{R}^3$ as follows.

$$T(\vec{x}) = \begin{bmatrix} 3 & 2 & 1 & 8 \\ 2 & 2 & -2 & 6 \\ 1 & 1 & -1 & 3 \end{bmatrix} \vec{x}$$

Find a basis for $\mathrm{im}(T)$. Also find a basis for $\ker(T)$.

Exercise $5.7E.11$

Define $T\colon \mathbb{R}^4 \to \mathbb{R}^3$ as follows.

$$T\vec{x} = \begin{bmatrix} 3 & 2 & 1 & 8 \\ 2 & 2 & -2 & 6 \\ 1 & 1 & -1 & 3 \end{bmatrix} \vec{x}$$

Find a basis for $\mathrm{im}(T)$. Also find a basis for $\ker(T)$

Answers

Exercise $5.7E.1$

In this case $\dim(W) = 1$ and a basis for W consisting of vectors in S can be obtained by taking any (nonzero) vector from S.

Exercise $5.7E.2$

A basis for $\ker(T)$ is $\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$ and a basis for $\mathrm{im}(T)$ is $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$. There are many other possibilities for the specific bases, but in this case $\dim(\ker(T)) = 1$ and $\dim(\mathrm{im}(T)) = 1$.

Exercise $5.7E.3$

In this case $\ker(T) = \{0\}$ and $\mathrm{im}(T) = \mathbb{R}^2$ (pick any basis of $\mathbb{R}^2$).

Exercise $5.7E.4$

There are many possible such extensions, one is (how do we know?):

$$\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$$

Exercise $5.7E.5$

We can easily see that $\dim(\mathrm{im}(T)) = 1$, and thus $\dim(\ker(T)) = 3 - \dim(\mathrm{im}(T)) = 3 - 1 = 2$.

Exercise $5.7E.7$

You might let A be such that

$$A\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, A\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

now find another vector $\vec{v} \in \mathbb{R}^3$ such that

$$\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \vec{v}\right\}$$

is a basis. You could pick

$$\vec{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

for example. Explain why this one works or one of your choice works. Then you could define $A\vec{v}$ to equal some vector in $\mathbb{R}^2$. Explain why there will be more than one such matrix A which will deliver the inverse isomorphism T^{-1} on $\mathrm{im}(T)$.

Exercise $5.7E.9$

Since the third vector is a linear combinations of the first two, then the image of the third vector will also be a linear combinations of the image of the first two. However the image of the first two vectors are linearly independent (check!), and hence form a basis of the image.

Thus a basis for $\mathrm{im}(T)$ is:

$$V = \mathrm{span}\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 4 \\ 5 \end{bmatrix} \right\}$$

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5.8: The Matrix of a Linear Transformation II

Learning Objectives

- Find the coordinates of a vector with respect to general bases.
- Find the matrix of a linear transformation with respect to general bases.

We begin this section with an important lemma.

Lemma 5.8.1: Mapping of a Basis

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^n$ be an isomorphism. Then T maps any basis of $\mathbb{R}^n$ to another basis for $\mathbb{R}^n$.

Conversely, if $T : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a linear transformation which maps a basis of $\mathbb{R}^n$ to another basis of $\mathbb{R}^n$, then it is an isomorphism.

Proof available at the end of this section.

Consider now an important definition.

Definition 5.8.1: Coordinate Vector

Let $B = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ be a basis for $\mathbb{R}^n$ and let $\vec{x}$ be an arbitrary vector in $\mathbb{R}^n$. Then $\vec{x}$ is uniquely represented as $\vec{x} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n$ for scalars $a_1, \cdots, a_n$.

The **coordinate vector** of $\vec{x}$ with respect to the basis B, written $C_B(\vec{x})$ or $[\vec{x}]_B$, is given by

$$C_B(\vec{x}) = C_B\left(a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n\right) = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

Consider the following example.

Example 5.8.1: Coordinate Vector

Let $B = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$ be a basis of $\mathbb{R}^2$ and let $\vec{x} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ be a vector in $\mathbb{R}^2$. Find $C_B(\vec{x})$.

Solution

First, note the order of the basis is important so label the vectors in the basis B as

$$B = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\} = \{\vec{v}_1, \vec{v}_2\}$$

Now we need to find a_1, a_2 such that $\vec{x} = a_1\vec{v}_1 + a_2\vec{v}_2$, that is:

$$\begin{bmatrix} 3 \\ -1 \end{bmatrix} = a_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + a_2\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Solving this system gives $a_1 = 2, a_2 = -1$. Therefore the coordinate vector of $\vec{x}$ with respect to the basis B is

$$C_B(\vec{x}) = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Given any basis B, one can easily verify that the coordinate function is actually an isomorphism.

Theorem 5.8.1: C_B is a Linear Transformation

For any basis B of $\mathbb{R}^n$, the coordinate function

$$C_B : \mathbb{R}^n \to \mathbb{R}^n$$

is a linear transformation, and moreover an isomorphism.

We now discuss the main result of this section, that is how to represent a linear transformation with respect to different bases.

Theorem 5.8.2: The Matrix of a Linear Transformation with Different Bases

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation, and let B_1 and B_2 be bases of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively.

Then the following holds

$$C_{B_2} T = M_{B_2 B_1} C_{B_1} \tag{5.8.1}$$

where $M_{B_2 B_1}$ is a unique $m \times n$ matrix.

If the basis B_1 is given by $B_1 = \{\vec{v}_1, \cdots, \vec{v}_n\}$ in this order, then

$$M_{B_2 B_1} = \left[C_{B_2}(T(\vec{v}_1)) \; C_{B_2}(T(\vec{v}_2)) \; \cdots C_{B_2}(T(\vec{v}_n)) \right]$$

Proof available at the end of this section.

Consider the following example.

Example 5.8.2: Matrix of a Linear Transformation with Different Bases

Let $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be a linear transformation defined by $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} b \\ a \end{bmatrix}$.

Consider the two bases

$$B_1 = \{\vec{v}_1, \vec{v}_2\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

and

$$B_2 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

Find the matrix $M_{B_2 B_1}$ of T with respect to the bases B_1 and B_2.

Solution

By Theorem 5.8.2, the columns of $M_{B_2 B_1}$ are the coordinate vectors of $T(\vec{v}_1), T(\vec{v}_2)$ with respect to B_2.

Since

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

a standard calculation yields

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \left(\frac{1}{2}\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left(-\frac{1}{2}\right)\begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

the first column of $M_{B_2B_1}$ is $\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$.

The second column is found in a similar way. We have

$$T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

and with respect to B_2 calculate:

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Hence the second column of $M_{B_2B_1}$ is given by $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. We thus obtain

$$M_{B_2B_1} = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \end{bmatrix}$$

We can verify that this is the correct matrix $M_{B_2B_1}$ on the specific example

$$\vec{v} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

First applying T gives

$$T(\vec{v}) = T\left(\begin{bmatrix} 3 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

and one can compute that

$$C_{B_2}\left(\begin{bmatrix} -1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

On the other hand, one compute $C_{B_1}(\vec{v})$ as

$$C_{B_1}\left(\begin{bmatrix} 3 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -1 \end{bmatrix},$$

and finally applying $M_{B_1B_2}$ gives

$$\begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

as above.

We see that the same vector results from either method, as suggested by Theorem 5.8.2.

If the bases B_1 and B_2 are equal, say B, then we write M_B instead of M_{BB}. The following example illustrates how to compute such a matrix. Note that this is what we did earlier when we considered only $B_1 = B_2$ to be the standard basis.

Example 5.8.3: Matrix of a Linear Transformation with respect to an Arbitrary Basis

Consider the basis B of $\mathbb{R}^3$ given by

$$B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right\}$$

And let $T : \mathbb{R}^3 \mapsto \mathbb{R}^3$ be the linear transformation defined on B as:

$$T\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}1\\-1\\1\end{bmatrix}, T\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1\\2\\-1\end{bmatrix}, T\begin{bmatrix}-1\\1\\0\end{bmatrix} = \begin{bmatrix}0\\1\\1\end{bmatrix}$$

1. Find the matrix M_B of T relative to the basis B.
2. Then find the usual matrix of T with respect to the standard basis of $\mathbb{R}^3$.

Solution

Equation (5.8.1) gives $C_B T = M_B C_B$, and thus $M_B = C_B T C_B^{-1}$.

Now $C_B(\vec{v}_i) = \vec{e}_i$, so the matrix of C_B^{-1} (with respect to the standard basis) is given by

$$\left[C_B^{-1}(\vec{e}_1)\ \ C_B^{-1}(\vec{e}_2)\ \ C_B^{-1}(\vec{e}_2)\right] = \begin{bmatrix}1&1&-1\\0&1&1\\1&1&0\end{bmatrix}$$

Moreover the matrix of TC_B^{-1} is given by

$$\left[TC_B^{-1}(\vec{e}_1)\ \ TC_B^{-1}(\vec{e}_2)\ \ TC_B^{-1}(\vec{e}_2)\right] = \begin{bmatrix}1&1&0\\-1&2&1\\1&-1&1\end{bmatrix}$$

Thus

$$\begin{aligned} M_B &= C_B T C_B^{-1} = [C_B^{-1}]^{-1}[TC_B^{-1}] \\ &= \begin{bmatrix}1&1&-1\\0&1&1\\1&1&0\end{bmatrix}^{-1}\begin{bmatrix}1&1&0\\-1&2&1\\1&-1&1\end{bmatrix} \\ &= \begin{bmatrix}2&-5&1\\-1&4&0\\0&-2&1\end{bmatrix} \end{aligned}$$

Consider how this works. Let $\vec{b} = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$ be an arbitrary vector in $\mathbb{R}^3$.

Apply C_B^{-1} to $\vec{b}$ to get

$$b_1\begin{bmatrix}1\\0\\1\end{bmatrix} + b_2\begin{bmatrix}1\\1\\1\end{bmatrix} + b_3\begin{bmatrix}-1\\1\\0\end{bmatrix}$$

Apply T to this linear combination to obtain

$$b_1\begin{bmatrix}1\\-1\\1\end{bmatrix} + b_2\begin{bmatrix}1\\2\\-1\end{bmatrix} + b_3\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}b_1+b_2\\-b_1+2b_2+b_3\\b_1-b_2+b_3\end{bmatrix}$$

Now take the matrix M_B of the transformation (as found above) and multiply it by $\vec{b}$.

$$\begin{bmatrix}2&-5&1\\-1&4&0\\0&-2&1\end{bmatrix}\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix} = \begin{bmatrix}2b_1-5b_2+b_3\\-b_1+4b_2\\-2b_2+b_3\end{bmatrix}$$

Is this the coordinate vector of the above relative to the given basis? We check as follows.

$$(2b_1-5b_2+b_3)\begin{bmatrix}1\\0\\1\end{bmatrix} + (-b_1+4b_2)\begin{bmatrix}1\\1\\1\end{bmatrix} + (-2b_2+b_3)\begin{bmatrix}-1\\1\\0\end{bmatrix}$$

$$= \begin{bmatrix} b_1 + b_2 \\ -b_1 + 2b_2 + b_3 \\ b_1 - b_2 + b_3 \end{bmatrix}$$

You see it is the same thing.

Now lets find the matrix of T with respect to the standard basis. Let A be this matrix. That is, multiplication by A is the same as doing T. Thus

$$A \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

Hence

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 3 & -3 \\ -3 & -2 & 4 \end{bmatrix}$$

Of course this is a very different matrix than the matrix of the linear transformation with respect to the non standard basis.

Answers

Lemma 5.8.1**: Mapping of a Basis**

First, suppose $T : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a linear transformation which is one to one and onto. Let $\{\vec{v}_1, \cdots, \vec{v}_n\}$ be a basis for $\mathbb{R}^n$. We wish to show that $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ is also a basis for $\mathbb{R}^n$.

First consider why it is linearly independent. Suppose $\sum_{k=1}^n a_k T(\vec{v}_k) = \vec{0}$. Then by linearity we have $T\left(\sum_{k=1}^n a_k \vec{v}_k\right) = \vec{0}$ and since T is one to one, it follows that $\sum_{k=1}^n a_k \vec{v}_k = \vec{0}$. This requires that each $a_k = 0$ because $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is independent, and it follows that $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ is linearly independent.

Next take $\vec{w} \in \mathbb{R}^n$. Since T is onto, there exists $\vec{v} \in \mathbb{R}^n$ such that $T(\vec{v}) = \vec{w}$. Since $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a basis, in particular it is a spanning set and there are scalars b_k such that $T\left(\sum_{k=1}^n b_k \vec{v}_k\right) = T(\vec{v}) = \vec{w}$. Therefore $\vec{w} = \sum_{k=1}^n b_k T(\vec{v}_k)$ which is in the $\operatorname{span}\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$. Therefore, $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ is a basis as claimed.

Suppose now that $T : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a linear transformation such that $T(\vec{v}_i) = \vec{w}_i$ where $\{\vec{v}_1, \cdots, \vec{v}_n\}$ and $\{\vec{w}_1, \cdots, \vec{w}_n\}$ are two bases for $\mathbb{R}^n$.

To show that T is one to one, let $T\left(\sum_{k=1}^n c_k \vec{v}_k\right) = \vec{0}$. Then $\sum_{k=1}^n c_k T(\vec{v}_k) = \sum_{k=1}^n c_k \vec{w}_k = \vec{0}$. It follows that each $c_k = 0$ because it is given that $\{\vec{w}_1, \cdots, \vec{w}_n\}$ is linearly independent. Hence $T\left(\sum_{k=1}^n c_k \vec{v}_k\right) = \vec{0}$ implies that $\sum_{k=1}^n c_k \vec{v}_k = \vec{0}$ and so T is one to one.

To show that T is onto, let $\vec{w}$ be an arbitrary vector in $\mathbb{R}^n$. This vector can be written as $\vec{w} = \sum_{k=1}^n d_k \vec{w}_k = \sum_{k=1}^n d_k T(\vec{v}_k) = T\left(\sum_{k=1}^n d_k \vec{v}_k\right)$. Therefore, T is also onto.

Theorem 5.8.2**: The Matrix of a Linear Transformation with Different Bases**

The above equation (5.8.1) can be represented by the following diagram.

$$\begin{array}{ccc} & T & \\ \mathbb{R}^n & \rightarrow & \mathbb{R}^m \\ C_{B_1} \downarrow & \circ & \downarrow C_{B_2} \\ \mathbb{R}^n & \rightarrow & \mathbb{R}^m \\ & M_{B_2 B_1} & \end{array}$$

Since C_{B_1} is an isomorphism, then the matrix we are looking for is the matrix of the linear transformation

$$C_{B_2} T C_{B_1}^{-1} : \mathbb{R}^n \mapsto \mathbb{R}^m.$$

By Theorem 5.2.2, the columns are given by the image of the standard basis $\{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n\}$. But since $C_{B_1}^{-1}(\vec{e}_i) = \vec{v}_i$, we readily obtain that

$$\begin{aligned} M_{B_2 B_1} &= \left[C_{B_2} T C_{B_1}^{-1}(\vec{e}_1) \;\; C_{B_2} T C_{B_1}^{-1}(\vec{2}_2) \;\; \cdots \;\; C_{B_2} T C_{B_1}^{-1}(\vec{e}_n)\right] \\ &= \left[C_{B_2}(T(\vec{v}_1)) \;\; C_{B_2}(T(\vec{v}_2)) \;\; \cdots \;\; C_{B_2}(T(\vec{v}_n))\right] \end{aligned}$$

and this completes the proof.

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5.8E: Exercises for Section 5.8

Exercise 5.8E.1

Let $B = \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}$ be a basis of $\mathbb{R}^2$ and let $\vec{x} = \begin{bmatrix} 5 \\ -7 \end{bmatrix}$ be a vector in $\mathbb{R}^2$. Find $C_B(\vec{x})$.

Exercise 5.8E.2

Let $B = \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right\}$ be a basis of $\mathbb{R}^3$ and let $\vec{x} = \begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix}$ be a vector in $\mathbb{R}^2$. Find $C_B(\vec{x})$.

Answer available at the end of this section.

Exercise 5.8E.3

Let T: $\mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by $T\left(\begin{bmatrix} a \\ b \end{bmatrix} \right) = \begin{bmatrix} a+b \\ a-b \end{bmatrix}$.

Consider the two bases

$$B_1 = \{\vec{v}_1, \vec{v}_2\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

and

$$B_2 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

Find the matrix $M_{B_2 B_1}$ of T with respect to the bases B_1 and B_2.

Answer available at the end of this section.

Exercise 5.8E.4

Consider the basis B of $\mathbb{R}^3$ given by

$$B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

And let $T : \mathbb{R}^3 \mapsto \mathbb{R}^3$ be the linear transformation defined on B as:

$$T\left(\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, T\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

1. Find the matrix M_B of T relative to the basis B.
2. Then find the usual matrix of T with respect to the standard basis of $\mathbb{R}^3$.

Exercise $5.8E.5$

Prove that for any basis B of $\mathbb{R}^n$, the coordinate function

$$C_B : \mathbb{R}^n \to \mathbb{R}^n$$

is a linear transformation.

Answers

Exercise $5.8E.2$

$$C_B(\vec{x}) = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

Exercise $5.8E.3$

$$M_{B_2B_1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$E

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5.9: The General Solution of a Linear System

Learning Objectives

- Use linear transformations to determine the particular solution and general solution to a system of equations.

Recall the definition of a linear transformation discussed above. T is a **linear transformation** if whenever $\vec{x}, \vec{y}$ are vectors and k, p are scalars,

$$T\left(k\vec{x} + p\vec{y}\right) = kT\left(\vec{x}\right) + pT\left(\vec{y}\right)$$

Thus linear transformations distribute across addition and pass scalars to the outside.

It turns out that we can use linear transformations to solve linear systems of equations. Indeed given a system of linear equations of the form $A\vec{x} = \vec{b}$, one may rephrase this as $T(\vec{x}) = \vec{b}$ where T is the linear transformation T_A induced by the coefficient matrix A. With this in mind consider the following definition.

Definition 5.9.1: Particular Solution of a System of Equations

Suppose a linear system of equations can be written in the form

$$T\left(\vec{x}\right) = \vec{b}$$

If $T\left(\vec{x}_p\right) = \vec{b}$, then $\vec{x}_p$ is called a **particular solution** of the linear system.

Recall that a system is called homogeneous if every equation in the system has the constant term 0. Suppose we represent a homogeneous system of equations by $T\left(\vec{x}\right) = \vec{0}$. It turns out that the $\vec{x}$ for which $T\left(\vec{x}\right) = \vec{0}$ are part of a special set called the **null space** of T. We may also refer to the null space as the **kernel** of T, and we write $\ker\left(T\right)$.

Consider the following definition.

Definition 5.9.2: Null Space or Kernel of a Linear Transformation

Let T be a linear transformation. Define

$$\ker\left(T\right) = \left\{\vec{x} : T\left(\vec{x}\right) = \vec{0}\right\}$$

The **kernel**, $\ker\left(T\right)$ consists of the set of all vectors $\vec{x}$ for which $T(\vec{x}) = \vec{0}$. This is also called the **null space** of T.

We may also refer to the kernel of T as the **solution space** of the equation $T\left(\vec{x}\right) = \vec{0}$.

Consider the following example.

Example 5.9.1: The Kernel of the Derivative

Let $\frac{d}{dx}$ denote the linear transformation defined on f, the functions which are defined on $\mathbb{R}$ and have a continuous derivative. Find $\ker\left(\frac{d}{dx}\right)$.

Solution

The example asks for functions f which has the property that $\frac{df}{dx} = 0$. As you may know from calculus, these functions are the constant functions. Thus $\ker\left(\frac{d}{dx}\right)$ is the set of constant functions.

Definition 5.9.2 states that $\ker(T)$ is the set of solutions to the equation,

$$T(\vec{x}) = \vec{0}$$

Since we can write $T(\vec{x})$ as $A\vec{x}$, you have been solving such equations for quite some time.

We have spent a lot of time finding solutions to systems of equations in general, as well as homogeneous systems. Suppose we look at a system given by $A\vec{x} = \vec{b}$, and consider the related homogeneous system. By this, we mean that we replace $\vec{b}$ by $\vec{0}$ and look at $A\vec{x} = \vec{0}$. It turns out that there is a very important relationship between the solutions of the original system and the solutions of the associated homogeneous system. In the following theorem, we use linear transformations to denote a system of equations. Remember that $T(\vec{x}) = A\vec{x}$.

Theorem 5.9.1: Particular Solution and General Solution

Suppose $\vec{x}_p$ is a solution to the linear system given by ,

$$T(\vec{x}) = \vec{b}$$

Then if $\vec{y}$ is any other solution to $T(\vec{x}) = \vec{b}$, there exists $\vec{x}_0 \in \ker(T)$ such that

$$\vec{y} = \vec{x}_p + \vec{x}_0$$

Hence, every solution to the linear system can be written as a sum of a particular solution, $\vec{x}_p$, and a solution $\vec{x}_0$ to the associated homogeneous system given by $T(\vec{x}) = \vec{0}$.

Proof available at the end of this section.

Sometimes people remember the above theorem in the following form. The solutions to the system $T(\vec{x}) = \vec{b}$ are given by $\vec{x}_p + \ker(T)$ where $\vec{x}_p$ is a particular solution to $T(\vec{x}) = \vec{b}$.

For now, we have been speaking about the kernel or null space of a linear transformation T. However, we know that every linear transformation T is determined by some matrix A. Therefore, we can also speak about the null space of a matrix. Consider the following example.

Example 5.9.2: The Null Space of a Matrix

Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 1 & 1 & 2 \\ 4 & 5 & 7 & 2 \end{bmatrix}$$

Find $\text{null}(A)$. Equivalently, find the solutions to the system of equations $A\vec{x} = \vec{0}$.

Solution

We are asked to find $\left\{\vec{x} : A\vec{x} = \vec{0}\right\}$. In other words we want to solve the system, $A\vec{x} = \vec{0}$. Let $\vec{x} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$. Then this amounts to solving

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 1 & 1 & 2 \\ 4 & 5 & 7 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This is the linear system

$$x+2y+3z=0$$
$$2x+y+z+2w=0$$
$$4x+5y+7z+2w=0$$

To solve, set up the augmented matrix and row reduce to find the reduced row-echelon form.

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 0 \\ 2 & 1 & 1 & 2 & 0 \\ 4 & 5 & 7 & 2 & 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & -\frac{1}{3} & \frac{4}{3} & 0 \\ 0 & 1 & \frac{5}{3} & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This yields $x=\frac{1}{3}z-\frac{4}{3}w$ and $y=\frac{2}{3}w-\frac{5}{3}z$. Since $\text{null}\,(A)$ consists of the solutions to this system, it consists vectors of the form,

$$\begin{bmatrix} \frac{1}{3}z-\frac{4}{3}w \\ \frac{2}{3}w-\frac{5}{3}z \\ z \\ w \end{bmatrix} = z\begin{bmatrix} \frac{1}{3} \\ -\frac{5}{3} \\ 1 \\ 0 \end{bmatrix} + w\begin{bmatrix} -\frac{4}{3} \\ \frac{2}{3} \\ 0 \\ 1 \end{bmatrix}$$

Consider the following example.

Example 5.9.3: A General Solution

The **general solution** of a linear system of equations is the set of all possible solutions. Find the general solution to the linear system,

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 1 & 1 & 2 \\ 4 & 5 & 7 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \\ 25 \end{bmatrix}$$

given that $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$ is one solution.

Solution

Note the matrix of this system is the same as the matrix in Example 5.9.2. Therefore, from Theorem 5.9.1, you will obtain all solutions to the above linear system by adding a particular solution $\vec{x}_p$ to the solutions of the associated homogeneous system, $\vec{x}$. One particular solution is given above by

$$\vec{x}_p = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

Using this particular solution along with the solutions found in Example 5.9.2, we obtain the following solutions,

$$z\begin{bmatrix} \frac{1}{3} \\ -\frac{5}{3} \\ 1 \\ 0 \end{bmatrix} + w\begin{bmatrix} -\frac{4}{3} \\ \frac{2}{3} \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

Hence, any solution to the above linear system is of this form.

Answers

Theorem $5.9.1$**: Particular Solution and General Solution**

Consider $\vec{y}-\vec{x}_p=\vec{y}+(-1)\vec{x}_p$. Then $T(\vec{y}-\vec{x}_p)=T(\vec{y})-T(\vec{x}_p)$. Since $\vec{y}$ and $\vec{x}_p$ are both solutions to the system, it follows that $T(\vec{y})=\vec{b}$ and $T(\vec{x}_p)=\vec{b}$. Hence, $T(\vec{y})-T(\vec{x}_p)=\vec{b}-\vec{b}=\vec{0}$. Let $\vec{x}_0=\vec{y}-\vec{x}_p$. Then, $T(\vec{x}_0)=\vec{0}$ so $\vec{x}_0$ is a solution to the associated homogeneous system and so is in $\ker(T)$.

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5.9E: Exercises for Section 5.9

Exercise $5.9E.1$

Write the solution set of the following system as a linear combination of vectors

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & -2 & 1 \\ 3 & -4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Answer available at the end of this section.

Exercise $5.9E.2$

Using Exercise $5.9E.1$ find the general solution to the following linear system.

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & -2 & 1 \\ 3 & -4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

Answer available at the end of this section.

Exercise $5.9E.3$

Write the solution set of the following system as a linear combination of vectors

$$\begin{bmatrix} 0 & -1 & 2 \\ 1 & -2 & 1 \\ 1 & -4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Answer available at the end of this section.

Exercise $5.9E.4$

Using Exercise $5.9E.3$ find the general solution to the following linear system.

$$\begin{bmatrix} 0 & -1 & 2 \\ 1 & -2 & 1 \\ 1 & -4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Answer available at the end of this section.

Exercise $5.9E.5$

Write the solution set of the following system as a linear combination of vectors.

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & -2 & 0 \\ 3 & -4 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Answer available at the end of this section.

Exercise $5.9E.6$

Using Exercise $5.9E.5$ find the general solution to the following linear system.

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & -2 & 0 \\ 3 & -4 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

Answer available at the end of this section.

Exercise $5.9E.7$

Write the solution set of the following system as a linear combination of vectors

$$\begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 1 \\ 1 & -2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Answer available at the end of this section.

Exercise $5.9E.8$

Using Exercise $5.9E.7$ find the general solution to the following linear system.

$$\begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 1 \\ 1 & -2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Answer available at the end of this section.

Exercise $5.9E.9$

Write the solution set of the following system as a linear combination of vectors

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & -1 & 1 & 0 \\ 3 & -1 & 3 & 2 \\ 3 & 3 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Answer available at the end of this section.

Exercise $5.9E.10$

Using Exercise $5.9E.9$ find the general solution to the following linear system.

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & -1 & 1 & 0 \\ 3 & -1 & 3 & 2 \\ 3 & 3 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 3 \end{bmatrix}$$

Answer available at the end of this section.

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Exercise $5.9E.11$

Write the solution set of the following system as a linear combination of vectors

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Answer available at the end of this section.

Exercise $5.9E.12$

Using Exercise $5.9E.11$ find the general solution to the following linear system.

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -3 \\ 0 \end{bmatrix}$$

Answer available at the end of this section.

Exercise $5.9E.13$

Write the solution set of the following system as a linear combination of vectors

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & -1 & 1 & 0 \\ 3 & 1 & 1 & 2 \\ 3 & 3 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Answer available at the end of this section.

Exercise $5.9E.14$

Using Exercise $5.9E.13$ find the general solution to the following linear system.

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & -1 & 1 & 0 \\ 3 & 1 & 1 & 2 \\ 3 & 3 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 3 \end{bmatrix}$$

Answer available at the end of this section.

Exercise $5.9E.15$

Write the solution set of the following system as a linear combination of vectors

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Answer available at the end of this section.

Exercise $5.9E.16$

Using Exercise $5.9E.15$ find the general solution to the following linear system.

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -3 \\ 1 \end{bmatrix}$$

Answer available at the end of this section.

Exercise $5.9E.17$

Suppose $A\vec{x} = \vec{b}$ has a solution. Explain why the solution is unique precisely when $A\vec{x} = \vec{0}$ has only the trivial solution.

Answer available at the end of this section.

Answers

Exercise $5.9E.1$

Solution is: $\begin{bmatrix} -3t \\ -t \\ t \end{bmatrix}$, $t \in \mathbb{R}$. A basis for the solution space is $\begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}$

Exercise $5.9E.2$

Note that this has the same matrix as the above problem. Solution is: $\begin{bmatrix} -3t \\ -t \\ t \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$, $t \in \mathbb{R}$

Exercise $5.9E.3$

Solution is: $\begin{bmatrix} 3t \\ 2t \\ t \end{bmatrix}$, A basis is $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

Exercise $5.9E.4$

Solution is: $\begin{bmatrix} 3t \\ 2t \\ t \end{bmatrix} + \begin{bmatrix} -3 \\ -1 \\ 0 \end{bmatrix}$, $\hat{t} \in \mathbb{R}$

Exercise $5.9E.5$

Solution is: $\begin{bmatrix} -4t \\ -2t \\ t \end{bmatrix}$. A basis is $\begin{bmatrix} -4 \\ -2 \\ 1 \end{bmatrix}$

Exercise $5.9E.6$

Solution is: $\begin{bmatrix} -4t \\ -2t \\ t \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$, $t \in \mathbb{R}$

Exercise $5.9E.7$

Solution is: $\begin{bmatrix} -t \\ 2t \\ t \end{bmatrix}$, $t \in \mathbb{R}$

Exercise $5.9E.8$

Solution is: $\begin{bmatrix} -t \\ 2t \\ t \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$

Exercise $5.9E.9$

Solution is: $\begin{bmatrix} 0 \\ -t \\ -t \\ t \end{bmatrix}$, $t \in \mathbb{R}$

Exercise $5.9E.10$

Solution is: $\begin{bmatrix} 0 \\ -t \\ -t \\ t \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ -1 \\ 0 \end{bmatrix}$

Exercise $5.9E.11$

Solution is: $\begin{bmatrix} -s-t \\ s \\ s \\ t \end{bmatrix}$, $s, t \in \mathbb{R}$. A basis is

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Exercise $5.9E.12$

Solution is:

$$\begin{bmatrix} -t \\ t \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} -8 \\ 5 \\ 0 \\ 5 \end{bmatrix}$$

Exercise $5.9E.13$

Solution is:

$$\begin{bmatrix} -\frac{1}{2}s - \frac{1}{2}t \\ \frac{1}{2}s - \frac{1}{2}t \\ s \\ t \end{bmatrix}$$

for $s, t \in \mathbb{R}$. A basis is

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Exercise $5.9E.14$

Solution is:

$$\begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2}s - \frac{1}{2}t \\ \frac{1}{2}s - \frac{1}{2}t \\ s \\ t \end{bmatrix}$$

Exercise $5.9E.15$

Solution is: $\begin{bmatrix} -t \\ t \\ t \\ 0 \end{bmatrix}$, a basis is $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$.

Exercise $5.9E.16$

Solution is: $\begin{bmatrix} -t \\ t \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} -9 \\ 5 \\ 0 \\ 6 \end{bmatrix}, t \in \mathbb{R}$.

Exercise $5.9E.17$

If not, then there would be a infintely many solutions to $A\vec{x} = \vec{0}$ and each of these added to a solution to $A\vec{x} = \vec{b}$ would be a solution to $A\vec{x} = \vec{b}$.

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6: SPECTRAL THEORY

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6.1: Eigenvalues and Eigenvectors of a Matrix

Learning Objectives

- Describe eigenvalues geometrically and algebraically.
- Find eigenvalues and eigenvectors for a square matrix.

Spectral Theory refers to the study of eigenvalues and eigenvectors of a matrix. It is of fundamental importance in many areas and is the subject of our study for this chapter.

Complex Numbers

In this section, we will work with the entire set of complex numbers, denoted by $\mathbb{C}$.

The complex numbers, $\mathbb{C}$, are numbers of the form $a+bi$ where a and b are real numbers and i is a root of the equation $x^2=-1$ For the number $z=a+bi$, a is called the real part of z, and b is called the imaginary part of z. $\mathbb{R}$ is the subset of numbers in $\mathbb{C}$ with imaginary part equal to 0.

The **sum** two complex numbers we get

$$(a+bi)+(c+di)=(a+c)+(b+d)i$$

That is, the real part of the sum of two numbers is the sum of the real parts of the numbers, and the imaginary part of the sum of two numbers is the sum of the imaginary parts of the numbers. When computing the **product** of two complex numbers, we can use the fact that $i^2=-1$ and get

$$(a+bi)(c+di)=(ac-bd)+(ad+bc)i$$

The **conjugate** of a complex number $z=a+bi$, denoted by $\bar{z}$, is the number $\bar{z}=a-bi$. Notice that

$$z+\bar{z}=2a \text{ and } z\bar{z}=a^2–b^2$$

Hence the sum of a complex number and its conjugate is a real number and the product of a complex number with its conjugate is a real number.

An important property of the complex numbers is that every polynomial of degree n can be completely factored into n polynomials of degree 1. If $p(x)$ is a polynomial of degree n and leading coefficient 1, $p(x)=(x-a_1)(x-a_2)\ldots(x-a_n)$. The roots of $p(x)=0$ are $a_1, a_2 \ldots a_n$. Of course, the a_i may not all be distinct. If the value a_i is repeated m times, then a_i said to be a root of multiplicity m. You can see from the factorization of $p(x)$ into n factors that the total of the multiplicity of the distinct roots will be n. If we consider only real numbers, we cannot always factor a polynomial into a product of first degree polynomials. For example $p(x)=x^2+1$ cannot be factored with real numbers and $x^2+1=0$ has no real roots. However, if we allow complex numbers, $x^2+1=(x–i)(x+i)$ and $x^2+1=0$ has two solutions, i, and $-i$.

Definition of Eigenvectors and Eigenvalues

To illustrate the idea behind what will be discussed, consider the following example.

Example 6.1.1: Eigenvectors and Eigenvalues

Let

$$A = \begin{bmatrix} 0 & 5 & -10 \\ 0 & 22 & 16 \\ 0 & -9 & -2 \end{bmatrix}$$

Compute the product $A\vec{x}$ for

$$\vec{x} = \begin{bmatrix} -5 \\ -4 \\ 3 \end{bmatrix}, \vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

What do you notice about $A\vec{x}$ in each of these products?

Solution

First, compute $A\vec{x}$ for

$$\vec{x} = \begin{bmatrix} -5 \\ -4 \\ 3 \end{bmatrix}$$

This product is given by

$$A\vec{x} = \begin{bmatrix} 0 & 5 & -10 \\ 0 & 22 & 16 \\ 0 & -9 & -2 \end{bmatrix} \begin{bmatrix} -5 \\ -4 \\ 3 \end{bmatrix} = \begin{bmatrix} -50 \\ -40 \\ 30 \end{bmatrix} = 10 \begin{bmatrix} -5 \\ -4 \\ 3 \end{bmatrix}$$

In this case, the product $A\vec{x}$ resulted in a vector which is equal to 10 times the vector $\vec{x}$. In other words, $A\vec{x} = 10\vec{x}$.

Let's see what happens in the next product. Compute $A\vec{x}$ for the vector

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

This product is given by

$$A\vec{x} = \begin{bmatrix} 0 & 5 & -10 \\ 0 & 22 & 16 \\ 0 & -9 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

In this case, the product $A\vec{x}$ resulted in a vector equal to 0 times the vector $\vec{x}$., $A\vec{x} = 0\vec{x}$.

Perhaps this matrix is such that $A\vec{x}$ results in $k\vec{x}$, for every vector $\vec{x}$. However, consider

$$\begin{bmatrix} 0 & 5 & -10 \\ 0 & 22 & 16 \\ 0 & -9 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 38 \\ -11 \end{bmatrix}$$

In this case, $A\vec{x}$ did not result in a vector of the form $k\vec{x}$ for some scalar k.

There is something special about the first two products calculated in Example $6.1.1$. Notice that for each, $A\vec{x} = k\vec{x}$ where k is some scalar. When this equation holds for some non-zero vector $\vec{x}$ and scalar k, we call the scalar k an **eigenvalue** of A. We often use the special symbol λ instead of k when referring to eigenvalues. In Example $6.1.1$, the values 10 and 0 are eigenvalues for the matrix A and we can label these as $\lambda_1 = 10$ and $\lambda_2 = 0$.

When $A\vec{x} = \lambda\vec{x}$ for some $\vec{x} \neq \vec{0}$, we call such an $\vec{x}$ an **eigenvector** of the matrix A. The eigenvectors of A are associated to an eigenvalue. Hence, if λ_1 is an eigenvalue of A and $A\vec{x} = \lambda_1\vec{x}$, we can label this eigenvector as $\vec{x}_1$. Note again that in order to be an eigenvector, $\vec{x}$ must be nonzero.

There is also a geometric significance to eigenvectors. When you have a **nonzero** vector which, when multiplied by a matrix results in another vector which is parallel to the first or equal to $\vec{0}$, this vector is called an eigenvector of the matrix. This is the meaning when the vectors are in $\mathbb{R}^n$.

The formal definition of eigenvalues and eigenvectors is as follows.

Definition 6.1.1: Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix and let $\vec{x} \in \mathbb{C}^n$ be a **nonzero vector** for which

$$A\vec{x} = \lambda\vec{x} \tag{6.1.1}$$

for some scalar λ. Then λ is called an **eigenvalue** of the matrix A and $\vec{x}$ is called an **eigenvector** of A associated with λ, or a λ-eigenvector of A.

The set of all eigenvalues of an $n \times n$ matrix A is denoted by $\sigma(A)$ and is referred to as the **spectrum** of A.

The eigenvectors of a matrix A are those vectors $\vec{x}$ for which multiplication by A results in a vector in the same direction or opposite direction to $\vec{x}$. Since the zero vector $\vec{0}$ has no direction this would make no sense for the zero vector. As noted above, $\vec{0}$ is never allowed to be an eigenvector.

Let's look at eigenvectors in more detail. Suppose $\vec{x}$ satisfies (6.1.1). Then

$$A\vec{x} - \lambda\vec{x} = \vec{0}$$
$$\text{or}$$
$$(A - \lambda I)\vec{x} = \vec{0}$$

for some $\vec{x} \neq \vec{x}$. Equivalently you could write $(\lambda I - A)\vec{x} = \vec{0}$, which is more commonly used. Hence, when we are looking for eigenvectors, we are looking for nontrivial solutions to this homogeneous system of equations!

Recall that the solutions to a homogeneous system of equations consist of basic solutions, and the linear combinations of those basic solutions. In this context, we call the basic solutions of the equation $(\lambda I - A)\vec{x} = \vec{x}$ **basic eigenvectors**. It follows that any (nonzero) linear combination of basic eigenvectors is again an eigenvector.

Suppose the matrix $(\lambda I - A)$ is invertible, so that $(\lambda I - A)^{-1}$ exists. Then the following equation would be true.

$$\begin{aligned} \vec{x} &= I\vec{x} \\ &= \left((\lambda I - A)^{-1}(\lambda I - A)\right)\vec{x} \\ &= (\lambda I - A)^{-1}\left((\lambda I - A)\vec{x}\right) \\ &= (\lambda I - A)^{-1}\vec{0} \\ &= \vec{0} \end{aligned} \tag{6.1.2}$$

This claims that $\vec{x} = \vec{0}$. However, we have required that $\vec{x} \neq \vec{0}$. Therefore $(\lambda I - A)$ cannot have an inverse!

Recall that if a matrix is not invertible, then its determinant is equal to 0. Therefore we can conclude that

$$\det(\lambda I - A) = 0 \tag{6.1.3}$$

Note that this is equivalent to $\det(A - \lambda I) = 0$.

The expression $\det(\lambda I - A)$ is a polynomial (in the variable λ) called the **characteristic polynomial of** A, and $\det(\lambda I - A) = 0$ is called the **characteristic equation**. For this reason we may also refer to the eigenvalues of A as **characteristic values**, but the former is often used for historical reasons.

The following theorem claims that the roots of the characteristic polynomial are the eigenvalues of A.

Theorem 6.1.1: The Existence of an Eigenvector

Let A be an $n \times n$ matrix and suppose $\det(\lambda I - A) = 0$ for some $\lambda \in \mathbb{C}$.
Then λ is an eigenvalue of A and thus there exists a nonzero vector $\vec{x} \in \mathbb{C}^n$ such that $A\vec{x} = \lambda\vec{x}$.

Proof available at the end of this section.

While the characteristic polynomial is guaranteed to have roots in the complex numbers, it may not have real roots. Hence, A may not have eigenvectors if we restrict ourself to real numbers.

Finding Eigenvectors and Eigenvalues

Now that eigenvalues and eigenvectors have been defined, we will study how to find them for a matrix A.

First, consider the following definition.

Definition 6.1.2: Algebraic Multiplicity of an Eigenvalue

Let A be an $n \times n$ matrix with characteristic polynomial given by $\det(\lambda I - A)$. Then, the **algebraic multiplicity** of an eigenvalue λ of A is the number of times λ occurs as a root of that characteristic polynomial.

In section 6.3 we will explore another concept of multiplicity called *geometric multiplicity*. In general, if we just refer to the multiplicity of an eigenvalue, we mean the algebraic multiplicity defined in 6.1.2.

For example, suppose the characteristic polynomial of A is given by $(\lambda - 2)^2$. Solving for the roots of this polynomial, we set $(\lambda - 2)^2 = 0$ and solve for λ. We find that $\lambda = 2$ is a root that occurs twice. Hence, in this case, $\lambda = 2$ is an eigenvalue of A of multiplicity equal to 2.

We will now look at how to find the eigenvalues and eigenvectors for a matrix A in detail. The steps used are summarized in the following procedure.

Procedure 6.1.1: Finding Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix.

1. First, find the eigenvalues λ of A by solving the equation $\det(\lambda I - A) = 0$.
2. For each λ, find the basic eigenvectors $\vec{x} \neq 0$ by finding the basic solutions to $(\lambda I - A)\vec{x} = 0$.

To verify your work, make sure that $A\vec{x} = \lambda\vec{x}$ for each λ and associated eigenvector $\vec{x}$.

We will explore these steps further in the following example.

Example 6.1.2: Find the Eigenvalues and Eigenvectors

Let $A = \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}$. Find its eigenvalues and eigenvectors.

Solution

We will use Procedure 6.1.1. First we find the eigenvalues of A by solving the equation

$$\det(\lambda I - A) = 0$$

This gives

$$\det\left(\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}\right) = 0$$

$$\det \begin{bmatrix} \lambda + 5 & -2 \\ 7 & \lambda - 4 \end{bmatrix} = 0 \tag{6.1.4}$$

Computing the determinant as usual, the result is

$$\lambda^2 + \lambda - 6 = 0$$

Solving this equation, we find that $\lambda_1 = 2$ and $\lambda_2 = -3$.

Now we need to find the basic eigenvectors for each λ. First we will find the eigenvectors for $\lambda_1 = 2$. We wish to find all vectors $\vec{x} \neq 0$ such that $A\vec{x} = 2\vec{x}$. These are the solutions to $(2I - A)\vec{x} = 0$.

$$\begin{aligned} \left(2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 7 & -2 \\ 7 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned} \tag{6.1.5}$$

The augmented matrix for this system and corresponding reduced row-echelon form are given by

$$\begin{bmatrix} 7 & -2 & 0 \\ 7 & -2 & 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & -\frac{2}{7} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The solution is any vector of the form

$$\begin{bmatrix} \frac{2}{7}s \\ s \end{bmatrix} = s\begin{bmatrix} \frac{2}{7} \\ 1 \end{bmatrix}$$

Multiplying this vector by 7 we obtain a simpler description for the solution to this system, given by

$$t\begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

This gives the basic eigenvector for $\lambda_1 = 2$ as

$$\begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

To check, we verify that $A\vec{x} = 2\vec{x}$ for this basic eigenvector.

$$\begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 4 \\ 14 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

This is what we wanted, so we know this basic eigenvector is correct.

Next we will repeat this process to find the basic eigenvector for $\lambda_2 = -3$. We wish to find all vectors $\vec{x} \neq 0$ such that $A\vec{x} = -3\vec{x}$. These are the solutions to $((-3)I - A)\vec{x} = 0$.

$$\begin{aligned} \left((-3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 2 & -2 \\ 7 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned} \tag{6.1.6}$$

The augmented matrix for this system and corresponding reduced row-echelon form are given by

$$\begin{bmatrix} 2 & -2 & 0 \\ 7 & -7 & 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The solution is any vector of the form

$$\begin{bmatrix} s \\ s \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

This gives the basic eigenvector for $\lambda_2 = -3$ as

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

To check, we verify that $A\vec{x} = -3\vec{x}$ for this basic eigenvector.

$$\begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

This is what we wanted, so we know this basic eigenvector is correct.

The following is an example using Procedure $6.1.1$ for a 3×3 matrix.

Example $6.1.3$: Find the Eigenvalues and Eigenvectors

Find the eigenvalues and eigenvectors for the matrix

$$A = \begin{bmatrix} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{bmatrix}$$

Solution

We will use Procedure $6.1.1$. First we need to find the eigenvalues of A. Recall that they are the solutions of the equation

$$\det(\lambda I - A) = 0$$

In this case the equation is

$$\det\left(\lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{bmatrix}\right) = 0$$

which becomes

$$\det \begin{bmatrix} \lambda - 5 & 10 & 5 \\ -2 & \lambda - 14 & -2 \\ 4 & 8 & \lambda - 6 \end{bmatrix} = 0$$

Using Laplace Expansion, compute this determinant and simplify. The result is the following equation.

$$(\lambda - 5)\left(\lambda^2 - 20\lambda + 100\right) = 0$$

Solving this equation, we find that the eigenvalues are $\lambda_1 = 5, \lambda_2 = 10$ and $\lambda_3 = 10$. Notice that 10 is a root of multiplicity two due to

$$\lambda^2 - 20\lambda + 100 = (\lambda - 10)^2$$

Therefore, $\lambda_2 = 10$ is an eigenvalue of multiplicity two.

Now that we have found the eigenvalues for A, we can compute the eigenvectors.

First we will find the basic eigenvectors for $\lambda_1 = 5$. In other words, we want to find all non-zero vectors $\vec{x}$ so that $A\vec{x} = 5\vec{x}$. This requires that we solve the equation $(5I - A)\vec{x} = 0$ for $\vec{x}$ as follows.

$$\left(5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{bmatrix}\right) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

That is you need to find the solution to

$$\begin{bmatrix} 0 & 10 & 5 \\ -2 & -9 & -2 \\ 4 & 8 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By now this is a familiar problem. You set up the augmented matrix and row reduce to get the solution. Thus the matrix you must row reduce is

$$\begin{bmatrix} 0 & 10 & 5 & 0 \\ -2 & -9 & -2 & 0 \\ 4 & 8 & -1 & 0 \end{bmatrix}$$

The reduced row-echelon form is

$$\begin{bmatrix} 1 & 0 & -\frac{5}{4} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and so the solution is any vector of the form

$$\begin{bmatrix} \frac{5}{4}s \\ -\frac{1}{2}s \\ s \end{bmatrix} = s \begin{bmatrix} \frac{5}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

where $s \in \mathbb{R}$. If we multiply this vector by 4, we obtain a simpler description for the solution to this system, as given by

$$t \begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix} \tag{6.1.7}$$

where $t \in \mathbb{R}$. Here, the basic eigenvector is given by

$$\vec{x}_1 = \begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix}$$

Notice that we cannot let $t = 0$ here, because this would result in the zero vector and eigenvectors are never equal to \vec{0}! Other than this value, every other choice of t in (6.1.7) results in an eigenvector.

It is a good idea to check your work! To do so, we will take the original matrix and multiply by the basic eigenvector $\vec{x}_1$. We check to see if we get $5\vec{x}_1$.

$$\begin{bmatrix} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 25 \\ -10 \\ 20 \end{bmatrix} = 5 \begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix}$$

This is what we wanted, so we know that our calculations were correct.

Next we will find the basic eigenvectors for $\lambda_2, \lambda_3 = 10$. These vectors are the basic solutions to the equation,

$$\left(10 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

That is you must find the solutions to

$$\begin{bmatrix} 5 & 10 & 5 \\ -2 & -4 & -2 \\ 4 & 8 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Consider the augmented matrix

$$\begin{bmatrix} 5 & 10 & 5 & 0 \\ -2 & -4 & -2 & 0 \\ 4 & 8 & 4 & 0 \end{bmatrix}$$

The reduced row-echelon form for this matrix is

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and so the eigenvectors are of the form

$$\begin{bmatrix} -2s-t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Note that you can't pick t and s both equal to zero because this would result in the zero vector and eigenvectors are never equal to zero.

Here, there are two basic eigenvectors, given by

$$\vec{x}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \vec{x}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Taking any (nonzero) linear combination of $\vec{x}_2$ and $\vec{x}_3$ will also result in an eigenvector for the eigenvalue $\lambda = 10$. As in the case for $\lambda = 5$, always check your work! For the first basic eigenvector, we can check $A\vec{x}_2 = 10\vec{x}_2$ as follows.

$$\begin{bmatrix} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -10 \\ 0 \\ 10 \end{bmatrix} = 10 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

This is what we wanted. Checking the second basic eigenvector, $\vec{x}_3$, is left as an exercise.

It is important to remember that for any eigenvector $\vec{x}$, $\vec{x} \neq 0$. However, it is possible to have eigenvalues equal to zero. This is illustrated in the following example.

Example 6.1.4: A Zero Eigenvalue

Let

$$A = \begin{bmatrix} 2 & 2 & -2 \\ 1 & 3 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

Find the eigenvalues and eigenvectors of A.

Solution

First we find the eigenvalues of A. We will do so using Definition 6.1.1.

In order to find the eigenvalues of A, we solve the following equation.

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - 2 & -2 & 2 \\ -1 & \lambda - 3 & 1 \\ 1 & -1 & \lambda - 1 \end{bmatrix} = 0$$

This reduces to $\lambda^3 - 6\lambda^2 + 8\lambda = 0$. You can verify that the solutions are $\lambda_1 = 0, \lambda_2 = 2, \lambda_3 = 4$. Notice that while eigenvectors can never equal $\vec{0}$, it is possible to have an eigenvalue equal to 0.

Now we will find the basic eigenvectors. For $\lambda_1 = 0$, we need to solve the equation $(0I - A)\vec{x} = \vec{0}$. This equation becomes $-A\vec{x} = \vec{0}$, and so the augmented matrix for finding the solutions is given by

$$\begin{bmatrix} -2 & -2 & 2 & 0 \\ -1 & -3 & 1 & 0 \\ 1 & -1 & -1 & 0 \end{bmatrix}$$

The reduced row-echelon form is

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, the eigenvectors are of the form $t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ where $t \neq 0$ and the basic eigenvector is given by

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

We can verify that this eigenvector is correct by checking that the equation $A\vec{x}_1 = 0\vec{x}_1$ holds. The product $A\vec{x}_1$ is given by

$$A\vec{x}_1 = \begin{bmatrix} 2 & 2 & -2 \\ 1 & 3 & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This clearly equals $0\vec{x}_1$, so the equation holds. Hence, $A\vec{x}_1 = 0\vec{x}_1$ and so 0 is an eigenvalue of A.

Computing the other basic eigenvectors is left as an exercise.

In the following section, we examine ways to simplify this process of finding eigenvalues and eigenvectors by using properties of special types of matrices.

Answers

Theorem $6.1.1$**: The Existence of an Eigenvector**

For A an $n \times n$ matrix, the method of Laplace Expansion demonstrates that $\det(\lambda I - A)$ is a polynomial of degree n. As such, the equation $(6.1.3)$ has a solution $\lambda \in \mathbb{C}$ by the Fundamental Theorem of Algebra. The fact that λ is an eigenvalue is left as an exercise.

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6.1E: Exercises for Section 6.1

Exercise $6.1E.1$

If A is an invertible $n \times n$ matrix, compare the eigenvalues of A and A^{-1}. More generally, for m an arbitrary integer, compare the eigenvalues of A and A^m.

Answer available at the end of this section.

Exercise $6.1E.2$

If A is an $n \times n$ matrix and c is a nonzero constant, compare the eigenvalues of A and cA.

Answer available at the end of this section.

Exercise $6.1E.3$

Let $A,\ B$ be invertible $n \times n$ matrices which commute. That is, $AB = BA$. Suppose $\vec{x}$ is an eigenvector of B. Show that then $A\vec{x}$ must also be an eigenvector for B.

Answer available at the end of this section.

Exercise $6.1E.4$

Suppose A is an $n \times n$ matrix and it satisfies $A^m = A$ for some m a positive integer larger than 1. Show that if λ is an eigenvalue of A then $|\lambda|$ equals either 0 or 1.

Answer available at the end of this section.

Exercise $6.1E.5$

Show that if $A\vec{x} = \lambda\vec{x}$ and $A\vec{y} = \lambda\vec{y}$, then whenever $k,\ p$ are scalars,

$$A(k\vec{x} + p\vec{y}) = \lambda(k\vec{x} + p\vec{y})$$

Does this imply that $k\vec{x} + p\vec{y}$ is an eigenvector? Explain.

Answer available at the end of this section.

Exercise $6.1E.6$

Suppose A is a 3×3 matrix and the following information is available.

$$A\begin{bmatrix}0\\-1\\-1\end{bmatrix}=0\begin{bmatrix}0\\-1\\-1\end{bmatrix}$$
$$A\begin{bmatrix}1\\1\\1\end{bmatrix}=-2\begin{bmatrix}1\\1\\1\end{bmatrix} \quad (6.1E.1)$$
$$A\begin{bmatrix}-2\\-3\\-2\end{bmatrix}=-2\begin{bmatrix}-2\\-3\\-2\end{bmatrix}$$

Find $A\begin{bmatrix}1\\-4\\3\end{bmatrix}$.

Exercise $6.1E.7$

Suppose A is a 3×3 matrix and the following information is available.

$$A\begin{bmatrix}-1\\-2\\-2\end{bmatrix}=1\begin{bmatrix}-1\\-2\\-2\end{bmatrix}$$
$$A\begin{bmatrix}1\\1\\1\end{bmatrix}=0\begin{bmatrix}1\\1\\1\end{bmatrix} \quad (6.1E.2)$$
$$A\begin{bmatrix}-1\\-4\\-3\end{bmatrix}=2\begin{bmatrix}-1\\-4\\-3\end{bmatrix}$$

Find $A\begin{bmatrix}3\\-4\\3\end{bmatrix}$.

Exercise $6.1E.8$

Suppose A is a 3×3 matrix and the following information is available.

$$A\begin{bmatrix}0\\-1\\-1\end{bmatrix}=2\begin{bmatrix}0\\-1\\-1\end{bmatrix}$$
$$A\begin{bmatrix}1\\1\\1\end{bmatrix}=1\begin{bmatrix}1\\1\\1\end{bmatrix} \quad (6.1E.3)$$
$$A\begin{bmatrix}-3\\-5\\-4\end{bmatrix}=-3\begin{bmatrix}-3\\-5\\-4\end{bmatrix}$$

Find $A\begin{bmatrix}2\\-3\\3\end{bmatrix}$.

Exercise $6.1E.9$

Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} -6 & -92 & 12 \\ 0 & 0 & 0 \\ -2 & -31 & 4 \end{bmatrix}$$

One eigenvalue is -2.

Exercise $6.1E.10$

Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} -2 & -17 & -6 \\ 0 & 0 & 0 \\ 1 & 9 & 3 \end{bmatrix}$$

One eigenvalue is 1.

Exercise $6.1E.11$

Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 9 & 2 & 8 \\ 2 & -6 & -2 \\ -8 & 2 & -5 \end{bmatrix}$$

One eigenvalue is -3.

Exercise $6.1E.12$

Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 6 & 76 & 16 \\ -2 & -21 & -4 \\ 2 & 64 & 17 \end{bmatrix}$$

One eigenvalue is -2.

Exercise $6.1E.13$

Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 3 & 5 & 2 \\ -8 & -11 & -4 \\ 10 & 11 & 3 \end{bmatrix}$$

One eigenvalue is -3.

Exercise $6.1E.14$

Is it possible for a nonzero matrix to have only 0 as an eigenvalue?

Answer available at the end of this section.

Exercise $6.1E.15$

If A is the matrix of a linear transformation which rotates all vectors in $\mathbb{R}^2$ through 60°, explain why A cannot have any real eigenvalues. Is there an angle such that rotation through this angle would have a real eigenvalue? What eigenvalues would be obtainable in this way?

Exercise $6.1E.16$

Let T be the linear transformation which rotates all vectors in $\mathbb{R}^2$ counterclockwise through an angle of $\pi/2$. Find a matrix of T and then find eigenvalues and eigenvectors.

Answer available at the end of this section.

Exercise $6.1E.17$

Let A be the 2×2 matrix of the linear transformation that rotates all vectors in $\mathbb{R}^2$ through an angle of θ. For which values of θ does A have a real eigenvalue?

Answer available at the end of this section.

Exercise $6.1E.18$

Let T be the linear transformation on $\mathbb{R}^2$ that reflects vectors about the x axis. Find a matrix for T and then find its eigenvalues and eigenvectors.

Answer available at the end of this section.

Exercise $6.1E.19$

Let T be the linear transformation on $\mathbb{R}^2$ that reflects vectors about the line $y = -x$. Find a matrix for T and then find its eigenvalues and eigenvectors.

Answer available at the end of this section.

Exercise $6.1E.20$

Let T be the linear transformation on $\mathbb{R}^2$ that projects vectors on to the line $y = x$. Find a matrix for T and then find its eigenvalues and eigenvectors.

Answer available at the end of this section.

Exercise $6.1E.21$

Let T be the linear transformation which reflects all vectors in $\mathbb{R}^3$ through the xy plane. Find a matrix for T and then obtain its eigenvalues and eigenvectors.

Answer available at the end of this section.

Exercise $6.1E.22$

Suppose the characteristic polynomial of an $n \times n$ matrix A is $1 - x^n$. Find A^{mn} where m is an integer.

Answer available at the end of this section.

Exercise $6.1E.23$

Suppose A is an $n \times n$ matrix consisting entirely of real entries but $a + ib$ is a complex eigenvalue having the eigenvector, $\vec{x} + i\vec{y}$ Here $\vec{x}$ and $\vec{y}$ are real vectors. Show that then $a - ib$ is also an eigenvalue with the eigenvector, $\vec{x} - i\vec{y}$. Hint: You should remember that the conjugate of a product of complex numbers equals the product of the conjugates. Here $a + ib$ is a complex number whose conjugate equals $a - ib$.

Answer available at the end of this section.

Answers

Exercise $6.1E.1$

$A^m\vec{x} = \lambda^m\vec{x}$ for any integer. In the case of A^{-1} : $A^{-1}\lambda\vec{x} = AA^{-1}\vec{x} = \vec{x}$ so $A^{-1}\vec{x} = \lambda^{-1}\vec{x}$. Thus the eigenvalues of A^{-1} are just λ^{-1} where λ is an eigenvalue of A.

Exercise $6.1E.2$

Say $A\vec{x} = \lambda\vec{x}$. Then $cA\vec{x} = c\lambda\vec{x}$ and so the eigenvalues of cA are just $c\lambda$ where λ is an eigenvalue of A.

Exercise $6.1E.3$

$BA\vec{x} = AB\vec{x} = A\lambda\vec{x} = \lambda A\vec{x}$. Here it is assumed that $B\vec{x} = \lambda\vec{x}$.

Exercise $6.1E.4$

Let $\vec{x}$ be the eigenvector. Then $A^m\vec{x} = \lambda^m\vec{x}$, $A^m\vec{x} = A\vec{x} = \lambda\vec{x}$ and so

$$\lambda^m = \lambda$$

Hence if $\lambda \neq 0$, then

$$\lambda^{m-1} = 1$$

and so $|\lambda| = 1$.

Exercise $6.1E.5$

The formula follows from properties of matrix multiplications. However, this vector might not be an eigenvector because it might equal 0 and eigenvectors cannot equal 0.

Exercise $6.1E.14$

Yes. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ works.

Exercise $6.1E.16$

The matrix of T is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. The eigenvectors and eigenvalues are:

$$\left\{\begin{bmatrix} -i \\ 1 \end{bmatrix}\right\} \leftrightarrow -i, \left\{\begin{bmatrix} i \\ 1 \end{bmatrix}\right\} \leftrightarrow i$$

Exercise $6.1E.17$

When you think of this geometrically, it is clear that the only two values of θ are 0 and π or these added to integer multiples of 2π.

Exercise $6.1E.18$

The matrix of T is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. The eigenvectors and eigenvalues are:

$$\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\} \leftrightarrow -1, \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\} \leftrightarrow 1$$

Exercise $6.1E.19$

Exercise $6.1E.20$

The matrix of T is $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$. The eigenvectors and eigenvalues are:

$$\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\} \leftrightarrow 0, \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\} \leftrightarrow 1$$

Exercise $6.1E.21$

The matrix of T is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. The eigenvectors and eigenvalues are:

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \leftrightarrow -1, \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \leftrightarrow 1$$

The eigenvalues are distinct because they are the nth roots of 1. Hence if $\vec{x}$ is a given vector with

$$\vec{x} = \sum_{j=1}^{n} a_j \vec{v}_j$$

then

$$A^{nm}\vec{x} = A^{nm} \sum_{j=1}^{n} a_j \vec{v}_j = \sum_{j=1}^{n} a_j A^{nm} \vec{v}_j = \sum_{j=1}^{n} a_j \vec{v}_j = \vec{x}$$

so $A^{nm} = I$.

Exercise $6.1E.23$

$A\vec{x} = (a+ib)\vec{x}$. Now take conjugates of both sides. Since A is real,

$$A\overline{\vec{x}} = (a-ib)\overline{\vec{x}}$$

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6.2: Eigenvalues and Eigenvectors for Special Matrices

Learning Objectives

- Find eigenvalues and eigenvectors for similar matrices, elementary matrices, and triangular matrices.

There are three special kinds of matrices which we can use to simplify the process of finding eigenvalues and eigenvectors. Throughout this section, we will discuss similar matrices, elementary matrices, as well as triangular matrices.

Similar Matrices

We begin with a definition.

Definition 6.2.3: Similar Matrices

Let A and B be $n \times n$ matrices. Suppose there exists an invertible matrix P such that

$$A = P^{-1}BP$$

Then A and B are called **similar matrices**.

It turns out that we can use the concept of similar matrices to help us find the eigenvalues of matrices. Consider the following lemma.

Lemma 6.2.1: Similar Matrices and Eigenvalues

Let A and B be similar matrices, so that $A = P^{-1}BP$ where A, B are $n \times n$ matrices and P is invertible. Then A, B have the same eigenvalues.

Proof available at the end of this section.

Note that this proof also demonstrates that the eigenvectors of A and B will (generally) be *different*. We see in the proof that $A\vec{x} = \lambda\vec{x}$, while $B(P\vec{x}) = \lambda(P\vec{x})$. Therefore, for an eigenvalue λ, A will have the eigenvector $\vec{x}$ while B will have the eigenvector $P\vec{x}$.

Elementary Matrices

The second special type of matrices we discuss in this section is elementary matrices. Recall that an elementary matrix E is obtained by applying one row operation to the identity matrix.

It is possible to use elementary matrices to find a simpler similar matrix before searching for its eigenvalues and eigenvectors. This is illustrated in the following example.

Example 6.2.5: Simplify Using Elementary Matrices

Find the eigenvalues for the matrix

$$A = \begin{bmatrix} 33 & 105 & 105 \\ 10 & 28 & 30 \\ -20 & -60 & -62 \end{bmatrix}$$

Solution

This matrix has big numbers and therefore we would like to simplify as much as possible before computing the eigenvalues.

We will do so using row operations. First, add 2 times the second row to the third row. To do so, left multiply A by $E(2\times 2+3)$. Then right multiply A by the inverse of $E(2\times 2+3)$ as illustrated.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 33 & 105 & 105 \\ 10 & 28 & 30 \\ -20 & -60 & -62 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 33 & -105 & 105 \\ 10 & -32 & 30 \\ 0 & 0 & -2 \end{bmatrix}$$

By Lemma $6.2.1$, the resulting matrix has the same eigenvalues as A where here, the matrix $E(2\times 2+3)$ plays the role of P.

We do this step again, as follows. In this step, we use the elementary matrix obtained by adding -3 times the second row to the first row, or $E(-3\times 2+1)$.

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 33 & -105 & 105 \\ 10 & -32 & 30 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 15 \\ 10 & -2 & 30 \\ 0 & 0 & -2 \end{bmatrix} \tag{6.2.1}$$

Again by Lemma $6.2.1$, this resulting matrix has the same eigenvalues as A. At this point, we can easily find the eigenvalues. Let

$$B = \begin{bmatrix} 3 & 0 & 15 \\ 10 & -2 & 30 \\ 0 & 0 & -2 \end{bmatrix}$$

Then, we find the eigenvalues of B (and therefore of A) by solving the equation $\det(\lambda I - B) = 0$. You should verify that this equation becomes

$$(\lambda+2)(\lambda+2)(\lambda-3) = 0$$

Solving this equation results in eigenvalues of $\lambda_1 = -2, \lambda_2 = -2$, and $\lambda_3 = 3$. Therefore, these are also the eigenvalues of A.

Through using elementary matrices, we were able to create a matrix for which finding the eigenvalues was easier than for A. At this point, you could go back to the original matrix A and solve $(\lambda I - A)\vec{x} = 0$ to obtain the eigenvectors of A.

Notice that when you multiply on the right by an elementary matrix, you are doing the column operation defined by the elementary matrix. In $(6.2.1)$ multiplication by the elementary matrix on the right merely involves taking three times the first column and adding to the second. Thus, without referring to the elementary matrices, the transition to the new matrix in $(6.2.1)$ can be illustrated by

$$\begin{bmatrix} 33 & -105 & 105 \\ 10 & -32 & 30 \\ 0 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 3 & -9 & 15 \\ 10 & -32 & 30 \\ 0 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 3 & 0 & 15 \\ 10 & -2 & 30 \\ 0 & 0 & -2 \end{bmatrix}$$

Triangular Matrices

The third special type of matrix we will consider in this section is the triangular matrix. Recall that an upper (lower) triangular matrix contains all zeros below (above) the main diagonal. Remember that finding the determinant of a triangular matrix is a simple procedure of taking the product of the entries on the main diagonal. It turns out that this leads to a simple way to find the eigenvalues of a triangular matrix.

In the next example we will demonstrate that the eigenvalues of a triangular matrix are the entries on the main diagonal.

Example $6.2.6$: Eigenvalues for a Triangular Matrix

Let $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 4 & 7 \\ 0 & 0 & 6 \end{bmatrix}$. Find the eigenvalues of A.

Solution

We need to solve the equation $\det(\lambda I - A) = 0$ as follows

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - 1 & -2 & -4 \\ 0 & \lambda - 4 & -7 \\ 0 & 0 & \lambda - 6 \end{bmatrix} = (\lambda - 1)(\lambda - 4)(\lambda - 6) = 0 \tag{6.2.2}$$

Solving the equation $(\lambda - 1)(\lambda - 4)(\lambda - 6) = 0$ for λ results in the eigenvalues $\lambda_1 = 1, \lambda_2 = 4$ and $\lambda_3 = 6$. Thus the eigenvalues are the entries on the main diagonal of the original matrix.

The same result is true for lower triangular matrices. For any triangular matrix, the eigenvalues are equal to the entries on the main diagonal. To find the eigenvectors of a triangular matrix, we use the usual procedure.

In the next section, we explore an important process involving the eigenvalues and eigenvectors of a matrix.

Answers

Lemma 6.2.1: Similar Matrices and Eigenvalues

Suppose $A = P^{-1}BP$ and λ is an eigenvalue of A, that is $A\vec{x} = \lambda\vec{x}$ for some $\vec{x} \neq \vec{0}$. Then

$$P^{-1}BP\vec{x} = \lambda\vec{x}$$

and so

$$BP\vec{x} = \lambda P\vec{x}$$

Since P is an invertible matrix (hence corresponds to a one-to-one transformation), and since $\vec{x} \neq 0$, it follows that $P\vec{x} \neq 0$. Here, $P\vec{x}$ plays the role of the eigenvector in this equation. Thus λ is also an eigenvalue of B. One can similarly verify that any eigenvalue of B is also an eigenvalue of A, and thus both matrices have the same eigenvalues as desired.

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6.2E: Exercises for Section 6.2

Exercise $6.2E.1$

Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} -6 & -40 & 12 \\ 0 & 4 & -5 \\ 0 & 0 & 3 \end{bmatrix}$$

Answer available at the end of this section.

Exercise $6.2E.2$

Using the matrix $P = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$, show that $A = \begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & 2 \\ -1 & 1 & 1 \end{bmatrix}$ is similar to the matrix $B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Exercise $6.2E.3$

Using the result of Exercise 2, find the eigenvalues of

$$A = \begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & 2 \\ -1 & 1 & 1 \end{bmatrix}$$

Exercise $6.2E.4$

Find the eigenvectors of

$$A = \begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & 2 \\ -1 & 1 & 1 \end{bmatrix}$$

Exercise $6.2E.5$

Find the eigenvectors of

$$B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Are they the same as the eigenvectors you found for matrix A in Exercise 4? What can you say about eigenvalues of similar matrices and eigenvectors of similar matrices?

Exercise $6.2E.6$

Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 24 & 72 & 48 \\ 6 & 18 & 12 \\ -12 & -36 & -38 \end{bmatrix}$$

using elementary matrices to create a similar triangular matrix.

Hint available at the end of this section.

Exercise $6.2E.7$

Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 17 & 29 & 41 \\ 8 & 15 & 21 \\ -12 & -25 & -37 \end{bmatrix}$$

Answer available at the end of this section.

Exercise $6.2E.6$

Let A and B be $n \times n$ matrices. Show that if there exists an invertible matrix P such that

$$A = P^{-1}BP$$

then there exists an invertible matrix U such that

$$B = U^{-1}AU$$

Answers

Exercise $6.2E.1$

$$\lambda_1 = -6, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_2 = 4, \quad \vec{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_3 = 3, \quad \vec{v}_3 = \begin{bmatrix} -21 \\ 5 \\ 1 \end{bmatrix}$$

Exercise $6.2E.6$

Begin by using the elementary matrix $E(-2 \times 2 + 3)$ to create a simpler similar matrix to A.

Exercise $6.2E.7$

The eigenvalues of A are $-5.5, -2.5, -11$.

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6.3: Diagonalization

Learning Objectives

- Determine when it is possible to diagonalize a matrix.
- When possible, diagonalize a matrix.

Similarity and Diagonalization

We begin this section by recalling the definition of similar matrices. Recall that if A, B are two $n \times n$ matrices, then they are **similar** if and only if there exists an invertible matrix P such that

$$A = P^{-1}BP$$

In this case we write $A \sim B$. The concept of similarity is an example of an **equivalence relation**.

Definition 6.3.1: Equivalence Relation

A relation on a set S is called an **equivalence relation** if for any elements $a, b, c \in S$ the following properties hold.

1. The relation is *reflexive*: a is related to a, written $a \sim a$.
2. The relation is *symmetric*: If a is related to b, then b is related to a. That is, if $a \sim b$ then $b \sim a$.
3. The relation is *transitive*: If a is related to b and b is related to c, then a is related to c. That is, if $a \sim b$ and $b \sim c$, then $a \sim c$.

We now show that similarity is an equivalence relation on the set of $n \times n$ matrices.

Lemma 6.3.1: Similarity is an Equivalence Relation

Similarity is an equivalence relation, i.e. for $n \times n$ matrices A, B, and C,

1. $A \sim A$ (reflexive)
2. If $A \sim B$, then $B \sim A$ (symmetric)
3. If $A \sim B$ and $B \sim C$, then $A \sim C$ (transitive)

Proof available at the end of this section.

Another important concept necessary to this section is the trace of a matrix. Consider the definition.

Definition 6.3.2: Trace of a Matrix

If $A = [a_{ij}]$ is an $n \times n$ matrix, then the **trace** of A is

$$\text{trace}(A) = \sum_{i=1}^{n} a_{ii}.$$

In words, the trace of a matrix is the sum of the entries on the main diagonal.

Lemma 6.3.2: Properties of Trace

For $n \times n$ matrices A and B, and any $k \in \mathbb{R}$,

1. $\text{trace}(A+B) = \text{trace}(A) + \text{trace}(B)$
2. $\text{trace}(kA) = k \cdot \text{trace}(A)$
3. $\text{trace}(AB) = \text{trace}(BA)$

The following theorem includes a reference to the characteristic polynomial of a matrix. Recall that for any $n \times n$ matrix A, the characteristic polynomial of A is $c_A(x) = \det(xI - A)$.

Theorem 6.3.1: Properties of Similar Matrices

If A and B are $n \times n$ matrices and $A \sim B$, then

1. $\det(A) = \det(B)$
2. $\text{rank}(A) = \text{rank}(B)$
3. $\text{trace}(A) = \text{trace}(B)$
4. $c_A(x) = c_B(x)$
5. A and B have the same eigenvalues

We now proceed to the main concept of this section. When a matrix is similar to a diagonal matrix, the matrix is said to be diagonalizable. We define a diagonal matrix D as a matrix containing a zero in every entry except those on the main diagonal. More precisely, if d_{ij} is the ij^{th} entry of a diagonal matrix D, then $d_{ij} = 0$ unless $i = j$. Such matrices look like the following.

$$D = \begin{bmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{bmatrix}$$

where $*$ is a number which might not be zero.

The following is the formal definition of a diagonalizable matrix.

Definition 6.3.3: Diagonalizable

Let A be an $n \times n$ matrix. Then A is said to be **diagonalizable** if there exists an invertible matrix P such that

$$P^{-1}AP = D$$

where D is a diagonal matrix.

Notice that the above equation can be rearranged as $A = PDP^{-1}$. Suppose we wanted to compute A^{100}. By diagonalizing A first it suffices to then compute $(PDP^{-1})^{100}$, which reduces to $PD^{100}P^{-1}$. This last computation is much simpler than A^{100}. While this process is described in detail later, it provides motivation for diagonalization.

Diagonalizing a Matrix

The most important theorem about diagonalizability is the following major result.

Theorem 6.3.2:Eigenvectors and Diagonalizable Matrices

An $n \times n$ matrix A is diagonalizable if and only if there is an invertible matrix P given by

$$P = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{bmatrix}$$

where the $\vec{x}_k$ are eigenvectors of A.

Moreover if A is diagonalizable, the corresponding eigenvalues of A are the diagonal entries of the diagonal matrix D.

Proof available at the end of this section.

Notice that because the matrix P defined above is invertible, it follows that the set of eigenvectors of A, $\{\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n\}$, form a basis of $\mathbb{R}^n$.

We demonstrate the concept given in the above theorem in the next example. Note that not only are the columns of the matrix P formed by eigenvectors, but P must be invertible so must consist of a wide variety of eigenvectors. We achieve this by using basic eigenvectors for the columns of P.

Example 6.3.1:Diagonalize a Matrix

Let

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & -1 \\ -2 & -4 & 4 \end{bmatrix}$$

Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

Solution

By Theorem 6.3.2 we use the eigenvectors of A as the columns of P, and the corresponding eigenvalues of A as the diagonal entries of D.

First, we will find the eigenvalues of A. To do so, we solve $\det(\lambda I - A) = 0$ as follows.

$$\det\left(\lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & -1 \\ -2 & -4 & 4 \end{bmatrix}\right) = 0$$

This computation is left as an exercise, and you should verify that the eigenvalues are $\lambda_1 = 2, \lambda_2 = 2$, and $\lambda_3 = 6$.

Next, we need to find the eigenvectors. We first find the eigenvectors for $\lambda_1, \lambda_2 = 2$. Solving $(2I - A)\vec{x} = 0$ to find the eigenvectors, we find that the eigenvectors are

$$t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

where t, s are scalars. Hence there are two basic eigenvectors which are given by

$$\vec{x}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

You can verify that the basic eigenvector for $\lambda_3 = 6$ is $\vec{x}_3 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$

Then, we construct the matrix P as follows.

$$P = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \vec{x}_3 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

That is, the columns of P are the basic eigenvectors of A. Then, you can verify that

$$P^{-1} = \begin{bmatrix} -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \end{bmatrix}$$

Thus,

$$\begin{aligned} P^{-1}AP &= \begin{bmatrix} -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & -1 \\ -2 & -4 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix} \end{aligned} \tag{6.3.1}$$

You can see that the result here is a diagonal matrix where the entries on the main diagonal are the eigenvalues of A. We expected this based on Theorem 6.3.2. Notice that eigenvalues on the main diagonal *must* be in the same order as the corresponding eigenvectors in P.

Below is a video example of diagonalizing a matrix.

Example 6.3.2: Diagonalize a Matrix

Let

$$A = \begin{bmatrix} -1 & -19 & -4 \\ 0 & -2 & 0 \\ 0 & 15 & 3 \end{bmatrix}$$

Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

Solution

- **Video Length**: 8 minutes 48 seconds
- **Context**: This video demonstrates how to diagonalize a matrix.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Consider the next important theorem.

Theorem 6.3.3: Linearly Independent Eigenvectors

Let A be an $n \times n$ matrix, and suppose that A has distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$. For each i, let $\vec{x}_i$ be a λ_i-eigenvector of A. Then $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m\}$ is linearly independent.

The corollary that follows from this theorem gives a useful tool in determining if A is diagonalizable.

Corollary 6.3.1: Distinct Eigenvalues

Let A be an $n \times n$ matrix and suppose it has n distinct eigenvalues. Then it follows that A is diagonalizable.

It is possible that a matrix A cannot be diagonalized. In other words, we cannot find an invertible matrix P so that $P^{-1}AP = D$.

Consider the following example.

Example 6.3.3: A Matrix which cannot be Diagonalized

Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

If possible, find an invertible matrix P and diagonal matrix D so that $P^{-1}AP = D$.

Solution

Through the usual procedure, we find that the eigenvalues of A are $\lambda_1 = 1, \lambda_2 = 1$. To find the eigenvectors, we solve the equation $(\lambda I - A)\vec{x} = 0$. The matrix $(\lambda I - A)$ is given by

$$\begin{bmatrix} \lambda - 1 & -1 \\ 0 & \lambda - 1 \end{bmatrix}$$

Substituting in $\lambda = 1$, we have the matrix

$$\begin{bmatrix} 1 - 1 & -1 \\ 0 & 1 - 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$

Then, solving the equation $(\lambda I - A)\vec{x} = 0$ involves carrying the following augmented matrix to its reduced row-echelon form.

$$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then the eigenvectors are of the form

$$t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and the basic eigenvector is

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

In this case, the matrix A has one eigenvalue of multiplicity two, but only one basic eigenvector. In order to diagonalize A, we need to construct an invertible 2×2 matrix P. However, because A only has one basic eigenvector, we cannot construct this P. Notice that if we were to use $\vec{x}_1$ as both columns of P, P would not be invertible. For this reason, we cannot repeat eigenvectors in P.

Hence this matrix cannot be diagonalized.

The idea that a matrix may not be diagonalizable suggests that conditions exist to determine when it is possible to diagonalize a matrix. We saw earlier in Corollary 6.3.1 that an $n \times n$ matrix with n distinct eigenvalues is diagonalizable. It turns out that there are other useful diagonalizability tests.

First we need the following definition.

Definition 6.3.4: Eigenspace

Let A be an $n \times n$ matrix and $\lambda \in \mathbb{R}$. The eigenspace of A corresponding to λ, written $E_\lambda(A)$ is the set of all eigenvectors corresponding to λ.

In other words, the eigenspace $E_\lambda(A)$ is all $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$. Notice that this set can be written $E_\lambda(A) = \text{null}(\lambda I - A)$, showing

that $E_\lambda(A)$ is a subspace of $\mathbb{R}^n$.

Recall that the algebraic multiplicity of an eigenvalue λ is the number of times that it occurs as a root of the characteristic polynomial. We refer to $\dim(E_\lambda(A))$ as the geometric multiplicity of λ.

Definition 6.3.5: Geometric Multiplicity

Let A be an $n \times n$ matrix with eigenvalue $\lambda \in \mathbb{R}$. The geometric multiplicity of λ is $\dim(E_\lambda(A))$.

Consider now the following lemma relating algebraic multiplicity and geometric multiplicity.

Lemma 6.3.3: Dimension of the Eigenspace

If A is an $n \times n$ matrix, then

$$\dim(E_\lambda(A)) \leq m$$

where λ is an eigenvalue of A of algebraic multiplicity m. In other words, the geometric multiplicity is less than or equal to the algebraic multiplicity.

This result tells us that if λ is an eigenvalue of A, then the number of linearly independent λ-eigenvectors is never more than the algebraic multiplicity of λ. We now use this fact to provide a useful diagonalizability condition.

Theorem 6.3.4: Diagonalizability Condition

Let A be an $n \times n$ matrix A. Then A is diagonalizable if and only if for each eigenvalue λ of A, $\dim(E_\lambda(A))$ is equal to the algebraic multiplicity of λ.

Complex Eigenvalues

In some applications, a matrix may have eigenvalues which are complex numbers. For example, this often occurs in differential equations. These questions are approached in the same way as above.

Consider the following example.

Example 6.3.4: A Real Matrix with Complex Eigenvalues

Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

Find the eigenvalues and eigenvectors of A.

Solution

We will first find the eigenvalues as usual by solving the following equation.

$$\det\left(\lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix}\right) = 0$$

This reduces to $(\lambda - 1)(\lambda^2 - 4\lambda + 5) = 0$. The solutions are $\lambda_1 = 1, \lambda_2 = 2 + i$ and $\lambda_3 = 2 - i$.

There is nothing new about finding the eigenvectors for $\lambda_1 = 1$ so this is left as an exercise.

Consider now the eigenvalue $\lambda_2 = 2 + i$. As usual, we solve the equation $(\lambda I - A)\vec{x} = 0$ as given by

$$\left((2+i) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix} \right) \vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

In other words, we need to solve the system represented by the augmented matrix

$$\begin{bmatrix} 1+i & 0 & 0 & 0 \\ 0 & i & 1 & 0 \\ 0 & -1 & i & 0 \end{bmatrix}$$

We now use our row operations to solve the system. Divide the first row by $(1 + i)$ and then take $-i$ times the second row and add to the third row. This yields

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now multiply the second row by $-i$ to obtain the reduced row-echelon form, given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -i & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, the eigenvectors are of the form

$$t \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$$

and the basic eigenvector is given by

$$\vec{x}_2 = \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$$

As an exercise, verify that the eigenvectors for $\lambda_3 = 2 - i$ are of the form

$$t \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}$$

Hence, the basic eigenvector is given by

$$\vec{x}_3 = \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}$$

As usual, be sure to check your answers! To verify, we check that $A\vec{x}_3 = (2 - i)\vec{x}_3$ as follows.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1-2i \\ 2-i \end{bmatrix} = (2-i) \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}$$

Therefore, we know that this eigenvector and eigenvalue are correct.

Notice that in Example 6.3.3, two of the eigenvalues were given by $\lambda_2 = 2 + i$ and $\lambda_3 = 2 - i$. You may recall that these two complex numbers are **conjugates**. It turns out that whenever a matrix containing real entries has a complex eigenvalue λ, it also has an eigenvalue equal to $\overline{\lambda}$, the conjugate of λ.

Answers

Lemma $6.3.1$**: Similarity is an Equivalence Relation**

It is clear that $A \sim A$, taking $P = I$.

Now, if $A \sim B$, then for some P invertible,

$$A = P^{-1}BP$$

and so

$$PAP^{-1} = B$$

But then

$$\left(P^{-1}\right)^{-1} AP^{-1} = B$$

which shows that $B \sim A$.

Now suppose $A \sim B$ and $B \sim C$. Then there exist invertible matrices P, Q such that

$$A = P^{-1}BP,\ B = Q^{-1}CQ$$

Then,

$$A = P^{-1}\left(Q^{-1}CQ\right)P = (QP)^{-1}C(QP)$$

showing that A is similar to C.

Theorem $6.3.2$**:Eigenvectors and Diagonalizable Matrices**

Suppose P is given as above as an invertible matrix whose columns are eigenvectors of A. Then P^{-1} is of the form

$$P^{-1} = \begin{bmatrix} \vec{w}_1^T \\ \vec{w}_2^T \\ \vdots \\ \vec{w}_n^T \end{bmatrix}$$

where $\vec{w}_k^T \vec{x}_j = \delta_{kj}$, which is the Kronecker's symbol defined by

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

Then

$$\begin{aligned} P^{-1}AP &= \begin{bmatrix} \vec{w}_1^T \\ \vec{w}_2^T \\ \vdots \\ \vec{w}_n^T \end{bmatrix} \begin{bmatrix} A\vec{x}_1 & A\vec{x}_2 & \cdots & A\vec{x}_n \end{bmatrix} \\ &= \begin{bmatrix} \vec{w}_1^T \\ \vec{w}_2^T \\ \vdots \\ \vec{w}_n^T \end{bmatrix} \begin{bmatrix} \lambda_1\vec{x}_1 & \lambda_2\vec{x}_2 & \cdots & \lambda_n\vec{x}_n \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \end{aligned} \tag{6.3.2}$$

Conversely, suppose A is diagonalizable so that $P^{-1}AP = D$. Let

$$P = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{bmatrix}$$

where the columns are the $\vec{x}_k$ and

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

Then

$$AP = PD = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

and so

$$\begin{bmatrix} A\vec{x}_1 & A\vec{x}_2 & \cdots & A\vec{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1\vec{x}_1 & \lambda_2\vec{x}_2 & \cdots & \lambda_n\vec{x}_n \end{bmatrix}$$

showing the $\vec{x}_k$ are eigenvectors of A and the λ_k are eigenvectors.

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6.3E: Exercises for Section 6.3

Exercise $6.3E.1$

Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 5 & -18 & -32 \\ 0 & 5 & 4 \\ 2 & -5 & -11 \end{bmatrix}$$

One eigenvalue is 1. Diagonalize if possible.

Answer available at the end of this section.

Exercise $6.3E.2$

Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} -12 & -28 & 28 \\ 4 & 9 & -8 \\ -4 & -8 & 9 \end{bmatrix}$$

One eigenvalue is 3. Diagonalize if possible.

Answer available at the end of this section.

Exercise $6.3E.3$

Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 89 & 38 & 268 \\ 14 & 2 & 40 \\ -30 & -12 & -90 \end{bmatrix}$$

One eigenvalue is -3. Diagonalize if possible.

Answer available at the end of this section.

Exercise $6.3E.4$

Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 1 & 90 & 0 \\ 0 & -2 & 0 \\ 3 & 89 & -2 \end{bmatrix}$$

One eigenvalue is 1. Diagonalize if possible.

Exercise $6.3E.5$

Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 11 & 45 & 30 \\ 10 & 26 & 20 \\ -20 & -60 & -44 \end{bmatrix}$$

One eigenvalue is 1. Diagonalize if possible.

Exercise $6.3E.6$

Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 95 & 25 & 24 \\ -196 & -53 & -48 \\ -164 & -42 & -43 \end{bmatrix}$$

One eigenvalue is 5. Diagonalize if possible.

Exercise $6.3E.7$

Suppose A is an $n \times n$ matrix and let V be an eigenvector such that $AV = \lambda V$. Also suppose the characteristic polynomial of A is

$$\det(xI - A) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

Explain why

$$(A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I)V = 0$$

If A is diagonalizable, give a proof of the Cayley Hamilton theorem based on this. This theorem says A satisfies its characteristic equation

$$A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I = 0$$

Exercise $6.3E.8$

Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 15 & -24 & 7 \\ -6 & 5 & -1 \\ -58 & 76 & -20 \end{bmatrix}$$

One eigenvalue is -2. Diagonalize if possible. Hint: This one has some complex eigenvalues.

Exercise $6.3E.9$

Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 15 & -25 & 6 \\ -13 & 23 & -4 \\ -91 & 155 & -30 \end{bmatrix}$$

One eigenvalue is 2. Diagonalize if possible. Hint: This one has some complex eigenvalues.

Exercise $6.3E.10$

Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} -11 & -12 & 4 \\ 8 & 17 & -4 \\ -4 & 28 & -3 \end{bmatrix}$$

One eigenvalue is 1. Diagonalize if possible. Hint: This one has some complex eigenvalues.

Exercise $6.3E.11$

Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 14 & -12 & 5 \\ -6 & 2 & -1 \\ -69 & 51 & -21 \end{bmatrix}$$

One eigenvalue is -3. Diagonalize if possible. Hint: This one has some complex eigenvalues.

Answers

Exercise $6.3E.1$

The eigenvalues are $-1, -1, 1$. The eigenvectors corresponding to the eigenvalues are:

$$\left\{\begin{bmatrix} 10 & -2 & 3 \end{bmatrix}\right\} \leftrightarrow -1, \left\{\begin{bmatrix} 7 \\ -2 \\ 2 \end{bmatrix}\right\} \leftrightarrow 1$$

Therefore this matrix is not diagonalizable.

Exercise $6.3E.2$

The eigenvectors and eigenvalues are:

$$\left\{\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\right\} \leftrightarrow 1, \left\{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}\right\} \leftrightarrow 1, \left\{\begin{bmatrix} 7 \\ -2 \\ 2 \end{bmatrix}\right\} \leftrightarrow 3$$

The matrix P needed to diagonalize the above matrix is

$$\begin{bmatrix} 2 & -2 & 7 \\ 0 & 1 & -2 \\ 1 & 0 & 2 \end{bmatrix}$$

and the diagonal matrix D is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Exercise $6.3E.3$

The eigenvectors and eigenvalues are:

$$\left\{\begin{bmatrix} -6 \\ -1 \\ -2 \end{bmatrix}\right\} \leftrightarrow 6, \left\{\begin{bmatrix} -5 \\ -2 \\ 2 \end{bmatrix}\right\} \leftrightarrow -3, \left\{\begin{bmatrix} -8 \\ -2 \\ 3 \end{bmatrix}\right\} \leftrightarrow 2$$

The matrix P needed to diagonalize the above matrix is

$$\begin{bmatrix} -6 & -5 & -8 \\ -1 & -2 & -2 \\ 2 & 2 & 3 \end{bmatrix}$$

and the diagonal matrix D is

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

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6.4: Applications of Spectral Theory

Learning Objectives

- Use diagonalization to find a high power of a matrix.
- Understand the matrix exponential and its application to differential equations

Raising a Matrix to a High Power

Suppose we have a matrix A and we want to find A^{50}. One could try to multiply A with itself 50 times, but this is computationally extremely intensive (try it!). However diagonalization allows us to compute high powers of a matrix relatively easily. Suppose A is diagonalizable, so that $P^{-1}AP = D$. We can rearrange this equation to write $A = PDP^{-1}$.

Now, consider A^2. Since $A = PDP^{-1}$, it follows that

$$A^2 = \left(PDP^{-1}\right)^2 = PDP^{-1}PDP^{-1} = PD^2P^{-1}$$

Similarly,

$$A^3 = \left(PDP^{-1}\right)^3 = PDP^{-1}PDP^{-1}PDP^{-1} = PD^3P^{-1}$$

In general,

$$A^n = \left(PDP^{-1}\right)^n = PD^nP^{-1}$$

Therefore, we have reduced the problem to finding D^n. In order to compute D^n, then because D is diagonal we only need to raise every entry on the main diagonal of D to the power of n.

Through this method, we can compute large powers of matrices. Consider the following example.

Example 6.4.1: Raising a Matrix to a High Power

Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$. Find A^{50}.

Solution

We will first diagonalize A. The steps are left as an exercise and you may wish to verify that the eigenvalues of A are $\lambda_1 = 1, \lambda_2 = 1$, and $\lambda_3 = 2$.

The basic eigenvectors corresponding to $\lambda_1, \lambda_2 = 1$ are

$$\vec{x}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

The basic eigenvector corresponding to $\lambda_3 = 2$ is

$$\vec{x}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Now we construct P by using the basic eigenvectors of A as the columns of P. Thus

$$P = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \vec{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Then also

$$P^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix}$$

which you may wish to verify.

Then,

$$\begin{aligned} P^{-1}AP &= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \\ &= D \end{aligned} \tag{6.4.1}$$

Now it follows by rearranging the equation that

$$A = PDP^{-1} = \begin{bmatrix} 0 & -1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix}$$

Therefore,

$$\begin{aligned} A^{50} &= PD^{50}P^{-1} \\ &= \begin{bmatrix} 0 & -1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}^{50} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix} \end{aligned} \tag{6.4.2}$$

By our discussion above, D^{50} is found as follows.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}^{50} = \begin{bmatrix} 1^{50} & 0 & 0 \\ 0 & 1^{50} & 0 \\ 0 & 0 & 2^{50} \end{bmatrix}$$

It follows that

$$\begin{aligned} A^{50} &= \begin{bmatrix} 0 & -1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1^{50} & 0 & 0 \\ 0 & 1^{50} & 0 \\ 0 & 0 & 2^{50} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2^{50} & -1+2^{50} & 0 \\ 0 & 1 & 0 \\ 1-2^{50} & 1-2^{50} & 1 \end{bmatrix} \end{aligned} \tag{6.4.3}$$

Through diagonalization, we can efficiently compute a high power of A. Without this, we would be forced to multiply this by hand!

The Matrix Exponential

The goal of this section is to use the concept of the matrix exponential to solve first order linear differential equations.

Suppose A is a diagonalizable matrix. Then the **matrix exponential**, written e^A, can be easily defined. Recall that if D is a diagonal matrix, then

$$P^{-1}AP = D$$

D is of the form

$$\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \tag{6.4.4}$$

and it follows that

$$D^m = \begin{bmatrix} \lambda_1^m & & 0 \\ & \ddots & \\ 0 & & \lambda_n^m \end{bmatrix}$$

Since A is diagonalizable,

$$A = PDP^{-1}$$

and

$$A^m = PD^mP^{-1}$$

Recall why this is true.

$$A = PDP^{-1}$$

and so

$$\begin{aligned} A^m &= \overbrace{PDP^{-1}PDP^{-1}PDP^{-1}\cdots PDP^{-1}}^{\text{m times}} \\ &= PD^mP^{-1} \end{aligned} \tag{6.4.5}$$

We now will examine what is meant by the matrix exponental e^A. Begin by formally writing the following power series for e^A:

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = \sum_{k=0}^{\infty} \frac{PD^kP^{-1}}{k!} = P\left(\sum_{k=0}^{\infty} \frac{D^k}{k!}\right)P^{-1}$$

If D is given above in (6.4.4), the above sum is of the form

$$P\left(\sum_{k=0}^{\infty} \begin{bmatrix} \frac{1}{k!}\lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \frac{1}{k!}\lambda_n^k \end{bmatrix}\right)P^{-1}$$

This can be rearranged as follows:

$$\begin{aligned} e^A &= P\begin{bmatrix} \sum_{k=0}^{\infty}\frac{1}{k!}\lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \sum_{k=0}^{\infty}\frac{1}{k!}\lambda_n^k \end{bmatrix}P^{-1} \\ &= P\begin{bmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{bmatrix}P^{-1} \end{aligned}$$

This justifies the following theorem.

Theorem 6.4.1:The Matrix Exponential

Let A be a diagonalizable matrix, with eigenvalues $\lambda_1, ..., \lambda_n$ and corresponding matrix of eigenvectors P. Then the matrix exponential, e^A, is given by

$$e^A = P\begin{bmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{bmatrix}P^{-1}$$

Example 6.4.2: Compute e^A for a Matrix A

Let

$$A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$

Find e^A.

Solution

The eigenvalues work out to be $1, 2, 3$ and eigenvectors associated with these eigenvalues are

$$\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \leftrightarrow 1, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \leftrightarrow 2, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \leftrightarrow 3$$

Then let

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, P = \begin{bmatrix} 0 & -1 & -1 \\ -1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

and so

$$P^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

Then the matrix exponential is

$$e^{At} = \begin{bmatrix} 0 & -1 & -1 \\ -1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^1 & 0 & 0 \\ 0 & e^2 & 0 \\ 0 & 0 & e^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -1 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} e^2 & e^2 - e^3 & e^2 - e^3 \\ e^2 - e & e^2 & e^2 - e \\ -e^2 + e & -e^2 + e^3 & -e^2 + e + e^3 \end{bmatrix}$$

The matrix exponential is a useful tool to solve autonomous systems of first order linear differential equations. These are equations which are of the form

$$\vec{x}' = A\vec{x}, \vec{x}(0) = C$$

where A is a diagonalizable $n \times n$ matrix and C is a constant vector. $\vec{x}$ is a vector of functions in one variable, t:

$$\vec{x} = \vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

Then $\vec{x}'$ refers to the first derivative of $\vec{x}$ and is given by

$$\vec{x}' = \vec{x}'(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}, \ x_i'(t) = \text{the derivative of } x_i(t)$$

Then it turns out that the solution to the above system of equations is $\vec{x}\,(t) = e^{At}C$. To see this, suppose A is diagonalizable so that

$$A = P \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} P^{-1}$$

Then

$$e^{At} = P \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} P^{-1}$$

$$e^{At}C = P \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} P^{-1}C$$

Differentiating $e^{At}C$ yields

$$\begin{aligned} \vec{x}' = \left(e^{At}C\right)' &= P \begin{bmatrix} \lambda_1 e^{\lambda_1 t} & & \\ & \ddots & \\ & & \lambda_n e^{\lambda_n t} \end{bmatrix} P^{-1}C \\ &= P \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} P^{-1}C \\ &= P \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} P^{-1} P \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} P^{-1}C \\ &= A\left(e^{At}C\right) = A\vec{x} \end{aligned} \tag{6.4.6}$$

Therefore $\vec{x} = \vec{x}(t) = e^{At}C$ is a solution to $\vec{x}' = A\vec{x}$.

To prove that $\vec{x}(0) = C$ if $\vec{x}(t) = e^{At}C$:

$$\vec{x}(0) = e^{A0}C = P \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} P^{-1}C = C$$

Example 6.4.3: Solving an Initial Value Problem

Solve the initial value problem

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Solution

The matrix is diagonalizable and can be written as

$$\begin{aligned} A &= PDP^{-1} \\ \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ -\frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -1 & -2 \end{bmatrix} \end{aligned} \tag{6.4.7}$$

Therefore, the matrix exponential is of the form

$$e^{At} = \begin{bmatrix} 1 & 1 \\ -\frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -1 & -2 \end{bmatrix}$$

The solution to the initial value problem is

$$\begin{aligned}\vec{x}(t) &= e^{At}C \\ \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ -\frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 4e^t - 3e^{2t} \\ 3e^{2t} - 2e^t \end{bmatrix}\end{aligned} \tag{6.4.8}$$

We can check that this works:

$$\begin{aligned}\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} &= \begin{bmatrix} 4e^0 - 3e^{2(0)} \\ 3e^{2(0)} - 2e^0 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}\end{aligned} \tag{6.4.9}$$

Lastly,

$$\vec{x}' = \begin{bmatrix} 4e^t - 3e^{2t} \\ 3e^{2t} - 2e^t \end{bmatrix}' = \begin{bmatrix} 4e^t - 6e^{2t} \\ 6e^{2t} - 2e^t \end{bmatrix}$$

and

$$A\vec{x} = \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 4e^t - 3e^{2t} \\ 3e^{2t} - 2e^t \end{bmatrix} = \begin{bmatrix} 4e^t - 6e^{2t} \\ 6e^{2t} - 2e^t \end{bmatrix}$$

which is the same thing. Thus this is the solution to the initial value problem.

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6.4E: Exercises for Section 6.4

Exercise 6.4E.1

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Diagonalize A to find A^{10}.

Answer available at the end of this section.

Exercise 6.4E.2

Let $A = \begin{bmatrix} 1 & 4 & 1 \\ 0 & 2 & 5 \\ 0 & 0 & 5 \end{bmatrix}$. Diagonalize A to find A^{50}.

Exercise 6.4E.3

Let $A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & -1 & 1 \\ -2 & 3 & 1 \end{bmatrix}$. Diagonalize A to find A^{100}.

Exercise 6.4E.4

Find the solution to the initial value problem

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 0 & -1 \\ 6 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \tag{6.4E.1}$$

Hint: form the matrix exponential e^{At} and then the solution is $e^{At}C$ where C is the initial vector,

Answer available at the end of this section.

Exercise 6.4E.5

Find the solution to the initial value problem

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} -4 & -3 \\ 6 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \tag{6.4E.2}$$

Hint available at the end of this section.

Exercise 6.4E.6

Find the solution to the initial value problem

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} -1 & 2 \\ -4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad (6.4\text{E}.3)$$
$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Hint available at the end of this section.

Answers

Exercise $6.4E.1$

First we write $A = PDP^{-1}$.

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Therefore $A^{10} = PD^{10}P^{-1}$.

$$\begin{aligned} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^{10} &= \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}^{10} \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (-1)^{10} & 0 \\ 0 & 3^{10} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} 29525 & 29524 \\ 29524 & 29525 \end{bmatrix} \end{aligned} \qquad (6.4\text{E}.4)$$

The solution is

$$e^{At}C = \begin{bmatrix} 8e^{2t} - 6e^{3t} \\ 18e^{3t} - 16e^{2t} \end{bmatrix}$$

Exercise $6.4E.5$

Form the matrix exponential e^{At} and then the solution is $e^{At}C$ where C is the initial vector.

Exercise $6.4E.6$

Form the matrix exponential e^{At} and then the solution is $e^{At}C$ where C is the initial vector.

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6.5: Markov Matrices

Learning Objectives

- Understand when a matrix is a Markov matrix.
- Understand applications of Markov matrices.

Markov Matrices

There are applications of great importance which feature a special type of matrix. Matrices whose columns consist of non-negative numbers that sum to one are called **Markov matrices**. An important application of Markov matrices is in population migration, as illustrated in the following definition.

Definition 6.5.1: Migration Matrices

Let m locations be denoted by the numbers $1, 2, \cdots, m$. Suppose it is the case that each year the proportion of residents in location j which move to location i is a_{ij}. Also suppose no one escapes or emigrates from without these m locations. This last assumption requires $\sum_i a_{ij} = 1$, and means that the matrix A, such that $A = [a_{ij}]$, is a Markov matrix. In this context, A is also called a **migration matrix**.

Consider the following example which demonstrates this situation.

Example 6.5.1: Migration Matrix

Let A be a Markov matrix given by

$$A = \begin{bmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{bmatrix}$$

Verify that A is a Markov matrix and describe the entries of A in terms of population migration.

Solution

The columns of A are comprised of non-negative numbers which sum to 1. Hence, A is a Markov matrix.

Now, consider the entries a_{ij} of A in terms of population. The entry $a_{11} = 0.4$ is the proportion of residents in location one which stay in location one in a given time period. Entry $a_{21} = 0.6$ is the proportion of residents in location 1 which move to location 2 in the same time period. Entry $a_{12} = 0.2$ is the proportion of residents in location 2 which move to location 1. Finally, entry $a_{22} = 0.8$ is the proportion of residents in location 2 which stay in location 2 in this time period.

Considered as a Markov matrix, these numbers are usually identified with probabilities. Hence, we can say that the probability that a resident of location one will stay in location one in the time period is 0.4.

Observe that in Example $6.5.1$ if there was initially say 15 thousand people in location 1 and 10 thousands in location 2, then after one year there would be $0.4 \times 15 + 0.2 \times 10 = 8$ thousands people in location 1 the following year, and similarly there would be $0.6 \times 15 + 0.8 \times 10 = 17$ thousands people in location 2 the following year.

More generally let $\vec{x}_n = [x_{1n} \cdots x_{mn}]^T$ where x_{in} is the population of location i at time period n. We call $\vec{x}_n$ the **state vector at period n**. In particular, we call $\vec{x}_0$ the initial state vector. Letting A be the migration matrix, we compute the population in each location i one time period later by $A\vec{x}_n$. In order to find the population of location i after k years, we compute the ith component of $A^k\vec{x}$. This discussion is summarized in the following theorem.

Theorem 6.5.1: State Vector

Let A be the migration matrix of a population and let $\vec{x}_n$ be the vector whose entries give the population of each location at time period n. Then $\vec{x}_n$ is the state vector at period n and it follows that

$$\vec{x}_{n+1} = A\vec{x}_n$$

The sum of the entries of $\vec{x}_n$ will equal the sum of the entries of the initial vector $\vec{x}_0$. Since the columns of A sum to 1, this sum is preserved for every multiplication by A as demonstrated below.

$$\sum_i \sum_j a_{ij} x_j = \sum_j x_j \left(\sum_i a_{ij} \right) = \sum_j x_j$$

Consider the following example.

Example 6.5.2: Using a Migration Matrix

Consider the migration matrix

$$A = \begin{bmatrix} 0.6 & 0 & 0.1 \\ 0.2 & 0.8 & 0 \\ 0.2 & 0.2 & 0.9 \end{bmatrix}$$

for locations $1, 2,$ and 3. Suppose initially there are 100 residents in location 1, 200 in location 2 and 400 in location 3. Find the population in the three locations after $1, 2,$ and 10 units of time.

Solution

Using Theorem $6.5.1$ we can find the population in each location using the equation $\vec{x}_{n+1} = A\vec{x}_n$. For the population after 1 unit, we calculate $\vec{x}_1 = A\vec{x}_0$ as follows.

$$\begin{aligned} \vec{x}_1 &= A\vec{x}_0 \\ \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} &= \begin{bmatrix} 0.6 & 0 & 0.1 \\ 0.2 & 0.8 & 0 \\ 0.2 & 0.2 & 0.9 \end{bmatrix} \begin{bmatrix} 100 \\ 200 \\ 400 \end{bmatrix} \\ &= \begin{bmatrix} 100 \\ 180 \\ 420 \end{bmatrix} \end{aligned} \tag{6.5.1}$$

Therefore after one time period, location 1 has 100 residents, location 2 has 180, and location 3 has 420. Notice that the **total** population is unchanged, it simply migrates within the given locations. We find the locations after two time periods in the same way.

$$\begin{aligned} \vec{x}_2 &= A\vec{x}_1 \\ \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} &= \begin{bmatrix} 0.6 & 0 & 0.1 \\ 0.2 & 0.8 & 0 \\ 0.2 & 0.2 & 0.9 \end{bmatrix} \begin{bmatrix} 100 \\ 180 \\ 420 \end{bmatrix} \\ &= \begin{bmatrix} 102 \\ 164 \\ 434 \end{bmatrix} \end{aligned} \tag{6.5.2}$$

We could progress in this manner to find the populations after 10 time periods. However from our above discussion, we can simply calculate $(A^n\vec{x}_0)_i$, where n denotes the number of time periods which have passed. Therefore, we compute the populations in each location after 10 units of time as follows.

$$\begin{aligned}\vec{x}_{10} &= A^{10}\vec{x}_0 \\ \begin{bmatrix} x_{110} \\ x_{210} \\ x_{310} \end{bmatrix} &= \begin{bmatrix} 0.6 & 0 & 0.1 \\ 0.2 & 0.8 & 0 \\ 0.2 & 0.2 & 0.9 \end{bmatrix}^{10} \begin{bmatrix} 100 \\ 200 \\ 400 \end{bmatrix} \\ &= \begin{bmatrix} 115.085\,829\,22 \\ 120.130\,672\,44 \\ 464.783\,498\,34 \end{bmatrix}\end{aligned} \tag{6.5.3}$$

Since we are speaking about populations, we would need to round these numbers to provide a logical answer. Therefore, we can say that after 10 units of time, there will be 115 residents in location one, 120 in location two, and 465 in location three.

A second important application of Markov matrices is the concept of random walks. Suppose a walker has m locations to choose from, denoted $1, 2, \cdots, m$. Let a_{ij} refer to the probability that the person will travel **to** location i **from** location j. Again, this requires that

$$\sum_{i=1}^{k} a_{ij} = 1$$

In this context, the vector $\vec{x}_n = [x_{1n} \cdots x_{mn}]^T$ contains the probabilities x_{in} the walker ends up in location $i, 1 \leq i \leq m$ at time n.

Example $6.5.3$: Random Walks

Suppose three locations exist, referred to as locations $1, 2$ and 3. The Markov matrix of probabilities $A = [a_{ij}]$ is given by

$$\begin{bmatrix} 0.4 & 0.1 & 0.5 \\ 0.4 & 0.6 & 0.1 \\ 0.2 & 0.3 & 0.4 \end{bmatrix}$$

If the walker starts in location 1, calculate the probability that he ends up in location 3 at time $n = 2$.

Solution

Since the walker begins in location 1, we have

$$\vec{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The goal is to calculate x_{32}. To do this we calculate $\vec{x}_2$, using $\vec{x}_{n+1} = A\vec{x}_n$.

$$\begin{aligned}\vec{x}_1 &= A\vec{x}_0 \\ &= \begin{bmatrix} 0.4 & 0.1 & 0.5 \\ 0.4 & 0.6 & 0.1 \\ 0.2 & 0.3 & 0.4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0.4 \\ 0.4 \\ 0.2 \end{bmatrix}\end{aligned} \tag{6.5.4}$$

$$\begin{aligned}\vec{x}_2 &= A\vec{x}_1 \\ &= \begin{bmatrix} 0.4 & 0.1 & 0.5 \\ 0.4 & 0.6 & 0.1 \\ 0.2 & 0.3 & 0.4 \end{bmatrix} \begin{bmatrix} 0.4 \\ 0.4 \\ 0.2 \end{bmatrix} \\ &= \begin{bmatrix} 0.3 \\ 0.42 \\ 0.28 \end{bmatrix}\end{aligned} \tag{6.5.5}$$

This gives the probabilities that our walker ends up in locations 1, 2, and 3. For this example we are interested in location 3, with a probability on 0.28.

Returning to the context of migration, suppose we wish to know how many residents will be in a certain location after a very long time. It turns out that if some power of the migration matrix has all positive entries, then there is a vector $\vec{x}_s$ such that $A^n\vec{x}_0$ approaches $\vec{x}_s$ as n becomes very large. Hence as more time passes and n increases, $A^n\vec{x}_0$ will become closer to the vector $\vec{x}_s$.

Consider Theorem $6.5.2$. Let n increase so that $\vec{x}_n$ approaches $\vec{x}_s$. As $\vec{x}_n$ becomes closer to $\vec{x}_s$, so too does $\vec{x}_{n+1}$. For sufficiently large n, the statement $\vec{x}_{n+1} = A\vec{x}_n$ can be written as $\vec{x}_s = A\vec{x}_s$.

This discussion motivates the following theorem.

Theorem $6.5.2$: Steady State Vector

Let A be a migration matrix. Then there exists a **steady state vector** written $\vec{x}_s$ such that

$$\vec{x}_s = A\vec{x}_s$$

where $\vec{x}_s$ has positive entries which have the same sum as the entries of $\vec{x}_0$.

As n increases, the state vectors $\vec{x}_n$ will approach $\vec{x}_s$.

Note that the condition in Theorem $6.5.2$ can be written as $(I - A)\vec{x}_s = 0$, representing a homogeneous system of equations.

Consider the following example. Notice that it is the same example as the Example $6.5.2$ but here it will involve a longer time frame.

Example $6.5.4$: Populations over the Long Run

Consider the migration matrix

$$A = \begin{bmatrix} 0.6 & 0 & 0.1 \\ 0.2 & 0.8 & 0 \\ 0.2 & 0.2 & 0.9 \end{bmatrix}$$

for locations $1, 2,$ and 3. Suppose initially there are 100 residents in location 1, 200 in location 2 and 400 in location 4. Find the population in the three locations after a long time.

Solution

By Theorem $6.5.2$ the steady state vector $\vec{x}_s$ can be found by solving the system $(I - A)\vec{x}_s = 0$.

Thus we need to find a solution to

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.6 & 0 & 0.1 \\ 0.2 & 0.8 & 0 \\ 0.2 & 0.2 & 0.9 \end{bmatrix} \right) \begin{bmatrix} x_{1s} \\ x_{2s} \\ x_{3s} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix and the resulting reduced row-echelon form are given by

$$\begin{bmatrix} 0.4 & 0 & -0.1 & 0 \\ -0.2 & 0.2 & 0 & 0 \\ -0.2 & -0.2 & 0.1 & 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & -0.25 & 0 \\ 0 & 1 & -0.25 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, the eigenvectors are

$$t \begin{bmatrix} 0.25 \\ 0.25 \\ 1 \end{bmatrix}$$

The initial vector $\vec{x}_0$ is given by

$$\begin{bmatrix} 100 \\ 200 \\ 400 \end{bmatrix}$$

Now all that remains is to choose the value of t such that

$$0.25t + 0.25t + t = 100 + 200 + 400$$

Solving this equation for t yields $t = \frac{1400}{3}$. Therefore the population in the long run is given by

$$\frac{1400}{3}\begin{bmatrix} 0.25 \\ 0.25 \\ 1 \end{bmatrix} = \begin{bmatrix} 116.666\,666\,666\,666\,7 \\ 116.666\,666\,666\,666\,7 \\ 466.666\,666\,666\,666\,7 \end{bmatrix}$$

Again, because we are working with populations, these values need to be rounded. The steady state vector $\vec{x}_s$ is given by

$$\begin{bmatrix} 117 \\ 117 \\ 466 \end{bmatrix}$$

We can see that the numbers we calculated in Example $6.5.2$ for the populations after the 10th unit of time are not far from the long term values.

Consider another example.

Example $6.5.5$: Populations After a Long Time

Suppose a migration matrix is given by

$$A = \begin{bmatrix} \frac{1}{5} & \frac{1}{2} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{11}{20} & \frac{1}{4} & \frac{3}{10} \end{bmatrix}$$

Find the comparison between the populations in the three locations after a long time.

Solution

In order to compare the populations in the long term, we want to find the steady state vector $\vec{x}_s$. Solve

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{5} & \frac{1}{2} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{11}{20} & \frac{1}{4} & \frac{3}{10} \end{bmatrix}\right)\begin{bmatrix} x_{1s} \\ x_{2s} \\ x_{3s} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix and the resulting reduced row-echelon form are given by

$$\begin{bmatrix} \frac{4}{5} & -\frac{1}{2} & -\frac{1}{5} & 0 \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{2} & 0 \\ -\frac{11}{20} & -\frac{1}{4} & \frac{7}{10} & 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & -\frac{16}{19} & 0 \\ 0 & 1 & -\frac{18}{19} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and so an eigenvector is

$$\begin{bmatrix} 16 \\ 18 \\ 19 \end{bmatrix}$$

Therefore, the proportion of population in location 2 to location 1 is given by $\frac{18}{16}$. The proportion of population 3 to location 2 is given by $\frac{19}{18}$.

Eigenvalues of Markov Matrices

The following is an important proposition.

Proposition 6.5.1: Eigenvalues of a Migration Matrix

Let $A = [a_{ij}]$ be a migration matrix. Then 1 is always an eigenvalue for A.

Proof available at the end of this section.

Answers

Proposition 6.5.1: Eigenvalues of a Migration Matrix

Remember that the determinant of a matrix always equals that of its transpose. Therefore,

$$\det(\lambda I - A) = \det\left((\lambda I - A)^T\right) = \det\left(\lambda I - A^T\right)$$

because $I^T = I$. Thus the characteristic equation for A is the same as the characteristic equation for A^T. Consequently, A and A^T have the same eigenvalues. We will show that 1 is an eigenvalue for A^T and then it will follow that 1 is an eigenvalue for A.

Remember that for a migration matrix, $\sum_i a_{ij} = 1$. Therefore, if $A^T = [b_{ij}]$ with $b_{ij} = a_{ji}$, it follows that

$$\sum_j b_{ij} = \sum_j a_{ji} = 1$$

Therefore, from matrix multiplication,

$$A^T \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} \sum_j b_{ij} \\ \vdots \\ \sum_j b_{ij} \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

Notice that this shows that $\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ is an eigenvector for A^T corresponding to the eigenvalue, $\lambda = 1$. As explained above, this shows that $\lambda = 1$ is an eigenvalue for A because A and A^T have the same eigenvalues.

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6.5E: Exercises for Section 6.5

Exercise $6.5E.1$

The following is a Markov (migration) matrix for three locations

$$\begin{bmatrix} \frac{7}{10} & \frac{1}{9} & \frac{1}{5} \\ \frac{1}{10} & \frac{7}{9} & \frac{2}{5} \\ \frac{1}{5} & \frac{1}{9} & \frac{2}{5} \end{bmatrix}$$

a. Initially, there are 90 people in location 1, 81 in location 2, and 85 in location 3. How many are in each location after one time period?
b. The total number of individuals in the migration process is 256. After a long time, how many are in each location?

Answer available at the end of this section.

Exercise $6.5E.2$

The following is a Markov (migration) matrix for three locations

$$\begin{bmatrix} \frac{1}{5} & \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{2}{5} & \frac{2}{5} \end{bmatrix}$$

a. Initially, there are 130 individuals in location 1, 300 in location 2, and 70 in location 3. How many are in each location after two time periods?
b. The total number of individuals in the migration process is 500. After a long time, how many are in each location?

Answer available at the end of this section.

Exercise $6.5E.3$

The following is a Markov (migration) matrix for three locations

$$\begin{bmatrix} \frac{3}{10} & \frac{3}{8} & \frac{1}{3} \\ \frac{1}{10} & \frac{3}{8} & \frac{1}{3} \\ \frac{3}{5} & \frac{1}{4} & \frac{1}{3} \end{bmatrix}$$

The total number of individuals in the migration process is 480. After a long time, how many are in each location?

Answer available at the end of this section.

Exercise $6.5E.4$

The following is a Markov (migration) matrix for three locations

$$\begin{bmatrix} \frac{3}{10} & \frac{1}{3} & \frac{1}{5} \\ \frac{3}{10} & \frac{1}{3} & \frac{7}{10} \\ \frac{2}{5} & \frac{1}{3} & \frac{1}{10} \end{bmatrix}$$

The total number of individuals in the migration process is 1155. After a long time, how many are in each location?

Exercise $6.5E.5$

The following is a Markov (migration) matrix for three locations

$$\begin{bmatrix} \frac{2}{5} & \frac{1}{10} & \frac{1}{8} \\ \frac{3}{10} & \frac{2}{5} & \frac{5}{8} \\ \frac{3}{10} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

The total number of individuals in the migration process is 704. After a long time, how many are in each location?

Exercise $6.5E.6$

A person sets off on a random walk with three possible locations. The Markov matrix of probabilities $A = [a_{ij}]$ is given by

$$\begin{bmatrix} 0.1 & 0.3 & 0.7 \\ 0.1 & 0.3 & 0.2 \\ 0.8 & 0.4 & 0.1 \end{bmatrix}$$

If the walker starts in location 2, what is the probability of ending back in location 2 at time $n = 3$?

Answer available at the end of this section.

Exercise $6.5E.7$

A person sets off on a random walk with three possible locations. The Markov matrix of probabilities $A = [a_{ij}]$ is given by

$$\begin{bmatrix} 0.5 & 0.1 & 0.6 \\ 0.2 & 0.9 & 0.2 \\ 0.3 & 0 & 0.2 \end{bmatrix}$$

It is unknown where the walker starts, but the probability of starting in each location is given by

$$\vec{x}_0 = \begin{bmatrix} 0.2 \\ 0.25 \\ 0.55 \end{bmatrix}$$

What is the probability of the walker being in location 1 at time $n = 2$?

Answer available at the end of this section.

Exercise $6.5E.8$

You own a trailer rental company in a large city and you have four locations, one in the South East, one in the North East, one in the North West, and one in the South West. Denote these locations by SE, NE, NW, and SW respectively. Suppose that the following table is observed to take place.

Table $6.5E.1$ Markov matrix of trailer rental company locations

	SE	NE	NW	SW
SE	$\frac{1}{3}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{5}$
NE	$\frac{1}{3}$	$\frac{7}{10}$	$\frac{1}{5}$	$\frac{1}{10}$
NW	$\frac{2}{9}$	$\frac{1}{10}$	$\frac{3}{5}$	$\frac{1}{5}$
SW	$\frac{1}{9}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{2}$

In this table, the probability that a trailer starting at NE ends in NW is $1/10$, the probability that a trailer starting at SW ends in NW is $1/5$, and so forth. Approximately how many will you have in each location after a long time if the total number of trailers is 413?

Answer available at the end of this section.

Exercise $6.5E.9$

You own a trailer rental company in a large city and you have four locations, one in the South East, one in the North East, one in the North West, and one in the South West. Denote these locations by SE, NE, NW, and SW respectively. Suppose that the following table is observed to take place.

Table $6.5E.2$ Markov matrix of trailer rental company locations

	SE	NE	NW	SW
SE	$\frac{1}{7}$	$\frac{1}{4}$	$\frac{1}{10}$	$\frac{1}{5}$
NE	$\frac{2}{7}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{10}$
NW	$\frac{1}{7}$	$\frac{1}{4}$	$\frac{3}{5}$	$\frac{1}{5}$
SW	$\frac{3}{7}$	$\frac{1}{4}$	$\frac{1}{10}$	$\frac{1}{2}$

In this table, the probability that a trailer starting at NE ends in NW is 1/10, the probability that a trailer starting at SW ends in NW is 1/5, and so forth. Approximately how many will you have in each location after a long time if the total number of trailers is 1469.

Exercise $6.5E.10$

The following table describes the transition probabilities between the states rainy, partly cloudy and sunny. The symbol p.c. indicates partly cloudy. Thus if it starts off p.c. it ends up sunny the next day with probability $\frac{1}{5}$. If it starts off sunny, it ends up sunny the next day with probability $\frac{2}{5}$ and so forth.

Table $6.5E.3$ Markov matrix of weather states

	rains	sunny	p.c.
rains	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{3}$
sunny	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{3}$
p.c.	$\frac{3}{5}$	$\frac{2}{5}$	$\frac{1}{3}$

Given this information, what are the probabilities that a given day is rainy, sunny, or partly cloudy?

Exercise $6.5E.11$

The following table describes the transition probabilities between the states rainy, partly cloudy and sunny. The symbol p.c. indicates partly cloudy. Thus if it starts off p.c. it ends up sunny the next day with probability $\frac{1}{10}$. If it starts off sunny, it ends up sunny the next day with probability $\frac{2}{5}$ and so forth.

Table 6.5E.4 Markov matrix of weather states

	rains	sunny	p.c.
rains	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{3}$
sunny	$\frac{1}{10}$	$\frac{2}{5}$	$\frac{4}{9}$
p.c.	$\frac{7}{10}$	$\frac{2}{5}$	$\frac{2}{9}$

Given this information, what are the probabilities that a given day is rainy, sunny, or partly cloudy?

Exercise 6.5E.12

You own a trailer rental company in a large city and you have four locations, one in the South East, one in the North East, one in the North West, and one in the South West. Denote these locations by SE, NE, NW, and SW respectively. Suppose that the following table is observed to take place.

Table 6.5E.5 Markov matrix of trailer rental company locations

	SE	NE	NW	SW
SE	$\frac{5}{11}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{5}$
NE	$\frac{1}{11}$	$\frac{7}{10}$	$\frac{1}{5}$	$\frac{1}{10}$
NW	$\frac{2}{11}$	$\frac{1}{10}$	$\frac{3}{5}$	$\frac{1}{5}$
SW	$\frac{3}{11}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{2}$

In this table, the probability that a trailer starting at NE ends in NW is 1/10, the probability that a trailer starting at SW ends in NW is 1/5, and so forth. Approximately how many will you have in each location after a long time if the total number of trailers is 407?

Exercise 6.5E.13

The University of Poohbah offers three degree programs, scouting education (SE), dance appreciation (DA), and engineering (E). It has been determined that the probabilities of transferring from one program to another are as in the following table.

Table 6.5E.6 Markov matrix of degree programs

	SE	DA	E
SE	.8	.1	.3
DA	.1	.7	.5
E	.1	.2	.2

where the number indicates the probability of transferring from the top program to the program on the left. Thus the probability of going from DA to E is .2. Find the probability that a student is enrolled in the various programs.

Exercise 6.5E.14

In the city of Nabal, there are three political persuasions, republicans (R), democrats (D), and neither one (N). The following table shows the transition probabilities between the political parties, the top row being the initial political party and the side row being the political affiliation the following year.

Table 6.5E.7 Markov matrix of political persuasions

	R	D	N
R	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{2}{7}$
D	$\frac{1}{5}$	$\frac{1}{3}$	$\frac{4}{7}$
N	$\frac{3}{5}$	$\frac{1}{2}$	$\frac{1}{7}$

Find the probabilities that a person will be identified with the various political persuasions. Which party will end up being most important?

Exercise 6.5E.15

The following table describes the transition probabilities between the states rainy, partly cloudy and sunny. The symbol p.c. indicates partly cloudy. Thus if it starts off p.c. it ends up sunny the next day with probability $\frac{1}{5}$. If it starts off sunny, it ends up sunny the next day with probability $\frac{2}{7}$ and so forth.

Table 6.5E.8 Markov matrix of weather states

	rains	sunny	p.c.
rains	$\frac{1}{5}$	$\frac{2}{7}$	$\frac{5}{9}$
sunny	$\frac{1}{5}$	$\frac{2}{7}$	$\frac{1}{3}$
p.c.	$\frac{3}{5}$	$\frac{3}{7}$	$\frac{1}{9}$

Given this information, what are the probabilities that a given day is rainy, sunny, or partly cloudy?

Answers

Exercise 6.5E.1

a. Multiply the given matrix by the initial state vector given by $\begin{bmatrix} 90 \\ 81 \\ 85 \end{bmatrix}$. After one time period there are 89 people in location 1, 106 in location 2, and 61 in location 3.

b. Solve the system given by $(I-A)\vec{x}_s = 0$ where A is the migration matrix and $\vec{x}_s = \begin{bmatrix} x_{1s} \\ x_{2s} \\ x_{3s} \end{bmatrix}$ is the steady state vector. The solution to this system is given by

$$\begin{aligned} x_{1s} &= \frac{8}{5}x_{3s} \\ x_{2s} &= \frac{63}{25}x_{3s} \end{aligned} \tag{6.5E.1}$$

Letting $x_{3s} = t$ and using the fact that there are a total of 256 individuals, we must solve

$$\frac{8}{5}t + \frac{63}{25}t + t = 256$$

We find that $t = 50$. Therefore after a long time, there are 80 people in location 1, 126 in location 2, and 50 in location 3.

Exercise 6.5E.2

Exercise 6.5E.3

We solve $(I-A)\vec{x}_s = 0$ to find the steady state vector $\vec{x}_s = \begin{bmatrix} x_{1s} \\ x_{2s} \\ x_{3s} \end{bmatrix}$. The solution to the system is given by

$$\begin{aligned} x_{1s} &= \frac{5}{6}x_{3s} \\ x_{2s} &= \frac{2}{3}x_{3s} \end{aligned} \tag{6.5E.2}$$

Letting $x_{3s} = t$ and using the fact that there are a total of 480 individuals, we must solve

$$\frac{5}{6}t + \frac{2}{3}t + t = 480$$

We find that $t = 192$. Therefore after a long time, there are 160 people in location 1, 128 in location 2, and 192 in location 3.

Exercise $6.5E.6$

$$\vec{x}_3 = \begin{bmatrix} 0.38 \\ 0.18 \\ 0.44 \end{bmatrix}$$

Therefore the probability of ending up back in location 2 is 0.18.

Exercise $6.5E.7$

$$\vec{x}_2 = \begin{bmatrix} 0.367 \\ 0.4625 \\ 0.1705 \end{bmatrix}$$

Therefore the probability of ending up in location 1 is 0.367.

Exercise $6.5E.8$

The migration matrix is

$$A = \begin{bmatrix} \frac{1}{3} & \frac{1}{10} & \frac{1}{10} & \frac{1}{5} \\ \frac{1}{3} & \frac{7}{10} & \frac{1}{5} & \frac{1}{10} \\ \frac{2}{9} & \frac{1}{10} & \frac{3}{5} & \frac{1}{5} \\ \frac{1}{9} & \frac{1}{10} & \frac{1}{10} & \frac{1}{2} \end{bmatrix}$$

To find the number of trailers in each location after a long time we solve system $(I - A)\vec{x}_s = 0$ for the steady state vector $\vec{x}_s = \begin{bmatrix} x_{1s} \\ x_{2s} \\ x_{3s} \\ x_{4s} \end{bmatrix}$. The solution to the system is

$$\begin{aligned} x_{1s} &= \frac{9}{10}x_{4s} \\ x_{2s} &= \frac{12}{5}x_{4s} \\ x_{3s} &= \frac{8}{5}x_{4s} \end{aligned} \tag{6.5E.3}$$

Letting $x_{4s} = t$ and using the fact that there are a total of 413 trailers we must solve

$$\frac{9}{10}t + \frac{12}{5}t + \frac{8}{5}t + t = 413$$

We find that $t = 70$. Therefore after a long time, there are 63 trailers in the SE, 168 in the NE, 112 in the NW and 70 in the SW.

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6.6: Dynamical Systems

Learning Objectives

- Understand how to solve dynamical systems.

Dynamical Systems

The migration matrices discussed in the previous section give an example of a discrete dynamical system. We call them discrete because they involve discrete values taken at a sequence of points rather than on a continuous interval of time.

An example of a situation which can be studied in this way is a predator prey model. Consider the following model where x is the number of prey and y the number of predators in a certain area at a certain time. These are functions of $n \in \mathbb{N}$ where $n = 1, 2, \cdots$ are the ends of intervals of time which may be of interest in the problem. In other words, $x(n)$ is the number of prey at the end of the nth interval of time. An example of this situation may be modeled by the following equation

$$\begin{bmatrix} x(n+1) \\ y(n+1) \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x(n) \\ y(n) \end{bmatrix}$$

This says that from time period n to $n+1$, x increases if there are more x and decreases as there are more y. In the context of this example, this means that as the number of predators increases, the number of prey decreases. As for y, it increases if there are more y and also if there are more x.

This is an example of a matrix recurrence which we define now.

Definition 6.6.1: Matrix Recurrence

Suppose a dynamical system is given by

$$\begin{aligned} x_{n+1} &= ax_n + by_n \\ y_{n+1} &= cx_n + dy_n \end{aligned} \tag{6.6.1}$$

This system can be expressed as $V_{n+1} = AV_n$ where $V_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix}$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

In this section, we will examine how to find solutions to a dynamical system given certain initial conditions. This process involves several concepts previously studied, including matrix diagonalization and Markov matrices. The procedure is given as follows. Recall that when diagonalized, we can write $A^n = PD^nP^{-1}$.

Procedure 6.6.1: Solving a Dynamical System

Suppose a dynamical system is given by

$$\begin{aligned} x_{n+1} &= ax_n + by_n \\ y_{n+1} &= cx_n + dy_n \end{aligned} \tag{6.6.2}$$

Given initial conditions x_0 and y_0, the solutions to the system are found as follows:

1. Express the dynamical system in the form $V_{n+1} = AV_n$.
2. Diagonalize A to be written as $A = PDP^{-1}$.
3. Then $V_n = PD^nP^{-1}V_0$ where V_0 is the vector containing the initial conditions.
4. If given specific values for n, substitute into this equation. Otherwise, find a general solution for n.

We will now consider an example in detail.

Example 6.6.1: Solutions of a Discrete Dynamical System

Suppose a dynamical system is given by

$$\begin{aligned} x_{n+1} &= 1.5x_n - 0.5y_n \\ y_{n+1} &= 1.0x_n \end{aligned} \tag{6.6.3}$$

Express this system as a matrix recurrence and find solutions to the dynamical system for initial conditions $x_0 = 20, y_0 = 10$.

Solution

First, we express the system as a matrix recurrence.

$$\begin{aligned} V_{n+1} &= AV_n \\ \begin{bmatrix} x(n+1) \\ y(n+1) \end{bmatrix} &= \begin{bmatrix} 1.5 & -0.5 \\ 1.0 & 0 \end{bmatrix} \begin{bmatrix} x(n) \\ y(n) \end{bmatrix} \end{aligned} \tag{6.6.4}$$

Then

$$A = \begin{bmatrix} 1.5 & -0.5 \\ 1.0 & 0 \end{bmatrix}$$

You can verify that the eigenvalues of A are 1 and $.5$. By diagonalizing, we can write A in the form

$$P^{-1}DP = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & .5 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

Now given an initial condition

$$V_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

the solution to the dynamical system is given by

$$\begin{aligned} V_n &= PD^nP^{-1}V_0 \\ \begin{bmatrix} x(n) \\ y(n) \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & .5 \end{bmatrix}^n \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (.5)^n \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \\ &= \begin{bmatrix} y_0((.5)^n - 1) - x_0((.5)^n - 2) \\ y_0(2(.5)^n - 1) - x_0(2(.5)^n - 2) \end{bmatrix} \end{aligned} \tag{6.6.5}$$

If we let n become arbitrarily large, this vector approaches

$$\begin{bmatrix} 2x_0 - y_0 \\ 2x_0 - y_0 \end{bmatrix}$$

Thus for large n,

$$\begin{bmatrix} x(n) \\ y(n) \end{bmatrix} \approx \begin{bmatrix} 2x_0 - y_0 \\ 2x_0 - y_0 \end{bmatrix}$$

Now suppose the initial condition is given by

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 20 \\ 10 \end{bmatrix}$$

Then, we can find solutions for various values of n. Here are the solutions for values of n between 1 and 5

$$n=1:\begin{bmatrix}25.0\\20.0\end{bmatrix},n=2:\begin{bmatrix}27.5\\25.0\end{bmatrix},n=3:\begin{bmatrix}28.75\\27.5\end{bmatrix}$$

$$n=4:\begin{bmatrix}29.375\\28.75\end{bmatrix},n=5:\begin{bmatrix}29.688\\29.375\end{bmatrix}$$

Notice that as n increases, we approach the vector given by

$$\begin{bmatrix}2x_0-y_0\\2x_0-y_0\end{bmatrix}=\begin{bmatrix}2(20)-10\\2(20)-10\end{bmatrix}=\begin{bmatrix}30\\30\end{bmatrix}$$

These solutions are graphed in the following figure.

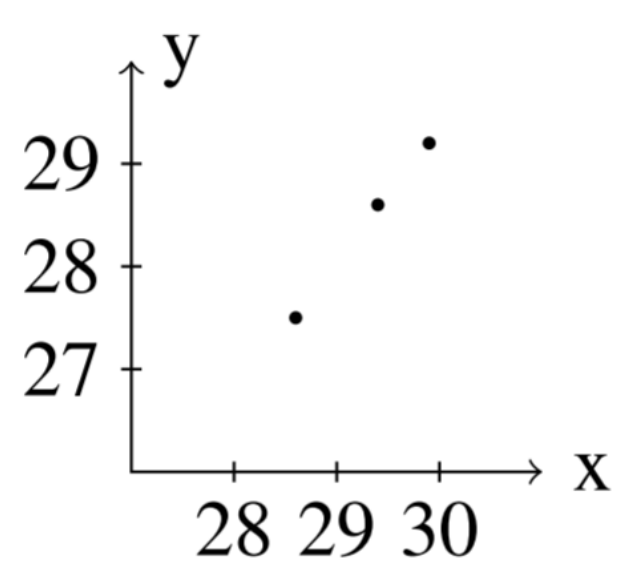


Figure 6.6.1 First quadrant of the XY-plane with three solution points plotted (CC BY-NC-SA 4.0; Kuttler via A First Course in LINEAR ALGEBRA)

The following example demonstrates another system which exhibits some interesting behavior. When we graph the solutions, it is possible for the ordered pairs to spiral around the origin.

Example 6.6.2: Finding Solutions to a Dynamical System

Suppose a dynamical system is of the form

$$\begin{bmatrix}x(n+1)\\y(n+1)\end{bmatrix}=\begin{bmatrix}0.7&0.7\\-0.7&0.7\end{bmatrix}\begin{bmatrix}x(n)\\y(n)\end{bmatrix}$$

Find solutions to the dynamical system for given initial conditions.

Solution

Let

$$A=\begin{bmatrix}0.7&0.7\\-0.7&0.7\end{bmatrix}$$

To find solutions, we must diagonalize A. You can verify that the eigenvalues of A are complex and are given by $\lambda_1=.7+.7i$ and $\lambda_2=.7-.7i$. The eigenvector for $\lambda_1=.7+.7i$ is

$$\begin{bmatrix}1\\i\end{bmatrix}$$

and that the eigenvector for $\lambda_2 = .7 - .7i$ is

$$\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Thus the matrix A can be written in the form

$$\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} .7+.7i & 0 \\ 0 & .7-.7i \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2}i \\ \frac{1}{2} & \frac{1}{2}i \end{bmatrix}$$

and so,

$$V_n = PD^nP^{-1}V_0$$

$$\begin{bmatrix} x(n) \\ y(n) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} (.7+.7i)^n & 0 \\ 0 & (.7-.7i)^n \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2}i \\ \frac{1}{2} & \frac{1}{2}i \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \tag{6.6.6}$$

The explicit solution is given by

$$\begin{bmatrix} x_0\left(\frac{1}{2}(0.7-0.7i)^n + \frac{1}{2}(0.7+0.7i)^n\right) + y_0\left(\frac{1}{2}i(0.7-0.7i)^n - \frac{1}{2}i(0.7+0.7i)^n\right) \\ y_0\left(\frac{1}{2}(0.7-0.7i)^n + \frac{1}{2}(0.7+0.7i)^n\right) - x_0\left(\frac{1}{2}i(0.7-0.7i)^n - \frac{1}{2}i(0.7+0.7i)^n\right) \end{bmatrix}$$

Suppose the initial condition is

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

Then one obtains the following sequence of values which are graphed below by letting $n = 1, 2, \cdots, 20$

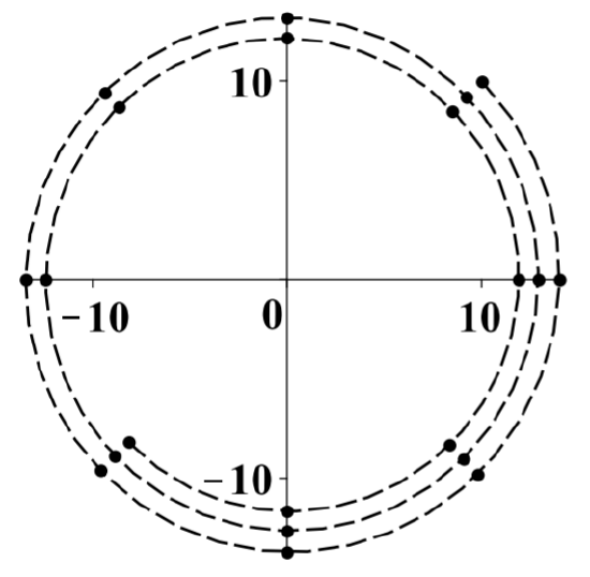


Figure 6.6.2 XY-plane with spiral starting in the third quadrant goes around 2 and a half times counter-clockwise to end in the first quadrant. Points plotted every 45°. (CC BY-NC-SA 4.0; Kuttler via A First Course in LINEAR ALGEBRA)

In this picture, the dots are the values and the dashed line is to help to picture what is happening.

These points are getting gradually closer to the origin, but they are circling the origin in the clockwise direction as they do so. As n increases, the vector $\begin{bmatrix} x(n) \\ y(n) \end{bmatrix}$ approaches $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

This type of behavior along with complex eigenvalues is typical of the deviations from an equilibrium point in the Lotka Volterra system of differential equations which is a famous model for predator-prey interactions. These differential equations are given by

$$\begin{aligned} x' &= x\left(a - by\right) \\ y' &= -y\left(c - dx\right) \end{aligned} \tag{6.6.7}$$

where a, b, c, d are positive constants. For example, you might have X be the population of moose and Y the population of wolves on an island.

Note that these equations make logical sense. The top says that the rate at which the moose population increases would be aX if there were no predators Y. However, this is modified by multiplying instead by $(a - bY)$ because if there are predators, these will militate against the population of moose. The more predators there are, the more pronounced is this effect. As to the predator equation, you can see that the equations predict that if there are many prey around, then the rate of growth of the predators would seem to be high. However, this is modified by the term $-cY$ because if there are many predators, there would be competition for the available food supply and this would tend to decrease Y'.

The behavior near an equilibrium point, which is a point where the right side of the differential equations equals zero, is of great interest. In this case, the equilibrium point is

$$x = \frac{c}{d}, y = \frac{a}{b}$$

Then one defines new variables according to the formula

$$x + \frac{c}{d} = x, \ y = y + \frac{a}{b}$$

In terms of these new variables, the differential equations become

$$\begin{aligned} x' &= \left(x + \frac{c}{d}\right)\left(a - b\left(y + \frac{a}{b}\right)\right) \\ y' &= -\left(y + \frac{a}{b}\right)\left(c - d\left(x + \frac{c}{d}\right)\right) \end{aligned} \tag{6.6.8}$$

Multiplying out the right sides yields

$$\begin{aligned} x' &= -bxy - b\frac{c}{d}y \\ y' &= dxy + \frac{a}{b}dx \end{aligned} \tag{6.6.9}$$

The interest is for x, y small and so these equations are essentially equal to

$$x' = -b\frac{c}{d}y, \ y' = \frac{a}{b}dx$$

Replace x' with the difference quotient $\frac{x(t+h)-x(t)}{h}$ where h is a small positive number and y' with a similar difference quotient. For example one could have h correspond to one day or even one hour. Thus, for h small enough, the following would seem to be a good approximation to the differential equations.

$$\begin{aligned} x\left(t + h\right) &= x\left(t\right) - hb\frac{c}{d}y \\ y\left(t + h\right) &= y\left(t\right) + h\frac{a}{b}dx \end{aligned} \tag{6.6.10}$$

Let $1, 2, 3, \cdots$ denote the ends of discrete intervals of time having length h chosen above. Then the above equations take the form

$$\begin{bmatrix} x\left(n+1\right) \\ y\left(n+1\right) \end{bmatrix} = \begin{bmatrix} 1 & -\frac{hbc}{d} \\ \frac{had}{b} & 1 \end{bmatrix} \begin{bmatrix} x\left(n\right) \\ y\left(n\right) \end{bmatrix}$$

Note that the eigenvalues of this matrix are always complex.

We are not interested in time intervals of length h for h very small. Instead, we are interested in much longer lengths of time. Thus, replacing the time interval with mh,

$$\begin{bmatrix} x\left(n+m\right) \\ y\left(n+m\right) \end{bmatrix} = \begin{bmatrix} 1 & -\frac{hbc}{d} \\ \frac{had}{b} & 1 \end{bmatrix}^m \begin{bmatrix} x\left(n\right) \\ y\left(n\right) \end{bmatrix}$$

For example, if $m = 2$, you would have

$$\begin{bmatrix} x\left(n+2\right) \\ y\left(n+2\right) \end{bmatrix} = \begin{bmatrix} 1 - ach^2 & -2b\frac{c}{d}h \\ 2\frac{a}{b}dh & 1 - ach^2 \end{bmatrix} \begin{bmatrix} x\left(n\right) \\ y\left(n\right) \end{bmatrix}$$

Note that most of the time, the eigenvalues of the new matrix will be complex.

You can also notice that the upper right corner will be negative by considering higher powers of the matrix. Thus letting $1, 2, 3, \cdots$ denote the ends of discrete intervals of time, the desired discrete dynamical system is of the form

$$\begin{bmatrix} x(n+1) \\ y(n+1) \end{bmatrix} = \begin{bmatrix} a & -b \\ c & d \end{bmatrix} \begin{bmatrix} x(n) \\ y(n) \end{bmatrix}$$

where a, b, c, d are positive constants and the matrix will likely have complex eigenvalues because it is a power of a matrix which has complex eigenvalues.

You can see from the above discussion that if the eigenvalues of the matrix used to define the dynamical system are less than 1 in absolute value, then the origin is stable in the sense that as $n \to \infty$, the solution converges to the origin. If either eigenvalue is larger than 1 in absolute value, then the solutions to the dynamical system will usually be unbounded, unless the initial condition is chosen very carefully. The next example exhibits the case where one eigenvalue is larger than 1 and the other is smaller than 1.

The following example demonstrates a familiar concept as a dynamical system.

Example 6.6.3: The Fibonacci Sequence

The Fibonacci sequence is the sequence given by

$$1, 1, 2, 3, 5, \cdots$$

which is defined recursively in the form

$$x(0) = 1 = x(1), \; x(n+2) = x(n+1) + x(n)$$

Show how the Fibonacci Sequence can be considered a dynamical system.

Solution

This sequence is extremely important in the study of reproducing rabbits. It can be considered as a dynamical system as follows. Let $y(n) = x(n+1)$. Then the above recurrence relation can be written as

$$\begin{bmatrix} x(n+1) \\ y(n+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x(n) \\ y(n) \end{bmatrix}, \; \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

The eigenvalues of the matrix A are $\lambda_1 = \frac{1}{2} - \frac{1}{2}\sqrt{5}$ and $\lambda_2 = \frac{1}{2}\sqrt{5} + \frac{1}{2}$. The corresponding eigenvectors are, respectively,

$$\vec{x}_1 = \begin{bmatrix} -\frac{1}{2}\sqrt{5} - \frac{1}{2} \\ 1 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} \frac{1}{2}\sqrt{5} - \frac{1}{2} \\ 1 \end{bmatrix}$$

You can see from a short computation that one of the eigenvalues is smaller than 1 in absolute value while the other is larger than 1 in absolute value. Now, diagonalizing A gives us

$$\begin{bmatrix} \frac{1}{2}\sqrt{5} - \frac{1}{2} & -\frac{1}{2}\sqrt{5} - \frac{1}{2} \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2}\sqrt{5} - \frac{1}{2} & -\frac{1}{2}\sqrt{5} - \frac{1}{2} \\ 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{2}\sqrt{5} + \frac{1}{2} & 0 \\ 0 & \frac{1}{2} - \frac{1}{2}\sqrt{5} \end{bmatrix}$$

Then it follows that for a given initial condition, the solution to this dynamical system is of the form

$$\begin{bmatrix} x(n) \\ y(n) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\sqrt{5} - \frac{1}{2} & -\frac{1}{2}\sqrt{5} - \frac{1}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \left(\frac{1}{2}\sqrt{5} + \frac{1}{2}\right)^n & 0 \\ 0 & \left(\frac{1}{2} - \frac{1}{2}\sqrt{5}\right)^n \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{5}\sqrt{5} & \frac{1}{10}\sqrt{5} + \frac{1}{2} \\ -\frac{1}{5}\sqrt{5} & \frac{1}{5}\sqrt{5}\left(\frac{1}{2}\sqrt{5} - \frac{1}{2}\right) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{6.6.11}$$

It follows that

$$x(n) = \left(\frac{1}{2}\sqrt{5} + \frac{1}{2}\right)^n \left(\frac{1}{10}\sqrt{5} + \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{2}\sqrt{5}\right)^n \left(\frac{1}{2} - \frac{1}{10}\sqrt{5}\right)$$

Here is a picture of the ordered pairs $(x(n), y(n))$ for $n = 0, 1, \cdots, n$.

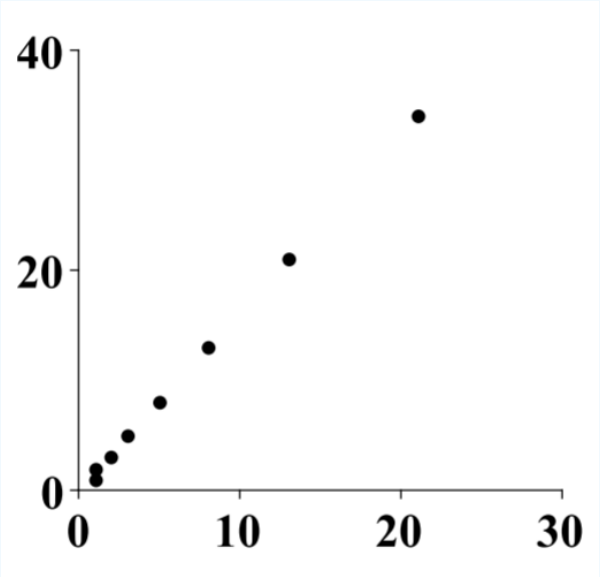


Figure 6.6.3: XY-plane with values (x, y) values representing consecutive values of the Fibonacci sequence (CC BY-NC-SA 4.0; Kuttler via A First Course in LINEAR ALGEBRA)There is so much more that can be said about dynamical systems. It is a major topic of study in differential equations and what is given above is just an introduction.

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6.6E: Exercises for Section 6.6

Exercise $6.6E.1$

Suppose in a certain area there are populations of coyotes and rabbits. The coyotes depend on the rabbits as their food source so the number of coyotes the next month depends on the beginning number and the number of rabbits available as prey. If we let x_n represent the number of coyotes and y_n the number of rabbits in month n, suppose

$$x_{n+1} = 0.7x_n + 0.1y_n \tag{6.6E.1}$$

The number of rabbits would naturally increase except that the coyotes decrease the number of rabbits by preying on them. Suppose

$$y_{n+1} = -0.5x_n + 1.3y_n \tag{6.6E.2}$$

Express this dynamical system in the form $V_{n+1} = AV_n$.

Answer available at the end of this section.

Exercise $6.6E.2$

For the system from Exercise $6.6E.1$, suppose there are 50 coyotes and 400 rabbits in month 0,

a. How many coyotes and rabbits will there be in month 1?
b. How many coyotes and rabbits will there be in month 2?

Exercise $6.6E.3$

Diagonalize the matrix A from Exercise $6.6E.1$ and use this to find:

a. The number of coyotes and rabbits in month 5.
b. The number of coyotes and rabbits in month n.

Example $6.6E.4$

Using the result from Exercise $6.6E.3$, describe what happens to the populations of coyotes and rabbits in the long run.

Exercise $6.6E.5$

For the system from Exercise $6.6E.1$, suppose the number of coyotes in month 0 is 100 and the number of rabbits in month 0 is 500.

a. How many coyotes and rabbits will there be in month 1?
b. How many coyotes and rabbits will there be in month 2?
c. How many coyotes and rabbits will there be in month n?

What is special about this initial value?

Answer available at the end of this section.

Exercise $6.6E.6$

In a forest ecosystem, the number of \textbf{foxes} and \textbf{mice} changes from month to month. Let f_n be the number of foxes and m_n the number of mice in month n.

Suppose the populations follow this system:

$$\begin{aligned} f_{n+1} &= 0.6f_n + 0.05m_n \\ m_{n+1} &= -0.4f_n + 1.1m_n \end{aligned} \tag{6.6E.3}$$

Let the initial populations be:

$$f_0 = 50, \quad m_0 = 200 \tag{6.6E.4}$$

a. Express the system in the form $V_{n+1} = AV_n$.
b. Find the populations in month 1 and month 2.
c. What is the long-term behavior of the system?

Answer available at the end of this section.

Exercise $6.6E.7$

In a valley, the number of \textbf{hawks} and \textbf{squirrels} evolves according to the system:

$$\begin{aligned} h_{n+1} &= 0.8h_n + 0.2s_n \\ s_{n+1} &= -0.3h_n + 1.2s_n \end{aligned}$$

Given initial populations:

$$h_0 = 30, \quad s_0 = 90$$

a. Express the system in matrix form.
b. Calculate the populations in months 1 and 2.
c. Does this system represent growth, decay, or stability

Answer available at the end of this section.

Answers

Exercise $6.6E.1$

$$V_{n+1} = AV_n \quad \text{where} \quad A = \begin{bmatrix} 0.7 & 0.1 \\ -0.5 & 1.3 \end{bmatrix}, \quad V_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix}$$

Exercise $6.6E.5$

Given $V_0 = \begin{bmatrix} 100 \\ 500 \end{bmatrix}, \quad A = \begin{bmatrix} 0.7 & 0.1 \\ -0.5 & 1.3 \end{bmatrix}$

$V_1 = AV_0 = \begin{bmatrix} 120 \\ 600 \end{bmatrix}$

$V_2 = AV_1 = \begin{bmatrix} 144 \\ 720 \end{bmatrix}$

$V_n = A^n V_0 = 100 \cdot (1.2)^n \cdot \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

What is special about this initial value? The initial vector $V_0 = \begin{bmatrix} 100 \\ 500 \end{bmatrix}$ is an eigenvector of the matrix A corresponding to the eigenvalue $\lambda = 1.2$. This means that the population vector grows exponentially while maintaining a constant ratio of coyotes to rabbits $1:5$. Each month, the population is scaled by a factor of 1.2 without changing direction.

Exercise $6.6E.6$

a. The system in matrix form is:

$$V_{n+1} = AV_n, \quad \text{where} \quad A = \begin{bmatrix} 0.6 & 0.05 \\ -0.4 & 1.1 \end{bmatrix}, \quad V_0 = \begin{bmatrix} 50 \\ 200 \end{bmatrix}$$

b. Month 1:

$$V_1 = \begin{bmatrix} 40 \\ 200 \end{bmatrix}$$

Month 2:

$$V_2 = \begin{bmatrix} 34 \\ 204 \end{bmatrix}$$

c. Long-Term Behavior: The number of foxes decreases while the number of mice gradually increases. The system will approach a direction aligned with the dominant eigenvector of A, suggesting eventual stability or unbounded growth along that vector.

Exercise $6.6E.7$

a. The system in matrix form is:

$$V_{n+1} = AV_n, \quad \text{where} \quad A = \begin{bmatrix} 0.8 & 0.2 \\ -0.3 & 1.2 \end{bmatrix}, \quad V_0 = \begin{bmatrix} 30 \\ 90 \end{bmatrix}$$

b. Month 1:

$$V_1 = \begin{bmatrix} 42 \\ 99 \end{bmatrix}$$

Month 2:

$$V_2 = \begin{bmatrix} 53.4 \\ 106.2 \end{bmatrix}$$

c. Long-Term Behavior: Both populations grow exponentially. The dominant eigenvalue of A is greater than 1, indicating the system represents exponential growth while maintaining a fixed population ratio over time.

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6.7: Orthogonal Diagonalization

Learning Objectives

- Understand the meaning of orthogonal diagonalization
- Determine if a matrix is orthogonally diagonalization

Orthogonal Diagonalization

We begin this section by recalling some important definitions. Recall that non-zero vectors are called orthogonal if their dot product equals 0. A set is orthonormal if it is orthogonal and each vector is a unit vector.

An orthogonal matrix U is one in which $UU^T = I$. In other words, the transpose of an orthogonal matrix is equal to its inverse. A key characteristic of orthogonal matrices, which will be essential in this section, is that the columns of an orthogonal matrix form an orthonormal set.

We now recall another important definition.

Definition 6.7.1: Symmetric and Skew Symmetric Matrices

A real $n \times n$ matrix A, is **symmetric** if $A^T = A$. If $A = -A^T$, then A is called **skew symmetric.**

Before proving an essential theorem, we first examine the following lemma which will be used below.

Lemma 6.7.1: The Dot Product

Let $A = (a_{ij})$ be a real symmetric $n \times n$ matrix, and let $\vec{x}, \vec{y} \in \mathbb{R}^n$. Then

$$A\vec{x} \cdot \vec{y} = \vec{x} \cdot A\vec{y}$$

Proof available at the end of this section.

We can now prove that the eigenvalues of a real symmetric matrix are real numbers. Consider the following important theorem.

Theorem 6.7.1: Orthogonal Eigenvectors

Let A be a real symmetric matrix. Then the eigenvalues of A are real numbers and eigenvectors corresponding to distinct eigenvalues are orthogonal.

Proof available at the end of this section.

The following theorem is proved in a similar manner.

Theorem 6.7.2: Eigenvalues of Skew Symmetric Matrix

The eigenvalues of a real skew symmetric matrix are either equal to 0 or are pure imaginary numbers.

Proof available at the end of this section.

Consider the following example.

Example 6.7.1: Eigenvalues of a Skew Symmetric Matrix

Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Find its eigenvalues.

Solution

First notice that A is skew symmetric. By Theorem 6.7.2, the eigenvalues will either equal 0 or be pure imaginary. The eigenvalues of A are obtained by solving the usual equation

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda & 1 \\ -1 & \lambda \end{bmatrix} = \lambda^2 + 1 = 0$$

Hence the eigenvalues are $\pm i$, pure imaginary.

Consider the following example.

Example 6.7.2: Eigenvalues of a Symmetric Matrix

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$. Find its eigenvalues.

Solution

First, notice that A is symmetric. By Theorem 6.7.1, the eigenvalues will all be real. The eigenvalues of A are obtained by solving the usual equation

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 3 \end{bmatrix} = \lambda^2 - 4\lambda - 1 = 0$$

The eigenvalues are given by $\lambda_1 = 2 + \sqrt{5}$ and $\lambda_2 = 2 - \sqrt{5}$ which are both real.

Recall that a diagonal matrix $D = (d_{ij})$ is one in which $d_{ij} = 0$ whenever $i \neq j$. In other words, all numbers not on the main diagonal are equal to zero.

If a matrix A is symmetric, then we can actually find a matrix U that is an orthogonal matrix of eigenvectors. Thus U^{-1} is simply its transpose U^T, and $U^T AU$ is diagonal. When this happens we say that A is orthogonally diagonalizable.

Definition: Orthogonally Diagonalizable

Let A be an $n \times n$ matrix. Then A is said to be **orthogonally diagonalizable** if there exists an orthogonal matrix U such that

$$U^{-1}AU = D$$

where D is a diagonal matrix. Note that since U is an orthogonal matrix, $U^{-1} = U^T$, so we have

$$U^T AU = D$$

Consider the following important theorem.

Theorem 6.7.3: Orthogonal Diagonalization

Let A be a real symmetric matrix. Then there exists an orthogonal matrix U such that

$$U^T AU = D$$

where D is a diagonal matrix. Moreover, the diagonal entries of D are the eigenvalues of A.

We can use this theorem to diagonalize a symmetric matrix, using orthogonal matrices. Consider the following corollary.

Corollary 6.7.1: Orthonormal Set of Eigenvectors

If A is a real $n \times n$ symmetric matrix, then there exists an orthonormal set of eigenvectors, $\{\vec{u}_1, \cdots, \vec{u}_n\}$.

Proof available at the end of this section.

Definition 6.7.2: Principal Axes

Let A be an $n \times n$ matrix. Then the **principal axes** of A is a set of orthonormal eigenvectors of A.

In the next example, we examine how to find such a set of orthonormal eigenvectors.

Example 6.7.3: Find an Orthonormal Set of Eigenvectors

Find an orthonormal set of eigenvectors for the symmetric matrix

$$A = \begin{bmatrix} 17 & -2 & -2 \\ -2 & 6 & 4 \\ -2 & 4 & 6 \end{bmatrix}$$

Solution

Recall Procedure 6.1.1 for finding the eigenvalues and eigenvectors of a matrix. You can verify that the eigenvalues are $18, 9, 2$. First find the eigenvector for 18 by solving the equation $(18I - A)\vec{x} = \vec{0}$. The appropriate augmented matrix is given by

$$\begin{bmatrix} 18-17 & 2 & 2 & 0 \\ 2 & 18-6 & -4 & 0 \\ 2 & -4 & 18-6 & 0 \end{bmatrix}$$

The reduced row-echelon form is

$$\begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore an eigenvector is

$$\begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}$$

Next find the eigenvector for $\lambda = 9$. The augmented matrix and resulting reduced row-echelon form are

$$\begin{bmatrix} 9-17 & 2 & 2 & 0 \\ 2 & 9-6 & -4 & 0 \\ 2 & -4 & 9-6 & 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus an eigenvector for $\lambda = 9$ is

$$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Finally find an eigenvector for $\lambda = 2$. The appropriate augmented matrix and reduced row-echelon form are

$$\begin{bmatrix} 2-17 & 2 & 2 & 0 \\ 2 & 2-6 & -4 & 0 \\ 2 & -4 & 2-6 & 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus an eigenvector for $\lambda = 2$ is

$$\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

The set of eigenvectors for A is given by

$$\left\{ \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

You can verify that these eigenvectors form an orthogonal set. By dividing each eigenvector by its magnitude, we obtain an orthonormal set:

$$\left\{ \frac{1}{\sqrt{18}} \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

Consider the following example.

Example 6.7.4: Repeated Eigenvalues

Find an orthonormal set of three eigenvectors for the matrix

$$A = \begin{bmatrix} 10 & 2 & 2 \\ 2 & 13 & 4 \\ 2 & 4 & 13 \end{bmatrix}$$

Solution

You can verify that the eigenvalues of A are 9 (with multiplicity two) and 18 (with multiplicity one). Consider the eigenvectors corresponding to $\lambda = 9$. The appropriate augmented matrix and reduced row-echelon form are given by

$$\begin{bmatrix} 9-10 & -2 & -2 & 0 \\ -2 & 9-13 & -4 & 0 \\ -2 & -4 & 9-13 & 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and so eigenvectors are of the form

$$\begin{bmatrix} -2y - 2z \\ y \\ z \end{bmatrix}$$

We need to find two of these which are orthogonal. Let one be given by setting $z = 0$ and $y = 1$, giving $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$.

In order to find an eigenvector orthogonal to this one, we need to satisfy

$$\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -2y - 2z \\ y \\ z \end{bmatrix} = 5y + 4z = 0$$

The values $y = -4$ and $z = 5$ satisfy this equation, giving another eigenvector corresponding to $\lambda = 9$ as

$$\begin{bmatrix} -2(-4)-2(5) \\ (-4) \\ 5 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ 5 \end{bmatrix}$$

Next find the eigenvector for $\lambda = 18$. The augmented matrix and the resulting reduced row-echelon form are given by

$$\begin{bmatrix} 18-10 & -2 & -2 & 0 \\ -2 & 18-13 & -4 & 0 \\ -2 & -4 & 18-13 & 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and so an eigenvector is

$$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Dividing each eigenvector by its length, the orthonormal set is

$$\left\{ \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \frac{\sqrt{5}}{15} \begin{bmatrix} -2 \\ -4 \\ 5 \end{bmatrix}, \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \right\}$$

In the above solution, the repeated eigenvalue implies that there would have been many other orthonormal bases which could have been obtained. While we chose to take $z = 0, y = 1$, we could just as easily have taken $y = 0$ or even $y = z = 1$. Any such change would have resulted in a different orthonormal set.

Recall that an $n \times n$ matrix A is said to be **non defective** or **diagonalizable** if there exists an invertible matrix P such that $P^{-1}AP = D$ where D is a diagonal matrix.

As indicated in Theorem $6.7.3$ if A is a real symmetric matrix, there exists an orthogonal matrix U such that $U^T AU = D$ where D is a diagonal matrix. Therefore, every symmetric matrix is diagonalizable because if U is an orthogonal matrix, it is invertible and its inverse is U^T. Therefore every symmetric matrix is in fact *orthogonally* diagonalizable. The next theorem provides another way to determine if a matrix is orthogonally diagonalizable.

Theorem $6.7.4$

Let A be an $n \times n$ matrix. Then A is orthogonally diagonalizable if and only if A has an orthonormal set of eigenvectors.

Recall from Corollary $6.7.1$ that every symmetric matrix has an orthonormal set of eigenvectors. In fact the following theorem shows these three conditions are equivalent.

Theorem $6.7.5$: Principal Axis Theorem

The following conditions are equivalent for an $n \times n$ matrix A:

1. A is symmetric.
2. A has an orthonormal set of eigenvectors.
3. A is orthogonally diagonalizable.

Proof available at the end of this section.

We demonstrate this in the following example.

In the following example, the orthogonal matrix U will be found to orthogonally diagonalize a matrix.

Example 6.7.5: Diagonalize a Symmetric Matrix

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{3}{2} \end{bmatrix}$. Find an orthogonal matrix U such that $U^T AU$ is a diagonal matrix.

Solution

In this case, the eigenvalues are 2 (with multiplicity one) and 1 (with multiplicity two). First we will find an eigenvector for the eigenvalue 2. The appropriate augmented matrix and resulting reduced row-echelon form are given by

$$\begin{bmatrix} 2-1 & 0 & 0 & 0 \\ 0 & 2-\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 2-\frac{3}{2} & 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and so an eigenvector is

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

However, it is desired that the eigenvectors be unit vectors and so dividing this vector by its length gives

$$\begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Next find the eigenvectors corresponding to the eigenvalue equal to 1. The appropriate augmented matrix and resulting reduced row-echelon form are given by:

$$\begin{bmatrix} 1-1 & 0 & 0 & 0 \\ 0 & 1-\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 1-\frac{3}{2} & 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, the eigenvectors are of the form

$$\begin{bmatrix} s \\ -t \\ t \end{bmatrix}$$

Two of these which are orthonormal are $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, choosing $s = 1$ and $t = 0$, and $\begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$, letting $s = 0$, $t = 1$ and normalizing the resulting vector.

To obtain the desired orthogonal matrix, we let the orthonormal eigenvectors computed above be the columns.

$$\begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

To verify, compute $U^T AU$ as follows:

$$U^T AU = \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = D$$

the desired diagonal matrix. Notice that the eigenvectors, which construct the columns of U, are in the same order as the eigenvalues in D.

We conclude this section with a Theorem that generalizes earlier results.

Theorem 6.7.6: Triangulation of a Matrix

Let A be an $n \times n$ matrix. If A has n real eigenvalues, then an orthogonal matrix U can be found to result in the upper triangular matrix $U^T AU$.

This Theorem provides a useful Corollary.

Corollary 6.7.2: Determinant and Trace

Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \cdots, \lambda_n$. Then it follows that $\det(A)$ is equal to the product of the λ_i, while $trace(A)$ is equal to the sum of the λ_i.

Proof available at the end of this section.

Answers

Lemma 6.7.1: The Dot Product

This result follows from the definition of the dot product together with properties of matrix multiplication, as follows:

$$\begin{aligned} A\vec{x} \cdot \vec{y} &= \sum_{k,l} a_{kl} x_l y_k \\ &= \sum_{k,l} (a_{lk})^T x_l y_k \\ &= \vec{x} \cdot A^T \vec{y} \\ &= \vec{x} \cdot A\vec{y} \end{aligned} \tag{6.7.1}$$

The last step follows from $A^T = A$, since A is symmetric.

Theorem 6.7.1: Orthogonal Eigenvectors

Recall that for a complex number $a + ib$, the complex conjugate, denoted by $\overline{a+ib}$ is given by $\overline{a+ib} = a - ib$. The notation, $\overline{\vec{x}}$ will denote the vector which has every entry replaced by its complex conjugate.

Suppose A is a real symmetric matrix and $A\vec{x} = \lambda\vec{x}$. Then

$$\overline{\lambda\vec{x}}^T \vec{x} = \left(\overline{A\vec{x}}\right)^T \vec{x} = \overline{\vec{x}}^T A^T \vec{x} = \overline{\vec{x}}^T A\vec{x} = \lambda \overline{\vec{x}}^T \vec{x}$$

Dividing by $\overline{\vec{x}}^T \vec{x}$ on both sides yields $\overline{\lambda} = \lambda$ which says λ is real. To do this, we need to ensure that $\overline{\vec{x}}^T \vec{x} \neq 0$. Notice that $\overline{\vec{x}}^T \vec{x} = 0$ if and only if $\vec{x} = \vec{0}$. Since we chose $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$, $\vec{x}$ is an eigenvector and therefore must be nonzero.

Now suppose A is real symmetric and $A\vec{x} = \lambda\vec{x}$, $A\vec{y} = \mu\vec{y}$ where $\mu \neq \lambda$. Then since A is symmetric, it follows from Lemma 6.7.1 about the dot product that

$$\lambda\vec{x} \cdot \vec{y} = A\vec{x} \cdot \vec{y} = \vec{x} \cdot A\vec{y} = \vec{x} \cdot \mu\vec{y} = \mu\vec{x} \cdot \vec{y}$$

Hence $(\lambda - \mu)\vec{x} \cdot \vec{y} = 0$. It follows that, since $\lambda - \mu \neq 0$, it must be that $\vec{x} \cdot \vec{y} = 0$. Therefore the eigenvectors form an orthogonal set.

Theorem 6.7.2: Eigenvalues of Skew Symmetric Matrix

First, note that if $A = 0$ is the zero matrix, then A is skew symmetric and has eigenvalues equal to 0.

Suppose $A = -A^T$ so A is skew symmetric and $A\vec{x} = \lambda\vec{x}$. Then

$$\overline{\lambda\vec{x}}^T \vec{x} = \left(\overline{A\vec{x}}\right)^T \vec{x} = \overline{\vec{x}}^T A^T \vec{x} = -\overline{\vec{x}}^T A\vec{x} = -\lambda \overline{\vec{x}}^T \vec{x}$$

and so, dividing by $\overline{\vec{x}}^T \vec{x}$ as before, $\overline{\lambda} = -\lambda$. Letting $\lambda = a + ib$, this means $a - ib = -a - ib$ and so $a = 0$. Thus λ is pure imaginary.

Corollary 6.7.1: Orthonormal Set of Eigenvectors

Since A is symmetric, then by Theorem 6.7.3, there exists an orthogonal matrix U such that $U^T AU = D$, a diagonal matrix whose diagonal entries are the eigenvalues of A. Therefore, since A is symmetric and all the matrices are real,

$$\overline{D} = \overline{D^T} = \overline{U^T A^T U} = U^T A^T U = U^T AU = D$$

showing D is real because each entry of D equals its complex conjugate.

Now let

$$U = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \end{bmatrix}$$

where the $\vec{u}_i$ denote the columns of U and

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

The equation, $U^T AU = D$ implies $AU = UD$ and

$$\begin{aligned} AU &= \begin{bmatrix} A\vec{u}_1 & A\vec{u}_2 & \cdots & A\vec{u}_n \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1\vec{u}_1 & \lambda_2\vec{u}_2 & \cdots & \lambda_n\vec{u}_n \end{bmatrix} \\ &= UD \end{aligned} \tag{6.7.2}$$

where the entries denote the columns of AU and UD respectively. Therefore, $A\vec{u}_i = \lambda_i\vec{u}_i$. Since the matrix U is orthogonal, the ij^{th} entry of $U^T U$ equals δ_{ij} and so

$$\delta_{ij} = \vec{u}_i^T\vec{u}_j = \vec{u}_i \cdot \vec{u}_j$$

This proves the corollary because it shows the vectors $\{\vec{u}_i\}$ form an orthonormal set.

Theorem 6.7.5: Principal Axis Theorem

The complete proof is beyond this course, but to give an idea assume that A has an orthonormal set of eigenvectors, and let P consist of these eigenvectors as columns. Then $P^{-1} = P^T$, and $P^T AP = D$ a diagonal matrix. But then $A = PDP^T$, and

$$A^T = (PDP^T)^T = (P^T)^T D^T P^T = PDP^T = A$$

so A is symmetric.

Now given a symmetric matrix A, one shows that eigenvectors corresponding to different eigenvalues are always orthogonal. So it suffices to apply the Gram-Schmidt process on the set of basic eigenvectors of each eigenvalue to obtain an orthonormal set of eigenvectors.

Corollary 6.7.2: Determinant and Trace

By Theorem 6.7.5, there exists an orthogonal matrix U such that $U^T AU = P$, where P is an upper triangular matrix. Since P is similar to A, the eigenvalues of P are $\lambda_1, \lambda_2, \ldots, \lambda_n$. Furthermore, since P is (upper) triangular, the entries on the main diagonal of P are its eigenvalues, so $\det(P) = \lambda_1\lambda_2\cdots\lambda_n$ and $trace(P) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$. Since P and A are similar, $\det(A) = \det(P)$ and $trace(A) = trace(P)$, and therefore the results follow.

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6.7E: Exercises for Section 6.7

Exercise $6.7E.1$

Find the eigenvalues and an orthonormal basis of eigenvectors for A.

$$A = \begin{bmatrix} 11 & -1 & -4 \\ -1 & 11 & -4 \\ -4 & -4 & 14 \end{bmatrix}$$

Hint: Two eigenvalues are 12 and 18.

Answer available at the end of this section.

Exercise $6.7E.2$

Find the eigenvalues and an orthonormal basis of eigenvectors for A.

$$A = \begin{bmatrix} 4 & 1 & -2 \\ 1 & 4 & -2 \\ -2 & -2 & 7 \end{bmatrix}$$

Hint: One eigenvalue is 3.

Answer available at the end of this section.

Exercise $6.7E.3$

Find the eigenvalues and an orthonormal basis of eigenvectors for A. Diagonalize A by finding an orthogonal matrix U and a diagonal matrix D such that $U^T AU = D$.

$$A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

Hint: One eigenvalue is -2.

Answer available at the end of this section.

Exercise $6.7E.4$

Find the eigenvalues and an orthonormal basis of eigenvectors for A. Diagonalize A by finding an orthogonal matrix U and a diagonal matrix D such that $U^T AU = D$.

$$A = \begin{bmatrix} 17 & -7 & -4 \\ -7 & 17 & -4 \\ -4 & -4 & 14 \end{bmatrix}$$

Hint: Two eigenvalues are 18 and 24.

Answer available at the end of this section.

Exercise $6.7E.5$

Find the eigenvalues and an orthonormal basis of eigenvectors for A. Diagonalize A by finding an orthogonal matrix U and a diagonal matrix D such that $U^TAU = D$.

$$A = \begin{bmatrix} 13 & 1 & 4 \\ 1 & 13 & 4 \\ 4 & 4 & 10 \end{bmatrix}$$

Hint: Two eigenvalues are 12 and 18.

Answer available at the end of this section.

Exercise $6.7E.6$

Find the eigenvalues and an orthonormal basis of eigenvectors for A. Diagonalize A by finding an orthogonal matrix U and a diagonal matrix D such that $U^TAU = D$.

$$A = \begin{bmatrix} -\frac{5}{3} & \frac{1}{15}\sqrt{6}\sqrt{5} & \frac{8}{15}\sqrt{5} \\ \frac{1}{15}\sqrt{6}\sqrt{5} & -\frac{14}{5} & -\frac{1}{15}\sqrt{6} \\ \frac{8}{15}\sqrt{5} & -\frac{1}{15}\sqrt{6} & \frac{7}{15} \end{bmatrix}$$

Hint: The eigenvalues are $-3, -2, 1$.

Answer available at the end of this section.

Exercise $6.7E.7$

Find the eigenvalues and an orthonormal basis of eigenvectors for A. Diagonalize A by finding an orthogonal matrix U and a diagonal matrix D such that $U^TAU = D$.

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

Answer available at the end of this section.

Exercise $6.7E.8$

Find the eigenvalues and an orthonormal basis of eigenvectors for A. Diagonalize A by finding an orthogonal matrix U and a diagonal matrix D such that $U^TAU = D$.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 1 \\ 0 & 1 & 5 \end{bmatrix}$$

Answer available at the end of this section.

Exercise $6.7E.9$

Find the eigenvalues and an orthonormal basis of eigenvectors for A. Diagonalize A by finding an orthogonal matrix U and a diagonal matrix D such that $U^TAU = D$.

$$A = \begin{bmatrix} \frac{4}{3} & \frac{1}{3}\sqrt{3}\sqrt{2} & \frac{1}{3}\sqrt{2} \\ \frac{1}{3}\sqrt{3}\sqrt{2} & 1 & -\frac{1}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{2} & -\frac{1}{3}\sqrt{3} & \frac{5}{3} \end{bmatrix}$$

Hint: The eigenvalues are $0, 2, 2$ where 2 is listed twice because it is a root of multiplicity 2.

Answer available at the end of this section.

Exercise $6.7E.10$

Find the eigenvalues and an orthonormal basis of eigenvectors for A. Diagonalize A by finding an orthogonal matrix U and a diagonal matrix D such that $U^T AU = D$.

$$A = \begin{bmatrix} 1 & \frac{1}{6}\sqrt{3}\sqrt{2} & \frac{1}{6}\sqrt{3}\sqrt{6} \\ \frac{1}{6}\sqrt{3}\sqrt{2} & \frac{3}{2} & \frac{1}{12}\sqrt{2}\sqrt{6} \\ \frac{1}{6}\sqrt{3}\sqrt{6} & \frac{1}{12}\sqrt{2}\sqrt{6} & \frac{1}{2} \end{bmatrix}$$

Hint: The eigenvalues are $2, 1, 0$.

Answer available at the end of this section.

Exercise $6.7E.11$

Find the eigenvalues and an orthonormal basis of eigenvectors for the matrix

$$A = \begin{bmatrix} \frac{1}{3} & \frac{1}{6}\sqrt{3}\sqrt{2} & -\frac{7}{18}\sqrt{3}\sqrt{6} \\ \frac{1}{6}\sqrt{3}\sqrt{2} & \frac{3}{2} & -\frac{1}{12}\sqrt{2}\sqrt{6} \\ -\frac{7}{18}\sqrt{3}\sqrt{6} & -\frac{1}{12}\sqrt{2}\sqrt{6} & -\frac{5}{6} \end{bmatrix}$$

Hint: The eigenvalues are $1, 2, -2$.

Answer available at the end of this section.

Exercise $6.7E.12$

Find the eigenvalues and an orthonormal basis of eigenvectors for the matrix

$$A = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{5}\sqrt{6}\sqrt{5} & \frac{1}{10}\sqrt{5} & \\ -\frac{1}{5}\sqrt{6}\sqrt{5} & \frac{7}{5} & -\frac{1}{5}\sqrt{6} & \\ \frac{1}{10} & \sqrt{5} & -\frac{1}{5}\sqrt{6} & -\frac{9}{10} \end{bmatrix}$$

Hint: The eigenvalues are $-1, 2, -1$ where -1 is listed twice because it has multiplicity 2 as a zero of the characteristic equation.

Answer available at the end of this section.

Exercise $6.7E.13$

Explain why a matrix A is symmetric if and only if there exists an orthogonal matrix U such that $A = U^T DU$ for D a diagonal matrix.

Answer available at the end of this section.

Exercise $6.7E.14$

Show that if A is a real symmetric matrix and λ and μ are two different eigenvalues, then if $\vec{x}$ is an eigenvector for λ and $\vec{y}$ is an eigenvector for μ, then $\vec{x} \bullet \vec{y} = 0$. Also all eigenvalues are real. Supply reasons for each step in the following argument. First

$$\lambda \vec{x}^T \overline{\vec{x}} = (A\vec{x})^T \overline{\vec{x}} = \vec{x}^T A \overline{\vec{x}} = \vec{x}^T \overline{A\vec{x}} = \vec{x}^T \overline{\lambda \vec{x}} = \overline{\lambda} \vec{x}^T \overline{\vec{x}}$$

and so $\lambda = \overline{\lambda}$. This shows that all eigenvalues are real. It follows all the eigenvectors are real. Why? Now let $\vec{x},\ \vec{y},\ \mu$ and λ be given as above.

$$\lambda(\vec{x} \bullet \vec{y}) = \lambda \vec{x} \bullet \vec{y} = A\vec{x} \bullet \vec{y} = \vec{x} \bullet A\vec{y} = \vec{x} \bullet \mu \vec{y} = \mu(\vec{x} \bullet \vec{y}) = \mu(\vec{x} \bullet \vec{y})$$

and so

$$(\lambda - \mu)\vec{x} \bullet \vec{y} = 0$$

Why does it follow that $\vec{x} \bullet \vec{y} = 0$?

Answer available at the end of this section.

Exercise $6.7E.15$

Prove that an $n \times n$ matrix U is orthogonal if and only if its columns are an orthonormal set of vectors.

Answers

Exercise $6.7E.1$

The eigenvectors and eigenvalues are:

$$\left\{ \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \leftrightarrow 6, \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\} \leftrightarrow 12, \left\{ \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \right\} \leftrightarrow 18$$

Exercise $6.7E.2$

The eigenvectors and eigenvalues are:

$$\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \leftrightarrow 3, \left\{ \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \right\} \leftrightarrow 9$$

Exercise $6.7E.3$

The eigenvectors and eigenvalues are:

$$\left\{ \begin{bmatrix} \frac{1}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{3} \end{bmatrix} \right\} \leftrightarrow 1, \left\{ \begin{bmatrix} -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{6}\sqrt{6} \\ -\frac{1}{6}\sqrt{6} \\ \frac{1}{3}\sqrt{2}\sqrt{3} \end{bmatrix} \right\} \leftrightarrow -2$$

$$\begin{bmatrix} \sqrt{3}/3 & -\sqrt{2}/2 & -\sqrt{6}/6 \\ \sqrt{3}/3 & \sqrt{2}/2 & -\sqrt{6}/6 \\ \sqrt{3}/3 & 0 & \frac{1}{3}\sqrt{2}\sqrt{3} \end{bmatrix}^T \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{3}/3 & -\sqrt{2}/2 & -\sqrt{6}/6 \\ \sqrt{3}/3 & \sqrt{2}/2 & -\sqrt{6}/6 \\ \sqrt{3}/3 & 0 & \frac{1}{3}\sqrt{2}\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Exercise $6.7E.4$

The eigenvectors and eigenvalues are:

$$\left\{ \begin{bmatrix} \frac{1}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{3} \end{bmatrix} \right\} \leftrightarrow 6, \left\{ \begin{bmatrix} -\frac{1}{6}\sqrt{6} \\ -\frac{1}{6}\sqrt{6} \\ \frac{1}{3}\sqrt{2}\sqrt{3} \end{bmatrix} \right\} \leftrightarrow 18, \left\{ \begin{bmatrix} -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \\ 0 \end{bmatrix} \right\} \leftrightarrow 24$$

The matrix U has these as its columns.

Exercise $6.7E.5$

The eigenvectors and eigenvalues are:

$$\left\{\begin{bmatrix} -\frac{1}{6}\sqrt{6} \\ -\frac{1}{6}\sqrt{6} \\ \frac{1}{3}\sqrt{2}\sqrt{3} \end{bmatrix}\right\} \leftrightarrow 6, \left\{\begin{bmatrix} -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \\ 0 \end{bmatrix}\right\} \leftrightarrow 12, \left\{\begin{bmatrix} \frac{1}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{3} \end{bmatrix}\right\} \leftrightarrow 18.$$

The matrix U has these as its columns.

Exercise $6.7E.6$

The eigenvectors and eigenvalues are:

$$\left\{\begin{bmatrix} \frac{1}{6}\sqrt{6} \\ 0 \\ \frac{1}{6}\sqrt{5}\sqrt{6} \end{bmatrix}\right\} \leftrightarrow 1, \left\{\begin{bmatrix} -\frac{1}{3}\sqrt{2}\sqrt{3} \\ -\frac{1}{5}\sqrt{5} \\ \frac{1}{15}\sqrt{2}\sqrt{15} \end{bmatrix}\right\} - 2, \left\{\begin{bmatrix} -\frac{1}{6}\sqrt{6} \\ \frac{2}{5}\sqrt{5} \\ \frac{1}{30}\sqrt{30} \end{bmatrix}\right\} \leftrightarrow -3$$

These vectors are the columns of U.

Exercise $6.7E.7$

The eigenvectors and eigenvalues are:

$$\left\{\begin{bmatrix} 0 \\ -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{bmatrix}\right\} \leftrightarrow 1, \left\{\begin{bmatrix} 0 \\ \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{bmatrix}\right\} \leftrightarrow 2, \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\} \leftrightarrow 3.$$

These vectors are the columns of the matrix U.

Exercise $6.7E.8$

The eigenvectors and eigenvalues are:

$$\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\} \leftrightarrow 2, \left\{\begin{bmatrix} 0 \\ -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{bmatrix}\right\} \leftrightarrow 4, \left\{\begin{bmatrix} 0 \\ \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{bmatrix}\right\} \leftrightarrow 6.$$

These vectors are the columns of U.

Exercise $6.7E.9$

The eigenvectors and eigenvalues are:

$$\left\{\begin{bmatrix} -\frac{1}{5}\sqrt{2}\sqrt{5} \\ \frac{1}{5}\sqrt{3}\sqrt{5} \\ \frac{1}{5}\sqrt{5} \end{bmatrix}\right\} \leftrightarrow 0, \left\{\begin{bmatrix} \frac{1}{3}\sqrt{3} \\ 0 \\ \frac{1}{3}\sqrt{2}\sqrt{3} \end{bmatrix}, \begin{bmatrix} \frac{1}{5}\sqrt{2}\sqrt{5} \\ \frac{1}{5}\sqrt{3}\sqrt{5} \\ -\frac{1}{5}\sqrt{5} \end{bmatrix}\right\} \leftrightarrow 2.$$

The columns are these vectors.

Exercise $6.7E.10$

The eigenvectors and eigenvalues are:

$$\left\{\begin{bmatrix} -\frac{1}{3}\sqrt{3} \\ 0 \\ \frac{1}{3}\sqrt{2}\sqrt{3} \end{bmatrix}\right\} \leftrightarrow 0, \left\{\begin{bmatrix} \frac{1}{3}\sqrt{3} \\ -\frac{1}{2}\sqrt{2} \\ \frac{1}{6}\sqrt{6} \end{bmatrix}\right\} \leftrightarrow 1, \left\{\begin{bmatrix} \frac{1}{3}\sqrt{3} \\ \frac{1}{2}\sqrt{2} \\ \frac{1}{6}\sqrt{6} \end{bmatrix}\right\} \leftrightarrow 2.$$

The columns are these vectors.

Exercise $6.7E.11$

The eigenvectors and eigenvalues are:

$$\left\{\begin{bmatrix} -\frac{1}{3}\sqrt{3} \\ \frac{1}{2}\sqrt{2} \\ \frac{1}{6}\sqrt{6} \end{bmatrix}\right\} \leftrightarrow 1, \left\{\begin{bmatrix} \frac{1}{3}\sqrt{3} \\ 0 \\ \frac{1}{3}\sqrt{2}\sqrt{3} \end{bmatrix}\right\} \leftrightarrow -2, \left\{\begin{bmatrix} \frac{1}{3}\sqrt{3} \\ \frac{1}{2}\sqrt{2} \\ -\frac{1}{6}\sqrt{6} \end{bmatrix}\right\} \leftrightarrow 2.$$

Then the columns of U are these vectors.

Exercise $6.7E.12$

The eigenvectors and eigenvalues are:

$$\left\{\begin{bmatrix} -\frac{1}{6}\sqrt{6} \\ 0 \\ \frac{1}{6}\sqrt{5}\sqrt{6} \end{bmatrix}, \begin{bmatrix} \frac{1}{3}\sqrt{2}\sqrt{3} \\ \frac{1}{5}\sqrt{5} \\ \frac{1}{15}\sqrt{2}\sqrt{15} \end{bmatrix}\right\} \leftrightarrow -1, \left\{\begin{bmatrix} \frac{1}{6}\sqrt{6} \\ -\frac{2}{5}\sqrt{5} \\ \frac{1}{30}\sqrt{30} \end{bmatrix}\right\} \leftrightarrow 2.$$

The columns of U are these vectors.

$$\begin{bmatrix} -\frac{1}{6}\sqrt{6} & \frac{1}{3}\sqrt{2}\sqrt{3} & \frac{1}{6}\sqrt{6} \\ 0 & \frac{1}{5}\sqrt{5} & -\frac{2}{5}\sqrt{5} \\ \frac{1}{6}\sqrt{5}\sqrt{6} & \frac{1}{15}\sqrt{2}\sqrt{15} & \frac{1}{30}\sqrt{30} \end{bmatrix}^T \begin{bmatrix} -\frac{1}{2} & -\frac{1}{5}\sqrt{6}\sqrt{5} & \frac{1}{10}\sqrt{5} \\ -\frac{1}{5}\sqrt{6}\sqrt{5} & \frac{7}{5} & -\frac{1}{5}\sqrt{6} \\ \frac{1}{10}\sqrt{5} & -\frac{1}{5}\sqrt{6} & -\frac{9}{10} \end{bmatrix}.$$

$$\begin{bmatrix} -\frac{1}{6}\sqrt{6} & \frac{1}{3}\sqrt{2}\sqrt{3} & \frac{1}{6}\sqrt{6} \\ 0 & \frac{1}{5}\sqrt{5} & -\frac{2}{5}\sqrt{5} \\ \frac{1}{6}\sqrt{5}\sqrt{6} & \frac{1}{15}\sqrt{2}\sqrt{15} & \frac{1}{30}\sqrt{30} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Exercise $6.7E.13$

If A is given by the formula, then

$$A^T = U^T D^T U = U^T D U = A$$

Next suppose $A = A^T$. Then by the theorems on symmetric matrices, there exists an orthogonal matrix U such that

$$UAU^T = D$$

for D diagonal. Hence

$$A = U^T D U$$

Exercise $6.7E.14$

Since $\lambda \neq \mu$, it follows $\vec{x} \bullet \vec{y} = 0$.

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6.8: Singular Value Decomposition

Learning Objectives

- Understand how to find the singular value decomposition of a matrix.

The Singular Value Decomposition

We begin this section with an important definition.

Definition 6.8.1: Singular Values

Let A be an $m \times n$ matrix. The **singular values** of A are the square roots of the positive eigenvalues of $A^T A$.

Singular Value Decomposition (SVD) can be thought of as a generalization of orthogonal diagonalization of a symmetric matrix to an arbitrary $m \times n$ matrix. This decomposition is the focus of this section.

The following is a useful result that will help when computing the SVD of matrices.

Proposition 6.8.1: Same Nonzero Eigenvalues

Let A be an $m \times n$ matrix. Then $A^T A$ and AA^T have the same **nonzero eigenvalues.**

Proof available at the end of this section.

Notice that AA^T and $A^T A$ are symmetric, so they will each have an orthonormal set of eigenvectors. Given an $m \times n$ matrix A, we will see how to express A as a product

$$A = U\Sigma V^T$$

where

- U is an $m \times m$ orthogonal matrix whose columns are eigenvectors of AA^T.
- V is an $n \times n$ orthogonal matrix whose columns are eigenvectors of $A^T A$.
- Σ is an $m \times n$ matrix whose only nonzero values lie on its main diagonal, and are the singular values of A.

How can we find such a decomposition? We are aiming to decompose A in the following form:

$$A = U \begin{bmatrix} \sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$$

where σ is of the form

$$\sigma = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_k \end{bmatrix}$$

Thus $A^T = V \begin{bmatrix} \sigma & 0 \\ 0 & 0 \end{bmatrix} U^T$ and it follows that

$$A^T A = V \begin{bmatrix} \sigma & 0 \\ 0 & 0 \end{bmatrix} U^T U \begin{bmatrix} \sigma & 0 \\ 0 & 0 \end{bmatrix} V^T = V \begin{bmatrix} \sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^T$$

and so $A^TAV = V\begin{bmatrix} \sigma^2 & 0 \\ 0 & 0 \end{bmatrix}$. Similarly, $AA^TU = U\begin{bmatrix} \sigma^2 & 0 \\ 0 & 0 \end{bmatrix}$. Therefore, you would find an orthonormal basis of eigenvectors for AA^T make them the columns of a matrix such that the corresponding eigenvalues are decreasing. This gives U. You could then do the same for A^TA to get V.

We formalize this discussion in the following theorem.

Theorem 6.8.1: Singular Value Decomposition

Let A be an $m \times n$ matrix. Then there exist orthogonal matrices U and V of the appropriate size such that $A = U\Sigma V^T$ where Σ is of the form

$$\Sigma = \begin{bmatrix} \sigma & 0 \\ 0 & 0 \end{bmatrix}$$

and σ is of the form

$$\sigma = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_k \end{bmatrix}$$

for the σ_i the singular values of A.

Proof available at the end of this section.

The singular value decomposition has as an immediate corollary which is given in the following interesting result.

Corollary 6.8.1: Rank and Singular Values

Let A be an $m \times n$ matrix. Then the rank of A and A^Tequals the number of singular values.

Let's compute the Singular Value Decomposition of a simple matrix.

Example 6.8.1: Singular Value Decomposition

Let $A = \begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix}$. Find the Singular Value Decomposition (SVD) of A.

Solution

To begin, we compute AA^T and A^TA.

$$AA^T = \begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -1 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 5 \\ 5 & 11 \end{bmatrix}.$$

$$A^TA = \begin{bmatrix} 1 & 3 \\ -1 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 2 & 6 \\ 2 & 2 & -2 \\ 6 & -2 & 10 \end{bmatrix}.$$

Since AA^T is 2×2 while A^TA is 3×3, and AA^T and A^TA have the same *nonzero* eigenvalues (by Proposition 6.8.1), we compute the characteristic polynomial $c_{AA^T}(x)$ (because it's easier to compute than $c_{A^TA}(x)$).

$$\begin{aligned} c_{AA^T}(x) = \det(xI - AA^T) &= \begin{vmatrix} x-11 & -5 \\ -5 & x-11 \end{vmatrix} \\ &= (x-11)^2 - 25 \\ &= x^2 - 22x + 121 - 25 \\ &= x^2 - 22x + 96 \\ &= (x-16)(x-6) \end{aligned} \tag{6.8.1}$$

Therefore, the eigenvalues of AA^T are $\lambda_1 = 16$ and $\lambda_2 = 6$.

The eigenvalues of A^TA are $\lambda_1 = 16$, $\lambda_2 = 6$, and $\lambda_3 = 0$, and the singular values of A are $\sigma_1 = \sqrt{16} = 4$ and $\sigma_2 = \sqrt{6}$. By convention, we list the eigenvalues (and corresponding singular values) in non increasing order (i.e., from largest to smallest).

To find the matrix V:

To construct the matrix V we need to find eigenvectors for A^TA. Since the eigenvalues of AA^T are distinct, the corresponding eigenvectors are orthogonal, and we need only normalize them.

$\lambda_1 = 16$: solve $(16I - A^TA)\vec{y} = 0$.

$$\begin{bmatrix} 6 & -2 & -6 & 0 \\ -2 & 14 & 2 & 0 \\ -6 & 2 & 6 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ so } \vec{y} = \begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, t \in \mathbb{R}.$$

$\lambda_2 = 6$: solve $(6I - A^TA)\vec{y} = 0$.

$$\begin{bmatrix} -4 & -2 & -6 & 0 \\ -2 & 4 & 2 & 0 \\ -6 & 2 & -4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ so } \vec{y} = \begin{bmatrix} -s \\ -s \\ s \end{bmatrix} = s \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, s \in \mathbb{R}.$$

$\lambda_3 = 0$: solve $(-A^TA)\vec{y} = 0$.

$$\begin{bmatrix} -10 & -2 & -6 & 0 \\ -2 & -2 & 2 & 0 \\ -6 & 2 & -10 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ so } \vec{y} = \begin{bmatrix} -r \\ 2r \\ r \end{bmatrix} = r \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, r \in \mathbb{R}.$$

Let

$$\vec{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \vec{v}_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}.$$

Then

$$V = \frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{3} & -\sqrt{2} & -1 \\ 0 & -\sqrt{2} & 2 \\ \sqrt{3} & \sqrt{2} & 1 \end{bmatrix}.$$

Also,

$$\Sigma = \begin{bmatrix} 4 & 0 & 0 \\ 0 & \sqrt{6} & 0 \end{bmatrix},$$

and we use A, V^T, and Σ to find U.

Since V is orthogonal and $A = U\Sigma V^T$, it follows that $AV = U\Sigma$. Let $V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}$, and let $U = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 \end{bmatrix}$, where $\vec{u}_1$ and $\vec{u}_2$ are the two columns of U.

Then we have

$$\begin{aligned} A\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} &= \begin{bmatrix} \vec{u}_1 & \vec{u}_2 \end{bmatrix}\Sigma \\ \begin{bmatrix} A\vec{v}_1 & A\vec{v}_2 & A\vec{v}_3 \end{bmatrix} &= \begin{bmatrix} \sigma_1\vec{u}_1 + 0\vec{u}_2 & 0\vec{u}_1 + \sigma_2\vec{u}_2 & 0\vec{u}_1 + 0\vec{u}_2 \end{bmatrix} \\ &= \begin{bmatrix} \sigma_1\vec{u}_1 & \sigma_2\vec{u}_2 & 0 \end{bmatrix} \end{aligned} \tag{6.8.2}$$

which implies that $A\vec{v}_1 = \sigma_1\vec{u}_1 = 4\vec{u}_1$ and $A\vec{v}_2 = \sigma_2\vec{u}_2 = \sqrt{6}\vec{u}_2$.

Thus,

$$\vec{u}_1 = \frac{1}{4}A\vec{v}_1 = \frac{1}{4}\begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{4\sqrt{2}}\begin{bmatrix} 4 \\ 4 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

and

$$\vec{u}_2 = \frac{1}{\sqrt{6}}A\vec{v}_2 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix}\frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \frac{1}{3\sqrt{2}}\begin{bmatrix} 3 \\ -3 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Therefore,

$$U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

and

$$\begin{aligned} A &= \begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix} \\ &= \left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right)\begin{bmatrix} 4 & 0 & 0 \\ 0 & \sqrt{6} & 0 \end{bmatrix}\left(\frac{1}{\sqrt{6}}\begin{bmatrix} \sqrt{3} & 0 & \sqrt{3} \\ -\sqrt{2} & -\sqrt{2} & \sqrt{2} \\ -1 & 2 & 1 \end{bmatrix}\right). \end{aligned} \tag{6.8.3}$$

Here is another example.

Example 6.8.2: Finding the SVD

Find an SVD for $A = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$.

Solution

Since A is 3×1, A^TA is a 1×1 matrix whose eigenvalues are easier to find than the eigenvalues of the 3×3 matrix AA^T.

$$A^TA = \begin{bmatrix} -1 & 2 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \end{bmatrix}.$$

Thus A^TA has eigenvalue $\lambda_1 = 9$, and the eigenvalues of AA^T are $\lambda_1 = 9$, $\lambda_2 = 0$, and $\lambda_3 = 0$. Furthermore, A has only one singular value, $\sigma_1 = 3$.

To find the matrix V: To do so we find an eigenvector for A^TA and normalize it. In this case, finding a unit eigenvector is trivial: $\vec{v}_1 = \begin{bmatrix} 1 \end{bmatrix}$, and

$$V = \begin{bmatrix} 1 \end{bmatrix}.$$

Also, $\Sigma = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$, and we use A, $\vec{v}^T$, and Σ to find U.

Now $AV = U\Sigma$, with $V = \begin{bmatrix} \vec{v}_1 \end{bmatrix}$, and $U = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{bmatrix}$, where $\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_3$ are the columns of U. Thus

$$\begin{aligned} A \begin{bmatrix} \vec{v}_1 \end{bmatrix} &= \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{bmatrix} \Sigma \\ \begin{bmatrix} A\vec{v}_1 \end{bmatrix} &= \begin{bmatrix} \sigma_1 \vec{u}_1 + 0\vec{u}_2 + 0\vec{u}_3 \end{bmatrix} \\ &= \begin{bmatrix} \sigma_1 \vec{u}_1 \end{bmatrix} \end{aligned} \tag{6.8.4}$$

This gives us $A\vec{v}_1 = \sigma_1 \vec{u}_1 = 3\vec{u}_1$, so

$$\vec{u}_1 = \frac{1}{3} AV_1 = \frac{1}{3} \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}.$$

The vectors $\vec{u}_2$ and $\vec{u}_3$ are eigenvectors of AA^T corresponding to the eigenvalue $\lambda_2 = \lambda_3 = 0$. Instead of solving the system $(0I - AA^T)\vec{x} = 0$ and then using the Gram-Schmidt process on the resulting set of two basic eigenvectors, the following approach may be used.

Find vectors $\vec{u}_2$ and $\vec{u}_3$ by first extending $\{\vec{u}_1\}$ to a basis of $\mathbb{R}^3$, then using the Gram-Schmidt algorithm to orthogonalize the basis, and finally normalizing the vectors.

Starting with $\{3\vec{u}_1\}$ instead of $\{\vec{u}_1\}$ makes the arithmetic a bit easier. It is easy to verify that

$$\left\{ \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

is a basis of $\mathbb{R}^3$. Set

$$\vec{e}_1 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{x}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

and apply the Gram-Schmidt algorithm to $\{\vec{e}_1, \vec{x}_2, \vec{x}_3\}$.

This gives us

$$\vec{e}_2 = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} \text{ and } \vec{e}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

Therefore,

$$\vec{u}_2 = \frac{1}{\sqrt{18}} \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}, \vec{u}_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix},$$

and

$$U = \begin{bmatrix} -\frac{1}{3} & \frac{4}{\sqrt{18}} & 0 \\ \frac{2}{3} & \frac{1}{\sqrt{18}} & \frac{1}{\sqrt{2}} \\ \frac{2}{3} & \frac{1}{\sqrt{18}} & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

Finally,

$$A = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & \frac{4}{\sqrt{18}} & 0 \\ \frac{2}{3} & \frac{1}{\sqrt{18}} & \frac{1}{\sqrt{2}} \\ \frac{2}{3} & \frac{1}{\sqrt{18}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix}.$$

Consider another example.

Example 6.8.3: Find the SVD

Find a singular value decomposition for the matrix

$$A = \begin{bmatrix} \frac{2}{5}\sqrt{2}\sqrt{5} & \frac{4}{5}\sqrt{2}\sqrt{5} & 0 \\ \frac{2}{5}\sqrt{2}\sqrt{5} & \frac{4}{5}\sqrt{2}\sqrt{5} & 0 \end{bmatrix}$$

Solution

First consider A^TA

$$\begin{bmatrix} \frac{16}{5} & \frac{32}{5} & 0 \\ \frac{32}{5} & \frac{64}{5} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

What are some eigenvalues and eigenvectors? Some computing shows these are

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{2}{5}\sqrt{5} \\ \frac{1}{5}\sqrt{5} \\ 0 \end{bmatrix} \right\} \leftrightarrow 0, \left\{ \begin{bmatrix} \frac{1}{5}\sqrt{5} \\ \frac{2}{5}\sqrt{5} \\ 0 \end{bmatrix} \right\} \leftrightarrow 16$$

Thus the matrix V is given by

$$V = \begin{bmatrix} \frac{1}{5}\sqrt{5} & -\frac{2}{5}\sqrt{5} & 0 \\ \frac{2}{5}\sqrt{5} & \frac{1}{5}\sqrt{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Next consider AA^T

$$\begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix}$$

Eigenvectors and eigenvalues are

$$\left\{ \begin{bmatrix} -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{bmatrix} \right\} \leftrightarrow 0, \left\{ \begin{bmatrix} \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{bmatrix} \right\} \leftrightarrow 16$$

Thus you can let U be given by

$$U = \begin{bmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix}$$

Lets check this. $U^TAV =$

$$\begin{bmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{2}{5}\sqrt{2}\sqrt{5} & \frac{4}{5}\sqrt{2}\sqrt{5} & 0 \\ \frac{2}{5}\sqrt{2}\sqrt{5} & \frac{4}{5}\sqrt{2}\sqrt{5} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{5}\sqrt{5} & -\frac{2}{5}\sqrt{5} & 0 \\ \frac{2}{5}\sqrt{5} & \frac{1}{5}\sqrt{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Answers

Proposition 6.8.1: Same Nonzero Eigenvalues

Suppose A is an $m \times n$ matrix, and suppose that λ is a nonzero eigenvalue of A^TA. Then there exists a nonzero vector $X \in \mathbb{R}^n$ such that

$$(A^TA)X = \lambda X. \tag{6.8.5}$$

Multiplying both sides of this equation by A yields:

$$\begin{aligned} A(A^TA)X &= A\lambda X \\ (AA^T)(AX) &= \lambda(AX). \end{aligned} \tag{6.8.6}$$

Since $\lambda \neq 0$ and $X \neq 0_n$, $\lambda X \neq 0_n$, and thus by equation (6.8.5), $(A^TA)X \neq 0_m$; thus $A^T(AX) \neq 0_m$, implying that $AX \neq 0_m$.

Therefore AX is an eigenvector of AA^T corresponding to eigenvalue λ. An analogous argument can be used to show that every nonzero eigenvalue of AA^T is an eigenvalue of A^TA, thus completing the proof.

Theorem 6.8.1: Singular Value Decomposition

There exists an orthonormal basis, $\{\vec{v}_i\}_{i=1}^n$ such that $A^TA\vec{v}_i = \sigma_i^2\vec{v}_i$ where $\sigma_i^2 > 0$ for $i = 1, \cdots, k, (\sigma_i > 0)$ and equals zero if $i > k$. Thus for $i > k$, $A\vec{v}_i = \vec{0}$ because

$$A\vec{v}_i \cdot A\vec{v}_i = A^TA\vec{v}_i \cdot \vec{v}_i = \vec{0} \cdot \vec{v}_i = 0.$$

For $i = 1, \cdots, k$, define $\vec{u}_i \in \mathbb{R}^m$ by

$$\vec{u}_i = \sigma_i^{-1}A\vec{v}_i.$$

Thus $A\vec{v}_i = \sigma_i\vec{u}_i$. Now

$$\begin{aligned} \vec{u}_i \cdot \vec{u}_j &= \sigma_i^{-1}A\vec{v}_i \cdot \sigma_j^{-1}A\vec{v}_j = \sigma_i^{-1}\vec{v}_i \cdot \sigma_j^{-1}A^TA\vec{v}_j \\ &= \sigma_i^{-1}\vec{v}_i \cdot \sigma_j^{-1}\sigma_j^2\vec{v}_j = \frac{\sigma_j}{\sigma_i}(\vec{v}_i \cdot \vec{v}_j) = \delta_{ij}. \end{aligned} \tag{6.8.7}$$

Thus $\{\vec{u}_i\}_{i=1}^k$ is an orthonormal set of vectors in $\mathbb{R}^m$. Also,

$$AA^T\vec{u}_i = AA^T\sigma_i^{-1}A\vec{v}_i = \sigma_i^{-1}AA^TA\vec{v}_i = \sigma_i^{-1}A\sigma_i^2\vec{v}_i = \sigma_i^2\vec{u}_i.$$

Now extend $\{\vec{u}_i\}_{i=1}^k$ to an orthonormal basis for all of $\mathbb{R}^m$, $\{\vec{u}_i\}_{i=1}^m$ and let

$$U = \begin{bmatrix} \vec{u}_1 & \cdots & \vec{u}_m \end{bmatrix}$$

while $V = (\vec{v}_1 \cdots \vec{v}_n)$. Thus U is the matrix which has the $\vec{u}_i$ as columns and V is defined as the matrix which has the $\vec{v}_i$ as columns. Then

$$U^TAV = \begin{bmatrix} \vec{u}_1^T \\ \vdots \\ \vec{u}_k^T \\ \vdots \\ \vec{u}_m^T \end{bmatrix} A \begin{bmatrix} \vec{v}_1 \cdots \vec{v}_n \end{bmatrix}$$

$$= \begin{bmatrix} \vec{u}_1^T \\ \vdots \\ \vec{u}_k^T \\ \vdots \\ \vec{u}_m^T \end{bmatrix} \begin{bmatrix} \sigma_1\vec{u}_1 & \cdots & \sigma_k\vec{u}_k & \vec{0} & \cdots & \vec{0} \end{bmatrix} = \begin{bmatrix} \sigma & 0 \\ 0 & 0 \end{bmatrix}$$

where σ is given in the statement of the theorem.

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6.8E: Exercises for Section 6.8

Exercise $6.8E.1$

Let $A = \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}$. Find the Singular Value Decomposition (SVD) of A.

Answer available at the end of this section.

Exercise $6.8E.2$

Let $A = \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix}$. Find the Singular Value Decomposition (SVD) of A.

Exercise $6.8E.3$

Let $A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}$. Find the Singular Value Decomposition (SVD) of A.

Answer available at the end of this section.

Exercise $6.8E.4$

Let $A = \begin{bmatrix} 2 & -2 & 1 \\ 4 & -8 & -8 \end{bmatrix}$. Find the Singular Value Decomposition (SVD) of A.

Exercise $6.8E.5$

Let A be an $n \times n$ matrix with rank n. Show the absolute value of det(A) equals the product of its singular values.

Answers

Exercise $6.8E.1$

$$U = \begin{bmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{10} \end{bmatrix}, \quad V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Exercise $6.8E.3$

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 2\sqrt{3} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{bmatrix}, \quad V^T = \begin{bmatrix} \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{30}} \\ \sqrt{\frac{2}{5}} & \frac{1}{\sqrt{5}} & -\sqrt{\frac{2}{5}} \\ \frac{1}{\sqrt{6}} & 0 & \sqrt{\frac{5}{6}} \end{bmatrix}$$

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6.9: Special Factorizations

Learning Objectives

- Understand what a positive definite matrix is.
- Understand Cholesky Factorization.
- Understand QR Factorization.

Thus far, we have looked some important factorizations (or decompositions). In Chapter 1, we used the LU factorization to solve linear systems. In this chapter, we have used diagonalization (or orthogonal diagonalization) to decompose some square matrices and the singular value decomposition to factor non-square matrices.

In this section, we look at some other factorizations.

Positive Definite Matrices and the Cholesky Factorization

Positive definite matrices are often encountered in applications such mechanics and statistics, and as we will see these matrices can be factored in a special way.

We begin with a definition.

Definition 6.9.1: Positive Definite Matrix

Let A be an $n \times n$ symmetric matrix. Then A is **positive definite** if all of its eigenvalues are positive. A is **negative definite** if all of its eigenvalues are negative.

The relationship between a negative definite matrix and positive definite matrix is as follows.

Lemma 6.9.1: Negative Definite Matrix

An $n \times n$ matrix A is negative definite if and only if $-A$ is positive definite

Consider the following lemma.

Lemma 6.9.2: Positive Definite Matrix and Invertibility

If A is positive definite, then it is invertible.

Proof available at the end of this section.

Notice that this lemma implies that if a matrix A is positive definite, then $\det(A) > 0$.

The following theorem provides another characterization of positive definite matrices. It gives a useful test for verifying if a matrix is positive definite.

Theorem 6.9.1: Positive Definite Matrix

Let A be a symmetric matrix. Then A is positive definite if and only if $\vec{x}^T A\vec{x}$ is positive for all nonzero $\vec{x} \in \mathbb{R}^n$.

Proof available at the end of this section.

There are some other very interesting consequences which result from a matrix being positive definite. First one can note that the property of being positive definite is transferred to each of the principal submatrices which we will now define.

Definition 6.9.2: The Submatrix A_k

Let A be an $n \times n$ matrix. Denote by A_k the $k \times k$ matrix obtained by deleting the $k+1, \cdots, n$ columns and the $k+1, \cdots, n$ rows from A. Thus $A_n = A$ and A_k is the $k \times k$ **submatrix** of A which occupies the upper left corner of A.

Lemma 6.9.3: Positive Definite and Submatrices

Let A be an $n \times n$ positive definite matrix. Then each submatrix A_k is also positive definite.

Proof available at the end of this section.

There is yet another way to recognize whether a matrix is positive definite which is described in terms of these submatrices. We state the result, the proof of which can be found in more advanced texts.

Theorem 6.9.2: Positive Matrix and Determinant of A_k

Let A be a symmetric matrix. Then A is positive definite if and only if $\det(A_k)$ is greater than 0 for every submatrix A_k, $k = 1, \cdots, n$.

Proof available at the end of this section.

Corollary 6.9.1: Symmetric and Negative Definite Matrix

Let A be symmetric. Then A is negative definite if and only if

$$(-1)^k \det(A_k) > 0$$

for every $k = 1, \cdots, n$.

Proof available at the end of this section.

Another important theorem is the existence of a specific factorization of positive definite matrices. It is called the Cholesky Factorization and factors the matrix into the product of an upper triangular matrix and its transpose.

Theorem 6.9.3: Cholesky Factorization

Let A be a positive definite matrix. Then there exists an upper triangular matrix U whose main diagonal entries are positive, such that A can be written

$$A = U^T U$$

This factorization is unique.

The process for finding such a matrix U relies on simple row operations.

Procedure 6.9.1: Finding the Cholesky Factorization

Let A be a positive definite matrix. The matrix U that creates the Cholesky Factorization can be found through two steps.

1. Using only type 3 elementary row operations (multiples of rows added to other rows) put A in upper triangular form. Call this matrix $\hat{U}$. Then $\hat{U}$ has positive entries on the main diagonal.
2. Divide each row of $\hat{U}$ by the square root of the diagonal entry in that row. The result is the matrix U.

Of course you can always verify that your factorization is correct by multiplying U and U^T to ensure the result is the original matrix A.

Consider the following example.

Example 6.9.1: Cholesky Factorization

Show that $A = \begin{bmatrix} 9 & -6 & 3 \\ -6 & 5 & -3 \\ 3 & -3 & 6 \end{bmatrix}$ is positive definite, and find the Cholesky factorization of A.

Solution

First we show that A is positive definite. By Theorem 6.9.2 it suffices to show that the determinant of each submatrix is positive.

$$A_1 = \begin{bmatrix} 9 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 9 & -6 \\ -6 & 5 \end{bmatrix},$$

so $\det(A_1) = 9$ and $\det(A_2) = 9$. Since $\det(A) = 36$, it follows that A is positive definite.

Now we use Procedure 6.9.1 to find the Cholesky Factorization. Row reduce (using only type 3 row operations) until an upper triangular matrix is obtained.

$$\begin{bmatrix} 9 & -6 & 3 \\ -6 & 5 & -3 \\ 3 & -3 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 9 & -6 & 3 \\ 0 & 1 & -1 \\ 0 & -1 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 9 & -6 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 4 \end{bmatrix}$$

Now divide the entries in each row by the square root of the diagonal entry in that row, to give

$$U = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

You can verify that $U^T U = A$.

Example 6.9.2: Cholesky Factorization

Let A be a positive definite matrix given by

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 4 & 2 \\ 1 & 2 & 5 \end{bmatrix}$$

Determine its Cholesky factorization.

Solution

You can verify that A is in fact positive definite.

To find the Cholesky factorization we first row reduce to an upper triangular matrix.

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 4 & 2 \\ 1 & 2 & 5 \end{bmatrix} \to \begin{bmatrix} 3 & 1 & 1 \\ 0 & \frac{11}{3} & \frac{5}{3} \\ 0 & \frac{5}{3} & \frac{14}{5} \end{bmatrix} \to \begin{bmatrix} 3 & 1 & 1 \\ 0 & \frac{11}{3} & \frac{5}{3} \\ 0 & 0 & \frac{43}{11} \end{bmatrix}$$

Now divide the entries in each row by the square root of the diagonal entry in that row and simplify.

$$U = \begin{bmatrix} \sqrt{3} & \frac{1}{3}\sqrt{3} & \frac{1}{3}\sqrt{3} \\ 0 & \frac{1}{3}\sqrt{3}\sqrt{11} & \frac{5}{33}\sqrt{3}\sqrt{11} \\ 0 & 0 & \frac{1}{11}\sqrt{11}\sqrt{43} \end{bmatrix}$$

The Cholesky Factorization has many applications. One example that you're familiar with is in solving linear systems. It can be used in the same way that the LU Factorization can be used, but when the Cholesky Factorization is available to use, it is more efficient than the LU Factorization.

QR Factorization

In this section, a reliable factorization of matrices is studied. Called the QR factorization of a matrix, it *always* exists. While much can be said about the QR factorization, this section will be limited to real matrices. Therefore we assume the dot product used below is the usual dot product. We begin with a definition.

Definition 6.9.3: QR Factorization

Let A be a real $m \times n$ matrix. Then a QR **Factorization** of A consists of two matrices, Q orthogonal and R upper triangular, such that $A = QR$.

The following theorem claims that such a factorization exists.

Theorem 6.9.4: Existence of QR Factorization

Let A be any real $m \times n$ matrix with linearly independent columns. Then there exists an orthogonal matrix Q and an upper triangular matrix R having non-negative entries on the main diagonal such that

$$A = QR$$

The procedure for obtaining the QR factorization for any matrix A is as follows.

Procedure 6.9.2: QR Factorization

Let A be an $m \times n$ matrix given by $A = \begin{bmatrix} A_1 & A_2 & \cdots & A_n \end{bmatrix}$ where the A_i are the linearly independent columns of A.

1. Apply the Gram-Schmidt Process 4.9.1 to the columns of A, writing B_i for the resulting columns.
2. Normalize the B_i, to find $C_i = \frac{1}{B_i} B_i$.
3. Construct the orthogonal matrix Q as $Q = \begin{bmatrix} C_1 & C_2 & \cdots & C_n \end{bmatrix}$.
4. Construct the upper triangular matrix R as

$$R = \begin{bmatrix} B_1 & A_2 \cdot C_1 & A_3 \cdot C_1 & \cdots & A_n \cdot C_1 \\ 0 & B_2 & A_3 \cdot C_2 & \cdots & A_n \cdot C_2 \\ 0 & 0 & B_3 & \cdots & A_n \cdot C_3 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & B_n \end{bmatrix}$$

5. Finally, write $A = QR$ where Q is the orthogonal matrix and R is the upper triangular matrix obtained above.

Notice that Q is an orthogonal matrix as the C_i form an orthonormal set. Since $B_i > 0$ for all i (since the length of a vector is always positive), it follows that R is an upper triangular matrix with positive entries on the main diagonal.

Consider the following example.

Example 6.9.3: Finding a QR Factorization

Let

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Find an orthogonal matrix Q and upper triangular matrix R such that $A = QR$.

Solution

First, observe that A_1, A_2, the columns of A, are linearly independent. Therefore we can use the Gram-Schmidt Process to create a corresponding orthogonal set $\{B_1, B_2\}$ as follows:

$$\begin{aligned} B_1 = A_1 &= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \\ B_2 = A_2 - \frac{A_2 \cdot B_1}{B_1^2} B_1 &= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \end{aligned} \tag{6.9.1}$$

Normalize each vector to create the set $\{C_1, C_2\}$ as follows:

$$\begin{aligned} C_1 = \frac{1}{B_1} B_1 &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \\ C_2 = \frac{1}{B_2} B_2 &= \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \end{aligned} \tag{6.9.2}$$

Now construct the orthogonal matrix Q as

$$\begin{aligned} Q &= \begin{bmatrix} C_1 & C_2 & \cdots & C_n \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \end{bmatrix} \end{aligned} \tag{6.9.3}$$

Finally, construct the upper triangular matrix R as

$$\begin{aligned} R &= \begin{bmatrix} B_1 & A_2 \cdot C_1 \\ 0 & B_2 \end{bmatrix} \\ &= \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix} \end{aligned} \tag{6.9.4}$$

It is left to the reader to verify that $A = QR$.

The *QR* Factorization and Eigenvalues

The QR factorization of a matrix has a very useful application. It turns out that it can be used repeatedly to estimate the eigenvalues of a matrix. Consider the following procedure.

Procedure 6.9.3: Using the QR Factorization to Estimate Eigenvalues

Let A be an invertible matrix. Define the matrices $A_1, A_2, \cdots$ as follows:

1. $A_1 = A$ factored as $A_1 = Q_1 R_1$
2. $A_2 = R_1 Q_1$ factored as $A_2 = Q_2 R_2$
3. $A_3 = R_2 Q_2$ factored as $A_3 = Q_3 R_3$

Continue in this manner, where in general $A_k = Q_k R_k$ and $A_{k+1} = R_k Q_k$.

Then it follows that this sequence of A_i converges to an upper triangular matrix which is similar to A. Therefore the eigenvalues of A can be approximated by the entries on the main diagonal of this upper triangular matrix.

Power Methods

While the QR algorithm can be used to compute eigenvalues, there is a useful and fairly elementary technique for finding the eigenvector and associated eigenvalue nearest to a given complex number which is called the **shifted inverse power method**. It tends to work extremely well provided you start with something which is fairly close to an eigenvalue.

Power methods are based on the consideration of powers of a given matrix. Let $\{\vec{x}_1, \cdots, \vec{x}_n\}$ be a basis of eigenvectors for $\mathbb{C}^n$ such that $A\vec{x}_n = \lambda_n \vec{x}_n$. Now let $\vec{u}_1$ be some nonzero vector. Since $\{\vec{x}_1, \cdots, \vec{x}_n\}$ is a basis, there exists unique scalars, c_i such that

$$\vec{u}_1 = \sum_{k=1}^{n} c_k \vec{x}_k$$

Assume you have not been so unlucky as to pick $\vec{u}_1$ in such a way that $c_n = 0$. Then let $A\vec{u}_k = \vec{u}_{k+1}$ so that

$$\vec{u}_m = A^m \vec{u}_1 = \sum_{k=1}^{n-1} c_k \lambda_k^m \vec{x}_k + \lambda_n^m c_n \vec{x}_n. \tag{6.9.5}$$

For large m the last term, $\lambda_n^m c_n \vec{x}_n$, determines quite well the direction of the vector on the right. This is because $|\lambda_n|$ is larger than $|\lambda_k|$ for \(kscaling factor. Then $\vec{u}_2$ will not be just $A\vec{u}_1$ but $A\vec{u}_1/S_2$. Next let S_3 denote the entry of $A\vec{u}_2$ which has largest absolute value and define $\vec{u}_3 \equiv A\vec{u}_2/S_3$. Continue this way. The scaling just described does not destroy the relative insignificance of the term involving a sum in (6.9.5). Indeed it amounts to nothing more than changing the units of length. Also note that from this scaling procedure, the absolute value of the largest element of $\vec{u}_k$ is always equal to 1. Therefore, for large m,

$$\vec{u}_m = \frac{\lambda_n^m c_n \vec{x}_n}{S_2 S_3 \cdots S_m} + (\text{relatively insignificant term}).$$

Therefore, the entry of $A\vec{u}_m$ which has the largest absolute value is essentially equal to the entry having largest absolute value of

$$A\left(\frac{\lambda_n^m c_n \vec{x}_n}{S_2 S_3 \cdots S_m}\right) = \frac{\lambda_n^{m+1} c_n \vec{x}_n}{S_2 S_3 \cdots S_m} \approx \lambda_n \vec{u}_m$$

and so for large m, it must be the case that $\lambda_n \approx S_{m+1}$. This suggests the following procedure.

Procedure 6.9.4: Finding the Largest Eigenvalue with its Eigenvector

1. Start with a vector $\vec{u}_1$ which you hope has a component in the direction of $\vec{x}_n$. The vector $(1, \cdots, 1)^T$ is usually a pretty good choice.

2. If $\vec{u}_k$ is known,
$$\vec{u}_{k+1} = \frac{A\vec{u}_k}{S_{k+1}}$$
where S_{k+1} is the entry of $A\vec{u}_k$ which has largest absolute value.
3. When the scaling factors, S_k are not changing much, S_{k+1} will be close to the eigenvalue and $\vec{u}_{k+1}$ will be close to an eigenvector.
4. Check your answer to see if it worked well.

The shifted inverse power method involves finding the eigenvalue closest to a given complex number along with the associated eigenvalue. If μ is a complex number and you want to find λ which is closest to μ, you could consider the eigenvalues and eigenvectors of $(A-\mu I)^{-1}$. Then $A\vec{x} = \lambda\vec{x}$ if and only if
$$(A-\mu I)\vec{x} = (\lambda-\mu)\vec{x}$$
If and only if
$$\frac{1}{\lambda-\mu}\vec{x} = (A-\mu I)^{-1}\vec{x}$$
Thus, if λ is the closest eigenvalue of A to μ then out of all eigenvalues of $(A-\mu I)^{-1}$, you would have $\frac{1}{\lambda-\mu}$ would be the largest. Thus all you have to do is apply the power method to $(A-\mu I)^{-1}$ and the eigenvector you get will be the eigenvector which corresponds to λ where λ is the closest to μ of all eigenvalues of A. You could use the eigenvector to determine this directly.

Example 6.9.4: Finding Eigenvalue and Eigenvector

Find the eigenvalue and eigenvector for
$$\begin{bmatrix} 3 & 2 & 1 \\ -2 & 0 & -1 \\ -2 & -2 & 0 \end{bmatrix}$$
which is closest to $0.9+0.9i$.

Solution
$$\left(\begin{bmatrix} 3 & 2 & 1 \\ -2 & 0 & -1 \\ -2 & -2 & 0 \end{bmatrix} - (0.9+0.9i)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)^{-1}$$
$$= \begin{bmatrix} -0.619\,19 - 10.545i & -5.524\,9 - 4.972\,4i & -0.370\,57 - 5.821\,3i \\ 5.524\,9 + 4.972\,4i & 5.276\,2 + 0.248\,62i & 2.762\,4 + 2.486\,2i \\ 0.741\,14 + 11.643i & 5.524\,9 + 4.972\,4i & 0.492\,52 + 6.918\,9i \end{bmatrix}$$
Then pick an initial guess and multiply by this matrix raised to a large power.
$$= \begin{bmatrix} -0.619\,19 - 10.545i & -5.524\,9 - 4.972\,4i & -0.370\,57 - 5.821\,3i \\ 5.524\,9 + 4.972\,4i & 5.276\,2 + 0.248\,62i & 2.762\,4 + 2.486\,2i \\ 0.741\,14 + 11.643i & 5.524\,9 + 4.972\,4i & 0.492\,52 + 6.918\,9i \end{bmatrix}^{15} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$
This equals
$$\begin{bmatrix} 1.562\,9 \times 10^{13} - 3.899\,3 \times 10^{12}i \\ -5.864\,5 \times 10^{12} + 9.764\,2 \times 10^{12}i \\ -1.562\,9 \times 10^{13} + 3.899\,9 \times 10^{12}i \end{bmatrix}$$
Now divide by an entry to make the vector have reasonable size. This yields

$$\begin{bmatrix} -0.999\,99 - 3.614\,0 \times 10^{-5} i \\ 0.499\,99 - 0.499\,99 i \\ 1.0 \end{bmatrix}$$

which is close to

$$\begin{bmatrix} -1 \\ 0.5 - 0.5i \\ 1.0 \end{bmatrix}$$

Then

$$\begin{bmatrix} 3 & 2 & 1 \\ -2 & 0 & -1 \\ -2 & -2 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0.5 - 0.5i \\ 1.0 \end{bmatrix} = \begin{bmatrix} -1.0 - 1.0i \\ 1.0 \\ 1.0 + 1.0i \end{bmatrix}$$

Now to determine the eigenvalue, you could just take the ratio of corresponding entries. Pick the two corresponding entries which have the largest absolute values. In this case, you would get the eigenvalue is $1 + i$ which happens to be the exact eigenvalue. Thus an eigenvector and eigenvalue are

$$\begin{bmatrix} -1 \\ 0.5 - 0.5i \\ 1.0 \end{bmatrix}, 1 + i$$

Usually it won't work out so well but you can still find what is desired. Thus, once you have obtained approximate eigenvalues using the QR algorithm, you can find the eigenvalue more exactly along with an eigenvector associated with it by using the shifted inverse power method.

Answers

Lemma $6.9.2$**: Positive Definite Matrix and Invertibility**

If $A\vec{v} = \vec{0}$, then 0 is an eigenvalue if $\vec{v}$ is nonzero, which does not happen for a positive definite matrix. Hence $\vec{v} = \vec{0}$ and so A is one to one. This is sufficient to conclude that it is invertible.

Theorem $6.9.1$**: Positive Definite Matrix**

Since A is symmetric, there exists an orthogonal matrix U so that

$$U^T AU = diag(\lambda_1, \lambda_2, \ldots, \lambda_n) = D,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the (not necessarily distinct) eigenvalues of A. Let $\vec{x} \in \mathbb{R}^n$, $\vec{x} \neq \vec{0}$, and define $\vec{y} = U^T \vec{x}$. Then

$$\vec{x}^T A\vec{x} = \vec{x}^T (UDU^T)\vec{x} = (\vec{x}^T U) D (U^T \vec{x}) = \vec{y}^T D\vec{y}.$$

Writing $\vec{y}^T = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix}$,

$$\begin{aligned} \vec{x}^T A\vec{x} &= \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} diag(\lambda_1, \lambda_2, \ldots, \lambda_n) \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \\ &= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots \lambda_n y_n^2. \end{aligned} \tag{6.9.6}$$

($\Rightarrow$) First we will assume that A is positive definite and prove that $\vec{x}^T A\vec{x}$ is positive.

Suppose A is positive definite, and $\vec{x} \in \mathbb{R}^n$, $\vec{x} \neq \vec{0}$. Since U^T is invertible, $\vec{y} = U^T \vec{x} \neq \vec{0}$, and thus $y_j \neq 0$ for some j, implying $y_j^2 > 0$ for some j. Furthermore, since all eigenvalues of A are positive, $\lambda_i y_i^2 \geq 0$ for all i and $\lambda_j y_j^2 > 0$. Therefore, $\vec{x}^T A\vec{x} > 0$.

($\Leftarrow$) Now we will assume $\vec{x}^T A\vec{x}$ is positive and show that A is positive definite.

If $\vec{x}^T A\vec{x} > 0$ whenever $\vec{x} \neq \vec{0}$, choose $\vec{x} = U\vec{e}_j$, where $\vec{e}_j$ is the jth column of I_n. Since U is invertible, $\vec{x} \neq \vec{0}$, and thus

$$\vec{y} = U^T \vec{x} = U^T (U\vec{e}_j) = \vec{e}_j.$$

Thus $y_j = 1$ and $y_i = 0$ when $i \neq j$, so

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots \lambda_n y_n^2 = \lambda_j,$$

i.e., $\lambda_j = \vec{x}^T A\vec{x} > 0$. Therefore, A is positive definite.

Lemma $6.9.3$**: Positive Definite and Submatrices**

This follows right away from the above definition. Let $\vec{x} \in \mathbb{R}^k$ be nonzero. Then

$$\vec{x}^T A_k \vec{x} = \begin{bmatrix} \vec{x}^T & 0 \end{bmatrix} A \begin{bmatrix} \vec{x} \\ 0 \end{bmatrix} > 0$$

by the assumption that A is positive definite.

Theorem $6.9.2$**: Positive Matrix and Determinant of** A_k

We prove this theorem by induction on n. It is clearly true if $n = 1$. Suppose then that it is true for $n-1$ where $n \geq 2$. Since $\det(A) = \det(A_n) > 0$, it follows that all the eigenvalues are nonzero. We need to show that they are all positive. Suppose not. Then there is some even number of them which are negative, even because the product of all the eigenvalues is known to be positive, equaling $\det(A)$. Pick two, λ_1 and λ_2 and let $A\vec{u}_i = \lambda_i \vec{u}_i$ where $\vec{u}_i \neq \vec{0}$ for $i = 1, 2$ and $\vec{u}_1 \cdot \vec{u}_2 = 0$. Now if $\vec{y} \equiv \alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2$ is an element of $span\{\vec{u}_1, \vec{u}_2\}$, then since these are eigenvalues and $\vec{u}_1 \cdot \vec{u}_2 = 0$, a short computation shows

$$\begin{aligned} &\left(\alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2\right)^T A \left(\alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2\right) \\ &= |\alpha_1|^2 \lambda_1 \vec{u}_1^2 + |\alpha_2|^2 \lambda_2 \vec{u}_2^2 < 0. \end{aligned}$$

Now letting $\vec{x} \in \mathbb{R}^{n-1}$, we can use the induction hypothesis to write

$$\begin{bmatrix} x^T & 0 \end{bmatrix} A \begin{bmatrix} \vec{x} \\ 0 \end{bmatrix} = \vec{x}^T A_{n-1} \vec{x} > 0.$$

Now the dimension of $\{\vec{z} \in \mathbb{R}^n : z_n = 0\}$ is $n-1$ and the dimension of $span\{\vec{u}_1, \vec{u}_2\} = 2$ and so there must be some nonzero $\vec{x} \in \mathbb{R}^n$ which is in both of these subspaces of $\mathbb{R}^n$. However, the first computation would require that $\vec{x}^T A \vec{x} < 0$ while the second would require that $\vec{x}^T A \vec{x} > 0$. This contradiction shows that all the eigenvalues must be positive. This proves the if part of the theorem. The converse can also be shown to be correct, but it is the direction which was just shown which is of most interest.

Corollary $6.9.1$**: Symmetric and Negative Definite Matrix**

This is immediate from the above theorem when we notice that A is negative definite if and only if $-A$ is positive definite. Therefore, if $\det(-A_k) > 0$ for all $k = 1, \cdots, n$, it follows that A is negative definite. However, $\det(-A_k) = (-1)^k \det(A_k)$.

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6.9E: Exercises for Section 6.9

Exercise 6.9E.1

Find the Cholesky factorization for the matrix

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 6 & 4 \\ 0 & 4 & 10 \end{bmatrix}$$

Exercise 6.9E.2

Find the Cholesky factorization for the matrix

$$\begin{bmatrix} 4 & 8 & 0 \\ 8 & 17 & 2 \\ 0 & 2 & 13 \end{bmatrix}$$

Exercise 6.9E.3

Find the Cholesky factorization for the matrix

$$\begin{bmatrix} 4 & 8 & 0 \\ 8 & 20 & 8 \\ 0 & 8 & 20 \end{bmatrix}$$

Exercise 6.9E.4

Find the Cholesky factorization for the matrix

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 8 & 10 \\ 1 & 10 & 18 \end{bmatrix}$$

Exercise 6.9E.5

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 8 & 10 \\ 1 & 10 & 26 \end{bmatrix}$$

a. Find the Cholesky factorization for A.

b. Use the Cholesky factorization to solve $A\vec{x} = \vec{b}$ for $\vec{b} = \begin{bmatrix} -1 \\ 2 \\ 7 \end{bmatrix}$.

Exercise $6.9E.6$

Suppose you have a lower triangular matrix L and it is invertible. Show that LL^T must be positive definite.

Exercise $6.9E.7$

Using the Gram Schmidt process or the QR factorization, find an orthonormal basis for the following span:

$$\text{span}\left\{\begin{bmatrix}1\\2\\1\end{bmatrix},\begin{bmatrix}2\\-1\\3\end{bmatrix},\begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$$

Answer available at the end of this section.

Exercise $6.9E.8$

Using the Gram Schmidt process or the QR factorization, find an orthonormal basis for the following span:

$$\text{span}\left\{\begin{bmatrix}1\\2\\1\\0\end{bmatrix},\begin{bmatrix}2\\-1\\3\\1\end{bmatrix},\begin{bmatrix}1\\0\\0\\1\end{bmatrix}\right\}$$

Answer available at the end of this section.

Exercise $6.9E.9$

Using a computer algebra system, find a QR factorization for the following matrices.

a. $\begin{bmatrix}1 & 1 & 2\\3 & -2 & 3\\2 & 1 & 1\end{bmatrix}$

b. $\begin{bmatrix}1 & 2 & 1 & 3\\4 & 5 & -4 & 3\\2 & 1 & 2 & 1\end{bmatrix}$

c. $\begin{bmatrix}1 & 2\\3 & 2\\1 & -4\end{bmatrix}$ Find the thin QR factorization of this one.

Exercise $6.9E.10$

Show that if U is an invertible $n \times n$ matrix then U^TU is positive definite.

Exercise $6.9E.11$

Show that if A is positive definite then kA is positive definite if $k > 0$ and negative definite if $k < 0$.

Answers

Exercise $6.9E.7$

Using the QR factorization, we have:

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 0 \\ 1 & 3 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{6}\sqrt{6} & \frac{3}{10}\sqrt{2} & \frac{7}{15}\sqrt{3} \\ \frac{1}{3}\sqrt{6} & -\frac{2}{5}\sqrt{2} & -\frac{1}{15}\sqrt{3} \\ \frac{1}{6}\sqrt{6} & \frac{1}{2}\sqrt{2} & -\frac{1}{3}\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{6} & \frac{1}{2}\sqrt{6} & \frac{1}{6}\sqrt{6} \\ 0 & \frac{5}{2}\sqrt{2} & \frac{3}{10}\sqrt{2} \\ 0 & 0 & \frac{7}{15}\sqrt{3} \end{bmatrix}$$

A solution is then

$$\begin{bmatrix} \frac{1}{6}\sqrt{6} \\ \frac{1}{3}\sqrt{6} \\ \frac{1}{6}\sqrt{6} \end{bmatrix}, \begin{bmatrix} \frac{3}{10}\sqrt{2} \\ -\frac{2}{5}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{bmatrix}, \begin{bmatrix} \frac{7}{15}\sqrt{3} \\ -\frac{1}{15}\sqrt{3} \\ -\frac{1}{3}\sqrt{3} \end{bmatrix}$$

Exercise $6.9E.8$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{6}\sqrt{6} & \frac{1}{6}\sqrt{2}\sqrt{3} & \frac{5}{111}\sqrt{3}\sqrt{37} & \frac{7}{111}\sqrt{111} \\ \frac{1}{3}\sqrt{6} & -\frac{2}{9}\sqrt{2}\sqrt{3} & \frac{1}{333}\sqrt{3}\sqrt{37} & -\frac{2}{111}\sqrt{111} \\ \frac{1}{6}\sqrt{6} & \frac{5}{18}\sqrt{2}\sqrt{3} & -\frac{17}{333}\sqrt{3}\sqrt{37} & -\frac{1}{37}\sqrt{111} \\ 0 & \frac{1}{9}\sqrt{2}\sqrt{3} & \frac{22}{333}\sqrt{3}\sqrt{37} & -\frac{7}{111}\sqrt{111} \end{bmatrix} \begin{bmatrix} \sqrt{6} & \frac{1}{2}\sqrt{6} & \frac{1}{6}\sqrt{6} \\ 0 & \frac{3}{2}\sqrt{2}\sqrt{3} & \frac{5}{18}\sqrt{2}\sqrt{3} \\ 0 & 0 & \frac{1}{9}\sqrt{3}\sqrt{37} \\ 0 & 0 & 0 \end{bmatrix}$$

Then a solution is

$$\begin{bmatrix} \frac{1}{6}\sqrt{6} & \\ \frac{1}{3}\sqrt{6} & \\ \frac{1}{6}\sqrt{6} & 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{6}\sqrt{2}\sqrt{3} \\ -\frac{2}{9}\sqrt{2}\sqrt{3} \\ \frac{5}{18}\sqrt{2}\sqrt{3} \\ \frac{1}{9}\sqrt{2}\sqrt{3} \end{bmatrix}, \begin{bmatrix} \frac{5}{111}\sqrt{3}\sqrt{37} \\ \frac{1}{333}\sqrt{3}\sqrt{37} \\ -\frac{17}{333}\sqrt{3}\sqrt{37} \\ \frac{22}{333}\sqrt{3}\sqrt{37} \end{bmatrix}$$

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6.10: Quadratic Forms

Learning Objectives

- Understand how to find the matrix of a quadratic form.
- Understand how to use matrix diagonalization to simplify quadratic forms.

Quadratic Forms

One of the applications of orthogonal diagonalization is that of quadratic forms and graphs of level curves of a quadratic form. This section has to do with rotation of axes so that with respect to the new axes, the graph of the level curve of a quadratic form is oriented parallel to the coordinate axes. This makes it much easier to understand. For example, we all know that $x_1^2 + x_2^2 = 1$ represents the equation in two variables whose graph in $\mathbb{R}^2$ is a circle of radius 1. But how do we know what the graph of the equation $5x_1^2 + 4x_1x_2 + 3x_2^2 = 1$ represents?

We first formally define what is meant by a quadratic form. In this section we will work with only *real* quadratic forms, which means that the coefficients will all be real numbers.

Definition 6.10.1: Quadratic Form

A **quadratic form** is a polynomial of degree two in n variables $x_1, x_2, \cdots, x_n$, written as a linear combination of x_i^2 terms and x_ix_j terms.

Consider the quadratic form $q = a_{11}x_1^2 + a_{22}x_2^2 + \cdots + a_{nn}x_n^2 + a_{12}x_1x_2 + \cdots$. We can write $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ as the vector whose entries are the variables contained in the quadratic form.

Similarly, let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ be the matrix whose entries are the coefficients of x_i^2 and x_ix_j from q. Note that the matrix A is not unique, and we will consider this further in the example below. Using this matrix A, the quadratic form can be written as $q = \vec{x}^T A\vec{x}$.

$$\begin{aligned} q &= \vec{x}^T A\vec{x} \\ &= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\ &= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} a_{11}x_1 + a_{21}x_2 + \cdots + a_{n1}x_n \\ a_{12}x_1 + a_{22}x_2 + \cdots + a_{n2}x_n \\ \vdots \\ a_{1n}x_1 + a_{2n}x_2 + \cdots + a_{nn}x_n \end{bmatrix} \\ &= a_{11}x_1^2 + a_{22}x_2^2 + \cdots + a_{nn}x_n^2 + a_{12}x_1x_2 + \cdots \end{aligned} \tag{6.10.1}$$

Let's explore how to find this matrix A. Consider the following example.

Example 6.10.1: Matrix of a Quadratic Form

Let a quadratic form q be given by

$$q = 6x_1^2 + 4x_1x_2 + 3x_2^2$$

Write q in the form $\vec{x}^T A\vec{x}$.

Solution

First, let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$.

Then, writing $q = \vec{x}^T A\vec{x}$ gives

$$\begin{aligned} q &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= a_{11}x_1^2 + a_{21}x_1x_2 + a_{12}x_1x_2 + a_{22}x_2^2 \end{aligned} \tag{6.10.2}$$

Notice that we have an x_1x_2 term as well as an x_2x_1 term. Since multiplication is commutative, these terms can be combined. This means that q can be written

$$q = a_{11}x_1^2 + (a_{21} + a_{12})\, x_1x_2 + a_{22}x_2^2$$

Equating this to q as given in the example, we have

$$a_{11}x_1^2 + (a_{21} + a_{12})\, x_1x_2 + a_{22}x_2^2 = 6x_1^2 + 4x_1x_2 + 3x_2^2$$

Therefore,

$$\begin{aligned} a_{11} &= 6 \\ a_{22} &= 3 \\ a_{21} + a_{12} &= 4 \end{aligned} \tag{6.10.3}$$

This demonstrates that the matrix A is not unique, as there are several correct solutions to $a_{21} + a_{12} = 4$. However, we will *always* choose the coefficients such that $a_{21} = a_{12} = \frac{1}{2}(a_{21} + a_{12})$. This results in $a_{21} = a_{12} = 2$. This choice is key, as it will ensure that A turns out to be a symmetric matrix.

Hence,

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$$

You can verify that $q = \vec{x}^T A\vec{x}$ holds for this choice of A.

The above procedure for choosing A to be symmetric applies for any quadratic form q. We will *always* choose coefficients such that $a_{ij} = a_{ji}$.

We now turn our attention to the focus of this section. Our goal is to start with a quadratic form q as given above and find a way to rewrite it to eliminate the x_ix_j terms. This is done through a change of variables. In other words, we wish to find y_i such that

$$q = d_{11}y_1^2 + d_{22}y_2^2 + \cdots + d_{nn}y_n^2$$

Letting $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ and $D = [d_{ij}]$, we can write $q = \vec{y}^T D\vec{y}$ where D is the matrix of coefficients from q. There is something special about this matrix D that is crucial. Since no y_iy_j terms exist in q, it follows that $d_{ij} = 0$ for all $i \neq j$. Therefore, D is a diagonal matrix. Through this change of variables, we find the **principal axes** $y_1, y_2, \cdots, y_n$ of the quadratic form.

This discussion sets the stage for the following essential theorem.

Theorem 6.10.1: Diagonalizing a Quadratic Form

Let q be a quadratic form in the variables $x_1, \cdots, x_n$. It follows that q can be written in the form $q = \vec{x}^T A \vec{x}$ where

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

and $A = [a_{ij}]$ is the symmetric matrix of coefficients of q.

New variables $y_1, y_2, \cdots, y_n$ can be found such that $q = \vec{y}^T D \vec{y}$ where

$$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

and $D = [d_{ij}]$ is a diagonal matrix. The matrix D contains the eigenvalues of A and is found by orthogonally diagonalizing A.

While not a formal proof, the following discussion should convince you that the above theorem holds. Let q be a quadratic form in the variables $x_1, \cdots, x_n$. Then, q can be written in the form $q = \vec{x}^T A \vec{x}$ for a symmetric matrix A. We can orthogonally diagonalize the matrix A such that $U^T A U = D$ for an orthogonal matrix U and diagonal matrix D.

Then, the vector $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ is found by $\vec{y} = U^T \vec{x}$. To see that this works, rewrite $\vec{y} = U^T \vec{x}$ as $\vec{x} = U\vec{y}$. Letting $q = \vec{x}^T A \vec{x}$, proceed as follows:

$$\begin{aligned} q &= \vec{x}^T A \vec{x} \\ &= (U\vec{y})^T A (U\vec{y}) \\ &= \vec{y}^T (U^T A U) \vec{y} \\ &= \vec{y}^T D \vec{y} \end{aligned} \tag{6.10.4}$$

The following procedure details the steps for the change of variables given in the above theorem.

Procedure 6.10.1: Diagonalizing a Quadratic Form

Let q be a quadratic form in the variables $x_1, \cdots, x_n$ given by

$$q = a_{11}x_1^2 + a_{22}x_2^2 + \cdots + a_{nn}x_n^2 + a_{12}x_1x_2 + \cdots$$

Then, q can be written as $q = d_{11}y_1^2 + \cdots + d_{nn}y_n^2$ as follows:

1. Write $q = \vec{x}^T A \vec{x}$ for a symmetric matrix A.
2. Orthogonally diagonalize A to be written as $U^T A U = D$ for an orthogonal matrix U and diagonal matrix D.
3. Write $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$. Then, $\vec{x} = U\vec{y}$.
4. The quadratic form q will now be given by
$$q = d_{11}y_1^2 + \cdots + d_{nn}y_n^2 = \vec{y}^T D \vec{y}$$
where $D = [d_{ij}]$ is the diagonal matrix found by orthogonally diagonalizing A.

Consider the following example.

Example 6.10.2: Choosing New Axes to Simplify a Quadratic Form

Consider the following level curve

$$6x_1^2 + 4x_1x_2 + 3x_2^2 = 7$$

shown in the following graph.

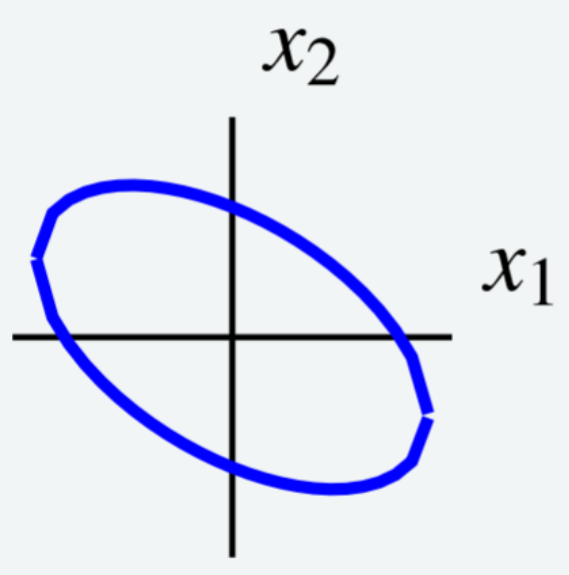


Figure 6.10.1 Graph of a rotated ellipse in the x_1x_2-plane, centered at the origin. (CC BY-NC-SA 4.0; Kuttler via A First Course in LINEAR ALGEBRA)

Use a change of variables to choose new axes such that the ellipse is oriented parallel to the new coordinate axes. In other words, use a change of variables to rewrite q to eliminate the x_1x_2 term.

Solution

Notice that the level curve is given by $q = 7$ for $q = 6x_1^2 + 4x_1x_2 + 3x_2^2$. This is the same quadratic form that we examined earlier in Example 6.10.1. Therefore we know that we can write $q = \vec{x}^T A\vec{x}$ for the matrix

$$A = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$$

Now we want to orthogonally diagonalize A to write $U^TAU = D$ for an orthogonal matrix U and diagonal matrix D. The details are left to the reader, and you can verify that the resulting matrices are

$$\begin{aligned} U &= \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \\ D &= \begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix} \end{aligned} \tag{6.10.5}$$

Next we write $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. It follows that $\vec{x} = U\vec{y}$.

We can now express the quadratic form q in terms of y, using the entries from D as coefficients as follows:

$$\begin{aligned} q &= d_{11}y_1^2 + d_{22}y_2^2 \\ &= 7y_1^2 + 2y_2^2 \end{aligned} \tag{6.10.6}$$

Hence the level curve can be written $7y_1^2 + 2y_2^2 = 7$. The graph of this equation is given by:

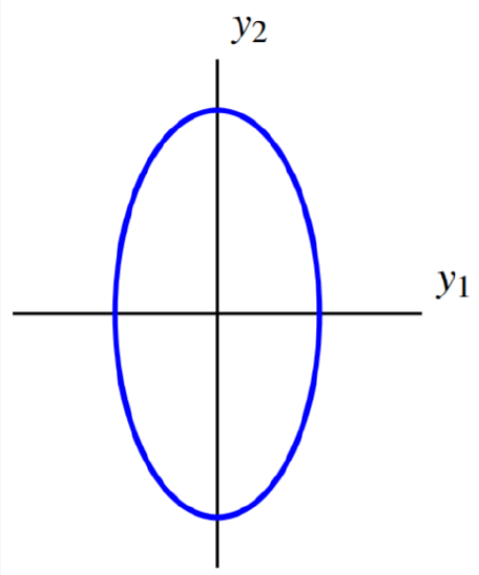


Figure 6.10.2 Graph of an ellipse in the y_1y_2-plane. (CC BY-NC-SA 4.0; Kuttler via A First Course in LINEAR ALGEBRA)

The change of variables results in new axes such that with respect to the new axes, the ellipse is oriented parallel to the coordinate axes. These are called the **principal axes** of the quadratic form.

The following is another example of diagonalizing a quadratic form.

Example 6.10.3: Choosing New Axes to Simplify a Quadratic Form

Consider the level curve

$$5x_1^2 - 6x_1x_2 + 5x_2^2 = 8$$

shown in the following graph.

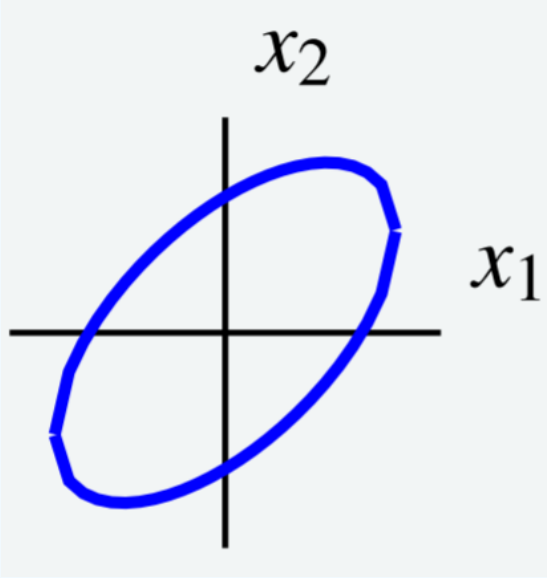


Figure 6.10.3 Graph of a rotated ellipse in the x_1x_2-plane, centered at the origin. (CC BY-NC-SA 4.0; Kuttler via A First Course in LINEAR ALGEBRA)

Use a change of variables to choose new axes such that the ellipse is oriented parallel to the new coordinate axes. In other words, use a change of variables to rewrite q to eliminate the x_1x_2 term.

Solution

First, express the level curve as $\vec{x}^T A\vec{x}$ where $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and A is symmetric. Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Then $q = \vec{x}^T A\vec{x}$ is given by

$$\begin{aligned} q &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= a_{11}x_1^2 + (a_{12} + a_{21})x_1x_2 + a_{22}x_2^2 \end{aligned} \tag{6.10.7}$$

Equating this to the given description for q, we have

$$5x_1^2 - 6x_1x_2 + 5x_2^2 = a_{11}x_1^2 + (a_{12} + a_{21})x_1x_2 + a_{22}x_2^2$$

This implies that $a_{11} = 5, a_{22} = 5$ and in order for A to be symmetric, $a_{12} = a_{22} = \frac{1}{2}(a_{12} + a_{21}) = -3$. The result is $A = \begin{bmatrix} 5 & -3 \\ -3 & 5 \end{bmatrix}$. We can write $q = \vec{x}^T A\vec{x}$ as

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 5 & -3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8$$

Next, orthogonally diagonalize the matrix A to write $U^T AU = D$. The details are left to the reader and the necessary matrices are given by

$$\begin{aligned} U &= \begin{bmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \end{bmatrix} \\ D &= \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix} \end{aligned} \tag{6.10.8}$$

Write $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, such that $\vec{x} = U\vec{y}$. Then it follows that q is given by

$$\begin{aligned} q &= d_{11}y_1^2 + d_{22}y_2^2 \\ &= 2y_1^2 + 8y_2^2 \end{aligned} \tag{6.10.9}$$

Therefore the level curve can be written as $2y_1^2 + 8y_2^2 = 8$.

This is an ellipse which is parallel to the coordinate axes. Its graph is of the form

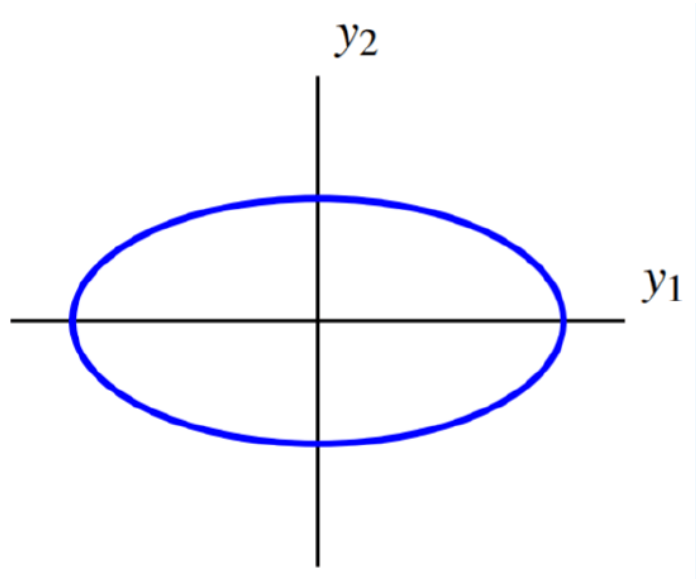


Figure $6.10.4$ Graph of an ellipse in the y_1y_2-plane, centered at the origin. (CC BY-NC-SA 4.0; Kuttler via A First Course in LINEAR ALGEBRA)

Thus this change of variables chooses new axes such that with respect to these new axes, the ellipse is oriented parallel to the coordinate axes.

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6.10E: Exercises for Section 6.10

Exercise $6.10E.1$

A quadratic form in three variables is an expression of the form $a_1x^2 + a_2y^2 + a_3z^2 + a_4xy + a_5xz + a_6yz$. Show that every such quadratic form may be written as

$$\begin{bmatrix} x & y & z \end{bmatrix} A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

where A is a symmetric matrix.

Answer available at the end of this section.

Exercise $6.10E.2$

Given a quadratic form in three variables, $x, y,$ and z, show there exists an orthogonal matrix U and variables x', y', z' such that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = U \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

with the property that in terms of the new varaibles, the quadratic form is

$$\lambda_1(x')^2 + \lambda_2(y')^2 + \lambda_3(z')^2$$

where the numbers, λ_1, λ_2, and λ_3 are the eigenvalues of the matrix A in Exercise $6.10E.1$.

Answer available at the end of this section.

Exercise $6.10E.3$

Consider the quadratic form q given by $q = 3x_1^2 - 12x_1x_2 - 2x_2^2$.

a. Write q in the form $\vec{x}^T A\vec{x}$ for an appropriate symmetric matrix A.
b. Use a change of variables to rewrite q to eliminate the x_1x_2 term.

Answer available at the end of this section.

Exercise $6.10E.4$

Consider the quadratic form q given by $q = -2x_1^2 + 2x_1x_2 - 2x_2^2$.

a. Write q in the form $\vec{x}^T A\vec{x}$ for an appropriate symmetric matrix A.
b. Use a change of variables to rewrite q to eliminate the x_1x_2 term.

Answer available at the end of this section.

Exercise $6.10E.5$

Consider the quadratic form q given by $q = 7x_1^2 + 6x_1x_2 - x_2^2$.

a. Write q in the form $\vec{x}^T A\vec{x}$ for an appropriate symmetric matrix A.
b. Use a change of variables to rewrite q to eliminate the x_1x_2 term.

Answer available at the end of this section.

Exercise $6.10E.6$

A quadratic form in three variables is an expression of the form $a_1x^2 + a_2y^2 + a_3z^2 + a_4xy + a_5xz + a_6yz$. Show that every such quadratic form may be written as

$$\begin{bmatrix} x & y & z \end{bmatrix} A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

where A is a symmetric matrix.

Answer available at the end of this section.

Answers

Exercise $6.10E.1$

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a_1 & a_4/2 & a_5/2 \\ a_4/2 & a_2 & a_6/2 \\ a_5/2 & a_6/2 & a_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Exercise $6.10E.2$

The quadratic form may be written as

$$\vec{x}^T A\vec{x}$$

where $A = A^T$. By the theorem about diagonalizing a symmetric matrix, there exists an orthogonal matrix U such that

$$U^T AU = D,\ A = UDU^T$$

Then the quadratic form is

$$\vec{x}^T UDU^T\vec{x} = (U^T\vec{x})^T D(U^T\vec{x})$$

where D is a diagonal matrix having the real eigenvalues of A down the main diagonal. Now simply let

$$\vec{x}' = U^T\vec{x}$$

Exercise $6.10E.3$

Exercise $6.10E.4$

Exercise $6.10E.6$

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a_1 & a_4/2 & a_5/2 \\ a_4/2 & a_2 & a_6/2 \\ a_5/2 & a_6/2 & a_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

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7: VECTOR SPACES

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7.1: Vector Space - Definition

Learning Objectives

- Develop the abstract concept of a vector space through axioms.
- Deduce basic properties of vector spaces.

This section introduces the abstract framework of vector spaces, extending beyond the familiar geometrical interpretation. A vector space is defined as a set of elements, called vectors, equipped with two operations—vector addition and scalar multiplication—that satisfy a specific set of axioms.

Definition 7.1.1: Vector Space

A vector space V is a set of vectors with two operations defined, addition and scalar multiplication, which satisfy the axioms of addition and scalar multiplication.

In the following definition we define two operations; vector addition, denoted by $+$ and scalar multiplication denoted by placing the scalar next to the vector. A vector space need not have usual operations, and for this reason the operations will always be given in the definition of the vector space. The below axioms for addition (written +) and scalar multiplication must hold for however addition and scalar multiplication are defined for the vector space.

It is important to note that we have seen much of this content before, in terms of $\mathbb{R}^n$. We will prove in this section that $\mathbb{R}^n$ is an example of a vector space and therefore all discussions in this chapter will pertain to $\mathbb{R}^n$. While it may be useful to consider all concepts of this chapter in terms of $\mathbb{R}^n$, it is also important to understand that these concepts apply to *all* vector spaces.

In the following definition, we will choose scalars a and b to be real numbers and are thus dealing with *real* vector spaces. However, we could also choose scalars which are complex numbers. In this case, we would call the vector space V complex.

Definition 7.1.2: Axioms of Addition

Let $\vec{v}, \vec{w}, \vec{z}$ be vectors in a vector space V. Then they satisfy the following axioms of addition:

- Closed under Addition

$$\text{If } \vec{v}, \vec{w} \text{ are in } V, \text{ then } \vec{v} + \vec{w} \text{ is also in } V.$$

- The Commutative Law of Addition

$$\vec{v} + \vec{w} = \vec{w} + \vec{v}$$

- The Associative Law of Addition

$$(\vec{v} + \vec{w}) + \vec{z} = \vec{v} + (\vec{w} + \vec{z})$$

- The Existence of an Additive Identity

$$\vec{v} + \vec{0} = \vec{v}$$

- The Existence of an Additive Inverse

$$\vec{v} + (-\vec{v}) = \vec{0}$$

Definition 7.1.3: Axioms of Scalar Multiplication

Let $a, b \in \mathbb{R}$ and let $\vec{v}, \vec{w}, \vec{z}$ be vectors in a vector space V. Then they satisfy the following axioms of scalar multiplication:

- Closed under Scalar Multiplication

$$\text{If } a \text{ is a real number, and } \vec{v} \text{ is in } V, \text{ then } a\vec{v} \text{ is in } V.$$

- $$a\left(\vec{v}+\vec{w}\right)=a\vec{v}+a\vec{w}$$
- $$\left(a+b\right)\vec{v}=a\vec{v}+b\vec{v}$$
- $$a\left(b\vec{v}\right)=\left(ab\right)\vec{v}$$
- $$1\vec{v}=\vec{v}$$

Consider the following example, in which we prove that $\mathbb{R}^n$ is in fact a vector space.

Example 7.1.1: $\mathbb{R}^n$

$\mathbb{R}^n$, under the usual operations of vector addition and scalar multiplication, is a vector space.

Solution

To show that $\mathbb{R}^n$ is a vector space, we need to show that the above axioms hold. Let $\vec{x},\vec{y},\vec{z}$ be vectors in $\mathbb{R}^n$. We first prove the axioms for vector addition.

- To show that $\mathbb{R}^n$ is closed under addition, we must show that for two vectors in $\mathbb{R}^n$ their sum is also in $\mathbb{R}^n$. The sum $\vec{x}+\vec{y}$ is given by:

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{bmatrix}$$

  The sum is a vector with n entries, showing that it is in $\mathbb{R}^n$. Hence $\mathbb{R}^n$ is closed under vector addition.
- To show that addition is commutative, consider the following:

$$\begin{aligned} \vec{x}+\vec{y} &= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \\ &= \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{bmatrix} \\ &= \begin{bmatrix} y_1+x_1 \\ y_2+x_2 \\ \vdots \\ y_n+x_n \end{bmatrix} \\ &= \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\ &= \vec{y}+\vec{x} \end{aligned} \tag{7.1.1}$$

  Hence addition of vectors in $\mathbb{R}^n$ is commutative.
- We will show that addition of vectors in $\mathbb{R}^n$ is associative in a similar way.

$$
\begin{aligned}
(\vec{x}+\vec{y})+\vec{z} &= \left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}\right) + \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \\
&= \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{bmatrix} + \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \\
&= \begin{bmatrix} (x_1+y_1)+z_1 \\ (x_2+y_2)+z_2 \\ \vdots \\ (x_n+y_n)+z_n \end{bmatrix} \\
&= \begin{bmatrix} x_1+(y_1+z_1) \\ x_2+(y_2+z_2) \\ \vdots \\ x_n+(y_n+z_n) \end{bmatrix} \\
&= \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{bmatrix} y_1+z_1 \\ y_2+z_2 \\ \vdots \\ y_n+z_n \end{bmatrix} \\
&= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \left(\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} + \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}\right) \\
&= \vec{x}+(\vec{y}+\vec{z})
\end{aligned}
\tag{7.1.2}
$$

Hence addition of vectors is associative.

- Next, we show the existence of an additive identity. Let $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

$$
\begin{aligned}
\vec{x}+\vec{0} &= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} x_1+0 \\ x_2+0 \\ \vdots \\ x_n+0 \end{bmatrix} \\
&= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\
&= \vec{x}
\end{aligned}
\tag{7.1.3}
$$

Hence the zero vector $\vec{0}$ is an additive identity.

- Next, we prove the existence of an additive inverse. Let $-\vec{x} = \begin{pmatrix} -x_1 \\ -x_2 \\ \vdots \\ -x_n \end{pmatrix}$.

$$\begin{aligned}\vec{x}+(-\vec{x}) &= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} -x_1 \\ -x_2 \\ \vdots \\ -x_n \end{bmatrix} \\ &= \begin{bmatrix} x_1 - x_1 \\ x_2 - x_2 \\ \vdots \\ x_n - x_n \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\ &= \vec{0}\end{aligned} \tag{7.1.4}$$

Hence $-\vec{x}$ is an additive inverse.

We now need to prove the axioms related to scalar multiplication. Let a, b be real numbers and let $\vec{x}, \vec{y}$ be vectors in $\mathbb{R}^n$.

- We first show that $\mathbb{R}^n$ is closed under scalar multiplication. To do so, we show that $a\vec{x}$ is also a vector with n entries.

$$a\vec{x} = a \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} ax_1 \\ ax_2 \\ \vdots \\ ax_n \end{bmatrix}$$

  The vector $a\vec{x}$ is again a vector with n entries, showing that $\mathbb{R}^n$ is closed under scalar multiplication.
- We wish to show that $a(\vec{x}+\vec{y}) = a\vec{x} + a\vec{y}$.

$$\begin{aligned}a(\vec{x}+\vec{y}) &= a\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \right) \\ &= a \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix} \\ &= \begin{bmatrix} a(x_1 + y_1) \\ a(x_2 + y_2) \\ \vdots \\ a(x_n + y_n) \end{bmatrix} \\ &= \begin{bmatrix} ax_1 + ay_1 \\ ax_2 + ay_2 \\ \vdots \\ ax_n + ay_n \end{bmatrix} \\ &= \begin{bmatrix} ax_1 \\ ax_2 \\ \vdots \\ ax_n \end{bmatrix} + \begin{bmatrix} ay_1 \\ ay_2 \\ \vdots \\ ay_n \end{bmatrix} \\ &= a\vec{x} + a\vec{y}\end{aligned} \tag{7.1.5}$$

- Next, we wish to show that $(a+b)\vec{x} = a\vec{x} + b\vec{x}$.

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- We wish to show that $a(b\vec{x}) = (ab)\vec{x}$.

$$\begin{aligned} a(b\vec{x}) &= a\left(b\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) \\ &= a\left(\begin{bmatrix} bx_1 \\ bx_2 \\ \vdots \\ bx_n \end{bmatrix}\right) \\ &= \begin{bmatrix} a(bx_1) \\ a(bx_2) \\ \vdots \\ a(bx_n) \end{bmatrix} \\ &= \begin{bmatrix} (ab)x_1 \\ (ab)x_2 \\ \vdots \\ (ab)x_n \end{bmatrix} \\ &= (ab)\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\ &= (ab)\vec{x} \end{aligned} \tag{7.1.6}$$

- Finally, we need to show that $1\vec{x} = \vec{x}$.

$$\begin{aligned} 1\vec{x} &= 1\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\ &= \begin{bmatrix} 1x_1 \\ 1x_2 \\ \vdots \\ 1x_n \end{bmatrix} \\ &= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\ &= \vec{x} \end{aligned} \tag{7.1.7}$$

By the above proofs, it is clear that $\mathbb{R}^n$ satisfies the vector space axioms. Hence, $\mathbb{R}^n$ is a vector space under the usual operations of vector addition and scalar multiplication.

Consider the following important theorem.

Theorem 7.1.1: Uniqueness

In any vector space, the following are true:

1. $\vec{0}$, the additive identity, is unique
2. $-\vec{x}$, the additive inverse, is unique
3. $0\vec{x} = \vec{0}$ for all vectors $\vec{x}$
4. $(-1)\,\vec{x} = -\vec{x}$ for all vectors $\vec{x}$

Proof available at the end of this section.

An important use of the additive inverse is the following theorem.

Theorem 7.1.2

Let V be a vector space. Then $\vec{v}+\vec{w}=\vec{v}+\vec{z}$ implies that $\vec{w}=\vec{z}$ for all $\vec{v},\vec{w},\vec{z}\in V$

Proof available at the end of this section.

Answers

Theorem 7.1.1: Uniqueness

1. When we say that the additive identity, $\vec{0}$, is unique, we mean that if a vector acts like the additive identity, then it **is** the additive identity. To prove this uniqueness, we want to show that another vector which acts like the additive identity is actually equal to $\vec{0}$. Suppose $\vec{0}'$ is also an additive identity. Then,
$$\vec{0}+\vec{0}'=\vec{0}$$
Now, for $\vec{0}$ the additive identity given above in the axioms, we have that
$$\vec{0}'+\vec{0}=\vec{0}'$$
So by the commutative property:
$$\vec{0}=\vec{0}+\vec{0}'=\vec{0}'+\vec{0}=\vec{0}' \tag{7.1.8}$$
This says that if a vector acts like an additive identity (such as $\vec{0}'$), it in fact equals $\vec{0}$. This proves the uniqueness of $\vec{0}$.
2. When we say that the additive inverse, $-\vec{x}$, is unique, we mean that if a vector acts like the additive inverse, then it **is** the additive inverse. Suppose that $\vec{y}$ acts like an additive inverse:
$$\vec{x}+\vec{y}=\vec{0}$$
Then the following holds:
$$\vec{y}=\vec{0}+\vec{y}=(-\vec{x}+\vec{x})+\vec{y}=-\vec{x}+(\vec{x}+\vec{y})=-\vec{x}+\vec{0}=-\vec{x}$$
Thus if $\vec{y}$ acts like the additive inverse, it is equal to the additive inverse $-\vec{x}$. This proves the uniqueness of $-\vec{x}$.
3. This statement claims that for all vectors $\vec{x}$, scalar multiplication by 0 equals the zero vector $\vec{0}$. Consider the following, using the fact that we can write $0=0+0$:
$$0\vec{x}=(0+0)\,\vec{x}=0\vec{x}+0\vec{x}$$
We use a small trick here: add $-0\vec{x}$ to both sides. This gives
$$\begin{aligned}0\vec{x}+(-0\vec{x})&=0\vec{x}+0\vec{x}+(-0\vec{x})\\ \vec{0}&=0\vec{x}+\vec{0}\\ \vec{0}&=0\vec{x}\end{aligned} \tag{7.1.9}$$
This proves that scalar multiplication of any vector by 0 results in the zero vector $\vec{0}$.
4. Finally, we wish to show that scalar multiplication of -1 and any vector $\vec{x}$ results in the additive inverse of that vector, $-\vec{x}$. Recall from $2.$ above that the additive inverse is unique. Consider the following:
$$\begin{aligned}(-1)\,\vec{x}+\vec{x}&=(-1)\,\vec{x}+1\vec{x}\\ &=(-1+1)\,\vec{x}\\ &=0\vec{x}\\ &=\vec{0}\end{aligned} \tag{7.1.10}$$
By the uniqueness of the additive inverse shown earlier, any vector which acts like the additive inverse must be equal to the additive inverse. It follows that $(-1)\,\vec{x}=-\vec{x}$.

Theorem 7.1.2

The proof follows from the vector space axioms, in particular the existence of an additive inverse $(-\vec{u})$. The proof is left as an exercise to the reader.

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7.1.E: Exercises for Section 7.1

Exercise 7.1.E.1

Suppose you have $\mathbb{R}^2$ and the $+$ operation is as follows:

$$(a,b)+(c,d)=(a+d,b+c).$$

Scalar multiplication is defined in the usual way. Is this a vector space? Explain why or why not.

Exercise 7.1.E.2

Suppose you have $\mathbb{R}^2$ and the $+$ operation is as follows:

$$(a,b)+(c,d)=(0,b+d)$$

Scalar multiplication is defined in the usual way. Is this a vector space? Explain why or why not.

Answer available at the end of this section.

Exercise 7.1.E.3

Suppose you have $\mathbb{R}^2$ and scalar multiplication is defined as $c(a,b)=(a,cb)$ while vector addition is defined as usual. Is this a vector space? Explain why or why not.

Exercise 7.1.E.4

Suppose you have $\mathbb{R}^2$ and the $+$ operation is defined as follows.

$$(a,b)+(c,d)=(a-c,b-d)$$

Scalar multiplication is same as usual. Is this a vector space? Explain why or why not.

Exercise 7.1.E.5

Consider the set of all vectors in $\mathbb{R}^2,(x,y)$ such that $x+y\geq 0$. Let the vector space operations be the usual ones. Is this a vector space? Is it a subspace of $\mathbb{R}^2$?

Answer available at the end of this section.

Exercise 7.1.E.6

Consider the vectors in $\mathbb{R}^2,(x,y)$ such that $xy=0$. Is this a subspace of $\mathbb{R}^2$? Is it a vector space? The addition and scalar multiplication are the usual operations.

Exercise 7.1.E.7

Define the operation of vector addition on $\mathbb{R}^2$ by $(x,y)+(u,v)=(x+u,y+v+1)$. Let scalar multiplication be the usual operation. Is this a vector space with these operations? Explain.

Answer available at the end of this section.

Exercise $7.1.E.8$

Let the vectors be real numbers. Define vector space operations in the usual way. That is $x+y$ means to add the two numbers and xy means to multiply them. Is $\mathbb{R}$ with these operations a vector space? Explain.

Answers

Exercise $7.1.E.2$

This is not a vector space. The axiom 'The Existence of an Additive Identity' is not satisfied, for example.

Exercise $7.1.E.5$

This is not a vector space. The axiom 'Closure under Scalar Multiplication' is not satisfied, for example.

Exercise $7.1.E.7$

This is not a vector space. The axiom ' $\square(\square^{\rightarrow}+\square^{\rightarrow})=\square\square^{\rightarrow}+\square\square^{\rightarrow}$ ' is not satisfied, for example.

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7.2: Vector Space - Examples

Learning Objectives

- Use the vector space axioms to determine if a set and its operations constitute a vector space.

We now consider some examples of vector spaces beyond $\mathbb{R}^n$.

Example 7.2.1: Vector Space of Polynomials

Let $\mathbb{P}_2$ be the set of all polynomials of at most degree 2 as well as the zero polynomial. Define addition to be the standard addition of polynomials, and scalar multiplication the usual multiplication of a polynomial by a number. Then $\mathbb{P}_2$ is a vector space.

Solution

We can write $\mathbb{P}_2$ explicitly as

$$\mathbb{P}_2 = \left\{a_2x^2 + a_1x + a_0 | a_i \in \mathbb{R} \text{ for all } i\right\}$$

To show that $\mathbb{P}_2$ is a vector space, we verify the axioms. Let $p(x), q(x), r(x)$ be polynomials in $\mathbb{P}_2$ and let a, b, c be real numbers. Write $p(x) = p_2x^2 + p_1x + p_0$, $q(x) = q_2x^2 + q_1x + q_0$, and $r(x) = r_2x^2 + r_1x + r_0$.

- We first prove that addition of polynomials in $\mathbb{P}_2$ is closed. For two polynomials in $\mathbb{P}_2$ we need to show that their sum is also a polynomial in $\mathbb{P}_2$. From the definition of $\mathbb{P}_2$, a polynomial is contained in $\mathbb{P}_2$ if it is of degree at most 2 or the zero polynomial.

$$\begin{aligned} p(x) + q(x) &= p_2x^2 + p_1x + p_0 + q_2x^2 + q_1x + q_0 \\ &= (p_2 + q_2)x^2 + (p_1 + q_1)x + (p_0 + q_0) \end{aligned} \tag{7.2.1}$$

  The sum is a polynomial of degree at most 2 and therefore is in $\mathbb{P}_2$. It follows that $\mathbb{P}_2$ is closed under addition.
- We need to show that addition is commutative, that is $p(x) + q(x) = q(x) + p(x)$.

$$\begin{aligned} p(x) + q(x) &= p_2x^2 + p_1x + p_0 + q_2x^2 + q_1x + q_0 \\ &= (p_2 + q_2)x^2 + (p_1 + q_1)x + (p_0 + q_0) \\ &= (q_2 + p_2)x^2 + (q_1 + p_1)x + (q_0 + p_0) \\ &= q_2x^2 + q_1x + q_0 + p_2x^2 + p_1x + p_0 \\ &= q(x) + p(x) \end{aligned} \tag{7.2.2}$$

- Next, we need to show that addition is associative. That is, that $(p(x) + q(x)) + r(x) = p(x) + (q(x) + r(x))$.

$$\begin{aligned} (p(x) + q(x)) + r(x) &= \left(p_2x^2 + p_1x + p_0 + q_2x^2 + q_1x + q_0\right) + r_2x^2 + r_1x + r_0 \\ &= (p_2 + q_2)x^2 + (p_1 + q_1)x + (p_0 + q_0) + r_2x^2 + r_1x + r_0 \\ &= (p_2 + q_2 + r_2)x^2 + (p_1 + q_1 + r_1)x + (p_0 + q_0 + r_0) \\ &= p_2x^2 + p_1x + p_0 + (q_2 + r_2)x^2 + (q_1 + r_1)x + (q_0 + r_0) \\ &= p_2x^2 + p_1x + p_0 + \left(q_2x^2 + q_1x + q_0 + r_2x^2 + r_1x + r_0\right) \\ &= p(x) + (q(x) + r(x)) \end{aligned} \tag{7.2.3}$$

- Next, we must prove that there exists an additive identity. Let $0(x) = 0x^2 + 0x + 0$.

$$\begin{aligned} p(x) + 0(x) &= p_2x^2 + p_1x + p_0 + 0x^2 + 0x + 0 \\ &= (p_2 + 0)x^2 + (p_1 + 0)x + (p_0 + 0) \\ &= p_2x^2 + p_1x + p_0 \\ &= p(x) \end{aligned} \tag{7.2.4}$$

  Hence an additive identity exists, specifically the zero polynomial.
- Next we must prove that there exists an additive inverse. Let $-p(x) = -p_2x^2 - p_1x - p_0$ and consider the following:

$$\begin{aligned} p(x)+(-p(x)) &= p_2x^2+p_1x+p_0+\left(-p_2x^2-p_1x-p_0\right) \\ &= (p_2-p_2)x^2+(p_1-p_1)x+(p_0-p_0) \\ &= 0x^2+0x+0 \\ &= 0(x) \end{aligned} \tag{7.2.5}$$

Hence an additive inverse $-p(x)$ exists such that $p(x)+(-p(x))=0(x)$.

We now need to verify the axioms related to scalar multiplication.

- First we prove that $\mathbb{P}_2$ is closed under scalar multiplication. That is, we show that $ap(x)$ is also a polynomial of degree at most 2.

  $$ap(x) = a\left(p_2x^2+p_1x+p_0\right) = ap_2x^2+ap_1x+ap_0$$

  Therefore $\mathbb{P}_2$ is closed under scalar multiplication.
- We need to show that $a(p(x)+q(x))=ap(x)+aq(x)$.

$$\begin{aligned} a(p(x)+q(x)) &= a\left(p_2x^2+p_1x+p_0+q_2x^2+q_1x+q_0\right) \\ &= a\left((p_2+q_2)x^2+(p_1+q_1)x+(p_0+q_0)\right) \\ &= a(p_2+q_2)x^2+a(p_1+q_1)x+a(p_0+q_0) \\ &= (ap_2+aq_2)x^2+(ap_1+aq_1)x+(ap_0+aq_0) \\ &= ap_2x^2+ap_1x+ap_0+aq_2x^2+aq_1x+aq_0 \\ &= ap(x)+aq(x) \end{aligned} \tag{7.2.6}$$

- Next we show that $(a+b)p(x)=ap(x)+bp(x)$.

$$\begin{aligned} (a+b)p(x) &= (a+b)(p_2x^2+p_1x+p_0) \\ &= (a+b)p_2x^2+(a+b)p_1x+(a+b)p_0 \\ &= ap_2x^2+ap_1x+ap_0+bp_2x^2+bp_1x+bp_0 \\ &= ap(x)+bp(x) \end{aligned} \tag{7.2.7}$$

- The next axiom which needs to be verified is $a(bp(x))=(ab)p(x)$.

$$\begin{aligned} a(bp(x)) &= a\left(b\left(p_2x^2+p_1x+p_0\right)\right) \\ &= a\left(bp_2x^2+bp_1x+bp_0\right) \\ &= abp_2x^2+abp_1x+abp_0 \\ &= (ab)(p_2x^2+p_1x+p_0) \\ &= (ab)p(x) \end{aligned} \tag{7.2.8}$$

- Finally, we show that $1p(x)=p(x)$.

$$\begin{aligned} 1p(x) &= 1\left(p_2x^2+p_1x+p_0\right) \\ &= 1p_2x^2+1p_1x+1p_0 \\ &= p_2x^2+p_1x+p_0 \\ &= p(x) \end{aligned} \tag{7.2.9}$$

Since the above axioms hold, we know that $\mathbb{P}_2$ as described above is a vector space.

Another important example of a vector space is the set of all matrices of the same size.

Example 7.2.2: Vector Space of Matrices

Let $\mathbb{M}_{2,3}$ be the set of all 2×3 matrices. Using the usual operations of matrix addition and scalar multiplication, show that $\mathbb{M}_{2,3}$ is a vector space.

Solution

Let A, B be 2×3 matrices in $\mathbb{M}_{2,3}$. We first prove the axioms for addition.

- In order to prove that $\mathbb{M}_{2,3}$ is closed under matrix addition, we show that the sum $A+B$ is in $\mathbb{M}_{2,3}$. This means showing that $A+B$ is a 2×3 matrix.

$$\begin{aligned} A+B &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} \\ &= \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{bmatrix} \end{aligned} \tag{7.2.10}$$

You can see that the sum is a 2×3 matrix, so it is in $\mathbb{M}_{2,3}$. It follows that $\mathbb{M}_{2,3}$ is closed under matrix addition.

- The remaining axioms regarding matrix addition follow from properties of matrix addition. Therefore $\mathbb{M}_{2,3}$ satisfies the axioms of matrix addition.

We now turn our attention to the axioms regarding scalar multiplication. Let A, B be matrices in $\mathbb{M}_{2,3}$ and let c be a real number.

- We first show that $\mathbb{M}_{2,3}$ is closed under scalar multiplication. That is, we show that cA a 2×3 matrix.

$$\begin{aligned} cA &= c \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \\ &= \begin{bmatrix} ca_{11} & ca_{12} & ca_{13} \\ ca_{21} & ca_{22} & ca_{23} \end{bmatrix} \end{aligned} \tag{7.2.11}$$

This is a 2×3 matrix in $\mathbb{M}_{2,3}$ which proves that the set is closed under scalar multiplication.

- The remaining axioms of scalar multiplication follow from properties of scalar multiplication of matrices. Therefore $\mathbb{M}_{2,3}$ satisfies the axioms of scalar multiplication.

In conclusion, $\mathbb{M}_{2,3}$ satisfies the required axioms and is a vector space.

While here we proved that the set of all 2×3 matrices is a vector space, there is nothing special about this choice of matrix size. In fact if we instead consider $\mathbb{M}_{m,n}$, the set of all $m \times n$ matrices, then $\mathbb{M}_{m,n}$ is a vector space under the operations of matrix addition and scalar multiplication.

We now examine an example of a set that does not satisfy all of the above axioms, and is therefore *not* a vector space.

Example 7.2.3: Not a Vector Space

Let V denote the set of 2×3 matrices. Let addition in V be defined by $A + B = A$ for matrices A, B in V. Let scalar multiplication in V be the usual scalar multiplication of matrices. Show that V is not a vector space.

Solution

In order to show that V is not a vector space, it suffices to find only one axiom which is not satisfied. We will begin by examining the axioms for addition until one is found which does not hold. Let A, B be matrices in V.

- We first want to check if addition is closed. Consider $A + B$. By the definition of addition in the example, we have that $A + B = A$. Since A is a 2×3 matrix, it follows that the sum $A + B$ is in V, and V is closed under addition.
- We now wish to check if addition is commutative. That is, we want to check if $A + B = B + A$ **for all** choices of A and B in V. From the definition of addition, we have that $A + B = A$ and $B + A = B$. Therefore, we can find A, B in V such that these sums are not equal. One example is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Using the operation defined by $A + B = A$, we have

$$\begin{aligned} A+B &= A \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ B+A &= B \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{aligned} \tag{7.2.12}$$

It follows that $A+B \neq B+A$. Therefore addition as defined for V is not commutative and V fails this axiom. Hence V is not a vector space.

Consider another example of a vector space.

Example 7.2.4: Vector Space of Functions

Let S be a nonempty set and define $\mathbb{F}_S$ to be the set of *real* functions defined on S. In other words, we write $\mathbb{F}_S : S \mapsto \mathbb{R}$. Letting a, b, c be scalars and f, g, h functions, the vector operations are defined as

$$\begin{aligned} (f+g)(x) &= f(x)+g(x) \\ (af)(x) &= a(f(x)) \end{aligned} \tag{7.2.13}$$

Show that $\mathbb{F}_S$ is a vector space.

Solution

To verify that $\mathbb{F}_S$ is a vector space, we must prove the axioms beginning with those for addition. Let f, g, h be functions in $\mathbb{F}_S$.

- First we check that addition is closed. For functions f, g defined on the set S, their sum given by
$$(f+g)(x) = f(x)+g(x)$$
is again a function defined on S. Hence this sum is in $\mathbb{F}_S$ and $\mathbb{F}_S$ is closed under addition.
- Secondly, we check the commutative law of addition:
$$(f+g)(x) = f(x)+g(x) = g(x)+f(x) = (g+f)(x)$$
Since x is arbitrary, $f+g = g+f$.
- Next we check the associative law of addition:
$$\begin{aligned} ((f+g)+h)(x) &= (f+g)(x)+h(x) = (f(x)+g(x))+h(x) \\ &= f(x)+(g(x)+h(x)) = (f(x)+(g+h)(x)) = (f+(g+h))(x) \end{aligned}$$
and so $(f+g)+h = f+(g+h)$.
- Next we check for an additive identity. Let 0 denote the function which is given by $0(x) = 0$. Then this is an additive identity because
$$(f+0)(x) = f(x)+0(x) = f(x)$$
and so $f+0 = f$.
- Finally, check for an additive inverse. Let $-f$ be the function which satisfies $(-f)(x) = -f(x)$. Then
$$(f+(-f))(x) = f(x)+(-f)(x) = f(x)+-f(x) = 0$$
Hence $f+(-f) = 0$.

Now, check the axioms for scalar multiplication.

- We first need to check that $\mathbb{F}_S$ is closed under scalar multiplication. For a function $f(x)$ in $\mathbb{F}_S$ and real number a, the function $(af)(x) = a(f(x))$ is again a function defined on the set S. Hence $a(f(x))$ is in $\mathbb{F}_S$ and $\mathbb{F}_S$ is closed under scalar multiplication.
- $$((a+b)f)(x) = (a+b)f(x) = af(x)+bf(x) = (af+bf)(x)$$
and so $(a+b)f = af+bf$.
- $$\begin{aligned} (a(f+g))(x) &= a(f+g)(x) = a(f(x)+g(x)) \\ &= af(x)+bg(x) = (af+bg)(x) \end{aligned}$$
and so $a(f+g) = af+bg$.
- $$((ab)f)(x) = (ab)f(x) = a(bf(x)) = (a(bf))(x)$$
so $(abf) = a(bf)$.
- Finally $(1f)(x) = 1f(x) = f(x)$ so $1f = f$.

It follows that V satisfies all the required axioms and is a vector space.

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7.2.E: Exercises for Section 7.2

Exercise 7.2.E.1

Let $\mathbb{P}_2$ be the set of all polynomials of degree 2 or less. That is, these are of the form $a+bx+cx^2$. Addition is defined as

$$(a+bx+cx^2)+(\hat{d}+\hat{b}x+\hat{c}x^2)=(a+\hat{a})+(b+\hat{b})x+(c+\hat{c})x^2$$

and scalar multiplication is defined as

$$d(a+bx+cx^2)=da+dbx+cdx^2$$

Show that, with this definition of the vector space operations that $\mathbb{P}_2$ is a vector space. Now let V denote those polynomials $a+bx+cx^2$ such that $a+b+c=0$. Is V a subspace of $\mathbb{P}_2$? Explain.

Exercise 7.2.E.2

Let X consist of the real valued functions which are defined on an interval $[a,b]$. For $f,g\in X$, $f+g$ is the name of the function which satisfies $(f+g)(x)=f(x)+g(x)$. For s a real number, $(sf)(x)=s(f(x))$. Show this is a vector space.

Answer available at the end of this section.

Exercise 7.2.E.3

Consider functions defined on $\{1,2,\cdots,n\}$ having values in $\mathbb{R}$. Explain how, if V is the set of all such functions, V can be considered as $\mathbb{R}^n$.

Answer available at the end of this section.

Exercise 7.2.E.4

Let the vectors be polynomials of degree no more than 3. Show that with the usual definitions of scalar multiplication and addition wherein, for $p(x)$ a polynomial, $(ap)(x)=ap(x)$ and for p,q polynomials $(p+q)(x)=p(x)+q(x)$, this is a vector space.

Answer available at the end of this section.

Exercise 7.2.E.5

Show that the set of all singular 3×3 matrices is not a vector space.

Exercise 7.2.E.6

Show that the set of second degree polynomials is not a vector space.

Exercise 7.2.E.7

Consider all the functions defined on a non empty set which have values in $\mathbb{R}$. Is this a vector space? Explain. The operations are defined as follows. Here f,g signify functions and a is a scalar

$$(f+g)(x) = f(x) + g(x)$$
$$(af)(x) = a(f(x)) \tag{7.2.E.1}$$

Answer available at the end of this section.

Exercise 7.2.E.8

Denote by $\mathbb{R}^{\mathbb{N}}$ the set of real valued sequences. For $\vec{a} \equiv \{a_n\}_{n=1}^{\infty}$, $\vec{b} \equiv \{b_n\}_{n=1}^{\infty}$ two of these, define their sum to be given by

$$\vec{a} + \vec{b} = \{a_n + b_n\}_{n=1}^{\infty}$$

and define scalar multiplication by

$$c\vec{a} = \{ca_n\}_{n=1}^{\infty} \text{ where } \vec{a} = \{a+n\}_{n=1}^{\infty}$$

Is this a special case of Exercise 7.2.E.5? Is this a vector space?

Exercise 7.2.E.9

Let V be the set of functions defined on a nonempty set which have values in a vector space W. Is this a vector space? Explain.

Exercise 7.2.E.0

Consider the space of $m \times n$ matrices with operation of addition and scalar multiplication defined the usual way. That is, if A, B are two $m \times n$ matrices and c a scalar,

$$(A+B)_{ij} = A_{ij} + B_{ij},\ (cA)_{ij} \equiv c(A_{ij})$$

Exercise 7.2.E.11

Consider the set of $n \times n$ symmetric matrices. That is, $A = A^T$. In other words, $A_{ij} = A_{ji}$. Show that this set of symmetric matrices is a vector space and a subspace of the vector space of $n \times n$ matrices.

Exercise 7.2.E.12

Let the scalars be the real numbers and let the vectors be real numbers which are the form $a + b\sqrt{2}$ for a, b real numbers. Show that with the usual operations, this is a vector space.

Answers

Exercise 7.2.E.2

The axioms of a vector space all hold because they hold for a vector space. The only thing left to verify is the assertions about the things which are supposed to exist. 0 would be the zero function which sends everything to 0. This is an additive identity. Now if f is a function, $-f(x) \equiv (-f(x))$. Then

$$(f + (-f))(x) \equiv f(x) + (-f)(x) \equiv f(x) + (-f(x)) = 0$$

Hence $f + -f = 0$. For each $x \in [a, b]$, let $f_x(x) = 1$ and $f_x(y) = 0$ if $y \neq x$. Then these vectors are obviously linearly independent.

Exercise 7.2.E.3

Let $f(i)$ be the ith component of a vector $\vec{x} \in \mathbb{R}^n$. Thus a typical element in $\mathbb{R}^n$ is $(f(1), \cdots, f(n))$.

Exercise 7.2.E.4

This is just a subspace of the vector space of functions because it is closed with respect to vector addition and scalar multiplication. Hence this is a vector space.

Exercise 7.2.E.7

Yes, this is a vector space. Show that all of the axioms are satisfied.

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7.3: Spanning Sets

Learning Objectives

- Determine if a vector is within a given span.

In this section we will examine the concept of spanning introduced earlier in terms of $\mathbb{R}^n$. Here, we will discuss these concepts in terms of abstract vector spaces.

Consider the following definition.

Definition 7.3.1: Subset

Let X and Y be two sets. If all elements of X are also elements of Y then we say that X is a **subset** of Y and we write

$$X \subseteq Y$$

In particular, we often speak of subsets of a vector space, such as $X \subseteq V$. By this we mean that every element in the set X is contained in the vector space V.

Definition 7.3.2: Linear Combination

Let V be a vector space and let $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n \subseteq V$. A vector $\vec{v} \in V$ is called a **linear combination** of the $\vec{v}_i$ if there exist scalars $c_i \in \mathbb{R}$ such that

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$

This definition leads to our next concept of span.

Definition 7.3.3: Span of Vectors

Let $\{\vec{v}_1, \cdots, \vec{v}_n\} \subseteq V$. Then, the **span** of the vectors $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is

$$\text{span}\left\{\vec{v}_1, \cdots, \vec{v}_n\right\} = \left\{\sum_{i=1}^{n} c_i\vec{v}_i : c_i \in \mathbb{R}\right\}$$

When we say that a vector $\vec{w}$ is in $\text{span}\left\{\vec{v}_1, \cdots, \vec{v}_n\right\}$ we mean that $\vec{w}$ can be written as a linear combination of the $\vec{v}_1$. We say that a collection of vectors $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a **spanning set** for V if $V = \text{span}\{\vec{v}_1, \cdots, \vec{v}_n\}$.

Consider the following example.

Example 7.3.1: Matrix Span

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Determine if A and B are in

$$\text{span}\left\{M_1, M_2\right\} = \text{span}\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

Solution

First consider A. We want to see if scalars s, t can be found such that $A = sM_1 + tM_2$.

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = s \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + t \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

The solution to this equation is given by

$$\begin{aligned} 1 &= s \\ 2 &= t \end{aligned} \tag{7.3.1}$$

and it follows that A is in $\text{span}\{M_1, M_2\}$.

Now consider B. Again we write $B = sM_1 + tM_2$ and see if a solution can be found for s, t.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = s \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + t \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Clearly no values of s and t can be found such that this equation holds. Therefore B is not in $\text{span}\{M_1, M_2\}$.

Consider another example.

Example 7.3.2: Polynomial Span

Show that $p(x) = 7x^2 + 4x - 3$ is in $\text{span}\{4x^2 + x, x^2 - 2x + 3\}$.

Solution

To show that $p(x)$ is in the given span, we need to show that it can be written as a linear combination of polynomials in the span. Suppose scalars a, b existed such that

$$7x^2 + 4x - 3 = a(4x^2 + x) + b(x^2 - 2x + 3)$$

If this linear combination were to hold, the following would be true:

$$\begin{aligned} 4a + b &= 7 \\ a - 2b &= 4 \\ 3b &= -3 \end{aligned} \tag{7.3.2}$$

You can verify that $a = 2, b = -1$ satisfies this system of equations. This means that we can write $p(x)$ as follows:

$$7x^2 + 4x - 3 = 2(4x^2 + x) - (x^2 - 2x + 3)$$

Hence $p(x)$ is in the given span.

Consider the following example.

Example 7.3.3: Spanning Set

Let $S = \{x^2 + 1, x - 2, 2x^2 - x\}$. Show that S is a spanning set for $\mathbb{P}_2$, the set of all polynomials of degree at most 2.

Solution

Let $p(x) = ax^2 + bx + c$ be an arbitrary polynomial in $\mathbb{P}_2$. To show that S is a spanning set, it suffices to show that $p(x)$ can be written as a linear combination of the elements of S. In other words, can we find r, s, t such that:

$$p(x) = ax^2 + bx + c = r(x^2 + 1) + s(x - 2) + t(2x^2 - x)$$

If a solution r, s, t can be found, then this shows that for any such polynomial $p(x)$, it can be written as a linear combination of the above polynomials and S is a spanning set.

$$\begin{aligned} ax^2+bx+c &= r(x^2+1)+s(x-2)+t(2x^2-x) \\ &= rx^2+r+sx-2s+2tx^2-tx \\ &= (r+2t)x^2+(s-t)x+(r-2s) \end{aligned} \tag{7.3.3}$$

For this to be true, the following must hold:

$$\begin{aligned} a &= r+2t \\ b &= s-t \\ c &= r-2s \end{aligned} \tag{7.3.4}$$

To check that a solution exists, set up the augmented matrix and row reduce:

$$\begin{bmatrix} 1 & 0 & 2 & a \\ 0 & 1 & -1 & b \\ 1 & -2 & 0 & c \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2}a+2b+\frac{1}{2}c \\ 0 & 1 & 0 & \frac{1}{4}a-\frac{1}{4}c \\ 0 & 0 & 1 & \frac{1}{4}a-b-\frac{1}{4}c \end{bmatrix}$$

Clearly a solution exists for any choice of a, b, c. Hence S is a spanning set for $\mathbb{P}_2$.

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7.3E: Exercises for Section 7.3

Exercise $7.3E.1$

Let V be a vector space and suppose $\{\vec{x}_1, \cdots, \vec{x}_l\}$ is a set of vectors in V. Show that $\vec{0}$ is in $\text{span}\{\vec{x}_1, \cdots, \vec{x}_k\}$.

Answer available at the end of this section.

Exercise $7.3E.2$

Determine if $p(x) = 4x^2 - x$ is in the span given by

$$\text{span}\{x^2 + x,\ x^2 - 1,\ -x + 2\}$$

Answer available at the end of this section.

Exercise $7.3E.3$

Determine if $p(x) = -x^2 + x + 2$ is in the span given by

$$\text{span}\{x^2 + x + 1,\ 2x^2 + x\}$$

Exercise $7.3E.3$

Determine if $A = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$ is in the span given by

$$\text{span}\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\right\}$$

Answer available at the end of this section.

Exercise $7.3E.5$

Show that the spanning set in Exercise $7.3E.4$ is a spanning set for M_{22}, the vector space of all 2×2 matrices.

Answers

Exercise $7.3E.1$

$\sum_{i=1}^{k} 0\vec{x}_k = \vec{0}$

Exercise $7.3E.2$

$p(x) = 4x^2 - x \in \text{span}\{x^2 + x,\ x^2 - 1,\ -x + 2\}$ with

$$p(x) = \frac{2}{3}(x^2 + x) + \frac{10}{3}(x^2 - 1) + \frac{5}{3}(-x + 2)$$

Exercise $7.3E.3$

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} \in \text{span}\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\right\}$$

with

$$\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} = 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 4\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} - 1\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

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7.4: Linear Independence

Learning Objectives

- Determine if a set is linearly independent.

In this section, we will again explore concepts introduced earlier in terms of $\mathbb{R}^n$ and extend them to apply to abstract vector spaces.

Definition 7.4.1: Linear Independence

Let V be a vector space. If $\{\vec{v}_1, \cdots, \vec{v}_n\} \subseteq V$, then it is linearly independent if

$$\sum_{i=1}^{n} a_i \vec{v}_i = \vec{0} \text{ implies } a_1 = \cdots = a_n = 0$$

where the a_i are real numbers.

The set of vectors is called linearly dependent if it is not linearly independent.

Example 7.4.1: Linear Independence

Let $S \subseteq \mathbb{P}_2$ be a set of polynomials given by

$$S = \left\{x^2 + 2x - 1, 2x^2 - x + 3\right\}$$

Determine if S is linearly independent.

Solution

To determine if this set S is linearly independent, we write

$$a(x^2 + 2x - 1) + b(2x^2 - x + 3) = 0x^2 + 0x + 0$$

If it is linearly independent, then $a = b = 0$ will be the only solution. We proceed as follows.

$$\begin{aligned} a(x^2 + 2x - 1) + b(2x^2 - x + 3) &= 0x^2 + 0x + 0 \\ ax^2 + 2ax - a + 2bx^2 - bx + 3b &= 0x^2 + 0x + 0 \\ (a + 2b)x^2 + (2a - b)x - a + 3b &= 0x^2 + 0x + 0 \end{aligned} \tag{7.4.1}$$

It follows that

$$\begin{aligned} a + 2b &= 0 \\ 2a - b &= 0 \\ -a + 3b &= 0 \end{aligned} \tag{7.4.2}$$

The augmented matrix and resulting reduced row-echelon form are given by

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ -1 & 3 & 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence the solution is $a = b = 0$ and the set is linearly independent.

The next example shows us what it means for a set to be dependent.

Example 7.4.2: Dependent Set

Determine if the set S given below is independent.

$$S = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \right\}$$

Solution

To determine if S is linearly independent, we look for solutions to

$$a \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Notice that this equation has nontrivial solutions, for example $a = 2$, $b = 3$ and $c = -1$. Therefore S is dependent.

The following is an important result regarding dependent sets.

Lemma 7.4.1: Dependent Sets

Let V be a vector space and suppose $W = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k\}$ is a subset of V. Then W is dependent if and only if $\vec{v}_i$ can be written as a linear combination of $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_{i-1}, \vec{v}_{i+1}, \cdots, \vec{v}_k\}$ for some $i \leq k$.

Revisit Example 7.4.2 with this in mind. Notice that we can write one of the three vectors as a combination of the others.

$$\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = 2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

By Lemma 7.4.1 this set is dependent.

If we know that one particular set is linearly independent, we can use this information to determine if a related set is linearly independent. Consider the following example.

Example 7.4.3: Related Independent Sets

Let V be a vector space and suppose $S \subseteq V$ is a set of linearly independent vectors given by $S = \{\vec{u}, \vec{v}, \vec{w}\}$. Let $R \subseteq V$ be given by $R = \left\{2\vec{u} - \vec{w}, \vec{w} + \vec{v}, 3\vec{v} + \frac{1}{2}\vec{u}\right\}$. Show that R is also linearly independent.

Solution

To determine if R is linearly independent, we write

$$a(2\vec{u} - \vec{w}) + b(\vec{w} + \vec{v}) + c(3\vec{v} + \frac{1}{2}\vec{u}) = \vec{0}$$

If the set is linearly independent, the only solution will be $a = b = c = 0$. We proceed as follows.

$$\begin{aligned} a(2\vec{u} - \vec{w}) + b(\vec{w} + \vec{v}) + c(3\vec{v} + \frac{1}{2}\vec{u}) &= \vec{0} \\ 2a\vec{u} - a\vec{w} + b\vec{w} + b\vec{v} + 3c\vec{v} + \frac{1}{2}c\vec{u} &= \vec{0} \\ (2a + \frac{1}{2}c)\vec{u} + (b + 3c)\vec{v} + (-a + b)\vec{w} &= \vec{0} \end{aligned} \tag{7.4.3}$$

We know that the set $S = \{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent, which implies that the coefficients in the last line of this equation must all equal 0. In other words:

$$\begin{aligned} 2a + \frac{1}{2}c &= 0 \\ b + 3c &= 0 \\ -a + b &= 0 \end{aligned} \tag{7.4.4}$$

The augmented matrix and resulting reduced row-echelon form are given by:

$$\begin{bmatrix} 2 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 3 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Hence the solution is $a = b = c = 0$ and the set is linearly independent.

The following theorem was discussed in terms in $\mathbb{R}^n$. We consider it here in the general case.

Theorem 7.4.1: Unique Representation

Let V be a vector space and let $U = \{\vec{v}_1, \cdots, \vec{v}_k\} \subseteq V$ be an independent set. If $\vec{v} \in \text{span } U$, then $\vec{v}$ can be written uniquely as a linear combination of the vectors in U.

Consider the span of a linearly independent set of vectors. Suppose we take a vector which is not in this span and add it to the set. The following lemma claims that the resulting set is still linearly independent.

Lemma 7.4.2: Adding to a Linearly Independent Set

Suppose $\vec{v} \notin \text{span}\{\vec{u}_1, \cdots, \vec{u}_k\}$ and $\{\vec{u}_1, \cdots, \vec{u}_k\}$ is linearly independent. Then the set

$$\{\vec{u}_1, \cdots, \vec{u}_k, \vec{v}\}$$

is also linearly independent.

Proof available at the end of this section.

Example 7.4.4: Adding to a Linearly Independent Set

Let $S \subseteq M_{22}$ be a linearly independent set given by

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$$

Show that the set $R \subseteq M_{22}$ given by

$$R = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

is also linearly independent.

Solution

Instead of writing a linear combination of the matrices which equals 0 and showing that the coefficients must equal 0, we can instead use Lemma 7.4.2.

To do so, we show that

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \notin \text{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$$

Write

$$\begin{aligned}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} &= a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}\end{aligned} \tag{7.4.5}$$

Clearly there are no possible a, b to make this equation true. Hence the new matrix does not lie in the span of the matrices in S. By Lemma $7.4.2$, R is also linearly independent.

Answers

Lemma $7.4.2$: Adding to a Linearly Independent Set

Suppose $\sum_{i=1}^{k} c_i\vec{u}_i + d\vec{v} = \vec{0}$. It is required to verify that each $c_i = 0$ and that $d = 0$. But if $d \neq 0$, then you can solve for $\vec{v}$ as a linear combination of the vectors, $\{\vec{u}_1, \cdots, \vec{u}_k\}$,

$$\vec{v} = -\sum_{i=1}^{k} \left(\frac{c_i}{d}\right) \vec{u}_i$$

contrary to the assumption that $\vec{v}$ is not in the span of the $\vec{u}_i$. Therefore, $d = 0$. But then $\sum_{i=1}^{k} c_i\vec{u}_i = \vec{0}$ and the linear independence of $\{\vec{u}_1, \cdots, \vec{u}_k\}$ implies each $c_i = 0$ also.

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7.4E: Exercises for Section 7.4

Exercise 7.4E.1

Determine if the following set is linearly independent. If it is linearly dependent, write one vector as a linear combination of the other vectors in the set.

$$\{x+1,\ x^2+2,\ x^2-x-3\}$$

Answer available at the end of this section.

Exercise 7.4E.2

Determine if the following set is linearly independent. If it is linearly dependent, write one vector as a linear combination of the other vectors in the set.

$$\{x^2+x,\ -2x^2-4x-6,\ 2x-2\}$$

Exercise 7.4E.3

Determine if the following set is linearly independent. If it is linearly dependent, write one vector as a linear combination of the other vectors in the set.

$$\left\{\begin{bmatrix}1&2\\0&1\end{bmatrix},\begin{bmatrix}-7&2\\-2&-3\end{bmatrix},\begin{bmatrix}4&0\\1&2\end{bmatrix}\right\}$$

Answer available at the end of this section.

Exercise 7.4E.4

Determine if the following set is linearly independent. If it is linearly dependent, write one vector as a linear combination of the other vectors in the set.

$$\left\{\begin{bmatrix}1&0\\0&1\end{bmatrix},\begin{bmatrix}0&1\\0&1\end{bmatrix},\begin{bmatrix}1&0\\1&0\end{bmatrix},\begin{bmatrix}0&0\\1&1\end{bmatrix}\right\}$$

Exercise 7.4E.5

Consider polynomials

$$\{a_ix^3+b_ix^2+c_ix+d_i|i=1,2,3,4\}$$

Show that this collection of polynomials is linearly independent on an interval $[s,t]$ if and only if

$$\begin{bmatrix}a_1&b_1&c_1&d_1\\a_2&b_2&c_2&d_2\\a_3&b_3&c_3&d_3\\a_4&b_4&c_4&d_4\end{bmatrix}$$

is an invertible matrix.

Answer available at the end of this section.

Exercise $7.4E.6$

If you have 3 polynomials in $\mathbb{P}_1$, is it possible they form a linearly independent set? Prove your answer.

Hint available at the end of this section.

Answers

Exercise $7.4E.1$

Check if the set is linearly independent by supposing $a(x+1)+b(x^2+2)+c(x^2-x-3)=0$ and solving for a, b, c. It turns out that $a=b=c=0$; hence, the set it linearly independent.

Exercise $7.4E.3$

The set is linearly dependent.

$$\begin{bmatrix}-7 & 2\\ -2 & -3\end{bmatrix}=\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}-2\begin{bmatrix}4 & 0\\ 1 & 2\end{bmatrix}$$

Exercise $7.4E.5$

Let $p_i(x)$ denote the ith of these polynomials. Suppose $\sum_i C_i p_i(x)=0$. Then collecting terms according to the exponent of x, you need to have

$$\begin{aligned} C_1a_1+C_2a_2+C_3a_3+C_4a_4&=0\\ C_1b_1+C_2b_2+C_3b_3+C_4b_4&=0\\ C_1c_1+C_2c_2+C_3c_3+C_4c_4&=0\\ C_1d_1+C_2d_2+C_3d_3+C_4d_4&=0\end{aligned} \tag{7.4E.1}$$

The matrix of coefficients is just the transpose of the above matrix. There exists a non trivial solution if and only if the determinant of this matrix equals 0.

Exercise $7.4E.6$

Begin by setting $a_1p_1(x)+a_2p_2(x)+a_3p_3(x)=0$ and examining the possible solutions for a_1, a_2, a_3.

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7.5: Subspaces

Learning Objectives

- Utilize the definition of a subspace and the subspace test to determine if a set is a subspace of a given vector space.

In this section we will examine the concept of subspaces introduced earlier in terms of $\mathbb{R}^n$. Here, we will discuss these concepts in terms of abstract vector spaces.

Consider the definition of a subspace.

Definition 7.5.1: Subspace

Let V be a vector space. A subset $W \subseteq V$ is said to be a **subspace** of V if $a\vec{x} + b\vec{y} \in W$ whenever $a, b \in \mathbb{R}$ and $\vec{x}, \vec{y} \in W$.

The following fundamental result says that subspaces are subsets of a vector space which are themselves vector spaces.

Theorem 7.5.1: Subspaces are Vector Spaces

Let W be a nonempty collection of vectors in a vector space V. Then W is a subspace if and only if W satisfies the vector space axioms, using the same operations as those defined on V.

Proof available at the end of this section.

Consider the following useful Corollary.

Corollary 7.5.1: Span is a Subspace

Let V be a vector space with $W \subseteq V$. If $W = \text{span}\{\vec{v}_1, \cdots, \vec{v}_n\}$ then W is a subspace of V.

When determining spanning sets the following theorem proves useful.

Theorem 7.5.2: Spanning Set

Let $W \subseteq V$ for a vector space V and suppose $W = \text{span}\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$.

Let $U \subseteq V$ be a subspace such that $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n \in U$. Then it follows that $W \subseteq U$.

In other words, this theorem claims that any subspace that contains a set of vectors must also contain the span of these vectors.

The following example will show that two spans, described differently, can in fact be equal.

Example 7.5.1: Equal Span

Let $p(x), q(x)$ be polynomials and suppose $U = \text{span}\{2p(x) - q(x), p(x) + 3q(x)\}$ and $W = \text{span}\{p(x), q(x)\}$. Show that $U = W$.

Solution

We will use Theorem 7.5.2 to show that $U \subseteq W$ and $W \subseteq U$. It will then follow that $U = W$.

1. $U \subseteq W$ Notice that $2p(x) - q(x)$ and $p(x) + 3q(x)$ are both in $W = span\{p(x), q(x)\}$. Then by Theorem 7.5.2 W must contain the span of these polynomials and so $U \subseteq W$.
2. $W \subseteq U$ Notice that

$$\begin{aligned} p(x) &= \frac{3}{7}(2p(x) - q(x)) + \frac{2}{7}(p(x) + 3q(x)) \\ q(x) &= -\frac{1}{7}(2p(x) - q(x)) + \frac{2}{7}(p(x) + 3q(x)) \end{aligned} \tag{7.5.1}$$

Hence $p(x),\ \ q(x)$ are in $span\{2p(x) - q(x),\ \ p(x) + 3q(x)\}$. By Theorem 7.5.2 U must contain the span of these polynomials and so $W \subseteq U$.

To prove that a set is a vector space, one must verify each of the axioms given in Definition 7.1.2 and 7.1.3. This is a cumbersome task, and therefore a shorter procedure is used to verify a subspace.

Procedure 7.5.1: Subspace Test

Suppose W is a subset of a vector space V. To determine if W is a subspace of V, it is sufficient to determine if the following three conditions hold, using the operations of V:

1. The additive identity $\vec{0}$ of V is contained in W.
2. For any vectors $\vec{w}_1, \vec{w}_2$ in W, $\vec{w}_1 + \vec{w}_2$ is also in W.
3. For any vector $\vec{w}_1$ in W and scalar a, the product $a\vec{w}_1$ is also in W.

Therefore it suffices to prove these three steps to show that a set is a subspace.

Consider the following example.

Example 7.5.2: Improper Subspaces

Let V be an arbitrary vector space. Then V is a subspace of itself. Similarly, the set $\left\{\vec{0}\right\}$ containing only the zero vector is also a subspace.

Solution

Using the subspace test in Procedure 7.5.1 we can show that V and $\left\{\vec{0}\right\}$ are subspaces of V.

Since V satisfies the vector space axioms it also satisfies the three steps of the subspace test. Therefore V is a subspace.

Let's consider the set $\left\{\vec{0}\right\}$.

1. The vector $\vec{0}$ is clearly contained in $\left\{\vec{0}\right\}$, so the first condition is satisfied.
2. Let $\vec{w}_1, \vec{w}_2$ be in $\left\{\vec{0}\right\}$. Then $\vec{w}_1 = \vec{0}$ and $\vec{w}_2 = \vec{0}$ and so

$$\vec{w}_1 + \vec{w}_2 = \vec{0} + \vec{0} = \vec{0}$$

It follows that the sum is contained in $\left\{\vec{0}\right\}$ and the second condition is satisfied.
3. Let $\vec{w}_1$ be in $\left\{\vec{0}\right\}$ and let a be an arbitrary scalar. Then

$$a\vec{w}_1 = a\vec{0} = \vec{0}$$

Hence the product is contained in $\left\{\vec{0}\right\}$ and the third condition is satisfied.

It follows that $\left\{\vec{0}\right\}$ is a subspace of V.

The two subspaces described above are called **improper subspaces**. Any subspace of a vector space V which is not equal to V or $\left\{\vec{0}\right\}$ is called a **proper subspace**.

Consider another example.

Example 7.5.3: Subspace of Polynomials

Let $\mathbb{P}_2$ be the vector space of polynomials of degree two or less. Let $W \subseteq \mathbb{P}_2$ be all polynomials of degree two or less which have 1 as a root. Show that W is a subspace of $\mathbb{P}_2$.

Solution

First, express W as follows:

$$W = \left\{p(x) = ax^2 + bx + c, a, b, c, \in \mathbb{R} | p(1) = 0\right\}$$

We need to show that W satisfies the three conditions of Procedure 7.5.1.

1. The zero polynomial of $\mathbb{P}_2$ is given by $0(x) = 0x^2 + 0x + 0 = 0$. Clearly $0(1) = 0$ so $0(x)$ is contained in W.
2. Let $p(x), q(x)$ be polynomials in W. It follows that $p(1) = 0$ and $q(1) = 0$. Now consider $p(x) + q(x)$. Let $r(x)$ represent this sum.

$$\begin{aligned} r(1) &= p(1) + q(1) \\ &= 0 + 0 \\ &= 0 \end{aligned} \tag{7.5.2}$$

   Therefore the sum is also in W and the second condition is satisfied.
3. Let $p(x)$ be a polynomial in W and let a be a scalar. It follows that $p(1) = 0$. Consider the product $ap(x)$.

$$\begin{aligned} ap(1) &= a(0) \\ &= 0 \end{aligned} \tag{7.5.3}$$

   Therefore the product is in W and the third condition is satisfied.

It follows that W is a subspace of $\mathbb{P}_2$.

Answers

Theorem 7.5.1: Subspaces are Vector Spaces

Suppose first that W is a subspace. It is obvious that all the algebraic laws hold on W because it is a subset of V and they hold on V. Thus $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ along with the other axioms. Does W contain $\vec{0}$? Yes because it contains $0\vec{u} = \vec{0}$.

Are the operations of V defined on W? That is, when you add vectors of W do you get a vector in W? When you multiply a vector in W by a scalar, do you get a vector in W? Yes. This is contained in the definition. Does every vector in W have an additive inverse? Yes by Theorem 7.1.1 because $-\vec{v} = (-1)\,\vec{v}$ which is given to be in W provided $\vec{v} \in W$.

Next suppose W is a vector space. Then by definition, it is closed with respect to linear combinations. Hence it is a subspace.

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7.5E: Exercises for Section 7.5

Exercise $7.5E.1$

Let $M = \{\vec{u} = (u_1,\ u_2,\ u_3,\ u_4) \in \mathbb{R}^4\ :\ |u_1| \leq 4\}$. Is M a subspace of $\mathbb{R}^4$?

Answer available at the end of this section.

Exercise $7.5E.2$

Let $M = \{\vec{u} = (u_1,\ u_2,\ u_3,\ u_4) \in \mathbb{R}^4\ :\ \sin(u_1) = 1\}$. Is M a subspace of $\mathbb{R}^4$?

Answer available at the end of this section.

Exercise $7.5E.3$

Let W be a subset of M_{22} given by

$$W = \{A | A \in M_{22}, A^T = A\}$$

In words, W is the set of all symmetric 2×2 matrices. Is W a subspace of M_{22}?

Exercise $7.5E.4$

Let W be a subset of M_{22} given by

$$W = \left\{ \left[\begin{array}{cc} a & b \\ c & d \end{array} \right] | a, b, c, d \in \mathbb{R},\ a + b = c + d \right\}$$

Is W a subspace of M_{22}?

Answer available at the end of this section.

Exercise $7.5E.5$

Let W be a subset of P_3 given by

$$W = \{ax^3 + bx^2 + cx + d |\ a, b, c, d \in \mathbb{R}, d = 0\}$$

Is W a subspace of P_3?

Exercise $7.5E.6$

Let W be a subset of P_3 given by

$$W = \{p(x) = ax^3 + bx^2 + cx + d |\ a, b, c, d \in \mathbb{R}, p(2) = 1\}$$

Is W a subspace of P_3?

Answers

Exercise $7.5E.1$

This is not a subspace. $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is in it, but $20 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is not.

Exercise $7.5E.2$

This is not a subspace. It doesn't satisfy *any* of the conditions in the Subspace Test.

Exercise $7.5E.4$

To show W is a subspace of M_{22}, we check the subspace criteria:

1. Zero matrix: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in W$ since $0 + 0 = 0 + 0$.

2. Closed under addition: If $A_1, B_1 \in W$, then

$$A_1 + A_2 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}$$

satisfies $(a_1 + a_2) + (b_1 + b_2) = (c_1 + c_2) + (d_1 + d_2)$, so $A_1 + A_2 \in W$.

3. Closed under scalar multiplication: If $A \in W$ and $\alpha \in \mathbb{R}$, then

$$\alpha A = \begin{bmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{bmatrix}$$

satisfies $\alpha(a + b) = \alpha(c + d)$, so $\alpha A \in W$.

Therefore, W is a subspace of M_{22}.

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7.6: Basis

Learning Objectives

- Extend a linearly independent set to a basis of a given vector space.
- Shrink a spanning set to a basis of a given vector space.

Definition 7.6.1: Basis

Let V be a vector space. Then $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is called a **basis** for V if the following conditions hold.

1. $\text{span}\{\vec{v}_1, \cdots, \vec{v}_n\} = V$
2. $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is linearly independent

Consider the following example.

Example 7.6.1: Polynomials of Degree Two

Let $\mathbb{P}_2$ be the set polynomials of degree no more than 2. We can write $\mathbb{P}_2 = \text{span}\{x^2, x, 1\}$. Is $\{x^2, x, 1\}$ a basis for $\mathbb{P}_2$?

Solution

It can be verified that $\mathbb{P}_2$ is a vector space defined under the usual addition and scalar multiplication of polynomials.

Now, since $\mathbb{P}_2 = \text{span}\{x^2, x, 1\}$, the set $\{x^2, x, 1\}$ is a basis if it is linearly independent. Suppose then that

$$ax^2 + bx + c = 0x^2 + 0x + 0$$

where a, b, c are real numbers. It is clear that this can only occur if $a = b = c = 0$. Hence the set is linearly independent and forms a basis of $\mathbb{P}_2$.

The next theorem is an essential result in linear algebra and is called the exchange theorem.

Theorem 7.6.1: Exchange Theorem

Let $\{\vec{x}_1, \cdots, \vec{x}_r\}$ be a linearly independent set of vectors such that each $\vec{x}_i$ is contained in span$\{\vec{y}_1, \cdots, \vec{y}_s\}$. Then $r \leq s$.

Proof available at the end of this section.

The following corollary follows from the exchange theorem.

Corollary 7.6.1: Two Bases of the Same Length

Let B_1, B_2 be two bases of a vector space V. Suppose B_1 contains m vectors and B_2 contains n vectors. Then $m = n$.

Proof available at the end of this section.

Given the result of the previous corollary, the following definition follows.

Definition 7.6.2: Dimension

A vector space V is of **dimension** n if it has a basis consisting of n vectors.

Notice that the dimension is well defined by Corollary 7.6.1. It is assumed here that $n < \infty$ and therefore such a vector space is said to be **finite dimensional**.

Example 7.6.2: Dimension of a Vector Space

Let $\mathbb{P}_2$ be the set of all polynomials of degree at most 2. Find the dimension of $\mathbb{P}_2$.

Solution

If we can find a basis of $\mathbb{P}_2$ then the number of vectors in the basis will give the dimension. Recall from Example 7.6.4 that a basis of $\mathbb{P}_2$ is given by

$$S = \left\{x^2, x, 1\right\}$$

There are three polynomials in S and hence the dimension of $\mathbb{P}_2$ is three.

It is important to note that a basis for a vector space is not unique. A vector space can have many bases. Consider the following example.

Example 7.6.3: A Different Basis for Polynomials of Degree Two

Let $\mathbb{P}_2$ be the polynomials of degree no more than 2. Is $\left\{x^2 + x + 1, 2x + 1, 3x^2 + 1\right\}$ a basis for $\mathbb{P}_2$?

Solution

Suppose these vectors are linearly independent but do not form a spanning set for $\mathbb{P}_2$. Then by Lemma 7.4.2, we could find a fourth polynomial in $\mathbb{P}_2$ to create a new linearly independent set containing four polynomials. However this would imply that we could find a basis of $\mathbb{P}_2$ of more than three polynomials. This contradicts the result of Example 7.6.2 in which we determined the dimension of $\mathbb{P}_2$ is three. Therefore if these vectors are linearly independent they must also form a spanning set and thus a basis for $\mathbb{P}_2$.

Suppose then that

$$\begin{aligned} a\left(x^2 + x + 1\right) + b\left(2x + 1\right) + c\left(3x^2 + 1\right) &= 0 \\ \left(a + 3c\right)x^2 + \left(a + 2b\right)x + \left(a + b + c\right) &= 0 \end{aligned} \tag{7.6.1}$$

We know that $\left\{x^2, x, 1\right\}$ is linearly independent, and so it follows that

$$\begin{aligned} a + 3c &= 0 \\ a + 2b &= 0 \\ a + b + c &= 0 \end{aligned} \tag{7.6.2}$$

and there is only one solution to this system of equations, $a = b = c = 0$. Therefore, these are linearly independent and form a basis for $\mathbb{P}_2$.

Consider the following theorem.

Theorem 7.6.2: Every Subspace has a Basis

Let W be a nonzero subspace of a finite dimensional vector space V. Suppose V has dimension n. Then W has a basis with no more than n vectors.

Proof available at the end of this section.

If in fact W has n vectors, then it follows that $W = V$.

Theorem 7.6.3: Subspace of Same Dimension

Let V be a vector space of dimension n and let W be a subspace. Then $W = V$ if and only if the dimension of W is also n.

Proof available at the end of this section.

Consider the following example.

Example 7.6.4: Basis of a Subspace

Let $U = \left\{ A \in \mathbb{M}_{22} \;\middle|\; A \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} A \right\}$. Then U is a subspace of $\mathbb{M}_{22}$ Find a basis of U, and hence $\dim(U)$.

Solution

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{M}_{22}$. Then

$$A \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} a+b & -b \\ c+d & -d \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ -c & -d \end{bmatrix}.$$

If $A \in U$, then $\begin{bmatrix} a+b & -b \\ c+d & -d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ -c & -d \end{bmatrix}$.

Equating entries leads to a system of four equations in the four variables a, b, c and d.

$$\begin{aligned} a+b &= a+c \\ -b &= b+d \\ c+d &= -c \\ -d &= -d \end{aligned}$$

or

$$\begin{aligned} b-c &= 0 \\ -2b-d &= 0 \\ 2c+d &= 0 \end{aligned}.$$

The solution to this system is $a = s$, $b = -\frac{1}{2}t$, $c = -\frac{1}{2}t$, $d = t$ for any $s, t \in \mathbb{R}$, and thus

$$A = \begin{bmatrix} s & \frac{t}{2} \\ -\frac{t}{2} & t \end{bmatrix} = s \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + t \begin{bmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}.$$

Let

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} \right\}.$$

Then $\text{span}(B) = U$, and it is routine to verify that B is an independent subset of $\mathbb{M}_{22}$. Therefore B is a basis of U, and $\dim(U) = 2$.

The following theorem claims that a spanning set of a vector space V can be shrunk down to a basis of V. Similarly, a linearly independent set within V can be enlarged to create a basis of V.

Theorem 7.6.4: Basis of V

If $V = \text{span}\{\vec{u}_1, \cdots, \vec{u}_n\}$ is a vector space, then some subset of $\{\vec{u}_1, \cdots, \vec{u}_n\}$ is a basis for V. Also, if $\{\vec{u}_1, \cdots, \vec{u}_k\} \subseteq V$ is linearly independent and the vector space is finite dimensional, then the set $\{\vec{u}_1, \cdots, \vec{u}_k\}$, can be enlarged to obtain a basis of V.

Proof available at the end of this section.

Recall Example 7.4.4 in which we added a matrix to a linearly independent set to create a larger linearly independent set. By Theorem 7.6.4 we can extend a linearly independent set to a basis.

Example 7.6.5: Adding to a Linearly Independent Set

Let $S \subseteq M_{22}$ be a linearly independent set given by

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$$

Enlarge S to a basis of M_{22}.

Solution

Recall from the solution of Example 9.3.4 that the set $R \subseteq M_{22}$ given by

$$R = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

is also linearly independent. However this set is still not a basis for M_{22} as it is not a spanning set. In particular, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ is not in $\text{span}R$. Therefore, this matrix can be added to the set by Lemma 9.3.2 to obtain a new linearly independent set given by

$$T = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

This set is linearly independent and now spans M_{22}. Hence T is a basis.

Next we consider the case where you have a spanning set and you want a subset which is a basis. The above discussion involved adding vectors to a set. The next theorem involves removing vectors.

Theorem 7.6.5: Basis from a Spanning Set

Let V be a vector space and let W be a subspace. Also suppose that $W = \text{span}\{\vec{w}_1, \cdots, \vec{w}_m\}$. Then there exists a subset of $\{\vec{w}_1, \cdots, \vec{w}_m\}$ which is a basis for W.

Proof available at the end of this section.

Consider the following example of this concept.

Example 7.6.6: Basis from a Spanning Set

Let V be the vector space of polynomials of degree no more than 3, denoted earlier as $\mathbb{P}_3$. Consider the following vectors in V.

$$2x^2+x+1, x^3+4x^2+2x+2, 2x^3+2x^2+2x+1, \\ x^3+4x^2-3x+2, x^3+3x^2+2x+1 \tag{7.6.3}$$

Then, as mentioned above, V has dimension 4 and so clearly these vectors are not linearly independent. A basis for V is $\{1, x, x^2, x^3\}$. Determine a linearly independent subset of these which has the same span. Determine whether this subset is a basis for V.

Solution

Consider an isomorphism which maps $\mathbb{R}^4$ to V in the obvious way. Thus

$$\begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}$$

corresponds to $2x^2+x+1$ through the use of this isomorphism. Then corresponding to the above vectors in V we would have the following vectors in $\mathbb{R}^4$.

$$\begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$$

Now if we obtain a subset of these which has the same span but which is linearly independent, then the corresponding vectors from V will also be linearly independent. If there are four in the list, then the resulting vectors from V must be a basis for V. The reduced row-echelon form for the matrix which has the above vectors as columns is

$$\begin{bmatrix} 1 & 0 & 0 & -15 & 0 \\ 0 & 1 & 0 & 11 & 0 \\ 0 & 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore, a basis for V consists of the vectors

$$2x^2+x+1, x^3+4x^2+2x+2, 2x^3+2x^2+2x+1, \\ x^3+3x^2+2x+1. \tag{7.6.4}$$

Note how this is a subset of the original set of vectors. If there had been only three pivot columns in this matrix, then we would not have had a basis for V but we would at least have obtained a linearly independent subset of the original set of vectors in this way.

Note also that, since all linear relations are preserved by an isomorphism,

$$\begin{aligned} &-15\left(2x^2+x+1\right)+11\left(x^3+4x^2+2x+2\right)+(-5)\left(2x^3+2x^2+2x+1\right) \\ =&x^3+4x^2-3x+2 \end{aligned} \tag{7.6.5}$$

Consider the following example.

Example 7.6.7: Shrinking a Spanning Set

Consider the set $S \subseteq \mathbb{P}_2$ given by

$$S = \left\{1, x, x^2, x^2+1\right\}$$

Show that S spans $\mathbb{P}_2$, then remove vectors from S until it creates a basis.

Solution

First we need to show that S spans $\mathbb{P}_2$. Let ax^2+bx+c be an arbitrary polynomial in $\mathbb{P}_2$. Write

$$ax^2+bx+c = r(1)+s(x)+t(x^2)+u(x^2+1)$$

Then,

$$\begin{aligned} ax^2+bx+c &= r(1)+s(x)+t(x^2)+u(x^2+1) \\ &= (t+u)x^2+s(x)+(r+u) \end{aligned} \tag{7.6.6}$$

It follows that

$$\begin{aligned} a &= t+u \\ b &= s \\ c &= r+u \end{aligned} \tag{7.6.7}$$

Clearly a solution exists for all a, b, c and so S is a spanning set for $\mathbb{P}_2$. By Theorem 7.6.4, some subset of S is a basis for $\mathbb{P}_2$.

Recall that a basis must be both a spanning set and a linearly independent set. Therefore we must remove a vector from S keeping this in mind. Suppose we remove x from S. The resulting set would be $\{1, x^2, x^2+1\}$. This set is clearly linearly dependent (and also does not span $\mathbb{P}_2$) and so is not a basis.

Suppose we remove x^2+1 from S. The resulting set is $\{1, x, x^2\}$ which is both linearly independent and spans $\mathbb{P}_2$. Hence this is a basis for $\mathbb{P}_2$. Note that removing any one of $1, x^2$, or x^2+1 will result in a basis.

Now the following is a fundamental result about subspaces.

Theorem 7.6.6: Basis of a Vector Space

Let V be a finite dimensional vector space and let W be a non-zero subspace. Then W has a basis. That is, there exists a linearly independent set of vectors $\{\vec{w}_1, \cdots, \vec{w}_r\}$ such that

$$span\{\vec{w}_1, \cdots, \vec{w}_r\} = W$$

Also if $\{\vec{w}_1, \cdots, \vec{w}_s\}$ is a linearly independent set of vectors, then W has a basis of the form $\{\vec{w}_1, \cdots, \vec{w}_s, \cdots, \vec{w}_r\}$ for $r \geq s$.

Proof available at the end of this section.

This also proves the following corollary. Let V play the role of W in the above theorem and begin with a basis for W, enlarging it to form a basis for V as discussed above.

Corollary 7.6.2: Basis Extension

Let W be any non-zero subspace of a vector space V. Then every basis of W can be extended to a basis for V.

Consider the following example.

Example 7.6.8: Basis Extension

Let $V = \mathbb{R}^4$ and let

$$W = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}\right\}$$

Extend this basis of W to a basis of V.

Solution

An easy way to do this is to take the reduced row-echelon form of the matrix

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.6.8}$$

Note how the given vectors were placed as the first two and then the matrix was extended in such a way that it is clear that the span of the columns of this matrix yield all of $\mathbb{R}^4$. Now determine the pivot columns. The reduced row-echelon form is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix} \tag{7.6.9}$$

These are

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

and now this is an extension of the given basis for W to a basis for $\mathbb{R}^4$.

Why does this work? The columns of (7.6.8) obviously span $\mathbb{R}^4$ the span of the first four is the same as the span of all six.

Answers

Theorem 7.6.1**: Exchange Theorem**

The proof will proceed as follows. First, we set up the necessary steps for the proof. Next, we will assume that $r > s$ and show that this leads to a contradiction, thus requiring that $r \leq s$.

Define span$\{\vec{y}_1, \cdots, \vec{y}_s\} = V$. Since each $\vec{x}_i$ is in span$\{\vec{y}_1, \cdots, \vec{y}_s\}$, it follows there exist scalars $c_1, \cdots, c_s$ such that

$$\vec{x}_1 = \sum_{i=1}^{s} c_i \vec{y}_i \tag{7.6.10}$$

Note that not all of these scalars c_i can equal zero. Suppose that all the $c_i = 0$. Then it would follow that $\vec{x}_1 = \vec{0}$ and so $\{\vec{x}_1, \cdots, \vec{x}_r\}$ would not be linearly independent. Indeed, if $\vec{x}_1 = \vec{0}$, $1\vec{x}_1 + \sum_{i=2}^{r} 0\vec{x}_i = \vec{x}_1 = \vec{0}$ and so there would exist a nontrivial linear combination of the vectors $\{\vec{x}_1, \cdots, \vec{x}_r\}$ which equals zero. Therefore at least one c_i is nonzero.

Say $c_k \neq 0$. Then solve (7.6.10) for $\vec{y}_k$ and obtain

$$\vec{y}_k \in \text{span}\left\{\vec{x}_1, \overbrace{\vec{y}_1, \cdots, \vec{y}_{k-1}, \vec{y}_{k+1}, \cdots, \vec{y}_s}^{\text{s-1 vectors here}}\right\}.$$

Define $\{\vec{z}_1, \cdots, \vec{z}_{s-1}\}$ to be

$$\{\vec{z}_1, \cdots, \vec{z}_{s-1}\} = \{\vec{y}_1, \cdots, \vec{y}_{k-1}, \vec{y}_{k+1}, \cdots, \vec{y}_s\}$$

Now we can write

$$\vec{y}_k \in \text{span}\{\vec{x}_1, \vec{z}_1, \cdots, \vec{z}_{s-1}\}$$

Therefore, $\text{span}\{\vec{x}_1, \vec{z}_1, \cdots, \vec{z}_{s-1}\} = V$. To see this, suppose $\vec{v} \in V$. Then there exist constants $c_1, \cdots, c_s$ such that

$$\vec{v} = \sum_{i=1}^{s-1} c_i \vec{z}_i + c_s \vec{y}_k.$$

Replace this $\vec{y}_k$ with a linear combination of the vectors $\{\vec{x}_1, \vec{z}_1, \cdots, \vec{z}_{s-1}\}$ to obtain $\vec{v} \in \text{span}\{\vec{x}_1, \vec{z}_1, \cdots, \vec{z}_{s-1}\}$. The vector $\vec{y}_k$, in the list $\{\vec{y}_1, \cdots, \vec{y}_s\}$, has now been replaced with the vector $\vec{x}_1$ and the resulting modified list of vectors has the same span as the original list of vectors, $\{\vec{y}_1, \cdots, \vec{y}_s\}$.

We are now ready to move on to the proof. Suppose that $r > s$ and that

$$\text{span}\{\vec{x}_1, \cdots, \vec{x}_l, \vec{z}_1, \cdots, \vec{z}_p\} = V$$

where the process established above has continued. In other words, the vectors $\vec{z}_1, \cdots, \vec{z}_p$ are each taken from the set $\{\vec{y}_1, \cdots, \vec{y}_s\}$ and $l + p = s$. This was done for $l = 1$ above. Then since $r > s$, it follows that \(l\leq s

Corollary 7.6.1**: Two Bases of the Same Length**

By Theorem 7.6.1, $m \leq n$ and $n \leq m$. Therefore $m = n$.

This corollary is very important so we provide another proof independent of the Exchange Theorem above.

Suppose $n > m$. Then since the vectors $\{\vec{u}_1, \cdots, \vec{u}_m\}$ span V, there exist scalars c_{ij} such that

$$\sum_{i=1}^{m} c_{ij} \vec{u}_i = \vec{v}_j.$$

Therefore,

$$\sum_{j=1}^{n} d_j \vec{v}_j = \vec{0} \text{ if and only if } \sum_{j=1}^{n}\sum_{i=1}^{m} c_{ij} d_j \vec{u}_i = \vec{0}$$

if and only if

$$\sum_{i=1}^{m} \left(\sum_{j=1}^{n} c_{ij} d_j \right) \vec{u}_i = \vec{0}$$

Now since $\{\vec{u}_1, \cdots, \vec{u}_n\}$ is independent, this happens if and only if

$$\sum_{j=1}^{n} c_{ij} d_j = 0, \ i = 1, 2, \cdots, m.$$

However, this is a system of m equations in n variables, $d_1, \cdots, d_n$ and \(m n\).

Theorem 7.6.2**: Every Subspace has a Basis**

Let $\vec{v}_1 \in V$ where $\vec{v}_1 \neq 0$. If $\text{span}\{\vec{v}_1\} = V$, then it follows that $\{\vec{v}_1\}$ is a basis for V. Otherwise, there exists $\vec{v}_2 \in V$ which is not in $\text{span}\{\vec{v}_1\}$. By Lemma 7.3.2 $\{\vec{v}_1, \vec{v}_2\}$ is a linearly independent set of vectors. Then $\{\vec{v}_1, \vec{v}_2\}$ is a basis for V and we are done. If $\text{span}\{\vec{v}_1, \vec{v}_2\} \neq V$, then there exists $\vec{v}_3 \notin \text{span}\{\vec{v}_1, \vec{v}_2\}$ and $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a larger linearly independent set of vectors. Continuing this way, the process must stop before $n+1$ steps because if not, it would be possible to obtain $n+1$ linearly independent vectors contrary to the exchange theorem, Theorem 7.6.1.

Theorem 7.6.3**: Subspace of Same Dimension**

First suppose $W = V$. Then obviously the dimension of $W = n$.

Now suppose that the dimension of W is n. Let a basis for W be $\{\vec{w}_1, \cdots, \vec{w}_n\}$. If W is not equal to V, then let $\vec{v}$ be a vector of V which is not contained in W. Thus $\vec{v}$ is not in $\text{span}\{\vec{w}_1, \cdots, \vec{w}_n\}$ and by Lemma 5.6.1, $\{\vec{w}_1, \cdots, \vec{w}_n, \vec{v}\}$ is linearly independent which contradicts Theorem 7.6.1 because it would be an independent set of $n+1$ vectors even though each of these vectors is in a spanning set of n vectors, a basis of V.

Theorem 7.6.4**: Basis of** V

Let

$$S = \{E \subseteq \{\vec{u}_1, \cdots, \vec{u}_n\} \text{ such that } \text{span}\{E\} = V\}.$$

For $E \in S$, let $|E|$ denote the number of elements of E. Let

$$m = \min\{|E| \text{ such that } E \in S\}.$$

Thus there exist vectors

$$\{\vec{v}_1, \cdots, \vec{v}_m\} \subseteq \{\vec{u}_1, \cdots, \vec{u}_n\}$$

such that

$$\text{span}\{\vec{v}_1, \cdots, \vec{v}_m\} = V$$

and m is as small as possible for this to happen. If this set is linearly independent, it follows it is a basis for V and the theorem is proved. On the other hand, if the set is not linearly independent, then there exist scalars, $c_1, \cdots, c_m$ such that

$$\vec{0} = \sum_{i=1}^{m} c_i \vec{v}_i$$

and not all the c_i are equal to zero. Suppose $c_k \neq 0$. Then solve for the vector $\vec{v}_k$ in terms of the other vectors. Consequently,

$$V = \text{span}\{\vec{v}_1, \cdots, \vec{v}_{k-1}, \vec{v}_{k+1}, \cdots, \vec{v}_m\}$$

contradicting the definition of m. This proves the first part of the theorem.

To obtain the second part, begin with $\{\vec{u}_1, \cdots, \vec{u}_k\}$ and suppose a basis for V is

$$\{\vec{v}_1, \cdots, \vec{v}_n\}$$

If

$$\text{span}\{\vec{u}_1, \cdots, \vec{u}_k\} = V,$$

then $k = n$. If not, there exists a vector

$$\vec{u}_{k+1} \notin \text{span}\{\vec{u}_1, \cdots, \vec{u}_k\}$$

Then from Lemma 7.4.2, $\{\vec{u}_1, \cdots, \vec{u}_k, \vec{u}_{k+1}\}$ is also linearly independent. Continue adding vectors in this way until n linearly independent vectors have been obtained. Then

$$\text{span}\{\vec{u}_1, \cdots, \vec{u}_n\} = V$$

because if it did not do so, there would exist $\vec{u}_{n+1}$ as just described and $\{\vec{u}_1, \cdots, \vec{u}_{n+1}\}$ would be a linearly independent set of vectors having $n+1$ elements. This contradicts the fact that $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a basis. In turn this would contradict Theorem 7.6.1. Therefore, this list is a basis.

Theorem 7.6.5**: Basis from a Spanning Set**

Let S denote the set of positive integers such that for $k \in S$, there exists a subset of $\{\vec{w}_1, \cdots, \vec{w}_m\}$ consisting of exactly k vectors which is a spanning set for W. Thus $m \in S$. Pick the smallest positive integer in S. Call it k. Then there exists $\{\vec{u}_1, \cdots, \vec{u}_k\} \subseteq \{\vec{w}_1, \cdots, \vec{w}_m\}$ such that $span\{\vec{u}_1, \cdots, \vec{u}_k\} = W$. If

$$\sum_{i=1}^{k} c_i \vec{w}_i = \vec{0}$$

and not all of the $c_i = 0$, then you could pick $c_j \neq 0$, divide by it and solve for $\vec{u}_j$ in terms of the others.

$$\vec{w}_j = \sum_{i \neq j} \left(-\frac{c_i}{c_j} \right) \vec{w}_i$$

Then you could delete $\vec{w}_j$ from the list and have the same span. In any linear combination involving $\vec{w}_j$, the linear combination would equal one in which $\vec{w}_j$ is replaced with the above sum, showing that it could have been obtained as a linear combination of $\vec{w}_i$ for $i \neq j$. Thus $k-1 \in S$ contrary to the choice of k. Hence each $c_i = 0$ and so $\{\vec{u}_1, \cdots, \vec{u}_k\}$ is a basis for W consisting of vectors of $\{\vec{w}_1, \cdots, \vec{w}_m\}$.

Theorem 7.6.6**: Basis of a Vector Space**

Let the dimension of V be n. Pick $\vec{w}_1 \in W$ where $\vec{w}_1 \neq \vec{0}$. If $\vec{w}_1, \cdots, \vec{w}_s$ have been chosen such that $\{\vec{w}_1, \cdots, \vec{w}_s\}$ is linearly independent, if $span\{\vec{w}_1, \cdots, \vec{w}_r\} = W$, stop. You have the desired basis. Otherwise, there exists $\vec{w}_{s+1} \notin span\{\vec{w}_1, \cdots, \vec{w}_s\}$ and $\{\vec{w}_1, \cdots, \vec{w}_s, \vec{w}_{s+1}\}$ is linearly independent. Continue this way until the process stops. It must stop since otherwise, you could obtain a linearly independent set of vectors having more than n vectors which is impossible.

The last claim is proved by following the above procedure starting with $\{\vec{w}_1, \cdots, \vec{w}_s\}$ as above.

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7.6.E: Exercises for Section 7.6

Exercise 7.6.E.1

Find a basis in $\mathbb{P}_2$ for the subspace

$$\text{span}\{1+x+x^2,\ 1+2x,\ 1+5x-3x^2\}$$

If the above three vectors do not yield a basis, exhibit one of them as a linear combination of the others.

Hint available at the end of this section.

Exercise 7.6.E.2

Find a basis in $\mathbb{P}_3$ for the subspace

$$\text{span}\{1+x-x^2+x^3,\ 1+2x+3x^3,\ -1+3x+5x^2+7x^3,\ 1+6x+4x^2+11x^3\}$$

If the above three vectors do not yield a basis, exhibit one of them as a linear combination of the others.

Answer available at the end of this section.

Exercise 7.6.E.3

Find a basis in $\mathbb{P}_3$ for the subspace

$$\text{span}\{1+x-x^2+x^3,\ 1+2x+3x^3,\ -1+3x+5x^2+7x^3,\ 1+6x+4x^2+11x^3\}$$

If the above three vectors do not yield a basis, exhibit one of them as a linear combination of the others.

Exercise 7.6.E.4

Find a basis in $\mathbb{P}_3$ for the subspace

$$\text{span}\{x^3-2x^2+x+2,\ 3x^3-x^2+2x+2,\ 7x^3+x^2+4x+2,\ 5x^3+3x+2\}$$

If the above three vectors do not yield a basis, exhibit one of them as a linear combination of the others.

Exercise 7.6.E.5

Find a basis in $\mathbb{P}_3$ for the subspace

$$\text{span}\{x^3+2x^2+x-2,\ 3x^3+3x^2+2x-2,\ 3x^3+x+2,\ 3x^3+x+2\}$$

If the above three vectors do not yield a basis, exhibit one of them as a linear combination of the others.

Exercise 7.6.E.6

Find a basis in $\mathbb{P}_3$ for the subspace

$$\text{span}\{x^3-5x^2+x+5,\ 3x^3-4x^2+2x+5,\ 5x^3+8x^2+2x-5,\ 11x^3+6x+5\}$$

If the above three vectors do not yield a basis, exhibit one of them as a linear combination of the others.

Exercise 7.6.E.7

Find a basis in $\mathbb{P}_3$ for the subspace

$$\text{span}\{x^3 - x^2 + x + 1,\ 3x^3 + 2x + 1,\ 4x^3 + x^2 + 2x + 1,\ 3x^3 + 2x - 1\}$$

If the above three vectors do not yield a basis, exhibit one of them as a linear combination of the others.

Answer available at the end of this section.

Exercise 7.6.E.8

Find a basis in $\mathbb{P}_3$ for the subspace

$$\text{span}\{x^3 - x^2 + x + 1,\ 3x^3 + 2x + 1,\ 13x^3 + x^2 + 8x + 4,\ 3x^3 + 2x - 1\}$$

If the above three vectors do not yield a basis, exhibit one of them as a linear combination of the others.

Exercise 7.6.E.9

Find a basis in $\mathbb{P}_3$ for the subspace

$$\text{span}\{x^3 - 3x^2 + x + 3,\ 3x^3 - 2x^2 + 2x + 3,\ -5x^3 + 5x^2 - 4x - 6,\ 7x^3 + 4x - 3\}$$

If the above three vectors do not yield a basis, exhibit one of them as a linear combination of the others.

Exercise 7.6.E.10

Find a basis in $\mathbb{P}_3$ for the subspace

$$\text{span}\{x^3 - 2x^2 + x + 2,\ 3x^3 - x^2 + 2x + 2,\ 7x^3 - x^2 + 4x + 4,\ 5x^3 + 3x - 2\}$$

If the above three vectors do not yield a basis, exhibit one of them as a linear combination of the others.

Exercise 7.6.E.11

Find a basis in $\mathbb{P}_3$ for the subspace

$$\text{span}\{x^3 - 2x^2 + x + 2,\ 3x^3 - x^2 + 2x + 2,\ 3x^3 + 4x^2 + x - 2,\ 7x^3 - x^2 + 4x + 4\}$$

If the above three vectors do not yield a basis, exhibit one of them as a linear combination of the others.

Exercise 7.6.E.12

Find a basis in $\mathbb{P}_3$ for the subspace

$$\text{span}\{x^3 - 4x^2 + x + 4,\ 3x^3 - 3x^2 + 2x + 4,\ -3x^3 + 3x^2 - 2x - 4,\ -2x^3 + 4x^2 - 2x - 4\}$$

If the above three vectors do not yield a basis, exhibit one of them as a linear combination of the others.

Exercise 7.6.E.13

Find a basis in $\mathbb{P}_3$ for the subspace

$$\text{span}\{x^3 + 2x^2 + x - 2,\ 3x^3 + 3x^2 + 2x - 2,\ 5x^3 + x^2 + 2x + 2,\ 10x^3 + 10x^2 + 6x - 6\}$$

If the above three vectors do not yield a basis, exhibit one of them as a linear combination of the others.

Exercise 7.6.E.14

Find a basis in $\mathbb{P}_3$ for the subspace

$$\text{span}\{x^3+x^2+x-1,\ 3x^3+2x^2+2x-1,\ x^3+1,\ 4x^3+3x^2+2x-1\}$$

If the above three vectors do not yield a basis, exhibit one of them as a linear combination of the others.

Exercise 7.6.E.15

Find a basis in $\mathbb{P}_3$ for the subspace

$$\text{span}\{x^3-x^2+x+1,\ 3x^3+2x+1,\ x^3+2x^2-1,\ 4x^3+x^2+2x+1\}$$

If the above three vectors do not yield a basis, exhibit one of them as a linear combination of the others.

Answer available at the end of this section.

Exercise 7.6.E.16

Here are some vectors.

$$\{x^3+x^2-x-1,\ 3x^3+2x^2+2x-1\}$$

If these are linearly independent, extend to a basis for all of $\mathbb{P}_3$.

Exercise 7.6.E.17

Here are some vectors.

$$\{x^3-2x^2-x+2,\ 3x^3-x^2+2x+2\}$$

If these are linearly independent, extend to a basis for all of $\mathbb{P}_3$.

Exercise 7.6.E.18

Here are some vectors.

$$\{x^3-3x^2-x+3,\ 3x^3-2x^2+2x+3\}$$

If these are linearly independent, extend to a basis for all of $\mathbb{P}_3$.

Exercise 7.6.E.19

Here are some vectors.

$$\{x^3-2x^2-3x+2,\ 3x^3-x^2-6x+2,\ -8x^3+18x+10\}$$

If these are linearly independent, extend to a basis for all of $\mathbb{P}_3$.

Exercise 7.6.E.20

Here are some vectors.

$$\{x^3-3x^2-3x+3,\ 3x^3-2x^2-6x+3,\ -8x^3+18x+40\}$$

If these are linearly independent, extend to a basis for all of $\mathbb{P}_3$.

Exercise $7.6.E.21$

Here are some vectors.

$$\{x^3 - x^2 + x + 1,\ 3x^3 + 2x + 1,\ 4x^3 + 2x + 2\}$$

If these are linearly independent, extend to a basis for all of $\mathbb{P}_3$.

Exercise $7.6.E.22$

Here are some vectors.

$$\{x^3 + x^2 + 2x - 1,\ 3x^3 + 2x^2 + 4x - 1,\ 7x^3 + 8x + 23\}$$

If these are linearly independent, extend to a basis for all of $\mathbb{P}_3$.

Exercise $7.6.E.23$

Let $\mathbb{P}_3$ be the polynomials of degree no more than 3. Determine which of the following are bases for this vector space.

a. $\{x + 1,\ x^3 + x^2 + 2x,\ x^2 + x,\ x^3 + x^2 + x\}$
b. $\{x^3 + 1,\ x^2 + x,\ 2x^3 + x^2,\ 2x^3 - x^2 - 3x + 1\}$

Answer available at the end of this section.

Exercise $7.6.E.24$

Suppose V is a finite dimensional vector space. Based on the Exchange Theorem, any two bases have the same number of vectors in them. Give a different proof of this fact using the earlier material in the book.

Hint available at the end of this section.

Answers

Exercise $7.6.E.1$

This is the situation in which you have a spanning set and you want to cut it down to form a linearly independent set which is also a spanning set. Use the same isomorphism above. Since T is an isomorphism, it preserves all linear relations so if such can be found in $\mathbb{R}^3$, the same linear relations will be present in $\mathbb{P}_2$.

Exercise $7.6.E.2$

Basis: $\{1 + x - x^2 + x^3,\ 1 + 2x + 3x^3\}$

$-1 + 3x + 5x^2 + 7x^3 = -5(1 + x - x^2 + x^3) + 4(1 + 2x + 3x^3)$

Exercise $7.6.E.7$

Basis: $\{x^3 - x^2 + x + 1,\ 3x^3 + 2x + 1,\ 4x^3 + x^2 + 2x + 1,\ 3x^3 + 2x - 1\}$. No vector is a linear combination of the others — they form a basis for a 4-dimensional subspace of $\mathbb{P}_3$, which in fact is $\mathbb{P}_3$.

Exercise $7.6.E.15$

Basis: $\{x^3 - x^2 + x + 1,\ 3x^3 + 2x + 1,\ 4x^3 + x^2 + 2x + 1\}$

$x^3 + 2x^2 - 1 = -2(x^3 - x^2 + x + 1) + 1(3x^3 + 2x + 1)$

Exercise $7.6.E.23$

a. The given vectors are linearly independent. Since $\dim(\mathbb{P}_3) = 4$ and we are given 4 linearly independent vectors, they form a basis.
b. This set of vectors are also linearly independent. Hence, by the same logic as in part a, they form a basis.

Exercise $7.6.E.24$

Suppose $\{\vec{x}_1, \cdots, \vec{x}_n\}$ and $\{\vec{y}_1, \cdots, \vec{y}_m\}$ are two bases with $m < n$. Then define

$$\varphi : \mathbb{R}^n \mapsto V,\ \psi : \mathbb{R}^m \mapsto V$$

by

$$\varphi(\vec{a}) = \sum_{k=1}^{n} a_k \vec{x}_k,\ \psi(\vec{b}) = \sum_{j=1}^{m} b_j \vec{y}_j$$

Consider the linear transformation, $\psi^{-1} \circ \varphi$. Argue it is a one to one and onto mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$. Now consider a matrix of this linear transformation and its reduced row-echelon form.

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7.7: Sums and Intersections

Learning Objectives

- Show that the sum of two subspaces is a subspace.
- Show that the intersection of two subspaces is a subspace.

We begin this section with a definition.

Definition 7.7.1: Sum and Intersection

Let V be a vector space, and let U and W be subspaces of V. Then

1. $U+W=\{\vec{u}+\vec{w} \mid \vec{u} \in U \text{ and } \vec{w} \in W\}$ and is called the **sum** of U and W.
2. $U \cap W=\{\vec{v} \mid \vec{v} \in U \text{ and } \vec{v} \in W\}$ and is called the **intersection** of U and W.

Therefore the intersection of two subspaces is all the vectors shared by both. If there are no vectors shared by both subspaces, meaning that $U \cap W=\left\{\vec{0}\right\}$, the sum $U+W$ takes on a special name.

Definition 7.7.2: Direct Sum

Let V be a vector space and suppose U and W are subspaces of V such that $U \cap W=\left\{\vec{0}\right\}$. Then the sum of U and W is called the **direct sum** and is denoted $U \oplus W$.

An interesting result is that both the sum $U+W$ and the intersection $U \cap W$ are subspaces of V.

Example 7.7.1: Intersection is a Subspace

Let V be a vector space and suppose U and W are subspaces. Then the intersection $U \cap W$ is a subspace of V.

Solution

By the subspace test, we must show three things:

1. $\vec{0} \in U \cap W$
2. For vectors $\vec{v}_1, \vec{v}_2 \in U \cap W, \vec{v}_1+\vec{v}_2 \in U \cap W$
3. For scalar a and vector $\vec{v} \in U \cap W, a\vec{v} \in U \cap W$

We proceed to show each of these three conditions hold.

1. Since U and W are subspaces of V, they each contain $\vec{0}$. By definition of the intersection, $\vec{0} \in U \cap W$.
2. Let $\vec{v}_1, \vec{v}_2 \in U \cap W$,. Then in particular, $\vec{v}_1, \vec{v}_2 \in U$. Since U is a subspace, it follows that $\vec{v}_1+\vec{v}_2 \in U$. The same argument holds for W. Therefore $\vec{v}_1+\vec{v}_2$ is in both U and W and by definition is also in $U \cap W$.
3. Let a be a scalar and $\vec{v} \in U \cap W$. Then in particular, $\vec{v} \in U$. Since U is a subspace, it follows that $a\vec{v} \in U$. The same argument holds for W so $a\vec{v}$ is in both U and W. By definition, it is in $U \cap W$.

Therefore $U \cap W$ is a subspace of V.

It can also be shown that $U+W$ is a subspace of V.

We conclude this section with an important theorem on dimension.

Theorem 7.7.1: Dimension of Sum

Let V be a vector space with subspaces U and W. Suppose U and W each have finite dimension. Then $U+W$ also has finite dimension which is given by

$$\dim(U+W)=\dim(U)+\dim(W)-\dim(U\cap W)$$

Notice that when $U\cap W=\left\{\vec{0}\right\}$, the sum becomes the direct sum and the above equation becomes

$$\dim(U\oplus W)=\dim(U)+\dim(W)$$

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7.7E: Exercises for Section 7.7

Exercise 7.7E.1

Let U and W be subspaces of $\mathbb{R}^3$, where

$$U = \text{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}, \quad W = \text{span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}\right\}.$$

a. Describe $U + W$ geometrically. Find a basis for $U + W$ and state its dimension.
b. Describe $U \cap W$ geometrically. Find a basis for $U \cap W$ and state its dimension.

Answer available at the end of this section.

Exercise 7.7E.2

Let

$$U = \left\{\begin{bmatrix}x\\y\\z\end{bmatrix} \in \mathbb{R}^3 \mid x + y + z = 0\right\}, \quad W = \left\{\begin{bmatrix}x\\y\\z\end{bmatrix} \in \mathbb{R}^3 \mid x = y\right\}.$$

a. Describe $U + W$ geometrically. Find a basis for $U + W$ and state its dimension.
b. Describe $U \cap W$ geometrically. Find a basis for $U \cap W$ and state its dimension.

Answer available at the end of this section.

Exercise 7.7E.3

Let

$$U = \text{span}\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\4\\6\end{bmatrix}\right\}, \quad W = \text{span}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}.$$

a. Find a basis for $U + W$ and its dimension.
b. Find a basis for $U \cap W$ and its dimension.

Exercise 7.7E.4

Let

$$U = \left\{\begin{bmatrix}x\\y\\z\\w\end{bmatrix} \in \mathbb{R}^4 \mid x + y = 0,\ z - w = 0\right\}, \quad W = \left\{\begin{bmatrix}x\\y\\z\\w\end{bmatrix} \in \mathbb{R}^4 \mid x - z = 0,\ y + w = 0\right\}. \tag{7.7E.1}$$

Find bases and dimensions for U, W, $U + W$, and $U \cap W$

Exercise $7.7E.5$

Let U, W be subspaces of a vector space V and consider $U+W$ defined as the set of all $u+w$ where $u \in U$ and $w \in W$. Show that $U+W$ is a subspace of V.

Exercise $7.7E.6$

Let M, N be subspaces of a vector space $\mathbb{R}^2$. Then $N \cup M$ consists of all vectors which are in either M or N. Show that $N \cup M$ is not necessarily a subspace of $\mathbb{R}^2$ by giving an example where $N \cup M$ fails to be a subspace.

Exercise $7.7E.7$

Suppose V and W both have dimension equal to 7 and they are subspaces of $\mathbb{R}^{10}$. What are the possibilities for the dimension of $V \cap W$?

Exercise $7.7E.8$

Suppose V has dimension p and W has dimension q and they are each contained in a subspace, U which has dimension equal to n where $n > \max(p, q)$. What are the possibilities for the dimension of $V \cap W$?

Answer available at the end of this section.

Exercise $7.7E.9$

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Show that

$$\dim(\ker(AB)) \leq \dim(\ker(A)) + \dim(\ker(B)).$$

Consider the subspace, $B(\mathbb{R}^p) \cap \ker(A)$ and suppose a basis for this subspace is $\{\vec{w}_1, \cdots, \vec{w}_k\}$. Now suppose $\{\vec{u}_1, \cdots, \vec{u}_r\}$ is a basis for $\ker(B)$. Let $\{\vec{z}_1, \cdots, \vec{z}_k\}$ be such that $B\vec{z}_1 = \vec{w}_i$ and argue that

$$\ker(AB) \subseteq span\{\vec{u}_1, \cdots, \vec{u}_r, \vec{z}_1, \cdots, \vec{z}_k\}.$$

Answer available at the end of this section.

Answers

Exercise $7.7E.1$

a. Since $W \subseteq U$, $U+W=U$, with basis $\operatorname{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$. Hence, $\dim(U+W)=2$.

b. Since $W \subseteq U$, $U \cap W = W$, with basis $\operatorname{span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$. Hence, $\dim(U \cap W)=1$.

Exercise $7.7E.2$

a. The sum $U+W$ is the entire space $\mathbb{R}^3$. Dimension: $\dim(U+W)=3$. Basis for $U+W$: $\left\{\begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$.

b. The intersection $U \cap W$ is a line in $\mathbb{R}^3$ through the origin. Dimension: $\dim(U \cap W)=1$. Basis for $U \cap W$: $\left\{\begin{bmatrix}1\\1\\-2\end{bmatrix}\right\}$.

Exercise $7.7E.8$

Let $\{x_1, \cdots, x_k\}$ be a basis for $V \cap W$. Then there is a basis for V and W which are respectively

$$\{x_1, \cdots, x_k, y_{k+1}, \cdots, y_p\},\ \{x_1, \cdots, x_k, z_{k+1}, \cdots z_q\}$$

It follows that you must have $k+p-k+q-k \leq n$ and so you must have

$$p+q-n \leq k$$

Exercise $7.7E.9$

Here is how you do this. Suppose $AB\vec{x}=\vec{0}$. Then $B\vec{x} \in \ker(A) \cap B(\mathbb{R}^p)$ and so $B\vec{x} = \sum_{i=1}^{k} B\vec{z}_i$ showing that

$$\vec{x} - \sum_{i=1}^{k} \vec{z}_i \in \ker(B)$$

Consider $B(\mathbb{R}^p) \cap \ker(A)$ and let a basis be $\{\vec{w}_1, \cdots, \vec{w}_k\}$. Then each $\vec{w}_i$ is of the form $B\vec{z}_i = \vec{w}_i$. Therefore, $\{\vec{z}_1, \cdots, \vec{z}_k\}$ is linearly independent and $AB\vec{z}_i = 0$. Now let $\{\vec{u}_1, \cdots, \vec{u}_r\}$ be a basis for $\ker(B)$. If $AB\vec{x}=\vec{0}$, then $B\vec{x} \in \ker(A) \cap B(\mathbb{R}^p)$ and so $B\vec{x} = \sum_{i=1}^{k} c_i B\vec{z}_1$ which implies

$$\vec{x} - \sum_{i=1}^{k} c_i \vec{z}_i \in \ker(B)$$

and so it is of the form

$$\vec{x} - \sum_{i=1}^{k} c_i \vec{z}_i = \sum_{j=1}^{r} d_j \vec{u}_j$$

It follows that if $AB\vec{x}=\vec{0}$ so that $\vec{x} \in \ker(AB)$, then

$$\vec{x} \in span(\vec{z}_1, \cdots, \vec{z}_k, \vec{u}_1, \cdots, \vec{u}_r).$$

Therefore,

$$\begin{array}{l} \dim(\ker(AB)) \leq k + r = \dim(B(\mathbb{R}^p) \cap \ker(A)) + \dim(\ker(B)) \\ \leq \dim(\ker(A)) + \dim(\ker(B)) \end{array} \tag{7.7E.2}$$

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7.8: Linear Transformations

Learning Objectives

- Understand the definition of a linear transformation in the context of vector spaces.

Recall that a function is simply a transformation of a vector to result in a new vector. Consider the following definition.

Definition 7.8.1: Linear Transformation

Let V and W be vector spaces. Suppose $T : V \mapsto W$ is a function, where for each $\vec{x} \in V, T(\vec{x}) \in W$. Then T is a **linear transformation** if whenever k, p are scalars and $\vec{v}_1$ and $\vec{v}_2$ are vectors in V

$$T(k\vec{v}_1 + p\vec{v}_2) = kT(\vec{v}_1) + pT(\vec{v}_2)$$

Several important examples of linear transformations include the zero transformation, the identity transformation, and the scalar transformation.

Example 7.8.1: Linear Transformations

Let V and W be vector spaces.

1. **The zero transformation**
   $0 : V \to W$ is defined by $0(\vec{v}) = \vec{0}$ for all $\vec{v} \in V$.
2. **The identity transformation**
   $1_V : V \to V$ is defined by $1_V(\vec{v}) = \vec{v}$ for all $\vec{v} \in V$.
3. **The scalar transformation** Let $a \in \mathbb{R}$.
   $s_a : V \to V$ is defined by $s_a(\vec{v}) = a\vec{v}$ for all $\vec{v} \in V$.

Solution

We will show that the scalar transformation s_a is linear, the rest are left as an exercise.

By Definition 7.8.1 we must show that for all scalars k, p and vectors $\vec{v}_1$ and $\vec{v}_2$ in V, $s_a(k\vec{v}_1 + p\vec{v}_2) = ks_a(\vec{v}_1) + ps_a(\vec{v}_2)$. Assume that a is also a scalar.

$$\begin{aligned} s_a(k\vec{v}_1 + p\vec{v}_2) &= a(k\vec{v}_1 + p\vec{v}_2) \\ &= ak\vec{v}_1 + ap\vec{v}_2 \\ &= k(a\vec{v}_1) + p(a\vec{v}_2) \\ &= ks_a(\vec{v}_1) + ps_a(\vec{v}_2) \end{aligned} \tag{7.8.1}$$

Therefore s_a is a linear transformation.

Theorem 7.8.1: Properties of Linear Transformations

Let V and W be vector spaces, and $T : V \mapsto W$ a linear transformation. Then

1. T preserves the zero vector.
   $$T(\vec{0}) = \vec{0}$$
2. T preserves additive inverses. For all $\vec{v} \in V$,

$$T(-\vec{v}) = -T(\vec{v})$$

3. T preserves linear combinations. For all $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m \in V$ and all $k_1, k_2, \ldots, k_m \in \mathbb{R}$,
$$T(k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_m\vec{v}_m) = k_1T(\vec{v}_1) + k_2T(\vec{v}_2) + \cdots + k_mT(\vec{v}_m).$$

Proof available at the end of this section.

Example 7.8.2: Linear Combination

Let $T : \mathbb{P}_2 \to \mathbb{R}$ be a linear transformation such that

$$T(x^2 + x) = -1; T(x^2 - x) = 1; T(x^2 + 1) = 3.$$

Find $T(4x^2 + 5x - 3)$.

We provide two solutions to this problem.

Solution 1:

Suppose $a(x^2 + x) + b(x^2 - x) + c(x^2 + 1) = 4x^2 + 5x - 3$. Then

$$(a + b + c)x^2 + (a - b)x + c = 4x^2 + 5x - 3.$$

Solving for a, b, and c results in the unique solution $a = 6$, $b = 1$, $c = -3$. Thus

$$\begin{aligned} T(4x^2 + 5x - 3) &= T(6(x^2 + x) + (x^2 - x) - 3(x^2 + 1)) \\ &= 6T(x^2 + x) + T(x^2 - x) - 3T(x^2 + 1) \\ &= 6(-1) + 1 - 3(3) = -14. \end{aligned} \tag{7.8.2}$$

Solution 2:

Notice that $S = \{x^2 + x, x^2 - x, x^2 + 1\}$ is a basis of $\mathbb{P}_2$, and thus x^2, x, and 1 can each be written as a linear combination of elements of S.

$$\begin{aligned} x^2 &= \tfrac{1}{2}(x^2 + x) + \tfrac{1}{2}(x^2 - x) \\ x &= \tfrac{1}{2}(x^2 + x) - \tfrac{1}{2}(x^2 - x) \\ 1 &= (x^2 + 1) - \tfrac{1}{2}(x^2 + x) - \tfrac{1}{2}(x^2 - x). \end{aligned} \tag{7.8.3}$$

Then

$$\begin{aligned} T(x^2) &= T\left(\tfrac{1}{2}(x^2 + x) + \tfrac{1}{2}(x^2 - x)\right) = \tfrac{1}{2}T(x^2 + x) + \tfrac{1}{2}T(x^2 - x) \\ &= \tfrac{1}{2}(-1) + \tfrac{1}{2}(1) = 0. \\ T(x) &= T\left(\tfrac{1}{2}(x^2 + x) - \tfrac{1}{2}(x^2 - x)\right) = \tfrac{1}{2}T(x^2 + x) - \tfrac{1}{2}T(x^2 - x) \\ &= \tfrac{1}{2}(-1) - \tfrac{1}{2}(1) = -1. \\ T(1) &= T\left((x^2 + 1) - \tfrac{1}{2}(x^2 + x) - \tfrac{1}{2}(x^2 - x)\right) \\ &= T(x^2 + 1) - \tfrac{1}{2}T(x^2 + x) - \tfrac{1}{2}T(x^2 - x) \\ &= 3 - \tfrac{1}{2}(-1) - \tfrac{1}{2}(1) = 3. \end{aligned} \tag{7.8.4}$$

Therefore,

$$\begin{aligned} T(4x^2 + 5x - 3) &= 4T(x^2) + 5T(x) - 3T(1) \\ &= 4(0) + 5(-1) - 3(3) = -14. \end{aligned} \tag{7.8.5}$$

The advantage of **Solution 2** over **Solution 1** is that if you were now asked to find $T(-6x^2 - 13x + 9)$, it is easy to use $T(x^2) = 0$, $T(x) = -1$ and $T(1) = 3$:

$$\begin{aligned} T(-6x^2 - 13x + 9) &= -6T(x^2) - 13T(x) + 9T(1) \\ &= -6(0) - 13(-1) + 9(3) = 13 + 27 = 40. \end{aligned} \tag{7.8.6}$$

More generally,

$$\begin{aligned} T(ax^2+bx+c) &= aT(x^2)+bT(x)+cT(1) \\ &= a(0)+b(-1)+c(3) = -b+3c. \end{aligned} \tag{7.8.7}$$

Suppose two linear transformations act in the same way on $\vec{v}$ for all vectors. Then we say that these transformations are equal.

Definition 7.8.2: Equal Transformations

Let S and T be linear transformations from V to W. Then S and T are **equal**, written $S = T$, if and only if for every $\vec{v} \in V$,

$$S(\vec{v}) = T(\vec{v})$$

The definition 7.8.2 requires that two transformations have the same action on every vector in order for them to be equal. The next theorem argues that it is only necessary to check the action of the transformations on basis vectors.

Theorem 7.8.2: Transformation of a Spanning Set

Let V and W be vector spaces and suppose that S and T are linear transformations from V to W. Then in order for S and T to be equal, it suffices that $S(\vec{v}_i) = T(\vec{v}_i)$ where $V = \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$.

Theorem 7.8.2 tells us that a linear transformation is completely determined by its actions on a spanning set. We can also examine the effect of a linear transformation on a basis.

Theorem 7.8.3: Transformation of a Basis

Suppose V and W are vector spaces and let $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n\}$ be any given vectors in W that may not be distinct. Then there exists a basis $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ of V and a unique linear transformation $T : V \mapsto W$ with $T(\vec{v}_i) = \vec{w}_i$.

Furthermore, if

$$\vec{v} = k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_n\vec{v}_n$$

is a vector of V, then

$$T(\vec{v}) = k_1\vec{w}_1 + k_2\vec{w}_2 + \cdots + k_n\vec{w}_n.$$

Answers

Theorem 7.8.1: Properties of Linear Transformations

1. Let $\vec{0}_V$ denote the zero vector of V and let $\vec{0}_W$ denote the zero vector of W. We want to prove that $T(\vec{0}_V) = \vec{0}_W$. Let $\vec{v} \in V$. Then $0\vec{v} = \vec{0}_V$ and
$$T(\vec{0}_V) = T(0\vec{v}) = 0T(\vec{v}) = \vec{0}_W.$$
2. Let $\vec{v} \in V$; then $-\vec{v} \in V$ is the additive inverse of $\vec{v}$, so $\vec{v} + (-\vec{v}) = \vec{0}_V$. Thus
$$\begin{aligned} T(\vec{v} + (-\vec{v})) &= T(\vec{0}_V) \\ T(\vec{v}) + T(-\vec{v})) &= \vec{0}_W \\ T(-\vec{v}) &= \vec{0}_W - T(\vec{v}) = -T(\vec{v}). \end{aligned} \tag{7.8.8}$$
3. This result follows from preservation of addition and preservation of scalar multiplication. A formal proof would be by induction on m.

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7.8E: Exercises for Section 7.8

Exercise 7.8E.1

Let T: $\mathbb{P}_2 \to \mathbb{R}$ be a linear transformation such that

$$T(x^2) = 1;\ T(x^2 + x) = 5;\ T(x^2 + x + 1) = -1.$$

Find $T(ax^2 + bx + c)$.

Answer available at the end of this section.

Exercise 7.8E.2

Let $T : \mathbb{P}_2 \to \mathbb{R}$ be a linear transformation such that

$$T(x^2) = 3; \quad T(x^2 + 2x) = 7; \quad T(x^2 + 2x + 4) = 1.$$

Find $T(ax^2 + bx + c)$.

Exercise 7.8E.3

Let $T : \mathbb{P}_3 \to \mathbb{R}^2$ be a linear transformation such that

$$T(x^3) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad T(x^3 + x^2) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \quad T(x^3 + x^2 + x) = \begin{bmatrix} 4 \\ 0 \end{bmatrix}, \quad T(x^3 + x^2 + x + 1) = \begin{bmatrix} 5 \\ 5 \end{bmatrix}.$$

Find $T(ax^3 + bx^2 + cx + d)$.

Answer available at the end of this section.

Exercise 7.8E.4

Let $T : \mathbb{P}_3 \to \mathbb{R}$ be a linear transformation such that

$$T(x^3) = 2, \quad T(x^3 + x^2) = 5, \quad T(x^3 + x^2 + x) = 4, \quad T(x^3 + x^2 + x + 1) = 6.$$

Find $T(ax^3 + bx^2 + cx + d)$.

Exercise 7.8E.5

Let $T : \mathbb{P}_1 \to \mathbb{R}^2$ be a linear transformation such that

$$T(x) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \quad T(x + 1) = \begin{bmatrix} 5 \\ 2 \end{bmatrix}.$$

Find $T(ax + b)$.

Exercise $7.8E.6$

Consider the following functions $T : \mathbb{P}_2 \to \mathbb{R}^3$. Explain why each of these functions T is not linear.

a. $T(ax^2+bx+c) = \begin{bmatrix} a+2b+c+1 \\ b-c \\ 3a \end{bmatrix}$

b. $T(ax^2+bx+c) = \begin{bmatrix} a^2+b+c \\ b-c \\ 3a \end{bmatrix}$

c. $T(ax^2+bx+c) = \begin{bmatrix} \cos(a)+b+c \\ b-c \\ 3a \end{bmatrix}$

d. $T(ax^2+bx+c) = \begin{bmatrix} a+b+c \\ b-c \\ \sqrt{a}+3a \end{bmatrix}$

Answer available at the end of this section.

Exercise $7.8E.7$

Let V and W be vector spaces and let T: $V \to W$ be a linear transformation. Suppose that $\{T\vec{v}_1, \cdots, T\vec{v}_r\}$ is linearly independent. Show that it must be the case that $\{\vec{v}_1, \cdots, \vec{v}_r\}$ is also linearly independent.

Answer available at the end of this section.

Answers

Exercise $7.8E.1$

By linearity we have $T(x^2) = 1$, $T(x) = T(x^2+x-x^2) = T(x^2+x) - T(x^2) = 5-1 = 5$, and $T(1) = T(x^2+x+1-(x^2+x)) = T(x^2+x+1) - T(x^2+x)) = -1-5 = -6$. Thus $T(ax^2+bx+c) = aT(x^2) + bT(x) + cT(1) = a+5b-6c$.

Exercise $7.8E.3$

$$T(ax^3+bx^2+cx+d) = a\begin{bmatrix}1\\2\end{bmatrix} + b\begin{bmatrix}2\\-3\end{bmatrix} + c\begin{bmatrix}1\\1\end{bmatrix} + d\begin{bmatrix}1\\5\end{bmatrix} = \begin{bmatrix} a+2b+c+d \\ 2a-3b+c+5d \end{bmatrix}$$

Exercise $7.8E.6$

Exercise $7.8E.7$

If $\sum_i^r a_i\vec{v}_r = 0$, then using linearity properties of T we get

$$0 = T(0) = T\left(\sum_i^r a_i\vec{v}_r\right) = \sum_i^r a_i T(\vec{v}_r).$$

Since we assume that $\{T\vec{v}_1, \cdots, T\vec{v}_r\}$ is linearly independent, we must have all $a_i = 0$, and therefore we conclude that $\{\vec{v}_1, \cdots, \vec{v}_r\}$ is also linearly independent.

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7.9: Isomorphisms

Outcomes

- Apply the concepts of one to one and onto to transformations of vector spaces.
- Determine if a linear transformation of vector spaces is an isomorphism.
- Determine if two vector spaces are isomorphic.

Recall the following definitions, given here in terms of vector spaces.

Definition 7.9.1: One to One Transformation

Let V, W be vector spaces with $\vec{v}_1, \vec{v}_2$ vectors in V. Then a linear transformation $T : V \mapsto W$ is called **one to one** if whenever $\vec{v}_1 \neq \vec{v}_2$ it follows that

$$T(\vec{v}_1) \neq T(\vec{v}_2)$$

Definition 7.9.2: Onto Transformation

Let V, W be vector spaces. Then a linear transformation $T : V \mapsto W$ is called **onto** if for all $\vec{w} \in \vec{W}$ there exists $\vec{v} \in V$ such that $T(\vec{v}) = \vec{w}$.

Recall that every linear transformation T has the property that $T(\vec{0}) = \vec{0}$. This will be necessary to prove the following useful lemma.

Lemma 7.9.1: One to One

The assertion that a linear transformation T is one to one is equivalent to saying that if $T(\vec{v}) = \vec{0}$, then $\vec{v} = 0$.

Proof available at the end of this section.

Consider the following example.

Example 7.9.1: One to One Transformation

Let $S : \mathbb{P}_2 \to \mathbb{M}_{22}$ be a linear transformation defined by

$$S(ax^2 + bx + c) = \begin{bmatrix} a+b & a+c \\ b-c & b+c \end{bmatrix}$$

for all $ax^2 + bx + c \in \mathbb{P}_2$.

Prove that S is one to one but not onto.

Solution

By definition,

$$\ker(S) = \{ax^2 + bx + c \in \mathbb{P}_2 \mid a + b = 0, a + c = 0, b - c = 0, b + c = 0\}.$$

Suppose $p(x) = ax^2 + bx + c \in \ker(S)$. This leads to a homogeneous system of four equations in three variables. Putting the augmented matrix in reduced row-echelon form:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The solution is $a = b = c = 0$. This tells us that if $S(p(x)) = 0$, then $p(x) = ax^2 + bx + c = 0x^2 + 0x + 0 = 0$. Therefore it is one to one.

To show that S is **not** onto, find a matrix $A \in \mathbb{M}_{22}$ such that for every $p(x) \in \mathbb{P}_2$, $S(p(x)) \neq A$. Let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix},$$

and suppose $p(x) = ax^2 + bx + c \in \mathbb{P}_2$ is such that $S(p(x)) = A$. Then

$$\begin{aligned} a + b &= 0 \quad a + c = 1 \\ b - c &= 0 \quad b + c = 2 \end{aligned}$$

Solving this system

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 2 \end{bmatrix}.$$

Since the system is inconsistent, there is no $p(x) \in \mathbb{P}_2$ so that $S(p(x)) = A$, and therefore S is not onto.

Example 7.9.2: An Onto Transformation

Let $T : \mathbb{M}_{22} \to \mathbb{R}^2$ be a linear transformation defined by

$$T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a + d \\ b + c \end{bmatrix} \text{ for all } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{M}_{22}.$$

Prove that T is onto but not one to one.

Solution

Let $\begin{bmatrix} x \\ y \end{bmatrix}$ be an arbitrary vector in $\mathbb{R}^2$. Since $T \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$, T is onto.

By Lemma 7.9.1 T is one to one if and only if $T(A) = \vec{0}$ implies that $A = 0$ the zero matrix. Observe that

$$T\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) = \begin{bmatrix} 1 + -1 \\ 0 + 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

There exists a nonzero matrix A such that $T(A) = \vec{0}$. It follows that T is not one to one.

The following example demonstrates that a one to one transformation preserves linear independence.

Example 7.9.3: One to One and Independence

Let V and W be vector spaces and $T : V \mapsto W$ a linear transformation. Prove that if T is one to one and $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is an independent subset of V, then $\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_k)\}$ is an independent subset of W.

Solution

Let $\vec{0}_V$ and $\vec{0}_W$ denote the zero vectors of V and W, respectively. Suppose that

$$a_1T(\vec{v}_1) + a_2T(\vec{v}_2) + \cdots + a_kT(\vec{v}_k) = \vec{0}_W$$

for some $a_1, a_2, \ldots, a_k \in \mathbb{R}$. Since linear transformations preserve linear combinations (addition and scalar multiplication),

$$T(a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k) = \vec{0}_W.$$

Now, since T is one to one, $\ker(T) = \{\vec{0}_V\}$, and thus

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k = \vec{0}_V.$$

However, $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is independent so $a_1 = a_2 = \cdots = a_k = 0$. Therefore, $\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_k)\}$ is independent.

A similar claim can be made regarding onto transformations. In this case, an onto transformation preserves a spanning set.

Example 7.9.4: Onto and Spanning

Let V and W be vector spaces and $T : V \to W$ a linear transformation. Prove that if T is onto and $V = \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$, then

$$W = \text{span}\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_k)\}.$$

Solution

Suppose that T is onto and let $\vec{w} \in W$. Then there exists $\vec{v} \in V$ such that $T(\vec{v}) = \vec{w}$. Since $V = \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$, there exist $a_1, a_2, \ldots a_k \in \mathbb{R}$ such that $\vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k$. Using the fact that T is a linear transformation,

$$\begin{aligned} \vec{w} = T(\vec{v}) &= T(a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k) \\ &= a_1T(\vec{v}_1) + a_2T(\vec{v}_2) + \cdots + a_kT(\vec{v}_k), \end{aligned} \tag{7.9.1}$$

i.e., $\vec{w} \in \text{span}\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_k)\}$, and thus

$$W \subseteq \text{span}\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_k)\}.$$

Since $T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_k) \in W$, it follows from that $span\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_k)\} \subseteq W$, and therefore $W = \text{span}\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_k)\}$.

Definition 7.9.3: Isomorphism

Let V and W be two vector spaces and let $T : V \mapsto W$ be a linear transformation. Then T is called an **isomorphism** if the following two conditions are satisfied.

- T is one to one.
- T is onto.

Definition 7.9.4: Isomorphic

Let V and W be two vector spaces and let $T : V \mapsto W$ be a linear transformation. Then if T is an isomorphism, we say that V and W are **isomorphic.**

Consider the following example of an isomorphism.

Example 7.9.5: Isomorphism

Let $T : \mathbb{M}_{22} \to \mathbb{R}^4$ be defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \text{ for all } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{M}_{22}.$$

Show that T is an isomorphism.

Solution

Notice that if we can prove T is an isomorphism, it will mean that $\mathbb{M}_{22}$ and $\mathbb{R}^4$ are isomorphic. It remains to prove that

1. T is a linear transformation;
2. T is one-to-one;
3. T is onto.

T is linear: Let k, p be scalars.

$$\begin{aligned} T\left(k\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + p\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) &= T\left(\begin{bmatrix} ka_1 & kb_1 \\ kc_1 & kd_1 \end{bmatrix} + \begin{bmatrix} pa_2 & pb_2 \\ pc_2 & pd_2 \end{bmatrix}\right) \\ &= T\left(\begin{bmatrix} ka_1 + pa_2 & kb_1 + pb_2 \\ kc_1 + pc_2 & kd_1 + pd_2 \end{bmatrix}\right) \\ &= \begin{bmatrix} ka_1 + pa_2 \\ kb_1 + pb_2 \\ kc_1 + pc_2 \\ kd_1 + pd_2 \end{bmatrix} \\ &= \begin{bmatrix} ka_1 \\ kb_1 \\ kc_1 \\ kd_1 \end{bmatrix} + \begin{bmatrix} pa_2 \\ pb_2 \\ pc_2 \\ pd_2 \end{bmatrix} \\ &= k\begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{bmatrix} + p\begin{bmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix} \\ &= kT\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + pT\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) \end{aligned} \tag{7.9.2}$$

Therefore T is linear.

T is one-to-one: By Lemma 7.9.1 we need to show that if $T(A) = 0$ then $A = 0$ for some matrix $A \in \mathbb{M}_{22}$.

$$T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This clearly only occurs when $a = b = c = d = 0$ which means that

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Hence T is one-to-one.

T is onto: Let

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4,$$

and define matrix $A \in \mathbb{M}_{22}$ as follows:

$$A = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}.$$

Then $T(A) = \vec{x}$, and therefore T is onto.

Since T is a linear transformation which is one-to-one and onto, T is an isomorphism. Hence $\mathbb{M}_{22}$ and $\mathbb{R}^4$ are isomorphic.

An important property of isomorphisms is that the inverse of an isomorphism is itself an isomorphism and the composition of isomorphisms is an isomorphism. We first recall the definition of composition.

Definition 7.9.5: Composition of Transformations

Let V, W, Z be vector spaces and suppose $T : V \mapsto W$ and $S : W \mapsto Z$ are linear transformations. Then the composite of S and T is

$$S \circ T : V \mapsto Z$$

and is defined by

$$(S \circ T)(\vec{v}) = S(T(\vec{v})) \text{ for all } \vec{v} \in V$$

Consider now the following proposition.

Proposition 7.9.1: Composite and Inverse Isomorphism

Let $T : V \to W$ be an isomorphism. Then $T^{-1} : W \to V$ is also an isomorphism. Also if $T : V \to W$ is an isomorphism and if $S : W \to Z$ is an isomorphism for the vector spaces V, W, Z, then $S \circ T$ defined by $(S \circ T)(v) = S(T(v))$ is also an isomorphism.

Proof available at the end of this section.

Suppose we say that two vector spaces V and W are related if there exists an isomorphism of one to the other, written as $V \sim W$. Then the above proposition suggests that $\sim$ is an equivalence relation. That is: $\sim$ satisfies the following conditions:

- $V \sim V$
- If $V \sim W$, it follows that $W \sim V$
- If $V \sim W$ and $W \sim Z$, then $V \sim Z$

We leave the proof of these to the reader.

The following fundamental lemma describes the relation between bases and isomorphisms.

Lemma 7.9.2: Bases and Isomorphisms

Let $T : V \to W$ be a linear map where V, W are vector spaces. Then a linear transformation T which is one to one has the property that if $\{\vec{u}_1, \cdots, \vec{u}_k\}$ is linearly independent, then so is $\{T(\vec{u}_1), \cdots, T(\vec{u}_k)\}$. More generally, T is an isomorphism if and only if whenever $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a basis for V, it follows that $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ is a basis for W.

Proof available at the end of this section.

The following theorem illustrates a very useful idea for defining an isomorphism. Basically, if you know what it does to a basis, then you can construct the isomorphism.

Theorem 7.9.1: Isomorphic Vector Spaces

Suppose V and W are two vector spaces. Then the two vector spaces are isomorphic if and only if they have the same dimension. In the case that the two vector spaces have the same dimension, then for a linear transformation $T : V \to W$, the following are equivalent.

1. T is one to one.
2. T is onto.
3. T is an isomorphism.

Proof available at the end of this section.

Note the interesting way of defining a linear transformation in the first part of the argument by describing what it does to a basis and then "extending it linearly".

Consider the following example.

Example 7.9.6

Let $V = \mathbb{R}^3$ and let W denote the polynomials of degree at most 2. Show that these two vector spaces are isomorphic.

Solution

First, observe that a basis for W is $\{1, x, x^2\}$ and a basis for V is $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$. Since these two have the same dimension, the two are isomorphic. An example of an isomorphism is this:

$$T(\vec{e}_1) = 1, T(\vec{e}_2) = x, T(\vec{e}_3) = x^2$$

and extend T linearly as in the above proof. Thus

$$T(a, b, c) = a + bx + cx^2$$

Answers

Lemma 7.9.1: One to One

Suppose first that T is one to one.

$$T(\vec{0}) = T\left(\vec{0} + \vec{0}\right) = T(\vec{0}) + T(\vec{0})$$

and so, adding the additive inverse of $T(\vec{0})$ to both sides, one sees that $T(\vec{0}) = \vec{0}$. Therefore, if $T(\vec{v}) = \vec{0}$, it must be the case that $\vec{v} = \vec{0}$ because it was just shown that $T(\vec{0}) = \vec{0}$.

Now suppose that if $T(\vec{v}) = \vec{0}$, then $\vec{v} = 0$. If $T(\vec{v}) = T(\vec{u})$, then $T(\vec{v}) - T(\vec{u}) = T(\vec{v} - \vec{u}) = \vec{0}$ which shows that $\vec{v} - \vec{u} = 0$ or in other words, $\vec{v} = \vec{u}$.

Proposition 7.9.1: Composite and Inverse Isomorphism

Consider the first claim. Since T is onto, a typical vector in W is of the form $T(\vec{v})$ where $\vec{v} \in V$. Consider then for a, b scalars,

$$T^{-1}(aT(\vec{v}_1) + bT(\vec{v}_2))$$

where $\vec{v}_1, \vec{v}_2 \in V$. Consider if this is equal to

$$aT^{-1}(T(\vec{v}_1)) + bT^{-1}(T(\vec{v}_2)) = a\vec{v}_1 + b\vec{v}_2?$$

Since T is one to one, this will be so if

$$T(a\vec{v}_1 + b\vec{v}_2) = T(T^{-1}(aT(\vec{v}_1) + bT(\vec{v}_2))) = aT(\vec{v}_1) + bT(\vec{v}_2)$$

However, the above statement is just the condition that T is a linear map. Thus T^{-1} is indeed a linear map. If $\vec{v} \in V$ is given, then $\vec{v} = T^{-1}(T(\vec{v}))$ and so T^{-1} is onto. If $T^{-1}(\vec{v}) = \vec{0}$, then

$$\vec{v} = T(T^{-1}(\vec{v})) = T(\vec{0}) = \vec{0}$$

and so T^{-1} is one to one.

Next suppose T and S are as described. Why is $S \circ T$ a linear map? Let for a, b scalars,

$$\begin{aligned} S \circ T\left(a\vec{v}_1 + b\vec{v}_2\right) &\equiv S\left(T\left(a\vec{v}_1 + b\vec{v}_2\right)\right) = S\left(aT(\vec{v}_1) + bT(\vec{v}_2)\right) \\ &= aS\left(T(\vec{v}_1)\right) + bS\left(T(\vec{v}_2)\right) \equiv a\left(S \circ T\right)(\vec{v}_1) + b\left(S \circ T\right)(\vec{v}_2) \end{aligned} \tag{7.9.3}$$

Hence $S \circ T$ is a linear map. If $(S \circ T)(\vec{v}) = 0$, then $S(T(\vec{v})) = \vec{0}$ and it follows that $T(\vec{v}) = \vec{0}$ and hence by this lemma again, $\vec{v} = \vec{0}$. Thus $S \circ T$ is one to one. It remains to verify that it is onto. Let $\vec{z} \in Z$. Then since S is onto, there exists $\vec{w} \in W$ such that $S(\vec{w}) = \vec{z}$. Also, since T is onto, there exists $\vec{v} \in V$ such that $T(\vec{v}) = \vec{w}$. It follows that $S(T(\vec{v})) = \vec{z}$ and so $S \circ T$ is also onto.

Lemma 7.9.2**: Bases and Isomorphisms**

First suppose that T is a linear map and is one to one and $\{\vec{u}_1, \cdots, \vec{u}_k\}$ is linearly independent. It is required to show that $\{T(\vec{u}_1), \cdots, T(\vec{u}_k)\}$ is also linearly independent. Suppose then that

$$\sum_{i=1}^{k} c_i T(\vec{u}_i) = \vec{0}$$

Then, since T is linear,

$$T\left(\sum_{i=1}^{n} c_i \vec{u}_i\right) = \vec{0}$$

Since T is one to one, it follows that

$$\sum_{i=1}^{n} c_i \vec{u}_i = 0$$

Now the fact that $\{\vec{u}_1, \cdots, \vec{u}_n\}$ is linearly independent implies that each $c_i = 0$. Hence $\{T(\vec{u}_1), \cdots, T(\vec{u}_n)\}$ is linearly independent.

Now suppose that T is an isomorphism and $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a basis for V. It was just shown that $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ is linearly independent. It remains to verify that the span of $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ is all of W. This is where T is onto is used. If $\vec{w} \in W$, there exists $\vec{v} \in V$ such that $T(\vec{v}) = \vec{w}$. Since $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a basis, it follows that there exists scalars $\{c_i\}_{i=1}^{n}$ such that

$$\sum_{i=1}^{n} c_i \vec{v}_i = \vec{v}.$$

Hence,

$$\vec{w} = T(\vec{v}) = T\left(\sum_{i=1}^{n} c_i \vec{v}_i\right) = \sum_{i=1}^{n} c_i T\vec{v}_i$$

which shows that the span of these vectors $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ is all of W showing that this set of vectors is a basis for W.

Next suppose that T is a linear map which takes a basis to a basis. Then for $\{\vec{v}_1, \cdots, \vec{v}_n\}$ a basis for V, it follows $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ is a basis for W. Then if $w \in W$, there exist scalars c_i such that $w = \sum_{i=1}^{n} c_i T(\vec{v}_i) = T\left(\sum_{i=1}^{n} c_i \vec{v}_i\right)$ showing that T is onto. If $T\left(\sum_{i=1}^{n} c_i \vec{v}_i\right) = 0$ then $\sum_{i=1}^{n} c_i T(\vec{v}_i) = \vec{0}$ and since the vectors $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ are linearly independent, it follows that each $c_i = 0$. Since $\sum_{i=1}^{n} c_i \vec{v}_i$ is a typical vector in V, this has shown that if $T(\vec{v}) = 0$ then $\vec{v} = \vec{0}$ and so T is also one to one. Thus T is an isomorphism.

Theorem 7.9.1**: Isomorphic Vector Spaces**

Suppose first these two vector spaces have the same dimension. Let a basis for V be $\{\vec{v}_1, \cdots, \vec{v}_n\}$ and let a basis for W be $\{\vec{w}_1, \cdots, \vec{w}_n\}$. Now define T as follows.

$$T(\vec{v}_i) = \vec{w}_i$$

for $\sum_{i=1}^{n} c_i \vec{v}_i$ an arbitrary vector of V,

$$T\left(\sum_{i=1}^{n} c_i \vec{v}_i\right) = \sum_{i=1}^{n} c_i T(\vec{v}_i) = \sum_{i=1}^{n} c_i \vec{w}_i.$$

It is necessary to verify that this is well defined. Suppose then that

$$\sum_{i=1}^{n} c_i \vec{v}_i = \sum_{i=1}^{n} \hat{c}_i \vec{v}_i$$

Then

$$\sum_{i=1}^{n} \left(c_i - \hat{c}_i\right) \vec{v}_i = 0$$

and since $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a basis, $c_i = \hat{c}_i$ for each i. Hence

$$\sum_{i=1}^{n} c_i \vec{w}_i = \sum_{i=1}^{n} \hat{c}_i \vec{w}_i$$

and so the mapping is well defined. Also if a, b are scalars,

$$\begin{aligned} T\left(a\sum_{i=1}^{n} c_i \vec{v}_i + b\sum_{i=1}^{n} \hat{c}_i \vec{v}_i\right) &= T\left(\sum_{i=1}^{n} \left(ac_i + b\hat{c}_i\right) \vec{v}_i\right) = \sum_{i=1}^{n} \left(ac_i + b\hat{c}_i\right) \vec{w}_i \\ &= a\sum_{i=1}^{n} c_i \vec{w}_i + b\sum_{i=1}^{n} \hat{c}_i \vec{w}_i \\ &= aT\left(\sum_{i=1}^{n} c_i \vec{v}_i\right) + bT\left(\sum_{i=1}^{n} \hat{c}_i \vec{v}_i\right) \end{aligned} \tag{7.9.4}$$

Thus T is a linear map.

Now if

$$T\left(\sum_{i=1}^{n} c_i \vec{v}_i\right) = \sum_{i=1}^{n} c_i \vec{w}_i = \vec{0},$$

then since the $\{\vec{w}_1, \cdots, \vec{w}_n\}$ are independent, each $c_i = 0$ and so $\sum_{i=1}^{n} c_i \vec{v}_i = \vec{0}$ also. Hence T is one to one. If $\sum_{i=1}^{n} c_i \vec{w}_i$ is a vector in W, then it equals

$$\sum_{i=1}^{n} c_i T\vec{v}_i = T\left(\sum_{i=1}^{n} c_i \vec{v}_i\right)$$

showing that T is also onto. Hence T is an isomorphism and so V and W are isomorphic.

Next suppose these two vector spaces are isomorphic. Let T be the name of the isomorphism. Then for $\{\vec{v}_1, \cdots, \vec{v}_n\}$ a basis for V, it follows that a basis for W is $\{T\vec{v}_1, \cdots, T\vec{v}_n\}$ showing that the two vector spaces have the same dimension.

Now suppose the two vector spaces have the same dimension.

First consider the claim that $1.) \Rightarrow 2.)$. If T is one to one, then if $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a basis for V, then $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ is linearly independent. If it is not a basis, then it must fail to span W. But then there would exist $\vec{w} \notin span\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ and it follows that $\{T(\vec{v}_1), \cdots, T(\vec{v}_n), \vec{w}\}$ would be linearly independent which is impossible because there exists a basis for W of n vectors. Hence

$$span\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\} = W$$

and so $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ is a basis. Hence, if $\vec{w} \in W$, there exist scalars c_i such that

$$\vec{w} = \sum_{i=1}^{n} c_i T(\vec{v}_i) = T\left(\sum_{i=1}^{n} c_i \vec{v}_i\right)$$

showing that T is onto. This shows that $1.) \Rightarrow 2.)$.

Next consider the claim that $2.) \Rightarrow 3.)$. Since $2.)$ holds, it follows that T is onto. It remains to verify that T is one to one. Since T is onto, there exists a basis of the form $\{T(\vec{v}_i), \cdots, T(\vec{v}_n)\}$. If $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is linearly independent, then this set of vectors must also be a basis for V because if not, there would exist $\vec{u} \notin span\{\vec{v}_1, \cdots, \vec{v}_n\}$ so $\{\vec{v}_1, \cdots, \vec{v}_n, \vec{u}\}$ would be a linearly independent set which is impossible because by assumption, there exists a basis which has n vectors. So why is $\{\vec{v}_1, \cdots, \vec{v}_n\}$ linearly independent? Suppose

$$\sum_{i=1}^{n} c_i \vec{v}_i = \vec{0}$$

Then

$$\sum_{i=1}^{n} c_i T\vec{v}_i = \vec{0}$$

Hence each $c_i = 0$ and so, as just discussed, $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a basis for V. Now it follows that a typical vector in V is of the form $\sum_{i=1}^{n} c_i \vec{v}_i$. If $T\left(\sum_{i=1}^{n} c_i \vec{v}_i\right) = \vec{0}$, it follows that

$$\sum_{i=1}^{n} c_i T(\vec{v}_i) = \vec{0}$$

and so, since $\{T(\vec{v}_i), \cdots, T(\vec{v}_n)\}$ is independent, it follows each $c_i = 0$ and hence $\sum_{i=1}^{n} c_i \vec{v}_i = \vec{0}$. Thus T is one to one as well as onto and so it is an isomorphism.

If T is an isomorphism, it is both one to one and onto by definition so $3.)$ implies both $1.)$ and $2.)$.

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7.9E: Exercises for Section 7.9

Exercise 7.9E.1

Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation given by

$T\vec{x} = A\vec{x}$
where A is a 3×3 matrix. Show that T is an isomorphism if and only if A is invertible.

Answer available at the end of this section.

Exercise 7.9E.2

Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation given by
$T\vec{x} = A\vec{x}$
where A is an $m \times n$ matrix. Show that T is never an isomorphism if $m \neq n$.

Hint available at the end of this section.

Exercise 7.9E.3

Let $T : \mathbb{R}^3 \to \mathbb{P}_2$ be a linear transformation defined by

$$T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = (a+b)x^2 + (b-c)x + (a-c)$$

for all $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3$.

a. Is T one to one? Prove your answer.
b. Is T onto? Prove your answer.

Hint available at the end of this section.

Exercise 7.9E.4

Let $T : \mathbb{M}_{22} \to \mathbb{R}^3$ be the linear transformation defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+b \\ b+c \\ c+d \end{bmatrix}$$

for all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{M}_{2\times 2}$.

a. Is T one to one? Prove your answer.
b. Is T onto? Prove your answer.

Hint available at the end of this section.

Answer available at the end of this section.

Exercise 7.9E.5

Let $T : \mathbb{P}_2 \to \mathbb{M}_{2\times 2}$ be the linear transformation defined by

$$T(ax^2 + bx + c) = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

for all $ax^2 + bx + c \in \mathbb{P}_2$.

a. Is T one to one? Prove your answer.
b. Is T onto? Prove your answer.

Answer available at the end of this section.

Exercise 7.9E.6

Let $T : \mathbb{M}_{22} \to \mathbb{P}_1$ be the linear transformation defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a + b)x + (c + d)$$

for all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{M}_{2\times 2}$.

a. Is T one to one? Prove your answer.
b. Is T onto? Prove your answer.

Exercise 7.9E.7

Let $T : \mathbb{P}_2 \to \mathbb{R}^3$ be defined by

$$T(ax^2 + bx + c) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

for all $ax^2 + bx + c \in \mathbb{P}_2$.

Prove that T is an isomorphism.

Hint available at the end of this section.

Exercise 7.9E.8

Let the following linear transformations be defined:

- $T : \mathbb{P}_2 \to \mathbb{R}^3$, given by

$$T(ax^2 + bx + c) = \begin{bmatrix} a - b \\ b + c \\ a + c \end{bmatrix}$$

- $S : \mathbb{R}^3 \to \mathbb{M}_{2\times 2}$, given by

$$S\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y & z \\ y-z & x \end{bmatrix}$$

- $R : \mathbb{M}_{22} \to \mathbb{R}^2$, given by

$$R\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+d \\ b-c \end{bmatrix}$$

a. Is T one to one? onto? isomorphism?
b. Is S one to one? onto? isomorphism?
c. Is R one to one? onto? isomorphism?
d. Find a formula for $S \circ T$.
e. Is $S \circ T$ one to one? onto? isomorphism?
f. Find a formula for $R \circ S$.
g. Is $R \circ S$ one to one? onto? isomorphism?
h. Find a formula for $R \circ S \circ T$.
i. Is $R \circ S \circ T$. one to one? onto? isomorphism?

Exercise $7.9E.9$

If $\{\vec{v}_1, \cdots, \vec{v}_r\}$ is linearly independent and T is a one to one linear transformation, show that $\{T\vec{v}_1, \cdots, T\vec{v}_r\}$ is also linearly independent. Give an example which shows that if T is only linear, it can happen hat, although $\{\vec{v}_1, \cdots, \vec{v}_r\}$ is linearly independent, $\{T\vec{v}_1, \cdots, T\vec{v}_r\}$ is not. In fact, show that it can happen that each of the $T\vec{v}_j$ equals 0

Exercise $7.9E.10$

Let V and W be subspaces of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively and let T: $V \to W$ be a linear transformation. Show that if T is onto W and if $\{\vec{v}_1, \cdots, \vec{v}_r\}$ is a basis for V, then $span\{T\vec{v}_1, \cdots, T\vec{v}_r\} = W$

Answers

Exercise $7.9E.1$

T is an isomorphism $\iff$ T is injective and surjective.
T is injective $\iff$ $T\vec{x} = \vec{0} \Rightarrow \vec{x} = \vec{0}$ $\iff$ $A\vec{x} = \vec{0} \Rightarrow \vec{x} = \vec{0}$ $\iff$ A is invertible.
T is surjective $\iff$ $\forall \vec{y} \in \mathbb{R}^3,\ \exists \vec{x} \in \mathbb{R}^3$ such that $T\vec{x} = \vec{y}$ $\iff$ $\forall \vec{y} \in \mathbb{R}^3,\ \exists \vec{x} \in \mathbb{R}^3$ such that $A\vec{x} = \vec{y}$ $\iff$ A is invertible.
Therefore, T is an isomorphism $\iff$ A is invertible.

Exercise $7.9E.2$

Show that:

- If $m > n$, then T cannot be onto, and
- If $m < n$, then T cannot be one-to-one

Exercise $7.9E.3$

a. Yes, T is one to one. To prove this, show that the $\ker(T) = \vec{0}$.
b. Yes, T is onto. To prove this, show that any polynomial in $\mathbb{P}_2$, i.e., any $px^2 + qx + r$ is in the image of T. This will give a consistent linear system (with a unique solution due to T being one to one).

Exercise $7.9E.4$

a. No, T is not one to one. To prove this, show that the $\ker(T)$ is more than just $\vec{0}$.
b. Yes, T is onto. To prove this, show that any vector in $\mathbb{R}^3$, i.e., any $\begin{bmatrix} p \\ q \\ r \end{bmatrix}$ is in the image of T. This will give a consistent linear system (with infinitely many solutions due to T being not one to one).

Exercise $7.9E.4$

Exercise $7.9E.5$

Exercise $7.9E.7$

Follow the method used in Example 7.9.5.

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7.10: The Kernel and Image of a Linear Map

Learning Objectives

- Describe the kernel and image of a linear transformation.
- Use the kernel and image to determine if a linear transformation is one to one or onto.

Here we consider the case where the linear map is not necessarily an isomorphism. First here is a definition of what is meant by the image and kernel of a linear transformation.

Definition 7.10.1: Kernel and Image

Let V and W be vector spaces and let $T : V \to W$ be a linear transformation. Then the **image** of T denoted as $\mathrm{im}\,(T)$ is defined to be the set

$$\{T(\vec{v}) : \vec{v} \in V\}$$

In words, it consists of all vectors in W that equal $T(\vec{v})$ for some $\vec{v} \in V$. The **kernel**, $\ker(T)$, consists of all $\vec{v} \in V$ such that $T(\vec{v}) = \vec{0}$. That is,

$$\ker(T) = \left\{\vec{v} \in V : T(\vec{v}) = \vec{0}\right\}$$

Then in fact, both $\mathrm{im}\,(T)$ and $\ker(T)$ are subspaces of W and V respectively.

Proposition 7.10.1: Kernel and Image as Subspaces

Let V, W be vector spaces and let $T : V \to W$ be a linear transformation. Then $\ker(T) \subseteq V$ and $\mathrm{im}\,(T) \subseteq W$. In fact, they are both subspaces.

Proof available at the end of this section.

Example 7.10.1: Kernel and Image of a Transformation

Let $T : \mathbb{P}_1 \to \mathbb{R}$ be the linear transformation defined by

$$T(p(x)) = p(1) \text{ for all } p(x) \in \mathbb{P}_1.$$

Find the kernel and image of T.

Solution

We will first find the kernel of T. It consists of all polynomials in $\mathbb{P}_1$ that have 1 for a root.

$$\begin{aligned} \ker(T) &= \{p(x) \in \mathbb{P}_1 \mid p(1) = 0\} \\ &= \{ax + b \mid a, b \in \mathbb{R} \text{ and } a + b = 0\} \\ &= \{ax - a \mid a \in \mathbb{R}\} \end{aligned} \tag{7.10.1}$$

Therefore a basis for $\ker(T)$ is

$$\{x - 1\}$$

Notice that this is a subspace of $\mathbb{P}_1$.

Now consider the image. It consists of all numbers which can be obtained by evaluating all polynomials in $\mathbb{P}_1$ at 1.

$$\begin{aligned} \mathrm{im}(T) &= \{p(1) \mid p(x) \in \mathbb{P}_1\} \\ &= \{a+b \mid ax+b \in \mathbb{P}_1\} \\ &= \{a+b \mid a,b \in \mathbb{R}\} \\ &= \mathbb{R} \end{aligned} \tag{7.10.2}$$

Therefore a basis for $\mathrm{im}(T)$ is

$$\{1\}$$

Notice that this is a subspace of $\mathbb{R}$, and in fact is the space $\mathbb{R}$ itself.

Example 7.10.2: Kernel and Image of a Linear Transformation

Let $T : \mathbb{M}_{22} \mapsto \mathbb{R}^2$ be defined by

$$T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a-b \\ c+d \end{bmatrix}$$

Then T is a linear transformation. Find a basis for $\ker(T)$ and $\mathrm{im}(T)$.

Solution

You can verify that T represents a linear transformation.

Now we want to find a way to describe all matrices A such that $T(A) = \vec{0}$, that is the matrices in $\ker(T)$. Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is such a matrix. Then

$$T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a-b \\ c+d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The values of a, b, c, d that make this true are given by solutions to the system

$$\begin{aligned} a-b &= 0 \\ c+d &= 0 \end{aligned} \tag{7.10.3}$$

The solution is $a = s, b = s, c = t, d = -t$ where s, t are scalars. We can describe $\ker(T)$ as follows.

$$\ker(T) = \left\{\begin{bmatrix} s & s \\ t & -t \end{bmatrix}\right\} = \mathrm{span}\left\{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}\right\}$$

It is clear that this set is linearly independent and therefore forms a basis for $\ker(T)$.

We now wish to find a basis for $\mathrm{im}(T)$. We can write the image of T as

$$\mathrm{im}(T) = \left\{\begin{bmatrix} a-b \\ c+d \end{bmatrix}\right\}$$

Notice that this can be written as

$$\mathrm{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$$

However this is clearly not linearly independent. By removing vectors from the set to create an independent set gives a basis of $\mathrm{im}(T)$.

$$\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$$

Notice that these vectors have the same span as the set above but are now linearly independent.

A major result is the relation between the dimension of the kernel and dimension of the image of a linear transformation. A special case was done earlier in the context of matrices. Recall that for an $m \times n$ matrix it was the case that the dimension of the kernel of A added to the rank of A equals n.

Theorem 7.10.1: Dimension of Kernel + Image

Let $T : V \to W$ be a linear transformation where V, W are vector spaces. Suppose the dimension of V is n. Then $n = \dim(\ker(T)) + \dim(\text{im}(T))$.

Proof available at the end of this section.

Definition 7.10.2: Rank of Linear Transformation

Let $T : V \to W$ be a linear transformation and suppose V, W are finite dimensional vector spaces. Then the **rank** of T denoted as $\text{rank}(T)$ is defined as the dimension of $\text{im}(T)$. The nullity of T is the dimension of $\ker(T)$. Thus the above theorem says that $\text{rank}(T) + (nullity(T)) = \dim(V)$.

Theorem 7.10.2: Subspace of Same Dimension

Let V be a vector space of dimension n and let W be a subspace. Then $W = V$ if and only if the dimension of W is also n.

Corollary 7.10.1: One to One and Onto Characterization

Let $T : V \to W$ be a linear map where the dimension of V is n and the dimension of W is m. Then T is one to one if and only if $\ker(T) = \left\{\vec{0}\right\}$ and T is onto if and only if $\text{rank}(T) = m$.

Proof available at the end of this section.

Example 7.10.3: One to One Transformation

Let $S : \mathbb{P}_2 \to \mathbb{M}_{22}$ be a linear transformation defined by

$$S(ax^2 + bx + c) = \begin{bmatrix} a+b & a+c \\ b-c & b+c \end{bmatrix} \text{ for all } ax^2 + bx + c \in \mathbb{P}_2.$$

Prove that S is one to one but not onto.

Solution

Here we will determine that S is one to one, but not onto, using the method provided in Corollary 7.10.1.

By definition,

$$\ker(S) = \{ax^2 + bx + c \in \mathbb{P}_2 \mid a + b = 0, a + c = 0, b - c = 0, b + c = 0\}.$$

Suppose $p(x) = ax^2 + bx + c \in \ker(S)$. This leads to a homogeneous system of four equations in three variables. Putting the augmented matrix in reduced row-echelon form:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the unique solution is $a = b = c = 0$, $\ker(S) = \{\vec{0}\}$, and thus S is one-to-one by Corollary 7.10.1.

Similarly, by Corollary 7.10.1, if S is onto it will have $\text{rank}(S) = \dim(\mathbb{M}_{22}) = 4$. The image of S is given by

$$\mathrm{im}(S) = \left\{ \begin{bmatrix} a+b & a+c \\ b-c & b+c \end{bmatrix} \right\} = \mathrm{span} \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \right\}$$

These matrices are linearly independent which means this set forms a basis for $\mathrm{im}(S)$. Therefore the dimension of $\mathrm{im}(S)$, also called $\mathrm{rank}(S)$, is equal to 3. It follows that S is not onto.

Answers

Proposition $7.10.1$**: Kernel and Image as Subspaces**

First consider $\ker(T)$. It is necessary to show that if $\vec{v}_1, \vec{v}_2$ are vectors in $\ker(T)$ and if a, b are scalars, then $a\vec{v}_1 + b\vec{v}_2$ is also in $\ker(T)$. But

$$T(a\vec{v}_1 + b\vec{v}_2) = aT(\vec{v}_1) + bT(\vec{v}_2) = a\vec{0} + b\vec{0} = \vec{0}$$

Thus $\ker(T)$ is a subspace of V.

Next suppose $T(\vec{v}_1), T(\vec{v}_2)$ are two vectors in $\mathrm{im}(T)$. Then if a, b are scalars,

$$aT(\vec{v}_2) + bT(\vec{v}_2) = T(a\vec{v}_1 + b\vec{v}_2)$$

and this last vector is in $\mathrm{im}(T)$ by definition.

Theorem $7.10.1$**: Dimension of Kernel + Image**

From Proposition $7.10.1$, $\mathrm{im}(T)$ is a subspace of W. By Theorem 7.6.6, there exists a basis for $\mathrm{im}(T), \{T(\vec{v}_1), \cdots, T(\vec{v}_r)\}$. Similarly, there is a basis for $\ker(T), \{\vec{u}_1, \cdots, \vec{u}_s\}$. Then if $\vec{v} \in V$, there exist scalars c_i such that

$$T(\vec{v}) = \sum_{i=1}^{r} c_i T(\vec{v}_i)$$

Hence $T\left(\vec{v} - \sum_{i=1}^{r} c_i \vec{v}_i\right) = 0$. It follows that $\vec{v} - \sum_{i=1}^{r} c_i \vec{v}_i$ is in $\ker(T)$. Hence there are scalars a_i such that

$$\vec{v} - \sum_{i=1}^{r} c_i \vec{v}_i = \sum_{j=1}^{s} a_j \vec{u}_j$$

Hence $\vec{v} = \sum_{i=1}^{r} c_i \vec{v}_i + \sum_{j=1}^{s} a_j \vec{u}_j$. Since $\vec{v}$ is arbitrary, it follows that

$$V = \mathrm{span}\{\vec{u}_1, \cdots, \vec{u}_s, \vec{v}_1, \cdots, \vec{v}_r\}$$

If the vectors $\{\vec{u}_1, \cdots, \vec{u}_s, \vec{v}_1, \cdots, \vec{v}_r\}$ are linearly independent, then it will follow that this set is a basis. Suppose then that

$$\sum_{i=1}^{r} c_i \vec{v}_i + \sum_{j=1}^{s} a_j \vec{u}_j = 0$$

Apply T to both sides to obtain

$$\sum_{i=1}^{r} c_i T(\vec{v}_i) + \sum_{j=1}^{s} a_j T(\vec{u}_j) = \sum_{i=1}^{r} c_i T(\vec{v}_i) = \vec{0}$$

Since $\{T(\vec{v}_1), \cdots, T(\vec{v}_r)\}$ is linearly independent, it follows that each $c_i = 0$. Hence $\sum_{j=1}^{s} a_j \vec{u}_j = 0$ and so, since the $\{\vec{u}_1, \cdots, \vec{u}_s\}$ are linearly independent, it follows that each $a_j = 0$ also. It follows that $\{\vec{u}_1, \cdots, \vec{u}_s, \vec{v}_1, \cdots, \vec{v}_r\}$ is a basis for V and so

$$n = s + r = \dim(\ker(T)) + \dim(\mathrm{im}(T))$$

Corollary $7.10.1$**: One to One and Onto Characterization**

The statement $\ker(T) = \left\{\vec{0}\right\}$ is equivalent to saying if $T(\vec{v}) = \vec{0}$, it follows that $\vec{v} = \vec{0}$. Thus by Proposition 5.5.1 T is one to one. If T is onto, then $\mathrm{im}(T) = W$ and so $\mathrm{rank}(T)$ which is defined as the dimension of $\mathrm{im}(T)$ is m. If $\mathrm{rank}(T) = m$, then by Theorem 7.6.3, since $\mathrm{im}(T)$ is a subspace of W, it follows that $\mathrm{im}(T) = W$.

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7.10E: Exercises for Section 7.10

Exercise 7.10E.1

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation given by

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Find a basis for $\ker(T)$ and $\text{im}(T)$.

Answer available at the end of this section.

Exercise 7.10E.2

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation given by

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

What is $\dim(\ker(T))$?

Answer available at the end of this section.

Exercise 7.10E.3

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation defined by

$$T(\vec{x}) = A\vec{x}, \quad \text{where} \quad A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}.$$

a. Find a basis for the kernel of T.
b. Find a basis for the image of T.
c. State the rank and nullity of T.

Answer available at the end of this section.

Exercise 7.10E.4

Define $T : \mathbb{P}_2 \to \mathbb{R}^2$ by

$$T(p(x)) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}.$$

a. Show that T is a linear transformation.
b. Find a basis for the kernel of T.
c. Find a basis for the image of T.

Answer available at the end of this section.

Exercise $7.10E.5$

Let $T : M_{22} \to \mathbb{R}^2$ be defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+d \\ b-c \end{bmatrix}.$$

a. Prove that T is linear.
b. Determine a basis for the kernel of T.
c. Determine a basis for the image of T.
d. What is the dimension of the kernel and image?

Answer available at the end of this section.

Exercise $7.10E.6$

Let $T : \mathbb{P}_3 \to \mathbb{P}_2$ be the derivative operator:

$$T(p(x)) = p'(x).$$

a. Show that T is linear.
b. Find a basis for the kernel of T.
c. Find a basis for the image of T.
d. Determine the nullity and rank of T.

Exercise $7.10E.7$

Which vector space is isomorphic to $\mathbb{R}^6$?

a. $M_{2,3}$

b. P_6

c. $C[0, 6]$

Answers

Exercise $7.10E.1$

In this case $\ker(T) = \{0\}$ and $\text{im}(T) = \mathbb{R}^2$ (pick any basis of $\mathbb{R}^2$).

Exercise $7.10E.2$

We can see that $\dim(\text{im}(T)) = 1$, and thus $\dim(\ker(T)) = 3 - \dim(Im(T)) = 3 - 1 = 2$.

Exercise $7.10E.3$

a. Basis for kernel:

$$\left\{\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}\right\}$$

b. Basis for image:

$$\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$$

c. nullity = 1, rank = 2

Exercise $7.10E.4$

a. Use the definition of linear transformation to show that T preserves linear combinations.
b. Basis for kernel:

$$\left\{x^2 - x\right\}, \quad \text{nullity} = 1$$

c. Basis for image:

$$\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\},\quad \text{rank}=2$$

Exercise $7.10E.5$

a. Use the definition of linear transformation to show that T preserves linear combinations.
b. Basis for kernel:

$$\left\{\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix},\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\right\}$$

c. Basis for image:

$$\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}$$

d. $\dim(\ker(T))=2$ and $\dim(\text{im}(T))=2$

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7.11: The Matrix of a Linear Transformation

Learning Objectives

- Find the coordinates of a vector with respect to general bases in vector spaces.
- Find the matrix of a linear transformation with respect to general bases in vector spaces.

You may recall from $\mathbb{R}^n$ that the matrix of a linear transformation depends on the bases chosen. This concept is explored in this section, where the linear transformation now maps from one arbitrary vector space to another.

Let $T : V \mapsto W$ be an isomorphism where V and W are vector spaces. Recall from Lemma 5.6.1 that T maps a basis in V to a basis in W. When discussing this Lemma, we were not specific on what this basis looked like. In this section we will make such a distinction.

Consider now an important definition.

Definition 7.11.1: Coordinate Isomorphism

Let V be a vector space with $\dim(V) = n$, let $B = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ be a fixed basis of V, and let $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ denote the standard basis of $\mathbb{R}^n$. We define a transformation $C_B : V \to \mathbb{R}^n$ by

$$C_B(a_1\vec{b}_1 + a_2\vec{b}_2 + \cdots + a_n\vec{b}_n) = a_1\vec{e}_1 + a_2\vec{e}_2 + \cdots + a_n\vec{e}_n = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}.$$

Then C_B is a linear transformation such that $C_B(\vec{b}_i) = \vec{e}_i$, $1 \leq i \leq n$.

C_B is an isomorphism, called the **coordinate isomorphism** corresponding to B.

We continue with another related definition.

Definition 7.11.2: Coordinate Vector

Let V be a finite dimensional vector space with $\dim(V) = n$, and let $B = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ be an ordered basis of V (meaning that the order that the vectors are listed is taken into account). The **coordinate vector** of $\vec{v}$ with respect to B is defined as $C_B(\vec{v})$.

Consider the following example.

Example 7.11.1: Coordinate Vector

Let $V = \mathbb{P}_2$ and $\vec{x} = -x^2 - 2x + 4$. Find $C_B(\vec{x})$ for the following bases B:

1. $B = \{1, x, x^2\}$
2. $B = \{x^2, x, 1\}$
3. $B = \{x + x^2, x, 4\}$

Solution

1. First, note the order of the basis is important. Now we need to find a_1, a_2, a_3 such that $\vec{x} = a_1(1) + a_2(x) + a_3(x^2)$, that is:

$$-x^2-2x+4=a_1(1)+a_2(x)+a_3(x^2)$$

Clearly the solution is

$$\begin{aligned} a_1&=4\\ a_2&=-2\\ a_3&=-1 \end{aligned} \tag{7.11.1}$$

Therefore the coordinate vector is

$$C_B(\vec{x})=\begin{bmatrix} 4\\ -2\\ -1 \end{bmatrix}$$

2. Again remember that the order of B is important. We proceed as above. We need to find a_1, a_2, a_3 such that $\vec{x}=a_1(x^2)+a_2(x)+a_3(1)$, that is:

$$-x^2-2x+4=a_1(x^2)+a_2(x)+a_3(1)$$

Here the solution is

$$\begin{aligned} a_1&=-1\\ a_2&=-2\\ a_3&=4 \end{aligned} \tag{7.11.2}$$

Therefore the coordinate vector is

$$C_B(\vec{x})=\begin{bmatrix} -1\\ -2\\ 4 \end{bmatrix}$$

3. Now we need to find a_1, a_2, a_3 such that $\vec{x}=a_1(x+x^2)+a_2(x)+a_3(4)$, that is:

$$\begin{aligned} -x^2-2x+4&=a_1(x+x^2)+a_2(x)+a_3(4)\\ &=a_1(x^2)+(a_1+a_2)(x)+a_3(4) \end{aligned} \tag{7.11.3}$$

The solution is

$$\begin{aligned} a_1&=-1\\ a_2&=-1\\ a_3&=1 \end{aligned} \tag{7.11.4}$$

and the coordinate vector is

$$C_B(\vec{x})=\begin{bmatrix} -1\\ -1\\ 1 \end{bmatrix}$$

Given that the coordinate transformation $C_B: V\to\mathbb{R}^n$ is an isomorphism, its inverse exists.

Theorem 7.11.1: Inverse of the Coordinate Isomorphism

Let V be a finite dimensional vector space with dimension n and ordered basis $B=\{\vec{b}_1,\vec{b}_2,\ldots,\vec{b}_n\}$. Then $C_B: V\to\mathbb{R}^n$ is an isomorphism whose inverse,

$$C_B^{-1}:\mathbb{R}^n\to V$$

is given by

$$C_B^{-1}\begin{bmatrix} a_1\\ a_2\\ \vdots\\ a_n \end{bmatrix}=a_1\vec{b}_1+a_2\vec{b}_2+\cdots+a_n\vec{b}_n \quad \text{for all} \quad \begin{bmatrix} a_1\\ a_2\\ \vdots\\ a_n \end{bmatrix}\in\mathbb{R}^n.$$

We now discuss the main result of this section, that is how to represent a linear transformation with respect to different bases.

Let V and W be finite dimensional vector spaces, and suppose

- $\dim(V) = n$ and $B_1 = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ is an ordered basis of V;
- $\dim(W) = m$ and B_2 is an ordered basis of W.

Let $T : V \to W$ be a linear transformation. If $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$, then we can find a matrix A so that $T_A = T$. For arbitrary vector spaces V and W, our goal is to represent T as a matrix., i.e., find a matrix A so that $T_A : \mathbb{R}^n \to \mathbb{R}^m$ and $T_A = C_{B_2} T C_{B_1}^{-1}$.

To find the matrix A:

$$T_A = C_{B_2} T C_{B_1}^{-1} \text{ implies that } T_A C_{B_1} = C_{B_2} T,$$

and thus for any $\vec{v} \in V$,

$$C_{B_2}[T(\vec{v})] = T_A[C_{B_1}(\vec{v})] = AC_{B_1}(\vec{v}).$$

Since $C_{B_1}(\vec{b}_j) = \vec{e}_j$ for each $\vec{b}_j \in B_1$, $AC_{B_1}(\vec{b}_j) = A\vec{e}_j$, which is simply the jth column of A. Therefore, the jth column of A is equal to $C_{B_2}[T(\vec{b}_j)]$.

The matrix of T corresponding to the ordered bases B_1 and B_2 is denoted $M_{B_2B_1}(T)$ and is given by

$$M_{B_2B_1}(T) = \left[\begin{array}{cccc} C_{B_2}[T(\vec{b}_1)] & C_{B_2}[T(\vec{b}_2)] & \cdots & C_{B_2}[T(\vec{b}_n)] \end{array} \right].$$

This result is given in the following theorem.

Theorem 7.11.2

Let V and W be vectors spaces of dimension n and m respectively, with $B_1 = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ an ordered basis of V and B_2 an ordered basis of W. Suppose $T : V \to W$ is a linear transformation. Then the unique matrix $M_{B_2B_1}(T)$ of T corresponding to B_1 and B_2 is given by

$$M_{B_2B_1}(T) = \left[\begin{array}{cccc} C_{B_2}[T(\vec{b}_1)] & C_{B_2}[T(\vec{b}_2)] & \cdots & C_{B_2}[T(\vec{b}_n)] \end{array} \right].$$

This matrix satisfies $C_{B_2}[T(\vec{v})] = M_{B_2B_1}(T)C_{B_1}(\vec{v})$ for all $\vec{v} \in V$.

We demonstrate this content in the following examples.

Example 7.11.2: Matrix of a Linear Transformation

Let $T : \mathbb{P}_3 \mapsto \mathbb{R}^4$ be an isomorphism defined by

$$T(ax^3 + bx^2 + cx + d) = \begin{bmatrix} a+b \\ b-c \\ c+d \\ d+a \end{bmatrix}$$

Suppose $B_1 = \{x^3, x^2, x, 1\}$ is an ordered basis of $\mathbb{P}_3$ and

$$B_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

be an ordered basis of $\mathbb{R}^4$. Find the matrix $M_{B_2B_1}(T)$.

Solution

To find $M_{B_2B_1}(T)$, we use the following definition.

$$M_{B_2B_1}(T) = \begin{bmatrix} C_{B_2}[T(x^3)] & C_{B_2}[T(x^2)] & C_{B_2}[T(x)] & C_{B_2}[T(x^2)] \end{bmatrix}$$

First we find the result of applying T to the basis B_1.

$$T(x^3) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, T(x^2) = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, T(x) = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, T(1) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Next we apply the coordinate isomorphism C_{B_2} to each of these vectors. We will show the first in detail.

$$C_{B_2}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right) = a_1 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} + a_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

This implies that

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 0 \\ a_1 - a_3 &= 0 \\ a_4 &= 1 \end{aligned} \tag{7.11.5}$$

which has a solution given by

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 0 \\ a_3 &= 1 \\ a_4 &= 1 \end{aligned} \tag{7.11.6}$$

Therefore $C_{B_2}[T(x^3)] = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$.

You can verify that the following are true.

$$C_{B_2}[T(x^2)] = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, C_{B_2}[T(x)] = \begin{bmatrix} 0 \\ -1 \\ -1 \\ 0 \end{bmatrix}, C_{B_2}[T(1)] = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Using these vectors as the columns of $M_{B_2B_1}(T)$ we have

$$M_{B_2B_1}(T) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 1 & -1 & -1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

The next example demonstrates that this method can be used to solve different types of problems. We will examine the above example and see if we can work backwards to determine the action of T from the matrix $M_{B_2B_1}(T)$.

Example 7.11.3: Finding the Action of a Linear Transformation

Let $T : \mathbb{P}_3 \mapsto \mathbb{R}^4$ be an isomorphism with

$$M_{B_2B_1}(T) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 1 & -1 & -1 \\ 1 & 0 & 0 & 1 \end{bmatrix},$$

where $B_1 = \{x^3, x^2, x, 1\}$ is an ordered basis of $\mathbb{P}_3$ and

$$B_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is an ordered basis of $\mathbb{R}^4$. If $p(x) = ax^3 + bx^2 + cx + d$, find $T(p(x))$.

Solution

Recall that $C_{B_2}[T(p(x))] = M_{B_2B_1}(T)C_{B_1}(p(x))$. Then we have

$$\begin{aligned} C_{B_2}[T(p(x))] &= M_{B_2B_1}(T)C_{B_1}(p(x)) \\ &= \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 1 & -1 & -1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \\ &= \begin{bmatrix} a+b \\ b-c \\ a+b-c-d \\ a+d \end{bmatrix} \end{aligned} \tag{7.11.7}$$

Therefore

$$\begin{aligned} T(p(x)) &= C_D^{-1} \begin{bmatrix} a+b \\ b-c \\ a+b-c-d \\ a+d \end{bmatrix} \\ &= (a+b) \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + (b-c) \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + (a+b-c-d) \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} + (a+d) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} a+b \\ b-c \\ c+d \\ a+d \end{bmatrix} \end{aligned} \tag{7.11.8}$$

You can verify that this was the definition of $T(p(x))$ given in the previous example.

We can also find the matrix of the composite of multiple transformations.

Theorem 7.11.3: Matrix of Composition

Let V, W and U be finite dimensional vector spaces, and suppose $T : V \mapsto W$, $S : W \mapsto U$ are linear transformations. Suppose V, W and U have ordered bases of B_1, B_2 and B_3 respectively. Then the matrix of the composite transformation $S \circ T$ (or ST) is given by

$$M_{B_3B_1}(ST) = M_{B_3B_2}(S)M_{B_2B_1}(T).$$

The next important theorem gives a condition on when T is an isomorphism.

Theorem 7.11.4: Isomorphism

Let V and W be vector spaces such that both have dimension n and let $T : V \mapsto W$ be a linear transformation. Suppose B_1 is an ordered basis of V and B_2 is an ordered basis of W.

Then the conditions that $M_{B_2B_1}(T)$ is invertible for **all** B_1 and B_2, and that $M_{B_2B_1}(T)$ is invertible for **some** B_1 and B_2 are equivalent. In fact, these occur if and only if T is an isomorphism.

If T is an isomorphism, the matrix $M_{B_2B_1}(T)$ is invertible and its inverse is given by $[M_{B_2B_1}(T)]^{-1} = M_{B_1B_2}(T^{-1})$.

Consider the following example.

Example 7.11.4

Suppose $T : \mathbb{P}_3 \to \mathbb{M}_{22}$ is a linear transformation defined by

$$T(ax^3 + bx^2 + cx + d) = \begin{bmatrix} a+d & b-c \\ b+c & a-d \end{bmatrix}$$

for all $ax^3 + bx^2 + cx + d \in \mathbb{P}_3$. Let $B_1 = \{x^3, x^2, x, 1\}$ and

$$B_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

be ordered bases of $\mathbb{P}_3$ and $\mathbb{M}_{22}$, respectively.

1. Find $M_{B_2B_1}(T)$.
2. Verify that T is an isomorphism by proving that $M_{B_2B_1}(T)$ is invertible.
3. Find $M_{B_1B_2}(T^{-1})$, and verify that $M_{B_1B_2}(T^{-1}) = [M_{B_2B_1}(T)]^{-1}$.
4. Use $M_{B_1B_2}(T^{-1})$ to find T^{-1}.

Solution

1.
$$\begin{aligned} M_{B_2B_1}(T) &= \begin{bmatrix} C_{B_2}[T(1)] & C_{B_2}[T(x)] & C_{B_2}[T(x^2)] & C_{B_2}[T(x^3)] \end{bmatrix} \\ &= \begin{bmatrix} C_{B_2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & C_{B_2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & C_{B_2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & C_{B_2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \end{aligned} \tag{7.11.9}$$

2. $\det(M_{B_2B_1}(T)) = 4$, so the matrix is invertible, and hence T is an isomorphism.

3.
$$T^{-1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1, T^{-1}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = x, T^{-1}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = x^2, T^{-1}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = x^3,$$

so

$$T^{-1}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \frac{1+x^3}{2}, T^{-1}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \frac{x-x^2}{2},$$
$$T^{-1}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \frac{x+x^2}{2}, T^{-1}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \frac{1-x^3}{2}.$$

Therefore,

$$M_{B_1B_2}(T^{-1}) = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$$

You should verify that $M_{B_2B_1}(T)M_{B_1B_2}(T^{-1}) = I_4$. From this it follows that $[M_{B_2B_1}(T)]^{-1} = M_{B_1B_2}(T^{-1})$.

4.

$$\begin{aligned} C_{B_1}\left(T^{-1}\begin{bmatrix} p & q \\ r & s \end{bmatrix}\right) &= M_{B_1B_2}(T^{-1})C_{B_2}\left(\begin{bmatrix} p & q \\ r & s \end{bmatrix}\right) \\ T^{-1}\begin{bmatrix} p & q \\ r & s \end{bmatrix} &= C_{B_1}^{-1}\left(M_{B_1B_2}(T^{-1})C_{B_2}\left(\begin{bmatrix} p & q \\ r & s \end{bmatrix}\right)\right) \\ &= C_{B_1}^{-1}\left(\frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix}\right) \\ &= C_{B_1}^{-1}\left(\frac{1}{2}\begin{bmatrix} p+s \\ q+r \\ r-q \\ p-s \end{bmatrix}\right) \\ &= \frac{1}{2}(p+s)x^3 + \frac{1}{2}(q+r)x^2 + \frac{1}{2}(r-q)x + \frac{1}{2}(p-s). \end{aligned} \tag{7.11.10}$$

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7.11E: Exercises for Section 7.11

Exercise 7.11E.1

Let $B = \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}$ be a basis of $\mathbb{R}^2$ and let $\vec{x} = \begin{bmatrix} 5 \\ -7 \end{bmatrix}$ be a vector in $\mathbb{R}^2$. Find $C_B(\vec{x})$.

Exercise 7.11E.2

Let $B = \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right\}$ be a basis of $\mathbb{R}^3$ and let $\vec{x} = \begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix}$ be a vector in $\mathbb{R}^2$. Find $C_B(\vec{x})$.

Answer available at the end of this section.

Exercise 7.11E.3

Let $V = \mathbb{P}_2$ and $\vec{p}(x) = 3x^2 - 5x + 2$. Find the coordinate vector $C_B(\vec{p}(x))$ for the following bases.

a. $B_1 = \{1, x, x^2\}$
b. $B_2 = \{x^2, x, 1\}$
c. $B_3 = \{x^2 + x, x, 1\}$

Answer available at the end of this section.

Exercise 7.11E.4

Let $V = M_{22}$ and let

$$A = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$$

Let the basis B be:

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Find the coordinates of A with respect to basis B, $C_B(A)$.

Answer available at the end of this section.

Exercise 7.11E.5

Let $T\colon \mathbb{R}^2 \mapsto \mathbb{R}^2$ be a linear transformation defined by $T\left(\begin{bmatrix} a \\ b \end{bmatrix} \right) = \begin{bmatrix} a+b \\ a-b \end{bmatrix}$.

Consider the two bases

$$B_1 = \{\vec{v}_1, \vec{v}_2\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

and

$$B_2 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

Find the matrix $M_{B_2,B_1}(T)$ with respect to the bases B_1 and B_2.

Answer available at the end of this section.

Exercise 7.11E.6

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation defined by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + 2y + z \\ y - z \end{bmatrix}.$$

a. Find the matrix of T with respect to the standard basis.
b. Find a basis for the kernel of T and determine its dimension.
c. Find a basis for the image of T and determine its dimension.

Answer available at the end of this section.

Exercise 7.11E.7

Let $T : \mathbb{P}_2 \to \mathbb{R}^3$ be defined by

$$T(p(x)) = \begin{bmatrix} p(0) \\ p(1) \\ p(2) \end{bmatrix}.$$

a. Find the matrix of T with respect to the standard bases of $\mathbb{P}_2$ and $\mathbb{R}^3$.
b. Find a basis for the kernel of T.
c. Find a basis for the image of T

Exercise 7.11E.8

Let $T : M_{22} \to \mathbb{R}^3$ be defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a + b \\ c + d \\ a - d \end{bmatrix}.$$

a. Find the matrix of T with respect to the standard basis of M_{22}.
b. Find a basis for the kernel of T and its dimension.
c. Find a basis for the image of T and its dimension.

Exercise 7.11E.9

Let $T : \mathbb{P}_3 \to \mathbb{P}_2$ be the derivative operator:

$$T(p(x)) = p'(x).$$

a. Find the matrix of T with respect to the standard bases of $\mathbb{P}_3$ and $\mathbb{P}_2$.
b. Find a basis and dimension of the kernel of T.
c. Find a basis and dimension of the image of T.

Answer available at the end of this section.

Exercise 7.11E.10

Let $T : \mathbb{P}_2 \to \mathbb{P}_1$ be the derivative operator:

$$T(p(x)) = p'(x)$$

Let the bases be:

$$B_1 = \{x^2 + 1, x, 1\} \quad \text{(domain)}, \qquad B_2 = \{x, 1\} \quad \text{(codomain)}$$

Find the matrix $M_{B_2,B_1}T$, the matrix of T with respect to B_1 and B_2.

Answer available at the end of this section.

Exercise 7.11E.11

Let $T : M_{22} \to \mathbb{R}^2$ be defined by:

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+b \\ c-d \end{bmatrix}$$

Let the basis of the domain be:

$$B_1 = \left\{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}\right\}$$

Let B_2 be the standard basis of $\mathbb{R}^2$. Find the matrix $M_{B_2,B_1}(T)$

Answers

Exercise 7.11E.2

$$C_B(\vec{x}) = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}.$$

Exercise 7.11E.3

Exercise 7.11E.4

The coordinate vector of

$$A = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} \quad \text{with respect to the basis} \quad B = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

is:

$$C_B(A) = \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

Exercise 7.11E.5

$$M_{B_2B_1}(T) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

Exercise 7.11E.6

a. Matrix of T (standard basis):

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

b. Basis for kernel:

$$\left\{\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}\right\}, \quad \text{nullity} = 1$$

c. Basis for image:

$$\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}2\\1\end{bmatrix}\right\},\quad \text{rank}=2$$

Exercise $7.11E.9$

a. Matrix of T (standard bases for $\mathbb{P}_3$ and $\mathbb{P}_2$:

$$\begin{bmatrix}0&1&0&0\\0&0&2&0\\0&0&0&3\end{bmatrix}$$

b. Basis for kernel:

$$\{1\},\quad \text{nullity}=1$$

c. Basis for image:

$$\left\{x^2,x,1\right\},\quad \text{rank}=3$$

Exercise $7.11E.10$

$$M_{B_2,B_1}(T)=\begin{bmatrix}2&1&0\\0&0&0\end{bmatrix}$$

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7.12: Inner Product Spaces

Learning Objectives

- Develop an understanding of inner product spaces through the definition of the inner product and its associated axioms, including linearity, symmetry, and positivity.
- Learn the concepts of orthogonality and orthonormality in vector spaces, and apply these ideas using the Gram-Schmidt process and theorems related to orthogonal projections and expansions.

The dot product was introduced in $\mathbb{R}^n$ to provide a natural generalization of the geometrical notions of length and orthogonality. The plan in this section is to define an inner product on an arbitrary real vector space V (of which the dot product is an example in $\mathbb{R}^n$) and use it to introduce these concepts in V.

Definition 7.12.1: Inner Product

An **inner product** on a real vector space V is a function that assigns a real number $\langle \vec{v}, \vec{w} \rangle$ to every pair $\vec{v}, \vec{w}$ of vectors in V in such a way that the following axioms are satisfied.

1. $\langle \vec{v}, \vec{w} \rangle$ is a real number for all $\vec{v}$ and $\vec{w}$ in V.
2. $\langle \vec{v}, \vec{w} \rangle = \langle \vec{w}, \vec{v} \rangle$ for all $\vec{v}$ and $\vec{w}$ in V.
3. $\langle \vec{v} + \vec{w}, \vec{u} \rangle = \langle \vec{v}, \vec{u} \rangle + \langle \vec{w}, \vec{u} \rangle$ for all $\vec{u}, \vec{v}$, and $\vec{w}$ in V.
4. $\langle r\vec{v}, \vec{w} \rangle = r\langle \vec{v}, \vec{w} \rangle$ for all $\vec{v}$ and $\vec{w}$ in V and all r in $\mathbb{R}$.
5. $\langle \vec{v}, \vec{v} \rangle > 0$ for all $\vec{v} \neq \vec{0}$ in V.

A real vector space V with an inner product $\langle , \rangle$ will be called an **inner product space**. Note that every subspace of an inner product space is again an inner product space using the same inner product.

Example 7.12.1

$\mathbb{R}^n$ is an inner product space with the dot product as inner product:

$$\langle \vec{v}, \vec{w} \rangle = \vec{v} \cdot \vec{w} \text{ for all } \vec{v}, \vec{w} \in \mathbb{R}^n \tag{7.12.1}$$

This is also called the Euclidean inner product, and $\mathbb{R}^n$, equipped with the dot product, is called Euclidean n-space.

Example 7.12.2

If A and B are $m \times n$ matrices, define $\langle A, B \rangle = \text{tr}\left(AB^T\right)$ where $\text{tr}(X)$ is the trace of the square matrix X. Show that $\langle , \rangle$ is an inner product in $\mathbb{M}_{mn}$

Solution

Axiom 1 is clear. Since $\text{tr}(P) = \text{tr}\left(P^T\right)$ for every $m \times n$ matrix P, we have Axiom 2:

$$\langle A, B \rangle = \text{tr}\left(AB^T\right) = \text{tr}\left[\left(AB^T\right)^T\right] = \text{tr}\left(BA^T\right) = \langle B, A \rangle. \tag{7.12.2}$$

Next, Axiom 3 and Axiom 4 follow because trace is a linear transformation $\vec{M}_{mn} \to \mathbb{R}$. Turning to Axiom 5, let $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_m$ denote the rows of the matrix A. Then the (i, j)-entry of AA^T is $\vec{r}_i \cdot \vec{r}_j$, so

$$\langle A, A \rangle = \text{tr}\left(AA^T\right) = \vec{r}_1 \cdot \vec{r}_1 + \vec{r}_2 \cdot \vec{r}_2 + \cdots + \vec{r}_m \cdot \vec{r}_m \tag{7.12.3}$$

But $\vec{r}_j \cdot \vec{r}_j$ is the sum of the squares of the entries of $\vec{r}_j$, so this shows that $\langle A, A \rangle$ is the sum of the squares of all nm entries of A. Axiom 5 follows.

The next example is important in analysis.

Example 7.12.3

Let $C[a, b]$ denote the vector space of continuous functions from $[a, b]$ to $\mathbb{R}$, a subspace of $F[a, b]$. Show that

$$\langle f, g \rangle = \int_a^b f(x)g(x)dx \tag{7.12.4}$$

defines an inner product on $C[a, b]$.

Solution

Axioms 1 and 2 are clear. As to Axiom 4,

$$\langle rf, g \rangle = \int_a^b rf(x)g(x)dx = r\int_a^b f(x)g(x)dx = r\langle f, g \rangle \tag{7.12.5}$$

Axiom 3 is similar. Finally, theorems of calculus show that $\langle f, f \rangle = \int_a^b f(x)^2 dx \geq 0$ and, if f is continuous, that this is zero if and only if f is the zero function. This gives Axiom 5.

If $\vec{v}$ is any vector, then, using Axiom 3, we get

$$\langle \vec{0}, \vec{v} \rangle = \langle \vec{0} + \vec{0}, \vec{v} \rangle = \langle \vec{0}, \vec{v} \rangle + \langle \vec{0}, \vec{v} \rangle \tag{7.12.6}$$

and it follows that the number $\langle \vec{0}, \vec{v} \rangle$ must be zero. This observation is recorded for reference in the following theorem, along with several other properties of inner products. The other proofs are left as Exercise.

Theorem 7.12.1

Let $\langle , \rangle$ be an inner product on a space V; let $\vec{v}, \vec{u}$, and $\vec{w}$ denote vectors in V; and let r denote a real number.

1. $\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$.
2. $\langle \vec{v}, r\vec{w} \rangle = r\langle \vec{v}, \vec{w} \rangle = \langle r\vec{v}, \vec{w} \rangle$.
3. $\langle \vec{v}, \vec{0} \rangle = 0 = \langle \vec{0}, \vec{v} \rangle$.
4. $\langle \vec{v}, \vec{v} \rangle = 0$ if and only if $\vec{v} = \vec{0}$.

Theorem 7.12.2

If A is any $n \times n$ positive definite matrix, then

$$\langle \vec{x}, \vec{y} \rangle = \vec{x}^T A\vec{y} \text{ for all columns } \vec{x}, \vec{y} \text{ in } \mathbb{R}^n \tag{7.12.7}$$

defines an inner product on $\mathbb{R}^n$, and every inner product on $\mathbb{R}^n$ arises in this way.

Proof

Given an inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$, let $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ be the standard basis of $\mathbb{R}^n$. If $\vec{x} = \sum_{i=1}^n x_i\vec{e}_i$ and $\vec{y} = \sum_{j=1}^n y_j\vec{e}_j$ are two vectors in $\mathbb{R}^n$, compute $\langle \vec{x}, \vec{y} \rangle$ by adding the inner product of each term $x_i\vec{e}_i$ to each term $y_j\vec{e}_j$. The result is a double sum.

$$\langle \vec{x}, \vec{y} \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} \langle x_i \vec{e}_i, y_j \vec{e}_j \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i \langle \vec{e}_i, \vec{e}_j \rangle y_j \tag{7.12.8}$$

As the reader can verify, this is a matrix product:

$$\langle \vec{x}, \vec{y} \rangle = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} \langle \vec{e}_1, \vec{e}_1 \rangle & \langle \vec{e}_1, \vec{e}_2 \rangle & \cdots & \langle \vec{e}_1, \vec{e}_n \rangle \\ \langle \vec{e}_2, \vec{e}_1 \rangle & \langle \vec{e}_2, \vec{e}_2 \rangle & \cdots & \langle \vec{e}_2, \vec{e}_n \rangle \\ \vdots & \vdots & \cdots & \vdots \\ \langle \vec{e}_n, \vec{e}_1 \rangle & \langle \vec{e}_n, \vec{e}_2 \rangle & \cdots & \langle \vec{e}_n, \vec{e}_n \rangle \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \tag{7.12.9}$$

Hence $\langle \vec{x}, \vec{y} \rangle = \vec{x}^T A \vec{y}$, where A is the $n \times n$ matrix whose (i, j)-entry is $\langle \vec{e}_i, \vec{e}_j \rangle$. The fact that $\langle \vec{e}_i, \vec{e}_j \rangle = \langle \vec{e}_j, \vec{e}_i \rangle$ shows that A is symmetric.

Thus, just as every linear operator $\mathbb{R}^n \to \mathbb{R}^n$ corresponds to an $n \times n$ matrix, every inner product on $\mathbb{R}^n$ corresponds to a positive definite $n \times n$ matrix. In particular, the dot product corresponds to the identity matrix I_n.

Note

If we refer to the inner product space $\mathbb{R}^n$ without specifying the inner product, we mean that the dot product is to be used.

Example 7.12.4

Let the inner product $\langle \mathbf{u}, \mathbf{v} \rangle$ be defined on $\mathbb{R}^2$ by

$$\left\langle \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right\rangle = 2v_1 w_1 - v_1 w_2 - v_2 w_1 + v_2 w_2 \tag{7.12.10}$$

Find a symmetric 2×2 matrix A such that $\langle \vec{x}, \vec{y} \rangle = \vec{x}^T A \vec{y}$ for all $\vec{x}, \vec{y}$ in $\mathbb{R}^2$.

Solution

The (i, j)-entry of the matrix A is the coefficient of $v_i w_j$ in the expression, so $A = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$. Incidentally, if $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, then

$$\langle \vec{x}, \vec{x} \rangle = 2x^2 - 2xy + y^2 = x^2 + (x - y)^2 \geq 0 \tag{7.12.11}$$

for all $\vec{x}$, so $\langle \vec{x}, \vec{x} \rangle = 0$ implies $\vec{x} = \vec{0}$. Hence $\langle , \rangle$ is indeed an inner product, so A is positive definite.

Definition 7.12.2: Norm

As in $\mathbb{R}^n$, if $\langle \cdot, \cdot \rangle$ is an inner product on a space V, the **norm** $\|\vec{v}\|$ of a vector $\vec{v}$ in V is defined by

$$\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}. \tag{7.12.12}$$

We define the distance between vectors $\vec{v}$ and $\vec{w}$ in an inner product space V to be

$$\mathrm{d}(\vec{v}, \vec{w}) = \|\vec{v} - \vec{w}\| \tag{7.12.13}$$

Note that Axiom 5 from the definition of the inner product guarantees that $\langle \vec{v}, \vec{v} \rangle \geq 0$, so $\|\vec{v}\|$ is a real number.

Example 7.12.5

The norm of a continuous function $f = f(x)$ in $C[a, b]$ with the inner product is given by

$$\|f\| = \sqrt{\int_a^b f(x)^2 dx} \tag{7.12.14}$$

Hence $\|f\|^2$ is the area beneath the graph of $y = f(x)^2$ between $x = a$ and $x = b$ (see Figure 7.12.1).

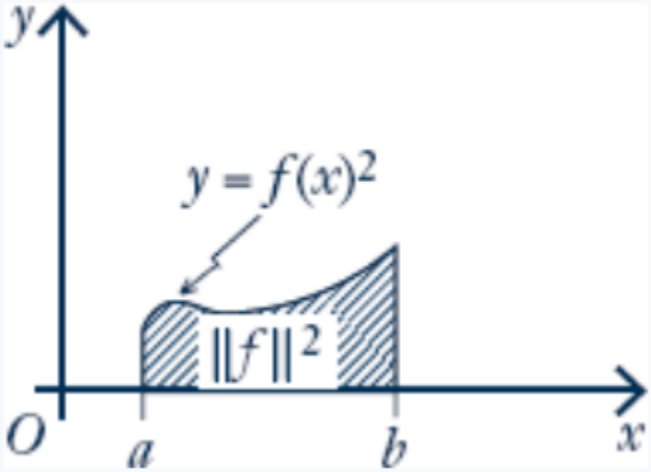


Figure 7.12.1: The area between the graph of $f(x)^2$ and the x-axis is shaded. The area is labeled '$\|f\|^2$'. (CC BY-NC-SA 4.0; Nicholson via LINEAR ALGEBRA With Application)

Example 7.12.6

Show that $\langle \vec{u} + \vec{v}, \vec{u} - \vec{v} \rangle = \|\vec{u}\|^2 - \|\vec{v}\|^2$ in any inner product space.

Solution

$$\begin{aligned} \langle \vec{u} + \vec{v}, \vec{u} - \vec{v} \rangle &= \langle \vec{u}, \vec{u} \rangle - \langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{u} \rangle - \langle \vec{v}, \vec{v} \rangle \\ &= \|\vec{u}\|^2 - \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{v} \rangle - \|\vec{v}\|^2 \\ &= \|\vec{u}\|^2 - \|\vec{v}\|^2 \end{aligned} \tag{7.12.15}$$

A vector $\vec{v}$ in an inner product space V is called a unit vector if $\|\vec{v}\| = 1$. The set of all unit vectors in V is called the unit ball in V. For example, if $V = \mathbb{R}^2$ (with the dot product) and $\vec{v} = (x, y)$, then

$$\|\vec{v}\|^2 = 1 \quad \text{if and only if} \quad x^2 + y^2 = 1 \tag{7.12.16}$$

Hence the unit ball in $\mathbb{R}^2$ is the unit circle $x^2 + y^2 = 1$ with centre at the origin and radius 1 . However, the shape of the unit ball varies with the choice of inner product.

Example 7.12.7

Let $a > 0$ and $b > 0$. If $\vec{v} = (x, y)$ and $\vec{w} = (x_1, y_1)$, define an inner product on $\mathbb{R}^2$ by

$$\langle \vec{v}, \vec{w} \rangle = \frac{xx_1}{a^2} + \frac{yy_1}{b^2} \tag{7.12.17}$$

The reader can verify that this is indeed an inner product. In this case

$$\|\vec{v}\|^2 = 1 \quad \text{if and only if} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{7.12.18}$$

so the unit ball is the ellipse shown in Figure 7.12.2.

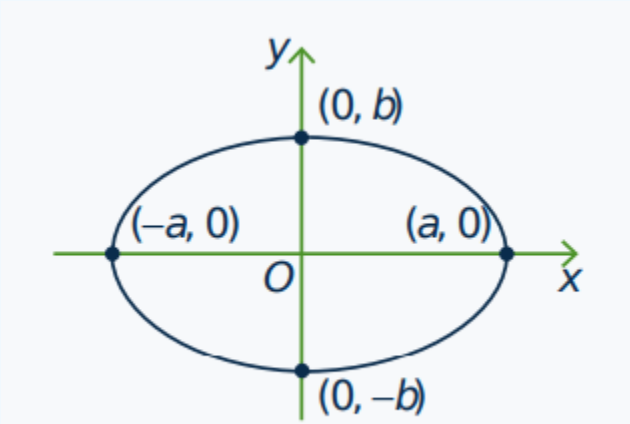


Figure 7.12.2: Figure illustrates the norms and distances in an inner product space. An ellipse in $\mathbb{R}^2$ passing through the points $(a,0),(0,b),(-a,0),(0,-b)$ is shown. (CC BY-NC-SA 4.0; Nicholson via LINEAR ALGEBRA With Application)

Example 7.11.7 graphically illustrates the fact that norms and distances in an inner product space V vary with the choice of inner product in V.

Theorem 7.12.3

If $\vec{v} \neq \vec{0}$ is any vector in an inner product space V, then $\frac{1}{\|\vec{v}\|}\vec{v}$ is the unique unit vector that is a positive multiple of $\vec{v}$.

The next theorem reveals an important and useful fact about the relationship between norms and inner products, extending the Cauchy inequality for $\mathbb{R}^n$.

Theorem 7.12.4 Cauchy-Schwartz Inequality

If $\vec{v}$ and $\vec{w}$ are two vectors in an inner product space V, then

$$\langle \vec{v}, \vec{w} \rangle^2 \leq \|\vec{v}\|^2 \|\vec{w}\|^2 \tag{7.12.19}$$

Moreover, equality occurs if and only if one of $\vec{v}$ and $\vec{w}$ is a scalar multiple of the other.

Proof

Write $\|\vec{v}\| = a$ and $\|\vec{w}\| = b$. We compute:

$$\begin{aligned} \|b\vec{v} - a\vec{w}\|^2 &= b^2\|\vec{v}\|^2 - 2ab\langle \vec{v}, \vec{w} \rangle + a^2\|\vec{w}\|^2 = 2ab(ab - \langle \vec{v}, \vec{w} \rangle) \\ \|b\vec{v} + a\vec{w}\|^2 &= b^2\|\vec{v}\|^2 + 2ab\langle \vec{v}, \vec{w} \rangle + a^2\|\vec{w}\|^2 = 2ab(ab + \langle \vec{v}, \vec{w} \rangle) \end{aligned} \tag{7.12.20}$$

It follows that $ab - \langle \vec{v}, \vec{w} \rangle \geq 0$ and $ab + \langle \vec{v}, \vec{w} \rangle \geq 0$, and hence that $-ab \leq \langle \vec{v}, \vec{w} \rangle \leq ab$. But then $|\ \langle \vec{v}, \vec{w}\ \rangle| \leq ab = \|\vec{v}\|\|\vec{w}\|$, as desired.

Conversely, if $|\langle \vec{v}, \vec{w} \rangle| = \|\vec{v}\|\|\vec{w}\| = ab$ then $\langle \vec{v}, \vec{w} \rangle = \pm ab$. Hence (10.1) shows that $b\vec{v} - a\vec{w} = \vec{0}$ or $b\vec{v} + a\vec{w} = \vec{0}$. It follows that one of $\vec{v}$ and $\vec{w}$ is a scalar multiple of the other, even if $a = 0$ or $b = 0$.

Example 7.12.8

If f and g are continuous functions on the interval $[a, b]$, then

$$\left\{\int_a^b f(x)g(x)dx\right\}^2 \leq \int_a^b f(x)^2dx \int_a^b g(x)^2dx \tag{7.12.21}$$

Another famous inequality, the so-called triangle inequality, also comes from the Cauchy-Schwarz inequality. It is included in the following list of basic properties of the norm of a vector.

Theorem 7.12.5

If V is an inner product space, the norm $\|\cdot\|$ has the following properties.

1. $\|\vec{v}\| \geq 0$ for every vector $\vec{v}$ in V.
2. $\|\vec{v}\| = 0$ if and only if $\vec{v} = \vec{0}$.
3. $\|r\vec{v}\| = |r|\|\vec{v}\|$ for every $\vec{v}$ in V and every r in $\mathbb{R}$.
4. $\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$ for all $\vec{v}$ and $\vec{w}$ in V (triangle inequality).

Proof

Because $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v}\rangle}$, properties (1) and (2) follow immediately from (3) and (4) of Theorem 7.11.1. As to (3), compute

$$\|r\vec{v}\|^2 = \langle r\vec{v}, r\vec{v}\rangle = r^2\langle \vec{v}, \vec{v}\rangle = r^2\|\vec{v}\|^2 \tag{7.12.22}$$

Hence (3) follows by taking positive square roots. Finally, the fact that $\langle \vec{v}, \vec{w}\rangle \leq \|\vec{v}\|\|\vec{w}\|$ by the CauchySchwarz inequality gives

$$\begin{aligned} \|\vec{v} + \vec{w}\|^2 = \langle \vec{v} + \vec{w}, \vec{v} + \vec{w}\rangle &= \|\vec{v}\|^2 + 2\langle \vec{v}, \vec{w}\rangle + \|\vec{w}\|^2 \\ &\leq \|\vec{v}\|^2 + 2\|\vec{v}\|\|\vec{w}\| + \|\vec{w}\|^2 \\ &= (\|\vec{v}\| + \|\vec{w}\|)^2 \end{aligned} \tag{7.12.23}$$

Hence (4) follows by taking positive square roots.
It is worth noting that the usual triangle inequality for absolute values,

$$|r + s| \leq |r| + |s| \text{ for all real numbers } r \text{ and } s, \tag{7.12.24}$$

is a special case of (4) where $V = \mathbb{R} = \mathbb{R}^1$ and the dot product $\langle r, s\rangle = rs$ is used.

In many calculations in an inner product space, it is required to show that some vector $\vec{v}$ is zero. This is often accomplished most easily by showing that its norm $\|\vec{v}\|$ is zero. Here is an example.

Example 7.12.9

Let $\{\vec{v}_1, \ldots, \vec{v}_n\}$ be a spanning set for an inner product space V. If $\vec{v}$ in V satisfies $\langle \vec{v}, \vec{v}_i\rangle = 0$ for each $i = 1, 2, \ldots, n$, show that $\vec{v} = \vec{0}$.

Solution

Write $\vec{v} = r_1\vec{v}_1 + \cdots + r_n\vec{v}_n, r_i$ in $\mathbb{R}$. To show that $\vec{v} = \vec{0}$, we show that $\|\vec{v}\|^2 = \langle \vec{v}, \vec{v}\rangle = 0$.
Compute:

$$\langle \vec{v}, \vec{v}\rangle = \langle \vec{v}, r_1\vec{v}_1 + \cdots + r_n\vec{v}_n\rangle = r_1\langle \vec{v}, \vec{v}_1\rangle + \cdots + r_n\langle \vec{v}, \vec{v}_n\rangle = 0 \tag{7.12.25}$$

by hypothesis, and the result follows.

Theorem 7.12.6

Let V be an inner product space.

1. $d(\vec{v},\vec{w}) \geq 0$ for all $\vec{v},\vec{w}$ in V.
2. $d(\vec{v},\vec{w}) = 0$ if and only if $\vec{v} = \vec{w}$.
3. $d(\vec{v},\vec{w}) = d(\vec{w},\vec{v})$ for all $\vec{v}$ and $\vec{w}$ in V.
4. $d(\vec{v},\vec{w}) \leq d(\vec{v},\vec{u}) + d(\vec{u},\vec{w})$ for all $\vec{v},\vec{u}$, and $\vec{w}$ in V.

The idea that two lines can be perpendicular is fundamental in geometry, and this section is devoted to introducing this notion into a general inner product space V. To motivate the definition, recall that two nonzero geometric vectors $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$ are perpendicular (or orthogonal) if and only if $\vec{x}\cdot\vec{y} = 0$. In general, two vectors $\vec{v}$ and $\vec{w}$ in an inner product space V are said to be orthogonal if

$$\langle \vec{v},\vec{w}\rangle = 0 \tag{7.12.26}$$

A set $\left\{\vec{f}_1,\vec{f}_2,\ldots,\vec{f}_n\right\}$ of vectors is called an orthogonal set of vectors if

1. Each $\vec{f}_i \neq \vec{0}$.
2. $\left\langle \vec{f}_i,\vec{f}_j\right\rangle = 0$ for all $i \neq j$.

If, in addition, $\left\|\vec{f}_i\right\| = 1$ for each i, the set $\left\{\vec{f}_1,\vec{f}_2,\ldots,\vec{f}_n\right\}$ is called an orthonormal set.

Example 7.12.10

$\{\sin x,\cos x\}$ is an orthogonalset in $C[-\pi,\pi]$, the set of all continuous functions on $[-\pi,\pi]$ because

$$\int_{-\pi}^{\pi}\sin x\cos x dx = \left[-\frac{1}{4}\cos 2x\right]_{-\pi}^{\pi} = 0 \tag{7.12.27}$$

The first result about orthogonal sets extends Pythagoras' theorem in $\mathbb{R}^n$ and the same proof works.

Theorem 7.12.7

If $\left\{\vec{f}_1,\vec{f}_2,\ldots,\vec{f}_n\right\}$ is an orthogonal set of vectors, then

$$\left\|\vec{f}_1+\vec{f}_2+\cdots+\vec{f}_n\right\|^2 = \left\|\vec{f}_1\right\|^2+\left\|\vec{f}_2\right\|^2+\cdots+\left\|\vec{f}_n\right\|^2 \tag{7.12.28}$$

The proof of the next result is left to the reader.

Theorem 7.12.8

Let $\left\{\vec{f}_1,\vec{f}_2,\ldots,\vec{f}_n\right\}$ be an orthogonal set of vectors.

1. $\left\{r_1\vec{f}_1,r_2\vec{f}_2,\ldots,r_n\vec{f}_n\right\}$ is also orthogonal for any $r_i \neq 0$ in $\mathbb{R}$.
2. $\left\{\frac{1}{\left\|\vec{f}_1\right\|}\vec{f}_1,\frac{1}{\left\|\vec{f}_2\right\|}\vec{f}_2,\ldots,\frac{1}{\left\|\vec{f}_n\right\|}\vec{f}_n\right\}$ is an orthonormal set.

As before, the process of passing from an orthogonal set to an orthonormal one is called normalizing the orthogonal set.

Theorem 7.12.9

Every orthogonal set of vectors is linearly independent.

Example 7.12.11

Show that $\left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \right\}$ is an orthogonal basis of $\mathbb{R}^3$ with inner product

$$\langle \vec{v}, \vec{w} \rangle = \vec{v}^T A \vec{w}, \text{ where } A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{7.12.29}$$

Solution

We have

$$\left\langle \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\rangle = \begin{bmatrix} 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 0 \tag{7.12.30}$$

and the reader can verify that the other pairs are orthogonal too. Hence the set is orthogonal, so it is linearly independent b. Because $\dim \mathbb{R}^3 = 3$, it is a basis.

Theorem 7.12.10

Let $\left\{ \vec{f}_1, \vec{f}_2, \ldots, \vec{f}_n \right\}$ be an orthogonal basis of an inner product space V. If $\vec{v}$ is any vector in V, then

$$\vec{v} = \frac{\left\langle \vec{v}, \vec{f}_1 \right\rangle}{\left\| \vec{f}_1 \right\|^2} \vec{f}_1 + \frac{\left\langle \vec{v}, \vec{f}_2 \right\rangle}{\left\| \vec{f}_2 \right\|^2} \vec{f}_2 + \cdots + \frac{\left\langle \vec{v}, \vec{f}_n \right\rangle}{\left\| \vec{f}_n \right\|^2} \vec{f}_n \tag{7.12.31}$$

is the expansion of $\vec{v}$ as a linear combination of the basis vectors.

The coefficients $\frac{\left\langle \vec{v}, \vec{f}_1 \right\rangle}{\left\| \vec{f}_1 \right\|^2}, \frac{\left\langle \vec{v}, \vec{f}_2 \right\rangle}{\left\| \vec{f}_2 \right\|^2}, \ldots, \frac{\left\langle \vec{v}, \vec{f}_n \right\rangle}{\left\| \vec{f}_n \right\|^2}$ in the expansion theorem are sometimes called the Fourier coefficients of $\vec{v}$ with respect to the orthogonal basis $\left\{ \vec{f}_1, \vec{f}_2, \ldots, \vec{f}_n \right\}$. This is in honour of the French mathematician J.B.J. Fourier (1768-1830). His original work was with a particular orthogonal set in the space $C[a, b]$.

Lemma 7.12.1 Orthogonal Lemma

Let $\left\{ \vec{f}_1, \vec{f}_2, \ldots, \vec{f}_m \right\}$ be an orthogonal set of vectors in an inner product space V, and let $\vec{v}$ be any vector not in $\operatorname{span} \left\{ \vec{f}_1, \vec{f}_2, \ldots, \vec{f}_m \right\}$. Define

$$\vec{f}_{m+1} = \vec{v} - \frac{\left\langle \vec{v}, \vec{f}_1 \right\rangle}{\left\| \vec{f}_1 \right\|^2} \vec{f}_1 - \frac{\left\langle \vec{v}, \vec{f}_2 \right\rangle}{\left\| \vec{f}_2 \right\|^2} \vec{f}_2 - \cdots - \frac{\left\langle \vec{v}, \vec{f}_m \right\rangle}{\left\| \vec{f}_m \right\|^2} \vec{f}_m \tag{7.12.32}$$

Then $\left\{ \vec{f}_1, \vec{f}_2, \ldots, \vec{f}_m, \vec{f}_{m+1} \right\}$ is an orthogonal set of vectors.

The proof of this result (and the next) is the same as for the dot product in $\mathbb{R}^n$.

Theorem 7.12.11

Let V be an inner product space and let $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be any basis of V. Define vectors $\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_n$ in V successively as follows:

$$\begin{aligned}
\vec{f}_1 &= \vec{v}_1 \\
\vec{f}_2 &= \vec{v}_2 - \frac{\left\langle \vec{v}_2, \vec{f}_1 \right\rangle}{\left\| \vec{f}_1 \right\|^2} \vec{f}_1 \\
\vec{f}_3 &= \vec{v}_3 - \frac{\left\langle \vec{v}_3, \vec{f}_1 \right\rangle}{\left\| \vec{f}_1 \right\|^2} \vec{f}_1 - \frac{\left\langle \vec{v}_3, \vec{f}_2 \right\rangle}{\left\| \vec{f}_2 \right\|^2} \vec{f}_2 \\
\vec{f}_k &= \vec{v}_k - \frac{\left\langle \vec{v}_k, \vec{f}_1 \right\rangle}{\left\| \vec{f}_1 \right\|^2} \vec{f}_1 - \frac{\left\langle \vec{v}_k, \vec{f}_2 \right\rangle}{\left\| \vec{f}_2 \right\|^2} \vec{f}_2 - \cdots - \frac{\left\langle \vec{v}_k, \vec{f}_{k-1} \right\rangle}{\left\| \vec{f}_{k-1} \right\|^2} \vec{f}_{k-1}
\end{aligned} \tag{7.12.33}$$

for each $k = 2, 3, \ldots, n$. Then

1. $\left\{\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_n\right\}$ is an orthogonal basis of V.
2. $\operatorname{span}\left\{\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_k\right\} = \operatorname{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ holds for each $k = 1, 2, \ldots, n$.

The purpose of the Gram-Schmidt algorithm is to convert a basis of an inner product space into an orthogonal basis. In particular, it shows that every finite dimensional inner product space has an orthogonal basis.

Example 7.12.12

Consider $V = \mathbf{P}_3$ with the inner product $\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)dx$. If the Gram-Schmidt algorithm is applied to the basis $\{1, x, x^2, x^3\}$, show that the result is the orthogonal basis

$$\left\{1, x, \frac{1}{3}\left(3x^2 - 1\right), \frac{1}{5}\left(5x^3 - 3x\right)\right\} \tag{7.12.34}$$

Solution

Take $\vec{f}_1 = 1$. Then the algorithm gives

$$\begin{aligned}
\vec{f}_2 &= x - \frac{\left\langle x, \vec{f}_1 \right\rangle}{\left\| \vec{f}_1 \right\|^2} \vec{f}_1 = x - \frac{0}{2}\vec{f}_1 = x \\
\vec{f}_3 &= x^2 - \frac{\left\langle x^2, \vec{f}_1 \right\rangle}{\left\| \vec{f}_1 \right\|^2} \vec{f}_1 - \frac{\left\langle x^2, \vec{f}_2 \right\rangle}{\left\| \vec{f}_2 \right\|^2} \vec{f}_2 \\
&= x^2 - \frac{\frac{2}{3}}{2} 1 - \frac{0}{\frac{2}{3}} x \\
&= \frac{1}{3}\left(3x^2 - 1\right)
\end{aligned} \tag{7.12.35}$$

The verification that $\vec{f}_4 = \frac{1}{5}\left(5x^3 - 3x\right)$ is omitted.

Definition 7.12.3: Orthogonal Projection

The projection on U with kernel $U^{\perp}$ is called the **orthogonal projection** on U (or simply the projection on U) and is denoted $\text{proj}_U : V \to V$.

Theorem 7.12.12

Let U be a finite dimensional subspace of an inner product space V and let $\vec{v}$ be a vector in V.

1. $\text{proj}_U : V \to V$ is a linear operator with image U and kernel $U^{\perp}$.
2. $\text{proj}_U \vec{v}$ is in U and $\vec{v} - \text{proj}_U \vec{v}$ is in $U^{\perp}$.
3. If $\left\{\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_m\right\}$ is any orthogonal basis of U, then

$$\text{proj}_U \vec{v} = \frac{\left\langle \vec{v}, \vec{f}_1 \right\rangle}{\left\| \vec{f}_1 \right\|^2} \vec{f}_1 + \frac{\left\langle \vec{v}, \vec{f}_2 \right\rangle}{\left\| \vec{f}_2 \right\|^2} \vec{f}_2 + \cdots + \frac{\left\langle \vec{v}, \vec{f}_m \right\rangle}{\left\| \vec{f}_m \right\|^2} \vec{f}_m \tag{7.12.36}$$

Proof

Only (3) remains to be proved. But since $\left\{\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_n\right\}$ is an orthogonal basis of U and since $\text{proj}_U \vec{v}$ is in U, the result follows from the expansion theorem applied to the finite dimensional space U.

The vectors $\vec{v}, \text{proj}_U \vec{v}$, and $\vec{v} - \text{proj}_U \vec{v}$ in Theorem 7.12.7 can be visualized geometrically (where U is shaded and $\dim U = 2$). This suggests that $\text{proj}_U \vec{v}$ is the vector in U closest to $\vec{v}$. This is, in fact, the case.

Theorem 7.12.13

Let U be a finite dimensional subspace of an inner product space V. If $\vec{v}$ is any vector in V, then $\text{proj}_U \vec{v}$ is the vector in U that is closest to $\vec{v}$. Here closest means that

$$\left\| \vec{v} - \text{proj}_U \vec{v} \right\| < \| \vec{v} - \vec{u} \| \tag{7.12.37}$$

for all $\vec{u}$ in $U, \vec{u} \neq \text{proj}_U \vec{v}$.

Proof

Write $\vec{p} = \text{proj}_U \vec{v}$, and consider $\vec{v} - \vec{u} = (\vec{v} - \vec{p}) + (\vec{p} - \vec{u})$. Because $\vec{v} - \vec{p}$ is in $U^{\perp}$ and $\vec{p} - \vec{u}$ is in U, Pythagoras' Theorem gives

$$\| \vec{v} - \vec{u} \|^2 = \| \vec{v} - \vec{p} \|^2 + \| \vec{p} - \vec{u} \|^2 > \| \vec{v} - \vec{p} \|^2 \tag{7.12.38}$$

because $\vec{p} - \vec{u} \neq 0$. The result follows.

Example 7.12.13

Consider the space $C[-1, 1]$ of real-valued continuous functions on the interval $[-1, 1]$ with inner product $\langle f, g \rangle = \int_{-1}^{1} f(x) g(x) dx$. Find the polynomial $p = p(x)$ of degree at most 2 that best approximates the absolute-value function f given by $f(x) = |x|$.

Solution

Here we want the vector p in the subspace $U = \mathbb{P}_2$ of $C[-1,1]$ that is closest to f. The Gram-Schmidt algorithm is applied to give an orthogonal basis $\left\{\vec{f}_1 = 1, \vec{f}_2 = x, \vec{f}_3 = 3x^2 - 1\right\}$ of $\mathbb{P}_2$ (where, for convenience, we have changed $\vec{f}_3$ by a numerical factor). Hence the required polynomial is

$$\begin{aligned} p &= \text{proj}_{\vec{P}_2} f \\ &= \frac{\left\langle f, \vec{f}_1 \right\rangle}{\left\| \vec{f}_1 \right\|^2} \vec{f}_1 + \frac{\left\langle f, \vec{f}_2 \right\rangle}{\left\| \vec{f}_2 \right\|^2} \vec{f}_2 + \frac{\left\langle f, \vec{f}_3 \right\rangle}{\left\| \vec{f}_3 \right\|^2} \vec{f}_3 \\ &= \frac{1}{2} \vec{f}_1 + 0 \vec{f}_2 + \frac{1/2}{8/5} \vec{f}_3 \\ &= \frac{3}{16} \left(5x^2 + 1\right) \end{aligned} \tag{7.12.39}$$

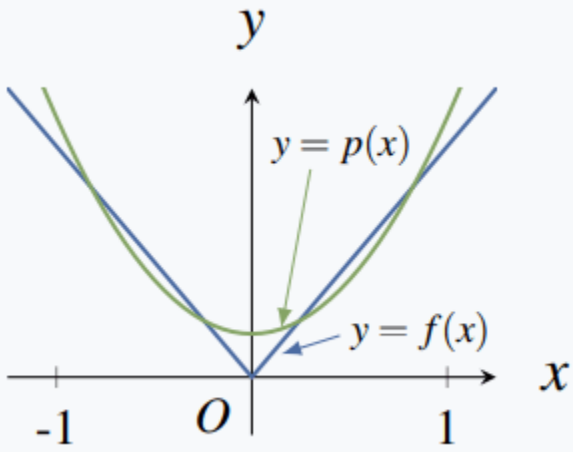


Figure 7.12.3: The graphs of $p(x)$ and $f(x)$ are given in the diagram. The graph of $f(x)$ is shaped like a 'V' as it's the graph of the absolute value function. The graph of $p(x)$ is a parabola that opens up with vertex above the origin. The graph of p crosses the graph of f four times. (CC BY-NC-SA 4.0; Nicholson via LINEAR ALGEBRA With Application)

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7.12E: Exercises for Section 7.12

Exercise 7.12E.1

In each case, determine which of axioms 1-5 fail to hold.

a. $V = \mathbb{R}^2, \langle (x_1, y_1), (x_2, y_2) \rangle = x_1 y_1 x_2 y_2$
b. $V = \mathbb{R}^3, \langle (x_1, x_2, x_3), (y_1, y_2, y_3) \rangle = x_1 y_1 - x_2 y_2 + x_3 y_3$
c. $V = \mathbb{C}, \langle z, w \rangle = z\bar{w}$, where $\bar{w}$ is complex conjugation
d. $V = \mathbb{P}_3$,$\langle p(x), q(x) \rangle = p(1)q(1)$
e. $V = \mathbb{M}_{2,2}$,$\langle A, B \rangle = \det(AB)$

Exercise 7.12E.2

Let V be an inner product space. If $U \subseteq V$ is a subspace, show that U is an inner product space using the same inner product.

Exercise 7.12E.3

In each case, find a scalar multiple of $\vec{v}$ that is a unit vector.

a. $\vec{v} = f$ in $C[0, 1]$ with $f(x) = x^2$, and $\langle \vec{f}, \vec{g} \rangle = \int_0^1 f(x)g(x)dx$
b. $\vec{v} = f$ in $C[-\pi, \pi]$ with $f(x) = \cos x$, and $\langle \vec{f}, \vec{g} \rangle = \int_{-\pi}^{\pi} f(x)g(x)dx$
c. $\vec{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ in $\mathbb{R}^2$, and $\langle \vec{v}, \vec{w} \rangle = \vec{v}^T \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \vec{w}$
d. $\vec{v} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ in $\mathbb{R}^2$, and $\langle \vec{v}, \vec{w} \rangle = \vec{v}^T \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \vec{w}$

Answer available at the end of this section.

Exercise 7.12E.4

In each case, find the distance between $\vec{u}$ and $\vec{v}$.

a. $\vec{u} = (3, -1, 2, 0), \vec{v} = (1, 1, 1, 3); \langle \vec{u}, \vec{v} \rangle = \vec{u} \cdot \vec{v}$
b. $\vec{u} = (1, 2, -1, 2), \vec{v} = (2, 1, -1, 3); \langle \vec{u}, \vec{v} \rangle = \vec{u}. \vec{v}$
c. $\vec{u} = f, \vec{v} = g$ in $C[0, 1]$ where $f(x) = x^2$ and $g(x) = 1 - x$; $\langle f, g \rangle = \int_0^1 f(x)g(x)dx$
d. $\vec{u} = f, \vec{v} = g$ in $C[-\pi, \pi]$ where $f(x) = 1$ and $g(x) = \cos x$; $\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)dx$

Answer available at the end of this section.

Exercise 7.12E.5

Let $a_1, a_2, \ldots, a_n$ be positive numbers. Given $\vec{v} = (v_1, v_2, \ldots, v_n)$ and $\vec{w} = (w_1, w_2, \ldots, w_n)$, define $\langle \vec{v}, \vec{w} \rangle = a_1 v_1 w_1 + \ldots + a_n v_n w_n$. Show that this is an inner product on $\mathbb{R}^n$.

Exercise 7.12E.6

If $\left\{\vec{b}_1, \ldots, \vec{b}_n\right\}$ is a basis of V and if $\vec{v} = v_1\vec{b}_1 + \cdots + v_n\vec{b}_n$ and $\vec{w} = w_1\vec{b}_1 + \cdots + w_n\vec{b}_n$ are vectors in V, define

$$\langle \vec{v}, \vec{w} \rangle = v_1w_1 + \cdots + v_nw_n.$$

Show that this is an inner product on V.

Answer available at the end of this section.

Exercise 7.12E.7

If $p = p(x)$ and $q = q(x)$ are polynomials in $\mathbb{P}_n$, define

$$\langle p, q \rangle = p(0)q(0) + p(1)q(1) + \cdots + p(n)q(n)$$

Show that this is an inner product on $\mathbb{P}_n$.

Exercise 7.12E.8

Let D_n denote the space of all functions from the set $\{1, 2, 3, \ldots, n\}$ to $\mathbb{R}$ with pointwise addition and scalar multiplication. Show that $\langle , \rangle$ is an inner product on D_n if $\langle f, g \rangle = f(1)g(1) + f(2)g(2) + \ldots + f(n)g(n)$.

Answer available at the end of this section.

Exercise 7.12E.9

Let re (z) denote the real part of the complex number z. Show that $\langle , \rangle$ is an inner product on $\mathbb{C}$ if $\langle z, w \rangle = re(z\bar{w})$.

Exercise 7.12E.10

If $T : V \to V$ is an isomorphism of the inner product space V, show that

$$\langle \vec{v}, \vec{w} \rangle_1 = \langle T(\vec{v}), T(\vec{w}) \rangle$$

defines a new inner product $\langle , \rangle_1$ on V.

Exercise 7.12E.11

Show that every inner product $\langle , \rangle$ on $\mathbb{R}^n$ has the form $\langle \vec{x}, \vec{y} \rangle = (U\vec{x}) \cdot (U\vec{y})$ for some upper triangular matrix U with positive diagonal entries.

Exercise 7.12E.12

In each case, show that $\langle \vec{v}, \vec{w} \rangle = \vec{v}^T A\vec{w}$ defines an inner product on $\mathbb{R}^2$ and hence show that A is positive definite.

a. $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

b. $A = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$
c. $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$
d. $A = \begin{bmatrix} 3 & 4 \\ 4 & 6 \end{bmatrix}$

Exercise 7.12E.13

In each case, find a symmetric matrix A such that $\langle \vec{v}, \vec{w} \rangle = \vec{v}^T A \vec{w}$.

a. $\left\langle \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right\rangle = v_1 w_1 + 2 v_1 w_2$
b. $\left\langle \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right\rangle = v_1 w_1 - v_1 w_2 - v_2 w_1$
c. $\left\langle \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \right\rangle = 2 v_1 w_1 + v_2 w_2$

Answer available at the end of this section.

Exercise 7.12E.14

If A is symmetric and $\vec{x}^T A \vec{x} = 0$ for all columns $\vec{x}$ in $\mathbb{R}^n$, show that $A = 0$. [Hint: Consider $\langle \vec{x} + \vec{y}, \vec{x} + \vec{y} \rangle$ where $\langle \vec{x}, \vec{y} \rangle = \vec{x}^T A \vec{y}$.]

Exercise 7.12E.15

Show that the sum of two inner products on V is again an inner product.

Exercise 7.12E.16

Let $\|\vec{u}\| = 1, \|\vec{v}\| = 2, \|\vec{w}\| = \sqrt{3}$, $\langle \vec{u}, \vec{v} \rangle = -1, \langle \vec{u}, \vec{w} \rangle = 0$ and $\langle \vec{v}, \vec{w} \rangle = 3$. Compute:

a. $\langle \vec{v} + \vec{w}, 2\vec{u} - \vec{v} \rangle$
b. $\langle \vec{u} - 2\vec{v} - \vec{w}, 3\vec{w} - \vec{v} \rangle$

Answer available at the end of this section.

Exercise 7.12E.17

Given the data in Exercise 16, show that $\vec{u} + \vec{v} = \vec{w}$.

Exercise 7.12E.18

Show that no vectors exist such that $\|\vec{u}\| = 1, \|\vec{v}\| = 2$, and $\langle \vec{u}, \vec{v} \rangle = -3$.

Exercise 7.12E.19

Let $\vec{u}$ and $\vec{v}$ be vectors in an inner product space V.

a. Expand $\langle 2\vec{u} - 7\vec{v}, 3\vec{u} + 5\vec{v} \rangle$.
b. Expand $\langle 3\vec{u} - 4\vec{v}, 5\vec{u} + \vec{v} \rangle$.
c. Show that $\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + 2\langle \vec{u}, \vec{v} \rangle + \|\vec{v}\|^2$.
d. Show that $\|\vec{u} - \vec{v}\|^2 = \|\vec{u}\|^2 - 2\langle \vec{u}, \vec{v} \rangle + \|\vec{v}\|^2$.

Exercise 7.12E.20

Show that $\|\vec{v}\|^2 + \|\vec{w}\|^2 = \frac{1}{2}\left\{\|\vec{v} + \vec{w}\|^2 + \|\vec{v} - \vec{w}\|^2\right\}$ for any $\vec{v}$ and $\vec{w}$ in an inner product space.

Answer available at the end of this section.

Exercise 7.12E.21

Let $\langle , \rangle$ be an inner product on a vector space V. Show that the corresponding distance function is translation invariant. That is, show that $\mathrm{d}(\vec{v}, \vec{w}) = \mathrm{d}(\vec{v} + \vec{u}, \vec{w} + \vec{u})$ for all $\vec{v}, \vec{w}$, and $\vec{u}$ in V.

Exercise 7.12E.22

Show that $\langle \vec{u}, \vec{v} \rangle = \frac{1}{4}\left[\|\vec{u} + \vec{v}\|^2 = \|\vec{u} - \vec{v}\|^2\right]$ for all $\vec{u}, \vec{v}$ in an inner product space V.

Answer available at the end of this section.

Exercise 7.12E.23

Let $\vec{v}$ denote a vector in an inner product space V.

a. Show that $W = \{\vec{w} \mid \vec{w} \text{ in } V, \langle \vec{v}, \vec{w} \rangle = 0\}$ is a subspace of V.
b. If $V = \mathbb{R}^3$ with the dot product, and if $\vec{v} = (1, -1, 2)$, find a basis for $W (W$ as in (a)).

Exercise 7.12E.24

Given vectors $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n$ and $\vec{v}$, assume that $\langle \vec{v}, \vec{w}_i \rangle = 0$ for each i. Show that $\langle \vec{v}, \vec{w} \rangle = 0$ for all $\vec{w}$ in $\operatorname{span}\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n\}$.

Exercise 7.12E.25

If $V = \operatorname{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ and $\langle \vec{v}, \vec{v}_i \rangle = \langle \vec{w}, \vec{v}_i \rangle$ holds for each i. Show that $\vec{v} = \vec{w}$.

Exercise 7.12E.26

Use the Cauchy-Schwarz inequality in an inner product space to show that:

a. If $\|\vec{u}\| \leq 1$, then $\langle \vec{u}, \vec{v} \rangle^2 \leq \|\vec{v}\|^2$ for all $\vec{v}$ in V.
b. $(x \cos\theta + y \sin\theta)^2 \leq x^2 + y^2$ for all real x, y, and θ.
c. $\left\|r_1\vec{v}_1 + \cdots + r_n\vec{v}_n\right\|^2 \leq \left[r_1\left\|\vec{v}_1\right\| + \cdots + r_n\left\|\vec{v}_n\right\|\right]^2$ for all vectors $\vec{v}_i$, and all $r_i > 0$ in $\mathbb{R}$.

Exercise $7.12E.27$

If A is a $2 \times n$ matrix, let $\vec{u}$ and $\vec{v}$ denote the rows of A.

a. Show that $AA^T = \begin{bmatrix} \|\vec{u}\|^2 & \vec{u} \cdot \vec{v} \\ \vec{u} \cdot \vec{v} & \|\vec{v}\|^2 \end{bmatrix}$.

b. Show that $\det(AA^T) \geq 0$.

Exercise $7.12E.28$

a. If $\vec{v}$ and $\vec{w}$ are nonzero vectors in an inner product space V, show that \(-1 \leq \frac{\langle\vec{v}, \vec{w}\rangle}{\|\vec{v}\|\|\vec{w}\|} \leq 1\), and hence that a unique angle θ exists such that \(\frac{\langle\vec{v}, \vec{w}\rangle}{\|\vec{v}\|\|\vec{w}\|}=\cos \theta\) and $0 \leq \theta \leq \pi$. This angle θ is called the angle between $\vec{v}$ and $\vec{w}$.

b. Find the angle between $\vec{v} = (1, 2, -1, 1, 3)$ and $\vec{w} = (2, 1, 0, 2, 0)$ in $\mathbb{R}^5$ with the dot product.

c. If θ is the angle between $\vec{v}$ and $\vec{w}$, show that the law of cosines is valid:

$$\|\vec{v} - \vec{w}\| = \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|\vec{v}\|\|\vec{w}\| \cos\theta.$$

Answer available at the end of this section.

Answers

Exercise $7.12E.3$

a. $\sqrt{5}x^2$

b. $\frac{1}{\sqrt{\pi}}\cos x$

c. $\begin{bmatrix} \frac{1}{5} \\ \frac{3}{5} \end{bmatrix}$

d. $\begin{bmatrix} \frac{3}{\sqrt{17}} \\ \frac{-1}{\sqrt{17}} \end{bmatrix}$

Exercise $7.12E.4$

a. $3\sqrt{2}$

b. $\sqrt{3}$

c. $\sqrt{\frac{11}{30}}$

d. $\sqrt{3\pi}$

Exercise $7.12E.6$

Let $\vec{v}, \vec{w}, \vec{u} \in V$, and $r \in \mathbb{R}$. Assume vectors are expressed in coordinates with respect to a basis of V.

Axiom 1 (Real-valued): The sum $\sum_{i=1}^n v_i w_i \in \mathbb{R}$.

Axiom 2: (Symmetric): $\langle \vec{v}, \vec{w} \rangle = \sum v_i w_i = \sum w_i v_i = \langle \vec{w}, \vec{v} \rangle$.

Axiom 3 (Linearity):

$$\langle \vec{v} + \vec{w}, \vec{u} \rangle = \sum (v_i + w_i)u_i = \sum v_i u_i + \sum w_i u_i = \langle \vec{v}, \vec{u} \rangle + \langle \vec{w}, \vec{u} \rangle.$$

Axiom 4 (Homogeneity):

$$\langle r\vec{v}, \vec{w} \rangle = \sum (rv_i)w_i = r \sum v_i w_i = r\langle \vec{v}, \vec{w} \rangle.$$

Axiom 5 (Positive-definite):

$$\langle \vec{v}, \vec{v} \rangle = \sum v_i^2 \geq 0,$$

with equality if and only if $\vec{v} = \vec{0}$.

Exercise $7.12E.8$

Let $f, g, h \in D_n$, and $r \in \mathbb{R}$. We verify the inner product axioms:

Axiom 1 (Real-valued): Since $f(i), g(i) \in \mathbb{R}$, the sum $\sum_{i=1}^n f(i)g(i) \in \mathbb{R}$.

Axiom 2 (Symmetry):

$$\langle f, g \rangle = \sum_{i=1}^n f(i)g(i) = \sum_{i=1}^n g(i)f(i) = \langle g, f \rangle.$$

Axiom 3 (Linearity in the first argument):

$$\langle f+g,h\rangle=\sum_{i=1}^{n}(f(i)+g(i))h(i)=\sum f(i)h(i)+\sum g(i)h(i)=\langle f,h\rangle+\langle g,h\rangle.$$

Axiom 4 (Homogeneity):

$$\langle rf,g\rangle=\sum_{i=1}^{n}(rf(i))g(i)=r\sum f(i)g(i)=r\langle f,g\rangle.$$

Axiom 5 (Positive-definiteness):

$$\langle f,f\rangle=\sum_{i=1}^{n}f(i)^2\geq 0,$$

and equality holds if and only if $f(i)=0$ for all i, i.e., $f=0$.

Exercise $7.12E.13$

a. $A=\begin{bmatrix}1&2\\2&0\end{bmatrix}$

b. $A=\begin{bmatrix}1&-1\\-1&0\end{bmatrix}$

c. $A=\begin{bmatrix}2&0&0\\0&1&0\\0&0&0\end{bmatrix}$

Exercise $7.12E.16$

a. -9

b. -15

Exercise $7.12E.20$

To prove $\|\vec{v}\|^2+\|\vec{w}\|^2=\frac{1}{2}\left(\|\vec{v}+\vec{w}\|^2+\|\vec{v}-\vec{w}\|^2\right)$, expand the right side using inner product properties:

$\frac{1}{2}\left(\langle\vec{v}+\vec{w},\vec{v}+\vec{w}\rangle+\langle\vec{v}-\vec{w},\vec{v}-\vec{w}\rangle\right).$

After expanding and simplifying, we have:

$\frac{1}{2}\left(2\langle\vec{v},\vec{v}\rangle+2\langle\vec{w},\vec{w}\rangle\right),$

which simplifies to $\|\vec{v}\|^2+\|\vec{w}\|^2$, thus proving the identity.

Exercise $7.12E.22$

Let $\vec{u}$ and $\vec{v}$ be vectors in V. Then

$$\begin{aligned}\|\vec{u}+\vec{v}\|^2&=\langle\vec{u}+\vec{v},\vec{u}+\vec{v}\rangle\\&=\|\vec{u}\|^2+2\langle\vec{u},\vec{v}\rangle+\|\vec{v}\|^2,\\\|\vec{u}-\vec{v}\|^2&=\langle\vec{u}-\vec{v},\vec{u}-\vec{v}\rangle\\&=\|\vec{u}\|^2-2\langle\vec{u},\vec{v}\rangle+\|\vec{v}\|^2.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{1}{4}(\|\vec{u}+\vec{v}\|^2-\|\vec{u}-\vec{v}\|^2)&=\frac{1}{4}(4\langle\vec{u},\vec{v}\rangle)\\&=\langle\vec{u},\vec{v}\rangle.\end{aligned}$$

Exercise $7.12E.28$

b. $\theta=\arccos\left(\frac{1}{2}\right)=\frac{\pi}{3}$ or 60°

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8: APPENDICES

8.1: Sets and Set Notation
8.2: Well Ordering and Induction

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8.1: Sets and Set Notation

A set is a collection of things called **elements**. For example $\{1,2,3,8\}$ would be a set consisting of the elements 1,2,3, and 8. To indicate that 3 is an element of $\{1,2,3,8\}$, it is customary to write $3 \in \{1,2,3,8\}$. We can also indicate when an element is not in a set, by writing $9 \notin \{1,2,3,8\}$ which says that 9 is not an element of $\{1,2,3,8\}$. Sometimes a rule specifies a set. For example you could specify a set as all integers larger than 2. This would be written as

$$S = \{x \in \mathbb{Z} : x > 2\}.$$

This notation says: S is the set of all integers, x, such that $x > 2$.

Suppose A and B are sets with the property that every element of A is an element of B. Then we say that A is a subset of B. For example, $\{1,2,3,8\}$ is a subset of $\{1,2,3,4,5,8\}$. In symbols, we write

$$\{1,2,3,8\} \subseteq \{1,2,3,4,5,8\}.$$

It is sometimes said that "A is contained in B" or even "B contains A". The same statement about the two sets may also be written as

$$\{1,2,3,4,5,8\} \supseteq \{1,2,3,8\}.$$

We can also talk about the **union** of two sets, which we write as $A \cup B$. This is the set consisting of everything which is an element of at least one of the sets, A or B. As an example of the union of two sets, consider

$$\{1,2,3,8\} \cup \{3,4,7,8\} = \{1,2,3,4,7,8\}.$$

This set is made up of the numbers which are in at least one of the two sets.

In general

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

Notice that an element which is in *both* A and B is also in the union, as well as elements which are in only one of A or B.

Another important set is the intersection of two sets A and B, written $A \cap B$. This set consists of everything which is in *both* of the sets. Thus $\{1,2,3,8\} \cap \{3,4,7,8\} = \{3,8\}$ because 3 and 8 are those elements the two sets have in common. In general,

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

If A and B are two sets, $A \setminus B$ denotes the set of things which are in A but not in B. Thus

$$A \setminus B = \{x \in A : x \notin B\}$$

For example, if $A = \{1,2,3,8\}$ and $B = \{3,4,7,8\}$, then $A \setminus B = \{1,2,3,8\} \setminus \{3,4,7,8\} = \{1,2\}$.

A special set which is very important in mathematics is the empty set denoted by $\emptyset$, which is defined as the set which has no elements in it. It follows that the empty set is a subset of every set. This is true because if it were not so, there would have to exist a set A, such that $\emptyset$ has something in it which is not in A. However, $\emptyset$ has nothing in it and so it must be that $\emptyset \subseteq A$.

We can also use brackets to denote sets which are intervals of numbers. Let a and b be real numbers. Then

- $[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\}$
- \(\left[a,b \right) = \{x \in \mathbb{R}: a\leq x
- **\(\left(a,b\right) = \{x \in \mathbb{R}: a**
- **\((a,b] = \{ x \in \mathbb{R}: a**
- $[a,\infty) = \{x \in \mathbb{R} : x \geq a\}$
- $(-\infty,a] = \{x \in \mathbb{R} : x \leq a\}$

These sorts of sets of real numbers are called intervals. The two points a and b are called endpoints, or bounds, of the interval. In particular, a is the *lower bound* while b is the *upper bound* of the above intervals, where applicable.

Other intervals such as $(-\infty, b)$ are defined by analogy to what was just explained. In general, the curved parenthesis, (, indicates the end point is not included in the interval, while the square parenthesis, [, indicates this end point is included. The reason that there will always be a curved parenthesis next to ∞ or $-\infty$ is that these are not real numbers and cannot be included in the interval in the way a real number can.

To illustrate the use of this notation relative to intervals consider three examples of inequalities. Their solutions will be written in the interval notation just described.

Example 8.1.1: Solving an Inequality

Solve the inequality $2x + 4 \leq x - 8$.

Solution

We need to find x such that $2x + 4 \leq x - 8$. Solving for x, we see that $x \leq -12$ is the answer. This is written in terms of an interval as $(-\infty, -12]$.

Consider the following example.

Example 8.1.2: Solving an Inequality

Solve the inequality $(x + 1)(2x - 3) \geq 0$.

Solution

We need to find x such that $(x + 1)(2x - 3) \geq 0$. The solution is given by $x \leq -1$ or $x \geq \frac{3}{2}$. Therefore, x which fit into either of these intervals gives a solution. In terms of set notation this is denoted by $(-\infty, -1] \cup [\frac{3}{2}, \infty)$.

Consider one last example.

Example 8.1.3: Solving an Inequality

Solve the inequality $x(x + 2) \geq -4$.

Solution

This inequality is true for any value of x where x is a real number. We can write the solution as $\mathbb{R}$ or $(-\infty, \infty)$.

In the next section, we examine another important mathematical concept.

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8.2: Well Ordering and Induction

We begin this section with some important notation. Summation notation, written $\sum_{i=1}^{j} i$, represents a sum. Here, i is called the index of the sum, and we add iterations until $i = j$. For example,

$$\sum_{i=1}^{j} i = 1 + 2 + \cdots + j$$

Another example:

$$a_{11} + a_{12} + a_{13} = \sum_{i=1}^{3} a_{1i}$$

The following notation is a specific use of summation notation.

Notation : Summation Notation

Let be real numbers, and suppose while . These numbers can be listed in a rectangular array as given by

Then means to first sum the numbers in each column (using as the index) and then to add the sums which result (using as the index). Similarly, means to sum the vectors in each row (using as the index) and then to add the sums which result (using as the index).

Notice that since addition is commutative,

We now consider the main concept of this section. Mathematical induction and well ordering are two extremely important principles in math. They are often used to prove significant things which would be hard to prove otherwise.

Definition : Well Ordered

A set is **well ordered** if every nonempty subset contains a smallest element having the property that for all

In particular, the set of natural numbers defined as is well ordered.

Consider the following proposition.

Proposition : Well Ordered Sets

Any set of integers larger than a given number is well ordered.

This proposition claims that if a set has a lower bound which is a real number, then this set is well ordered.

Further, this proposition implies the principle of mathematical induction. The symbol denotes the set of all integers. Note that if is an integer, then there are no integers between and

Theorem : Mathematical Induction

A set having the property that and whenever , contains all integers such that

Proof available at the end of this section.

Mathematical induction is a very useful device for proving theorems about the integers. The procedure is as follows.

Procedure : Proof by Mathematical Induction

Suppose is a statement which is a function of the number , for , and we wish to show that is true for all . To do so using mathematical induction, use the following steps.

1. **Base Case:** Show is true.
2. Assume is true for some , which is the **induction hypothesis**. Then, using this assumption, show that is true.

Proving these two steps shows that is true for all .

We can use this procedure to solve the following examples.

Example : Proving by Induction

Prove by induction that .

Solution

By Procedure , we first need to show that this statement is true for . When , the statement says that

The sum on the left hand side also equals , so this equation is true for .

Now suppose this formula is valid for some where is an integer. Hence, the following equation is true.

We want to show that this is true for .

Suppose we add to both sides of Equation .

The step going from the first to the second line is based on the assumption that the formula is true for Now simplify the expression in the second line,

This equals and

Therefore, showing the formula holds for whenever it holds for This proves the formula by mathematical induction. In other words, this formula is true for all .

Consider another example.

Example : Proving an Inequality by Induction

Show that for all ,

Solution

Again we will use the procedure given in Procedure to prove that this statement is true for all . Suppose . Then the statement says which is true.

Suppose then that the inequality holds for In other words, is true.

Now multiply both sides of this inequality by . This yields

The theorem will be proved if this last expression is less than This happens if and only if which occurs if and only if and this is clearly true which may be seen from expanding both sides. This proves the inequality.

Let's review the process just used. If is the set of integers at least as large as for which the formula holds, the first step was to

show and then that whenever it follows Therefore, by the principle of mathematical induction, contains all positive integers. In doing an inductive proof of this sort, the set is normally not mentioned. One just verifies the steps above.

Answers

Theorem 8.2.1**: Mathematical Induction**

Let T consist of all integers larger than or equal to a which are not in S. The theorem will be proved if $T = \emptyset$. If $T \neq \emptyset$ then by the well ordering principle, there would have to exist a smallest element of T, denoted as b. It must be the case that $b > a$ since by definition, $a \notin T$. Thus $b \geq a + 1$, and so $b - 1 \geq a$ and $b - 1 \notin S$ because if $b - 1 \in S$, then $b - 1 + 1 = b \in S$ by the assumed property of S. Therefore, $b - 1 \in T$ which contradicts the choice of b as the smallest element of T. ($b - 1$ is smaller.) Since a contradiction is obtained by assuming $T \neq \emptyset$, it must be the case that $T = \emptyset$ and this says that every integer at least as large as a is also in S.

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