Calculus III: Series and Vector Calculus

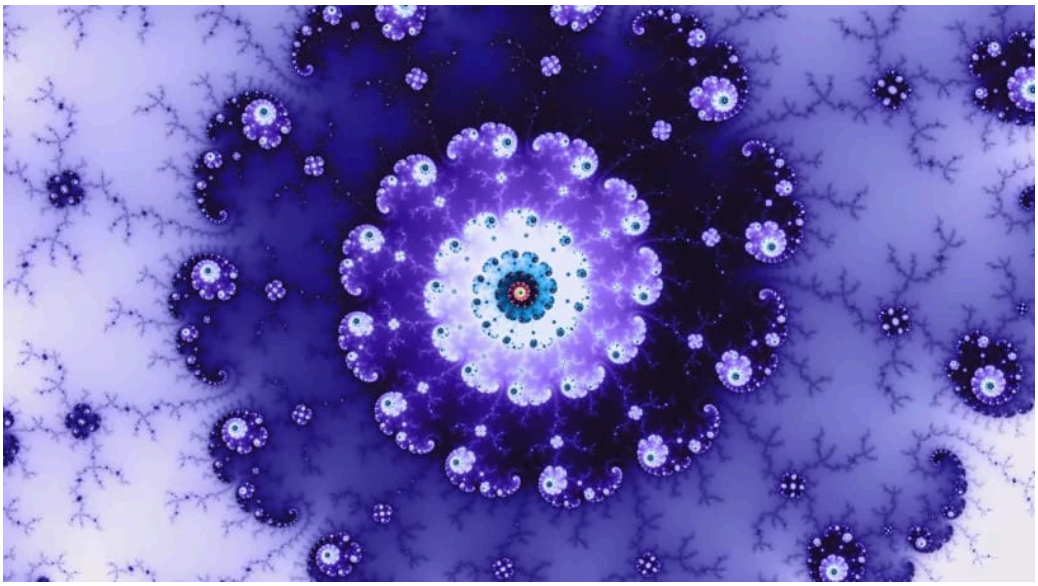
Doli Bambhania, Vinh Kha Nguyen & Jyothsna Viswanadha

De Anza College

Mathematics

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About the Book

Publication Date: August, 2025

This **Calculus textbook** is part of the **De Anza College Zero Textbook Cost (ZTC) Pathways** initiative, supporting both students and instructors with free, adaptable Open Educational Resources (OER).

This series builds upon the OpenStax Calculus foundation and acknowledges the contributions of Lake Tahoe Community College's derivative version, incorporating videos and interactive elements. De Anza's version enhances accessibility, adds multimedia resources, and emphasizes critical thinking and problem-solving for both classroom and independent study.

This third volume in a four-part calculus series is designed for both beginners and advanced students, with a focus on building strong conceptual understanding. It explores essential topics in depth to help students develop a solid foundation in the principles and techniques of calculus. Covering the essence of differential and integral calculus in one variable, the construction and use of Taylor series, and the study of vector calculus with its applications, the book connects theory to real-world problems across physics, economics, and engineering.

With numerous examples, practical applications, and videos for visual demonstrations, the series is crafted to engage learners and support multiple ways of understanding. Structured to foster critical thinking and problem-solving skills, it is a valuable resource for both individual study and classroom use. With its blend of open access, practical applications, and rich multimedia support, this textbook reflects De Anza College's commitment to student equity, innovation, and leadership in OER.

In addition to the core text, we provide ancillary materials to enhance teaching and learning.

Included Ancillary Materials

- **PowerPoint Slide Decks**: Accessible chapter-aligned slides for lectures, presentations, or flipped classrooms.
- **Instructor Notes**: Guidance for educators with teaching tips, common misconceptions, and activity ideas.

These ancillaries can be found under **"assets"** on the De Anza Commons page for Calculus III: Series and Vector Calculus.

Online Homework

Aligned online homework is available on **MyOpenMath**. After logging into an instructor account, select *Add New Course* → *Copy a template or promoted course* → choose *Calculus III: Series and Vector Calculus (De Anza ZTC)*.

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About the Authors

Information about the authors

Author Images	Author Bios
	Doli Bambhania is a Mathematics Instructor at De Anza College in Cupertino, CA, with over two decades of teaching experience. She brings a vibrant energy and a student-centered approach to her work, striving to ensure success for students not just in her class but throughout their entire math journey. Doli has been involved in several programs aimed at removing barriers to math education, such as De Anza's Math Performance Success (MPS) and Learning in Communities (LinC) programs, and Statway. Additionally, she has developed open educational resources (OER) workbooks for algebra and statistics, reflecting her strong commitment to using OER materials in all her classes. Outside of teaching, Doli enjoys spending time with her family, playing games, hiking, dancing, and music.Doli received her Bachelors degree in Mathematics from University of California, San Diego, and Master degrees in Mathematics and Computer Science from University of California, Santa Barbara.
	Vinh Kha Nguyen is a dedicated math instructor at De Anza College who is passionate about making mathematics accessible to everyone. With a deep commitment to education, he strives to help students develop a strong foundation and appreciation for the subject. Outside the classroom, Vinh enjoys hiking and traveling with his family, embracing new adventures together. He is especially grateful to his wife for her unwavering support in caring for their child, allowing him the time to focus on completing his Open Educational Resources (OER) textbook to benefit students and educators alike.

Author Images	Author Bios
	Jyothsna Viswanadha is a Mathematics Instructor at De Anza College with experience in teaching community college-level mathematics courses. Passionate about student-centered and equity-minded pedagogies, she has taught courses ranging from developmental math to calculus and business calculus. Jyothsna has taught in several programs designed to bridge the equity gap in Math education, including the Math Performance Success (MPS) program. Jyothsna is committed to adopting Open Educational Resources (OER) materials for her classes and has been involved in the development of OER workbooks for Calculus III. She is also committed to community service, volunteering with organizations like Vision Literacy, and Assistance League, where she supports underserved populations in achieving their educational goals.

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1: SEQUENCES AND SERIES

The topic of infinite series may seem unrelated to differential and integral calculus. In fact, an infinite series whose terms involve powers of a variable is a powerful tool that we can use to express functions as “infinite polynomials.” We can use infinite series to evaluate complicated functions, approximate definite integrals, and create new functions. In addition, infinite series are used to solve differential equations that model physical behavior, from tiny electronic circuits to Earth-orbiting satellites.

The Koch snowflake is constructed from an infinite number of nonoverlapping equilateral triangles. Consequently, we can express its area as a sum of infinitely many terms. How do we add an infinite number of terms? Can a sum of an infinite number of terms be finite? To answer these questions, we need to introduce the concept of an infinite series, a sum with infinitely many terms. Having defined the necessary tools, we will be able to calculate the area of the Koch snowflake.

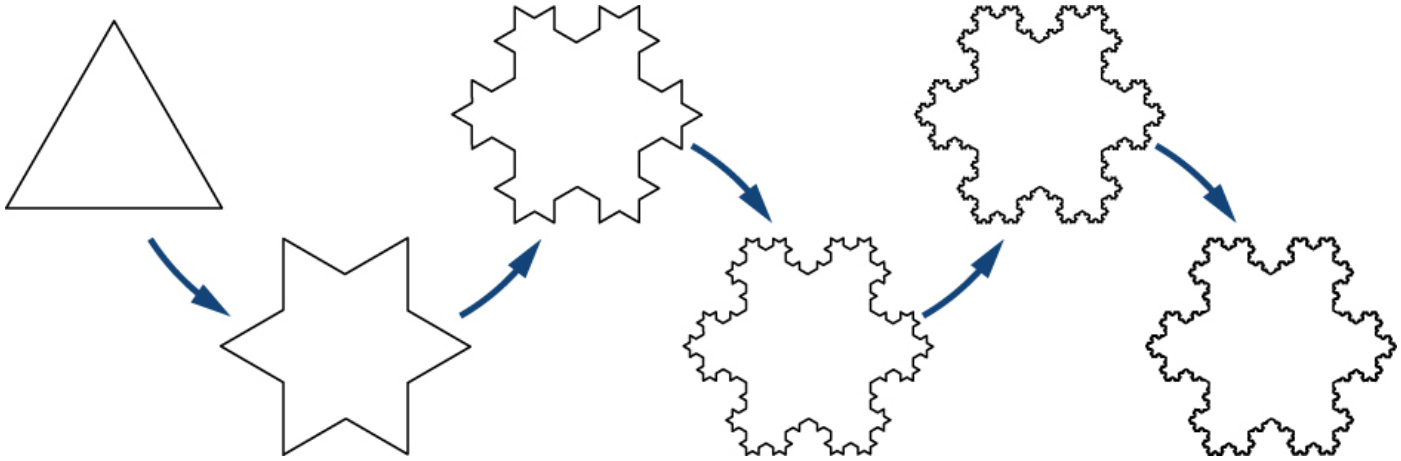


Figure 1: The Koch snowflake is constructed by using an iterative process. Starting with an equilateral triangle, at each step of the process the middle third of each line segment is removed and replaced with an equilateral triangle pointing outward.

1.1: Sequences

1.1E: Exercises for Section 1.1

1.2: Infinite Series

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1.1: Sequences

Learning Objectives

- Find the formula for the general term of a sequence.
- Calculate the limit of a sequence if it exists.
- Determine the convergence or divergence of a given sequence.

In this section, we introduce sequences and define what it means for a sequence to converge or diverge. We show how to find limits of sequences that converge, often by using the properties of limits for functions discussed earlier. We close this section with the Monotone Convergence Theorem, a tool we can use to prove that certain types of sequences converge.

Terminology of Sequences

To work with this new topic, we need some new terms and definitions. First, an infinite sequence is an ordered list of numbers of the form

$$a_1, a_2, a_3, \ldots, a_n, \ldots.$$

Each of the numbers in the sequence is called a term. The symbol n is called the index variable for the sequence. We use the notation

$$\{a_n\}_{n=1}^{\infty},$$

or simply $\{a_n\}$, to denote this sequence. A similar notation is used for sets, but a sequence is an ordered list, whereas a set is not ordered. Because a particular number a_n exists for each positive integer n, we can also define a sequence as a function whose domain is the set of positive integers.

Let's consider the infinite, ordered list

$$2, 4, 8, 16, 32, \ldots.$$

This is a sequence in which the first, second, and third terms are given by $a_1 = 2, a_2 = 4,$ and $a_3 = 8.$ You can probably see that the terms in this sequence have the following pattern:

$$a_1 = 2^1,\ a_2 = 2^2,\ a_3 = 2^3,\ a_4 = 2^4 \text{ and } a_5 = 2^5$$

Assuming this pattern continues, we can write the nth term in the sequence by the explicit formula $a_n = 2^n.$ Using this notation, we can write this sequence as

$$\{2^n\}_{n=1}^{\infty}$$

or

$$\{2^n\}.$$

Alternatively, we can describe this sequence in a different way. Since each term is twice the previous term, this sequence can be defined recursively by expressing the nth term a_n in terms of the previous term a_{n-1}. In particular, we can define this sequence as the sequence $\{a_n\}$ where $a_1 = 2$ and for all $n \geq 2$, each term an is defined by the **recurrence relation**

$$a_n = 2a_{n-1}. \tag{1.1.1}$$

Definition: Infinite Sequence

An infinite sequence $\{a_n\}$ is an ordered list of numbers of the form

$$a_1,\ a_2,\ \ldots,\ a_n,\ \ldots.$$

The subscript n is called the **index variable** of the sequence. Each number a_n is a term of the sequence. Sometimes sequences are defined by **explicit formulas,** in which case $a_n = f(n)$ for some function $f(n)$ defined over the positive integers. In other cases, sequences are defined by using a **recurrence relation**. In a recurrence relation, one term (or more) of the sequence is given explicitly, and subsequent terms are defined in terms of earlier terms in the sequence.

Note that the index does not have to start at $n = 1$ but could start with other integers. For example, a sequence given by the explicit formula $a_n = f(n)$ could start at $n = 0$, in which case the sequence would be

$$a_0,\ a_1,\ a_2, \ldots.$$

Similarly, for a sequence defined by a recurrence relation, the term a_0 may be given explicitly, and the terms a_n for $n \geq 1$ may be defined in terms of a_{n-1}. Since a sequence $\{a_n\}$ has exactly one value for each positive integer n, it can be described as a function whose domain is the set of positive integers. As a result, it makes sense to discuss the graph of a sequence. The graph of a sequence $\{a_n\}$ consists of all points (n, a_n) for all positive integers n. Figure shows the graph of 2^n.

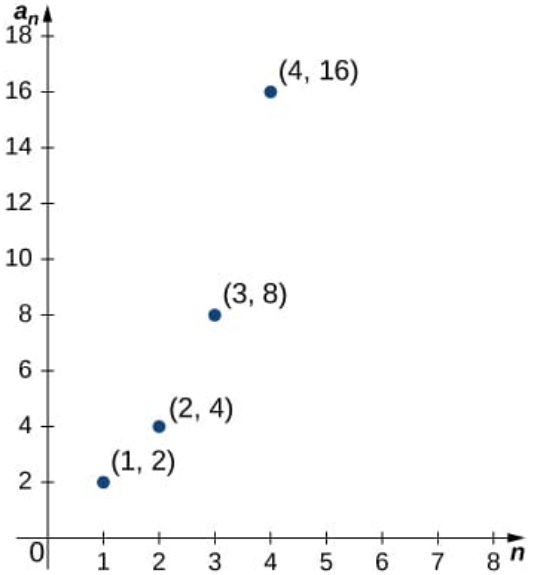


Figure 1.1.1: The image is a graph in the Cartesian coordinate system. The horizontal axis is labeled "n" and the vertical axis is labeled "a_n". The graph depicts a series of dots plotted at coordinates (1, 2), (2, 4), (3, 8), and (4, 16). The horizontal axis ranges from 0 to 8 and the vertical axis ranges from 0 to 18. The dots form an upward curve as they increase in both n and a_n values.

Two types of sequences occur often and are given special names: arithmetic sequences and geometric sequences. In an arithmetic sequence, the difference between every pair of consecutive terms is the same. For example, consider the sequence

$$3,\ 7,\ 11,\ 15, 19,\ \ldots$$

You can see that the difference between every consecutive pair of terms is 4. Assuming that this pattern continues, this sequence is an arithmetic sequence. It can be described by using the recurrence relation

$$\begin{cases} a_1 = 3 \\ a_n = a_{n-1} + 4, \text{ for } n \geq 2 \end{cases}.$$

Note that

$$a_2 = 3 + 4$$

$$a_3 = 3 + 4 + 4 = 3 + 2 \cdot 4$$

$$a_4 = 3 + 4 + 4 + 4 = 3 + 3 \cdot 4.$$

Thus the sequence can also be described using the explicit formula

$$a_n = 3 + 4(n-1) = 4n - 1.$$

In general, an arithmetic sequence is any sequence of the form $a_n = cn + b$.

In a geometric sequence**,** the ratio of every pair of consecutive terms is the same. For example, consider the sequence

$$2, -\frac{2}{3}, \frac{2}{9}, -\frac{2}{27}, \frac{2}{81}, \ldots.$$

We see that the ratio of any term to the preceding term is $-\frac{1}{3}$. Assuming this pattern continues, this sequence is a geometric sequence. It can be defined recursively as

$$a_1 = 2$$

$$a_n = -\frac{1}{3} \cdot a_{n-1}, \text{ for } n \geq 2.$$

Alternatively, since

$$a_2 = -\frac{1}{3} \cdot 2$$

$$a_3 = \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)(2) = \left(-\frac{1}{3}\right)^2 \cdot 2$$

$$a_4 = \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)(2) = \left(-\frac{1}{3}\right)^3 \cdot 2,$$

we see that the sequence can be described by using the explicit formula

$$a_n = 2\left(-\frac{1}{3}\right)^{n-1}.$$

The sequence $\{2^n\}$ that we discussed earlier is a geometric sequence, where the ratio of any term to the previous term is 2. In general, a geometric sequence is any sequence of the form $a_n = cr^n$.

Example 1.1.1: Finding Explicit Formulas

For each of the following sequences, find an explicit formula for the nth term of the sequence.

a. $-\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, -\frac{5}{6}, \ldots$

b. $\frac{3}{4}, \frac{9}{7}, \frac{27}{10}, \frac{81}{13}, \frac{243}{16}, \ldots$.

Solution

a. First, note that the sequence is alternating from negative to positive. The odd terms in the sequence are negative, and the even terms are positive. Therefore, the nth term includes a factor of $(-1)^n$. Next, consider the sequence of numerators $1, 2, 3, \ldots$ and the sequence of denominators $2, 3, 4, \ldots$. We can see that both of these sequences are arithmetic sequences. The nth term in the sequence of numerators is n, and the nth term in the sequence of denominators is $n + 1$. Therefore, the sequence can be described by the explicit formula

$$a_n = \frac{(-1)^n n}{n+1}. \tag{1.1.2}$$

b. The sequence of numerators $3, 9, 27, 81, 243, \ldots$ is a geometric sequence. The numerator of the nth term is 3^n The sequence of denominators $4, 7, 10, 13, 16, \ldots$ is an arithmetic sequence. The denominator of the nth term is $4 + 3(n-1) = 3n + 1$. Therefore, we can describe the sequence by the explicit formula $a_n = \dfrac{3^n}{3n+1.}$

Exercise 1.1.1

Find an explicit formula for the nth term of the sequence $\left\{\frac{1}{5}, -\frac{1}{7}, \frac{1}{9}, -\frac{1}{11}, \ldots\right\}$.

Hint available at the end of this section.

Answer available at the end of this section.

Example 1.1.2: Defined by Recurrence Relations

For each of the following recursively defined sequences, find an explicit formula for the sequence.

a. $a_1 = 2, a_n = -3a_{n-1}$ for $n \geq 2$
b. $a_1 = \left(\frac{1}{2}\right), a_n = a_{n-1} + \left(\frac{1}{2}\right)^n$ for $n \geq 2$

Solution

a. Writing out the first few terms, we have

$$\begin{aligned} a_1 &= 2 \\ a_2 &= -3a_1 = -3(2) \\ a_3 &= -3a_2 = (-3)^2 2 \\ a_4 &= -3a_3 = (-3)^3 2. \end{aligned}$$

In general,

$$a_n = 2(-3)^{n-1}.$$

b. Write out the first few terms:

$$a_1 = \frac{1}{2}$$

$$a_2 = a_1 + \left(\frac{1}{2}\right)^2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$a_3 = a_2 + \left(\frac{1}{2}\right)^3 = \frac{3}{4} + \frac{1}{8} = \frac{7}{8}$$

$$a_4 = a_3 + \left(\frac{1}{2}\right)^4 = \frac{7}{8} + \frac{1}{16} = \frac{15}{16}.$$

From this pattern, we derive the explicit formula

$$a_n = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}.$$

Exercise 1.1.2

Find an explicit formula for the sequence defined recursively such that $a_1 = -4$ and $a_n = a_{n-1} + 6$.

Hint available at the end of this section.

Answer available at the end of this section.

Limit of a Sequence

A fundamental question that arises regarding infinite sequences is the behavior of the terms as n gets larger. Since a sequence is a function defined on the positive integers, it makes sense to discuss the limit of the terms as $n \to \infty$. For example, consider the following four sequences and their different behaviors as $n \to \infty$ (Figure 1.1.2):

a. $\{1 + 3n\} = \{4, 7, 10, 13, ...\}$. The terms $1 + 3n$ become arbitrarily large as $n \to \infty$. In this case, we say that $1 + 3n \to \infty$ as $n \to \infty$.

b. $\left\{1 - \left(\frac{1}{2}\right)^n\right\} = \left\{\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}\ ...\right\}$. The terms $1 - \left(\frac{1}{2}\right)^n \to 1$ as $n \to \infty$.

c. $\{(-1)^n\} = \{-1, 1, -1, 1, ...\}$. The terms alternate but do not approach one single value as $n \to \infty$.

d. $\left\{\frac{(-1)^n}{n}\right\} = \left\{-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, ...\right\}$. The terms alternate for this sequence as well, but $\frac{(-1)^n}{n} \to 0$ as $n \to \infty$.

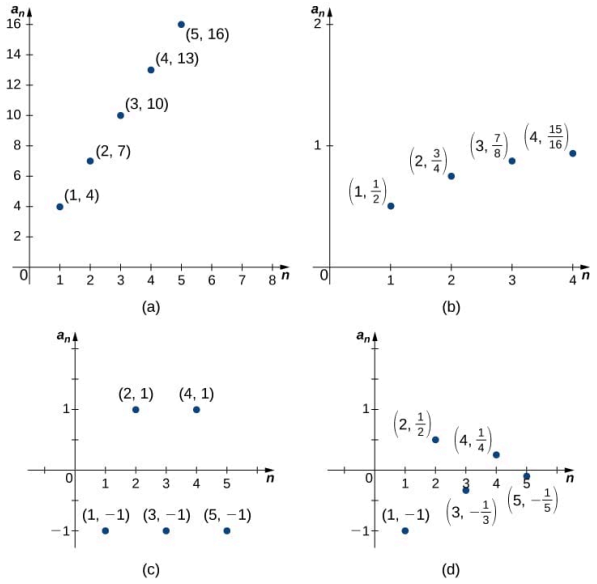


Figure 1.1.2: (a) The terms in the sequence become arbitrarily large as $n \to \infty$. (b) The terms in the sequence approach 1 as $n \to \infty$. (c) The terms in the sequence alternate between 1 and -1 as $n \to \infty$. (d) The terms in the sequence alternate between positive and negative values but approach 0 as $n \to \infty$.

From these examples, we see several possibilities for the behavior of the terms of a sequence as $n \to \infty$. In two of the sequences, the terms approach a finite number as $n \to \infty$. In the other two sequences, the terms do not. If the terms of a sequence approach a finite number L as $n \to \infty$, we say that the sequence is a convergent sequence and the real number L is the limit of the sequence. We can give an informal definition here.

Definition: Convergent and Divergent Sequences

Given a sequence a_n, if the terms become arbitrarily close to a finite number L as n becomes sufficiently large, we say $\{a_n\}$ is a **convergent sequence** and L is the **limit of the sequence.** In this case, we write

$$\lim_{n\to\infty} a_n = L. \tag{1.1.3}$$

If a sequence $\{a_n\}$ is not convergent, we say it is a **divergent sequence.**

From Figure $1.1.2b$, we see that the terms in the sequence $\left\{1-\left(\frac{1}{2}\right)^n\right\}$ are becoming arbitrarily close to 1 as n becomes very large. We conclude that $\left\{1-\left(\frac{1}{2}\right)^n\right\}$ is a convergent sequence and its limit is 1. In contrast, from Figure $1.1.2a$, we see that the terms in the sequence $1+3n$ are not approaching a finite number as n becomes larger. We say that $\{1+3n\}$ is a divergent sequence.

In the informal definition for the limit of a sequence, we used the terms “arbitrarily close” and “sufficiently large.” Although these phrases help illustrate the meaning of a converging sequence, they are somewhat vague. To be more precise, we now present the more formal definition of limit for a sequence and show these ideas graphically in Figure $1.1.3$.

Definition: Convergence

A sequence $\{a_n\}$ converges to a real number L if for all $\varepsilon > 0$, there exists an integer N such that for all $n \geq N$ $|a_n - L| < \varepsilon$. The number L is the limit of the sequence and we write

$$\lim_{n\to\infty} a_n = L \text{ or } a_n \to L. \tag{1.1.4}$$

In this case, we say the sequence $\{a_n\}$ is a **convergent sequence**. If a sequence does not converge, it is a **divergent sequence**, and we say the limit does not exist.

We remark that the convergence or divergence of a sequence $\{a_n\}$ depends only on what happens to the terms a_n as $n \to \infty$. Therefore, if a finite number of terms $b_1, b_2, \ldots, b_N$ are placed before a_1 to create a new sequence

$$b_1,\ b_2,\ \ldots,\ b_N,\ a_1,\ a_2,\ \ldots,$$

this new sequence will converge if $\{a_n\}$ converges and diverge if $\{a_n\}$ diverges. Further, if the sequence $\{a_n\}$ converges to L, this new sequence will also converge to L.

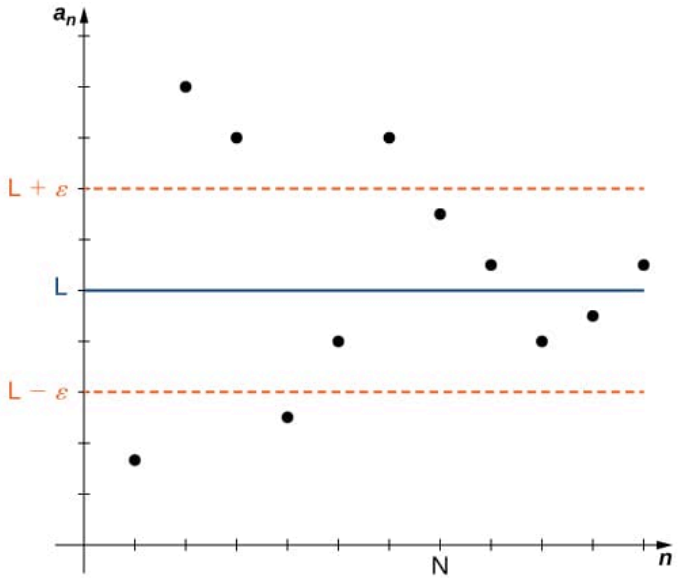


Figure $1.1.3$: The image is a graph depicting a series of points representing the terms $\{a_n\}$ as n increases. The x-axis is labeled "n" and the y-axis is labeled "$\{a_n\}$". As n approaches N, the points increasingly cluster closer to L, demonstrating the concept as n increases, the terms a_n become closer to L. For values of $n \geq N$, the distance between each point (n, a_n) and the line $y = L$ is less than ε.

As defined above, if a sequence does not converge, it is said to be a divergent sequence. For example, the sequences $\{1+3n\}$

and $\{(-1)^n\}$ shown in Figure diverge. However, different sequences can diverge in different ways. The sequence $\{(-1)^n\}$ diverges because the terms alternate between 1 and -1, but do not approach one value as $n \to \infty$. On the other hand, the sequence $\{1 + 3n\}$ diverges because the terms $1 + 3n \to \infty$ as $n \to \infty$. We say the sequence $\{1 + 3n\}$ diverges to infinity and write $\lim_{n\to\infty}(1 + 3n) = \infty$. It is important to recognize that this notation does not imply the limit of the sequence $\{1 + 3n\}$ exists. The sequence is, in fact, divergent. Writing that the limit is infinity is intended only to provide more information about why the sequence is divergent. A sequence can also diverge to negative infinity. For example, the sequence $\{-5n + 2\}$ diverges to negative infinity because $-5n + 2 \to -\infty$ as $n \to -\infty$. We write this as $\lim_{n\to\infty}(-5n + 2) =\to -\infty$.

Because a sequence is a function whose domain is the set of positive integers, we can use properties of limits of functions to determine whether a sequence converges. For example, consider a sequence $\{a_n\}$ and a related function f defined on all positive real numbers such that $f(n) = a_n$ for all integers $n \geq 1$. Since the domain of the sequence is a subset of the domain of f, if $\lim_{x\to\infty} f(x)$ exists, then the sequence converges and has the same limit. For example, consider the sequence $\left\{\frac{1}{n}\right\}$ and the related function $f(x) = \frac{1}{x}$. Since the function f defined on all real numbers $x > 0$ satisfies $f(x) = \frac{1}{x} \to 0$ as $x \to \infty$, the sequence $\left\{\frac{1}{n}\right\}$ must satisfy $\frac{1}{n} \to 0$ as $n \to \infty$.

Limit of a Sequence Defined by a Function

Consider a sequence $\{a_n\}$ such that $a_n = f(n)$ for all $n \geq 1$. If there exists a real number L such that

$$\lim_{x\to\infty} f(x) = L, \tag{1.1.5}$$

then $\{a_n\}$ converges and

$$\lim_{n\to\infty} a_n = L. \tag{1.1.6}$$

We can use this theorem to evaluate $\lim_{n\to\infty} r^n$ for $0 \leq r \leq 1$. For example, consider the sequence $\{(1/2)^n\}$ and the related exponential function $f(x) = (1/2)^x$. Since $\lim_{x\to\infty}(1/2)^x = 0$, we conclude that the sequence $\{(1/2)^n\}$ converges and its limit is 0. Similarly, for any real number r such that $0 \leq r < 1$, $\lim_{x\to\infty} r^x = 0$, and therefore the sequence $\{r^n\}$ converges. On the other hand, if $r = 1$, then $\lim_{x\to\infty} r^x = 1$, and therefore the limit of the sequence $\{1^n\}$ is 1. If $r > 1$, $\lim_{x\to\infty} r^x = \infty$, and therefore we cannot apply this theorem. However, in this case, just as the function r^x grows without bound as $n \to \infty$, the terms r^n in the sequence become arbitrarily large as $n \to \infty$, and we conclude that the sequence $\{r^n\}$ diverges to infinity if $r > 1$.

We summarize these results regarding the geometric sequence r^n:

$$r^n \to 0 \text{ if } r < 1$$

$$r^n \to 1 \text{ if } r = 1$$

$$r^n \to \infty \text{ if } r > 1$$

Later in this section we consider the case when $r < 0$.

We now consider slightly more complicated sequences. For example, consider the sequence $\{(2/3)^n + (1/4)^n\}$. The terms in this sequence are more complicated than other sequences we have discussed, but luckily the limit of this sequence is determined by the limits of the two sequences $\{(2/3)^n\}$ and $\{(1/4)^n\}$. As we describe in the following algebraic limit laws, since $\{(2/3)^n\}$ and $\{1/4)^n\}$ both converge to 0, the sequence $\{(2/3)^n + (1/4)^n\}$ converges to $0 + 0 = 0$. Just as we were able to evaluate a limit involving an algebraic combination of functions f and g by looking at the limits of f and g (see Introduction to Limits), we are able to evaluate the limit of a sequence whose terms are algebraic combinations of a_n and b_n by evaluating the limits of $\{a_n\}$ and $\{b_n\}$.

Algebraic Limit Laws

Given sequences $\{a_n\}$ and $\{b_n\}$ and any real number c, if there exist constants A and B such that $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, then

i. $\lim_{n\to\infty} c = c$
ii. $\lim_{n\to\infty} ca_n = cA$
iii. $\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n = A \pm B$
iv. $\lim_{n\to\infty} (a_n \cdot b_n) = \left(\lim_{n\to\infty} a_n\right) \cdot \left(\lim_{n\to\infty} b_n\right) = A \cdot B$
v. $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{A}{B}$, provided $B \neq 0$ and each $b_n \neq 0$.

Proof

We prove part iii.

Let $\epsilon > 0$. Since $\lim_{n\to\infty} a_n = A$, there exists a constant positive integer N_1 such that for all $n \geq N_1$. Since $\lim_{n\to\infty} b_n = B$, there exists a constant N_2 such that $|b_n - B| > \epsilon/2$ for all $n \geq N_2$. Let N be the larger of N_1 and N_2. Therefore for all $n \geq N$,

$$|(a_n + b_n) - (A + B)| \leq |a_n - A| + |b_n - B| < \epsilon/2 + \epsilon/2 = \epsilon.$$

The algebraic limit laws allow us to evaluate limits for many sequences. For example, consider the sequence $a_n = \frac{1}{n^2}$. As shown earlier, $\lim_{n\to\infty} \frac{1}{n} = 0$. Similarly, for any positive integer k, we can conclude that

$$\lim_{n\to\infty} \frac{1}{n^k} = 0. \tag{1.1.7}$$

In the next example, we make use of this fact along with the limit laws to evaluate limits for other sequences.

Example 1.1.3: Determining Convergence and Finding Limits

For each of the following sequences, determine whether or not the sequence converges. If it converges, find its limit.

a. $\left\{5 - \frac{3}{n^2}\right\}$
b. $\left\{\frac{3n^4 - 7n^2 + 5}{6 - 4n^4}\right\}$
c. $\left\{\frac{2^n}{n^2}\right\}$
d. $\left\{\left(1 + \frac{4}{n}\right)^n\right\}$

Solution

a. We know that $\lim_{n\to\infty} \frac{1}{n} = 0$. Using this fact, we conclude that

$$\lim_{n\to\infty} \frac{1}{n^2} = \lim_{n\to\infty} \frac{1}{n} \cdot \lim_{n\to\infty} \frac{1}{n} = 0.$$

Therefore,

$$\lim_{n\to\infty} \left(5 - \frac{3}{n^2}\right) = \lim_{n\to\infty} 5 - 3 \lim_{n\to\infty} \frac{1}{n^2} = 5 - 3.0 = 5.$$

The sequence converges and its limit is 5.

b. By factoring n^4 out of the numerator and denominator and using the limit laws above, we have

$$\begin{aligned}\lim_{n\to\infty}\frac{3n^4-7n^2+5}{6-4n^4} &= \lim_{n\to\infty}\frac{3-\frac{7}{n^2}+\frac{5}{n^4}}{\frac{6}{n^4}-4}\\ &= \frac{\lim_{n\to\infty}(3-\frac{7}{n^2}+\frac{5}{n^4})}{\lim_{n\to\infty}(\frac{6}{n^4}-4)}\\ &= \frac{\lim_{n\to\infty}(3)-\lim_{n\to\infty}\frac{7}{n^2}+\lim_{n\to\infty}\frac{5}{n^4}}{\lim_{n\to\infty}\frac{6}{n^4}-\lim_{n\to\infty}(4)}\\ &= \frac{\lim_{n\to\infty}(3)-7\cdot\lim_{n\to\infty}\frac{1}{n^2}+5\cdot\lim_{n\to\infty}\frac{1}{n^4}}{6\cdot\lim_{n\to\infty}\frac{1}{n^4}-\lim_{n\to\infty}(4)}\\ &= \frac{3-7\cdot 0+5\cdot 0}{6\cdot 0-4}=-\frac{3}{4}.\end{aligned}$$

The sequence converges and its limit is $-3/4$.

c. Consider the related function $f(x)=2^x/x^2$ defined on all real numbers $x>0$. Since $2^x\to\infty$ and $x^2\to\infty$ as $x\to\infty$, apply L'Hôpital's rule and write

$$\begin{aligned}\lim_{x\to\infty}\frac{2^x}{x^2} &= \lim_{x\to\infty}\frac{2^x\ln 2}{2x} && \text{Take the derivatives of the numerator and denominator.}\\ &= \lim_{x\to\infty}\frac{2^x(\ln 2)^2}{2} && \text{Take the derivatives again.}\\ &= \infty.\end{aligned}$$

We conclude that the sequence diverges.

d. Consider the function $f(x)=\left(1+\frac{4}{x}\right)^x$ defined on all real numbers $x>0$. This function has the indeterminate form 1^∞ as $x\to\infty$. Let

$$y=\lim_{x\to\infty}\left(1+\frac{4}{x}\right)^x.$$

Now taking the natural logarithm of both sides of the equation, we obtain

$$\ln(y)=\ln\left[\lim_{x\to\infty}\left(1+\frac{4}{x}\right)^x\right].$$

Since the function $f(x)=\ln(x)$ is continuous on its domain, we can interchange the limit and the natural logarithm. Therefore,

$$\ln(y)=\lim_{x\to\infty}\left[\ln\left(1+\frac{4}{x}\right)^x\right].$$

Using properties of logarithms, we write

$$\lim_{x\to\infty}\left[\ln\left(1+\frac{4}{x}\right)^x\right]=\lim_{x\to\infty}x\ln\left(1+\frac{4}{x}\right).$$

Since the right-hand side of this equation has the indeterminate form $\infty \cdot 0$, rewrite it as a fraction to apply L'Hôpital's rule. Write

$$\lim_{x\to\infty} x\ln\left(1+\frac{4}{x}\right) = \lim_{x\to\infty} \frac{\ln(1+4/x)}{1/x}.$$

Since the right-hand side is now in the indeterminate form 0/0, we are able to apply L'Hôpital's rule. We conclude that

$$\lim_{x\to\infty} \frac{\ln(1+4/x)}{1/x} = \lim_{x\to\infty} \frac{4}{1+4/x} = 4.$$

Therefore, $\ln(y) = 4$ and $y = e^4$. Therefore, since $\lim_{x\to\infty}\left(1+\frac{4}{x}\right)^x = e^4$, we can conclude that the sequence $\left\{\left(1+\frac{4}{n}\right)^n\right\}$ converges to e^4.

Watch the accompanying video for a visual demonstration of this concept.

- **Video Length:** 3 minute 15 seconds.
- **Context**: This video demonstrates the process involved in determining whether the given series converges or diverges. If converges, then how to find the limit.

$$a_n = \left(1+\frac{2}{n}\right)^n.$$

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Exercise 1.1.3

Consider the sequence $\left\{(5n^2+1)/e^n\right\}$. Determine whether or not the sequence converges. If it converges, find its limit.

Hint available at the end of this section.

Answer available at the end of this section.

Recall that if f is a continuous function at a value L, then $f(x) \to f(L)$ as $x \to L$. This idea applies to sequences as well. Suppose a sequence $a_n \to L$, and a function f is continuous at L. Then $f(a_n) \to f(L)$. This property often enables us to find limits for complicated sequences. For example, consider the sequence $\sqrt{5-\frac{3}{n^2}}$. From the above example we know the sequence $5-\frac{3}{n^2} \to 5$. Since $\sqrt{x}$ is a continuous function at $x = 5$,

$$\lim_{n\to\infty}\sqrt{5-\frac{3}{n^2}} = \sqrt{\lim_{n\to\infty}(5-\frac{3}{n^2})} = \sqrt{5}.$$

Continuous Functions Defined on Convergent Sequences

Consider a sequence $\{a_n\}$ and suppose there exists a real number L such that the sequence $\{a_n\}$ converges to L. Suppose f is a continuous function at L. Then there exists an integer N such that f is defined at all values for $n \geq N$, and the sequence $\{f(a_n)\}$ converges to $f(L)$ (Figure 1.1.4).

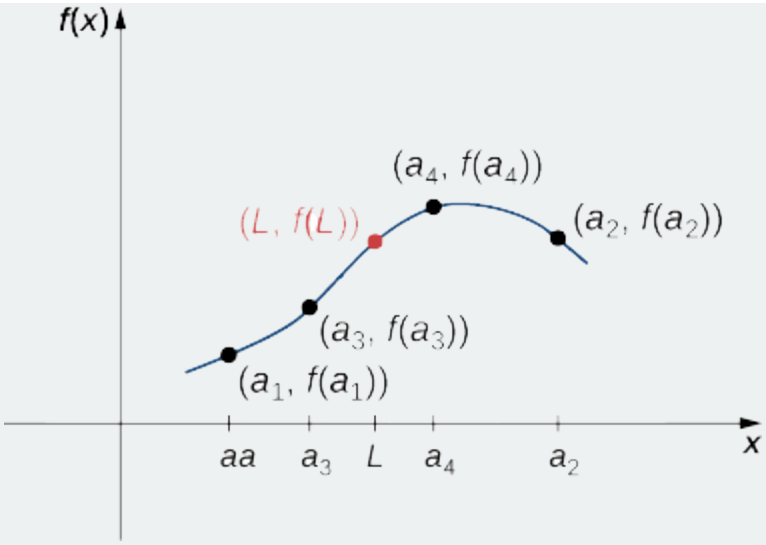


Figure 1.1.4: A graph of f is a continuous function with the inputs $a_1,\ a_2,\ a_3,\ \ldots$ approach L, the outputs $f(a_1),\ f(a_2),\ f(a_3),\ \ldots$ approach $f(L)$.

Proof

Let $\epsilon > 0$. Since f is continuous at L, there exists $\delta > 0$ such that $|f(x) - f(L)| < \varepsilon$ if $|x - L| < \delta$. Since the sequence a_n converges to L, there exists N such that $|a_n - L| < \delta$ for all $n \geq N$. Therefore, for all $n \geq N$, $|a_n - L| < \delta$, which implies $|f(a_n) - f(L)| < \varepsilon$. We conclude that the sequence $f(a_n)$ converges to $f(L)$.

Example 1.1.4: Limits Involving Continuous Functions Defined on Convergent Sequences

Determine whether the sequence $\{\cos(3/n^2)\}$ converges. If it converges, find its limit.

Solution

Since the sequence $\{3/n^2\}$ converges to 0 and $\cos x$ is continuous at $x = 0$, we can conclude that the sequence $\{\cos(3/n^2)\}$ converges and

$$\lim_{n\to\infty} \cos\left(\frac{3}{n^2}\right) = \cos 0 = 1.$$

Exercise 1.1.4

Determine if the sequence $\left\{\sqrt{\frac{2n+1}{3n+5}}\right\}$ converges. If it converges, find its limit.

Hint available at the end of this section.

Answer available at the end of this section.

Another theorem involving limits of sequences is an extension of the Squeeze Theorem for limits discussed in Introduction to Limits.

Theorem: Squeeze Theorem for Sequences

Consider sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$. Suppose there exists an integer N such that

$$a_n \leq b_n \leq c_n \text{ for all } n \geq N.$$

If there exists a real number L such that

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n, \tag{1.1.8}$$

then $\{b_n\}$ converges and $\lim_{n\to\infty} b_n = L$ (Figure 1.1.5).

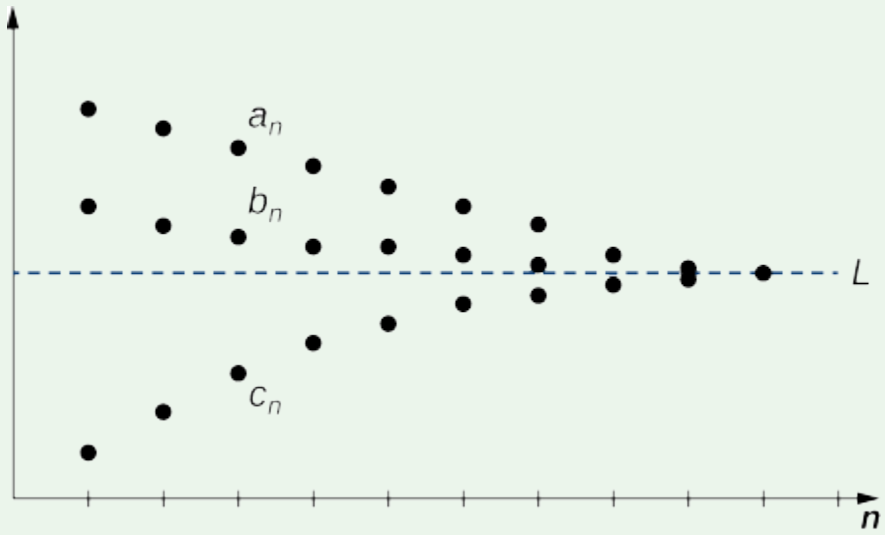


Figure 1.1.5: A scatter plot of a sequence $\{a_n\}$ on the coordinate plane, with the horizontal axis labeled n and the vertical axis labeled a. The terms of the sequence are plotted as open circles scattered above and below the horizontal line $a = 1$. A few terms are labeled specifically: $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$. A blue dashed line represents the value $a = 1$, suggesting it is the limit that the sequence terms are approaching as n increases.

Proof

Let $\varepsilon > 0$. Since the sequence $\{a_n\}$ converges to L, there exists an integer N_1 such that $|a_n - L| < \varepsilon$ for all $n \geq N_1$.

Similarly, since $\{c_n\}$ converges to L, there exists an integer N_2 such that $|c_n - L| < \varepsilon$ for all $n \geq N_2$.

By assumption, there exists an integer N such that $a_n \leq b_n \leq c_n$ for all $n \geq N$. Let M be the largest of N_1,N_2, and N. We must show that $|b_n - L| < \varepsilon$ for all $n \geq M$.

For all $n \geq M$,

$$-\varepsilon < -|a_n - L| \leq a_n - L \leq b_n - L \leq c_n - L \leq |c_n - L| < \varepsilon$$

Therefore, \(–ε

Example 1.1.5: Using the Squeeze Theorem

Use the Squeeze Theorem to find the limit of each of the given sequences.

a. $\left\{\frac{\cos\, n}{n^2}\right\}$
b. $\left\{\left(-\frac{1}{2}\right)^n\right\}$

Solution

a. Since $-1 \leq \cos n \leq 1$ for all integers n, we have

$$-\frac{1}{n^2} \leq \frac{\cos n}{n^2} \leq \frac{1}{n^2}.$$

Since $-1/n^2 \to 0$ and $1/n^2 \to 0$, we conclude that $\cos n/n^2 \to 0$ as well.

b. Since

$$-\frac{1}{2^n} \leq \left(-\frac{1}{2}\right)^n \leq \frac{1}{2^n}$$

for all positive integers n, $-1/2^n \to 0$ and $1/2^n \to 0$, we can conclude that $(-1/2)^n \to 0$.

Exercise 1.1.5

Find $\lim\limits_{n\to\infty} \frac{2n - \sin\, n}{n}$.

Hint available at the end of this section.

Answer available at the end of this section.

Using the idea from Example $1.1.5$ b we conclude that $r^n \to 0$ for any real number r such that \(–1

$$r^n \to 0 \text{ if } |r| < 1$$

$$r^n \to 1 \text{ if } r = 1$$

$$r^n \to \infty \text{ if } r > 1$$

$$\{r^n\} \text{ diverges if } r \leq -1$$

Bounded Sequences

We now turn our attention to one of the most important theorems involving sequences: the Monotone Convergence Theorem. Before stating the theorem, we need to introduce some terminology and motivation. We begin by defining what it means for a sequence to be bounded.

Definition: Bounded Sequence

A sequence $\{a_n\}$ is **bounded above** if there exists a real number M such that

$$a_n \leq M$$

for all positive integers n.

A sequence $\{a_n\}$ is **bounded below** if there exists a real number m such that

$$m \leq a_n$$

for all positive integers n.

A sequence $\{a_n\}$ is a **bounded sequence** if it is bounded above and bounded below.

If a sequence is not bounded, it is an **unbounded sequence.**

For example, the sequence $\{1/n\}$ is bounded above because $1/n \leq 1$ for all positive integers n. It is also bounded below because $1/n \geq 0$ for all positive integers n. Therefore, $\{1/n\}$ is a bounded sequence. On the other hand, consider the sequence $\{2^n\}$. Because $2^n \geq 2$ for all $n \geq 1$, the sequence is bounded below. However, the sequence is not bounded above. Therefore, $\{2^n\}$ is an unbounded sequence.

We now discuss the relationship between boundedness and convergence. Suppose a sequence $\{a_n\}$ is unbounded. Then it is not bounded above, or not bounded below, or both. In either case, there are terms an that are arbitrarily large in magnitude as n gets larger. As a result, the sequence $\{a_n\}$ cannot converge. Therefore, being bounded is a necessary condition for a sequence to converge.

Convergent Sequences Are Bounded

If a sequence $\{a_n\}$ converges, then it is bounded.

Note that a sequence being bounded is not a sufficient condition for a sequence to converge. For example, the sequence $\{(-1)^n\}$ is bounded, but the sequence diverges because the sequence oscillates between 1 and -1 and never approaches a finite number. We now discuss a sufficient (but not necessary) condition for a bounded sequence to converge.

Consider a bounded sequence $\{a_n\}$. Suppose the sequence $\{a_n\}$ is increasing. That is, $a_1 \leq a_2 \leq a_3 \ldots$. Since the sequence is increasing, the terms are not oscillating. Therefore, there are two possibilities. The sequence could diverge to infinity, or it could converge. However, since the sequence is bounded, it is bounded above and the sequence cannot diverge to infinity. We conclude that $\{a_n\}$ converges. For example, consider the sequence

$$\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right\}. \tag{1.1.9}$$

Since this sequence is increasing and bounded above, it converges. Next, consider the sequence

$$\left\{2, 0, 3, 0, 4, 0, 1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \ldots\right\}. \tag{1.1.10}$$

Even though the sequence is not increasing for all values of n, we see that $-1/2 < -1/3 < -1/4 < \cdots$. Therefore, starting with the eighth term, $a_8 = -1/2$, the sequence is increasing. In this case, we say the sequence is eventually increasing. Since the sequence is bounded above, it converges. It is also true that if a sequence is decreasing (or eventually decreasing) and bounded below, it also converges.

Definition: Monotone Sequence

A sequence $\{a_n\}$ is **increasing** for all $n \geq n_0$ if

$$a_n \leq a_{n+1} \text{ for all } n \geq n_0.$$

A sequence $\{a_n\}$ is **decreasing** for all $n \geq n_0$ if

$$a_n \geq a_{n+1} \text{ for all } n \geq n_0.$$

A sequence $\{a_n\}$ is a **monotone sequence** for all $n \geq n_0$ if it is increasing for all $n \geq n_0$ or decreasing for all $n \geq n_0$.

We now have the necessary definitions to state the Monotone Convergence Theorem, which gives a sufficient condition for convergence of a sequence.

Theorem: Monotone Convergence Theorem

If $\{a_n\}$ is a bounded sequence and there exists a positive integer n_0 such that $\{a_n\}$ is monotone for all $n \geq n_0$, then $\{a_n\}$ converges.

The proof of this theorem is beyond the scope of this text. Instead, we provide a graph to show intuitively why this theorem makes sense (Figure 1.1.6).

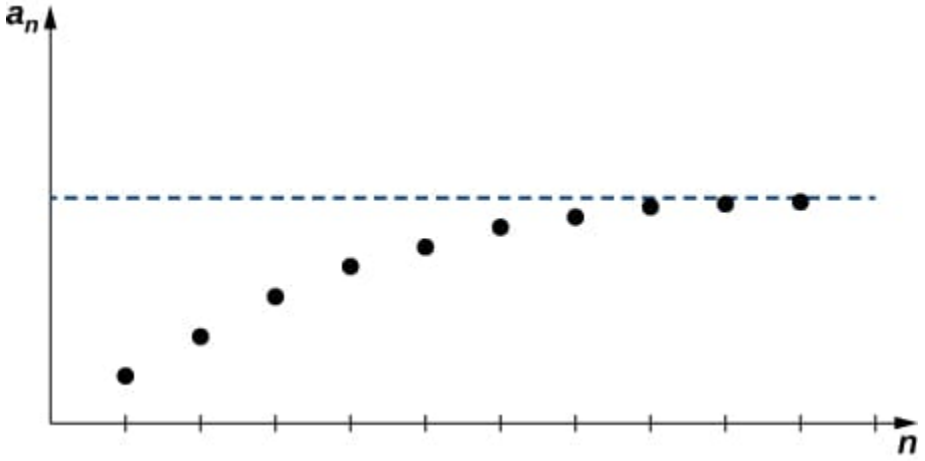


Figure 1.1.6: The image is a line graph illustrating the relationship between two variables, labeled "$\{a_n\}$" on the vertical axis and "n" on the horizontal axis. A series of black dots form a curve that starts lower on the left and progresses upward to the right, suggesting an increasing trend. Since the sequence $\{a_n\}$ is increasing and bounded above, it must converge.

In the following example, we show how the Monotone Convergence Theorem can be used to prove convergence of a sequence.

Example 1.1.6: Using the Monotone Convergence Theorem

For each of the following sequences, use the Monotone Convergence Theorem to show the sequence converges and find its limit.

a. $\left\{\frac{4^n}{n!}\right\}$
b. $\{a_n\}$ defined recursively such that

$a_1 = 2$ and $a_{n+1} = \frac{a_n}{2} + \frac{1}{2a_n}$ for all $n \geq 2$.

Solution

a. Writing out the first few terms, we see that

$$\left\{\frac{4^n}{n!}\right\} = \left\{4, 8, \frac{32}{3}, \frac{32}{3}, \frac{128}{15}, \ldots\right\}.$$

At first, the terms increase. However, after the third term, the terms decrease. In fact, the terms decrease for all $n \geq 3$. We can show this as follows.

$$a_{n+1} = \frac{4^{n+1}}{(n+1)!} = \frac{4}{n+1} \cdot \frac{4^n}{n!} = \frac{4}{n+1} \cdot a_n \leq a_n \text{ if } n \geq 3.$$

Therefore, the sequence is decreasing for all $n \geq 3$. Further, the sequence is bounded below by 0 because $4n/n! \geq 0$ for all positive integers n. Therefore, by the Monotone Convergence Theorem, the sequence converges.

To find the limit, we use the fact that the sequence converges and let $L = \lim_{n\to\infty} a_n$. Now note this important observation. Consider $\lim_{n\to\infty} a_{n+1}$. Since

$$\{a_{n+1}\} = \{a_2,\ a_3,\ a_4,\ ...\},$$

the only difference between the sequences $\{a_{n+1}\}$ and $\{a_n\}$ is that $\{a_{n+1}\}$ omits the first term. Since a finite number of terms does not affect the convergence of a sequence,

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = L.$$

Combining this fact with the equation

$$a_{n+1} = \frac{4}{n+1} a_n$$

and taking the limit of both sides of the equation

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{4}{n+1} a_n,$$

we can conclude that

$$L = 0 \cdot L = 0.$$

b. Writing out the first several terms,

$$\left\{2,\ \frac{5}{4},\ \frac{41}{40},\ \frac{3281}{3280},\ ...\right\}.$$

we can conjecture that the sequence is decreasing and bounded below by 1. To show that the sequence is bounded below by 1, we can show that

$$\frac{a_n}{2} + \frac{1}{2a_n} \geq 1.$$

To show this, first rewrite

$$\frac{a_n}{2} + \frac{1}{2a_n} = \frac{a_n^2 + 1}{2a_n}.$$

Since $a_1 > 0$ and a_2 is defined as a sum of positive terms, $a_2 > 0$. Similarly, all terms $a_n > 0$. Therefore,

$$\frac{a^2 n + 1}{2a_n} \geq 1$$

if and only if

$$a_n^2 + 1 \geq 2a_n.$$

Rewriting the inequality $a_n^2 + 1 \geq 2a_n$ as $a_n^2 - 2a_n + 1 \geq 0$, and using the fact that

$$a_n^2 - 2a_n + 1 = (a_n - 1)^2 \geq 0$$

because the square of any real number is nonnegative, we can conclude that

$$\frac{a^n}{2} + \frac{1}{2a_n} \geq 1.$$

To show that the sequence is decreasing, we must show that $a_{n+1} \leq a_n$ for all $n \geq 1$. Since $1 \leq a_n^2$, it follows that

$$a_n^2 + 1 \leq 2a_n^2.$$

Dividing both sides by $2a_n$, we obtain

$$\frac{a_n}{2} + \frac{1}{2a_n} \leq a_n.$$

Using the definition of a_{n+1}, we conclude that

$$a_{n+1} = \frac{a_n}{2} + \frac{1}{2a_n} \leq a_n.$$

Since $\{a_n\}$ is bounded below and decreasing, by the Monotone Convergence Theorem, it converges.

To find the limit, let $L = \lim_{n\to\infty} a_n$. Then using the recurrence relation and the fact that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$, we have

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} (\frac{a_n}{2} + \frac{1}{2a_n}),$$

and therefore

$$L = \frac{L}{2} + \frac{1}{2L}.$$

Multiplying both sides of this equation by $2L$, we arrive at the equation

$$2L^2 = L^2 + 1.$$

Solving this equation for L, we conclude that $L^2 = 1$, which implies $L = \pm 1$. Since all the terms are positive, the limit $L = 1$.

Exercise 1.1.6

Consider the sequence $\{a_n\}$ defined recursively such that $a_1 = 1$, $a_n = a_{n-1}/2$. Use the Monotone Convergence Theorem to show that this sequence converges and find its limit.

Hint available at the end of this section.

Answer available at the end of this section.

Definition: Fibonacci Numbers

The **Fibonacci numbers** are defined recursively by the sequence $\{F_n\}$ where $F_0 = 0,\ \ F_1 = 1$ and for $n \geq 2$,

$$F_n = F_{n-1} + F_{n-2}.$$

Here we look at properties of the Fibonacci numbers.

1. Write out the first twenty Fibonacci numbers.
2. Find a closed formula for the Fibonacci sequence by using the following steps.
    a. Consider the recursively defined sequence x_n where $x_0 = c$ and $x_{n+1} = ax_n$. Show that this sequence can be described by the closed formula $x_n = ca^n$ for all $n \geq 0$.
    b. Using the result from part a. as motivation, look for a solution of the equation $F_n = F_{n-1} + F_{n-2}$ of the form $F_n = c\lambda^n$. Determine what two values for λ will allow F_n to satisfy this equation.
    c. Consider the two solutions from part b.: λ_1 and λ_2. Let $F_n = c_1\lambda_1^n + c_2\lambda_2^n$. Use the initial conditions F_0 and F_1 to determine the values for the constants c_1 and c_2 and write the closed formula F_n.

3. Use the answer in 2c. to show that

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}.$$

The number $\phi = (1+\sqrt{5})/2$ is known as the **golden ratio** (Figure 1.1.7 and Figure Figure 1.1.8).



Figure 1.1.7: The seeds in a sunflower exhibit spiral patterns curving to the left and to the right. The number of spirals in each direction is always a Fibonacci number—always. (CC-BY-NC-SA modification of work by Esdras Calderan, Wiki media Commons).

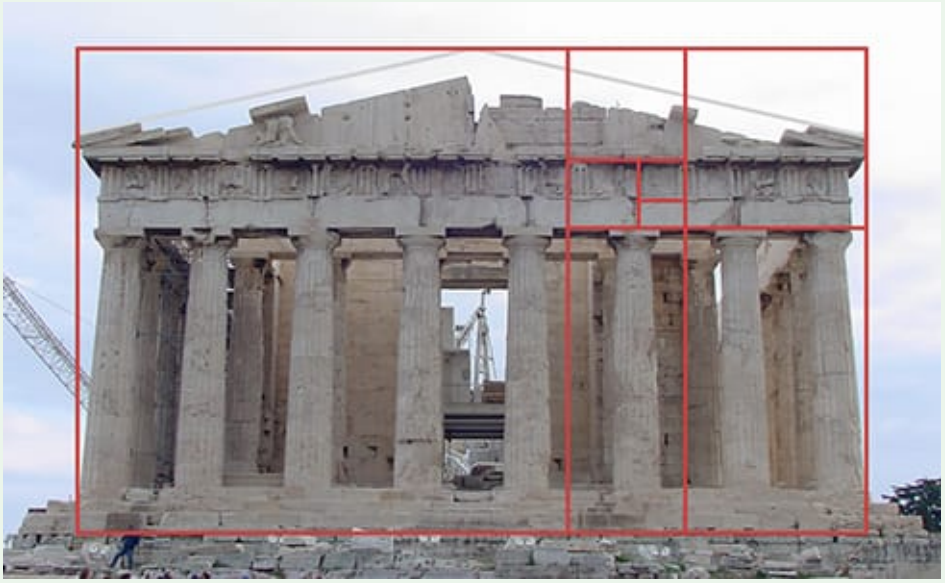


Figure 1.1.8: The proportion of the golden ratio appears in many famous examples of art and architecture. The ancient Greek temple known as the Parthenon was designed with these proportions, and the ratio appears again in many of the smaller details. (CC-BY-NC-SA modification of work by TravelingOtter, Flickr, Wiki Media Commons).

Key Concepts

- To determine the convergence of a sequence given by an explicit formula $a_n = f(n)$, we use the properties of limits for functions.
- If $\{a_n\}$ and $\{b_n\}$ are convergent sequences that converge to A and B, respectively, and c is any real number, then the sequence $\{ca_n\}$converges to $c \cdot A$, the sequences $\{a_n \pm b_n\}$ converge to $A \pm B$, the sequence $\{a_n \cdot b_n\}$ converges to $A \cdot B$, and the sequence $\{a_n/b_n\}$ converges to A/B, provided $B \neq 0$.
- If a sequence is bounded and monotone, then it converges, but not all convergent sequences are monotone.

- If a sequence is unbounded, it diverges, but not all divergent sequences are unbounded.
- The geometric sequence $\{r^n\}$ converges if and only if $|r| < 1$ or $r = 1$.

Glossary

Arithmetic Sequence
A sequence in which the difference between every pair of consecutive terms is the same is called an arithmetic sequence.

Bounded Above
A sequence $\{a_n\}$ is bounded above if there exists a constant M such that $a_n \leq M$ for all positive integers n.

Bounded Below
A sequence $\{a_n\}$ is bounded below if there exists a constant M such that $M \leq a_n$ for all positive integers n.

Bounded Sequence
A sequence $\{a_n\}$ is bounded if there exists a constant M such that $|a_n| \leq M$ for all positive integers n.

Convergent Sequence
A convergent sequence is a sequence $\{a_n\}$ for which there exists a real number L such that a_n is arbitrarily close to L as long as n is sufficiently large.

Divergent Sequence
A sequence that is not convergent is divergent.

Explicit Formula
A sequence may be defined by an explicit formula such that $a_n = f(n)$.

Geometric Sequence
A sequence $\{a_n\}$ in which the ratio a_{n+1}/a_n is the same for all positive integers n is called a geometric sequence.

Index Variable
The subscript used to define the terms in a sequence is called the index.

Limit of a Sequence
The real number L to which a sequence converges is called the limit of the sequence.

Monotone Sequence
An increasing or decreasing sequence.

Recurrence Relation
A recurrence relation is a relationship in which a term a_n in a sequence is defined in terms of earlier terms in the sequence.

Sequence
An ordered list of numbers of the form $a_1,\ a_2,\ a_3,\ \ldots$ is a sequence.

Term
The number a_n in the sequence $\{a_n\}$ is called the nth term of the sequence.

Unbounded Sequence
A sequence that is not bounded is called unbounded.

Answers

Exercise 1.1.1

The denominators form an arithmetic sequence.

Exercise 1.1.1

$$a_n = \frac{(-1)^{n+1}}{3+2n}$$

Exercise 1.1.2

This is an arithmetic sequence.

Exercise $1.1.2$

$a_n = 6n - 10$

Exercise $1.1.3$

Use L'Hôpital's rule.

Exercise $1.1.3$

The sequence converges, and its limit is 0

Exercise $1.1.4$

Consider the sequence $\left\{\frac{2n+1}{3n+5}\right\}$.

Exercise $1.1.4$

The sequence converges, and its limit is $\sqrt{2/3}$.

Exercise $1.1.5$

Use the fact that $-1 \leq \sin n \leq 1$.

Exercise $1.1.5$

2

Exercise $1.1.6$

Show the sequence is decreasing and bounded below.

Exercise $1.1.6$

0.

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1.1E: Exercises for Section 1.1

Exercises 1 - 4

Find the first six terms of each sequence, starting with $n = 1$.

1. $a_n = 1 + (-1)^n$ for $n \geq 1$

   Answer

   $a_n = 0$ if n is odd and $a_n = 2$ if n is even

2. $a_n = n^2 - 1$ for $n \geq 1$
3. $a_1 = 1$ and $a_n = a_{n-1} + n$ for $n \geq 2$

   Answer

   $a_n = 1, 3, 6, 10, 15, 21, ...$

4. $a_1 = 1, a_2 = 1$ and $a_n + 2 = a_n + a_{n+1}$ for $n \geq 1$
5. Find an explicit formula for a_n where $a_1 = 1$ and $a_n = a_{n-1} + n$ for $n \geq 2$.

   Answer

   $$a_n = \frac{n(n+1)}{2}$$

6. Find a formula a_n for the nth term of the arithmetic sequence whose first term is $a_1 = 1$ such that $a_{n-1} - a_n = 17$ for $n \geq 1$.
7. Find a formula a_n for the nth term of the arithmetic sequence whose first term is $a_1 = -3$ such that $a_{n-1} - a_n = 4$ for $n \geq 1$.

   Answer

   $a_n = 4n - 7$

8. Find a formula a_n for the nth term of the geometric sequence whose first term is $a_1 = 1$ such that $\frac{a_{n+1}}{a_n} = 10$ for $n \geq 1$.
9. Find a formula a_n for the nth term of the geometric sequence whose first term is $a_1 = 3$ such that $\frac{a_{n+1}}{a_n} = 1/10$ for $n \geq 1$.

   Answer

   $a_n = 3.10^{1-n} = 30.10^{-n}$

10. Find an explicit formula for the nth term of the sequence whose first several terms are $0, 3, 8, 15, 24, 35, 48, 63, 80, 99, ...$ (Hint: First add one to each term.)
11. Find an explicit formula for the nth term of the sequence satisfying $a_1 = 0$ and $a_n = 2a_{n-1} + 1$ for $n \geq 2$.

    Answer

    $a_n = 2^n - 1$

Exercises 12 and 13

Find a formula for the general term a_n of each of the following sequences.

12. $1, 0, -1, 0, 1, 0, -1, 0, ...$ (Hint: Find where $\sin x$ takes these values)
13. $1, -1/3, 1/5, -1/7, ...$

    Answer

    $$a_n = \frac{(-1)^{n-1}}{2n-1}$$

Exercises 14-18

Find a function $f(n)$ that identifies the nth term a_n of the following recursively defined sequences, as $a_n = f(n)$.

14. $a_1 = 1$ and $a_{n+1} = -a_n$ for $n \geq 1$
15. $a_1 = 2$ and $a_{n+1} = 2a_n$ for $n \geq 1$

Answer

$$f(n) = 2^n$$

16. $a_1 = 1$ and $a_{n+1} = (n+1)a_n$ for $n \geq 1$
17. $a_1 = 2$ and $a_{n+1} = (n+1)a_n/2$ for $n \geq 1$

Answer

$$f(n) = \frac{n!}{2^{n-2}}$$

18. $a_1 = 1$ and $a_{n+1} = a_n/2^n$ for $n \geq 1$

Exercises 19 - 22

Plot the first N terms of the given sequence. State whether the graphical evidence suggests that the sequence converges or diverges.

19. [T] $a_1 = 1, a_2 = 2$, and for $n \geq 2, a_n = \frac{1}{2}(a_{n-1} + a_{n-2})$; $N = 30$

Answer

Terms oscillate above and below $5/3$ and appear to converge to $5/3$.

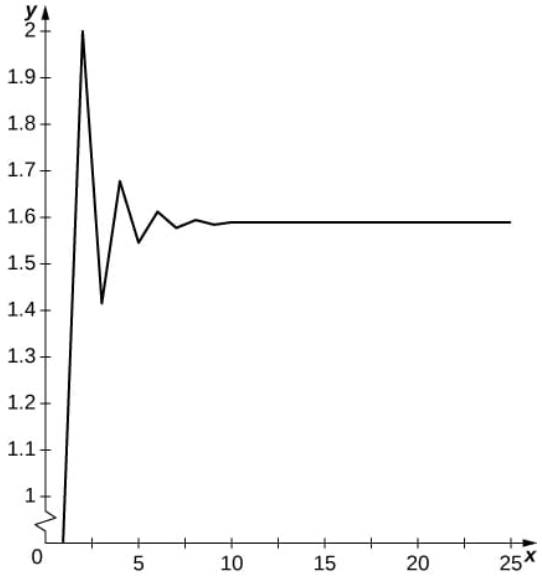


Figure $1.1E.1$: The image shows a graph of a function that appears to represent a sequence or a function with oscillatory behavior converging to a value around ◈ = 1.6. The ◈-axis ranges from 0 to 25, and the ◈-axis ranges from 0 to 2. The graph includes a jagged initial rise, some oscillations, and then stabilizes into a straight line as ◈ increases. The vertical axis is labeled as ◈ and the horizontal axis as ◈.

20. [T] $a_1 = 1, a_2 = 2, a_3 = 3$ and for $n \geq 4, a_n = \frac{1}{3}(a_{n-1} + a_{n-2} + a_{n-3}), N = 30$
21. [T] $a_1 = 1, a_2 = 2$, and for $n \geq 3, a_n = \sqrt{a_{n-1}a_{n-2}}; N = 30$

Answer

Terms oscillate above and below $y \approx 1.57..$ and appear to converge to a limit.

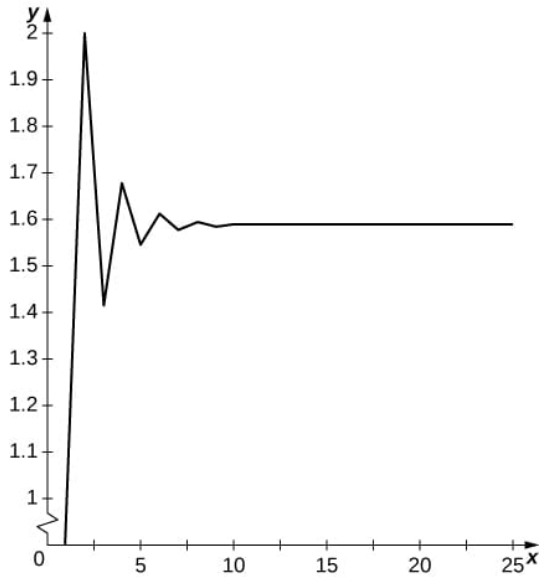


Figure $1.1E.2$: The image is a line graph depicting a function that starts at the origin and initially spikes upwards, reaching a peak above 1.9 on the y-axis. After the peak, the line fluctuates, showing minor oscillations above and below 1.6 before stabilizing into a horizontal line. The x-axis ranges from 0 to 25, while the y-axis is between 0 and 2, with a small break represented by a zigzag on the y-axis near the origin. The main feature of the graph is the transition from an initial sharp rise to a steady state.

22. [T] $a_1 = 1, a_2 = 2, a_3 = 3$, and for $n \geq 4, a_n = \sqrt{a_{n-1}a_{n-2}a_{n-3}}; N = 30$

Exercises 23 - 26

Suppose that $\lim_{n\to\infty} a_n = 1$, $\lim_{n\to\infty} b_n = -1$, and \(0<–b_n for all n.

Using this information, evaluate each of the following limits, state that the limit does not exist, or state that there is not enough information to determine whether the limit exists.

23. $\lim_{n\to\infty} 3a_n - 4b_n$

Answer

$$\lim_{n\to\infty} 3a_n - 4b_n \quad = \quad 7$$

24. $\lim_{n\to\infty} \frac{1}{2}b_n - \frac{1}{2}a_n$
25. $\lim_{n\to\infty} \frac{a_n + b_n}{a_n - b_n}$

Answer

$$\lim_{n\to\infty} \frac{a_n + b_n}{a_n - b_n} \quad = \quad 0$$

26. $\lim_{n\to\infty} \frac{a_n - b_n}{a_n + b_n}$

Exercises 27 - 30

Find the limit of each of the following sequences, using L'Hôpital's rule when appropriate.

27. $\dfrac{n^2}{2^n}$

Answer

$$\lim_{n\to\infty} \frac{n^2}{2^n} \quad = \quad 0$$

28. $\dfrac{(n-1)^2}{(n+1)^2}$
29. $\dfrac{\sqrt{n}}{\sqrt{n+1}}$

Answer

$$\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} \quad = \quad 1$$

30. $n^{1/n}$ (Hint: $n^{1/n} = e^{\frac{1}{n}\ln n}$)

Exercises 31 - 37

State whether each sequence is bounded and whether it is eventually monotone, increasing, or decreasing.

31. $n/2^n, n \geq 2$

Answer

bounded, decreasing for $n \geq 1$

32. $\ln\left(1+\dfrac{1}{n}\right)$
33. $\sin n$

Answer

bounded, not monotone

34. $\cos(n^2)$
35. $n^{1/n}, \quad n \geq 3$

Answer

bounded, decreasing

36. $n^{-1/n}, \quad n \geq 3$
37. $\tan n$

Answer

not monotone, not bounded

Exercises 38 - 39

Determine whether the given sequence has a limit. If it does, find the limit.

38. $a_1 = \sqrt{2}, a_2 = \sqrt{2\sqrt{2}}.a_3 = \sqrt{2\sqrt{2\sqrt{2}}}$ etc.
39. $a_1 = 3, a_n = \sqrt{2a_{n-1}}, n = 2, 3, \ldots .$

Answer

a_n is decreasing and bounded below by 2. The limit a must satisfy $a = \sqrt{2a}$ so $a = 2$, independent of the initial value.

Exercises 40 - 43

Use the Squeeze Theorem to find the limit of each sequence

40. $n\sin(1/n)$
41. $\dfrac{\cos(1/n)-1}{1/n}$

Answer

0

42. $a_n = \dfrac{n!}{n^n}$
43. $a_n = \sin n \sin(1/n)$

Answer

0 since $|\sin x| \leq |x|$ and $|\sin x| \leq 1$ so $-\dfrac{1}{n} \leq a_n \leq \dfrac{1}{n}$).

Exercises 44 and 45

For the sequences given plot the first 25 terms of the sequence and state whether the graphical evidence suggests that the sequence converges or diverges.

44. [T] $a_n = \sin n$
45. [T] $a_n = \cos n$

Answer

Graph oscillates and suggests no limit.

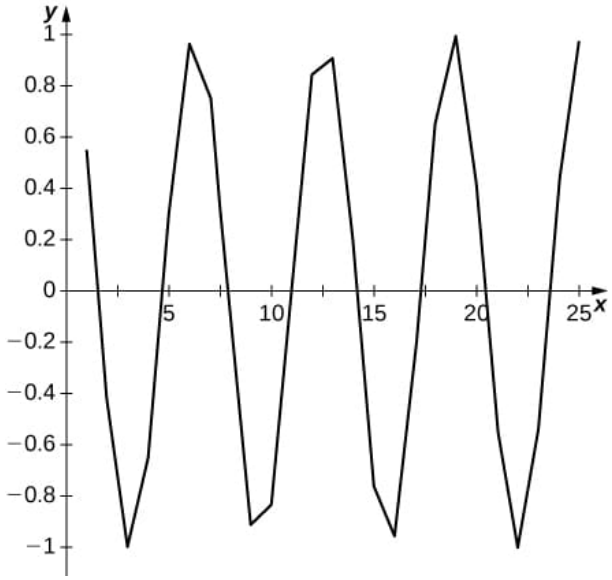


Figure $1.1E.3$: The image is a graph displaying a periodic waveform pattern plotted on a Cartesian coordinate system. The x-axis ranges from 0 to 25 and the y-axis ranges from -1 to 1. The waveform consists of sharp peaks and troughs, repeating approximately four times across the x-axis. Each cycle rises sharply to a peak near 0.8 on the y-axis and falls to a trough near -1. The axis lines are labeled with "x" for the horizontal axis and "y" for the vertical axis, with tick marks and numeric labels provided at regular intervals. The design suggests a continuous sharp triangular wave pattern.

Exercises 46 - 53

Determine the limit of the sequence or show that the sequence diverges. If it converges, find its limit.

46. $a_n = \tan^{-1}(n^2)$
47. $a_n = (2n)^{1/n} - n^{1/n}$

Answer

$n^{1/n} \to 1$ and $2^{1/n} \to 1$, so $a_n \to 0$

48) $a_n = \dfrac{\ln(n^2)}{\ln(2n)}$

49) $a_n = \left(1 - \frac{2}{n}\right)^n$

Answer

Since $(1 + 1/n)^n \to e$, one has $(1 - 2/n)^n \approx (1 + k)^{-2k} \to e^{-2}$ as $k \to \infty$.

50. $a_n = \ln\left(\dfrac{n+2}{n^2-3}\right)$
51. $a_n = \dfrac{2^n + 3^n}{4^n}$

Answer

$2^n + 3^n \le 2 \cdot 3^n$ and $3^n/4^n \to 0$ as $n \to \infty$, so $a_n \to 0$ as $n \to \infty$.

52. $a_n = \dfrac{(1000)^n}{n!}$
53. $a_n = \dfrac{(n!)^2}{(2n)!}$

Answer

$\dfrac{a_{n+1}}{a_n} = n!/(n+1)(n+2)\cdots(2n) = \dfrac{1 \cdot 2 \cdot 3 \cdots n}{(n+1)(n+2)\cdots(2n)} < 1/2^n$. In particular, $a_{n+1}/a_n \le 1/2$, so $a_n \to 0$ as $n \to \infty$.

Exercises 54 - 57

Newton's method seeks to approximate a solution $f(x) = 0$ that starts with an initial approximation x_0 and successively defines a sequence $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$. For the given choice of f and x_0, write out the formula for x_{n+1}. If the sequence appears to converge, give an exact formula for the solution x, then identify the limit x accurate to four decimal places and the smallest n such that x_n agrees with x up to four decimal places.

54. [T] $f(x) = x^2 - 2, \quad x_0 = 1$
55. [T] $f(x) = (x-1)^2 - 2, \quad x_0 = 2$

Answer

$x_{n+1} = x_n - ((x_n - 1)^2 - 2)/2(x_n - 1); \;\; x = 1 + \sqrt{2}, \;\; x \approx 2.4142, \;\; n = 5$

56. [T] $f(x) = e^x - 2, \quad x_0 = 1$
57. [T] $f(x) = \ln x - 1, \quad x_0 = 2$

Answer

$x_{n+1} = x_n - x_n(\ln(x_n) - 1); \;\; x = e, \;\; x \approx 2.7183, \;\; n = 5$

58. [T] Suppose you start with one liter of vinegar and repeatedly remove 0.1 L, replace with water, mix, and repeat.
    a. Find a formula for the concentration after n steps.
    b. After how many steps does the mixture contain less than 10% vinegar?
59. [T] A lake initially contains 2000 fish. Suppose that in the absence of predators or other causes of removal, the fish population increases by 6% each month. However, factoring in all causes, 150 fish are lost each month.

a. Explain why the fish population after n months is modeled by $P_n = 1.06P_{n-1} - 150$ with $P_0 = 2000$.
b. How many fish will be in the pond after one year?

Answer

a. Without losses, the population would obey $P_n = 1.06P_{n-1}$. The subtraction of 150 accounts for fish losses.

b. After 12 months, we have $P_{12} \approx 1494$.

60. [T] A bank account earns 5% interest compounded monthly. Suppose that $\$1000$ is initially deposited into the account, but that $\$10$ is withdrawn each month.
a. Show that the amount in the account after n months is $A_n = (1 + .05/12)A_{n-1} - 10; \quad A_0 = 1000.$
b. How much money will be in the account after 1 year?
c. Is the amount increasing or decreasing?
d. Suppose that instead of $\$10$, a fixed amount d dollars is withdrawn each month. Find a value of d such that the amount in the account after each month remains $\$1000$.
e. What happens if d is greater than this amount?

61. [T] A student takes out a college loan of $\$10,000$ at an annual percentage rate of 6%, compounded monthly.
a. If the student makes payments of $\$100$ per month, how much does the student owe after 12 months?
b. After how many months will the loan be paid off?

Answer

a. The student owes $\$9383$ after 12 months.

b. The loan will be paid in full after 139 months or eleven and a half years.

62. [T] Consider a series combining geometric growth and arithmetic decrease. Let $a_1 = 1$. Fix $a > 1$ and \(0

63. [T] The binary representation $x = 0.b_1b_2b_3...$ of a number x between 0 and 1 can be defined as follows. Let $b_1 = 0$ if $x < 1/2$ and $b_1 = 1$ if $1/2 \leq x < 1$. Let $x_1 = 2x - b_1$. Let $b_2 = 0$ if $x_1 < 1/2$ and $b_2 = 1$ if $1/2 \leq x < 1$. Let $x_2 = 2x_1 - b_2$ and in general, $x_n = 2x_{n-1} - b_n$ and $b_n–1 = 0$ if $x_n < 1/2$ and $b_{n-1} = 1$ if $1/2 \leq x_n < 1$. Find the binary expansion of $1/3$.

Answer

$b_1 = 0, x_1 = 2/3, b_2 = 1, x_2 = 4/3 - 1 = 1/3$, so the pattern repeats, and $1/3 = 0.010101\ldots$.

64. [T] To find an approximation for π, set $a_0 = \sqrt{2+1}, a_1 = \sqrt{2+a_0}$, and, in general, $a_{n+1} = \sqrt{2+a_n}$. Finally, set $p_n = 3.2^n\sqrt{2-a_n}$. Find the first ten terms of p_n and compare the values to π.

For the following two exercises, assume that you have access to a computer program or Internet source that can generate a list of zeros and ones of any desired length. Pseudo-random number generators (PRNGs) play an important role in simulating random noise in physical systems by creating sequences of zeros and ones that appear like the result of flipping a coin repeatedly. One of the simplest types of PRNGs recursively defines a random-looking sequence of N integers $a_1, a_2, \ldots, a_N$ by fixing two special integers (K and M and letting a_{n+1} be the remainder after dividing $K.a_n$ into M, then creates a bit sequence of zeros and ones whose n^{th} term b_n is equal to one if a_n is odd and equal to zero if a_n is even. If the bits b_n are pseudo-random, then the behavior of their average $(b_1 + b_2 + \cdots + b_N)/N$ should be similar to behavior of averages of truly randomly generated bits.

65. [T] Starting with $K = 16,807$ and $M = 2,147,483,647$, using ten different starting values of a_1, compute sequences of bits b_n up to $n = 1000$, and compare their averages to ten such sequences generated by a random bit generator.

Answer

For the starting values $a_1 = 1, a_2 = 2, \ldots, a_1 = 10$, the corresponding bit averages calculated by the method indicated are $0.5220, 0.5000, 0.4960, 0.4870, 0.4860, 0.4680, 0.5130, 0.5210, 0.5040,$ and 0.4840. Here is an example of ten corresponding averages of strings of 1000 bits generated by a random number generator: $0.4880, 0.4870, 0.5150, 0.5490, 0.5130, 0.5180, 0.4860, 0.5030, 0.5050, 0.4980.$ There is no real pattern in either type of average. The random-number-generated averages range between 0.4860 and 0.5490, a range of 0.0630, whereas the calculated PRNG bit averages range between 0.4680 and 0.5220, a range of 0.0540.

66. [T] Find the first 1000 digits of π using either a computer program or Internet resource. Create a bit sequence b_n by letting $b_n = 1$ if the nth digit of π is odd and $b_n = 0$ if the nth digit of π is even. Compute the average value of b_n and the average value of $d_n = |b_{n+1} - b_n|, n = 1, \ldots, 999$. Does the sequence b_n appear random? Do the differences between successive elements of b_n appear random?

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1.2: Infinite Series

Learning Objectives

- Explain the meaning of the sum of an infinite series.
- Calculate the sum of a geometric series.
- Evaluate a telescoping series.

We have seen that a sequence is an ordered set of terms. If you add these terms together, you get a series. In this section we define an infinite series and show how series are related to sequences. We also define what it means for a series to converge or diverge. We introduce one of the most important types of series: the geometric series. We will use geometric series in the next chapter to write certain functions as polynomials with an infinite number of terms. This process is important because it allows us to evaluate, differentiate, and integrate complicated functions by using polynomials that are easier to handle. We also discuss the harmonic series, arguably the most interesting divergent series because it just fails to converge.

Sums and Series

An infinite series is a sum of infinitely many terms and is written in the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots.$$

But what does this mean? We cannot add an infinite number of terms in the same way we can add a finite number of terms. Instead, the value of an infinite series is defined in terms of the limit of partial sums. A partial sum of an infinite series is a finite sum of the form

$$\sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k.$$

To see how we use partial sums to evaluate infinite series, consider the an example. Suppose oil is seeping into a lake such that 1000 gallons enters the lake the first week. During the second week, an additional 500 gallons of oil enters the lake. The third week, 250 more gallons enters the lake. Assume this pattern continues such that each week half as much oil enters the lake as did the previous week. If this continues forever, what can we say about the amount of oil in the lake? Will the amount of oil continue to get arbitrarily large, or is it possible that it approaches some finite amount? To answer this question, we look at the amount of oil in the lake after k weeks. Letting S_k denote the amount of oil in the lake (measured in thousands of gallons) after k weeks, we see that

$$S_1 = 1$$

$$S_2 = 1 + 0.5 = 1 + \tfrac{1}{2}$$

$$S_3 = 1 + 0.5 + 0.25 = 1 + \tfrac{1}{2} + \tfrac{1}{4}$$

$$S_4 = 1 + 0.5 + 0.25 + 0.125 = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8}$$

$$S_5 = 1 + 0.5 + 0.25 + 0.125 + 0.0625 = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16}.$$

Looking at this pattern, we see that the amount of oil in the lake (in thousands of gallons) after k weeks is

$$S_k = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^{k-1}} = \sum_{n=1}^{k} \left(\frac{1}{2}\right)^{n-1}.$$

We are interested in what happens as $k \to \infty$. Symbolically, the amount of oil in the lake as $k \to \infty$ is given by the infinite series

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-1}=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots.$$

At the same time, as $k\to\infty$, the amount of oil in the lake can be calculated by evaluating $\lim_{k\to\infty} S_k$. Therefore, the behavior of the infinite series can be determined by looking at the behavior of the sequence of partial sums S_k. If the sequence of partial sums S_k converges, we say that the infinite series converges, and its sum is given by $\lim_{k\to\infty} S_k$. If the sequence S_k diverges, we say the infinite series diverges. We now turn our attention to determining the limit of this sequence S_k.

First, simplifying some of these partial sums, we see that

$$S_1=1$$

$$S_2=1+\tfrac{1}{2}=\tfrac{3}{2}$$

$$S_3=1+\tfrac{1}{2}+\tfrac{1}{4}=\tfrac{7}{4}$$

$$S_4=1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}=\tfrac{15}{8}$$

$$S_5=1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{16}=\tfrac{31}{16}.$$

Plotting some of these values in Figure 1.2.1 it appears that the sequence S_k could be approaching 2.

This is a graph in quadrant 1with the x and y axes labeled n and S_n, respectively. From 1 to 5, points are plotted. They increase and seem to converge to 2 and n goes to infinity.

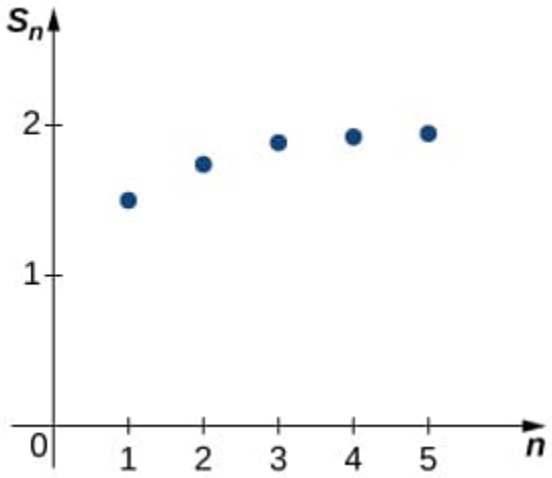


Figure 1.2.1: The graph shows the sequence of partial sums S_k. It appears that the sequence is approaching the value 2.

Let's look for more convincing evidence. In Table 1 titled Approximations we list the values of S_k for several values of k.

Table 1: Approximations

k	5	10	15	20
S_k	1.9375	1.998	1.999939	1.999998

These data supply more evidence suggesting that the sequence S_k converges to 2. Later we will provide an analytic argument that can be used to prove that $\lim_{k\to\infty} S_k=2$. For now, we rely on the numerical and graphical data to convince ourselves that the sequence of partial sums does actually converge to 2. Since this sequence of partial sums converges to 2, we say the infinite series converges to 2 and write

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-1}=2.$$

Returning to the question about the oil in the lake, since this infinite series converges to 2, we conclude that the amount of oil in the lake will get arbitrarily close to 2000 gallons as the amount of time gets sufficiently large.

This series is an example of a geometric series. We discuss geometric series in more detail later in this section. First, we summarize what it means for an infinite series to converge.

Definition

An **infinite series** is an expression of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots. \tag{1.2.1}$$

For each positive integer k, the sum

$$S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k \tag{1.2.2}$$

is called the kth **partial sum** of the infinite series. The partial sums form a sequence S_k. If the sequence of partial sums converges to a real number S, the infinite series converges. If we can describe the **convergence of a series** to S, we call S the sum of the series, and we write

$$\sum_{n=1}^{\infty} a_n = S. \tag{1.2.3}$$

If the sequence of partial sums diverges, we have the **divergence of a series.**

Note that the index for a series need not begin with $n = 1$ but can begin with any value. For example, the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1}$$

can also be written as

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n} \text{ or } \sum_{n=5}^{\infty} \left(\frac{1}{2}\right)^{n-5}.$$

Often it is convenient for the index to begin at 1, so if for some reason it begins at a different value, we can re-index by making a change of variables. For example, consider the series

$$\sum_{n=2}^{\infty} \frac{1}{n^2}.$$

By introducing the variable $m = n - 1$, so that $n = m + 1$, we can rewrite the series as

$$\sum_{m=1}^{\infty} \frac{1}{(m+1)^2}.$$

Example 1.2.1: Evaluating Limits of Sequences of Partial Sums

For each of the given series, use the sequence of partial sums to determine whether the series converges or diverges.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}$

b. $\sum_{n=1}^{\infty}(-1)^n$

c. $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$

Solution

a. The sequence of partial sums S_k satisfies

$$S_1=\frac{1}{2}$$

$$S_2=\frac{1}{2}+\frac{2}{3}$$

$$S_3=\frac{1}{2}+\frac{2}{3}+\frac{3}{4}$$

$$S_4=\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}.$$

Notice that each term added is greater than $1/2$. As a result, we see that

$$S_1=\frac{1}{2}$$

$$S_2=\frac{1}{2}+\frac{2}{3}>\frac{1}{2}+\frac{1}{2}=2\left(\frac{1}{2}\right)$$

$$S_3=\frac{1}{2}+\frac{2}{3}+\frac{3}{4}>\frac{1}{2}+\frac{1}{2}+\frac{1}{2}=3\left(\frac{1}{2}\right)$$

$$S_4=\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}>\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}=4\left(\frac{1}{2}\right).$$

From this pattern we can see that $S_k>k\left(\frac{1}{2}\right)$ for every integer k. Therefore, S_k is unbounded and consequently, diverges. Therefore, the infinite series $\sum_{n=1}^{\infty}\frac{n}{n+1}$ diverges.

b. The sequence of partial sums S_k satisfies

$$S_1=-1$$

$$S_2=-1+1=0$$

$$S_3=-1+1-1=-1$$

$$S_4=-1+1-1+1=0.$$

From this pattern we can see the sequence of partial sums is

$$S_k=-1,0,-1,0,....$$

Since this sequence diverges, the infinite series $\sum_{n=1}^{\infty}(-1)^n$ diverges.

c. The sequence of partial sums S_k satisfies

$$S_1=\frac{1}{1\cdot 2}=\frac{1}{2}$$

$$S_2=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}=\frac{1}{2}+\frac{1}{6}=\frac{2}{3}$$

$$S_3 = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{3}{4}$$

$$S_4 = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} = \frac{4}{5}$$

$$S_5 = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \frac{1}{5\cdot 6} = \frac{5}{6}.$$

From this pattern, we can see that the kth partial sum is given by the explicit formula

$$S_k = \frac{k}{k+1}$$

.

Since $k/(k+1) \to 1$, we conclude that the sequence of partial sums converges, and therefore the infinite series converges to 1. We have

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

Exercise 1.2.1

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n}$ converges or diverges.

Hint available at the end of this section.

Answer available at the end of this section.

The Harmonic Series

A useful series to know about is the harmonic series.

Definition: Harmonic Series

The **harmonic series** is defined as

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots. \tag{1.2.4}$$

This series is interesting because it diverges, but it diverges very slowly. By this we mean that the terms in the sequence of partial sums S_k approach infinity, but do so very slowly. We will show that the series diverges, but first we illustrate the slow growth of the terms in the sequence S_k in the table.

Table illustrates the slow growth of the terms in the sequence S_k

k	10	100	1000	10,00	100,000	1,000,000
S_k	2.92897	5.18738	7.48547	9.78761	12.09015	14.39273

Even after $1,000,000$ terms, the partial sum is still relatively small. From this table, it is not clear that this series actually diverges. However, we can show analytically that the sequence of partial sums diverges, and therefore the series diverges.

To show that the sequence of partial sums diverges, we show that the sequence of partial sums is unbounded. We begin by writing the first several partial sums:

$$S_1 = 1$$

$$S_2 = 1 + \frac{1}{2}$$

$$S_3 = 1 + \frac{1}{2} + \frac{1}{3}$$

$$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}.$$

Notice that for the last two terms in S_4,

$$\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4}$$

Therefore, we conclude that

$$S_4 > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{1}{2} + \frac{1}{2} = 1 + 2\left(\frac{1}{2}\right).$$

Using the same idea for S_8, we see that

$$S_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + 3\left(\frac{1}{2}\right).$$

From this pattern, we see that $S_1 = 1, S_2 = 1 + 1/2, S_4 > 1 + 2(1/2)$, and $S_8 > 1 + 3(1/2)$. More generally, it can be shown that $S_{2^j} > 1 + j(1/2)$ for all $j > 1$. Since $1 + j(1/2) \to \infty$, we conclude that the sequence S_k is unbounded and therefore diverges. In the previous section, we stated that convergent sequences are bounded. Consequently, since S_k is unbounded, it diverges. Thus, the harmonic series diverges.

Algebraic Properties of Convergent Series

Since the sum of a convergent infinite series is defined as a limit of a sequence, the algebraic properties for series listed below follow directly from the algebraic properties for sequences.

Algebraic Properties of Convergent Series

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent series. The algebraic properties hold.

i. The series $\sum_{n=1}^{\infty}(a_n + b_n)$ converges, and $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$. (Sum Rule)

ii. The series $\sum_{n=1}^{\infty}(a_n - b_n)$ converges, and $\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$. (Difference Rule)

iii. For any real number c, the series $\sum_{n=1}^{\infty} ca_n$ converges, and $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$. (Constant Multiple Rule)

Example 1.2.2: Using Algebraic Properties of Convergent Series

Evaluate $\sum_{n=1}^{\infty}\left[\frac{3}{n(n+1)} + \left(\frac{1}{2}\right)^{n-2}\right]$.

Solution

We showed earlier that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

and

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = 2.$$

Since both of those series converge, we can apply the properties of Note 1.2.1 to evaluate

$$\sum_{n=1}^{\infty} \left[\frac{3}{n(n+1)} + \left(\frac{1}{2}\right)^{n-2}\right].$$

Using the sum rule, write

$$\sum_{n=1}^{\infty} \left[\frac{3}{n(n+1)} + \left(\frac{1}{2}\right)^{n-2}\right] = \sum_{n=1}^{\infty} \frac{3}{n(n+1)} + \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-2}.$$

Then, using the constant multiple rule and the sums above, we can conclude that

$$\sum_{n=1}^{\infty} \frac{3}{n(n+1)} + \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-2} = 3\sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \left(\frac{1}{2}\right)^{-1} \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = 3(1) + \left(\frac{1}{2}\right)^{-1} (2) = 3 + 2(2) = 7.$$

Exercise 1.2.2

Evaluate $\sum_{n=1}^{\infty} \frac{5}{2^{n-1}}$.

Hint available at the end of this section.

Answer available at the end of this section.

Geometric Series

Definition: Geometric Series

A **geometric series** is any series that we can write in the form

$$a + ar + ar^2 + ar^3 + \cdots = \sum_{n=1}^{\infty} ar^{n-1}. \tag{1.2.5}$$

Because the ratio of each term in this series to the previous term is *r*, the number *r* is called the ratio. We refer to a as the initial term because it is the first term in the series. For example, the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

is a geometric series with initial term $a = 1$ and ratio $r = 1/2$.

In general, when does a geometric series converge? Consider the geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} \tag{1.2.6}$$

when $a > 0$. Its sequence of partial sums S_k is given by

$$S_k = \sum_{n=1}^{k} ar^{n-1} = a + ar + ar^2 + \cdots + ar^{k-1}. \tag{1.2.7}$$

Consider the case when $r = 1$. In that case,

$$S_k = a + a(1) + a(1)^2 + \cdots + a(1)^{k-1} = ak. \tag{1.2.8}$$

Since $a > 0$, we know $ak \to \infty$ as $k \to \infty$. Therefore, the sequence of partial sums is unbounded and thus diverges. Consequently, the infinite series diverges for $r = 1$. For $r \neq 1$, to find the limit of S_k, multiply Equation by $1 - r$. Doing so, we see that

$$(1-r)S_k = a(1-r)(1 + r + r^2 + r^3 + \cdots + r^{k-1}) = a[(1 + r + r^2 + r^3 + \cdots + r^{k-1}) - (r + r^2 + r^3 + \cdots + r^k)] = a(1 - r^k). \quad (1.2.9$$

All the other terms cancel out.

Therefore,

$$S_k = \frac{a(1-r^k)}{1-r} \text{ for } r \neq 1.$$

From our discussion in the previous section, we know that the geometric sequence $r^k \to 0$ if $|r| < 1$ and that r^k diverges if $|r| > 1$ or $r = \pm 1$. Therefore, for $|r| < 1, S_k \to \frac{a}{1-r}$ and we have

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \text{ if } |r| < 1.$$

If $|r| \geq 1, S_k$ diverges, and therefore

$$\sum_{n=1}^{\infty} ar^{n-1} \text{ diverges if } |r| \geq 1.$$

Note

Consider a geometric series, a series of the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots. \tag{1.2.10}$$

If $|r| < 1$, the series converges, and

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \text{ for } |r| < 1. \tag{1.2.11}$$

If $|r| \geq 1$, the series diverges.

Using the above fact to determine if an infinite geometric series converges is sometimes known as the **Geometric Series Test**.

Geometric series sometimes appear in slightly different forms. For example, sometimes the index begins at a value other than $n = 1$ or the exponent involves a linear expression for n other than $n - 1$. As long as we can rewrite the series in the form given by Equation, it is a geometric series. For example, consider the series

$$\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{n+2}.$$

To see that this is a geometric series, we write out the first several terms:

$$\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^{n+2}=\left(\frac{2}{3}\right)^{2}+\left(\frac{2}{3}\right)^{3}+\left(\frac{2}{3}\right)^{4}+\cdots=\frac{4}{9}+\frac{4}{9}\cdot\left(\frac{2}{3}\right)+\frac{4}{9}\cdot\left(\frac{2}{3}\right)^{2}+\cdots.$$

We see that the initial term is $a=4/9$ and the ratio is $r=2/3$. Therefore, the series can be written as

$$\sum_{n=1}^{\infty}\frac{4}{9}\cdot\left(\frac{2}{3}\right)^{n-1}.$$

Since $r=2/3<1$, this series converges, and its sum is given by

$$\sum_{n=1}^{\infty}\frac{4}{9}\cdot\left(\frac{2}{3}\right)^{n-1}=\frac{4/9}{1-2/3}=\frac{4}{3}.$$

Example 1.2.3: Determining Convergence or Divergence of a Geometric Series

Determine whether each of the geometric series converges or diverges, and if it converges, find its sum.

a. $\displaystyle\sum_{n=1}^{\infty}\frac{(-3)^{n+1}}{4^{n-1}}$

b. $\displaystyle\sum_{n=1}^{\infty}e^{2n}$

Solution

a. Writing out the first several terms in the series, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-3)^{n+1}}{4^{n-1}}&=\frac{(-3)^{2}}{4^{0}}+\frac{(-3)^{3}}{4}+\frac{(-3)^{4}}{4^{2}}+\cdots\\&=(-3)^{2}+(-3)^{2}\cdot\left(\frac{-3}{4}\right)+(-3)^{2}\cdot\left(\frac{-3}{4}\right)^{2}+\cdots\\&=9+9\cdot\left(\frac{-3}{4}\right)+9\cdot\left(\frac{-3}{4}\right)^{2}+\cdots.\end{aligned}$$

The initial term $a=-3$ and the ratio $r=-3/4$. Since $|r|=3/4<1$, the series converges to

$$\frac{9}{1-(-3/4)}=\frac{9}{7/4}=\frac{36}{7}.$$

b. Writing this series as

$$e^{2}\sum_{n=1}^{\infty}(e^{2})^{n-1}$$

we can see that this is a geometric series where $r=e^{2}>1$. Therefore, the series diverges.

Watch the accompanying video for a visual demonstration of this concept.

- **Video Length**: 3 minutes and 6 seconds.
- **Context**: This video explains how to determine whether a given $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{n}2}{3^{n+1}}$ geometric series converges or diverges.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Exercise 1.2.3

Determine whether the series $\sum_{n=1}^{\infty}\left(\frac{-2}{5}\right)^{n-1}$ converges or diverges. If it converges, find its sum.

Hint available at the end of this section.

Answer available at the end of this section.

We now turn our attention to a nice application of geometric series. We show how they can be used to write repeating decimals as fractions of integers.

Example 1.2.4:Writing Repeating Decimals as Fractions of Integers

Use a geometric series to write $3.2\overline{6}$ as a fraction of integers.

Solution

Since $3.2\overline{6}— = 3.262626\ldots$, first we write

$$\begin{aligned} 3.262626\ldots &= 3+\frac{26}{100}+\frac{26}{1000}+\frac{26}{100,000}+\cdots \\ &= 3+\frac{26}{10^2}+\frac{26}{10^4}+\frac{26}{10^6}+\cdots. \end{aligned}$$

Ignoring the term 3, the rest of this expression is a geometric series with initial term $a=26/10^2$ and ratio $r=1/10^2$. Therefore, the sum of this series is

$$\frac{26/10^2}{1-(1/10^2)}=\frac{26/10^2}{99/10^2}=\frac{26}{99}.$$

Thus,

$$3.262626\ldots = 3+\tfrac{26}{99}=\tfrac{323}{99}.$$

Exercise 1.2.4

Write $5.2\overline{7}$ as a fraction of integers.

Hint available at the end of this section.

Answer available at the end of this section.

Example 1.2.5:Finding the Area of the Koch Snowflake

Define a sequence of figures $\{F_n\}$ recursively as follows (Figure 1.2.2). Let F_0 be an equilateral triangle with sides of length 1 . For $n \geq 1$, let F_n be the curve created by removing the middle third of each side of F_{n-1} and replacing it with an equilateral triangle pointing outward. The limiting figure as $n \to \infty$ is known as **Koch's snowflake.** Koch Snowflake is created through iterations. The base case is an equilateral triangle. In each iteration, the middle third of each line segment is replaced with another equilateral triangle pointing outward.

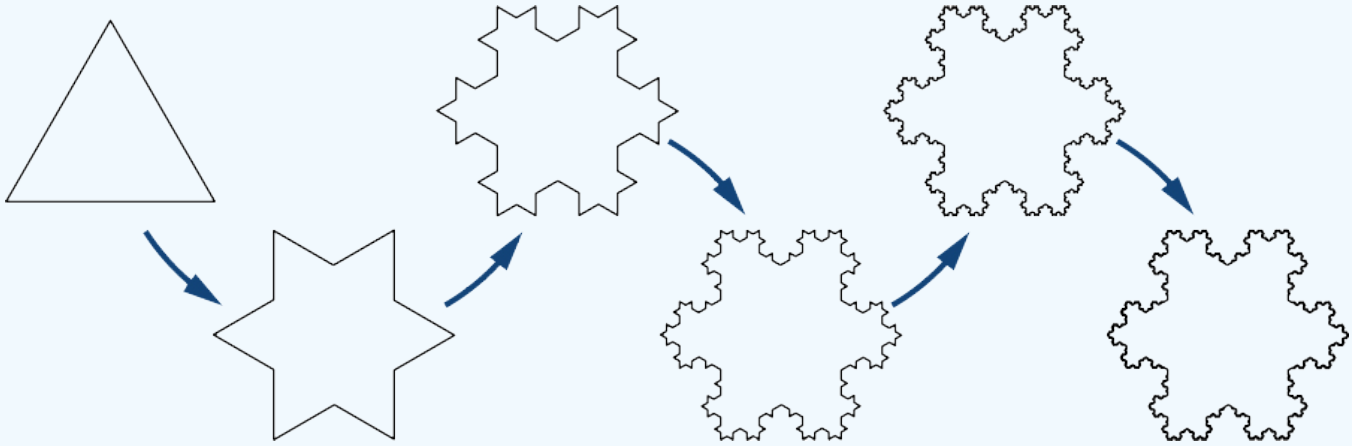


Figure 1.2.2: An illustration showing the step-by-step construction of the Koch snowflake. It begins with an equilateral triangle. In the first step, each side of the triangle is divided into three segments, and an outward-pointing triangular spike is added to the middle segment, forming a six-pointed star. In subsequent steps, this process is repeated for each line segment, progressively adding smaller triangular spikes to create more intricate patterns. The final stages depict the increasing complexity of the fractal structure.

a. Find the length L_n of the perimeter of F_n. Evaluate $\lim\limits_{n\to\infty} L_n$ to find the length of the perimeter of Koch's snowflake.
b. Find the area A_n of figure F_n. Evaluate $\lim\limits_{n\to\infty} A_n$ to find the area of Koch's snowflake.

Solution

a. Let N_n denote the number of sides of figure F_n. Since F_0 is a triangle, $N_0 = 3$. Let ln denote the length of each side of F_n . Since F_0 is an equilateral triangle with sides of length $l_0 = 1$, we now need to determine N_1 and l_1. Since F_1 is created by removing the middle third of each side and replacing that line segment with two line segments, for each side of F_0, we get four sides in F_1. Therefore, the number of sides for F_1 is

$$N_1 = 4 \cdot 3.$$

Since the length of each of these new line segments is $1/3$ the length of the line segments in F_0, the length of the line segments for F_1 is given by

$$l_1 = \tfrac{1}{3} \cdot 1 = \tfrac{1}{3}.$$

Similarly, for F_2, since the middle third of each side of F_1 is removed and replaced with two line segments, the number of sides in F_2 is given by

$$N_2 = 4N_1 = 4(4 \cdot 3) = 4^2 \cdot 3.$$

Since the length of each of these sides is $1/3$ the length of the sides of F_1, the length of each side of figure F_2 is given by

$$l_2 = \tfrac{1}{3} \cdot l_1 = \tfrac{1}{3} \cdot \tfrac{1}{3} = \left(\tfrac{1}{3}\right)^2.$$

More generally, since F_n is created by removing the middle third of each side of F_{n-1} and replacing that line segment with two line segments of length $\frac{1}{3}l_{n-1}$ in the shape of an equilateral triangle, we know that $N_n = 4N_{n-1}$ and $l_n = \dfrac{l_{n-1}}{3}$. Therefore, the number of sides of figure F_n is

$$N_n = 4^n \cdot 3$$

and the length of each side is

$$l_n = \left(\frac{1}{3}\right)^n. \tag{1.2.12}$$

Therefore, to calculate the perimeter of F_n, we multiply the number of sides N_n and the length of each side l_n. We conclude that the perimeter of F_n is given by

$$L_n = N_n \cdot l_n = 3 \cdot \left(\frac{4}{3}\right)^n \tag{1.2.13}$$

Therefore, the length of the perimeter of Koch's snowflake is

$$L = \lim_{n\to\infty} L_n = \infty. \tag{1.2.14}$$

b. Let T_n denote the area of each new triangle created when forming F_n. For $n = 0, T_0$ is the area of the original equilateral triangle. Therefore, $T_0 = A_0 = \sqrt{3}/4$. For $n \geq 1$, since the lengths of the sides of the new triangle are $1/3$ the length of the sides of F_{n-1}, we have

$$T_n = \left(\frac{1}{3}\right)^2 \cdot T_{n-1} = \frac{1}{9} \cdot T_{n-1}. \tag{1.2.15}$$

Therefore, $T_n = \left(\frac{1}{9}\right)^n \cdot \frac{\sqrt{3}}{4}$. Since a new triangle is formed on each side of F_{n-1},

$$A_n = A_{n-1} + N_{n-1} \cdot T_n = A_{n-1} + (3 \cdot 4_{n-1}) \cdot \left(\frac{1}{9}\right)^n \cdot \frac{\sqrt{3}}{4} = A_{n-1} + \frac{3}{4} \cdot \left(\frac{4}{9}\right)^n \cdot \frac{\sqrt{3}}{4}. \tag{1.2.16}$$

Writing out the first few terms A_0, A_1, A_2, we see that

$A_0 = \frac{\sqrt{3}}{4}$

$A_1 = A_0 + \frac{3}{4} \cdot \left(\frac{4}{9}\right) \cdot \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{4} + \frac{3}{4} \cdot \left(\frac{4}{9}\right) \cdot \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{4}\left[1 + \frac{3}{4} \cdot \left(\frac{4}{9}\right)\right]$

$A_2 = A_1 + \frac{3}{4} \cdot \left(\frac{4}{9}\right)^2 \cdot \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{4}\left[1 + \frac{3}{4} \cdot \left(\frac{4}{9}\right)\right] + \frac{3}{4} \cdot \left(\frac{4}{9}\right)^2 \cdot \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{4}[1 + \frac{3}{4} \cdot \left(\frac{4}{9}\right) + \frac{3}{4} \cdot \left(\frac{4}{9}\right)^2]$.

More generally,

$$A_n = \frac{\sqrt{3}}{4}\left[1 + \frac{3}{4}\left(\frac{4}{9} + \left(\frac{4}{9}\right)^2 + \cdots + \left(\frac{4}{9}\right)^n\right)\right].$$

Factoring $4/9$ out of each term inside the inner parentheses, we rewrite our expression as

$$A_n = \frac{\sqrt{3}}{4}\left[1 + \frac{1}{3}\left(1 + \frac{4}{9} + \left(\frac{4}{9}\right)^2 + \cdots + \left(\frac{4}{9}\right)^{n-1}\right)\right].$$

The expression $1 + \left(\frac{4}{9}\right) + \left(\frac{4}{9}\right)^2 + \cdots + \left(\frac{4}{9}\right)^{n-1}$ is a geometric sum. As shown earlier, this sum satisfies

$$1 + \frac{4}{9} + \left(\frac{4}{9}\right)^2 + \cdots + \left(\frac{4}{9}\right)^{n-1} = \frac{1 - (4/9)^n}{1 - (4/9)}.$$

Substituting this expression into the expression above and simplifying, we conclude that

$$A_n = \frac{\sqrt{3}}{4}\left[1 + \frac{1}{3}\left(\frac{1 - (4/9)^n}{1 - (4/9)}\right)\right] = \frac{\sqrt{3}}{4}\left[\frac{8}{5} - \frac{3}{5}\left(\frac{4}{9}\right)^n\right].$$

Therefore, the area of Koch's snowflake is

$$A = \lim_{n\to\infty} A_n = \frac{2\sqrt{3}}{5}.$$

Analysis

The Koch snowflake is interesting because it has finite area, yet infinite perimeter. Although at first this may seem impossible, recall that you have seen similar examples earlier in the text. For example, consider the region bounded by the curve $y = 1/x^2$ and the x-axis on the interval $[1, \infty)$. Since the improper integral

$$\int_1^\infty \frac{1}{x^2}\, dx$$

converges, the area of this region is finite, even though the perimeter is infinite.

Koch's Snowflake

You can move it around and also change its equation to see the effect on the graph. The interactive app is unfortunately not accessible.

Koch's Snowflake Maker

By clicking on Next or Previous, you can visualize Koch's Snowflake for increased or decreased values of n.

Next Previous n = 1

Telescoping Series

Consider the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. We discussed this series in Example, showing that the series converges by writing out the first several partial sums $S_1, S_2, \ldots, S_6$ and noticing that they are all of the form $S_k = \frac{k}{k+1}$. Here we use a different technique to show that this series converges. By using partial fractions, we can write

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

Therefore, the series can be written as

$$\sum_{n=1}^{\infty} \left[\frac{1}{n} - \frac{1}{n+1}\right] = \left(1 + \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots.$$

Writing out the first several terms in the sequence of partial sums S_k, we see that

$$S_1 = 1 - \tfrac{1}{2}$$

$$S_2 = \left(1 - \tfrac{1}{2}\right) + \left(\tfrac{1}{2} - \tfrac{1}{3}\right) = 1 - \tfrac{1}{3}$$

$$S_3 = \left(1 - \tfrac{1}{2}\right) + \left(\tfrac{1}{2} - \tfrac{1}{3}\right) + \left(\tfrac{1}{3} - \tfrac{1}{4}\right) = 1 - \tfrac{1}{4}.$$

In general,

$$S_k = \left(1 - \tfrac{1}{2}\right) + \left(\tfrac{1}{2} - \tfrac{1}{3}\right) + \left(\tfrac{1}{3} - \tfrac{1}{4}\right) + \cdots + \left(\tfrac{1}{k} - \tfrac{1}{k+1}\right) = 1 - \frac{1}{k+1}.$$

We notice that the middle terms cancel each other out, leaving only the first and last terms. In a sense, the series collapses like a spyglass with tubes that disappear into each other to shorten the telescope. For this reason, we call a series that has this property a telescoping series. For this series, since $S_k = 1 - 1/(k+1)$ and $1/(k+1) \to 0$ as $k \to \infty$, the sequence of partial sums converges to 1, and therefore the series converges to 1.

Definition: Telescoping Series

A **telescoping series** is a series in which most of the terms cancel in each of the partial sums, leaving only some of the first terms and some of the last terms.

For example, any series of the form

$$\sum_{n=1}^{\infty}[b_n - b_{n+1}] = (b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + \cdots$$

is a telescoping series. We can see this by writing out some of the partial sums. In particular, we see that

$$S_1 = b_1 - b_2$$

$$S_2 = (b_1 - b_2) + (b_2 - b_3) = b_1 - b_3$$

$$S_3 = (b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) = b_1 - b_4.$$

In general, the *kth* partial sum of this series is

$$S_k = b_1 - b_{k+1}.$$

Since the *kth* partial sum can be simplified to the difference of these two terms, the sequence of partial sums S_k will converge if and only if the sequence b_{k+1} converges. Moreover, if the sequence b_{k+1} converges to some finite number B, then the sequence of partial sums converges to $b_1 - B$, and therefore

$$\sum_{n=1}^{\infty}[b_n - b_{n+1}] = b_1 - B.$$

In the next example, we show how to use these ideas to analyze a telescoping series of this form.

Example 1.2.6: Evaluating a Telescoping Series

Determine whether the telescoping series

$$\sum_{n=1}^{\infty}\left[\cos\left(\frac{1}{n}\right) - \cos\left(\frac{1}{n+1}\right)\right]$$

converges or diverges. If it converges, find its sum.

Solution

By writing out terms in the sequence of partial sums, we can see that

$$S_1 = \cos(1) - \cos(\tfrac{1}{2})$$

$$S_2 = (\cos(1) - \cos(\tfrac{1}{2})) + (\cos(\tfrac{1}{2}) - \cos(\tfrac{1}{3})) = \cos(1) - \cos(\tfrac{1}{3})$$

$$S_3 = (\cos(1) - \cos(\tfrac{1}{2})) + (\cos(\tfrac{1}{2}) - \cos(\tfrac{1}{3})) + (\cos(\tfrac{1}{3}) - \cos(\tfrac{1}{4}))$$

$$= \cos(1) - \cos(\tfrac{1}{4}).$$

In general,

$$S_k = \cos(1) - \cos\left(\tfrac{1}{k+1}\right).$$

Since $1/(k+1) \to 0$ as $k \to \infty$ and $\cos x$ is a continuous function, $\cos(1/(k+1)) \to \cos(0) = 1$. Therefore, we conclude that $S_k \to \cos(1) - 1$. The telescoping series converges and the sum is given by

$$\sum_{n=1}^{\infty} \left[\cos\left(\frac{1}{n}\right) - \cos\left(\frac{1}{n+1}\right)\right] = \cos(1) - 1.$$

Watch the accompanying video for a visual demonstration of this concept.

- **Video Length**: 3 minutes and 44 seconds.
- **Context:** This video demonstrates how to find the sum of a telescoping series $\sum_{n=1}^{\infty} \frac{2}{n(n+2)}$.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Exercise 1.2.5

Determine whether $\sum_{n=1}^{\infty}[e^{1/n} - e^{1/(n+1)}]$ converges or diverges. If it converges, find its sum.

Hint available at the end of this section.

Answer available at the end of this section.

Euler's Constant

We have shown that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Here we investigate the behavior of the partial sums S_k as $k \to \infty$. In particular, we show that they behave like the natural logarithm function by showing that there exists a constant γ such that

$$\sum_{n=1}^{k} \left(\frac{1}{n} - \ln k\right) \to \gamma \text{ as } k \to \infty.$$

This constant γ is known as **Euler's constant.**

1. Let $T_k = \sum_{n=1}^{k} \left(\frac{1}{n} - \ln k\right)$. Evaluate T_k for various values of k.
2. For T_k as defined in part 1. show that the sequence T_k converges.
   a. Show that the sequence T_k is monotone decreasing. (Hint: Show that $\ln(1 + 1/k) > 1/(k+1)$)
   b. Show that the sequence T_k is bounded below by zero. (Hint: Express $\ln k$ as a definite integral.)
   c. Use the Monotone Convergence Theorem to conclude that the sequence T_k converges. The limit γ is Euler's constant.
3. Now estimate how far T_k is from γ for a given integer k. Prove that for \(k≥1, 0
   a. Show that $\ln(k+1) - \ln k < 1/k$.
   b. Use the result from part a. to show that for any integer k,

$$T_k - T_{k+1} < \frac{1}{k} - \frac{1}{k+1}.$$

   c. For any integers k and j such that $j > k$, express $T_k - T_j$ as a telescoping sum by writing

$$T_k - T_j = (T_k - T_{k+1}) + (T_{k+1} - T_{k+2}) + (T_{k+2} - T_{k+3}) + \cdots + (T_{j-1} - T_j).$$

Use the result from part b. combined with this telescoping sum to conclude that

$$T_k - T_j < \frac{1}{k} - \frac{1}{j}.$$

4. Apply the limit to both sides of the inequality in part c. to conclude that

$$T_k - \gamma \leq \frac{1}{k}.$$

5. Estimate γ to an accuracy of within **0.001**.

Key Concepts

- Given the infinite series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

and the corresponding sequence of partial sums S_k where

$$S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k,$$

the series converges if and only if the sequence S_k converges.

- The geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ converges if $|r| < 1$ and diverges if $|r| \geq 1$. For $|r| < 1$,

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}.$$

- The harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

diverges.

- A series of the form $\sum_{n=1}^{\infty} [b_n - b_{n+1}] = [b_1 - b_2] + [b_2 - b_3] + [b_3 - b_4] + \cdots + [b_n - b_{n+1}] + \cdots$ is a telescoping series. The kth partial sum of this series is given by $S_k = b_1 - b_{k+1}$. The series will converge if and only if $\lim_{k\to\infty} b_{k+1}$ exists. In that case,

$$\sum_{n=1}^{\infty} [b_n - b_{n+1}] = b_1 - \lim_{k\to\infty} (b_{k+1}).$$

Key Equations

- **Harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

- **Sum of a geometric series**

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \text{ for } |r| < 1$$

Glossary

Convergence of a Series
A series converges if the sequence of partial sums for that series converges.

Divergence of a Series
A series diverges if the sequence of partial sums for that series diverges.

Geometric Series
A geometric series is a series that can be written in the form.

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots$$

Harmonic Series
The harmonic series takes the form.

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

Infinite Series
An infinite series is an expression of the form.

$$a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n.$$

Partial Sum

The kth partial sum of the infinite series $\sum_{n=1}^{\infty} a_n$ is the finite sum

$$S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k.$$

Telescoping Series
A telescoping series is one in which most of the terms cancel in each of the partial sums.

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Answers

Exercise 1.2.1
Look at the sequence of partial sums.

Exercise 1.2.1
The series diverges because the kth partial sum $S_k > k$.

Exercise 1.2.2
Rewrite as $\sum_{n=1}^{\infty} 5\left(\frac{1}{2}\right)^{n-1}$.

Exercise 1.2.2
10

Exercise $1.2.3$

$r = -2/5$

Exercise $1.2.3$

$5/7$

Exercise $1.2.4$

By expressing this number as a series, find a geometric series with initial term $a = 7/100$ and ratio $r = 1/10$.

Exercise $1.2.4$

$475/90$

Exercise $1.2.5$

Write out the sequence of partial sums to see which terms cancel.

Exercise $1.2.5$

$e - 1$

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1.2E: Exercises for Section 1.2

Exercises 1 - 4

Use sigma notation to write each expressions as an infinite series.

1. $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$

   Answer

   $$\sum_{n=1}^{\infty}\frac{1}{n}$$

2. $1-1+1-1+\cdots$
3. $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$

   Answer

   $$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$$

4. $\sin 1+\sin\frac{1}{2}+\sin\frac{1}{3}+\sin\frac{1}{4}+\cdots$

Exercises 5 - 8

Compute the first four partial sums $S_1,\ldots,S_4$ for the series having n^{th} term a_n starting with $n=1$ as follows.

5. $a_n=n$

   Answer

   $1,3,6,10$

6. $a_n=1/n$
7. $a_n=\sin\frac{n\pi}{2}$

   Answer

   $1,1,0,0$

8. $a_n=(-1)^n$

Exercises 9 - 12

Compute the general term a_n of the series with the given partial sum S_n. If the sequence of partial sums converges, find its limit S.

9. $S_n=1-\frac{1}{n},\quad n\geq 2$

   Answer

   $a_n=S_n-S_{n-1}=\dfrac{1}{n-1}-\dfrac{1}{n}$. Since $S=\lim\limits_{n\to\infty}S_n=\lim\limits_{n\to\infty}\left(1-\dfrac{1}{n}\right)=1$, the series converges to $S=1$.

10. $S_n=\dfrac{n(n+1)}{2},\quad n\geq 1$
11. $S_n=\sqrt{n},\quad n\geq 2$

Answer

$a_n = S_n - S_{n-1} = \sqrt{n} - \sqrt{n-1} = \dfrac{1}{\sqrt{n-1}+\sqrt{n}}.$
The series diverges because the partial sums are unbounded.
That is, $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} \sqrt{n} = \infty.$

12. $S_n = 2 - \dfrac{n+2}{2^n}, \quad n \geq 1$

Exercises 13 - 16

For each series given, use the sequence of partial sums to determine whether the series converges or diverges.

13. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+2}$

Answer

$S_1 = 1/3,$
$S_2 = 1/3 + 2/4 > 1/3 + 1/3 = 2/3,$
$S_3 = 1/3 + 2/4 + 3/5 > 3 \cdot (1/3) = 1.$
In general $S_k > k/3$, so the series diverges.
Note that the nth Term Test for Divergence could also be used to prove that this series diverges.

14. $\displaystyle\sum_{n=1}^{\infty}(1-(-1)^n))$
15. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$ Hint: Use a partial fraction decomposition like that for $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}.$)

Answer

$$S_1 = 1/(2\cdot 3) = 1/6 = 2/3 - 1/2,$$

$$S_2 = 1/(2\cdot 3) + 1/(3\cdot 4) = 2/12 + 1/12 = 1/4 = 3/4 - 1/2,$$

$$S_3 = 1/(2\cdot 3) + 1/(3\cdot 4) + 1/(4\cdot 5) = 10/60 + 5/60 + 3/60 = 3/10 = 4/5 - 1/2,$$

$$S_4 = 1/(2\cdot 3) + 1/(3\cdot 4) + 1/(4\cdot 5) + 1/(5\cdot 6) = 10/60 + 5/60 + 3/60 + 2/60 = 1/3 = 5/6 - 1/2.$$

The pattern is $S_k = \dfrac{k+1}{k+2} - \dfrac{1}{2}.$
Then $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} \left(\dfrac{k+1}{k+2} - \dfrac{1}{2}\right) = \dfrac{1}{2}$, so the series converges to $1/2$.

16. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n+1}$ Hint: Follow the reasoning for $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}.$)

Exercises 17 - 20.

Suppose that $\displaystyle\sum_{n=1}^{\infty} a_n = 1$, that $\displaystyle\sum_{n=1}^{\infty} b_n = -1$, that $a_1 = 2$, and $b_1 = -3$. Use this information to find the sum of the indicated series.

17. $\displaystyle\sum_{n=1}^{\infty}(a_n + b_n)$

Answer

$$\sum_{n=1}^{\infty}(a_n + b_n) \quad = \quad \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n \quad = \quad 1 + (-1) \quad = \quad 0$$

18. $\sum_{n=1}^{\infty}(a_n - 2b_n)$

19. $\sum_{n=2}^{\infty}(a_n - b_n)$

Answer

$$\sum_{n=2}^{\infty}(a_n - b_n) = \sum_{n=2}^{\infty} a_n - \sum_{n=2}^{\infty} b_n = \left(\sum_{n=1}^{\infty} a_n - a_1\right) - \left(\sum_{n=1}^{\infty} b_n - b_1\right) = (1-2) - (-1-(-3)) = -1 - 2 = -3$$

20. $\sum_{n=1}^{\infty}(3a_{n+1} - 4b_{n+1})$

Exercises 21 - 26

State whether the given series converges or diverges and explain why.

21. $\sum_{n=1}^{\infty} \frac{1}{n+1000}$ (Hint: Rewrite using a change of index.)

Answer

The series diverges, $\sum_{n=1001}^{\infty} \frac{1}{n}$

22. $\sum_{n=1}^{\infty} \frac{1}{n+10^{80}}$ (Hint: Rewrite using a change of index.)

23. $1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots$

Answer

This is a convergent geometric series, since $r = \frac{1}{10} < 1$

24. $1 + \frac{e}{\pi} + \frac{e^2}{\pi^2} + \frac{e^3}{\pi^3} + \cdots$

25. $1 + \frac{\pi}{e^2} + \frac{\pi^2}{e^4} + \frac{\pi^3}{e^6} + \frac{\pi^4}{e^8} + \cdots$

Answer

This is a convergent geometric series, since $r = \pi/e^2 < 1$

26. $1 - \sqrt{\frac{\pi}{3}} + \sqrt{\frac{\pi^2}{9}} - \sqrt{\frac{\pi^3}{27}} + \cdots$

Exercises 27 - 30

For each a_n in the given exercises, write its sum as a geometric series of the form $\sum_{n=1}^{\infty} ar^n$. State whether the series converges and if it does, find the exact value of its sum.

27. $a_1 = -1$ and $\frac{a_n}{a_{n+1}} = -5$ for $n \geq 1$.

Answer

$\sum_{n=1}^{\infty} 5 \cdot (-1/5)^n$, converges to $-5/6$

28. $a_1 = 2$ and $\frac{a_n}{a_{n+1}} = 1/2$ for $n \geq 1$.

29. $a_1 = 10$ and $\frac{a_n}{a_{n+1}} = 10$ for $n \geq 1$.

Answer

$\sum_{n=1}^{\infty} 100 \cdot (1/10)^n$, converges to $\frac{100}{9}$

30. $a_1 = \frac{1}{10}$ and $a_n/a_{n+1} = -10$ for $n \geq 1$.

Exercises 31 - 34

Use the identity $\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n$ (which is true for $|y| < 1$) to express each function as a geometric series in the indicated term.

31. $\frac{x}{1+x}$ in x

Answer

$$x \sum_{n=0}^{\infty} (-x)^n = \sum_{n=1}^{\infty} (-1)^{n-1} x^n$$

32. $\frac{\sqrt{x}}{1-x^{3/2}}$ in $\sqrt{x}$

33. $\frac{1}{1+\sin^2 x}$ in $\sin x$

Answer

$$\sum_{n=0}^{\infty} (-1)^n \sin^{2n}(x)$$

34. $\sec^2 x$ in $\sin x$

Exercises 35 - 38

Evaluate the telescoping series or state whether the series diverges.

35. $\sum_{n=1}^{\infty} 2^{1/n} - 2^{1/(n+1)}$

Answer

$S_k = 2 - 2^{1/(k+1)} \to 1$ as $k \to \infty$.

36. $\sum_{n=1}^{\infty} \frac{1}{n^{13}} - \frac{1}{(n+1)^{13}}$

37. $\sum_{n=1}^{\infty} (\sqrt{n} - \sqrt{n+1})$

Answer

$S_k = 1 - \sqrt{k+1}$ diverges

38. $\sum_{n=1}^{\infty} (\sin n - \sin(n+1))$

Exercises 39 - 42

Express each given series as a telescoping sum and evaluate its nth partial sum.

39. $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$

Answer

$$\sum_{n=1}^{\infty}[\ln n - \ln(n+1)],$$
$$S_k = -\ln(k+1)$$

10. $\displaystyle\sum_{n=1}^{\infty}\frac{2n+1}{(n^2+n)^2}$ (Hint: Factor denominator and use partial fractions.)
11. $\displaystyle\sum_{n=2}^{\infty}\frac{\ln\left(1+\frac{1}{n}\right)}{(\ln n)\ln(n+1)}$

Answer

$a_n = \frac{1}{\ln n} - \frac{1}{\ln(n+1)}$ and $S_k = \frac{1}{\ln(2)} - \frac{1}{\ln(k+1)} \to \frac{1}{\ln(2)}$

12. $\displaystyle\sum_{n=1}^{\infty}\frac{(n+2)}{n(n+1)2^{n+1}}$ (Hint: Look at $1/(n2^n)$.

A general telescoping series is one in which all but the first few terms cancel out after summing a given number of successive terms.

13. Let $a_n = f(n) - 2f(n+1) + f(n+2)$, in which $f(n) \to 0$ as $n \to \infty$. Find $\displaystyle\sum_{n=1}^{\infty} a_n$.

Answer

$$\sum_{n=1}^{\infty} a_n = f(1) - f(2)$$

14. $a_n = f(n) - f(n+1) - f(n+2) + f(n+3)$, in which $f(n) \to 0$ as $n \to \infty$. Find $\displaystyle\sum_{n=1}^{\infty} a_n$.
15. Suppose that $a_n = c_0 f(n) + c_1 f(n+1) + c_2 f(n+2) + c_3 f(n+3) + c_4 f(n+4)$, where $f(n) \to 0$ as $n \to \infty$. Find a condition on the coefficients $c_0, \ldots, c_4$ that make this a general telescoping series.

Answer

$$c_0 + c_1 + c_2 + c_3 + c_4 = 0$$

16. Evaluate $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)}$ (Hint: $\displaystyle\frac{1}{n(n+1)(n+2)} = \frac{1}{2n} - \frac{1}{n+1} + \frac{1}{2(n+2)}$)
17. Evaluate $\displaystyle\sum_{n=2}^{\infty}\frac{2}{n^3-n}$.

Answer

$$\frac{2}{n^3-1} = \frac{1}{n-1} - \frac{2}{n} + \frac{1}{n+1},$$
$$S_n = (1 - 1 + 1/3) + (1/2 - 2/3 + 1/4) + (1/3 - 2/4 + 1/5) + (1/4 - 2/5 + 1/6) + \cdots = 1/2$$

18. Find a formula for $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+N)}$ where N is a positive integer.
19. [T] Define a sequence $t_k = \displaystyle\sum_{n=1}^{k-1}(1/k) - \ln k$. Use the graph of $1/x$ to verify that t_k is increasing. Plot t_k for $k = 1 \ldots 100$ and state whether it appears that the sequence converges.

Answer

t_k converges to $0.57721 \ldots$ t_k is a sum of rectangles of height $1/k$ over the interval $[k, k+1]$ which lie above the graph of $1/x$.

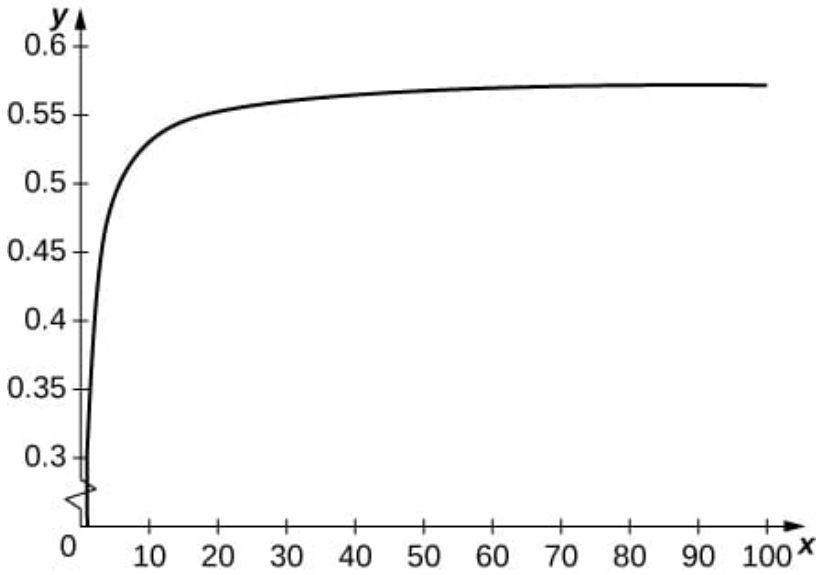


Figure $1.2E.1$: Graph of a function showing a curve that starts at approximately $y = 0.3$ when $x = 0$ and rises steeply before leveling off around $y = 0.55$ as x approaches 100. The x-axis is labeled from 0 to 100, and the $y - axis$ is labeled from 0.3 to 0.6, indicating the behavior of a function that converges to a horizontal asymptote.

50. [T] Suppose that N equal uniform rectangular blocks are stacked one on top of the other, allowing for some overhang. Archimedes' law of the lever implies that the stack of N blocks is stable as long as the center of mass of the top $(N-1)$ blocks lies at the edge of the bottom block. Let x denote the position of the edge of the bottom block, and think of its position as relative to the center of the next-to-bottom block. This implies that $(N-1)x = \left(\frac{1}{2} - x\right)$ or $x = 1/(2N)$. Use this expression to compute the maximum overhang (the position of the edge of the top block over the edge of the bottom block.) See the following figure.

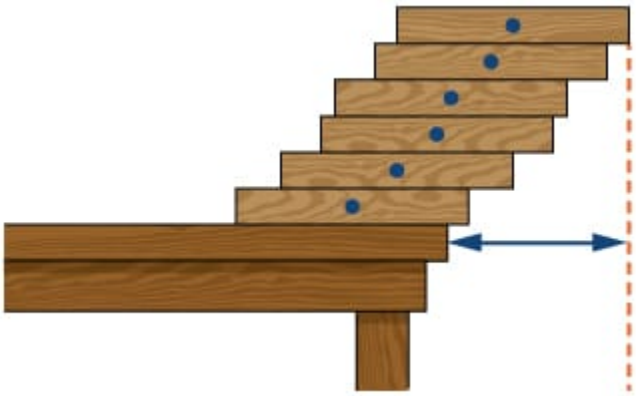


Figure $1.2E.2$: An illustration of a wooden staircase, viewed from the side, showing overlapping wooden planks. Blue dots indicate points along the edges of the steps. A horizontal arrow marks the distance from the edge of the bottom plank to a vertical dashed red line, representing the total overhang of the steps.

Each of the following infinite series converges to the given multiple of π or $1/\pi$.

In each case, find the minimum value of N such that the Nth partial sum of the series accurately approximates the left-hand side to the given number of decimal places, and give the desired approximate value. Up to 15 decimals place, $\pi = 3.141592653589793\ldots$.

51. [T] $\pi = -3 + \sum_{n=1}^{\infty} \frac{n2^n n!^2}{(2n)!}$, error < 0.0001

Answer

$N = 22,$
$S_N = 6.1415$

52. [T] $\frac{\pi}{2} = \sum_{k=0}^{\infty} \frac{k!}{(2k+1)!!} = \sum_{k=0}^{\infty} \frac{2^k k!^2}{(2k+1)!}$, error $< 10^{-4}$

53. [T] $\frac{9801}{2\pi} = \frac{4}{9801} \sum_{k=0}^{\infty} \frac{(4k)!(1103+26390k)}{(k!)^4 396^{4k}}$, error $< 10^{-12}$

Answer

$N = 3,$
$S_N = 1.559877597243667...$

54. [T] $\frac{1}{12\pi} = \sum_{k=0}^{\infty} \frac{(-1)^k (6k)!(13591409+545140134k)}{(3k)!(k!)^3 640320^{3k+3/2}}$, error $< 10^{-15}$

55. [T] A fair coin is one that has probability $1/2$ of coming up heads when flipped.
   a. What is the probability that a fair coin will come up tails n times in a row?
   b. Find the probability that a coin comes up heads for the first time after an even number of coin flips.

Answer

a. The probability of any given ordered sequence of outcomes for n coin flips is $1/2^n$.

b. The probability of coming up heads for the first time on the n th flip is the probability of the sequence $TT...TH$ which is $1/2^n$. The probability of coming up heads for the first time on an even flip is $\sum_{n=1}^{\infty} 1/2^{2n}$ or $1/3$.

56. [T] Find the probability that a fair coin is flipped a multiple of three times before coming up heads.

57. [T] Find the probability that a fair coin will come up heads for the second time after an even number of flips.

Answer

$5/9$

58. [T] Find a series that expresses the probability that a fair coin will come up heads for the second time on a multiple of three flips.

59. [T] The expected number of times that a fair coin will come up heads is defined as the sum over $n = 1, 2, ...$ of n times the probability that the coin will come up heads exactly n times in a row, or $\frac{n}{2^{n+1}}$. Compute the expected number of consecutive times that a fair coin will come up heads.

Answer

$E = \sum_{n=1}^{\infty} \frac{n}{2^{n+1}} = 1,$ as can be shown using summation by parts

60. [T] A person deposits $\$10$ at the beginning of each quarter into a bank account that earns 4 annual interest compounded quarterly (four times a year).
   a. Show that the interest accumulated after n quarters is $\$10(\frac{1.01^{n+1}-1}{0.01} - n)$.
   b. Find the first eight terms of the sequence.
   c. How much interest has accumulated after 2 years?

61. [T] Suppose that the amount of a drug in a patient's system diminishes by a multiplicative factor $r < 1$ each hour. Suppose that a new dose is administered every N hours. Find an expression that gives the amount $A(n)$ in the patient's system after n hours for each n in terms of the dosage d and the ratio r. (Hint: Write $n = mN + k$, where \(0≤k

Answer

The part of the first dose after n hours is dr^n, the part of the second dose is dr^{n-N}, and, in general, the part remaining of the mth dose is dr^{n-mN}, so $A(n) = \sum_{l=0}^{m} dr^{n-lN} = \sum_{l=0}^{m} dr^{k+(m-l)N} = \sum_{q=0}^{m} dr^{k+qN} = dr^k \sum_{q=0}^{m} r^{Nq}$
$= dr^k \frac{1-r^{(m+1)N}}{1-r^N}, \quad \text{where } n = k + mN.$

62. [T] A certain drug is effective for an average patient only if there is at least 1 mg per kg in the patient's system, while it is safe only if there is at most 2 mg per kg in an average patient's system. Suppose that the amount in a patient's system diminishes by a multiplicative factor of 0.9 each hour after a dose is administered. Find the maximum interval N of hours between doses,

and corresponding dose range d (in mg/kg) for this N that will enable use of the drug to be both safe and effective in the long term.

53. Suppose that $a_n \geq 0$ is a sequence of numbers. Explain why the sequence of partial sums of a_n is increasing.

Answer

$S_{N+1} = a_{N+1} + S_N \geq S_N$

54. [T] Suppose that a_n is a sequence of positive numbers and the sequence S_n of partial sums of a_n is bounded above. Explain why $\sum_{n=1}^{\infty} a_n$ converges. Does the conclusion remain true if we remove the hypothesis $a_n \geq 0$?

55. [T] Suppose that $a_1 = S_1 = 1$ and that, for given numbers $S > 1$ and \(0

Answer

Since $S > 1, a_2 > 0$, and since $k < 1, S_2 = 1 + a_2 < 1 + (S - 1) = S$. If $S_n > S$ for some n, then there is a smallest n. For this $n, S > S_{n-1}$, so \(S_n=S_{n−1}+k(S−S_{n−1})=kS+(1−k)S_{n−1}0\) for all n, so S_n is increasing and bounded by S. Let $S_* = \lim S_n$. If \(S_∗0\), but we can find n such that $S_* - S_n < \delta/2$, which implies that $S_{n+1} = S_n + k(S - S_n) > S_* + \delta/2$, contradicting that Sn is increasing to S_*. Thus $S_n \to S$.

56. [T] A version of **von Bertalanffy growth** can be used to estimate the age of an individual in a homogeneous species from its length if the annual increase in year $n + 1$ satisfies $a_{n+1} = k(S - S_n)$, with S_n as the length at year n, S as a limiting length, and k as a relative growth constant. If $S_1 = 3, S = 9$, and $k = 1/2$, numerically estimate the smallest value of n such that $S_n \geq 8$. Note that $S_{n+1} = S_n + a_{n+1}$. Find the corresponding n when $k = 1/4$.

57. [T] Suppose that $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive terms. Explain why $\lim_{N\to\infty} \sum_{n=N+1}^{\infty} a_n = 0$.

Answer

Let $S_k = \sum_{n=1}^{k} a_n$ and $S_k \to L$. Then S_k eventually becomes arbitrarily close to L, which means that $L - S_N = \sum_{n=N+1}^{\infty} a_n$ becomes arbitrarily small as $N \to \infty$.

58. [T] Find the length of the dashed zig-zag path in the following figure.

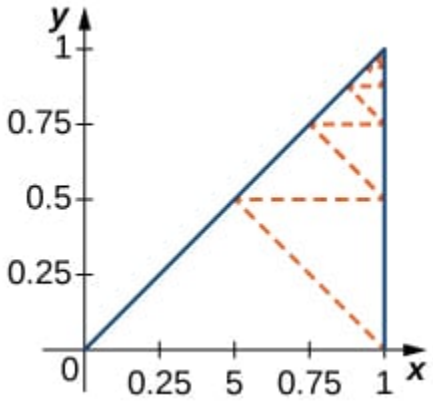


Figure $1.2E.3$: A graph of a right triangle with its hypotenuse as a blue diagonal line from (0, 0) to (1, 1). The x-axis and y-axis are labeled from 0 to 1. Inside the triangle, orange dashed lines form a staircase pattern, connecting points along the hypotenuse to the axes, illustrating a stepwise approximation of the hypotenuse.

59. [T] Find the total length of the dashed path in the following figure.

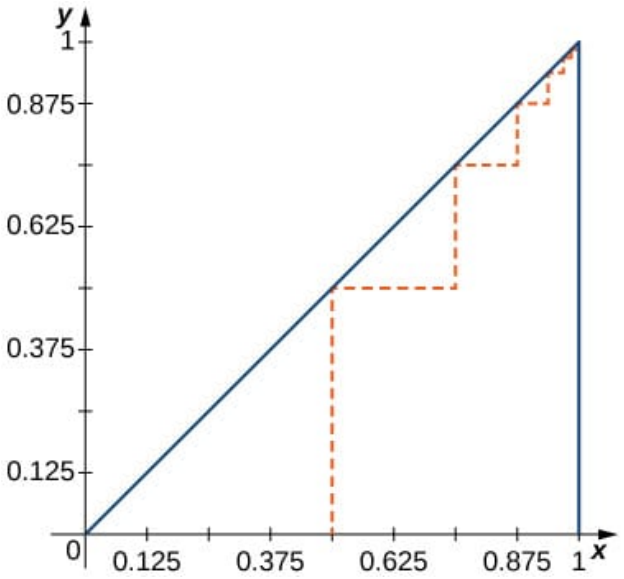


Figure $1.2E.4$: A graph of a right triangle with its hypotenuse as a blue diagonal line from $(0,0)$ to $(1,1)$. The $x-axis$ and $y-axis$ are labeled from 0 to 1, with smaller increments shown $(e.g., 0.125, 0.375)$. Inside the triangle, orange dashed lines create a staircase pattern, connecting points along the hypotenuse to the axes at finer intervals, illustrating a more detailed stepwise approximation of the hypotenuse.

Answer

$$L = \left(1 + \frac{1}{2}\right) \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{3}{2}.$$

70. [T] The **Sierpinski triangle** is obtained from a triangle by deleting the middle fourth as indicated in the first step, by deleting the middle fourths of the remaining three congruent triangles in the second step, and in general deleting the middle fourths of the remaining triangles in each successive step. Assuming that the original triangle is shown in the figure, find the areas of the remaining parts of the original triangle after N steps and find the total length of all of the boundary triangles after N steps.

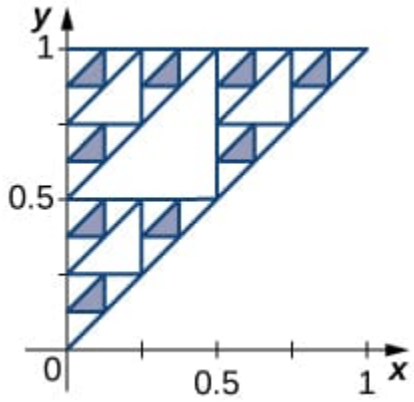


Figure $1.2E.5$: A graph of a right triangle with its hypotenuse running diagonally from $(0,0)$ to $(1,1)$. The triangle is subdivided into smaller, repeating right triangles and empty spaces, creating a fractal-like pattern. The x-axis and y-axis are labeled from $0 to 1$, with increments of 0.5, illustrating a geometric recursive subdivision of the original triangle.

71. [T] The Sierpinski gasket is obtained by dividing the unit square into nine equal sub-squares, removing the middle square, then doing the same at each stage to the remaining sub-squares. The figure shows the remaining set after four iterations. Compute the total area removed after N stages, and compute the length the total perimeter of the remaining set after N stages.

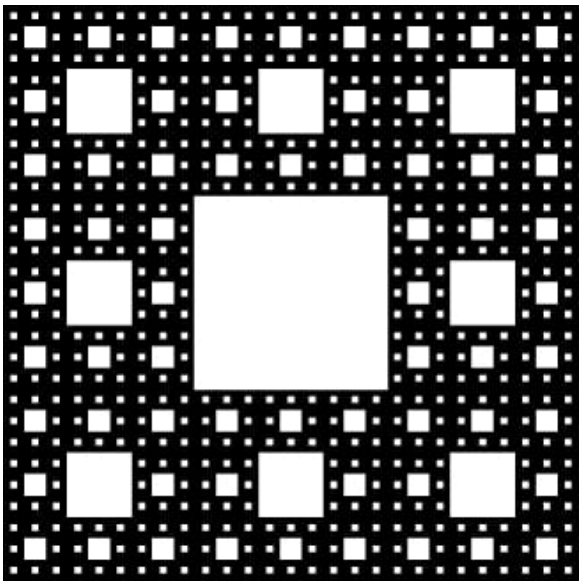


Figure $1.2E.6$: An illustration of a Sierpiński carpet, a fractal generated by repeatedly subdividing a square into nine smaller squares and removing the center square at each iteration. The pattern shows alternating black and white regions, with smaller squares nested within larger squares, creating a complex and self-similar structure.

Answer

At stage one a square of area $1/9$ is removed, at stage 2 one removes 8 squares of area $1/9^2$, at stage three one removes 8^2 squares of area $1/9^3$, and so on. The total removed area after N stages is $\sum_{n=0}^{N-1} \frac{8^N}{9^{N+1}} = \frac{1}{8} \cdot \frac{1-(8/9)^N}{1-8/9} \to 1$ as $N \to \infty$. The total perimeter is $4 + 4\sum_{n=0}^{\infty} \frac{8^N}{3^{N+1}} \to \infty$.

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1.3: The Divergence and Integral Tests

Learning Objectives

- Use the divergence test to determine whether a series converges or diverges.
- Use the integral test to determine the convergence of a series.
- Estimate the value of a series by finding bounds on its remainder term.

In the previous section, we determined the convergence or divergence of several series by explicitly calculating the limit of the sequence of partial sums S_k. In practice, explicitly calculating this limit can be difficult or impossible. Luckily, several tests exist that allow us to determine convergence or divergence for many types of series. In this section, we discuss two of these tests: the divergence test and the integral test. We will examine several other tests in the rest of this chapter and then summarize how and when to use them.

Divergence Test

For a series $\sum_{n=1}^{\infty} a_n$ to converge, the nth term a_n must satisfy $a_n \to 0$ as $n \to \infty$. Therefore, from the algebraic limit properties of sequences,

$$\begin{aligned} \lim_{k\to\infty} a_k &= \lim_{k\to\infty} (S_k - S_{k-1}) \\ &= \lim_{k\to\infty} S_k - \lim_{k\to\infty} S_{k-1} \\ &= S - S = 0. \end{aligned}$$

Therefore, if $\sum_{n=1}^{\infty} a_n$ converges, the nth term $a_n \to 0$ as $n \to \infty$. An important consequence of this fact is the statement:

$$\text{If } a_n \nrightarrow 0 \text{ as } n \to \infty, \sum_{n=1}^{\infty} a_n \text{ diverges.}$$

This test is known as the **divergence test** because it provides a way of proving that a series diverges.

Theorem: Divergence Test

If $\lim_{n\to\infty} a_n = c \neq 0$ or $\lim_{n\to\infty} a_n$ does not exist, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

It is important to note that the converse of this theorem is not true. That is, if $\lim_{n\to\infty} a_n = 0$, we cannot make any conclusion about the convergence of $\sum_{n=1}^{\infty} a_n$.

For example, $\lim_{n\to 0} \frac{1}{n} = 0$, but the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. In this section and the remaining sections of this chapter, we show many more examples of such series. Consequently, although we can use the divergence test to show that a series diverges, we cannot use it to prove that a series converges. Specifically, if $a_n \to 0$, the divergence test is inconclusive.

Example 1.3.1: Using the divergence

For each of the given series, apply the divergence test. If the divergence test proves that the series diverges, state so. Otherwise, indicate that the divergence test is inconclusive.

a. $\sum_{n=1}^{\infty} \frac{n}{3n-1}$

b. $\sum_{n=1}^{\infty} \frac{1}{n^3}$

c. $\sum_{n=1}^{\infty} e^{1/n^2}$

Solution

a. Since $\lim_{n\to\infty} \frac{n}{3n-1} = \frac{1}{3} \neq 0$, by the divergence test, we can conclude that $\sum_{n=1}^{\infty} \frac{n}{3n-1}$ diverges.

b. Since $\lim_{n\to\infty} \frac{1}{n^3} = 0$, the divergence test is inconclusive.

c. Since $\lim_{n\to\infty} e^{1/n^2} = 1 \neq 0$, by the divergence test, the series $\sum_{n=1}^{\infty} e^{1/n^2}$ diverges.

Exercise 1.3.1

What does the divergence test tell us about the series $\sum_{n=1}^{\infty} \cos(1/n^2)$?

Hint available at the end of this section.

Integral Test

In the previous section, we proved that the harmonic series diverges by looking at the sequence of partial sums S_k and showing that $S_{2^k} > 1 + k/2$ for all positive integers k. In this section we use a different technique to prove the divergence of the harmonic series. This technique is important because it is used to prove the divergence or convergence of many other series. This test, called the **integral test,** compares an infinite sum to an improper integral. It is important to note that this test can only be applied when we are considering a series whose terms are all positive.

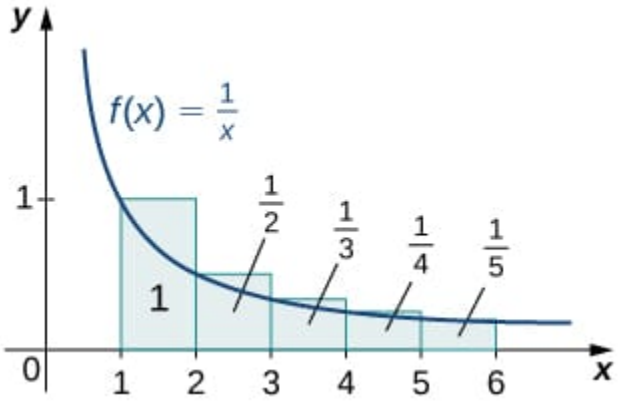


Figure 1.3.1: The sum of the areas of the rectangles is greater than the area between the curve $f(x) = 1/x$ and the x-axis for $x \geq 1$. Since the area bounded by the curve is infinite (as calculated by an improper integral), the sum of the areas of the rectangles is also infinite.

To illustrate how the integral test works, use the harmonic series as an example. In Figure 1.3.1, we depict the harmonic series by sketching a sequence of rectangles with areas $1, 1/2, 1/3, 1/4, \ldots$ along with the function $f(x) = 1/x$. From the graph, we see that

$$\sum_{n=1}^{k} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} > \int_1^{k+1} \frac{1}{x}\, dx. \tag{1.3.1}$$

Therefore, for each k, the kth partial sum S_k satisfies

$$\begin{aligned} S_k = \sum_{n=1}^{k} \frac{1}{n} &> \int_1^{k+1} \frac{1}{x}\, dx = \ln x \Big|_1^{k+1} \\ &= \ln(k+1) - \ln(1) \\ &= \ln(k+1). \end{aligned}$$

Since $\lim_{k\to\infty} \ln(k+1) = \infty$, we see that the sequence of partial sums S_k is unbounded. Therefore, S_k diverges, and, consequently, the series $\sum_{n=1}^{\infty} \frac{1}{n}$ also diverges.

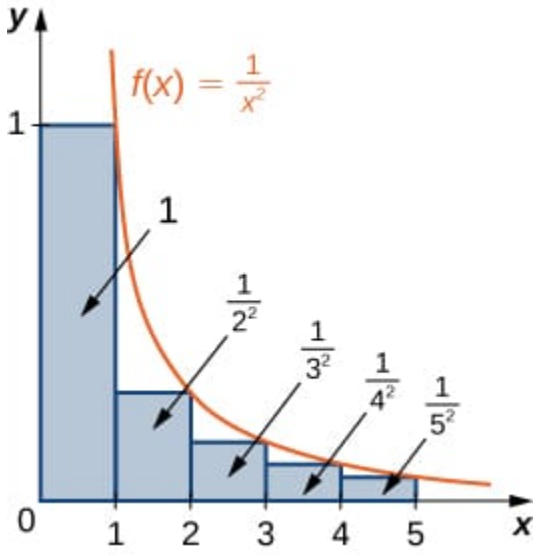


Figure 1.3.2: The sum of the areas of the rectangles is less than the sum of the area of the first rectangle and the area between the curve $f(x) = 1/x^2$ and the x-axis for $x \geq 1$. Since the area bounded by the curve is finite, the sum of the areas of the rectangles is also finite.

Now consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. We show how an integral can be used to prove that this series converges. In Figure 1.3.2, we sketch a sequence of rectangles with areas $1, 1/2^2, 1/3^2, \ldots$ along with the function $f(x) = \frac{1}{x^2}$. From the graph we see that

$$\sum_{n=1}^{k} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} < 1 + \int_1^{k} \frac{1}{x^2}\, dx. \tag{1.3.2}$$

Therefore, for each k, the kth partial sum S_k satisfies

$$\begin{aligned} S_k = \sum_{n=1}^{k} \frac{1}{n^2} &< 1 + \int_1^{k} \frac{1}{x^2}\, dx = 1 - \frac{1}{x}\Big|_1^{k} \\ &= 1 - \frac{1}{k} + 1 \\ &= 2 - \frac{1}{k} < 2. \end{aligned}$$

We conclude that the sequence of partial sums S_k is bounded. We also see that S_k is an increasing sequence:

$$S_k = S_{k-1} + \frac{1}{k^2} \tag{1.3.3}$$

for $k \geq 2$.

Since S_k is increasing and bounded, by the **Monotone Convergence Theorem**, it converges. Therefore, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

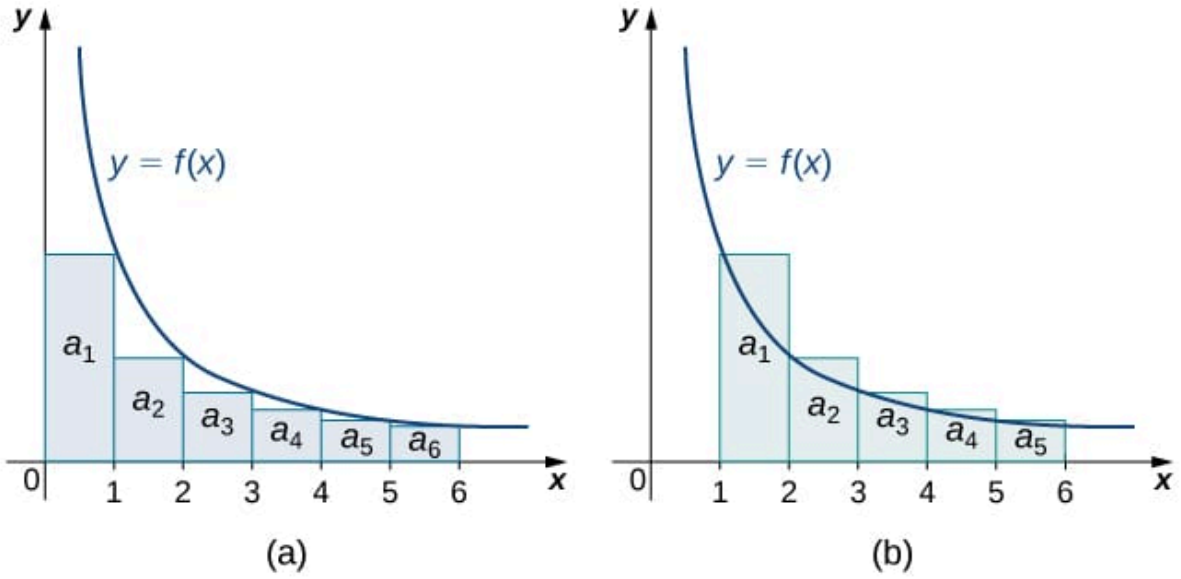


Figure 1.3.3: (a) If we can inscribe rectangles inside a region bounded by a curve $y = f(x)$ and the x-axis, and the area bounded by those curves for $x \geq 1$ is finite, then the sum of the areas of the rectangles is also finite. (b) If a set of rectangles circumscribes the region bounded by $y = f(x)$ and the x axis for $x \geq 1$ and the region has infinite area, then the sum of the areas of the rectangles is also infinite.

We can extend this idea to prove convergence or divergence for many different series. Suppose $\sum_{n=1}^{\infty} a_n$ is a series with positive terms a_n such that there exists a continuous, positive, decreasing function f where $f(n) = a_n$ for all positive integers. Then, as in Figure 1.3.3*a*, for any integer k, the kth partial sum S_k satisfies

\[S_k=a_1+a_2+a_3+\cdots+a_k

Therefore, if $\int_1^{\infty} f(x)\ dx$ converges, then the sequence of partial sums S_k is bounded. Since S_k is an increasing sequence, if it is also a bounded sequence, then by the Monotone Convergence Theorem, it converges. We conclude that if $\int_1^{\infty} f(x)\ dx$ converges, then the series $\sum_{n=1}^{\infty} a_n$ also converges. On the other hand, from Figure 1.3.3*b*, for any integer k, the kth partial sum S_k satisfies

$$S_k = a_1 + a_2 + a_3 + \cdots + a_k > \int_1^{k+1} f(x)\, dx. \tag{1.3.4}$$

If

$$\lim_{k\to\infty} \int_1^{k+1} f(x)\, dx = \infty, \tag{1.3.5}$$

then S_k is an unbounded sequence and therefore diverges. As a result, the series $\sum_{n=1}^{\infty} a_n$ also diverges. Since f is a positive function, if $\int_1^{\infty} f(x)\ dx$ diverges, then

$$\lim_{k\to\infty} \int_1^{k+1} f(x)\, dx = \infty. \tag{1.3.6}$$

We conclude that if $\int_1^{\infty} f(x)\ dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Theorem: Integral Test

Suppose $\sum_{n=1}^{\infty} a_n$ is a series with positive terms a_n. Suppose there exists a function f and a positive integer N such that the three conditions are satisfied:

i. f is continuous,
ii. f is decreasing, and
iii. $f(n) = a_n$ for all integers $n \geq N$.

Then

$$\sum_{n=1}^{\infty} a_n$$

and

$$\int_{N}^{\infty} f(x)\,dx$$

both converge or both diverge (Figure $1.3.3$).

You will be shown a graph and the result of integrating from 1 to ∞. Your task is to decide whether the integral test gives convergence or divergence of the corresponding sum or if the integral test cannot be used. Click on the Start button to begin.

Use of Integral Test

You can move it around and also change its equation to see the effect on the graph. The interactive app is unfortunately not accessible.

You will be shown a graph and the result of integrating from 1 to ∞. Your task is to decide whether the integral test gives convergence or divergence of the corresponding sum or if the integral test cannot be used. Click on the Start button to begin.

$\int_1^{\infty} e^{-x}dx = e^1, \sum_{n=1}^{\infty} e^{-x}$

$\int_1^{\infty} \frac{1}{\sqrt{x}}dx = \infty, \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

$\int_1^{\infty} \frac{\sin(4x)}{x}dx \sim -0.19, \sum_{n=1}^{\infty} \frac{\sin(4n)}{n}$

$\int_1^{\infty} (x+1)^{\frac{-3}{2}}dx = \sqrt{2}, \sum_{n=1}^{\infty} (n+1)^{\frac{-3}{2}}$

$\int_1^{\infty} \frac{1}{(x+1)\ln(x+1)}dx = \infty, \sum_{n=1}^{\infty} \frac{1}{(n+1)\ln(n+1)}$

$\int_1^{\infty} \frac{x}{x+1}dx = \infty, \sum_{n=1}^{\infty} \frac{n}{n+1}$

$\int_1^{\infty} \frac{x}{e^{x^2}}dx = \frac{1}{2e}, \sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$

$\int_1^\infty \frac{1}{\sqrt{1+x^2}}dx = \infty, \sum_{n=1}^\infty \frac{1}{\sqrt{1+n^2}}$

$\int_1^\infty \frac{\cos(3x)}{e^x}dx \sim 0.052, \sum_{n=1}^\infty \frac{\cos(3n)}{e^n}$

$\int_1^\infty \frac{1}{1+x^2}dx = \frac{\pi}{4}, \sum_{n=1}^\infty \frac{1}{1+n^2}$

$\int_1^\infty \frac{x}{1+x^2}dx = \infty, \sum_{n=1}^\infty \frac{n}{1+n^2}$

$\int_1^\infty \frac{1+2\cos(2x)}{x+\sin(2x)}dx = \infty, \sum_{n=1}^\infty \frac{1+2\cos(2n)}{n+\sin(2n)}$

○ Diverges ○ Converges ○ Integral Test Cannot be Used

Start

Although convergence of $\int_N^\infty f(x)\ dx$ implies convergence of the related series $\sum_{n=1}^\infty a_n$, it does not imply that the value of the integral and the series are the same. They may be different, and often are. For example,

$$\sum_{n=1}^\infty \left(\frac{1}{e}\right)^n = \frac{1}{e} + \left(\frac{1}{e}\right)^2 + \left(\frac{1}{e}\right)^3 + \cdots \tag{1.3.7}$$

is a geometric series with initial term $a = 1/e$ and ratio $r = 1/e$, which converges to

$$\frac{1/e}{1-(1/e)} = \frac{1/e}{(e-1)/e} = \frac{1}{e-1}. \tag{1.3.8}$$

However, the related integral $\int_1^\infty (1/e)^x\ dx$ satisfies

$$\int_1^\infty \left(\frac{1}{e}\right)^x dx = \int_1^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_1^b e^{-x}\,dx = \lim_{b\to\infty} -e^{-x}\Big|_1^b = \lim_{b\to\infty}[-e^{-b}+e^{-1}] = \frac{1}{e}. \tag{1.3.9}$$

Example 1.3.2: Using the Integral Test

For each of the given series, use the integral test to determine whether the series converges or diverges.

a. $\sum_{n=1}^\infty \frac{1}{n^3}$

b. $\sum_{n=1}^\infty \frac{1}{\sqrt{2n-1}}$

Solution

a. Compare

$$\sum_{n=1}^\infty \frac{1}{n^3} \text{ and } \int_1^\infty \frac{1}{x^3}\ dx.$$

We have

$$\int_1^\infty \frac{1}{x^3}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^3}\,dx = \lim_{b\to\infty}\left[-\frac{1}{2x^2}\Big|_1^b\right] = \lim_{b\to\infty}\left[-\frac{1}{2b^2}+\frac{1}{2}\right] = \frac{1}{2}.$$

Thus the integral $\int_1^\infty \frac{1}{x^3}\,dx$ converges, and therefore so does the series

$$\sum_{n=1}^\infty \frac{1}{n^3}.$$

b. Compare

$$\sum_{n=1}^\infty \frac{1}{\sqrt{2n-1}} \text{ and } \int_1^\infty \frac{1}{\sqrt{2x-1}}\,dx.$$

Since

$$\int_1^\infty \frac{1}{\sqrt{2x-1}}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{\sqrt{2x-1}}\,dx = \lim_{b\to\infty}\sqrt{2x-1}\Big|_1^b = \lim_{b\to\infty}\left[\sqrt{2b-1}-1\right] = \infty,$$

the integral $\int_1^\infty \frac{1}{\sqrt{2x-1}}\,dx$ diverges, and therefore diverges.

$$\sum_{n=1}^\infty \frac{1}{\sqrt{2n-1}}$$

Watch the accompanying video for a visual demonstration of this concept.

- **Video Length:** 2 minute 6 seconds.
- **Context:** This video demonstrates the use of the Integral Test to show how a series $\sum_{n=0}^\infty \frac{1}{4+n^2}$ converges.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

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Exercise 1.3.2

Use the integral test to determine whether the series $\sum_{n=1}^\infty \frac{n}{3n^2+1}$ converges or diverges.

Hint available at the end of this section.

The p-Series

The harmonic series $\sum_{n=1}^\infty 1/n$ and the series $\sum_{n=1}^\infty 1/n^2$ are both examples of a type of series called a p-series.

Definition: p-series

For any real number p, the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \tag{1.3.10}$$

is called a p**-series.**

We know the p-series converges if $p = 2$ and diverges if $p = 1$. What about other values of p? In general, it is difficult, if not impossible, to compute the exact value of most p-series. However, we can use the tests presented thus far to prove whether a p -series converges or diverges.

If $p < 0$, then $1/n^p \to \infty$, and if $p = 0$, then $1/n^p \to 1$. Therefore, by the divergence test,

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \tag{1.3.11}$$

diverges if $p \leq 0$.

If $p > 0$, then $f(x) = 1/x^p$ is a positive, continuous, decreasing function. Therefore, for $p > 0$, we use the integral test, comparing

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \tag{1.3.12}$$

and

$$\int_1^{\infty} \frac{1}{x^p}\, dx. \tag{1.3.13}$$

We have already considered the case when $p = 1$. Here we consider the case when $p > 0, p \neq 1$. For this case,

$$\int_1^{\infty} \frac{1}{x^p}\, dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^p}\, dx = \lim_{b\to\infty} \frac{1}{1-p} x^{1-p} \,|_1^b = \lim_{b\to\infty} \frac{1}{1-p}[b^{1-p} - 1]. \tag{1.3.14}$$

Because

$$b^{1-p} \to 0 \text{ if } p > 1 \text{ and } b^{1-p} \to \infty \text{ if } p < 1,$$

we conclude that

$$\int_1^{\infty} \frac{1}{x^p}\, dx = \begin{cases} \frac{1}{p-1}, \text{if } p > 1 \\ \infty, \text{if } p < 1. \end{cases} \tag{1.3.15}$$

Therefore, $\sum_{n=1}^{\infty} 1/n^p$ converges if $p > 1$ and diverges if \(0

In summary,

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \leq 1 \end{cases}. \tag{1.3.16}$$

Using the above fact to determine if a p-series converges is sometimes called the p**-Series Test** or simply the p**-Test**.

Example 1.3.3: Testing for Convergence of p-series

For each of the given series, determine whether it converges or diverges.

a. $\sum_{n=1}^{\infty} \frac{1}{n^4}$

b. $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$

Solution

a. This is a p-series with $p = 4 > 1$,so the series converges.
b. Since $p = 2/3 < 1$, the series diverges.

Watch the accompanying video for a visual demonstration of this concept.

- **Video Length**: 5 minute 35 seconds.
- **Context**: This video demonstrates the use of the p series to show a series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$diverges.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online



Exercise 1.3.3

Does the series $\sum_{n=1}^{\infty} \frac{1}{n^{5/4}}$ converge or diverge?

Hint available at the end of this section.

Answer available at the end of this section.

Estimating the Value of a Series

Suppose we know that a series $\sum_{n=1}^{\infty} a_n$ converges and we want to estimate the sum of that series. Certainly we can approximate that sum using any finite sum $\sum_{n=1}^{N} a_n$ where N is any positive integer. The question we address here is, for a convergent series $\sum_{n=1}^{\infty} a_n$, how good is the approximation $\sum_{n=1}^{N} a_n$?

More specifically, if we let

$$R_N = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n \tag{1.3.17}$$

be the remainder when the sum of an infinite series is approximated by the Nth partial sum, how large is R_N? For some types of series, we are able to use the ideas from the integral test to estimate R_N.

Remainder Estimate from the Integral Test

Suppose $\sum_{n=1}^{\infty} a_n$ is a convergent series with positive terms. Suppose there exists a function f satisfying the three conditions:

i. f is continuous,
ii. f is decreasing, and
iii. $f(n) = a_n$ for all integers $n \geq 1$.

Let S_N be the Nth partial sum of $\sum_{n=1}^{\infty} a_n$. For all positive integers N,

\[S_N+∫^∞_{N+1}f(x)\,dx<\sum_{n=1}^∞a_n

In other words, the remainder $R_N = \sum_{n=1}^{\infty} a_n - S_N = \sum_{n=N+1}^{\infty} a_n$ satisfies the reminder estimate:

\[∫^∞_{N+1}f(x)\,dx

This is known as the **remainder estimate.**

We illustrate Note 1.3.1 in Figure 1.3.4. In particular, by representing the remainder $R_N = a_{N+1} + a_{N+2} + a_{N+3} + \cdots$ as the sum of areas of rectangles, we see that the area of those rectangles is bounded above by $\int_N^{\infty} f(x)\ dx$ and bounded below by $\int_{N+1}^{\infty} f(x)\ dx$. In other words,

$$R_N = a_{N+1} + a_{N+2} + a_{N+3} + \cdots > \int_{N+1}^{\infty} f(x)\,dx \tag{1.3.18}$$

and

$$R_N = a_{N+1} + a_{N+2} + a_{N+3} + \cdots < \int_{N}^{\infty} f(x)\,dx. \tag{1.3.19}$$

We conclude that

\[∫^∞_{N+1}f(x)\,dx

Since

$$\sum_{n=1}^{\infty} a_n = S_N + R_N, \tag{1.3.20}$$

where S_N is the Nth partial sum, we conclude that

\[S_N+∫^∞_{N+1}f(x)\,dx<\sum_{n=1}^∞a_n

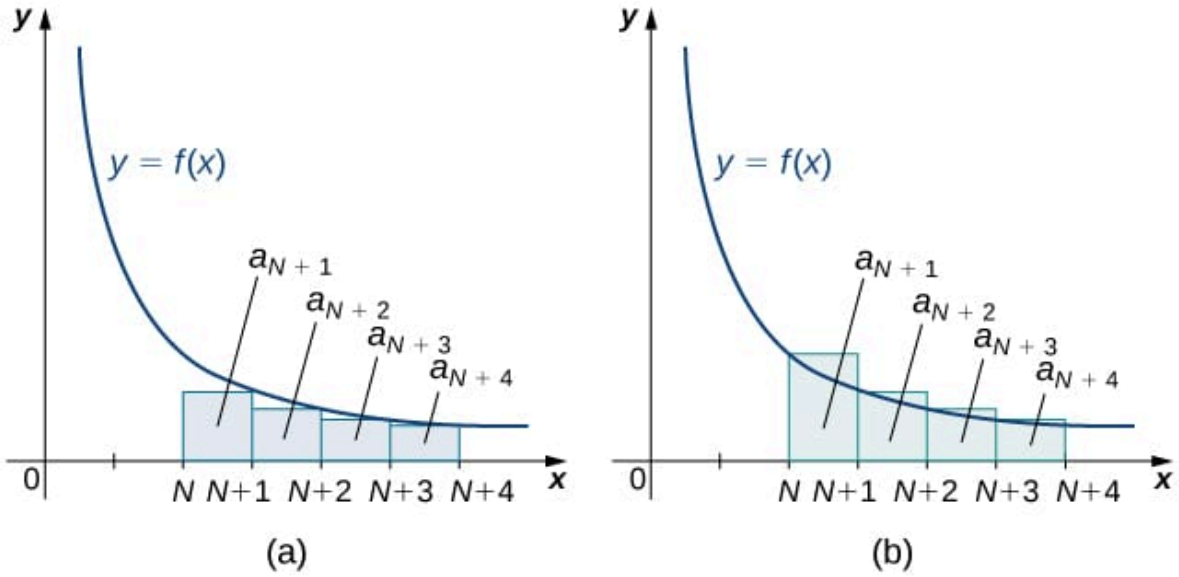


Figure 1.3.4: Given a continuous, positive, decreasing function f and a sequence of positive terms a_n such that $a_n = f(n)$ for all positive integers n, (a) the areas $a_{N+1} + a_{N+2} + a_{N+3} + \cdots < \int_N^\infty f(x)\ dx$, or (b) the areas $a_{N+1} + a_{N+2} + a_{N+3} + \cdots > \int_{N+1}^\infty f(x)\ dx$. Therefore, the integral is either an overestimate or an underestimate of the error.

Example 1.3.4

Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$.

a. Calculate $S_{10} = \sum_{n=1}^{10} \frac{1}{n^3}$ and estimate the error.

b. Determine the least value of N necessary such that S_N will estimate $\sum_{n=1}^{\infty} \frac{1}{n^3}$ to within 0.001.

Solution

a. Using a calculating utility, we have

$$S_{10} = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots + \frac{1}{10^3} \approx 1.19753. \tag{1.3.21}$$

By the remainder estimate, we know

$$R_N < \int_N^\infty \frac{1}{x^3}\, dx. \tag{1.3.22}$$

We have

$$\int_{10}^\infty \frac{1}{x^3}\, dx = \lim_{b\to\infty} \int_{10}^b \frac{1}{x^3}\, dx = \lim_{b\to\infty} \left[-\frac{1}{2x^2}\right]_N^b = \lim_{b\to\infty} \left[-\frac{1}{2b^2} + \frac{1}{2N^2}\right] = \frac{1}{2N^2}. \tag{1.3.23}$$

Therefore, the error is $R_{10} < 1/2(10)^2 = 0.005$.

b. Find N such that $R_N < 0.001$. In part a. we showed that $R_N < 1/2N^2$. Therefore, the remainder $R_N < 0.001$ as long as $1/2N^2 < 0.001$. That is, we need $2N^2 > 1000$. Solving this inequality for N, we see that we need $N > 22.36$. To ensure that the remainder is within the desired amount, we need to round up to the nearest integer. Therefore, the minimum necessary value is $N = 23$.

Exercise 1.3.4

For $\sum_{n=1}^{\infty} \frac{1}{n^4}$, calculate S_5 and estimate the error R_5.

Hint available at the end of this section.

Key Concepts

- If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.
- If $\lim_{n\to\infty} a_n = 0$, the series $\sum_{n=1}^{\infty} a_n$ may converge or diverge.
- If $\sum_{n=1}^{\infty} a_n$ is a series with positive terms a_n and f is a continuous, decreasing function such that $f(n) = a_n$ for all positive integers n, then

$$\sum_{n=1}^{\infty} a_n \tag{1.3.24}$$

and

$$\int_1^{\infty} f(x)\,dx \tag{1.3.25}$$

either both converge or both diverge. Furthermore, if $\sum_{n=1}^{\infty} a_n$ converges, then the Nth partial sum approximation S_N is accurate up to an error R_N where \(\displaystyle ∫^∞_{N+1}f(x)\,dx

- The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Key Equations

- **Divergence Test**

  If $a_n \nrightarrow 0$ as $n \to \infty$, $\sum_{n=1}^{\infty} a_n$ diverges.

- **p-series**

  $\sum_{n=1}^{\infty} \frac{1}{n^p}$ $\begin{cases} \text{converges, if } p > 1 \\ \text{diverges, if } p \leq 1 \end{cases}$

- **Remainder Estimate from The Integral Test**

  \(\displaystyle ∫^∞_{N+1}f(x)\,dx

Glossary

Divergence Test

If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Integral Test

For a series $\sum_{n=1}^{\infty} a_n$ with positive terms a_n, if there exists a continuous, decreasing function f such that $f(n) = a_n$ for all positive integers n, then

$$\sum_{n=1}^{\infty} a_n \tag{1.3.26}$$

and

$$\int_1^{\infty} f(x)\,dx \tag{1.3.27}$$

either both converge or both diverge.

***p*-series**

A series of the form $\sum_{n=1}^{\infty} 1/n^p$.

Remainder Estimate

For a series $\sum_{n=}^{\infty} 1a_n$ with positive terms a_n and a continuous, decreasing function f such that $f(n) = a_n$ for all positive integers n, the remainder $R_N = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n$ satisfies the following estimate:

\[∫^∞_{N+1}f(x)\,dx

Answers

Exercise 1.3.1

Look at $\lim_{n\to\infty} \cos(1/n^2)$.

The series diverges.

Exercise 1.3.2

Compare to the integral $\int_1^{\infty} \frac{x}{3x^2+1}\,dx$.

The series diverges.

Exercise 1.3.3

$p = 5/4$

Exercise 1.3.3

The series converges.

Exercise 1.3.4

Use the remainder estimate $R_N < \int_N^{\infty} \frac{1}{x^4}\,dx$.

$S_5 \approx 1.080351, R_5 < 0.00267$

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1.3E: Exercises for Section 1.3

Exercises 1 - 14 - Divergence Test Problems

Consider the sequence for each series if the divergence test applies, either state that $\lim_{n\to\infty} a_n$ does not exist or find $\lim_{n\to\infty} a_n$. If the divergence test does not apply, state why.

1. $\sum_{n=1}^{\infty} \frac{n}{n+2}$
2. $\sum_{n=1}^{\infty} \frac{n}{5n^2-3}$

Answer

$\lim_{n\to\infty} a_n = 0$. The Divergence Test does not apply.

3. $\sum_{n=1}^{\infty} \frac{n}{\sqrt{3n^2+2n+1}}$
4. $\sum_{n=1}^{\infty} \frac{(2n+1)(n-1)}{(n+1)^2}$

Answer

$\lim_{n\to\infty} a_n = 2$. So the series diverges by the nth-Term Test for Divergence.

5. $\sum_{n=1}^{\infty} \frac{(2n+1)^{2n}}{(3n^2+1)^n}$
6. $\sum_{n=1}^{\infty} \frac{2^n}{3^{n/2}}$

Answer

$\lim_{n\to\infty} a_n = \infty$ (does not exist). So the series diverges by the nth-Term Test for Divergence.

7. $\sum_{n=1}^{\infty} \frac{2^n+3^n}{10^{n/2}}$
8. $\sum_{n=1}^{\infty} e^{-2/n}$

Answer

$\lim_{n\to\infty} a_n = 1$. So the series diverges by the nth-Term Test for Divergence.

9. $\sum_{n=1}^{\infty} \cos n$
10. $\sum_{n=1}^{\infty} \tan n$

Answer

$\lim_{n\to\infty} a_n$ does not exist. So the series diverges by the nth-Term Test for Divergence.

11. $\sum_{n=1}^{\infty} \frac{1-\cos^2(1/n)}{\sin^2(2/n)}$
12. $\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^{2n}$

Answer

$\lim_{n\to\infty} a_n = 1/e^2$. So the series diverges by the nth-Term Test for Divergence.

13. $\sum_{n=1}^{\infty} \frac{\ln n}{n}$
14. $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{\sqrt{n}}$

Answer

$\lim_{n\to\infty} a_n = 0$. The Divergence Test does not apply.

Exercises 15 - 20 - p-Series Problems & Integral Test Problems

State whether the given p-series converges.

15. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$
16. $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$

Answer

The series converges, since $p = 3/2 > 1$.

17. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2}}$
18. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^4}}$

Answer

The series converges, since $p = 4/3 > 1$.

19. $\sum_{n=1}^{\infty} \frac{n^e}{n^\pi}$
20. $\sum_{n=1}^{\infty} \frac{n^\pi}{n^{2e}}$

Answer

The series converges, since $p = 2e - \pi > 1$.

Exercises 21 - 27

Use the integral test to determine whether the following sums converge.

21. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+5}}$
22. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n+5}}$

Answer

The series diverges by the Integral Test since $\int_1^{\infty} \frac{dx}{(x+5)^{1/3}}$ can be shown to diverge.

23. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

24. $\sum_{n=1}^{\infty} \frac{n}{1+n^2}$

Answer

The series diverges by the Integral Test since $\int_1^{\infty} \frac{x}{1+x^2}\ dx$ can be shown to diverge.

25. $\sum_{n=1}^{\infty} \frac{e^n}{1+e^{2n}}$

26. $\sum_{n=1}^{\infty} \frac{2n}{1+n^4}$

Answer

The series converges by the Integral Test since $\int_1^{\infty} \frac{2x}{1+x^4}\ dx$ converges.

27. $\sum_{n=2}^{\infty} \frac{1}{n \ln^2 n}$

Exercises 28 - 31

Express the sums as p-series and determine whether each converges.

28. $\sum_{n=1}^{\infty} 2^{-\ln n}$ Hint: $2^{-\ln n} = \frac{1}{n^{\ln 2}}.$)

Answer

$2^{-\ln n} = 1/n^{\ln 2}$. Since $p = \ln 2 < 1$, this series diverges by the p-series test.

29. $\sum_{n=1}^{\infty} 3^{-\ln n}$ Hint: $3^{-\ln n} = \frac{1}{n^{\ln 3}}.$)

30. $\sum_{n=1}^{n} 2^{-2\ln n}$

Answer

$2^{-2\ln n} = 1/n^{2\ln 2}$. Since $p = 2\ln 2 - 1 < 1$, this series diverges by the p-series test.

31. $\sum_{n=1}^{\infty} n3^{-2\ln n}$

Exercises 32 - 35

Use the estimate $R_N \le \int_N^{\infty} f(t)\ dt$ to find a bound for the remainder $R_N = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n$ where $a_n = f(n)$.

32. $\sum_{n=1}^{1000} \frac{1}{n^2}$

Answer

$$R_{1000} \le \int_{1000}^{\infty} \frac{dt}{t^2} = \lim_{b\to\infty} -\frac{1}{t}\Big|_{1000}^{b} = \lim_{b\to\infty} \left(-\frac{1}{b} + \frac{1}{1000}\right) = 0.001$$

33. $\sum_{n=1}^{1000} \frac{1}{n^3}$

34. $\sum_{n=1}^{1000} \frac{1}{1+n^2}$

Answer

$$R_{1000} \le \int_{1000}^{\infty} \frac{dt}{1+t^2} = \lim_{b\to\infty}\left(\tan^{-1} b - \tan^{-1}(1000)\right) = \pi/2 - \tan^{-1}(1000) \approx 0.000999$$

35. $\displaystyle\sum_{n=1}^{100} \frac{n}{2^n}$

[T] Exercises 36 - 40

Find the minimum value of N such that the remainder estimate \(\displaystyle ∫^∞_{N+1}f(x)\,dx guarantees that $\displaystyle\sum_{n=1}^{N} a_n$ **estimates** $\displaystyle\sum_{n=1}^{\infty} a_n$**, accurate to within the given error.**

36. $a_n = \dfrac{1}{n^2}$, error $< 10^{-4}$

Answer

$$R_N < \int_N^{\infty} \frac{dx}{x^2} = 1/N, \text{ for } N > 10^4$$

37. $a_n = \dfrac{1}{n^{1.1}}$, error $< 10^{-4}$

38. $a_n = \dfrac{1}{n^{1.01}}$, error $< 10^{-4}$

Answer

$$R_N < \int_N^{\infty} \frac{dx}{x^{1.01}} = 100N^{-0.01}, \text{ for } N > 10^{600}$$

39. $a_n = \dfrac{1}{n\ln^2 n}$, error $< 10^{-3}$

40. $a_n = \dfrac{1}{1+n^2}$, error $< 10^{-3}$

Answer

$$R_N < \int_N^{\infty} \frac{dx}{1+x^2} = \pi/2 - \tan^{-1}(N), \text{ for } N > \tan(\pi/2 - 10^{-3}) \approx 1000$$

Exercises 41 - 45

Find a value of N such that R_N is smaller than the desired error. Compute the corresponding sum $\displaystyle\sum_{n=1}^{N} a_n$ and compare it to the given estimate of the infinite series.

41. $a_n = \dfrac{1}{n^{11}}$, error $< 10^{-4}, \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{11}} = 1.000494\ldots$

42. $a_n = \dfrac{1}{e^n}$, error $< 10^{-5}, \displaystyle\sum_{n=1}^{\infty} \frac{1}{e^n} = \frac{1}{e-1} = 0.581976\ldots$

Answer

$R_N < \displaystyle\int_N^{\infty} \frac{dx}{e^x} = e^{-N}$, for $N > 5\ln(10)$, okay if $N = 12$; $\displaystyle\sum_{n=1}^{12} e^{-n} = 0.581973\ldots$. Estimate agrees with $1/(e-1)$ to five decimal places.

43. $a_n = \dfrac{1}{e^{n^2}}$, error $< 10^{-5}, \displaystyle\sum_{n=1}^{\infty} \frac{1}{e^{n^2}} = 0.40488139857\ldots$

14. $a_n = \dfrac{1}{n^4}$, error $< 10^{-4}, \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} = 1.08232...$

Answer

$R_N < \displaystyle\int_N^{\infty} \frac{dx}{x^4} = 4/N^3,$ for $N > (4.10^4)^{1/3}$, okay if $N = 35$; $\displaystyle\sum_{n=1}^{35} \frac{1}{n^4} = 1.08231\ldots$. Estimate agrees with the sum to four decimal places.

15. $a_n = \dfrac{1}{n^6}$, error $< 10^{-6}, \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945} = 1.01734306...,$

16. Find the limit as $n \to \infty$ of $\dfrac{1}{n} + \dfrac{1}{n+1} + \cdots + \dfrac{1}{2n}$. Hint: Compare to $\displaystyle\int_n^{2n} \frac{1}{t}\ dt.\Big)$

Answer

$\ln(2)$

17. Find the limit as $n \to \infty$ of $\dfrac{1}{n} + \dfrac{1}{n+1} + \cdots + \dfrac{1}{3n}$

The next few exercises are intended to give a sense of applications in which partial sums of the harmonic series arise.

18. In certain applications of probability, such as the so-called Watterson estimator for predicting mutation rates in population genetics, it is important to have an accurate estimate of the number $H_k = (1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k})$. Recall that $T_k = H_k - \ln k$ is decreasing. Compute $T = \lim\limits_{k\to\infty} T_k$ to four decimal places.

Hint: $\dfrac{1}{k+1} < \displaystyle\int_k^{k+1} \frac{1}{x}\ dx.\Big)$

Answer

$T = 0.5772...$

19. [T] Complete sampling with replacement, sometimes called the **coupon collector's problem**, is phrased as follows: Suppose you have N unique items in a bin. At each step, an item is chosen at random, identified, and put back in the bin. The problem asks what is the expected number of steps $E(N)$ that it takes to draw each unique item at least once. It turns out that $E(N) = N.H_N = N\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N}\right)$. Find $E(N)$ for $N = 10, 20,$ and 50.

20. [T] The simplest way to shuffle cards is to take the top card and insert it at a random place in the deck, called top random insertion, and then repeat. We will consider a deck to be randomly shuffled once enough top random insertions have been made that the card originally at the bottom has reached the top and then been randomly inserted. If the deck has n cards, then the probability that the insertion will be below the card initially at the bottom (call this card B) is $1/n$. Thus the expected number of top random insertions before B is no longer at the bottom is n. Once one card is below B, there are two places below B and the probability that a randomly inserted card will fall below B is $2/n$. The expected number of top random insertions before this happens is $n/2$. The two cards below B are now in random order. Continuing this way, find a formula for the expected number of top random insertions needed to consider the deck to be randomly shuffled.

Answer

The expected number of random insertions to get B to the top is $n + n/2 + n/3 + \cdots + n/(n-1)$. Then one more insertion puts B back in at random. Thus, the expected number of shuffles to randomize the deck is $n(1 + 1/2 + \cdots + 1/n)$.

21. Suppose a scooter can travel 100 km on a full tank of fuel. Assuming that fuel can be transferred from one scooter to another but can only be carried in the tank, present a procedure that will enable one of the scooters to travel $100H_N$ km, where $H_N = 1 + 1/2 + \cdots + 1/N$.

22. Show that for the remainder estimate to apply on $[N, \infty)$ it is sufficient that $f(x)$ be decreasing on $[N, \infty)$, but f need not be decreasing on $[1, \infty)$.

Answer

Set $b_n = a_{n+N}$ and $g(t) = f(t + N)$ such that f is decreasing on $[t, \infty)$.

53. [T] Use the remainder estimate and integration by parts to approximate $\sum_{n=1}^{\infty} \frac{n}{e^n}$ within an error smaller than 0.0001.

54. Does $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converge if p is large enough? If so, for which p?

Answer

The series converges for $p > 1$ by integral test using change of variable.

55. [T] Suppose a computer can sum one million terms per second of the divergent series $\sum_{n=1}^{N} \frac{1}{n}$. Use the integral test to approximate how many seconds it will take to add up enough terms for the partial sum to exceed 100.

56. [T] A fast computer can sum one million terms per second of the divergent series $\sum_{n=2}^{N} \frac{1}{n \ln n}$. Use the integral test to approximate how many seconds it will take to add up enough terms for the partial sum to exceed 100.

Answer

$N = e^{e^{100}} \approx e^{10^{43}}$ terms are needed.

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1.4: Comparison Tests

Learning Objectives

- Use the comparison test to test a series for convergence.
- Use the limit comparison test to determine convergence of a series.

We have seen that the integral test allows us to determine the convergence or divergence of a series by comparing it to a related improper integral. In this section, we show how to use comparison tests to determine the convergence or divergence of a series by comparing it to a series whose convergence or divergence is known. Typically these tests are used to determine convergence of series that are similar to geometric series or p-series.

Comparison Test

In the preceding two sections, we discussed two large classes of series: geometric series and p-series. We know exactly when these series converge and when they diverge. Here we show how to use the convergence or divergence of these series to prove convergence or divergence for other series, using a method called the **comparison test.**

For example, consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}. \tag{1.4.1}$$

This series looks similar to the convergent series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \tag{1.4.2}$$

Since the terms in each of the series are positive, the sequence of partial sums for each series is monotone increasing. Furthermore, since

$$0 < \frac{1}{n^2+1} < \frac{1}{n^2} \tag{1.4.3}$$

for all positive integers n, the kth partial sum S_k of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ satisfies

$$S_k = \sum_{n=1}^{k} \frac{1}{n^2+1} < \sum_{n=1}^{k} \frac{1}{n^2} < \sum_{n=1}^{\infty} \frac{1}{n^2}. \tag{1.4.4}$$

(See Figure $1.4.1a$ and Table $1.4.1$.) Since the series on the right converges, the sequence S_k is bounded above. We conclude that S_k is a monotone increasing sequence that is bounded above. Therefore, by the Monotone Convergence Theorem, S_k converges, and thus

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1} \tag{1.4.5}$$

converges.

Similarly, consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n-1/2}. \tag{1.4.6}$$

This series looks similar to the divergent series

$$\sum_{n=1}^{\infty} \frac{1}{n}. \tag{1.4.7}$$

The sequence of partial sums for each series is monotone increasing and

$$\frac{1}{n-1/2} > \frac{1}{n} > 0 \tag{1.4.8}$$

for every positive integer n. Therefore, the kth partial sum S_k of

$$\sum_{n=1}^{\infty} \frac{1}{n-1/2} \tag{1.4.9}$$

satisfies

$$S_k = \sum_{n=1}^{k} \frac{1}{n-1/2} > \sum_{n=1}^{k} \frac{1}{n}. \tag{1.4.10}$$

(See Figure 1.4.1b and Table 1.4.1). Since the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to infinity, the sequence of partial sums $\sum_{n=1}^{k} \frac{1}{n}$ is unbounded. Consequently, S_k is an unbounded sequence, and therefore diverges. We conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n-1/2} \tag{1.4.11}$$

diverges.

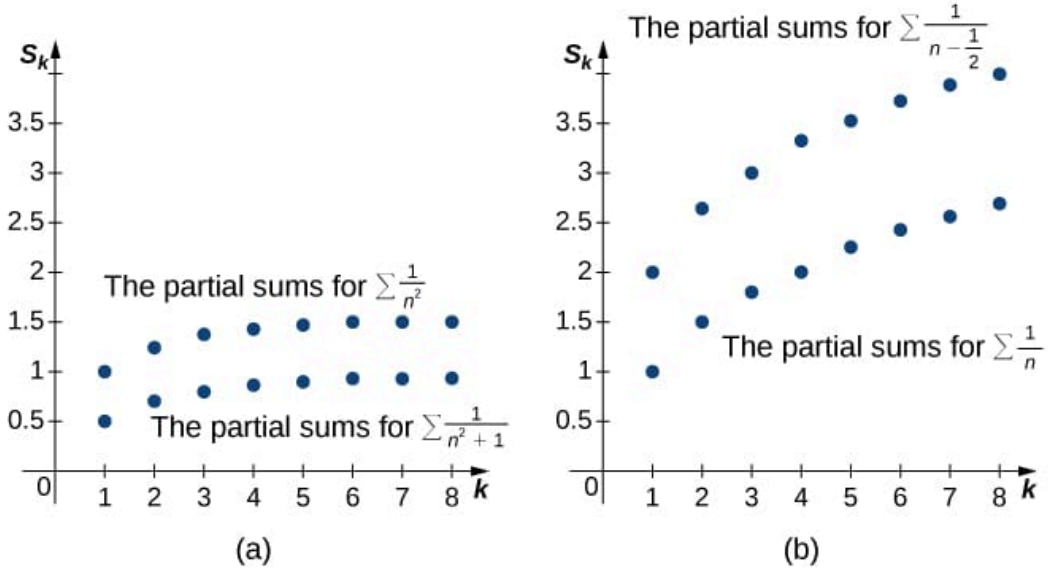


Figure 1.4.1: (a) Each of the partial sums for the given series is less than the corresponding partial sum for the converging $p-series$. (b) Each of the partial sums for the given series is greater than the corresponding partial sum for the diverging harmonic series.

Table 1.4.1: Comparing a series with a p-series ($p = 2$)

k	1	2	3	4	5	6	7	8
$\sum_{n=1}^{k} \frac{1}{n^2+1}$	0.5	0.7	0.8	0.8588	0.8973	0.9243	0.9443	0.9597

$\sum_{n=1}^{k} \frac{1}{n^2}$	1	1.25	1.3611	1.4236	1.4636	1.4914	1.5118	1.5274

Table 1.4.2: Comparing a series with the harmonic series

k	1	2	3	4	5	6	7	8
$\sum_{n=1}^{k} \frac{1}{n-1/2}$	2	2.6667	3.0667	3.3524	3.5746	3.7564	3.9103	4.0436
$\sum_{n=1}^{k} \frac{1}{n}$	1	1.5	1.8333	2.0933	2.2833	2.45	2.5929	2.7179

Theorem: Comparison Test

i. Suppose there exists an integer N such that $0 \le a_n \le b_n$ for all $n \ge N$. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

ii. Suppose there exists an integer N such that $a_n \ge b_n \ge 0$ for all $n \ge N$. If $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

The **Comparison Test** is sometimes also referred to as the **Direct Comparison Test**.

Proof

We prove part i. The proof of part ii. is the contrapositive of part i. Let S_k be the sequence of partial sums associated with $\sum_{n=1}^{\infty} a_n$, and let $L = \sum_{n=1}^{\infty} b_n$. Since the terms $a_n \ge 0$,

$$S_k = a_1 + a_2 + \cdots + a_k \le a_1 + a_2 + \cdots + a_k + a_{k+1} = S_{k+1}.$$

Therefore, the sequence of partial sums is increasing. Further, since $a_n \le b_n$ for all $n \ge N$, then

$$\sum_{n=N}^{k} a_n \le \sum_{n=N}^{k} b_n \le \sum_{n=1}^{\infty} b_n = L.$$

Therefore, for all $k \ge 1$,

$$S_k = (a_1 + a_2 + \cdots + a_{N-1}) + \sum_{n=N}^{k} a_n \le (a_1 + a_2 + \cdots + a_{N-1}) + L.$$

Since $a_1 + a_2 + \cdots + a_{N-1}$ is a finite number, we conclude that the sequence S_k is bounded above. Therefore, S_k is an increasing sequence that is bounded above. By the Monotone Convergence Theorem, we conclude that S_k converges, and therefore the series $\sum_{n=1}^{\infty} a_n$ converges.

□

To use the comparison test to determine the convergence or divergence of a series $\sum_{n=1}^{\infty} a_n$, it is necessary to find a suitable series with which to compare it. Since we know the convergence properties of geometric series and p-series, these series are often used. If there exists an integer N such that for all $n \ge N$, each term an is less than each corresponding term of a known convergent series, then $\sum_{n=1}^{\infty} a_n$ converges. Similarly, if there exists an integer N such that for all $n \ge N$, each term an is greater than each

corresponding term of a known divergent series, then $\sum_{n=1}^{\infty} a_n$ diverges.

Example 1.4.1: Using the Comparison Test

For each of the series, use the comparison test to determine whether the series converges or diverges.

a. $\sum_{n=1}^{\infty} \frac{1}{n^3+3n+1}$
b. $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$
c. $\sum_{n=2}^{\infty} \frac{1}{\ln n}$

Solution

a. Compare to $\sum_{n=1}^{\infty} \frac{1}{n^3}$. Since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a p-series with $p=3$, it converges. Further,

$$\frac{1}{n^3+3n+1} < \frac{1}{n^3}$$

for every positive integer n. Therefore, we can conclude that $\sum_{n=1}^{\infty} \frac{1}{n^3+3n+1}$ converges.

b. Compare to $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$. Since $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ is a geometric series with $r=\frac{1}{2}$ and $\left|\frac{1}{2}\right| < 1$, it converges. Also,

$$\frac{1}{2^n+1} < \frac{1}{2^n}$$

for every positive integer n. Therefore, we see that $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ converges.

c. Compare to $\sum_{n=2}^{\infty} \frac{1}{n}$. Since

$$\frac{1}{\ln n} > \frac{1}{n}$$

for every integer $n \geq 2$ and $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges, we have that $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges.

Watch the accompanying video for a visual demonstration of this concept.

- **Video Length:** 2 minute 6 seconds.
- **Context:** This video demonstrates the use of Direct Comparison to show series $\sum_{n=0}^{\infty} \frac{1}{1+3^n}$ converge.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

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Exercise 1.4.1

Use the comparison test to determine if the series $\sum_{n=1}^{\infty} \frac{n}{n^3+n+1}$ converges or diverges.

Hint available at the end of this section.

Answer available at the end of this section.

Limit Comparison Test

The comparison test works nicely if we can find a comparable series satisfying the hypothesis of the test. However, sometimes finding an appropriate series can be difficult. Consider the series

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1}. \tag{1.4.12}$$

It is natural to compare this series with the convergent series

$$\sum_{n=2}^{\infty} \frac{1}{n^2}. \tag{1.4.13}$$

However, this series does not satisfy the hypothesis necessary to use the comparison test because

$$\frac{1}{n^2-1} > \frac{1}{n^2} \tag{1.4.14}$$

for all integers $n \geq 2$. Although we could look for a different series with which to compare $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$, instead we show how we can use the **limit comparison test** to compare

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1} \tag{1.4.15}$$

and

$$\sum_{n=2}^{\infty} \frac{1}{n^2}. \tag{1.4.16}$$

Let us examine the idea behind the limit comparison test. Consider two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$. with positive terms a_n and b_n and evaluate

$$\lim_{n\to\infty} \frac{a_n}{b_n}. \tag{1.4.17}$$

If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L \neq 0, \tag{1.4.18}$$

then, for n sufficiently large, $a_n \approx Lb_n$. Therefore, either both series converge or both series diverge. For the series $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ and $\sum_{n=2}^{\infty} \frac{1}{n^2}$, we see that

$$\lim_{n\to\infty}\frac{1/(n^2-1)}{1/n^2}=\lim_{n\to\infty}\frac{n^2}{n^2-1}=1. \tag{1.4.19}$$

Since $\sum_{n=2}^{\infty}\frac{1}{n^2}$ converges, we conclude that

$$\sum_{n=2}^{\infty}\frac{1}{n^2-1} \tag{1.4.20}$$

converges.

The limit comparison test can be used in two other cases. Suppose

$$\lim_{n\to\infty}\frac{a_n}{b_n}=0. \tag{1.4.21}$$

In this case, a_n/b_n is a bounded sequence. As a result, there exists a constant M such that $a_n\le Mb_n$. Therefore, if $\sum_{n=1}^{\infty}b_n$ converges, then $\sum_{n=1}^{\infty}a_n$ converges. On the other hand, suppose

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\infty. \tag{1.4.22}$$

In this case, a_n/b_n is an unbounded sequence. Therefore, for every constant M there exists an integer N such that $a_n\ge Mb_n$ for all $n\ge N$. Therefore, if $\sum_{n=1}^{\infty}b_n$ diverges, then $\sum_{n=1}^{\infty}a_n$ diverges as well.

Theorem: Limit Comparison Test

Let $a_n,b_n\ge 0$ for all $n\ge 1$.

i. If $\lim_{n\to\infty}\frac{a_n}{b_n}=L\ne 0$, then $\sum_{n=1}^{\infty}a_n$ and $\sum_{n=1}^{\infty}b_n$ both converge or both diverge.
ii. If $\lim_{n\to\infty}\frac{a_n}{b_n}=0$ and $\sum_{n=1}^{\infty}b_n$ converges, then $\sum_{n=1}^{\infty}a_n$ converges.
iii. If $\lim_{n\to\infty}\frac{a_n}{b_n}=\infty$ and $\sum_{n=1}^{\infty}b_n$ diverges, then $\sum_{n=1}^{\infty}a_n$ diverges.

Note that if $\frac{a_n}{b_n}\to 0$ and $\sum_{n=1}^{\infty}b_n$ diverges, the limit comparison test gives no information. Similarly, if $\frac{a_n}{b_n}\to\infty$ and $\sum_{n=1}^{\infty}b_n$ converges, the test also provides no information. For example, consider the two series $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ and $\sum_{n=1}^{\infty}\frac{1}{n^2}$. These series are both p-series with $p=\frac{1}{2}$ and $p=2$, respectively. Since $p=\frac{1}{2}<1$, the series $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges. On the other hand, since $p=2>1$, the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges. However, suppose we attempted to apply the limit comparison test, using the convergent **p–series** $\sum_{n=1}^{\infty}\frac{1}{n^3}$ as our comparison series. First, we see that

$$\frac{1/\sqrt{n}}{1/n^3}=\frac{n^3}{\sqrt{n}}=n^{5/2}\to\infty\ \text{ as }\ n\to\infty. \tag{1.4.23}$$

Similarly, we see that

$$\frac{1/n^2}{1/n^3}=n\to\infty\ \text{ as }\ n\to\infty. \tag{1.4.24}$$

Therefore, if $\frac{a_n}{b_n} \to \infty$ when $\sum_{n=1}^{\infty} b_n$ converges, we do not gain any information on the convergence or divergence of $\sum_{n=1}^{\infty} a_n$.

Example 1.4.2: Using the Limit Comparison Test

For each of the series, use the limit comparison test to determine whether the series converges or diverges. If the test does not apply, say so.

a. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1}$
b. $\sum_{n=1}^{\infty} \frac{2^n+1}{3^n}$
c. $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$

Solution

a. Compare this series to $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$. Calculate

$$\lim_{n\to\infty} \frac{1/(\sqrt{n}+1)}{1/\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}+1} = \lim_{n\to\infty} \frac{1/\sqrt{n}}{1+1/\sqrt{n}} = 1.$$

By the limit comparison test, since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges, then $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1}$ diverges.

b. Compare this series to $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$. We see that

$$\lim_{n\to\infty} \frac{(2^n+1)/3^n}{2^n/3^n} = \lim_{n\to\infty} \frac{2^n+1}{3^n} \cdot \frac{3^n}{2^n} = \lim_{n\to\infty} \frac{2^n+1}{2^n} = \lim_{n\to\infty} \left[1+\left(\tfrac{1}{2}\right)^n\right] = 1.$$

Therefore,

$$\lim_{n\to\infty} \frac{(2^n+1)/3^n}{2^n/3^n} = 1.$$

Since $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ converges, we conclude that $\sum_{n=1}^{\infty} \frac{2^n+1}{3^n}$ converges.

Watch the accompanying video for a visual demonstration of this concept.

- **Video Length:** 2 minute 6 seconds.
- **Context:** This video demonstrates the use of Direct Comparison to show series $\sum_{n=0}^{\infty} \frac{3^n+2}{4^n+1}$converge.

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c. Since \(\ln n

$$\lim_{n\to\infty}\frac{\ln n/n^2}{1/n}=\lim_{n\to\infty}\frac{\ln n}{n^2}\cdot\frac{n}{1}=\lim_{n\to\infty}\frac{\ln n}{n}.$$

In order to evaluate $\lim_{n\to\infty}\ln n/n$, evaluate the limit as $x\to\infty$ of the real-valued function $\ln(x)/x$. These two limits are equal, and making this change allows us to use L'Hôpital's rule. We obtain

$$\lim_{x\to\infty}\frac{lnx}{x}=\lim_{x\to\infty}\frac{1}{x}=0.$$

Therefore, $\lim_{n\to\infty}\frac{\ln n}{n}=0$, and, consequently,

$$\lim_{n\to\infty}\frac{(\ln n)/n^2}{1/n}=0.$$

Since the limit is 0 but $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, the limit comparison test does not provide any information.

Compare with $\sum_{n=1}^{\infty}\frac{1}{n^2}$ instead. In this case,

$$\lim_{n\to\infty}\frac{(\ln n)/n^2}{1/n^2}=\lim_{n\to\infty}\frac{\ln n}{n^2}\cdot\frac{n^2}{1}=\lim_{n\to\infty}\ln n=\infty.$$

Since the limit is ∞ but $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, the test still does not provide any information.

So now we try a series between the two we already tried. Choosing the series $\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$, we see that

$$\lim_{n\to\infty}\frac{(\ln n)/n^2}{1/n^{3/2}}=\lim_{n\to\infty}\frac{\ln n}{n^2}\cdot\frac{n^{3/2}}{1}=\lim_{n\to\infty}\frac{\ln n}{\sqrt{n}}.$$

As above, in order to evaluate $\lim_{n\to\infty}\frac{\ln n}{\sqrt{n}}$, evaluate the limit as $x\to\infty$ of the real-valued function $\frac{\ln n}{\sqrt{n}}$. Using L'Hôpital's rule,

$$\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}=\lim_{x\to\infty}\frac{2\sqrt{x}}{x}=\lim_{x\to\infty}\frac{2}{\sqrt{x}}=0.$$

Since the limit is 0 and $\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ converges, we can conclude that $\sum_{n=1}^{\infty}\frac{\ln n}{n^2}$ converges.

Exercise 1.4.2

Use the limit comparison test to determine whether the series $\sum_{n=1}^{\infty}\frac{5^n}{3^n+2}$ converges or diverges.

Hint available at the end of this section.

Answer available at the end of this section.

Key Concepts

- The comparison tests are used to determine convergence or divergence of series with positive terms.
- When using the comparison tests, a series $\sum_{n=1}^{\infty}a_n$ is often compared to a geometric or p-series.

Glossary

Comparison Test

If $0 \leq a_n \leq b_n$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges; if $a_n \geq b_n \geq 0$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Limit Comparison Test

Suppose $a_n, b_n \geq 0$ for all $n \geq 1$. If $\lim_{n\to\infty} a_n/b_n \to L \neq 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge; if $\lim_{n\to\infty} a_n/b_n \to 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. If $\lim_{n\to\infty} a_n/b_n \to \infty$, and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Answers

Exercise 1.4.1

Find a value p such that $\frac{n}{n^3+n+1} \leq \frac{1}{n^p}$.

Exercise 1.4.1

The series converges

Exercise 1.4.2

Compare with a geometric series.

Exercise 1.4.2

The series diverges.

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1.4E: Exercises for Section 1.4

Exercises 1 - 13

Use the Comparison Test to determine whether each series in converges or diverges.

1. $\sum_{n=1}^{\infty} a_n$ where $a_n = \dfrac{2}{n(n+1)}$
2. $\sum_{n=1}^{\infty} a_n$ where $a_n = \dfrac{1}{n(n+1/2)}$

Answer

Converges by comparison with $\dfrac{1}{n^2}$.

3. $\sum_{n=1}^{\infty} \dfrac{1}{2(n+1)}$
4. $\sum_{n=1}^{\infty} \dfrac{1}{2n-1}$

Answer

Since $2n-1 < 2n$, we have $\dfrac{1}{2n} < \dfrac{1}{2n-1} \rightarrow \sum_{n=1}^{\infty} \dfrac{1}{2n} < \sum_{n=1}^{\infty} \dfrac{1}{2n-1}$.

But the series on the left is just half of the harmonic series, since $\sum_{n=1}^{\infty} \dfrac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \dfrac{1}{n}$. This clearly diverges, so the given series also diverges, by the Comparison Test.

5. $\sum_{n=2}^{\infty} \dfrac{1}{(n \ln n)^2}$
6. $\sum_{n=1}^{\infty} \dfrac{n!}{(n+2)!}$

Answer

$a_n = 1/(n+1)(n+2) < 1/n^2$. Converges by comparison with p-series, $p = 2 > 1$.

7. $\sum_{n=1}^{\infty} \dfrac{1}{n!}$
8. $\sum_{n=1}^{\infty} \dfrac{\sin(1/n)}{n}$

Answer

$\sin(1/n) \leq 1/n$, so converges by comparison with p-series, $p = 2 > 1$.

9. $\sum_{n=1}^{\infty} \dfrac{\sin^2 n}{n^2}$
10. $\sum_{n=1}^{\infty} \dfrac{\sin(1/n)}{\sqrt{n}}$

Answer

$\sin(1/n) \leq 1$, so converges by comparison with p-series, $p = 3/2 > 1$.

11. $\sum_{n=1}^{\infty} \dfrac{n^{1.2} - 1}{n^{2.3} + 1}$

12. $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n}$

Answer

Since $\sqrt{n+1}-\sqrt{n} = 1/(\sqrt{n+1}+\sqrt{n}) \leq 2/\sqrt{n}$, series converges by comparison with p-series for $p = 1.5 > 1$.

13. $\sum_{n=1}^{\infty} \frac{\sqrt[4]{n}}{\sqrt[3]{n^4+n^2}}$

Exercises 14 - 28

Use the Limit Comparison Test to determine whether each series in converges or diverges.

14. $\sum_{n=1}^{\infty} \left(\frac{\ln n}{n}\right)^2$

Answer

Converges by limit comparison with p-series for any value of p with \(1

15. $\sum_{n=1}^{\infty} \left(\frac{\ln n}{n^{0.6}}\right)^2$

16. $\sum_{n=1}^{\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{n}$

Answer

Converges by limit comparison with p-series, $p = 2 > 1.$

17. $\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n^2}\right)$

18. $\sum_{n=1}^{\infty} \frac{1}{4^n-3^n}$

Answer

Converges by limit comparison with 4^{-n}.

19. $\sum_{n=1}^{\infty} \frac{1}{n^2-n\sin n}$

20. $\sum_{n=1}^{\infty} \frac{1}{e^{(1.1)n}-3^n}$

Answer

Converges by limit comparison with $1/e^{1.1n}$.

21. $\sum_{n=1}^{\infty} \frac{1}{e^{(1.01)n}-3^n}$

22. $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$

Answer

Diverges by limit comparison with harmonic series.

23. $\sum_{n=1}^{\infty} \frac{1}{2^{1+1/n}} n^{1+1/n}$

24. $\sum_{n=1}^{\infty} \left(\frac{1}{n}-\sin\left(\frac{1}{n}\right)\right)$

Answer

Converges by limit comparison with p-series, $p = 3 > 1$.

25. $\sum_{n=1}^{\infty} \left(1 - \cos\left(\frac{1}{n}\right)\right)$

26. $\sum_{n=1}^{\infty} \frac{1}{n}\left(\tan^{-1} n - \frac{\pi}{2}\right)$

Answer

Converges by limit comparison with p-series, $p = 3 > 1$.

27. $\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^{n.n}$ (Hint: $\left(1 - \frac{1}{n}\right)^{n} \to 1/e$.)

28. $\sum_{n=1}^{\infty} \left(1 - e^{-1/n}\right)$ (Hint: $1/e \approx (1 - 1/n)^n$, so $1 - e^{-1/n} \approx 1/n$.)

Answer

Diverges by limit comparison with $1/n$.

29. Does $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^p}$ converge if p is large enough? If so, for which p?

30. Does $\sum_{n=1}^{\infty} \left(\frac{\ln n}{n}\right)^p$ converge if p is large enough? If so, for which p?

Answer

Converges for $p > 1$ by comparison with a p series for slightly smaller p.

31. For which p does the series $\sum_{n=1}^{\infty} 2^{pn}/3^n$ converge?

32. For which $p > 0$ does the series $\sum_{n=1}^{\infty} \frac{n^p}{2^n}$ converge?

Answer

Converges for all $p > 0$.

33. For which $r > 0$ does the series $\sum_{n=1}^{\infty} \frac{r^{n^2}}{2^n}$ converge?

34. For which $r > 0$ does the series $\sum_{n=1}^{\infty} \frac{2^n}{r^{n^2}}$ converge?

Answer

Converges for all $r > 1$. If $r > 1$ then $r^n > 4$, say, once $n > \ln(2)/\ln(r)$ and then the series converges by limit comparison with a geometric series with ratio $1/2$.

35. Find all values of p and q such that $\sum_{n=1}^{\infty} \frac{n^p}{(n!)^q}$ converges.

36. Does $\sum_{n=1}^{\infty} \frac{\sin^2(nr/2)}{n}$ converge or diverge? Explain.

Answer

The numerator is equal to 1 when n is odd and 0 when n is even, so the series can be rewritten $\sum_{n=1}^{\infty} \frac{1}{2n+1}$, which diverges by limit comparison with the harmonic series.

37. Explain why, for each n, at least one of $|\sin n|, |\sin(n+1)|, ..., |\sin(n+6)|$ is larger than $1/2$. Use this relation to test convergence of $\sum_{n=1}^{\infty} \frac{|\sin n|}{\sqrt{n}}$.

38. Suppose that $a_n \geq 0$ and $b_n \geq 0$ and that $\sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} b_n^2$ converge. Prove that $\sum_{n=1}^{\infty} a_n b_n$ converges and $\sum_{n=1}^{\infty} a_n b_n \leq \frac{1}{2}\left(\sum_{n=1}^{\infty} a_n^2 + \sum_{n=1}^{\infty} b_n^2\right)$.

Answer

$(a-b)^2 = a^2 - 2ab + b^2$ or $a^2 + b^2 \geq 2ab$, so convergence follows from comparison of $2a_n b_n$ with $a_n^2 + b_n^2$. Since the partial sums on the left are bounded by those on the right, the inequality holds for the infinite series.

39. Does $\sum_{n=1}^{\infty} 2^{-\ln \ln n}$ converge? (Hint: Write $2^{\ln \ln n}$ as a power of $\ln n$.)

40. Does $\sum_{n=1}^{\infty} (\ln n)^{-\ln n}$ converge? (Hint: Use $t = e^{\ln(t)}$ to compare to a p–series.)

Answer

$(\ln n)^{-\ln n} = e^{-\ln(n)\ln\ln(n)}$. If n is sufficiently large, then $\ln \ln n > 2$, so $(\ln n)^{-\ln n} < 1/n^2$, and the series converges by comparison to a p–series.

41. Does $\sum_{n=2}^{\infty} (\ln n)^{-\ln \ln n}$ converge? (Hint: Compare a_n to $1/n$.)

42. Show that if $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n^2$ converges. If $\sum_{n=1}^{\infty} a_n^2$ converges, does $\sum_{n=1}^{\infty} a_n$ necessarily converge?

Answer

$a_n \to 0$, so $a_n^2 \leq |a_n|$ for large n. Convergence follows from limit comparison. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, but $\sum_{n=1}^{\infty} \frac{1}{n}$ does not, so the fact that $\sum_{n=1}^{\infty} a_n^2$ converges does not imply that $\sum_{n=1}^{\infty} a_n$ converges.

43. Suppose that $a_n > 0$ for all n and that $\sum_{n=1}^{\infty} a_n$ converges. Suppose that b_n is an arbitrary sequence of zeros and ones. Does $\sum_{n=1}^{\infty} a_n b_n$ necessarily converge?

44. Suppose that $a_n > 0$ for all n and that $\sum_{n=1}^{\infty} a_n$ diverges. Suppose that b_n is an arbitrary sequence of zeros and ones with infinitely many terms equal to one. Does $\sum_{n=1}^{\infty} a_n b_n$ necessarily diverge?

Answer

No. $\sum_{n=1}^{\infty} 1/n$ diverges. Let $b_k = 0$ unless $k = n^2$ for some n. Then $\sum_k b_k/k = \sum 1/k^2$ converges.

45. Complete the details of the following argument: If $\sum_{n=1}^{\infty} \frac{1}{n}$ converges to a finite sum s, then $\frac{1}{2}s = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots$ and $s - \frac{1}{2}s = 1 + \frac{1}{3} + \frac{1}{5} + \cdots$. Why does this lead to a contradiction?

46. Show that if $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n^2$ converges, then $\sum_{n=1}^{\infty} \sin^2(a_n)$ converges.

Answer

$|\sin t| \leq |t|$, so the result follows from the comparison test.

47. Suppose that $a_n/b_n \to 0$ in the comparison test, where $a_n \geq 0$ and $b_n \geq 0$. Prove that if $\sum b_n$ converges, then $\sum a_n$ converges.

48. Let b_n be an infinite sequence of zeros and ones. What is the largest possible value of $x = \sum_{n=1}^{\infty} b_n/2^n$?

Answer

By the comparison test, $x = \sum_{n=1}^{\infty} b_n/2^n \leq \sum_{n=1}^{\infty} 1/2^n = 1$.

49. Let d_n be an infinite sequence of digits, meaning d_n takes values in $\{0, 1, \ldots, 9\}$. What is the largest possible value of $x = \sum_{n=1}^{\infty} d_n/10^n$ that converges?

50. Explain why, if $x > 1/2$, then x cannot be written $x = \sum_{n=2}^{\infty} \frac{b_n}{2^n} \quad (b_n = 0 \text{ or } 1, \ b_1 = 0)$.

Answer

If $b_1 = 0$, then, by comparison, $x \leq \sum_{n=2}^{\infty} 1/2^n = 1/2$.

51. [T] Evelyn has a perfect balancing scale, an unlimited number of 1-kg weights, and one each of $1/2$-kg, $1/4$-kg, $1/8$-kg, and so on weights. She wishes to weigh a meteorite of unspecified origin to arbitrary precision. Assuming the scale is big enough, can she do it? What does this have to do with infinite series?

52. [T] Robert wants to know his body mass to arbitrary precision. He has a big balancing scale that works perfectly, an unlimited collection of 1-kg weights, and nine each of 0.1-kg, 0.01-kg, 0.001-kg, and so on weights. Assuming the scale is big enough, can he do this? What does this have to do with infinite series?

Answer

Yes. Keep adding 1-kg weights until the balance tips to the side with the weights. If it balances perfectly, with Robert standing on the other side, stop. Otherwise, remove one of the 1-kg weights, and add 0.1-kg weights one at a time. If it balances after adding some of these, stop. Otherwise if it tips to the weights, remove the last 0.1-kg weight. Start adding 0.01-kg weights. If it balances, stop. If it tips to the side with the weights, remove the last 0.01-kg weight that was added. Continue in this way for the 0.001-kg weights, and so on. After a finite number of steps, one has a finite series of the form $A + \sum_{n=1}^{N} s_n/10^n$ where A is the number of full kg weights and d_n is the number of $1/10^n$-kg weights that were added. If at some state this series is Robert's exact weight, the process will stop. Otherwise it represents the Nth partial sum of an infinite series that gives Robert's exact weight, and the error of this sum is at most $1/10^N$.

53. The series $\sum_{n=1}^{\infty} \frac{1}{2n}$ is half the harmonic series and hence diverges. It is obtained from the harmonic series by deleting all terms in which n is odd. Let $m > 1$ be fixed. Show, more generally, that deleting all terms $1/n$ where $n = mk$ for some integer k also results in a divergent series.

54. In view of the previous exercise, it may be surprising that a subseries of the harmonic series in which about one in every five terms is deleted might converge. A depleted harmonic series is a series obtained from $\sum_{n=1}^{\infty} \frac{1}{n}$ by removing any term $1/n$ if a given digit, say 9, appears in the decimal expansion of n. Argue that this depleted harmonic series converges by answering the following questions.
    a. How many whole numbers n have d digits?
    b. How many d-digit whole numbers $h(d)$. do not contain 9 as one or more of their digits?
    c. What is the smallest d-digit number $m(d)$?
    d. Explain why the deleted harmonic series is bounded by $\sum_{d=1}^{\infty} \frac{h(d)}{m(d)}$.
    e. Show that $\sum_{d=1}^{\infty} \frac{h(d)}{m(d)}$ converges.

Answer

a. $10^d - 10^{d-1} < 10^d$

b. $h(d) < 9^d$

c. $m(d) = 10^{d-1} + 1$

d. Group the terms in the deleted harmonic series together by number of digits. $h(d)$ bounds the number

of terms, and each term is at most $\frac{1}{m(d)}$.

Then $\sum_{d=1}^{\infty} h(d)/m(d) \leq \sum_{d=1}^{\infty} 9^d/(10)^{d-1} \leq 90$. One can actually use comparison to estimate the value to smaller than 80. The actual value is smaller than 23.

55. Suppose that a sequence of numbers $a_n > 0$ has the property that $a_1 = 1$ and $a_{n+1} = \frac{1}{n+1} S_n$, where $S_n = a_1 + \cdots + a_n$. Can you determine whether $\sum_{n=1}^{\infty} a_n$ converges? (Hint: S_n is monotone.)

56. Suppose that a sequence of numbers $a_n > 0$ has the property that $a_1 = 1$ and $a_{n+1} = \frac{1}{(n+1)^2} S_n$, where $S_n = a_1 + \cdots + a_n$. Can you determine whether $\sum_{n=1}^{\infty} a_n$ converges? (Hint: $S_2 = a_2 + a_1 = a_2 + S_1 = a_2 + 1 = 1 + 1/4 = (1 + 1/4)S_1, S_3 = \frac{1}{3^2} S_2 + S_2 = (1 + 1/9)S_2 = (1 + 1/9)(1 + 1/4)S_1$, etc. Look at $\ln(S_n)$, and use $\ln(1 + t) \leq t, t > 0$.)

Answer

Continuing the hint gives $S_N = (1 + 1/N^2)(1 + 1/(N-1)^2 \ldots (1 + 1/4))$. Then $\ln(S_N) = \ln(1 + 1/N^2) + \ln(1 + 1/(N-1)^2) + \cdots + \ln(1 + 1/4)$. Since $\ln(1 + t)$ is bounded by a constant times t, when \(0

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1.5: Alternating Series

Learning Objectives

- Use the alternating series test to test an alternating series for convergence.
- Estimate the sum of an alternating series.
- Explain the meaning of absolute convergence and conditional convergence.

So far in this chapter, we have primarily discussed series with positive terms. In this section we introduce alternating series—those series whose terms alternate in sign. We will show in a later chapter that these series often arise when studying power series. After defining alternating series, we introduce the alternating series test to determine whether such a series converges.

The Alternating Series Test

A series whose terms alternate between positive and negative values is an **alternating series.** For example, the series

$$\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^n = -\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-... \tag{1.5.1}$$

and

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+... \tag{1.5.2}$$

are both alternating series.

Definition: Alternating Series

Any series whose terms alternate between positive and negative values is called an **alternating series**. An alternating series can be written in the form

$$\sum_{n=1}^{\infty}(-1)^{n+1}b_n = b_1-b_2+b_3-b_4+... \tag{1.5.3}$$

or

$$\sum_{n=1}^{\infty}(-1)^{n}b_n = -b_1+b_2-b_3+b_4-... \tag{1.5.4}$$

Where $b_n \geq 0$ for all positive integers n.

Series (1), shown in Equation 1.5.1, is a *geometric series*. Since $|r| = |-1/2| < 1$, the series converges.

Series (2), shown in Equation 1.5.2, is called the *alternating harmonic series*. We will show that whereas the harmonic series diverges, the alternating harmonic series converges.

Theorem

The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots,$$

known as the **alternating harmonic series**, converges.

Proof

Consider the odd terms S_{2k+1} for $k \geq 0$. Since $1/(2k+1) < 1/2k$,

\[S_{2k+1}=S_{2k−1}−\dfrac{1}{2k}+\dfrac{1}{2k+1}

Therefore, $\{S_{2k+1}\}$ is a decreasing sequence. Also,

$$S_{2k+1} = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{2k-1} - \frac{1}{2k}\right) + \frac{1}{2k+1} > 0. \tag{1.5.5}$$

Therefore, $\{S_{2k+1}\}$ is bounded below. Since $\{S_{2k+1}\}$ is a decreasing sequence that is bounded below, by the Monotone Convergence Theorem, $\{S_{2k+1}\}$ converges. Similarly, the even terms $\{S_{2k}\}$ form an increasing sequence that is bounded above because

$$S_{2k} = S_{2k-2} + \frac{1}{2k-1} - \frac{1}{2k} > S_{2k-2} \tag{1.5.6}$$

and

$$S_{2k} = 1 + \left(-\frac{1}{2} + \frac{1}{3}\right) + \ldots + \left(-\frac{1}{2k-2} + \frac{1}{2k-1}\right) - \frac{1}{2k} < 1. \tag{1.5.7}$$

Therefore, by the Monotone Convergence Theorem, the sequence $\{S_{2k}\}$ also converges. Since

$$S_{2k+1} = S_{2k} + \frac{1}{2k+1}, \tag{1.5.8}$$

we know that

$$\lim_{k\to\infty} S_{2k+1} = \lim_{k\to\infty} S_{2k} + \lim_{k\to\infty} \frac{1}{2k+1}. \tag{1.5.9}$$

Letting $S = \lim_{k\to\infty} S_{2k+1}$ and using the fact that $1/(2k+1) \to 0$, we conclude that $\lim_{k\to\infty} S_{2k} = S$. Since the odd terms and the even terms in the sequence of partial sums converge to the same limit S, it can be shown that the sequence of partial sums converges to S, and therefore the alternating harmonic series converges to S.

It can also be shown that $S = \ln 2$, and we can write

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + a \ldots = \ln(2). \tag{1.5.10}$$

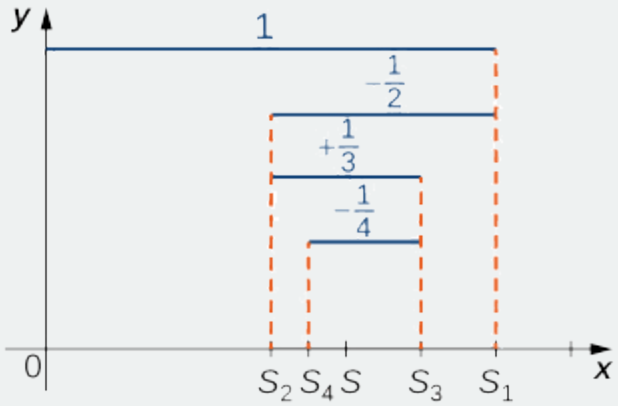


Figure 1.5.1: For the alternating harmonic series, the odd terms S_{2k+1} in the sequence of partial sums are decreasing and bounded below. The even terms S_{2k} are increasing and bounded above.

□

More generally, any alternating series of form (3) (Equation 1.5.3) or (4) (Equation 1.5.4) converges as long as $b_1 \geq b_2 \geq b_3 \geq \cdots$ and $b_n \to 0$. The proof is similar to the proof for the alternating harmonic series.

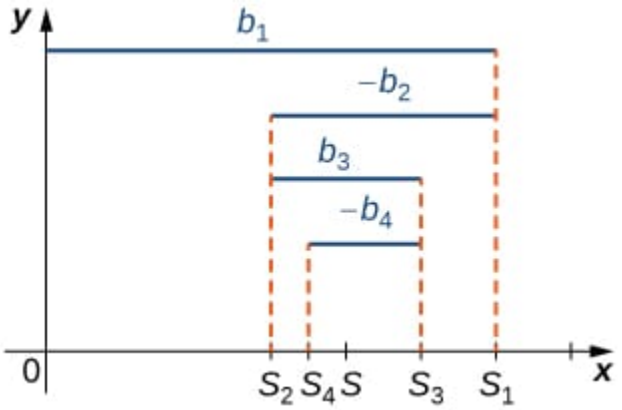


Figure 1.5.2: For an alternating series $b_1 - b_2 + b_3 - \cdots$ in which $b_1 > b_2 > b_3 > \cdots$, the odd terms S_{2k+1} in the sequence of partial sums are decreasing and bounded below. The even terms S_{2k} are increasing and bounded above.

Theorem: Alternating Series Test

An alternating series of the form

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n \tag{1.5.11}$$

or

$$\sum_{n=1}^{\infty} (-1)^{n} b_n \tag{1.5.12}$$

converges if

i. $0 \leq b_{n+1} \leq b_n$ for all $n \geq 1$ and
ii. $\lim_{n\to\infty} b_n = 0$.

This is known as the **alternating series test**.

We remark that this theorem is true more generally as long as there exists some integer N such that $0 \leq b_{n+1} \leq b_n$ for all $n \geq N$.

Example 1.5.1: Convergence of Alternating Series

For each of the alternating series, determine whether the series converges or diverges.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

b. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n+1}$

Solution

a. Since $\dfrac{1}{(n+1)^2} < \dfrac{1}{n^2}$ and $\dfrac{1}{n^2} \to 0$, the series converges.

b. Since $n/(n+1) \nrightarrow 0$ as $n \to \infty$, we cannot apply the alternating series test. Instead, we use the nth term test for divergence. Since $\displaystyle\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$, the series diverges.

Exercise 1.5.1

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{2^n}$ converges or diverges.

Hint available at the end of this section.

Answer available at the end of this section.

Remainder of an Alternating Series

It is difficult to explicitly calculate the sum of most alternating series, so typically the sum is approximated by using a partial sum. When doing so, we are interested in the amount of error in our approximation. Consider an alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n \tag{1.5.13}$$

satisfying the hypotheses of the alternating series test. Let S denote the sum of this series and S_k be the corresponding sequence of partial sums. From Figure 1.5.2, we see that for any integer $N \geq 1$, the remainder R_N satisfies

$$|R_N| = |S - S_N| \leq |S_{N+1} - S_N| = b_{n+1}. \tag{1.5.14}$$

Remainder in Alternating Series

Consider an alternating series of the form

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n \tag{1.5.15}$$

or

$$\sum_{n=1}^{\infty} (-1)^{n} b_n \tag{1.5.16}$$

that satisfies the hypotheses of the alternating series test. Let S denote the sum of the series and S_N denote the Nth partial sum. For any integer $N \geq 1$, the remainder $R_N = S - S_N$ satisfies

$$|R_N| \leq b_{N+1}. \tag{1.5.17}$$

In other words, if the conditions of the alternating series test apply, then the error in approximating the infinite series by the Nth partial sum S_N is in magnitude at most the size of the next term b_{N+1}.

Example 1.5.2: Estimating the Remainder of an Alternating Series

Consider the alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}. \tag{1.5.18}$$

Use the remainder estimate to determine a bound on the error R_{10} if we approximate the sum of the series by the partial sum S_{10}.

Solution

From the theorem stated above,

$$|R_{10}| \leq b_{11} = \frac{1}{11^2} \approx 0.008265. \tag{1.5.19}$$

Watch the accompanying video for a visual demonstration of this concept.

- **Video Length:** 4 minutes 8 seconds.
- **Context:** This video demonstrates how to find the number of terms needed to approximate an alternating series $\sum_{n=1}^{\infty} \frac{4(-1)^{n-1}}{2n-1}$.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Exercise 1.5.2

Find a bound for R_{20} when approximating $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ by S_{20}.

Hint available at the end of this section.

Answer available at the end of this section.

Absolute and Conditional Convergence

Consider a series $\sum_{n=1}^{\infty} a_n$ and the related series $\sum_{n=1}^{\infty} |a_n|$. Here we discuss possibilities for the relationship between the convergence

of these two series. For example, consider the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. The series whose terms are the absolute value of these terms is the harmonic series, since $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$. Since the alternating harmonic series converges, but the harmonic series diverges, we say the alternating harmonic series exhibits conditional convergence.

By comparison, consider the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$. The series whose terms are the absolute values of the terms of this series is the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Since both of these series converge, we say the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ exhibits absolute convergence.

Definition: Absolute Convergence

A series $\sum_{n=1}^{\infty} a_n$ exhibits **absolute convergence** if $\sum_{n=1}^{\infty} |a_n|$ converges. A series $\sum_{n=1}^{\infty} a_n$ exhibits **conditional convergence** if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

As shown by the alternating harmonic series, a series $\sum_{n=1}^{\infty} a_n$ may converge, but $\sum_{n=1}^{\infty} |a_n|$ may diverge. In the next theorem, however, we show that if $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Absolute Convergence Implies Convergence

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Proof

Suppose that $\sum_{n=1}^{\infty} |a_n|$ converges. We show this by using the fact that $a_n = |a_n|$ or $a_n = -|a_n|$ and therefore $|a_n| + a_n = 2|a_n|$ or $|a_n| + a_n = 0$. Therefore, $0 \leq |a_n| + a_n \leq 2|a_n|$. Consequently, by the comparison test, since $2\sum_{n=1}^{\infty} |a_n|$ converges, the series

$$\sum_{n=1}^{\infty} (|a_n| + a_n) \tag{1.5.20}$$

converges. By using the algebraic properties for convergent series, we conclude that

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (|a_n| + a_n) - \sum_{n=1}^{\infty} |a_n| \tag{1.5.21}$$

converges.

□

Example 1.5.3: Absolute versus Conditional Convergence

For each of the series, determine whether the series converges absolutely, converges conditionally, or diverges.

a. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3n+1}$

b. $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$

Solution

a. We can see that

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{3n+1} \right| = \sum_{n=1}^{\infty} \frac{1}{3n+1}$$

diverges by using the limit comparison test with the harmonic series. In fact,

$$\lim_{n\to\infty} \frac{1/(3n+1)}{1/n} = \frac{1}{3}.$$

Therefore, the series does not converge absolutely. However, since

$$\frac{1}{3(n+1)+1} < \frac{1}{3n+1} \text{ and } \frac{1}{3n+1} \to 0,$$

the series converges. We can conclude that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3n+1}$ converges conditionally.

b. Noting that $|\cos n| \leq 1$, to determine whether the series converges absolutely, compare

$$\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right|$$

with the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, by the comparison test, $\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right|$ converges, and therefore $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ converges absolutely.

Watch the accompanying video for a visual demonstration of this concept.

- **Video Length:** 2 minute 40 seconds.
- **Context:** This video demonstrates how to determine the divergence of an Alternating Series $\sum_{n=1}^{\infty} \frac{(-1)^n e^n}{n}$

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Exercise 1.5.3

Determine whether the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{2n^3+1}$ converges absolutely, converges conditionally, or diverges.

Hint available at the end of this section.

Answer available at the end of this section.

To see the difference between absolute and conditional convergence, look at what happens when we rearrange the terms of the alternating harmonic series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$. We show that we can rearrange the terms so that the new series diverges. Certainly if we rearrange the terms of a finite sum, the sum does not change. When we work with an infinite sum, however, interesting things can happen.

Begin by adding enough of the positive terms to produce a sum that is larger than some real number $M = 10$ For example, let $M = 10$, and find an integer k such that

$$1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2k-1}>10 \tag{1.5.22}$$

(We can do this because the series $\sum_{n=1}^{\infty}\frac{1}{2n-1}$ diverges to infinity.) Then subtract $1/2$. Then add more positive terms until the sum reaches 100. That is, find another integer $j > k$ such that

$$(1+\frac{1}{3}+\cdots+\frac{1}{2k-1}-\frac{1}{2}+\frac{1}{2k+1}+\ldots+\frac{1}{2j+1}>100. \tag{1.5.23}$$

Then subtract $1/4$. Continuing in this way, we have found a way of rearranging the terms in the alternating harmonic series so that the sequence of partial sums for the rearranged series is unbounded and therefore diverges.

The terms in the alternating harmonic series can also be rearranged so that the new series converges to a different value. In Example, we show how to rearrange the terms to create a new series that converges to $3\ln(2)/2$. We point out that the alternating harmonic series can be rearranged to create a series that converges to any real number r; however, the proof of that fact is beyond the scope of this text.

In general, any series $\sum_{n=1}^{\infty}a_n$ that converges conditionally can be rearranged so that the new series diverges or converges to a different real number. A series that converges absolutely does not have this property. For any series $\sum_{n=1}^{\infty}a_n$ that converges absolutely, the value of $\sum_{n=1}^{\infty}a_n$ is the same for any rearrangement of the terms. This result is known as the **Riemann Rearrangement Theorem**, which is beyond the scope of this book.

Example 1.5.4: Rearranging Series

Use the fact that

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots=\ln 2 \tag{1.5.24}$$

to rearrange the terms in the alternating harmonic series so the sum of the rearranged series is $3\ln(2)/2$.

Solution

Let

$$\sum_{n=1}^{\infty}a_n=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\cdots. \tag{1.5.25}$$

Since $\sum_{n=1}^{\infty}a_n=\ln(2)$, by the algebraic properties of convergent series,

$$\sum_{n=1}^{\infty}\frac{1}{2}a_n=\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\cdots=\frac{1}{2}\sum_{n=1}^{\infty}a_n=\frac{\ln 2}{2}. \tag{1.5.26}$$

Now introduce the series $\sum_{n=1}^{\infty} b_n$ such that for all $n \geq 1, b_{2n-1} = 0$ and $b_{2n} = a_n/2$. Then

$$\sum_{n=1}^{\infty} b_n = 0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} + \cdots = \frac{\ln 2}{2}. \tag{1.5.27}$$

Then using the algebraic limit properties of convergent series, since $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge, the series $\sum_{n=1}^{\infty}(a_n + b_n)$ converges and

$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n = \ln 2 + \frac{\ln 2}{2} = \frac{3\ln 2}{2}. \tag{1.5.28}$$

Now adding the corresponding terms, a_n and b_n, we see that

$$\sum_{n=1}^{\infty}(a_n + b_n) = (1+0) + \left(-\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{3} + 0\right) + \left(-\frac{1}{4} - 14\right) + \left(\frac{1}{5} + 0\right) + \left(-\frac{1}{6} + \frac{1}{6}\right) + \left(\frac{1}{7} + 0\right) + \left(\frac{1}{8} - \frac{1}{8}\right) + \cdots =$$

We notice that the series on the right side of the equal sign is a rearrangement of the alternating harmonic series. Since $\sum_{n=1}^{\infty}(a_n + b_n) = 3\ln(2)/2$, we conclude that

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots = \frac{3\ln(2)}{2}. \tag{1.5.30}$$

Therefore, we have found a rearrangement of the alternating harmonic series having the desired property.

Key Concepts

- For an alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$, if $b_{k+1} \leq b_k$ for all k and $b_k \to 0$ as $k \to \infty$, the alternating series converges.
- If $\sum_{n=1}^{\infty}|a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Key Equations

- **Alternating Series**

  $\sum_{n=1}^{\infty}(-1)^{n+1}b_n = b_1 - b_2 + b_3 - b_4 + \cdots$ or

  $\sum_{n=1}^{\infty}(-1)^{n}b_n = -b_1 + b_2 - b_3 + b_4 - \cdots$

Glossary

Absolute Convergence
: If the series $\sum_{n=1}^{\infty}|a_n|$ converges, the series $\sum_{n=1}^{\infty} a_n$ is said to converge absolutely.

Alternating Series
: A series of the form $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ or $\sum_{n=1}^{\infty}(-1)^{n}b_n$, where $b_n \geq 0$, is called an alternating series.

Alternating Series Test
: For an alternating series of either form, if $b_{n+1} \leq b_n$ for all integers $n \geq 1$ and $b_n \to 0$, then an alternating series

converges.

Conditional Convergence

If the series $\sum_{n=1}^{\infty} a_n$ converges, but the series $\sum_{n=1}^{\infty} |a_n|$ diverges, the series $\sum_{n=1}^{\infty} a_n$ is said to converge conditionally.

Answers

Exercise $1.5.1$

Is $\{\frac{n}{2^n}\}$ decreasing? What is $\lim_{n\to\infty} \frac{n}{2^n}$?

Exercise $1.5.1$

The series converges.

Exercise $1.5.2$

$|R_{20}| \leq b_{21}$

Exercise $1.5.2$

0.04762

Exercise $1.5.3$

Check for absolute convergence first.

Exercise $1.5.3$

The series converges absolutely.

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1.5E: Exercises for Section 1.5

Exercises 1 - 30

State whether each of the following series converges absolutely, conditionally, or not at all.

1. $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{n}{n+3}$
2. $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{\sqrt{n}+1}{\sqrt{n}+3}$

Answer

This series diverges by the divergence test. Terms do not tend to zero.

3. $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{1}{\sqrt{n+3}}$
4. $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{\sqrt{n+3}}{n}$

Answer

Converges conditionally by alternating series test, since $\sqrt{n+3}/n$ is decreasing and its limit is 0. Does not converge absolutely by comparison with p-series, $p=1/2$.

5. $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{1}{n!}$
6. $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{3^n}{n!}$

Answer

Converges absolutely by limit comparison to $3^n/4^n$, for example.

7. $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\dfrac{n-1}{n}\right)^n$
8. $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\dfrac{n+1}{n}\right)^n$

Answer

Diverges by divergence test since $\lim_{n\to\infty}|a_n| = e$ and not 0.

9. $\sum_{n=1}^{\infty}(-1)^{n+1}\sin^2 n$
10. $\sum_{n=1}^{\infty}(-1)^{n+1}\cos^2 n$

Answer

Diverges by the divergence test, since its terms do not tend to zero. The limit of the sequence of its terms does not exist.

11. $\sum_{n=1}^{\infty}(-1)^{n+1}\sin^2(1/n)$
12. $\sum_{n=1}^{\infty}(-1)^{n+1}\cos^2(1/n)$

Answer

$\lim_{n\to\infty} \cos^2(1/n) = 1$. Diverges by divergence test.

13. $\sum_{n=1}^{\infty} (-1)^{n+1} \ln(1/n)$
14. $\sum_{n=1}^{\infty} (-1)^{n+1} \ln(1+\frac{1}{n})$

Answer

Converges by alternating series test.

15. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{1+n^4}$
16. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^e}{1+n^\pi}$

Answer

Converges conditionally by alternating series test. Does not converge absolutely by limit comparison with p -series, $p = \pi - e$.

17. $\sum_{n=1}^{\infty} (-1)^{n+1} 2^{1/n}$
18. $\sum_{n=1}^{\infty} (-1)^{n+1} n^{1/n}$

Answer

Diverges; terms do not tend to zero.

19. $\sum_{n=1}^{\infty} (-1)^n (1 - n^{1/n})$ (Hint: $n^{1/n} \approx 1 + \ln(n)/n$ for large n.)
20. $\sum_{n=1}^{\infty} (-1)^{n+1} n \left(1 - \cos\left(\frac{1}{n}\right)\right)$ (Hint: $\cos(1/n) \approx 1 - 1/n^2$ for large n.)

Answer

Converges by alternating series test. Does not converge absolutely by limit comparison with harmonic series.

21. $\sum_{n=1}^{\infty} (-1)^{n+1} (\sqrt{n+1} - \sqrt{n})$ (Hint: Rationalize the numerator.)
22. $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$ (Hint: Cross-multiply then rationalize numerator.)

Answer

Converges absolutely by limit comparison with p-series, $p = 3/2$, after applying the hint.

23. $\sum_{n=1}^{\infty} (-1)^{n+1} (\ln(n+1) - \ln n)$
24. $\sum_{n=1}^{\infty} (-1)^{n+1} n(\tan^{-1}(n+1) - \tan^{-1} n)$ (Hint: Use Mean Value Theorem.)

Answer

Converges by alternating series test since $n(\tan^{-1}(n+1) - \tan^{-1} n)$ is decreasing to zero for large n.Does not converge absolutely by limit comparison with harmonic series after applying hint.

25. $\sum_{n=1}^{\infty} (-1)^{n+1} ((n+1)^2 - n^2)$
26. $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{1}{n} - \frac{1}{n+1}\right)$

Answer

Converges absolutely, since $a_n = \dfrac{1}{n} - \dfrac{1}{n+1}$ are terms of a telescoping series.

27. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n}$
28. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n^{1/n}}$

Answer

Terms do not tend to zero. Series diverges by divergence test.

29. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}\sin(\frac{n\pi}{2})$
30. $\displaystyle\sum_{n=1}^{\infty} \sin(n\pi/2)\sin(1/n)$

Answer

Converges by alternating series test. Does not converge absolutely by limit comparison with harmonic series.

Exercises 31 - 36

Use the estimate $|R_N| \le b_{N+1}$ to find a value of N that guarantees that the sum of the first N terms of the alternating series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ differs from the infinite sum by at most the given error. Calculate the partial sum S_N for this N.

31. [T] $b_n = 1/n$, error $< 10^{-5}$
32. [T] $b_n = 1/\ln(n), n \ge 2$, error $< 10^{-1}$

Answer

$\ln(N+1) > 10, N+1 > e^{10}, N \ge 22026; S_{22026} = 0.0257\ldots$

33. [T] $b_n = 1/\sqrt{n}$, error $< 10^{-3}$
34. [T] $b_n = 1/2^n$, error $< 10^{-6}$

Answer

$2^{N+1} > 10^6$ or $N+1 > 6\ln(10)/\ln(2) = 19.93.$ or $N \ge 19; S_{19} = 0.333333969\ldots$

35. [T] $b_n = ln(1+\dfrac{1}{n})$, error $< 10^{-3}$
36. [T] $b_n = 1/n^2$, error $< 10^{-6}$

Answer

$(N+1)^2 > 10^6$ or $N > 999; S_{1000} \approx 0.822466.$

Exercises 37 - 45

Indicate whether each of the following statements is true or false. If the statement is false, provide an example in which it is false.

37. If $b_n \ge 0$ is decreasing and $\lim\limits_{n\to\infty} b_n = 0$, then $\displaystyle\sum_{n=1}^{\infty}(b_{2n-1} - b_{2n})$ converges absolutely.
38. If $b_n \ge 0$ is decreasing, then $\displaystyle\sum_{n=1}^{\infty}(b_{2n-1} - b_{2n})$ converges absolutely.

Answer

True. b_n need not tend to zero since if $c_n = b_n - \lim b_n$, then $c_{2n-1} - c_{2n} = b_{2n-1} - b_{2n}$.

39. If $b_n \geq 0$ and $\lim_{n\to\infty} b_n = 0$ then $\sum_{n=1}^{\infty}(\frac{1}{2}(b_{3n-2}+b_{3n-1})-b_{3n})$ converges.

40. If $b_n \geq 0$ is decreasing and $\sum_{n=1}^{\infty}(b_{3n-2}+b_{3n-1}-b_{3n})$ converges then $\sum_{n=1}^{\infty} b_{3n-2}$ converges.

Answer

True. $b_{3n-1}-b_{3n}\geq 0$, so convergence of $\sum b_{3n-2}$ follows from the comparison test.

41. If $b_n \geq 0$ is decreasing and $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ converges conditionally but not absolutely, then b_n does not tend to zero.

42. Let $a_n^+ = a_n$ if $a_n \geq 0$ and $a_n^- = -a_n$ if $a_n < 0$. (Also, $a_n^+ = 0$ if $a_n < 0$ and $a_n^- = 0$ if $a_n \geq 0$.) If $\sum_{n=1}^{\infty} a_n$ converges conditionally but not absolutely, then neither $\sum_{n=1}^{\infty} a_n^+$ nor $\sum_{n=1}^{\infty} a_n^-$ converge.

Answer

True. If one converges, then so must the other, implying absolute convergence.

43. Suppose that a_n is a sequence of positive real numbers and that $\sum_{n=1}^{\infty} a_n$ converges.

44. Suppose that b_n is an arbitrary sequence of ones and minus ones. Does $\sum_{n=1}^{\infty} a_n b_n$ necessarily converge?

45. Suppose that a_n is a sequence such that $\sum_{n=1}^{\infty} a_n b_n$ converges for every possible sequence b_n of zeros and ones. Does $\sum_{n=1}^{\infty} a_n$ converge absolutely?

Answer

Yes. Take $b_n = 1$ if $a_n \geq 0$ and $b_n = 0$ if $a_n < 0$. Then $\sum_{n=1}^{\infty} a_n b_n = \sum_{n:a_n\geq 0} a_n$ converges. Similarly, one can show $\sum_{n:a_n<0} a_n$ converges. Since both series converge, the series must converge absolutely.

Exercises 46 - 49

The series do not satisfy the hypotheses of the alternating series test as stated. In each case, state which hypothesis is not satisfied. State whether the series converges absolutely.

46. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin^2 n}{n}$

47. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos^2 n}{n}$

Answer

Not decreasing. Does not converge absolutely.

48. $1+\frac{1}{2}-\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+\frac{1}{6}-\frac{1}{7}-\frac{1}{8}+\cdots$

49. $1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}+\frac{1}{8}-\frac{1}{9}+\cdots$

Answer

Not alternating. Can be expressed as $\sum_{n=1}^{\infty}\left(\frac{1}{3n-2}+\frac{1}{3n-1}-\frac{1}{3n}\right)$, which diverges by comparison with $\sum_{n=1}^{\infty}\frac{1}{3n-2}$.

50. Show that the alternating series $1-\frac{1}{2}+\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}+\frac{1}{4}-\frac{1}{8}+\cdots$ does not converge. What hypothesis of the alternating

series test is not met?

51. Suppose that $\sum a_n$ converges absolutely. Show that the series consisting of the positive terms a_n also converges.

Answer

Let $a_n^+ = a_n$ if $a_n \geq 0$ and $a_n^+ = 0$ if $a_n < 0$. Then $a_n^+ \leq |a_n|$ for all n so the sequence of partial sums of a_n^+ is increasing and bounded above by the sequence of partial sums of $|a_n|$, which converges; hence, $\sum_{n=1}^{\infty} a_n^+$ converges.

52. Show that the alternating series $\frac{2}{3} - \frac{3}{5} + \frac{4}{7} - \frac{5}{9} + \cdots$ does not converge. What hypothesis of the alternating series test is not met?

53. The formula $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$ will be derived in the next chapter. Use the remainder $|R_N| \leq b_{N+1}$ to find a bound for the error in estimating $\cos\theta$ by the fifth partial sum $1 - \theta^2/2! + \theta^4/4! - \theta^6/6! + \theta^8/8!$ for $\theta = 1, \theta = \pi/6$, and $\theta = \pi$.

Answer

For $N = 5$ one has $\mid R_N \mid b_6 = \theta^{10}/10!$. When $\theta = 1, R_5 \leq 1/10! \approx 2.75 \times 10^{-7}$. When $\theta = \pi/6$, $R_5 \leq (\pi/6)^{10}/10! \approx 4.26 \times 10^{-10}$. When $\theta = \pi, R_5 \leq \pi^{10}/10! = 0.0258$.

54. The formula $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$ will be derived in the next chapter. Use the remainder $|R_N| \leq b_{N+1}$ to find a bound for the error in estimating $\sin\theta$ by the fifth partial sum $\theta - \theta^3/3! + \theta^5/5! - \theta^7/7! + \theta^9/9!$ for $\theta = 1, \theta = \pi/6$, and $\theta = \pi$.

55. How many terms in $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$ are needed to approximate $\cos 1$ accurate to an error of at most 0.00001?

Answer

Let $b_n = 1/(2n-2)!$. Then $R_N \leq 1/(2N)! < 0.00001$ when $(2N)! > 10^5$ or $N = 5$ and $1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \frac{1}{8!} = 0.540325\ldots$, whereas $\cos 1 = 0.5403023\ldots$

56. How many terms in $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$ are needed to approximate $\sin 1$ accurate to an error of at most 0.00001?

57. Sometimes the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ converges to a certain fraction of an absolutely convergent series $\sum_{n=1}^{\infty} b_n$ at a faster rate. Given that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, find $S = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$. Which of the series $6\sum_{n=1}^{\infty} \frac{1}{n^2}$ and $S\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ gives a better estimation of π^2 using 1000 terms?

Answer

Let $T = \sum \frac{1}{n^2}$. Then $T - S = \frac{1}{2}T$, so $S = T/2$. $\sqrt{6 \times \sum_{n=1}^{1000} 1/n^2} = 3.140638\ldots$; $\sqrt{\frac{1}{2} \times \sum_{n=1}^{1000} (-1)^{n-1}/n^2} = 3.141591\ldots$; $\pi = 3.141592\ldots$. The alternating series is more accurate for 1000 terms.

Exercises 58 & 59

The alternating series converge to given multiples of π. Find the value of N predicted by the remainder estimate such that the Nth partial sum of the series accurately approximates the left-hand side to within the given error. Find the minimum N for which the error bound holds, and give the desired approximate value in each case. Up to 15 decimals places, $\pi = 3.141592653589793\ldots$.

58. [T] $\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$, error < 0.0001

59. [T] $\frac{\pi}{\sqrt{12}} = \sum_{k=0}^{\infty} \frac{(-3)^{-k}}{2k+1}$, error < 0.0001

Answer

$N = 6, S_N = 0.9068$

60. [T] The series $\sum_{n=0}^{\infty} \frac{\sin(x+\pi n)}{x+\pi n}$ plays an important role in signal processing. Show that $\sum_{n=0}^{\infty} \frac{\sin(x+\pi n)}{x+\pi n}$ converges whenever \(0

61. [T] If $\sum_{n=1}^{N} (-1)^{n-1}\frac{1}{n} \to ln2$, what is $1+\frac{1}{3}+\frac{1}{5}-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\frac{1}{8}-\frac{1}{10}-\frac{1}{12}+\cdots$?

Answer

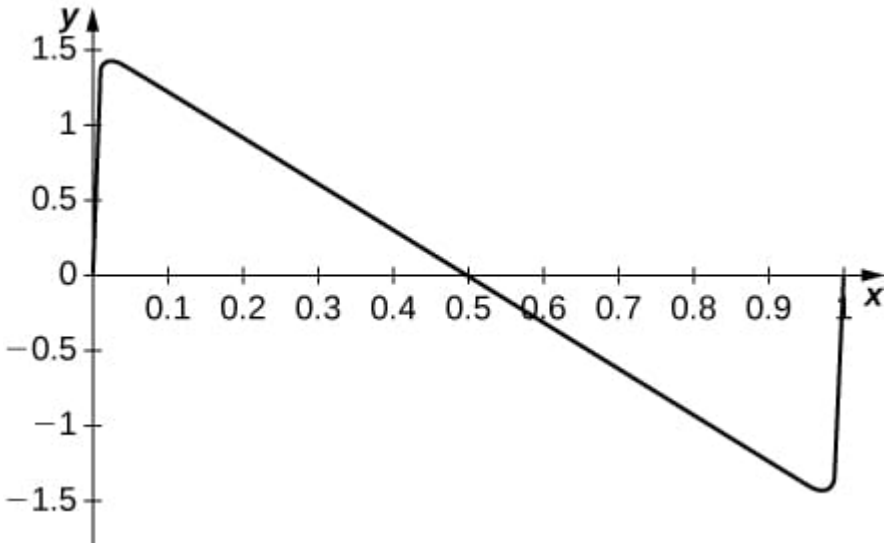
$\ln(2)$. The nth partial sum is the same as that for the alternating harmonic series.

62. [T] Plot the series $\sum_{n=1}^{100} \frac{\cos(2\pi nx)}{n}$ for $0 \le x < 1$. Explain why $\sum_{n=1}^{100} \frac{\cos(2\pi nx)}{n}$ diverges when $x = 0, 1$. How does the series behave for other x?

63. [T] Plot the series $\sum_{n=1}^{100} \frac{\sin(2\pi nx)}{n}$ for $0 \le x < 1$ and comment on its behavior

Answer

The series jumps rapidly near the endpoints. For x away from the endpoints, the graph looks like $\pi(1/2 - x)$.

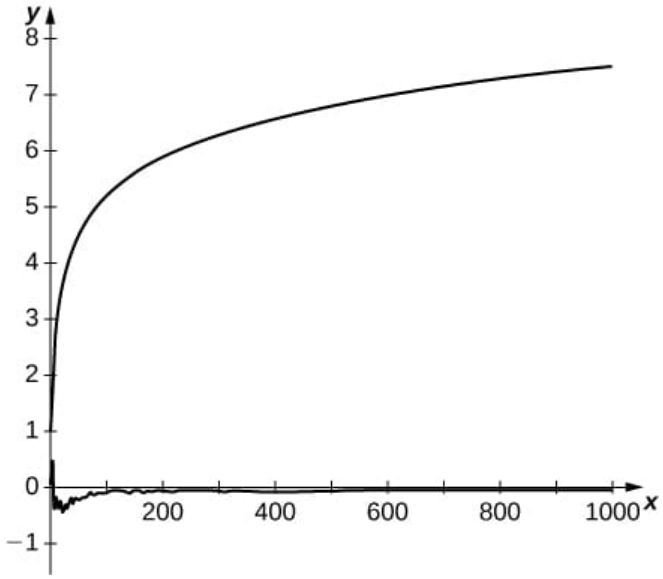


64. [T] Plot the series $\sum_{n=1}^{100} \frac{\cos(2\pi nx)}{n^2}$ for $0 \le x < 1$ and describe its graph.

65. [T] The alternating harmonic series converges because of cancelation among its terms. Its sum is known because the cancelation can be described explicitly. A random harmonic series is one of the form $\sum_{n=1}^{\infty} \frac{S_n}{n}$, where s_n is a randomly generated sequence of $\pm 1's$ in which the values ± 1 are equally likely to occur. Use a random number generator to produce 1000 random $\pm 1's$ and plot the partial sums $S_N = \sum_{n=1}^{N} \frac{s_n}{n}$ of your random harmonic sequence for $N = 1$ to 1000. Compare to a plot of the first 1000 partial sums of the harmonic series.

Answer

Here is a typical result. The top curve consists of partial sums of the harmonic series. The bottom curve plots partial sums of a random harmonic series.



56. [T] Estimates of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ can be accelerated by writing its partial sums as $\sum_{n=1}^{N} \frac{1}{n^2} = \sum_{n=1}^{N} \frac{1}{n(n+1)} + \sum_{n=1}^{N} \frac{1}{n^2(n+1)}$ and recalling that $\sum_{n=1}^{N} \frac{1}{n(n+1)} = 1 - \frac{1}{N+1}$ converges to one as $N \to \infty$. Compare the estimate of $\pi^2/6$ using the sums $\sum_{n=1}^{1000} \frac{1}{n^2}$ with the estimate using $1 + \sum_{n=1}^{1000} \frac{1}{n^2(n+1)}$.

57. [T] The **Euler transform** rewrites $S = \sum_{n=0}^{\infty} (-1)^n b_n$ as $S = \sum_{n=0}^{\infty} (-1)^n 2^{-n-1} \sum_{m=0}^{n} \binom{n}{m} b_{n-m}$. For the alternating harmonic series, it takes the form $\ln(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \sum_{n=1}^{\infty} \frac{1}{n2^n}$. Compute partial sums of $\sum_{n=1}^{\infty} \frac{1}{n2^n}$ until they approximate $\ln(2)$ accurate to within 0.0001. How many terms are needed? Compare this answer to the number of terms of the alternating harmonic series are needed to estimate $\ln(2)$.

Answer

By the alternating series test, $|S_n - S| \leq b_{n+1}$, so one needs 10^4 terms of the alternating harmonic series to estimate $\ln(2)$ to within 0.0001. The first 10 partial sums of the series $\sum_{n=1}^{\infty} \frac{1}{n2^n}$ are (up to four decimals) $0.5000, 0.6250, 0.6667, 0.6823, 0.6885, 0.6911, 0.6923, 0.6928, 0.6930, 0.6931$ and the tenth partial sum is within 0.0001 of $\ln(2) = 0.6931 \ldots$.

58. [T] In the text it was stated that a conditionally convergent series can be rearranged to converge to any number. Here is a slightly simpler, but similar, fact. If $a_n \geq 0$ is such that $a_n \to 0$ as $n \to \infty$ but $\sum_{n=1}^{\infty} a_n$ diverges, then, given any number A there is a sequence s_n of $\pm 1's$ such that $\sum_{n=1}^{\infty} a_n s_n \to A$. Show this for $A > 0$ as follows.

a. Recursively define s_n by $s_n = 1$ if \(\displaystyle S_{n−1}=\sum_{k=1}^{n−1}a_ks_k
b. Explain why eventually $S_n \geq A$, and for any m larger than this n, $A - a_m \leq S_m \leq A + a_m$.
c. Explain why this implies that $S_n \to A$ as $n \to \infty$.

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1.6: Ratio and Root Tests

Learning Objectives

- Use the ratio test to determine absolute convergence of a series.
- Use the root test to determine absolute convergence of a series.
- Describe a strategy for testing the convergence of a given series.

In this section, we prove the last two series convergence tests: the ratio test and the root test. These tests are particularly nice because they do not require us to find a comparable series. The ratio test will be especially useful in the discussion of power series in the next chapter. Throughout this chapter, we have seen that no single convergence test works for all series. Therefore, at the end of this section we discuss a strategy for choosing which convergence test to use for a given series.

Ratio Test

Consider a series $\displaystyle \sum_{n=1}^∞a_n$. From our earlier discussion and examples, we know that $\displaystyle \lim_{n→∞}a_n=0$ is not a sufficient condition for the series to converge. Not only do we need $a_n→0$, but we need $a_n→0$ quickly enough. For example, consider the series $\displaystyle \sum_{n=1}^∞\frac{1}{n}$ and the series $\displaystyle \sum_{n=1}^∞\frac{1}{n^2}$. We know that $\frac{1}{n}→0$ and $\frac{1}{n^2}→0$. However, only the series $\displaystyle \sum_{n=1}^∞ \frac{1}{n^2}$ converges. The series $\displaystyle \sum_{n=1}^∞\frac{1}{n}$ diverges because the terms in the sequence $\left\{\frac{1}{n}\right\}$ do not approach zero fast enough as $n→∞$. Here we introduce the **ratio test,** which provides a way of measuring how fast the terms of a series approach zero.

Theorem: Ratio Test

Let $\displaystyle \sum^∞_{n=1}a_n$ be a series with nonzero terms. Let

$$ρ=\lim_{n→∞} \left|\frac{a_{n+1}}{a_n}\right|.$$

i. If $0≤ρ<1,$ then $\displaystyle \sum^∞_{n=1}a_n$ converges absolutely.
ii. If $ρ>1$ or $ρ=∞$, then $\displaystyle \sum^∞_{n=1}a_n$ diverges.
iii. If $ρ=1,$ the test does not provide any information.

Proof

Let $\displaystyle \sum_{n=1}^∞a_n$ be a series with nonzero terms.

We begin with the proof of part i. In this case, $ρ=\lim_{n→∞}|\frac{a_{n+1}}{a_n}|<1.$ Since $0≤ρ<1$, there exists R such that $0≤ρ0$. By the definition of limit of a sequence, there exists some integer N such that

$$\left|\left|\frac{a_{n+1}}{a_n}\right|−ρ\right|<ε,\;\text{for all}\; n≥N.$$

Therefore,

$$\left|\frac{a_{n+1}}{a_n}\right|<ρ+ε=R, \;\text{for all}\; n≥N$$

and, thus,

$$\begin{align*} |a_{N+1}| & |a_{N+2}|& |a_{N+3}|& |a_{N+4}|& ⋮. \end{align*}$$

Since $R<1,$ the geometric series

$$R|a_N|+R^2|a_N|+R^3|a_N|+\cdots$$

converges. Given the inequalities above, we can apply the comparison test and conclude that the series

$$|a_{N+1}|+|a_{N+2}|+|a_{N+3}|+|a_{N+4}|+\cdots$$

converges. Therefore, since

$$\sum_{n=1}^{\infty}|a_n|=\sum_{n=1}^{N}|a_n|+\sum_{n=N+1}^{\infty}|a_n|$$

where $\sum_{n=1}^{N}|a_n|$ is a finite sum and $\sum_{n=N+1}^{\infty}|a_n|$ converges, we conclude that $\sum_{n=1}^{\infty}|a_n|$ converges.

For part ii.

$$\rho=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|>1.$$

Since $\rho>1,$ there exists R such that $\rho>R>1$. Let $\varepsilon=\rho-R>0$. By the definition of the limit of a sequence, there exists an integer N such that

$$\left|\left|\frac{a_{n+1}}{a_n}\right|-\rho\right|<\varepsilon, \;\text{for all}\; n\geq N.$$

Therefore,

$$R=\rho-\varepsilon<\left|\frac{a_{n+1}}{a_n}\right|, \;\text{for all}\; n\geq N,$$

and, thus,

$$\begin{aligned}
|a_{N+1}| &>R|a_N|\\
|a_{N+2}| &>R|a_{N+1}|>R^2|a_N|\\
|a_{N+3}| &>R|a_{N+2}|>R^2|a_{N+1}|>R^3|a_N|\\
|a_{N+4}| &>R|a_{N+3}|>R^2|a_{N+2}|>R^3|a_{N+1}|>R^4|a_N|.
\end{aligned}$$

Since $R>1,$ the geometric series

$$R|a_N|+R^2|a_N|+R^3|a_N|+\cdots$$

diverges. Applying the comparison test, we conclude that the series

$$|a_{N+1}|+|a_{N+2}|+|a_{N+3}|+\cdots$$

diverges, and therefore the series $\sum_{n=1}^{\infty}|a_n|$ diverges.

For part iii. we show that the test does not provide any information if $\rho=1$ by considering the p–series $\sum^{\infty}_{n=1}\frac{1}{n^p}$. For any real number p,

$$\rho=\lim_{n\to\infty}\frac{1/(n+1)^p}{1/n^p}=\lim_{n\to\infty}\frac{n^p}{(n+1)^p}=1.$$

However, we know that if $p\leq 1,$ the p–series $\sum^{\infty}_{n=1}\frac{1}{n^p}$ diverges, whereas $\sum^{\infty}_{n=1}\frac{1}{n^p}$ converges if $p>1$.

□

The ratio test is particularly useful for series whose terms contain factorials or exponential, where the ratio of terms simplifies the expression. The ratio test is convenient because it does not require us to find a comparative series. The drawback is that the test sometimes does not provide any information regarding convergence.

Example \(\PageIndex{1}\)

Before working on applications of root test, go through the example on how to simplify factorials.

Simplify \(\frac{(3n+1)}{(3n-2)!}\)

Watch the accompanying video for a visual demonstration of this concept.

- **Video Length:** 1 minute 57 seconds.
- **Context:** This video demonstrates an example to show how to simplify a factorial.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Example \(\PageIndex{2}\): Using the Ratio Test

For each of the series, use the ratio test to determine whether the series converges or diverges.

a. \(\displaystyle \sum^∞_{n=1}\frac{2^n}{n!}\)
b. \(\displaystyle \sum^∞_{n=1}\frac{n^n}{n!} \)
c. \(\displaystyle \sum_{n=1}^∞\frac{(−1)^n(n!)^2}{(2n)!}\)

Solution

a. From the ratio test, we can see that

\[ρ=\lim_{n→∞}\frac{2^{n+1}/(n+1)!}{2^n/n!}=\lim_{n→∞}\frac{2^{n+1}}{(n+1)!}⋅\frac{n!}{2^n}.\]

Since \((n+1)!=(n+1)⋅n!,\)

\[ρ=\lim_{n→∞}\frac{2}{n+1}=0.\]

Since \(ρ<1,\) the series converges.

b. We can see that

\[ρ=\lim_{n→∞}\frac{(n+1)^{n+1}/(n+1)!}{n^n/n!}=\lim_{n→∞}\frac{(n+1)^{n+1}}{(n+1)!}⋅\frac{n!}{n^n}=\lim_{n→∞}(\frac{n+1}{n})^n=\lim_{n→∞}(1+\frac{1}{n})^n=e.\]

Since \(ρ>1,\) the series diverges.

c. Since

\[\left|\frac{(−1)^{n+1}((n+1)!)^2/(2(n+1))!}{(−1)^n(n!)^2/(2n)!}\right|=\frac{(n+1)!(n+1)!}{(2n+2)!}⋅\frac{(2n)!}{n!n!}=\frac{(n+1)(n+1)}{(2n+

we see that

\[ρ=\lim_{n→∞}\frac{(n+1)(n+1)}{(2n+2)(2n+1)}=\frac{1}{4}.\]

Since \(ρ<1\), the series converges.

Watch the accompanying video for a visual demonstration of this concept.

- **Video Length:**3 minute 24 seconds.
- **Context:** This video demonstrates how to apply Ratio Test to show a series converges \(\displaystyle \sum^∞_{n=0}\frac{n2^{n}}{3^{n}}\).

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Exercise \(\PageIndex{1}\)

Use the ratio test to determine whether the series \(\displaystyle \sum^∞_{n=1}\frac{n^3}{3^n}\) converges or diverges.

Hint available at the end of this section.

Answer available at the end of this section.

Root Test

The approach of the **root test** is similar to that of the ratio test. Consider a series \(\displaystyle \sum^∞_{n=1}a_n\) such that \(\displaystyle \lim_{n→∞}\sqrt[n]{|a_n|}=ρ\) for some real number \(ρ\). Then for \(N\) sufficiently large, \(|a_N|≈ρN.\) Therefore, we can approximate \(\displaystyle \sum_{n=N}^∞|a_n|\) by writing

\[|a_N|+|a_{N+1}|+|a_{N+2}|+⋯≈ρ^N+ρ^{N+1}+ρ^{N+2}+⋯.\]

The expression on the right-hand side is a geometric series. As in the ratio test, the series \(\displaystyle \sum^∞_{n=1}a_n\) converges absolutely if \(0≤ρ<1\) and the series diverges if \(ρ≥1\). If \(ρ=1\), the test does not provide any information. For example, for any \(p\)-series, \(\displaystyle \sum_{n=1}^∞\frac{1}{n^p}\), we see that

\[ρ=\lim_{n→∞}\sqrt[n]{|\frac{1}{n^p}|}=\lim_{n→∞}\frac{1}{n^{p/n}}.\]

To evaluate this limit, we use the natural logarithm function. Doing so, we see that

\(\ln ρ=\ln(\lim_{n→∞}\frac{1}{n^{p/n}})=\lim_{n→∞}\ln(\frac{1}{n})^{p/n}=\lim_{n→∞}\frac{p}{n}·\ln(\frac{1}{n})=\lim_{n→∞}\frac{p\ln(1/n)}{n}.\)

Using L'Hôpital's rule, it follows that \(\ln ρ=0\), and therefore \(ρ=1\) for all \(p\). However, we know that the \(p\)-series only converges if \(p>1\) and diverges if \(p<1\).

Theorem: Root Test

Consider the series \(\displaystyle \sum^∞_{n=1}a_n\). Let

\[ρ=\lim_{n→∞}\sqrt[n]{|a_n|}\].

i. If \(0≤ρ<1,\) then \(\displaystyle \sum^∞_{n=1}a_n\) converges absolutely.
ii. If \(ρ>1\) or \(ρ=∞\), then \(\displaystyle \sum^∞_{n=1}a_n\) diverges.
iii. If \(ρ=1\), the test does not provide any information.

The root test is useful for series whose terms involve exponentials. In particular, for a series whose terms \(a_n\) satisfy \(|a_n|=(b_n)^n\), then \(\sqrt[n]{|a_n|}=b_n\) and we need only evaluate \(\displaystyle \lim_{n→∞}b_n\).

Example \(\PageIndex{2}\): Using the Root Test

For each of the series, use the root test to determine whether the series converges or diverges.

a. \(\displaystyle \sum^∞_{n=1}\frac{(n^2+3n)^n}{(4n^2+5)^n}\)
b. \(\displaystyle \sum^∞_{n=1}\frac{n^n}{(\ln(n))^n}\)

Solution

a. To apply the root test, we compute \[ρ=\lim_{n→∞}\sqrt[n]{\dfrac{(n^2+3n)^n}{(4n^2+5)^n}}=\lim_{n→∞}\frac{n^2+3n}{4n^2+5}=\frac{1}{4}.\] Since \(ρ<1,\) the series converges absolutely.
b. We have \[ρ=\lim_{n→∞}\sqrt[n]{\dfrac{n^n}{(\ln n)^n}}=\lim_{n→∞}\frac{n}{\ln n}=∞\quad \text{by L'Hôpital's rule.}\] Since \(ρ=∞\), the series diverges.

Watch the accompanying video for a visual demonstration of this concept.

- **Video Length**: 2 minute 26 seconds.
- **Context**: This video demonstrates how to apply Ratio Test to show a series converges \(\displaystyle \sum^∞_{n=0}(\frac{n+1}{2n+1})^{n}\)

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Exercise \(\PageIndex{2}\)

Use the root test to determine whether the series \(\displaystyle \sum^∞_{n=1}1/n^n\) converges or diverges.

Hint available at the end of this section.

Answer available at the end of this section.

Choosing a Convergence Test

At this point, we have a long list of convergence tests. However, not all tests can be used for all series. When given a series, we must determine which test is the best to use. Here is a strategy for finding the best test to apply.

Problem-Solving Strategy: Choosing a Convergence Test for a Series

Consider a series \(\displaystyle \sum_{n=1}^∞a_n.\) In the steps below, we outline a strategy for determining whether the series converges.

1. Is \(\displaystyle \sum_{n=1}^∞a_n\) a familiar series? For example, is it the harmonic series (which diverges) or the alternating harmonic series (which converges)? Is it a \(p\)–series or geometric series? If so, check the power \(p\) or the ratio \(r\) to determine if the series converges.
2. Is it an alternating series? Are we interested in absolute convergence or just convergence? If we are just interested in whether the series converges, apply the alternating series test. If we are interested in absolute convergence, proceed to step \(3\), considering the series of absolute values \(\displaystyle \sum_{n=1}^∞|a_n|.\)
3. Is the series similar to a \(p\)–series or geometric series? If so, try the comparison test or limit comparison test.

4. Do the terms in the series contain a factorial or power? If the terms are powers such that \(a_n=(b_n)^n,\) try the root test first. Otherwise, try the ratio test first.
5. Use the divergence test. If this test does not provide any information, try the integral test.

Example \(\PageIndex{3}\): Using Convergence Tests

For each of the series, determine which convergence test is the best to use and explain why. Then determine if the series converges or diverges. If the series is an alternating series, determine whether it converges absolutely, converges conditionally, or diverges.

a. \(\displaystyle \sum^∞_{n=1}\frac{n^2+2n}{n^3+3n^2+1}\)
b. \(\displaystyle \sum^∞_{n=1}\frac{(−1)^{n+1}(3n+1)}{n!}\)
c. \(\displaystyle \sum^∞_{n=1}\frac{e^n}{n^3}\)
d. \(\displaystyle \sum^∞_{n=1}\frac{3^n}{(n+1)^n}\)

Solution a

Step 1. The series is not a harmonic or geometric series.

Step 2. The series is not alternating.

Step 3. For large values of \(n\), we approximate the series by the expression

\[\frac{n^2+2n}{n^3+3n^2+1}≈\frac{n^2}{n^3}=\frac{1}{n}. \nonumber\]

Therefore, it seems reasonable to apply the comparison test or limit comparison test using the series \(\displaystyle \sum_{n=1}^∞1/n\). Using the limit comparison test, we see that

\[\displaystyle \lim_{n→∞}\frac{(n^2+2n)/(n^3+3n^2+1)}{1/n}=\lim_{n→∞}\frac{n^3+2n^2}{n^3+3n^2+1}=1. \nonumber\]

Since the series \(\displaystyle \sum_{n=1}^∞1/n\) diverges, this series **diverges** as well.

Solution b

Step 1.The series is not a familiar series.

Step 2. The series is alternating. Since we are interested in absolute convergence, consider the series

\[\displaystyle \sum_{n=1}^∞\frac{3n}{(n+1)!}. \nonumber\]

Step 3. The series is not similar to a \(p\)-series or geometric series.

Step 4. Since each term contains a factorial, apply the ratio test. We see that

\[\displaystyle \lim_{n→∞}\frac{(3(n+1))/(n+1)!}{(3n+1)/n!}=\lim_{n→∞}\frac{3n+3}{(n+1)!}⋅\frac{n!}{3n+1}=\lim_{n→∞}\frac{3n+3}{(n+1)(3 \nonumber\]

Therefore, this series **converges**, and we conclude that the original series converges absolutely, and thus converges.

Solution c

Step 1. The series is not a familiar series.

Step 2. It is not an alternating series.

Step 3. There is no obvious series with which to compare this series.

Step 4. There is no factorial. There is a power, but it is not an ideal situation for the root test.

Step 5. To apply the divergence test, we calculate that

$$\lim_{n\to\infty}\frac{e^n}{n^3}=\infty.$$

Therefore, by the divergence test, the series **diverges**.

Solution d

Step 1. This series is not a familiar series.

Step 2. It is not an alternating series.

Step 3. There is no obvious series with which to compare this series.

Step 4. Since each term is a power of n,we can apply the root test. Since

$$\lim_{n\to\infty}\sqrt[n]{\left(\frac{3}{n+1}\right)^n}=\lim_{n\to\infty}\frac{3}{n+1}=0,$$

by the root test, we conclude that the series **converges**.

Exercise \(\PageIndex{3}\)

For the series $\sum^\infty_{n=1}\frac{2^n}{3^n+n}$, determine which convergence test is the best to use and explain why.

Hint available at the end of this section.

Answer available at the end of this section.

In Table, we summarize the convergence tests and when each can be applied. Note that while the comparison test, limit comparison test, and integral test require the series $\sum_{n=1}^\infty a_n$ to have nonnegative terms, if $\sum_{n=1}^\infty a_n$ has negative terms, these tests can be applied to $\sum_{n=1}^\infty |a_n|$ to test for absolute convergence.

Table \(\PageIndex{1}\): Table Summarizes Convergence Tests

Series or Test	Conclusions	Comments
Divergence Test For any series $\sum^\infty_{n=1}a_n$, evaluate $\lim_{n\to\infty}a_n$.	If $\lim_{n\to\infty}a_n=0$, the test is inconclusive.	This test cannot prove convergence of a series.
	If $\lim_{n\to\infty}a_n\neq 0$, the series diverges.	
Geometric Series $\sum^\infty_{n=1}ar^{n-1}$	If $\lvert r\rvert<1$, the series converges to $a/(1-r)$.	Any geometric series can be reindexed to be written in the form $a+ar+ar^2+\cdots$, where a is the initial term and r is the ratio.
	If $\lvert r\rvert\geq 1$, the series diverges.	

Series or Test	Conclusions	Comments
p-Series $\displaystyle \sum^∞_{n=1}\frac{1}{n^p}$	If $p>1$, the series converges.	For $p=1$, we have the harmonic series $\displaystyle \sum^∞_{n=1}1/n$.
	If $p≤1$, the series diverges.	
Direct Comparison Test For $\displaystyle \sum^∞_{n=1}a_n$ with nonnegative terms, compare with a known series $\displaystyle \sum^∞_{n=1}b_n$.	If $a_n≤b_n$ for all $n≥N$ and $\displaystyle \sum^∞_{n=1}b_n$ converges, then $\displaystyle \sum^∞_{n=1}a_n$ converges.	Typically used for a series similar to a geometric or p-series. It can sometimes be difficult to find an appropriate series.
	If $a_n≥b_n$ for all $n≥N$ and $\displaystyle \sum^∞_{n=1}b_n$ diverges, then $\displaystyle \sum^∞_{n=1}a_n$ diverges.	
Limit Comparison Test For $\displaystyle \sum^∞_{n=1}a_n$ with positive terms, compare with a series $\displaystyle \sum^∞_{n=1}b_n$ by evaluating $L=\displaystyle \lim_{n→∞}\frac{a_n}{b_n}.$	If L is a real number and $L≠0$, then $\displaystyle \sum^∞_{n=1}a_n$ and $\displaystyle \sum^∞_{n=1}b_n$ both converge or both diverge.	Typically used for a series similar to a geometric or p-series. Often easier to apply than the comparison test.
	If $L=0$ and $\displaystyle \sum^∞_{n=1}b_n$ converges, then $\displaystyle \sum^∞_{n=1}a_n$ converges.	
	If $L=∞$ and $\displaystyle \sum^∞_{n=1}b_n$ diverges, then $\displaystyle \sum^∞_{n=1}a_n$ diverges.	
Integral Test If there exists a positive, continuous, decreasing function f such that $a_n=f(n)$ for all $n≥N$, evaluate $\displaystyle ∫^∞_Nf(x)dx.$	$∫^∞_Nf(x)dx$ and $\displaystyle \sum^∞_{n=1}a_n$ both converge or both diverge.	Limited to those series for which the corresponding function can be easily integrated.
Alternating Series $\displaystyle \sum^∞_{n=1}(−1)^{n+1}b_n$ or $\displaystyle \sum^∞_{n=1}(−1)^nb_n$	If $b_{n+1}≤b_n$ for all $n≥1$ and $b_n→0$, then the series converges.	Only applies to alternating series.
Ratio Test For any series $\displaystyle \sum^∞_{n=1}a_n$ with nonzero terms, let $\displaystyle ρ=\lim_{n→∞}\left\|\frac{a_{n+1}}{a_n}\right\|$	If $0≤ρ<1$, the series converges absolutely.	Often used for series involving factorials or exponentials.
	If $ρ>1$ or $ρ=∞$, the series diverges.	
	If $ρ=1$, the test is inconclusive.	

Series or Test	Conclusions	Comments
Root Test For any series $\displaystyle \sum^{\infty}_{n=1}a_n$, let $\displaystyle \rho=\lim_{n\to\infty}\sqrt[n]{\lvert a_n\rvert}$.	If $0\le\rho<1$, the series converges absolutely.	Often used for series where $\lvert a_n\rvert=(b_n)^n$.
	If $\rho>1$ or $\rho=\infty$, the series diverges.	
	If $\rho=1$, the test is inconclusive.	

Series Converging to π and $1/\pi$

Dozens of series exist that converge to π or an algebraic expression containing π. Here we look at several examples and compare their rates of convergence. By rate of convergence, we mean the number of terms necessary for a partial sum to be within a certain amount of the actual value. The series representations of π in the first two examples can be explained using Maclaurin series, which are discussed in the next chapter. The third example relies on material beyond the scope of this text.

1. The series

$$\pi=4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}=4-\frac{4}{3}+\frac{4}{5}-\frac{4}{7}+\frac{4}{9}-\cdots$$

was discovered by Gregory and Leibniz in the late 1600s. This result follows from the Maclaurin series for $f(x)=\tan^{-1}x$. We will discuss this series in the next chapter.

   a. Prove that this series converges.
   b. Evaluate the partial sums S_n for $n=10,20,50,100$.
   c. Use the remainder estimate for alternating series to get a bound on the error R_n.
   d. What is the smallest value of N that guarantees $\lvert R_N\rvert<0.01$? Evaluate S_N.

2. The series

$$\pi=6\sum^{\infty}_{n=0}\frac{(2n)!}{2^{4n+1}(n!)^2(2n+1)}=6\left(\frac{1}{2}+\frac{1}{2\cdot 3}\left(\frac{1}{2}\right)^3+\frac{1\cdot 3}{2\cdot 4\cdot 5}\cdot\left(\frac{1}{2}\right)^5+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}\le$$

has been attributed to Newton in the late 1600s. The proof of this result uses the Maclaurin series for $f(x)=\sin^{-1}x$.

   a. Prove that the series converges.
   b. Evaluate the partial sums S_n for $n=5,10,20$.
   c. Compare S_n to π for $n=5,10,20$ and discuss the number of correct decimal places.

3. The series

$$\frac{1}{\pi}=\frac{\sqrt{8}}{9801}\sum_{n=0}^{\infty}\frac{(4n)!(1103+26390n)}{(n!)^4396^{4n}}$$

was discovered by **Ramanujan** in the early 1900s. William Gosper, Jr., used this series to calculate π to an accuracy of more than 17 million digits in the $mid-1980$s. At the time, that was a world record. Since that time, this series and others by Ramanujan have led mathematicians to find many other series representations for π and $1/\pi$.

   a. Prove that this series converges.
   b. Evaluate the first term in this series. Compare this number with the value of π from a calculating utility. To how many decimal places do these two numbers agree? What if we add the first two terms in the series?

c. Investigate the life of Srinivasa Ramanujan \((1887–1920)\) and write a brief summary. Ramanujan is one of the most fascinating stories in the history of mathematics. He was basically self-taught, with no formal training in mathematics, yet he contributed in highly original ways to many advanced areas of mathematics.

Key Concepts

- For the ratio test, we consider

\[ρ=\lim_{n→∞}|\frac{a_{n+1}}{a_n}|.\]

If \(ρ<1\), the series \(\displaystyle \sum_{n=1}^∞a_n\) converges absolutely. If \(ρ>1\), the series diverges. If \(ρ=1\), the test does not provide any information. This test is useful for series whose terms involve factorials.

- For the root test, we consider

\[ρ=\lim_{n→∞}\sqrt[n]{|a_n|}\].

If \(ρ<1\), the series \(\displaystyle \sum_{n=1}^∞a_n\) converges absolutely. If \(ρ>1\), the series diverges. If \(ρ=1\), the test does not provide any information. The root test is useful for series whose terms involve powers.

- For a series that is similar to a geometric series or **p–series,** consider one of the comparison tests.

Glossary

Ratio Test

For a series \(\displaystyle \sum^∞_{n=1}a_n\) with nonzero terms, let \(\displaystyle ρ=\lim_{n→∞}|a_{n+1}/a_n|\); if \(0≤ρ<1\), the series converges absolutely; if \(ρ>1\), the series diverges; if \(ρ=1\), the test is inconclusive.

Root Test

For a series \(\displaystyle \sum^∞_{n=1}a_n,\) let \(\displaystyle ρ=\lim_{n→∞}\sqrt[n]{|a_n|}\); if \(0≤ρ<1\), the series converges absolutely; if \(ρ>1\), the series diverges; if \(ρ=1\), the test is inconclusive.

Answers

Exercise \(\PageIndex{1}\)

Evaluate \(\displaystyle \lim_{n→∞}\frac{(n+1)^3}{3^{n+1}}·\frac{3^n}{n^3}.\)

Exercise \(\PageIndex{1}\)

The series converges.

Exercise \(\PageIndex{2}\)

Evaluate \(\displaystyle \lim_{n→∞}\sqrt[n]{\frac{1}{n^n}}\).

Exercise \(\PageIndex{2}\)

The series converges.

Exercise \(\PageIndex{3}\)

The series is similar to the geometric series \(\displaystyle \sum^∞_{n=1}\left(\frac{2}{3}\right)^n\).

Exercise \(\PageIndex{3}\)

The comparison test because \(\dfrac{2^n}{3^n+n}<\dfrac{2^n}{3^n}\) for all positive integers \(n\). The limit comparison test could also be used.

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1.6E: Exercises for Section 1.6

Exercises 1 - 11

Use the ratio test to determine whether each series $\sum_{n=1}^{\infty} a_n$ converges or diverges. State if the ratio test is inconclusive.

1. $\sum_{n=1}^{\infty} \frac{1}{n!}$

Answer

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0$. Converges by the Ratio Test.

2. $\sum_{n=1}^{\infty} \frac{10^n}{n!}$
3. $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

Answer

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{2}\left(\frac{n+1}{n}\right)^2 = \frac{1}{2} < 1$. Converges by the Ratio Test.

4. $\sum_{n=1}^{\infty} \frac{n^{10}}{2^n}$
5. $\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!}$

Answer

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{27} < 1$. Converges by the Ratio Test.

6. $\sum_{n=1}^{\infty} \frac{2^{3n}(n!)^3}{(3n)!}$
7. $\sum_{n=1}^{\infty} \frac{(2n)!}{n^{2n}}$

Answer

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{4}{e^2} < 1$. Converges by the Ratio Test.

8. $\sum_{n=1}^{\infty} \frac{(2n)!}{(2n)^n}$
9. $\sum_{n=1}^{\infty} \frac{n!}{(n/e)^n}$

Answer

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$. Ratio test is inconclusive.

10. $\sum_{n=1}^{\infty} \frac{(2n)!}{(n/e)^{2n}}$
11. $\sum_{n=1}^{\infty} \frac{(2^n n!)^2}{(2n)^{2n}}$

Answer

$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \frac{1}{e^2} < 1$. Converges by the Ratio Test.

Exercises 12 - 21

Use the root test to determine whether $\sum_{n=1}^{\infty} a_n$ converges, where a_n is as follows.

12. $a_k = \left(\frac{k-1}{2k+3}\right)^k$
13. $a_k = \left(\frac{2k^2-1}{k^2+3}\right)^k$

Answer

$\lim_{k\to\infty} (a_k)^{1/k} = 2 > 1$. Diverges by the Root Test.

14. $a_n = \frac{(\ln n)^{2n}}{n^n}$
15. $a_n = n/2^n$

Answer

$\lim_{n\to\infty} (a_n)^{1/n} = 1/2 < 1$. Converges by the Root Test.

16. $a_n = n/e^n$
17. $a_k = \frac{k^e}{e^k}$

Answer

$\lim_{k\to\infty} (a_k)^{1/k} = 1/e < 1$. Converges by the Root Test.

18. $a_k = \frac{\pi^k}{k^\pi}$
19. $a_n = \left(\frac{1}{e} + \frac{1}{n}\right)^n$

Answer

$\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \frac{1}{e} + \frac{1}{n} = \frac{1}{e} < 1$. Converges by the Root Test.

20. $a_k = \frac{1}{(1+\ln k)^k}$
21. $a_n = \frac{(\ln(1+\ln n))^n}{(\ln n)^n}$

Answer

$\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \frac{(\ln(1+\ln n))}{(\ln n)} = 0$ by L'Hôpital's rule. Converges by the Root Test.

Exercises 22 - 28

Use either the ratio test or the root test as appropriate to determine whether the series $\sum_{k=1}^{\infty} a_k$ with given terms a_k converges, or state if the test is inconclusive.

22. $a_k = \frac{k!}{1\cdot 3\cdot 5\cdots(2k-1)}$
23. $a_k = \frac{2\cdot 4\cdot 6\cdots 2k}{(2k)!}$

Answer

$\lim_{k\to\infty}\frac{a_{k+1}}{a_k}=\lim_{k\to\infty}\frac{1}{2k+1}=0$. Converges by the Ratio Test.

24. $a_k=\dfrac{1\cdot 4\cdot 7\cdots(3k-2)}{3^k k!}$

25. $a_n=\left(1-\dfrac{1}{n}\right)^{n^2}$

Answer

$\lim_{n\to\infty}(a_n)^{1/n}=1/e$. Converges by the Root Test.

26. $a_k=\left(\dfrac{1}{k+1}+\dfrac{1}{k+2}+\cdots+\dfrac{1}{2k}\right)^k$ (Hint: Compare $a_k^{1/k}$ to $\int_k^{2k}\frac{dt}{t}$.)

27. $a_k=\left(\dfrac{1}{k+1}+\dfrac{1}{k+2}+\cdots+\dfrac{1}{3k}\right)^k$

Answer

$\lim_{k\to\infty}a_k^{1/k}=\ln(3)>1$. Diverges by the Root Test.

28. $a_n=\left(n^{1/n}-1\right)^n$

Exercises 29 - 30

Use the ratio test to determine whether $\sum_{n=1}^{\infty}a_n$ converges, or state if the ratio test is inconclusive.

29. $\sum_{n=1}^{\infty}\dfrac{3^{n^2}}{2^{n^3}}$

Answer

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{3^{2n+1}}{2^{3n^2+3n+1}}=0$. Converges by the Ratio Test.

30. $\sum_{n=1}^{\infty}\dfrac{2^{n^2}}{n^n n!}$

Exercises 31

Use the root and limit comparison tests to determine whether $\sum_{n=1}^{\infty}a_n$ converges.

31. $\sum_{n=1}^{\infty}\dfrac{1}{x_n^n}$ where $x_{n+1}=\frac{1}{2}x_n+\dfrac{1}{x_n}, x_1=1$ (Hint: Find limit of x_n.)

Answer

Converges by the Root Test and Limit Comparison Test since $\lim_{n\to\infty}x_n=\sqrt{2}$.

Exercises 32 - 43

Use an appropriate test to determine whether the series converges.

32. $\sum_{n=1}^{\infty}\dfrac{n+1}{n^3+n^2+n+1}$

33. $\sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}(n+1)}{n^3+3n^2+3n+1}$

Answer

Converges absolutely by limit comparison with p**–series,** $p = 2$.

34. $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)^2}{n^3 + (1.1)^n}$
35. $\displaystyle\sum_{n=1}^{\infty} \frac{(n-1)^n}{(n+1)^n}$

Answer

$\lim_{n\to\infty} a_n = 1/e^2 \neq 0$. Series diverges by the Divergence Test.

36. $a_n = \left(1 + \dfrac{1}{n^2}\right)^n$ $\left(\text{Hint: } \left(1 + \dfrac{1}{n^2}\right)^{n^2} \approx e.\right)$
37. $a_k = 1/2^{\sin^2 k}$

Answer

Terms do not tend to zero: $a_k \geq 1/2$, since $\sin^2 x \leq 1$.

38. $a_k = 2^{-\sin(1/k)}$
39. $a_n = 1/\binom{n+2}{n}$ where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Answer

$a_n = \dfrac{2}{(n+1)(n+2)}$, which converges by comparison with p**–series** for $p = 2$.

40. $a_k = 1/\binom{2k}{k}$
41. $a_k = 2^k/\binom{3k}{k}$

Answer

$a_k = \dfrac{2^k 1 \cdot 2 \cdots k}{(2k+1)(2k+2)\cdots 3k} \leq (2/3)^k$ converges by comparison with geometric series.

42. $a_k = \left(\dfrac{k}{k + \ln k}\right)^k$ $\left(\text{Hint: } a_k = \left(1 + \dfrac{\ln k}{k}\right)^{-(k/\ln k)\ln k} \approx e^{-\ln k}.\right)$
43. $a_k = \left(\dfrac{k}{k + \ln k}\right)^{2k}$ $\left(\text{Hint: } a_k = \left(1 + \dfrac{\ln k}{k}\right)^{-(k/\ln k)\ln k^2}.\right)$

Answer

$a_k \approx e^{-\ln k^2} = 1/k^2$. Series converges by limit comparison with p**–series**, $p = 2$.

Exercises 44 - 47

The series in converge by the ratio test. Use summation by parts, $\displaystyle\sum_{k=1}^{n} a_k(b_{k+1} - b_k) = [a_{n+1}b_{n+1} - a_1 b_1] - \sum_{k=1}^{n} b_{k+1}(a_{k+1} - a_k)$**, to find the sum of the given series.**

44. $\displaystyle\sum_{k=1}^{\infty} \frac{k}{2^k}$ (Hint: Take $a_k = k$ and $b_k = 2^{1-k}$.)
45. $\displaystyle\sum_{k=1}^{\infty} \frac{k}{c^k}$, where $c > 1$ (Hint: Take $a_k = k$ and $b_k = c^{1-k}/(c-1)$.)

Answer

If $b_k = c^{1-k}/(c-1)$ and $a_k = k$, then $b_{k+1} - b_k = -c^{-k}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{k}{c^k} = a_1 b_1 + \frac{1}{c-1}\sum_{k=1}^{\infty} c^{-k} = \frac{c}{(c-1)^2}$.

46. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

47. $\sum_{n=1}^{\infty} \frac{(n+1)^2}{2^n}$

Answer

$6+4+1=11$

Exercises 48 - 59

The kth term of each of the following series has a factor x^k. Find the range of x for which the ratio test implies that the series converges.

48. $\sum_{k=1}^{\infty} \frac{x^k}{k^2}$
49. $\sum_{k=1}^{\infty} \frac{x^{2k}}{k^2}$

Answer

$|x| \leq 1$

50. $\sum_{k=1}^{\infty} \frac{x^{2k}}{3^k}$
51. $\sum_{k=1}^{\infty} \frac{x^k}{k!}$

Answer

$|x| < \infty$

52. Does there exist a number p such that $\sum_{n=1}^{\infty} \frac{2^n}{n^p}$ converges?
53. Let \(0

Answer

All real numbers p by the Ratio Test.

54. Suppose that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = p$. For which values of p must $\sum_{n=1}^{\infty} 2^n a_n$ converge?
55. Suppose that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = p$. For which values of $r > 0$ is $\sum_{n=1}^{\infty} r^n a_n$ guaranteed to converge?

Answer

$r < 1/p$

56. Suppose that $\left|\frac{a_{n+1}}{a_n}\right| \leq (n+1)^p$ for all $n = 1, 2, \ldots$ where p is a fixed real number. For which values of p is $\sum_{n=1}^{\infty} n! a_n$ guaranteed to converge?
57. For which values of $r > 0$, if any, does $\sum_{n=1}^{\infty} r^{\sqrt{n}}$ converge? $\left(\text{Hint: } sum_{n=1}^{\infty} a_n = \sum_{k=1}^{\infty} \sum_{n=k^2}^{(k+1)^2-1} a_n.\right)$

Answer

\(0

58. Suppose that $\left|\frac{a_{n+2}}{a_n}\right| \leq r < 1$ for all n. Can you conclude that $\sum_{n=1}^{\infty} a_n$ converges?
59. Let $a_n = 2^{-[n/2]}$ where $[x]$ is the greatest integer less than or equal to x. Determine whether $\sum_{n=1}^{\infty} a_n$ converges and justify your answer.

Answer

One has $a_1 = 1, a_2 = a_3 = 1/2, \ldots a_{2n} = a_{2n+1} = 1/2^n$. The ratio test does not apply because $a_{n+1}/a_n = 1$ if n is even. However, $a_{n+2}/a_n = 1/2$, so the series converges according to the previous exercise. Of course, the series is just a duplicated geometric series.

Exercises 60 - 62

The following *advanced exercises* use a generalized ratio test to determine convergence of some series that arise in particular applications when tests in this chapter, including the ratio and root test, are not powerful enough to determine their convergence. The test states that if $\lim_{n\to\infty} \frac{a_{2n}}{a_n} < 1/2$, then $\sum a_n$ converges, while if $\lim_{n\to\infty} \frac{a_{2n+1}}{a_n} > 1/2$, then $\sum a_n$ diverges.

60. Let $a_n = \frac{1}{4}\frac{3}{6}\frac{5}{8}\cdots\frac{2n-1}{2n+2} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n(n+1)!}$. Explain why the ratio test cannot determine convergence of $\sum_{n=1}^{\infty} a_n$. Use the fact that $1 - 1/(4k)$ is increasing k to estimate $\lim_{n\to\infty} \frac{a_{2n}}{a_n}$.

61. Let $a_n = \frac{1}{1+x}\frac{2}{2+x}\cdots\frac{n}{n+x}\frac{1}{n} = \frac{(n-1)!}{(1+x)(2+x)\cdots(n+x).}$ Show that $a_{2n}/a_n \le e^{-x/2}/2$. For which $x > 0$ does the generalized ratio test imply convergence of $\sum_{n=1}^{\infty} a_n$? (Hint: Write $2a_{2n}/a_n$ as a product of n factors each smaller than $1/(1 + x/(2n)).$)

Answer

$a_{2n}/a_n = \frac{1}{2}\cdot\frac{n+1}{n+1+x}\frac{n+2}{n+2+x}\cdots\frac{2n}{2n+x}$. The inverse of the kth factor is $(n+k+x)/(n+k) > 1 + x/(2n)$ so the product is less than $(1 + x/(2n))^{-n} \approx e^{-x/2}$. Thus for $x > 0, \frac{a_{2n}}{a_n} \le \frac{1}{2}e^{-x/2}$. The series converges for $x > 0$.

62. Let $a_n = \frac{n^{\ln n}}{(\ln n)^n}$. Show that $\frac{a_{2n}}{a_n} \to 0$ as $n \to \infty$.

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1.7: Chapter 1 Review Exercises

Exercises 1 - 4

True or False? Justify your answer with a proof or a counterexample.

1. If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

   Answer

   false

2. If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.
3. If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

   Answer

   true

4. If $\sum_{n=1}^{\infty} 2^n a_n$ converges, then $\sum_{n=1}^{\infty} (-2)^n a_n$ converges.

Exercises 5 - 9

Is the sequence bounded, monotone, and convergent or divergent? If it is convergent, find the limit.

5. $a_n = \dfrac{3+n^2}{1-n}$

   Answer

   unbounded, not monotone, divergent

6. $a_n = \ln\left(\frac{1}{n}\right)$
7. $a_n = \dfrac{\ln(n+1)}{\sqrt{n+1}}$

   Answer

   bounded, monotone, convergent, 0

8. $a_n = \dfrac{2^{n+1}}{5^n}$
9. $a_n = \dfrac{\ln(\cos n)}{n}$

   Answer

   unbounded, not monotone, divergent

Exercises 10 - 14

Is the series convergent or divergent?

10. $\sum_{n=1}^{\infty} \dfrac{1}{n^2+5n+4}$

11. $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$

Answer

diverges

12. $\sum_{n=1}^{\infty} \frac{2^n}{n^4}$
13. $\sum_{n=1}^{\infty} \frac{e^n}{n!}$

Answer

converges

14. $\sum_{n=1}^{\infty} n^{-(n+1/n)}$

Exercises 15 - 19

Is the series convergent or divergent? If convergent, is it absolutely convergent?

15. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

Answer

converges, but not absolutely

16. $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{3^n}$
17. $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{n^n}$

Answer

converges absolutely

18. $\sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{2}\right)$
19. $\sum_{n=1}^{\infty} \cos(\pi n) e^{-n}$

Answer

converges absolutely

Exercises 20 - 22

Evaluate.

20. $\sum_{n=1}^{\infty} \frac{2^{n+4}}{7^n}$
21. $\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$

Answer

$\frac{1}{2}$

22. A legend from India tells that a mathematician invented chess for a king. The king enjoyed the game so much he allowed the

mathematician to demand any payment. The mathematician asked for one grain of rice for the first square on the chessboard, two grains of rice for the second square on the chessboard, and so on. Find an exact expression for the total payment (in grains of rice) requested by the mathematician. Assuming there are $30,000$ grains of rice in 1 pound, and 2000 pounds in 1 ton, how many tons of rice did the mathematician attempt to receive?

Exercises 23 - 26

The following problems consider a simple population model of the housefly, which can be exhibited by the recursive formula $x_{n+1} = bx_n$, where x_n is the population of houseflies at generation n, and b is the average number of offspring per housefly who survive to the next generation. Assume a starting population x_0.

23. Find $\lim_{n\to\infty} x_n$ if $b > 1$, $b < 1$, and $b = 1$.

    Answer

    $$\infty, \quad 0, \quad x_0$$

24. Find an expression for $S_n = \sum_{i=0}^{n} x_i$ in terms of b and x_0. What does it physically represent?

25. If $b = \frac{3}{4}$ and $x_0 = 100$, find S_{10} and $\lim_{n\to\infty} S_n$

    Answer

    $$S_{10} \approx 383, \qquad \lim_{n\to\infty} S_n = 400$$

26. For what values of b will the series converge and diverge? What does the series converge to?

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2: POWER SERIES

A power series (in one variable) is an infinite series. Any polynomial can be easily expressed as a power series around any center c, although most of the coefficients will be zero since a power series has infinitely many terms by definition. One can view power series as being like "polynomials of infinite degree," although power series are not polynomials. The content in this Textmap's chapter is complemented by Guichard's Calculus Textmap.

When winning a lottery, sometimes an individual has an option of receiving winnings in one lump-sum payment or receiving smaller payments over fixed time intervals. For example, you might have the option of receiving 20 million dollars today or receiving 1.5 million dollars each year for the next 20 years. Which is the better deal? Certainly 1.5 million dollars over 20 years is equivalent to 30 million dollars. However, receiving the 20 million dollars today would allow you to invest the money.



Figure : An image showing multiple U.S. $\$100$ bills arranged in a grid-like pattern. The bills feature the portrait of Benjamin Franklin and other recognizable details, such as the green treasury seal and serial numbers (Credit:CC-BY-NC-SA,work by Ervins Strauhmanis's,www.flickr.com)

Alternatively, what if you were guaranteed to receive 1 million dollars every year indefinitely (extending to your heirs) or receive 20 million dollars today. Which would be the better deal? To answer these questions, you need to know how to use infinite series to calculate the value of periodic payments over time in terms of today's dollars.

An infinite series of the form

$$\sum_{n=0}^{\infty} c_n x^n$$

is known as a **power series**. Since the terms contain the variable x, power series can be used to define functions. They can be used to represent given functions, but they are also important because they allow us to write functions that cannot be expressed any other way than as "infinite polynomials." In addition, power series can be easily differentiated and integrated, thus being useful in solving differential equations and integrating complicated functions. An infinite series can also be truncated, resulting in a finite polynomial that we can use to approximate functional values. Power series have applications in a variety of fields, including physics, chemistry, biology, and economics. As we will see in this chapter, representing functions using power series allows us to solve mathematical problems that cannot be solved with other techniques.

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- **2.3: Taylor and Maclaurin Series**
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2.1: Power Series and Functions

Learning Objectives

- Identify a power series and provide examples of them.
- Determine the radius of convergence and interval of convergence of a power series.
- Use a power series to represent a function.

A power series is a type of series with terms involving a variable. More specifically, if the variable is x, then all the terms of the series involve powers of x. As a result, a power series can be thought of as an infinite polynomial. Power series are used to represent common functions and also to define new functions. In this section we define power series and show how to determine when a power series converges and when it diverges. We also show how to represent certain functions using power series.

Form of a Power Series

A series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots, \tag{2.1.1}$$

where x is a variable and the coefficients c_n are constants, is known as a **power series.** The series

$$1 + x + x^2 + \dots = \sum_{n=0}^{\infty} x^n \tag{2.1.2}$$

is an example of a power series. Since this series is a geometric series with ratio $r = |x|$, we know that it converges if $|x| < 1$ and diverges if $|x| \geq 1$.

Definition: Power Series

A series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots \tag{2.1.3}$$

is a power series centered at $x = 0$. A series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \dots \tag{2.1.4}$$

is a **power series** centered at $x = a$.

To make this definition precise, we stipulate that $x^0 = 1$ and $(x-a)^0 = 1$ even when $x = 0$ and $x = a$, respectively.

The series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \tag{2.1.5}$$

and

$$\sum_{n=0}^{\infty} n!x^n = 1 + x + 2!x^2 + 3!x^3 + \ldots \tag{2.1.6}$$

are both power series centered at $x = 0$. The series

$$\sum_{n=0}^{\infty} \frac{(x-2)^n}{(n+1)3^n} = 1 + \frac{x-2}{2 \cdot 3} + \frac{(x-2)^2}{3 \cdot 3^2} + \frac{(x-2)^3}{4 \cdot 3^3} + \ldots \tag{2.1.7}$$

is a power series centered at $x = 2$.

Convergence of a Power Series

Since the terms in a power series involve a variable x, the series may converge for certain values of x and diverge for other values of x. For a power series centered at $x = a$, the value of the series at $x = a$ is given by c_0. Therefore, a power series always converges at its center. Some power series converge only at that value of x. Most power series, however, converge for more than one value of x. In that case, the power series either converges for all real numbers x or converges for all x in a finite interval. For example, the geometric series $\sum_{n=0}^{\infty} x^n$ converges for all x in the interval $(-1, 1)$, but diverges for all x outside that interval. We now summarize these three possibilities for a general power series.

Theorem: Convergence of a Power Series

Consider the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$. The series satisfies exactly one of the properties:

i. The series converges at $x = a$ and diverges for all $x \neq a$.
ii. The series converges for all real numbers x.
iii. There exists a real number $R > 0$ such that the series
   a. converges if \(|x–a|
   b. diverges if $|x - a| > R$.
   c. At the values x where |x–a|=R, the series may converge or diverge.

Proof

Suppose that the power series is centered at $a = 0$. (For a series centered at a value of a other than zero, the result follows by letting $y = x - a$ and considering the series

$$\sum_{n=1}^{\infty} c_n y^n.$$

If there exists a real number $d \neq 0$ such that $\sum_{n=0}^{\infty} c_n d^n$ converges, then the series $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely for all x such that $|x| < |d|$.

Since $\sum_{n=0}^{\infty} c_n d^n$ converges, the nth term $c_n d^n \to 0$ as $n \to \infty$. Therefore, there exists an integer N such that $|c_n d^n| \leq 1$ for all $n \geq N$. Writing

$$|c_n x^n| = |c_n d^n| \left|\frac{x}{d}\right|^n,$$

we conclude that, for all n≥N,

$$|c_n x^n| \leq \left|\frac{x}{d}\right|^n.$$

The series

$$\sum_{n=N}^{\infty} \left|\frac{x}{d}\right|^n$$

is a geometric series that converges if $|\frac{x}{d}| < 1$. Therefore, by the comparison test, we conclude that $\sum_{n=N}^{\infty} c_n x^n$ also converges for $|x| < |d|$. Since we can add a finite number of terms to a convergent series, we conclude that $\sum_{n=0}^{\infty} c_n x^n$ converges for $|x| < |d|$.

With this result, we can now prove the theorem. Consider the series

$$\sum_{n=0}^{\infty} a_n x^n$$

and let S be the set of real numbers for which the series converges. Suppose that the set $S = 0$. Then the series falls under case i.

Suppose that the set S is the set of all real numbers. Then the series falls under case ii. Suppose that $S \neq 0$ and S is not the set of real numbers. Then there exists a real number $x* \neq 0$ such that the series does not converge. Thus, the series cannot converge for any x such that $|x| > |x*|$. Therefore, the set S must be a bounded set, which means that it must have a smallest upper bound. (This fact follows from the **Least Upper Bound Property** for the real numbers, which is beyond the scope of this text and is covered in real analysis courses.) Call that smallest upper bound R. Since $S \neq 0$, the number $R > 0$. Therefore, the series converges for all x such that \(|x|

□

If a series $\sum_{n=0}^{\infty} c_n(x-a)^n$ falls into case iii. of Note, then the series converges for all x such that \(|x–a|9\), and diverges for all x such that $|x-a| > R$. The series may converge or diverge at the values x where $|x-a| = R$. The set of values x for which the series $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges is known as the interval of convergence. Since the series diverges for all values x where $|x-a| > R$, the length of the interval is $2R$, and therefore, the radius of the interval is R. The value R is called the radius of convergence. For example, since the series $\sum_{n=0}^{\infty} x^n$ converges for all values x in the interval $(-1,1)$ and diverges for all values x such that $|x| \geq 1$, the interval of convergence of this series is $(-1,1)$. Since the length of the interval is 2, the radius of convergence is 1.

Definition: Interval and Radius of Convergence

Consider the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$. The set of real numbers x where the series converges is the **interval of convergence**. If there exists a real number $R > 0$ such that the series converges for \(|x–a|R,\) then R is the **radius of convergence**. If the series converges only at $x = a$, we say the **radius of convergence is** $R = 0$. If the series converges for all real numbers x, we say the **radius of convergence** is $R = \infty$ (Figure $2.1.1$).

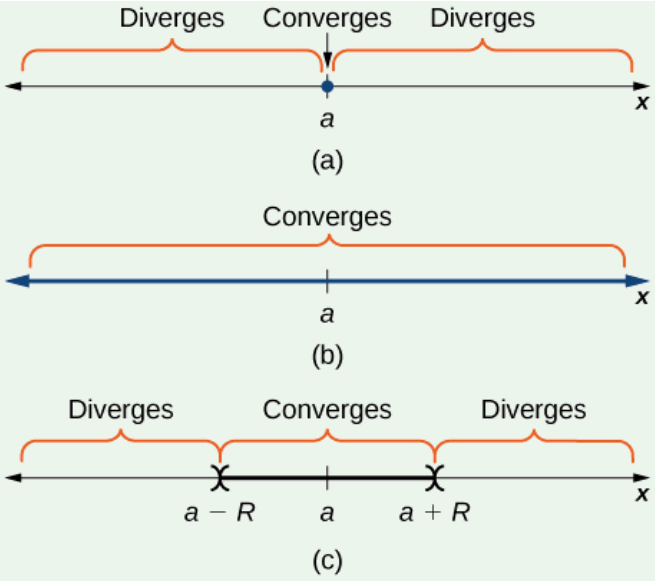


Figure 2.1.1: For a series $\sum_{n=0}^{\infty} c_n(x-a)^n$ graph (a) shows a radius of convergence at $R=0$, graph (b) shows a radius of convergence at $R=\infty$, and graph (c) shows a radius of convergence at R. For graph (c) we note that the series may or may not converge at the endpoints $x=a+R$ and $x=a-R$.

To determine the interval of convergence for a power series, we typically apply the ratio test. In Example 2.1.1, we show the three different possibilities illustrated in Figure 2.1.1.

Example 2.1.1: Finding the Interval and Radius of Convergenc

For each of the series, find the interval and radius of convergence.

a. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

b. $\sum_{n=0}^{\infty} n!x^n$

c. $\sum_{n=0}^{\infty} \frac{(x-2)^n}{(n+1)3^n}$

Solution

a. To check for convergence, apply the ratio test. We have

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| \\
&= \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| \\
&= \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)\cdot n!} \cdot \frac{n!}{x^n} \right| \\
&= \lim_{n\to\infty} \left| \frac{x}{n+1} \right| \\
&= |x| \lim_{n\to\infty} \frac{1}{n+1} \\
&= 0 < 1
\end{aligned}$$

for all values of x. Therefore, the series converges for all real numbers x. The interval of convergence is $(-\infty, \infty)$ and the radius of convergence is $R = \infty$.

Watch the accompanying video for a visual demonstration of this concept.

- **Video Length:** 4 minute 12 seconds.
- **Context:** This video demonstrates how to use Ratio test to find interval of convergence when Radius of Convergence is infinite. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(n+1)!}$

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

b. Apply the ratio test. For $x \neq 0$, we see that

$$\begin{aligned} \rho &= \lim_{n\to\infty} \left| \frac{(n+1)!x^{n+1}}{n!x^n} \right| \\ &= \lim_{n\to\infty} |(n+1)x| \\ &= |x| \lim_{n\to\infty} (n+1) \\ &= \infty. \end{aligned}$$

Therefore, the series diverges for all $x \neq 0$. Since the series is centered at $x = 0$, it must converge there, so the series converges only for $x = 0$. The interval of convergence is the single value $x = 0$ and the radius of convergence is $R = 0$.

c. In order to apply the ratio test, consider

$$\begin{aligned} \rho &= \lim_{n\to\infty} \left| \frac{\frac{(x-2)^{n+1}}{(n+2)3^{n+1}}}{\frac{(x-2)^n}{(n+1)3^n}} \right| \\ &= \lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{(n+2)3^{n+1}} \cdot \frac{(n+1)3^n}{(x-2)^n} \right| \\ &= \lim_{n\to\infty} \left| \frac{(x-2)(n+1)}{3(n+2)} \right| \\ &= \frac{|x-2|}{3}. \end{aligned}$$

The ratio $\rho < 1$ if $|x-2| < 3$. Since $|x-2| < 3$ implies that \(–31\) if $|x-2| > 3$. Therefore, the series diverges if $x < -1$ or $x > 5$. The ratio test is inconclusive if $\rho = 1$. The ratio $\rho = 1$ if and only if $x = -1$ or $x = 5$. We need to test these values of x separately. For $x = -1$, the series is given by

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots.$$

Since this is the alternating harmonic series, it converges. Thus, the series converges at $x = -1$. For $x = 5$, the series is given by

$$\sum_{n=0}^{\infty} \frac{1}{n+1} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots.$$

This is the harmonic series, which is divergent. Therefore, the power series diverges at $x = 5$. We conclude that the interval of convergence is $[-1, 5)$ and the radius of convergence is $R = 3$.

Watch the accompanying video for a visual demonstration of this concept.

- **Video Length:** 8 minute 2 seconds.
- **Context:** This video uses the ratio test to analyze the series' convergence and explore its behavior at the endpoints of the interval. $\sum_{n=0}^{\infty} \frac{(-1)^n (x+3)^n}{(n+1)5^n}$

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

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Exercise 2.1.1

Find the interval and radius of convergence for the series

$$\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}.$$

Hint available at the end of this section.

Answer available at the end of this section.

Representing Functions as Power Series

Being able to represent a function by an "infinite polynomial" is a powerful tool. Polynomial functions are the easiest functions to analyze, since they only involve the basic arithmetic operations of addition, subtraction, multiplication, and division. If we can represent a complicated function by an infinite polynomial, we can use the polynomial representation to differentiate or integrate it. In addition, we can use a truncated version of the polynomial expression to approximate values of the function. So, the question is, when can we represent a function by a power series?

Consider again the geometric series

$$1 + x + x^2 + x^3 + ... = \sum_{n=0}^{\infty} x^n. \tag{2.1.8}$$

Recall that the geometric series

$$a + ar + ar^2 + ar^3 + ... \tag{2.1.9}$$

converges if and only if $|r| < 1$. In that case, it converges to $\frac{a}{1-r}$. Therefore, if $|x| < 1$, the series in Example 2.1.1 converges to $\frac{1}{1-x}$ and we write

$$1 + x + x^2 + x^3 + ... = \frac{1}{1-x} for |x| < 1. \tag{2.1.10}$$

As a result, we are able to represent the function $f(x) = \frac{1}{1-x}$ by the power series

$$1 + x + x^2 + x^3 + ... when |x| < 1. \tag{2.1.11}$$

We now show graphically how this series provides a representation for the function $f(x) = \dfrac{1}{1-x}$ by comparing the graph of f with the graphs of several of the partial sums of this infinite series.

Example 2.1.2: Graphing a Function and Partial Sums of its Power Series

Sketch a graph of $f(x) = \dfrac{1}{1-x}$ and the graphs of the corresponding partial sums $S_N(x) = \sum_{n=0}^{N} x^n$ for $N = 2, 4, 6$ on the interval $(-1, 1)$. Comment on the approximation S_N as N increases.

Solution

From the graph in Figure 2.1.2 you see that as N increases, S_N becomes a better approximation for $f(x) = \dfrac{1}{1-x}$ for x in the interval $(-1, 1)$.

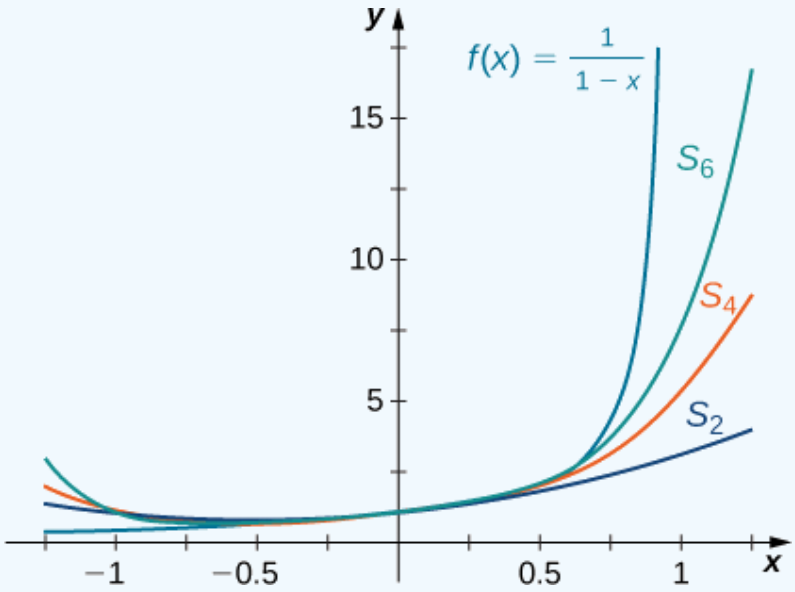


Figure 2.1.2: The graph shows a function and three approximations of it by partial sums of a power series. The $x - axis$ ranges from -1.5 to 2, and the $y - axis$ ranges from -5 to 20. The main function depicted is $f(x) = \frac{1}{1-x}$. The three curves represent different approximations of this function: S_2, S_4, and S_6. As the degree of the polynomials increases, the curves more closely approach the steep ascent near $x = 1$, aligning more closely with $f(x)$. The graph shows how each approximation behaves differently around $x = 1$.

Interactive Element

You can move it around and also change its equation to see the effect on the graph. The interactive app is unfortunately not accessible.

Visualize the Power Series Approximation of $f(x) = \frac{1}{1-x}$

Move the slider to see how well $\sum_{n=0}^{N} x^n$ approximates $f(x) = \frac{1}{1-x}$.

2

N = 2

$S_0(x) = 1$

$S_1(x) = 1 + x$

$S_2(x) = 1 + x + x^2$

$S_3(x) = 1 + x + x^2 + x^3$

$S_4(x) = 1 + x + x^2 + x^3 + x^4$

$S_5(x) = 1 + x + x^2 + x^3 + x^4 + x^5$

$S_6(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6$

$S_7(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7$

$S_8(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8$

$S_9(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9$

$S_{10}(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10}$

$S_{11}(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11}$

$S_{12}(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12}$

$S_{13}(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12} + x^{13}$

$S_{14}(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12} + x^{13} + x^{14}$

$S_{15}(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12} + x^{13} + x^{14} + x^{15}$

$S_{16}(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12} + x^{13} + x^{14} + x^{15} + x^{16}$

$S_{17}(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12} + x^{13} + x^{14} + x^{15} + x^{16} + x^{17}$

$S_{18}(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12} + x^{13} + x^{14} + x^{15} + x^{16} + x^{17} + x^{18}$

$S_{19}(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12} + x^{13} + x^{14} + x^{15} + x^{16} + x^{17} + x^{18} + x^{19}$

$S_{20}(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12} + x^{13} + x^{14} + x^{15} + x^{16} + x^{17} + x^{18} + x^{19} + x^{20}$

Exercise 2.1.2

Sketch a graph of $f(x) = \dfrac{1}{1 - x^2}$ and the corresponding partial sums $S_N(x) = \displaystyle\sum_{n=0}^{N} x^{2n}$ for $N = 2, 4, 6$ on the interval $(-1, 1)$.

Hint available at the end of this section.

Answer available at the end of this section.

Next we consider functions involving an expression similar to the sum of a geometric series and show how to represent these functions using power series.

Example 2.1.3: Representing a Function with a Power Series

Use a power series to represent each of the functions f. Find the interval of convergence.

a. $f(x) = \dfrac{1}{1+x^3}$

b. $f(x) = \dfrac{x^2}{4-x^2}$

Solution

a. You should recognize this function f as the sum of a geometric series, because

$$\frac{1}{1+x^3} = \frac{1}{1-(-x^3)}.$$

Using the fact that, for $|r|<1$, $\dfrac{a}{1-r}$ is the sum of the geometric series

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \dots,$$

we see that, for $|-x^3|<1$,

$$\begin{aligned}\frac{1}{1+x^3} &= \frac{1}{1-(-x^3)} \\ &= \sum_{n=0}^{\infty}(-x^3)^n \\ &= 1 - x^3 + x^6 - x^9 + \dots.\end{aligned}$$

Since this series converges if and only if $|-x^3|<1$, the interval of convergence is $(-1,1)$, and we have

$$\frac{1}{1+x^3} = 1 - x^3 + x^6 - x^9 + \dots \; for |x|<1.$$

b. This function is not in the exact form of a sum of a geometric series. However, with a little algebraic manipulation, we can relate f to a geometric series. By factoring 4 out of the two terms in the denominator, we obtain

$$\begin{aligned}\frac{x^2}{4-x^2} &= \frac{x^2}{4(\frac{1-x^2}{4})} \\ &= \frac{x^2}{4(1-(\frac{x}{2})^2)}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\frac{x^2}{4-x^2} &= \frac{x^2}{4(1-(\frac{x}{2})^2)} \\ &= \frac{\frac{x^2}{4}}{1-(\frac{x}{2})^2} \\ &= \sum_{n=0}^{\infty}\frac{x^2}{4}(\frac{x}{2})^{2n}.\end{aligned}$$

The series converges as long as $|(\frac{x}{2})^2| < 1$ (note that when $|(\frac{x}{2})^2| = 1$ the series does not converge). Solving this inequality, we conclude that the interval of convergence is $(-2, 2)$ and

$$\begin{aligned}\frac{x^2}{4-x^2} &= \sum_{n=0}^{\infty} \frac{x^{2n+2}}{4^{n+1}} \\ &= \frac{x^2}{4} + \frac{x^4}{4^2} + \frac{x^6}{4^3} + \ldots\end{aligned}$$

for $|x| < 2$.

Exercise 2.1.3

Represent the function $f(x) = \dfrac{x^3}{2-x}$ using a power series and find the interval of convergence.

Hint available at the end of this section.

Answer available at the end of this section.

In the remaining sections of this chapter, we will show ways of deriving power series representations for many other functions, and how we can make use of these representations to evaluate, differentiate, and integrate various functions.

Key Concepts

- For a power series centered at $x = a$, one of the following three properties hold:
  i. The power series converges only at $x = a$. In this case, we say that the radius of convergence is $R = 0$.
  ii. The power series converges for all real numbers x. In this case, we say that the radius of convergence is $R = \infty$.
  iii. There is a real number R such that the series converges for \(|x–a|R\). In this case, the radius of convergence is R.
- If a power series converges on a finite interval, the series may or may not converge at the endpoints.
- The ratio test may often be used to determine the radius of convergence.
- The geometric series $\sum_{n=0}^{\infty} x^n = \dfrac{1}{1-x}$ for $|x| < 1$ allows us to represent certain functions using geometric series.

Key Equations

- **Power series centered at** $x = 0$

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots n$$

- **Power series centered at** $x = a$

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \ldots$$

Glossary

Interval of Convergence
The set of real numbers x for which a power series converges.

Power Series
A series of the form $\sum_{n=0}^{\infty} c_n x^n$ is a power series centered at $x = 0$; a series of the form $\sum_{n=0}^{\infty} c_n (x-a)^n$ is a power series centered at $x = a$.

Radius of Convergence

If there exists a real number $R > 0$ such that a power series centered at $x = a$ converges for \(|x–a|R\), then R is the radius of convergence; if the power series only converges at $x = a$, the radius of convergence is $R = 0$; if the power series converges for all real numbers x, the radius of convergence is $R = \infty$.

Answers

Exercise 2.1.1

Apply the ratio test to check for absolute convergence.

Exercise 2.1.1

The interval of convergence is $[-1, 1)$. The radius of convergence is $R = 1$.

Exercise 2.1.2

$$S_N(x) = 1 + x^2 + ... + x^{2N} = \frac{1 - x^{2(N+1)}}{1 - x^2}$$

Exercise 2.1.2

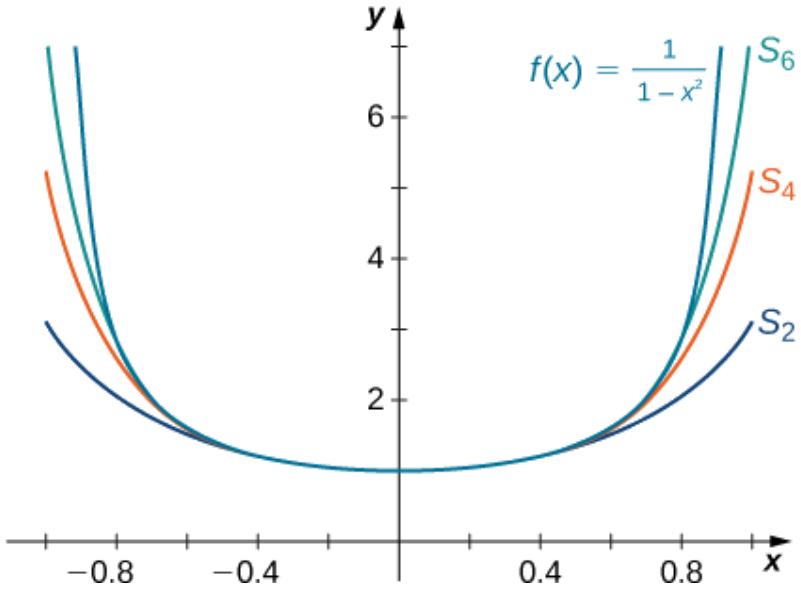


Figure 2.1.3: The graph shows a function and three approximations of it by partial sums of $f(x) = \frac{1}{1 - x^2}$ power series with three approximating curves labeled S_2, S_4, and S_6. The graph is plotted on a coordinate plane with the x-axis ranging from -1 to 1 and the y-axis ranging from 0 to 7. The function $f(x)$ and the curves are upward-opening, getting steeper closer to the vertical asymptotes as x approaches -1 or 1. The graph highlights how the series approximations S_N become closer to the actual function $f(x)$ as the value of n increases.

Exercise 2.1.3

Rewrite f in the form $f(x) = \frac{g(x)}{1 - h(x)}$ for some functions g and h.

Exercise 2.1.3

$\sum_{n=0}^{\infty} \frac{x^{n+3}}{2^{n+1}}$ with interval of convergence $(-2, 2)$

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2.1E: Exercises for Section 2.1

Exercises 1 - 4

State whether each statement is true, or give an example to show that it is false.

1. If $\sum_{n=1}^{\infty} a_n x^n$ converges, then $a_n x^n \to 0$ as $n \to \infty$.

   Answer

   True. If a series converges then its terms tend to zero.

2. $\sum_{n=1}^{\infty} a_n x^n$ converges at $x = 0$ for any real numbers a_n.
3. Given any sequence a_n, there is always some $R > 0$, possibly very small, such that $\sum_{n=1}^{\infty} a_n x^n$ converges on $(-R, R)$.

   Answer

   False. It would imply that $a_n x^n \to 0$ for \(|x|

4. If $\sum_{n=1}^{\infty} a_n x^n$ has radius of convergence $R > 0$ and if $|b_n| \leq |a_n|$ for all n, then the radius of convergence of $\sum_{n=1}^{\infty} b_n x^n$ is greater than or equal to R.
5. Suppose that $\sum_{n=0}^{\infty} a_n (x-3)^n$ converges at $x = 6$. At which of the following points must the series also converge? Use the fact that if $\sum a_n (x-c)^n$ converges at x, then it converges at any point closer to c than x.
   a. $x = 1$
   b. $x = 2$
   c. $x = 3$
   d. $x = 0$
   e. $x = 5.99$
   f. $x = 0.000001$

   Answer

   It must converge on $(0, 6]$ and hence at: a. $x = 1$; b. $x = 2$; c. $x = 3$; d. $x = 0$; e. $x = 5.99$; and f. $x = 0.000001$.

6. Suppose that $\sum_{n=0}^{\infty} a_n (x+1)^n$ converges at $x = -2$. At which of the following points must the series also converge? Use the fact that if $\sum a_n (x-c)^n$ converges at x, then it converges at any point closer to c than x.
   a. $x = 2$
   b. $x = -1$
   c. $x = -3$
   d. $x = 0$
   e. $x = 0.99$
   f. $x = 0.000001$

Exercises 7 - 12

In the following exercises, suppose that $\left|\frac{a_{n+1}}{a_n}\right| \to 1$ as $n \to \infty$. Find the radius of convergence for each series.

7. $\sum_{n=0}^{\infty} a_n 2^n x^n$

Answer

$$\left|\frac{a_{n+1}2^{n+1}x^{n+1}}{a_n 2^n x^n}\right| = 2|x|\left|\frac{a_{n+1}}{a_n}\right| \to 2|x| \text{ so } R = \tfrac{1}{2}$$

8. $\sum_{n=0}^{\infty} \frac{a_n x^n}{2^n}$
9. $\sum_{n=0}^{\infty} \frac{a_n \pi^n x^n}{e^n}$

Answer

$$\left|\frac{a_{n+1}(\frac{\pi}{e})^{n+1}x^{n+1}}{a_n(\frac{\pi}{e})^n x^n}\right| = \frac{\pi|x|}{e}\left|\frac{a_{n+1}}{a_n}\right| \to \frac{\pi|x|}{e} \text{ so } R = \tfrac{e}{\pi}$$

10. $\sum_{n=0}^{\infty} \frac{a_n(-1)^n x^n}{10^n}$
11. $\sum_{n=0}^{\infty} a_n(-1)^n x^{2n}$

Answer

$$\left|\frac{a_{n+1}(-1)^{n+1}x^{2n+2}}{a_n(-1)^n x^{2n}}\right| = |x^2|\left|\frac{a_{n+1}}{a_n}\right| \to |x^2| \text{ so } R = 1$$

12. $\sum_{n=0}^{\infty} a_n(-4)^n x^{2n}$

Exercises 13 - 22

Find the radius of convergence R and interval of convergence for $\sum a_n x^n$ with the given coefficients a_n.

13. $\sum_{n=1}^{\infty} \frac{(2x)^n}{n}$

Answer

$a_n = \frac{2^n}{n}$ so $\frac{a_{n+1}x}{a_n} \to 2x$. so $R = \frac{1}{2}$. When $x = \frac{1}{2}$ the series is harmonic and diverges. When $x = -\frac{1}{2}$ the series is alternating harmonic and converges. The interval of convergence is $I = \left[-\frac{1}{2}, \frac{1}{2}\right)$.

14. $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{\sqrt{n}}$
15. $\sum_{n=1}^{\infty} \frac{nx^n}{2^n}$

Answer

$a_n = \frac{n}{2^n}$ so $\frac{a_{n+1}x}{a_n} \to \frac{x}{2}$ so $R = 2$. When $x = \pm 2$ the series diverges by the divergence test. The interval of convergence is $I = (-2, 2)$.

16. $\sum_{n=1}^{\infty} \frac{nx^n}{e^n}$
17. $\sum_{n=1}^{\infty} \frac{n^2 x^n}{2^n}$

Answer

$a_n = \frac{n^2}{2^n}$ so $R = 2$. When $x = \pm 2$ the series diverges by the divergence test. The interval of convergence is $I = (-2, 2)$.

18. $\sum_{k=1}^{\infty} \frac{k^e x^k}{e^k}$
19. $\sum_{k=1}^{\infty} \frac{\pi^k x^k}{k^\pi}$

Answer

$a_k = \frac{\pi^k}{k^\pi}$ so $R = \frac{1}{\pi}$. When $x = \pm\frac{1}{\pi}$ the series is an absolutely convergent p-series. The interval of convergence is $I = \left[-\frac{1}{\pi}, \frac{1}{\pi}\right]$.

20. $\sum_{n=1}^{\infty} \frac{x^n}{n!}$
21. $\sum_{n=1}^{\infty} \frac{10^n x^n}{n!}$

Answer

$a_n = \frac{10^n}{n!}, \frac{a_{n+1}x}{a_n} = \frac{10x}{n+1} \to 0 < 1$ so the series converges for all x by the ratio test and $I = (-\infty, \infty)$.

22. $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{\ln(2n)}$

Exercises 23 - 28

Find the radius of convergence of each series.

23. $\sum_{k=1}^{\infty} \frac{(k!)^2 x^k}{(2k)!}$

Answer

$a_k = \frac{(k!)^2}{(2k)!}$ so $\frac{a_{k+1}}{a_k} = \frac{(k+1)^2}{(2k+2)(2k+1)} \to \frac{1}{4}$ so $R = 4$

24. $\sum_{n=1}^{\infty} \frac{(2n)! x^n}{n^{2n}}$
25. $\sum_{k=1}^{\infty} \frac{k!}{1 \cdot 3 \cdot 5 \cdots (2k-1)} x^k$

Answer

$a_k = \frac{k!}{1 \cdot 3 \cdot 5 \cdots (2k-1)}$ so $\frac{a_{k+1}}{a_k} = \frac{k+1}{2k+1} \to \frac{1}{2}$ so $R = 2$

26. $\sum_{k=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \cdots 2k}{(2k)!} x^k$
27. $\sum_{n=1}^{\infty} \frac{x^n}{\binom{2n}{n}}$ where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Answer

$a_n = \frac{1}{\binom{2n}{n}}$ so $\frac{a_{n+1}}{a_n} = \frac{\left((n+1)!\right)^2}{(2n+2)!} \frac{2n!}{(n!)^2} = \frac{(n+1)^2}{(2n+2)(2n+1)} \to \frac{1}{4}$ so $R = 4$

28. $\sum_{n=1}^{\infty} \sin^2 n x^n$

Exercises 29 - 32

Use the ratio test to determine the radius of convergence of each series.

29. $\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!} x^n$

Answer

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^3}{(3n+3)(3n+2)(3n+1)} \to \frac{1}{27} \text{ so } R = 27$$

30. $\sum_{n=1}^{\infty} \frac{2^{3n}(n!)^3}{(3n)!} x^n$

31. $\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$

Answer

$$a_n = \frac{n!}{n^n} \text{ so } \frac{a_{n+1}}{a_n} = \frac{(n+1)!}{n!} \frac{n^n}{(n+1)^{n+1}} = (\frac{n}{n+1})^n \to \frac{1}{e} \text{ so } R = e$$

32. $\sum_{n=1}^{\infty} \frac{(2n)!}{n^{2n}} x^n$

Exercises 33 - 42

In the following exercises, given that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ with convergence in $(-1, 1)$, find the power series for each function with the given center a, and identify its interval of convergence.

33. $f(x) = \frac{1}{x}; a = 1$ (Hint: $\frac{1}{x} = \frac{1}{1-(1-x)}$)

Answer

$$f(x) = \sum_{n=0}^{\infty} (1-x)^n \text{ on } I = (0, 2)$$

34. $f(x) = \frac{1}{1-x^2}; a = 0$

35. $f(x) = \frac{x}{1-x^2}; a = 0$

Answer

$$\sum_{n=0}^{\infty} x^{2n+1} \text{ on } I = (-1, 1)$$

36. $f(x) = \frac{1}{1+x^2}; a = 0$

37. $f(x) = \frac{x^2}{1+x^2}; a = 0$

Answer

$$\sum_{n=0}^{\infty} (-1)^n x^{2n+2} \text{ on } I = (-1, 1)$$

38. $f(x) = \frac{1}{2-x}; a = 1$

39. $f(x) = \frac{1}{1-2x}; a = 0.$

Answer

$$\sum_{n=0}^{\infty} 2^n x^n \text{ on } \left(-\tfrac{1}{2}, \tfrac{1}{2}\right)$$

40. $f(x) = \dfrac{1}{1-4x^2}; a = 0$

41. $f(x) = \dfrac{x^2}{1-4x^2}; a = 0$

Answer

$$\sum_{n=0}^{\infty} 4^n x^{2n+2} \text{ on } \left(-\tfrac{1}{2}, \tfrac{1}{2}\right)$$

42. $f(x) = \dfrac{x^2}{5-4x+x^2}; a = 2$

Exercises 43 - 51

Use the result of exercise 43 to find the radius of convergence of the given series in the subsequent exercises.

43. Explain why, if $|a_n|^{1/n} \to r > 0$, then $|a_n x^n|^{1/n} \to |x| r < 1$ whenever $|x| < \frac{1}{r}$ and, therefore, the radius of convergence of $\sum_{n=1}^{\infty} a_n x^n$ is $R = \frac{1}{r}$.

Answer

$|a_n x^n|^{1/n} = |a_n|^{1/n}|x| \to |x| r$ as $n \to \infty$ and $|x| r < 1$ when $|x| < \frac{1}{r}$. Therefore, $\sum_{n=1}^{\infty} a_n x^n$ converges when $|x| < \frac{1}{r}$ by the nth root test.

44. $\sum_{n=1}^{\infty} \dfrac{x^n}{n^n}$

45. $\sum_{k=1}^{\infty} \left(\dfrac{k-1}{2k+3}\right)^k x^k$

Answer

$a_k = \left(\dfrac{k-1}{2k+3}\right)^k$ so $(a_k)^{1/k} \to \frac{1}{2} < 1$ so $R = 2$

46. $\sum_{k=1}^{\infty} (\dfrac{2k^2-1}{k^2+3})^k x^k$

47. $\sum_{n=1}^{\infty} a_n = (n^{1/n} - 1)^n x^n$

Answer

$a_n = (n^{1/n} - 1)^n$ so $(a_n)^{1/n} \to 0$ so $R = \infty$

48. Suppose that $p(x) = \sum_{n=0}^{\infty} a_n x^n$ such that $a_n = 0$ if n is even. Explain why $p(x) = p(-x)$.

49. Suppose that $p(x) = \sum_{n=0}^{\infty} a_n x^n$ such that $a_n = 0$ if n is odd. Explain why $p(x) = -p(-x)$.

Answer

We can rewrite $p(x) = \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1}$ and $p(x) = p(-x)$ since $x^{2n+1} = -(-x)^{2n+1}$.

50. Suppose that $p(x) = \sum_{n=0}^{\infty} a_n x^n$ converges on $(-1, 1]$. Find the interval of convergence of $p(Ax)$.

51. Suppose that $p(x) = \sum_{n=0}^{\infty} a_n x^n$ converges on $(-1, 1]$. Find the interval of convergence of $p(2x-1)$.

Answer

If $x \in [0, 1]$, then $y = 2x - 1 \in [-1, 1]$ so $p(2x-1) = p(y) = \sum_{n=0}^{\infty} a_n y^n$ converges.

Exercises 52 - 62

In the following exercises, suppose that $p(x) = \sum_{n=0}^{\infty} a_n x^n$ satisfies $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ where $a_n \geq 0$ for each n. State whether each series converges on the full interval $(-1, 1)$, or if there is not enough information to draw a conclusion. Use the comparison test when appropriate.

52. $\sum_{n=0}^{\infty} a_n x^{2n}$
53. $\sum_{n=0}^{\infty} a_{2n} x^{2n}$

Answer

Converges on $(-1, 1)$ by the ratio test

54. $\sum_{n=0}^{\infty} a_{2n} x^n$ (Hint:$x = \pm\sqrt{x^2}$)
55. $\sum_{n=0}^{\infty} a_{n^2} x^{n^2}$ (Hint: Let $b_k = a_k$ if $k = n^2$ for some n, otherwise $b_k = 0$.)

Answer

Consider the series $\sum b_k x^k$ where $b_k = a_k$ if $k = n^2$ and $b_k = 0$ otherwise. Then $b_k \leq a_k$ and so the series converges on $(-1, 1)$ by the comparison test.

56. Suppose that $p(x)$ is a polynomial of degree N. Find the radius and interval of convergence of $\sum_{n=1}^{\infty} p(n) x^n$.
57. [T] Plot the graphs of $\frac{1}{1-x}$ and of the partial sums $S_N = \sum_{n=0}^{N} x^n$ for $n = 10, 20, 30$ on the interval $[-0.99, 0.99]$. Comment on the approximation of $\frac{1}{1-x}$ by S_N near $x = -1$ and near $x = 1$ as N increases.

Answer

The approximation is more accurate near $x = -1$. The partial sums follow $\frac{1}{1-x}$ more closely as N increases but are never accurate near $x = 1$ since the series diverges there.

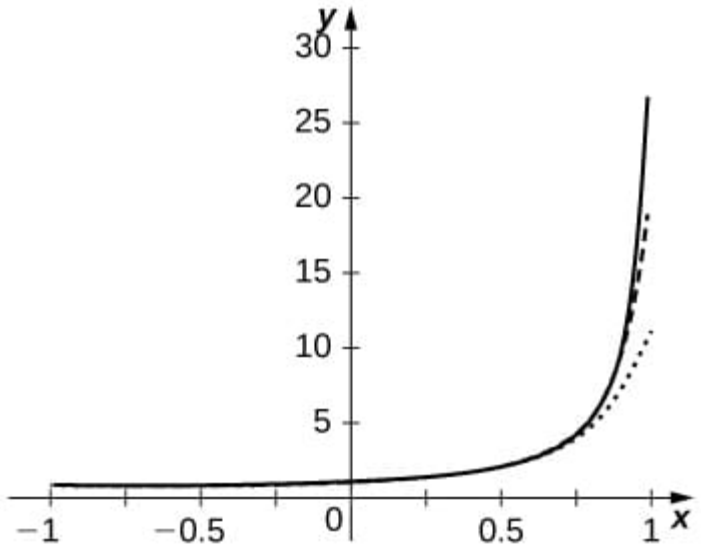


Figure $2.1E.1$: A graph showing a function with a rapid exponential increase as \(x\) approaches 1 from the left. The y-axis ranges from 0 to 30, while the $x-axis$ spans from -1 to 1. Two curves are displayed: a solid line representing the primary function and a dotted line slightly diverging as xx approaches 1, possibly representing an approximation or a related function.

58. [T] Plot the graphs of $-\ln(1-x)$ and of the partial sums $S_N = \sum_{n=1}^{N} \frac{x^n}{n}$ for $n = 10, 50, 100$ on the interval $[-0.99, 0.99]$. Comment on the behavior of the sums near $x = -1$ and near $x = 1$ as N increases.

59. [T] Plot the graphs of the partial sums $S_n = \sum_{n=1}^{N} \frac{x^n}{n^2}$ for $n = 10, 50, 100$ on the interval $[-0.99, 0.99]$. Comment on the behavior of the sums near $x = -1$ and near $x = 1$ as N increases.

Answer

The approximation appears to stabilize quickly near both $x = \pm 1$.

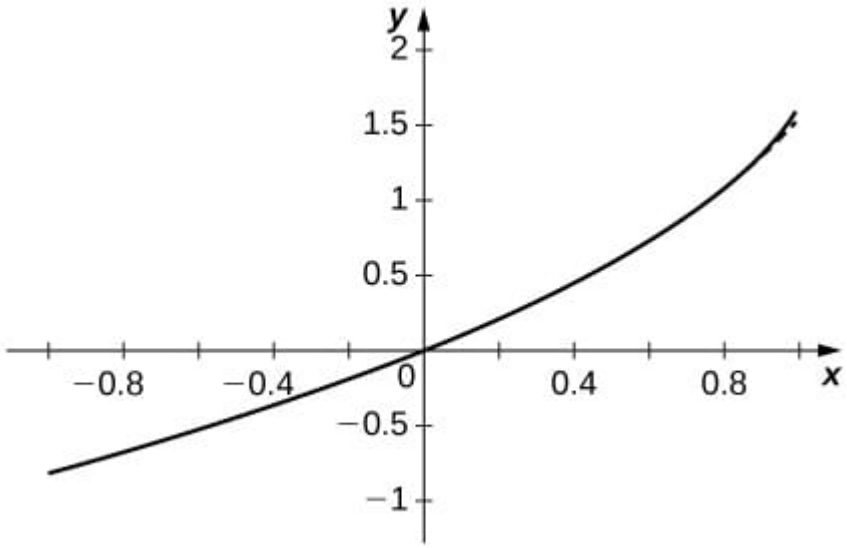


Figure $2.1E.2$: A graph of a function that curves upward, starting below the origin and crossing the $y-axis$ at 0. The $x-axis$ ranges from -1 to 1, and the $y-axis$ ranges from -1 to 2. The curve exhibits a gradual slope for negative $xvalues$ and increases more steeply for positive x values, showing an exponential growth behavior as x approaches 1.

60. [T] Plot the graphs of the partial sums $S_N = \sum_{n=1}^{N} (\sin n) x^n$ for $n = 10, 50, 100$ on the interval $[-0.99, 0.99]$. Comment on the

behavior of the sums near $x = -1$ and near $x = 1$ as N increases.

61. [T] Plot the graphs of the partial sums $S_N = \sum_{n=0}^{N} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for $n = 3, 5, 10$ on the interval $[-2\pi, 2\pi]$. Comment on how these plots approximate $\sin x$ as N increases.

Answer

The polynomial curves have roots close to those of $\sin x$ up to their degree and then the polynomials diverge from $\sin x$.

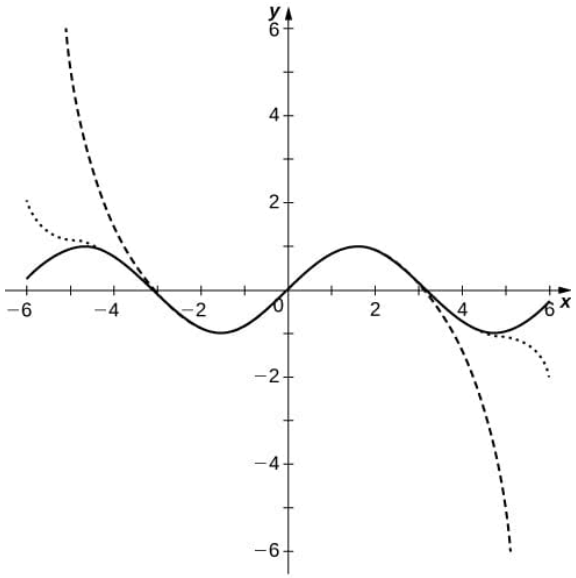


Figure $2.1E.3$:This figure is the graph of the partial sums of $(-1)^n$ times x^{2n+1} divided by $(2n+1)!$ For $n = 3, 5, 10$. The curves approximate the sine curve close to the origin and then separate as the curves move away from the origin.

62. [T] Plot the graphs of the partial sums $S_N = \sum_{n=0}^{N} (-1)^n \frac{x^{2n}}{(2n)!}$ for $n = 3, 5, 10$ on the interval $[-2\pi, 2\pi]$. Comment on how these plots approximate $\cos x$ as N increases.

Contributors

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2.2: Properties of Power Series

Learning Objectives

- Combine power series by addition or subtraction.
- Create a new power series by multiplication by a power of the variable or a constant, or by substitution.
- Multiply two power series together.
- Differentiate and integrate power series term-by-term.

In the preceding section on power series and functions we showed how to represent certain functions using power series. In this section we discuss how power series can be combined, differentiated, or integrated to create new power series. This capability is particularly useful for a couple of reasons. First, it allows us to find power series representations for certain elementary functions, by writing those functions in terms of functions with known power series. For example, given the power series representation for $f(x) = \dfrac{1}{1-x}$, we can find a power series representation for $f\,'(x) = \dfrac{1}{(1-x)^2}$. Second, being able to create power series allows us to define new functions that cannot be written in terms of elementary functions. This capability is particularly useful for solving differential equations for which there is no solution in terms of elementary functions.

Combining Power Series

If we have two power series with the same interval of convergence, we can add or subtract the two series to create a new power series, also with the same interval of convergence. Similarly, we can multiply a power series by a power of x or evaluate a power series at x^m for a positive integer m to create a new power series. Being able to do this allows us to find power series representations for certain functions by using power series representations of other functions. For example, since we know the power series representation for $f(x) = \frac{1}{1-x}$, we can find power series representations for related functions, such as

$$y = \frac{3x}{1-x^2}$$

and

$$y = \frac{1}{(x-1)(x-3)}.$$

In Theorem 2.2.1, we state results regarding addition or subtraction of power series, composition of a power series, and multiplication of a power series by a power of the variable. For simplicity, we state the theorem for power series centered at $x = 0$. Similar results hold for power series centered at $x = a$.

Theorem 2.2.1: Combining Power Series

Suppose that the two power series $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$ converge to the functions f and g, respectively, on a common interval I.

i. The power series $\sum_{n=0}^{\infty} (c_n x^n \pm d_n x^n)$ converges to $f \pm g$ on I.

ii. For any integer $m \geq 0$ and any real number b, the power series $\sum_{n=0}^{\infty} b x^m c_n x^n$ converges to $b x^m f(x)$ on I.

iii. For any integer $m \geq 0$ and any real number b, the series $\sum_{n=0}^{\infty} c_n (b x^m)^n$ converges to $f(bx^m)$ for all x such that bx^m is in I.

Proof

We prove $i.$ in the case of the series $\sum_{n=0}^{\infty}(c_nx^n + d_nx^n)$. Suppose that $\sum_{n=0}^{\infty}c_nx^n$ and $\sum_{n=0}^{\infty}d_nx^n$ converge to the functions f and g, respectively, on the interval I. Let x be a point in I and let $S_N(x)$ and $T_N(x)$ denote the Nth partial sums of the series $\sum_{n=0}^{\infty}c_nx^n$ and $\sum_{n=0}^{\infty}d_nx^n$, respectively. Then the sequence $S_N(x)$ converges to $f(x)$ and the sequence $T_N(x)$ converges to $g(x)$. Furthermore, the Nth partial sum of $\sum_{n=0}^{\infty}(c_nx^n + d_nx^n)$ is

$$\begin{aligned}\sum_{n=0}^{N}(c_nx^n + d_nx^n) &= \sum_{n=0}^{N}c_nx^n + \sum_{n=0}^{N}d_nx^n \\ &= S_N(x) + T_N(x).\end{aligned}$$

Because

$$\begin{aligned}\lim_{N\to\infty}(S_N(x) + T_N(x)) &= \lim_{N\to\infty}S_N(x) + \lim_{N\to\infty}T_N(x) \\ &= f(x) + g(x),\end{aligned}$$

we conclude that the series $\sum_{n=0}^{\infty}(c_nx^n + d_nx^n)$ converges to $f(x) + g(x)$.

□

We examine products of power series in a later theorem. First, we show several applications of Theorem $2.2.1$ and how to find the interval of convergence of a power series given the interval of convergence of a related power series.

Example $2.2.1$: Combining Power Series

Suppose that $\sum_{n=0}^{\infty}a_nx^n$ is a power series whose interval of convergence is $(-1, 1)$, and suppose that $\sum_{n=0}^{\infty}b_nx^n$ is a power series whose interval of convergence is $(-2, 2)$.

a. Find the interval of convergence of the series $\sum_{n=0}^{\infty}(a_nx^n + b_nx^n)$.

b. Find the interval of convergence of the series $\sum_{n=0}^{\infty}a_n3^nx^n$.

Solution

a. Since the interval $(-1, 1)$ is a common interval of convergence of the series $\sum_{n=0}^{\infty}a_nx^n$ and $\sum_{n=0}^{\infty}b_nx^n$, the interval of convergence of the series $\sum_{n=0}^{\infty}(a_nx^n + b_nx^n)$ is $(-1, 1)$.

b. Since $\sum_{n=0}^{\infty}a_nx^n$ is a power series centered at zero with radius of convergence 1, it converges for all x in the interval $(-1, 1)$. By Theorem $2.2.1$, the series

$$\sum_{n=0}^{\infty}a_n3^nx^n = \sum_{n=0}^{\infty}a_n(3x)^n$$

converges if $3x$ is in the interval $(-1, 1)$. Therefore, the series converges for all x in the interval $\left(-\frac{1}{3}, \frac{1}{3}\right)$.

Exercise 2.2.1

Suppose that $\sum_{n=0}^{\infty} a_n x^n$ has an interval of convergence of $(-1,1)$. Find the interval of convergence of $\sum_{n=0}^{\infty} a_n \left(\frac{x}{2}\right)^n$.

Hint available at the end of this section.

Answer available at the end of this section.

In the next example, we show how to use Theorem 2.2.1 and the power series for a function f to construct power series for functions related to f. Specifically, we consider functions related to the function $f(x) = \frac{1}{1-x}$ and we use the fact that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots$$

for $|x| < 1$.

Example 2.2.2: Constructing Power Series from Known Power Series

Use the power series representation for $f(x) = \frac{1}{1-x}$ combined with Theorem 2.2.1 to construct a power series for each of the functions. Find the interval of convergence of the power series.

a. $f(x) = \frac{3x}{1+x^2}$

b. $f(x) = \frac{1}{(x-1)(x-3)}$

Solution

a. First write $f(x)$ as

$$f(x) = 3x\left(\frac{1}{1-(-x^2)}\right).$$

Using the power series representation for $f(x) = \frac{1}{1-x}$ and parts ii. and iii. of Theorem 2.2.1, we find that a power series representation for f is given by

$$\sum_{n=0}^{\infty} 3x(-x^2)^n = \sum_{n=0}^{\infty} 3(-1)^n x^{2n+1}.$$

Since the interval of convergence of the series for $\frac{1}{1-x}$ is $(-1,1)$, the interval of convergence for this new series is the set of real numbers x such that $|\ x^2\ | < 1$. Therefore, the interval of convergence is $(-1,1)$.

b. To find the power series representation, use partial fractions to write $f(x) = \frac{1}{(x-1)(x-3)}$ as the sum of two fractions. We have

$$\frac{1}{(x-1)(x-3)} = \frac{-1/2}{x-1} + \frac{1/2}{x-3} = \frac{1/2}{1-x} - \frac{1/2}{3-x} = \frac{1/2}{1-x} - \frac{1/6}{1-\frac{x}{3}}.$$

First, using part ii. of Theorem 2.2.1, we obtain

$$\frac{1/2}{1-x} = \sum_{n=0}^{\infty} \frac{1}{2} x^n \quad \text{for } |x| < 1.$$

Then, using parts ii. and iii. of Theorem 2.2.1, we have

$$\frac{1/6}{1-x/3} = \sum_{n=0}^{\infty} \frac{1}{6}\left(\frac{x}{3}\right)^n \quad \text{for } |x| < 3.$$

Since we are combining these two power series, the interval of convergence of the difference must be the smaller of these two intervals. Using this fact and part i. of Theorem 2.2.1, we have

$$\frac{1}{(x-1)(x-3)} = \sum_{n=0}^{\infty} \left(\frac{1}{2} - \frac{1}{6 \cdot 3^n}\right) x^n$$

where the interval of convergence is $(-1, 1)$.

Exercise 2.2.2

Use the series for $f(x) = \dfrac{1}{1-x}$ on $|x| < 1$ to construct a series for $\dfrac{1}{(1-x)(x-2)}$. Determine the interval of convergence.

Hint available at the end of this section.

Answer available at the end of this section.

In Example 2.2.2, we showed how to find power series for certain functions. In Example 2.2.3 we show how to do the opposite: given a power series, determine which function it represents.

Example 2.2.3: Finding the Function Represented by a Given Power Series

Consider the power series $\sum_{n=0}^{\infty} 2^n x^n$. Find the function f represented by this series. Determine the interval of convergence of the series.

Solution

Writing the given series as

$$\sum_{n=0}^{\infty} 2^n x^n = \sum_{n=0}^{\infty} (2x)^n,$$

we can recognize this series as the power series for

$$f(x) = \frac{1}{1-2x}.$$

Since this is a geometric series, the series converges if and only if $|2x| < 1$. Therefore, the interval of convergence is $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

Exercise 2.2.3

Find the function represented by the power series $\sum_{n=0}^{\infty} \frac{1}{3^n} x^n$.

Determine its interval of convergence.

Hint available at the end of this section.

Answer available at the end of this section.

Recall the questions posed in the chapter opener about which is the better way of receiving payouts from lottery winnings. We now revisit those questions and show how to use series to compare values of payments over time with a lump sum payment today. We will compute how much future payments are worth in terms of today's dollars, assuming we have the ability to invest winnings and earn interest. The value of future payments in terms of today's dollars is known as the *present value* of those payments.

Example 2.2.4: Present Value of Future Winnings

Suppose you win the lottery and are given three options:

- Receive 20 million dollars today;
- Receive 1.5 million dollars per year over the next 20 years; or
- Receive 1 million dollars per year indefinitely (being passed on to your heirs).

Which is the best deal, assuming that the annual interest rate is 5%? We answer this by working through sequence of questions.

a. How much is the 1.5 million dollars received annually over the course of 20 years worth in terms of today's dollars, assuming an annual interest rate of 5%?
b. Use the answer to part a. to find a general formula for the **present value** of payments of C dollars received each year over the next n years, assuming an average annual interest rate r.
c. Find a formula for the present value if annual payments of C dollars continue indefinitely, assuming an average annual interest rate r.
d. Use the answer to part c. to determine the present value of 1 million dollars paid annually indefinitely.
e. Use your answers to parts a. and d. to determine which of the three options is best.



Figure 2.2.1: An image showing multiple U.S. $100 bills arranged in a grid-like pattern. The bills feature the portrait of Benjamin Franklin and other recognizable details, such as the green treasury seal and serial numbers (Credit:CC-BY-NC-SA,work by Ervins Strauhmanis's,www.flickr.com)

Solution

a. Consider the payment of 1.5 million dollars made at the end of the first year. If you were able to receive that payment today instead of one year from now, you could invest that money and earn 5% interest. Therefore, the present value of that money P_1 satisfies $P_1(1+0.05) = 1.5$ **million dollars.** We conclude that

$$P_1 = \frac{1.5}{1.05} = 1.429 \text{ million dollars.}$$

Similarly, consider the payment of 1.5 million dollars made at the end of the second year. If you were able to receive that payment today, you could invest that money for two years, earning 5% interest, compounded annually. Therefore, the present value of that money P_2 satisfies $P_2(1+0.05)^2 = 1.5$ **million dollars**. We conclude that

$$P_2 = 1.5(1.05)^2 = 1.361 \textbf{ million dollars.}$$

The value of the future payments today is the sum of the present values $P_1, P_2, \ldots, P_{20}$ of each of those annual payments. The present value P_k satisfies

$$P_k = \frac{1.5}{(1.05)^k}.$$

Therefore,

$$P = \frac{1.5}{1.05} + \frac{1.5}{(1.05)^2} + \ldots + \frac{1.5}{(1.05)^{20}} = 18.693 \textbf{ million dollars.}$$

b. Using the result from part a. we see that the present value P of C dollars paid annually over the course of n years, assuming an annual interest rate r, is given by

$$P = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \ldots + \frac{C}{(1+r)^n} \textbf{ dollars.}$$

c. Using the result from part b. we see that the present value of an annuity that continues indefinitely is given by the infinite series

$$P = \sum_{n=0}^{\infty} \frac{C}{(1+r)^{n+1}}.$$

We can view the present value as a power series in r, which converges as long as $\left|\frac{1}{1+r}\right| < 1$. Since $r > 0$, this series converges. Rewriting the series as

$$P = \frac{C}{(1+r)} \sum_{n=0}^{\infty} \left(\frac{1}{1+r}\right)^n,$$

we recognize this series as the power series for

$$f(r) = \frac{1}{1-\left(\frac{1}{1+r}\right)} = \frac{1}{\left(\frac{r}{1+r}\right)} = \frac{1+r}{r}.$$

We conclude that the present value of this annuity is

$$P = \frac{C}{1+r} \cdot \frac{1+r}{r} = \frac{C}{r}.$$

d. From the result to part c. we conclude that the present value P of $C = 1$ **million dollars** paid out every year indefinitely, assuming an annual interest rate $r = 0.05$, is given by

$$P = \frac{1}{0.05} = 20 \textbf{ million dollars.}$$

e. From part a. we see that receiving \$1.5 million dollars over the course of 20 years is worth \$18.693 million dollars in today's dollars. From part d. we see that receiving \$1 million dollars per year indefinitely is worth \$20 million dollars in today's dollars. Therefore, either receiving a lump-sum payment of \$20 million dollars today or receiving \$1 million dollars indefinitely have the same present value.

Multiplication of Power Series

We can also create new power series by multiplying power series. Being able to multiply two power series provides another way of finding power series representations for functions. The way we multiply them is similar to how we multiply polynomials. For example, suppose we want to multiply

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots$$

and

$$\sum_{n=0}^{\infty} d_n x^n = d_0 + d_1 x + d_2 x^2 + \ldots .$$

It appears that the product should satisfy

$$\left(\sum_{n=0}^{\infty} c_n x^n\right)\left(\sum_{n=-0}^{\infty} d_n x^n\right) = (c_0 + c_1 x + c_2 x^2 + \ldots)\cdot(d_0 + d_1 x + d_2 x^2 + \ldots) = c_0 d_0 + (c_1 d_0 + c_0 d_1)x + (c_2 d_0 + c_1 d_1 + c_0 d_2)x^2 + \ldots .$$

In Theorem 2.2.2, we state the main result regarding multiplying power series, showing that if $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$ converge on a common interval I, then we can multiply the series in this way, and the resulting series also converges on the interval I.

Theorem 2.2.2: Multiplying Power Series

Suppose that the power series $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$ converge to f and g, respectively, on a common interval I. Let

$$e_n = c_0 d_n + c_1 d_{n-1} + c_2 d_{n-2} + \ldots + c_{n-1} d_1 + c_n d_0 = \sum_{k=0}^{n} c_k d_{n-k}.$$

Then

$$\left(\sum_{n=0}^{\infty} c_n x^n\right)\left(\sum_{n=0}^{\infty} d_n x^n\right) = \sum_{n=0}^{\infty} e_n x^n$$

and

$$\sum_{n=0}^{\infty} e_n x^n \text{ converges to } f(x)\cdot g(x) \text{ on } I.$$

The series $\sum_{n=0}^{\infty} e_n x^n$ is known as the **Cauchy product** of the series $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$.

We omit the proof of this theorem, as it is beyond the level of this text and is typically covered in a more advanced course. We now provide an example of this theorem by finding the power series representation for

$$f(x) = \frac{1}{(1-x)(1-x^2)}$$

using the power series representations for

$$y = \frac{1}{1-x} \text{ and } y = \frac{1}{1-x^2}$$

.

Example 2.2.5: Multiplying Power Series

Multiply the power series representation

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots$$

for $|x| < 1$ with the power series representation

$$\frac{1}{1-x^2} = \sum_{n=0}^{\infty} \left(x^2\right)^n = 1 + x^2 + x^4 + x^6 + \ldots$$

for $|x| < 1$ to construct a power series for $f(x) = \dfrac{1}{(1-x)(1-x^2)}$ on the interval $(-1, 1)$.

Solution

We need to multiply

$$(1 + x + x^2 + x^3 + \ldots)(1 + x^2 + x^4 + x^6 + \ldots).$$

Writing out the first several terms, we see that the product is given by

$$(1 + x^2 + x^4 + x^6 + \ldots) + (x + x^3 + x^5 + x^7 + \ldots) + (x^2 + x^4 + x^6 + x^8 + \ldots) + (x^3 + x^5 + x^7 + x^9 + \ldots) = 1 + x + (1+1)x^2 + (1 +$$

Since the series for $y = \dfrac{1}{1-x}$ and $y = \dfrac{1}{1-x^2}$ both converge on the interval $(-1, 1)$, the series for the product also converges on the interval $(-1, 1)$.

Exercise 2.2.4

Multiply the series $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ by itself to construct a series for $\dfrac{1}{(1-x)(1-x)}$.

Hint available at the end of this section.

Answer available at the end of this section.

Differentiating and Integrating Power Series

Consider a power series $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots$ that converges on some interval I, and let f be the function defined by this series. Here we address two questions about f.

- Is f differentiable, and if so, how do we determine the derivative $f\,'$?
- How do we evaluate the indefinite integral $\int f(x)\ dx$?

We know that, for a polynomial with a finite number of terms, we can evaluate the derivative by differentiating each term separately. Similarly, we can evaluate the indefinite integral by integrating each term separately. Here we show that we can do the same thing for convergent power series. That is, if

$$f(x) = c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots$$

converges on some interval I, then

$$f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + \ldots$$

and

$$\int f(x)\,dx = C + c_0 x + c_1 \frac{x^2}{2} + c_2 \frac{x^3}{3} + \ldots.$$

Evaluating the derivative and indefinite integral in this way is called **term-by-term differentiation of a power series** and **term-by-term integration of a power series**, respectively. The ability to differentiate and integrate power series term-by-term also allows us to use known power series representations to find power series representations for other functions. For example, given the power series for $f(x) = \dfrac{1}{1-x}$, we can differentiate term-by-term to find the power series for $f\,'(x) = \dfrac{1}{(1-x)^2}$. Similarly, using the power series for $g(x) = \dfrac{1}{1+x}$, we can integrate term-by-term to find the power series for $G(x) = \ln(1+x)$, an antiderivative of g. We show how to do this in Example 2.2.6 and Example 2.2.7. First, we state Theorem 2.2.3, which provides the main result regarding differentiation and integration of power series.

Theorem 2.2.3: Term-by-Term Differentiation and Integration for Power Series

Suppose that the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges on the interval $(a-R, a+R)$ for some $R > 0$. Let f be the function defined by the series

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots$$

for \(|x–a|

$$f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1} = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \ldots$$

for \(|x–a|

$$\int f(x)\,dx = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \ldots$$

for \(|x–a|

The proof of this result is beyond the scope of the text and is omitted. Note that although Theorem 2.2.3 guarantees the same radius of convergence when a power series is differentiated or integrated term-by-term, it says nothing about what happens at the endpoints. It is possible that the differentiated and integrated power series have different behavior at the endpoints than does the original series. We see this behavior in the next examples.

Example 2.2.6: Differentiating Power Series

a. Use the power series representation

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots$$

for $|x| < 1$ to find a power series representation for

$$g(x) = \frac{1}{(1-x)^2}$$

on the interval $(-1, 1)$. Determine whether the resulting series converges at the endpoints.

b. Use the result of part a. to evaluate the sum of the series $\sum_{n=0}^{\infty} \dfrac{n+1}{4^n}$.

Solution

a. Since $g(x) = \dfrac{1}{(1-x)^2}$ is the derivative of $f(x) = \dfrac{1}{1-x}$, we can find a power series representation for g by differentiating the power series for f term-by-term. The result is

$$g(x)=\frac{1}{(1-x)^2}=\frac{d}{dx}\left(\frac{1}{1-x}\right)=\sum_{n=0}^{\infty}\frac{d}{dx}(x^n)=\frac{d}{dx}(1+x+x^2+x^3+...)=0+1+2x+3x^2+4x^3+...=\sum_{n=0}^{\infty}(n+1)x^n$$

for $|x|<1$.

Theorem 2.2.1 does not guarantee anything about the behavior of this series at the endpoints. Testing the endpoints by using the divergence test, we find that the series diverges at both endpoints $x=\pm 1$. Note that this is the same result found in Example 2.2.5.

b. From part a. we know that

$$\sum_{n=0}^{\infty}(n+1)x^n=\frac{1}{(1-x)^2}.$$

Therefore,

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{n+1}{4^n}&=\sum_{n=0}^{\infty}(n+1)\left(\frac{1}{4}\right)^n\\&=\frac{1}{\left(1-\frac{1}{4}\right)^2}\\&=\frac{1}{\left(\frac{3}{4}\right)^2}\\&=\frac{16}{9}\end{aligned}$$

Exercise 2.2.5

Differentiate the series $\frac{1}{(1-x)^2}=\sum_{n=0}^{\infty}(n+1)x^n$ term-by-term to find a power series representation for $\frac{2}{(1-x)^3}$ on the interval $(-1,1)$.

Hint available at the end of this section.

Answer available at the end of this section.

Example 2.2.7: Integrating Power Series

For each of the functions f, find a power series representation for f by integrating the power series for f' and find its interval of convergence.

a. $f(x)=\ln(1+x)$
b. $f(x)=\tan^{-1}x$

Solution:

a. For $f(x)=\ln(1+x)$,the derivative is $f'(x)=\frac{1}{1+x}$. We know that

$$\frac{1}{1+x}=\frac{1}{1-(-x)}=\sum_{n=0}^{\infty}(-x)^n=1-x+x^2-x^3+...$$

for $|x|<1$. To find a power series for $f(x)=\ln(1+x)$, we integrate the series term-by-term.

$$\int f'(x)\,dx = \int (1 - x + x^2 - x^3 + \ldots)\,dx = C + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

Since $f(x) = \ln(1+x)$ is an antiderivative of $\dfrac{1}{1+x}$, it remains to solve for the constant C. Since $\ln(1+0) = 0$, we have $C = 0$. Therefore, a power series representation for $f(x) = \ln(1+x)$ is

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{x^n}{n} \text{ for } |x| < 1.$$

Theorem 2.2.3 does not guarantee anything about the behavior of this power series at the endpoints. However, checking the endpoints, we find that at $x = 1$ the series is the alternating harmonic series, which converges. Also, at $x = -1$, the series is the harmonic series, which diverges. It is important to note that, even though this series converges at $x = 1$, Theorem 2.2.3 does not guarantee that the series actually converges to $\ln(2)$. In fact, the series does converge to $\ln(2)$, but showing this fact requires more advanced techniques. (Abel's theorem, covered in more advanced texts, deals with this more technical point.) The interval of convergence is $(-1, 1]$.

b. The derivative of $f(x) = \tan^{-1} x$ is $f\,'(x) = \dfrac{1}{1+x^2}$. We know that

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = 1 - x^2 + x^4 - x^6 + \ldots$$

for $|x| < 1$. To find a power series for $f(x) = \tan^{-1} x$, we integrate this series term-by-term.

$$\int f'(x)\,dx = \int (1 - x^2 + x^4 - x^6 + \ldots)\,dx = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

Since $\tan^{-1}(0) = 0$, we have $C = 0$. Therefore, a power series representation for $f(x) = \tan^{-1} x$ is

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

for $|x| < 1$. Again, Theorem 2.2.3 does not guarantee anything about the convergence of this series at the endpoints. However, checking the endpoints and using the alternating series test, we find that the series converges at $x = 1$ and $x = -1$. As discussed in part a., using Abel's theorem, it can be shown that the series actually converges to $\tan^{-1}(1)$ and $\tan^{-1}(-1)$ at $x = 1$ and $x = -1$, respectively. Thus, the interval of convergence is $[-1, 1]$.

Exercise 2.2.6

Integrate the power series $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{x^n}{n}$ term-by-term to evaluate $\int \ln(1+x)\ dx$.

Hint available at the end of this section.

Answer available at the end of this section.

Up to this point, we have shown several techniques for finding power series representations for functions. However, how do we know that these power series are unique? That is, given a function f and a power series for f at a, is it possible that there is a different power series for f at a that we could have found if we had used a different technique? The answer to this question is no. This fact should not seem surprising if we think of power series as polynomials with an infinite number of terms. Intuitively, if

$$c_0 + c_1 x + c_2 x^2 + \ldots = d_0 + d_1 x + d_2 x^2 + \ldots$$

for all values x in some open interval I about zero, then the coefficients c_n should equal d_n for $n \geq 0$. We now state this result formally.

Theorem 2.2.4: Uniqueness of Power Series

Let $\sum_{n=0}^{\infty} c_n(x-a)^n$ and $\sum_{n=0}^{\infty} d_n(x-a)^n$ be two convergent power series such that

$$\sum_{n=0}^{\infty} c_n(x-a)^n = \sum_{n=0}^{\infty} d_n(x-a)^n$$

for all x in an open interval containing a. Then $c_n = d_n$ for all $n \geq 0$.

Proof

Let

$$\begin{aligned} f(x) &= c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots \\ &= d_0 + d_1(x-a) + d_2(x-a)^2 + d_3(x-a)^3 + \ldots . \end{aligned}$$

Then $f(a) = c_0 = d_0$. By Theorem 2.2.3, we can differentiate both series term-by-term. Therefore,

$$\begin{aligned} f'(x) &= c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \ldots \\ &= d_1 + 2d_2(x-a) + 3d_3(x-a)^2 + \ldots , \end{aligned}$$

and thus, $f\,'(a) = c_1 = d_1$. Similarly,

$$\begin{aligned} f''(x) &= 2c_2 + 3 \cdot 2c_3(x-a) + \ldots \\ &= 2d_2 + 3 \cdot 2d_3(x-a) + \ldots \end{aligned}$$

implies that $f''(a) = 2c_2 = 2d_2$, and therefore, $c_2 = d_2$. More generally, for any integer $n \geq 0, f^{(n)}(a) = n!c_n = n!d_n$, and consequently, $c_n = d_n$ for all $n \geq 0$.

□

In this section we have shown how to find power series representations for certain functions using various algebraic operations, differentiation, or integration. At this point, however, we are still limited as to the functions for which we can find power series representations. Next, we show how to find power series representations for many more functions by introducing Taylor series.

Key Concepts

- Given two power series $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$ that converge to functions f and g on a common interval I, the sum and difference of the two series converge to $f \pm g$, respectively, on I. In addition, for any real number b and integer $m \geq 0$, the series $\sum_{n=0}^{\infty} bx^m c_n x^n$ converges to $bx^m f(x)$ and the series $\sum_{n=0}^{\infty} c_n(bx^m)^n$ converges to $f(bx^m)$ whenever bx^m is in the interval I.
- Given two power series that converge on an interval $(-R, R)$, the Cauchy product of the two power series converges on the interval $(-R, R)$.
- Given a power series that converges to a function f on an interval $(-R, R)$, the series can be differentiated term-by-term and the resulting series converges to $f\,'$ on $(-R, R)$. The series can also be integrated term-by-term and the resulting series converges to $\int f(x)\ dx$ on $(-R, R)$.

Glossary

Term-by-Term Differentiation of a Power Series

A technique for evaluating the derivative of a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ by evaluating the derivative of each term

separately to create the new power series $\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$.

Term-by-Term Integration of a Power Series

A technique for integrating a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ by integrating each term separately to create the new power series $C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$.

Answers

Exercise 2.2.1

Find the values of x such that $\frac{x}{2}$ is in the interval $(-1, 1)$.

Exercise 2.2.1

Interval of convergence is $(-2, 2)$.

Exercise 2.2.2

Use partial fractions to rewrite $\frac{1}{(1-x)(x-2)}$ as the difference of two fractions.

Exercise 2.2.2

$\sum_{n=0}^{\infty} \left(-1 + \frac{1}{2^{n+1}}\right) x^n$. The interval of convergence is $(-1, 1)$.

Exercise 2.2.3

Write $\frac{1}{3^n}x^n = \left(\frac{x}{3}\right)^n$.

Exercise 2.2.3

$f(x) = \frac{3}{3-x}$. The interval of convergence is $(-3, 3)$.

Exercise 2.2.4

Multiply the first few terms of $(1 + x + x^2 + x^3 + ...)(1 + x + x^2 + x^3 + ...)$

Exercise 2.2.4

$1 + 2x + 3x^2 + 4x^3 + ...$

Exercise 2.2.5

Write out the first several terms and apply the power rule.

Exercise 2.2.5

$\sum_{n=0}^{\infty} (n+2)(n+1)x^n$

Exercise 2.2.6

Use the fact that $\frac{x^{n+1}}{(n+1)n}$ is an antiderivative of $\frac{x^n}{n}$.

Exercise 2.2.6

$\sum_{n=2}^{\infty} \frac{(-1)^n x^n}{n(n-1)}$

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2.2E: Exercises for Section 2.2

Exercises 1 - 2

1. If $f(x)=\sum_{n=0}^{\infty}\frac{x^n}{n!}$ and $g(x)=\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n!}$, find the power series of $\frac{1}{2}\big(f(x)+g(x)\big)$ and of $\frac{1}{2}\big(f(x)-g(x)\big)$.

   Answer

   $$\frac{1}{2}\big(f(x)+g(x)\big)=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!} \text{ and } \frac{1}{2}\big(f(x)-g(x)\big)=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}.$$

2. If $C(x)=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$ and $S(x)=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$, find the power series of $C(x)+S(x)$ and of $C(x)-S(x)$.

Exercises 3 - 6

Use partial fractions to find the power series of each function.

3. $\dfrac{4}{(x-3)(x+1)}$

   Answer

   $$\frac{4}{(x-3)(x+1)} = \frac{1}{x-3} - \frac{1}{x+1} = -\frac{1}{3(1-\frac{x}{3})} - \frac{1}{1-(-x)} = -\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n - \sum_{n=0}^{\infty}(-1)^n x^n$$
   $$= \sum_{n=0}^{\infty}\left((-1)^{n+1}-\frac{1}{3n+1}\right)x^n$$

4. $\dfrac{3}{(x+2)(x-1)}$
5. $\dfrac{5}{(x^2+4)(x^2-1)}$

   Answer

   $$\frac{5}{(x^2+4)(x^2-1)} = \frac{1}{x^2-1} - \frac{1}{4}\frac{1}{1+\left(\frac{x}{2}\right)^2} = -\sum_{n=0}^{\infty}x^{2n} - \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\left((-1)+(-1)^{n+1}\frac{1}{2^{n+2}}\right)x^{2n}$$

6. $\dfrac{30}{(x^2+1)(x^2-9)}$

Exercises 7 - 10

Express each series as a rational function.

7. $\sum_{n=1}^{\infty}\frac{1}{x^n}$

   Answer

   $$\frac{1}{x}\sum_{n=0}^{\infty}\frac{1}{x^n}=\frac{1}{x}\cdot\frac{1}{1-\frac{1}{x}}=\frac{1}{x-1}$$

8. $\sum_{n=1}^{\infty}\frac{1}{x^{2n}}$

9. $\sum_{n=1}^{\infty} \frac{1}{(x-3)^{2n-1}}$

Answer

$$\frac{1}{x-3} \cdot \frac{1}{1-\frac{1}{(x-3)^2}} = \frac{x-3}{(x-3)^2-1}$$

10. $\sum_{n=1}^{\infty} \left(\frac{1}{(x-3)^{2n-1}} - \frac{1}{(x-2)^{2n-1}} \right)$

Exercises 11 - 16

Explore applications of annuities.

11. Calculate the present values P of an annuity in which \$10,000 is to be paid out annually for a period of 20 years, assuming interest rates of $r = 0.03,\ \ r = 0.05$, and $r = 0.07$.

Answer

$P = P_1 + \cdots + P_{20}$ where $P_k = 10,000\frac{1}{(1+r)^k}$. Then $P = 10,000\sum_{k=1}^{20} \frac{1}{(1+r)^k} = 10,000\frac{1-(1+r)^{-20}}{r}$. When $r = 0.03,\ \ P \approx 10,000 \times 14.8775 = 148,775$. When $r = 0.05,\ \ P \approx 10,000 \times 12.4622 = 124,622$. When $r = 0.07,\ \ P \approx 105,940$.

12. Calculate the present values P of annuities in which \$9,000 is to be paid out annually perpetually, assuming interest rates of $r = 0.03,\ \ r = 0.05$ and $r = 0.07$.
13. Calculate the annual payouts C to be given for 20 years on annuities having present value \$100,000 assuming respective interest rates of $r = 0.03,\ \ r = 0.05$, and $r = 0.07$.

Answer

In general, $P = \frac{C(1-(1+r)^{-N})}{r}$ for N years of payouts, or $C = \frac{Pr}{1-(1+r)^{-N}}$. For $N = 20$ and $P = 100,000$, one has $C = 6721.57$ when $r = 0.03;\ \ C = 8024.26$ when $r = 0.05$; and $C \approx 9439.29$ when $r = 0.07$.

14. Calculate the annual payouts C to be given perpetually on annuities having present value \$100,000 assuming respective interest rates of $r = 0.03,\ \ r = 0.05$, and $r = 0.07$.
15. Suppose that an annuity has a present value $P = 1$ million dollars. What interest rate r would allow for perpetual annual payouts of \$50,000?

Answer

In general, $P = \frac{C}{r}$. Thus, $r = \frac{C}{P} = 5 \times \frac{10^4}{10^6} = 0.05$.

16. Suppose that an annuity has a present value $P = 10$ million dollars. What interest rate r would allow for perpetual annual payouts of \$100,000?

Exercises 17 - 20

Express the sum of each power series in terms of geometric series, and then express the sum as a rational function.

17. $x + x^2 - x^3 + x^4 + x^5 - x^6 + \cdots$ (Hint: Group powers $x^{3k},\ \ x^{3k-1}$, and x^{3k-2}.)

Answer

$$(x + x^2 - x^3)(1 + x^3 + x^6 + \cdots) = \frac{x + x^2 - x^3}{1 - x^3}$$

18. $x + x^2 - x^3 - x^4 + x^5 + x^6 - x^7 - x^8 + \cdots$ (Hint: Group powers $x^{4k},\ \ x^{4k-1}$, etc.)
19. $x - x^2 - x^3 + x^4 - x^5 - x^6 + x^7 - \cdots$ (Hint: Group powers $x^{3k},\ \ x^{3k-1}$, and x^{3k-2}.)

Answer

$$(x - x^2 - x^3)(1 + x^3 + x^6 + \cdots) = \frac{x - x^2 - x^3}{1 - x^3}$$

20. $\frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \frac{x^4}{16} + \frac{x^5}{32} - \frac{x^6}{64} + \cdots$ (Hint: Group powers $\left(\frac{x}{2}\right)^{3k}$, $\left(\frac{x}{2}\right)^{3k-1}$, and $\left(\frac{x}{2}\right)^{3k-2}$.)

Exercises 21 - 24

Find the power series of $f(x)g(x)$ given f and g as defined.

21. $f(x) = 2\sum_{n=0}^{\infty} x^n, g(x) = \sum_{n=0}^{\infty} nx^n$

Answer

$a_n = 2,\ b_n = n$ so $c_n = \sum_{k=0}^{n} b_k a_{n-k} = 2\sum_{k=0}^{n} k = (n)(n+1)$ and $f(x)g(x) = \sum_{n=1}^{\infty} n(n+1)x^n$

22. $f(x) = \sum_{n=1}^{\infty} x^n,\ \ g(x) = \sum_{n=1}^{\infty} \frac{1}{n} x^n$. Express the coefficients of $f(x)g(x)$ in terms of $H_n = \sum_{k=1}^{n} \frac{1}{k}$.

23. $f(x) = g(x) = \sum_{n=1}^{\infty} \left(\frac{x}{2}\right)^n$

Answer

$a_n = b_n = 2^{-n}$ so $c_n = \sum_{k=1}^{n} b_k a_{n-k} = 2^{-n}\sum_{k=1}^{n} 1 = \frac{n}{2^n}$ and $f(x)g(x) = \sum_{n=1}^{\infty} n\left(\frac{x}{2}\right)^n$

24. $f(x) = g(x) = \sum_{n=1}^{\infty} nx^n$

Exercises 25 - 26

Differentiate the given series expansion of f term-by-term to obtain the corresponding series expansion for the derivative of f.

25. $f(x) = \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$

Answer

The derivative of f is $-\frac{1}{(1+x)^2} = -\sum_{n=0}^{\infty} (-1)^n (n+1)x^n$.

26. $f(x) = \frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n}$

Exercises 27 - 28

Integrate the given series expansion of f term-by-term from zero to x to obtain the corresponding series expansion for the indefinite integral of f.

27. $f(x) = \frac{2x}{(1+x^2)^2} = \sum_{n=1}^{\infty} (-1)^n (2n) x^{2n-1}$

Answer

The indefinite integral of f is $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$.

28. $f(x) = \dfrac{2x}{1+x^2} = 2\sum_{n=0}^{\infty}(-1)^n x^{2n+1}$

Exercises 29 - 32

Evaluate each infinite series by identifying it as the value of a derivative or integral of geometric series.

29. Evaluate $\sum_{n=1}^{\infty}\dfrac{n}{2^n}$ as $f'(\frac{1}{2})$ where $f(x) = \sum_{n=0}^{\infty}x^n$.

Answer

$$f(x) = \sum_{n=0}^{\infty}x^n = \frac{1}{1-x};\quad f'(\frac{1}{2}) = \sum_{n=1}^{\infty}\frac{n}{2^{n-1}} = \frac{d}{dx}(1-x)^{-1}\Big|_{x=1/2} = \frac{1}{(1-x)^2}\Big|_{x=1/2} = 4 \text{ so } \sum_{n=1}^{\infty}\frac{n}{2^n} = 2.$$

30. Evaluate $\sum_{n=1}^{\infty}\dfrac{n}{3^n}$ as $f'(\frac{1}{3})$ where $f(x) = \sum_{n=0}^{\infty}x^{6n}$.

31. Evaluate $\sum_{n=2}^{\infty}\dfrac{n(n-1)}{2^n}$ as $f''\left(\frac{1}{2}\right)$ where $f(x) = \sum_{n=0}^{\infty}x^n$.

Answer

$$f(x) = \sum_{n=0}^{\infty}x^n = \frac{1}{1-x};\quad f''\left(\frac{1}{2}\right) = \sum_{n=2}^{\infty}\frac{n(n-1)}{2^{n-2}} = \frac{d^2}{dx^2}(1-x)^{-1}\Big|_{x=1/2} = \frac{2}{(1-x)^3}\Big|_{x=1/2} = 16 \text{ so } \sum_{n=2}^{\infty}n\frac{(n-1)}{2^n} = 4.$$

32. Evaluate $\sum_{n=0}^{\infty}\dfrac{(-1)^n}{n+1}$ as $\int_0^1 f(t)\ dt$ where $f(x) = \sum_{n=0}^{\infty}(-1)^n x^{2n} = \dfrac{1}{1+x^2}$.

Exercises 33 - 39

Given that $\dfrac{1}{1-x} = \sum_{n=0}^{\infty}x^n$, use term-by-term differentiation or integration to find power series for each function centered at the given point.

33. $f(x) = \ln x$ centered at $x = 1$ (Hint: $x = 1-(1-x)$)

Answer

$$\int\sum(1-x)^n\ dx = \int\sum(-1)^n(x-1)^n\ dx = \sum\frac{(-1)^n(x-1)^{n+1}}{n+1}$$

34. $\ln(1-x)$ at $x = 0$

35. $\ln(1-x^2)$ at $x = 0$

Answer

$$-\int_{t=0}^{x^2}\frac{1}{1-t}dt = -\sum_{n=0}^{\infty}\int_0^{x^2}t^n dx - \sum_{n=0}^{\infty}\frac{x^{2(n+1)}}{n+1} = -\sum_{n=1}^{\infty}\frac{x^{2n}}{n}$$

36. $f(x) = \dfrac{2x}{(1-x^2)^2}$ at $x = 0$

37. $f(x) = \tan^{-1}(x^2)$ at $x = 0$

Answer

$$\int_0^{x^2}\frac{dt}{1+t^2} = \sum_{n=0}^{\infty}(-1)^n\int_0^{x^2}t^{2n}dt = \sum_{n=0}^{\infty}(-1)^n\frac{t^{2n+1}}{2n+1}\Big|_{t=0}^{x^2} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+2}}{2n+1}$$

38. $f(x) = \ln(1+x^2)$ at $x = 0$

39. $f(x) = \int_0^x \ln t \; dt$ where $\ln(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-1)^n}{n}$

Answer

Term-by-term integration gives $\int_0^x \ln t \; dt = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-1)^{n+1}}{n(n+1)} = \sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{1}{n} - \frac{1}{n+1} \right) (x-1)^{n+1}$
$= (x-1)\ln x + \sum_{n=2}^{\infty} (-1)^n \frac{(x-1)^n}{n} = x \ln x - x.$

40. [T] Evaluate the power series expansion $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ at $x = 1$ to show that $\ln(2)$ is the sum of the alternating harmonic series. Use the alternating series test to determine how many terms of the sum are needed to estimate $\ln(2)$ accurate to within 0.001, and find such an approximation.

41. [T] Subtract the infinite series of $\ln(1-x)$ from $\ln(1+x)$ to get a power series for $\ln\left(\frac{1+x}{1-x}\right)$. Evaluate at $x = \frac{1}{3}$. What is the smallest N such that the Nth partial sum of this series approximates $\ln(2)$ with an error less than 0.001?

Answer

We have $\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$ so $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$. Thus, $\ln\left(\frac{1+x}{1-x}\right) = \sum_{n=1}^{\infty} \left(1 + (-1)^{n-1}\right) \frac{x^n}{n}$
$= 2\sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1}$. When $x = \frac{1}{3}$ we obtain $\ln(2) = 2\sum_{n=1}^{\infty} \frac{1}{3^{2n-1}(2n-1)}$. We have $2\sum_{n=1}^{3} \frac{1}{3^{2n-1}(2n-1)} = 0.69300\ldots$,
while $2\sum_{n=1}^{4} \frac{1}{3^{2n-1}(2n-1)} = 0.69313\ldots$ and $\ln(2) = 0.69314\ldots$; therefore, $N = 4$.

Exercises 42 - 45

Using a substitution if indicated, express each series in terms of elementary functions and find the radius of convergence of the sum.

42. $\sum_{k=0}^{\infty} (x^k - x^{2k+1})$

43. $\sum_{k=1}^{\infty} \frac{x^{3k}}{6k}$

Answer

$\sum_{k=1}^{\infty} \frac{x^k}{k} = -\ln(1-x)$ so $\sum_{k=1}^{\infty} \frac{x^{3k}}{6k} = -\frac{1}{6}\ln(1-x^3)$. The radius of convergence is equal to 1 by the ratio test.

44. $\sum_{k=1}^{\infty} (1+x^2)^{-k}$ using $y = \frac{1}{1+x^2}$

45. $\sum_{k=1}^{\infty} 2^{-kx}$ using $y = 2^{-x}$

Answer

If $y = 2^{-x}$, then $\sum_{k=1}^{\infty} y^k = \frac{y}{1-y} = \frac{2^{-x}}{1-2^{-x}} = \frac{1}{2^x - 1}$. If $a_k = 2^{-kx}$, then $\frac{a_{k+1}}{a_k} = 2^{-x} < 1$ when $x > 0$. So the series converges for all $x > 0$.

46. Show that, up to powers x^3 and y^3, $E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ satisfies $E(x+y) = E(x)E(y)$.

47. Differentiate the series $E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ term-by-term to show that $E(x)$ is equal to its derivative.

Answer

Answers will vary.

48. Show that if $f(x)=\sum_{n=0}^{\infty} a_n x^n$ is a sum of even powers, that is, $a_n=0$ if n is odd, then $F=\int_0^x f(t)\ dt$ is a sum of odd powers, while if I is a sum of odd powers, then F is a sum of even powers.

49. [T] Suppose that the coefficients an of the series $\sum_{n=0}^{\infty} a_n x^n$ are defined by the recurrence relation $a_n=\frac{a_{n-1}}{n}+\frac{a_{n-2}}{n(n-1)}$. For $a_0=0$ and $a_1=1$, compute and plot the sums $S_N=\sum_{n=0}^{N} a_n x^n$ for $N=2,3,4,5$ on $[-1,1]$.

Answer

The solid curve is S_5. The dashed curve is S_2, dotted is S_3, and dash-dotted is S_4

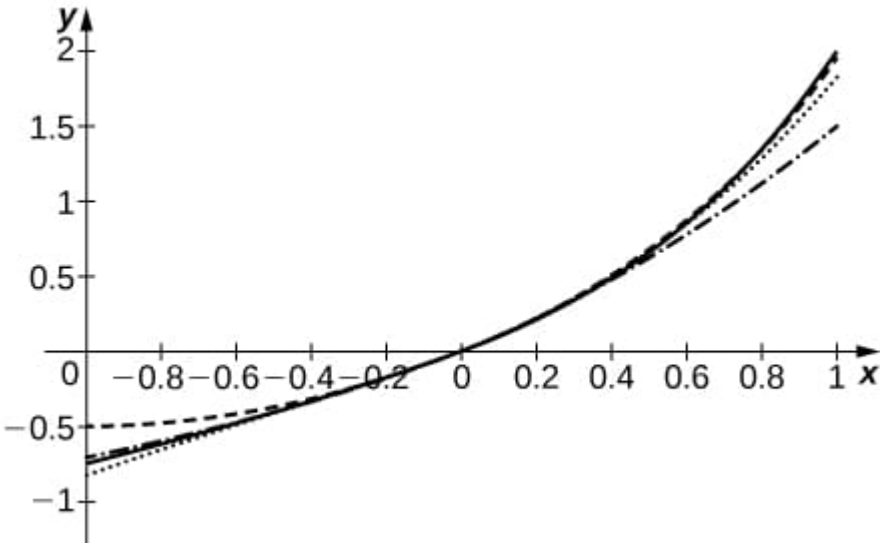


Figure $2.2E.1$: This is a graph of three curves. They are all increasing and become very close as the curves approach $x=0$. Then they separate as x moves away from 0.

50. [T] Suppose that the coefficients an of the series $\sum_{n=0}^{\infty} a_n x^n$ are defined by the recurrence relation $a_n=\frac{a_{n-1}}{\sqrt{n}}-\frac{a_{n-2}}{\sqrt{n(n-1)}}$. For $a_0=1$ and $a_1=0$, compute and plot the sums $S_N=\sum_{n=0}^{N} a_n x^n$ for $N=2,3,4,5$ on $[-1,1]$.

51. [T] Given the power series expansion $\ln(1+x)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$, determine how many terms N of the sum evaluated at $x=-1/2$ are needed to approximate $\ln(2)$ accurate to within $1/1000$. Evaluate the corresponding partial sum $\sum_{n=1}^{N}(-1)^{n-1}\frac{x^n}{n}$.

Answer

When $x=-\frac{1}{2}$, $-\ln(2)=\ln\left(\frac{1}{2}\right)=-\sum_{n=1}^{\infty}\frac{1}{n2^n}$. Since $\sum_{n=11}^{\infty}\frac{1}{n2^n}<\sum_{n=11}^{\infty}\frac{1}{2^n}=\frac{1}{2^{10}}$, one has $\sum_{n=1}^{10}\frac{1}{n2^n}=0.69306\ldots$ whereas $\ln(2)=0.69314\ldots$; therefore, $N=10$.

52. [T] Given the power series expansion $\tan^{-1}(x)=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{2k+1}$, use the alternating series test to determine how many terms N of the sum evaluated at $x=1$ are needed to approximate $\tan^{-1}(1)=\frac{\pi}{4}$ accurate to within $1/1000$. Evaluate the corresponding partial sum $\sum_{k=0}^{N}(-1)^k\frac{x^{2k+1}}{2k+1}$.

53. [T] Recall that $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=\frac{\pi}{6}$. Assuming an exact value of $\frac{1}{\sqrt{3}}$), estimate $\frac{\pi}{6}$ by evaluating partial sums $S_N\left(\frac{1}{\sqrt{3}}\right)$ of the power series expansion $\tan^{-1}(x)=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{2k+1}$ at $x=\frac{1}{\sqrt{3}}$. What is the smallest number N such that $6S_N\left(\frac{1}{\sqrt{3}}\right)$ approximates π accurately to within 0.001? How many terms are needed for accuracy to within 0.00001?

Answer

$6S_N\left(\frac{1}{\sqrt{3}}\right) = 2\sqrt{3}\sum_{n=0}^{N}(-1)^n\frac{1}{3^n(2n+1).}$ One has $\pi - 6S_4\left(\frac{1}{\sqrt{3}}\right) = 0.00101\ldots$ and $\pi - 6S_5\left(\frac{1}{\sqrt{3}}\right) = 0.00028\ldots$ so $N = 5$ is the smallest partial sum with accuracy to within $0.001.$ Also, $\pi - 6S_7\left(\frac{1}{\sqrt{3}}\right) = 0.00002\ldots$ while $\pi - 6S_8\left(\frac{1}{\sqrt{3}}\right) = -0.000007\ldots$ so $N = 8$ is the smallest N to give accuracy to within $0.00001.$

Contributors

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2.3: Taylor and Maclaurin Series

Learning Objectives

- Describe the procedure for finding a Taylor polynomial of a given order for a function.
- Explain the meaning and significance of Taylor's theorem with remainder.
- Estimate the remainder for a Taylor series approximation of a given function.

In the previous two sections we discussed how to find power series representations for certain types of functions--specifically, functions related to geometric series. Here we discuss power series representations for other types of functions. In particular, we address the some questions: Which functions can be represented by power series and how do we find such representations? If we can find a power series representation for a particular function f and the series converges on some interval, how do we prove that the series actually converges to f?

Overview of Taylor/Maclaurin Series

Consider a function f that has a power series representation at $x = a$. Then the series has the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots . \tag{2.3.1}$$

What should the coefficients be? For now, we ignore issues of convergence, but instead focus on what the series should be, if one exists. We return to discuss convergence later in this section. If the series (Equation 2.3.1) is a representation for f at $x = a$, we certainly want the series to equal $f(a)$ at $x = a$. Evaluating the series at $x = a$, we see that

Thus, the series equals $f(a)$ if the coefficient $c_0 = f(a)$. In addition, we would like the first derivative of the power series to equal $f\,'(a)$ at $x = a$. Differentiating (Equation 2.3.1) term-by-term, we see that

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \ldots . \tag{2.3.2}$$

Therefore, at $x = a$, the derivative is

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) = c_1 + 2c_2(a-a) + 3c_3(a-a)^2 + \cdots = c_1. \tag{2.3.3}$$

Therefore, the derivative of the series equals $f\,'(a)$ if the coefficient $c_1 = f\,'(a)$. Continuing in this way, we look for coefficients c_n such that all the derivatives of the power series (Equation 2.3.3) will agree with all the corresponding derivatives of f at $x = a$. The second and third derivatives of (Equation 2.3.2) are given by

$$\frac{d^2}{dx^2}\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) = 2c_2 + 3\cdot 2c_3(x-a) + 4\cdot 3c_4(x-a)^2 + \ldots \tag{2.3.4}$$

and

$$\frac{d^3}{dx^3}\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) = 3\cdot 2c_3 + 4\cdot 3\cdot 2c_4(x-a) + 5\cdot 4\cdot 3c_5(x-a)^2 + \cdots . \tag{2.3.5}$$

Therefore, at $x = a$, the second and third derivatives

$$\frac{d^2}{dx^2}\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) = 2c_2 + 3\cdot 2c_3(a-a) + 4\cdot 3c_4(a-a)^2 + \cdots = 2c_2 \tag{2.3.6}$$

and

$$\frac{d^3}{dx^3}\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) = 3\cdot 2c_3 + 4\cdot 3\cdot 2c_4(a-a) + 5\cdot 4\cdot 3c_5(a-a)^2 + \cdots = 3\cdot 2c_3 \tag{2.3.7}$$

equal $f''(a)$ and $f'''(a)$, respectively, if $c_2 = \dfrac{f''(a)}{2}$ and $c_3 = \dfrac{f'''(a)}{3\cdot 2}$. More generally, we see that if f has a power series representation at $x = a$, then the coefficients should be given by $c_n = \dfrac{f^{(n)}(a)}{n!}$. That is, the series should be

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \tag{2.3.8}$$

This power series for f is known as the Taylor series for f at a. If $x = 0$, then this series is known as the Maclaurin series for f.

Definition: Maclaurin and Taylor Series

If f has derivatives of all orders at $x = a$, then the **Taylor series** for the function f at a is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots \tag{2.3.9}$$

The Taylor series for f at 0 is known as the **Maclaurin series** for f.

Later in this section, we will show examples of finding Taylor series and discuss conditions under which the Taylor series for a function will converge to that function. Here, we state an important result. Recall that power series representations are unique. Therefore, if a function f has a power series at a, then it must be the Taylor series for f at a.

Uniqueness of Taylor Series

If a function f has a power series at a that converges to f on some open interval containing a, then that power series is the Taylor series for f at a.

The proof follows directly from that discussed previously.

To determine if a Taylor series converges, we need to look at its sequence of partial sums. These partial sums are finite polynomials, known as **Taylor polynomials.**

Taylor Polynomials

The nth partial sum of the Taylor series for a function f at a is known as the nth-degree Taylor polynomial. For example, the 0th, 1st, 2nd, and 3rd partial sums of the Taylor series are given by

$$\begin{aligned} p_0(x) &= f(a) \\ p_1(x) &= f(a) + f'(a)(x-a) \\ p_2(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 \\ p_3(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 \end{aligned}$$

respectively. These partial sums are known as the 0th, 1st, 2nd, and 3rd degree Taylor polynomials of f at a, respectively. If $x = a$, then these polynomials are known as *Maclaurin polynomials* for f. We now provide a formal definition of Taylor and Maclaurin polynomials for a function f.

Definition: Taylor and Maclaurin Polynomial

If f has n derivatives at $x = a$, then the nth-degree **Taylor polynomial** of f at a is

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n. \tag{2.3.10}$$

The nth-degree Taylor polynomial for f at 0 is known as the nth-degree **Maclaurin polynomial** for f.

We now show how to use this definition to find several Taylor polynomials for $f(x) = \ln x$ at $x = 1$.

Example 2.3.1: Finding Taylor Polynomials

Find the Taylor polynomials p_0, p_1, p_2 and p_3 for $f(x) = \ln x$ at $x = 1$. Use a graphing utility to compare the graph of f with the graphs of p_0, p_1, p_2 and p_3.

Solution

To find these Taylor polynomials, we need to evaluate f and its first three derivatives at $x = 1$.

$$f(x) = \ln x \; f(1) = 0$$

$$f\,'(x) = \frac{1}{x} \; f\,'(1) = 1$$

$$f''(x) = -\frac{1}{x^2} \; f''(1) = -1$$

$$f'''(x) = \frac{2}{x^3} \; f'''(1) = 2$$

Therefore,

$$\begin{aligned}
p_0(x) &= f(1) = 0,\\
p_1(x) &= f(1) + f'(1)(x-1) = x - 1,\\
p_2(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 = (x-1) - \frac{1}{2}(x-1)^2\\
p_3(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3
\end{aligned}$$

The graphs of $y = f(x)$ and the first three Taylor polynomials are shown in Figure 2.3.1.

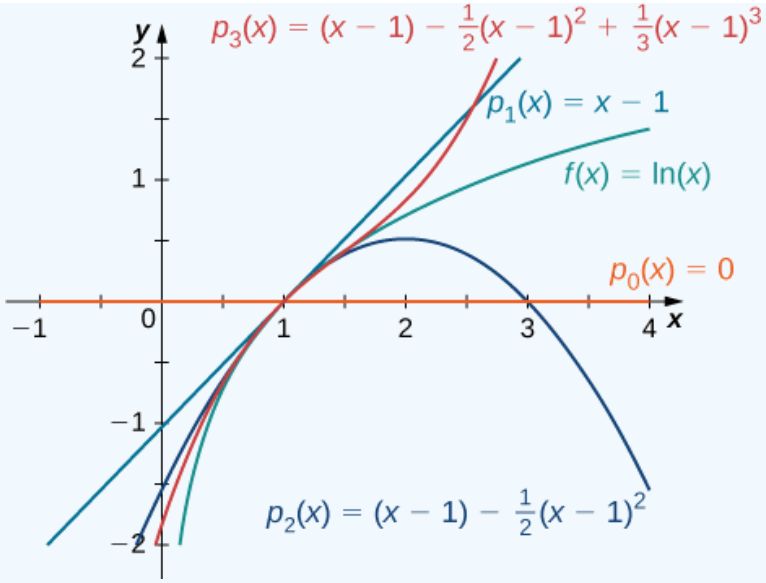


Figure 2.3.1: The graph illustrates the function $y = \ln x$ and it's Taylor polynomials p_0, p_1, p_2 and p_3 at $x = 1$. Each polynomial increasingly aligns with $y = \ln x$ near $x = 1$, demonstrating improved accuracy as the degree increases.

The Taylor Polynomial for $\cos(x)$

You can move it around and also change its equation to see the effect on the graph. The interactive app is unfortunately not accessible.

Below is the Nth degree Taylor polynomial for $f(x) = \cos(x)$ at $x = a$. Move the sliders to change N and a.

xxxxx

2

N = 2

50

a = 1

Exercise 2.3.1

Find the Taylor polynomials p_0, p_1, p_2 and p_3 for $f(x) = \dfrac{1}{x^2}$ at $x = 1$.

Hint available at the end of this section.

Answer available at the end of this section.

We now show how to find Maclaurin polynomials for $e^x, \sin x,$ and $\cos x$. As stated above, Maclaurin polynomials are Taylor polynomials centered at zero.

Example 2.3.2:Finding Maclaurin Polynomials

For each of the functions, find formulas for the Maclaurin polynomials p_0, p_1, p_2 and p_3. Find a formula for the nth-degree Maclaurin polynomial and write it using sigma notation. Use a graphing utility to compare the graphs of p_0, p_1, p_2 and p_3 with f.

a. $f(x) = e^x$
b. $f(x) = \sin x$
c. $f(x) = \cos x$

Solution

a. Since $f(x) = e^x$,we know that $f(x) = f\,'(x) = f''(x) = \cdots = f^{(n)}(x) = e^x$ for all positive integers n. Therefore,

$$f(0) = f'(0) = f''(0) = \cdots = f^{(n)}(0) = 1$$

for all positive integers n. Therefore, we have

$$p_0(x) = f(0) = 1,$$

$$p_1(x) = f(0) + f\,'(0)x = 1 + x,$$

$$p_2(x) = f(0) + f\,'(0)x + \frac{f''(0)}{2!}x^2 = 1 + x + \frac{1}{2}x^2,$$

$$p_3(x) = f(0) + f\,'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3,$$

$$p_n(x) = f(0) + f\,'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} = \sum_{k=0}^{n} \frac{x^k}{k!}.$$

The function and the first three Maclaurin polynomials are shown in Figure 2.

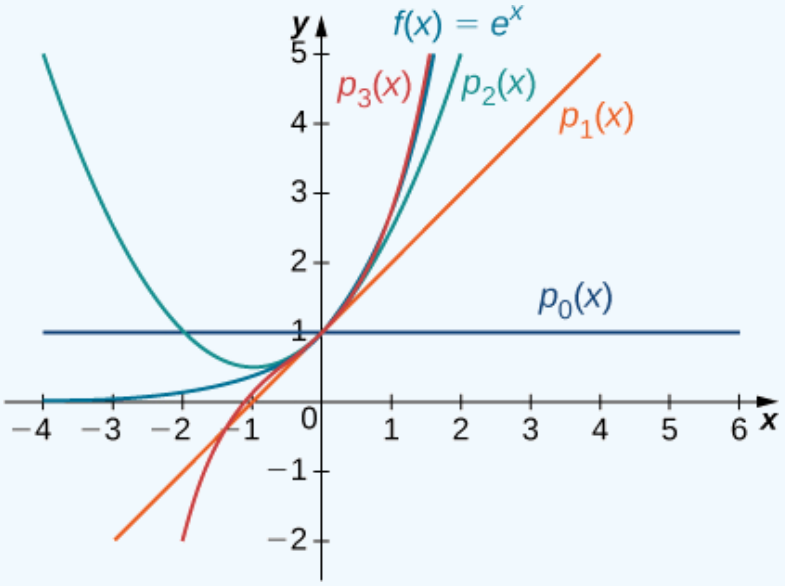


Figure 2.3.2: The graph shows the function $y = e^x$ and the Maclaurin polynomials $p_0,\ p_1,\ p_2$ and p_3 at $x = 0$. Each polynomial increasingly aligns with $y = e^x$ near $x = 0$, demonstrating improved accuracy as the degree increases.

b. For $f(x) = \sin x$, the values of the function and its first four derivatives at $x = 0$ are given as follows:

$$f(x) = \sin x \; f(0) = 0$$

$$f'(x) = \cos x \; f'(0) = 1$$

$$f''(x) = -\sin x \; f''(0) = 0$$

$$f'''(x) = -\cos x \; f'''(0) = -1$$

$$f^{(4)}(x) = \sin x \; f^{(4)}(0) = 0.$$

Since the fourth derivative is $\sin x$, the pattern repeats. That is, $f^{(2m)}(0) = 0$ and $f^{(2m+1)}(0) = (-1)^m$ for $m \geq 0$. Thus, we have

$$p_0(x) = 0,$$

$$p_1(x) = 0 + x = x,$$

$$p_2(x) = 0 + x + 0 = x,$$

$$p_3(x) = 0 + x + 0 - \frac{1}{3!}x^3 = x - \frac{x^3}{3!},$$

$$p_4(x) = 0 + x + 0 - \frac{1}{3!}x^3 + 0 = x - \frac{x^3}{3!},$$

$$p_5(x) = 0 + x + 0 - \frac{1}{3!}x^3 + 0 + \frac{1}{5!}x^5 = x - \frac{x^3}{3!} + \frac{x^5}{5!},$$

and for $m \geq 0$,

$$p_{2m+1}(x) = p_{2m+2}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^m \frac{x^{2m+1}}{(2m+1)!} = \sum_{k=0}^{m} (-1)^k \frac{x^{2k+1}}{(2k+1)!}.$$

Graphs of the function and its Maclaurin polynomials are shown in Figure 2.3.3.

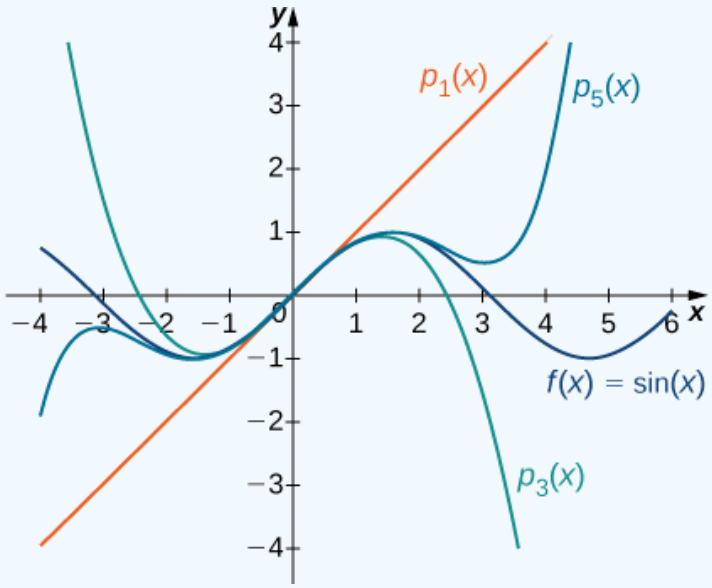


Figure 2.3.3: The graph shows the function $y = \sin x$ and the Maclaurin polynomials p_1, p_3 and p_5. Each polynomial increasingly aligns with $y = \sin x$ near $x = 0$, demonstrating improved accuracy as the degree increases.

c. For $f(x) = \cos x$, the values of the function and its first four derivatives at $x = 0$ are given as follows:

$$f(x) = \cos x \; f(0) = 1$$

$$f'(x) = -\sin x \; f'(0) = 0$$

$$f''(x) = -\cos x \; f''(0) = -1$$

$$f'''(x) = \sin x \; f'''(0) = 0$$

$$f^{(4)}(x) = \cos x \; f^{(4)}(0) = 1.$$

Since the fourth derivative is $\sin x$, the pattern repeats. In other words, $f^{(2m)}(0) = (-1)^m$ and $f^{(2m+1)} = 0$ for $m \geq 0$. Therefore,

$$p_0(x) = 1,$$

$$p_1(x) = 1 + 0 = 1,$$

$$p_2(x) = 1 + 0 - \frac{1}{2!}x^2 = 1 - \frac{x^2}{2!},$$

$$p_3(x) = 1 + 0 - \frac{1}{2!}x^2 + 0 = 1 - \frac{x^2}{2!},$$

$$p_4(x) = 1 + 0 - \frac{1}{2!}x^2 + 0 + \frac{1}{4!}x^4 = 1 - \frac{x^2}{2!} + \frac{x^4}{4!},$$

$$p_5(x) = 1 + 0 - \frac{1}{2!}x^2 + 0 + \frac{1}{4!}x^4 + 0 = 1 - \frac{x^2}{2!} + \frac{x^4}{4!},$$

and for $n \geq 0$,

$$p_{2m}(x) = p_{2m+1}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^m \frac{x^{2m}}{(2m)!} = \sum_{k=0}^{m} (-1)^k \frac{x^{2k}}{(2k)!}.$$

Graphs of the function and the Maclaurin polynomials appear in Figure 4.

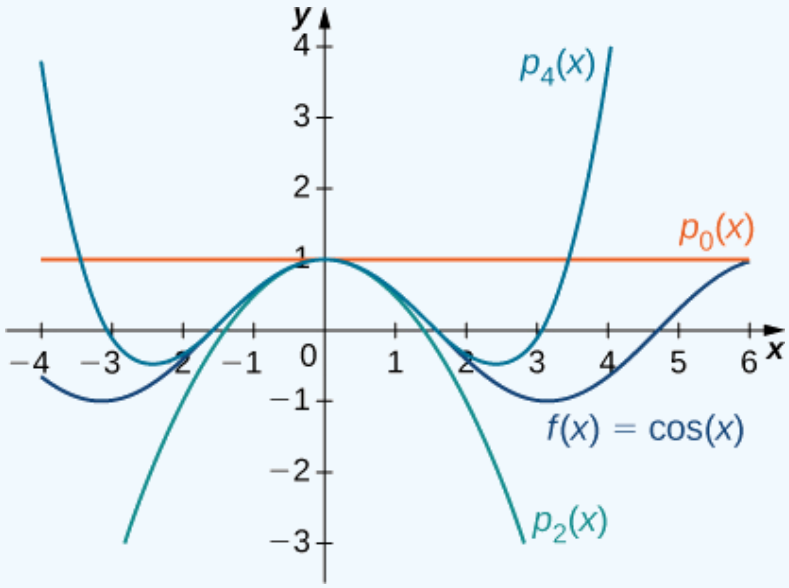


Figure 2.3.4: The function $y = \cos x$ and the Maclaurin polynomials p_0, p_2 and p_4 are plotted on this graph. Each polynomial increasingly aligns with $y = cosx$ near $x = 0$, demonstrating improved accuracy as the degree increases.

Exercise 2.3.2

Find formulas for the Maclaurin polynomials $p_0,\ p_1,\ p_2$ and p_3 for $f(x) = \dfrac{1}{1+x}$.

Find a formula for the nth-degree Maclaurin polynomial. Write your answer using sigma notation.

Hint available at the end of this section.

Answer available at the end of this section.

Recall that the nth-degree Taylor polynomial for a function f at a is the nth partial sum of the Taylor series for f at a. Therefore, to determine if the Taylor series converges, we need to determine whether the sequence of Taylor polynomials p_n converges. However, not only do we want to know if the sequence of Taylor polynomials converges, we want to know if it converges to f. To answer this question, we define the remainder $R_n(x)$ as

$$R_n(x) = f(x) - p_n(x). \tag{2.3.11}$$

For the sequence of Taylor polynomials to converge to f, we need the remainder R_n to converge to zero. To determine if R_n converges to zero, we introduce **Taylor's theorem with remainder**. Not only is this theorem useful in proving that a Taylor series converges to its related function, but it will also allow us to quantify how well the nth-degree Taylor polynomial approximates the function.

Here we look for a bound on $|R_n|$. Consider the simplest case: $n = 0$. Let p_0 be the 0th Taylor polynomial at a for a function f. The remainder R_0 satisfies

$$R_0(x) = f(x) - p_0(x) = f(x) - f(a).$$

If f is differentiable on an interval I containing a and x, then by the Mean Value Theorem there exists a real number c between a and x such that $f(x) - f(a) = f\,'(c)(x - a)$. Therefore,

$$R_0(x) = f'(c)(x - a). \tag{2.3.12}$$

Using the Mean Value Theorem in a similar argument, we can show that if f is n times differentiable on an interval I containing a and x, then the nth remainder R_n satisfies

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - a)^{n+1} \tag{2.3.13}$$

for some real number c between a and x. It is important to note that the value c in the numerator above is not the center a, but rather an unknown value c between a and x. This formula allows us to get a bound on the remainder R_n. If we happen to know that $|\, f^{(n+1)}(x)\,|$ is bounded by some real number M on this interval I, then

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x - a|^{n+1} \tag{2.3.14}$$

for all x in the interval I.

We now state Taylor's theorem, which provides the formal relationship between a function f and its nth-degree Taylor polynomial $p_n(x)$. This theorem allows us to bound the error when using a Taylor polynomial to approximate a function value, and will be important in proving that a Taylor series for f converges to f.

Theorem 2.3.1: Taylor's Theorem with Remainder

Let f be a function that can be differentiated $n + 1$ times on an interval I containing the real number a. Let p_n be the nth -degree Taylor polynomial of f at a and let

$$R_n(x) = f(x) - p_n(x) \tag{2.3.15}$$

be the nth remainder. Then for each x in the interval I, there exists a real number c between a and x such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - a)^{n+1} \tag{2.3.16}$$

.

If there exists a real number M such that $|f^{(n+1)}(x)| \leq M$ for all $x \in I$, then

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \tag{2.3.17}$$

for all x in I.

Proof

Fix a point $x \in I$ and introduce the function g such that

$$g(t) = f(x) - f(t) - f'(t)(x-t) - \frac{f''(t)}{2!}(x-t)^2 - \cdots - \frac{f^{(n)}(t)}{n!}(x-t)^n - R_n(x)\frac{(x-t)^{n+1}}{(x-a)^{n+1}}. \tag{2.3.18}$$

We claim that g satisfies the criteria of Rolle's theorem. Since g is a polynomial function (in t), it is a differentiable function. Also, g is zero at $t=a$ and $t=x$ because

$$\begin{aligned} g(a) &= f(x) - f(a) - f'(a)(x-a) - \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n - R_n(x) \\ &= f(x) - p_n(x) - R_n(x) \\ &= 0, \\ g(x) &= f(x) - f(x) - 0 - \cdots - 0 \\ &= 0. \end{aligned}$$

Therefore, g satisfies Rolle's theorem, and consequently, there exists c between a and x such that $g\,'(c) = 0$. We now calculate $g\,'$. Using the product rule, we note that

$$\frac{d}{dt}\left[\frac{f^{(n)}(t)}{n!}(x-t)^n\right] = -\frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} + \frac{f^{(n+1)}(t)}{n!}(x-t)^n. \tag{2.3.19}$$

Consequently,

$$g'(t) = -f'(t) + [f'(t) - f''(t)(x-t)] + \left[f''(t)(x-t) - \frac{f'''(t)}{2!}(x-t)^2\right] + \cdots + \left[\frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} - \frac{f^{(n+1)}(t)}{n!}(x-t)^n\right] +$$

.

Notice that there is a telescoping effect. Therefore,

$$g'(t) = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n + (n+1)R_n(x)\frac{(x-t)^n}{(x-a)^{n+1}} \tag{2.3.21}$$

.

By Rolle's theorem, we conclude that there exists a number c between a and x such that $g\,'(c) = 0$. Since

$$g'(c) = -\frac{f^{(n+1)}(c)}{n!}(x-c)^n + (n+1)R_n(x)\frac{(x-c)^n}{(x-a)^{n+1}} \tag{2.3.22}$$

we conclude that

$$-\frac{f^{(n+1)}(c)}{n!}(x-c)^n + (n+1)R_n(x)\frac{(x-c)^n}{(x-a)^{n+1}} = 0. \tag{2.3.23}$$

Adding the first term on the left-hand side to both sides of the equation and dividing both sides of the equation by $n+1$, we conclude that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \tag{2.3.24}$$

as desired. From this fact, it follows that if there exists M such that $|\, f^{(n+1)}(x)\,| \leq M$ for all x in I, then

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \tag{2.3.25}$$

.

Not only does Taylor's theorem allow us to prove that a Taylor series converges to a function, but it also allows us to estimate the accuracy of Taylor polynomials in approximating function values. We begin by looking at linear and quadratic approximations of $f(x) = \sqrt[3]{x}$ at $x = 8$ and determine how accurate these approximations are at estimating $\sqrt[3]{11}$.

Example 2.3.3: Using Linear and Quadratic Approximations to Estimate Function Values

Consider the function $f(x) = \sqrt[3]{x}$.

a. Find the first and second Taylor polynomials for f at $x = 8$. Use a graphing utility to compare these polynomials with f near $x = 8$.
b. Use these two polynomials to estimate $\sqrt[3]{11}$.
c. Use Taylor's theorem to bound the error.

Solution

a. For $f(x) = \sqrt[3]{x}$, the values of the function and its first two derivatives at $x = 8$ are as follows:

$$f(x) = \sqrt[3]{x}, f(8) = 2$$

$$f\,'(x) = \frac{1}{3x^{2/3}},\ f\,'(8) = \frac{1}{12}$$

$$f''(x) = \frac{-2}{9x^{5/3}},\ f''(8) = -\frac{1}{144.}$$

Thus, the first and second Taylor polynomials at $x = 8$ are given by

$$\begin{aligned} p_1(x) &= f(8) + f\,'(8)(x-8) \\ &= 2 + \frac{1}{12}(x-8) \\ p_2(x) &= f(8) + f\,'(8)(x-8) + \frac{f''(8)}{2!}(x-8)^2 \\ &= 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2. \end{aligned}$$

The function and the Taylor polynomials are shown in Figure 4.

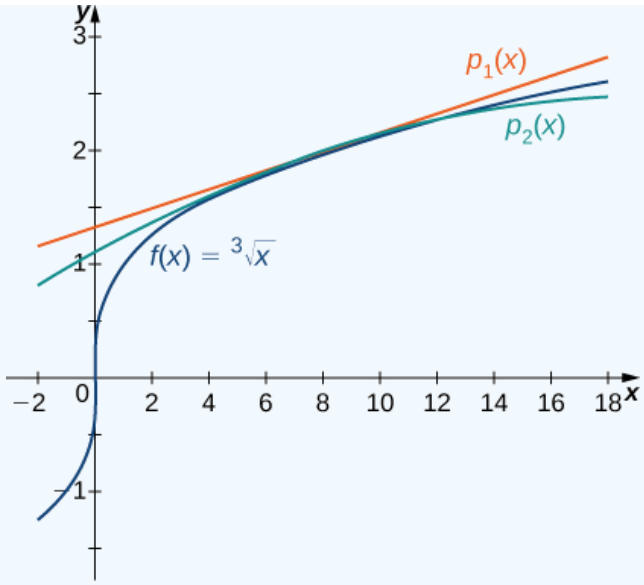


Figure 2.3.5: The graphs of $f(x) = \sqrt[3]{x}$ and the linear and quadratic approximations $p_1(x)$ and $p_2(x)$ centered at $x = 0$.

b. Using the first Taylor polynomial at $x = 8$, we can estimate

$$\sqrt[3]{11} \approx p_1(11) = 2 + \frac{1}{12}(11 - 8) = 2.25. \tag{2.3.26}$$

c. Using the second Taylor polynomial at $x = 8$, we obtain

$$\sqrt[3]{11} \approx p_2(11) = 2 + \frac{1}{12}(11 - 8) - \frac{1}{288}(11 - 8)^2 = 2.21875. \tag{2.3.27}$$

By Note, there exists a c in the interval $(8, 11)$ such that the remainder when approximating $\sqrt[3]{11}$ by the first Taylor polynomial satisfies

$$R_1(11) = \frac{f''(c)}{2!}(11 - 8)^2. \tag{2.3.28}$$

We do not know the exact value of c, so we find an upper bound on $R_1(11)$ by determining the maximum value of f'' on the interval $(8, 11)$. Since $f''(x) = -\frac{2}{9x^{5/3}}$, the largest value for $|f''(x)|$ on that interval occurs at $x = 8$. Using the fact that $f''(8) = -\frac{1}{144}$, we obtain

$$|R_1(11)| \leq \frac{1}{144 \cdot 2!}(11 - 8)^2 = 0.03125.$$

Similarly, to estimate $R_2(11)$, we use the fact that

$$R_2(11) = \frac{f'''(c)}{3!}(11 - 8)^3.$$

Since $f'''(x) = \frac{10}{27x^{8/3}}$, the maximum value of f''' on the interval $(8, 11)$ is $f'''(8) \approx 0.0014468$. Therefore, we have

$$|R_2(11)| \leq \frac{0.0011468}{3!}(11 - 8)^3 \approx 0.0065104.$$

Watch the accompanying video for a visual demonstration of the concept.

- **Video Length:** 4 minute 5 seconds.

- **Context:** This video demonstrates how to find upper bound for the error in approximating using a fifth-degree Taylor polynomial:$sin(0.2) \approx \frac{(0.2)^1}{1!} - \frac{(0.2)^3}{3!} + \frac{(0.2)^5}{5!}$

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Exercise 2.3.3

Find the first and second Taylor polynomials for $f(x) = \sqrt{x}$ at $x = 4$. Use these polynomials to estimate $\sqrt{6}$. Use Taylor's theorem to bound the error.

Hint available at the end of this section.

Answer available at the end of this section.

Example 2.3.4: Approximating using Maclaurin Polynomials.

From Example 2.3.2b, the Maclaurin polynomials for $\sin x$ are given by

$$p_{2m+1}(x) = p_{2m+2}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^m \frac{x^{2m+1}}{(2m+1)!}$$

for $m = 0, 1, 2, \ldots$.

a. Use the fifth Maclaurin polynomial for $\sin x$ to approximate $\sin\left(\frac{\pi}{18}\right)$ and bound the error.
b. For what values of x does the fifth Maclaurin polynomial approximate $\sin x$ to within 0.0001?

Solution

a. The fifth Maclaurin polynomial is

$$p_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} \tag{2.3.29}$$

.

Using this polynomial, we can estimate as follows:

$$\sin\left(\frac{\pi}{18}\right) \approx p_5\left(\frac{\pi}{18}\right) = \frac{\pi}{18} - \frac{1}{3!}\left(\frac{\pi}{18}\right)^3 + \frac{1}{5!}\left(\frac{\pi}{18}\right)^5 \approx 0.173648. \tag{2.3.30}$$

To estimate the error, use the fact that the sixth Maclaurin polynomial is $p_6(x) = p_5(x)$ and calculate a bound on $R_6(\frac{\pi}{18})$. By Note, the remainder is

$$R_6\left(\frac{\pi}{18}\right) = \frac{f^{(7)}(c)}{7!}\left(\frac{\pi}{18}\right)^7 \tag{2.3.31}$$

for some c between 0 and $\frac{\pi}{18}$. Using the fact that $|\, f^{(7)}(x)\,| \leq 1$ for all x, we find that the magnitude of the error is at most

$$\frac{1}{7!} \cdot \left(\frac{\pi}{18}\right)^7 \leq 9.8 \times 10^{-10}. \tag{2.3.32}$$

b. We need to find the values of x such that

$$\frac{1}{7!}|x|^7 \leq 0.0001. \tag{2.3.33}$$

Solving this inequality for x, we have that the fifth Maclaurin polynomial gives an estimate to within 0.0001 as long as $|x| < 0.907$.

Watch the accompanying video for a visual demonstration of the concept.

- **Video Length:** 3 minute 38 seconds.
- **Context:** This video demonstrates how to find the degree of the Maclaurin polynomial to approximate e with an error less than 0.001.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online



Exercise 2.3.4

Use the fourth Maclaurin polynomial for $\cos x$ to approximate $\cos\left(\frac{\pi}{12}\right)$.

Hint available at the end of this section.

Answer available at the end of this section.

Now that we are able to bound the remainder $R_n(x)$, we can use this bound to prove that a Taylor series for f at a converges to f.

Representing Functions with Taylor and Maclaurin Series

We now discuss issues of convergence for Taylor series. We begin by showing how to find a Taylor series for a function, and how to find its interval of convergence.

Example 2.3.5:Finding a Taylor Series

Find the Taylor series for $f(x) = \frac{1}{x}$ at $x = 1$. Determine the interval of convergence.

Solution

For $f(x) = \frac{1}{x}$, the values of the function and its first four derivatives at $x = 1$ are

$$f(x) = \frac{1}{x}\ f(1) = 1$$

$$f\,'(x) = -\frac{1}{x^2}\ f\,'(1) = -1$$

$$f''(x) = \frac{2}{x^3}\ f''(1) = 2!$$

$$f'''(x) = -\frac{3 \cdot 2}{x^4}\ f'''(1) = -3!$$

$$f^{(4)}(x) = \frac{4 \cdot 3 \cdot 2}{x^5} \quad f^{(4)}(1) = 4!.$$

That is, we have $f^{(n)}(1) = (-1)^n n!$ for all $n \geq 0$. Therefore, the Taylor series for f at $x = 1$ is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = \sum_{n=0}^{\infty} (-1)^n (x-1)^n.$$

To find the interval of convergence, we use the ratio test. We find that

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|\,(-1)^{n+1}(x-1)n^{+1}\,|}{|(-1)^n(x-1)^n|} = |x-1|.$$

Thus, the series converges if $|x-1| < 1$. That is, the series converges for \(0

$$\sum_{n=0}^{\infty} (-1)^n (2-1)^n = \sum_{n=0}^{\infty} (-1)^n$$

diverges by the divergence test. Similarly, at $x = 0$,

$$\sum_{n=0}^{\infty} (-1)^n (0-1)^n = \sum_{n=0}^{\infty} (-1)^{2n} = \sum_{n=0}^{\infty} 1$$

diverges. Therefore, the interval of convergence is $(0, 2)$.

Exercise 2.3.5

Find the Taylor series for $f(x) = \frac{1}{x}$ at $x = 2$ and determine its interval of convergence.

Hint available at the end of this section.

Answer available at the end of this section.

We know that the Taylor series found in this example converges on the interval $(0, 2)$, but how do we know it actually converges to f? We consider this question in more generality in a moment, but for this example, we can answer this question by writing

$$f(x) = \frac{1}{x} = \frac{1}{1-(1-x)}. \tag{2.3.34}$$

That is, f can be represented by the geometric series $\sum_{n=0}^{\infty} (1-x)^n$. Since this is a geometric series, it converges to $\frac{1}{x}$ as long as $|1-x| < 1$. Therefore, the Taylor series found in Example does converge to $f(x) = \frac{1}{x}$ on $(0, 2)$.

We now consider the more general question: if a Taylor series for a function f converges on some interval, how can we determine if it actually converges to f? To answer this question, recall that a series converges to a particular value if and only if its sequence of partial sums converges to that value. Given a Taylor series for f at a, the nth partial sum is given by the nth-degree Taylor polynomial p_n. Therefore, to determine if the Taylor series converges to f, we need to determine whether

$$\lim_{n \to \infty} p_n(x) = f(x).$$

Since the remainder $R_n(x) = f(x) - p_n(x)$, the Taylor series converges to f if and only if

$$\lim_{n \to \infty} R_n(x) = 0.$$

We now state this theorem formally.

Convergence of Taylor Series

Suppose that f has derivatives of all orders on an interval I containing a. Then the Taylor series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{2.3.35}$$

converges to $f(x)$ for all x in I if and only if

$$\lim_{n\to\infty} R_n(x) = 0 \tag{2.3.36}$$

for all x in I.

With this theorem, we can prove that a Taylor series for f at a converges to f if we can prove that the remainder $R_n(x) \to 0$. To prove that $R_n(x) \to 0$, we typically use the bound

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1} \tag{2.3.37}$$

from Taylor's theorem with remainder.

In the next example, we find the Maclaurin series for e^x and $\sin x$ and show that these series converge to the corresponding functions for all real numbers by proving that the remainders $R_n(x) \to 0$ for all real numbers x.

Example 2.3.6: Finding Maclaurin Series

For each of the functions, find the Maclaurin series and its interval of convergence. Use Note to prove that the Maclaurin series for f converges to f on that interval.

a. e^x
b. $\sin x$

Solution

a. Using the nth-degree Maclaurin polynomial for e^x found in Example a., we find that the Maclaurin series for e^x is given by

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

To determine the interval of convergence, we use the ratio test. Since

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|x|^{n+1}}{(n+1)!} \cdot \frac{n!}{|x|^n} = \frac{|x|}{n+1},$$

we have

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x|}{n+1} = 0$$

for all x. Therefore, the series converges absolutely for all x, and thus, the interval of convergence is $(-\infty, \infty)$. To show that the series converges to e^x for all x, we use the fact that $f^{(n)}(x) = e^x$ for all $n \ge 0$ and e^x is an increasing function on $(-\infty, \infty)$. Therefore, for any real number b, the maximum value of e^x for all $|x| \le b$ is e^b. Thus,

$$|R_n(x)| \le \frac{e^b}{(n+1)!}|x|^{n+1}.$$

Since we just showed that

$$\sum_{n=0}^{\infty} \frac{|x|^n}{n!}$$

converges for all x, by the divergence test, we know that

$$\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$$

for any real number x. By combining this fact with the squeeze theorem, the result is $\lim_{n\to\infty} R_n(x) = 0$.

b. Using the nth-degree Maclaurin polynomial for $\sin x$ found in Example b., we find that the Maclaurin series for $\sin x$ is given by

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

In order to apply the ratio test, consider

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|x|^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{|x|^{2n+1}} = \frac{|x|^2}{(2n+3)(2n+2)}.$$

Since

$$\lim_{n\to\infty} \frac{|x|^2}{(2n+3)(2n+2)} = 0$$

for all x, we obtain the interval of convergence as $(-\infty, \infty)$. To show that the Maclaurin series converges to $\sin x$, look at $R_n(x)$. For each x there exists a real number c between 0 and x such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}.$$

Since $| f^{(n+1)}(c) | \leq 1$ for all integers n and all real numbers c, we have

$$|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}$$

for all real numbers x. Using the same idea as in part a., the result is $\lim_{n\to\infty} R_n(x) = 0$ for all x, and therefore, the Maclaurin series for $\sin x$ converges to $\sin x$ for all real x.

Exercise 2.3.6

Find the Maclaurin series for $f(x) = \cos x$. Use the ratio test to show that the interval of convergence is $(-\infty, \infty)$. Show that the Maclaurin series converges to $\cos x$ for all real numbers x.

Hint available at the end of this section.

Answer available at the end of this section.

Proving that e is Irrational

In this project, we use the Maclaurin polynomials for e^x to prove that e is irrational. The proof relies on supposing that e is rational and arriving at a contradiction. Therefore, in the given steps, we suppose $e = r/s$ for some integers r and s where $s \neq 0$.

1. Write the Maclaurin polynomials $p_0(x), p_1(x), p_2(x), p_3(x), p_4(x)$ for e^x. Evaluate $p_0(1), p_1(1), p_2(1), p_3(1), p_4(1)$ to estimate e.
2. Let $R_n(x)$ denote the remainder when using $p_n(x)$ to estimate e^x. Therefore, $R_n(x) = e^x - p_n(x)$, and $R_n(1) = e - p_n(1)$. Assuming that $e = \frac{r}{s}$ for integers r and s, evaluate $R_0(1), R_1(1), R_2(1), R_3(1), R_4(1)$.
3. Using the results from part 2, show that for each remainder $R_0(1), R_1(1), R_2(1), R_3(1), R_4(1)$, we can find an integer k such that $kR_n(1)$ is an integer for $n = 0, 1, 2, 3, 4$.
4. Write down the formula for the nth-degree Maclaurin polynomial $p_n(x)$ for e^x and the corresponding remainder $R_n(x)$. Show that $sn!R_n(1)$ is an integer.
5. Use Taylor's theorem to write down an explicit formula for $R_n(1)$. Conclude that $R_n(1) \neq 0$, and therefore, $sn!R_n(1) \neq 0$.
6. Use Taylor's theorem to find an estimate on $R_n(1)$. Use this estimate combined with the result from part 5 to show that $|sn!R_n(1)| < \frac{se}{n+1}$. Conclude that if n is large enough, then $|sn!R_n(1)| < 1$. Therefore, $sn!R_n(1)$ is an integer with magnitude less than 1. Thus, $sn!R_n(1) = 0$. But from part 5, we know that $sn!R_n(1) \neq 0$. We have arrived at a contradiction, and consequently, the original supposition that e is rational must be false.

Key Concepts

- Taylor polynomials are used to approximate functions near a value $x = a$. Maclaurin polynomials are Taylor polynomials at $x = 0$.
- The nth-degree Taylor polynomials for a function f are the partial sums of the Taylor series for f.
- If a function f has a power series representation at $x = a$, then it is given by its Taylor series at $x = a$.
- A Taylor series for f converges to f if and only if $\lim_{n\to\infty} R_n(x) = 0$ where $R_n(x) = f(x) - p_n(x)$.
- The Taylor series for $e^x, \sin x$, and $\cos x$ converge to the respective functions for all real x.

Key Equations

- **Taylor series for the function f at the point $x = a$**

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f\,'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

Glossary

Maclaurin Polynomial
A Taylor polynomial centered at 0; the nth-degree Taylor polynomial for f at 0 is the nth-degree Maclaurin polynomial for f.

Maclaurin Series
A Taylor series for a function f at $x = 0$ is known as a Maclaurin series for f.

Taylor Polynomials
The nth-degree Taylor polynomial for f at $x = a$ is $p_n(x) = f(a) + f\,'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$.

Taylor Series
A power series at a that converges to a function f on some open interval containing a.

Taylor's Theorem with Remainder

For a function f and the nth-degree Taylor polynomial for f at $x = a$, the remainder $R_n(x) = f(x) - p_n(x)$ satisfies $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$

For some$c$ between x and a; if there exists an interval I containing a and a real number M such that $|\ f^{(n+1)}(x)$

$|\leq M$ for all x in I, then $|R_n(x)| \leq \dfrac{M}{(n+1)!}|x-a|^{n+1}$.

Answers

Exercise 2.3.1

Find the first three derivatives of f and evaluate them at $x = 1$.

Exercise 2.3.1

$$p_0(x) = 1 \tag{2.3.38}$$

$$p_1(x) = 1 - 2(x-1) \tag{2.3.39}$$

$$p_2(x) = 1 - 2(x-1) + 3(x-1)^2 \tag{2.3.40}$$

$$p_3(x) = 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 \tag{2.3.41}$$

Exercise 2.3.2

Evaluate the first four derivatives of f and look for a pattern.

Exercise 2.3.2

$p_0(x) = 1;\ \ p_1(x) = 1 - x;\ \ p_2(x) = 1 - x + x^2;\ \ p_3(x) = 1 - x + x^2 - x^3;\ \ p_n(x) = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n = \sum_{k=0}^{n} (-1)^k x^k$

Exercise 2.3.3

Evaluate $f(4), f\,'(4)$, and $f''(4)$.

Exercise 2.3.3

$p_1(x) = 2 + \dfrac{1}{4}(x-4); p_2(x) = 2 + \dfrac{1}{4}(x-4) - \dfrac{1}{64}(x-4)^2; p_1(6) = 2.5; p_2(6) = 2.4375;$

$|R_1(6)| \leq 0.0625; |R_2(6)| \leq 0.015625$

Exercise 2.3.4

The fourth Maclaurin polynomial is $p_4(x) = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!}$.

Exercise 2.3.4

0.96593

Exercise 2.3.5

$f^{(n)}(2) = \dfrac{(-1)^n n!}{2^{n+1}}$

Exercise 2.3.5

$\dfrac{1}{2}\sum_{n=0}^{\infty} (\dfrac{2-x}{2})^n$. The interval of convergence is $(0, 4)$.

Exercise 2.3.6

Use the Maclaurin polynomials for $\cos x$.

Exercise 2.3.6

$\sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n}}{(2n)!}$

By the ratio test, the interval of convergence is $(-\infty, \infty)$. Since $|R_n(x)| \leq \dfrac{|x|^{n+1}}{(n+1)!}$, the series converges to $\cos x$ for all real x.

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2.3E: Exercises for Section 2.3

Exercises 1 - 8

Find the Taylor polynomials of degree two approximating the given function centered at the given point.

1. $f(x) = 1 + x + x^2$ at $a = 1$
2. $f(x) = 1 + x + x^2$ at $a = -1$
3. $f(x) = \sin(2x)$ at $a = \frac{\pi}{2}$

Answer

$$f\,'(x) = 2\cos(2x);\quad f''(x) = -4\sin(2x);\qquad p_2(x) = -2(x - \tfrac{\pi}{2})$$

5. $f(x) = \sqrt{x}$ at $a = 4$
6. $f(x) = \ln x$ at $a = 1$

Answer

$$f\,'(x) = \frac{1}{x};\quad f''(x) = -\frac{1}{x^2};\qquad p_2(x) = 0 + (x-1) - \tfrac{1}{2}(x-1)^2$$

7. $f(x) = \dfrac{1}{x}$ at $a = 1$
8. $f(x) = e^x$ at $a = 1$

Answer

$$p_2(x) = e + e(x-1) + \frac{e}{2}(x-1)^2$$

Exercises 9 - 14 - Taylor Remainder Theorem

Verify that the given choice of n in the remainder estimate $|R_n| \le \dfrac{M}{(n+1)!}(x-a)^{n+1}$, where M is the maximum value of $|\,f^{(n+1)}(z)\,|$ on the interval between a and the indicated point, yields $|R_n| \le \frac{1}{1000}$. Find the value of the Taylor polynomial p_n of f at the indicated point.

9. [T] $\sqrt{10};\ \ a = 9,\ \ n = 3$
10. [T] $(28)^{1/3};\ \ a = 27,\ \ n = 1$

Answer

$\dfrac{d^2}{dx^2}x^{1/3} = -\dfrac{2}{9x^{5/3}} \ge -0.00092\ldots$ when $x \ge 28$ so the remainder estimate applies to the linear approximation $x^{1/3} \approx p_1(27) = 3 + \dfrac{x-27}{27}$, which gives $(28)^{1/3} \approx 3 + \frac{1}{27} = 3.0\bar{3}7$, while $(28)^{1/3} \approx 3.03658$.

11. [T] $\sin(6);\ \ a = 2\pi,\ \ n = 5$
12. [T] $e^2;\ \ a = 0,\ \ n = 9$

Answer

Using the estimate $\dfrac{2^{10}}{10!} < 0.000283$ we can use the Taylor expansion of order 9 to estimate e^x at $x = 2$. as $e^2 \approx p_9(2) = 1 + 2 + \frac{2^2}{2} + \frac{2^3}{6} + \cdots + \frac{2^9}{9!} = 7.3887\ldots$ whereas $e^2 \approx 7.3891$.

13. [T] $\cos(\frac{\pi}{5});\ \ a = 0,\ \ n = 4$
14. [T] $\ln(2);\ \ a = 1,\ \ n = 1000$

Answer

Since $\frac{d^n}{dx^n}(\ln x) = (-1)^{n-1}\frac{(n-1)!}{x^n}, R_{1000} \approx \frac{1}{1001}$. One has $p_{1000}(1) = \sum_{n=1}^{1000}\frac{(-1)^{n-1}}{n} \approx 0.6936$ whereas $\ln(2) \approx 0.6931\cdots$.

Exercises 15-16

Approximating Definite Integrals Using Taylor Series

15. Integrate the approximation $\sin t \approx t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{5040}$ evaluated at πt to approximate $\int_0^1 \frac{\sin \pi t}{\pi t}\, dt$.
16. Integrate the approximation $e^x \approx 1 + x + \frac{x^2}{2} + \cdots + \frac{x^6}{720}$ evaluated at $-x^2$ to approximate $\int_0^1 e^{-x^2}\, dx$.

Answer

$\int_0^1 \left(1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24} - \frac{x^{10}}{120} + \frac{x^{12}}{720}\right)\, dx = 1 - \frac{1^3}{3} + \frac{1^5}{10} - \frac{1^7}{42} + \frac{1^9}{9\cdot 24} - \frac{1^{11}}{120\cdot 11} + \frac{1^{13}}{720\cdot 13} \approx 0.74683$ whereas $\int_0^1 e^{-x^2}dx \approx 0.74682$.

Exercises 17 - 20

Find the smallest value of n such that the remainder estimate $|R_n| \le \frac{M}{(n+1)!}(x-a)^{n+1}$, where M is the maximum value of $|\, f^{(n+1)}(z)\,|$ on the interval between a and the indicated point, yields $|R_n| \le \frac{1}{1000}$ on the indicated interval.

17. $f(x) = \sin x$ on $[-\pi, \pi]$, $a = 0$
18. $f(x) = \cos x$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$, $a = 0$

Answer

Since $f^{(n+1)}(z)$ is $\sin z$ or $\cos z$, we have $M = 1$. Since $|x - 0| \le \frac{\pi}{2}$, we seek the smallest n such that $\frac{\pi^{n+1}}{2^{n+1}(n+1)!} \le 0.001$. The smallest such value is $n = 7$. The remainder estimate is $R_7 \le 0.00092$.

19. $f(x) = e^{-2x}$ on $[-1, 1], a = 0$
20. $f(x) = e^{-x}$ on $[-3, 3], a = 0$

Answer

Since $f^{(n+1)}(z) = \pm e^{-z}$ one has $M = e^3$. Since $|x - 0| \le 3$, one seeks the smallest n such that $\frac{3^{n+1}e^3}{(n+1)!} \le 0.001$. The smallest such value is $n = 14$. The remainder estimate is $R_{14} \le 0.000220$.

Exercises 21 - 24

The maximum of the right-hand side of the remainder estimate $|R_1| \le \frac{max|f''(z)|}{2}R^2$ on $[a - R, a + R]$ occurs at a or $a \pm R$. Estimate the maximum value of R such that $\frac{max|f''(z)|}{2}R^2 \le 0.1$ on $[a - R, a + R]$ by plotting this maximum as a function of R.

21. [T] e^x approximated by $1 + x$, $a = 0$
22. [T] $\sin x$ approximated by x, $a = 0$

Answer

Since $\sin x$ is increasing for small x and since $\frac{d^2}{dx^2}(\sin x) = -\sin x$, the estimate applies whenever $R^2 \sin(R) \le 0.2$, which applies up to $R = 0.596$.

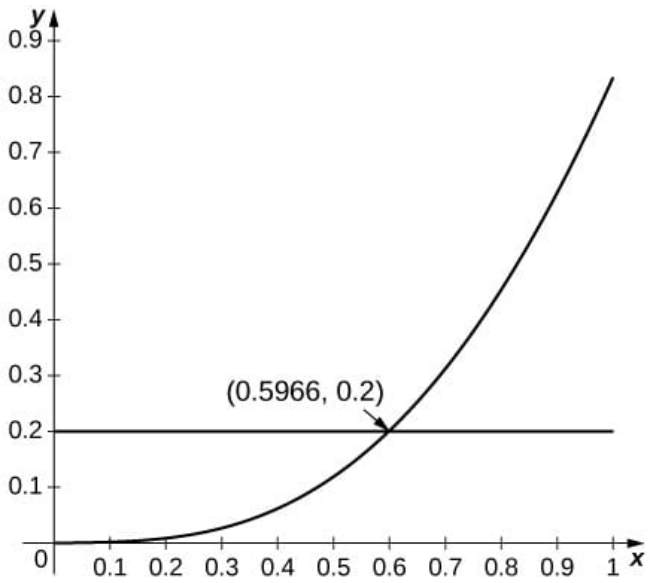


Figure $2.3E.1$: A graph showing an exponential curve that begins near the origin and increases as x approaches 1. The curve intersects a horizontal line at \(y=0.2\), with the intersection point clearly marked at $(0.5966, 0.2)$. The $x-axis$ is labeled from 0 to 1, and the $y-axis$ is labeled from 0 to 0.9. The graph highlights the relationship between the curve and the horizontal line, emphasizing their point of intersection.

23. [T] $\ln x$ approximated by $x-1, \quad a=1$
24. [T] $\cos x$ approximated by $1, \quad a=0$

Answer

Since the second derivative of $\cos x$ is $-\cos x$ and since $\cos x$ is decreasing away from $x=0$, the estimate applies when $R^2 \cos R \leq 0.2$ or $R \leq 0.447$.

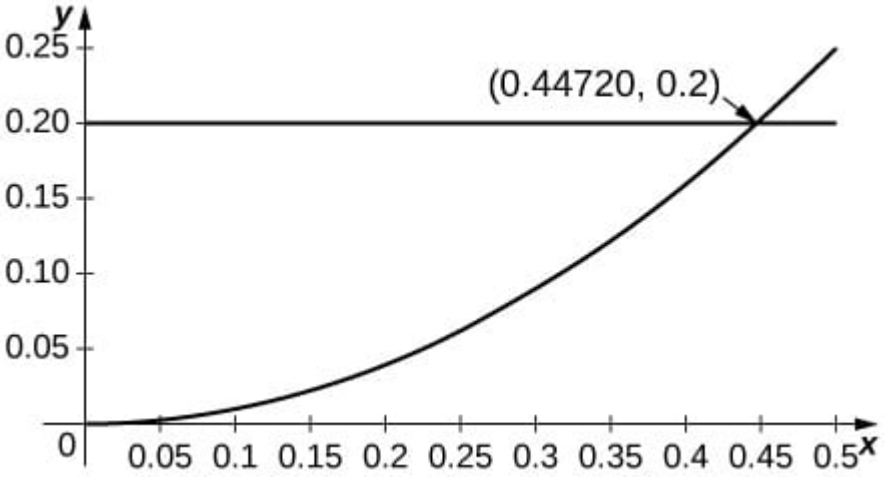


Figure $2.3E.2$: A graph showing an exponential curve starting at $(0,0)$ and increasing gradually as x approaches 0.5. A horizontal line at $y=0.2$ intersects the curve at the point $(0.44720, 0.2)$ which is marked with coordinates. The x-axis is labeled from 0 to 0.5, and the $y-axis$ is labeled from 0 to 0.25, illustrating the relationship between the function and the horizontal line.

Exercises 25 - 35

Find the Taylor series of the given function centered at the indicated point.

25. $f(x)=x^4$ at $a=-1$
26. $f(x)=1+x+x^2+x^3$ at $a=-1$

Answer

$(x+1)^3 - 2(x+1)^2 + 2(x+1)$

27. $f(x) = \sin x$ at $a = \pi$
28. $f(x) = \cos x$ at $a = 2\pi$

Answer

Values of derivatives are the same as for $x = 0$ so $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{(x-2\pi)^{2n}}{(2n)!}$

29. $f(x) = \sin x$ at $x = \frac{\pi}{2}$
30. $f(x) = \cos x$ at $x = \frac{\pi}{2}$

Answer

$\cos(\frac{\pi}{2}) = 0, \ -\sin(\frac{\pi}{2}) = -1$ so $\cos x = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-\frac{\pi}{2})^{2n+1}}{(2n+1)!}$, which is also $-\cos(x - \frac{\pi}{2})$.

31. $f(x) = e^x$ at $a = -1$
32. $f(x) = e^x$ at $a = 1$

Answer

The derivatives are $f^{(n)}(1) = e$, so $e^x = e \sum_{n=0}^{\infty} \frac{(x-1)^n}{n!}$.

33. $f(x) = \dfrac{1}{(x-1)^2}$ at $a = 0$ (Hint: Differentiate the Taylor Series for $\dfrac{1}{1-x}$.)
34. $f(x) = \dfrac{1}{(x-1)^3}$ at $a = 0$

Answer

$$\frac{1}{(x-1)^3} = -\frac{1}{2}\frac{d^2}{dx^2}\left(\frac{1}{1-x}\right) = -\sum_{n=0}^{\infty}\left(\frac{(n+2)(n+1)x^n}{2}\right)$$

35. $F(x) = \int_0^x \cos(\sqrt{t})\ dt; \quad$ where $f(t) = \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{(2n)!}$ at a=0 (Note: f is the Taylor series of $\cos(\sqrt{t})$.)

Exercises 36 - 44

Compute the Taylor series of each function around $x = 1$.

36. $f(x) = 2 - x$

Answer

$2 - x = 1 - (x-1)$

37. $f(x) = x^3$
38. $f(x) = (x-2)^2$

Answer

$((x-1)-1)^2 = (x-1)^2 - 2(x-1) + 1$

39. $f(x) = \ln x$
40. $f(x) = \dfrac{1}{x}$

Answer

$$\frac{1}{1-(1-x)} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$$

41. $f(x) = \dfrac{1}{2x - x^2}$
42. $f(x) = \dfrac{x}{4x - 2x^2 - 1}$

Answer

$$x \sum_{n=0}^{\infty} 2^n (1-x)^{2n} = \sum_{n=0}^{\infty} 2^n (x-1)^{2n+1} + \sum_{n=0}^{\infty} 2^n (x-1)^{2n}$$

43. $f(x) = e^{-x}$
44. $f(x) = e^{2x}$

Answer

$$e^{2x} = e^{2(x-1)+2} = e^2 \sum_{n=0}^{\infty} \frac{2^n (x-1)^n}{n!}$$

Exercises 45 - 48

[T] Identify the value of x such that the given series $\sum_{n=0}^{\infty} a_n$ is the value of the Maclaurin series of $f(x)$ at x. Approximate the value of $f(x)$ using $S_{10} = \sum_{n=0}^{10} a_n$.

45. $\sum_{n=0}^{\infty} \dfrac{1}{n!}$
46. $\sum_{n=0}^{\infty} \dfrac{2^n}{n!}$

Answer

$$x = e^2; \quad S_{10} = \frac{34,913}{4725} \approx 7.3889947$$

47. $\sum_{n=0}^{\infty} \dfrac{(-1)^n (2\pi)^{2n}}{(2n)!}$
48. $\sum_{n=0}^{\infty} \dfrac{(-1)^n (2\pi)^{2n+1}}{(2n+1)!}$

Answer

$$\sin(2\pi) = 0; \quad S_{10} = 8.27 \times 10^{-5}$$

Exercises 49 - 52

Use the functions $S_5(x) = x - \dfrac{x^3}{6} + \dfrac{x^5}{120}$ and $C_4(x) = 1 - \dfrac{x^2}{2} + \dfrac{x^4}{24}$ on $[-\pi, \pi]$.

49. [T] Plot $\sin^2 x - (S_5(x))^2$ on $[-\pi, \pi]$. Compare the maximum difference with the square of the Taylor remainder estimate for $\sin x$.
50. [T] Plot $\cos^2 x - (C_4(x))^2$ on $[-\pi, \pi]$. Compare the maximum difference with the square of the Taylor remainder estimate for $\cos x$.

Answer

The difference is small on the interior of the interval but approaches 1 near the endpoints. The remainder estimate is $|R_4| = \frac{\pi^5}{120} \approx 2.552$.

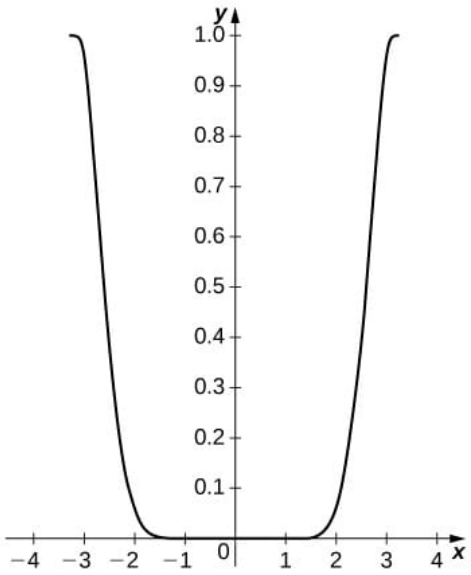


Figure $2.3E.3$: A graph of a function with a U-shaped curve that dips near $y = 0$ around $x = 0$ and rises steeply as x moves away from 0 in both the positive and negative directions. The $y - axis$ ranges from 0 to 1, and the $x - axis$ ranges from -4 to 4, illustrating a symmetric function with a minimum point near the origin.

51. [T] Plot $|2S_5(x)C_4(x) - \sin(2x)|$ on $[-\pi, \pi]$.

52. [T] Compare $\dfrac{S_5(x)}{C_4(x)}$ on $[-1, 1]$ to $\tan x$. Compare this with the Taylor remainder estimate for the approximation of $\tan x$ by $x + \dfrac{x^3}{3} + \dfrac{2x^5}{15}$.

Answer

The difference is on the order of 10^{-4} on $[-1, 1]$ while the Taylor approximation error is around 0.1 near ± 1. The top curve is a plot of $\tan^2 x - \left(\dfrac{S_5(x)}{C_4(x)}\right)^2$ and the lower dashed plot shows $t^2 - \left(\dfrac{S_5}{C_4}\right)^2$.

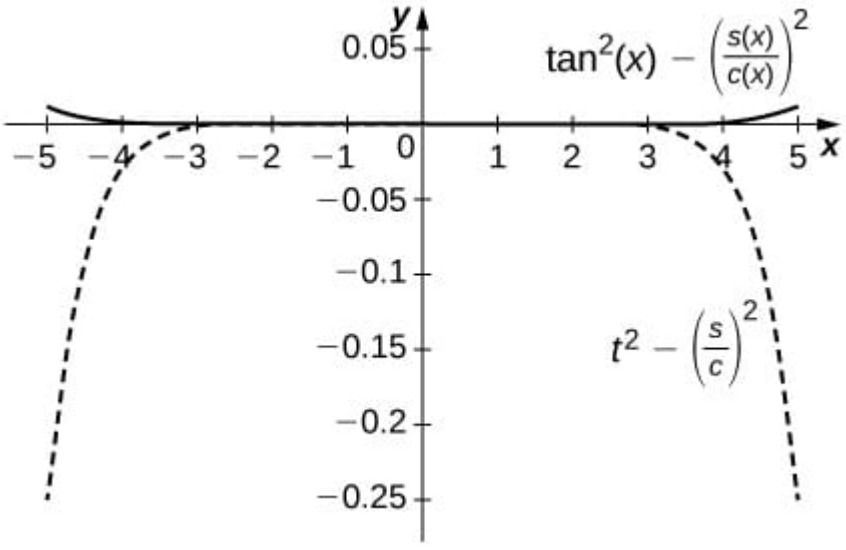


Figure $2.3E.4$:A graph depicting the function $\tan^2 x - \left(\dfrac{S_(x)}{C_(x)}\right)^2$ with the x-axis ranging from -5 to 5 and the y-axis ranging from -0.25 to 0.05. The solid curve near the x-axis represents small variations, while dashed curves diverge sharply for larger magnitudes of x, indicating asymptotic behavior.

53. [T] Plot $e^x - e_4(x)$ where $e_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$ on $[0, 2]$. Compare the maximum error with the Taylor remainder estimate.

54. (Taylor approximations and root finding.) Recall that Newton's method $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ approximates solutions of $f(x) = 0$ near the input x_0.
    a. If f and g are inverse functions, explain why a solution of $g(x) = a$ is the value $f(a)$ of f.
    b. Let $p_N(x)$ be the Nth degree Maclaurin polynomial of e^x. Use Newton's method to approximate solutions of $p_N(x) - 2 = 0$ for $N = 4, 5, 6$.
    c. Explain why the approximate roots of $p_N(x) - 2 = 0$ are approximate values of $\ln(2)$.

Answer

a. Answers will vary.

b. The following are the x_n values after 10 iterations of Newton's method to approximation a root of $p_N(x) - 2 = 0$: for $N = 4, x = 0.6939...$; for $N = 5, x = 0.6932...$; for $N = 6, x = 0.69315...$; . (Note: $\ln(2) = 0.69314...$)

c. Answers will vary.

Exercises 55 - 58

Use the fact that if $q(x) = \sum_{n=1}^{\infty} a_n(x-c)^n$ converges in an interval containing c, then $\lim_{x\to c} q(x) = a_0$ to evaluate each limit using Taylor series.

55. $\lim_{x\to 0} \frac{\cos x - 1}{x^2}$

56. $\lim_{x\to 0} \frac{\ln(1-x^2)}{x^2}$

Answer

$$\frac{\ln(1-x^2)}{x^2} \to -1$$

57. $\lim_{x\to 0} \frac{e^{x^2} - x^2 - 1}{x^4}$

58. $\lim_{x\to 0^+} \frac{\cos(\sqrt{x}) - 1}{2x}$

Answer

$$\frac{\cos(\sqrt{x}) - 1}{2x} \approx \frac{(1 - \frac{x}{2} + \frac{x^2}{4!} - \cdots) - 1}{2x} \to -\frac{1}{4}$$

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2.4: Working with Taylor Series

Learning Objectives

- Write the terms of the binomial series.
- Recognize the Taylor series expansions of common functions.
- Recognize and apply techniques to find the Taylor series for a function.
- Use Taylor series to solve differential equations.
- Use Taylor series to evaluate non-elementary integrals.

In the preceding section, we defined Taylor series and showed how to find the Taylor series for several common functions by explicitly calculating the coefficients of the Taylor polynomials. In this section we show how to use those Taylor series to derive Taylor series for other functions. We then present two common applications of power series. First, we show how power series can be used to solve differential equations. Second, we show how power series can be used to evaluate integrals when the anti derivative of the integrand cannot be expressed in terms of elementary functions. In one example, we consider $\int e^{-x^2}dx$, an integral that arises frequently in probability theory.

The Binomial Series

Our first goal in this section is to determine the Maclaurin series for the function $f(x) = (1+x)^r$ for all real numbers r. The Maclaurin series for this function is known as the **binomial series.** We begin by considering the simplest case: r is a nonnegative integer. We recall that, for $r = 0, 1, 2, 3, 4, f(x) = (1+x)^r$ can be written as

$$\begin{aligned} f(x) = (1+x)^0 &= 1, \\ f(x) = (1+x)^1 &= 1+x, \\ f(x) = (1+x)^2 &= 1+2x+x^2, \\ f(x) = (1+x)^3 &= 1+3x+3x^2+x^3 \\ f(x) = (1+x)^4 &= 1+4x+6x^2+4x^3+x^4. \end{aligned}$$

The expressions on the right-hand side are known as binomial expansions and the coefficients are known as binomial coefficients. More generally, for any nonnegative integer r, the binomial coefficient of x^n in the binomial expansion of $(1+x)^r$ is given by

$$\binom{r}{n} = \frac{r!}{n!(r-n)!} \tag{2.4.1}$$

and

$$f(x) = (1+x)^r = \binom{r}{0}1 + \binom{r}{1}x + \binom{r}{2}x^2 + \binom{r}{3}x^3 + \cdots + \binom{r}{r-1}x^{r-1} + \binom{r}{r}x^r = \sum_{n=0}^{r}\binom{r}{n}x^n. \tag{2.4.2}$$

For example, using this formula for $r = 5$, we see that

$$\begin{aligned} f(x) = (1+x)^5 &= \tbinom{5}{0}1+\tbinom{5}{1}x+\tbinom{5}{2}x^2+\tbinom{5}{3}x^3+\tbinom{5}{4}x^4+\tbinom{5}{5}x^5 \\ &= \frac{5!}{0!5!}1 + \frac{5!}{1!4!}x + \frac{5!}{2!3!}x^2 + \frac{5!}{3!2!}x^3 + \frac{5!}{4!1!}x^4 + \frac{5!}{5!0!}x^5 \\ &= 1+5x+10x^2+10x^3+5x^4+x^5. \end{aligned}$$

We now consider the case when the exponent r

is any real number, not necessarily a nonnegative integer. If r is not a nonnegative integer, then $f(x) = (1+x)^r$ cannot be written as a finite polynomial. However, we can find a power series for f. Specifically, we look for the Maclaurin series for f. To do this, we find the derivatives of f and evaluate them at $x = 0$.

$$
\begin{aligned}
f(x) &= (1+x)^r \\
f(0) &= 1 \\
f'(x) &= r(1+x)^{r-1} \\
f'(0) &= r \\
f''(x) &= r(r-1)(1+x)^{r-2} \\
f''(0) &= r(r-1) \\
f'''(x) &= r(r-1)(r-2)(1+x)^{r-3} \\
f'''(0) &= r(r-1)(r-2) \\
f(n)(x) &= r(r-1)(r-2)\cdots(r-n+1)(1+x)^{r-n} \\
f^{(n)}(0) &= r(r-1)(r-2)\cdots(r-n+1)
\end{aligned}
$$

We conclude that the coefficients in the binomial series are given by

$$\frac{f^{(n)}(0)}{n!} = \frac{r(r-1)(r-2)\cdots(r-n+1)}{n!}. \tag{2.4.3}$$

We note that if r is a nonnegative integer, then the $(r+1)st$ derivative $f^{(r+1)}$ is the zero function, and the series terminates. In addition, if r is a nonnegative integer, then Equation for the coefficients agrees with Equation for the coefficients, and the formula for the binomial series agrees with Equation for the finite binomial expansion. More generally, to denote the binomial coefficients for any real number r, we define

$$\binom{r}{n} = \frac{(r-1)(r-2)\cdots(r-n+1)}{n!}. \tag{2.4.4}$$

With this notation, we can write the binomial series for $(1+x)^r$ as

$$\sum_{n=0}^{\infty}\binom{r}{n}x^n = 1 + rx + \frac{r(r-1)}{2!}x^2 + \cdots + \frac{r(r-1)\cdots(r-n+1)}{n!}x^n + \cdots. \tag{2.4.5}$$

We now need to determine the interval of convergence for the binomial series Equation 2.4.5. We apply the ratio test. Consequently, we consider

$$
\begin{aligned}
\frac{|a_{n+1}|}{|a_n|} = \frac{|r(r-1)(r-2)\cdots(r-n)||x||^{n+1}}{(n+1)!} \cdot \frac{n}{|r(r-1)(r-2)\cdots(r-n+1)||x|^n} \\
= \frac{|r-n||x|}{|n+1|}
\end{aligned}
$$

.

Since

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = |x| < 1 \tag{2.4.6}$$

if and only if $|x| < 1$, we conclude that the interval of convergence for the binomial series is $(-1, 1)$. The behavior at the endpoints depends on r. It can be shown that for $r \geq 0$ the series converges at both endpoints; for \(–1

Definition: Binomial Series

For any real number r, the Maclaurin series for $f(x) = (1+x)^r$ is the **binomial series**. It converges to f for $|x| < 1$, and we write

$$(1+x)^r = \sum_{n=0}^{\infty}\binom{r}{n}x^n = 1 + rx + \frac{r(r-1)}{2!}x^2 + \cdots + r\frac{(r-1)\cdots(r-n+1)}{n!}x^n + \cdots \tag{2.4.7}$$

for $|x| < 1$.

We can use this definition to find the binomial series for $f(x) = \sqrt{1+x}$ and use the series to approximate $\sqrt{1.5}$.

Example 2.4.1: Finding Binomial Series

a. Find the binomial series for $f(x) = \sqrt{1+x}$.
b. Use the third-order Maclaurin polynomial $p_3(x)$ to estimate $\sqrt{1.5}$. Use Taylor's theorem to bound the error. Use a graphing utility to compare the graphs of f and p_3.

Solution

a. Here $r = \dfrac{1}{2}$. Using the definition for the binomial series, we obtain

$$\begin{aligned}\sqrt{1+x} &= 1 + \frac{1}{2}x + \frac{(1/2)(-1/2)}{2!}x^2 + \frac{(1/2)(-1/2)(-3/2)}{3!}x^3 + \cdots \\ &= 1 + \frac{1}{2}x - \frac{1}{2!}\frac{1}{2^2}x^2 + \frac{1}{3!}\frac{1\cdot 3}{2^3}x^3 - \cdots + \frac{(-1)^{n+1}}{n!}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n}x^n + \cdots \\ &= 1 + \textstyle\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n!}\dfrac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n}x^n.\end{aligned}$$

b. From the result in part a. the third-order Maclaurin polynomial is

$$p_3(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3.$$

Therefore,

$$\sqrt{1.5} = \sqrt{1+0.5} \approx 1 + \frac{1}{2}(0.5) - \frac{1}{8}(0.5)^2 + \frac{1}{16}(0.5)^3 \approx 1.2266.$$

From Taylor's theorem, the error satisfies

$$R_3(0.5) = \frac{f^{(4)}(c)}{4!}(0.5)^4$$

for some c between 0 and 0.5. Since $f^{(4)}(x) = -\dfrac{15}{2^4(1+x)^{7/2}}$, and the maximum value of $|\, f^{(4)}(x)\,|$ on the interval $(0, 0.5)$ occurs at $x = 0$, we have

$$|R_3(0.5)| \le \frac{15}{4!2^4}(0.5)^4 \approx 0.00244.$$

The function and the Maclaurin polynomial p_3 are graphed in the Figure 2.4.1.

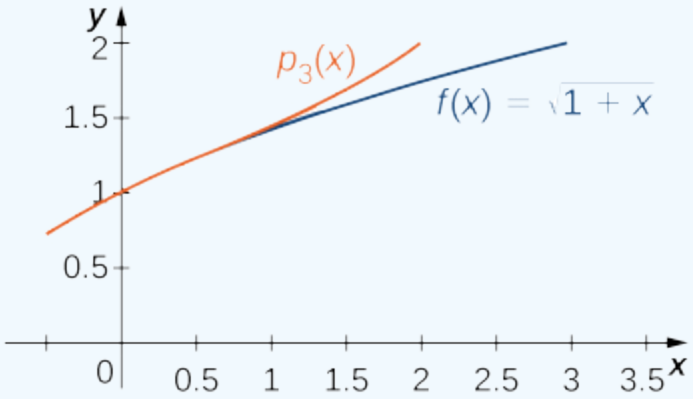


Figure 2.4.1: The third-order Maclaurin polynomial $p_3(x)$ provides a good approximation for $f(x) = \sqrt{1+x}$ for x near zero.

Exercise 2.4.1

Find the binomial series for $f(x) = \dfrac{1}{(1+x)^2}$.

Hint available at the end of this section.

Answer available at the end of this section.

Common Functions Expressed as Taylor Series

At this point, we have derived Maclaurin series for exponential, trigonometric, and logarithmic functions, as well as functions of the form $f(x) = (1+x)^r$. In Table 2.4.1, we summarize the results of these series. We remark that the convergence of the Maclaurin series for $f(x) = ln(1+x)$ at the endpoint $x = 1$ and the Maclaurin series for $f(x) = tan^{-1}x$ at the endpoints $x = 1$ and $x = -1$ relies on a more advanced theorem than we present here. (Refer to Abel's theorem for a discussion of this more technical point.)

Table 2.4.1: Maclaurin Series for Common Functions

Function	Maclaurin Series	Interval of Convergence
$f(x) = \dfrac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	\(−1
$f(x) = e^x$	$\sum_{n=0}^{\infty} \dfrac{x^n}{n!}$	\(−∞
$f(x) = \sin x$	$\sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!}$	\(−∞
$f(x) = \cos x$	$\sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!}$	\(−∞
$f(x) = \ln(1+x)$	$\sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{x^n}{n}$	\(−1
$f(x) = tan^{-1}x$	$\sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1}$	\(−1
$f(x) = (1+x)^r$	$\sum_{n=0}^{\infty} \binom{r}{n} x^n$	\(−1

Earlier in the chapter, we showed how you could combine power series to create new power series. Here we use these properties, combined with the Maclaurin series in Table 2.4.1, to create Maclaurin series for other functions.

Example 2.4.2: Deriving Maclaurin Series from Known Series

Find the Maclaurin series of each of the functions by using one of the series listed in Table 2.4.1.

a. $f(x) = \cos\sqrt{x}$
b. $f(x) = \sinh x$

Solution

a. Using the Maclaurin series for $\cos x$ we find that the Maclaurin series for $\cos\sqrt{x}$ is given by

$$\sum_{n=0}^{\infty} \frac{(-1)^n(\sqrt{x})^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!} = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \frac{x^4}{8!} - \cdots.$$

This series converges to $cos\sqrt{x}$ for all x in the domain of $cos\sqrt{x}$; that is, for all $x \geq 0$.

b. To find the Maclaurin series for $sinhx$, we use the fact that

$$\sinh x = \frac{e^x - e^{-x}}{2}.$$

Using the Maclaurin series for e^x, we see that the nth term in the Maclaurin series for $sinhx$ is given by

$$\frac{x^n}{n!} - \frac{(-x)^n}{n!}.$$

For n even, this term is zero. For n odd, this term is $\frac{2x^n}{n!}$. Therefore, the Maclaurin series for $sinhx$ has only odd-order terms and is given by

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots.$$

Watch the accompanying video for a visual demonstration of this concept.

- **Video Length:** 2 minutes.
- **Context:** This video demonstrates how to use a known Maclaurin Series $f(x) = x^2\ cosx^{3/2}$ to find a more complicated Maclaurin Series.

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Exercise 2.4.2

Find the Maclaurin series for $\sin(x^2)$.

Hint available at the end of this section.

Answer available at the end of this section.

We also showed previously in this chapter how power series can be differentiated term by term to create a new power series. In Example, we differentiate the binomial series for $\sqrt{1+x}$ term by term to find the binomial series for $\frac{1}{\sqrt{1+x}}$. Note that we

could construct the binomial series for $\frac{1}{\sqrt{1+x}}$ directly from the definition, but differentiating the binomial series for $\sqrt{1+x}$ is an easier calculation.

Example 2.4.3: Differentiating a Series to Find a New Series

Use the binomial series for $\sqrt{1+x}$ to find the binomial series for $\frac{1}{\sqrt{1+x}}$.

Solution

The two functions are related by

$$\frac{d}{dx}\sqrt{1+x} = \frac{1}{2\sqrt{1+x}},$$

so the binomial series for $\frac{1}{\sqrt{1+x}}$ is given by

$$\frac{1}{\sqrt{1+x}} = 2\frac{d}{dx}\sqrt{1+x} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n}x^n.$$

Exercise 2.4.3

Find the binomial series for $f(x) = \frac{1}{(1+x)^{3/2}}$

Hint available at the end of this section.

Answer available at the end of this section.

In this example, we differentiated a known Taylor series to construct a Taylor series for another function. The ability to differentiate power series term by term makes them a powerful tool for solving differential equations. We now show how this is accomplished.

Solving Differential Equations with Power Series

Consider the differential equation

$$y'(x) = y. \tag{2.4.8}$$

Recall that this is a first-order separable equation and its solution is $y = Ce^x$. This equation is easily solved using techniques discussed earlier in the text. For most differential equations, however, we do not yet have analytical tools to solve them. Power series are an extremely useful tool for solving many types of differential equations. In this technique, we look for a solution of the form $y = \sum_{n=0}^{\infty} c_n x^n$ and determine what the coefficients would need to be. Shifting the index of summation is an important tool used in solving differential equations with power series.

In the next example, we consider an initial-value problem involving $y' = y$ to illustrate the technique.

Example 2.4.4: Power Series Solution of a Differential Equation

Use power series to solve the initial-value problem

$$y' = y, y(0) = 3. \tag{2.4.9}$$

Solution

Suppose that there exists a power series solution

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots.$$

Differentiating this series term by term, we obtain

$$y' = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots.$$

If y satisfies the differential equation, then

$$c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots = c_1 + 2c_2 x + 3c_3 x^2 + 4c_3 x^3 + \cdots.$$

Using the uniqueness of power series representations, we know that these series can only be equal if their coefficients are equal. Therefore,

$$c_0 = c_1,$$

$$c_1 = 2c_2,$$

$$c_2 = 3c_3,$$

$$c_3 = 4c_4,$$

$$\vdots.$$

Using the initial condition $y(0) = 3$ combined with the power series representation

$$y(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots,$$

we find that $c_0 = 3$. We are now ready to solve for the rest of the coefficients. Using the fact that $c_0 = 3$, we have

$$c_1 = c_0 = 3 = \frac{3}{1!},$$

$$c_2 = \frac{c_1}{2} = \frac{3}{2} = \frac{3}{2!},$$

$$c_3 = \frac{c_2}{3} = \frac{3}{3 \cdot 2} = \frac{3}{3!},$$

$$c_4 = \frac{c_3}{4} = \frac{3}{4 \cdot 3 \cdot 2} = \frac{3}{4!}.$$

Therefore,

$$y = 3[1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 \frac{1}{4!}x^4 + \cdots] = 3\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

You might recognize

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

as the Taylor series for e^x. Therefore, the solution is $y = 3e^x$.

Watch the accompanying video for a visual demonstration of this concept.

- **Video Length:** 6 minute 26 seconds.
- **Context:** This video demonstrates how to verify that a given power series is a solution to a differential equation y'' $= \sum_{n=1}^{\infty} \frac{(-1)^n 4^n (2n)(2n-1)x^{2n-1}}{2n!}$

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Exercise 2.4.4

Use power series to solve $y' = 2y, y(0) = 5$.

Hint available at the end of this section.

Answer available at the end of this section.

We now consider an example involving a differential equation that we cannot solve using previously discussed methods. This differential equation

$$y'-xy = 0 \tag{2.4.10}$$

is known as **Airy's equation**. It has many applications in mathematical physics, such as modeling the diffraction of light. Here we show how to solve it using power series.

Example 2.4.5: Power Series Solution of Airy's Equation

Use power series to solve

$$y'' - xy = 0 \tag{2.4.11}$$

with the initial conditions $y(0) = a$ and $y'(0) = b$.

Solution

We look for a solution of the form

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots.$$

Differentiating this function term by term, we obtain

$$y' = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots,$$

$$y'' = 2 \cdot 1c_2 + 3 \cdot 2c_3 x + 4 \cdot 3c_4 x^2 + \cdots.$$

If y satisfies the equation $y'' = xy$, then

$$2 \cdot 1c_2 + 3 \cdot 2c_3 x + 4 \cdot 3c_4 x^2 + \cdots = x(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots).$$

Using [link] on the uniqueness of power series representations, we know that coefficients of the same degree must be equal. Therefore,

$$2 \cdot 1c_2 = 0,$$

$$3 \cdot 2c_3 = c_0,$$

$$4 \cdot 3c_4 = c_1,$$

$$5 \cdot 4c_5 = c_2,$$

$$\vdots.$$

More generally, for $n \geq 3$, we have $n \cdot (n-1)c_n = c_{n-3}$. In fact, all coefficients can be written in terms of c_0 and c_1. To see this, first note that $c_2 = 0$. Then

$$c_3 = \frac{c_0}{3 \cdot 2},$$

$$c_4 = \frac{c_1}{4 \cdot 3}.$$

For c_5, c_6, c_7, we see that

$$c_5 = \frac{c_2}{5 \cdot 4} = 0,$$

$$c_6 = \frac{c_3}{6 \cdot 5} = \frac{c_0}{6 \cdot 5 \cdot 3 \cdot 2},$$

$$c_7 = \frac{c_4}{7 \cdot 6} = \frac{c_1}{7 \cdot 6 \cdot 4 \cdot 3}.$$

Therefore, the series solution of the differential equation is given by

$$y = c_0 + c_1 x + 0 \cdot x^2 + \frac{c_0}{3 \cdot 2}x^3 + \frac{c_1}{4 \cdot 3}x^4 + 0 \cdot x^5 + \frac{c_0}{6 \cdot 5 \cdot 3 \cdot 2}x^6 + \frac{c_1}{7 \cdot 6 \cdot 4 \cdot 3}x^7 + \cdots.$$

The initial condition $y(0) = a$ implies $c_0 = a$. Differentiating this series term by term and using the fact that $y\,'(0) = b$, we conclude that $c_1 = b$.

Therefore, the solution of this initial-value problem is

$$y = a(1 + \frac{x^3}{3 \cdot 2} + \frac{x}{6 \cdot 5 \cdot 3 \cdot 2} + \cdots) + b(x + \frac{x^4}{4 \cdot 3} + \frac{x^7}{7 \cdot 6 \cdot 4 \cdot 3} + \cdots).$$

Exercise 2.4.5

Use power series to solve $y'' + x^2 y = 0$ with the initial condition $y(0) = a$ and $y\,'(0) = b$.

Hint available at the end of this section.

Answer available at the end of this section.

Evaluating Nonelementary Integrals

Solving differential equations is one common application of power series. We now turn to a second application. We show how power series can be used to evaluate integrals involving functions whose antiderivatives cannot be expressed using elementary functions.

One integral that arises often in applications in probability theory is $\int e^{-x^2} dx$. Unfortunately, the antiderivative of the integrand e^{-x^2} is not an elementary function. By elementary function, we mean a function that can be written using a finite number of algebraic combinations or compositions of exponential, logarithmic, trigonometric, or power functions. We remark that the term "elementary function" is not synonymous with noncomplicated function. For example, the function $f(x) = \sqrt{x^2 - 3x} + e^{x^3} - sin(5x + 4)$ is an elementary function, although not a particularly simple-looking function. Any integral of the form $\int f(x)dx$ where the antiderivative of f cannot be written as an elementary function is considered a **nonelementary integral.**

Nonelementary integrals cannot be evaluated using the basic integration techniques discussed earlier. One way to evaluate such integrals is by expressing the integrand as a power series and integrating term by term. We demonstrate this technique by considering $\int e^{-x^2} dx$.

Example 2.4.6: Using Taylor Series to Evaluate a Definite Integral

a. Express $\int e^{-x^2}dx$ as an infinite series.
b. Evaluate $\int_0^1 e^{-x^2}dx$ to within an error of 0.01.

Solution

a. The Maclaurin series for e^{-x^2} is given by

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + (-1)^n \frac{x^{2n}}{n!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}.$$

Therefore,

$$\int e^{-x^2}dx = \int (1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + (-1)^n \frac{x^{2n}}{n!} + \cdots)dx = C + x - \frac{x^3}{3} + \frac{x^5}{5.2!} - \frac{x^7}{7.3!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)n!} + \cdots.$$

b. Using the result from part a. we have

$$\int_0^1 e^{-x^2}dx = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \cdots.$$

The sum of the first four terms is approximately 0.74. By the alternating series test, this estimate is accurate to within an error of less than $\frac{1}{216} \approx 0.0046296 < 0.01$.

Watch the accompanying video for a visual demonstration of this concept.

- **Video Length:** 5 minute 9 seconds.
- **Context:** This video demonstrates how to use Power Series to integrate and approximate the integral with an error less than 0.001.

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Exercise 2.4.6

Express $\int cos\sqrt{x}dx$ as an infinite series. Evaluate $\int_0^1 cos\sqrt{x}dx$ to within an error of 0.01.

Hint available at the end of this section.

Answer available at the end of this section.

As mentioned above, the integral $\int e^{-x^2}dx$ arises often in probability theory. Specifically, it is used when studying data sets that are normally distributed, meaning the data values lie under a bell-shaped curve. For example, if a set of data values is normally distributed with mean μ and standard deviation σ, then the probability that a randomly chosen value lies between $x = a$ and $x = b$ is given by

$$\frac{1}{\sigma\sqrt{2\pi}} \int_a^b e^{-(x-\mu)^2/(2\sigma^2)}dx. \tag{2.4.12}$$

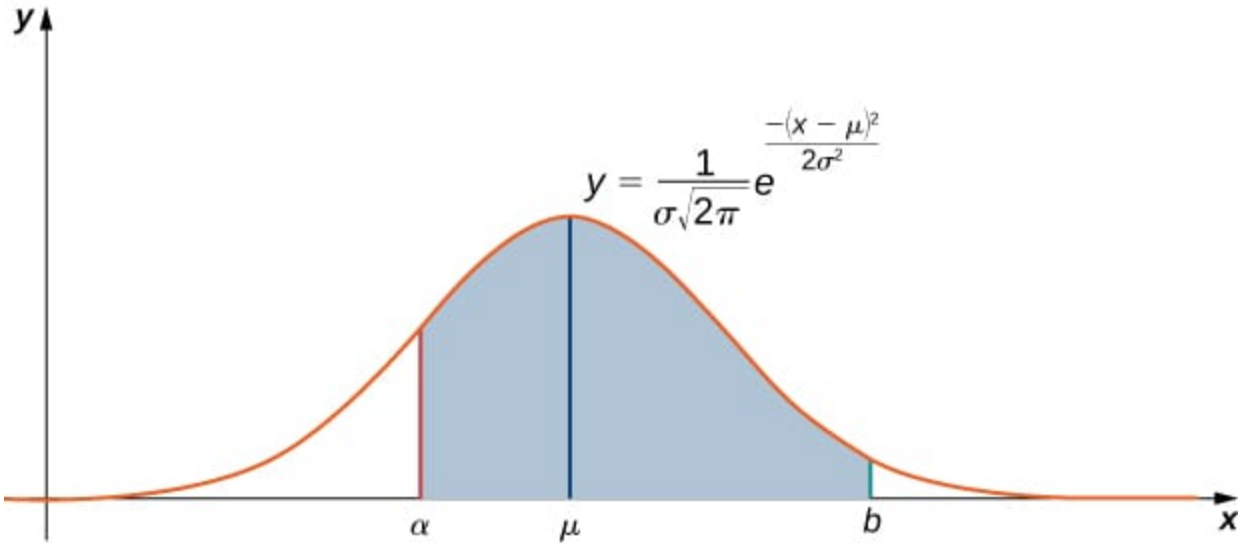


Figure 2.4.2: If data values are normally distributed with mean μ and standard deviation σ, the probability that a randomly selected data value is between a and b is the area under the curve $y = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$ between $x = a$ and $x = b$.

To simplify this integral, we typically let $z = \frac{x-\mu}{\sigma}$. This quantity z is known as the z score of a data value. With this simplification, integral Equation becomes

$$\frac{1}{\sqrt{2\pi}} \int_{(a-\mu)/\sigma}^{(b-\mu)/\sigma} e^{-z^2/2} dz. \tag{2.4.13}$$

In the next example, we show how we can use this integral in calculating probabilities.

Example 2.4.7: Using Maclaurin Series to Approximate a Probability

Suppose a set of standardized test scores are normally distributed with mean $\mu = 100$ and standard deviation $\sigma = 50$. Use Equation and the first six terms in the Maclaurin series for $e^{-x^2/2}$ to approximate the probability that a randomly selected test score is between $x = 100$ and $x = 200$. Use the alternating series test to determine how accurate your approximation is.

Solution

Since $\mu = 100, \sigma = 50,$ and we are trying to determine the area under the curve from $a = 100$ to $b = 200$, integral Equation becomes

$$\frac{1}{\sqrt{2\pi}} \int_0^2 e^{-z^2/2} dz.$$

The Maclaurin series for $e^{-x^2/2}$ is given by

$$e^{-x^2/2} = \sum_{n=0}^{\infty} \frac{(-\frac{x^2}{2})^n}{n!} = 1 - \frac{x^2}{2^1 \cdot 1!} + \frac{x^4}{2^2 \cdot 2!} - \frac{x^6}{2^3 \cdot 3!} + \cdots + (-1)^n \frac{x^{2n}}{2^n \cdot n}! + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^n \cdot n!}.$$

Therefore,

$$\frac{1}{\sqrt{2\pi}} \int e^{-z^2/2} dz = \frac{1}{\sqrt{2\pi}} \int (1 - \frac{z^2}{2^1 \cdot 1!} + \frac{z^4}{2^2 \cdot 2!} - \frac{z^6}{2^3 \cdot 3!} + \cdots + (-1)^n \frac{z^{2n}}{2^n \cdot n!} + \cdots) dz = \frac{1}{\sqrt{2\pi}} (C + z - \frac{z^3}{3 \cdot 2^1 \cdot 1!} + \frac{z^5}{5 \cdot 2^2 \cdot 2!} - \frac{z^7}{7 \cdot 2^3 \cdot 3!} + \cdots + (-1)^n \frac{z^{2n+1}}{(2n+1)2^n \cdot n!} + \cdots)$$

$$\frac{1}{\sqrt{2\pi}} \int_0^2 e^{-z^2/2} dz = \frac{1}{\sqrt{2\pi}} (2 - \frac{8}{6} + \frac{32}{40} - \frac{128}{336} + \frac{512}{3456} - \frac{2^{11}}{11 \cdot 2^5 \cdot 5!} + \cdots)$$

Using the first five terms, we estimate that the probability is approximately **0.4922.** By the alternating series test, we see that this estimate is accurate to within

$$\frac{1}{\sqrt{2\pi}}\frac{2^{13}}{13\cdot 2^6\cdot 6!}\approx 0.00546. \tag{2.4.14}$$

Analysis

If you are familiar with probability theory, you may know that the probability that a data value is within two standard deviations of the mean is approximately 95 Here we calculated the probability that a data value is between the mean and two standard deviations above the mean, so the estimate should be around 47.5. The estimate, combined with the bound on the accuracy, falls within this range.

Exercise 2.4.7

Use the first five terms of the Maclaurin series for $e^{-x^2/2}$ to estimate the probability that a randomly selected test score is between 100 and 150. Use the alternating series test to determine the accuracy of this estimate.

Hint available at the end of this section.

Answer available at the end of this section.

Another application in which a nonelementary integral arises involves the period of a pendulum. The integral is

$$\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2sin^2\theta}} \tag{2.4.15}$$

.

An integral of this form is known as an **elliptic integral** of the first kind. Elliptic integrals originally arose when trying to calculate the arc length of an ellipse. We now show how to use power series to approximate this integral.

Example 2.4.8: Period of a Pendulum

The period of a pendulum is the time it takes for a pendulum to make one complete back-and-forth swing. For a pendulum with length L that makes a maximum angle θ_{max} with the vertical, its period T is given by

$$T=4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2sin^2\theta}}$$

where g is the acceleration due to gravity and $k=sin(\frac{\theta_{max}}{2})$. (We note that this formula for the period arises from a non-linearized model of a pendulum. In some cases, for simplification, a linearized model is used and $sin\theta$ is approximated by θ.)

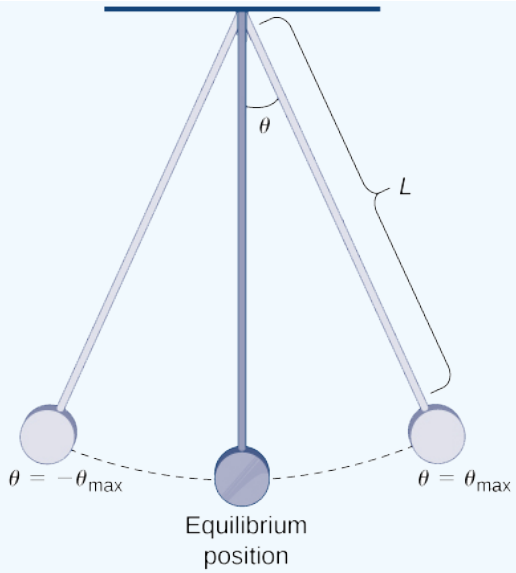


Figure 2.4.3: This pendulum has length and makes a maximum angle θ_{max} with the vertical.The image shows a diagram of a pendulum with three positions. The pendulum consists of a rod of length L attached to a pivot at the top. At the bottom is a circular bob. The central position is labeled "Equilibrium Position", where the rod hangs vertically down. On each side, dashed lines show the pendulum swung to it maximum angles, labeled θ=$-\theta_{max}$ on the left and $\theta = \theta_{max}$ on the right. An angle θ is marked from the vertical to indicate the swing of the pendulum. The length L of the pendulum is also indicated along the line of the rod.

Use the binomial series

$$\frac{1}{\sqrt{1+x}} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n} x^n$$

to estimate the period of this pendulum. Specifically, approximate the period of the pendulum if

a. you use only the first term in the binomial series, and
b. you use the first two terms in the binomial series.

Solution

We use the binomial series, replacing x with $-k^2 sin^2\theta$. Then we can write the period as

$$T = 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} (1 + \frac{1}{2}k^2 sin^2\theta + \frac{1 \cdot 3}{2!2^2} k^4 sin^4\theta + \cdots) d\theta.$$

a. Using just the first term in the integrand, the first-order estimate is

$$T \approx 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} d\theta = 2\pi\sqrt{\frac{L}{g}}.$$

If θ_{max} is small, then $k = sin(\frac{\theta_{max}}{2})$ is small. We claim that when k is small, this is a good estimate. To justify this claim, considerN$\int_0^{\pi/2} (1 + \frac{1}{2}k^2 sin^2\theta + \frac{1 \cdot 3}{2!2^2} k^4 sin^4\theta + \cdots) d\theta.$

Since $|sinx| \leq 1$, this integral is bounded by

$$\int_0^{\pi/2} (\frac{1}{2}k^2 + \frac{1.3}{2!2^2} k^4 + \cdots) d\theta < \frac{\pi}{2} (\frac{1}{2}k^2 + \frac{1 \cdot 3}{2!2^2} k^4 + \cdots).$$

Furthermore, it can be shown that each coefficient on the right-hand side is less than 1 and, therefore, that this expression is bounded by

$$\frac{\pi k^2}{2}(1+k^2+k^4+\cdots)=\frac{\pi k^2}{2}\cdot\frac{1}{1-k^2},$$

which is small for k small.

b. For larger values of θ_{max}, we can approximate T by using more terms in the integrand. By using the first two terms in the integral, we arrive at the estimate

$$T\approx 4\sqrt{fracLg}\int_0^{\pi/2}(1+\frac{1}{2}k^2sin^2\theta)d\theta=2\pi\sqrt{\frac{L}{g}}(1+\frac{k^2}{4}).$$

The applications of Taylor series in this section are intended to highlight their importance. In general, Taylor series are useful because they allow us to represent known functions using polynomials, thus providing us a tool for approximating function values and estimating complicated integrals. In addition, they allow us to define new functions as power series, thus providing us with a powerful tool for solving differential equations.

Key Concepts

- The binomial series is the Maclaurin series for $f(x)=(1+x)^r$. It converges for $|x|<1$.
- Taylor series for functions can often be derived by algebraic operations with a known Taylor series or by differentiating or integrating a known Taylor series.
- Power series can be used to solve differential equations.
- Taylor series can be used to help approximate integrals that cannot be evaluated by other means.

Glossary

Binomial Series

The Maclaurin series for $f(x)=(1+x)^r$; it is given by $(1+x)^r=\sum_{n=0}^{\infty}\binom{r}{n}x^n=1+rx+\frac{r(r-1)}{2!}x^2+\cdots+\frac{r(r-1)\cdots(r-n+1)}{n!}x^n+\cdots$ for $|x|<1$.

Non-Elementary Integral

An integral for which the antiderivative of the integrand cannot be expressed as an elementary function.

Answers

Exercise 2.4.1

Use the definition of binomial series for $r=-2$.

Exercise 2.4.1

$\sum_{n=0}^{\infty}(-1)^n(n+1)x^n$

Exercise 2.4.2

Use the Maclaurin series for $\sin x$.

Exercise 2.4.2

$\sum_{n=0}^{\infty}\frac{(-1)^nx^{4n+2}}{(2n+1)!}$

Exercise 2.4.3

Differentiate the series for $\frac{1}{\sqrt{1+x}}$

Exercise 2.4.3

$\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{1\cdot3\cdot5\cdots(2n+1)}{2^n}x^n$

Exercise 2.4.4

The equations for the first several coefficients c_n will satisfy $c_0=2c_1, c_1=2\cdot2c_2, c_2=2\cdot3c_3,\ldots$. In general, for all $n\geq0, c_n=2(n+1)C_{n+1}$.

Exercise 2.4.4

$y=5e^{2x}$

Exercise 2.4.5

The coefficients satisfy $c_0=a, c_1=b, c_2=0, c_3=0$, and for $n\geq4, n(n-1)c_n=-c_{n-4}$.

Exercise $2.4.5$

$y = a(1 - \frac{x^4}{3 \cdot 4} + \frac{x^8}{3 \cdot 4 \cdot 7 \cdot 8} - \cdots) + b(x - \frac{x^5}{4 \cdot 5} + \frac{x^9}{4 \cdot 5 \cdot 8 \cdot 9} - \cdots)$

Exercise $2.4.6$

Use the series found in Example.

Exercise $2.4.6$

$C + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n(2n-2)!}$ The definite integral is approximately 0.514 to within an error of 0.01.

Exercise $2.4.7$

Evaluate $\int_0^1 e^{-z^2/2} dz$ using the first five terms of the Maclaurin series for $e^{-z^2/2}$.

Exercise $2.4.7$

The estimate is approximately 0.3414. This estimate is accurate to within 0.0000094.

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2.4E: Exercises for Section 2.4

Exercises 1 - 4

Use appropriate substitutions to write down the Maclaurin series for the given binomial.

1. $(1-x)^{1/3}$
2. $(1+x^2)^{-1/3}$

Answer

$$(1+x^2)^{-1/3} = \sum_{n=0}^{\infty} \left(n^{-\frac{1}{3}}\right) x^{2n}$$

3. $(1-x)^{1.01}$
4. $(1-2x)^{2/3}$

Answer

$$(1-2x)^{2/3} = \sum_{n=0}^{\infty} (-1)^n 2^n \left(n^{\frac{2}{3}}\right) x^n$$

Exercises 5 - 12

Use the substitution $(b+x)^r = (b+a)^r \left(1+\dfrac{x-a}{b+a}\right)^r$ in the binomial expansion to find the Taylor series of each function with the given center.

5. $\sqrt{x+2}$ at $a=0$
6. $\sqrt{x^2+2}$ at $a=0$

Answer

$$\sqrt{2+x^2} = \sum_{n=0}^{\infty} 2^{(1/2)-n} \left(n^{\frac{1}{2}}\right) x^{2n}; (|x^2|<2)$$

7. $\sqrt{x+2}$ at $a=1$
8. $\sqrt{2x-x^2}$ at $a=1$ (Hint: $2x-x^2 = 1-(x-1)^2$)

Answer

$$\sqrt{2x-x^2} = \sqrt{1-(x-1)^2} \text{ so } \sqrt{2x-x^2} = \sum_{n=0}^{\infty} (-1)^n \left(n^{\frac{1}{2}}\right) (x-1)^{2n}$$

9. $(x-8)^{1/3}$ at $a=9$
10. $\sqrt{x}$ at $a=4$

Answer

$$\sqrt{x} = 2\sqrt{1+\tfrac{x-4}{4}} \text{ so } \sqrt{x} = \sum_{n=0}^{\infty} 2^{1-2n} \left(n^{\frac{1}{2}}\right) (x-4)^n$$

11. $x^{1/3}$ at $a=27$
12. $\sqrt{x}$ at $x=9$

Answer

$$\sqrt{x} = \sum_{n=0}^{\infty} 3^{1-3n} \left(n^{\frac{1}{2}}\right) (x-9)^n$$

Exercises 13 - 14

Use the binomial theorem to estimate each number, computing enough terms to obtain an estimate accurate to an error of at most 1/1000.

13. [T] $(15)^{1/4}$ using $(16-x)^{1/4}$
14. [T] $(1001)^{1/3}$ using $(1000+x)^{1/3}$

Answer

$10(1+\frac{x}{1000})^{1/3} = \sum_{n=0}^{\infty} 10^{1-3n}\binom{\frac{1}{3}}{n}x^n$. Using, for example, a fourth-degree estimate at $x=1$ gives $(1001)^{1/3}$ $\approx 10\left(1+\left(1^{\frac{1}{3}}\right)10^{-3}+\left(2^{\frac{1}{3}}\right)10^{-6}+\left(3^{\frac{1}{3}}\right)10^{-9}+\left(3^{\frac{1}{3}}\right)10^{-12}\right)$ $= 10\left(1+\frac{1}{3.10^3}-\frac{1}{9.10^6}+\frac{5}{81.10^9}-\frac{10}{243.10^{12}}\right)$ $= 10.00333222\ldots$ whereas $(1001)^{1/3} = 10.00332222839093\ldots$. Two terms would suffice for three-digit accuracy.

Exercises 15 - 20

Use the binomial approximation $\sqrt{1-x} \approx 1-\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{16}-\frac{5x^4}{128}-\frac{7x^5}{256}$ for $|x|<1$ to approximate each number. Compare this value to the value given by a scientific calculator.

15. [T] $\frac{1}{\sqrt{2}}$ using $x=\frac{1}{2}$ in $(1-x)^{1/2}$
16. [T] $\sqrt{5}=5\times\frac{1}{\sqrt{5}}$ using $x=\frac{4}{5}$ in $(1-x)^{1/2}$

Answer

The approximation is 2.3152; the CAS value is $2.23\ldots$.

17. [T] $\sqrt{3}=\frac{3}{\sqrt{3}}$ using $x=\frac{2}{3}$ in $(1-x)^{1/2}$
18. [T] $\sqrt{6}$ using $x=\frac{5}{6}$ in $(1-x)^{1/2}$

Answer

The approximation is $2.583\ldots$; the CAS value is $2.449\ldots$.

19. Integrate the binomial approximation of $\sqrt{1-x}$ to find an approximation of $\int_0^x \sqrt{1-t}\ dt$.
20. [T] Recall that the graph of $\sqrt{1-x^2}$ is an upper semicircle of radius 1. Integrate the binomial approximation of $\sqrt{1-x^2}$ up to order 8 from $x=-1$ to $x=1$ to estimate $\frac{\pi}{2}$.

Answer

$\sqrt{1-x^2}=1-\frac{x^2}{2}-\frac{x^4}{8}-\frac{x^6}{16}-\frac{5x^8}{128}+\cdots$. Thus $\int_{-1}^{1}\sqrt{1-x^2}\ dx = \left[x-\frac{x^3}{6}-\frac{x^5}{40}-\frac{x^7}{7\cdot 16}-\frac{5x^9}{9\cdot 128}+\cdots\right]\Big|_{-1}^{1} \approx 2$ $-\frac{1}{3}-\frac{1}{20}-\frac{1}{56}-\frac{10}{9\cdot 128}+error = 1.590\ldots$ whereas $\frac{\pi}{2}=1.570\ldots$

Exercises 21 - 28

Use the expansion $(1+x)^{1/3}=1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3-\frac{10}{243}x^4+\cdots$ to write the first five terms (not necessarily a quartic polynomial) of each expression.

21. $(1+4x)^{1/3};\ \ a=0$
22. $(1+4x)^{4/3};\ \ a=0$

Answer

$(1+x)^{4/3}=(1+x)(1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3-\frac{10}{243}x^4+\cdots)=1+\frac{4x}{3}+\frac{2x^2}{9}-\frac{4x^3}{81}+\frac{5x^4}{243}+\cdots$

23. $(3+2x)^{1/3};\ \ a=-1$
24. $(x^2+6x+10)^{1/3};\ \ a=-3$

Answer

$$(1+(x+3)^2)^{1/3} = 1 + \tfrac{1}{3}(x+3)^2 - \tfrac{1}{9}(x+3)^4 + \tfrac{5}{81}(x+3)^6 - \tfrac{10}{243}(x+3)^8 + \cdots$$

25. Use $(1+x)^{1/3} = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 - \frac{10}{243}x^4 + \cdots$ with $x = 1$ to approximate $2^{1/3}$.
26. Use the approximation $(1-x)^{2/3} = 1 - \frac{2x}{3} - \frac{x^2}{9} - \frac{4x^3}{81} - \frac{7x^4}{243} - \frac{14x^5}{729} + \cdots$ for $|x| < 1$ to approximate $2^{1/3} = 2.2^{-2/3}$.

Answer

Twice the approximation is $1.260\ldots$ whereas $2^{1/3} = 1.2599\ldots$.

27. Find the 25th derivative of $f(x) = (1+x^2)^{13}$ at $x = 0$.
28. Find the 99th derivative of $f(x) = (1+x^4)^{25}$.

Answer

$$f^{(99)}(0) = 0$$

Exercises 29 - 36

Find the Maclaurin series of each function.

29. $f(x) = xe^{2x}$
30. $f(x) = 2^x$

Answer

$$\sum_{n=0}^{\infty} \frac{(\ln(2)x)^n}{n!}$$

31. $f(x) = \dfrac{\sin x}{x}$
32. $f(x) = \dfrac{\sin(\sqrt{x})}{\sqrt{x}}, (x > 0),$

Answer

For $x > 0$, $\sin(\sqrt{x}) = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{(2n+1)/2}}{\sqrt{x}(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(2n+1)!}.$

33. $f(x) = \sin(x^2)$
34. $f(x) = e^{x^3}$

Answer

$$e^{x^3} = \sum_{n=0}^{\infty} \frac{x^{3n}}{n!}$$

35. $f(x) = \cos^2 x$ using the identity $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x)$
36. $f(x) = \sin^2 x$ using the identity $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$

Answer

$$\sin^2 x = -\sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k-1} x^{2k}}{(2k)!}$$

Exercises 37 - 44

Find the Maclaurin series of $F(x) = \displaystyle\int_0^x f(t)\ dt$ by integrating the Maclaurin series of f term by term. If f is not strictly defined at zero, you may substitute the value of the Maclaurin series at zero.

37. $F(x) = \displaystyle\int_0^x e^{-t^2}\ dt;\ \ f(t) = e^{-t^2} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{n!}$

38. $F(x) = \tan^{-1} x;\ \ f(t) = \dfrac{1}{1+t^2} = \displaystyle\sum_{n=0}^{\infty} (-1)^n t^{2n}$

Answer

$$\tan^{-1} x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}$$

39. $F(x) = \tanh^{-1} x;\ \ f(t) = \dfrac{1}{1-t^2} = \displaystyle\sum_{n=0}^{\infty} t^{2n}$

40. $F(x) = \sin^{-1} x;\ \ f(t) = \dfrac{1}{\sqrt{1-t^2}} = \displaystyle\sum_{k=0}^{\infty} \left(k^{\frac{1}{2}}\right) \frac{t^{2k}}{k!}$

Answer

$$\sin^{-1} x = \sum_{n=0}^{\infty} \left(n^{\frac{1}{2}}\right) \frac{x^{2n+1}}{(2n+1)n!}$$

41. $F(x) = \displaystyle\int_0^x \frac{\sin t}{t}\ dt;\ \ f(t) = \frac{\sin t}{t} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n+1)!}$

42. $F(x) = \displaystyle\int_0^x \cos\left(\sqrt{t}\right)\ dt;\ \ f(t) = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(2n)!}$

Answer

$$F(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{(n+1)(2n)!}$$

43. $F(x) = \displaystyle\int_0^x \frac{1-\cos t}{t^2}\ dt;\ \ f(t) = \frac{1-\cos t}{t^2} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n+2)!}$

44. $F(x) = \displaystyle\int_0^x \frac{\ln(1+t)}{t}\ dt;\ \ f(t) = \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{n+1}$

Answer

$$F(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n^2}$$

Exercises 45 - 52

Compute at least the first three nonzero terms (not necessarily a quadratic polynomial) of the Maclaurin series of f.

45. $f(x) = \sin\left(x + \frac{\pi}{4}\right) = \sin x \cos\left(\frac{\pi}{4}\right) + \cos x \sin\left(\frac{\pi}{4}\right)$

46. $f(x) = \tan x$

Answer

$$x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$$

47. $f(x) = \ln(\cos x)$

48. $f(x) = e^x \cos x$

Answer

$$1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \cdots$$

49. $f(x) = e^{\sin x}$

50. $f(x) = \sec^2 x$

Answer

$$1 + x^2 + \frac{2x^4}{3} + \frac{17x^6}{45} + \cdots$$

51. $f(x) = \tanh x$
52. $f(x) = \dfrac{\tan\sqrt{x}}{\sqrt{x}}$ (see expansion for $\tan x$)

Answer

Using the expansion for $\tan x$ gives $1 + \dfrac{x}{3} + \dfrac{2x^2}{15}$.

Exercises 53 - 56

Find the radius of convergence of the Maclaurin series of each function.

53. $\ln(1+x)$
54. $\dfrac{1}{1+x^2}$

Answer

$\dfrac{1}{1+x^2} = \displaystyle\sum_{n=0}^{\infty} (-1)^n x^{2n}$ so $R = 1$ by the ratio test.

55. $\tan^{-1} x$
56. $\ln(1+x^2)$

Answer

$\ln(1+x^2) = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^{2n}$ so $R = 1$ by the ratio test.

57. Find the Maclaurin series of $\sinh x = \dfrac{e^x - e^{-x}}{2}$.
58. Find the Maclaurin series of $\cosh x = \dfrac{e^x + e^{-x}}{2}$.

Answer

Add series of e^x and e^{-x} term by term. Odd terms cancel and $\cosh x = \displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$.

59. Differentiate term by term the Maclaurin series of $\sinh x$ and compare the result with the Maclaurin series of $\cosh x$.
60. [T] Let $S_n(x) = \displaystyle\sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$ and $C_n(x) = \displaystyle\sum_{n=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!}$ denote the respective Maclaurin polynomials of degree $2n+1$ of $\sin x$ and degree $2n$ of $\cos x$. Plot the errors $\dfrac{S_n(x)}{C_n(x)} - \tan x$ for $n = 1, .., 5$ and compare them to $x + \dfrac{x^3}{3} + \dfrac{2x^5}{15} + \dfrac{17x^7}{315} - \tan x$ on $(-\frac{\pi}{4}, \frac{\pi}{4})$.

Answer

The ratio $\dfrac{S_n(x)}{C_n(x)}$ approximates $\tan x$ better than does $p_7(x) = x + \dfrac{x^3}{3} + \dfrac{2x^5}{15} + \dfrac{17x^7}{315}$ for $N \geq 3$. The dashed curves are $\dfrac{S_n}{C_n} - \tan x$ for $n = 1,\ 2$. The dotted curve corresponds to $n = 3$, and the dash-dotted curve corresponds to $n = 4$. The solid curve is $p_7 - \tan x$.

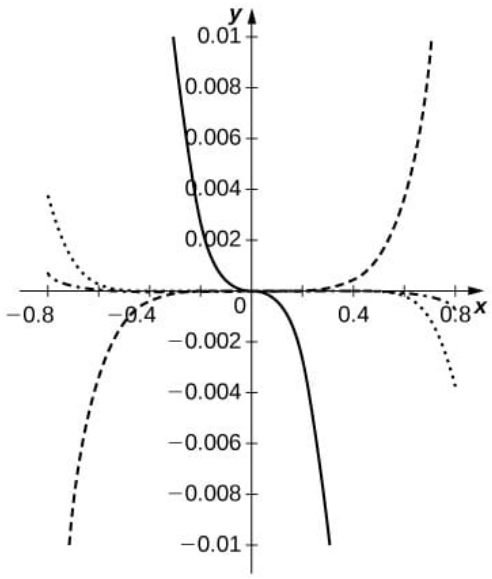


Figure $2.4E.1$:The image is a graph displaying two functions on a Cartesian plane. The horizontal axis is labeled $"x"$ and the vertical axis is labeled $"y"$. The graph features a solid curve that appears to form the shape of a hyperbola centered on the origin $(0, 0)$, with the solid line approaching but never intersecting the axes. There are also two sets of dashed curves representing asymptotes. The solid curve extends vertically and slightly curves inwards on both sides of the y -axis. The dashed curves move towards the horizontal asymptote as the $x - values$ approach positive and negative limits. Tick marks on both axes are labeled with values, increasing from -0.8 to 0.8 in intervals of 0.4.

61. Use the identity $2\sin x\cos x = \sin(2x)$ to find the power series expansion of $\sin^2 x$ at $x = 0$. (Hint: Integrate the Maclaurin series of $\sin(2x)$ term by term.)

62. If $y = \sum_{n=0}^{\infty} a_n x^n$, find the power series expansions of xy' and x^2y''.

Answer

By the term-by-term differentiation theorem, $y' = \sum_{n=1}^{\infty} na_n x^{n-1}$ so $y' = \sum_{n=1}^{\infty} na_n x^{n-1}$ $xy' = \sum_{n=1}^{\infty} na_n x^n$, whereas $y' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$ so $xy'' = \sum_{n=2}^{\infty} n(n-1)a_n x^n$.

63. [T] Suppose that $y = \sum_{k=0}^{\infty} a^k x^k$ satisfies $y' = -2xy$ and $y(0) = 0$. Show that $a_{2k+1} = 0$ for all k and that $a_{2k+2} = \dfrac{-a_{2k}}{k+1}$. Plot the partial sum S_{20} of y on the interval $[-4, 4]$.

64. [T] Suppose that a set of standardized test scores is normally distributed with mean $\mu = 100$ and standard deviation $\sigma = 10$. Set up an integral that represents the probability that a test score will be between 90 and 110 and use the integral of the degree 10 Maclaurin polynomial of $\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ to estimate this probability.

Answer

The probability is $p = \dfrac{1}{\sqrt{2\pi}} \int_{(a-\mu)/\sigma}^{(b-\mu)/\sigma} e^{-x^2/2}\ dx$ where $a = 90$ and $b = 100$, that is, $p = \dfrac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-x^2/2}\ dx$ $= \dfrac{1}{\sqrt{2\pi}} \int_{-1}^{1} \sum_{n=0}^{5} (-1)^n \dfrac{x^{2n}}{2^n n!}\ dx = \dfrac{2}{\sqrt{2\pi}} \sum_{n=0}^{5} (-1)^n \dfrac{1}{(2n+1)2^n n!} \approx 0.6827.$

65. [T] Suppose that a set of standardized test scores is normally distributed with mean $\mu = 100$ and standard deviation $\sigma = 10$. Set up an integral that represents the probability that a test score will be between 70 and 130 and use the integral of the degree 50 Maclaurin polynomial of $\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ to estimate this probability.

66. [T] Suppose that $\sum_{n=0}^{\infty} a_n x^n$ converges to a function $f(x)$ such that $f(0) = 1,\ f'(0) = 0$, and $f''(x) = -f(x)$. Find a formula for a_n and plot the partial sum S_N for $N = 20$ on $[-5, 5]$.

Answer

As in the previous problem one obtains $a_n = 0$ if n is odd and $a_n = -(n+2)(n+1)a_{n+2}$ if n is even, so $a_0 = 1$ leads to $a_{2n} = \dfrac{(-1)^n}{(2n)!}$.

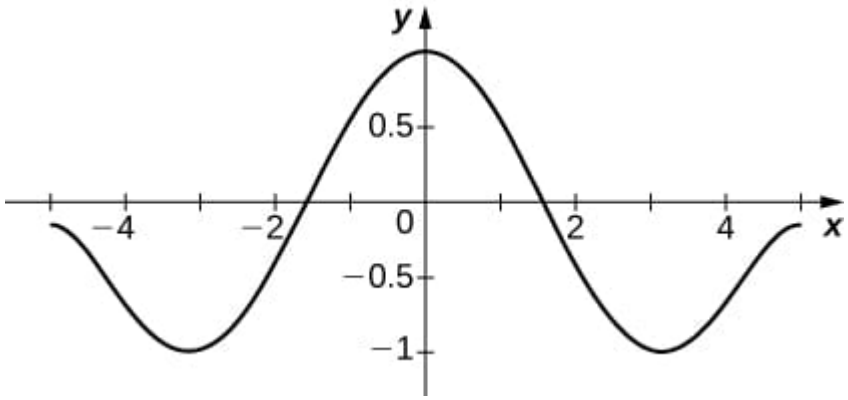


Figure $2.4E.2$:The image shows a graph of a mathematical sine function plotted on a coordinate plane. The $x-axis$ and $y-axis$ are both labeled, with the $x-axisranging$ from -5 to 5 and the $y-axis$ ranging from -1.5 to 1.5. The graph depicts a smooth, continuous wave that oscillates above and below the x-axis. The wave reaches its maximum height at approximately $y = 1$ and its minimum at $y = -1$, peaking at $x = 0$ and oscillating symmetrically towards the edges of the graph.

57. [T] Suppose that $\sum_{n=0}^{\infty} a_n x^n$ converges to a function $f(x)$ such that $f(0) = 0, \quad f\,'(0) = 1$, and $f''(x) = -f(x)$. Find a formula for an and plot the partial sum S_N for $N = 10$ on $[-5, 5]$.

58. Suppose that $\sum_{n=0}^{\infty} a_n x^n$ converges to a function y such that $y'' - y\,' + y = 0$ where $y(0) = 1$ and $y'(0) = 0$. Find a formula that relates $a_{n+2}, \quad a_{n+1}$, and an and compute $a_0, \ldots, a_5$.

Answer

$y'' = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$ and $y\,' = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$ so $y'' - y\,' + y = 0$ implies that $(n+2)(n+1)a_{n+2} - (n+1)a_{n+1} + a_n = 0$ or $a_n = \dfrac{a_{n-1}}{n} - \dfrac{a_{n-2}}{n(n-1)}$ for all $n \cdot y(0) = a_0 = 1$ and $y\,'(0) = a_1 = 0$, so $a_2 = \frac{1}{2}, \quad a_3 = \frac{1}{6}$, $a_4 = 0$, and $a_5 = -\frac{1}{120}$.

59. Suppose that $\sum_{n=0}^{\infty} a_n x^n$ converges to a function y such that $y'' - y\,' + y = 0$ where $y(0) = 0$ and $y\,'(0) = 1$. Find a formula that relates $a_{n+2}, \quad a_{n+1}$, and an and compute $a_1, \ldots, a_5$.

Exercises 70 - 71

The error in approximating the integral $\int_a^b f(t)\ dt$ by that of a Taylor approximation $\int_a^b Pn(t)\ dt$ is at most $\int_a^b R_n(t)\ dt$. The Taylor remainder estimate $R_n \le \frac{M}{(n+1)!}|x-a|^{n+1}$ guarantees that the integral of the Taylor polynomial of the given order approximates the integral of f with an error less than $\frac{1}{10}$.

a. Evaluate the integral of the appropriate Taylor polynomial and verify that it approximates the CAS value with an error less than $\frac{1}{100}$.
b. Compare the accuracy of the polynomial integral estimate with the remainder estimate.

70. [T] $\int_0^{\pi} \dfrac{\sin t}{t}\ dt; \quad P_s = 1 - \dfrac{x^2}{3!} + \dfrac{x^4}{5!} - \dfrac{x^6}{7!} + \dfrac{x^8}{9!}$ (You may assume that the absolute value of the ninth derivative of $\frac{\sin t}{t}$ is bounded by 0.1.)

Answer

a. (Proof)

b. We have $R_s \leq \frac{0.1}{(9)!}\pi^9 \approx 0.0082 < 0.01$. We have $\int_0^{\pi}\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\frac{x^8}{9!}\right)\,dx = \pi - \frac{\pi^3}{3\cdot 3!}+\frac{\pi^5}{5\cdot 5!} - \frac{\pi^7}{7\cdot 7!}+\frac{\pi^9}{9\cdot 9!} = 1.852...$, whereas $\int_0^{\pi}\frac{\sin t}{t}\,dt = 1.85194...$, so the actual error is approximately 0.00006.

71. [T] $\int_0^2 e^{-x^2}\,dx;\ p_{11} = 1-x^2+\frac{x^4}{2}-\frac{x^6}{3!}+\cdots-\frac{x^{22}}{11!}$ (You may assume that the absolute value of the 23rd derivative of e^{-x^2} is less than 2×10^{14}.)

Exercises 72-73

The following deal with *Fresnel integrals*.

72. The Fresnel integrals are defined by $C(x)=\int_0^x \cos(t^2)\,dt$ and $S(x)=\int_0^x \sin(t^2)\,dt$. Compute the power series of $C(x)$ and $S(x)$ and plot the sums $C_N(x)$ and $S_N(x)$ of the first $N=50$ nonzero terms on $[0,2\pi]$.

Answer

Since $\cos(t^2)=\sum_{n=0}^{\infty}(-1)^n\frac{t^{4n}}{(2n)!}$ and $\sin(t^2)=\sum_{n=0}^{\infty}(-1)^n\frac{t^{4n+2}}{(2n+1)!}$, one has $S(x)=\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+3}}{(4n+3)(2n+1)!}$ and $C(x)=\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+1}}{(4n+1)(2n)!}$. The sums of the first 50 nonzero terms are plotted below with $C_{50}(x)$ the solid curve and $S_{50}(x)$ the dashed curve.

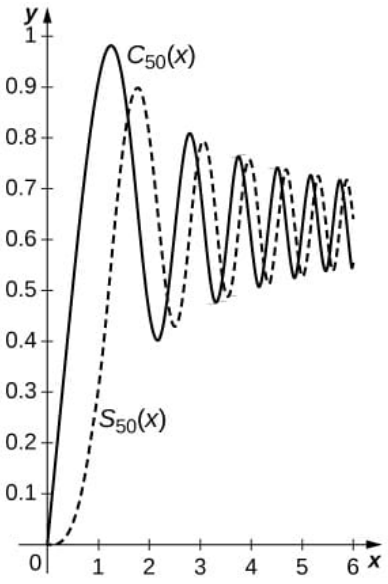


Figure $2.4E.3$: This graph has two curves. The first one is a solid curve labeled $C_50(x)$. It begins at the origin and is a wave that gradually decreases in amplitude. The highest it reaches is $y=1$. The second curve is labeled $S_50(x)$. It is a wave that gradually decreases in amplitude. The highest it reaches is 0.9. It is very close to the pattern of the first curve with a slight shift to the right.

73. [T] The Fresnel integrals are used in design applications for roadways and railways and other applications because of the curvature properties of the curve with coordinates $(C(t),S(t))$. Plot the curve (C_{50},S_{50}) for $0\leq t\leq 2\pi$, the coordinates of which were computed in the previous exercise.

74. Estimate $\int_0^{1/4}\sqrt{x-x^2}\,dx$ by approximating $\sqrt{1-x}$ using the binomial approximation $1-\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{16}-\frac{5x^4}{2128}-\frac{7x^5}{256}$.

Answer

$\int_0^{1/4} \sqrt{x}\left(1-\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{16}-\frac{5x^4}{128}-\frac{7x^5}{256}\right)\ dx = \frac{2}{3}2^{-3} - \frac{1}{2}\frac{2}{5}2^{-5} - \frac{1}{8}\frac{2}{7}2^{-7} - \frac{1}{16}\frac{2}{9}2^{-9} - \frac{5}{128}\frac{2}{11}2^{-11} - \frac{7}{256}\frac{2}{13}2^{-13} = 0.0767732\ldots$ whereas $\int_0^{1/4} \sqrt{x-x^2}\ dx = 0.076773.$

75. [T] Use Newton's approximation of the binomial $\sqrt{1-x^2}$ to approximate π as follows. The circle centered at $(\frac{1}{2}, 0)$ with radius $\frac{1}{2}$ has upper semicircle $y = \sqrt{x}\sqrt{1-x}$. The sector of this circle bounded by the x-axis between $x = 0$ and $x = \frac{1}{2}$ and by the line joining $(\frac{1}{4}, \frac{\sqrt{3}}{4})$ corresponds to $\frac{1}{6}$ of the circle and has area $\frac{\pi}{24}$. This sector is the union of a right triangle with height $\frac{\sqrt{3}}{4}$ and base $\frac{1}{4}$ and the region below the graph between $x = 0$ and $x = \frac{1}{4}$. To find the area of this region you can write $y = \sqrt{x}\sqrt{1-x}$ $= \sqrt{x} \times (\text{binomial expansion of} \sqrt{1-x})$ and integrate term by term. Use this approach with the binomial approximation from the previous exercise to estimate π.

76. Use the approximation $T \approx 2\pi\sqrt{\frac{L}{g}}(1+\frac{k^2}{4})$ to approximate the period of a pendulum having length 10 meters and maximum angle $\theta_{max} = \frac{\pi}{6}$ where $k = \sin\left(\frac{\theta_{max}}{2}\right)$. Compare this with the small angle estimate $T \approx 2\pi\sqrt{\frac{L}{g}}$.

Answer

$T \approx 2\pi\sqrt{\frac{10}{9.8}}\left(1+\frac{\sin^2(\theta/12)}{4}\right) \approx 6.453$ seconds. The small angle estimate is $T \approx 2\pi\sqrt{\frac{10}{9.8}} \approx 6.347$. The relative error is around 2 percent.

77. Suppose that a pendulum is to have a period of 2 seconds and a maximum angle of $\theta_{max} = \frac{\pi}{6}$. Use $T \approx 2\pi\sqrt{\frac{L}{g}}\left(1+\frac{k^2}{4}\right)$ to approximate the desired length of the pendulum. What length is predicted by the small angle estimate $T \approx 2\pi\sqrt{\frac{L}{g}}$?

78. Evaluate $\int_0^{\pi/2} \sin^4\theta\ d\theta$ in the approximation $T = 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\left(1+\frac{1}{2}k^2\sin^2\theta+\frac{3}{8}k^4\sin^4\theta+\cdots\right)\ d\theta$ to obtain an improved estimate for T.

Answer

$\int_0^{\pi/2} \sin^4\theta\ d\theta = \frac{3\pi}{16}$. Hence $T \approx 2\pi\sqrt{\frac{L}{g}}\left(1+\frac{k^2}{4}+\frac{9}{256}k^4\right).$

79. [T] An equivalent formula for the period of a pendulum with amplitude θ_{max} is $T(\theta_{max}) = 2\sqrt{2}\sqrt{\frac{L}{g}}\int_0^{\theta_{max}} \frac{d\theta}{\sqrt{\cos\theta - \cos(\theta_{max})}}$ where L is the pendulum length and g is the gravitational acceleration constant. When $\theta_{max} = \frac{\pi}{3}$ we get $\frac{1}{\sqrt{\cos t - 1/2}}$ $\approx \sqrt{2}\left(1+\frac{t^2}{2}+\frac{t^4}{3}+\frac{181t^6}{720}\right)$. Integrate this approximation to estimate $T(\frac{\pi}{3})$ in terms of L and g. Assuming $g = 9.806$ meters per second squared, find an approximate length L such that $T(\frac{\pi}{3}) = 2$ seconds.

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2.5: Chapter 2 Review Exercises

Exercises 1 - 4

True or False? Justify your answer with a proof or a counterexample.

1. If the radius of convergence for a power series $\sum_{n=0}^{\infty} a_n x^n$ is 5, then the radius of convergence for the series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ is also 5.

   Answer

   True

2. Power series can be used to show that the derivative of e^x is e^x. (Hint: Recall that $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$.)
3. For small values of x, $\sin x \approx x$.

   Answer

   True

4. The radius of convergence for the Maclaurin series of $f(x) = 3^x$ is 3.

Exercises 5 - 8

Find the radius of convergence and the interval of convergence for the given series.

5. $\sum_{n=0}^{\infty} n^2 (x-1)^n$

   Answer

   ROC: 1; IOC: $(0, 2)$

6. $\sum_{n=0}^{\infty} \frac{x^n}{n^n}$
7. $\sum_{n=0}^{\infty} \frac{3n x^n}{12^n}$

   Answer

   ROC: 12; IOC: $(-16, 8)$

8. $\sum_{n=0}^{\infty} \frac{2^n}{e^n} (x-e)^n$

Exercises 9 - 10

Find the power series representation for the given function. Determine the radius of convergence and the interval of convergence for that series.

9. $f(x) = \frac{x^2}{x+3}$

Answer

$\sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} x^n$; ROC: 3; IOC: $(-3, 3)$

10. $f(x) = \dfrac{8x+2}{2x^2-3x+1}$

Exercises 11 - 12

Find the power series for the given function using term-by-term differentiation or integration.

11. $f(x) = \tan^{-1}(2x)$

Answer

integration: $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} (2x)^{2n+1}$

12. $f(x) = \dfrac{x}{(2+x^2)^2}$

Exercises 13 - 14

Evaluate the Taylor series expansion of degree four for the given function at the specified point. What is the error in the approximation?

13. $f(x) = x^3 - 2x^2 + 4, \quad a = -3$

Answer

$p_4(x) = (x+3)^3 - 11(x+3)^2 + 39(x+3) - 41$; exact

14. $f(x) = e^{x/4}, \quad a = 4$

Exercises 15 - 16

Find the Maclaurin series for the given function.

15. $f(x) = \cos(3x)$

Answer

$\sum_{n=0}^{\infty} \frac{(-1)^n (3x)^{2n}}{2n!}$

16. $f(x) = \ln(x+1)$

Exercises 17 - 18

Find the Taylor series at the given value.

17. $f(x) = \sin x, \quad a = \frac{\pi}{2}$

Answer

$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(x - \frac{\pi}{2}\right)^{2n}$

18. $f(x) = \dfrac{3}{x}, \quad a = 1$

Exercises 19 - 20

Find the Maclaurin series for the given function.

19. $f(x) = e^{-x^2} - 1$

Answer

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} x^{2n}$$

20. $f(x) = \cos x - x \sin x$

Exercises 21 - 23

Find the Maclaurin series for $F(x) = \int_0^x f(t)\ dt$ by integrating the Maclaurin series of $f(x)$ term by term.

21. $f(x) = \dfrac{\sin x}{x}$

Answer

$$F(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!} x^{2n+1}$$

22. $f(x) = 1 - e^x$
23. Use power series to prove **Euler's formula:** $e^{ix} = cosx + isinx$

Answer

Answers may vary.

Exercises 24 - 26

Consider problems of *annuity payments*.

24. For annuities with a present value of $\$1$ million, calculate the annual payouts given over 25 years assuming interest rates of $1\%, 5\%$, and 10%.
25. A lottery winner has an annuity that has a present value of $\$10$ million. What interest rate would they need to live on perpetual annual payments of $\$250,000$?

Answer

2.5%

26. Calculate the necessary present value of an annuity in order to support annual payouts of $\$15,000$ given over 25 years assuming interest rates of $1\%, 5\%$,and 10%.

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3: PARAMETRIC EQUATIONS AND POLAR COORDINATES

Parametric equations define a group of quantities as functions of one or more independent variables called parameters. Parametric equations are commonly used to express the coordinates of the points that make up a geometric object such as a curve or surface, in which case the equations are collectively called a parametric representation or parameterization. The polar coordinate system is a two-dimensional coordinate system in which each point on a plane is determined by a distance from a reference point and an angle from a reference direction. The reference point (analogous to the origin of a Cartesian system) is called the pole, and the ray from the pole in the reference direction is the polar axis. The distance from the pole is called the radial coordinate or radius, and the angle is called the angular coordinate, polar angle, or azimuth.

The **chambered nautilus** is a fascinating creature. This animal feeds on hermit crabs, fish, and other crustaceans. It has a hard outer shell with many chambers connected in a spiral fashion, and it can retract into its shell to avoid predators. When part of the shell is cut away, a perfect spiral is revealed, with chambers inside that are somewhat similar to growth rings in a tree.

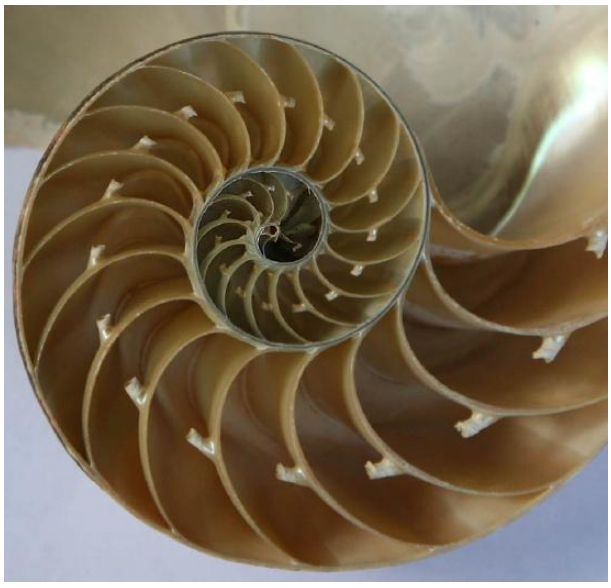


Figure 1: The chambered nautilus is a marine animal that lives in the tropical Pacific Ocean. Scientists think they have existed mostly unchanged for about 500 million years. (Credit:CC-BY-NC-SA modification of work by Jitze Couperus, Flickr)

The mathematical function that describes a spiral can be expressed using rectangular (or Cartesian) coordinates. However, if we change our coordinate system to something that works a bit better with circular patterns, the function becomes much simpler to describe. The polar coordinate system is well suited for describing curves of this type. How can we use this coordinate system to describe spirals and other radial figures?

In this chapter we also study parametric equations, which give us a convenient way to describe curves, or to study the position of a particle or object in two dimensions as a function of time. We will use parametric equations and polar coordinates for describing many topics later in this text.

3.1: Parametric Equations

3.1E: Exercises for Section 3.1

3.2: Calculus of Parametric Curves

3.2E: Exercises for Section 3.2

3.3: Polar Coordinates

3.3E: Exercises for Section 3.3

3.4: Area and Arc Length in Polar Coordinates

3.4E: Exercises for Section 3.4

3.5: Chapter 3 Review Exercises

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3.1: Parametric Equations

Learning Objectives

- Plot a curve described by parametric equations.
- Convert the parametric equations of a curve into the form $y = f(x)$.
- Recognize the parametric equations of basic curves, such as a line and a circle.
- Recognize the parametric equations of a cycloid.

In this section we examine parametric equations and their graphs. In the two-dimensional coordinate system, parametric equations are useful for describing curves that are not necessarily functions. The parameter is an independent variable that both x and y depend on, and as the parameter increases, the values of x and y trace out a path along a plane curve. For example, if the parameter is t (a common choice), then t might represent time. Then x and y are defined as functions of time, and $(x(t), y(t))$ can describe the position in the plane of a given object as it moves along a curved path.

Parametric Equations and Their Graphs

Consider the orbit of Earth around the Sun. Our year lasts approximately 365.25 days, but for this discussion we will use 365 days. On January 1 of each year, the physical location of Earth with respect to the Sun is nearly the same, except for leap years, when the lag introduced by the extra $\frac{1}{4}$ day of orbiting time is built into the calendar. We call January 1 "day 1" of the year. Then, for example, day 31 is January 31, day 59 is February 28, and so on.

The number of the day in a year can be considered a variable that determines Earth's position in its orbit. As Earth revolves around the Sun, its physical location changes relative to the Sun. After one full year, we are back where we started, and a new year begins. According to Kepler's laws of planetary motion, the shape of the orbit is elliptical, with the Sun at one focus of the ellipse. We study this idea in more detail in Conic Sections.

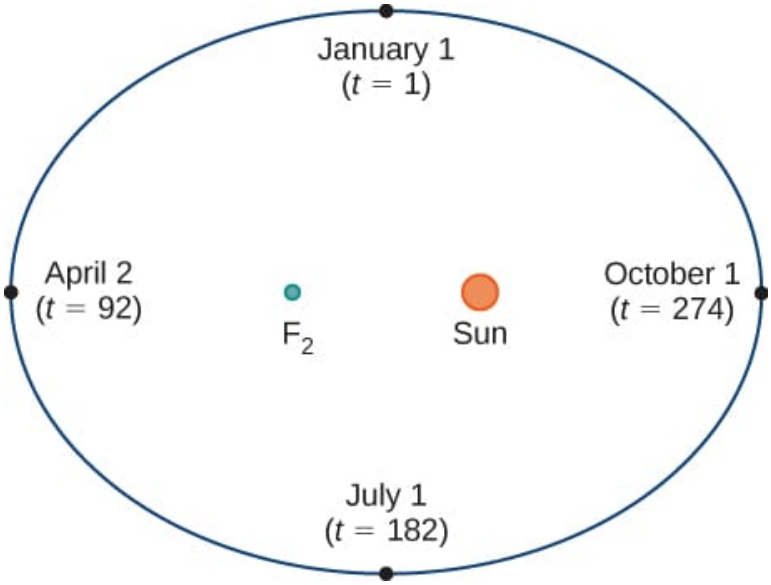


Figure 3.1.1: Earth's orbit around the Sun in one year. An ellipse with January 1 ($t = 1$) at the top, April 2 ($t = 92$) on the left, July 1 ($t = 182$) on the bottom, and October 1 ($t = 274$) on the right. The focal points of the ellipse have F_2 on the left and the Sun on the right.

Figure 3.1.1 depicts **Earth's orbit** around the Sun during one year. The point labeled F_2 is one of the foci of the ellipse; the other focus is occupied by the Sun. If we superimpose coordinate axes over this graph, then we can assign ordered pairs to each point on the ellipse (Figure 3.1.2). Then each x value on the graph is a value of position as a function of time, and each y value is also a value of position as a function of time. Therefore, each point on the graph corresponds to a value of Earth's position as a function of time.

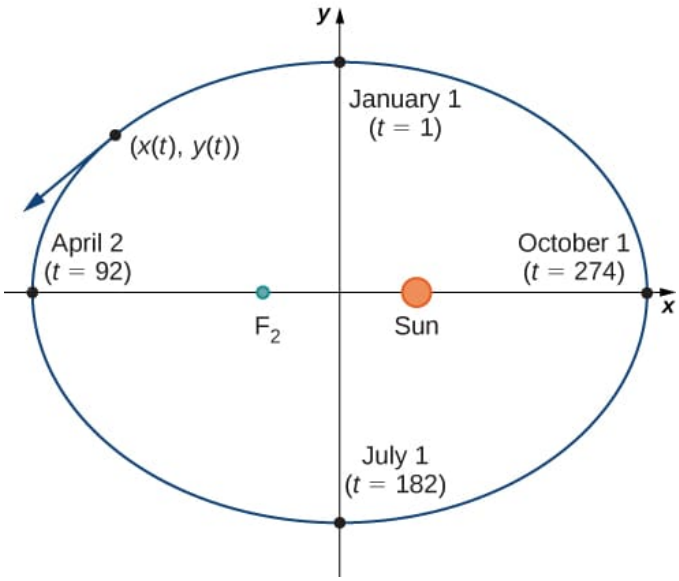


Figure 3.1.2: Coordinate axes superimposed on the orbit of Earth.An ellipse with January 1 ($t = 1$) at the top, April 2 ($t = 92$) on the left, July 1 ($t = 182$) on the bottom, and October 1 ($t = 274$) on the right. The focal points of the ellipse have F_2 on the left and the Sun on the right. There is also a line going from $t = 92$ to $t = 274$ that passes through F_2 and the Sun. On the upper left side, there is a point marked $(x(t), y(t))$ with a tangent line pointing down and to the left.

We can determine the functions for $x(t)$ and $y(t)$, thereby parameterize the orbit of Earth around the Sun. The variable t is called an independent parameter and, in this context, represents time relative to the beginning of each year.

A curve in the (x, y) plane can be represented parametrically. The equations that are used to define the curve are called *parametric equations*.

Definition: Parametric Equations

If x and y are continuous functions of t on an interval I, then the equations

$$x = x(t) \tag{3.1.1}$$

and

$$y = y(t) \tag{3.1.2}$$

are called **parametric equations** and t is called the parameter. The set of points (x, y) obtained as t varies over the interval I is called the graph of the parametric equations. The graph of parametric equations is called a **parametric curve** or plane curve, and is denoted by C.

Notice in this definition that x and y are used in two ways. The first is as functions of the independent variable t. As t varies over the interval I, the functions $x(t)$ and $y(t)$ generate a set of ordered pairs (x, y). This set of ordered pairs generates the graph of the parametric equations. In this second usage, to designate the ordered pairs, x and y are variables. It is important to distinguish the variables x and y from the functions $x(t)$ and $y(t)$.

Example 3.1.1: Graphing a Parametrically Defined Curve

Sketch the curves described by the given parametric equations:

a. $x(t) = t - 1, \quad y(t) = 2t + 4, \quad \text{for } -3 \le t \le 2$
b. $x(t) = t^2 - 3, \quad y(t) = 2t + 1, \quad \text{for } -2 \le t \le 3$
c. $x(t) = 4\cos t, \quad y(t) = 4\sin t, \quad \text{for } 0 \le t \le 2\pi$

Solution

a. To create a graph of this curve, first set up a table of values. Since the independent variable in both $x(t)$ and $y(t)$ is t, let t appear in the first column. Then $x(t)$ and $y(t)$ will appear in the second and third columns of the table.

Table 3.1.1: Generating points to graph the curve.

t	$x(t)$	$y(t)$
–3	–4	–2
–2	–3	0
–1	–2	2
0	–1	4
1	0	6
2	1	8

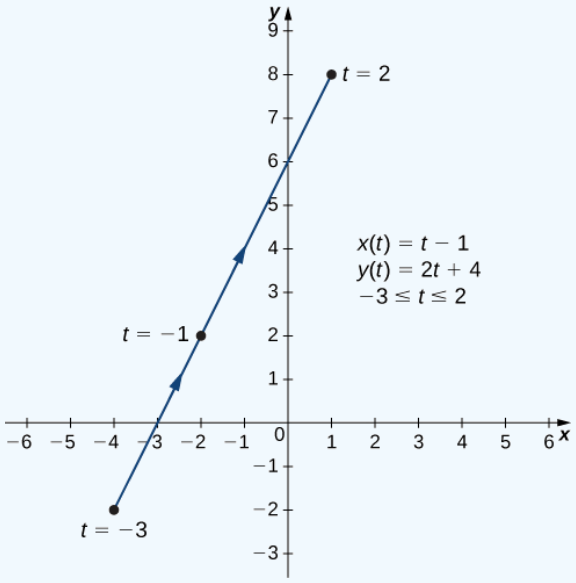


Figure 3.1.3: A straight line going from $(-4,-2)$ through $(-3,0)$, $(-2,2)$, and $(0,6)$ to $(1,8)$ with arrow pointed up and to the right. The point $(-4,-2)$ is marked $t=-3$, the point $(-2,2)$ is marked $t=-1$, and the point $(1,8)$ is marked $t=2$. On the graph there are also written three equations: $x(t)=t-1$, $y(t)=2t+4$, and $-3\le t\le 2$.

b. To create a graph of this curve, again set up a table of values.

Table 3.1.2: Generating points to graph the curve.

t	$x(t)$	$y(t)$
–2	1	–3
–1	–2	–1
0	–3	1
1	–2	3
2	1	5
3	6	7

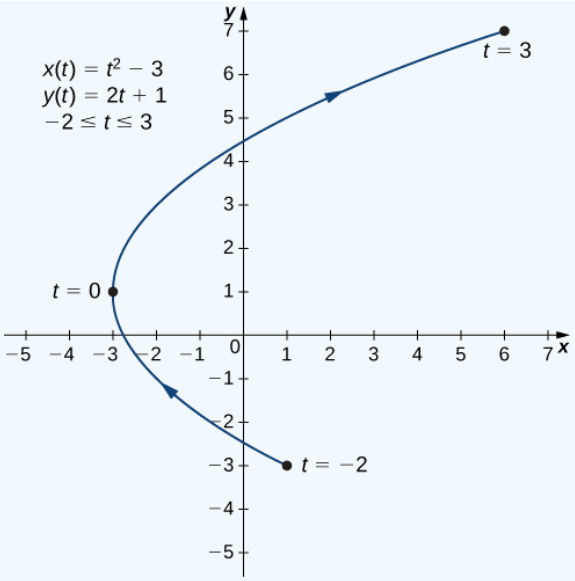


Figure 3.1.4: Graph of the plane curve described by the parametric equations in part b. A curved line going from $(1, -3)$ through $(-3, 1)$ to $(6, 7)$ with arrow pointing in that order. The point $(1, -3)$ is marked $t = -2$, the point $(-3, 1)$ is marked $t = 0$, and the point $(6, 7)$ is marked $t = 3$. On the graph there are also written three equations: $x(t) = t^2 - 3$, $y(t) = 2t + 1$, and $-2 \leq t \leq 3$.

c. In this case, use multiples of $\pi/6$ for t and create another table of values:

Table 3.1.3: Generating values to graph the given curve.

t	$x(t)$	$y(t)$	t	$x(t)$	$y(t)$
0	4	0	$\frac{7\pi}{6}$	$-2\sqrt{3} \approx -3.5$	-2
$\frac{\pi}{6}$	$2\sqrt{3} \approx 3.5$	2	$\frac{4\pi}{3}$	–2	$-2\sqrt{3} \approx -3.5$
$\frac{\pi}{3}$	2	$2\sqrt{3} \approx 3.5$	$\frac{3\pi}{2}$	0	–4
$\frac{\pi}{2}$	0	4	$\frac{5\pi}{3}$	2	$-2\sqrt{3} \approx -3.5$
$\frac{2\pi}{3}$	–2	$2\sqrt{3} \approx 3.5$	$\frac{11\pi}{6}$	$2\sqrt{3} \approx 3.5$	-2
$\frac{5\pi}{6}$	$-2\sqrt{3} \approx -3.5$	2	2π	4	0
π	–4	0			

The graph of this plane curve appears in the Figure 3.1.5.

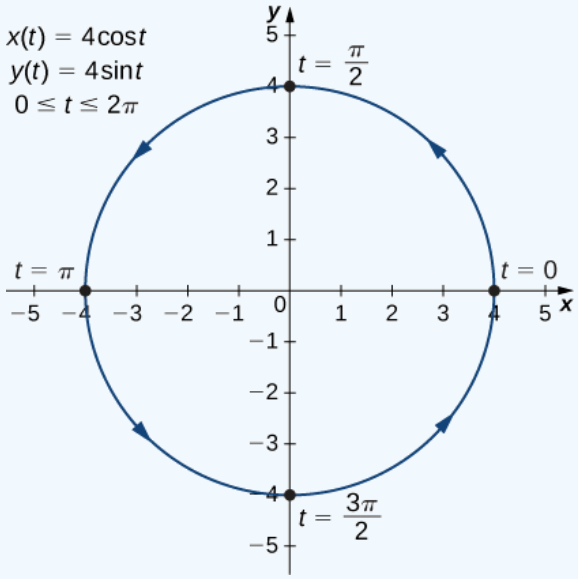


Figure 3.1.5: Graph of the plane curve described by the parametric equations in part c. A circle with radius 4 centered at the origin is graphed with arrow going counterclockwise. The point $(4,0)$ is marked $t=0$, the point $(4,0)$ is marked $t=\pi/2$, the point $(-4,0)$ is marked $t=\pi$, and the point $(0,-4)$ is marked $t=3\pi/2$. On the graph there are also written three equations: $x(t)=4cos(t)$, $y(t)=4sin(t)$, and $0\leq t\leq 2\pi$.

Exercise 3.1.1

Sketch the curve described by the parametric equations

$$x(t)=3t+2, \quad y(t)=t^2-1, \quad \text{for } -3\leq t\leq 2.$$

Hint available at the end of this section.

Answer available at the end of this section.

Interactive Element

You can move it around and also change its equation to see the effect on the graph. The interactive app is unfortunately not accessible.

Visualizing a Parametric Curves As Time Elapses

Unlike curves that are described in the form $y=f(x)$ that just give the static path of an object, parametrically described curves, $x=x(t), y=y(t)$, give the dynamic motion of an object. The visualization below shows the graph of the static curve defined parametrically, but if you click on the "Move" button, you can observe its dynamic motion by observing the dynamic motion of an ant as it travels according to the motion described by the parametrically defined path. If you click on the Next button, it will take you to a different parametrically defined curve.

\(x(t)=\sin(t), y(t)=2\sin(t), 0

\(x(t)=t^2, y(t)=t^3, -2

\(x(t)=\cos(-t), y(t)=\sin(-t), 0

\(x(t)=e^t, y(t)=e^{2t}, -2

\(x(t)=t^3-6t, y(t)=t^2-5, 0

\(x(t)=t-\sin(t), y(t)=1-\cos(t), 0

\(x(t)=2\cos(t^2), y(t)=2\sin(t^2), 0

\(x(t)=e^t\cos(8t), y(t)=e^t\sin(8t), -2

\(x(t)=\frac{1}{t}, y(t)=\frac{1}{t^2}, -4

\(x(t)=t+t^2, y(t)=t-t^2, -3

Move Next





Eliminating the Parameter

To better understand the graph of a curve represented parametrically, it is useful to rewrite the two equations as a single equation relating the variables x and y. Then we can apply any previous knowledge of equations of curves in the plane to identify the curve. For example, the equations describing the plane curve in Example $3.1.1b$ are

$$x(t) = t^2 - 3 \tag{3.1.3}$$
$$y(t) = 2t + 1 \tag{3.1.4}$$

over the region $-2 \leq t \leq 3$.

Solving Equation $3.1.4$ for t gives

$$t = \frac{y-1}{2}.$$

This can be substituted into Equation $3.1.3$:

$$x = \left(\frac{y-1}{2}\right)^2 - 3 \tag{3.1.5}$$
$$= \frac{y^2 - 2y + 1}{4} - 3 \tag{3.1.6}$$
$$= \frac{y^2 - 2y - 11}{4}. \tag{3.1.7}$$

Equation $3.1.7$ describes x as a function of y. These steps give an example of eliminating the parameter. The graph of this function is a parabola opening to the right (Figure $3.1.4$). Recall that the plane curve started at $(1, -3)$ and ended at $(6, 7)$. These terminations were due to the restriction on the parameter t.

Example $3.1.2$: Eliminating the Parameter

Eliminate the parameter for each of the plane curves described by the given parametric equations and describe the resulting graph.

a. $x(t) = \sqrt{2t+4}, \quad y(t) = 2t + 1, \quad \text{for } -2 \leq t \leq 6$

b. $x(t) = 4\cos t, \quad y(t) = 3\sin t, \quad \text{for } 0 \le t \le 2\pi$

Solution

a. To eliminate the parameter, we can solve either of the equations for t. For example, solving the first equation for t gives

$$\begin{aligned} x &= \sqrt{2t+4} \\ x^2 &= 2t+4 \\ x^2 - 4 &= 2t \\ t &= \frac{x^2-4}{2}. \end{aligned}$$

Note that when we square both sides it is important to observe that $x \ge 0$. Substituting $t = \dfrac{x^2-4}{2}$ into $y(t)$ yields

$$y(t) = 2t + 1 \tag{3.1.8}$$

$$y = 2\left(\frac{x^2-4}{2}\right) + 1 \tag{3.1.9}$$

$$y = x^2 - 4 + 1 \tag{3.1.10}$$

$$y = x^2 - 3. \tag{3.1.11}$$

This is the equation of a parabola opening upward. There is, however, a domain restriction because of the limits on the parameter t. When $t = -2$, $x = \sqrt{2(-2)+4} = 0$, and when $t = 6$, $x = \sqrt{2(6)+4} = 4$. The graph of this plane curve follows.

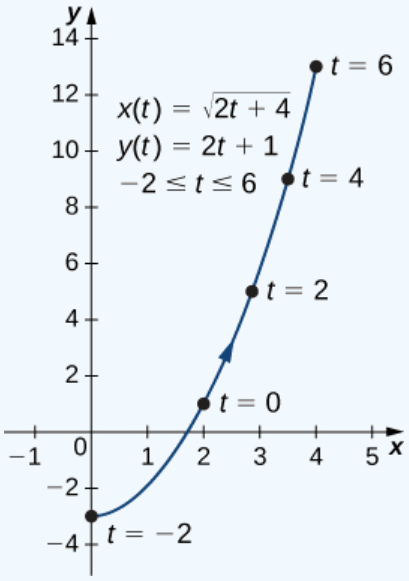


Figure 3.1.7: Graph of the plane curve described by the parametric equations in part a $x(t) = \sqrt{2t+4}, \quad y(t) = 2t+1, \quad \text{for } -2 \le t \le 6$. Points ranging from $t = -2$ to $t = 6$.

Watch the accompanying video for a visual demonstration of this concept.

- **Video Length:** 1 minute 44 seconds.
- **Context:** This video demonstrates how to eliminate the parameter to graph the curve represented by parametric equations $x = e^t$,$y = e^{2t}$.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

b. Sometimes it is necessary to be a bit creative in eliminating the parameter. The parametric equations for this example are

$$x(t) = 4\cos t$$

and

$$y(t) = 3\sin t$$

Solving either equation for t directly is not advisable because sine and cosine are not one-to-one functions. However, dividing the first equation by 4 and the second equation by 3 (and suppressing the t) gives us

$$\cos t = \frac{x}{4}$$

and

$$\sin t = \frac{y}{3}.$$

Now use the Pythagorean identity $\cos^2 t + \sin^2 t = 1$ and replace the expressions for $\sin t$ and $\cos t$ with the equivalent expressions in terms of x and y. This gives

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$

$$\frac{x^2}{16} + \frac{y^2}{9} = 1.$$

This is the equation of a horizontal ellipse centered at the origin, with semi-major axis 4 and semi-minor axis 3.

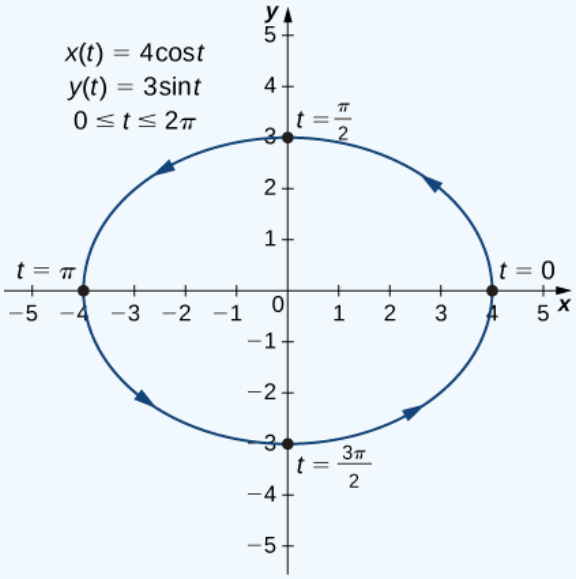


Figure 3.1.8: Graph of the plane curve described by the parametric equations in part b. An ellipse with major axis horizontal and of length 8 and with minor radius vertical and of length 6 that is centered at the origin with arrow going counterclockwise. The point $(4, 0)$ is marked $t = 0$, the point $(0, 3)$ is marked $t = \pi/2$, the point $(-4, 0)$ is marked $t = \pi$, and the point $(0, -3)$ is marked $t = 3\pi/2$. On the graph there are also written three equations: $x(t) = 4cos(t)$, $y(t) = 3sin(t)$, and $0 \leq t \leq 2\pi$.

As *t* progresses from 0 to 2π, a point on the curve traverses the ellipse once, in a counterclockwise direction. Recall from the section opener that the orbit of Earth around the Sun is also elliptical. This is a perfect example of using parametrized curves to model a real-world phenomenon.

Watch the accompanying video for a visual demonstration of this concept.

- **Video Length:** 2 minute 24 seconds.
- **Context:** This video demosntrates how to eliminate the parameter to find the corresponding rectangular equation $x = 2sin\theta$,$y = 2cos\theta$.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Exercise 3.1.2

Eliminate the parameter for the plane curve defined by the given parametric equations and describe the resulting graph.

$$x(t) = 2 + \frac{3}{t}, \quad y(t) = t - 1, \quad \text{for } 2 \leq t \leq 6$$

Hint available at the end of this section.

Answer available at the end of this section.

So far we have seen the method of eliminating the parameter, assuming we know a set of parametric equations that describe a plane curve. What if we would like to start with the equation of a curve and determine a pair of parametric equations for that curve? This is certainly possible, and in fact it is possible to do so in many different ways for a given curve. The process is known as **parametrization of a curve.**

Example 3.1.3: Parameterizing a Curve

Find two different pairs of parametric equations to represent the graph of $y = 2x^2 - 3$.

Solution

First, it is always possible to parameterize a curve by defining $x(t) = t$, then replacing x with t in the equation for $y(t)$. This gives the parametrization

$$x(t) = t, \quad y(t) = 2t^2 - 3.$$

Since there is no restriction on the domain in the original graph, there is no restriction on the values of t.

We have complete freedom in the choice for the second parametrization. For example, we can choose $x(t) = 3t - 2$. The only thing we need to check is that there are no restrictions imposed on x; that is, the range of $x(t)$ is all real numbers. This is the case for $x(t) = 3t - 2$. Now since $y = 2x^2 - 3$, we can substitute $x(t) = 3t - 2$ for x. This gives

$$y(t) = 2(3t - 2)^2 - 2 = 2(9t^2 - 12t + 4) - 2 = 18t^2 - 24t + 8 - 2 = 18t^2 - 24t + 6.$$

Therefore, a second parameterization of the curve can be written as

$$x(t) = 3t - 2 \text{ and } y(t) = 18t^2 - 24t + 6.$$

Answer available at the end of this section.

Exercise 3.1.3

Find two different sets of parametric equations to represent the graph of $y = x^2 + 2x$.

Hint available at the end of this section.

Answer available at the end of this section.

Cycloids and Other Parametric Curves

Imagine going on a bicycle ride through the country. The tires stay in contact with the road and rotate in a predictable pattern. Now suppose a very determined ant is tired after a long day and wants to get home. So he hangs onto the side of the tire and gets a free ride. The path that this ant travels down a straight road is called a *cycloid* (Figure 3.1.8). A cycloid generated by a circle (or bicycle wheel) of radius a is given by the parametric equations

$$x(t) = a(t - \sin t), \quad y(t) = a(1 - \cos t).$$

To see why this is true, consider the path that the center of the wheel takes. The center moves along the x-axis at a constant height equal to the radius of the wheel. If the radius is a, then the coordinates of the center can be given by the equations

$$x(t) = at, \quad y(t) = a$$

for any value of t. Next, consider the ant, which rotates around the center along a circular path. If the bicycle is moving from left to right then the wheels are rotating in a clockwise direction. A possible parameterization of the circular motion of the ant (relative to the center of the wheel) is given by

$$x(t) = -a \sin t$$
$$y(t) = -a \cos t.$$

(The negative sign is needed to reverse the orientation of the curve. If the negative sign were not there, we would have to imagine the wheel rotating counterclockwise.) Adding these equations together gives the equations for the cycloid.

$$x(t) = a(t - \sin t)$$
$$y(t) = a(1 - \cos t)$$

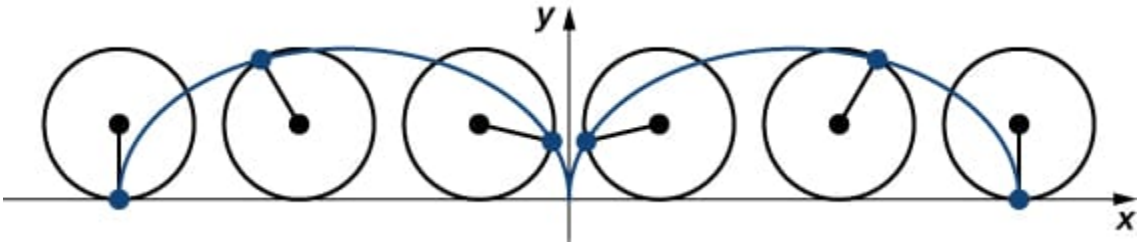


Figure 3.1.10: A series of circles with center marked and a point on the circle drawing out a curve as if the circle was rolling along a plane. The shape made seems to be half an ellipse with height the diameter of the original circle and with major axis the circumference of the circle.

Now suppose that the bicycle wheel doesn't travel along a straight road but instead moves along the inside of a larger wheel, as in Figure 3.1.9. In this graph, the green circle is traveling around the blue circle in a counterclockwise direction. A point on the edge of the green circle traces out the red graph, which is called a *hypocycloid*.

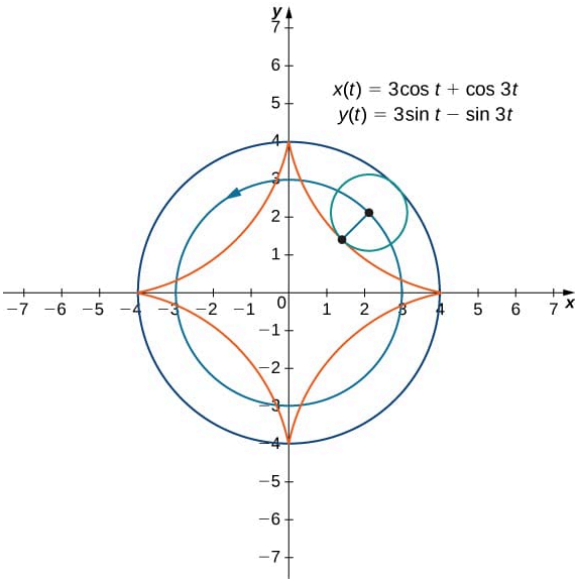


Figure 3.1.11: Two circles are drawn both with center at the origin and with radii 3 and 4, respectively; the circle with radius 3 has an arrow pointing in the counterclockwise direction. There is a third circle drawn with center on the circle with radius 3 and touching the circle with radius 4 at one point. That is, this third circle has radius 1. A point is drawn on this third circle, and if it were to roll along the other two circles, it would draw out a four pointed star with points at $(4, 0)$, $(0, 4)$, $(-4, 0)$, and $(0, -4)$. On the graph there are also written two equations: $x(t) = 3cos(t) + cos(3t)$ and $y(t) = 3sin(t) – sin(3t)$.

The general parametric equations for a hypocycloid are

$$x(t) = (a - b)\cos t + b\cos(\frac{a - b}{b})t$$

$$y(t) = (a - b)\sin t - b\sin(\frac{a - b}{b})t.$$

These equations are a bit more complicated, but the derivation is somewhat similar to the equations for the cycloid. In this case we assume the radius of the larger circle is a and the radius of the smaller circle is b. Then the center of the wheel travels along a circle of radius $a - b$. This fact explains the first term in each equation above. The period of the second trigonometric function in both $x(t)$ and $y(t)$ is equal to $\frac{2\pi b}{a - b}$.

The ratio $\frac{a}{b}$ is related to the number of **cusps** on the graph (cusps are the corners or pointed ends of the graph), as illustrated in Figure 3.1.10. This ratio can lead to some very interesting graphs, depending on whether or not the ratio is rational. Figure 3.1.9 corresponds to $a = 4$ and $b = 1$. The result is a hypocycloid with four cusps. Figure 3.1.10 shows some other possibilities. The last

two hypocycloids have irrational values for $\frac{a}{b}$. In these cases the hypocycloids have an infinite number of cusps, so they never return to their starting point. These are examples of what are known as **space-filling curves.**

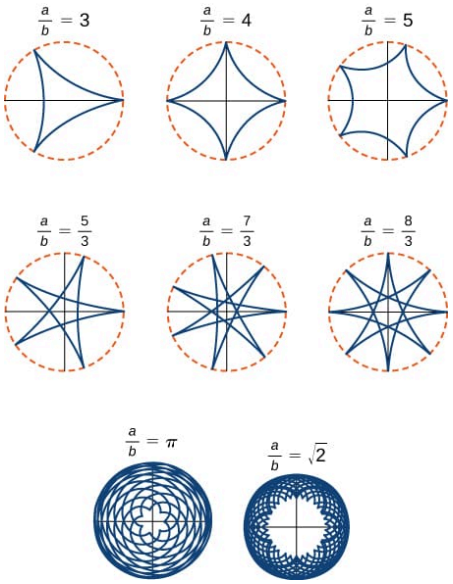


Figure $3.1.12$: A series of hypocycloids is given. The first is a three pointed star marked $a/b = 3$. The second is a four pointed star marked $a/b = 4$. The third is a five pointed star marked $a/b = 5$. None of these first three figures has lines that cross each other. The fourth figure is a five pointed star but this one has lines which cross each other and looks like the star that children first learn to draw; it is marked $a/b = 5/3$. A similar sort of star with seven points is next and is marked $a/b = 7/3$. Then a similar star with eight points is next and is marked $a/b = 8/3$. The next figure is a complicated series of curves that ultimately creates a small rosette in the middle; this is marked $a/b = \pi$. Lastly, there is an even more complicated series of curves that creates a large rosette with sharper florets marked $a/b = sqrt2$.

The Witch of Agnesi

Many plane curves in mathematics are named after the people who first investigated them, like the folium of Descartes or the spiral of Archimedes. However, perhaps the strangest name for a curve is the **witch of Agnesi.** Why a witch?

Maria Gaetana Agnesi (1718–1799) was one of the few recognized women mathematicians of eighteenth-century Italy. She wrote a popular book on analytic geometry, published in 1748, which included an interesting curve that had been studied by Fermat in 1630. The mathematician Guido Grandi showed in 1703 how to construct this curve, which he later called the "versoria," a Latin term for a rope used in sailing. Agnesi used the Italian term for this rope, "versiera," but in Latin, this same word means a "female goblin." When Agnesi's book was translated into English in 1801, the translator used the term "witch" for the curve, instead of rope. The name "witch of Agnesi" has stuck ever since.

The witch of Agnesi is a curve defined as follows: Start with a circle of radius a so that the points $(0, 0)$ and $(0, 2a)$ are points on the circle (Figure $3.1.11$). Let O denote the origin. Choose any other point A on the circle, and draw the secant line OA. Let B denote the point at which the line OA intersects the horizontal line through $(0, 2a)$. The vertical line through B intersects the horizontal line through A at the point P. As the point A varies, the path that the point P travels is the witch of Agnesi curve for the given circle.

Witch of Agnesi curves have applications in physics, including modeling water waves and distributions of spectral lines. In probability theory, the curve describes the probability density function of the *Cauchy distribution*. In this project you will parameterize these curves.

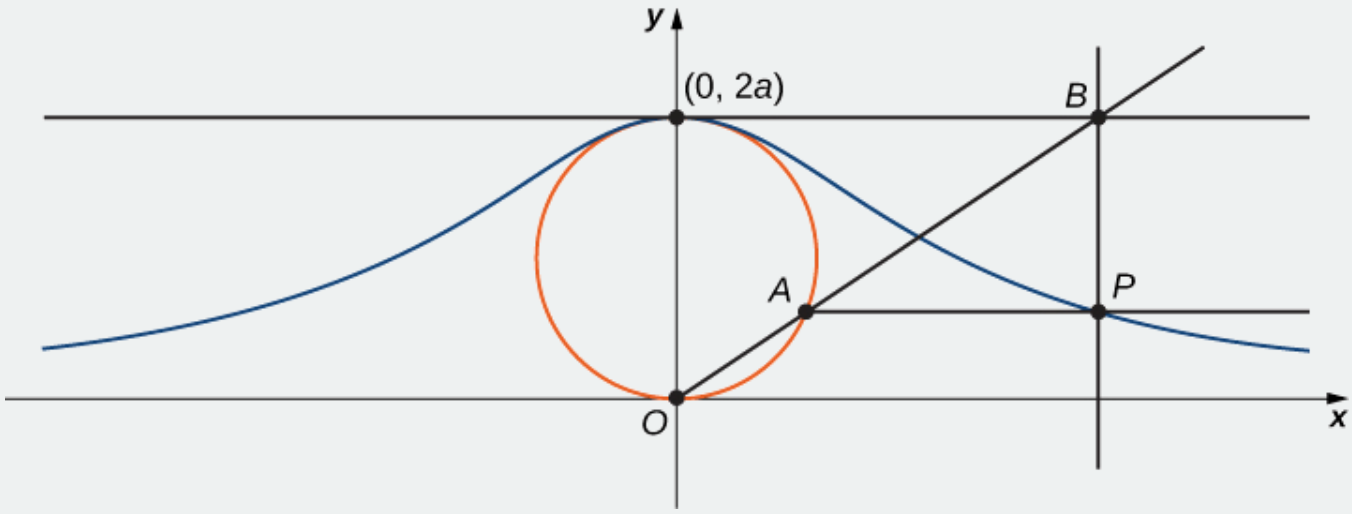


Figure 3.1.13: As the point A moves around the circle, the point P traces out the witch of Agnesi curve for the given circle. A circle with bottom at point O (the origin) and top at point $(0, 2a)$ is drawn. The x axis is drawn from point O, and the y axis is drawn up from point O through $(0, 2a)$. Parallel to the x axis is a line drawn from $(0, 2a)$; it has point B marked to the right. A line from point B to point O passes through the circle at point A. A line is drawn parallel to the x axis from point A, and it forms a right angle with a line drawn down from point B; these lines intersect at point P. There is a curve that is symmetric about the y axis that passes through the point P. This curve has its maximum at $(0, 2a)$ and gently decreases through the point P.

1. On the figure, label the given points, lengths, and angle:
   a. C is the point on the x-axis with the same x-coordinate as A.
   b. x is the x-coordinate of P, and y is the y-coordinate of P.
   c. E is the point $(0, a)$.
   d. F is the point on the line segment OA such that the line segment EF is perpendicular to the line segment OA.
   e. b is the distance from O to F.
   f. c is the distance from F to A.
   g. d is the distance from O to C.
   h. θ is the measure of angle $\angle COA$.

The goal of this project is to parameterize the witch using θ as a parameter. To do this, write equations for x and y in terms of only θ.

2. Show that $d = \dfrac{2a}{\sin\theta}$.
3. Note that $x = d\cos\theta$. Show that $x = 2a\cot\theta$. When you do this, you will have parametrized the x-coordinate of the curve with respect to θ. If you can get a similar equation for y, you will have parametrized the curve.
4. In terms of θ, what is the angle \(∠EOA\)?
5. Show that $b + c = 2a\cos\left(\frac{\pi}{2} - \theta\right)$.
6. Show that $y = 2a\cos\left(\frac{\pi}{2} - \theta\right)\sin\theta$.
7. Show that $y = 2a\sin^2\theta$. You have now parametrized the y-coordinate of the curve with respect to θ.
8. Conclude that a parametrization of the given witch curve is

$$x = 2a\cot\theta, \quad y = 2a\sin^2\theta, \quad \text{for } -\infty < \theta < \infty. \tag{3.1.12}$$

9. Use your parametrization to show that the given witch curve is the graph of the function $f(x) = \dfrac{8a^3}{x^2 + 4a^2}$.

Travels with My Ant: The Curtate and Prolate Cycloids

Earlier in this section, we looked at the parametric equations for a cycloid, which is the path a point on the edge of a wheel traces as the wheel rolls along a straight path. In this project we look at two different variations of the cycloid, called the curtate and prolate cycloids.

First, let's revisit the derivation of the parametric equations for a cycloid. Recall that we considered a tenacious ant trying to get home by hanging onto the edge of a bicycle tire. We have assumed the ant climbed onto the tire at the very edge, where the tire touches the ground. As the wheel rolls, the ant moves with the edge of the tire (Figure 3.1.12).

As we have discussed, we have a lot of flexibility when parametrizing a curve. In this case we let our parameter t represent the angle the tire has rotated through. Looking at Figure 3.1.12, we see that after the tire has rotated through an angle of t, the position of the center of the wheel, $C = (x_C, y_C)$, is given by

$$x_C = at \text{ and } y_C = a.$$

Furthermore, letting $A = (x_A, y_A)$ denote the position of the ant, we note that

$$x_C - x_A = a\sin t \text{ and } y_C - y_A = a\cos t$$

Then

$$x_A = x_C - a\sin t = at - a\sin t = a(t - \sin t) \tag{3.1.13}$$

$$y_A = y_C - a\cos t = a - a\cos t = a(1 - \cos t). \tag{3.1.14}$$

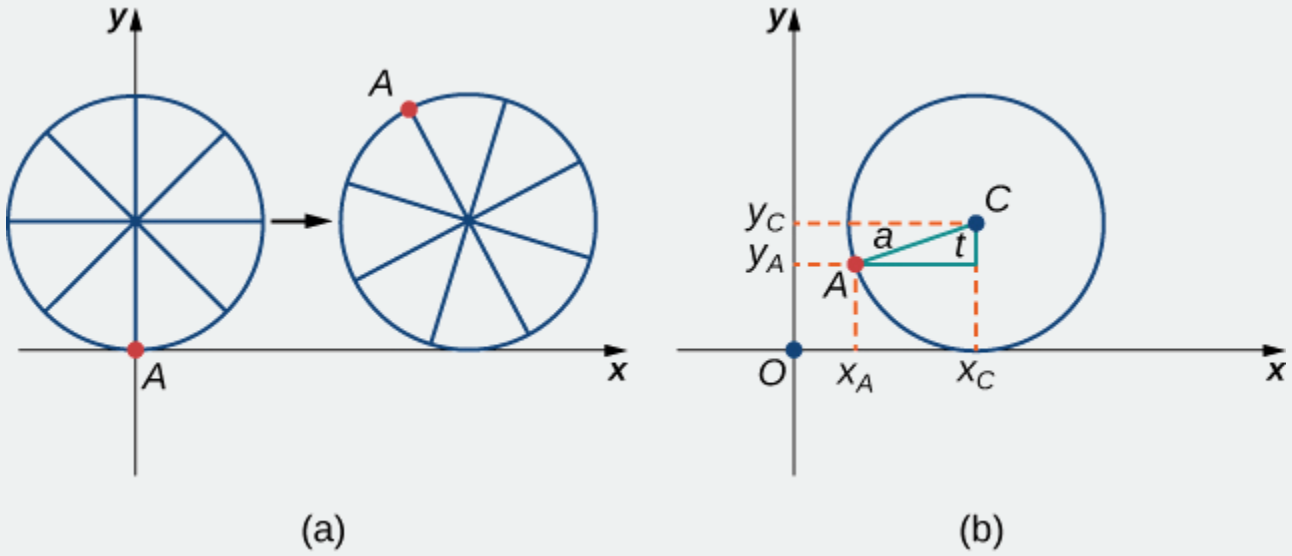


Figure 3.1.14: (a) The ant clings to the edge of the bicycle tire as the tire rolls along the ground. (b) Using geometry to determine the position of the ant after the tire has rotated through an angle of t. There are two figures marked (a) and (b). Figure a has a circle with point A on the circle at the origin. The circle has "spokes," with point A being at the end of one of these spokes. The circle appears to be traveling to the right on the x axis, with point A being up above the x axis in a second image of the circle drawn slightly to the right. Figure b has a circle in the first quadrant with center C. It touches the x axis at x_c. A point A is drawn on the circle and a right triangle is made from this point and point C. The hypotenuse is marked a and the angle at C between A and x_c is marked t. Lines are drawn to give the x and y values of A as x_A and y_A, respectively. Similarly, a line is drawn to give the y value of C as y_C.

Note that these are the same parametric representations we had before, but we have now assigned a physical meaning to the parametric variable t.

After a while the ant is getting dizzy from going round and round on the edge of the tire. So he climbs up one of the spokes toward the center of the wheel. By climbing toward the center of the wheel, the ant has changed his path of motion. The new path has less up-and-down motion and is called a **curtate cycloid** (Figure 3.1.13). We let b denote the distance along the spoke from the center of the wheel to the ant. As before, we let t represent the angle the tire has rotated through. Additionally, we let $C = (x_C, y_C)$ represent the position of the center of the wheel and $A = (x_A, y_A)$ represent the position of the ant.

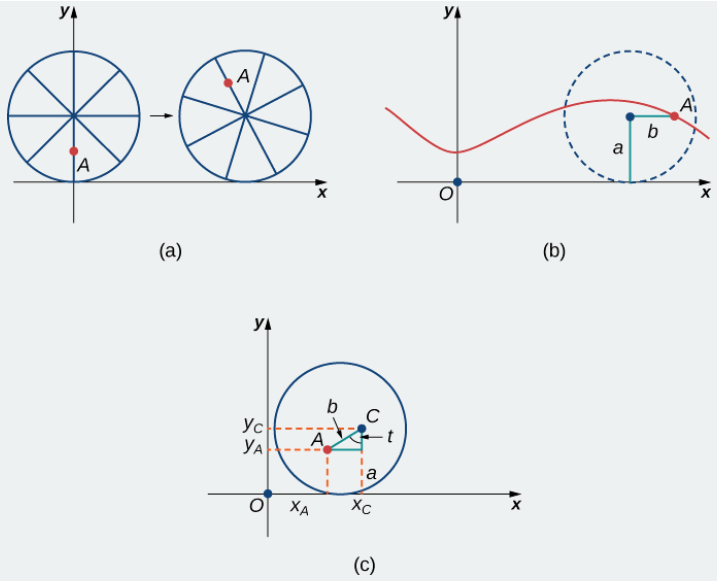


Figure 3.1.15: (a) The ant climbs up one of the spokes toward the center of the wheel. (b) The ant's path of motion after he climbs closer to the center of the wheel. This is called a curtate cycloid. (c) The new setup, now that the ant has moved closer to the center of the wheel.

1. What is the position of the center of the wheel after the tire has rotated through an angle of t?
2. Use geometry to find expressions for $x_C - x_A$ and for $y_C - y_A$.
3. On the basis of your answers to parts 1 and 2, what are the parametric equations representing the curtate cycloid?

Once the ant's head clears, he realizes that the bicyclist has made a turn, and is now traveling away from his home. So he drops off the bicycle tire and looks around. Fortunately, there is a set of train tracks nearby, headed back in the right direction. So the ant heads over to the train tracks to wait. After a while, a train goes by, heading in the right direction, and he manages to jump up and just catch the edge of the train wheel (without getting squished!).

The ant is still worried about getting dizzy, but the train wheel is slippery and has no spokes to climb, so he decides to just hang on to the edge of the wheel and hope for the best. Now, train wheels have a flange to keep the wheel running on the tracks. So, in this case, since the ant is hanging on to the very edge of the flange, the distance from the center of the wheel to the ant is actually greater than the radius of the wheel (Figure 3.1.14).

The setup here is essentially the same as when the ant climbed up the spoke on the bicycle wheel. We let b denote the distance from the center of the wheel to the ant, and we let t represent the angle the tire has rotated through. Additionally, we let $C = (x_C, y_C)$ represent the position of the center of the wheel and $A = (x_A, y_A)$ represent the position of the ant (Figure 3.1.14).

When the distance from the center of the wheel to the ant is greater than the radius of the wheel, his path of motion is called a prolate cycloid. A graph of a **prolate cycloid** is given in the Figure 3.1.16.

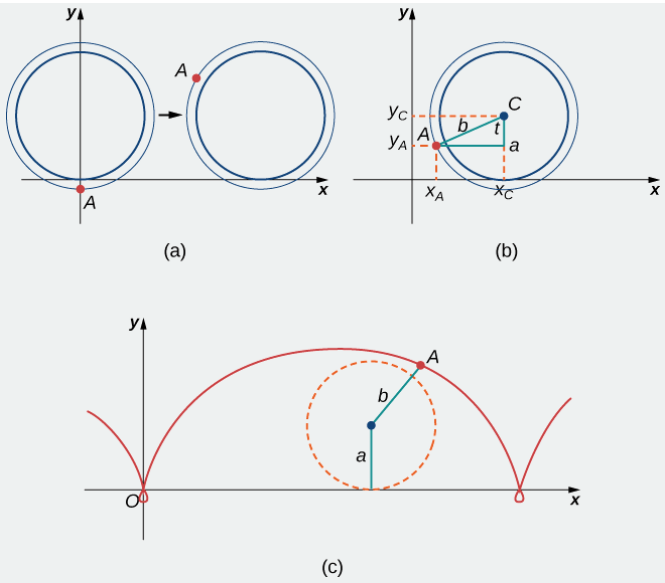


Figure 3.1.16: (a) The ant is hanging onto the flange of the train wheel. (b) The new setup, now that the ant has jumped onto the train wheel. (c) The ant travels along a prolate cycloid.

4. Using the same approach you used in parts 1– 3, find the parametric equations for the path of motion of the ant.
5. What do you notice about your answer to part 3 and your answer to part 4?

Notice that the ant is actually traveling backward at times (the "loops" in the graph), even though the train continues to move forward. He is probably going to be really dizzy by the time he gets home!

Key Concepts

- Parametric equations provide a convenient way to describe a curve. A parameter can represent time or some other meaningful quantity.
- It is often possible to eliminate the parameter in a parameterized curve to obtain a function or relation describing that curve.
- There is always more than one way to parameterize a curve.
- Parametric equations can describe complicated curves that are difficult or perhaps impossible to describe using rectangular coordinates.

Glossary

Cycloid
The curve traced by a point on the rim of a circular wheel as the wheel rolls along a straight line without slippage.

Cusp
A pointed end or part where two curves meet.

Orientation
The direction that a point moves on a graph as the parameter increases.

Parameter
An independent variable that both x and y depend on in a parametric curve; usually represented by the variable t.

Parametric Curve
The graph of the parametric equations $x(t)$ and $y(t)$ over an interval $a \leq t \leq b$ combined with the equations.

Parametric Equations
The equations $x = x(t)$ and $y = y(t)$ that define a parametric curve.

Parametrization of a Curve

Rewriting the equation of a curve defined by a function $y = f(x)$ as parametric equations.

Answers

Exercise 3.1.1

Make a table of values for $x(t)$ and $y(t)$ using t values from -3 to 2.

Exercise 3.1.1

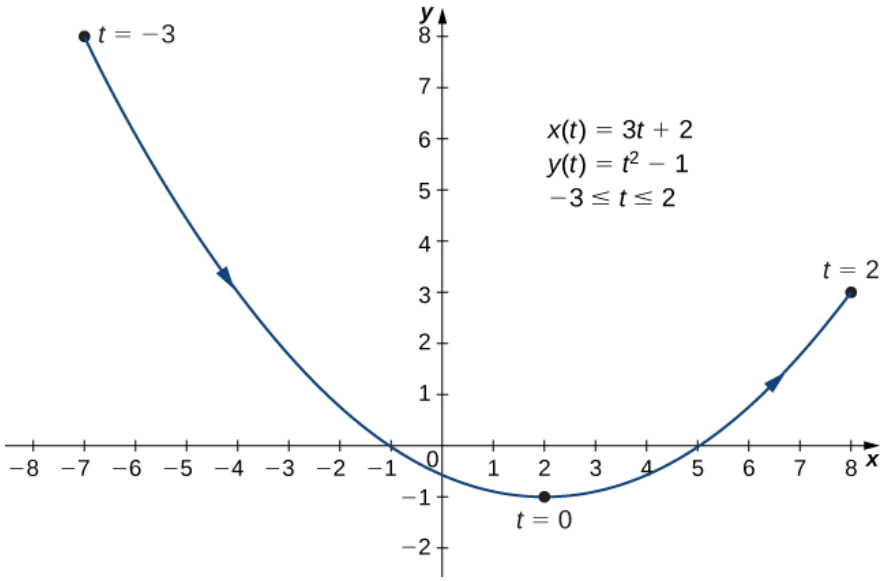


Figure 3.1.6:A curved line going from $(-7, 8)$ through $(-1, 0)$ and $(2, -1)$ to $(8, 3)$ with arrow going in that order. The point $(-7, 8)$ is marked $t = -3$, the point $(2, -1)$ is marked $t = 0$, and the point $(8, 3)$ is marked $t = 2$. On the graph there are also written three equations: $x(t) = 3t + 2$, $y(t) = t^2 - 1$, and $-3 \le t \le 2$.

Exercise 3.1.2

Solve one of the equations for t and substitute into the other equation.

Exercise 3.1.2

$x = 2 + \frac{3}{y+1}$, or $y = -1 + \frac{3}{x-2}$. This equation describes a portion of a rectangular hyperbola centered at $(2, -1)$.

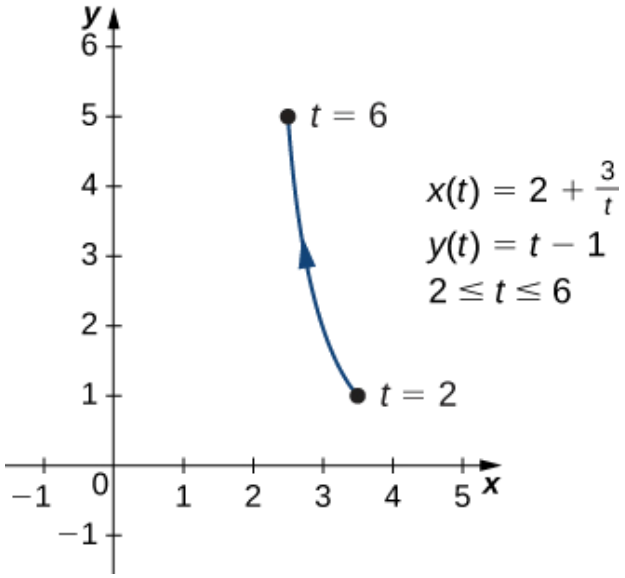


Figure 3.1.9: A curved line going from $(3.5, 1)$ to $(2.5, 5)$ with arrow going in that order. The point $(3.5, 1)$ is marked $t = 2$ and the point $(2.5, 5)$ is marked $t = 6$. On the graph there are also written three equations: $x(t) = 2 + 3/t$, $y(t) = t - 1$, and $2 \le t \le 6$.

Example 3.1.3**: Parameterizing a Curve**

Watch the accompanying video for a visual demonstration of this concept.

- **Video Length:** 3 minute 28 seconds.
- **Context:** This video demonstrates how to find parametric equations of a line that passes through two given points $(3, 4$ and $1, 7$.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Exercise 3.1.3

Follow the steps in Example 3.1.3. Remember we have freedom in choosing the parameterization for $x(t)$.

Exercise 3.1.3

One possibility is $x(t) = t, \quad y(t) = t^2 + 2t$. Another possibility is $x(t) = 2t - 3, \quad y(t) = (2t-3)^2 + 2(2t-3) = 4t^2 - 8t + 3$. There are, in fact, an infinite number of possibilities.

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3.1E: Exercises for Section 3.1

Exercises 1 - 4

Sketch the curves below by eliminating the parameter t. Give the orientation of the curve.

1. $x = t^2 + 2t, \quad y = t + 1$

Answer

Orientation: bottom to top

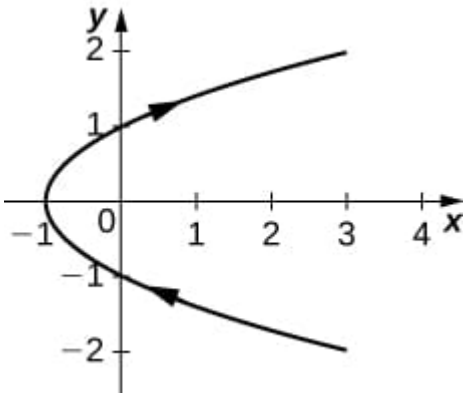


Figure $3.1E.1$: A graph of a parametric curve resembling a hyperbolic shape. The curve starts at negative values of y, passes through the origin, and opens outward as x increases. Arrows on the curve indicate the direction of increasing parameter values, moving from the lower left through the origin to the upper right. The $x - axis$ ranges from -1 to 4, and the $y - axis$ ranges from -2 to 2, highlighting the symmetry and progression of the curve.

2. $x = \cos(t), \quad y = \sin(t), \quad \text{for } (0, 2\pi]$
3. $x = 2t + 4, \quad y = t - 1$

Answer

Orientation: left to right

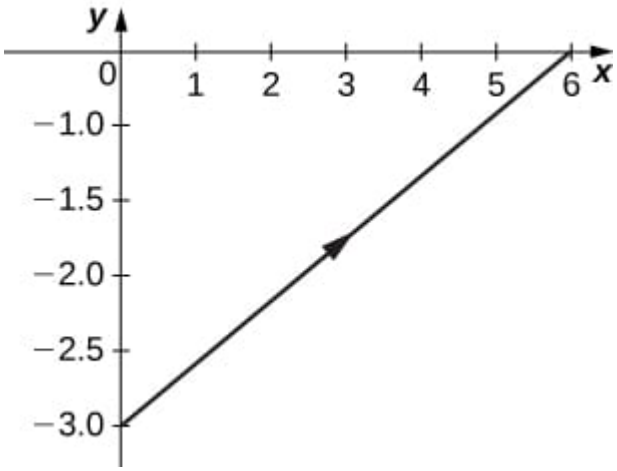


Figure $3.1E.2$:A graph showing a straight diagonal line with an arrow pointing upward to the right, indicating direction. The line starts at approximately $(0, -3)$ and passes through the x-axis as it moves toward $(6, 0)$. The x-axis ranges from 0 to 6, and the y-axis ranges from -3 to 0.

4. $x = 3 - t, \quad y = 2t - 3, \quad \text{for } 1.5 \leq t \leq 3$

Exercise 5

Eliminate the parameter and sketch the graph.

5. $x = 2t^2, \quad y = t^4 + 1$

Answer

$$y = \frac{x^2}{4} + 1$$

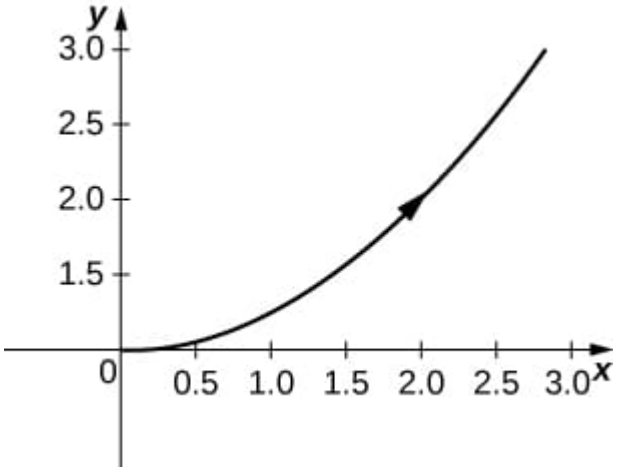


Figure $3.1E.3$: A graph showing an exponential curve starting near $(0, 0)$ and increasing as x progresses from 0 to 3. The arrow on the curve points upward and to the right, indicating the direction of increasing values. The x-axis ranges from 0 to 3, and the y-axis ranges from 0 to 3.

Exercises 6 - 9

Use technology (CAS or calculator) to sketch the parametric equations.

6. [T] $x = t^2 + t, \quad y = t^2 - 1$
7. [T] $x = e^{-t}, \quad y = e^{2t} - 1$

Answer

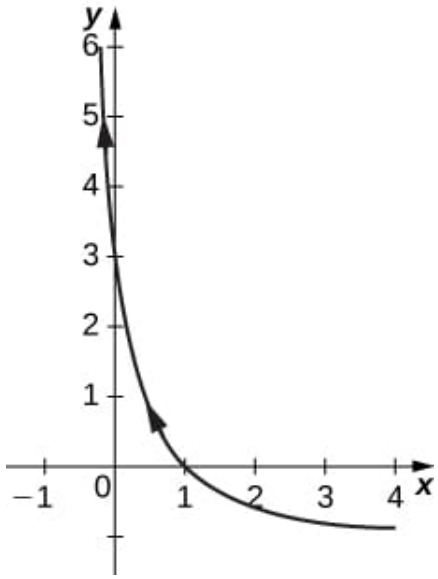


Figure $3.1E.4$: A graph showing a curve that starts at $y = 6$ when $x$is near 0, decreasing rapidly and leveling off as x approaches 4. Arrows along the curve indicate the direction of decreasing values from $(0, 6)$ to lower y-values as x increases. The x-axis ranges from -1 to 4, and the y-axis ranges from 0 to 6.

8. [T] $x = 3\cos t, \quad y = 4\sin t$
9. [T] $x = \sec t, \quad y = \cos t$

Answer

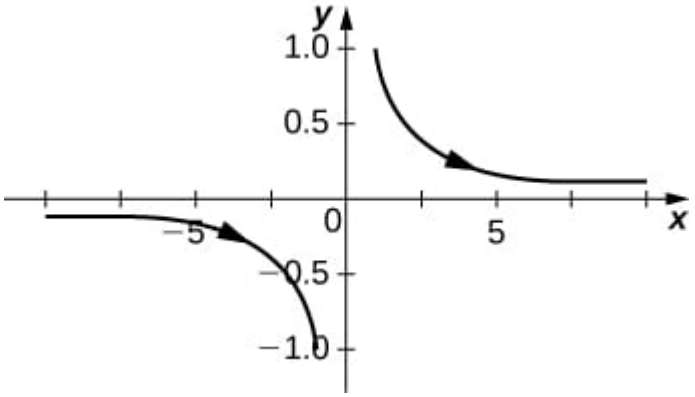


Figure $3.1E.5$: A graph showing two branches of a hyperbolic curve. The right branch starts near $y = 1.0$ and decreases toward $y = 0$ as x increases. The left branch starts near $y = -1.0$ and increases toward $y = 0$ as x becomes more negative. Arrows indicate the direction of the curve. The x-axis ranges from -5 to 5, and the y-axis ranges from -1.0 to 1.0.

Exercises 10 - 20

Sketch the parametric equations by eliminating the parameter. Indicate any asymptotes of the graph.

10. $x = e^t, \quad y = e^{2t} + 1$
11. $x = 6\sin(2\theta), \quad y = -4\cos(2\theta)$

Answer

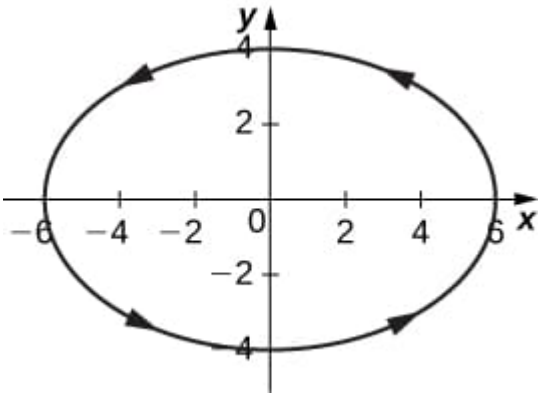


Figure $3.1E.6$: A graph of an ellipse centered at the origin, with a horizontal major axis of length 12 and a vertical minor axis of length 8. Arrows along the ellipse indicate a counterclockwise orientation, showing the direction of motion around the curve.

12. $x = \cos\theta, \quad y = 2\sin(2\theta)$
13. $x = 3 - 2\cos\theta, \quad y = -5 + 3\sin\theta$

Answer

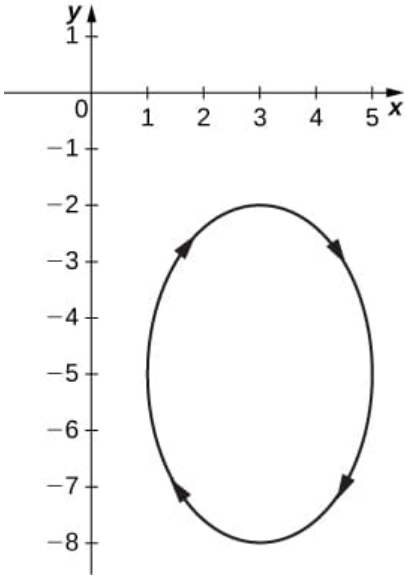


Figure $3.1E.7$: A graph of a vertical ellipse centered at approximately $(2.5, -5)$ with the x values ranging from 1 to 5 and the y values ranging from -8 to 2. Arrows along the ellipse indicate a clockwise orientation, showing the direction of motion around the curve.

14. $x = 4 + 2\cos\theta, \quad y = -1 + \sin\theta$
15. $x = \sec t, \quad y = \tan t$

Answer

Asymptotes are $y = x$ and $y = -x$

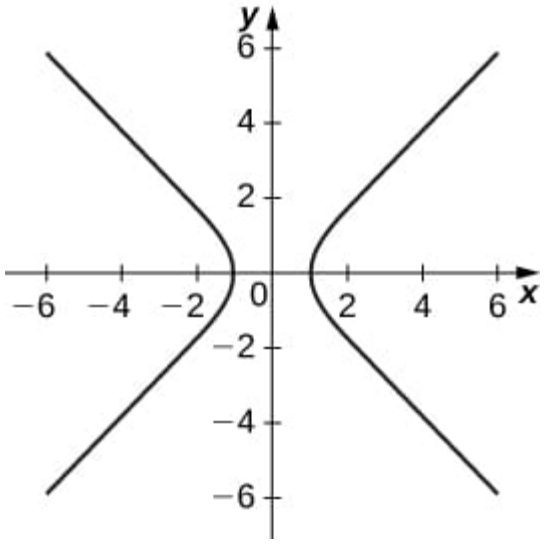


Figure $3.1E.8$: A graph of a hyperbola centered at the origin, opening outward along the x-axis. The x-axis and y-axis both range from -6 to 6, with the two branches of the hyperbola extending symmetrically in opposite directions. The curve approaches but does not touch the axes, showing asymptotic behavior.

16. $x = \ln(2t), \quad y = t^2$
17. $x = e^t, \quad y = e^{2t}$

Answer

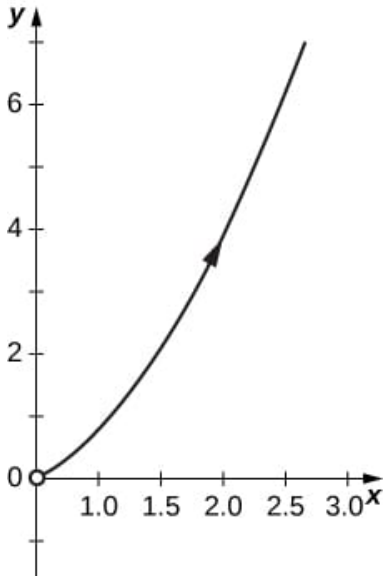


Figure $3.1E.9$: A graph showing an exponential curve starting at the origin $(0, 0)$ and increasing steeply as x progresses from 1 to 3. An arrow on the curve points upward, indicating the direction of growth. The y-axis ranges from 0 to 6, and the x-axis ranges from 1 to 3, with a hollow circle at $(1, 0)$ indicating a specific point on the curve.

18. $x = e^{-2t}, \quad y = e^{3t}$
19. $x = t^3, \quad y = 3 \ln t$

Answer

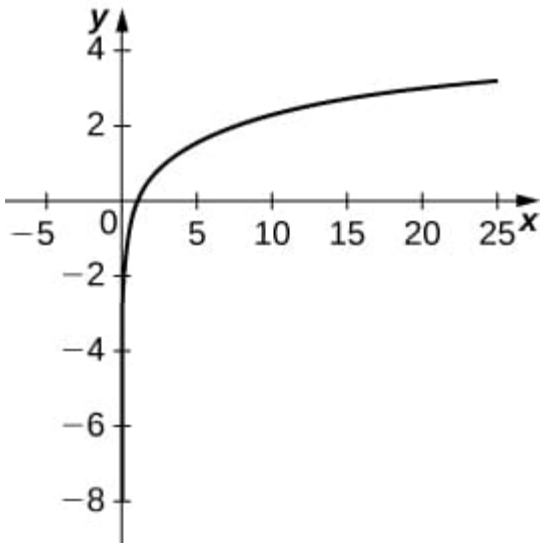


Figure $3.1E.10$: A graph showing a curve that starts steeply decreasing from $(0, -\infty)$ crosses the$y - axis$ at $y = 0$, and gradually levels off as x approaches 25. The $x - axis$ ranges from -5 to 25, and the y-axis ranges from -8 to 4, illustrating the behavior of a function with an asymptote.

20. $x = 4\sec\theta, \quad y = 3\tan\theta$

Exercises 21 - 38

Convert the parametric equations of a curve into rectangular form. No sketch is necessary. State the domain of the rectangular form.

21. $x = t^2 - 1, \quad y = \dfrac{t}{2}$

Answer

$x = 4y^2 - 1$; domain: $x \in [-1, \infty)$.

22. $x = \dfrac{1}{\sqrt{t+1}}, \quad y = \dfrac{t}{1+t}, \quad \text{for } t > -1$
23. $x = 4\cos\theta, \quad y = 3\sin\theta, \quad \text{for } t \in (0, 2\pi]$

Answer

$\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$; domain $x \in [-4, 4]$.

24. $x = \cosh t, \quad y = \sinh t$
25. $x = 2t - 3, \quad y = 6t - 7$

Answer

$y = 3x + 2$; domain: all real numbers.

26. $x = t^2, \quad y = t^3$
27. $x = 1 + \cos t, \quad y = 3 - \sin t$

Answer

$(x-1)^2 + (y-3)^2 = 1$; domain: $x \in [0, 2]$.

28. $x = \sqrt{t}, \quad y = 2t + 4$
29. $x = \sec t, \quad y = \tan t, \quad \text{for } \pi \le t < \frac{3\pi}{2}$

Answer

$y = \sqrt{x^2 - 1}$; domain: $x \in (-\infty, -1]$.

30. $x = 2\cosh t, \quad y = 4\sinh t$
31. $x = \cos(2t), \quad y = \sin t$

Answer

$y^2 = \dfrac{1-x}{2}$; domain: $x \in [-1, 1]$.

32. $x = 4t + 3, \quad y = 16t^2 - 9$
33. $x = t^2, \quad y = 2\ln t, \quad \text{for } t \geq 1$

Answer

$y = \ln x$; domain: $x \in [1, \infty)$.

34. $x = t^3, \quad y = 3\ln t, \quad \text{for } t \geq 1$
35. $x = t^n, \quad y = n\ln t, \quad \text{for } t \geq 1$, where n is a natural number

Answer

$y = \ln x$; domain: $x \in (0, \infty)$.

36. $x = \ln(5t), \quad y = \ln(t^2)$ where $1 \leq t \leq e$
37. $x = 2\sin(8t), \quad y = 2\cos(8t)$

Answer

$x^2 + y^2 = 4$; domain: $x \in [-2, 2]$.

38. $x = \tan t, \quad y = \sec^2 t - 1$

Exercises 39 - 50

The pairs of parametric equations represent lines, parabolas, circles, ellipses, or hyperbolas. Name the type of basic curve that each pair of equations represents.

39. $x = 3t + 4, \quad y = 5t - 2$

Answer

line

40. $x - 4 = 5t, \quad y + 2 = t$
41. $x = 2t + 1, \quad y = t^2 - 3$

Answer

parabola

42. $x = 3\cos t, \quad y = 3\sin t$
43. $x = 2\cos(3t), \quad y = 2\sin(3t)$

Answer

circle

44. $x = \cosh t, \quad y = \sinh t$
45. $x = 3\cos t, \quad y = 4\sin t$

Answer

ellipse

46. $x = 2\cos(3t), \quad y = 5\sin(3t)$
47. $x = 3\cosh(4t) \quad y = 4\sinh(4t)$

Answer

the right branch of a horizontally opening hyperbola

48. $x = 2\cosh t, \quad y = 2\sinh t$
49. Show that $x = h + r\cos\theta, \quad y = k + r\sin\theta$ represents the equation of a circle.
50. Use the equations in the preceding problem to find a set of parametric equations for a circle whose radius is 5 and whose center is $(-2, 3)$.

Exercises 51 - 53

Use a graphing utility to graph the curve represented by the parametric equations and identify the curve from its equation.

51. [T] $x = \theta + \sin\theta, \quad y = 1 - \cos\theta$

Answer

The equations represent a cycloid.

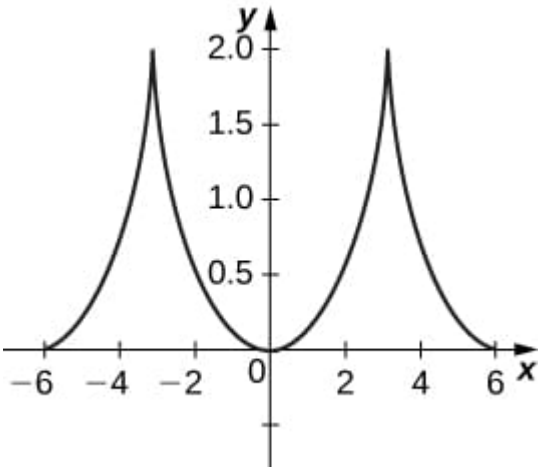


Figure $3.1E.11$: A graph showing two bell-shaped curves, one centered at $x = -3$ and the other at $x = 3$, both peaking at $y = 2$. The curves descend symmetrically toward $y = 0$ as x approaches 0 and ± 6. The x-axis ranges from -6 to 6, and the y-axis ranges from 0 to 2, illustrating a periodic or symmetric function.

52. [T] $x = 2t - 2\sin t, \quad y = 2 - 2\cos t$
53. [T] $x = t - 0.5\sin t, \quad y = 1 - 1.5\cos t$

Answer

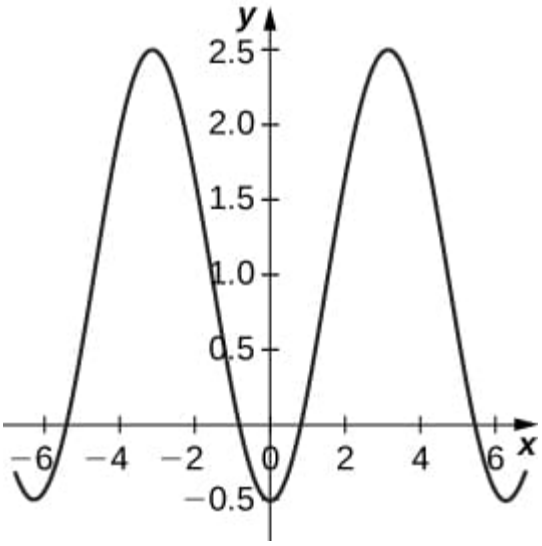


Figure $3.1E.12$: A graph showing a sinusoidal wave with two peaks and two troughs. The peaks occur at $x = -4$ reaching a maximum of $y = 2.5$. The troughs occur at $x = -2$ and $x = 2$, reaching a minimum of $y = -0.5$. The curve oscillates symmetrically around the x-axis, with the x-axis ranging from -6 to 6 and the y-axis ranging from -0.5 to 2.5.

54. An airplane traveling horizontally at 100 m/s over flat ground at an elevation of 4000 meters must drop an emergency package on a target on the ground. The trajectory of the package is given by $x = 100t, \quad y = -4.9t^2 + 4000, \quad \text{where } t \geq 0$ where the origin is the point on the ground directly beneath the plane at the moment of release. How many horizontal meters before the target should the package be released in order to hit the target?
55. The trajectory of a bullet is given by $x = v_0(\cos\alpha)t, \quad y = v_0(\sin\alpha)t - \frac{1}{2}gt^2$ where $v_0 = 500$ m/s, $g = 9.8 = 9.8 \text{ m/s}^2$, and $\alpha = 30$ degrees. When will the bullet hit the ground? How far from the gun will the bullet hit the ground?

Answer

22,092 meters at approximately 51 seconds.

56. [T] Use technology to sketch the curve represented by $x = \sin(4t), \quad y = \sin(3t), \quad \text{for } 0 \leq t \leq 2\pi$.
57. [T] Use technology to sketch \(x=2\tan(t), \quad y=3\sec(t), \quad \text{for }–π

Answer

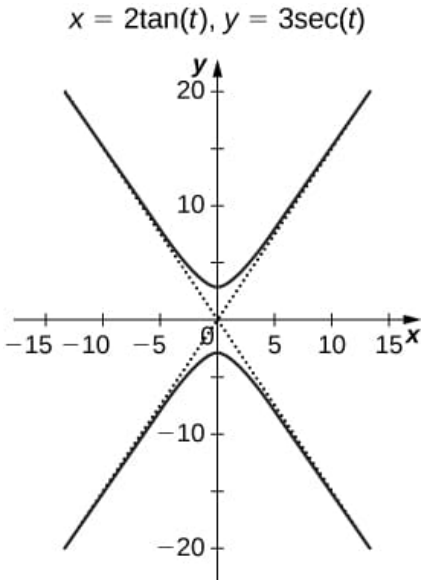


Figure $3.1E.13$: A graph of a parametric curve defined by \(x=2\tan(t), \quad y=3\sec(t), \quad \text{for }–π

58. Sketch the curve known as an **epitrochoid**, which gives the path of a point on a circle of radius b as it rolls on the outside of a circle of radius a. The equations are

$$x = (a+b)\cos t - c \cdot \cos\left[\tfrac{(a+b)t}{b}\right], \quad y = (a+b)\sin t - c \cdot \sin\left[\tfrac{(a+b)t}{b}\right].$$

Let $a = 1, \ b = 2, \ c = 1$.

59. [T] Use technology to sketch the spiral curve given by $x = t\cos(t), \quad y = t\sin(t)$ for $-2\pi \le t \le 2\pi$.

Answer

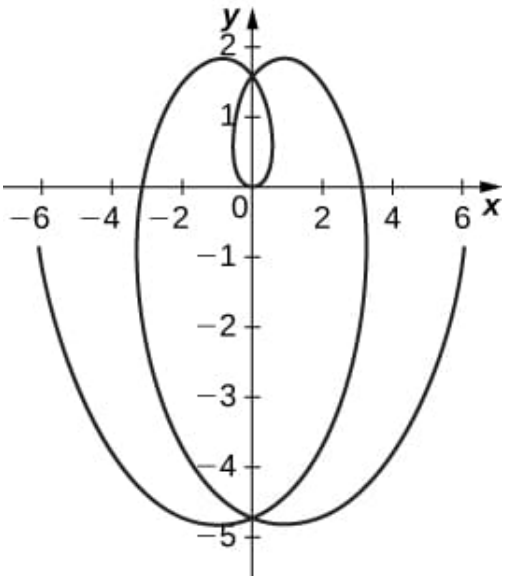


Figure $3.1E.14$: A graph showing a series of nested, vertically aligned loops. The outer loop opens downward, reaching a minimum at $y = -5$, while a smaller inner loop is centered around $y = 1$. The loops intersect at the origin and extend symmetrically along the x-axis, which ranges from -6 to 6. The y-axis ranges from -5 to 2, highlighting the layered structure of the loops.

60. [T] Use technology to graph the curve given by the parametric equations $x = 2\cot(t), \quad y = 1 - \cos(2t), \quad \text{for } -\pi/2 \le t \le \pi/2$. This curve is known as the witch of Agnesi.

61. [T] Sketch the curve given by parametric equations $x = \cosh(t), \quad y = \sinh(t)$, for $-2 \le t \le 2$.

Answer

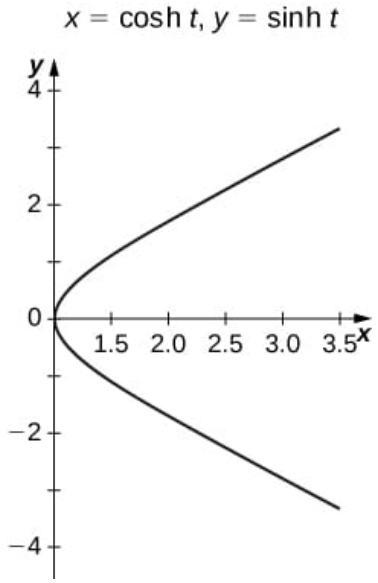


Figure $3.1E.14$:A graph of a parametric curve defined by $x = \cosh(t), \quad y = \sinh(t)$, forming a hyperbolic shape. The curve originates at $(1, 0)$, opening outward symmetrically as x and y increase in magnitude. The x-axis ranges from 1 to 3.5, and the y-axis ranges from -4 to 4, illustrating the characteristic hyperbolic structure.

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3.2: Calculus of Parametric Curves

Learning Objectives

- Determine derivatives and equations of tangents for parametric curves.
- Find the area under a parametric curve.
- Use the equation for arc length of a parametric curve.
- Apply the formula for surface area to a volume generated by a parametric curve.

Now that we have introduced the concept of a parameterize curve, our next step is to learn how to work with this concept in the context of calculus. For example, if we know a parametrization of a given curve, is it possible to calculate the slope of a tangent line to the curve? How about the arc length of the curve? Or the area under the curve?

Another scenario: Suppose we would like to represent the location of a baseball after the ball leaves a pitcher's hand. If the position of the baseball is represented by the plane curve $(x(t), y(t))$ then we should be able to use calculus to find the speed of the ball at any given time. Furthermore, we should be able to calculate just how far that ball has traveled as a function of time.

Derivatives of Parametric Equations

We start by asking how to calculate the slope of a line tangent to a parametric curve at a point. Consider the plane curve defined by the parametric equations

$$x(t) = 2t + 3 \tag{3.2.1}$$
$$y(t) = 3t - 4 \tag{3.2.2}$$

within $-2 \leq t \leq 3$.

The graph of this curve appears in Figure 3.2.1. It is a line segment starting at $(-1, -10)$ and ending at $(9, 5)$.

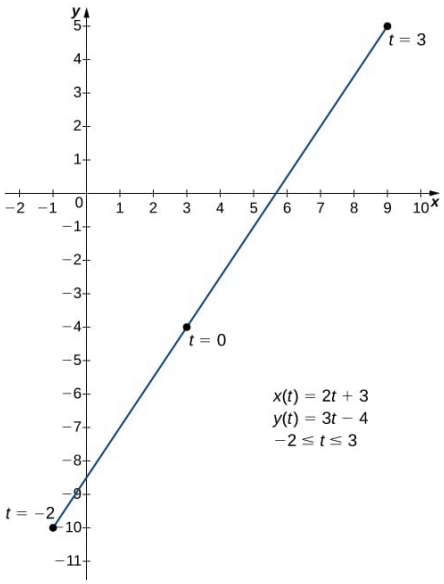


Figure 3.2.1: A straight line from $(-1, -10)$ to $(9, 5)$. The point $(-1, -10)$ is marked $t = -2$, the point $(3, -4)$ is marked $t = 0$, and the point $(9, 5)$ is marked $t = 3$. There are three equations marked: $x(t) = 2t + 3$, $y(t) = 3t$–4, and $-2 \leq t \leq 3$.

We can eliminate the parameter by first solving (Equation 3.2.1) for t:

$$x(t) = 2t + 3$$

$$x - 3 = 2t$$

$$t = \frac{x-3}{2}.$$

Substituting this into $y(t)$ (Equation 3.2.2), we obtain

$$y(t) = 3t - 4$$

$$y = 3\left(\frac{x-3}{2}\right) - 4$$

$$y = \frac{3x}{2} - \frac{9}{2} - 4$$

$$y = \frac{3x}{2} - \frac{17}{2}.$$

The slope of this line is given by $\frac{dy}{dx} = \frac{3}{2}$. Next we calculate $x\,'(t)$ and $y\,'(t)$. This gives $x\,'(t) = 2$ and $y\,'(t) = 3$. Notice that

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3}{2}.$$

This is no coincidence, as outlined in the next theorem.

Derivative of Parametric Equations

Consider the plane curve defined by the parametric equations $x = x(t)$ and $y = y(t)$. Suppose that $x\,'(t)$ and $y\,'(t)$ exist, and assume that $x\,'(t) \neq 0$. Then the derivative $\frac{dy}{dx}$ is given by

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}. \tag{3.2.3}$$

Proof

This theorem can be proven using the Chain Rule. In particular, assume that the parameter t can be eliminated, yielding a differentiable function $y = F(x)$. Then $y(t) = F(x(t))$. Differentiating both sides of this equation using the Chain Rule yields

$$y'(t) = F'\big(x(t)\big)x'(t),$$

so

$$F'\big(x(t)\big) = \frac{y'(t)}{x'(t)}.$$

But $F\,'\big(x(t)\big) = \frac{dy}{dx}$, which proves the theorem.

□

(Equation 3.2.3) can be used to calculate derivatives of plane curves, as well as critical points. Recall that a critical point of a differentiable function $y = f(x)$ is any point $x = x_0$ such that either $f\,'(x_0) = 0$ or $f\,'(x_0)$ does not exist. (Equation 3.2.3) gives a formula for the slope of a tangent line to a curve defined parametrically regardless of whether the curve can be described by a function $y = f(x)$ or not.

Example 3.2.1: Finding the Derivative of a Parametric Curv

Calculate the derivative $\frac{dy}{dx}$ for each of the given parametrically defined plane curves, and locate any critical points on their respective graphs.

a. $x(t) = t^2 - 3, \quad y(t) = 2t - 1, \quad \text{for } -3 \le t \le 4$
b. $x(t) = 2t + 1, \quad y(t) = t^3 - 3t + 4, \quad \text{for } -2 \le t \le 2$
c. $x(t) = 5\cos t, \quad y(t) = 5\sin t, \quad \text{for } 0 \le t \le 2\pi$

Solution

a. To apply (Equation 3.2.3), first calculate $x\,'(t)$ and $y\,'(t)$:

$$x\,'(t) = 2t$$

$$y\,'(t) = 2.$$

Next substitute these into the equation:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{dy}{dx} = \frac{2}{2t}$$

$$\frac{dy}{dx} = \frac{1}{t}.$$

This derivative is undefined when $t = 0$. Calculating $x(0)$ and $y(0)$ gives $x(0) = (0)^2 - 3 = -3$ and $y(0) = 2(0) - 1 = -1$, which corresponds to the point $(-3, -1)$ on the graph.

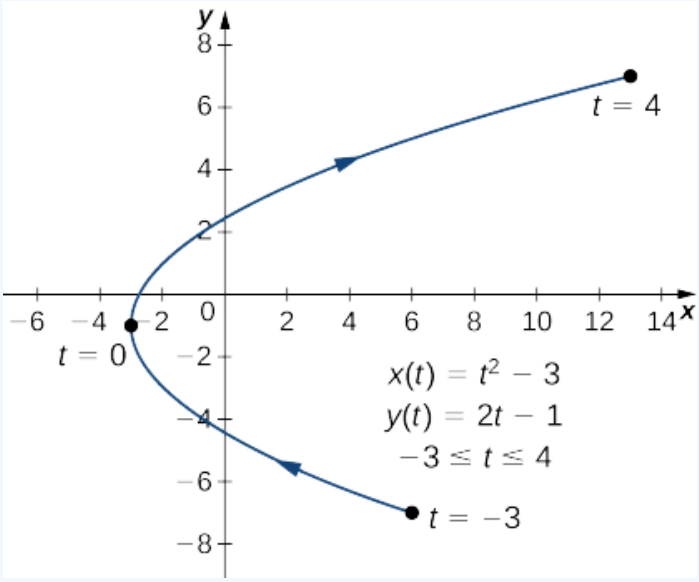


Figure 3.2.2: A curved line going from $(6, -7)$ through $(-3, -1)$ to $(13, 7)$ with arrow pointing in that order. The point $(6, -7)$ is marked $t = -3$, the point $(-3, -1)$ is marked $t = 0$, and the point $(13, 7)$ is marked $t = 4$. On the graph there are also written three equations: $x(t) = t^2 - 3$, $y(t) = 2t - 1$, and $-3 \le t \le 4$.

b. To apply (Equation 3.2.3), first calculate $x\,'(t)$ and $y\,'(t)$:

$$x\,'(t) = 2$$

$$y'(t) = 3t^2 - 3.$$

Next substitute these into the equation:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{dy}{dx} = \frac{3t^2 - 3}{2}.$$

This derivative is zero when $t = \pm 1$. When $t = -1$ we have

$$x(-1) = 2(-1) + 1 = -1 \text{ and } y(-1) = (-1)^3 - 3(-1) + 4 = -1 + 3 + 4 = 6,$$

which corresponds to the point $(-1, 6)$ on the graph. When $t = 1$ we have

$$x(1) = 2(1) + 1 = 3 \text{ and } y(1) = (1)^3 - 3(1) + 4 = 1 - 3 + 4 = 2,$$

which corresponds to the point $(3, 2)$ on the graph. The point $(3, 2)$ is a relative minimum and the point $(-1, 6)$ is a relative maximum, as seen in the Figure 3.2.3.

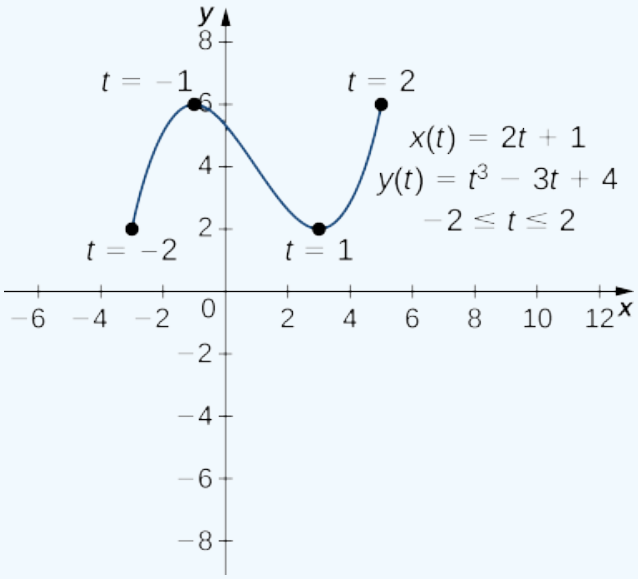


Figure 3.2.3: A vaguely sinusoidal curve going from $(-3, 2)$ through $(-1, 6)$ and $(3, 2)$ to $(5, 6)$. The point $(-3, 2)$ is marked $t = -2$, the point $(-1, 6)$ is marked $t = -1$, the point $(3, 2)$ is marked $t = 1$, and the point $(5, 6)$ is marked $t = 2$. On the graph there are also written three equations: $x(t) = 2t + 1$, $y(t) = t^3 - 3t + 4$, and $-2 \leq t \leq 2$.

c. To apply (Equation 3.2.3), first calculate $x'(t)$ and $y'(t)$:

$$x'(t) = -5 \sin t$$

$$y'(t) = 5 \cos t.$$

Next substitute these into the equation:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{dy}{dx} = \frac{5 \cos t}{-5 \sin t}$$

$$\frac{dy}{dx} = -\cot t.$$

This derivative is zero when $\cos t = 0$ and is undefined when $\sin t = 0$. This gives $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and 2π as critical points for *t*. Substituting each of these into $x(t)$ and $y(t)$, we obtain

Table 3.2.1*: Values generated to graph* $x(t) = 5cost$ *and* $y(t) = 5sint$.

t	$x(t)$	$y(t)$
0	5	0
$\frac{\pi}{2}$	0	5
π	–5	0
$\frac{3\pi}{2}$	0	–5
2π	5	0

These points correspond to the sides, top, and bottom of the circle that is represented by the parametric equations (Figure 3.2.4). On the left and right edges of the circle, the derivative is undefined, and on the top and bottom, the derivative equals zero.

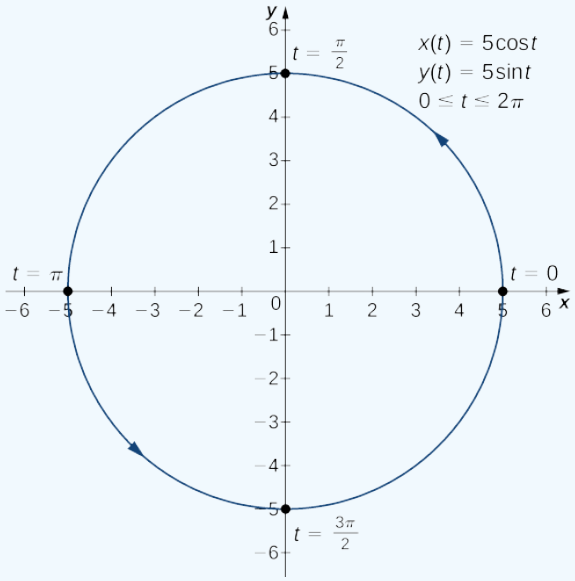


Figure 3.2.4: A circle with radius 5 centered at the origin is graphed with arrow going counterclockwise. The point $(5, 0)$ is marked $t = 0$, the point $(0, 5)$ is marked $t = \pi/2$, the point $(-5, 0)$ is marked $t = \pi$, and the point $(0, -5)$ is marked $t = 3\pi/2$. On the graph there are also written three equations: $x(t) = 5cos(t)$, $y(t) = 5sin(t)$, and $0 \leq t \leq 2\pi$.

Exercise 3.2.1

Calculate the derivative dy/dx for the plane curve defined by the equations

$$x(t) = t^2 - 4t, \quad y(t) = 2t^3 - 6t, \quad \text{for } -2 \leq t \leq 3$$

and locate any critical points on its graph.

Hint available at the end of this section.

Answer available at the end of this section.

Example 3.2.2: Finding a Tangent Line

Find the equation of the tangent line to the curve defined by the equations

$$x(t) = t^2 - 3, \quad y(t) = 2t - 1, \quad \text{for } -3 \leq t \leq 4 \tag{3.2.4}$$

when $t = 2$.

Solution

First find the slope of the tangent line using (Equation 3.2.3), which means calculating $x\,'(t)$ and $y\,'(t)$:

$$x\,'(t) = 2t$$

$$y\,'(t) = 2.$$

Next substitute these into the equation:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{dy}{dx} = \frac{2}{2t}$$

$$\frac{dy}{dx} = \frac{1}{t}.$$

When $t = 2$, $\frac{dy}{dx} = \frac{1}{2}$, so this is the slope of the tangent line. Calculating $x(2)$ and $y(2)$ gives

$$x(2) = (2)^2 - 3 = 1 \text{ and } y(2) = 2(2) - 1 = 3,$$

which corresponds to the point $(1, 3)$ on the graph (Figure 3.2.5). Now use the point-slope form of the equation of a line to find the equation of the tangent line:

$$y - y_0 = m(x - x_0)$$

$$y - 3 = \frac{1}{2}(x - 1)$$

$$y - 3 = \frac{1}{2}x - \frac{1}{2}$$

$$y = \frac{1}{2}x + \frac{5}{2}.$$

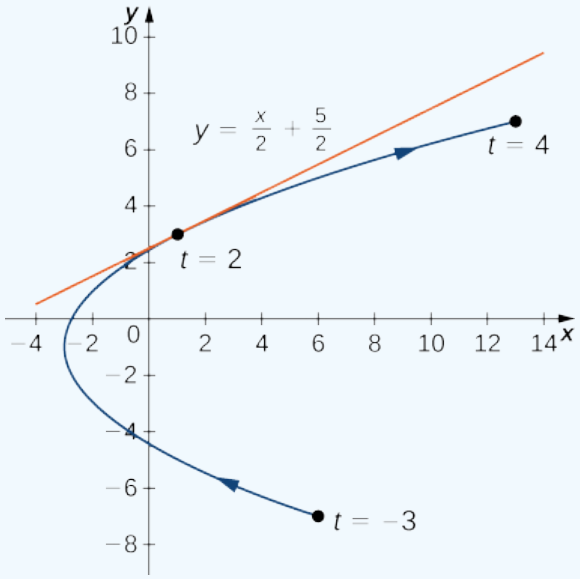


Figure 3.2.5: A curved line going from $(6, -7)$ through $(-3, -1)$ to $(13, 7)$ with arrow pointing in that order. The point $(6, -7)$ is marked$t = -3$, the point $(-3, -1)$ is marked $t = 0$, and the point $(13, 7)$ is marked $t = 4$. On the graph there are also written three equations: $x(t) = t2 - 3$, $y(t) = 2t - 1$, and $-3 \leq t \leq 4$. At the point $(1, 3)$, which is marked $t = 2$, there is a tangent line with equation $y = x/2 + 5/2$.

Example 3.2.3

Find the equation of the tangent lines where the curve crosses itself at $(3, 2)$ and the curve is defined by the equations

$$x(t) = t^2 - t + 1, \quad y(t) = t^3 - 3t \tag{3.2.5}$$

Watch the accompanying video for a visual demonstration of this concept.

- **Video Length:** 3 minutes 33 seconds.
- **Context:** This video demonstrates how to find the equation of a tangent line where a curve crosses itself. Given equations are $x = t^2 - t + 1$, $y = t^3 - 3t$

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Exercise 3.2.2

Find the equation of the tangent line to the curve defined by the equations

$$x(t) = t^2 - 4t, \quad y(t) = 2t^3 - 6t, \quad \text{for } -2 \leq t \leq 6 \text{ when } t = 5.$$

Hint available at the end of this section.

Answer available at the end of this section.

Second-Order Derivatives

Our next goal is to see how to take the second derivative of a function defined parametrically. The second derivative of a function $y = f(x)$ is defined to be the derivative of the first derivative; that is,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right]. \tag{3.2.6}$$

Since

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt},$$

we can replace the y on both sides of (Equation 3.2.6) with $\frac{dy}{dx}$. This gives us

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{(d/dt)(dy/dx)}{dx/dt}. \tag{3.2.7}$$

If we know dy/dx as a function of t, then this formula is straightforward to apply

Example 3.2.4: Finding a Second Derivative

Calculate the second derivative d^2y/dx^2 for the plane curve defined by the parametric equations $x(t) = t^2 - 3, \quad y(t) = 2t - 1,$ for $-3 \le t \le 4$.

Solution

From Example 3.2.1 we know that $\frac{dy}{dx} = \frac{2}{2t} = \frac{1}{t}$. Using (Equation 3.2.7), we obtain

$$\frac{d^2y}{dx^2} = \frac{(d/dt)(dy/dx)}{dx/dt} = \frac{(d/dt)(1/t)}{2t} = \frac{-t^{-2}}{2t} = -\frac{1}{2t^3}.$$

Exercise 3.2.3

Calculate the second derivative d^2y/dx^2 for the plane curve defined by the equations

$$x(t) = t^2 - 4t, \quad y(t) = 2t^3 - 6t, \quad \text{for } -2 \le t \le 3$$

and locate any critical points on its graph.

Hint available at the end of this section.

Answer available at the end of this section.

Integrals Involving Parametric Equations

Now that we have seen how to calculate the derivative of a plane curve, the next question is this: How do we find the area under a curve defined parametrically? Recall the cycloid defined by these parametric equations

$$x(t) = t - \sin t$$
$$y(t) = 1 - \cos t.$$

Suppose we want to find the area of the shaded region in the Figure 3.2.6.

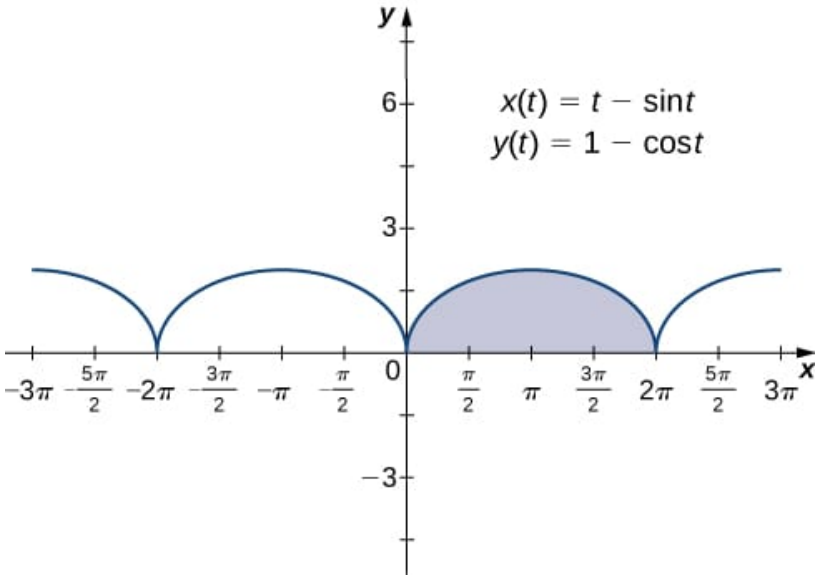


Figure 3.2.6: A series of half circles drawn above the x axis with x intercepts being multiples of 2π. The half circle between 0 and 2π is highlighted. On the graph there are also written two equations: $x(t) = t–sin(t)$ and $y(t) = 1–cos(t)$.

To derive a formula for the area under the curve defined by the functions

$$x = x(t)$$
$$y = y(t)$$

where $a \leq t \leq b$.

We assume that $x(t)$ is differentiable and start with an equal partition of the interval $a \leq t \leq b$. Suppose $t_0 = a < t_1 < t_2 < \cdots < t_n = b$ and="" consider="" the="" figure="" $3.2.\eatspaces7 =″″\endspaces.<="" p=""> </t_1<t_2<⋯<t_n=b\)>

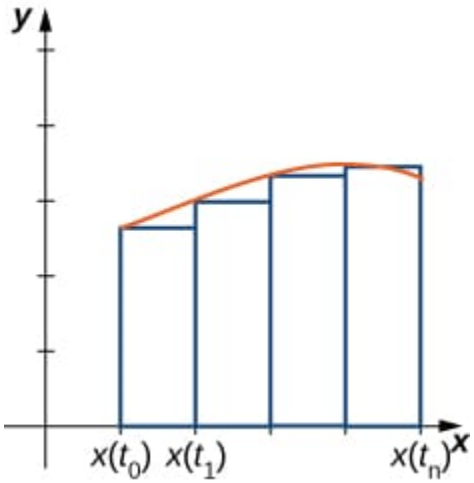


Figure 3.2.7: A curved line is drawn in the first quadrant. Below it are a series of rectangles marked that begin at the x axis and reach up to the curved line; the rectangle's height is determined by the location of the curved line at the leftmost point of the rectangle. These lines are noted as $x(t_0)$, $x(t_1)$, ..., $x(t_n)$.

We use rectangles to approximate the area under the curve. The height of a typical rectangle in this parametrization is $y(x(\bar{t_i}))$ for some value $\bar{t_i}$ in the ith subinterval, and the width can be calculated as $x(t_i) - x(t_{i-1})$. Thus the area of the ith rectangle is given by

$$A_i = y(x(\bar{t_i}))(x(t_i) - x(t_{i-1})).$$

Then a Riemann sum for the area is

$$A_n = \sum_{i=1}^{n} y(x(\bar{t}_i))(x(t_i) - x(t_{i-1})).$$

Multiplying and dividing each area by $t_i - t_{i-1}$ gives

$$\begin{aligned} A_n &= \sum_{i=1}^{n} y(x(\bar{t}_i)) \left(\frac{x(t_i) - x(t_{i-1})}{t_i - t_{i-1}} \right) (t_i - t_{i-1}) \\ &= \sum_{i=1}^{n} y(x(\bar{t}_i)) \left(\frac{x(t_i) - x(t_{i-1})}{\Delta t} \right) \Delta t. \end{aligned}$$

Taking the limit as n approaches infinity gives

$$A = \lim_{n \to \infty} A_n = \int_a^b y(t)x'(t)\, dt.$$

This leads to the next theorem.

Area under a Parametric Curve

Consider the non-self-intersecting plane curve defined by the parametric equations

$$x = x(t), \quad y = y(t), \quad \text{for } a \le t \le b$$

and assume that $x(t)$ is differentiable. The area under this curve is given by

$$A = \int_a^b y(t)x'(t)\, dt. \tag{3.2.8}$$

Example 3.2.5: Finding the Area under a Parametric Curve

Find the area under the curve of the cycloid defined by the equations

$$x(t) = t - \sin t, \quad y(t) = 1 - \cos t, \quad \text{for } 0 \le t \le 2\pi. \tag{3.2.9}$$

Solution

Using (Equation 3.2.8), we have

$$\begin{aligned} A &= \int_a^b y(t)x'(t)\, dt \\ &= \int_0^{2\pi} (1 - \cos t)(1 - \cos t)\, dt \\ &= \int_0^{2\pi} (1 - 2\cos t + \cos^2 t)\, dt \\ &= \int_0^{2\pi} \left(1 - 2\cos t + \frac{1 + \cos(2t)}{2}\right) dt \\ &= \int_0^{2\pi} \left(\frac{3}{2} - 2\cos t + \frac{\cos(2t)}{2}\right) dt \\ &= \frac{3t}{2} - 2\sin t + \frac{\sin(2t)}{4} \Big|_0^{2\pi} \\ &= 3\pi \end{aligned}$$

Exercise 3.2.4

Find the area under the curve of the hypocycloid defined by the equations

$$x(t) = 3\cos t + \cos(3t), \quad y(t) = 3\sin t - \sin(3t), \quad \text{for } 0 \le t \le \pi.$$

Hint available at the end of this section.

Answer available at the end of this section.

Arc Length of a Parametric Curve

In addition to finding the area under a parametric curve, we sometimes need to find the arc length of a parametric curve. In the case of a line segment, arc length is the same as the distance between the endpoints. If a particle travels from point A to point B along a curve, then the distance that particle travels is the arc length. To develop a formula for arc length, we start with an approximation by line segments.

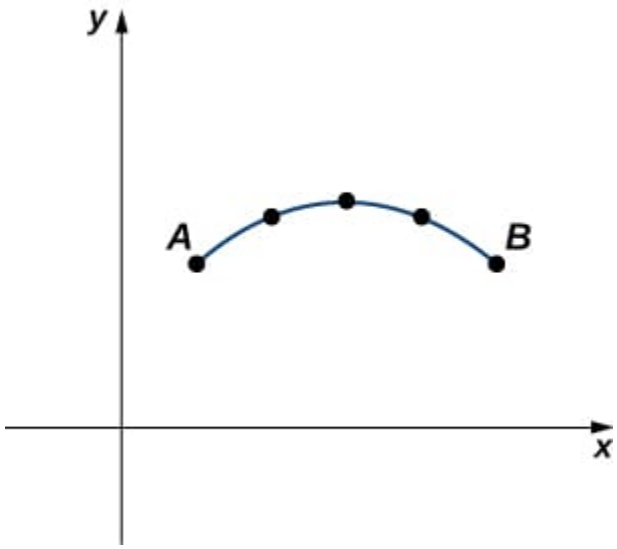


Figure 3.2.8: A curved line in the first quadrant with points marked for x = 1, 2, 3, 4, and 5. These points have values roughly 2.1, 2.7, 3, 2.7, and 2.1, respectively. The points for x = 1 and 5 are marked A and B, respectively.

Given a plane curve defined by the functions $x = x(t), \quad y = y(t), \quad \text{for } a \le t \le b$, we start by partitioning the interval $[a, b]$ into n equal subintervals: $t_0 = a < t_1 < t_2 < \cdots < t_n = b$. the="" width="" of="" each="" subinterval="" is="" given="" by="" $\Delta t = \prime(b - a)/n$." we="" can="" calculate="" length="" line="" segment:<="" p=""> </t_1<t_2<⋯<t_n=b\).>

$$d_1 = \sqrt{(x(t_1) - x(t_0))^2 + (y(t_1) - y(t_0))^2}$$

$$d_2 = \sqrt{(x(t_2) - x(t_1))^2 + (y(t_2) - y(t_1))^2}$$

etc.

Then add these up. We let s denote the exact arc length and s_n denote the approximation by n line segments:

$$s \approx \sum_{k=1}^{n} s_k = \sum_{k=1}^{n} \sqrt{(x(t_k) - x(t_{k-1}))^2 + (y(t_k) - y(t_{k-1}))^2}. \tag{3.2.10}$$

If we assume that $x(t)$ and $y(t)$ are differentiable functions of t, then the Mean Value Theorem applies, so in each subinterval $[t_{k-1}, t_k]$ there exist $\hat{t_k}$ and $\tilde{t_k}$ such that

$$x(t_k) - x(t_{k-1}) = x'(\hat{t_k})(t_k - t_{k-1}) = x'(\hat{t_k})\,\Delta t$$

$$y(t_k) - y(t_{k-1}) = y'(\tilde{t_k})(t_k - t_{k-1}) = y'(\tilde{t_k})\,\Delta t.$$

Therefore (Equation 3.2.10) becomes

$$\begin{aligned} s \approx \sum_{k=1}^{n} s_k &= \sum_{k=1}^{n} \sqrt{(x'(\hat{t_k})\Delta t)^2 + (y'(\tilde{t_k})\Delta t)^2} \\ &= \sum_{k=1}^{n} \sqrt{(x'(\hat{t_k}))^2(\Delta t)^2 + (y'(\tilde{t_k}))^2(\Delta t)^2} \\ &= \sum_{k=1}^{n} \sqrt{(x'(\hat{t_k}))^2 + (y'(\tilde{t_k}))^2}\Delta t. \end{aligned}$$

This is a Riemann sum that approximates the arc length over a partition of the interval $[a, b]$. If we further assume that the derivatives are continuous and let the number of points in the partition increase without bound, the approximation approaches the exact arc length. This gives

$$\begin{aligned} s &= \lim_{n\to\infty} \sum_{k=1}^{n} s_k \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{(x'(\hat{t_k}))^2 + (y'(\tilde{t_k}))^2}\Delta t \\ &= \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt. \end{aligned}$$

When taking the limit, the values of $\hat{t_k}$ and $\tilde{t_k}$ are both contained within the same ever-shrinking interval of width Δt, so they must converge to the same value.

We can summarize this method in the next theorem.

Arc Length of a Parametric Curve

Consider the plane curve defined by the parametric equations

$$x = x(t), \quad y = y(t), \quad \text{for } t_1 \le t \le t_2$$

and assume that $x(t)$ and $y(t)$ are differentiable functions of t. Then the arc length of this curve is given by

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt. \tag{3.2.11}$$

At this point a side derivation leads to a previous formula for arc length. In particular, suppose the parameter can be eliminated, leading to a function $y = F(x)$. Then $y(t) = F(x(t))$ and the Chain Rule gives

$$y'(t) = F'\big(x(t)\big)x'(t).$$

Substituting this into (Equation 3.2.11) gives

$$\begin{aligned} s &= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(F'(x)\frac{dx}{dt}\right)^2}\, dt \\ &= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 \left(1 + (F'(x))^2\right)}\, dt \\ &= \int_{t_1}^{t_2} x'(t)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dt. \end{aligned}$$

Here we have assumed that $x'(t) > 0$, which is a reasonable assumption. The Chain Rule gives $dx = x'(t)\ dt$, and letting $a = x(t_1)$ and $b = x(t_2)$ we obtain the formula

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx,$$

which is the formula for arc length obtained in the Introduction to the Applications of Integration.

Example 3.2.6: Finding the Arc Length of a Parametric Curve

Find the arc length of the semicircle defined by the equations

$$x(t) = 3\cos t, \quad y(t) = 3\sin t, \quad \text{for } 0 \le t \le \pi.$$

Solution

The values $t = 0$ to $t = \pi$ trace out the blue curve in Figure 3.2.8. To determine its length, use (Equation 3.2.11):

$$\begin{aligned} s &= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_0^{\pi} \sqrt{(-3\sin t)^2 + (3\cos t)^2}\, dt \\ &= \int_0^{\pi} \sqrt{9\sin^2 t + 9\cos^2 t}\, dt \\ &= \int_0^{\pi} \sqrt{9(\sin^2 t + \cos^2 t)}\, dt \\ &= \int_0^{\pi} 3\, dt = 3t\Big|_0^{\pi} \\ &= 3\pi \text{ units.} \end{aligned}$$

Note that the formula for the arc length of a semicircle is πr and the radius of this circle is 3. This is a great example of using calculus to derive a known formula of a geometric quantity.

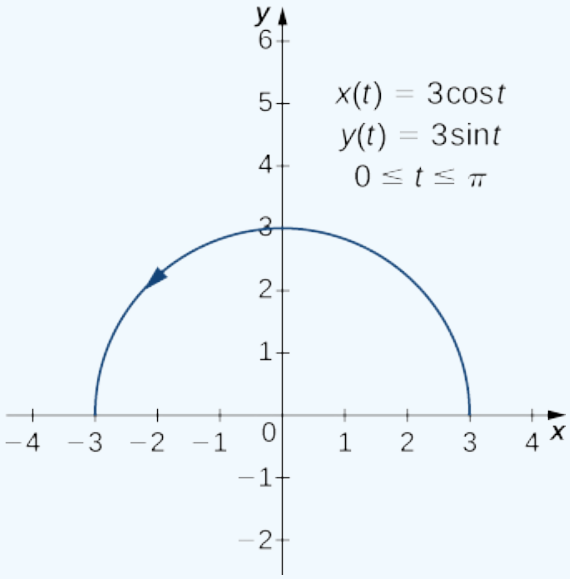


Figure 3.2.9: A semicircle is drawn with radius 3. There is an arrow pointing counterclockwise. On the graph there are also written three equations: $x(t) = 3cos(t)$, $y(t) = 3sin(t)$, and $0 \le t \le \pi$.

Watch the accompanying video for a visual demonstration of this concept.

- **Video Length:** 1 minute 45 seconds.
- **Context:** This video demonstrates how to find the arc length of a parametrically defined curve $x = t^2$, $y = e^{2t}$

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Exercise 3.2.5

Find the arc length of the curve defined by the equations

$$x(t) = 3t^2, \quad y(t) = 2t^3, \quad \text{for } 1 \leq t \leq 3.$$

Hint available at the end of this section.

Answer available at the end of this section.

We now return to the problem posed at the beginning of the section about a baseball leaving a pitcher's hand. Ignoring the effect of air resistance (unless it is a curve ball!), the ball travels a parabolic path. Assuming the pitcher's hand is at the origin and the ball travels left to right in the direction of the positive x-axis, the parametric equations for this curve can be written as

$$x(t) = 140t, \quad y(t) = -16t^2 + 2t$$

where t represents time. We first calculate the distance the ball travels as a function of time. This distance is represented by the arc length. We can modify the arc length formula slightly. First rewrite the functions $x(t)$ and $y(t)$ using *v* as an independent variable, so as to eliminate any confusion with the parameter *t*:

$$x(v) = 140v, \quad y(v) = -16v^2 + 2v.$$

Then we write the arc length formula as follows:

$$\begin{aligned} s(t) &= \int_0^t \sqrt{(\frac{dx}{dv})^2 + (\frac{dy}{dv})^2}\, dv \\ &= \int_0^t \sqrt{140^2 + (-32v + 2)^2}\, dv \end{aligned}$$

The variable v acts as a dummy variable that disappears after integration, leaving the arc length as a function of time t. To integrate this expression we can use a formula from Appendix A,

$$\int \sqrt{a^2 + u^2}\, du = \frac{u}{2}\sqrt{a^2 + u^2} + \frac{a^2}{2}\ln \mid u + \sqrt{a^2 + u^2} \mid + C.$$

We set $a = 140$ and $u = -32v + 2$. This gives $du = -32\ dv$, so $dv = -\frac{1}{32}\ du$. Therefore

$$\begin{aligned} \int \sqrt{140^2 + (-32v + 2)^2}\, dv &= -\frac{1}{32}\int \sqrt{a^2 + u^2}\, du \\ &= -\frac{1}{32}\left[\frac{(-32v + 2)}{2}\sqrt{140^2 + (-32v + 2)^2} + \frac{140^2}{2}\ln \mid (-32v + 2) + \sqrt{140^2 + (-32v + 2)^2} \mid + C\right] \end{aligned}$$

and

$$s(t) = -\frac{1}{32}\left[\frac{(-32t+2)}{2}\sqrt{140^2+(-32t+2)^2} + \frac{140^2}{2}\ln\left|(-32t+2)+\sqrt{140^2+(-32t+2)^2}\right|\right] + \frac{1}{32}\left[\sqrt{140^2+2^2} + \frac{140^2}{2}\ln\left|2+\sqrt{140^2+2^2}\right|\right]$$
$$= \left(\frac{t}{2}-\frac{1}{32}\right)\sqrt{1024t^2-128t+19604} - \frac{1225}{4}\ln\left|(-32t+2)+\sqrt{1024t^2-128t+19604}\right| + \frac{\sqrt{19604}}{32} + \frac{1225}{4}\ln(2+\sqrt{19604})$$

This function represents the distance traveled by the ball as a function of time. To calculate the speed, take the derivative of this function with respect to t. While this may seem like a daunting task, it is possible to obtain the answer directly from the Fundamental Theorem of Calculus:

$$\frac{d}{dx}\int_a^x f(u)\,du = f(x).$$

Therefore

$$\begin{aligned} s'(t) &= \frac{d}{dt}\Big[s(t)\Big] \\ &= \frac{d}{dt}\left[\int_0^t \sqrt{140^2+(-32v+2)^2}\,dv\right] \\ &= \sqrt{140^2+(-32t+2)^2} \\ &= \sqrt{1024t^2-128t+19604} \\ &= 2\sqrt{256t^2-32t+4901}. \end{aligned}$$

One third of a second after the ball leaves the pitcher's hand, the distance it travels is equal to

$$\begin{aligned} s\left(\frac{1}{3}\right) &= \left(\frac{1/3}{2}-\frac{1}{32}\right)\sqrt{1024\left(\frac{1}{3}\right)^2 - 128\left(\frac{1}{3}\right)+19604} \\ &\quad - \frac{1225}{4}\ln\left|\left(-32\left(\frac{1}{3}\right)+2\right)+\sqrt{1024\left(\frac{1}{3}\right)^2-128\left(\frac{1}{3}\right)+19604}\right| \\ &\quad + \frac{\sqrt{19604}}{32} + \frac{1225}{4}\ln(2+\sqrt{19604}) \\ &\approx 46.69 \text{ feet}. \end{aligned}$$

This value is just over three quarters of the way to home plate. The speed of the ball is

$$s'\left(\tfrac{1}{3}\right) = 2\sqrt{256\left(\tfrac{1}{3}\right)^2 - 32\left(\tfrac{1}{3}\right)+4901} \approx 140.27 \text{ ft/s}.$$

This speed translates to approximately 95 mph—a major-league fastball.

Surface Area Generated by a Parametric Curve

Recall the problem of finding the surface area of a volume of revolution. In Curve Length and Surface Area, we derived a formula for finding the surface area of a volume generated by a function $y = f(x)$ from $x = a$ to $x = b$, revolved around the x-axis:

$$S = 2\pi\int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx. \tag{3.2.12}$$

We now consider a volume of revolution generated by revolving a parametrically defined curve $x = x(t), \quad y = y(t), \quad \text{for } a \le t \le b$ around the x-axis as represented in Figure 3.2.9.

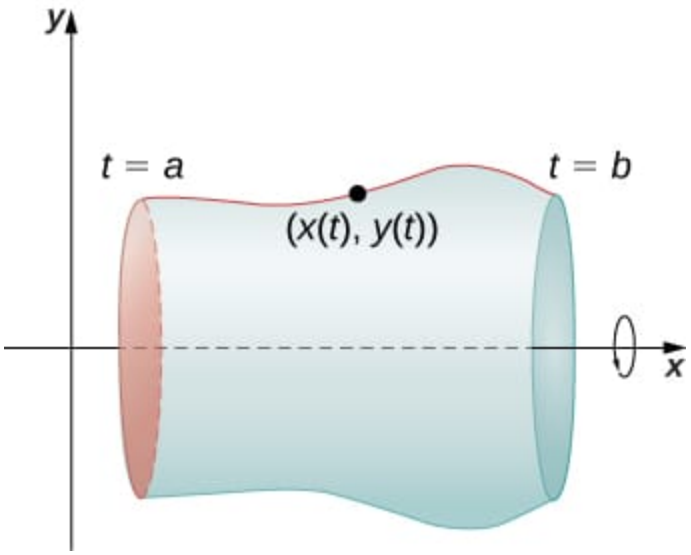


Figure 3.2.10: A curve is drawn in the first quadrant with endpoints marked $t = a$ and $t = b$. On this curve, there is a point marked $(x(t), y(t))$. There is a circle with an arrow drawn around the x axis that seems to indicate a rotation about the x axis, and there is a shape that accompanies that curve that seems to be what you would obtain if you rotated the curve about the $x - axis$.

The analogous formula for a parametrically defined curve is

$$S = 2\pi \int_a^b y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\, dt \tag{3.2.13}$$

provided that $y(t)$ is not negative on $[a, b]$.

Example 3.2.7: Finding Surface Area

Find the surface area of a sphere of radius r centered at the origin.

Solution

We start with the curve defined by the equations

$$x(t) = r\cos t, \quad y(t) = r\sin t, \quad \text{for } 0 \le t \le \pi.$$

This generates an upper semicircle of radius r centered at the origin is graphed in the Figure 3.2.11.

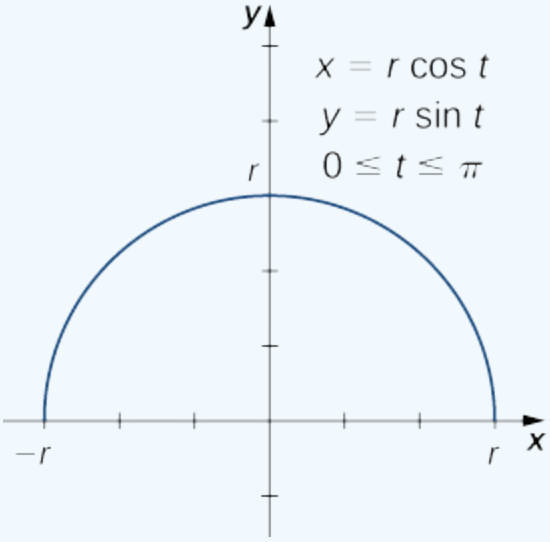


Figure 3.2.11: A semicircle is drawn with radius r. On the graph there are also written three equations: $x(t) = rcos(t)$, $y(t) = rsin(t)$, and $0 \leq t \leq \pi$.

When this curve is revolved around the x-axis, it generates a sphere of radius r. To calculate the surface area of the sphere, we use (Equation 3.2.13):

$$\begin{aligned} S &= 2\pi \int_a^b y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\, dt \\ &= 2\pi \int_0^\pi r\sin t \sqrt{(-r\sin t)^2 + (r\cos t)^2}\, dt \\ &= 2\pi \int_0^\pi r\sin t \sqrt{r^2\sin^2 t + r^2\cos^2 t}\, dt \\ &= 2\pi \int_0^\pi r\sin t \sqrt{r^2(\sin^2 t + \cos^2 t)}\, dt \\ &= 2\pi \int_0^\pi r^2 \sin t\, dt \\ &= 2\pi r^2 \left(-\cos t\Big|_0^\pi\right) \\ &= 2\pi r^2(-\cos\pi + \cos 0) \\ &= 4\pi r^2 \text{ units}^2. \end{aligned}$$

This is, in fact, the formula for the surface area of a sphere.

Watch the accompanying video for a visual demonstration of this concept.

- **Video Length:** 3 minutes 48 seconds.
- **Context:** This video demonstrates how to find the surface are of revolution for a parametrically defined curve $x = t^2 + 1$, $y = t^3 + 1$, $0 \leq t \leq 1$.

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Exercise 3.2.6

Find the surface area generated when the plane curve defined by the equations

$$x(t) = t^3, \quad y(t) = t^2, \quad \text{for } 0 \le t \le 1$$

is revolved around the x-axis.

Hint available at the end of this section.

Answer available at the end of this section.

Key Concepts

- The derivative of the parametrically defined curve $x = x(t)$ and $y = y(t)$ can be calculated using the formula $\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$. Using the derivative, we can find the equation of a tangent line to a parametric curve.
- The area between a parametric curve and the x-axis can be determined by using the formula $A = \int_{t_1}^{t_2} y(t)x\,'(t)\ dt$.
- The arc length of a parametric curve can be calculated by using the formula

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

- The surface area of a volume of revolution revolved around the x-axis is given by

$$S = 2\pi \int_a^b y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\, dt.$$

- If the curve is revolved around the y-axis, then the formula is

$$S = 2\pi \int_a^b x(t)\sqrt{(x'(t))^2 + (y'(t))^2}\, dt.$$

Key Equations

- **Derivative of parametric equations**

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}$$

- **Second-order derivative of parametric equations**

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{(d/dt)(dy/dx)}{dx/dt}$$

- **Area under a parametric curve**

$$A = \int_a^b y(t)x'(t)\, dt$$

- **Arc length of a parametric curve**

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

- **Surface area generated by a parametric curve**

$$S = 2\pi \int_a^b y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

Glossary

Parametric Equations

The equations $x = x(t)$ and $y = y(t)$ that define a parametric curve.

Tangent Line

A line that touches a curve at a single point without crossing it. The slope of the tangent line to a parametric curve can be found using the derivatives of the parametric equations.

Arc Length

The distance along a curve between two points. For parametric curves, the arc length can be calculated using an integral involving the derivatives of the parametric equations.

Curvature

A measure of how quickly a curve changes direction at a given point. For parametric curves, curvature can be calculated using the derivatives of the parametric equations.

Velocity and Acceleration

The graph of the parametric equations $x(t)$ and $y(t)$ over an interval $a \le t \le b$ combined with the equations.

Answers

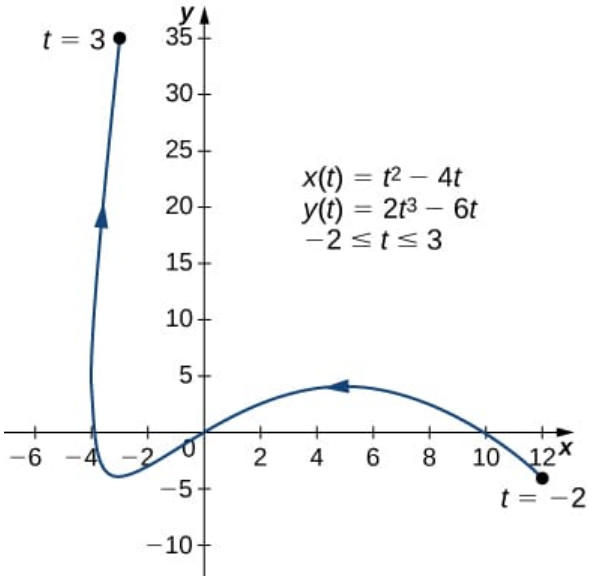
Exercise 3.2.1

Calculate $x'(t)$ and $y'(t)$ and use (Equation 3.2.3).

Exercise 3.2.1

$x'(t) = 2t - 4$ and $y'(t) = 6t^2 - 6$, so $\frac{dy}{dx} = \frac{6t^2 - 6}{2t - 4} = \frac{3t^2 - 3}{t - 2}$.

This expression is undefined when $t = 2$ and equal to zero when $t = \pm 1$.



A curve going from $(12, -4)$ through the origin and $(-4, 0)$ to $(-3, 36)$ with arrows in that order. The point $(12, -4)$ is marked $t = -2$ and the point $(-3, 36)$ is marked $t = 3$. On the graph there are also written three equations: $x(t) = t2–4t$, $y(t) = 2t^3–6t$, and $-2 \le t \le 3$.

Exercise 3.2.2

Calculate $x'(t)$ and $y'(t)$ and use (Equation 3.2.3).

Exercise 3.2.2

The equation of the tangent line is $y = 24x + 100$.

Exercise 3.2.3

Start with the solution from the previous exercise, and use (Equation 3.2.7).

Exercise 3.2.3

$\frac{d^2y}{dx^2} = \frac{3t^2 - 12t + 3}{2(t-2)^3}$. Critical points $(5, 4),\ (-3, -4)$, and $(-4, 4)$.

Exercise 3.2.4

Use (Equation 3.2.8), along with the identities $\sin\alpha \sin\beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$ and $\sin^2 t = \frac{1 - \cos(2t)}{2}$.

Exercise 3.2.4

$A = 3\pi$ (Note that the integral formula actually yields a negative answer. This is due to the fact that $x(t)$ is a decreasing function over the interval $[0, \pi]$; that is, the curve is traced from right to left.)

Exercise 3.2.5

Use (Equation 3.2.11).

Exercise 3.2.5

$s = 2(10^{3/2} - 2^{3/2}) \approx 57.589$ units

Exercise 3.2.6

Use (Equation 3.2.13). When evaluating the integral, use a u-substitution.

Exercise 3.2.6

$A = \frac{\pi(494\sqrt{13} + 128)}{1215}$ units^2

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3.2E: Exercises for Section 3.2

Exercises 1 - 4

Each set of parametric equations represents a line. Without eliminating the parameter, find the slope of each line.

1. $x = 3 + t, \quad y = 1 - t$
2. $x = 8 + 2t, \quad y = 1$

Answer

$m = 0$

3. $x = 4 - 3t, \quad y = -2 + 6t$
4. $x = -5t + 7, \quad y = 3t - 1$

Answer

$m = -\frac{3}{5}$

Exercises 5 - 9

Determine the slope of the tangent line, then find the equation of the tangent line at the given value of the parameter.

5. $x = 3\sin t, \quad y = 3\cos t, \quad \text{for } t = \frac{\pi}{4}$
6. $x = \cos t, \quad y = 8\sin t, \quad \text{for } t = \frac{\pi}{2}$

Answer

Slope $= 0; y = 8.$

7. $x = 2t, \quad y = t^3, \quad \text{for } t = -1$
8. $x = t + \dfrac{1}{t}, \quad y = t - \dfrac{1}{t}, \quad \text{for } t = 1$

Answer

Slope is undefined; $x = 2.$

9. $x = \sqrt{t}, \quad y = 2t, \quad \text{for } t = 4$

Exercises 10 - 13

Find all points on the curve that have the given slope.

10. $x = 4\cos t, \quad y = 4\sin t$, slope = 0.5

Solution

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{4\cos t}{-4\sin t} = -\cot t.$

Setting this derivative equal to 0.5, we obtain the equation, $\tan t = -2.$

$\tan t = -2 \implies \dfrac{y}{x} = -2 \implies y = -2x.$

Note also that this pair of parametric equations represents the circle $x^2 + y^2 = 16.$

By substitution, we find that this curve has a slope of 0.5 at the points:

$\left(\frac{4\sqrt{5}}{5}, \frac{-8\sqrt{5}}{5}\right)$ and $\left(\frac{-4\sqrt{5}}{5}, \frac{8\sqrt{5}}{5}\right)$.

11. $x = 2\cos t, \quad y = 8\sin t$, slope= -1

12. $x = t + \dfrac{1}{t}, \quad y = t - \dfrac{1}{t}$, slope= 1

Answer

No points possible; undefined expression.

13. $x = 2 + \sqrt{t}, \quad y = 2 - 4t$, slope= 0

Exercises 14 - 16

Write the equation of the tangent line in Cartesian coordinates for the given parameter t.

14. $x = e^{\sqrt{t}}, \quad y = 1 - \ln t^2, \quad$ for $t = 1$

Answer

$y = -\left(\frac{4}{e}\right)x + 5$

15. $x = t \ln t, \quad y = \sin^2 t, \quad$ for $t = \frac{\pi}{4}$
16. $x = e^t, \quad y = (t-1)^2$, at $(1, 1)$

Answer

$y = -2x + 3$

17. For $x = \sin(2t), \quad y = 2\sin t$ where $0 \le t < 2\pi$. Find all values of t at which a horizontal tangent line exists.
18. For $x = \sin(2t), \quad y = 2\sin t$ where $0 \le t < 2\pi$. Find all values of t at which a vertical tangent line exists.

Answer

A vertical tangent line exists at $t = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$

19. Find all points on the curve $x = 4\cos(t), \quad y = 4\sin(t)$ that have the slope of $\frac{1}{2}$.
20. Find $\dfrac{dy}{dx}$ for $x = \sin(t), \quad y = \cos(t)$.

Answer

$$\frac{dy}{dx} = -\tan(t)$$

21. Find the equation of the tangent line to $x = \sin(t), \quad y = \cos(t)$ at $t = \frac{\pi}{4}$.
22. For the curve $x = 4t, \quad y = 3t - 2$, find the slope and concavity of the curve at $t = 3$.

Answer

$\dfrac{dy}{dx} = \dfrac{3}{4}$ and $\dfrac{d^2y}{dx^2} = 0$, so the curve is neither concave up nor concave down at $t = 3$. Therefore the graph is linear and has a constant slope but no concavity.

23. For the parametric curve whose equation is $x = 4\cos\theta, \quad y = 4\sin\theta$, find the slope and concavity of the curve at $\theta = \frac{\pi}{4}$.
24. Find the slope and concavity for the curve whose equation is $x = 2 + \sec\theta, \quad y = 1 + 2\tan\theta$ at $\theta = \frac{\pi}{6}$.

Answer

$\dfrac{dy}{dx} = 4, \quad \dfrac{d^2y}{dx^2} = -4\sqrt{3}$; the curve is concave down at $\theta = \frac{\pi}{6}$.

25. Find all points on the curve $x = t + 4, \quad y = t^3 - 3t$ at which there are vertical and horizontal tangents.
26. Find all points on the curve $x = \sec\theta, \quad y = \tan\theta$ at which horizontal and vertical tangents exist.

Answer

No horizontal tangents. Vertical tangents at $(1, 0)$ and $(-1, 0)$.

Exercises 27 - 29

Find d^2y/dx^2.

27. $x = t^4 - 1, \quad y = t - t^2$
28. $x = \sin(\pi t), \quad y = \cos(\pi t)$

Answer

$$d^2y/dx^2 = -\sec^3(\pi t)$$

29. $x = e^{-t}, \quad y = te^{2t}$

Exercises 30 - 31

Find points on the curve at which tangent line is horizontal or vertical.

30. $x = t(t^2 - 3), \quad y = 3(t^2 - 3)$

Answer

Horizontal $(0, -9)$;
Vertical $(\pm 2, -6)$.

31. $x = \dfrac{3t}{1 + t^3}, \quad y = \dfrac{3t^2}{1 + t^3}$

Exercises 32 - 34

Find dy/dx at the value of the parameter.

32. $x = \cos t, y = \sin t, \quad \text{for } t = \frac{3\pi}{4}$

Answer

$$dy/dx = 1$$

33. $x = \sqrt{t}, \quad y = 2t + 4, t = 9$
34. $x = 4\cos(2\pi s), \quad y = 3\sin(2\pi s), \quad \text{for } s = -\frac{1}{4}$

Answer

$$dy/dx = 0$$

Exercises 35 - 36

Find d^2y/dx^2 at the given point without eliminating the parameter.

35. $x = \frac{1}{2}t^2, \quad y = \frac{1}{3}t^3, \quad \text{for } t = 2$
36. $x = \sqrt{t}, \quad y = 2t + 4, \quad \text{for } t = 1$

Answer

$$d^2y/dx^2 = 4$$

37. Find intervals for t on which the curve $x = 3t^2, \quad y = t^3 - t$ is concave up as well as concave down.
38. Determine the concavity of the curve $x = 2t + \ln t, \quad y = 2t - \ln t$.

Answer

Concave up on $t > 0$.

39. Sketch and find the area under one arch of the cycloid $x = r(\theta - \sin\theta), \quad y = r(1 - \cos\theta)$.

40. Find the area bounded by the curve $x = \cos t, \quad y = e^t, \quad \text{for } 0 \le t \le \frac{\pi}{2}$ and the lines $y = 1$ and $x = 0$.

Answer

1 unit^2

41. Find the area enclosed by the ellipse $x = a\cos\theta, \quad y = b\sin\theta, \quad \text{for } 0 \le \theta < 2\pi.$
42. Find the area of the region bounded by $x = 2\sin^2\theta, \quad y = 2\sin^2\theta\tan\theta$, for $0 \le \theta \le \frac{\pi}{2}$.

Answer

$\frac{3\pi}{2} \text{ units}^2$

Exercises 43 - 46

Find the area of the regions bounded by the parametric curves and the indicated values of the parameter.

43. $x = 2\cot\theta, \quad y = 2\sin^2\theta, \quad \text{for } 0 \le \theta \le \pi$
44. [T] $x = 2a\cos t - a\cos(2t), \quad y = 2a\sin t - a\sin(2t), \quad \text{for } 0 \le t < 2\pi$

Answer

$6\pi a^2 \text{ units}^2$

45. [T] $x = a\sin(2t), \quad y = b\sin(t), \quad \text{for } 0 \le t < 2\pi$ (the “hourglass”)
46. [T] $x = 2a\cos t - a\sin(2t), \quad y = b\sin t, \quad \text{for } 0 \le t < 2\pi$ (the “teardrop”)

Answer

$2\pi ab \text{ units}^2$

Exercises 47 - 57

Find the arc length of the curve on the indicated interval of the parameter.

47. $x = 4t + 3, \quad y = 3t - 2, \quad \text{for } 0 \le t \le 2$
48. $x = \frac{1}{3}t^3, \quad y = \frac{1}{2}t^2, \quad \text{for } 0 \le t \le 1$

Answer

$s = \frac{1}{3}(2\sqrt{2} - 1) \text{ units}$

49. $x = \cos(2t), \quad y = \sin(2t), \quad \text{for } 0 \le t \le \frac{\pi}{2}$
50. $x = 1 + t^2, \quad y = (1 + t)^3, \quad \text{for } 0 \le t \le 1$

Answer

$s = 7.075 \text{ units}$

51. $x = e^t\cos t, \quad y = e^t\sin t, \quad \text{for } 0 \le t \le \frac{\pi}{2}$ (express answer as a decimal rounded to three places)
52. $x = a\cos^3\theta, \quad y = a\sin^3\theta$ on the interval $[0, 2\pi)$ (the hypocycloid)

Answer

$s = 6a \text{ units}$

53. Find the length of one arch of the cycloid $x = 4(t - \sin t), \quad y = 4(1 - \cos t).$
54. Find the distance traveled by a particle with position (x, y) as t varies in the given time interval: $x = \sin^2 t, \quad y = \cos^2 t, \quad \text{for } 0 \le t \le 3\pi.$

Answer

$6\sqrt{2} \text{ units}$

55. Find the length of one arch of the cycloid $x = \theta - \sin\theta, \quad y = 1 - \cos\theta.$

56. Show that the total length of the ellipse $x = 4\sin\theta, \quad y = 3\cos\theta$ is $L = 16\int_0^{\pi/2} \sqrt{1 - e^2\sin^2\theta}\ d\theta$, where $e = \frac{c}{a}$ and $c = \sqrt{a^2 - b^2}$.

57. Find the length of the curve $x = e^t - t, \quad y = 4e^{t/2}, \quad \text{for } -8 \le t \le 3.$

Exercises 58 - 59

Find the area of the surface obtained by rotating the given curve about the x-axis.

58. $x = t^3, \quad y = t^2, \quad \text{for } 0 \le t \le 1$

Answer

$$\frac{2\pi(247\sqrt{13} + 64)}{1215} \text{ units}^2$$

59. $x = a\cos^3\theta, \quad y = a\sin^3\theta, \quad \text{for } 0 \le \theta \le \frac{\pi}{2}$

60. [T] Use a CAS to find the area of the surface generated by rotating $x = t + t^3, \quad y = t - \frac{1}{t^2}, \quad \text{for } 1 \le t \le 2$ about the x-axis. (Answer to three decimal places.)

Answer

$$59.101 \text{ units}^2$$

61. Find the surface area obtained by rotating $x = 3t^2, \quad y = 2t^3, \quad \text{for } 0 \le t \le 5$ about the y-axis.

62. Find the area of the surface generated by revolving $x = t^2, \quad y = 2t, \quad \text{for } 0 \le t \le 4$ about the x-axis.

Answer

$$\frac{8\pi}{3}(17\sqrt{17} - 1) \text{ units}^2$$

63. Find the surface area generated by revolving $x = t^2, \quad y = 2t^2, \quad \text{for } 0 \le t \le 1$ about the y-axis.

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3.3: Polar Coordinates

Learning Objectives

- Locate points in a plane by using polar coordinates.
- Convert points between rectangular and polar coordinates.
- Sketch polar curves from given equations.
- Convert equations between rectangular and polar coordinates.
- Identify symmetry in polar curves and equations.

The rectangular coordinate system (or Cartesian plane) provides a means of mapping points to ordered pairs and ordered pairs to points. This is called a *one-to-one mapping* from points in the plane to ordered pairs. The polar coordinate system provides an alternative method of mapping points to ordered pairs. In this section we see that in some circumstances, polar coordinates can be more useful than rectangular coordinates.

Defining Polar Coordinates

To find the coordinates of a point in the polar coordinate system, consider Figure 3.3.1. The point P has Cartesian coordinates (x, y). The line segment connecting the origin to the point P measures the distance from the origin to P and has length r. The angle between the positive x-axis and the line segment has measure θ. This observation suggests a natural correspondence between the coordinate pair (x, y) and the values r and θ. This correspondence is the basis of the **polar coordinate system**. Note that every point in the Cartesian plane has two values (hence the term ordered pair) associated with it. In the polar coordinate system, each point also has two values associated with it: r and θ.

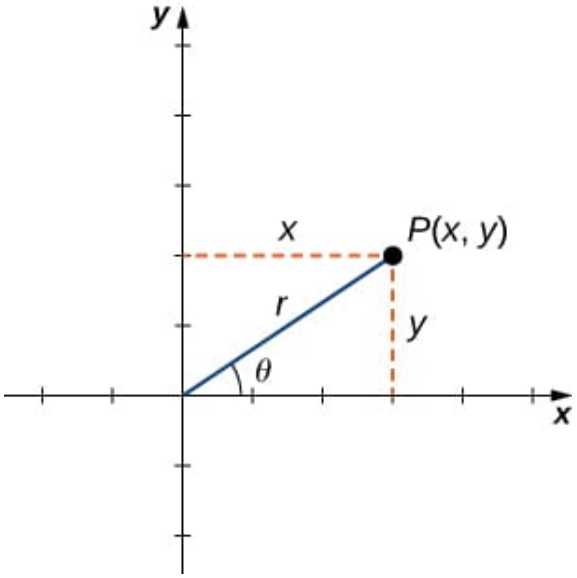


Figure 3.3.1: A point $P(x, y)$ is given in the first quadrant with lines drawn to indicate its x and y values. There is a line from the origin to $P(x, y)$ marked r and this line make an angle θ with the x axis.

Using right-triangle trigonometry, the equations are true for the point P:

$$\cos\theta = \frac{x}{r} \text{ so } x = r\cos\theta \tag{3.3.1}$$

$$\sin\theta = \frac{y}{r} \text{ so } y = r\sin\theta. \tag{3.3.2}$$

Furthermore,

$$r^2 = x^2 + y^2 \tag{3.3.3}$$

and

$$\tan\theta = \frac{y}{x}. \tag{3.3.4}$$

Each point (x, y) in the Cartesian coordinate system can therefore be represented as an ordered pair (r, θ) in the polar coordinate system. The first coordinate is called the **radial coordinate** and the second coordinate is called the **angular coordinate**. Every point in the plane can be represented in this form.

Note that the equation $\tan\theta = y/x$ has an infinite number of solutions for any ordered pair (x, y). However, if we restrict the solutions to values between 0 and 2π then we can assign a unique solution to the quadrant in which the original point (x, y) is located. Then the corresponding value of r is positive, so $r^2 = x^2 + y^2$.

Converting Points between Coordinate Systems

Given a point P in the plane with Cartesian coordinates (x, y) and polar coordinates (r, θ), the conversion formulas hold true:

$$x = r\cos\theta \tag{3.3.5}$$

$$y = r\sin\theta \tag{3.3.6}$$

and

$$r^2 = x^2 + y^2 \tag{3.3.7}$$

$$\tan\theta = \frac{y}{x} \tag{3.3.8}$$

.

These formulas can be used to convert from rectangular to polar or from polar to rectangular coordinates. Notice that Equation 3.3.7 is the *Pythagorean theorem*. (Figure 3.3.1).

Example 3.3.1: Converting between Rectangular and Polar Coordinates

Convert each of the given points into polar coordinates.

a. $(1, 1)$
b. $(-3, 4)$
c. $(0, 3)$
d. $(5\sqrt{3}, -5)$

Convert each of the given points into rectangular coordinates.

e. $(3, \pi/3)$
f. $(2, 3\pi/2)$
g. $(6, -5\pi/6)$

Solution

a. Use $x = 1$ and $y = 1$ in (Equation 3.3.7):

$$\begin{aligned} r^2 &= x^2 + y^2 \\ &= 1^2 + 1^2 \\ r &= \sqrt{2} \end{aligned}$$

and via (Equation 3.3.8)

$$\tan\theta = \frac{y}{x} = \frac{1}{1} = 1$$
$$\theta = \frac{\pi}{4}.$$

Therefore this point can be represented as $(\sqrt{2}, \frac{\pi}{4})$ in polar coordinates.

b. Use $x = -3$ and $y = 4$ in (Equation 3.3.7):

$$r^2 = x^2 + y^2 = (-3)^2 + (4)^2$$
$$r = 5$$

and via (Equation 3.3.8)

$$\tan\theta = \frac{y}{x} = -\frac{4}{3}$$

$$\theta = \arctan(-\frac{4}{3}) + \pi \approx 2.21.$$

Therefore this point can be represented as $(5, 2.21)$ in polar coordinates.

c. Use $x = 0$ and $y = 3$ in (Equation 3.3.7):

$$r^2 = x^2 + y^2 = (3)^2 + (0)^2 = 9 + 0\ r = 3$$

and via (Equation 3.3.8)

$$\tan\theta = \frac{y}{x} = \frac{3}{0}.$$

Direct application of the second equation leads to division by zero. Graphing the point $(0, 3)$ on the rectangular coordinate system reveals that the point is located on the positive y-axis. The angle between the positive x-axis and the positive y-axis is $\frac{\pi}{2}$. Therefore this point can be represented as $(3, \frac{\pi}{2})$ in polar coordinates.

d. Use $x = 5\sqrt{3}$ and $y = -5$ in (Equation 3.3.7):

$$r^2 = x^2 + y^2 = (5\sqrt{3})^2 + (-5)^2 = 75 + 25$$

$$r = 10$$

and via (Equation 3.3.8)

$$\tan\theta = \frac{y}{x} = \frac{-5}{5\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

$$\theta = -\frac{\pi}{6}.$$

Therefore this point can be represented as $(10, -\frac{\pi}{6})$ in polar coordinates.

e. Use $r = 3$ and $\theta = \frac{\pi}{3}$ in (Equation 3.3.5):

$$x = r\cos\theta = 3\cos(\frac{\pi}{3}) = 3(\frac{1}{2}) = \frac{3}{2}$$

and

$$y = r\sin\theta = 3\sin(\frac{\pi}{3}) = 3(\frac{\sqrt{3}}{2}) = \frac{3\sqrt{3}}{2}.$$

Therefore this point can be represented as $(\frac{3}{2}, \frac{3\sqrt{3}}{2})$ in rectangular coordinates.

f. Use $r = 2$ and $\theta = \frac{3\pi}{2}$ in (Equation 3.3.5):

$$x = r\cos\theta = 2\cos(\frac{3\pi}{2}) = 2(0) = 0$$

and

$$y = r\sin\theta = 2\sin(\frac{3\pi}{2}) = 2(-1) = -2.$$

Therefore this point can be represented as $(0, -2)$ in rectangular coordinates.

g. Use $r = 6$ and $\theta = -\frac{5\pi}{6}$ in (Equation 3.3.5):

$$x = r\cos\theta = 6\cos(-\frac{5\pi}{6}) = 6(-\frac{\sqrt{3}}{2}) = -3\sqrt{3}$$

and

$$y = r\sin\theta = 6\sin(-\frac{5\pi}{6}) = 6(-\frac{1}{2}) = -3.$$

Therefore this point can be represented as $(-3\sqrt{3}, -3)$ in rectangular coordinates.

Exercise 3.3.1

Convert $(-8, -8)$ into polar coordinates and $(4, \frac{2\pi}{3})$ into rectangular coordinates.

Hint available at the end of this section.

Answer available at the end of this section.

The polar representation of a point is not unique. For example, the polar coordinates $(2, \frac{\pi}{3})$ and $(2, \frac{7\pi}{3})$ both represent the point $(1, \sqrt{3})$ in the rectangular system. Also, the value of r can be negative. Therefore, the point with polar coordinates $(-2, \frac{4\pi}{3})$ also represents the point $(1, \sqrt{3})$ in the rectangular system, as we can see by using (Equation 3.3.5):

$$x = r\cos\theta = -2\cos(\frac{4\pi}{3}) = -2(-\frac{1}{2}) = 1 \tag{3.3.9}$$

and

$$y = r\sin\theta = -2\sin(\frac{4\pi}{3}) = -2(-\frac{\sqrt{3}}{2}) = \sqrt{3}. \tag{3.3.10}$$

Every point in the plane has an infinite number of representations in polar coordinates. However, each point in the plane has only one representation in the rectangular coordinate system.

Note that the polar representation of a point in the plane also has a visual interpretation. In particular, r is the directed distance that the point lies from the origin, and θ measures the angle that the line segment from the origin to the point makes with the positive x-axis. Positive angles are measured in a counterclockwise direction and negative angles are measured in a clockwise direction. The polar coordinate system appears in Figure 3.3.2.

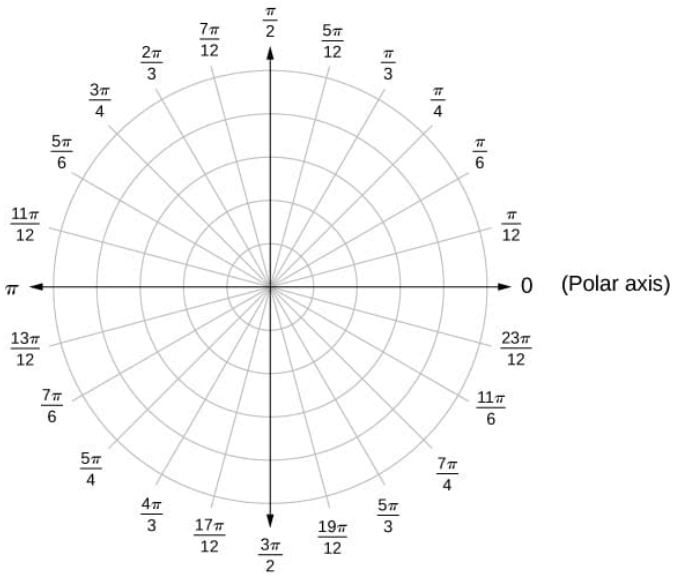


Figure 3.3.2: A series of concentric circles is drawn with spokes indicating different values between 0 and 2π in increments of $\pi/12$. The first quadrant starts with 0 where the x axis would be, then the next spoke is marked $\pi/12$, then $\pi/6$, $\pi/4$, $\pi/3$, $5\pi/12$, $\pi/2$, and so on into the second, third, and fourth quadrants. The polar axis is noted near the former x axis line.

The line segment starting from the center of the graph going to the right (called the positive x-axis in the Cartesian system) is the **polar axis.** The center point is the **pole**, or origin, of the coordinate system, and corresponds to $r = 0$. The innermost circle represented in Figure 3.3.2 contains all points a distance of 1 unit from the pole, and is represented by the equation $r = 1$. Then $r = 2$ is the set of points 2 units from the pole, and so on. The line segments emanating from the pole correspond to fixed angles. To plot a point in the polar coordinate system, start with the angle. If the angle is positive, then measure the angle from the polar axis in a counterclockwise direction. If it is negative, then measure it clockwise. If the value of r is positive, move that distance along the terminal ray of the angle. If it is negative, move along the ray that is opposite the terminal ray of the given angle.

Example 3.3.2: Plotting Points in the Polar Plane

Plot each of the given points on the polar plane.

a. $(2, \frac{\pi}{4})$
b. $(-3, \frac{2\pi}{3})$
c. $(4, \frac{5\pi}{4})$

Solution

The three points are plotted in Figure 3.3.3.

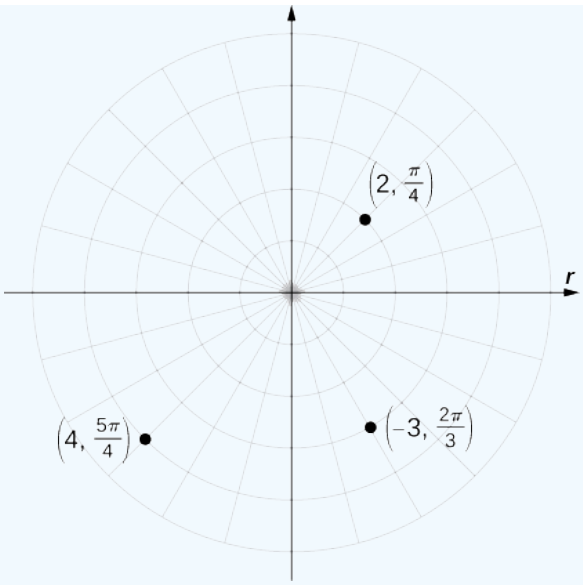


Figure 3.3.3:Three points are marked on a polar coordinate plane. The point $(2, \pi/4)$ is in the first quadrant, distance 2 units from the pole and at angle $\pi/4$ from the polar axis. The point $(-3, 2\pi/3)$ is in the fourth quadrant, distance 3 units from the pole and at angle $-2\pi/3$ from the polar axis. The point \((4,5π/4)\) is in the third quadrant, distance 4 units from the pole and at angle $5\pi/4$ from the polar axis.

Exercise 3.3.2

Plot $(4, \frac{5\pi}{3})$ and $(-3, -\frac{7\pi}{2})$ on the polar plane.

Hint available at the end of this section.

Answer available at the end of this section.

Polar Curves

Now that we know how to plot points in the polar coordinate system, we can discuss how to plot curves. In the rectangular coordinate system, we can graph a function $y = f(x)$ and create a curve in the Cartesian plane. In a similar fashion, we can graph a curve that is generated by a function $r = f(\theta)$.

The general idea behind graphing a function in polar coordinates is the same as graphing a function in rectangular coordinates. Start with a list of values for the independent variable (θ in this case) and calculate the corresponding values of the dependent variable r. This process generates a list of ordered pairs, which can be plotted in the polar coordinate system. Finally, connect the points, and take advantage of any patterns that may appear. The function may be periodic, for example, which indicates that only a limited number of values for the independent variable are needed.

Problem-Solving Strategy: Plotting a Curve in Polar Coordinates

1. Create a table with two columns. The first column is for θ, and the second column is for r.
2. Create a list of values for θ.
3. Calculate the corresponding r values for each θ.
4. Plot each ordered pair (r, θ) on the coordinate axes.
5. Connect the points and look for a pattern.

Example 3.3.3: Graphing a Function in Polar Coordinates

Graph the curve defined by the function $r = 4\sin\theta$. Identify the curve and rewrite the equation in rectangular coordinates.

Solution

Because the function is a multiple of a sine function, it is periodic with period 2π, so use values for θ between 0 and 2π.

Figure 3.3.5 shows the graph based on this table.

The result of steps 1–3 appear in the table.

θ	$r = 4\sin\theta$	θ	$r = 4\sin\theta$
0	0	π	0
$\frac{\pi}{6}$	2	$\frac{7\pi}{6}$	-2
$\frac{\pi}{4}$	$2\sqrt{2} \approx 2.8$	$\frac{5\pi}{4}$	$-2\sqrt{2} \approx -2.8$
$\frac{\pi}{3}$	$2\sqrt{3} \approx 3.4$	$\frac{4\pi}{3}$	$-2\sqrt{3} \approx -3.4$
$\frac{\pi}{2}$	4	$\frac{3\pi}{2}$	-4
$\frac{2\pi}{3}$	$2\sqrt{3} \approx 3.4$	$\frac{5\pi}{3}$	$-2\sqrt{3} \approx -3.4$
$\frac{3\pi}{4}$	$2\sqrt{2} \approx 2.8$	$\frac{7\pi}{4}$	$-2\sqrt{2} \approx -2.8$
$\frac{5\pi}{6}$	2	$\frac{11\pi}{6}$	–2
		2π	0

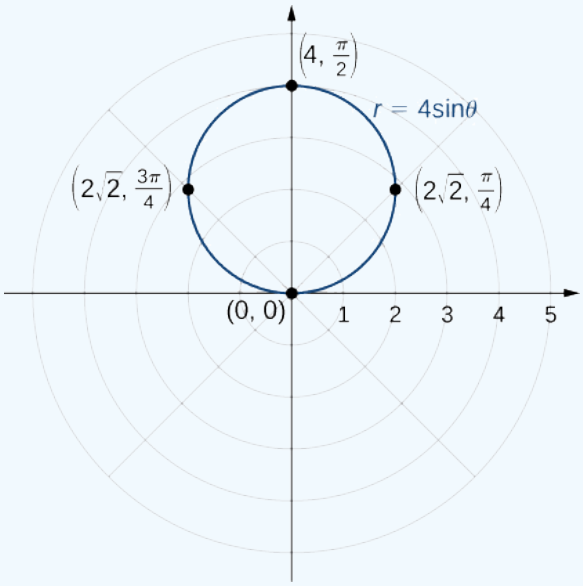


Figure 3.3.5: On the polar coordinate plane, a circle is drawn with radius 2. It passes through the pole and the polar points $(2\sqrt{2}, \pi/4)$, $(4, \pi/2)$, and $(2\sqrt{2}, 3\pi/4)$.

This is the graph of a circle. The equation $r = 4\sin\theta$ can be converted into rectangular coordinates by first multiplying both sides by r. This gives the equation $r^2 = 4r\sin\theta$. Next use the facts that $r^2 = x^2 + y^2$ and $y = r\sin\theta$. This gives $x^2 + y^2 = 4y$. To put this equation into standard form, subtract $4y$ from both sides of the equation and complete the square:

$$x^2 + y^2 - 4y = 0$$
$$x^2 + (y^2 - 4y) = 0$$
$$x^2 + (y^2 - 4y + 4) = 0 + 4$$
$$x^2 + (y - 2)^2 = 4$$

This is the equation of a circle with radius 2 and center $(0, 2)$ in the rectangular coordinate system.

Exercise 3.3.3

Create a graph of the curve defined by the function $r = 4 + 4\cos\theta$.

Hint available at the end of this section.

Answer available at the end of this section.

The graph in Example 3.3.3 was that of a circle. The equation of the circle can be transformed into rectangular coordinates using the coordinate transformation formulas in (Equation 3.3.5). Example 3.3.4 gives some more examples of functions for transforming from polar to rectangular coordinates.

Example 3.3.4: Transforming Polar Equations to Rectangular Coordinates

Rewrite each of the given equations in rectangular coordinates and identify the graph.

a. $\theta = \dfrac{\pi}{3}$
b. $r = 3$
c. $r = 6\cos\theta - 8\sin\theta$

Solution:

a. Take the tangent of both sides. This gives $\tan\theta = \tan(\pi/3) = \sqrt{3}$. Since $\tan\theta = y/x$ we can replace the left-hand side of this equation by y/x. This gives $y/x = \sqrt{3}$, which can be rewritten as $y = x\sqrt{3}$. This is the equation of a straight line passing through the origin with slope $\sqrt{3}$. In general, any polar equation of the form $\theta = K$ represents a straight line through the pole with slope equal to $\tan K$.
b. First, square both sides of the equation. This gives $r^2 = 9$. Next replace r^2 with $x^2 + y^2$. This gives the equation $x^2 + y^2 = 9$, which is the equation of a circle centered at the origin with radius 3. In general, any polar equation of the form $r = k$ where *k* is a positive constant represents a circle of radius *k* centered at the origin. (Note: when squaring both sides of an equation it is possible to introduce new points unintentionally. This should always be taken into consideration. However, in this case we do not introduce new points. For example, $(-3, \dfrac{\pi}{3})$ is the same point as $(3, \dfrac{4\pi}{3})$.)
c. Multiply both sides of the equation by r. This leads to $r^2 = 6r\cos\theta - 8r\sin\theta$. Next use the formulas

$$r^2 = x^2 + y^2, x = r\cos\theta, y = r\sin\theta.$$

This gives

$$r^2 = 6(r\cos\theta) - 8(r\sin\theta)$$

$$x^2 + y^2 = 6x - 8y.$$

To put this equation into standard form, first move the variables from the right-hand side of the equation to the left-hand side, then complete the square.

$$x^2 + y^2 = 6x - 8y$$

$$x^2 - 6x + y^2 + 8y = 0$$

$$(x^2 - 6x) + (y^2 + 8y) = 0$$

$$(x^2 - 6x + 9) + (y^2 + 8y + 16) = 9 + 16$$

$$(x - 3)^2 + (y + 4)^2 = 25.$$

This is the equation of a circle with center at $(3, -4)$ and radius 5. Notice that the circle passes through the origin since the center is 5 units away.

Watch the accompanying video for a visual demonstration of this concept.

- **Video Length:** 2 minutes 26 seconds.
- **Context:** This video demonstrates how to convert a given polar equation to a rectangular equation.$r = 2sec\theta tan\theta$.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Exercise 3.3.4

Rewrite the equation $r = \sec\theta \tan\theta$ in rectangular coordinates and identify its graph.

Hint available at the end of this section.

Answer available at the end of this section.

We have now seen several examples of drawing graphs of curves defined by **polar equations.** A summary of some common curves is given in the tables below. In each equation, *a* and *b* are arbitrary constants.

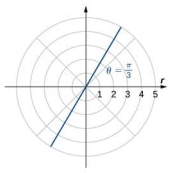
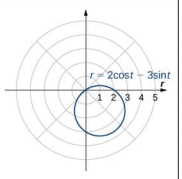
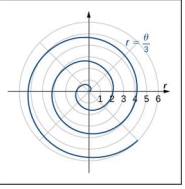
Name	Equation	Example
Line passing through the pole with slope tan *K*	$\theta = K$	
Circle	$r = a\cos\theta + b\sin\theta$	
Spiral	$r = a + b\theta$	

Figure 3.3.7: This table has three columns and three rows. The first row is a header row and is given from left to right as Name, Equation, and Example. The second row is Line passing through the pole with slope $\tan K$; $\theta = K$; and a picture of a straight line on the polar coordinate plane with $\theta = \pi/3$. The third row is Circle; $r = a\cos\theta + b\sin\theta$; and a picture of a circle on the polar coordinate plane with equation $r = 2\cos(t) - 3\sin(t)$: the circle touches the origin but has center in the third quadrant. The third row is Spiral; $r = a + b\theta$; and a picture of a spiral starting at the origin with equation $r = \theta/3$.

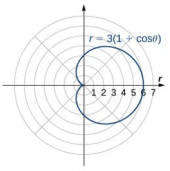
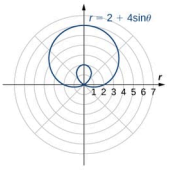
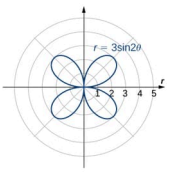
Name	Equation	Example
Cardioid	$r = a(1 + \cos\theta)$ $r = a(1 - \cos\theta)$ $r = a(1 + \sin\theta)$ $r = a(1 - \sin\theta)$	
Limaçon	$r = a\cos\theta + b$ $r = a\sin\theta + b$	
Rose	$r = a\cos(b\theta)$ $r = a\sin(b\theta)$	

Figure 3.3.8 This table has three columns and three rows. The first row is Cardioid; $r = a(1 + \cos\theta)$, $r = a(1-\cos\theta)$, $r = a(1 + \sin\theta)$, $r = a(1-\sin\theta)$; and a picture of a cardioid with equation $r = 3(1 + \cos\theta)$: the cardioid looks like a heart turned on its side with a rounded bottom instead of a pointed one. The second row is Limaçon; $r = a\cos\theta + b$, $r = a\sin\theta + b$; and a picture of a limaçon with equation $r = 2 + 4\sin\theta$: the figure looks like a deformed circle with a loop inside of it. The third row is Rose; $r = a\cos(b\theta)$, $r = a\sin(b\theta)$; and a picture of a rose with equation $r = 3\sin(2\theta)$: the rose looks like a flower with four petals, one petal in each quadrant, each with length 3 and reaching to the origin between each petal.

A **cardioid** is a special case of a **limaçon** (pronounced “lee-mah-son”), in which $a = b$ or $a = -b$. The **rose i**s a very interesting curve. Notice that the graph of $r = 3\sin 2\theta$ has four petals. However, the graph of $r = 3\sin 3\theta$ has three petals.

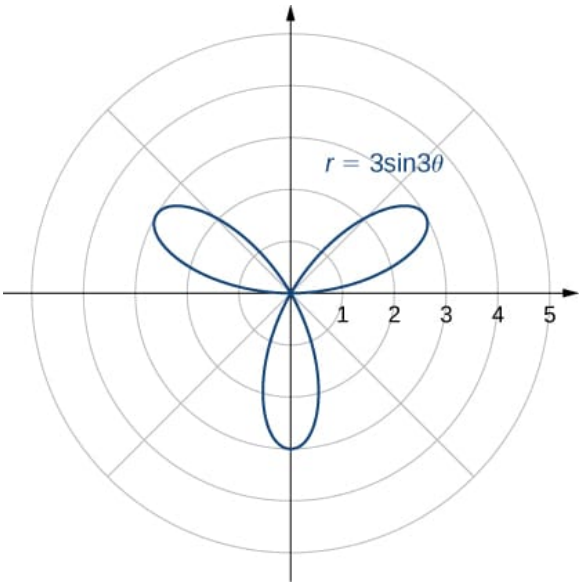


Figure 3.3.9: A rose with three petals, one in the first quadrant, another in the second quadrant, and the third in both the third and fourth quadrants, each with length 3. Each petal starts and ends at the origin.

If the coefficient of θ is even, the graph has twice as many petals as the coefficient. If the coefficient of θ is odd, then the number of petals equals the coefficient. You are encouraged to explore why this happens.

Exploring Graphs of Multiple Petaled Roses.

You can move it around and also change its equation to see the effect on the graph. The interactive app is unfortunately not accessible.

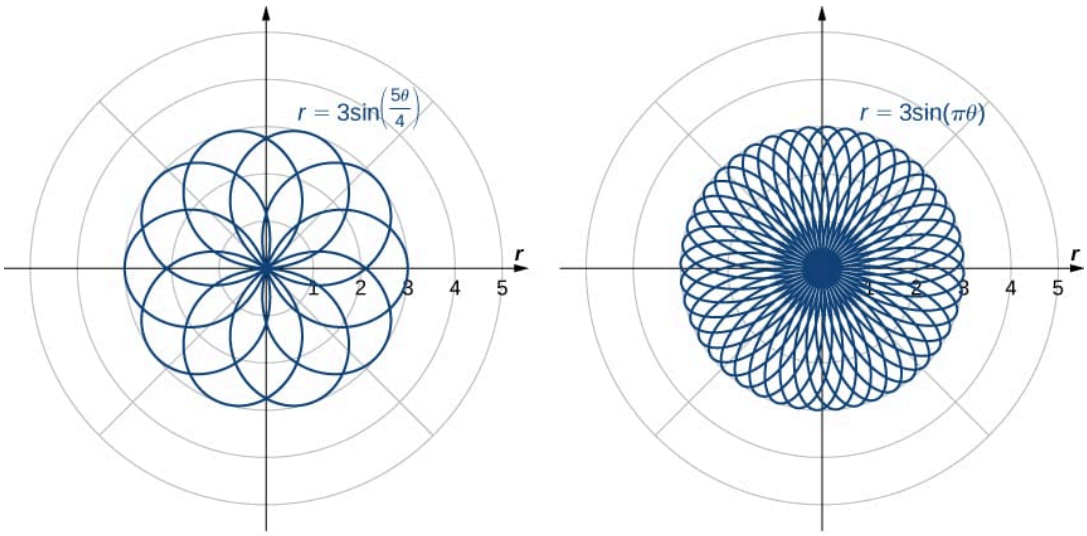
Decide whether you want to graph a rose that comes from a sin or cos polar curve. Then move the slider to see how changing the factor in front of x changes the rose.

2

r = cos(2x)

Button sin Button cos

Even more interesting graphs emerge when the coefficient of θ is not an integer. For example, if it is rational, then the curve is closed; that is, it eventually ends where it started (Figure $3.3.10a$). However, if the coefficient is irrational, then the curve never closes (Figure $3.3.10b$). Although it may appear that the curve is closed, a closer examination reveals that the petals just above the positive x axis are slightly thicker. This is because the petal does not quite match up with the starting point.



(a) (b)

Figure $3.3.10$: There are two graphs. The first graph, (a), is a rose with so many overlapping petals that there are a few patterns that develop, starting with a sharp 10 pointed star in the center and moving out to an increasingly rounded set of petals. The second graph, (b), is a rose with even more overlapping petals, so many so that it is impossible to tell what is happening in the center, but on the outer edges are a number of sharply rounded petals. Note that the rose in part (b) would actually fill the entire circle of radius 3 if plotted in full.

Since the curve defined by the graph of $r = 3\sin(\pi\theta)$ never closes, the curve depicted in Figure $3.3.10b$ is only a partial depiction. In fact, this is an example of a **space-filling curve.** A space-filling curve is one that in fact occupies a two-dimensional subset of the real plane. In this case the curve occupies the circle of radius 3 centered at the origin.

Example $3.3.5$: Describing a Spiral

Recall the **chambered nautilus** introduced in the chapter prelude. This creature displays a spiral when half the outer shell is cut away. It is possible to describe a spiral using rectangular coordinates. Figure $3.3.11$ shows a spiral in rectangular coordinates. How can we describe this curve mathematically?

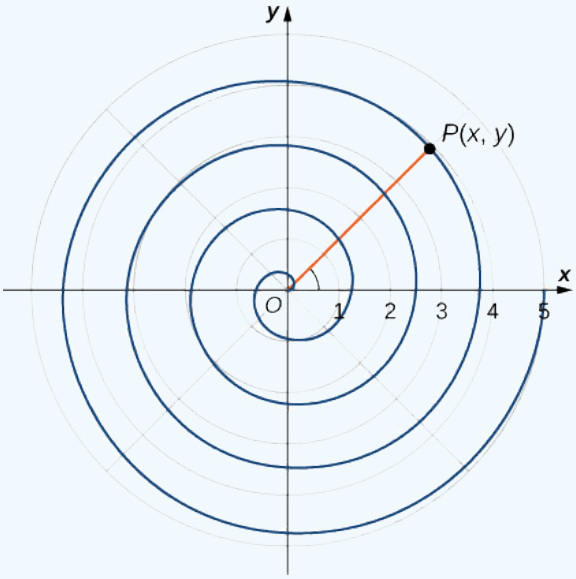


Figure 3.3.11: A spiral starting at the origin and continually increasing its radius to a point $P(x, y)$. A uniformly spaced out spiral moves in counterclockwise direction.

Solution

As the point *P* travels around the spiral in a counterclockwise direction, its distance d from the origin increases. Assume that the distance d is a constant multiple k of the angle θ that the line segment OP makes with the positive x-axis. Therefore $d(P, O) = k\theta$, where O is the origin. Now use the distance formula and some trigonometry:

$$d(P, O) = k\theta$$

$$\sqrt{(x-0)^2 + (y-0)^2} = k \arctan(\frac{y}{x})$$

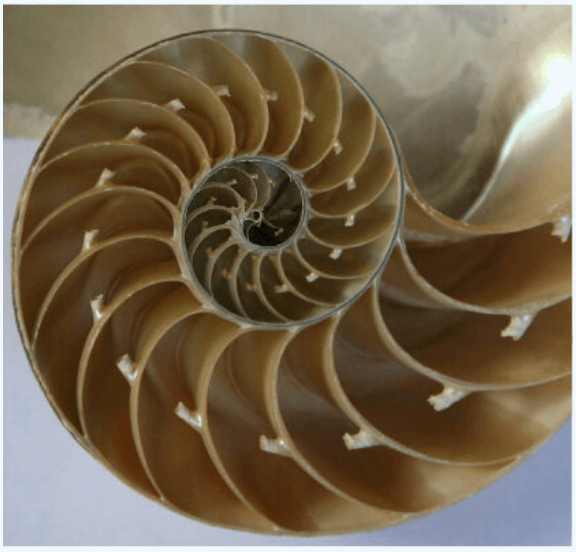
$$\sqrt{x^2 + y^2} = k \arctan(\frac{y}{x})$$

$$\arctan(\frac{y}{x}) = \frac{\sqrt{x^2 + y^2}}{k}$$

$$y = x \tan(\frac{\sqrt{x^2 + y^2}}{k}).$$

Although this equation describes the spiral, it is not possible to solve it directly for either *x* or *y*. However, if we use polar coordinates, the equation becomes much simpler. In particular, $d(P, O) = r$, and θ is the second coordinate. Therefore the equation for the spiral becomes $r = k\theta$. Note that when $\theta = 0$ we also have $r = 0$, so the spiral emanates from the origin. We can remove this restriction by adding a constant to the equation. Then the equation for the spiral becomes $r = a + k\theta$ for arbitrary constants a and k. This is referred to as an **Archimedean spiral**, after the Greek mathematician Archimedes.

Another type of spiral is the logarithmic spiral, described by the function $r = a \cdot b^{\theta}$. A graph of the function $r = 1.2(1.25^{\theta})$ is given in Figure 3.3.12. This spiral describes the shell shape of the chambered nautilus.



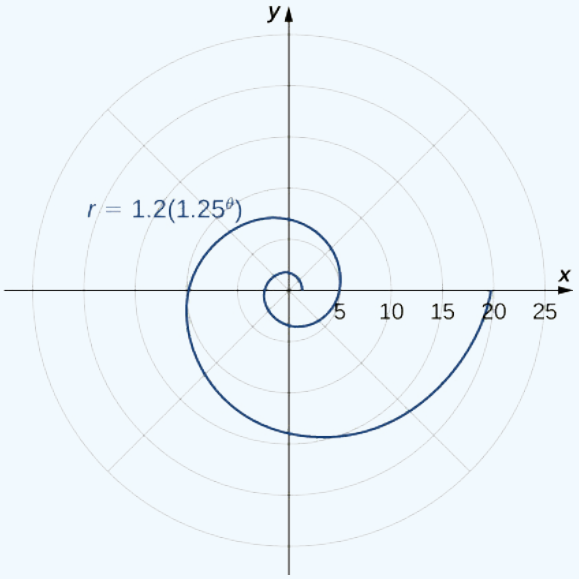


Figure 3.3.12: This figure has two figures. The first is a shell with many chambers that increase in size from the center out. The second is a spiral with equation $r = 1.2(1.25\theta)$. A logarithmic spiral is similar to the shape of the chambered nautilus shell. (Credit: CC-BY-NC-SA modification of work by Jitze Couperus, Flickr)

Suppose a curve is described in the polar coordinate system via the function $r = f(\theta)$. Since we have conversion formulas from polar to rectangular coordinates given by

$$x = r\cos\theta \tag{3.3.11}$$

$$y = r\sin\theta \tag{3.3.12}$$

,

it is possible to rewrite these formulas using the function

$$x = f(\theta)\cos\theta \tag{3.3.13}$$

$$y = f(\theta)\sin\theta. \tag{3.3.14}$$

This step gives a parameterization of the curve in rectangular coordinates using θ as the parameter. For example, the spiral formula $r = a + b\theta$ from Figure becomes

$$x = (a + b\theta)\cos\theta \tag{3.3.15}$$

$$y = (a + b\theta)\sin\theta. \tag{3.3.16}$$

Letting θ range from $-\infty$ to ∞ generates the entire spiral.

Symmetry in Polar Coordinates

When studying **symmetry** of functions in rectangular coordinates (i.e., in the form $y = f(x)$), we talk about symmetry with respect to the *y*-axis and symmetry with respect to the origin. In particular, if $f(-x) = f(x)$ for all x in the domain of f, then f is an even function and its graph is symmetric with respect to the *y*-axis. If $f(-x) = -f(x)$ for all *x* in the domain of f, then f is an odd function and its graph is symmetric with respect to the origin. By determining which types of symmetry a graph exhibits, we can learn more about the shape and appearance of the graph. Symmetry can also reveal other properties of the function that generates the graph. Symmetry in polar curves works in a similar fashion.

Symmetry in Polar Curves and Equations

Consider a curve generated by the function $r = f(\theta)$ in polar coordinates.

i. The curve is symmetric about the polar axis if for every point (r, θ) on the graph, the point $(r, -\theta)$ is also on the graph. Similarly, the equation $r = f(\theta)$ is unchanged by replacing θ with $-\theta$.
ii. The curve is symmetric about the pole if for every point (r, θ) on the graph, the point $(r, \pi + \theta)$ is also on the graph. Similarly, the equation $r = f(\theta)$ is unchanged when replacing r with $-r$, or θ with $\pi + \theta$.
iii. The curve is symmetric about the vertical line $\theta = \dfrac{\pi}{2}$ if for every point (r, θ) on the graph, the point $(r, \pi - \theta)$ is also on the graph. Similarly, the equation $r = f(\theta)$ is unchanged when θ is replaced by $\pi - \theta$.

Figure 3.3.13 has examples of each type of symmetry.

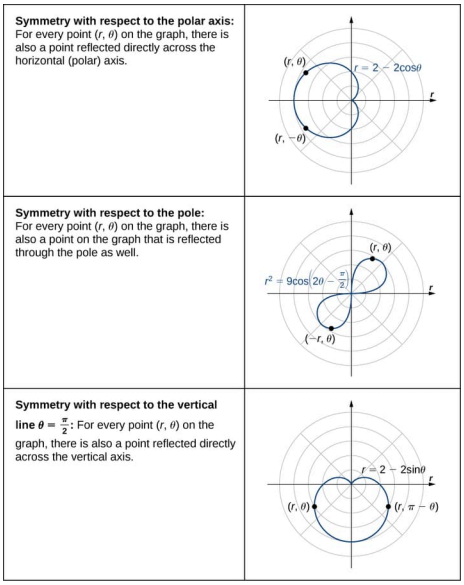


Figure 3.3.13: This table has two columns and three rows. The first row is a graph represented by the equation $r = 2 - 2cos\theta$, which is a "sideways" cardioid and is symmetric with respect to the polar axis. The second row is a graph represented by the equation $r^2 = 9cos(2\theta - \dfrac{\pi}{2})$, which is shaped like a slanted "8" with axis of symmetry at the $45°$. This graph demonstrates symmetry with respect to the pole. The third row is a graph represented by the equation $r = 2 - 2sin(\theta)$, which is an "upright" cardioid and is symmetric with respect to the line $\theta = \pi/2$.

Example 3.3.6: Using Symmetry to Graph a Polar Equation

Find the symmetry of the rose defined by the equation $r = 3\sin(2\theta)$ and create a graph.

Solution

Suppose the point (r, θ) is on the graph of $r = 3\sin(2\theta)$.

i. To test for symmetry about the polar axis, first try replacing θ with $-\theta$. This gives $r = 3\sin(2(-\theta)) = -3\sin(2\theta)$. Since this changes the original equation, this test is not satisfied. However, returning to the original equation and replacing r with $-r$ and θ with $\pi - \theta$ yields

$$\begin{aligned} -r &= 3\sin(2(\pi - \theta)) \\ -r &= 3\sin(2\pi - 2\theta) \\ -r &= 3\sin(-2\theta) \\ -r &= -3\sin 2\theta. \end{aligned}$$

Multiplying both sides of this equation by -1 gives $r = 3\sin 2\theta$, which is the original equation. This demonstrates that the graph is symmetric with respect to the polar axis.

ii. To test for symmetry with respect to the pole, first replace r with $-r$, which yields $-r = 3\sin(2\theta)$. Multiplying both sides by -1 gives $r = -3\sin(2\theta)$, which does not agree with the original equation. Therefore the equation does not pass the test for this symmetry. However, returning to the original equation and replacing θ with $\theta + \pi$ gives

$$\begin{aligned} r &= 3\sin(2(\theta+\pi)) \\ &= 3\sin(2\theta + 2\pi) \\ &= 3(\sin 2\theta \cos 2\pi + \cos 2\theta \sin 2\pi) \\ &= 3\sin 2\theta. \end{aligned}$$

Since this agrees with the original equation, the graph is symmetric about the pole.

iii. To test for symmetry with respect to the vertical line $\theta = \dfrac{\pi}{2}$, first replace both r with $-r$ and θ with $-\theta$.

$$\begin{aligned} -r &= 3\sin(2(-\theta)) \\ -r &= 3\sin(-2\theta) \\ -r &= -3\sin 2\theta. \end{aligned}$$

Multiplying both sides of this equation by -1 gives $r = 3\sin 2\theta$, which is the original equation. Therefore the graph is symmetric about the vertical line $\theta = \dfrac{\pi}{2}$.

This graph has symmetry with respect to the polar axis, the origin, and the vertical line going through the pole. To graph the function, tabulate values of θ between 0 and $\pi/2$ and then reflect the resulting graph.

Table of values of θ between 0 and $\pi/2$.

0	0
$\frac{\pi}{6}$	$\frac{3\sqrt{3}}{2} \approx 2.6$
$\frac{\pi}{4}$	3
$\frac{\pi}{3}$	$\frac{3\sqrt{3}}{2} \approx 2.6$
$\frac{\pi}{2}$	0

This gives one petal of the rose.

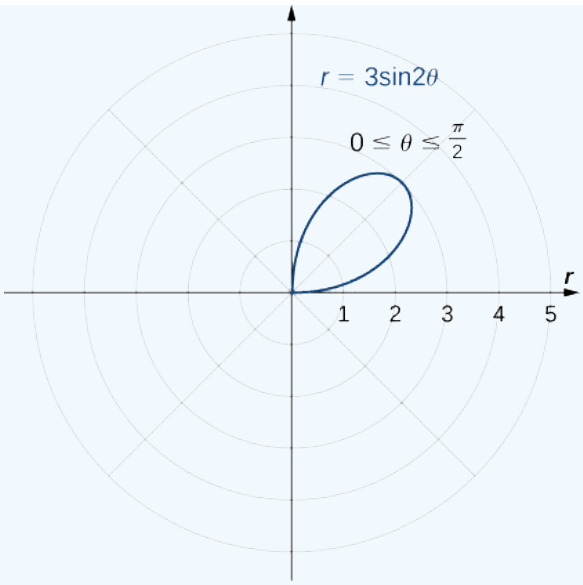


Figure 3.3.14: A single petal is graphed with equation $r = 3sin(2\theta)$ for $0 \leq \theta \leq \pi/2$. It starts at the pole and reaches a maximum distance from the origin of 3 at angle $\theta = \pi/4$. Then, over $\pi/4 \leq \theta \leq \pi/2$, the graph returns to the pole.

Reflecting this image into the other three quadrants gives the entire graph of $r = 3sin(2\theta)$.

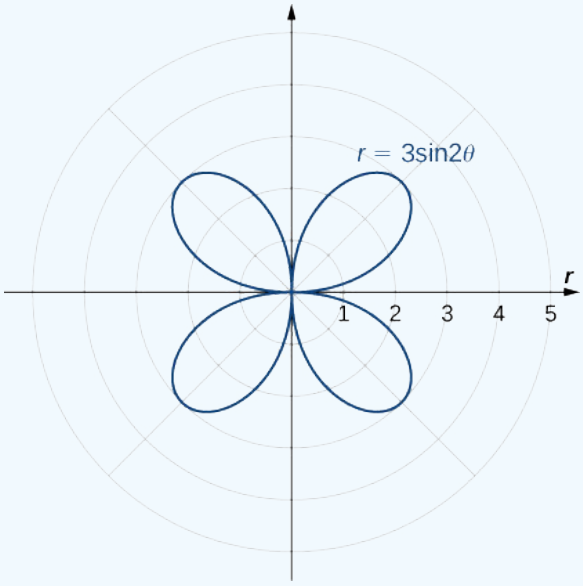


Figure 3.3.15: A four-petaled rose with equation $r = 3sin(2\theta)$ is graphed. Each petal starts at the pole and reaches a maximum distance from the pole of 3. The maximum distance occurs at angles $\pi/4, 3\pi/4, 5\pi/4, 7\pi/4$.

Exercise 3.3.5Symmetry

Determine the symmetry of the graph determined by the equation $r = 2\cos(3\theta)$ and create a graph.

Hint available at the end of this section.

Answer available at the end of this section.

Key Concepts

- The polar coordinate system provides an alternative way to locate points in the plane.
- Convert points between rectangular and polar coordinates using the formulas

$$x = r\cos\theta \text{ and } y = r\sin\theta \tag{3.3.17}$$

and

$$r = \sqrt{x^2 + y^2} \text{ and } \tan\theta = \frac{y}{x}. \tag{3.3.18}$$

- To sketch a polar curve from a given polar function, make a table of values and take advantage of periodic properties.
- Use the conversion formulas to convert equations between rectangular and polar coordinates.
- Identify symmetry in polar curves, which can occur through the pole, the horizontal axis, or the vertical axis.

Glossary

Angular Coordinate
: θ the angle formed by a line segment connecting the origin to a point in the polar coordinate system with the positive radial (x) axis, measured counterclockwise.

Cardioid
: A plane curve traced by a point on the perimeter of a circle that is rolling around a fixed circle of the same radius; the equation of a cardioid is $r = a(1 + \sin\theta)$ or $r = a(1 + \cos\theta)$.

Limaçon
: The graph of the equation $r = a + b\sin\theta$ or $r = a + b\cos\theta$. If $a = b$ then the graph is a cardioid.

Polar Axis
: The horizontal axis in the polar coordinate system corresponding to $r \geq 0$.

Polar Coordinate System
: A system for locating points in the plane. The coordinates are r, the radial coordinate, and θ, the angular coordinate.

Polar Equation
: An equation or function relating the radial coordinate to the angular coordinate in the polar coordinate system.

Pole
: The central point of the polar coordinate system, equivalent to the origin of a Cartesian system.

Radial Coordinate
: r the coordinate in the polar coordinate system that measures the distance from a point in the plane to the pole.

Rose
: Graph of the polar equation $r = a\cos 2\theta$ or $r = a\sin 2\theta$for a positive constant a.

Space-Filling Curve
: A curve that completely occupies a two-dimensional subset of the real plane.

Answers

Exercise 3.3.1

Use (Equation 3.3.7) and (Equation 3.3.5). Make sure to check the quadrant when calculating θ.

Exercise 3.3.1

$(8\sqrt{2}, \frac{5\pi}{4})$ and $(-2, 2\sqrt{3})$

Exercise 3.3.2

Start with θ, then use r.

Exercise 3.3.2

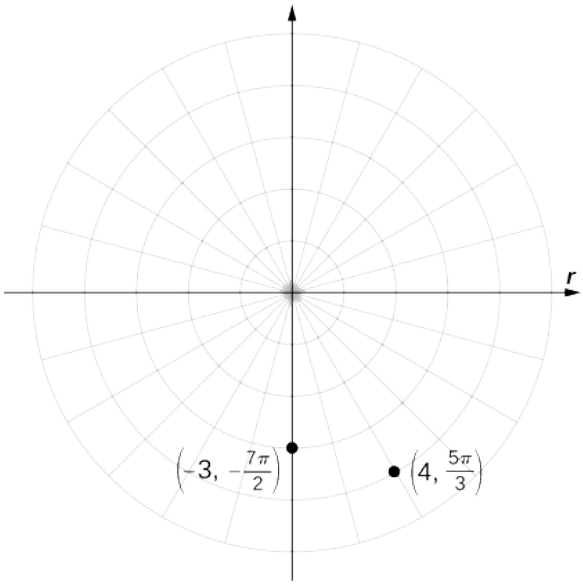


Figure 3.3.4: Two points are marked on a polar coordinate plane. The point $(-3, 7\pi/2)$ is on the y-axis at distance 3 units from the pole. The point $(4, 5\pi/3)$ is in the fourth quadrant, distance 4 units from the pole and at angle $5\pi/3$ from the polar axis.

Exercise 3.3.3

Follow the problem-solving strategy for creating a graph in polar coordinates.

Exercise 3.3.3

The name of this shape is a cardioid, which we will study further later in this section.

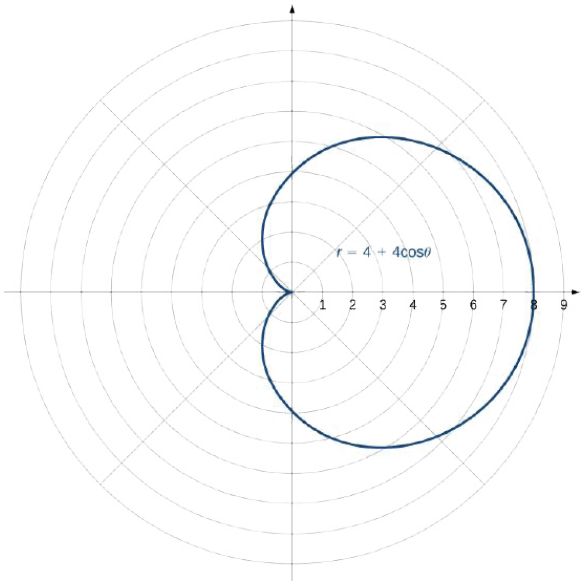


Figure 3.3.6: This image is a polar plot of the equation \(r=4+4 cosθ\), which forms a cardioid. The graph's shape is roughly a sideways "heart". The graph passes through the pole with a cusp at that point, and it is symmetric about the polar axis. The graph is smooth at all of the other points, passing through the polar points $(8, 0)$, $(4, \pi/2)$, $(0, \pi)$, $(4, 3\pi/2)$.

Exercise 3.3.4

Convert to sine and cosine, then multiply both sides by cosine.

Exercise 3.3.4

$y = x^2$, which is the equation of a parabola opening upward.

Exercise 3.3.5**Symmetry**

Use Note.

Exercise 3.3.5**Symmetry**

Symmetric with respect to the polar axis.

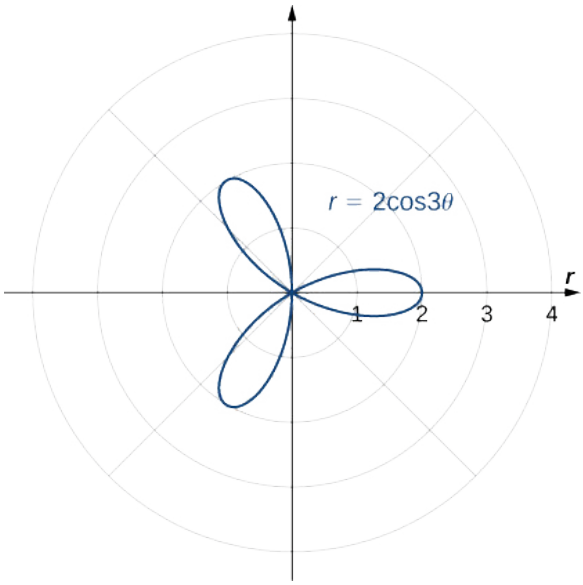


Figure 3.3.16: This shows the polar graph of the equation \(r=2cos(3θ)\), which forms a rose curve with three evenly spaced petals. The graph is symmetric about the polar axis (horizontal axis) due to the cosine term. Each petal reaches its maximum length of \(r=2\), and the petals are oriented at equal angular intervals of-frac{2\pi}{3} $\frac{2\pi}{3}$ radians.

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3.3E: Exercises for Section 3.3

Exercises 1 - 7

Plot the point whose polar coordinates are given by first constructing the angle θ and then marking off the distance r along the ray.

1. $\left(3, \frac{\pi}{6}\right)$

Answer

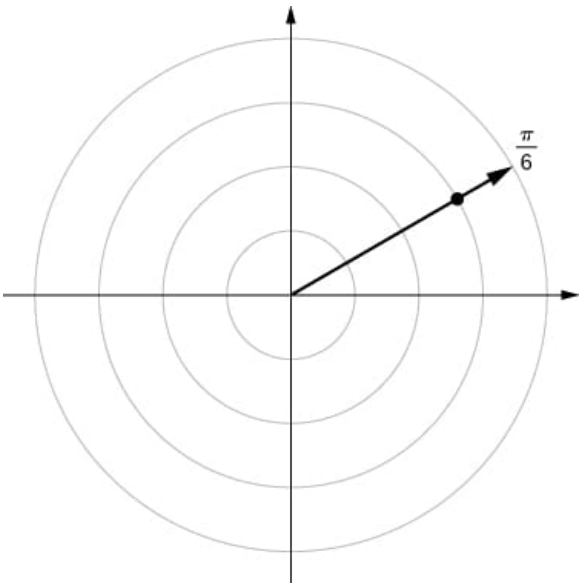


Figure $3.3E.1$: A polar graph with concentric circles centered at the origin, representing radial distances. The graph includes a bold arrow starting at the origin and extending outward at an angle of 30 degrees from the positive $x-axis$. A small point is marked on the arrow to indicate a specific location along its length. The grid includes radial lines and labeled axes.

2. $\left(-2, \frac{5\pi}{3}\right)$
3. $\left(0, \frac{7\pi}{6}\right)$

Answer

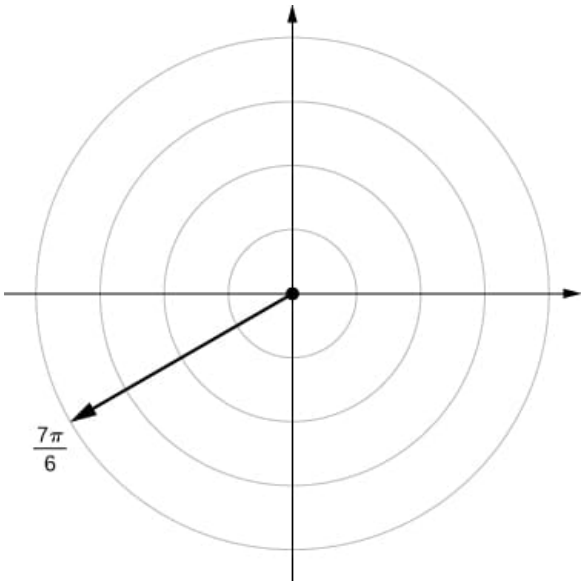


Figure $3.3E.2$: A polar graph with concentric circles centered at the origin, representing radial distances. The graph includes a bold arrow starting at the origin and extending outward at an angle of [redacted] 210 degrees from the positive $x-axis$. The arrow is drawn toward the third quadrant. The grid includes radial lines and labeled axes.

4. $\left(-4, \frac{3\pi}{4}\right)$
5. $\left(1, \frac{\pi}{4}\right)$

Answer

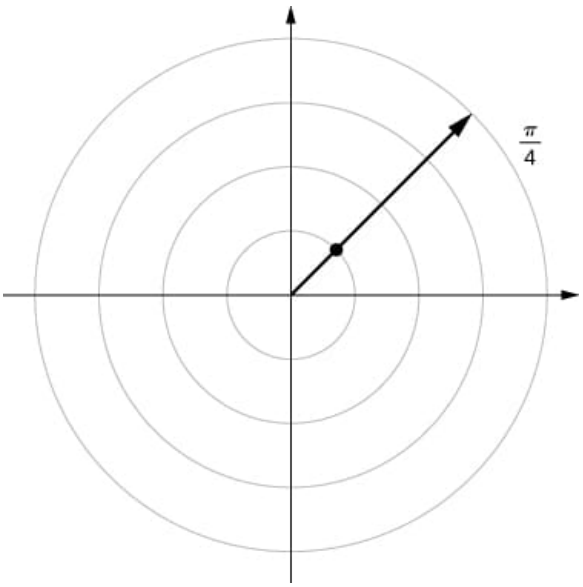


Figure $3.3E.3$:A polar graph with concentric circles centered at the origin, representing radial distances. The graph includes a bold arrow starting at the origin and extending outward at an angle of 45 degrees from the positive x-axis. A small point is marked on the arrow to indicate a specific location along its length. The grid includes radial lines and labeled axes.

6. $\left(2, \frac{5\pi}{6}\right)$
7. $\left(1, \frac{\pi}{2}\right)$

Answer

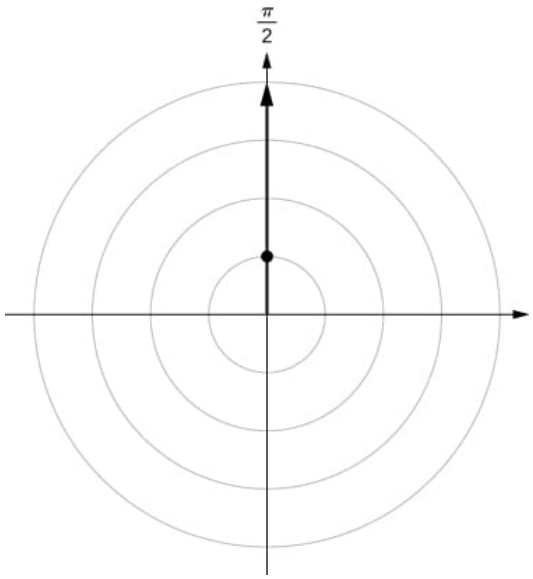


Figure $3.3E.4$:A polar graph with concentric circles centered at the origin, representing radial distances. The graph includes a bold arrow starting at the origin and extending upward along the positive y-axis, corresponding to an angle of [redacted]90 degrees. A small point is marked on the arrow to indicate a specific location along its length. The grid includes radial lines and labeled axes.

Exercises 8 - 11

Consider the polar graph below. Give two sets of polar coordinates for each point.

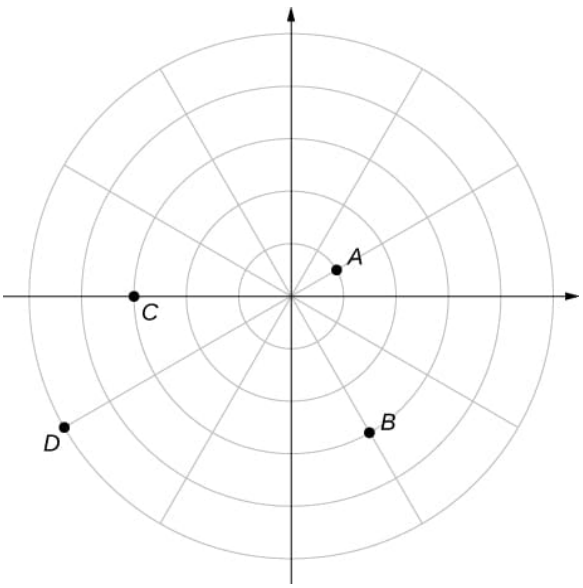


Figure $3.3E.5$:A polar graph with concentric circles and radial lines centered at the origin, representing angular and radial coordinates. Four points labeled A, B, C and D are marked on the graph.

8. Coordinates of point A.
9. Coordinates of point B.

Answer

$B\left(3, \frac{-\pi}{3}\right)\, B\left(-3, \frac{2\pi}{3}\right)$

10. Coordinates of point C.
11. Coordinates of point D.

Answer

$D\left(5, \frac{7\pi}{6}\right)\, D\left(-5, \frac{\pi}{6}\right)$

Exercises 12 - 17

The rectangular coordinates of a point are given. Find two sets of polar coordinates for the point in $(0, 2\pi]$. Round to three decimal places.

12. $(2, 2)$
13. $(3, -4)$

Answer

$(5, -0.927),\ \ (-5, -0.927 + \pi)$

14. $(8, 15)$
15. $(-6, 8)$

Answer

$(10, -0.927),\ \ (-10, -0.927 + \pi)$

16. $(4, 3)$
17. $(3, -\sqrt{3})$

Answer

$(2\sqrt{3}, -0.524),\ \ (-2\sqrt{3}, -0.524 + \pi)$

Exercises 18 - 24

Find rectangular coordinates for the given point in polar coordinates.

18. $\left(2, \frac{5\pi}{4}\right)$
19. $\left(-2, \frac{\pi}{6}\right)$

Answer

$(-\sqrt{3}, -1)$

20. $\left(5, \frac{\pi}{3}\right)$
21. $\left(1, \frac{7\pi}{6}\right)$

Answer

$\left(-\frac{\sqrt{3}}{2}, \frac{-1}{2}\right)$

22. $\left(-3, \frac{3\pi}{4}\right)$
23. $\left(0, \frac{\pi}{2}\right)$

Answer

$(0, 0)$

24. $(-4.5, 6.5)$

Exercises 25 - 29

Determine whether the graphs of the polar equation are symmetric with respect to the x-axis, the y-axis, or the origin.

25. $r = 3\sin(2\theta)$

Answer

Symmetry with respect to the *x*-axis, *y*-axis, and origin.

26. $r^2 = 9\cos\theta$
27. $r = \cos\left(\frac{\theta}{5}\right)$

Answer

Symmetric with respect to *x*-axis only.

28. $r = 2\sec\theta$
29. $r = 1 + \cos\theta$

Answer

Symmetry with respect to *x*-axis only.

Exercises 30 - 33

Describe the graph of each polar equation. Confirm each description by converting into a rectangular equation.

30. $r = 3$
31. $\theta = \frac{\pi}{4}$

Answer

Line $y = x$

32. $r = \sec\theta$
33. $r = \csc\theta$

Answer

$y = 1$

Exercises 34 - 36

Convert the rectangular equation to polar form and sketch its graph.

34. $x^2 + y^2 = 16$
35. $x^2 - y^2 = 16$

Answer

Hyperbola; polar form $r^2\cos(2\theta) = 16$ or $r^2 = 16\sec\theta$.

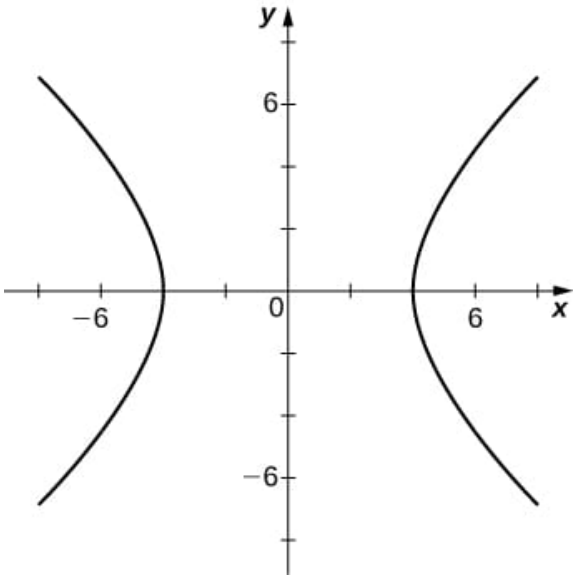


Figure $3.3E.6$:A graph showing a hyperbola on a Cartesian coordinate system. The hyperbola consists of two symmetric branches opening outward along the $x-axis$. The center of the hyperbola is at the origin $(0,0)$. The $x-axis$ ranges from -6 to 6, and the $y-axis$ ranges from -6 to 6. The axes are labeled "x" (horizontal) and "y" (vertical). The branches curve away from each other as they extend further from the center.

36. $x=8$

Exercises 37 - 38

Convert the rectangular equation to polar form and sketch its graph.

37. $3x-y=2$

Answer

$r=\frac{2}{3\cos\theta-\sin\theta}$

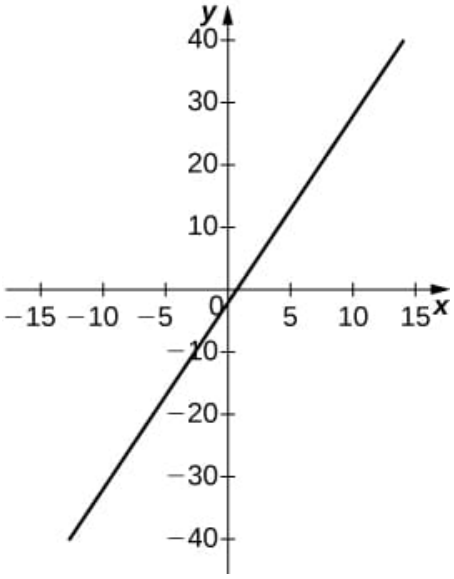


Figure $3.3E.7$:A graph showing a straight line on a Cartesian coordinate system. The line passes through the origin $(0,0)$ with a positive slope, extending into the first and third quadrants. The $x-axis$ ranges from -15 to 15, and the $y-axis$ ranges from -40 to 40. The axes are labeled "x" (horizontal) and "y" (vertical).

38. $y^2 = 4x$

Exercises 39 - 43

Convert the polar equation to rectangular form and sketch its graph.

39. $r = 4 \sin \theta$
40. $x^2 + y^2 = 4y$

Figure $3.3E.8$:

Answer

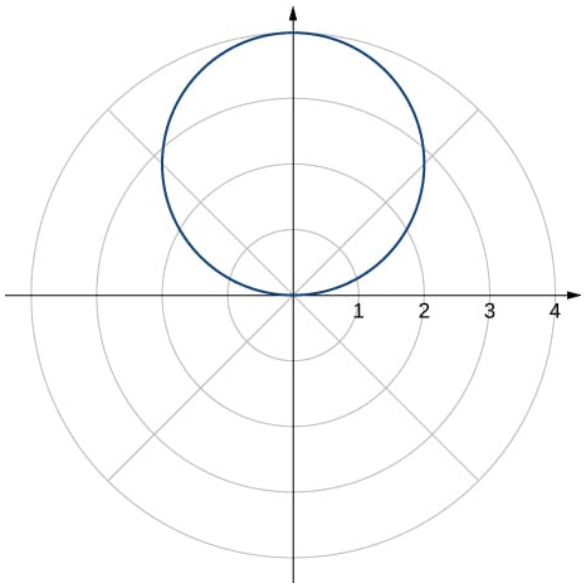


Figure $3.3E.8$: This image displays a polar graph of the equation \/(r=2\), which represents a **circle** in polar coordinates. The circle is centered at the origin (pole) with a constant radius of\(r=2\). The graph illustrates that the radial distance remains the same regardless of the angle \(θ\), resulting in a perfect circle.

41. $r = 6 \cos \theta$
42. $r = \theta$

Answer

$$x \tan \sqrt{x^2 + y^2} = y$$

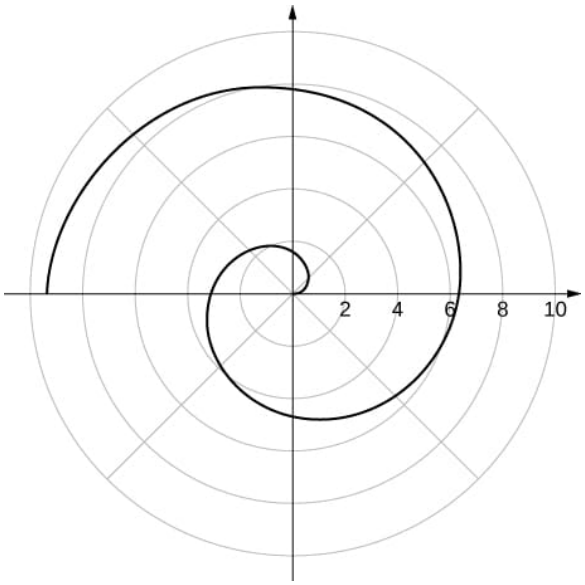


Figure $3.3E.9$:A polar graph with concentric circles and radial lines, representing angular and radial coordinates. The graph features a blue circle centered at the origin, with a radius of approximately 3 units. The circle is drawn within the polar grid, and the axes are labeled, with the positive x-axis extending to 4 for reference.

13. $r = \cot\theta \csc\theta$

Exercises 44 - 54

Sketch a graph of the polar equation and identify any symmetry.

14. $r = 1 + \sin\theta$

Answer

y-axis symmetry

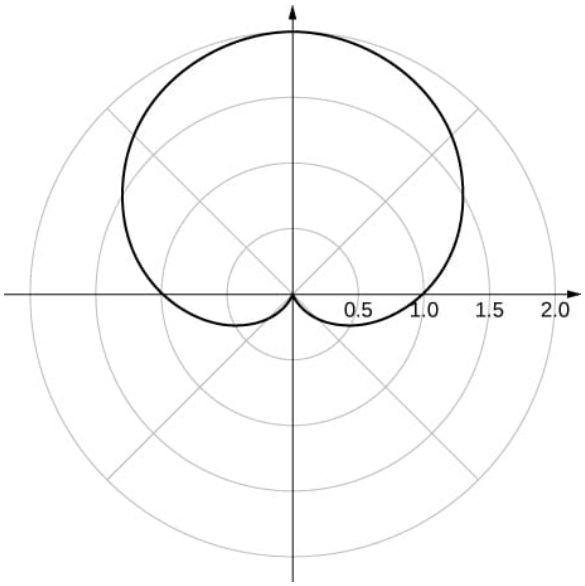


Figure $3.3E.10$:A polar graph with concentric circles and radial lines, representing angular and radial coordinates. The graph features a cardioid curve centered at the origin. The cardioid has a cusp at the origin and a maximum radial distance of approximately 2 units along the positive x-axis. The curve is symmetric about the polar axis (horizontal axis). The positive x-axis is labeled and extends to 2 for reference.

15. $r = 3 - 2\cos\theta$
16. $r = 2 - 2\sin\theta$

Answer

y-axis symmetry

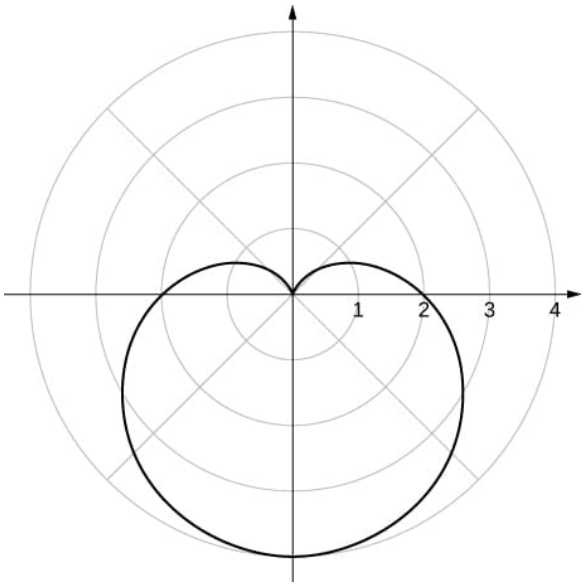


Figure $3.3E.11$:A polar graph with concentric circles and radial lines, representing angular and radial coordinates. The graph displays a limacon curve with an inner loop. The curve is centered at the origin, with its maximum radial distance extending to approximately 3 units along the positive x-axis. The inner loop is centered near the origin, and the curve is symmetric about the polar axis (horizontal axis). The positive x-axis is labeled and extends to 4 for reference.

17. $r = 5 - 4\sin\theta$
18. $r = 3\cos(2\theta)$

Answer

x-and y-axis symmetry and symmetry about the pole

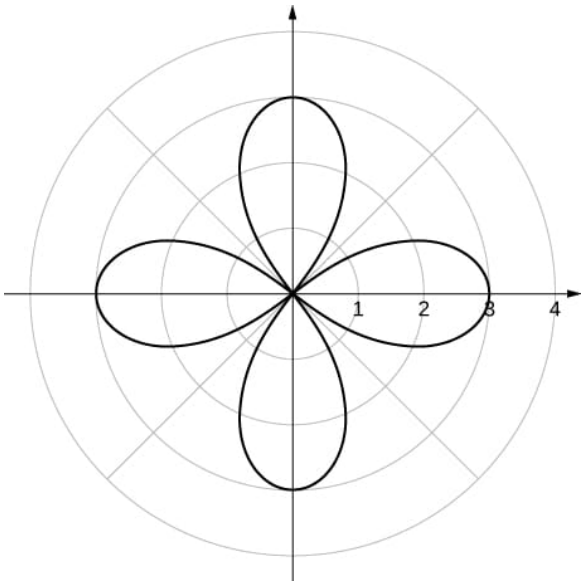


Figure $3.3E.12$:A polar graph with concentric circles and radial lines, representing angular and radial coordinates. The graph shows a four-petal rose curve, centered at the origin. The petals are symmetrically distributed, extending outward along angles of 0°, 90°, 180°, and 270°. Each petal reaches a maximum radial distance of approximately 3 units. The positive x-axis is labeled and extends to 4 for reference.

49. $r = 3\sin(2\theta)$
50. $r = 2\cos(3\theta)$

Answer

x-axis symmetry

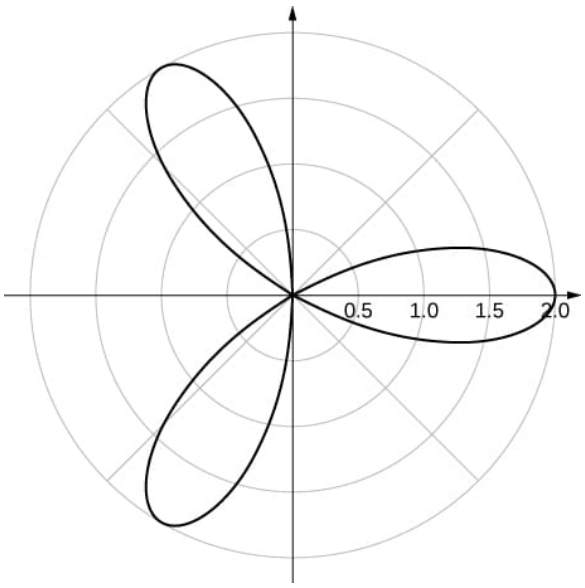


Figure $3.3E.13$:A polar graph with concentric circles and radial lines, representing angular and radial coordinates. The graph shows a three-petal rose curve, centered at the origin. The petals are symmetrically distributed, with one petal along the positive x-axis and the others in the second and fourth quadrants. Each petal reaches a maximum radial distance of approximately 2 units. The positive x-axis is labeled and extends to 2 for reference.

51. $r = 3\cos\left(\frac{\theta}{2}\right)$
52. $r^2 = 4\cos\left(\frac{2}{\theta}\right)$

Answer

x-and y-axis symmetry and symmetry about the pole

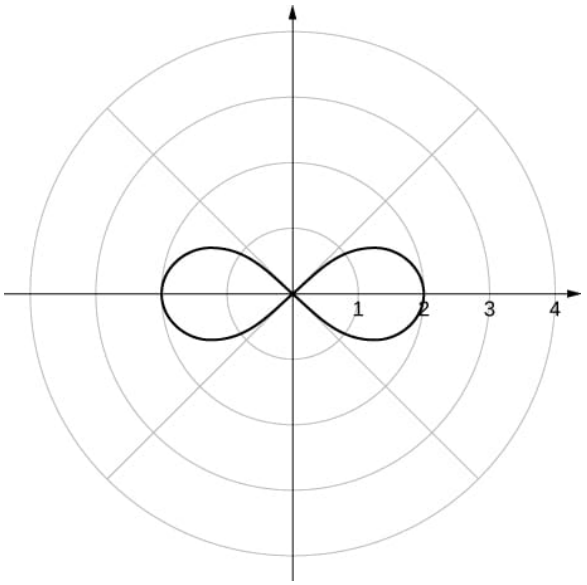


Figure $3.3E.14$:A polar graph with concentric circles and radial lines, representing angular and radial coordinates. The graph displays a lemniscate curve, resembling a figure-eight shape centered at the origin. The curve has two symmetrical loops, with one extending into the first quadrant and the other into the third quadrant. The maximum radial distance of the loops is approximately 2 units. The positive x-axis is labeled and extends to 4 for reference.

53. $r^2 = 4\sin\theta$
54. $r = 2\theta$

Answer

no symmetry

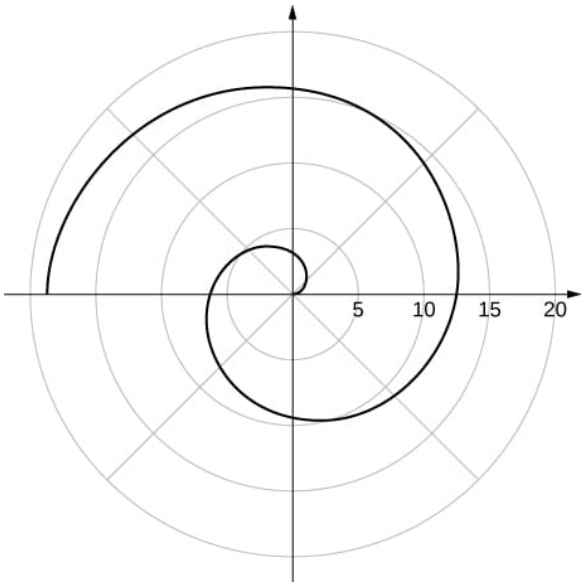


Figure $3.3E.15$:A polar graph with concentric circles and radial lines, representing angular and radial coordinates. The graph displays a spiral curve starting at the origin and winding outward in a counterclockwise direction. The spiral expands gradually, crossing through increasing radial distances, and extends to a maximum radial distance of approximately 20 units. The positive x-axis is labeled and extends to 20 for reference.

55. [T] The graph of $r = 2\cos(2\theta)\sec(\theta).$ is called a strophoid. Use a graphing utility to sketch the graph, and, from the graph, determine the asymptote.
56. [T] Use a graphing utility and sketch the graph of $r = \dfrac{6}{2\sin\theta - 3\cos\theta}$.

Answer

a line

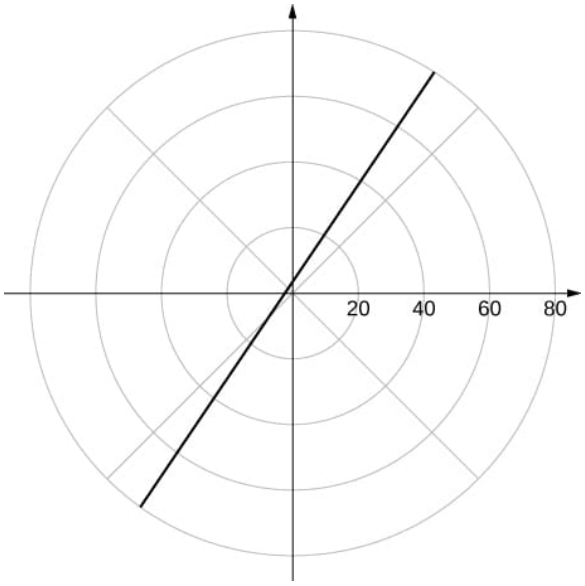


Figure $3.3E.16$:A polar graph with concentric circles and radial lines, representing angular and radial coordinates. The graph displays a straight line passing through the origin, extending diagonally from the third quadrant to the first quadrant. The line has a positive slope and crosses radial distances labeled up to 80 units. The positive x-axis is labeled and extends to 80 for reference.

57. [T] Use a graphing utility to graph $r = \frac{1}{1-\cos\theta}$.
58. [T] Use technology to graph $r = e^{\sin(\theta)} - 2\cos(4\theta)$.

Answer

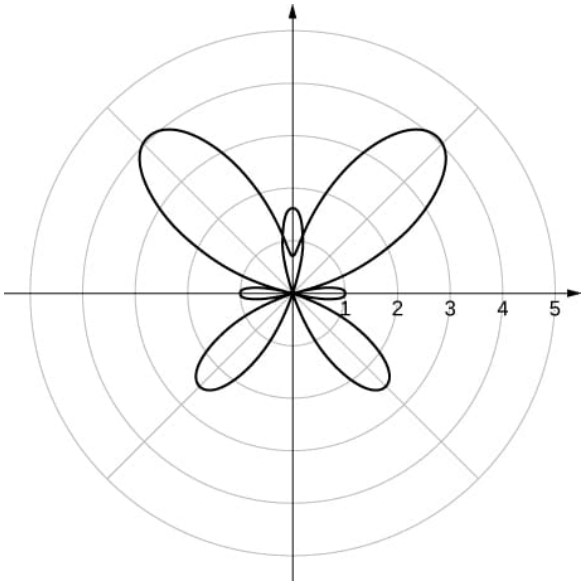


Figure $3.3E.17$:A polar graph with concentric circles and radial lines, representing angular and radial coordinates. The graph features an eight-petal rose curve centered at the origin. The petals are symmetrically distributed, extending outward in pairs in all directions. Each petal reaches a maximum radial distance of approximately 4 units. The positive x-axis is labeled and extends to 5 for reference.

59. [T] Use technology to plot $r = \sin(\frac{3\theta}{7})$ (use the interval $0 \leq \theta \leq 14\pi$).
60. Without using technology, sketch the polar curve $\theta = \frac{2\pi}{3}$.

Answer

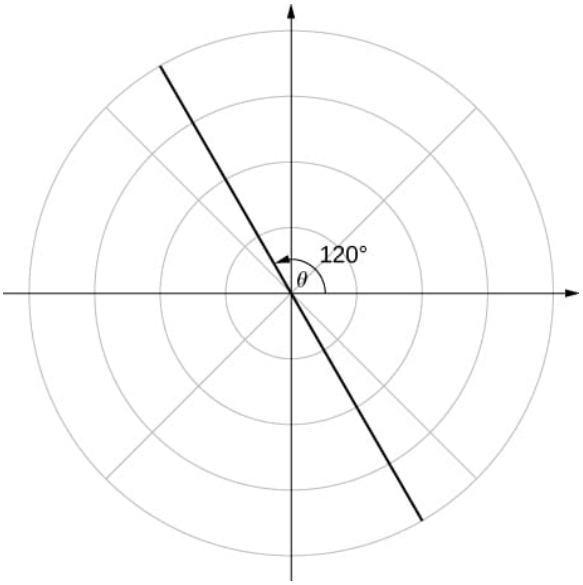


Figure $3.3E.18$:A polar graph with concentric circles and radial lines, representing angular and radial coordinates. The graph shows a straight line passing through the origin, forming an angle of \($\theta = 120^\circ$ \) with the positive x-axis. The angle \($\theta = 120^\circ$ \)is marked near the origin, and the line extends diagonally through the second and fourth quadrants. The positive x-axis is labeled for reference.

61. [T] Use a graphing utility to plot $r = \theta \sin \theta$ for $-\pi \leq \theta \leq \pi$.
62. [T] Use technology to plot $r = e^{-0.1\theta}$ for $-10 \leq \theta \leq 10$.

Answer

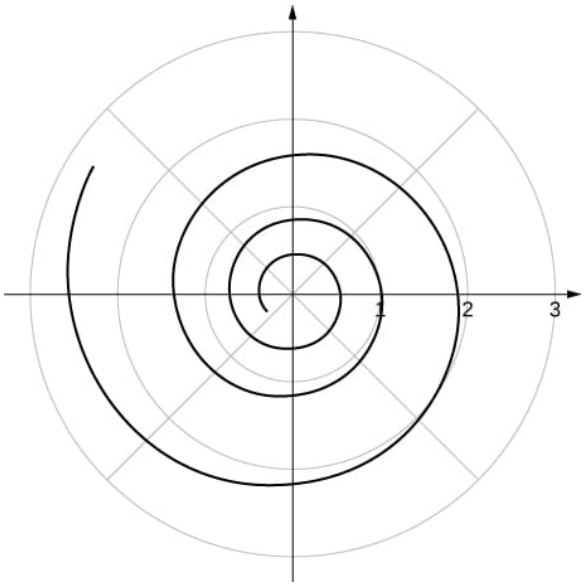


Figure $3.3E.19$:A polar graph with concentric circles and radial lines, representing angular and radial coordinates. The graph displays a spiral curve starting near the origin and winding outward in a counterclockwise direction. The spiral gradually increases in radial distance as it expands, reaching up to approximately 3 units. The positive x-axis is labeled for reference.

63. [T] There is a curve known as the “Black Hole.” Use technology to plot $r = e^{-0.01\theta}$ for $-100 \leq \theta \leq 100$.
64. [T] Use the results of the preceding two problems to explore the graphs of $r = e^{-0.001\theta}$ and $r = e^{-0.0001\theta}$ for $|\theta| > 100$.

Answer

Answers vary. One possibility is the spiral lines become closer together and the total number of spirals increases.

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3.4: Area and Arc Length in Polar Coordinates

Learning Objectives

- Apply the formula for area of a region in polar coordinates.
- Determine the arc length of a polar curve.

In the rectangular coordinate system, the definite integral provides a way to calculate the area under a curve. In particular, if we have a function $y = f(x)$ defined from $x = a$ to $x = b$ where $f(x) > 0$ on this interval, the area between the curve and the x-axis is given by

$$A = \int_a^b f(x)dx.$$

This fact, along with the formula for evaluating this integral, is summarized in the Fundamental Theorem of Calculus. Similarly, the arc length of this curve is given by

$$L = \int_a^b \sqrt{1 + (f'(x))^2}dx.$$

In this section, we study analogous formulas for area and arc length in the polar coordinate system.

Areas of Regions Bounded by Polar Curves

We have studied the formulas for area under a curve defined in rectangular coordinates and parametrically defined curves. Now we turn our attention to deriving a formula for the area of a region bounded by a polar curve. Recall that the proof of the Fundamental Theorem of Calculus used the concept of a Riemann sum to approximate the area under a curve by using rectangles. For polar curves we use the Riemann sum again, but the rectangles are replaced by sectors of a circle.

Consider a curve defined by the function $r = f(\theta)$, where $\alpha \leq \theta \leq \beta$. Our first step is to partition the interval $[\alpha, \beta]$ into *n* equal-width subintervals. The width of each subinterval is given by the formula $\Delta\theta = (\beta - \alpha)/n$, and the *i*th partition point θ_i is given by the formula $\theta_i = \alpha + i\Delta\theta$. Each partition point $\theta = \theta_i$ defines a line with slope $\tan\theta_i$ passing through the pole as represented in the Figure $3.4.1$.

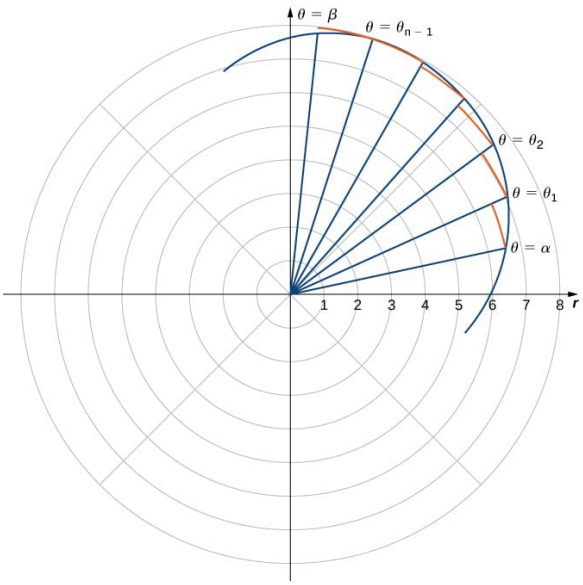


Figure 3.4.1: On the polar coordinate plane, a curve is drawn in the first quadrant, and there are rays from the pole that intersect this curve at a regular interval. Every time one of these rays intersects the curve, a perpendicular line is made from the ray to the next ray. The first instance of a ray-curve intersection is labeled $\theta = \alpha$; the last instance is labeled $\theta = \beta$. The intervening ones are marked θ_1,θ_2, ..., θ_{n-1}.

The line segments are connected by arcs of constant radius. This defines sectors whose areas can be calculated by using a geometric formula. The area of each sector is then used to approximate the area between successive line segments. We then sum the areas of the sectors to approximate the total area. This approach gives a Riemann sum approximation for the total area. The formula for the area of a sector of a circle is illustrated in the Figure 3.4.2.

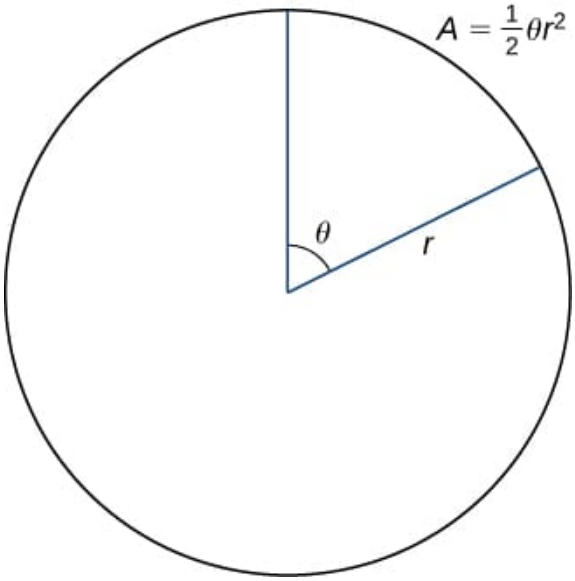


Figure 3.4.2: The area of a sector of a circle is given by $A = \frac{1}{2}\theta r^2$. The image features a circle with a sector highlighted. The circle's boundary is drawn in black. The sector is outlined by two blue lines radiating from the circle's center to its perimeter, forming an angle marked as theta (θ). The radius of the circle is labeled as "r" along one of these radii.

Recall that the area of a circle is $A = \pi r^2$. When measuring angles in radians, 360 degrees is equal to 2π radians. Therefore a fraction of a circle can be measured by the central angle θ. The fraction of the circle is given by $\frac{\theta}{2\pi}$, so the area of the sector is this fraction multiplied by the total area:

$$A = (\frac{\theta}{2\pi})\pi r^2 = \frac{1}{2}\theta r^2. \tag{3.4.1}$$

Since the radius of a typical sector in Figure 3.4.1 is given by $r_i = f(\theta_i)$, the area of the ith sector is given by

$$A_i = \frac{1}{2}(\Delta\theta)(f(\theta_i))^2. \tag{3.4.2}$$

Therefore a Riemann sum that approximates the area is given by

$$A_n = \sum_{i=1}^{n} A_i \approx \sum_{i=1}^{n} \frac{1}{2}(\Delta\theta)(f(\theta_i))^2. \tag{3.4.3}$$

We take the limit as $n \to \infty$ to get the exact area:

$$A = \lim_{n\to\infty} A_n = \frac{1}{2}\int_{\alpha}^{\beta} (f(\theta))^2 d\theta. \tag{3.4.4}$$

Area of a Region Bounded by a Polar Curve

Suppose f is continuous and nonnegative on the interval $\alpha \le \theta \le \beta$ with $0 < \beta - \alpha \le 2\pi$. The area of the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 d\theta \tag{3.4.5}$$

$$= \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta. \tag{3.4.6}$$

Example 3.4.1: Finding an Area of a Polar Region

Find the area of one petal of the rose defined by the equation $r = 3\sin(2\theta)$.

Solution

The graph of $r = 3\sin(2\theta)$ follows.

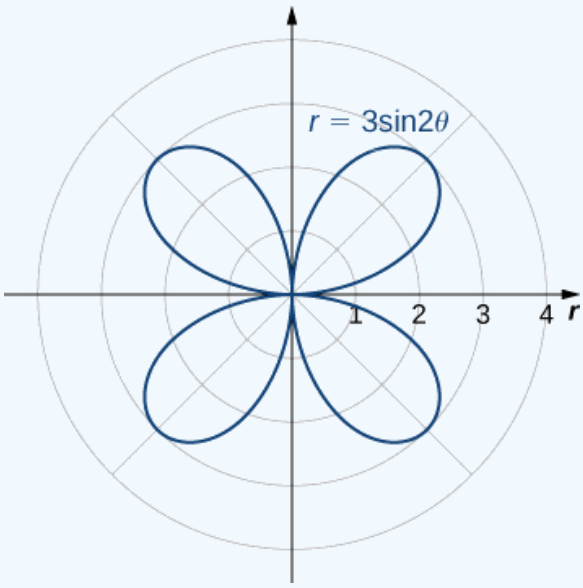


Figure 3.4.3: The graph of $r = 3\sin(2\theta)$. A four-petaled rose with furthest point 3 units from the pole occurring at $\pi/4$, $3\pi/4$, $5\pi/4$, and $7\pi/4$. The petals are evenly distributed around the center, forming a symmetrical shape.

When $\theta = 0$ we have $r = 3\sin(2(0)) = 0$. The next value for which $r = 0$ is $\theta = \pi/2$. This can be seen by solving the equation $3\sin(2\theta) = 0$ for θ. Therefore the values $\theta = 0$ to $\theta = \pi/2$ trace out the first petal of the rose. To find the area inside this petal, use (Equation 3.4.6) with $f(\theta) = 3\sin(2\theta), \alpha = 0$, and $\beta = \pi/2$:

$$\begin{aligned} A &= \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 d\theta \\ &= \frac{1}{2}\int_{0}^{\pi/2} [3\sin(2\theta)]^2 d\theta \\ &= \frac{1}{2}\int_{0}^{\pi/2} 9\sin^2(2\theta) d\theta. \end{aligned}$$

To evaluate this integral, use the formula $\sin^2\alpha = (1 - \cos(2\alpha))/2$ with $\alpha = 2\theta$:

$$\begin{aligned} A &= \frac{1}{2}\int_{0}^{\pi/2} 9\sin^2(2\theta) d\theta \\ &= \frac{9}{2}\int_{0}^{\pi/2} \frac{(1-\cos(4\theta))}{2} d\theta \\ &= \frac{9}{4}(\int_{0}^{\pi/2} 1 - \cos(4\theta) d\theta) \\ &= \frac{9}{4}(\theta - \frac{\sin(4\theta)}{4} \; |_{0}^{\pi/2} \\ &= \frac{9}{4}(\frac{\pi}{2} - \frac{\sin 2\pi}{4}) - \frac{9}{4}(0 - \frac{\sin 4(0)}{4}) \\ &= \frac{9\pi}{8} \end{aligned}$$

Watch the accompanying video for a visual demonstration of this concept.

- **Video Length:** 2 minutes 3 seconds.
- **Context:** This video demonstrates how to find the area of one petal of the given equation $r = 3sin(4\theta)$.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Exercise 3.4.1

Find the area inside the cardioid defined by the equation $r = 1 - \cos\theta$.

Hint available at the end of this section.

Answer available at the end of this section.

Example 3.4.1 involved finding the area inside one curve. We can also use (Equation 3.4.6) to find the area between two polar curves. However, we often need to find the points of intersection of the curves and determine which function defines the outer curve or the inner curve between these two points.

Example 3.4.2: Finding the Area between Two Polar Curves

Find the area outside the cardioid $r = 2 + 2\sin\theta$ and inside the circle $r = 6\sin\theta$.

Solution

First draw a graph containing both curves.

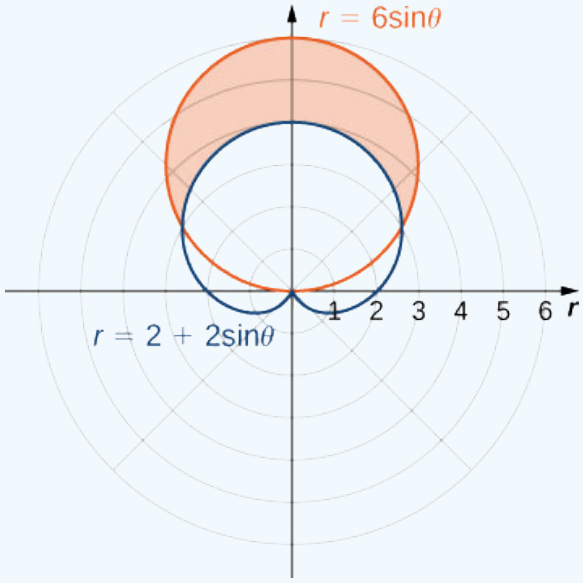


Figure 3.4.4: A cardioid with equation $r = 2 + 2\sin\theta$ is graphed, so it has its upper heart part at the pole and the rest of the cardioid is pointed up. There is a circle with radius 6 centered at $(3, \pi/2)$. The area above the cardioid but below the circle is shaded orange.

To determine the limits of integration, first find the points of intersection by setting the two functions equal to each other and solving for θ:

$$\begin{aligned} 6\sin\theta &= 2 + 2\sin\theta \\ 4\sin\theta &= 2 \\ \sin\theta &= \frac{1}{2} \end{aligned}$$

This gives the solutions $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$, which are the limits of integration. The circle $r = 3\sin\theta$ is the red graph, which is the outer function, and the cardioid $r = 2 + 2\sin\theta$ is the blue graph, which is the inner function. To calculate the area between the curves, start with the area inside the circle between $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$, then subtract the area inside the cardioid between $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$:

$$\begin{aligned} A &= \text{circle} - \text{cardioid} \\ &= \frac{1}{2}\int_{\pi/6}^{5\pi/6} [6\sin\theta]^2 d\theta - \frac{1}{2}\int_{\pi/6}^{5\pi/6} [2 + 2\sin\theta]^2 d\theta \\ &= \frac{1}{2}\int_{\pi/6}^{5\pi/6} 36\sin^2\theta \; d\theta - \frac{1}{2}\int_{\pi/6}^{5\pi/6} 4 + 8\sin\theta + 4\sin^2\theta \; d\theta \\ &= 18\int_{\pi/6}^{5\pi/6} \frac{1 - \cos(2\theta)}{2} d\theta - 2\int_{\pi/6}^{5\pi/6} 1 + 2\sin\theta + \frac{1 - \cos(2\theta)}{2} d\theta \\ &= 9[\theta - \frac{\sin(2\theta)}{2}]_{\pi/6}^{5\pi/6} - 2[\frac{3\theta}{2} - 2\cos\theta - \frac{\sin(2\theta)}{4}]_{\pi/6}^{5\pi/6} \end{aligned}$$

$$= 9(\frac{5\pi}{6} - \frac{\sin(10\pi/6)}{2}) - 9(\frac{\pi}{6} - \frac{\sin(2\pi/6)}{2}) - (3(\frac{5\pi}{6}) - 4\cos\frac{5\pi}{6} - \frac{\sin(10\pi/6)}{2}) + (3(\frac{\pi}{6}) - 4\cos\frac{\pi}{6} - \frac{\sin(2\pi/6)}{2})$$

$$= 4\pi.$$

Exercise 3.4.2

Find the area inside the circle $r = 4\cos\theta$ and outside the circle $r = 2$.

Hint available at the end of this section.

Answer available at the end of this section.

In Example 3.4.2 we found the area inside the circle and outside the cardioid by first finding their intersection points. Notice that solving the equation directly for θ yielded two solutions: $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$. However, in the graph there are three intersection points. The third intersection point is the pole. The reason why this point did not show up as a solution is because the pole is on both graphs but for different values of θ. For example, for the cardioid we get

$$\begin{aligned} 2 + 2\sin\theta &= 0 \\ \sin\theta &= -1, \end{aligned}.$$

so the values for θ that solve this equation are $\theta = \frac{3\pi}{2} + 2n\pi$, where n is any integer. For the circle we get

$$6\sin\theta = 0. \tag{3.4.7}$$

The solutions to this equation are of the form $\theta = n\pi$ for any integer value of n. These two solution sets have no points in common. Regardless of this fact, the curves intersect at the pole. This case must always be taken into consideration.

Arc Length in Polar Curves

Here we derive a formula for the arc length of a curve defined in polar coordinates. In rectangular coordinates, the arc length of a parameterized curve $(x(t), y(t))$ for $a \le t \le b$ is given by

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt. \tag{3.4.8}$$

In polar coordinates we define the curve by the equation $r = f(\theta)$, where $\alpha \le \theta \le \beta$. In order to adapt the arc length formula for a polar curve, we use the equations

$$x = r\cos\theta = f(\theta)\cos\theta \tag{3.4.9}$$

and

$$y = r\sin\theta = f(\theta)\sin\theta, \tag{3.4.10}$$

and we replace the parameter t by θ. Then

$$\frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta \tag{3.4.11}$$

$$\frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta. \tag{3.4.12}$$

We replace dt by $d\theta$, and the lower and upper limits of integration are α and β, respectively. Then the arc length formula becomes

$$
\begin{aligned}
L &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\
&= \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta \\
&= \int_\alpha^\beta \sqrt{(f'(\theta)\cos\theta - f(\theta)\sin\theta)^2 + (f'(\theta)\sin\theta + f(\theta)\cos\theta)^2}\, d\theta \\
&= \int_\alpha^\beta \sqrt{(f'(\theta))^2(\cos^2\theta + \sin^2\theta) + (f(\theta))^2(\cos^2\theta + \sin^2\theta)}\, d\theta \\
&= \int_\alpha^\beta \sqrt{(f'(\theta))^2 + (f(\theta))^2}\, d\theta \\
&= \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta
\end{aligned}
$$

Arc Length of a Curve Defined by a Polar Function

Let f be a function whose derivative is continuous on an interval $\alpha \le \theta \le \beta$. The length of the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is

$$L = \int_\alpha^\beta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta \tag{3.4.13}$$

$$= \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta. \tag{3.4.14}$$

Example 3.4.3: Finding the Arc Length of a cardioid

Find the arc length of the cardioid $r = 2 + 2\cos\theta$.

Solution

When $\theta = 0, r = 2 + 2\cos 0 = 4$. Furthermore, as θ goes from 0 to 2π, the cardioid is traced out exactly once. Therefore these are the limits of integration. Using $f(\theta) = 2 + 2\cos\theta, \alpha = 0$, and $\beta = 2\pi$, (Equation 3.4.13) becomes

$$
\begin{aligned}
L &= \int_\alpha^\beta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta \\
&= \int_0^{2\pi} \sqrt{[2 + 2\cos\theta]^2 + [-2\sin\theta]^2}\, d\theta \\
&= \int_0^{2\pi} \sqrt{4 + 8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}\, d\theta \\
&= \int_0^{2\pi} \sqrt{4 + 8\cos\theta + 4(\cos^2\theta + \sin^2\theta)}\, d\theta \\
&= \int_0^{2\pi} \sqrt{8 + 8\cos\theta}\, d\theta \\
&= 2\int_0^{2\pi} \sqrt{2 + 2\cos\theta}\, d\theta.
\end{aligned}
$$

Next, using the identity $\cos(2\alpha) = 2\cos^2\alpha - 1$, add 1 to both sides and multiply by 2. This gives $2 + 2\cos(2\alpha) = 4\cos^2\alpha$. Substituting $\alpha = \theta/2$ gives $2 + 2\cos\theta = 4\cos^2(\theta/2)$, so the integral becomes

$$L = 2\int_0^{2\pi} \sqrt{2+2\cos\theta}\, d\theta$$
$$= 2\int_0^{2\pi} \sqrt{4\cos^2(\frac{\theta}{2})}\, d\theta$$
$$= 4\int_0^{2\pi} |\cos(\frac{\theta}{2})|\, d\theta.$$

The absolute value is necessary because the cosine is negative for some values in its domain. To resolve this issue, change the limits from 0 to π and double the answer. This strategy works because cosine is positive between 0 and $\frac{\pi}{2}$. Thus,

$$L = 4\int_0^{2\pi} |\cos(\frac{\theta}{2})|\, d\theta$$
$$= 8\int_0^{\pi} \cos(\frac{\theta}{2})\, d\theta$$
$$= 8(2\sin(\frac{\theta}{2}))\,|_0^{\pi}$$
$$= 16$$

Watch the accompanying video for a visual demonstration of this concept.

- **Video Length:** 1 minute 36 seconds.
- **Context:** This video demonstrates how to find the arc length of a given polar curve $r = 1 + cos\theta, 0 \leq \theta \leq 2\pi$.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Exercise 3.4.3

Find the total arc length of $r = 3\sin\theta$.

Hint available at the end of this section.

Answer available at the end of this section.

Key Concepts

- The area of a region in polar coordinates defined by the equation $r = f(\theta)$ with $\alpha \leq \theta \leq \beta$ is given by the integral $A = \frac{1}{2}\int_\alpha^\beta [f(\theta)]^2 d\theta$.
- To find the area between two curves in the polar coordinate system, first find the points of intersection, then subtract the corresponding areas.
- The arc length of a polar curve defined by the equation $r = f(\theta)$ with $\alpha \leq \theta \leq \beta$ is given by the integral $L = \int_\alpha^\beta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} d\theta = \int_\alpha^\beta \sqrt{r^2 + (\frac{dr}{d\theta})^2} d\theta$.

Key Equations

- **Area of a region bounded by a polar curve**

$$A = \frac{1}{2}\int_{\alpha}^{\beta}[f(\theta)]^2 d\theta = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta$$

- **Arc length of a polar curve**

$$L = \int_{\alpha}^{\beta}\sqrt{[f(\theta)]^2 + [f'(\theta)]^2}d\theta = \int_{\alpha}^{\beta}\sqrt{r^2 + (\frac{dr}{d\theta})^2}d\theta$$

Glossary

Polar Coordinates

A coordinate system where each point on a plane is determined by an angle and a distance from a reference point.

Polar Equation

An equation that expresses the relationship between the radius r and the angle \(θ\) in polar coordinates.

Area in Polar Coordinates

The area enclosed by a curve in polar coordinates can be found using an integral involving the polar equation.

Arc Length in Polar Coordinates

The length of a curve in polar coordinates can be calculated using an integral involving the polar equation.

Answers

Exercise 3.4.1

Use (Equation 3.4.6). Be sure to determine the correct limits of integration before evaluating.

Exercise 3.4.1

$A = 3\pi/2$

Exercise 3.4.2

Use (Equation 3.4.6) and take advantage of symmetry.

Exercise 3.4.2

$A = \frac{4\pi}{3} + 2\sqrt{3}$

Exercise 3.4.3

Use (Equation 3.4.13). To determine the correct limits, make a table of values.

Exercise 3.4.3

$s = 3\pi$

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3.4E: Exercises for Section 3.4

Exercises 1 -13

Determine a definite integral that represents the area.

1. Region enclosed by $r = 4$
2. Region enclosed by $r = 3\sin\theta$

   Answer
   $$\frac{9}{2}\int_0^{\pi} \sin^2\theta \; d\theta$$

3. Region in the first quadrant within the cardioid $r = 1 + \sin\theta$
4. Region enclosed by one petal of $r = 8\sin(2\theta)$

   Answer
   $$\frac{3}{2}\int_0^{\pi/2} \sin^2(2\theta) \; d\theta$$

5. Region enclosed by one petal of $r = cos(3\theta)$
6. Region below the polar axis and enclosed by $r = 1 - \sin\theta$

   Answer
   $$\frac{1}{2}\int_{\pi}^{2\pi} (1 - \sin\theta)^2 \; d\theta$$

7. Region in the first quadrant enclosed by $r = 2 - \cos\theta$
8. Region enclosed by the inner loop of $r = 2 - 3\sin\theta$

   Answer
   $$\int_{\sin^{-1}(2/3)}^{\pi/2} (2 - 3\sin\theta)^2 \; d\theta$$

9. Region enclosed by the inner loop of $r = 3 - 4\cos\theta$
10. Region enclosed by $r = 1 - 2\cos\theta$ and outside the inner loop

    Answer
    $$\int_0^{\pi} (1 - 2\cos\theta)^2 \; d\theta - \int_0^{\pi/3} (1 - 2\cos\theta)^2 \; d\theta$$

11. Region common to $r = 3\sin\theta$ and $r = 2 - \sin\theta$
12. Region common to $r = 2$ and $r = 4\cos\theta$

    Answer
    $$4\int_0^{\pi/3} \; d\theta + 16\int_{\pi/3}^{\pi/2} (\cos^2\theta) \; d\theta$$

13. Region common to $r = 3\cos\theta$ and $r = 3\sin\theta$

Exercises 14 -26

Find the area of the described region.

14. Enclosed by $r = 6\sin\theta$

Answer

$9\pi \text{ units}^2$

15. Above the polar axis enclosed by $r = 2 + \sin\theta$
16. Below the polar axis and enclosed by $r = 2 - \cos\theta$

Answer

$\frac{9\pi}{4} \text{ units}^2$

17. Enclosed by one petal of $r = 4\cos(3\theta)$
18. Enclosed by one petal of $r = 3\cos(2\theta)$

Answer

$\frac{9\pi}{8} \text{ units}^2$

19. Enclosed by $r = 1 + \sin\theta$
20. Enclosed by the inner loop of $r = 3 + 6\cos\theta$

Answer

$\frac{18\pi - 27\sqrt{3}}{2} \text{ units}^2$

21. Enclosed by $r = 2 + 4\cos\theta$ and outside the inner loop
22. Common interior of $r = 4\sin(2\theta)$ and $r = 2$

Answer

$\frac{4}{3}(4\pi - 3\sqrt{3}) \text{ units}^2$

23. Common interior of $r = 3 - 2\sin\theta$ and $r = -3 + 2\sin\theta$
24. Common interior of $r = 6\sin\theta$ and $r = 3$

Answer

$\frac{3}{2}(4\pi - 3\sqrt{3}) \text{ units}^2$

25. Inside $r = 1 + \cos\theta$ and outside $r = \cos\theta$
26. Common interior of $r = 2 + 2\cos\theta$ and $r = 2\sin\theta$

Answer

$(2\pi - 4) \text{ units}^2$

Exercises 27 - 30

Find a definite integral that represents the arc length.

27. $r = 4\cos\theta$ on the interval $0 \le \theta \le \frac{\pi}{2}$
28. $r = 1 + \sin\theta$ on the interval $0 \le \theta \le 2\pi$

Answer

$$\int_0^{2\pi} \sqrt{(1 + \sin\theta)^2 + \cos^2\theta}\ d\theta$$

29. $r = 2\sec\theta$ on the interval $0 \le \theta \le \frac{\pi}{3}$
30. $r = e^\theta$ on the interval $0 \le \theta \le 1$

Answer

$$\sqrt{2}\int_0^1 e^\theta\ d\theta$$

Exercises 31 - 35

Find the length of the curve over the given interval.

31. $r = 6$ on the interval $0 \leq \theta \leq \frac{\pi}{2}$
32. $r = e^{3\theta}$ on the interval $0 \leq \theta \leq 2$

Answer

$\frac{\sqrt{10}}{3}(e^6 - 1)$ units

33. $r = 6\cos\theta$ on the interval $0 \leq \theta \leq \frac{\pi}{2}$
34. $r = 8 + 8\cos\theta$ on the interval $0 \leq \theta \leq \pi$

Answer

32 units

35. $r = 1 - \sin\theta$ on the interval $0 \leq \theta \leq 2\pi$

Exercises 36 - 40

Use the integration capabilities of a calculator to approximate the length of the curve.

36. [T] $r = 3\theta$ on the interval $0 \leq \theta \leq \frac{\pi}{2}$

Answer

6.238 units

37. [T] $r = \dfrac{2}{\theta}$ on the interval $\pi \leq \theta \leq 2\pi$
38. [T] $r = \sin^2\left(\frac{\theta}{2}\right)$ on the interval $0 \leq \theta \leq \pi$

Answer

2 units

39. [T] $r = 2\theta^2$ on the interval $0 \leq \theta \leq \pi$
40. [T] $r = \sin(3\cos\theta)$ on the interval $0 \leq \theta \leq \pi$

Answer

4.39 units

Exercises 41 - 43

Use the familiar formula from geometry to find the area of the region described and then confirm by using the definite integral.

41. $r = 3\sin\theta$ on the interval $0 \leq \theta \leq \pi$
42. $r = \sin\theta + \cos\theta$ on the interval $0 \leq \theta \leq \pi$

Answer

$A = \pi\left(\frac{\sqrt{2}}{2}\right)^2 = \frac{\pi}{2}$ units^2 and $\frac{1}{2}\int_0^{\pi}(1 + 2\sin\theta\cos\theta)\ d\theta = \frac{\pi}{2}$ units^2

43. $r = 6\sin\theta + 8\cos\theta$ on the interval $0 \leq \theta \leq \pi$

Exercises 44 - 46

Use the familiar formula from geometry to find the length of the curve and then confirm using the definite integral.

14. $r = 3\sin\theta$ on the interval $0 \le \theta \le \pi$

Answer

$C = 2\pi\left(\frac{3}{2}\right) = 3\pi$ units and $\int_0^{\pi} 3\ d\theta = 3\pi$ units

15. $r = \sin\theta + \cos\theta$ on the interval $0 \le \theta \le \pi$
16. $r = 6\sin\theta + 8\cos\theta$ on the interval $0 \le \theta \le \pi$

Answer

$C = 2\pi(5) = 10\pi$ units and $\int_0^{\pi} 10\ d\theta = 10\pi$ units

17. Verify that if $y = r\sin\theta = f(\theta)\sin\theta$ then $\frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta$.

Exercises 48 - 58

Find the slope of a tangent line to a polar curve $r = f(\theta)$. Let $x = r\cos\theta = f(\theta)\cos\theta$ and $y = r\sin\theta = f(\theta)\sin\theta$, so the polar equation $r = f(\theta)$ is now written in parametric form.

18. Use the definition of the derivative $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$ and the product rule to derive the derivative of a polar equation.

Answer

$$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

19. $r = 1 - \sin\theta;\ \ \left(\frac{1}{2}, \frac{\pi}{6}\right)$
20. $r = 4\cos\theta;\ \ \left(2, \frac{\pi}{3}\right)$

Answer

The slope is $\frac{1}{\sqrt{3}}$.

21. $r = 8\sin\theta;\ \ \left(4, \frac{5\pi}{6}\right)$
22. $r = 4 + \sin\theta;\ \ \left(3, \frac{3\pi}{2}\right)$

Answer

The slope is 0.

23. $r = 6 + 3\cos\theta;\ \ (3, \pi)$
24. $r = 4\cos(2\theta)$; tips of the leaves

Answer

At $(4, 0)$, the slope is undefined. At $\left(-4, \frac{\pi}{2}\right)$, the slope is 0.

25. $r = 2\sin(3\theta)$; tips of the leaves
26. $r = 2\theta;\ \ \left(\frac{\pi}{2}, \frac{\pi}{4}\right)$

Answer

The slope is undefined at $\theta = \frac{\pi}{4}$.

27. Find the points on the interval $-\pi \le \theta \le \pi$ at which the cardioid $r = 1 - \cos\theta$ has a vertical or horizontal tangent line.
28. For the cardioid $r = 1 + \sin\theta$, find the slope of the tangent line when $\theta = \frac{\pi}{3}$.

Answer

Slope = –1.

Exercises 59 - 62

Find the slope of the tangent line to the given polar curve at the point given by the value of θ.

59. $r = 3\cos\theta, \quad \theta = \frac{\pi}{3}$
60. $r = \theta, \quad \theta = \frac{\pi}{2}$

Answer

Slope is $\frac{-2}{\pi}$.

61. $r = \ln\theta, \quad \theta = e$
62. [T] Use technology: $r = 2 + 4\cos\theta$ at $\theta = \frac{\pi}{6}$

Answer

Calculator answer: –0.836.

Exercises 63 - 66

Find the points at which the following polar curves have a horizontal or vertical tangent line.

63. $r = 4\cos\theta$
64. $r^2 = 4\cos(2\theta)$

Answer

Horizontal tangent at $\left(\pm\sqrt{2}, \frac{\pi}{6}\right), \quad \left(\pm\sqrt{2}, -\frac{\pi}{6}\right)$.

65. $r = 2\sin(2\theta)$
66. The cardioid $r = 1 + \sin\theta$

Answer

Horizontal tangents at $\frac{\pi}{2}, \quad \frac{7\pi}{6}, \quad \frac{11\pi}{6}$.
Vertical tangents at $\frac{\pi}{6}, \quad \frac{5\pi}{6}$ and also at the pole $(0, 0)$.

67. Show that the curve $r = \sin\theta\tan\theta$ (called a **cissoid of Diocles**) has the line $x = 1$ as a vertical asymptote.

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3.5: Chapter 3 Review Exercises

Exercises 1 - 4

True or False? Justify your answer with a proof or a counterexample.

1. The rectangular coordinates of the point $\left(4, \frac{5\pi}{6}\right)$ are $\left(2\sqrt{3}, -2\right)$.
2. The equations $x = \cosh(3t), \;\; y = 2\sinh(3t)$ represent a hyperbola.

Answer

True

3. The arc length of the spiral given by $r = \dfrac{\theta}{2}$ for $0 \leq \theta \leq 3\pi$ is $\frac{9}{4}\pi^3$ units.
4. Given $x = f(t)$ and $y = g(t)$, if $\dfrac{dx}{dy} = \dfrac{dy}{dx}$, then $f(t) = g(t) + C$, where C is a constant.

Answer

False. Imagine $y = t + 1, \;\; x = -t + 1.$

Exercises 5 -8

Sketch the parametric curve and eliminate the parameter to find the Cartesian equation of the curve.

5. $x = 1 + t, \;\; y = t^2 - 1, \quad -1 \leq t \leq 1$
6. $x = e^t, \;\; y = 1 - e^{3t}, \quad 0 \leq t \leq 1$

Answer

$y = 1 - x^3$

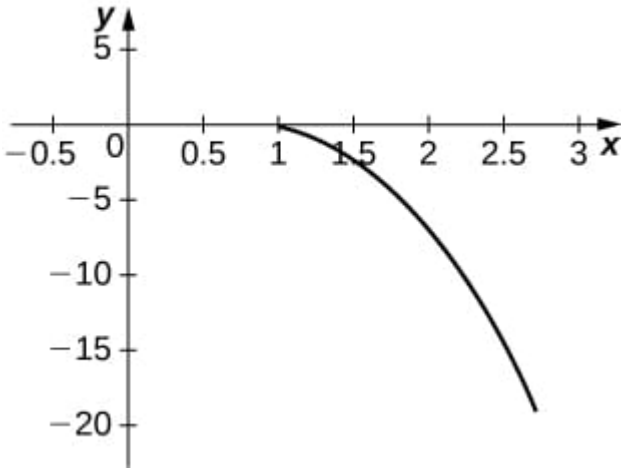


Figure 3.5.1:A graph showing a curve on a Cartesian coordinate system. The $x - axis$ ranges from -0.5 to 3, and the $y - axis$ ranges from -20 to 5. The curve starts at approximately $(1, 0)$, slopes downward, and curves to the right, ending near $(2.5, -20)$. The graph is labeled with "x" on the horizontal axis and "y" on the vertical axis.

7. $x = \sin\theta, \;\; y = 1 - \csc\theta, \quad 0 \leq \theta \leq 2\pi$
8. $x = 4\cos\phi, \;\; y = 1 - \sin\phi, \quad 0 \leq \phi \leq 2\pi$

Answer

$$\frac{x^2}{16} + (y-1)^2 = 1$$

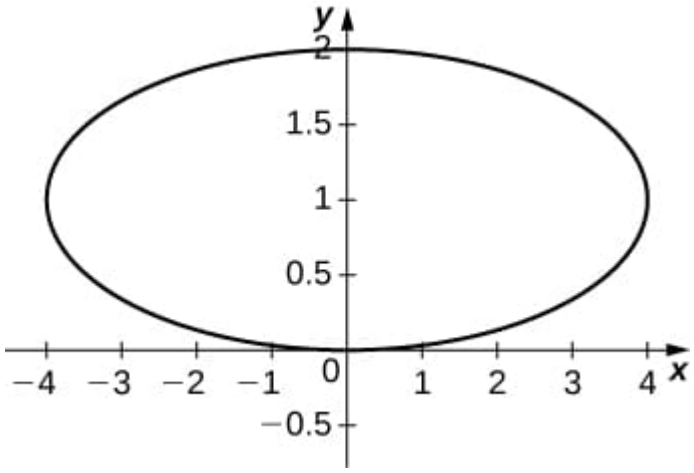


Figure 3.5.2:A graph showing an ellipse on a Cartesian coordinate system. The ellipse is centered at the origin $(0,0)$ with its major axis along the $x-axis$ and minor axis along the $y-axis$. The $x-axis$ ranges from -4 to 4, and the $y-axis$ ranges from -0.5 to 2. The ellipse stretches horizontally to approximately $(-4,0)$ and $(4,0)$ and vertically to $(0,2)$ and $(0,-0.5)$. The axes are labeled "x" (horizontal) and "y" (vertical).

Exercises 9 - 10

Sketch the polar curve and determine what type of symmetry exists, if any.

9. $r = 4\sin\left(\frac{\theta}{3}\right)$
10. $r = 5\cos(5\theta)$

Answer

Symmetric about polar axis

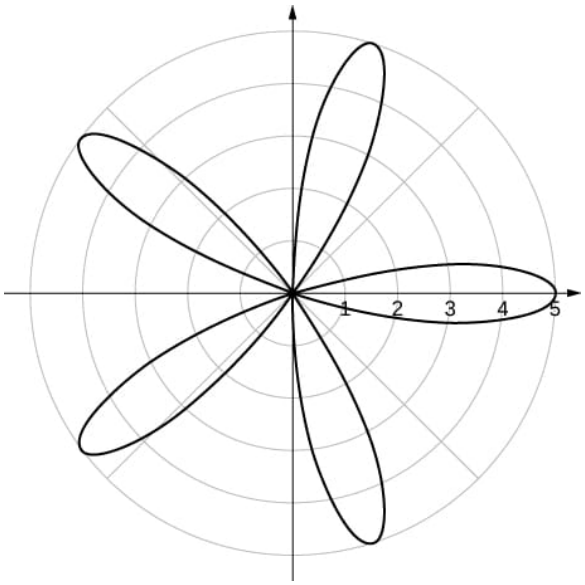


Figure 3.5.3:A polar graph displaying a four-petal rose curve. The petals are symmetric around the origin, extending outward to approximately 5 units from the origin. The petals are oriented along angles of 0°, 90°, 180°, and 270°, with smooth curves connecting them. Concentric circles and radial lines provide a grid for reference, and the radial distance is labeled numerically along the positive x-axis.

Exercises 11 - 12

Find the polar equation for the curve given as a Cartesian equation.

11. $x + y = 5$
12. $y^2 = 4 + x^2$

Answer

$$r^2 = \frac{4}{\sin^2\theta - \cos^2\theta}$$

Exercises 13 - 14

Find the equation of the tangent line to the given curve. Graph both the function and its tangent line.

13. $x = \ln(t),\ \ y = t^2 - 1,\ \ t = 1$
14. $r = 3 + \cos(2\theta),\ \ \theta = \frac{3\pi}{4}$

Answer

$$y = \frac{3\sqrt{2}}{2} + \frac{1}{5}\left(x + \frac{3\sqrt{2}}{2}\right)$$

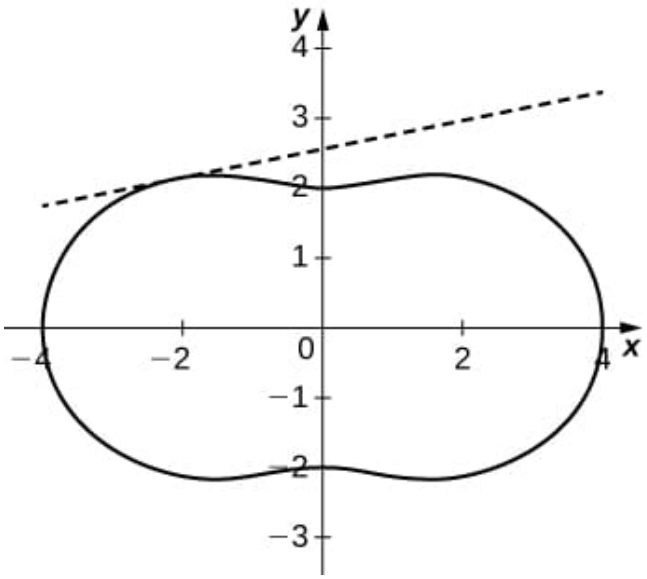


Figure 3.5.4:A graph showing a closed curve resembling a "figure-eight" or a squashed lemniscate on a Cartesian coordinate system. The curve is symmetric about the x-axis, extending horizontally to approximately -4 and 4, and vertically between -3 and 3. Above the curve, there is a dashed line sloping upward, indicating a tangent or related reference line. The x-axis and y-axis are labeled "x" and "y," respectively. The x-axis ranges from -4 to 4, and the y-axis ranges from -3 to 4.

15. Find $\frac{dy}{dx}$, $\frac{dx}{dy}$, and $\frac{d^2x}{dy^2}$ of $y = (2 + e^{-t}),\ \ x = 1 - \sin t$

Exercises 16 -17

Find the area of the region.

16. $x = t^2,\ \ y = \ln(t),\quad 0 \le t \le e$

Answer

$$\frac{e^2}{2} \text{ units}^2$$

17. $r = 1 - \sin\theta$ in the first quadrant

Exercises 18 - 19

Find the arc length of the curve over the given interval.

18. $x = 3t + 4,\ \ y = 9t - 2,\quad 0 \le t \le 3$

Answer

$9\sqrt{10}$ units

19. $r = 6\cos\theta,\quad 0 \le \theta \le 2\pi$. Check your answer by geometry.

Exercises 20 - 22

Find the Cartesian equation describing the given shapes.

20. A parabola with focus $(2, -5)$ and directrix $x = 6$

Answer

$$(y+5)^2 = -8x + 32$$

21. An ellipse with a major axis length of 10 and foci at $(-7, 2)$ and $(1, 2)$
22. A hyperbola with vertices at $(3, -2)$ and $(-5, -2)$ and foci at $(-2, -6)$ and $(-2, 4)$

Answer

$$\frac{(y+1)^2}{16} - \frac{(x+2)^2}{9} = 1$$

Exercises 23 - 25

Determine the eccentricity and identify the conic. Sketch the conic.

23. $r = \dfrac{6}{1 + 3\cos\theta}$
24. $r = \dfrac{4}{3 - 2\cos\theta}$

Answer

$e = \frac{2}{3}$, ellipse

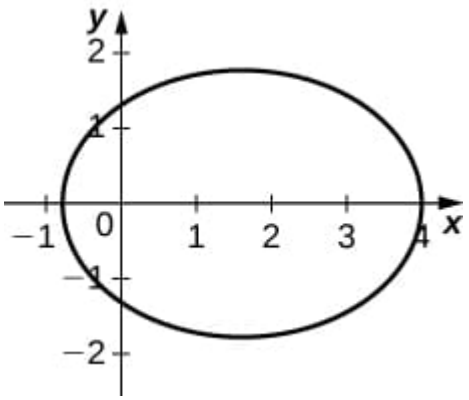


Figure 3.5.5:A graph showing an ellipse on a Cartesian coordinate system. The ellipse is centered at the origin (0, 0) with its major axis along the x-axis and minor axis along the y-axis. The x-axis ranges from -1 to 4, and the y-axis ranges from -2 to 2. The ellipse stretches horizontally to approximately (3.5, 0) and (-1, 0) and vertically to (0, 2) and (0, -2). The axes are labeled "x" (horizontal) and "y" (vertical).

25. $r = \dfrac{7}{5 - 5\cos\theta}$
26. Determine the Cartesian equation describing the orbit of Pluto, the most eccentric orbit around the Sun. The length of the major axis is 39.26 AU and minor axis is 38.07 AU. What is the eccentricity?

Answer

$$\frac{y^2}{19.03^2} + \frac{x^2}{19.63^2} = 1, \quad e = 0.2447$$

27. The C/1980 E1 comet was observed in 1980. Given an eccentricity of 1.057 and a perihelion (point of closest approach to the Sun) of 3.364 AU, find the Cartesian equations describing the comet's trajectory. Are we guaranteed to see this comet again? (Hint: Consider the Sun at point $(0, 0)$.)

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4: VECTORS IN SPACE

A quantity that has magnitude and direction is called a **vector**. Vectors have many real-life applications, including situations involving force or velocity. For example, consider the forces acting on a boat crossing a river. The boat's motor generates a force in one direction, and the current of the river generates a force in another direction. Both forces are vectors. We must take both the magnitude and direction of each force into account if we want to know where the boat will go.

Modern astronomical observatories often consist of a large number of parabolic reflectors, connected by computers, used to analyze radio waves. Each dish focuses the incoming parallel beams of radio waves to a precise focal point, where they can be synchronized by computer. If the surface of one of the parabolic reflectors is described by the equation $\frac{x^2}{100} + \frac{y^2}{100} = \frac{z}{4}$, where is the focal point of the reflector?



Figure 4.1: The Karl G. Jansky Very Large Array, located in Socorro, New Mexico, consists of a large number of radio telescopes that can collect radio waves and collate them as if they were gathering waves over a huge area with no gaps in coverage. (CC-BY-2.0 modification of work by CGP Grey, Wikimedia Commons)

We are now about to begin a new part of the calculus course, when we study functions of two or three independent variables in multidimensional space. Many of the computations are similar to those in the study of single-variable functions, but there are also a lot of differences. In this first chapter, we examine coordinate systems for working in three-dimensional space, along with vectors, which are a key mathematical tool for dealing with quantities in more than one dimension. Let's start here with the basic ideas and work our way up to the more general and powerful tools of mathematics in later chapters.

4.1: Vectors in the Plane

4.1E: Exercises for Section 4.1

4.2: Vectors in Three Dimensions

4.2E: Exercises for Section 4.2

4.3: The Dot Product

4.3E: Exercises for Section 4.3

4.4: The Cross Product

4.4E: Exercises for Section 4.4

4.5: Equations of Lines and Planes in Space

4.5E: Exercises for Section 4.5

4.6: Quadric Surfaces

4.6E: Exercises for Section 4.6

4.7: Chapter 4 Review Exercises

Contributors and Attributions

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4.1: Vectors in the Plane

Learning Objectives

- Describe a plane vector, using correct notation.
- Perform basic vector operations (scalar multiplication, addition, subtraction).
- Express a vector in component form.
- Explain the formula for the magnitude of a vector.
- Express a vector in terms of unit vectors.
- Give two examples of vector quantities.

When describing the movement of an airplane in flight, it is important to communicate two pieces of information: the direction in which the plane is traveling and the plane's speed. When measuring a force, such as the thrust of the plane's engines, it is important to describe not only the strength of that force, but also the direction in which it is applied. Some quantities, such as velocity or force, are defined in terms of both size (also called magnitude) and direction. A quantity that has magnitude and direction is called a **vector**. In this text, we denote vectors by boldface letters, such as $\mathbf{v}$. Many times, we will also include an arrow or harpoon above the boldface letter, giving us $\vec{v}$ or $\mathbf{v}$. This is how we will write vectors by hand, since it's hard to write in boldface.

Definition: Vector

A **vector** is a quantity that has both magnitude and direction.

Vector Representation

A vector in a plane is represented by a directed line segment (an arrow). The endpoints of the segment are called the **initial point** and the **terminal point** of the vector. An arrow from the initial point to the terminal point indicates the direction of the vector. The length of the line segment represents its **magnitude**. We use the notation $\|\mathbf{v}\|$ to denote the magnitude of the vector $\mathbf{v}$. A vector with an initial point and terminal point that are the same is called the **zero vector**, denoted $\mathbf{0}$. The zero vector is the only vector without a direction, and by convention can be considered to have any direction convenient to the problem at hand.

Vectors with the same magnitude and direction are called equivalent vectors. We treat equivalent vectors as equal, even if they have different initial points. Thus, if $\mathbf{v}$ and $\mathbf{w}$ are equivalent, we write

$$\mathbf{v} = \mathbf{w}. \tag{4.1.1}$$

Definition: Equivalent Vectors

Vectors are said to be **equivalent vectors** if they have the same magnitude and direction.

The arrows in Figure $4.1.1(b)$ are equivalent. Each arrow has the same length and direction. A closely related concept is the idea of parallel vectors. Two vectors are said to be parallel if they have the same or opposite directions. We explore this idea in more detail later in the chapter. A vector is defined by its magnitude and direction, regardless of where its initial point is located.

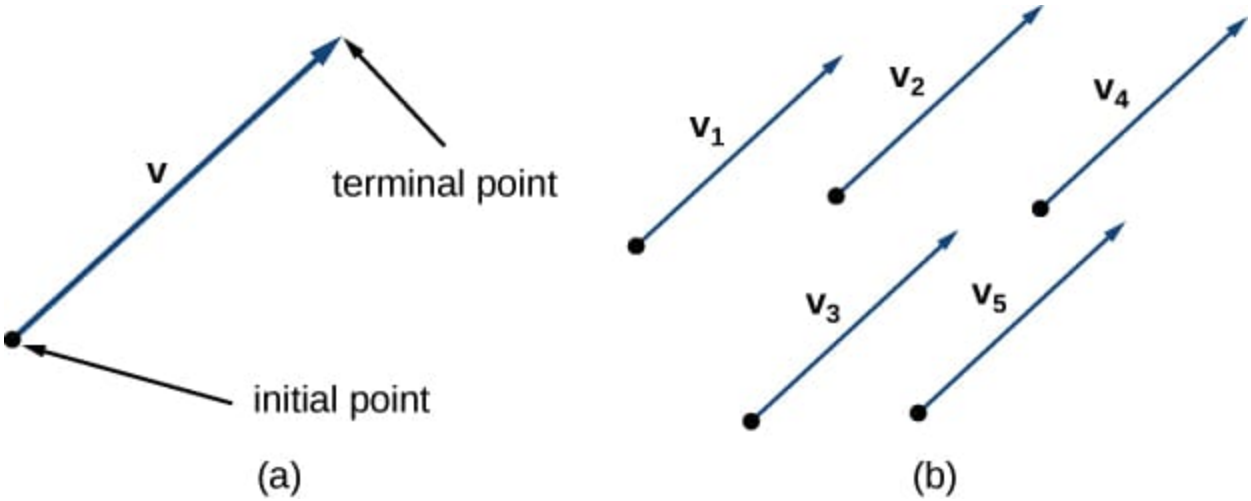


Figure 4.1.1: (a) A vector is represented by a directed line segment from its initial point to its terminal point. (b) Vectors $\mathbf{v}_1$ through $\mathbf{v}_5$ are equivalent.

The use of boldface, lowercase letters to name vectors is a common representation in print, but there are alternative notations. When writing the name of a vector by hand, for example, it is easier to sketch an arrow over the variable than to show it is a vector: $\vec{v}$. When a vector has initial point P and terminal point Q, the notation $\vec{PQ}$ is useful because it indicates the direction and location of the vector.

Example 4.1.1: Sketching Vectors

Sketch a vector in the plane from initial point $P(1,1)$ to terminal point $Q(8,5)$.

Solution

See Figure 4.1.2. Because the vector goes from point P to point Q, we name it $\vec{PQ}$.

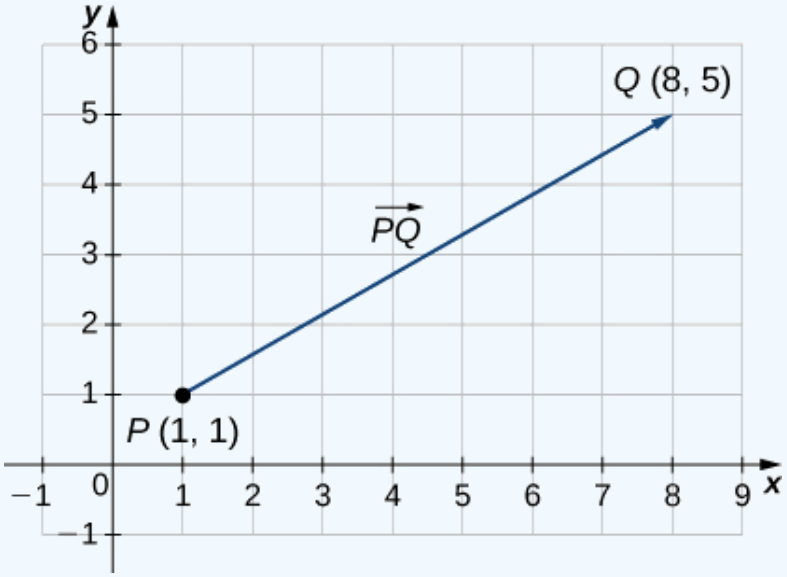


Figure 4.1.2: The vector with initial point $(1,1)$ and terminal point $(8,5)$ is named $\vec{PQ}$.

Exercise 4.1.1

Sketch the vector $\vec{ST}$ where S is point $(3,-1)$ and T is point $(-2,3)$.

Hint available at the end of this section.

Answer available at the end of this section.

Answer available at the end of this section.

Combining Vectors

Vectors have many real-life applications, including situations involving force or velocity. For example, consider the forces acting on a boat crossing a river. The boat's motor generates a force in one direction, and the current of the river generates a force in another direction. Both forces are vectors. We must take both the magnitude and direction of each force into account if we want to know where the boat will go.

A second example that involves vectors is a quarterback throwing a football. The quarterback does not throw the ball parallel to the ground; instead, he aims up into the air. The velocity of his throw can be represented by a vector. If we know how hard he throws the ball (magnitude—in this case, speed), and the angle (direction), we can tell how far the ball will travel down the field.

A real number is often called a **scalar** in mathematics and physics. Unlike vectors, scalars are generally considered to have a magnitude only, but no direction. Multiplying a vector by a scalar changes the vector's magnitude. This is called scalar multiplication. Note that changing the magnitude of a vector does not indicate a change in its direction. For example, wind blowing from north to south might increase or decrease in speed while maintaining its direction from north to south.

Definition: Scalar Multiplication

The product $k\mathbf{v}$ of a vector $\mathbf{v}$ and a scalar k is a vector with a magnitude that is $|k|$ times the magnitude of $\mathbf{v}$, and with a direction that is the same as the direction of $\mathbf{v}$ if $k > 0$, and opposite the direction of $\mathbf{v}$ if $k < 0$. This is called **scalar multiplication.** If $k = 0$ or $\mathbf{v} = \mathbf{0}$, then $k\mathbf{v} = \mathbf{0}$.

As you might expect, if $k = -1$, we denote the product $k\mathbf{v}$ as

$$k\mathbf{v} = (-1)\mathbf{v} = -\mathbf{v}. \tag{4.1.2}$$

Note that $-\mathbf{v}$ has the same magnitude as $\mathbf{v}$, but has the opposite direction (Figure $4.1.4$).

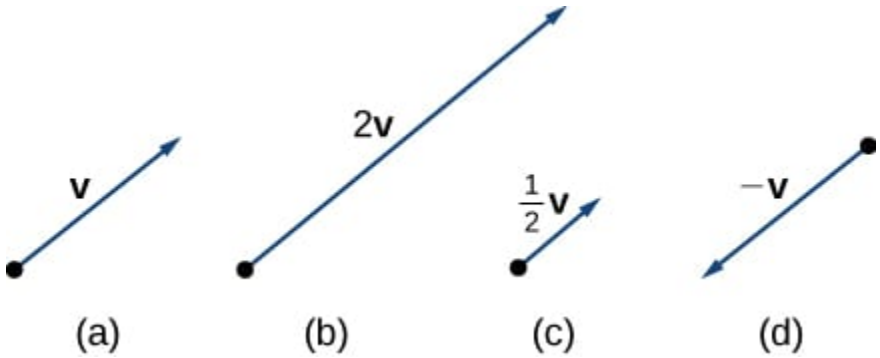


Figure $4.1.4$: (a) The original vector $\mathbf{v}$ has length n units. (b) The length of $2\mathbf{v}$ equals $2n$ units. (c) The length of $\mathbf{v}/2$ is $n/2$ units. (d) The vectors $\mathbf{v}$ and $-\mathbf{v}$ have the same length but opposite directions.

Another operation we can perform on vectors is to add them together in vector addition, but because each vector may have its own direction, the process is different from adding two numbers. The most common graphical method for adding two vectors is to place the initial point of the second vector at the terminal point of the first, as in Figure $4.1.4(a)$. To see why this makes sense, suppose, for example, that both vectors represent displacement. If an object moves first from the initial point to the terminal point of vector $\mathbf{v}$, then from the initial point to the terminal point of vector $\mathbf{w}$, the overall displacement is the same as if the object had made just one movement from the initial point to the terminal point of the vector $\mathbf{v} + \mathbf{w}$. For obvious reasons, this approach is called the **triangle method**. Notice that if we had switched the order, so that $\mathbf{w}$ was our first vector and $\mathbf{v}$ was our second vector,

we would have ended up in the same place. (Again, see Figure $4.1.4(a)$.) Thus,

$$\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}. \tag{4.1.3}$$

A second method for adding vectors is called the **parallelogram method**. With this method, we place the two vectors so they have the same initial point, and then we draw a parallelogram with the vectors as two adjacent sides, as in Figure $4.1.5(b)$. The length of the diagonal of the parallelogram is the sum. Comparing Figure $4.1.4(b)$ and Figure $4.1.5(a)$, we can see that we get the same answer using either method. The vector $\mathbf{v} + \mathbf{w}$ is called the **vector sum.**

Definition: Vector Addition

The sum of two vectors $\mathbf{v}$ and $\mathbf{w}$ can be constructed graphically by placing the initial point of $\mathbf{w}$ at the terminal point of $\mathbf{v}$. Then, the vector sum, $\mathbf{v} + \mathbf{w}$, is the vector with an initial point that coincides with the initial point of $\mathbf{v}$ and has a terminal point that coincides with the terminal point of $\mathbf{w}$. This operation is known as **vector addition.**

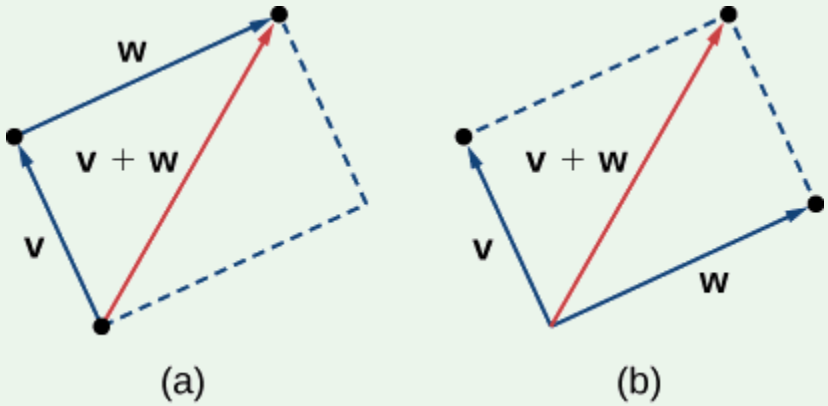


Figure $4.1.5$: (a) When adding vectors by the triangle method, the initial point of $\mathbf{w}$ is the terminal point of $\mathbf{v}$. (b) When adding vectors by the parallelogram method, the vectors$\mathbf{v}$ and $\mathbf{w}$ have the same initial point.

It is also appropriate here to discuss vector subtraction. We define $\mathbf{v} - \mathbf{w}$ as $\mathbf{v} + (-\mathbf{w}) = \mathbf{v} + (-1)\mathbf{w}$. The vector $\mathbf{v} - \mathbf{w}$ is called the **vector difference**. Graphically, the vector $\mathbf{v} - \mathbf{w}$ is depicted by drawing a vector from the terminal point of $\mathbf{w}$ to the terminal point of $\mathbf{v}$ (Figure $4.1.6$).

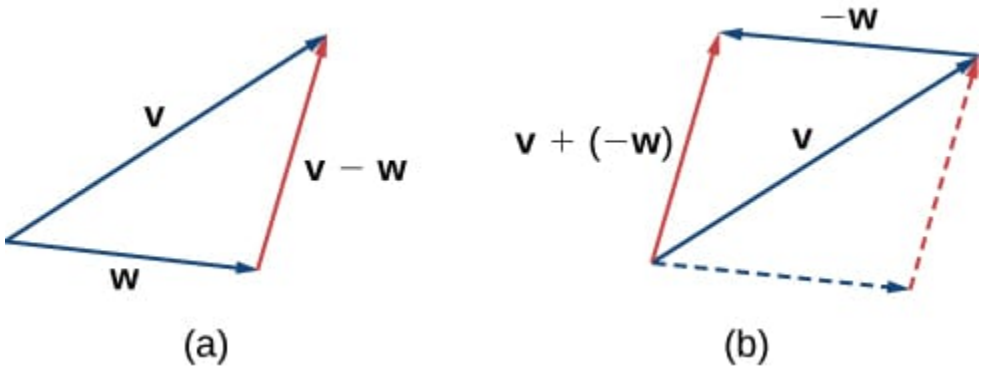


Figure $4.1.6$: (a) The vector difference $\mathbf{v} - \mathbf{w}$ is depicted by drawing a vector from the terminal point of $\mathbf{w}$ to the terminal point of $\mathbf{v}$. (b) The vector$\mathbf{v} - \mathbf{w}$ is equivalent to the vector $\mathbf{v} + (-\mathbf{w})$.

In Figure $4.1.6(a)$, the initial point of $\mathbf{v} + \mathbf{w}$ is the initial point of $\mathbf{v}$. The terminal point of $\mathbf{v} + \mathbf{w}$ is the terminal point of $\mathbf{w}$. These three vectors form the sides of a triangle. It follows that the length of any one side is less than the sum of the lengths of the remaining sides. So we have

$$\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|. \tag{4.1.4}$$

This is known more generally as the **triangle inequality**. There is one case, however, when the resultant vector $\mathbf{u} + \mathbf{v}$ has the same magnitude as the sum of the magnitudes of $\mathbf{u}$ and $\mathbf{v}$. This happens only when $\mathbf{u}$ and $\mathbf{v}$ have the same direction.

Example 4.1.2: Combining Vectors

Given the vectors v and w shown in Figure 4.1.7, sketch the vectors

a. $3\mathrm{w}$
b. $\mathrm{v}+\mathrm{w}$
c. $2\mathrm{v}-\mathrm{w}$

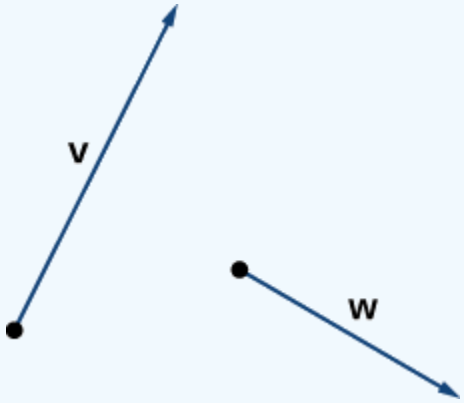


Figure 4.1.7: Vectors v *and* w *lie in the same plane.*

Solution

a. The vector $3\mathrm{w}$ has the same direction as w; it is three times as long as w.

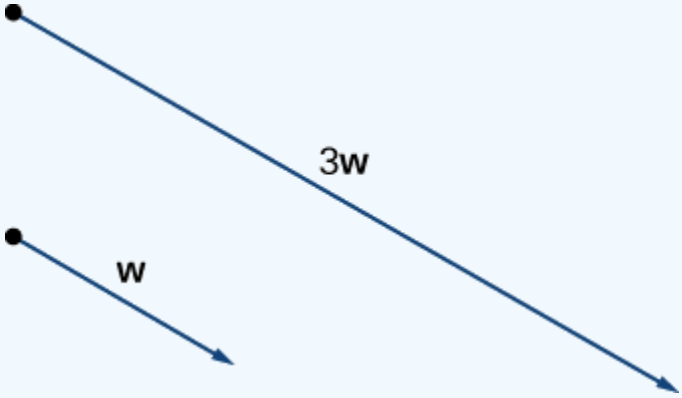


Figure 4.1.8: Vector $3\mathrm{w}$ has the same direction as w and is three times as long.

b. Use either addition method to find $\mathrm{v}+\mathrm{w}$.

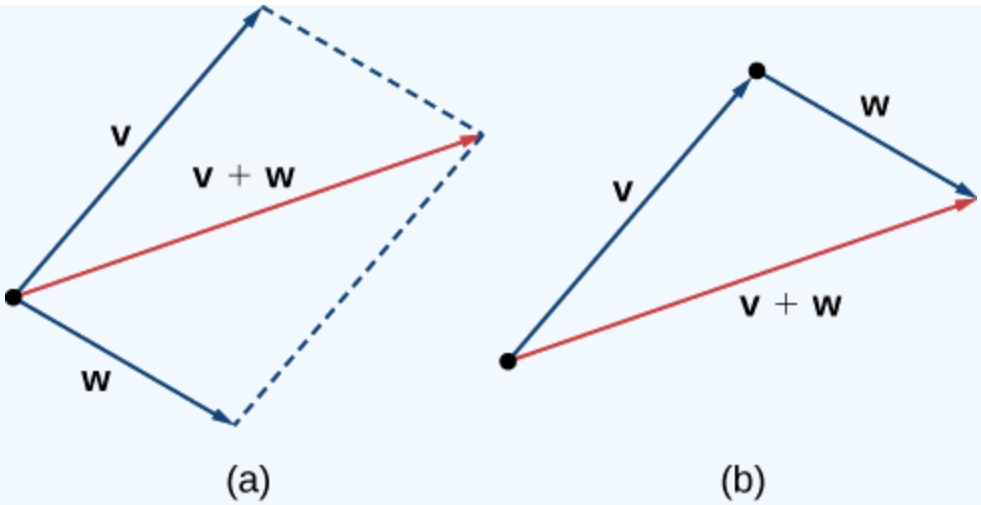


Figure 4.1.9: To find $\mathrm{v}+\mathrm{w}$, align the vectors at their initial points or place the initial point of one vector at the terminal point of the other. (a) The vector$\mathrm{v}+\mathrm{w}$ is the diagonal of the parallelogram with sides v and w. (b) The vector $\mathrm{v}+\mathrm{w}$ is the third side of a triangle formed with w placed at the terminal point of v.

c. To find $2\mathrm{v}-\mathrm{w}$, we can first rewrite the expression as $2\mathrm{v}+(-\mathrm{w})$. Then we can draw the vector $-\mathrm{w}$, then add it to the vector $2\mathrm{v}$.

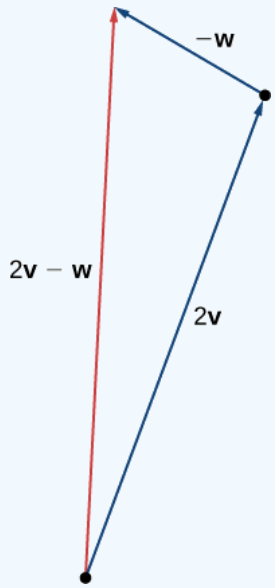


Figure 4.1.10: To find $2\mathrm{v}-\mathrm{w}$, simply add $2\mathrm{v}+(-\mathrm{w})$ using the triangular method.

Watch the accompanying video for a visual demonstration of the concept.

- **Video length:** 2 minutes 59 seconds
- **Context:** This video demonstrates the concept of arithmetic of vectors algebraically and geometrically. Find $\mathrm{u}+2\mathrm{v}$

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Exercise 4.1.2

Using vectors $\mathbf{v}$ and $\mathbf{w}$ from Example 4.1.2, sketch the vector $2\mathbf{w} - \mathbf{v}$.

Hint available at the end of this section.

Answer available at the end of this section.

Vector Components

Working with vectors in a plane is easier when we are working in a coordinate system. When the initial points and terminal points of vectors are given in Cartesian coordinates, computations become straightforward.

Example 4.1.3: Comparing Vectors

Are $\mathbf{v}$ and $\mathbf{w}$ equivalent vectors?

a.

- $\mathbf{v}$ has initial point $(3, 2)$ and terminal point $(7, 2)$
- $\mathbf{w}$ has initial point $(1, -4)$ and terminal point $(1, 0)$

b.

- $\mathbf{v}$ has initial point $(0, 0)$ and terminal point $(1, 1)$
- $\mathbf{w}$ has initial point $(-2, 2)$ and terminal point $(-1, 3)$

Solution

a. The vectors are each 4 units long, but they are oriented in different directions. So $\mathbf{v}$ and $\mathbf{w}$ are not equivalent (Figure 4.1.12).

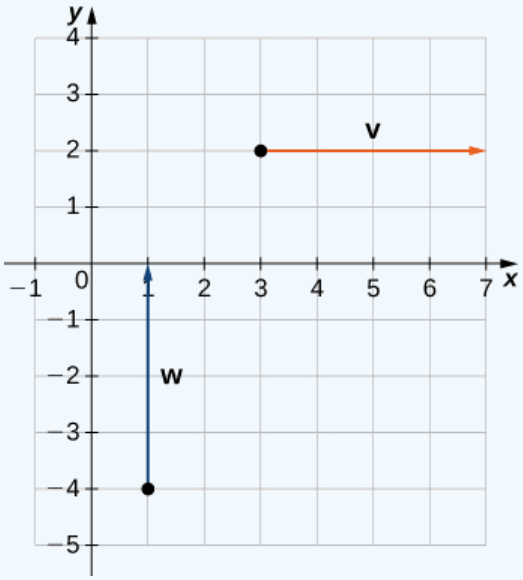


Figure 4.1.12: The blue vector $\mathbf{w}$ pointing upward and the orange vector $\mathbf{v}$ pointing to the right are not equivalent.

b. Based on Figure 4.1.13, and using a bit of geometry, it is clear these vectors have the same length and the same direction, so $\mathbf{v}$ and $\mathbf{w}$ are equivalent.

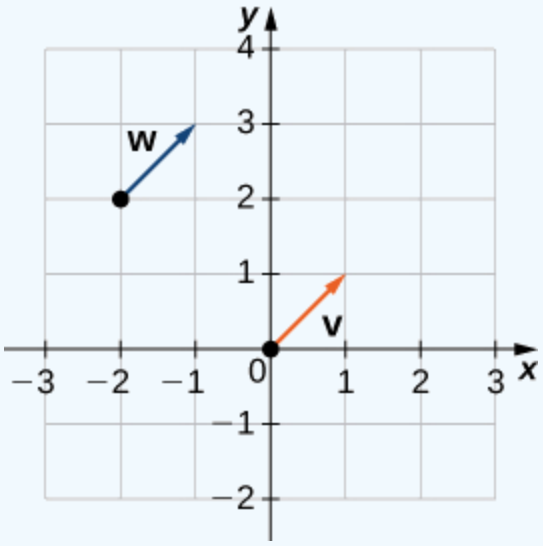


Figure 4.1.13: The blue vector $\mathbf{w}$ pointing in the same direction as the orange vector $\mathbf{v}$. They are equivalent.

Exercise 4.1.3

Which of the following vectors are equivalent?

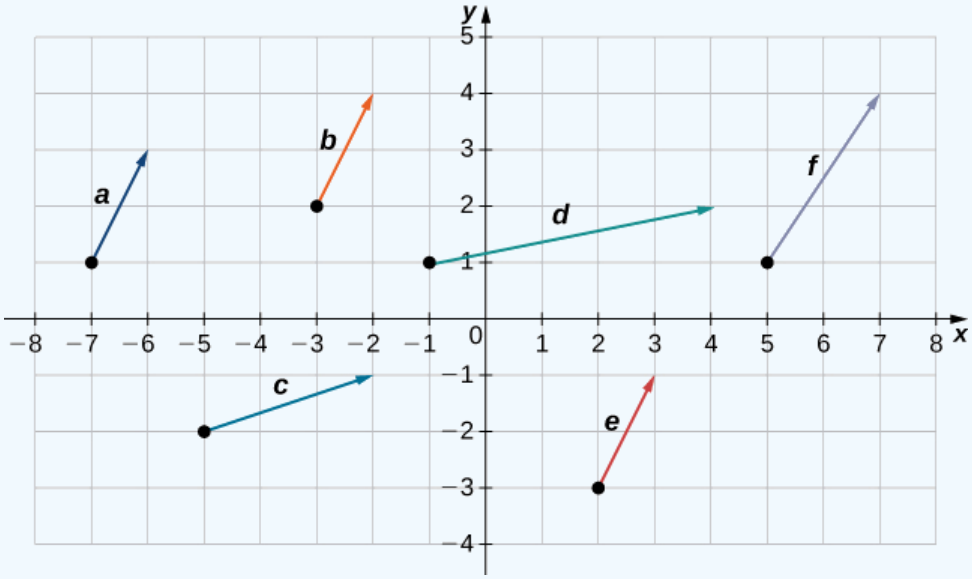


Figure 4.1.14: A list of 6 vectors to determine which ones are equivalent.

Hint available at the end of this section.

Answer available at the end of this section.

We have seen how to plot a vector when we are given an initial point and a terminal point. However, because a vector can be placed anywhere in a plane, it may be easier to perform calculations with a vector when its initial point coincides with the origin. We call a vector with its initial point at the origin a **standard-position vector.** Because the initial point of any vector in standard position is known to be $(0, 0)$, we can describe the vector by looking at the coordinates of its terminal point. Thus, if vector $\mathbf{v}$ has its initial point at the origin and its terminal point at (x, y), we write the vector in component form as

$$\mathbf{v} = \langle x, y \rangle. \tag{4.1.5}$$

When a vector is written in component form like this, the scalars x and y are called the **components** of $\mathbf{v}$.

Definition: Vector Components

The vector with initial point $(0, 0)$ and terminal point (x, y) can be written in component form as

$$\mathbf{v} = \langle x, y \rangle. \tag{4.1.6}$$

The scalars x and y are called the **components** of $\mathbf{v}$.

Recall that vectors are named with lowercase letters in bold type or by drawing an arrow over their name. We have also learned that we can name a vector by its component form, with the coordinates of its terminal point in angle brackets. However, when writing the component form of a vector, it is important to distinguish between $\langle x, y \rangle$ and (x, y). The first ordered pair uses angle brackets to describe a vector, whereas the second uses parentheses to describe a point in a plane. The initial point of $\langle x, y \rangle$ is $(0, 0)$; the terminal point of $\langle x, y \rangle$ is (x, y).

When we have a vector not already in standard position, we can determine its component form in one of two ways. We can use a geometric approach, in which we sketch the vector in the coordinate plane, and then sketch an equivalent standard-position vector. Alternatively, we can find it algebraically, using the coordinates of the initial point and the terminal point. To find it algebraically, we subtract the x-coordinate of the initial point from the x-coordinate of the terminal point to get the x-component, and we subtract the y-coordinate of the initial point from the y-coordinate of the terminal point to get the y-component.

Rule: Component Form of a Vector

Let $\mathbf{v}$ be a vector with initial point (x_i, y_i) and terminal point (x_t, y_t). Then we can express $\mathbf{v}$ in component form as $\mathbf{v} = \langle x_t - x_i, y_t - y_i \rangle$.

Example 4.1.4: Expressing Vectors in Component Form

Express vector $\mathbf{v}$ with initial point $(-3, 4)$ and terminal point $(1, 2)$ in component form.

Solution:

a. Geometric
   1. Sketch the vector in the coordinate plane (Figure 4.1.15).
   2. The terminal point is 4 units to the right and 2 units down from the initial point.
   3. Find the point that is 4 units to the right and 2 units down from the origin.
   4. In standard position, this vector has initial point $(0, 0)$ and terminal point $(4, -2)$:

$$\mathbf{v} = \langle 4, -2 \rangle.$$

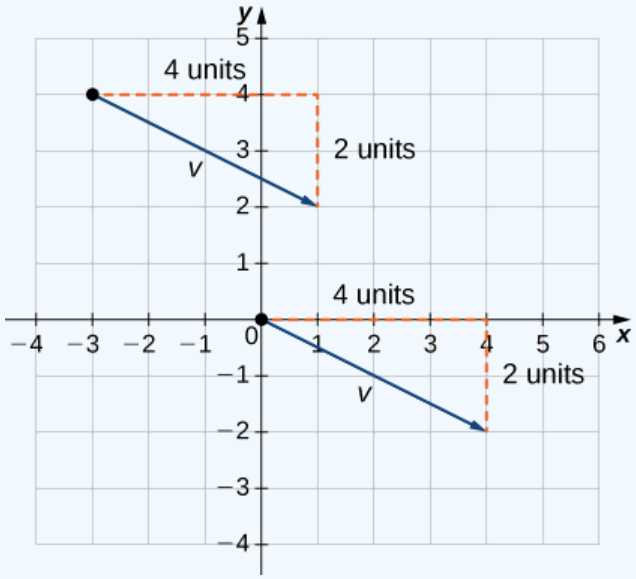


Figure 4.1.15: The first vector has initial point at the origin and terminal point at (4, -2). The horizontal distance from the initial to the terminal point for the vector is labeled as “4 units.” The vertical distance from the initial to the terminal point is labeled as “2 units.” The second vector has initial point at (-3, 4) and terminal point at (1, 2). The horizontal distance from the initial to the terminal point for the vector is labeled as “4 units.” The vertical distance from the initial to the terminal point is labeled as “2 units”. These vectors are equivalent.

b. Algebraic

In the first solution, we used a sketch of the vector to see that the terminal point lies 4 units to the right. We can accomplish this algebraically by finding the difference of the x-coordinates:

$$x_t - x_i = 1 - (-3) = 4.$$

Similarly, the difference of the y-coordinates shows the vertical length of the vector.

$$y_t - y_i = 2 - 4 = -2.$$

So, in component form,

$$\mathbf{v} = \langle x_t - x_i, y_t - y_i \rangle = \langle 1 - (-3), 2 - 4 \rangle = \langle 4, -2 \rangle.$$

Exercise 4.1.4

Vector $\mathbf{w}$ has initial point $(-4, -5)$ and terminal point $(-1, 2)$. Express $\mathbf{w}$ in component form.

Hint available at the end of this section.

Answer available at the end of this section.

To find the magnitude of a vector, we calculate the distance between its initial point and its terminal point. The **magnitude** of vector $\mathbf{v} = \langle x, y \rangle$ is denoted $\|\mathbf{v}\|$, or $|\mathbf{v}|$, and can be computed using the formula

$$\|\mathbf{v}\| = \sqrt{x^2 + y^2}. \tag{4.1.7}$$

Note that because this vector is written in component form, it is equivalent to a vector in standard position, with its initial point at the origin and terminal point (x, y). Thus, it suffices to calculate the magnitude of the vector in standard position. Using the distance formula to calculate the distance between initial point $(0, 0)$ and terminal point (x, y), we have

$$\|\mathbf{v}\| = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}. \tag{4.1.8}$$

Based on this formula, it is clear that for any vector $\mathbf{v}$, $\|\mathbf{v}\| \geq 0$, and $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = \mathbf{0}$.

The magnitude of a vector can also be derived using the Pythagorean theorem, as in the following figure.

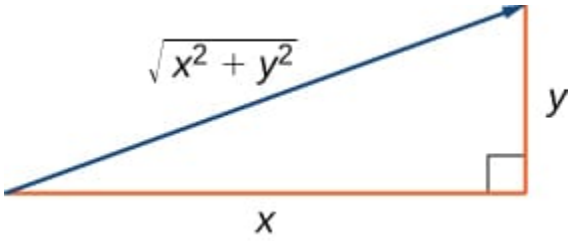


Figure 4.1.16: A right triangle with two sides label as x and y. The hypotenuse is represented as a vector and is labeled $\sqrt{x^2 + y^2}$.

We have defined scalar multiplication and vector addition geometrically. Expressing vectors in component form allows us to perform these same operations algebraically.

Definition: Scalar Multiplication and Vector Addition

Let $\mathbf{v} = \langle x_1, y_1 \rangle$ and $\mathbf{w} = \langle x_2, y_2 \rangle$ be vectors, and let k be a scalar.

- **Scalar multiplication**:

$$k\mathbf{v} = \langle kx_1, ky_1 \rangle \tag{4.1.9}$$

- **Vector addition**:

$$\mathbf{v} + \mathbf{w} = \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle = \langle x_1 + x_2, y_1 + y_2 \rangle \tag{4.1.10}$$

Example 4.1.5: Performing Operations in Component Form

Let $\mathbf{v}$ be the vector with initial point $(2, 5)$ and terminal point $(8, 13)$, and let $\mathbf{w} = \langle -2, 4 \rangle$.

a. Express $\mathbf{v}$ in component form and find $\|\mathbf{v}\|$. Then, using algebra, find
b. $\mathbf{v} + \mathbf{w}$,
c. $3\mathbf{v}$, and
d. $\mathbf{v} - 2\mathbf{w}$.

Solution

a. To place the initial point of $\mathbf{v}$ at the origin, we must translate the vector **2** units to the left and **5** units down (Figure 4.1.17). Using the algebraic method, we can express $\mathbf{v}$ as $\mathbf{v} = \langle 8 - 2, 13 - 5 \rangle = \langle 6, 8 \rangle$:

$$\|\mathbf{v}\| = \sqrt{6^2 + 8^2} = \sqrt{36 + 64} = \sqrt{100} = 10.$$

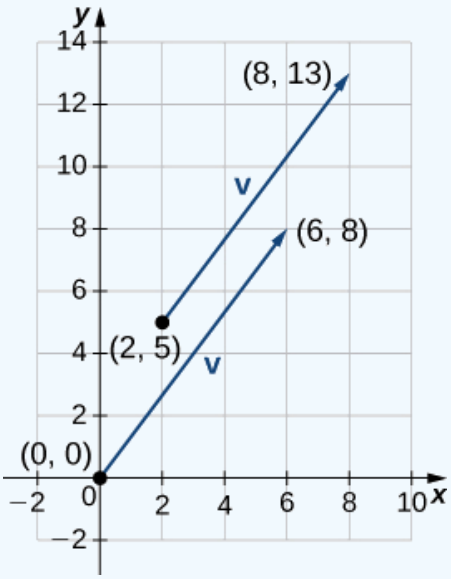


Figure 4.1.17: The first vector has initial point at (2, 5) and terminal point (8, 13). The second vector has initial point at the origin and terminal point at (6, 8). In component form, $\mathbf{v} = \langle 6, 8 \rangle$.

b. To find $\mathbf{v} + \mathbf{w}$, add the x-components and the y-components separately:

$$\mathbf{v} + \mathbf{w} = \langle 6, 8 \rangle + \langle -2, 4 \rangle = \langle 4, 12 \rangle.$$

c. To find $3\mathbf{v}$, multiply $\mathbf{v}$ by the scalar $k = 3$:

$$3\mathbf{v} = 3 \cdot \langle 6, 8 \rangle = \langle 3 \cdot 6, 3 \cdot 8 \rangle = \langle 18, 24 \rangle.$$

d. To find $\mathbf{v} - 2\mathbf{w}$, find $-2\mathbf{w}$ and add it to $\mathbf{v}$:

$$\mathbf{v} - 2\mathbf{w} = \langle 6, 8 \rangle - 2 \cdot \langle -2, 4 \rangle = \langle 6, 8 \rangle + \langle 4, -8 \rangle = \langle 10, 0 \rangle.$$

Exercise 4.1.5

Let $\mathbf{a} = \langle 7, 1 \rangle$ and let $\mathbf{b}$ be the vector with initial point $(3, 2)$ and terminal point $(-1, -1)$.

a. Find $\|\mathbf{a}\|$.
b. Express $\mathbf{b}$ in component form.
c. Find $3\mathbf{a} - 4\mathbf{b}$.

Hint available at the end of this section.

Answer a available at the end of this section.

Now that we have established the basic rules of vector arithmetic, we can state the properties of vector operations. We will prove two of these properties. The others can be proved in a similar manner.

Properties of Vector Operations

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in a plane. Let r and s be scalars.

i. *Commutative property*

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} \tag{4.1.11}$$

ii. *Associative property*

$$(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w}) \tag{4.1.12}$$

iii. *Additive identity property*

$$\mathbf{u}+\mathbf{0}=\mathbf{u} \tag{4.1.13}$$

iv. *Additive inverse property*

$$\mathbf{u}+(-\mathbf{u})=\mathbf{0} \tag{4.1.14}$$

v. *Associativity of scalar multiplication*

$$r(s\mathbf{u})=(rs)\mathbf{u} \tag{4.1.15}$$

vi. *Distributive property*

$$(r+s)\mathbf{u}=r\mathbf{u}+s\mathbf{u} \tag{4.1.16}$$

vii. *Distributive property*

$$r(\mathbf{u}+\mathbf{v})=r\mathbf{u}+r\mathbf{v} \tag{4.1.17}$$

viii. *Identity and zero properties*

$$1\mathbf{u}=\mathbf{u},\ 0\mathbf{u}=\mathbf{0} \tag{4.1.18}$$

Proof of Commutative Property

Let $\mathbf{u}=\langle x_1,y_1\rangle$ and $\mathbf{v}=\langle x_2,y_2\rangle$. Apply the commutative property (Equation 4.1.11) for real numbers:

$$\begin{aligned}\mathbf{u}+\mathbf{v}&=\langle x_1+x_2,y_1+y_2\rangle\\&=\langle x_2+x_1,y_2+y_1\rangle\\&=\mathbf{v}+\mathbf{u}.\end{aligned}$$

□

Proof of Distributive Property

Apply the distributive property (Equation 4.1.16) for real numbers:

$$\begin{aligned}r(\mathbf{u}+\mathbf{v})&=r\cdot\langle x_1+x_2,y_1+y_2\rangle\\&=\langle r(x_1+x_2),r(y_1+y_2)\rangle\\&=\langle rx_1+rx_2,ry_1+ry_2\rangle\\&=\langle rx_1,ry_1\rangle+\langle rx_2,ry_2\rangle\\&=r\mathbf{u}+r\mathbf{v}.\end{aligned}$$

We have found the components of a vector given its initial and terminal points. In some cases, we may only have the magnitude and direction of a vector, not the points. For these vectors, we can identify the horizontal and vertical components using trigonometry (Figure 4.1.18).

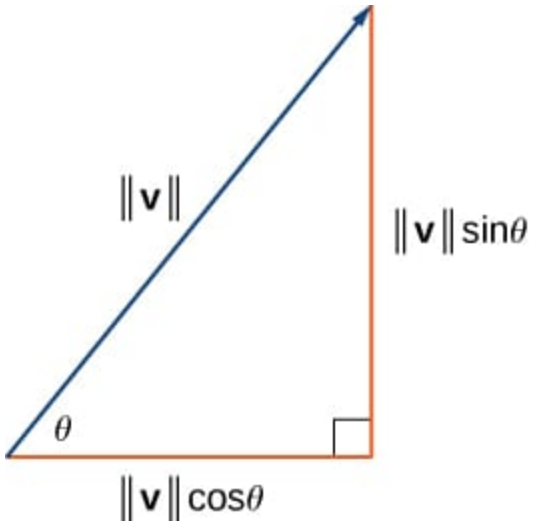


Figure 4.1.18: The components of a vector form the legs of a right triangle, with the vector as the hypotenuse. The two sides are labeled $\langle \|\mathbf{v}\| \cos\theta$ and $\|\mathbf{v}\| \sin\theta \rangle$. The hypotenuse is labeled $\|\mathbf{v}\|$.

Consider the angle θ formed by the vector $\mathbf{v}$ and the positive x-axis. We can see from the triangle that the components of vector $\mathbf{v}$ are $\langle \|\mathbf{v}\| \cos\theta, \ \|\mathbf{v}\| \sin\theta \rangle$. Therefore, given an angle and the magnitude of a vector, we can use the cosine and sine of the angle to find the components of the vector.

Example 4.1.6: Finding the Component Form of a Vector Using Trigonometry

Find the component form of a vector with magnitude 4 that forms an angle of $-45\,^\circ$ with the x-axis.

Solution

Let x and y represent the components of the vector (Figure 4.1.19). Then $x = 4\cos(-45\,^\circ) = 2\sqrt{2}$ and $y = 4\sin(-45\,^\circ) = -2\sqrt{2}$. The component form of the vector is $\langle 2\sqrt{2}, -2\sqrt{2} \rangle$.

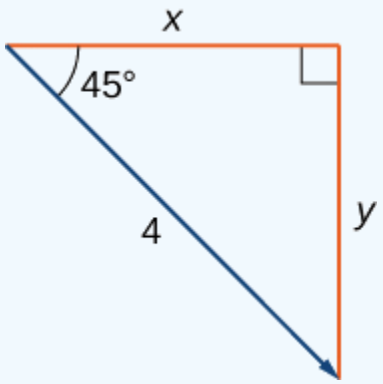


Figure 4.1.19: Use trigonometric ratios, $x = \|\mathbf{v}\| \cos\theta$ *and* $y = \|\mathbf{v}\| \sin\theta$, to identify the components of the vector.

Exercise 4.1.6

Find the component form of vector $\mathbf{v}$ with magnitude **10** that forms an angle of $120\,^\circ$ with the positive x-axis.

Hint available at the end of this section.

Unit Vectors

A **unit vector** is a vector with magnitude 1. For any nonzero vector $\mathbf{v}$, we can use scalar multiplication to find a unit vector $\mathbf{u}$ that has the same direction as $\mathbf{v}$. To do this, we multiply the vector by the reciprocal of its magnitude:

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}. \tag{4.1.19}$$

Recall that when we defined scalar multiplication, we noted that $\|k\mathbf{v}\| = |k| \cdot \|\mathbf{v}\|$. For $\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$, it follows that $\|\mathbf{u}\| = \frac{1}{\|\mathbf{v}\|}(\|\mathbf{v}\|) = 1$. We say that $\mathbf{u}$ is the *unit vector in the direction of* $\mathbf{v}$ (Figure $4.1.20$). The process of using scalar multiplication to find a unit vector with a given direction is called **normalization.**



Figure $4.1.20$: The vector $\mathbf{v}$ and associated unit vector $\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$. In this case, $\|\mathbf{v}\| > 1$.

Example $4.1.7$: Finding a Unit Vector

Let $\mathbf{v} = \langle 1, 2\rangle$.

a. Find a unit vector with the same direction as $\mathbf{v}$.
b. Find a vector $\mathbf{w}$ with the same direction as $\mathbf{v}$ such that $\|\mathbf{w}\| = 7$.

Solution:

a. First, find the magnitude of $\mathbf{v}$, then divide the components of $\mathbf{v}$ by the magnitude:

$$\|\mathbf{v}\| = \sqrt{1^2 + 2^2} = \sqrt{1+4} = \sqrt{5}$$

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}v = \frac{1}{\sqrt{5}}\langle 1, 2\rangle = \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\rangle.$$

b. The vector $\mathbf{u}$ is in the same direction as $\mathbf{v}$ and $\|\mathbf{u}\| = 1$. Use scalar multiplication to increase the length of $\mathbf{u}$ without changing direction:

$$\mathbf{w} = 7\mathbf{u} = 7\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\rangle = \langle \frac{7}{\sqrt{5}}, \frac{14}{\sqrt{5}}\rangle.$$

Watch the accompanying video for a visual demonstration of the concept.

- **Video length:** 1 minute 31 seconds.
- **Context:** This video demonstrates how to find the unit vector in the direction of a vector.

$$\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}.$$

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Exercise 4.1.7

Let $\mathbf{v} = \langle 9, 2 \rangle$. Find a vector with magnitude 5 in the opposite direction as $\mathbf{v}$.

Hint available at the end of this section.

We have seen how convenient it can be to write a vector in component form. Sometimes, though, it is more convenient to write a vector as a sum of a horizontal vector and a vertical vector. To make this easier, let's look at standard unit vectors. The **standard unit vectors** are the vectors $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$ (Figure 4.1.21).

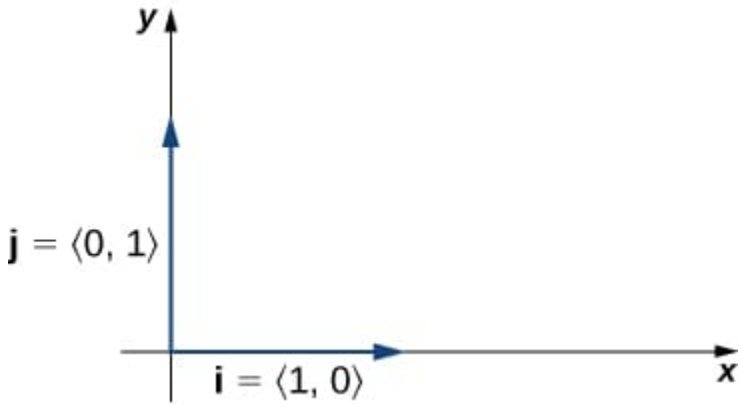


Figure 4.1.21: The standard unit vectors $\mathbf{i} = \langle 1, 0 \rangle$ is on the x-axis and $\mathbf{j} = \langle 0, 1 \rangle$ is on the y-axis.

By applying the properties of vectors, it is possible to express any vector in terms of $\mathbf{i}$ and $\mathbf{j}$ in what we call a *linear combination:*

$$\mathbf{v} = \langle x, y \rangle = \langle x, 0 \rangle + \langle 0, y \rangle = x\langle 1, 0 \rangle + y\langle 0, 1 \rangle = x\mathbf{i} + y\mathbf{j}. \tag{4.1.20}$$

Thus, $\mathbf{v}$ is the sum of a horizontal vector with magnitude x, and a vertical vector with magnitude y, as in Figure 4.1.22.

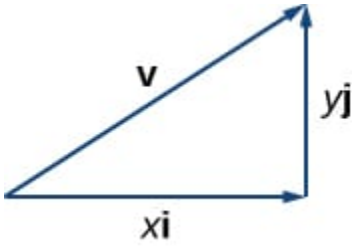


Figure 4.1.22: The vector $\mathbf{v}$ *is the sum of* $x\mathbf{i}$ *and* $y\mathbf{j}$.

Example 4.1.8: Using Standard Unit Vectors

a. Express the vector $\mathbf{w} = \langle 3, -4 \rangle$ in terms of standard unit vectors.
b. Vector $\mathbf{u}$ is a unit vector that forms an angle of $60°$ with the positive x-axis. Use standard unit vectors to describe $\mathbf{u}$.

Solution:

a. Resolve vector $\mathbf{w}$ into a vector with a zero y-component and a vector with a zero x-component:

$$\mathbf{w} = \langle 3, -4 \rangle = 3\mathbf{i} - 4\mathbf{j}.$$

b. Because $\mathbf{u}$ is a unit vector, the terminal point lies on the unit circle when the vector is placed in standard position (Figure 4.1.23).

$$\begin{aligned}\mathbf{u} &= \langle \cos 60°, \sin 60° \rangle \\ &= \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle \\ &= \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}.\end{aligned}$$

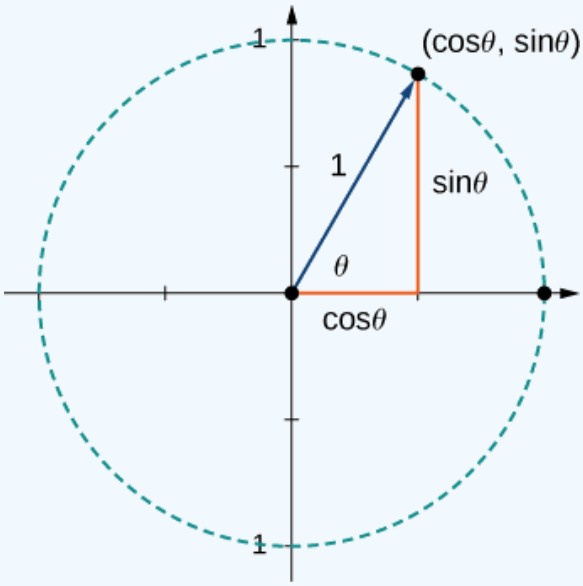


Figure 4.1.23: The terminal point of the blue vector of length 1 is $(\cos\theta, \sin\theta)$. It lies on the unit circle.

Exercise 4.1.8

Let $\mathbf{a} = \langle 16, -11 \rangle$ and let $\mathbf{b}$ be a unit vector that forms an angle of $225\,°$ with the positive x-axis. Express $\mathbf{a}$ and $\mathbf{b}$ in terms of the standard unit vectors.

Hint available at the end of this section.

Applications of Vectors

Because vectors have both direction and magnitude, they are valuable tools for solving problems involving such applications as motion and **force.** Recall the boat example and the quarterback example we described earlier. Here we look at two other examples in detail.

Example 4.1.9A: Finding Resultant Force

Jane's car is stuck in the mud. Lisa and Jed come along in a truck to help pull her out. They attach one end of a tow strap to the front of the car and the other end to the truck's trailer hitch, and the truck starts to pull. Meanwhile, Jane and Jed get behind the car and push. The truck generates a horizontal force of **300** lb on the car. Jane and Jed are pushing at a slight upward angle and generate a force of **150** lb on the car. These forces can be represented by vectors, as shown in Figure 4.1.24. The angle between these vectors is **15°**. Find the resultant force (the vector sum) and give its magnitude to the nearest tenth of a pound and its direction angle from the positive x-axis.



Figure 4.1.24: Two forces acting on a car in different directions. From the front of the car there is a horizontal vector labeled "300 pounds." Also, from the front of the automobile there is another vector labeled "150 pounds." The angle between the two vectors is 15 degrees.

Solution

To find the effect of combining the two forces, add their representative vectors. First, express each vector in component form or in terms of the standard unit vectors. For this purpose, it is easiest if we align one of the vectors with the positive x-axis. The horizontal vector, then, has initial point $(0, 0)$ and terminal point $(300, 0)$. It can be expressed as $\langle 300, 0\rangle$ or $300\mathbf{i}$.

The second vector has magnitude 150 and makes an angle of $15\,^\circ$ with the first, so we can express it as $\langle 150\cos(15\,^\circ), 150\sin(15\,^\circ)\rangle$, or $150\cos(15\,^\circ)\mathbf{i} + 150\sin(15\,^\circ)\mathbf{j}$. Then, the sum of the vectors, or resultant vector, is $\mathbf{r} = \langle 300, 0\rangle + \langle 150\cos(15\,^\circ), 150\sin(15\,^\circ)\rangle$, and we have

$$\|\mathbf{r}\| = \sqrt{(300 + 150\cos(15°))^2 + (150\sin(15°))^2} \approx 446.6.$$

The angle θ made by $\mathbf{r}$ and the positive x-axis has $\tan\theta = \dfrac{150\sin 15°}{(300 + 150\cos 15°)} \approx 0.09$, so $\theta \approx \tan^{-1}(0.09) \approx 5\,^\circ$, which means the resultant force $\mathbf{r}$ has an angle of $5\,^\circ$ above the horizontal axis.

Example $4.1.9B$: Finding Resultant Velocity

An airplane flies due west at an airspeed of 425 mph. The wind is blowing from the northeast at 40 mph. What is the ground speed of the airplane? What is the bearing of the airplane?

Solution

Let's start by sketching the situation described (Figure $4.1.25$).

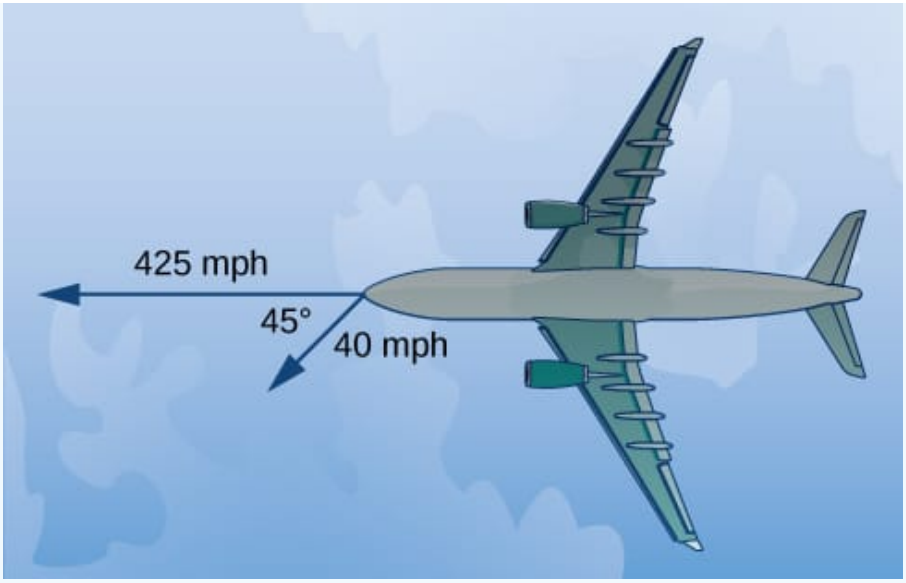


Figure 4.1.25: Initially, the plane travels due west. The wind is from the northeast, so it is blowing to the southwest. Coming out of the front of the airplane are two vectors. The first vector is labeled “425 mph” and the second vector is labeled “40 mph”. The angle between the vectors is 45 degrees. In another word, the angle between the plane’s course and the wind is $45\,^\circ$.

Set up a sketch so that the initial points of the vectors lie at the origin. Then, the plane’s velocity vector is $\mathbf{p} = -425\mathbf{i}$. The vector describing the wind makes an angle of $225\,^\circ$ with the positive x-axis:

$$\mathbf{w} = \langle 40\cos(225°), 40\sin(225°)\rangle = \langle -\frac{40}{\sqrt{2}}, -\frac{40}{\sqrt{2}}\rangle = -\frac{40}{\sqrt{2}}\mathbf{i} - \frac{40}{\sqrt{2}}\mathbf{j}.$$

When the airspeed and the wind act together on the plane, we can add their vectors to find the resultant force:

$$\mathbf{p} + \mathbf{w} = -425\mathbf{i} + (-\frac{40}{\sqrt{2}}\mathbf{i} - \frac{40}{\sqrt{2}}\mathbf{j}) = (-425 - \frac{40}{\sqrt{2}})\mathbf{i} - \frac{40}{\sqrt{2}}\mathbf{j}.$$

The magnitude of the resultant vector shows the effect of the wind on the ground speed of the airplane:

$$\|\mathbf{p} + \mathbf{w}\| = \sqrt{(-425 - \frac{40}{\sqrt{2}})^2 + (-\frac{40}{\sqrt{2}})^2} \approx 454.17\ mph$$

As a result of the wind, the plane is traveling at approximately 454 mph relative to the ground.

To determine the bearing of the airplane, we want to find the direction of the vector $\mathbf{p} + \mathbf{w}$:

$$\tan\theta = \frac{-\frac{40}{\sqrt{2}}}{(-425 - \frac{40}{\sqrt{2}})} \approx 0.06$$

$$\theta \approx 3.57\,^\circ.$$

The overall direction of the plane is $3.57\,^\circ$ south of west.

Watch the accompanying video for a visual demonstration of the concept.

- **Video length:** 4 minutes 59 seconds
- **Context:** This video demonstrates an airplane application of vector arithmetic.

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Exercise 4.1.9

An airplane flies due north at an airspeed of 550 mph. The wind is blowing from the northwest at 50 mph. What is the ground speed of the airplane?

Hint available at the end of this section.

Key Concepts

- Vectors are used to represent quantities that have both magnitude and direction.
- We can add vectors by using the parallelogram method or the triangle method to find the sum. We can multiply a vector by a scalar to change its length or give it the opposite direction.
- Subtraction of vectors is defined in terms of adding the negative of the vector.
- A vector is written in component form as $\mathbf{v} = \langle x, y \rangle$.
- The magnitude of a vector is a scalar: $\|\mathbf{v}\| = \sqrt{x^2 + y^2}$.
- A unit vector $\mathbf{u}$ has magnitude 1 and can be found by dividing a vector by its magnitude: $\mathbf{u} = \dfrac{1}{\|\mathbf{v}\|}\mathbf{v}$. The standard unit vectors are $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$. A vector $\mathbf{v} = \langle x, y \rangle$ can be expressed in terms of the standard unit vectors as $\mathbf{v} = x\mathbf{i} + y\mathbf{j}$.
- Vectors are often used in physics and engineering to represent forces and velocities, among other quantities.

Glossary

Component
A scalar that describes either the vertical or horizontal direction of a vector.

Equivalent Vectors
Vectors that have the same magnitude and the same direction.

Initial Point
The starting point of a vector.

Magnitude
The length of a vector.

Normalization
Using scalar multiplication to find a unit vector with a given direction.

Parallelogram Method
A method for finding the sum of two vectors; position the vectors so they share the same initial point; the vectors then form two adjacent sides of a parallelogram; the sum of the vectors is the diagonal of that parallelogram.

Scalar
A real number.

Scalar Multiplication
A vector operation that defines the product of a scalar and a vector.

Standard-Position Vector
A vector with initial point $(0, 0)$.

Standard Unit Vectors
Unit vectors along the coordinate axes: $\mathbf{i} = \langle 1, 0 \rangle,\ \mathbf{j} = \langle 0, 1 \rangle$.

Terminal Point
The endpoint of a vector.

Triangle Inequality
The length of any side of a triangle is less than the sum of the lengths of the other two sides.

Triangle Method
A method for finding the sum of two vectors; position the vectors so the terminal point of one vector is the initial point of the other; these vectors then form two sides of a triangle; the sum of the vectors is the vector that forms the third side; the initial point of the sum is the initial point of the first vector; the terminal point of the sum is the terminal point of the second vector.

Unit Vector
A vector with magnitude 1.

Vector
A mathematical object that has both magnitude and direction.

Vector Addition
A vector operation that defines the sum of two vectors.

Vector Difference
The vector difference $\mathbf{v} - \mathbf{w}$ is defined as $\mathbf{v} + (-\mathbf{w}) = \mathbf{v} + (-1)\mathbf{w}$.

Vector Sum
The sum of two vectors, $\mathbf{v}$ and $\mathbf{w}$, can be constructed graphically by placing the initial point of $\mathbf{w}$ at the terminal point of $\mathbf{v}$; then the vector sum $\mathbf{v} + \mathbf{w}$ is the vector with an initial point that coincides with the initial point of $\mathbf{v}$, and with a terminal point that coincides with the terminal point of $\mathbf{w}$.

Zero Vector
The vector with both initial point and terminal point $(0, 0)$.

Answers

Exercise $4.1.1$

The first point listed in the name of the vector is the initial point of the vector.

Exercise $4.1.1$

Exercise $4.1.1$

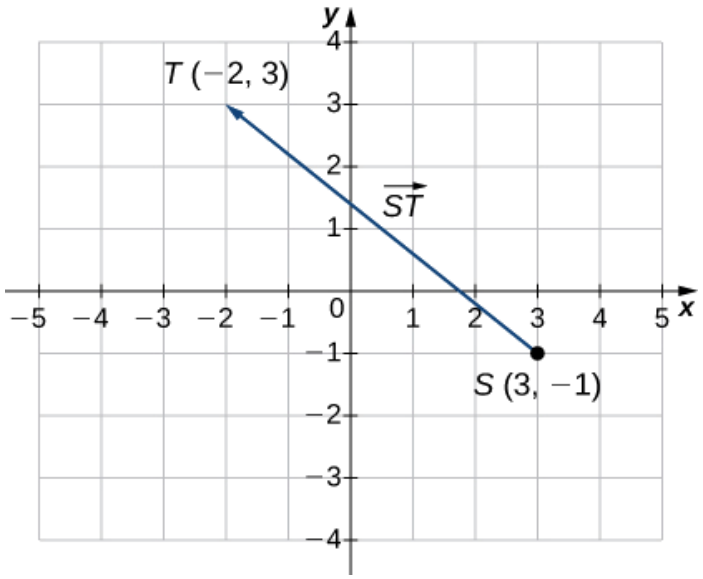


Figure 4.1.3: The vector with initial point $(3, -1)$ and terminal point $(-2, 3)$ is named $\vec{ST}$.

Exercise 4.1.2

First sketch vectors $2\mathbf{w}$ and $-\mathbf{v}$.

Exercise 4.1.2

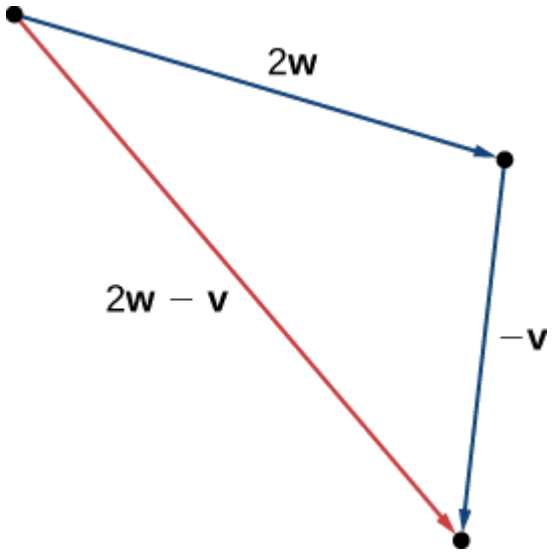


Figure 4.1.11: To find $2\mathbf{w} - \mathbf{v}$, *simply add* $2\mathbf{w} + (-\mathbf{v})$.

Exercise 4.1.3

Equivalent vectors have both the same magnitude and the same direction.

Exercise 4.1.3

Vectors $\mathbf{a}, \mathbf{b}$, and $\mathbf{e}$ are equivalent.

Exercise 4.1.4

You may use either geometric or algebraic method.

Exercise 4.1.4

$\langle 3, 7 \rangle$

Exercise 4.1.5

Use the Pythagorean Theorem to find $\|\mathbf{a}\|$. To find $3\mathbf{a} - 4\mathbf{b}$, start by finding the scalar multiples $3\mathbf{a}$ and $-4\mathbf{b}$.

Exercise 4.1.5

$\|\mathbf{a}\| = 5\sqrt{2}$,
$\mathbf{b} = \langle -4, -3 \rangle$,
$3\mathbf{a} - 4\mathbf{b} = \langle 37, 15 \rangle$

Exercise 4.1.6

$x = \|\mathbf{v}\| \cos\theta$ and $y = \|\mathbf{v}\| \sin\theta$

$v = \langle -5, 5\sqrt{3} \rangle$

Exercise 4.1.7

First, find a unit vector in the same direction as $\mathbf{v}$.

$\langle -\frac{45}{\sqrt{85}}, -\frac{10}{\sqrt{85}} \rangle$

Exercise 4.1.8

Use sine and cosine to find the components of $\mathbf{b}$.

$\mathbf{a} = 16\mathbf{i} - 11\mathbf{j}, \quad \mathbf{b} = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$

Exercise 4.1.9

Sketch the vectors with the same initial point and find their sum.

Approximately 516 mph

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4.1E: Exercises for Section 4.1

Exercises 1 - 13

Consider points $P(-1,3),\ Q(1,5),$ and $R(-3,7)$. Determine the requested vectors and express each of them

a. **in component form**
b. **by using standard unit vectors.**

1. $\vecd{PQ}$

Answer

a. $\vecd{PQ} = \langle 2, 2\rangle$
b. $\vecd{PQ} = 2\mathbf{i} + 2\mathbf{j}$

2. $\vecd{PR}$
3. $\vecd{QP}$

Answer

a. $\vecd{QP} = \langle -2, -2\rangle$
b. $\vecd{QP} = -2\mathbf{i} - 2\mathbf{j}$

4. $\vecd{RP}$
5. $\vecd{PQ} + \vecd{PR}$

Answer

a. $\vecd{PQ} + \vecd{PR} = \langle 0, 6\rangle$
b. $\vecd{PQ} + \vecd{PR} = 6\mathbf{j}$

6. $\vecd{PQ} - \vecd{PR}$
7. $2\vecd{PQ} - 2\vecd{PR}$

Answer

a. $2\vecd{PQ} \to -2\vecd{PR} = \langle 8, -4\rangle$
b. $2\vecd{PQ} - 2\vecd{PR} = 8\mathbf{i} - 4\mathbf{j}$

8. $2\vecd{PQ} + \frac{1}{2}\vecd{PR}$
9. The unit vector in the direction of $\vecd{PQ}$

Answer

a. $\left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$
b. $\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$

10. The unit vector in the direction of $\vecd{PR}$
11. A vector $\vec{\mathbf{v}}$ has initial point $(-1,-3)$ and terminal point $(2,1)$. Find the unit vector in the direction of $\mathbf{v}$. Express the answer in component form.

Answer

$\langle \frac{3}{5}, \frac{4}{5}\rangle$

12. A vector $\mathbf{v}$ has initial point $(-2,5)$ and terminal point $(3,-1)$. Find the unit vector in the direction of $\mathbf{v}$. Express the answer in component form.
13. The vector $\mathbf{v}$ has initial point $P(1,0)$ and terminal point Q that is on the y-axis and above the initial point. Find the coordinates of terminal point Q such that the magnitude of the vector $\mathbf{v}$ is $\sqrt{5}$.

Answer

$Q(0,2)$

14. The vector $\mathbf{v}$ has initial point $P(1,1)$ and terminal point Q that is on the x-axis and left of the initial point. Find the coordinates of terminal point Q such that the magnitude of the vector $\mathbf{v}$ is $\sqrt{10}$.

Exercises 15-16

Use the given vectors a and b.

a. **Determine the vector sum $\mathrm{a}+\mathrm{b}$ and express it in both the component form and by using the standard unit vectors.**
b. **Find the vector difference $\mathrm{a}-\mathrm{b}$ and express it in both the component form and by using the standard unit vectors.**
c. **Verify that the vectors a, b, and $\mathrm{a}+\mathrm{b}$, and, respectively, a, b, and $\mathrm{a}-\mathrm{b}$ satisfy the triangle inequality.**
d. **Determine the vectors $2\mathrm{a}$, $-\mathrm{b}$, and $2\mathrm{a}-\mathrm{b}$. Express the vectors in both the component form and by using standard unit vectors.**

15. $\mathbf{a}=2\mathbf{i}+\mathbf{j}, \quad \mathbf{b}=\mathbf{i}+3\mathbf{j}$

Answer

a. \mathbf a+\mathbf b=⟨3,4⟩, \quad \mathbf a+\mathbf b=3{\mathbf i}+4{\mathbf j}\)
b. \mathbf a–\mathbf b=⟨1,–2⟩, \quad \mathbf a–\mathbf b={\mathbf i}–2{\mathbf j}\)
c. Answers will vary
d. 2\mathbf a=⟨4,2⟩, \quad 2\mathbf a=4{\mathbf i}+2{\mathbf j}, \quad –\mathbf b=⟨–1,–3⟩, \quad –\mathbf b=–{\mathbf i}–3{\mathbf j}, \quad 2\mathbf a–\mathbf b=⟨3,–1⟩, \quad 2\mathbf a–\mathbf b=3{\mathbf i}–{\mathbf j}\)

16. $\mathbf{a}=2\mathbf{i}, \quad \mathbf{b}=-2\mathbf{i}+2\mathbf{j}$
17. Let $\mathbf{a}$ be a standard-position vector with terminal point $(-2,-4)$. Let $\mathbf{b}$ be a vector with initial point $(1,2)$ and terminal point $(-1,4)$. Find the magnitude of vector $-3\mathbf{a}+\mathbf{b}-4\mathbf{i}+\mathbf{j}$.

Answer

15

18. Let $\mathbf{a}$ be a standard-position vector with terminal point at $(2,5)$. Let $\mathbf{b}$ be a vector with initial point $(-1,3)$ and terminal point $(1,0)$. Find the magnitude of vector $\mathbf{a}-3\mathbf{b}+14\mathbf{i}-14\mathbf{j}$.
19. Let $\mathbf{u}$ and $\mathbf{v}$ be two nonzero vectors that are nonequivalent. Consider the vectors $\mathbf{a}=4\mathbf{u}+5\mathbf{v}$ and $\mathbf{b}=\mathbf{u}+2\mathbf{v}$ defined in terms of $\mathbf{u}$ and $\mathbf{v}$. Find the scalar λ such that vectors $\mathbf{a}+\lambda\mathbf{b}$ and $\mathbf{u}-\mathbf{v}$ are equivalent.

Answer

$\lambda=-3$

20. Let $\mathbf{u}$ and $\mathbf{v}$ be two nonzero vectors that are nonequivalent. Consider the vectors $\mathbf{a}=2\mathbf{u}-4\mathbf{v}$ and $\mathbf{b}=3\mathbf{u}-7\mathbf{v}$ defined in terms of $\mathbf{u}$ and $\mathbf{v}$. Find the scalars α and β such that vectors $\alpha\mathbf{a}+\beta\mathbf{b}$ and $\mathbf{u}-\mathbf{v}$ are equivalent.
21. Consider the vector $\mathbf{a}(t)=\langle\cos t,\sin t\rangle$ with components that depend on a real number t. As the number t varies, the components of $\mathbf{a}(t)$ change as well, depending on the functions that define them.
    a. Write the vectors $\mathbf{a}(0)$ and $\mathbf{a}(\pi)$ in component form.
    b. Show that the magnitude $\|\,\mathbf{a}(t)\,\|$ of vector $\mathbf{a}(t)$ remains constant for any real number t.
    c. As t varies, show that the terminal point of vector $\mathbf{a}(t)$ describes a circle centered at the origin of radius 1.

Answer

a. $\mathbf{a}(0)=\langle 1,0\rangle, \quad \mathbf{a}(\pi)=\langle -1,0\rangle$
b. Answers may vary
c. Answers may vary

22. Consider vector $\mathbf{a}(x)=\langle x,\sqrt{1-x^2}\rangle$ with components that depend on a real number $x\in[-1,1]$. As the number x varies, the components of $\mathbf{a}(x)$ change as well, depending on the functions that define them.

a. Write the vectors $\mathbf{a}(0)$ and $\mathbf{a}(1)$ in component form.
b. Show that the magnitude $\| \mathbf{a}(x) \|$ of vector $\mathbf{a}(x)$ remains constant for any real number x.
c. As x varies, show that the terminal point of vector $\mathbf{a}(x)$ describes a circle centered at the origin of radius 1.

23. Show that vectors $\mathbf{a}(t) = \langle \cos t, \sin t \rangle$ and $\mathbf{a}(x) = \langle x, \sqrt{1-x^2} \rangle$ are equivalent for $x = 1$ and $t = 2k\pi$, where k is an integer.

Answer Answers may vary

24. Show that vectors $\mathbf{a}(t) = \langle \cos t, \sin t \rangle$ and $\mathbf{a}(x) = \langle x, \sqrt{1-x^2} \rangle$ are opposite for $x = 1$ and $t = \pi + 2k\pi$, where k is an integer.

Exercises 25-28

Find a vector v with the given magnitude and in the same direction as the vector u.

25. $\|\mathbf{v}\| = 7, \quad \mathbf{u} = \langle 3, 4 \rangle$

Answer

$$\mathbf{v} = \langle \tfrac{21}{5}, \tfrac{28}{5} \rangle$$

26. $\|\mathbf{v}\| = 3, \quad \mathbf{u} = \langle -2, 5 \rangle$
27. $\|\mathbf{v}\| = 7, \quad \mathbf{u} = \langle 3, -5 \rangle$

Answer

$$\mathbf{v} = \langle \tfrac{21\sqrt{34}}{34}, -\tfrac{35\sqrt{34}}{34} \rangle$$

28. $\|\mathbf{v}\| = 10, \quad \mathbf{u} = \langle 2, -1 \rangle$

Exercises 29-34

Find the component form of vector u, given its magnitude and the angle the vector makes with the positive x-axis. Give exact answers when possible.

29. $\|\mathbf{u}\| = 2, \theta = 30\,^\circ$

Answer

$$\mathbf{u} = \langle \sqrt{3}, 1 \rangle$$

30. $\|\mathbf{u}\| = 6, \theta = 60\,^\circ$
31. $\|\mathbf{u}\| = 5, \theta = \frac{\pi}{2}$

Answer

$$\mathbf{u} = \langle 0, 5 \rangle$$

32. $\|\mathbf{u}\| = 8, \theta = \pi$
33. $\|\mathbf{u}\| = 10, \theta = \frac{5\pi}{6}$

Answer

$$\mathbf{u} = \langle -5\sqrt{3}, 5 \rangle$$

34. $\|\mathbf{u}\| = 50, \theta = \frac{3\pi}{4}$

Exercises 35-36

Vector u is given. Find the angle $\theta \in [0, 2\pi)$ that vector u makes with the positive direction of the x-axis, in a counter-clockwise direction.

35. $\mathbf{u} = 5\sqrt{2}\mathbf{i} - 5\sqrt{2}\mathbf{j}$

Answer

$\theta = \frac{7\pi}{4}$

36. $\mathbf{u} = -\sqrt{3}\mathbf{i} - \mathbf{j}$
37. Let $\mathbf{a} = \langle a_1, a_2 \rangle, \mathbf{b} = \langle b_1, b_2 \rangle$, and $\mathbf{c} = \langle c_1, c_2 \rangle$ be three nonzero vectors. If $a_1 b_2 - a_2 b_1 \neq 0$, then show there are two scalars, α and β, such that $\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b}$.

Answer Answers may vary

38. Consider vectors $\mathbf{a} = \langle 2, -4 \rangle, \mathbf{b} = \langle -1, 2 \rangle$, and $\mathbf{c} = \mathbf{0}$ Determine the scalars α and β such that $\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b}$.
39. Let $P(x_0, f(x_0))$ be a fixed point on the graph of the differentiable function f with a domain that is the set of real numbers.
    a. Determine the real number z_0 such that point $Q(x_0 + 1, z_0)$ is situated on the line tangent to the graph of f at point P.
    b. Determine the unit vector $\mathbf{u}$ with initial point P and terminal point Q.

Answer

$a. \quad z_0 = f(x_0) + f\,'(x_0); \quad b. \quad \mathbf{u} = \frac{1}{\sqrt{1+[f'(x_0)]^2}} \langle 1, f\,'(x_0) \rangle$

40. Consider the function $f(x) = x^4$, where $x \in R$.
    a. Determine the real number z_0 such that point $Q(2, z_0)$ s situated on the line tangent to the graph of f at point $P(1, 1)$.
    b. Determine the unit vector $\mathbf{u}$ with initial point P and terminal point Q.
41. Consider f and g two functions defined on the same set of real numbers D. Let $\mathbf{a} = \langle x, f(x) \rangle$ and $\mathbf{b} = \langle x, g(x) \rangle$ be two vectors that describe the graphs of the functions, where $x \in D$. Show that if the graphs of the functions f and g do not intersect, then the vectors $\mathbf{a}$ and $\mathbf{b}$ are not equivalent.
42. Find $x \in R$ such that vectors $\mathbf{a} = \langle x, \sin x \rangle$ and $\mathbf{b} = \langle x, \cos x \rangle$ are equivalent.
43. Calculate the coordinates of point D such that $ABCD$ is a parallelogram, with $A(1, 1), B(2, 4)$, and $C(7, 4)$.

Answer

$D(6, 1)$

44. Consider the points $A(2, 1), B(10, 6), C(13, 4)$, and $D(16, -2)$. Determine the component form of vector $\vec{AD}$.
45. The **speed** of an object is the magnitude of its related velocity vector. A football thrown by a quarterback has an initial speed of 70 mph and an angle of elevation of $30\,°$ Figure $4.1E.1$. Determine the velocity vector in mph and express it in component form. (Round to two decimal places.)

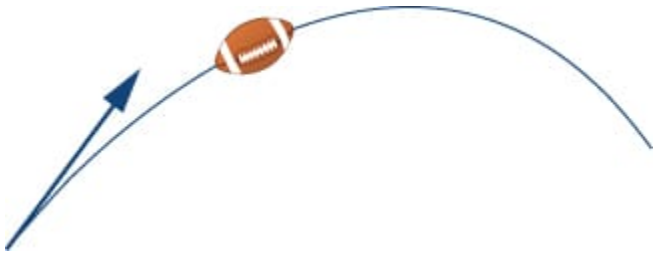


Figure $4.1E.1$: A football is following a curved trajectory, characteristic of projectile motion. The path indicates that the ball was thrown or kicked with an initial velocity, represented by a vector arrow pointing upward and to the right. The curve demonstrates the influence of gravity, causing the ball to rise, reach a peak height, and then descend. This path is a parabola, typical in physics when analyzing objects under the influence of gravity in the absence of air resistance.

Answer

$\langle 60.62, 35 \rangle$

46. A baseball player throws a baseball at an angle of $30\,°$ with the horizontal. If the initial speed of the ball is 100 mph, find the horizontal and vertical components of the initial velocity vector of the baseball. (Round to two decimal places.)
47. A bullet is fired with an initial velocity of 1500 ft/sec at an angle of $60\,°$ with the horizontal Figure $4.1E.2$. Find the horizontal and vertical components of the velocity vector of the bullet. (Round to two decimal places.)

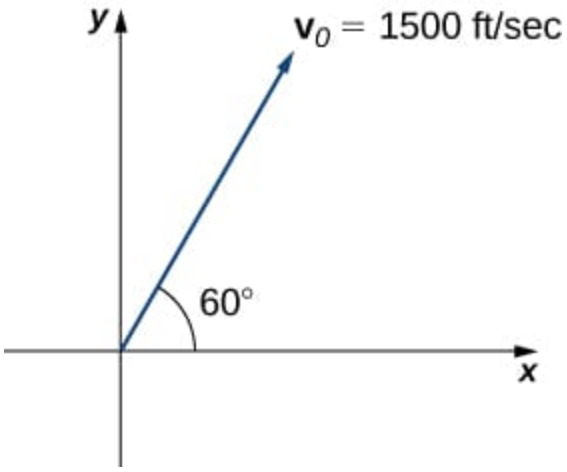


Figure $4.1E.2$: A projectile motion setup. It shows a vector originating from the origin of an xy-coordinate system, representing the initial velocity v_0 of the projectile. The vector has a magnitude of 1500 ft/sec, and is inclined at an angle of $60\,^\circ$ to the horizontal axis (x-axis). The diagram indicates the initial conditions for analyzing the projectile's trajectory, including its vertical and horizontal components.

Answer

The horizontal and vertical components are 750 ft/sec and 1299.04 ft/sec, respectively.

18. [T] A 65-kg sprinter exerts a force of 798 N at a $19\,^\circ$ angle with respect to the ground on the starting block at the instant a race begins. Find the horizontal component of the force. (Round to two decimal places.)
19. [T] Two forces, a horizontal force of 45 lb and another of 52 lb, act on the same object. The angle between these forces is $25\,^\circ$ Figure $4.1E.3$. Find the magnitude and direction angle from the positive x-axis of the resultant force that acts on the object. (Round to two decimal places.)

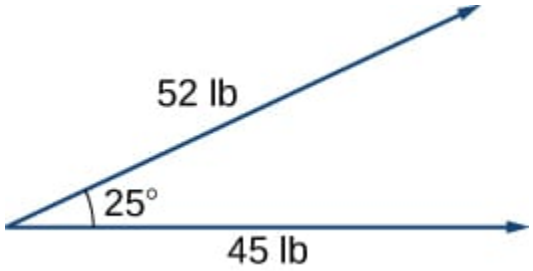


Figure $4.1E.3$: A force triangle. It shows two forces represented as vectors. A horizontal vector with a magnitude of 45 lb. A diagonal vector inclined at an angle of $25\,^\circ$ to the horizontal, with a magnitude of 52 lb.

Answer

The magnitude of resultant force is 94.71 lb; the direction angle is $13.42\,^\circ$.

50. [T] Two forces, a vertical force of 26 lb and another of 45 lb, act on the same object. The angle between these forces is $55\,^\circ$. Find the magnitude and direction angle from the positive x-axis of the resultant force that acts on the object. (Round to two decimal places.)
51. [T] Three forces act on object. Two of the forces have the magnitudes 58 N and 27 N, and make angles $53\,^\circ$ and $152\,^\circ$, respectively, with the positive x-axis. Find the magnitude and the direction angle from the positive x-axis of the third force such that the resultant force acting on the object is zero. (Round to two decimal places.)

Answer

The magnitude of the third vector is 60.03 N; the direction angle is $259.38\,^\circ$.

52. Three forces with magnitudes **80** lb, **120** lb, and **60** lb act on an object at angles of $45\,^\circ, 60\,^\circ$ and $30\,^\circ$, respectively, with the positive x-axis. Find the magnitude and direction angle from the positive x-axis of the resultant force Figure $4.1E.3$. (Round to two decimal places.)

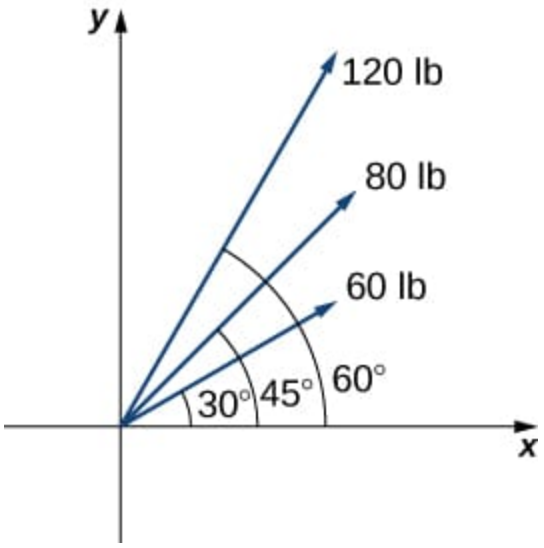


Figure $4.1E.3$: Three force vectors in a 2D coordinate system, each originating from the origin and inclined at different angles relative to the horizontal axis (x-axis). 120 lb vector: Inclined at an angle of $60\,^\circ$ to the x-axis. 80 lb vector: Inclined at an angle of $45\,^\circ$ to the x-axis. 60 lb vector: Inclined at an angle of $30\,^\circ$ to the x-axis.

53. [T] An airplane is flying in the direction of $43\,^\circ$ east of north (also abbreviated as $N43E$ at a speed of 550 mph. A wind with speed 25 mph comes from the southwest at a bearing of $N15E$. What are the ground speed and new direction of the airplane?

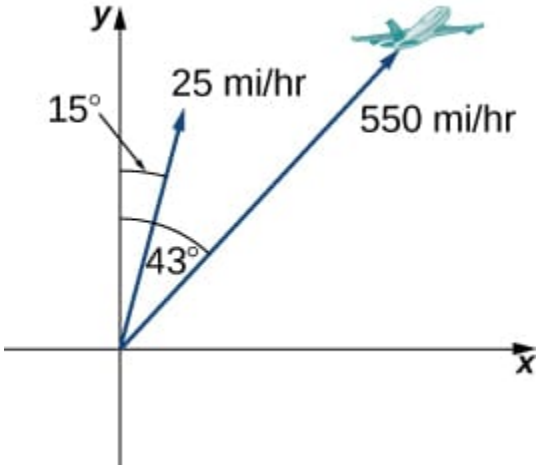


Figure $4.1E.4$: The velocity vectors of an airplane in a 2D coordinate system in quadrant 1. Wind velocity is vector with a magnitude of 25 mi/hr, inclined at an angle of $43\,^\circ$ below the y-axis. Airplane velocity relative to the air is a vector with a magnitude of 550 mi/hr, inclined at an angle of $15\,^\circ$ below the y-axis.

Answer

The new ground speed of the airplane is 572.19 mph; the new direction is $N41.82E$.

54. [T] A boat is traveling in the water at 30 mph in a direction of $N20E$ (that is, $20\,^\circ$ east of north). A strong current is moving at 15 mph in a direction of $N45E$. What are the new speed and direction of the boat?

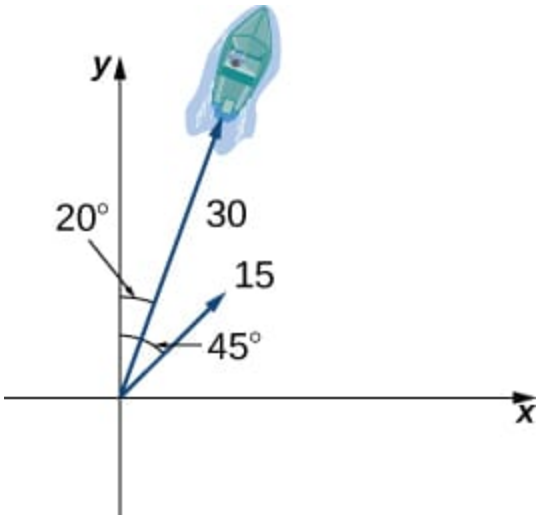


Figure $4.1E.5$: The velocity vectors for a boat navigating in a 2D coordinate system in quadrant 1. Water current velocity is a vector with a magnitude of 15, inclined at an angle of 45 ° below the y-axis. Boat velocity relative to the water is vector with a magnitude of 30, inclined at an angle of 45 ° below the y-axis.

55. [T] A 50-lb weight is hung by a cable so that the two portions of the cable make angles of 40 ° and 53 °, respectively, with the horizontal. Find the magnitudes of the forces of tension $\mathbf{T}_1$ and $\mathbf{T}_2$ in the cables if the resultant force acting on the object is zero. (Round to two decimal places.)

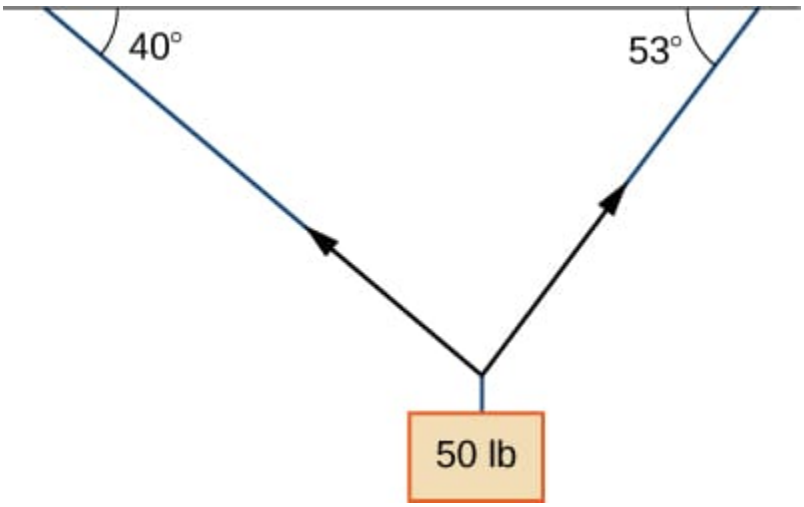


Figure $4.1E.6$: A static equilibrium setup involving a 50 lb weight suspended by two cables at angles to the horizontal. The left cable forms an angle of 40 ° with the horizontal. The right cable forms an angle of 53 ° with the horizontal. The weight of 50 lb is hanging directly downward due to gravity.

Answer

$$\|\mathbf{T}_1\| = 30.13 \ lb, \quad \|\mathbf{T}_2\| = 38.35 \ lb$$

56. [T] A 62-lb weight hangs from a rope that makes the angles of 29 ° and 61 °, respectively, with the horizontal. Find the magnitudes of the forces of tension $\mathbf{T}_1$ and $\mathbf{T}_2$ in the cables if the resultant force acting on the object is zero. (Round to two decimal places.)

57. [T] A 1500-lb boat is parked on a ramp that makes an angle of 30 ° with the horizontal. The boat's weight vector points downward and is a sum of two vectors: a horizontal vector $\mathbf{v}_1$ that is parallel to the ramp and a vertical vector $\mathbf{v}_2$ that is perpendicular to the inclined surface. The magnitudes of vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are the horizontal and vertical component, respectively, of the boat's weight vector. Find the magnitudes of $\mathbf{v}_1$ and $\mathbf{v}_2$. (Round to the nearest integer.)

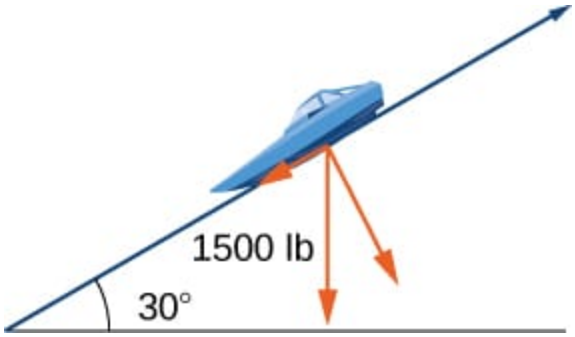


Figure $4.1E.7$: A boat on an inclined ramp with a weight of 1500 lb acting vertically downward due to gravity. The inclined plane is at an angle of $30°$ with the horizontal. The diagram also shows two force components. Parallel component is one component of the weight acts along the incline, contributing to the tendency of the boat to slide down the ramp. Perpendicular component is the other component acts perpendicular to the inclined plane, representing the normal force exerted by the surface on the boat.

Answer

$$\|\mathbf{v}_1\| = 750 \ lb, \quad \|\mathbf{v}_2\| = 1299 \ lb$$

58. [T] An 85-lb box is at rest on a $26°$ incline. Determine the magnitude of the force parallel to the incline necessary to keep the box from sliding. (Round to the nearest integer.)
59. A guy-wire supports a pole that is 75 ft high. One end of the wire is attached to the top of the pole and the other end is anchored to the ground 50 ft from the base of the pole. Determine the horizontal and vertical components of the force of tension in the wire if its magnitude is 50 lb. (Round to the nearest integer.)

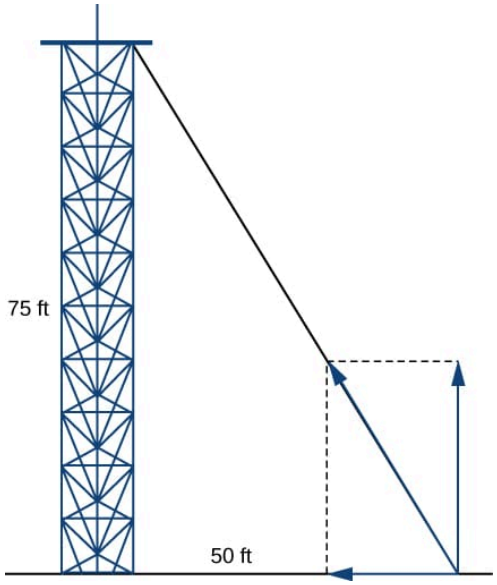


Figure $4.1E.8$: A structural setup involving a pole and a support wire. It has three key elements. Pole height is the vertical structure has a height of 75 ft. Cable length and orientation is a cable is attached to the top of the tower and anchored to the ground at a horizontal distance of 50 ft from the base of the tower. Triangle representation by the tower, the cable, and the horizontal ground form a right triangle. The hypotenuse is the cable, the vertical leg represents the tower, and the horizontal leg represents the ground distance.

Answer

The two horizontal and vertical components of the force of tension are 28 lb and 42 lb, respectively.

60. A telephone pole guy-wire has an angle of elevation of $35°$ with respect to the ground. The force of tension in the guy-wire is 120 lb. Find the horizontal and vertical components of the force of tension. (Round to the nearest integer.)

Contributors

Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

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4.2: Vectors in Three Dimensions

Learning Objectives

- Describe three-dimensional space mathematically.
- Locate points in space using coordinates.
- Write the distance formula in three dimensions.
- Write the equations for simple planes and spheres.
- Perform vector operations in $\mathbb{R}^3$.

Vectors are useful tools for solving two-dimensional problems. Life, however, happens in three dimensions. To expand the use of vectors to more realistic applications, it is necessary to create a framework for describing three-dimensional space. For example, although a two-dimensional map is a useful tool for navigating from one place to another, in some cases the topography of the land is important. Does your planned route go through the mountains? Do you have to cross a river? To appreciate fully the impact of these geographic features, you must use three dimensions. This section presents a natural extension of the two-dimensional Cartesian coordinate plane into three dimensions.

Three-Dimensional Coordinate Systems

As we have learned, the two-dimensional rectangular coordinate system contains two perpendicular axes: the horizontal x-axis and the vertical y-axis. We can add a third dimension, the z-axis, which is perpendicular to both the x-axis and the y-axis. We call this system the three-dimensional rectangular coordinate system. It represents the three dimensions we encounter in real life.

Definition: Three-dimensional Rectangular Coordinate System

The **three-dimensional rectangular coordinate system** consists of three perpendicular axes: the x-axis, the y-axis, and the z-axis. Because each axis is a number line representing all real numbers in $\mathbb{R}$, the three-dimensional system is often denoted by $\mathbb{R}^3$.

In Figure $4.2.1a$, the positive z-axis is shown above the plane containing the x- and y-axes. The positive x-axis appears to the left and the positive y-axis is to the right. A natural question to ask is: How was this arrangement determined? The system displayed follows the **right-hand rule**. If we take our right hand and align the fingers with the positive x-axis, then curl the fingers so they point in the direction of the positive y-axis, our thumb points in the direction of the positive z-axis (Figure $4.2.1b$). In this text, we always work with coordinate systems set up in accordance with the right-hand rule. Some systems do follow a left-hand rule, but the right-hand rule is considered the standard representation.

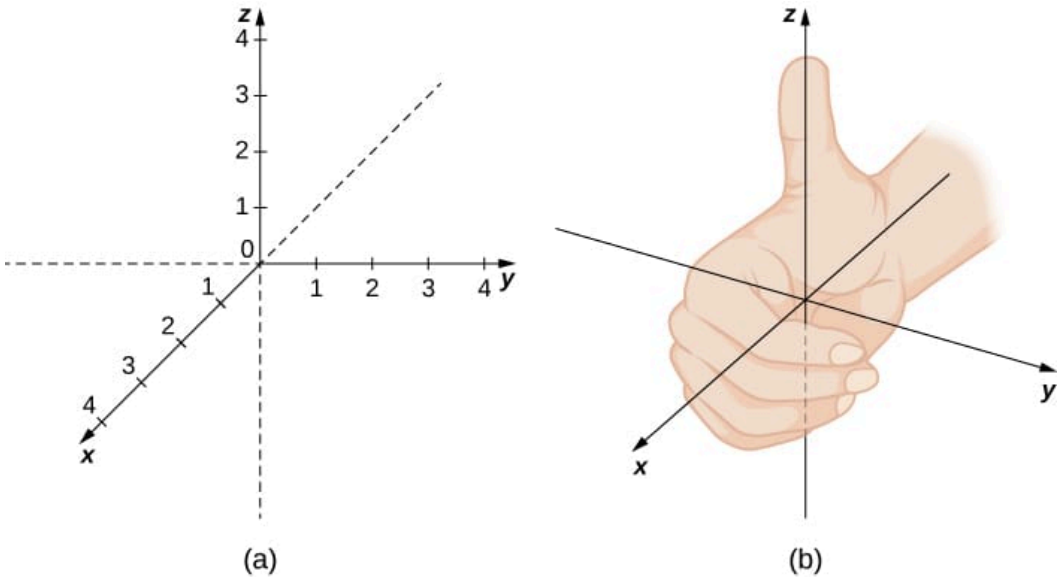


Figure 4.2.1: (a) Extension of the two-dimensional rectangular coordinate system by adding a third axis, the z-axis, that is perpendicular to both the x-axis and the y-axis. (b) The right-hand rule is used to determine the placement of the coordinate axes in the standard Cartesian plane.

In two dimensions, we describe a point in the plane with the coordinates (x, y). Each coordinate describes how the point aligns with the corresponding axis. In three dimensions, a new coordinate, z, is appended to indicate alignment with the z-axis: (x, y, z) . A point in space is identified by all three coordinates (Figure 4.2.2). To plot the point (x, y, z), go x units along the x-axis, then y units in the direction of the y-axis, then z units in the direction of the z-axis.

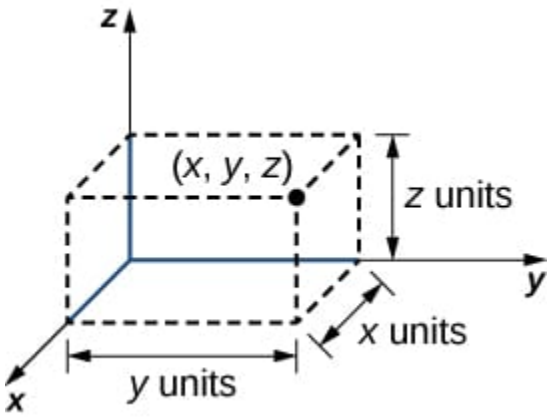


Figure 4.2.2: To plot the point (x, y, z) go x units along the x-axis, then y units in the direction of the y-axis, then z units in the direction of the z -axis.

Interactive Element

Sketching the Vector <2,1,3>

Below is an example of sketching the vector <2,1,3>. Notice that to draw <2,1,3> you start at the origin, (0,0,0), move to (2,0,0), then move to (2,1,0), and then finally arrive at the point (2,1,3). The interactive app is unfortunately not accessible.

Start Drawing

Example 4.2.1: Locating Points in Space

Sketch the point $(1, -2, 3)$ in three-dimensional space.

Solution

To sketch a point, start by sketching three sides of a rectangular prism along the coordinate axes: one unit in the positive x direction, 2 units in the negative y direction, and 3 units in the positive z direction. Complete the prism to plot the point (Figure 4.2.3).

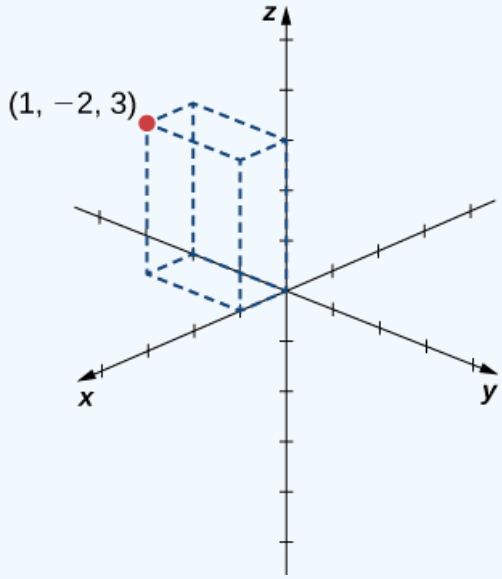


Figure 4.2.3: A 3D coordinate system with a point labeled $(1, -2, 3)$. To plot the point, go 1 unit from the origin in the direction of the x-axis, -2 units from the origin in the direction of the y-axis, and 3 units from the origin in the direction of z-axis.

Watch the accompanying video for a visual demonstration of the concept.

- **Video length:** 2 minutes 01 seconds.
- **Context:** This video demonstartes how to plot a point $(2, -1, 3)$ in the xyz-space.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Exercise 4.2.1

Sketch the point $(-2, 3, -1)$ in three-dimensional space.

Hint available at the end of this section.

In two-dimensional space, the coordinate plane is defined by a pair of perpendicular axes. These axes allow us to name any location within the plane. In three dimensions, we define **coordinate planes** by the coordinate axes, just as in two dimensions. There are three axes now, so there are three intersecting pairs of axes. Each pair of axes forms a coordinate plane: the xy-plane, the xz-plane, and the yz-plane (Figure 4.2.5). We define the xy-plane formally as the following set: $\{(x, y, 0) : x, y \in \mathbb{R}\}$. Similarly, the xz-plane and the yz-plane are defined as $\{(x, 0, z) : x, z \in \mathbb{R}\}$ and $\{(0, y, z) : y, z \in \mathbb{R}\}$, respectively.

To visualize this, imagine you're building a house and are standing in a room with only two of the four walls finished. (Assume the two finished walls are adjacent to each other.) If you stand with your back to the corner where the two finished walls meet, facing out into the room, the floor is the xy-plane, the wall to your right is the xz-plane, and the wall to your left is the yz-plane.

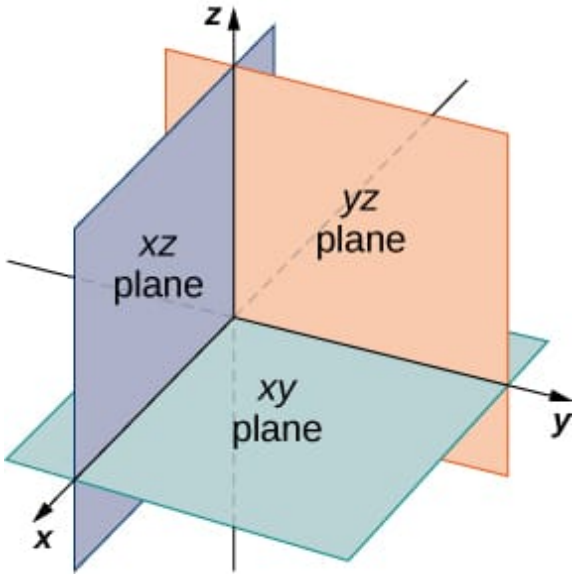


Figure $4.2.5$: The plane containing the x- and y-axes is called the xy-plane. The plane containing the x- and z-axes is called the xz-plane, and the y- and z-axes define the yz-plane.

In two dimensions, the coordinate axes partition the plane into four quadrants. Similarly, the coordinate planes divide space between them into eight regions about the origin, called **octants.** The octants fill $\mathbb{R}^3$ in the same way that quadrants fill $\mathbb{R}^2$, as shown in Figure $4.2.6$.

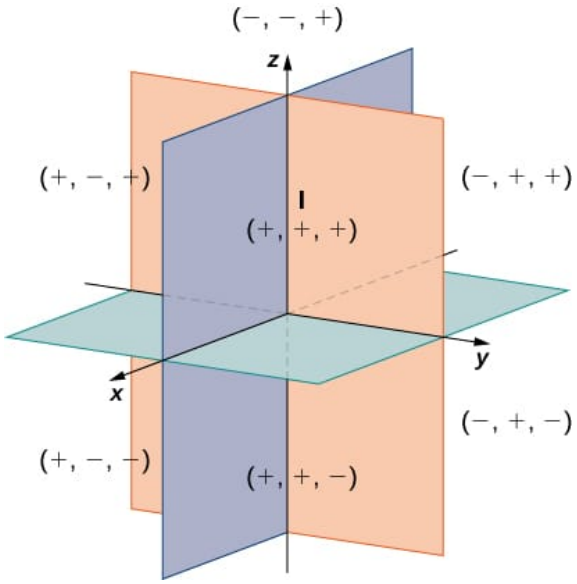


Figure $4.2.6$: 3D coordinate system with eight octants highlighted. Points that lie in octants have three nonzero coordinates. Also, for each quadrant there are the signs of the values of x, y, and z. They are: I (+, +, +); 2nd (-, +, +); 3rd (-, -, +); 4th (+, -, +); 5th (+, +, -); 6th (-, +, -); 7th (-, -, -); and 8th (+, -, -).

Most work in three-dimensional space is a comfortable extension of the corresponding concepts in two dimensions. In this section, we use our knowledge of circles to describe spheres, then we expand our understanding of vectors to three dimensions. To accomplish these goals, we begin by adapting the distance formula to three-dimensional space.

If two points lie in the same coordinate plane, then it is straightforward to calculate the distance between them. We know that the distance d between two points (x_1, y_1) and (x_2, y_2) in the xy-coordinate plane is given by the formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \tag{4.2.1}$$

The formula for the distance between two points in space is a natural extension of this formula.

The Distance between Two Points in Space

The distance d between points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by the formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}. \tag{4.2.2}$$

The proof of this theorem is left as an exercise. (Hint: First find the distance d_1 between the points (x_1, y_1, z_1) and (x_2, y_2, z_1) as shown in Figure 4.2.7.)

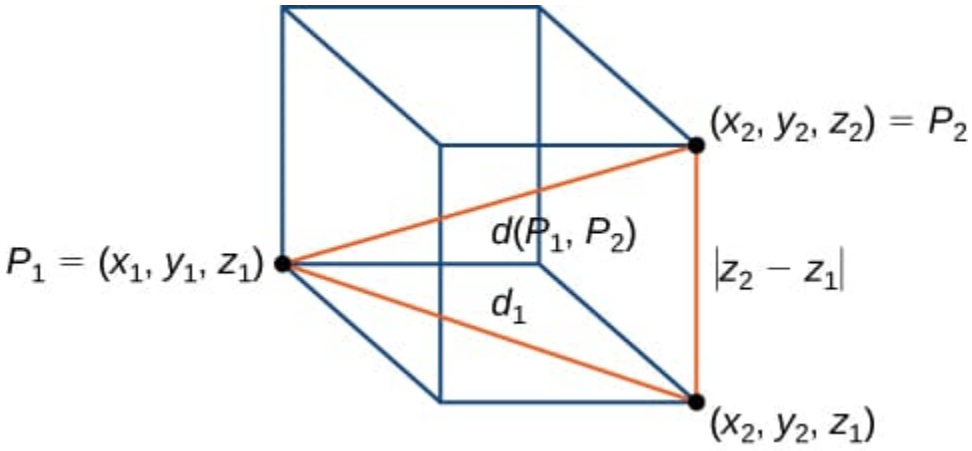


Figure 4.2.7: The distance between P_1 and P_2 is the length of the diagonal of the rectangular prism having $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_1)$ as opposite corners.

Example 4.2.2: Distance in Space

Find the distance between points $P_1 = (3, -1, 5)$ and $P_2 = (2, 1, -1)$.

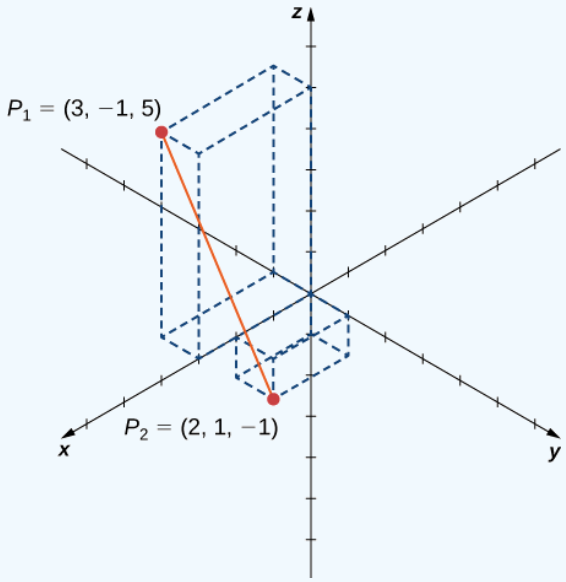


Figure 4.2.8: A 3D coordinate system with two points labeled $P_1 = (3, -1, 5)$ and $P_2 = (2, 1, -1)$ connected by an organe line segment.

Solution

Substitute values directly into the distance formula (Equation 4.2.2):

$$\begin{aligned} d(P_1, P_2) &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \\ &= \sqrt{(2-3)^2 + (1-(-1))^2 + (-1-5)^2} \\ &= \sqrt{(-1)^2 + 2^2 + (-6)^2} \\ &= \sqrt{41}. \end{aligned}$$

Exercise 4.2.2

Find the distance between points $P_1 = (1, -5, 4)$ and $P_2 = (4, -1, -1)$.

Hint available at the end of this section.

Before moving on to the next section, let's get a feel for how $\mathbb{R}^3$ differs from $\mathbb{R}^2$. For example, in $\mathbb{R}^2$, lines that are not parallel must always intersect. This is not the case in $\mathbb{R}^3$. For example, consider the line shown in Figure 4.2.9. These two lines are not parallel, nor do they intersect.

Figure 4.2.9: *The red line and blue line are not parallel, but still do not intersect.*

You can also have circles that are interconnected but have no points in common, as in Figure 4.2.10.

Figure 4.2.10: The red circle and blue circle are interconnected, but have no points in common.

We have a lot more flexibility working in three dimensions than we do if we stuck with only two dimensions.

Writing Equations in $\mathbb{R}^3$

Now that we can represent points in space and find the distance between them, we can learn how to write equations of geometric objects such as lines, planes, and curved surfaces in $\mathbb{R}^3$. First, we start with a simple equation. Compare the graphs of the equation $x = 0$ in $\mathbb{R}$, $\mathbb{R}^2$,and $\mathbb{R}^3$ (Figure 4.2.11). From these graphs, we can see the same equation can describe a point, a line, or a plane.

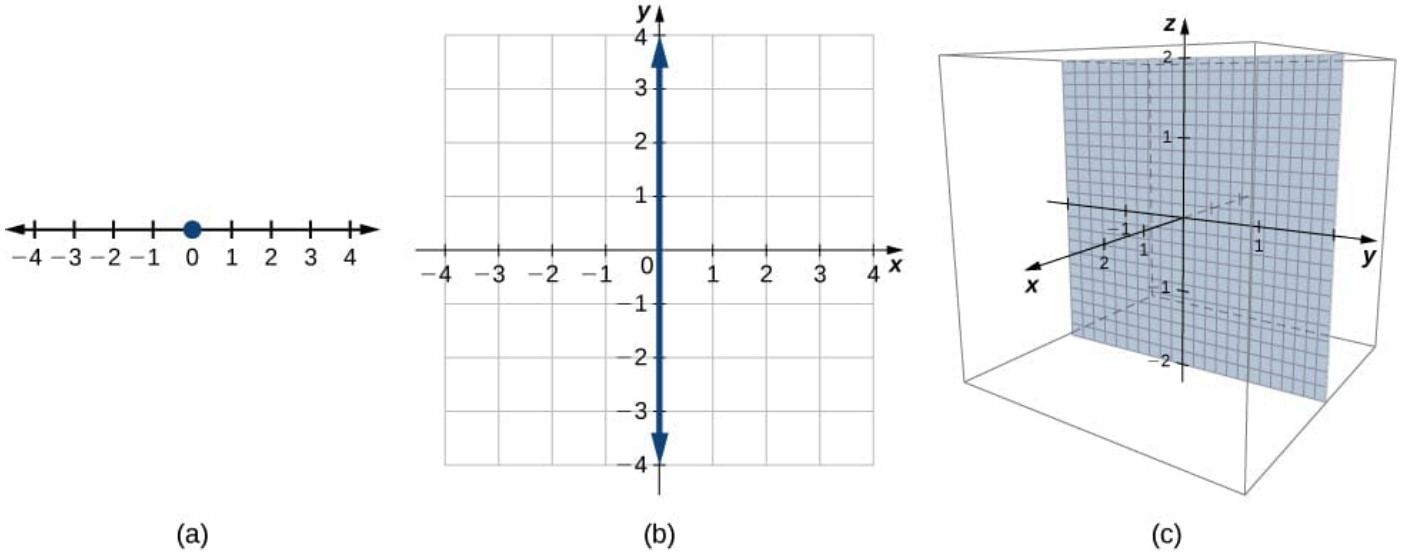


Figure 4.2.11: (a) In $\mathbb{R}$, the equation $x = 0$ describes a single point. (b) In $\mathbb{R}^2$, the equation $x = 0$ describes a line, the y-axis. (c) In $\mathbb{R}^3$, the equation $x = 0$ describes a plane, the yz-plane.

In space, the equation $x = 0$ describes all points $(0, y, z)$. This equation defines the yz-plane. Similarly, the xy-plane contains all

points of the form $(x, y, 0)$. The equation $z = 0$ defines the xy-plane and the equation $y = 0$ describes the xz-plane (Figure $4.2.12$).

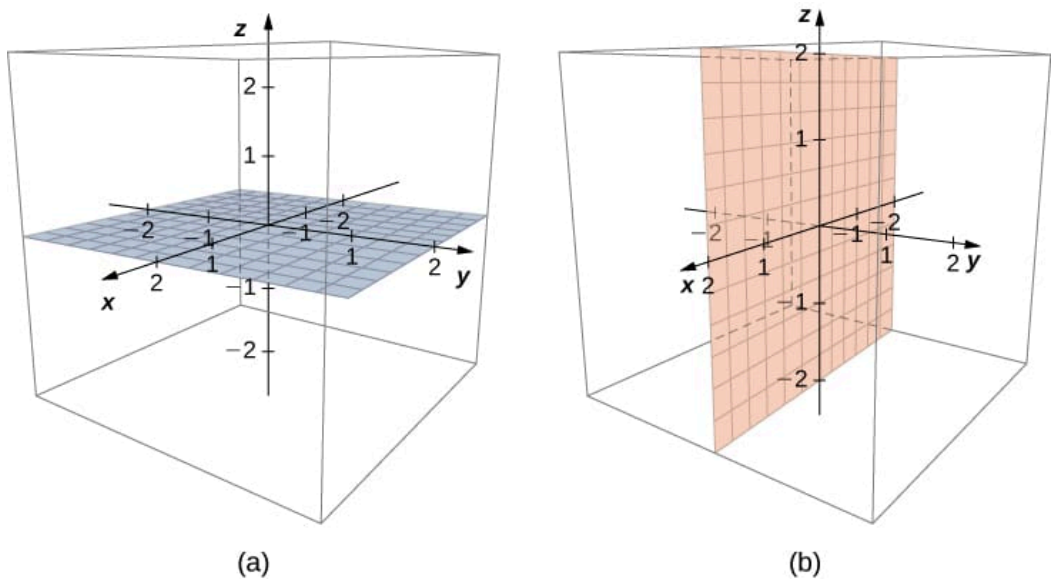


Figure $4.2.12$: (a) In space, the equation $z = 0$ describes the xy-plane. (b) All points in the xz-plane satisfy the equation $y = 0$.

Understanding the equations of the coordinate planes allows us to write an equation for any plane that is parallel to one of the coordinate planes. When a plane is parallel to the xy-plane, for example, the z-coordinate of each point in the plane has the same constant value. Only the x- and y-coordinates of points in that plane vary from point to point.

Equations of Planes Parallel to Coordinate Planes

1. The plane in space that is parallel to the xy-plane and contains point (a, b, c) can be represented by the equation $z = c$.
2. The plane in space that is parallel to the xz-plane and contains point (a, b, c) can be represented by the equation $y = b$.
3. The plane in space that is parallel to the yz-plane and contains point (a, b, c) can be represented by the equation $x = a$.

Example $4.2.3$: Writing Equations of Planes Parallel to Coordinate Planes

a. Write an equation of the plane passing through point $(3, 11, 7)$ that is parallel to the yz-plane.
b. Find an equation of the plane passing through points $(6, -2, 9), (0, -2, 4)$, and $(1, -2, -3)$.

Solution

a. When a plane is parallel to the yz-plane, only the y- and z-coordinates may vary. The x-coordinate has the same constant value for all points in this plane, so this plane can be represented by the equation $x = 3$.
b. Each of the points $(6, -2, 9), (0, -2, 4)$, and $(1, -2, -3)$ has the same y-coordinate. This plane can be represented by the equation $y = -2$.

Watch the accompanying video for a visual demonstration of the concept.

- **Video length:** 2 minutes 19 seconds.
- **Context**: This video desmonstrates how to sketch a plane $4x + 12y + 3z = 12$.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Exercise 4.2.3

Write an equation of the plane passing through point $(1, -6, -4)$ that is parallel to the xy-plane.

Hint available at the end of this section.

As we have seen, in $\mathbb{R}^2$ the equation $x = 5$ describes the vertical line passing through point $(5, 0)$. This line is parallel to the y -axis. In a natural extension, the equation $x = 5$ in $\mathbb{R}^3$ describes the plane passing through point $(5, 0, 0)$, which is parallel to the yz-plane. Another natural extension of a familiar equation is found in the equation of a sphere.

Definition: Sphere

A **sphere** is the set of all points in space equidistant from a fixed point, the center of the sphere (Figure 4.2.13), just as the set of all points in a plane that are equidistant from the center represents a circle. In a sphere, as in a circle, the distance from the center to a point on the sphere is called the *radius*.

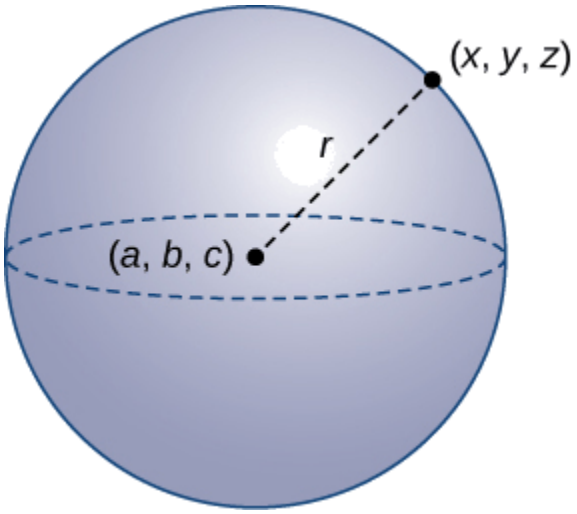


Figure 4.2.13: Each point (x, y, z) on the surface of a sphere is r units away from the center (a, b, c).

The equation of a circle is derived using the distance formula in two dimensions. In the same way, the equation of a sphere is based on the three-dimensional formula for distance.

Standard Equation of a Sphere

The sphere with center (a, b, c) and radius r can be represented by the equation

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2. \tag{4.2.3}$$

This equation is known as the **standard equation of a sphere**.

Example 4.2.4: Finding an Equation of a Sphere

Find the standard equation of the sphere with center $(10, 7, 4)$ and point $(-1, 3, -2)$, as shown in Figure 4.2.14.

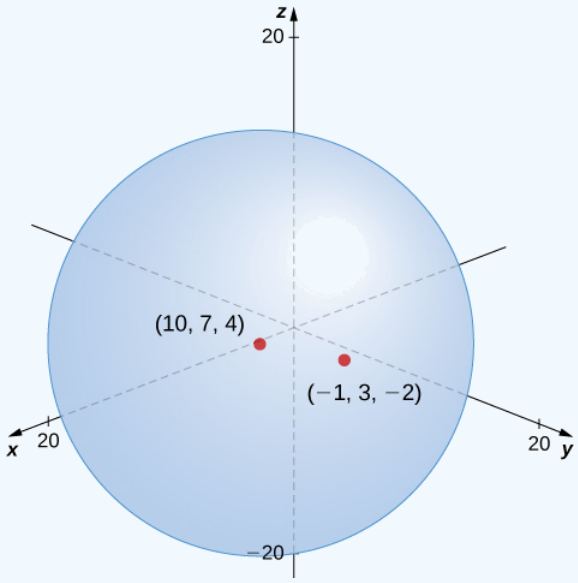


Figure 4.2.14: The sphere centered at $(10, 7, 4)$ containing point $(-1, 3, -2)$.

Solution

Use the distance formula to find the radius r of the sphere:

$$\begin{aligned} r &= \sqrt{(-1-10)^2 + (3-7)^2 + (-2-4)^2} \\ &= \sqrt{(-11)^2 + (-4)^2 + (-6)^2} \\ &= \sqrt{173} \end{aligned}$$

The standard equation of the sphere is

$$(x-10)^2 + (y-7)^2 + (z-4)^2 = 173.$$

Exercise 4.2.4

Find the standard equation of the sphere with center $(-2, 4, -5)$ containing point $(4, 4, -1)$.

Hint available at the end of this section.

Example 4.2.5: Finding the Equation of a Sphere

Let $P = (-5, 2, 3)$ and $Q = (3, 4, -1)$, and suppose line segment $\overline{PQ}$ forms the diameter of a sphere (Figure 4.2.15). Find the equation of the sphere.

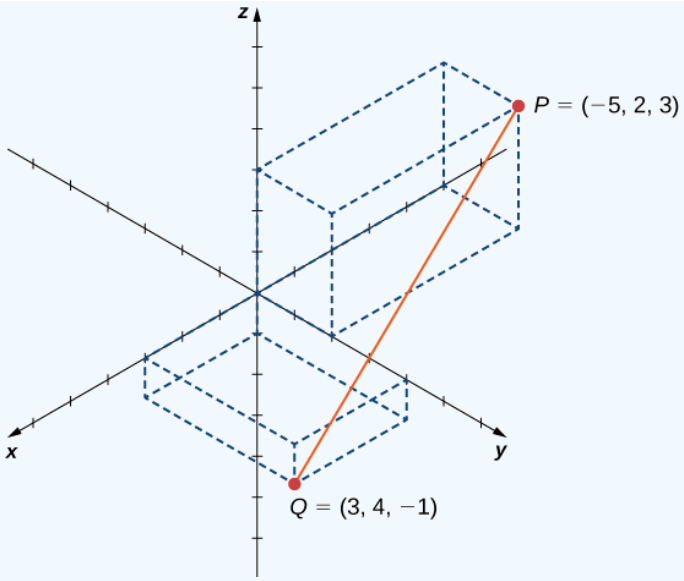


Figure 4.2.15: Line segment $\overline{PQ}$ between points $P = (-5, 2, 3)$ and $Q = (3, 4, -1)$.

Solution:

Since $\overline{PQ}$ is a diameter of the sphere, we know the center of the sphere is the midpoint of $\overline{PQ}$.Then,

$$C = \left(\frac{-5+3}{2}, \frac{2+4}{2}, \frac{3+(-1)}{2}\right) = (-1, 3, 1).$$

Furthermore, we know the radius of the sphere is half the length of the diameter. This gives

$$\begin{aligned} r &= \frac{1}{2}\sqrt{(-5-3)^2 + (2-4)^2 + (3-(-1))^2} \\ &= \frac{1}{2}\sqrt{64+4+16} \\ &= \sqrt{21} \end{aligned}$$

Then, the equation of the sphere is $(x+1)^2 + (y-3)^2 + (z-1)^2 = 21$.

Exercise 4.2.5

Find the equation of the sphere with diameter $\overline{PQ}$, where $P = (2, -1, -3)$ and $Q = (-2, 5, -1)$.

Hint available at the end of this section.

Example 4.2.6: Graphing Other Equations in Three Dimensions

Describe the set of points that satisfies $(x-4)(z-2) = 0$, and graph the set.

Solution

We must have either $x - 4 = 0$ or $z - 2 = 0$, so the set of points forms the two planes $x = 4$ and $z = 2$ (Figure 4.2.16).

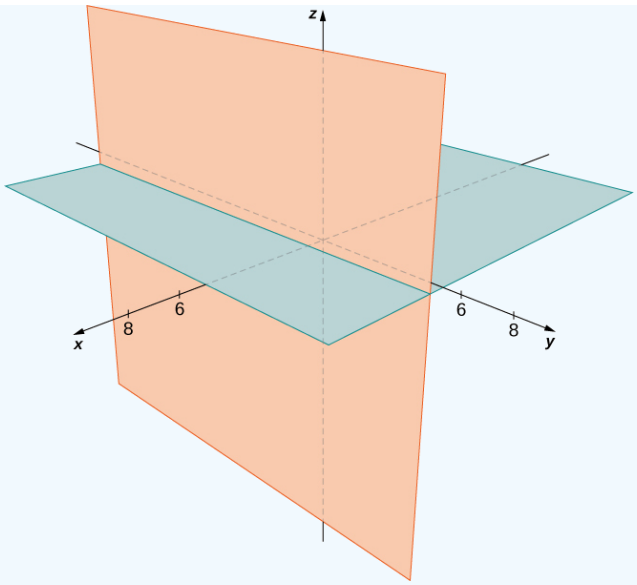


Figure 4.2.16: The set of points satisfying $(x-4)(z-2)=0$ forms the two intersecting planes $x=4$ in orange color and $z=2$ in green color.

Exercise 4.2.6

Describe the set of points that satisfies $(y+2)(z-3)=0$, and graph the set.

Hint available at the end of this section.

Example 4.2.7: Graphing Other Equations in Three Dimensions

Describe the set of points in three-dimensional space that satisfies $(x-2)^2+(y-1)^2=4$, and graph the set.

Solution

The x- and y-coordinates form a circle in the xy-plane of radius 2, centered at $(2,1)$. Since there is no restriction on the z-coordinate, the three-dimensional result is a circular cylinder of radius 2 centered on the line with $x=2$ and $y=1$. The cylinder extends indefinitely in the z-direction (Figure 4.2.18).

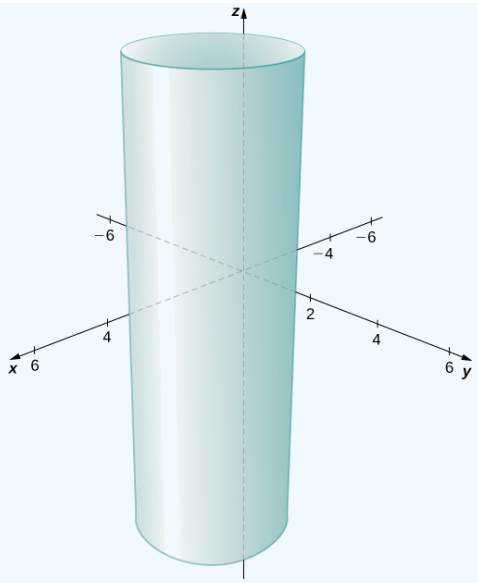


Figure 4.2.18: The set of points satisfying $(x-2)^2+(y-1)^2=4$. This is a cylinder of radius 2 centered on the line with $x=2$ and $y=1$.

Exercise 4.2.7

Describe the set of points in three dimensional space that satisfies $x^2+(z-2)^2=16$, and graph the surface.

Hint available at the end of this section.

Working with Vectors in $\mathbb{R}^3$

Just like two-dimensional vectors, three-dimensional vectors are quantities with both magnitude and direction, and they are represented by directed line segments (arrows). With a three-dimensional vector, we use a three-dimensional arrow.

Three-dimensional vectors can also be represented in component form. The notation $\mathbf{v}=\langle x,y,z\rangle$ is a natural extension of the two-dimensional case, representing a vector with the initial point at the origin, $(0,0,0)$, and terminal point (x,y,z). The zero vector is $\mathbf{0}=\langle 0,0,0\rangle$. So, for example, the three dimensional vector $\mathbf{v}=\langle 2,4,1\rangle$ is represented by a directed line segment from point $(0,0,0)$ to point $(2,4,1)$ (Figure 4.2.20).

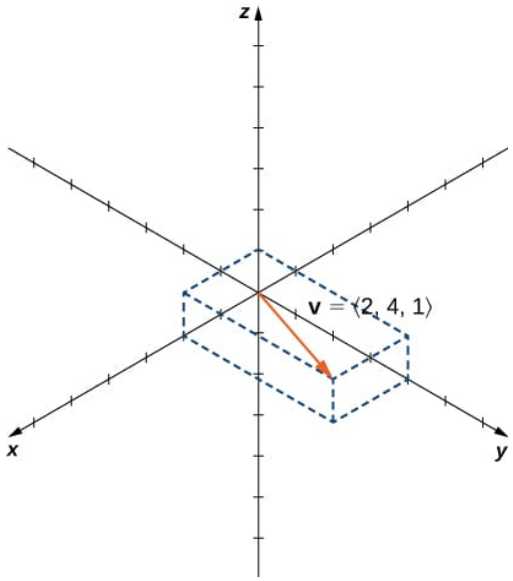


Figure 4.2.20: Vector $\mathbf{v} = \langle 2, 4, 1 \rangle$ is represented by a directed line segment from point $(0, 0, 0)$ to point $(2, 4, 1)$.

Vector addition and scalar multiplication are defined analogously to the two-dimensional case. If $\mathbf{v} = \langle x_1, y_1, z_1 \rangle$ and $\mathbf{w} = \langle x_2, y_2, z_2 \rangle$ are vectors, and k is a scalar, then

$$\mathbf{v} + \mathbf{w} = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle \tag{4.2.4}$$

and

$$k\mathbf{v} = \langle kx_1, ky_1, kz_1 \rangle. \tag{4.2.5}$$

If $k = -1$, then $k\mathbf{v} = (-1)\mathbf{v}$ is written as $-\mathbf{v}$, and vector subtraction is defined by $\mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w}) = \mathbf{v} + (-1)\mathbf{w}$.

The standard unit vectors extend easily into three dimensions as well, $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, and $\mathbf{k} = \langle 0, 0, 1 \rangle$, and we use them in the same way we used the standard unit vectors in two dimensions. Thus, we can represent a vector in $\mathbb{R}^3$ in the following ways:

$$\mathbf{v} = \langle x, y, z \rangle = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{4.2.6}$$

.

Example 4.2.8: Vector Representations

Let $\vecd{PQ}$ be the vector with initial point $P = (3, 12, 6)$ and terminal point $Q = (-4, -3, 2)$ as shown in Figure 4.2.21. Express $\vecd{PQ}$ in both component form and using standard unit vectors.

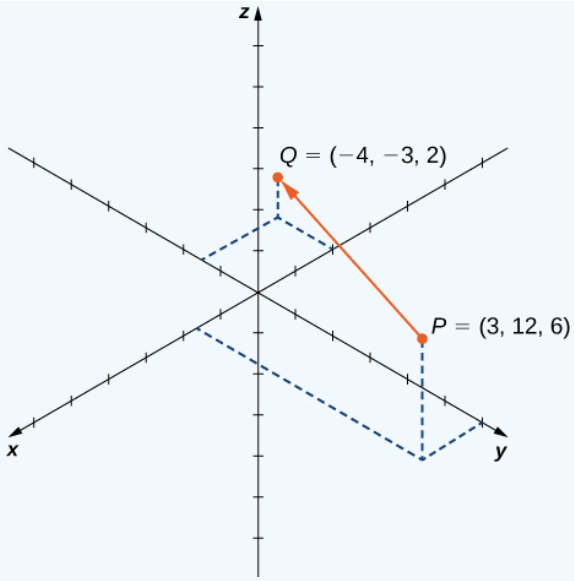


Figure 4.2.21: The vector with initial point $P = (3, 12, 6)$ and terminal point $Q = (-4, -3, 2)$ in xyz-space. The vector is colored in orange.

Solution

In component form,

$$\begin{aligned} \vec{PQ} &= \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle \\ &= \langle -4 - 3, -3 - 12, 2 - 6 \rangle \\ &= \langle -7, -15, -4 \rangle. \end{aligned}$$

In standard unit form,

$$\vec{PQ} = -7\mathbf{i} - 15\mathbf{j} - 4\mathbf{k}.$$

Exercise 4.2.8

Let $S = (3, 8, 2)$ and $T = (2, -1, 3)$. Express $\vec{ST}$ in component form and in standard unit form.

Hint available at the end of this section.

As described earlier, vectors in three dimensions behave in the same way as vectors in a plane. The geometric interpretation of vector addition, for example, is the same in both two- and three-dimensional space (Figure 4.2.22).

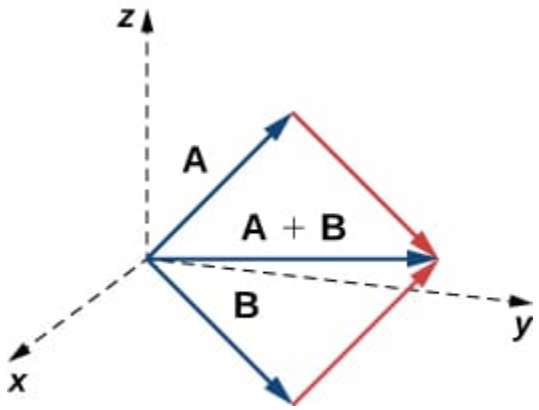


Figure 4.2.22: A 3D coordinate system with two vectors, A and B, and their sum, A+B in xyz-space. Vector A and vector B forms a parallelogram with vector A+B as its diagonal.

We have already seen how some of the algebraic properties of vectors, such as vector addition and scalar multiplication, can be extended to three dimensions. Other properties can be extended in similar fashion. They are summarized here for our reference.

Properties of Vectors in Space

Let $\mathbf{v} = \langle x_1, y_1, z_1 \rangle$ and $\mathbf{w} = \langle x_2, y_2, z_2 \rangle$ be vectors, and let k be a scalar.

- **Scalar multiplication**:

$$k\mathbf{v} = \langle kx_1, ky_1, kz_1 \rangle \tag{4.2.7}$$

- **Vector addition**:

$$\mathbf{v} + \mathbf{w} = \langle x_1, y_1, z_1 \rangle + \langle x_2, y_2, z_2 \rangle = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle \tag{4.2.8}$$

- **Vector subtraction**:

$$\mathbf{v} - \mathbf{w} = \langle x_1, y_1, z_1 \rangle - \langle x_2, y_2, z_2 \rangle = \langle x_1 - x_2, y_1 - y_2, z_1 - z_2 \rangle \tag{4.2.9}$$

- **Vector magnitude**:

$$\|\mathbf{v}\| = \sqrt{x_1^2 + y_1^2 + z_1^2} \tag{4.2.10}$$

- **Unit vector in the direction of** $\mathbf{v}$:

$$\frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{\|\mathbf{v}\|}\langle x_1, y_1, z_1 \rangle = \langle \frac{x_1}{\|\mathbf{v}\|}, \frac{y_1}{\|\mathbf{v}\|}, \frac{z_1}{\|\mathbf{v}\|} \rangle, \quad \text{if } \mathbf{v} \neq \mathbf{0} \tag{4.2.11}$$

We have seen that vector addition in two dimensions satisfies the commutative, associative, and additive inverse properties. These properties of vector operations are valid for three-dimensional vectors as well. Scalar multiplication of vectors satisfies the distributive property, and the zero vector acts as an additive identity. The proofs to verify these properties in three dimensions are straightforward extensions of the proofs in two dimensions.

Example 4.2.9: Vector Operations in Three Dimensions

Let $\mathbf{v} = \langle -2, 9, 5 \rangle$ and $\mathbf{w} = \langle 1, -1, 0 \rangle$ (Figure 4.2.23). Find the following vectors.

a. $3\mathbf{v} - 2\mathbf{w}$
b. $5\|\mathbf{w}\|$
c. $\|5\mathbf{w}\|$
d. A unit vector in the direction of $\mathbf{v}$

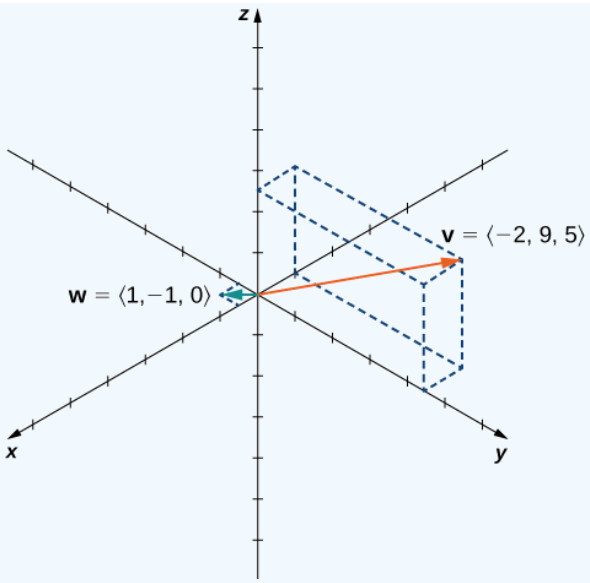


Figure 4.2.23: The vectors $\mathbf{v} = \langle -2, 9, 5 \rangle$ in orange pointing in the direction of the yz-plane *and* $\mathbf{w} = \langle 1, -1, 0 \rangle$ in green pointing in the direction of xy-plane.

Solution

a. First, use scalar multiplication of each vector, then subtract:

$$\begin{aligned} 3\mathbf{v} - 2\mathbf{w} &= 3\langle -2, 9, 5 \rangle - 2\langle 1, -1, 0 \rangle \\ &= \langle -6, 27, 15 \rangle - \langle 2, -2, 0 \rangle \\ &= \langle -6 - 2, 27 - (-2), 15 - 0 \rangle \\ &= \langle -8, 29, 15 \rangle. \end{aligned}$$

b. Write the equation for the magnitude of the vector, then use scalar multiplication:

$$5\|\mathbf{w}\| = 5\sqrt{1^2 + (-1)^2 + 0^2} = 5\sqrt{2}.$$

c. First, use scalar multiplication, then find the magnitude of the new vector. Note that the result is the same as for part b.:

$$\|5\mathbf{w}\| = \| \langle 5, -5, 0 \rangle \| = \sqrt{5^2 + (-5)^2 + 0^2} = \sqrt{50} = 5\sqrt{2}$$

d. Recall that to find a unit vector in two dimensions, we divide a vector by its magnitude. The procedure is the same in three dimensions:

$$\begin{aligned} \frac{\mathbf{v}}{\|\mathbf{v}\|} &= \frac{1}{\|\mathbf{v}\|}\langle -2, 9, 5 \rangle \\ &= \frac{1}{\sqrt{(-2)^2 + 9^2 + 5^2}}\langle -2, 9, 5 \rangle \\ &= \frac{1}{\sqrt{110}}\langle -2, 9, 5 \rangle \\ &= \left\langle \frac{-2}{\sqrt{110}}, \frac{9}{\sqrt{110}}, \frac{5}{\sqrt{110}} \right\rangle. \end{aligned}$$

Exercise 4.2.9:

Let $\mathbf{v} = \langle -1, -1, 1 \rangle$ and $\mathbf{w} = \langle 2, 0, 1 \rangle$. Find a unit vector in the direction of $5\mathbf{v} + 3\mathbf{w}$.

Hint available at the end of this section.

Example 4.2.10: Throwing a Forward Pass

A quarterback is standing on the football field preparing to throw a pass. His receiver is standing 20 yd down the field and 15 yd to the quarterback's left. The quarterback throws the ball at a velocity of 60 mph toward the receiver at an upward angle of $30°$ Figure 4.2.24. Write the initial velocity vector of the ball, $\mathbf{v}$, in component form.

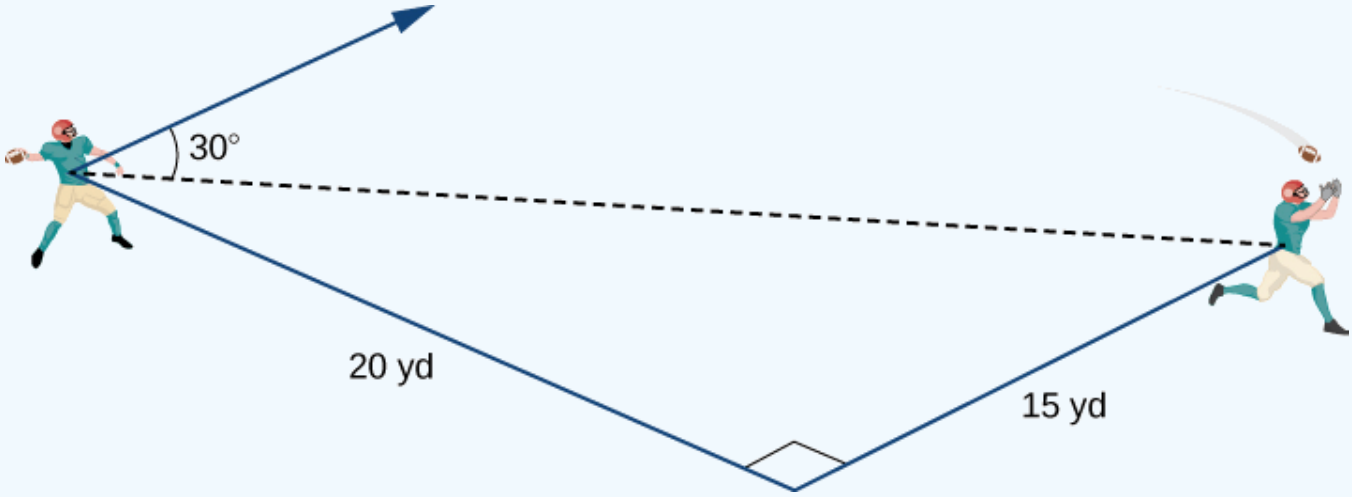


Figure 4.2.24: Two football players with the first one throwing the football at an upward angle of $30°$ to the second one. There is a line segment from each player to the bottom of the image. The distance from the first player to the bottom of the image is 20 yards. The distance from the second player to the same point on the bottom of the image is 15 yards. The two line segments are perpendicular. There is a broken line segment from the first player to the second player.

Solution

The first thing we want to do is find a vector in the same direction as the velocity vector of the ball. We then scale the vector appropriately so that it has the right magnitude. Consider the vector $\mathbf{w}$ extending from the quarterback's arm to a point directly above the receiver's head at an angle of $30°$ Figure 4.2.25. This vector would have the same direction as $\mathbf{v}$, but it may not have the right magnitude.

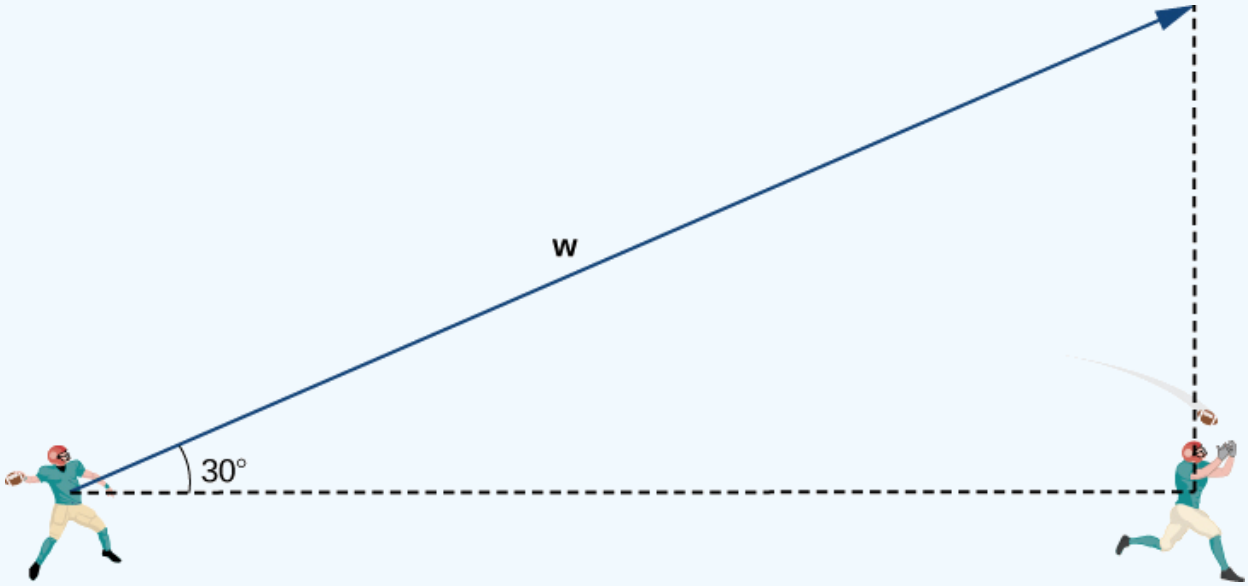


Figure 4.2.25: Two football players with the first player throwing the football to the second player. The distance between the two players is represented with a broken line segment. There is a vector $\mathbf{w}$ at 30 degrees from the first player. A right triangle formed from the two broken line segments and the vector from the first player.

The receiver is 20 yd down the field and 15 yd to the quarterback's left. Therefore, the straight-line distance from the quarterback to the receiver is

$$\textbf{Dist from QB to receiver} = \sqrt{15^2 + 20^2} = \sqrt{225 + 400} = \sqrt{625} = 25 \text{ yd.}$$

We have $\dfrac{25}{\|\mathbf{w}\|} = \cos 30\,^\circ$. Then the magnitude of $\mathbf{w}$ is given by

$$\|\mathbf{w}\| = \frac{25}{\cos 30^\circ} = \frac{25 \cdot 2}{\sqrt{3}} = \frac{50}{\sqrt{3}} \text{ yd}$$

and the vertical distance from the receiver to the terminal point of $\mathbf{w}$ is

$$\textbf{Vert dist from receiver to terminal point of } \mathbf{w} = \|\mathbf{w}\| \sin 30\,^\circ = \frac{50}{\sqrt{3}} \cdot \frac{1}{2} = \frac{25}{\sqrt{3}} \text{ yd.}$$

Then $\mathbf{w} = \langle 20, 15, \dfrac{25}{\sqrt{3}} \rangle$, and has the same direction as $\mathbf{v}$.

Recall, though, that we calculated the magnitude of $\mathbf{w}$ to be $\|\mathbf{w}\| = \dfrac{50}{\sqrt{3}}$, and $\mathbf{v}$ has magnitude 60 mph. So, we need to multiply vector $\mathbf{w}$ by an appropriate constant, k. We want to find a value of k so that $\| k\mathbf{w} \| = 60$ mph. We have

$$\|k\mathbf{w}\| = k\|\mathbf{w}\| = k\frac{50}{\sqrt{3}} \text{ mph,}$$

so we want

$$k\frac{50}{\sqrt{3}} = 60$$

$$k = \frac{60\sqrt{3}}{50}$$

$$k = \frac{6\sqrt{3}}{5}.$$

Then

$$\mathbf{v} = k\mathbf{w} = k\langle 20, 15, \frac{25}{\sqrt{3}} \rangle = \frac{6\sqrt{3}}{5}\langle 20, 15, \frac{25}{\sqrt{3}} \rangle = \langle 24\sqrt{3}, 18\sqrt{3}, 30 \rangle.$$

Let's double-check that $\|\mathbf{v}\| = 60$. We have

$$\|\mathbf{v}\| = \sqrt{(24\sqrt{3})^2 + (18\sqrt{3})^2 + (30)^2} = \sqrt{1728 + 972 + 900} = \sqrt{3600} = 60 \text{ mph.}$$

So, we have found the correct components for $\mathbf{v}$.

Exercise 4.2.10

Assume the quarterback and the receiver are in the same place as in the previous example. This time, however, the quarterback throws the ball at velocity of 40 mph and an angle of $45\,^\circ$. Write the initial velocity vector of the ball, $\mathbf{v}$, in component form.

Hint available at the end of this section.

Key Concepts

- The three-dimensional coordinate system is built around a set of three axes that intersect at right angles at a single point, the origin. Ordered triples (x, y, z) are used to describe the location of a point in space.
- The distance d between points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by the formula
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$
- In three dimensions, the equations $x = a$, $y = b$, and $z = c$ describe planes that are parallel to the coordinate planes.
- The standard equation of a sphere with center (a, b, c) and radius r is
$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2.$$
- In three dimensions, as in two, vectors are commonly expressed in component form, $\mathbf{v} = \langle x, y, z \rangle$, or in terms of the standard unit vectors, $\mathbf{v} = x\ \hat{\mathbf{i}} + y\ \hat{\mathbf{j}} + z\ \hat{\mathbf{k}}$.
- Properties of vectors in space are a natural extension of the properties for vectors in a plane. Let $\mathbf{v} = \langle x_1, y_1, z_1 \rangle$ and $\mathbf{w} = \langle x_2, y_2, z_2 \rangle$ be vectors, and let k be a scalar.

Key Equations

- **Scalar multiplication:**
$$k\mathbf{v} = \langle kx_1, ky_1, kz_1 \rangle$$
- **Vector addition:**
$$\mathbf{v} + \mathbf{w} = \langle x_1, y_1, z_1 \rangle + \langle x_2, y_2, z_2 \rangle = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle$$
- **Vector subtraction:**
$$\mathbf{v} - \mathbf{w} = \langle x_1, y_1, z_1 \rangle - \langle x_2, y_2, z_2 \rangle = \langle x_1 - x_2, y_1 - y_2, z_1 - z_2 \rangle$$
- **Vector magnitude:**
$$\|\mathbf{v}\| = \sqrt{x_1^2 + y_1^2 + z_1^2}$$
- **Unit vector in the direction of $\mathbf{v}$:**
$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\|\mathbf{v}\|}\langle x_1, y_1, z_1 \rangle = \langle \frac{x_1}{\|\mathbf{v}\|}, \frac{y_1}{\|\mathbf{v}\|}, \frac{z_1}{\|\mathbf{v}\|} \rangle,\ \mathbf{v} \neq \mathbf{0}$$
- **Distance between two points in space:**
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$
- **Sphere with center (a, b, c) and radius r:**
$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$$

Glossary

Coordinate Plane
A plane containing two of the three coordinate axes in the three-dimensional coordinate system, named by the axes it contains: the xy-plane, xz-plane, or the yz-plane.

Right-Hand Rule
A common way to define the orientation of the three-dimensional coordinate system; when the right hand is curved around the z-axis in such a way that the fingers curl from the positive x-axis to the positive y-axis, the thumb points in the direction of the positive z-axis.

Octants

The eight regions of space created by the coordinate planes.

Sphere

The set of all points equidistant from a given point known as the *center*.

Standard Equation of a Sphere

$(x-a)^2+(y-b)^2+(z-c)^2=r^2$ describes a sphere with center (a,b,c) and radius r.

Three-Dimensional Rectangular Coordinate System

A coordinate system defined by three lines that intersect at right angles; every point in space is described by an ordered triple (x,y,z) that plots its location relative to the defining axes.

Answers

Exercise 4.2.1

Start by sketching the coordinate axes. e.g., Figure 4.2.3. Then sketch a rectangular prism to help find the point in space.

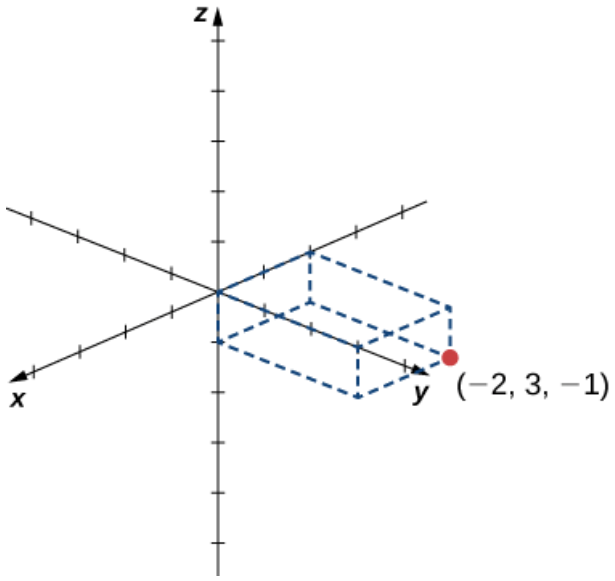


Figure 4.2.4: A 3D coordinate system with a point labeled $(-2,3,-1)$. To plot the point, go -2 unit from the origin in the direction of the x-axis, 3 units from the origin in the direction of the y-axis, and -1 units from the origin in the direction of z-axis.

Exercise 4.2.2

$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

$$5\sqrt{2}$$

Exercise 4.2.3

If a plane is parallel to the xy-plane, the z-coordinates of the points in that plane do not vary.

$$z=-4$$

Exercise 4.2.4

First use the distance formula to find the radius of the sphere.

$$(x+2)^2+(y-4)^2+(z+5)^2=52$$

Exercise 4.2.5

Find the midpoint of the diameter first.

$$x^2+(y-2)^2+(z+2)^2=14$$

Exercise 4.2.6

One of the factors must be zero.

The set of points forms the two planes $y=-2$ and $z=3$.

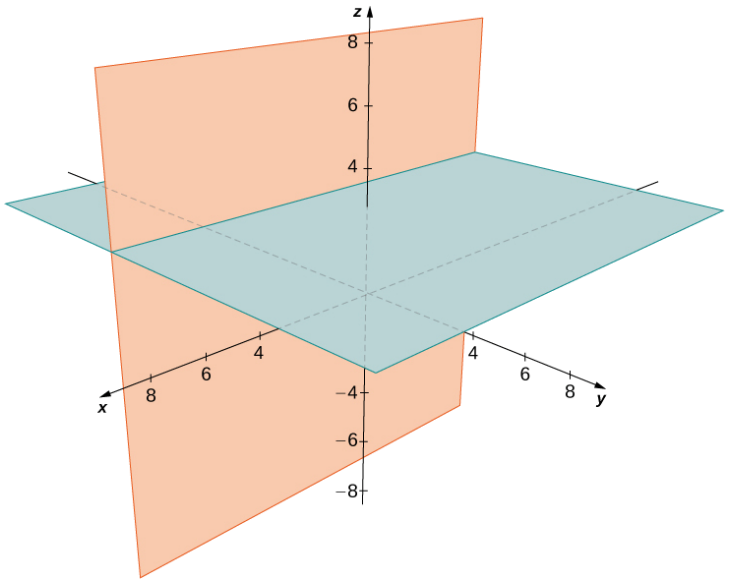


Figure 4.2.17: The set of points satisfying $(y+2)(z-3)=0$, forms the two intersecting planes $y=-2$ in orange color and $z=3$ in green color.

Exercise 4.2.7

Think about what happens if you plot this equation in two dimensions in the xz-plane.

A cylinder of radius 4 centered on the line with $x=0$ and $z=2$.

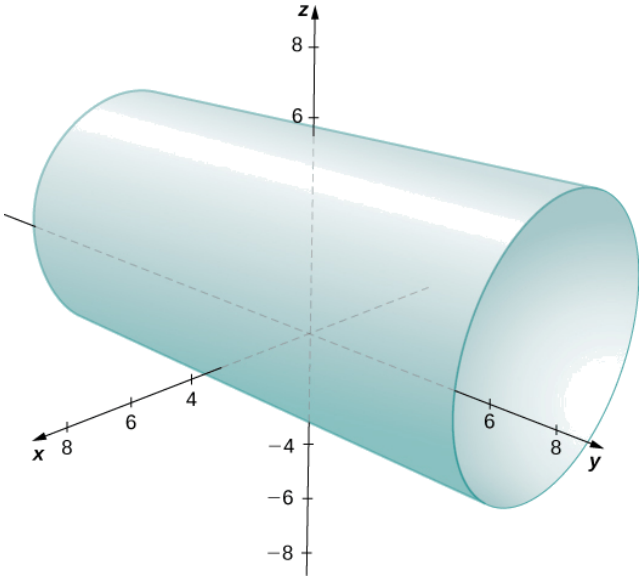


Figure 4.2.19: The set of points satisfying $x^2+(z-2)^2=16$. This is a a cylinder of radius 4 centered on the line with $x=0$ and $z=2$.

Exercise 4.2.8

Write $\vecd{ST}$ in component form first. T is the terminal point of $\vecd{ST}$.

$\vecd{ST}=\langle -1,-9,1\rangle=-\mathbf{i}-9\mathbf{j}+\mathbf{k}$

Exercise 4.2.9**:**

Start by writing $5\mathbf{v}+3\mathbf{w}$ in component form.

$\langle \frac{1}{3\sqrt{10}},-\frac{5}{3\sqrt{10}},\frac{8}{3\sqrt{10}}\rangle$

Exercise 4.2.10

Follow the process used in the previous example.

$v=\langle 16\sqrt{2},12\sqrt{2},20\sqrt{2}\rangle$

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4.2E: Exercises for Section 4.2

1. Consider a rectangular box with one of the vertices at the origin, as shown in the following figure. If point $A(2,3,5)$ is the opposite vertex to the origin, then find
   a. the coordinates of the other six vertices of the box and
   b. the length of the diagonal of the box determined by the vertices O and A.

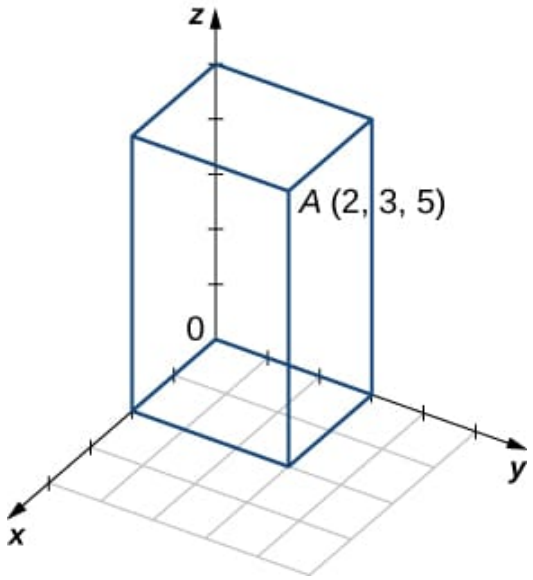


Figure $4.2E.1$: A rectangular box in 3D coordinate system. One vertex is labeled A(2,3,5). The box is sketched in the first octant.

Answer

a. $(2,0,5),(2,0,0),(2,3,0),(0,3,0),(0,3,5),(0,0,5)$ b. $\sqrt{38}$

2. Find the coordinates of point P and determine its distance to the origin.

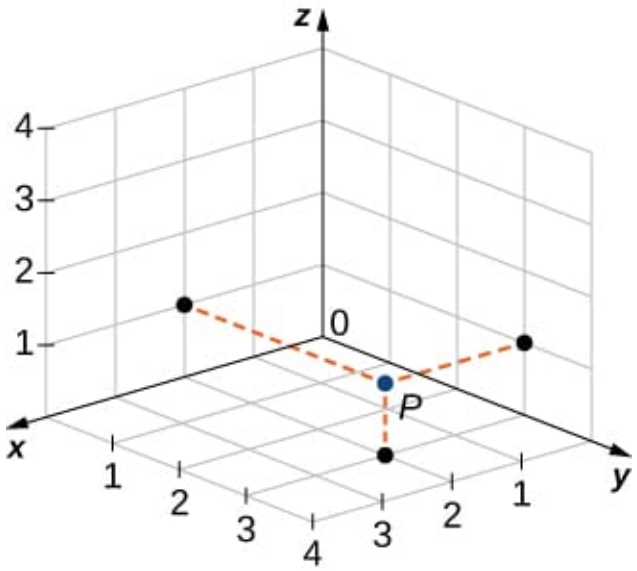


Figure $4.2E.2$: A point P in 3D coordinate system. It has three dash lines connected to coordinates (2,3) on xy-plane, (2,1) on xz-plane, and (3,1) on yz-plane.

Exercises 3-6

Describe and graph the set of points that satisfies the given equation.

3. $(y-5)(z-6)=0$

Answer

A union of two planes: $y=5$ (a plane parallel to the xz-plane) and $z=6$ (a plane parallel to the xy-plane)

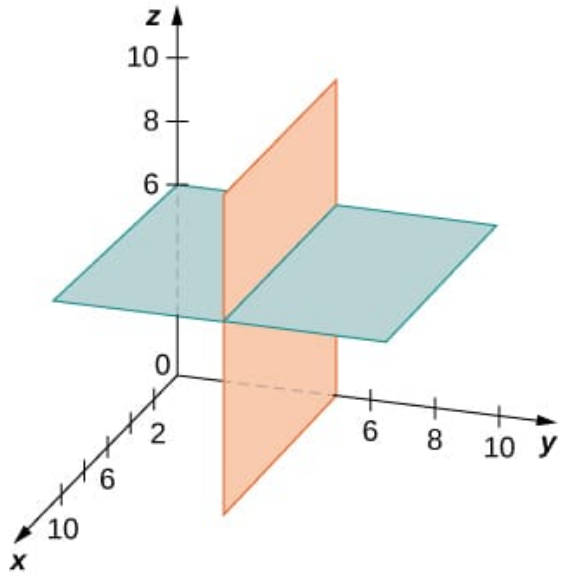


Figure $4.2E.3$: Plane $y=5$ is intersecting plane $z=6$ in 3d coordinate system.

4. $(z-2)(z-5)=0$
5. $(y-1)^2+(z-1)^2=1$

Answer

A cylinder of radius 1 centered on the line $y=1, z=1$

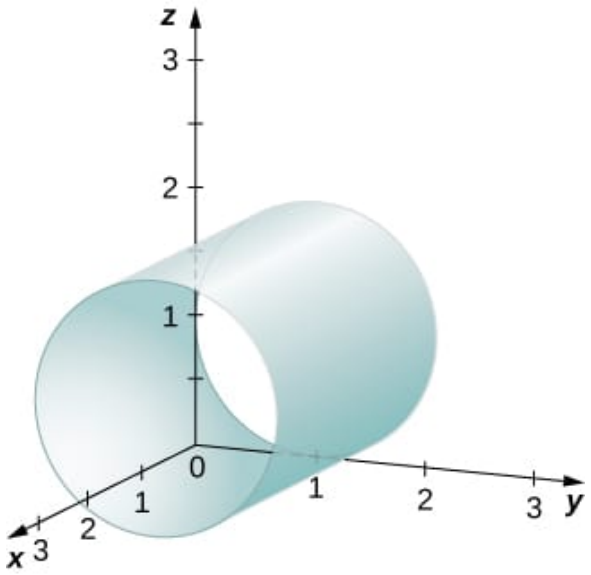


Figure $4.2E.4$: A cylinder of radius 1 and centered on the line y=1 and z=1. It is parallel to the x-axis.

6. $(x-2)^2+(z-5)^2=4$
7. Write the equation of the plane passing through point $(1,1,1)$ that is parallel to the xy-plane.

Answer

$z = 1$

8. Write the equation of the plane passing through point $(1, -3, 2)$ that is parallel to the xz-plane.
9. Find an equation of the plane passing through points $(1, -3, -2)$, $(0, 3, -2)$, and $(1, 0, -2)$.

Answer

$z = -2$

10. Find an equation of the plane passing through points $(1, 9, 2)$, $(1, 3, 6)$, and $(1, -7, 8)$.

Exercises 11-14

Find the equation of the sphere in standard form that satisfies the given conditions.

11. Center $C(-1, 7, 4)$ and radius 4

Answer

$(x + 1)^2 + (y - 7)^2 + (z - 4)^2 = 16$

12. Center $C(-4, 7, 2)$ and radius 6
13. Diameter PQ, where $P(-1, 5, 7)$ and $Q(-5, 2, 9)$

Answer

$x + 3)^2 + (y - 3.5)^2 + (z - 8)^2 = \frac{29}{4}$

14. Diameter PQ, where $P(-16, -3, 9)$ and $Q(-2, 3, 5)$

Exercises 15 and 16

Find the center and radius of the sphere with an equation in general form that is given.

15. $x^2 + y^2 + z^2 - 4z + 3 = 0$

Answer

Center $C(0, 0, 2)$ and radius 1

16. $x^2 + y^2 + z^2 - 6x + 8y - 10z + 25 = 0$

Exercises 17-20

Express vector $\vecd{PQ}$ with the initial point at P and the terminal point at Q

a. **a. in component form**
b. **by using standard unit vectors.**

17. $P(3, 0, 2)$ and $Q(-1, -1, 4)$

Answer

a. $\vecd{PQ} = \langle -4, -1, 2 \rangle$
b. $\vecd{PQ} = -4\mathbf{i} - \mathbf{j} + 2\mathbf{k}$

18. $P(0, 10, 5)$ and $Q(1, 1, -3)$
19. $P(-2, 5, -8)$ and $M(1, -7, 4)$, where M is the midpoint of the line segment $\overline{PQ}$

Answer

a. $\vecd{PQ} = \langle 6, -24, 24 \rangle$

b. $\vecd{PQ} = 6\mathbf{i} - 24\mathbf{j} + 24\mathbf{k}$

20. $Q(0, 7, -6)$ and $M(-1, 3, 2)$, where M is the midpoint of the line segment $\overline{PQ}$
21. Find terminal point Q of vector $\vecd{PQ} = \langle 7, -1, 3\rangle$ with the initial point at $P(-2, 3, 5)$.

Answer

$Q(5, 2, 8)$

22. Find initial point P of vector $\vecd{PQ} = \langle -9, 1, 2\rangle$ with the terminal point at $Q(10, 0, -1)$.

Exercises 23-26

Use the given vectors a and b to find and express the vectors $\mathrm{a} + \mathrm{b}$, $4\mathrm{a}$, and $-5\mathrm{a} + 3\mathrm{b}$ in component form.

23. $\mathbf{a} = \langle -1, -2, 4\rangle, \quad \mathbf{b} = \langle -5, 6, -7\rangle$

Answer

$\mathbf{a} + \mathbf{b} = \langle -6, 4, -3\rangle, 4\mathbf{a} = \langle -4, -8, 16\rangle, -5\mathbf{a} + 3\mathbf{b} = \langle -10, 28, -41\rangle$

24. $\mathbf{a} = \langle 3, -2, 4\rangle, \quad \mathbf{b} = \langle -5, 6, -9\rangle$
25. $\mathbf{a} = -\mathbf{k}, \quad \mathbf{b} = -\mathbf{i}$

Answer

$\mathbf{a} + \mathbf{b} = \langle -1, 0, -1\rangle, 4\mathbf{a} = \langle 0, 0, -4\rangle, -5\mathbf{a} + 3\mathbf{b} = \langle -3, 0, 5\rangle$

26. $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}, \quad \mathbf{b} = 2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$

Exercises 27-30

Vectors u and v are given. Find the magnitudes of vectors $\mathrm{u} - \mathrm{v}$ and $-2\mathrm{u}$.

27. $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}, \quad \mathbf{v} = -\mathbf{i} + 5\mathbf{j} - \mathbf{k}$

Answer

$\|\mathbf{u} - \mathbf{v}\| = \sqrt{38}, \quad \|-2\mathbf{u}\| = 2\sqrt{29}$

28. $\mathbf{u} = \mathbf{i} + \mathbf{j}, \quad \mathbf{v} = \mathbf{j} - \mathbf{k}$
29. $\mathbf{u} = \langle 2\cos t, -2\sin t, 3\rangle, \quad \mathbf{v} = \langle 0, 0, 3\rangle, \quad$ where t is a real number.

Answer

$\|\mathbf{u} - \mathbf{v}\| = 2, \quad \|-2\mathbf{u}\| = 2\sqrt{13}$

30. $\mathbf{u} = \langle 0, 1, \sinh t\rangle, \quad \mathbf{v} = \langle 1, 1, 0\rangle, \quad$ where t is a real number.

Exercises 31-36

Find the unit vector in the direction of the given vector a and express it using standard unit vectors.

31. $\mathbf{a} = 3\mathbf{i} - 4\mathbf{j}$

Answer

$\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$

32. $\mathbf{a} = \langle 4, -3, 6\rangle$
33. $\mathbf{a} = \vecd{PQ}$, where $P(-2, 3, 1)$ and $Q(0, -4, 4)$

Answer

$\frac{\sqrt{62}}{31}\mathbf{i} - \frac{7\sqrt{62}}{62}\mathbf{j} + \frac{3\sqrt{62}}{62}\mathbf{k}$

34. $\mathbf{a} = \vecd{OP}$, where $P(-1,-1,1)$
35. $\mathbf{a} = \mathbf{u} - \mathbf{v} + \mathbf{w}$, where $\mathbf{u} = \mathbf{i} - \mathbf{j} - \mathbf{k}, \quad \mathbf{v} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}, \quad$ and $\mathbf{w} = -\mathbf{i} + \mathbf{j} + 3\mathbf{k}$

Answer

$-\frac{\sqrt{6}}{3}\mathbf{i} + \frac{\sqrt{6}}{6}\mathbf{j} + \frac{\sqrt{6}}{6}\mathbf{k}$

36. $\mathbf{a} = 2\mathbf{u} + \mathbf{v} - \mathbf{w}, \quad$ where $\mathbf{u} = \mathbf{i} - \mathbf{k}, \quad \mathbf{v} = 2\mathbf{j} \quad$, and $\mathbf{w} = \mathbf{i} - \mathbf{j}$
37. Determine whether $\vecd{AB}$ and $\vecd{PQ}$ are equivalent vectors, where $A(1,1,1), \ B(3,3,3), \ P(1,4,5)$, and $Q(3,6,7)$.

Answer

Equivalent vectors

38. Determine whether the vectors $\vecd{AB}$ and $\vecd{PQ}$ are equivalent, where $A(1,4,1), \ B(-2,2,0), \ P(2,5,7)$, and $Q(-3,2,1)$.

Exercises 39-42

Find vector $\mathbf{u}$ with a magnitude that is given and satisfies the given conditions.

39. $\mathbf{v} = \langle 7,-1,3\rangle, \ \|\mathbf{u}\| = 10$, and $\mathbf{u}$ and $\mathbf{v}$ have the same direction

Answer

$\mathbf{u} = \langle \frac{70\sqrt{59}}{59}, -\frac{10\sqrt{59}}{59}, \frac{30\sqrt{59}}{59}\rangle$

40. $\mathbf{v} = \langle 2,4,1\rangle, \ \|\mathbf{u}\| = 15$, and $\mathbf{u}$ and $\mathbf{v}$ have the same direction
41. $\mathbf{v} = \langle 2\sin t, \ 2\cos t, 1\rangle, \|\mathbf{u}\| = 2, \mathbf{u}$ and $\mathbf{v}$ have opposite directions for any t, where t is a real number

Answer

$\mathbf{u} = \langle -\frac{4\sqrt{5}}{5}\sin t, -\frac{4\sqrt{5}}{5}\cos t, -\frac{2\sqrt{5}}{5}\rangle$

42. $\mathbf{v} = \langle 3\sinh t, 0, 3\rangle, \ \|\mathbf{u}\| = 5$, and $\mathbf{u}$ and $\mathbf{v}$ have opposite directions for any t, where t is a real number
43. Determine a vector of magnitude 5 in the direction of vector $\vecd{AB}$, where $A(2,1,5)$ and $B(3,4,-7)$.

Answer

$\langle \frac{5\sqrt{154}}{154}, \frac{15\sqrt{154}}{154}, -\frac{30\sqrt{154}}{77}\rangle$

44. Find a vector of magnitude 2 that points in the opposite direction than vector $\vecd{AB}$, where $A(-1,-1,1)$ and $B(0,1,1)$. Express the answer in component form.
45. Consider the points $A(2,\alpha,0), \ B(0,1,\beta)$, and $C(1,1,\beta)$, where α and β are negative real numbers. Find α and β such that $\|\vecd{OA} - \vecd{OB} + \vecd{OC}\| = \|\vecd{OB}\| = 4$.

Answer

$\alpha = -\sqrt{7}, \ \beta = -\sqrt{15}$

46. Consider points $A(\alpha,0,0), \ B(0,\beta,0)$, and $C(\alpha,\beta,\beta)$, where α and β are positive real numbers. Find α and β such that $\|\overline{OA} + \overline{OB}\| = \sqrt{2}$ and $\|\overline{OC}\| = \sqrt{3}$.
47. Let $P(x,y,z)$ be a point situated at an equal distance from points $A(1,-1,0)$ and $B(-1,2,1)$. Show that point P lies on the plane of equation $-2x + 3y + z = 2$.
48. Let $P(x,y,z)$ be a point situated at an equal distance from the origin and point $A(4,1,2)$. Show that the coordinates of point P satisfy the equation $8x + 2y + 4z = 21$.
49. The points A, B, and C are collinear (in this order) if the relation $\|\vecd{AB}\| + \|\vecd{BC}\| = \|\vecd{AC}\|$ is satisfied. Show that $A(5,3,-1), \ B(-5,-3,1)$, and $C(-15,-9,3)$ are collinear points.
50. Show that points $A(1,0,1), \ B(0,1,1)$, and $C(1,1,1)$ are not collinear.
51. [T] A force $\mathbf{F}$ of $50 \ N$ acts on a particle in the direction of the vector $\vecd{OP}$, where $P(3,4,0)$.
    a. Express the force as a vector in component form.
    b. Find the angle between force $\mathbf{F}$ and the positive direction of the x-axis. Express the answer in degrees rounded to the

nearest integer.

Answer

$a.\mathbf{F} = \langle 30, 40, 0\rangle; \quad b.53\,°$

52. [T] A force $\mathbf{F}$ of $40\ N$ acts on a box in the direction of the vector $\vec{OP}$, where $P(1, 0, 2)$.
    a. Express the force as a vector by using standard unit vectors.
    b. Find the angle between force $\mathbf{F}$ and the positive direction of the x-axis.
53. If $\mathbf{F}$ is a force that moves an object from point $P_1(x_1, y_1, z_1)$ to another point $P_2(x_2, y_2, z_2)$, then the displacement vector is defined as $\mathbf{D} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$. A metal container is lifted 10 m vertically by a constant force $\mathbf{F}$. Express the displacement vector $\mathbf{D}$ by using standard unit vectors.

Answer

$\mathbf{D} = 10\mathbf{k}$

54. A box is pulled 4 yd horizontally in the x-direction by a constant force $\mathbf{F}$. Find the displacement vector in component form.
55. The sum of the forces acting on an object is called the resultant or net force. An object is said to be in static equilibrium if the resultant force of the forces that act on it is zero. Let $\mathbf{F}_1 = \langle 10, 6, 3\rangle$, $\mathbf{F}_2 = \langle 0, 4, 9\rangle$, and $\mathbf{F}_3 = \langle 10, -3, -9\rangle$ be three forces acting on a box. Find the force $\mathbf{F}_4$ acting on the box such that the box is in static equilibrium. Express the answer in component form.

Answer

$\mathbf{F}_4 = \langle -20, -7, -3\rangle$

56. [T] Let $\mathbf{F}_k = \langle 1, k, k^2\rangle, k = 1, ..., n$ be n forces acting on a particle, with $n \geq 2$.
    a. Find the net force $\mathbf{F} = \sum_{k=1}^{n} \mathbf{F}_k$. Express the answer using standard unit vectors.
    b. Use a computer algebra system (CAS) to find n such that $\|\mathbf{F}\| < 100$.
57. The force of gravity $\mathbf{F}$ acting on an object is given by $\mathbf{F} = m\mathbf{g}$, where m is the mass of the object (expressed in kilograms) and $\mathbf{g}$ is acceleration resulting from gravity, with $\|\mathbf{g}\| = 9.8\ N/kg$. A 2-kg disco ball hangs by a chain from the ceiling of a room.
    a. Find the force of gravity $\mathbf{F}$ acting on the disco ball and find its magnitude. Express the answers using standard unit vectors.
    b. Find the force of tension $\mathbf{T}$ in the chain and its magnitude. Express the answers using standard unit vectors.

Answer

a. $\mathbf{F} = -19.6\mathbf{k}, \quad \|\mathbf{F}\| = 19.6\ N$
b. $\mathbf{T} = 19.6\mathbf{k}, \quad \|\mathbf{T}\| = 19.6\ N$

58. A 5-kg pendant chandelier is designed such that the alabaster bowl is held by four chains of equal length, as shown in Figure $4.2E.5$.
    a. Find the magnitude of the force of gravity acting on the chandelier.
    b. Find the magnitudes of the forces of tension for each of the four chains (assume chains are essentially vertical).



Figure $4.2E.5$: A pendant chandelier where the alabaster bowl is held by four chains of equal length.

59. [T] A 30-kg block of cement is suspended by three cables of equal length that are anchored at points $P(-2,0,0), Q(1,\sqrt{3},0)$, and $R(1,-\sqrt{3},0)$. The load is located at $S(0,0,-2\sqrt{3})$, as shown in the following figure. Let $\mathbf{F}_1, \mathbf{F}_2$, and $\mathbf{F}_3$ be the forces of tension resulting from the load in cables RS, QS, and PS, respectively.
    a. Find the gravitational force $\mathbf{F}$ acting on the block of cement that counterbalances the sum $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$ of the forces of tension in the cables.
    b. Find forces $\mathbf{F}_1, \mathbf{F}_2$, and $\mathbf{F}_3$. Express the answer in component form.

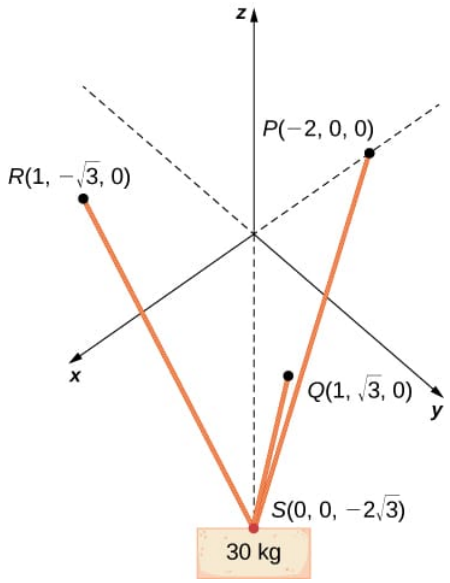


Figure $4.2E.6$: A 30-kg block of cement is suspended by three cables of equal length that are anchored at points $P(-2,0,0), Q(1,\sqrt{3},0)$, and $R(1,-\sqrt{3},0)$. Each orange line segment from the block to the point represents tension in each cable. The block and the three cables are drawn in the 3d coordinate system.

Answer

a. $\mathbf{F} = -294\mathbf{k}$ N;
b. $\mathbf{F}_1 = \langle -\frac{49\sqrt{3}}{3}, 49, -98\rangle, \mathbf{F}_2 = \langle -\frac{49\sqrt{3}}{3}, -49, -98\rangle$, and $\mathbf{F}_3 = \langle \frac{98\sqrt{3}}{3}, 0, -98\rangle$ (each component is expressed in newtons)

60. Two soccer players are practicing for an upcoming game. One of them runs 10 m from point A to point B. She then turns left at $90°$ and runs 10 m until she reaches point C. Then she kicks the ball with a speed of 10 m/sec at an upward angle of $45°$ to her teammate, who is located at point A. Write the velocity of the ball in component form.

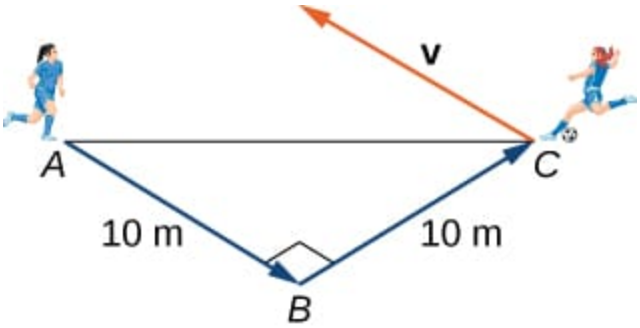


Figure $4.2E.7$: A soccer scenario involving three points (A, B, C) and two players. The three points form a right triangle. The lengths of the legs AB and BC are each 10 m, and the right angle is at B. Player at point A appears to be running towards C. Player at point C is about to kick the ball with an initial velocity $\mathbf{v}$, which points away from C at an upward angle. The velocity vector $\mathbf{v}$ indicates trajectory of the ball as it is being kicked in a specific direction, likely to send it into play.

61. Let $\mathbf{r}(t) = \langle x(t),\ y(t),\ z(t)\rangle$ be the position vector of a particle at the time $t \in [0, T]$, where x, y, and z are smooth functions on $[0, T]$. The instantaneous velocity of the particle at time t is defined by vector $\mathbf{v}(t) = \langle x'(t),\ y'(t),\ z'(t)\rangle$, with components that are the derivatives with respect to t, of the functions x, y, and z, respectively. The magnitude $\| \mathbf{v}(t) \|$ of the instantaneous velocity vector is called the speed of the particle at time t. Vector $\mathbf{a}(t) = \langle x''(t),\ y''(t),\ z''(t)\rangle$, with components that are the second derivatives with respect to t, of the functions x, y, and z, respectively, gives the acceleration of the particle at time t. Consider $\mathbf{r}(t) = \langle \cos t,\ \sin t,\ 2t\rangle$ the position vector of a particle at time $t \in [0, 30]$, where the components of $\mathbf{r}$ are expressed in centimeters and time is expressed in seconds.
    a. Find the instantaneous velocity, speed, and acceleration of the particle after the first second. Round your answer to two decimal places.
    b. Use a CAS to visualize the path of the particle—that is, the set of all points of coordinates $(\cos t, \sin t, 2t)$, where $t \in [0, 30]$.

Answer

a. $\mathbf{v}(1) = \langle -0.84, 0.54, 2\rangle$ (each component is expressed in centimeters per second); $\| \mathbf{v}(1) \| = 2.24$ (expressed in centimeters per second); $\mathbf{a}(1) = \langle -0.54, -0.84, 0\rangle$ (each component expressed in centimeters per second squared);

b.

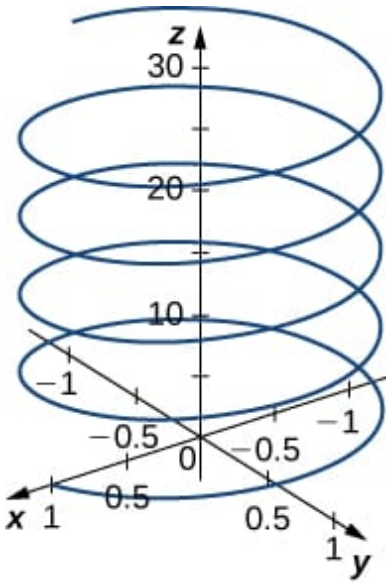


Figure $4.2E.8$: A three-dimensional helical spiral, a structure that elegantly combines symmetry and motion in space. The helix rises along the z-axis, forming smooth, regular coils as it progresses upward. Its base traces a circular path in the xy-plane, with a radius of approximately 1, giving it a beautifully balanced appearance. As the spiral ascends to z=30, it maintains a consistent pitch, revealing a harmony between vertical height and rotational movement.

62. [T] Let $\mathbf{r}(t) = \langle t, 2t^2, 4t^2\rangle$ be the position vector of a particle at time t (in seconds), where $t \in [0, 10]$ (here the components of $\mathbf{r}$ are expressed in centimeters).

a. Find the instantaneous velocity, speed, and acceleration of the particle after the first two seconds. Round your answer to two decimal places.
b. Use a CAS to visualize the path of the particle defined by the points $(t,\ 2t^2,\ 4t^2)$, where $t \in [0,\ 60]$.

Contributors

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4.3: The Dot Product

Learning Objectives

- Calculate the dot product of two given vectors.
- Determine whether two given vectors are perpendicular.
- Find the direction cosines of a given vector.
- Explain what is meant by the vector projection of one vector onto another vector, and describe how to compute it.
- Calculate the work done by a given force.

If we apply a force to an object so that the object moves, we say that work is done by the force. Previously, we looked at a constant force and we assumed the force was applied in the direction of motion of the object. Under those conditions, work can be expressed as the product of the force acting on an object and the distance the object moves. In this chapter, however, we have seen that both force and the motion of an object can be represented by vectors.

In this section, we develop an operation called the dot product, which allows us to calculate work in the case when the force vector and the motion vector have different directions. The dot product essentially tells us how much of the force vector is applied in the direction of the motion vector. The dot product can also help us measure the angle formed by a pair of vectors and the position of a vector relative to the coordinate axes. It even provides a simple test to determine whether two vectors meet at a right angle.

The Dot Product and Its Properties

We have already learned how to add and subtract vectors. In this chapter, we investigate two types of vector multiplication. The first type of vector multiplication is called the dot product, based on the notation we use for it, and it is defined as follows:

Definition: Dot Product

The **dot product** of vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is given by the sum of the products of the components

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3. \tag{4.3.1}$$

Note that if u and v are two-dimensional vectors, we calculate the dot product in a similar fashion. Thus, if $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$, then

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2. \tag{4.3.2}$$

When two vectors are combined under addition or subtraction, the result is a vector. When two vectors are combined using the dot product, the result is a scalar. For this reason, the dot product is often called the *scalar product*. It may also be called the *inner product*.

Example 4.3.1: Calculating Dot Products

a. Find the dot product of $\mathbf{u} = \langle 3, 5, 2 \rangle$ and $\mathbf{v} = \langle -1, 3, 0 \rangle$.
b. Find the scalar product of $\mathbf{p} = 10\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$ and $\mathbf{q} = -2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 6\hat{\mathbf{k}}$.

Solution:

a. Substitute the vector components into the formula for the dot product:

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= u_1v_1 + u_2v_2 + u_3v_3\\ &= 3(-1) + 5(3) + 2(0)\\ &= -3 + 15 + 0\\ &= 12.\end{aligned}$$

b. The calculation is the same if the vectors are written using standard unit vectors. We still have three components for each vector to substitute into the formula for the dot product:

$$\begin{aligned}\mathbf{p}\cdot\mathbf{q} &= u_1v_1 + u_2v_2 + u_3v_3\\ &= 10(-2) + (-4)(1) + (7)(6)\\ &= -20 - 4 + 42\\ &= 18.\end{aligned}$$

Exercise 4.3.1

Find $\mathbf{u}\cdot\mathbf{v}$, where $\mathbf{u} = \langle 2, 9, -1\rangle$ and $\mathbf{v} = \langle -3, 1, -4\rangle$.

Hint available at the end of this section.

Like vector addition and subtraction, the dot product has several algebraic properties. We prove three of these properties and leave the rest as exercises.

Properties of the Dot Product

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors, and let c be a scalar.

i. *Commutative property*

$$\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u} \tag{4.3.3}$$

ii. *Distributive property*

$$\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w} \tag{4.3.4}$$

iii. *Associative property*

$$c(\mathbf{u}\cdot\mathbf{v}) = (c\mathbf{u})\cdot\mathbf{v} = \mathbf{u}\cdot(c\mathbf{v}) \tag{4.3.5}$$

iv. *Property of magnitude*

$$\mathbf{v}\cdot\mathbf{v} = \|\mathbf{v}\|^2 \tag{4.3.6}$$

Proof

Let $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$. Then

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= \langle u_1, u_2, u_3\rangle\cdot\langle v_1, v_2, v_3\rangle\\ &= u_1v_1 + u_2v_2 + u_3v_3\\ &= v_1u_1 + v_2u_2 + v_3u_3\\ &= \langle v_1, v_2, v_3\rangle\cdot\langle u_1, u_2, u_3\rangle\\ &= \mathbf{v}\cdot\mathbf{u}.\end{aligned}$$

The associative property looks like the associative property for real-number multiplication, but pay close attention to the difference between scalar and vector objects:

$$\begin{aligned} c(\mathbf{u}\cdot\mathbf{v}) &= c(u_1v_1+u_2v_2+u_3v_3) \\ &= c(u_1v_1)+c(u_2v_2)+c(u_3v_3) \\ &= (cu_1)v_1+(cu_2)v_2+(cu_3)v_3 \\ &= \langle cu_1,cu_2,cu_3\rangle\cdot\langle v_1,v_2,v_3\rangle \\ &= c\langle u_1,u_2,u_3\rangle\cdot\langle v_1,v_2,v_3\rangle \\ &= (c\mathbf{u})\cdot\mathbf{v}. \end{aligned}$$

The proof that $c(\mathbf{u}\cdot\mathbf{v}) = \mathbf{u}\cdot(c\mathbf{v})$ is similar.

The fourth property shows the relationship between the magnitude of a vector and its dot product with itself:

$$\begin{aligned} \mathbf{v}\cdot\mathbf{v} &= \langle v_1,v_2,v_3\rangle\cdot\langle v_1,v_2,v_3\rangle \\ &= (v_1)^2+(v_2)^2+(v_3)^2 \\ &= \left[\sqrt{(v_1)^2+(v_2)^2+(v_3)^2}\right]^2 \\ &= \|\mathbf{v}\|^2. \end{aligned}$$

□

Note that by property iv. we have $\mathbf{0}\cdot\mathbf{v} = 0$. Also by property iv. if $\mathbf{v}\cdot\mathbf{v} = 0$, then $\mathbf{v} = \mathbf{0}$.

Example 4.3.2: Using Properties of the Dot Product

Let $\mathbf{a} = \langle 1,2,-3\rangle$, $\mathbf{b} = \langle 0,2,4\rangle$, and $\mathbf{c} = \langle 5,-1,3\rangle$.

Find each of the following products.

a. $(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$
b. $\mathbf{a}\cdot(2\mathbf{c})$
c. $\|\mathbf{b}\|^2$

Solution

a. Note that this expression asks for the scalar multiple of $\mathbf{c}$ by $\mathbf{a}\cdot\mathbf{b}$:

$$\begin{aligned} (\mathbf{a}\cdot\mathbf{b})\mathbf{c} &= (\langle 1,2,-3\rangle\cdot\langle 0,2,4\rangle)\langle 5,-1,3\rangle \\ &= (1(0)+2(2)+(-3)(4))\langle 5,-1,3\rangle \\ &= -8\langle 5,-1,3\rangle \\ &= \langle -40,8,-24\rangle. \end{aligned}$$

b. This expression is a dot product of vector $\mathbf{a}$ and scalar multiple $2\mathbf{c}$:

$$\begin{aligned} \mathbf{a}\cdot(2\mathbf{c}) &= 2(\mathbf{a}\cdot\mathbf{c}) \\ &= 2(\langle 1,2,-3\rangle\cdot\langle 5,-1,3\rangle) \\ &= 2(1(5)+2(-1)+(-3)(3)) \\ &= 2(-6) = -12. \end{aligned}$$

c. Simplifying this expression is a straightforward application of the dot product:

$$\begin{aligned}\|\mathbf{b}\|^2 &= \mathbf{b}\cdot\mathbf{b}\\ &= \langle 0,2,4\rangle\cdot\langle 0,2,4\rangle\\ &= 0^2+2^2+4^2\\ &= 0+4+16\\ &= 20.\end{aligned}$$

Exercise 4.3.2

Find the following products for $\mathbf{p}=\langle 7,0,2\rangle$, $\mathbf{q}=\langle -2,2,-2\rangle$, and $\mathbf{r}=\langle 0,2,-3\rangle$.

a. $(\mathbf{r}\cdot\mathbf{p})\mathbf{q}$
b. $\|\mathbf{p}\|^2$

Hint available at the end of this section.

Using the Dot Product to Find the Angle between Two Vectors

When two nonzero vectors are placed in standard position, whether in two dimensions or three dimensions, they form an angle between them (Figure 4.3.1). The dot product provides a way to find the measure of this angle. This property is a result of the fact that we can express the dot product in terms of the cosine of the angle formed by two vectors.

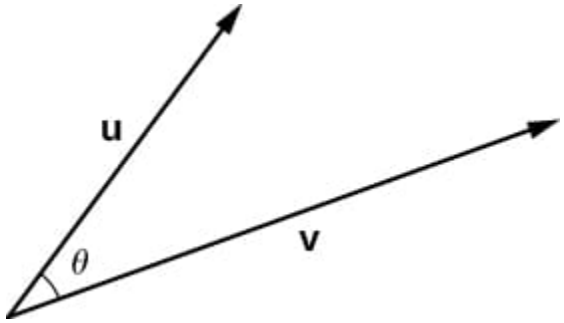


Figure 4.3.1: θ is the angle between two nonzero vectors $\mathbf{u}$ *and* $\mathbf{v}$ *such that* $0\le\theta\le\pi$.

Evaluating a Dot Product

The dot product of two vectors is the product of the magnitude of each vector and the cosine of the angle between them:

$$\mathbf{u}\cdot\mathbf{v}=\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta. \tag{4.3.7}$$

Proof

Place vectors $\mathbf{u}$ and $\mathbf{v}$ in standard position and consider the vector $\mathbf{v}-\mathbf{u}$ (Figure 4.3.2). These three vectors form a triangle with side lengths $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, and $\|\mathbf{v}-\mathbf{u}\|$.

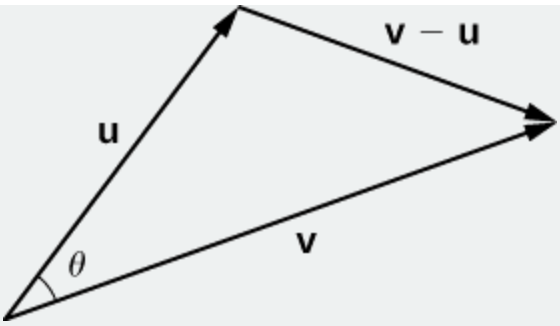


Figure 4.3.2: Vector diagram illustrating the relationship between vectors $\mathbf{u}$, $\mathbf{v}$, and their difference $\mathbf{v}-\mathbf{u}$, with an angle θ between $\mathbf{u}$ and $\mathbf{v}$.

Recall from trigonometry that the law of cosines describes the relationship among the side lengths of the triangle and the angle θ. Applying the law of cosines here gives

$$\|\mathbf{v}-\mathbf{u}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta. \tag{4.3.8}$$

The dot product provides a way to rewrite the left side of Equation 4.3.8:

$$\begin{aligned}\|\mathbf{v}-\mathbf{u}\|^2 &= (\mathbf{v}-\mathbf{u})\cdot(\mathbf{v}-\mathbf{u})\\ &= (\mathbf{v}-\mathbf{u})\cdot\mathbf{v} - (\mathbf{v}-\mathbf{u})\cdot\mathbf{u}\\ &= \mathbf{v}\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{u}\\ &= \mathbf{v}\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{u}\\ &= \|\mathbf{v}\|^2 - 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{u}\|^2.\end{aligned}$$

Substituting into the law of cosines yields

$$\begin{aligned}\|\mathbf{v}-\mathbf{u}\|^2 &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta\\ \|\mathbf{v}\|^2 - 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{u}\|^2 &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta\\ -2\mathbf{u}\cdot\mathbf{v} &= -2\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta\\ \mathbf{u}\cdot\mathbf{v} &= \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta.\end{aligned}$$

□

We can use the form of the dot product in Equation 4.3.7 to find the measure of the angle between two nonzero vectors by rearranging Equation 4.3.7 to solve for the cosine of the angle:

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}. \tag{4.3.9}$$

Using this equation, we can find the cosine of the angle between two nonzero vectors. Since we are considering the smallest angle between the vectors, we assume $0° \le \theta \le 180°$ (or $0 \le \theta \le \pi$ if we are working in radians). The inverse cosine is unique over this range, so we are then able to determine the measure of the angle θ.

Example 4.3.3: Finding the Angle between Two Vectors

Find the measure of the angle between each pair of vectors.

a. $\mathbf{i}+\mathbf{j}+\mathbf{k}$ and $2\mathbf{i}-\mathbf{j}-3\mathbf{k}$
b. $\langle 2,5,6\rangle$ and $\langle -2,-4,4\rangle$

Solution

a. To find the cosine of the angle formed by the two vectors, substitute the components of the vectors into Equation 4.3.9:

$$\begin{aligned}\cos\theta &= \frac{(\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot(2\mathbf{i}-\mathbf{j}-3\mathbf{k})}{\|\,\mathbf{i}+\mathbf{j}+\mathbf{k}\,\|\cdot\|\,2\mathbf{i}-\mathbf{j}-3\mathbf{k}\,\|}\\ &= \frac{1(2)+(1)(-1)+(1)(-3)}{\sqrt{1^2+1^2+1^2}\sqrt{2^2+(-1)^2+(-3)^2}}\\ &= \frac{-2}{\sqrt{3}\sqrt{14}} = \frac{-2}{\sqrt{42}}.\end{aligned}$$

Therefore, $\theta = \arccos\dfrac{-2}{\sqrt{42}}$ rad.

b. Start by finding the value of the cosine of the angle between the vectors:

$$\begin{aligned}\cos\theta &= \frac{\langle 2,5,6\rangle\cdot\langle -2,-4,4\rangle}{\|\,\langle 2,5,6\rangle\,\|\cdot\|\,\langle -2,-4,4\rangle\,\|}\\ &= \frac{2(-2)+(5)(-4)+(6)(4)}{\sqrt{2^2+5^2+6^2}\sqrt{(-2)^2+(-4)^2+4^2}}\\ &= \frac{0}{\sqrt{65}\sqrt{36}} = 0.\end{aligned}$$

Now, $\cos\theta = 0$ and $0 \le \theta \le \pi$, so $\theta = \pi/2$.

Watch the accompanying video for a visual demonstration of the concept.

- **Video length:** 2 minutes 32 seconds.
- **Context:** This video demonstrates how to find the angle between two vectors. $\mathbf{u} = 3\mathbf{ik}$ and $\mathbf{v} = 2\mathbf{i}+\mathbf{j}+3\mathbf{k}$

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Exercise 4.3.3

Find the measure of the angle, in radians, formed by vectors $\mathbf{a} = \langle 1,2,0\rangle$ and $\mathbf{b} = \langle 2,4,1\rangle$. Round to the nearest hundredth.

Hint available at the end of this section.

The angle between two vectors can be acute $(0 < \cos\theta < 1)$, obtuse $(-1 < \cos\theta < 0)$, or straight $(\cos\theta = -1)$. If $\cos\theta = 1$, then both vectors have the same direction. If $\cos\theta = 0$, then the vectors, when placed in standard position, form a right angle (Figure 4.3.3). We can formalize this result into a theorem regarding orthogonal (perpendicular) vectors.

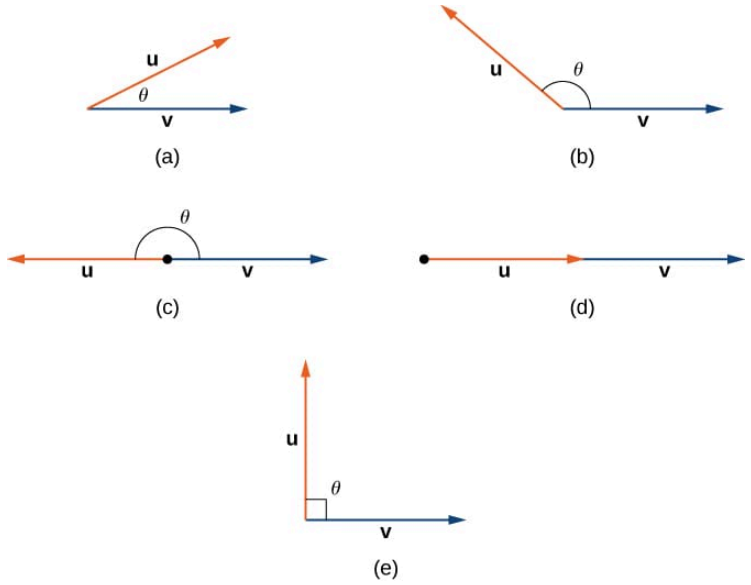


Figure 4.3.3: (a) An acute angle has $0 < \cos\theta < 1$. (b) An obtuse angle has $-1 < \cos\theta < 0$. (c) A straight line has $\cos\theta = -1$. (d) If the vectors have the same direction, $\cos\theta = 1$. (e) If the vectors are orthogonal (perpendicular), $\cos\theta = 0$.

Example 4.3.4 Determine Triangle Types Using Vectors

Determine if the triangle with vertices (1,2,0), (3,5,-1), and (0,2,3) is an acute triangle, an obtuse triangle, or a right triangle.

Solution

- Video length: 4 minutes 23 seconds.
- Context: This video demonstrates how to use vectors to determine if a triangle is acute, obtuse, or a right triangle given the vertices $(1, 2, 0)$, $(3, 5, -1)$, and $(0, 2, 3)$

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

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Orthogonal Vectors

The nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal vectors** if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

Proof

Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors, and let θ denote the angle between them. First, assume $\mathbf{u} \cdot \mathbf{v} = 0$.Then

$$\|\mathbf{u}\|\|\mathbf{v}\| \cos\theta = 0. \tag{4.3.10}$$

However, $\|\mathbf{u}\| \neq 0$ and $\|\mathbf{v}\| \neq 0$, so we must have $\cos\theta = 0$. Hence, $\theta = 90\,°$, and the vectors are orthogonal.

Now assume $\mathbf{u}$ and $\mathbf{v}$ are orthogonal. Then $\theta = 90\,°$ and we have

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta \\ &= \|\mathbf{u}\|\|\mathbf{v}\|\cos 90° \\ &= \|\mathbf{u}\|\|\mathbf{v}\|(0) \\ &= 0.\end{aligned}$$

□

The terms **orthogonal, perpendicular,** and **normal** each indicate that mathematical objects are intersecting at right angles. The use of each term is determined mainly by its context. We say that vectors are orthogonal and lines are perpendicular. The term *normal* is used most often when measuring the angle made with a plane or other surface.

Example 4.3.5: Identifying Orthogonal Vectors

Determine whether $\mathbf{p} = \langle 1, 0, 5\rangle$ and $\mathbf{q} = \langle 10, 3, -2\rangle$ are orthogonal vectors.

Solution

Using the definition, we need only check the dot product of the vectors:

$$\mathbf{p}\cdot\mathbf{q} = 1(10) + (0)(3) + (5)(-2) = 10 + 0 - 10 = 0.$$

Because $\mathbf{p}\cdot\mathbf{q} = 0$, the vectors are orthogonal (Figure 4.3.4).

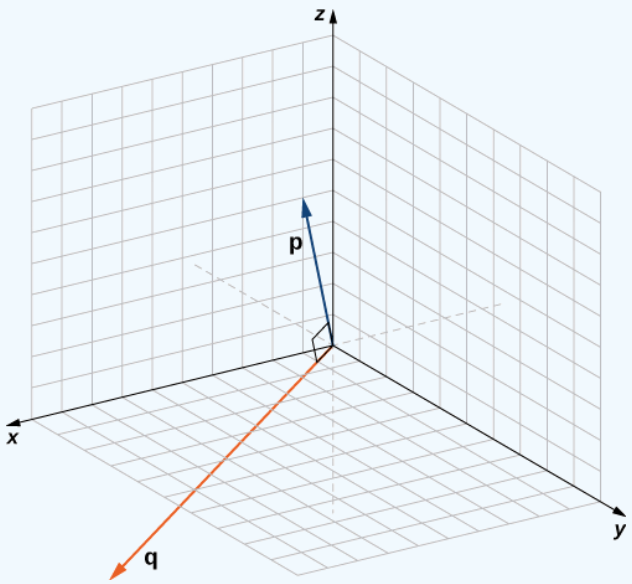


Figure 4.3.4: Vectors $\mathbf{p}$ and $\mathbf{q}$ form a right angle when their initial points are aligned.

Watch the accompanying video for a visual demonstration of the concept.

- **Video length:** 2 minutes 40 seconds.
- **Context:** This video demonstrates how to find two vectors in opposite directions that are orthogonal to the given third vector $\mathbf{u} = \langle 2, -1, -3\rangle$

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Exercise 4.3.4

For which value of x is $\mathbf{p} = \langle 2, 8, -1 \rangle$ orthogonal to $\mathbf{q} = \langle x, -1, 2 \rangle$?

Hint available at the end of this section.

Example 4.3.6: Measuring the Angle Formed by Two Vectors

Let $\mathbf{v} = \langle 2, 3, 3 \rangle$. Find the measures of the angles formed by the following vectors.

a. $\mathbf{v}$ and $\mathbf{i}$
b. $\mathbf{v}$ and $\mathbf{j}$
c. $\mathbf{v}$ and $\mathbf{k}$

Solution

a. Let α be the angle formed by $\mathbf{v}$ and $\mathbf{i}$:

$$\cos\alpha = \frac{\mathbf{v}\cdot\mathbf{i}}{\|\mathbf{v}\|\cdot\|\mathbf{i}\|} = \frac{\langle 2,3,3\rangle\cdot\langle 1,0,0\rangle}{\sqrt{2^2+3^2+3^2}\sqrt{1}} = \frac{2}{\sqrt{22}}$$

$$\alpha = \arccos\frac{2}{\sqrt{22}} \approx 1.130\,\text{rad}.$$

b. Let β represent the angle formed by $\mathbf{v}$ and $\mathbf{j}$:

$$\cos\beta = \frac{\mathbf{v}\cdot\mathbf{j}}{\|\mathbf{v}\|\cdot\|\mathbf{j}\|} = \frac{\langle 2,3,3\rangle\cdot\langle 0,1,0\rangle}{\sqrt{2^2+3^2+3^2}\sqrt{1}} = \frac{3}{\sqrt{22}}$$

$$\beta = \arccos\frac{3}{\sqrt{22}} \approx 0.877\,\text{rad}.$$

c. Let γ represent the angle formed by $\mathbf{v}$ and $\mathbf{k}$:

$$\cos\gamma = \frac{\mathbf{v}\cdot\mathbf{k}}{\|\mathbf{v}\|\cdot\|\mathbf{k}\|} = \frac{\langle 2,3,3\rangle\cdot\langle 0,0,1\rangle}{\sqrt{2^2+3^2+3^2}\sqrt{1}} = \frac{3}{\sqrt{22}}$$

$$\gamma = \arccos\frac{3}{\sqrt{22}} \approx 0.877\,\text{rad}.$$

Exercise 4.3.5

Let $\mathbf{v} = \langle 3, -5, 1 \rangle$. Find the measure of the angles formed by each pair of vectors.

a. $\mathbf{v}$ and $\mathbf{i}$
b. $\mathbf{v}$ and $\mathbf{j}$
c. $\mathbf{v}$ and $\mathbf{k}$

Hint available at the end of this section.

The angle a vector makes with each of the coordinate axes, called a direction angle, is very important in practical computations, especially in a field such as engineering. For example, in astronautical engineering, the angle at which a rocket is launched must be determined very precisely. A very small error in the angle can lead to the rocket going hundreds of miles off course. Direction angles are often calculated by using the dot product and the cosines of the angles, called the direction cosines. Therefore, we define both these angles and their cosines.

Definition: Direction Angles

The angles formed by a nonzero vector and the coordinate axes are called the **direction angles** for the vector (Figure $4.3.5$). The cosines for these angles are called the **direction cosines.**

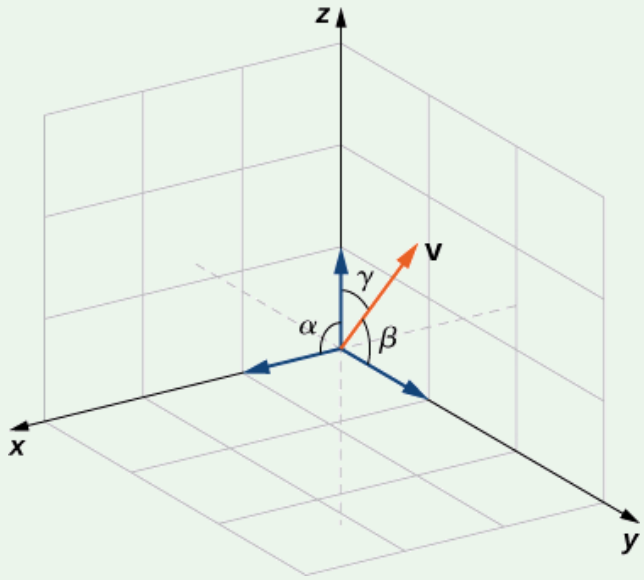


Figure $4.3.5$: Angle α is formed by vector $\mathbf{v}$ and unit vector $\mathbf{i}$. Angle β is formed by vector $\mathbf{v}$ and unit vector $\mathbf{j}$. Angle γ is formed by vector $\mathbf{v}$ and unit vector $\mathbf{k}$.

In Example, the direction cosines of $\mathbf{v} = \langle 2, 3, 3 \rangle$ are $\cos\alpha = \dfrac{2}{\sqrt{22}}$, $\cos\beta = \dfrac{3}{\sqrt{22}}$, and $\cos\gamma = \dfrac{3}{\sqrt{22}}$. The direction angles of $\mathbf{v}$ are $\alpha = 1.130$ rad, $\beta = 0.877$ rad, and $\gamma = 0.877$ rad.

So far, we have focused mainly on vectors related to force, movement, and position in three-dimensional physical space. However, vectors are often used in more abstract ways. For example, suppose a fruit vendor sells apples, bananas, and oranges. On a given day, he sells 30 apples, 12 bananas, and 18 oranges. He might use a quantity vector, $\mathbf{q} = \langle 30, 12, 18 \rangle$, to represent the quantity of fruit he sold that day. Similarly, he might want to use a price vector, $\mathbf{p} = \langle 0.50, 0.25, 1 \rangle$, to indicate that he sells his apples for 50¢ each, bananas for 25¢ each, and oranges for $1 apiece. In this example, although we could still graph these vectors, we do not interpret them as literal representations of position in the physical world. We are simply using vectors to keep track of particular pieces of information about apples, bananas, and oranges.

This idea might seem a little strange, but if we simply regard vectors as a way to order and store data, we find they can be quite a powerful tool. Going back to the fruit vendor, let's think about the dot product, $\mathbf{q} \cdot \mathbf{p}$. We compute it by multiplying the number of apples sold (30) by the price per apple (50¢), the number of bananas sold by the price per banana, and the number of oranges sold by the price per orange. We then add all these values together. So, in this example, the dot product tells us how much money the fruit vendor had in sales on that particular day.

When we use vectors in this more general way, there is no reason to limit the number of components to three. What if the fruit vendor decides to start selling grapefruit? In that case, he would want to use four-dimensional quantity and price vectors to represent the number of apples, bananas, oranges, and grapefruit sold, and their unit prices. As you might expect, to calculate the dot product of four-dimensional vectors, we simply add the products of the components as before, but the sum has four terms instead of three.

Example 4.3.7: Using Vectors in an Economic Context

AAA Party Supply Store sells invitations, party favors, decorations, and food service items such as paper plates and napkins. When AAA buys its inventory, it pays 25¢ per package for invitations and party favors. Decorations cost AAA 50¢ each, and food service items cost 20¢ per package. AAA sells invitations for $2.50 per package and party favors for $1.50 per package. Decorations sell for $4.50 each and food service items for $1.25 per package.

During the month of May, AAA Party Supply Store sells 1258 invitations, 342 party favors, 2426 decorations, and 1354 food service items. Use vectors and dot products to calculate how much money AAA made in sales during the month of May. How much did the store make in profit?

Solution

The cost, price, and quantity vectors are

$$\begin{aligned}\mathbf{c} &= \langle 0.25, 0.25, 0.50, 0.20\rangle \\ \mathbf{p} &= \langle 2.50, 1.50, 4.50, 1.25\rangle \\ \mathbf{q} &= \langle 1258, 342, 2426, 1354\rangle.\end{aligned}$$

AAA sales for the month of May can be calculated using the dot product $\mathbf{p}\cdot\mathbf{q}$. We have

$$\begin{aligned}\mathbf{p}\cdot\mathbf{q} &= \langle 2.50, 1.50, 4.50, 1.25\rangle \cdot \langle 1258, 342, 2426, 1354\rangle \\ &= 3145 + 513 + 10917 + 1692.5 \\ &= 16267.5.\end{aligned}$$

So, AAA took in $16,267.50 during the month of May. To calculate the profit, we must first calculate how much AAA paid for the items sold. We use the dot product $c\cdot q$ to get

$$\begin{aligned}\mathbf{c}\cdot\mathbf{q} &= \langle 0.25, 0.25, 0.50, 0.20\rangle \cdot \langle 1258, 342, 2426, 1354\rangle \\ &= 314.5 + 85.5 + 1213 + 270.8 \\ &= 1883.8.\end{aligned}$$

So, AAA paid $1,883.30 for the items they sold. Their profit, then, is given by

$$\mathbf{p}\cdot\mathbf{q} - \mathbf{c}\cdot\mathbf{q} = 16267.5 - 1883.8 = 14383.7.$$

Therefore, AAA Party Supply Store made $14,383.70 in May.

Exercise 4.3.6

On June 1, AAA Party Supply Store decided to increase the price they charge for party favors to $2 per package. They also changed suppliers for their invitations, and are now able to purchase invitations for only 10¢ per package. All their other costs and prices remain the same. If AAA sells 1408 invitations, 147 party favors, 2112 decorations, and 1894 food service items in the month of June, use vectors and dot products to calculate their total sales and profit for June.

Hint available at the end of this section.

Projections

As we have seen, addition combines two vectors to create a resultant vector. But what if we are given a vector and we need to find its component parts? We use vector projections to perform the opposite process; they can break down a vector into its components. The magnitude of a vector projection is a scalar projection. For example, if a child is pulling the handle of a wagon at a 55° angle, we can use projections to determine how much of the force on the handle is actually moving the wagon forward (4.3.6). We return to this example and learn how to solve it after we see how to calculate projections.

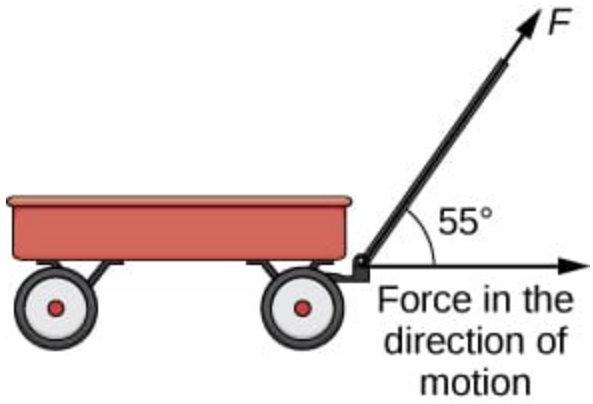


Figure 4.3.6: A wagon is pulled with a force at a 55° angle over a distance along a horizontal surface. *The force in the direction of motion is horizontal.*

Definition: Vector and Scalar Projection

The **vector projection** of $\mathbf{v}$ onto $\mathbf{u}$ is the vector labeled $\text{proj}_{\mathbf{u}}\mathbf{v}$ in Figure 4.3.7. It has the same initial point as $\mathbf{u}$ and $\mathbf{v}$ and the same direction as $\mathbf{u}$, and represents the component of $\mathbf{v}$ that acts in the direction of $\mathbf{u}$. Let θ represent the angle between $\mathbf{u}$ and $\mathbf{v}$. Then, $\|\mathbf{v}\|\cos\theta$ represents the "length" of the shadow of $\mathbf{v}$ on $\mathbf{u}$. Note that when θ is an obtuse angle, $\|\mathbf{v}\|\cos\theta$ will be negative. This quantity is known as the **scalar projection** of $\mathbf{v}$ onto $\mathbf{u}$:

$$\text{comp}_{\mathbf{u}}\mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|.} \tag{4.3.11}$$

Multiplying this by a unit vector in the direction of $\mathbf{u}$, namely $\frac{\mathbf{u}}{\|\mathbf{u}\|}$, we get the **vector projection** of $\mathbf{v}$ onto $\mathbf{u}$:

$$\text{proj}_{\mathbf{u}}\mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|}\left(\frac{1}{\|\mathbf{u}\|}\mathbf{u}\right) = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u}. \tag{4.3.12}$$

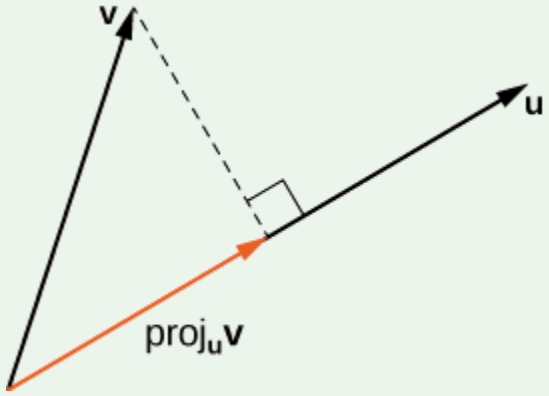


Figure 4.3.7: The projection of $\mathbf{v}$ onto $\mathbf{u}$ shows the component of vector $\mathbf{v}$ in the direction of $\mathbf{u}$.

Figure 4.3.8 below shows that $\text{proj}_{\mathbf{u}}\mathbf{v}$ points in the direction opposite of $\mathbf{u}$ when the angle between $\mathbf{u}$ and $\mathbf{v}$ is obtuse. In this case, $\text{comp}_{\mathbf{u}}\mathbf{v}$ is negative.

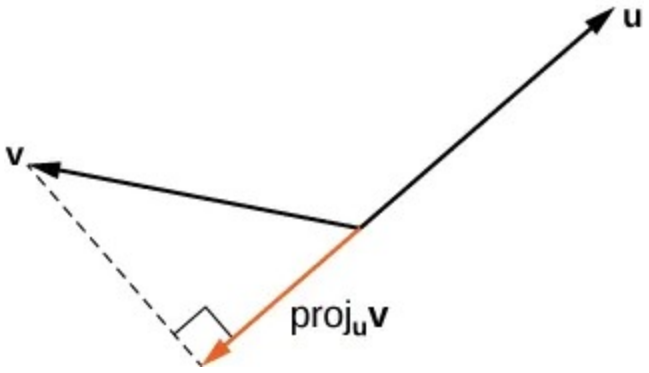


Figure 4.3.8: The projection of $\mathbf{v}$ onto $\mathbf{u}$ when the angle between $\mathbf{u}$ and $\mathbf{v}$ is obtuse, making the direction of $\text{proj}_{\mathbf{u}}\mathbf{v}$ opposite of $\mathbf{u}$ and $\text{comp}_{\mathbf{u}}\mathbf{v}$ negative.

Example 4.3.8: Finding Projections

Find the projection of $\mathbf{v}$ onto $\mathbf{u}$.

a. $\mathbf{v} = \langle 3, 5, 1\rangle$ and $\mathbf{u} = \langle -1, 4, 3\rangle$
b. $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j}$ and $\mathbf{u} = \mathbf{i} + 6\mathbf{j}$

Solution

a. Substitute the components of $\mathbf{v}$ and $\mathbf{u}$ into the formula for the projection:

$$\begin{aligned}\text{proj}_{\mathbf{u}}\mathbf{v} &= \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u}\\ &= \frac{\langle -1,4,3\rangle\cdot\langle 3,5,1\rangle}{\|\langle -1,4,3\rangle\|^2}\langle -1,4,3\rangle\\ &= \frac{-3+20+3}{(-1)^2+4^2+3^2}\langle -1,4,3\rangle\\ &= \frac{20}{26}\langle -1,4,3\rangle\\ &= \langle -\frac{10}{13}, \frac{40}{13}, \frac{30}{13}\rangle.\end{aligned}$$

b. To find the two-dimensional projection, simply adapt the formula to the two-dimensional case:

$$\begin{aligned}\text{proj}_{\mathbf{u}}\mathbf{v} &= \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u}\\ &= \frac{(\mathbf{i}+6\mathbf{j})\cdot(3i-2j)}{\|\mathbf{i}+6\mathbf{j}\|^2}(i+6j)\\ &= \frac{1(3)+6(-2)}{1^2+6^2}(\mathbf{i}+6\mathbf{j})\\ &= -\frac{9}{37}(\mathbf{i}+6\mathbf{j})\\ &= -\frac{9}{37}\mathbf{i} - \frac{54}{37}\mathbf{j}.\end{aligned}$$

Watch the accompanying video for a visual demonstration of the concept.

- **Video length:** 2 minutes 52 seconds.
- **Context:** This video demonstrates how to find the projection of a vector $\mathbf{u}$ onto a vector $\mathbf{v}$ and the component of $\mathbf{u}$ orthogonal to $\mathbf{v}$.

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Sometimes it is useful to decompose vectors—that is, to break a vector apart into a sum. This process is called the **resolution of a vector into components.** Projections allow us to identify two orthogonal vectors having a desired sum. For example, let $\mathbf{v} = \langle 6, -4 \rangle$ and let $\mathbf{u} = \langle 3, 1 \rangle$. We want to decompose the vector $\mathbf{v}$ into orthogonal components such that one of the component vectors has the same direction as $\mathbf{u}$.

We first find the component that has the same direction as $\mathbf{u}$ by projecting $\mathbf{v}$ onto $\mathbf{u}$. Let $\mathbf{p} = \text{proj}_{\mathbf{u}}\mathbf{v}$. Then, we have

$$\begin{aligned} \mathbf{p} &= \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u} \\ &= \frac{18 - 4}{9 + 1}\mathbf{u} \\ = \frac{7}{5}\mathbf{u} = \frac{7}{5}\langle 3, 1 \rangle &= \langle \frac{21}{5}, \frac{7}{5} \rangle. \end{aligned}$$

Now consider the vector $\mathbf{q} = \mathbf{v} - \mathbf{p}$. We have

$$\begin{aligned} \mathbf{q} &= \mathbf{v} - \mathbf{p} \\ &= \langle 6, -4 \rangle - \langle \frac{21}{5}, \frac{7}{5} \rangle \\ &= \langle \frac{9}{5}, -\frac{27}{5} \rangle. \end{aligned}$$

Clearly, by the way we defined $\mathbf{q}$, we have $\mathbf{v} = \mathbf{q} + \mathbf{p}$, and

$$\begin{aligned} \mathbf{q} \cdot \mathbf{p} &= \langle \frac{9}{5}, -\frac{27}{5} \rangle \cdot \langle \frac{21}{5}, \frac{7}{5} \rangle \\ &= \frac{9(21)}{25} + -\frac{27(7)}{25} \\ &= \frac{189}{25} - \frac{189}{25} = 0. \end{aligned}$$

Therefore, $\mathbf{q}$ and $\mathbf{p}$ are orthogonal.

Example 4.3.9: Resolving Vectors into Components

Express $\mathbf{v} = \langle 8, -3, -3 \rangle$ as a sum of orthogonal vectors such that one of the vectors has the same direction as $\mathbf{u} = \langle 2, 3, 2 \rangle$.

Solution

Let $\mathbf{p}$ represent the projection of $\mathbf{v}$ onto $\mathbf{u}$:

$$\begin{aligned}\mathbf{p} &= \text{proj}_{\mathbf{u}}\mathbf{v} \\ &= \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u} \\ &= \frac{\langle 2,3,2\rangle\cdot\langle 8,-3,-3\rangle}{\|\ \langle 2,3,2\rangle\ \|^2}\langle 2,3,2\rangle \\ &= \frac{16-9-6}{2^2+3^2+2^2}\langle 2,3,2\rangle \\ &= \frac{1}{17}\langle 2,3,2\rangle \\ &= \langle \frac{2}{17},\frac{3}{17},\frac{2}{17}\rangle.\end{aligned}$$

Then,

$$\begin{aligned}\mathbf{q} = \mathbf{v}-\mathbf{p} &= \langle 8,-3,-3\rangle - \langle \frac{2}{17},\frac{3}{17},\frac{2}{17}\rangle \\ &= \langle \frac{134}{17},-\frac{54}{17},-\frac{53}{17}\rangle.\end{aligned}$$

To check our work, we can use the dot product to verify that $\mathbf{p}$ and $\mathbf{q}$ are orthogonal vectors:

$$\begin{aligned}\mathbf{p}\cdot\mathbf{q} &= \langle \frac{2}{17},\frac{3}{17},\frac{2}{17}\rangle\cdot\langle \frac{134}{17},-\frac{54}{17},-\frac{53}{17}\rangle \\ &= \frac{268}{17}-\frac{162}{17}-\frac{106}{17}=0.\end{aligned}$$

Then,

$$\mathbf{v} = \mathbf{p}+\mathbf{q} = \langle \frac{2}{17},\frac{3}{17},\frac{2}{17}\rangle + \langle \frac{134}{17},-\frac{54}{17},-\frac{53}{17}\rangle.$$

Exercise 4.3.7

Express $\mathbf{v}=5\mathbf{i}-\mathbf{j}$ as a sum of orthogonal vectors such that one of the vectors has the same direction as $\mathbf{u}=4\mathbf{i}+2\mathbf{j}$.

Hint available at the end of this section.

Example 4.3.10: Scalar Projection of Velocity

A container ship leaves port traveling $15\,^\circ$ north of east. Its engine generates a speed of 20 knots along that path Figure 4.3.9 . In addition, the ocean current moves the ship northeast at a speed of 2 knots. Considering both the engine and the current, how fast is the ship moving in the direction $15\,^\circ$ north of east? Round the answer to two decimal places.

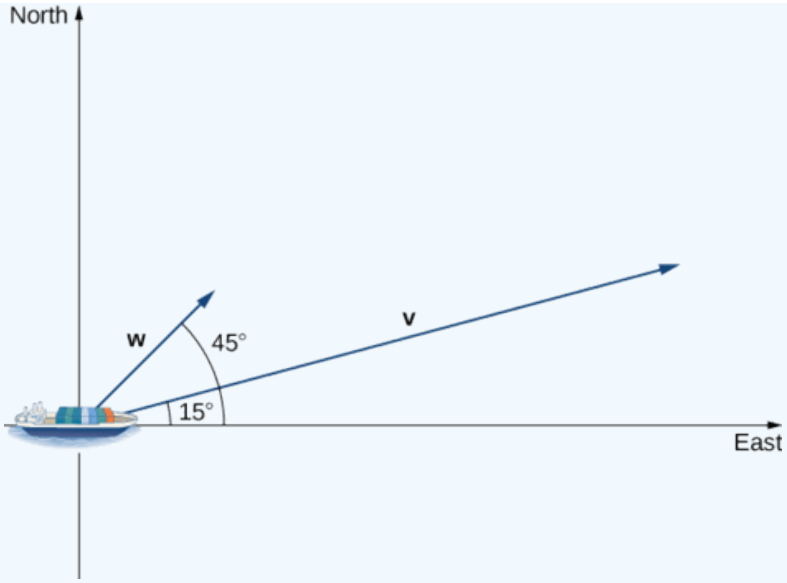


Figure 4.3.9: A ship moving in the direction 15 ° north of east with current moving in northeast direction at 45 °.

Solution

Let $\mathbf{v}$ be the velocity vector generated by the engine, and let $\mathbf{w}$ be the velocity vector of the current. We already know $\|\mathbf{v}\| = 20$ along the desired route. We just need to add in the scalar projection of $\mathbf{w}$ onto $\mathbf{v}$. We get

$$\text{comp}_{\mathbf{v}}\mathbf{w} = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|}$$

$$= \frac{\|\mathbf{v}\|\|\mathbf{w}\|\cos(30°)}{\|\mathbf{v}\|} = \|\mathbf{w}\|\cos(30°) = 2\frac{\sqrt{3}}{2} = \sqrt{3} \approx 1.73 \text{ knots}.$$

The ship is moving at 21.73 knots in the direction 15 ° north of east.

Exercise 4.3.8

Repeat the previous example, but assume the ocean current is moving southeast instead of northeast, as shown in Figure 4.3.10.

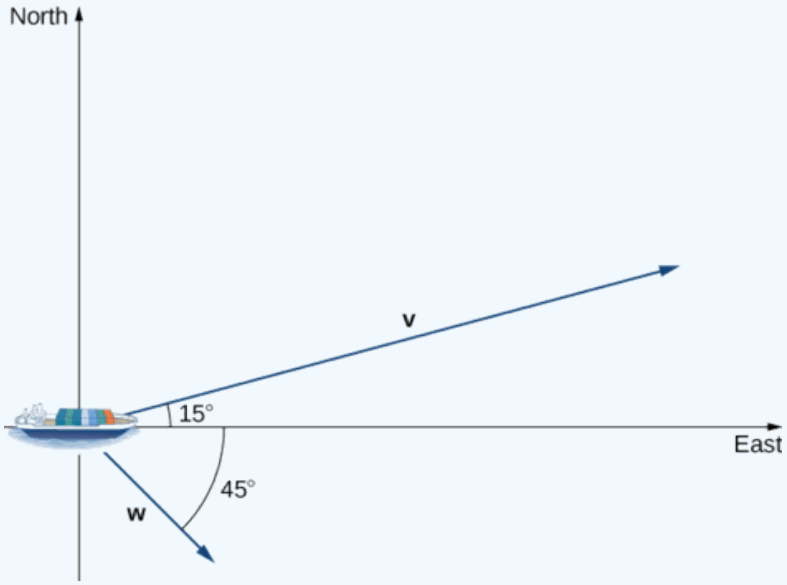


Figure 4.3.10: A ship moving in the direction 15 ° north of east with current moving in southeast direction at 45 °.

Hint available at the end of this section.

Work

Now that we understand dot products, we can see how to apply them to real-life situations. The most common application of the dot product of two vectors is in the calculation of work.

From physics, we know that work is done when an object is moved by a force. When the force is constant and applied in the same direction the object moves, then we define the work done as the product of the force and the distance the object travels: $W = Fd$. We saw several examples of this type in earlier chapters. Now imagine the direction of the force is different from the direction of motion, as with the example of a child pulling a wagon. To find the work done, we need to multiply the component of the force that acts in the direction of the motion by the magnitude of the displacement. The dot product allows us to do just that. If we represent an applied force by a vector $\mathbf{F}$ and the displacement of an object by a vector $\mathbf{s}$, then the **work done by the force** is the dot product of $\mathbf{F}$ and $\mathbf{s}$.

Definition: Work

When a constant force is applied to an object so the object moves in a straight line from point P to point Q, the **work** W done by the force $\mathbf{F}$, acting at an angle θ from the line of motion, is given by

$$W = \mathbf{F} \cdot \vec{PQ} = \| \mathbf{F} \| \| \vec{PQ} \| \cos\theta. \tag{4.3.13}$$

Let's revisit the problem of the child's wagon introduced earlier. Suppose a child is pulling a wagon with a force having a magnitude of 8 lb on the handle at an angle of 55°. If the child pulls the wagon 50 ft, find the work done by the force (Figure 4.3.11).

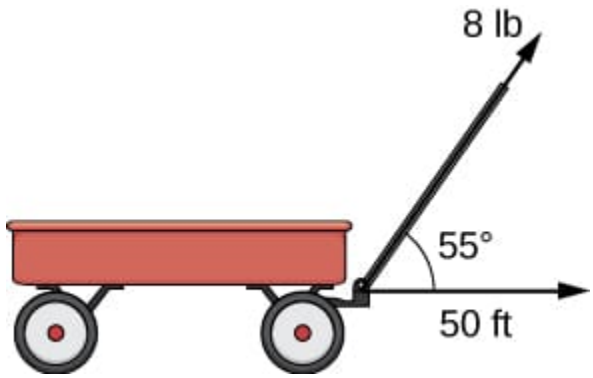


Figure 4.3.11: A wagon is pulled with a force of 8 lb at a 55° angle over a distance of 50 ft along a horizontal surface.

We have

$$W = \| \mathbf{F} \| \| \vec{PQ} \| \cos\theta = 8(50)(\cos(55°)) \approx 229\,\text{ft·lb}.$$

In U.S. standard units, we measure the magnitude of force $\| \mathbf{F} \|$ in pounds. The magnitude of the displacement vector $\| \vec{PQ} \|$ tells us how far the object moved, and it is measured in feet. The customary unit of measure for work, then, is the foot-pound. One foot-pound is the amount of work required to move an object weighing 1 lb a distance of 1 ft straight up. In the metric system, the unit of measure for force is the newton (N), and the unit of measure of magnitude for work is a newton-meter (N·m), or a joule (J).

Example 4.3.11: Calculating Work

A conveyor belt generates a force $\mathbf{F} = 5\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ that moves a suitcase from point $(1, 1, 1)$ to point $(9, 4, 7)$ along a straight line. Find the work done by the conveyor belt. The distance is measured in meters and the force is measured in newtons.

Solution

The displacement vector $\vecd{PQ}$ has initial point $(1, 1, 1)$ and terminal point $(9, 4, 7)$:

$$\vecd{PQ} = \langle 9-1, 4-1, 7-1 \rangle = \langle 8, 3, 6 \rangle = 8\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}.$$

Work is the dot product of force and displacement:

$$\begin{aligned} W &= \mathbf{F} \cdot \vecd{PQ} \\ &= (5\mathbf{i} - 3\mathbf{j} + \mathbf{k}) \cdot (8\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}) \\ &= 5(8) + (-3)(3) + 1(6) \\ &= 37\,\text{N·m} \\ &= 37\,\text{J} \end{aligned}$$

Watch the accompanying video for a visual demonstration of the concept.

- **Video length:** 1 minute 36 seconds.
- **Context**: This video demonstrates how to use a dot product to find work done.

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Exercise 4.3.9

A constant force of 30 lb is applied at an angle of 60° to pull a handcart 10 ft across the ground. What is the work done by this force?

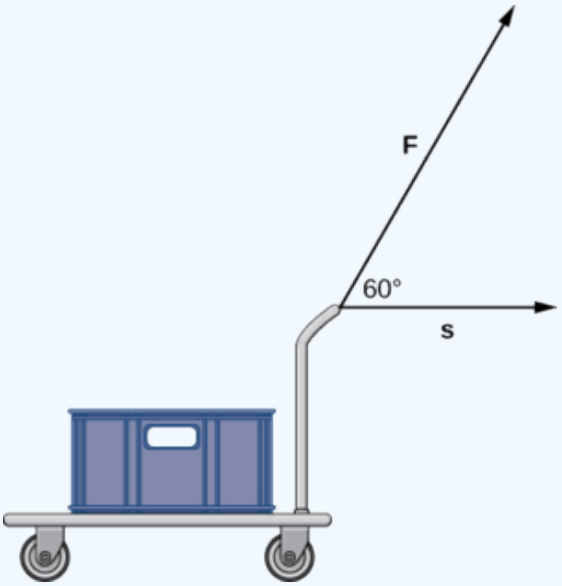


Figure 4.3.12: A force of 30 lb is applied at an angle of 60° above the horizon to pull a handcart.

Hint available at the end of this section.

Key Concepts

- The dot product, or scalar product, of two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$.
- The dot product satisfies the following properties:
  - $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
  - $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
  - $c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v})$
  - $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$
- The dot product of two vectors can be expressed, alternatively, as $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos\theta$. This form of the dot product is useful for finding the measure of the angle formed by two vectors.
- Vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$.
- The angles formed by a nonzero vector and the coordinate axes are called the *direction angles* for the vector. The cosines of these angles are known as the *direction cosines*.
- The vector projection of $\mathbf{v}$ onto $\mathbf{u}$ is the vector $\text{proj}_{\mathbf{u}}\mathbf{v} = \dfrac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u}$. The magnitude of this vector is known as the *scalar projection* of $\mathbf{v}$ onto $\mathbf{u}$, given by $\text{comp}_{\mathbf{u}}\mathbf{v} = \dfrac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|}$.
- Work is done when a force is applied to an object, causing displacement. When the force is represented by the vector $\mathbf{F}$ and the displacement is represented by the vector $\mathbf{s}$, then the work done W is given by the formula $W = \mathbf{F} \cdot \mathbf{s} = \| \mathbf{F} \| \, \|\mathbf{s}\| \cos\theta$.

Key Equations

- **Dot product of $\mathbf{u}$ and $\mathbf{v}$**

  $$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = \|\mathbf{u}\| \|\mathbf{v}\| \cos\theta$$

- **Cosine of the angle formed by $\mathbf{u}$ and $\mathbf{v}$**

  $$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}$$

- **Vector projection of $\mathbf{v}$ onto $\mathbf{u}$**

  $$\text{proj}_{\mathbf{u}}\mathbf{v} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u}$$

- **Scalar projection of $\mathbf{v}$ onto $\mathbf{u}$**

  $$\text{comp}_{\mathbf{u}}\mathbf{v} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|}$$

- **Work done by a force $\mathbf{F}$ to move an object through displacement vector $\vecd{PQ}$**

  $$W = \mathbf{F} \cdot \vecd{PQ} = \| \mathbf{F} \| \| \vecd{PQ} \| \cos\theta$$

Glossary

Direction Angles
: The angles formed by a nonzero vector and the coordinate axes.

Direction Cosines
: The cosines of the angles formed by a nonzero vector and the coordinate axes.

Dot Product or Scalar Product
: $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$ where $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$.

Scalar Projection
: The magnitude of the vector projection of a vector.

Orthogonal Vectors
Vectors that form a right angle when placed in standard position.

Vector Projection
The component of a vector that follows a given direction.

Work Done by a Force
Work is generally thought of as the amount of energy it takes to move an object; if we represent an applied force by a vector $\mathbf{F}$ and the displacement of an object by a vector $\mathbf{s}$, then the work done by the force is the dot product of $\mathbf{F}$ and $\mathbf{s}$.

Answers

Exercise 4.3.1
Multiply corresponding components and then add their products.
7

Exercise 4.3.2
$\mathbf{r} \cdot \mathbf{p}$ is a scalar.
a. $(\mathbf{r} \cdot \mathbf{p})\mathbf{q} = \langle 12, -12, 12 \rangle$; *b.* $\|\mathbf{p}\|^2 = 53$

Exercise 4.3.3
Use the Equation 4.3.9.
$\theta \approx 0.22$ rad

Exercise 4.3.4
Vectors $\mathbf{p}$ and $\mathbf{q}$ are orthogonal if and only if $\mathbf{p} \cdot \mathbf{q} = 0$.
$x = 5$

Exercise 4.3.5
$\mathbf{i} = \langle 1, 0, 0 \rangle, \mathbf{j} = \langle 0, 1, 0 \rangle$, and $\mathbf{k} = \langle 0, 0, 1 \rangle$
a. $\alpha \approx 1.04$ rad; b. $\beta \approx 2.58$ rad; c. $\gamma \approx 1.40$ rad

Exercise 4.3.6
Use four-dimensional vectors for cost, price, and quantity sold.
Sales = $15,685.50; profit = $14,073.15

Exercise 4.3.7
Start by finding the projection of $\mathbf{v}$ onto $\mathbf{u}$.
$\mathbf{v} = \mathbf{p} + \mathbf{q}$, where $\mathbf{p} = \frac{18}{5}\mathbf{i} + \frac{9}{5}\mathbf{j}$ and $\mathbf{q} = \frac{7}{5}\mathbf{i} - \frac{14}{5}\mathbf{j}$

Exercise 4.3.8
Compute the scalar projection of $\mathbf{w}$ onto $\mathbf{v}$.
21 knots

Exercise 4.3.9
Use the definition of work as the dot product of force and distance.
150 ft-lb

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4.3E: Exercises for Section 4.3

Exercises 1-4

The vectors u and v are given. Calculate the dot product $\mathrm{u} \cdot \mathrm{v}$.

1. $\mathbf{u} = \langle 3, 0 \rangle, \quad \mathbf{v} = \langle 2, 2 \rangle$

   Answer

   $\mathbf{u} \cdot \mathbf{v} = 6$

2. $\mathbf{u} = \langle 3, -4 \rangle, \quad \mathbf{v} = \langle 4, 3 \rangle$
3. $\mathbf{u} = \langle 2, 2, -1 \rangle, \quad \mathbf{v} = \langle -1, 2, 2 \rangle$

   Answer

   $\mathbf{u} \cdot \mathbf{v} = 0$

4. $\mathbf{u} = \langle 4, 5, -6 \rangle, \quad \mathbf{v} = \langle 0, -2, -3 \rangle$

Exercises 5-8

The vectors a, b, and c are given. Determine the vectors $(\mathrm{a} \cdot \mathrm{b})\mathrm{c}$ and $(\mathrm{a} \cdot \mathrm{c})\mathrm{b}$. Express the vectors in component form.

5. $\mathbf{a} = \langle 2, 0, -3 \rangle, \quad \mathbf{b} = \langle -4, -7, 1 \rangle, \quad \mathbf{c} = \langle 1, 1, -1 \rangle$

   Answer

   $(\mathbf{a} \cdot \mathbf{b})\mathbf{c} = \langle -11, -11, 11 \rangle; (\mathbf{a} \cdot \mathbf{c})\mathbf{b} = \langle -20, -35, 5 \rangle$

6. $\mathbf{a} = \langle 0, 1, 2 \rangle, \quad \mathbf{b} = \langle -1, 0, 1 \rangle, \quad \mathbf{c} = \langle 1, 0, -1 \rangle$
7. $\mathbf{a} = \mathbf{i} + \mathbf{j}, \quad \mathbf{b} = \mathbf{i} - \mathbf{k}, \quad \mathbf{c} = \mathbf{i} - 2\mathbf{k}$

   Answer

   $(\mathbf{a} \cdot \mathbf{b})\mathbf{c} = \langle 1, 0, -2 \rangle; (\mathbf{a} \cdot \mathbf{c})\mathbf{b} = \langle 1, 0, -1 \rangle$

8. $\mathbf{a} = \mathbf{i} - \mathbf{j} + \mathbf{k}, \quad \mathbf{b} = \mathbf{j} + 3\mathbf{k}, \quad \mathbf{c} = -\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$

Exercises 9-12

Two vectors are given.

a. **Find the measure of the angle θ between these two vectors. Express the answer in radians rounded to two decimal places, if it is not possible to express it exactly.**
b. **Is θ an acute angle?**

9. [T] $\mathbf{a} = \langle 3, -1 \rangle, \quad \mathbf{b} = \langle -4, 0 \rangle$

   Answer

   a. $\theta = 2.82$ rad;
   b. θ is not acute.

10. [T] $\mathbf{a} = \langle 2, 1 \rangle, \quad \mathbf{b} = \langle -1, 3 \rangle$
11. $\mathbf{u} = 3\mathbf{i}, \quad \mathbf{v} = 4\mathbf{i} + 4\mathbf{j}$

Answer

a. $\theta = \frac{\pi}{4}$ rad;
b. θ is acute.

12. $\mathbf{u} = 5\mathbf{i}, \quad \mathbf{v} = -6\mathbf{i} + 6\mathbf{j}$

Exercises 13-18

Find the measure of the angle between the three-dimensional vectors $\mathbf{a}$ and $\mathbf{b}$. Express the answer in radians rounded to two decimal places, if it is not possible to express it exactly.

13. $\mathbf{a} = \langle 3, -1, 2\rangle, \quad \mathbf{b} = \langle 1, -1, -2\rangle$

Answer

$\theta = \frac{\pi}{2}$

14. $\mathbf{a} = \langle 0, -1, -3\rangle, \quad \mathbf{b} = \langle 2, 3, -1\rangle$
15. $\mathbf{a} = \mathbf{i} + \mathbf{j}, \quad \mathbf{b} = \mathbf{j} - \mathbf{k}$

Answer

$\theta = \frac{\pi}{3}$

16. $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}, \quad \mathbf{b} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$
17. [T] $\mathbf{a} = 3\mathbf{i} - \mathbf{j} - 2\mathbf{k}, \quad \mathbf{b} = \mathbf{v} + \mathbf{w}$, where $\mathbf{v} = -2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ and $\mathbf{w} = \mathbf{i} + 2\mathbf{k}$

Answer

$\theta = 2$ rad

18. [T] $\mathbf{a} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}, \quad \mathbf{b} = \mathbf{v} - \mathbf{w}$, where $\mathbf{v} = 2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$ and $\mathbf{w} = 6\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

Exercises 19-22

Determine whether the given vectors are orthogonal.

19. $\mathbf{a} = \langle x, y\rangle, \quad \mathbf{b} = \langle -y, x\rangle$, where x and y are nonzero real numbers

Answer

Orthogonal

20. $\mathbf{a} = \langle x, x\rangle, \quad \mathbf{b} = \langle -y, y\rangle$, where x and y are nonzero real numbers
21. $\mathbf{a} = 3\mathbf{i} - \mathbf{j} - 2\mathbf{k}, \quad \mathbf{b} = -2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$

Answer

Not orthogonal

22. $\mathbf{a} = \mathbf{i} - \mathbf{j}, \quad \mathbf{b} = 7\mathbf{i} + 2\mathbf{j} - \mathbf{k}$
23. Find all two-dimensional vectors $\mathbf{a}$ orthogonal to vector $\mathbf{b} = \langle 3, 4\rangle$. Express the answer in component form.

Answer

$\mathbf{a} = \langle -\frac{4\alpha}{3}, \alpha\rangle$, where $\alpha \neq 0$ is a real number

24. Find all two-dimensional vectors $\mathbf{a}$ orthogonal to vector $\mathbf{b} = \langle 5, -6\rangle$. Express the answer by using standard unit vectors.
25. Determine all three-dimensional vectors $\mathbf{u}$ orthogonal to vector $\mathbf{v} = \langle 1, 1, 0\rangle$. Express the answer by using standard unit vectors.

Answer

$\mathbf{u} = -\alpha\mathbf{i} + \alpha\mathbf{j} + \beta\mathbf{k}$, where α and β are real numbers such that $\alpha^2 + \beta^2 \neq 0$

26. Determine all three-dimensional vectors $\mathbf{u}$ orthogonal to vector $\mathbf{v} = \mathbf{i} - \mathbf{j} - \mathbf{k}$. Express the answer in component form.
27. Determine the real number α such that vectors $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{b} = 9\mathbf{i} + \alpha\mathbf{j}$ are orthogonal.

Answer

$\alpha = -6$

28. Determine the real number α such that vectors $\mathbf{a} = -3\mathbf{i} + 2\mathbf{j}$ and $\mathbf{b} = 2\mathbf{i} + \alpha\mathbf{j}$ are orthogonal.
29. [T] Consider the points $P(4, 5)$ and $Q(5, -7)$, and note that O represents the origin.
    a. Determine vectors $\vecd{OP}$ and $\vecd{OQ}$. Express the answer by using standard unit vectors.
    b. Determine the measure of angle O in triangle OPQ. Express the answer in degrees rounded to two decimal places.

Answer

$a. \vecd{OP} = 4\mathbf{i} + 5\mathbf{j}, \quad \vecd{OQ} = 5\mathbf{i} - 7\mathbf{j}; \quad b. 105.8\,^\circ$

30. [T] Consider points $A(1, 1), B(2, -7)$, and $C(6, 3)$.
    a. Determine vectors $\vecd{BA}$ and $\vecd{BC}$. Express the answer in component form.
    b. Determine the measure of angle B in triangle ABC. Express the answer in degrees rounded to two decimal places.
31. Determine the measure of angle A in triangle ABC, where $A(1, 1, 8), B(4, -3, -4)$, and $C(-3, 1, 5)$. Express your answer in degrees rounded to two decimal places.

Answer

$68.33\,^\circ$

32. Consider points $P(3, 7, -2)$ and $Q(1, 1, -3)$. Determine the angle between vectors $\vecd{OP}$ and $\vecd{OQ}$. [Remember that O represents the origin.] Express the answer in degrees rounded to two decimal places.

Exercises 33-34

Determine which (if any) pairs of the following vectors are orthogonal.

33. $\mathbf{u} = \langle 3, 7, -2\rangle, \quad \mathbf{v} = \langle 5, -3, -3\rangle, \quad \mathbf{w} = \langle 0, 1, -1\rangle$

Answer

$\mathbf{u}$ and $\mathbf{v}$ are orthogonal; $\mathbf{v}$ and $\mathbf{w}$ are orthogonal, but $\mathbf{u}$ and $\mathbf{w}$ are **not** orthogonal.

34. $\mathbf{u} = \mathbf{i} - \mathbf{k}, \quad \mathbf{v} = 5\mathbf{j} - 5\mathbf{k}, \quad \mathbf{w} = 10\mathbf{j}$
35. Use vectors to show that a parallelogram with equal diagonals is a **rectangle**.
36. Use vectors to show that the diagonals of a **rhombus** are perpendicular.
37. Show that $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ is true for any vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$.
38. Verify the identity $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ for vectors $\mathbf{u} = \langle 1, 0, 4\rangle, \mathbf{v} = \langle -2, 3, 5\rangle$, and $\mathbf{w} = \langle 4, -2, 6\rangle$.

Exercises 39-41

Determine $\mathbf{u} \cdot \mathbf{v}$ using the given information.

39. $\|\mathbf{u}\| = 5, \quad \|\mathbf{v}\| = 3$, and the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\pi/6$ rad.

Answer

$\mathbf{u} \cdot \mathbf{v} = (5)(3)\cos\frac{\pi}{6} = 15 \cdot \frac{\sqrt{3}}{2} = \frac{15\sqrt{3}}{2}$

40. $\|\mathbf{u}\| = 20, \quad \|\mathbf{v}\| = 15$, and the angle between $\mathbf{u}$ and $\mathbf{v}$ is $5\pi/4$ rad.
41. $\|\mathbf{u}\| = 7, \quad \|\mathbf{v}\| = 12$, and the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\pi/2$ rad.

Answer

$\mathbf{u} \cdot \mathbf{v} = (7)(12)\cos\frac{\pi}{2} = 84 \cdot 0 = 0$

42. Considering the definition of the dot product, $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, what can you say about the angle θ between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ if:

a. $\mathbf{u} \cdot \mathbf{v} > 0$?
b. $\mathbf{u} \cdot \mathbf{v} < 0$?
c. $\mathbf{u} \cdot \mathbf{v} = 0$?

Exercises 43-45

You are given the vertices of a triangle. Use dot products to determine whether each triangle is *acute*, *obtuse*, or *right*.

43. $(2,3,0),\ (3,1,-2),\ (-1,4,5)$

Answer

Label the points P, Q and R. Then determine the vectors $\vecd{PQ}$ and $\vecd{PR}$.
Here we see that, if $P=(2,3,0),\ Q=(3,1,-2)$ and $R=(-1,4,5)$, so we have $\vecd{PQ} =< 1,-2,-2 >$ and $\vecd{PR} =< -3,1,5 >$.
Then $\vecd{PQ} \cdot \vecd{PR} = -15 < 0$, so angle QPR is obtuse. Therefore this triangle must be an ***obtuse triangle***.

44. $(5,1,0),\ (7,1,1),\ (6,3,2)$
45. $(6,1,4),\ (3,2,-1),\ (2,3,1)$

Solution:

Label the points P, Q and R. Then determine the vectors $\vecd{PQ}$ and $\vecd{PR}$.
Here we see that, if $P=(6,1,4),\ Q=(3,2,-1)$ and $R=(2,3,1)$, so we have $\vecd{PQ} =< -3,1,-5 >$ and $\vecd{PR} =< -4,2,-3 >$.
Then $\vecd{PQ} \cdot \vecd{PR} = 29 > 0$, so angle QPR is **acute**. This triangle could still be any of the three types, so we need to check another angle.
Considering angle PQR we find $\vecd{QP} =< 3,-1,5 >$ and $\vecd{QR} =< -1,1,2 >$, so $\vecd{QP} \cdot \vecd{QR} = 6 > 0$, showing angle PQR is also **acute**.
We then must check the third angle to determine whether this is a right triangle or an acute triangle.
To check angle PRQ we find $\vecd{RP} =< 4,-2,3 >$ and $\vecd{RQ} =< 1,-1,-2 >$, giving us $\vecd{RP} \cdot \vecd{RQ} = 0$. Angle PRQ is thus a **right** angle.
Therefore this triangle is a ***right triangle***.

46. Which of the following operations are allowed for nonzero vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, and which are not. Explain your reasoning.
a. $\mathbf{u} + (\mathbf{v} \cdot \mathbf{w})$ b. $(\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w}$ c. $(\mathbf{u} \cdot \mathbf{v})\,\mathbf{w}$
b. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w}$ e. $(\mathbf{u} + \mathbf{v}) \cdot \|\mathbf{w}\|$ f. $\|\mathbf{u} + \mathbf{v}\|\|\mathbf{w}\|$

Exercises 47-50

Given the vectors $\mathbf{u}$ and $\mathbf{v}$:

a. **Find the vector projection $\text{Proj}_{\mathbf{u}}\mathbf{v}$ of vector $\mathbf{v}$ onto vector $\mathbf{u}$ and the component of $\mathbf{v}$ that is orthogonal to $\mathbf{u}$, i.e., $\mathbf{v}_{\text{perp}}$. Express your answers in component form.**
b. **Find the scalar projection $\text{comp}_{\mathbf{u}}\mathbf{v} = \|\text{Proj}_{\mathbf{u}}\mathbf{v}\|$ of vector $\mathbf{v}$ onto vector $\mathbf{u}$.**
c. **Find the vector projection $\text{Proj}_{\mathbf{v}}\mathbf{u}$ of vector $\mathbf{u}$ onto vector $\mathbf{v}$ and the component of $\mathbf{u}$ that is orthogonal to $\mathbf{v}$, i.e., $\mathbf{u}_{\text{perp}}$. Express your answers in unit vector form.**
d. **Find the scalar projection $\text{comp}_{\mathbf{v}}\mathbf{u}$ of vector $\mathbf{u}$ onto vector $\mathbf{v}$.**

47. $\mathbf{u} = 5\mathbf{i} + 2\mathbf{j},\quad \mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$

Answer

a. $\text{Proj}_{\mathbf{u}}\mathbf{v} = \langle \frac{80}{29}, \frac{32}{29} \rangle$ and $\mathbf{v}_{\text{perp}} =< -\frac{22}{29}, \frac{55}{29} >$;
b. b. $\text{comp}_{\mathbf{u}}\mathbf{v} = \frac{16}{\sqrt{29}} = \frac{16\sqrt{29}}{29}$;
c. $\text{Proj}_{\mathbf{v}}\mathbf{u} = \frac{32}{13}\mathbf{i} + \frac{48}{13}\mathbf{j}$ and $\mathbf{u}_{\text{perp}} = \frac{33}{13}\mathbf{i} - \frac{22}{13}\mathbf{j}$;
d. d. $\text{comp}_{\mathbf{v}}\mathbf{u} = \frac{16}{\sqrt{13}} = \frac{16\sqrt{13}}{13}$

48. $\mathbf{u} = \langle -4, 7 \rangle,\quad \mathbf{v} = \langle 3, 5 \rangle$

49. $\mathbf{u} = 3\mathbf{i} + 2\mathbf{k}, \quad \mathbf{v} = 2\mathbf{j} + 4\mathbf{k}$

Answer

a. $\text{Proj}_{\mathbf{u}}\mathbf{v} = \langle \frac{24}{13}, 0, \frac{16}{13} \rangle$ and $\mathbf{v}_{\text{perp}} =< -\frac{24}{13}, 2, \frac{36}{13} >$;
b. b. $\text{comp}_{\mathbf{u}}\mathbf{v} = \frac{8}{\sqrt{13}} = \frac{8\sqrt{13}}{13}$
c. $\text{Proj}_{\mathbf{v}}\mathbf{u} = \frac{4}{5}\mathbf{j} + \frac{8}{5}\mathbf{k}$ and $\mathbf{u}_{\text{perp}} = 3\mathbf{i} - \frac{4}{5}\mathbf{j} + \frac{2}{5}\mathbf{k}$;
d. d. $\text{comp}_{\mathbf{v}}\mathbf{u} = \frac{\sqrt{80}}{5} = \frac{4\sqrt{5}}{5}$;

50. $\mathbf{u} = \langle 4, 4, 0 \rangle, \quad \mathbf{v} = \langle 0, 4, 1 \rangle$

51. Consider the vectors $\mathbf{u} = 4\mathbf{i} - 3\mathbf{j}$ and $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j}$.
   a. Find the component form of vector $\text{Proj}_{\mathbf{u}}\mathbf{v}$ that represents the projection of $\mathbf{v}$ onto $\mathbf{u}$.
   b. Write the decomposition $\mathbf{v} = \mathbf{w} + \mathbf{q}$ of vector $\mathbf{v}$ into the orthogonal components $\mathbf{w}$ and $\mathbf{q}$, where $\mathbf{w}$ is the projection of $\mathbf{v}$ onto $\mathbf{u}$ and $\mathbf{q}$ is the vector component of $\mathbf{v}$ orthogonal to the direction of $\mathbf{u}$. That is, $\mathbf{q} = \mathbf{v}_{\text{perp}}$.

Answer

a. $\text{Proj}_{\mathbf{u}}\mathbf{v} = \langle \frac{24}{25}, -\frac{18}{25} \rangle$; b. $\mathbf{q} = \mathbf{v}_{\text{perp}} = \langle \frac{51}{25}, \frac{68}{25} \rangle$,

$\mathbf{v} = \mathbf{w} + \mathbf{q} = \text{Proj}_{\mathbf{u}}\mathbf{v} + \mathbf{v}_{\text{perp}} = \langle \frac{24}{25}, -\frac{18}{25} \rangle + \langle \frac{51}{25}, \frac{68}{25} \rangle$
So we have that, $\mathbf{v} = \langle \frac{24}{25}, -\frac{18}{25} \rangle + \langle \frac{51}{25}, \frac{68}{25} \rangle$.

52. Consider vectors $\mathbf{u} = 2\mathbf{i} + 4\mathbf{j}$ and $\mathbf{v} = 4\mathbf{j} + 2\mathbf{k}$.
   a. Find the component form of vector $\mathbf{w} = \text{Proj}_{\mathbf{u}}\mathbf{v}$ that represents the projection of $\mathbf{v}$ onto $\mathbf{u}$.
   b. Write the decomposition $\mathbf{v} = \mathbf{w} + \mathbf{q}$ of vector $\mathbf{v}$ into the orthogonal components $\mathbf{w}$ and $\mathbf{q}$, where $\mathbf{w}$ is the projection of $\mathbf{v}$ onto $\mathbf{u}$ and $\mathbf{q}$ is a vector orthogonal to the direction of $\mathbf{u}$.

53. A $50,000$-pound truck is parked on a hill with a 5 ° slope above the horizontal (in the positive x-direction). Considering only the force due to gravity, find
   a. the component of the weight of the truck that is pulling the truck down the hill (aligned with the road)
   b. the component of the weight of the truck that is perpendicular to the road.
   c. the magnitude of the force needed to keep the truck from rolling down the hill
   d. the magnitude of the force that is perpendicular to the road.

Solution:

Let $\mathbf{F}$ be the force due to gravity on the truck (its weight) and let $\mathbf{r}$ be the unit vector pointing up the hill to the right.

Then

a. the component of the weight of the truck that is pulling the truck down the hill is $\text{Proj}_{\mathbf{r}}\mathbf{F}$. Now $\mathbf{F} = -50,000\mathbf{j}$ and $\mathbf{r} = \cos 5\,^\circ\ \mathbf{i} + \sin 5\,^\circ\ \mathbf{j}$ and $\mathbf{F} \cdot \mathbf{r} = -50,000 \sin 5\,^\circ \approx -4357.78714$
   Note that $\|\mathbf{r}\| = 1$.
   Then: $\text{Proj}_{\mathbf{r}}\mathbf{F} = \left(\frac{\mathbf{F}\cdot\mathbf{r}}{\|\mathbf{r}\|^2}\right)\mathbf{r} = (-50,000 \sin 5°)\,\mathbf{r} = -50000 \sin 5\,^\circ \cos 5\,^\circ\ \mathbf{i} - 50000 \sin^2 5\,^\circ\ \mathbf{j}$.
   Using the identity, $\sin 2\theta = 2\cos\theta \sin\theta$, we can simplify this as
   $\text{Proj}_{\mathbf{r}}\mathbf{F} = -25000 \sin 10\,^\circ\ \mathbf{i} - 50000 \sin^2 5\,^\circ\ \mathbf{j} \approx -4341.2\ \mathbf{i} - 379.8\ \mathbf{j}$.
b. the component of the weight of the truck that is perpendicular to the road is $\mathbf{F}_{\text{perp}}$. Since we know that $\mathbf{F} = \text{Proj}_{\mathbf{r}}\mathbf{F} + \mathbf{F}_{\text{perp}}$, we have that $\mathbf{F}_{\text{perp}} = \mathbf{F} - \text{Proj}_{\mathbf{r}}\mathbf{F}$.
   So $\mathbf{F}_{\text{perp}} = -50000\mathbf{j} - (-25000 \sin 10°\,\mathbf{i} - 50000 \sin^2 5°\,\mathbf{j})$
   $= 25000 \sin 10\,^\circ\ \mathbf{i} + (50000 \sin^2 5° - 50000)\,\mathbf{j} = 25000 \sin 10\,^\circ\ \mathbf{i} - 50000 \cos^2 5\,^\circ\ \mathbf{j} \approx 4341.2\ \mathbf{i} - 49620.2\ \mathbf{j}$.
c. the magnitude of the force needed to keep the truck from rolling down the hill is $\|\text{Proj}_{\mathbf{r}}\mathbf{F}\| = 50,000 \sin 5$ ° lbs ≈ 4357.8 lbs. Note that this force is actually the negative of the projection vector, but its magnitude is the same.
d. the magnitude of the truck's weight that is perpendicular to the road is $\|\mathbf{F}_{\text{perp}}\| \approx 49809.7$ lbs.

54. Given the vectors $\mathbf{u}$ and $\mathbf{v}$ shown in Figure $4.3E.1$ and Figure $4.3E.2$, draw in $\text{Proj}_{\mathbf{v}}\mathbf{u}$ and $\mathbf{u}_{\text{perp}}$.

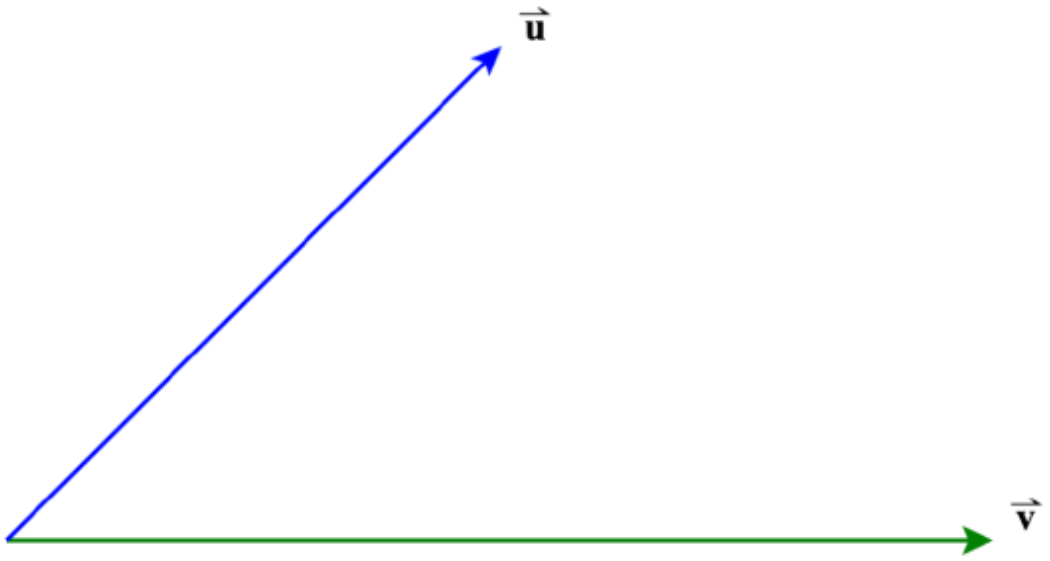


Figure $4.3E.1$: An acute angle formed by two vectors. Vector $\mathbf{v}$ is horizonal (parallel to x-axis). Vector $\mathbf{u}$ is less than $90\,^\circ$ from the horizon.

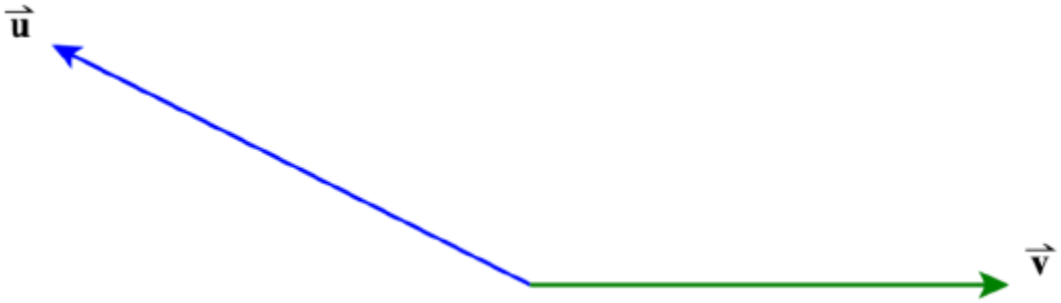


Figure $4.3E.2$: An obtuse angle formed by two vectors. Vector $\mathbf{v}$ is horizonal (parallel to x-axis). Vector $\mathbf{u}$ is between $90\,^\circ$ and $180\,^\circ$from the horizon.

55. Find the work done by force $\mathbf{F} = \langle 5, 6, -2 \rangle$ (measured in Newtons) that moves a particle from point $P(3, -1, 0)$ to point $Q(2, 3, 1)$ along a straight line (the distance is measured in meters).

Answer

$17\ \mathrm{N \cdot m}$

56. [T] A sled is pulled by exerting a force of 100 N on a rope that makes an angle of $25\,^\circ$ with the horizontal. Find the work done in pulling the sled 40 m. (Round the answer to one decimal place.)
57. [T] A father is pulling his son on a sled at an angle of $20\,^\circ$ with the horizontal with a force of 25 lb (see Figure $4.3E.3$. He pulls the sled in a straight path of 50 ft. How much work was done by the man pulling the sled? (Round the answer to the nearest integer.)

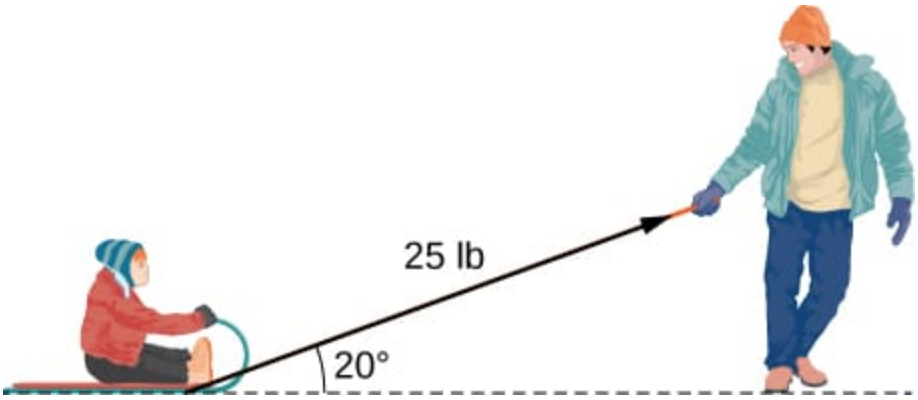


Figure $4.3E.3$: A father is pulling his son on a sled at an angle of $20\,^\circ$ with the horizontal with a force of 25 lb.

Answer

1175 ft · lb

58. [T] A car is towed using a force of 1600 N. The rope used to pull the car makes an angle of 25° with the horizontal. Find the work done in towing the car 2 km. Express the answer in joules (1 J $= 1$ N · m) rounded to the nearest integer.
59. [T] A boat sails north aided by a wind blowing in a direction of $N30\,^\circ E$ with a magnitude of 500 lb. How much work is performed by the wind as the boat moves 100 ft? (Round the answer to two decimal places.)

Answer

$25000\sqrt{3}$ ft-lbs $\approx 43,301.27$ ft-lbs

Solution:

Vector representing the wind: $\mathbf{w} = 500\cos 60^\circ \mathbf{i} + 500\sin 60^\circ \mathbf{j}$
Vector representing the displacement to the north: $\mathbf{d} = 100\mathbf{j}$
Work done by the wind: $W = \mathbf{w} \cdot \mathbf{d} = 25000\sqrt{3}$ ft-lbs $\approx 43,301.27$ ft-lbs

60. [T] Find the vectors that join the center of a clock to the hours 1:00, 2:00, and 3:00. Assume the clock is circular with a radius of 1 unit.
61. A methane molecule has a carbon atom situated at the origin and four hydrogen atoms located at points $P(1,1,-1), Q(1,-1,1), R(-1,1,1)$, and $S(-1,-1,-1)$ (see Figure $4.3E.3$).
    a. Find the distance between the hydrogen atoms located at $P$and R.
    b. Find the angle between vectors $\vec{OS}$ and $\vec{OR}$ that connect the carbon atom with the hydrogen atoms located at S and R, which is also called the *bond angle*. Express the answer in degrees rounded to two decimal places.

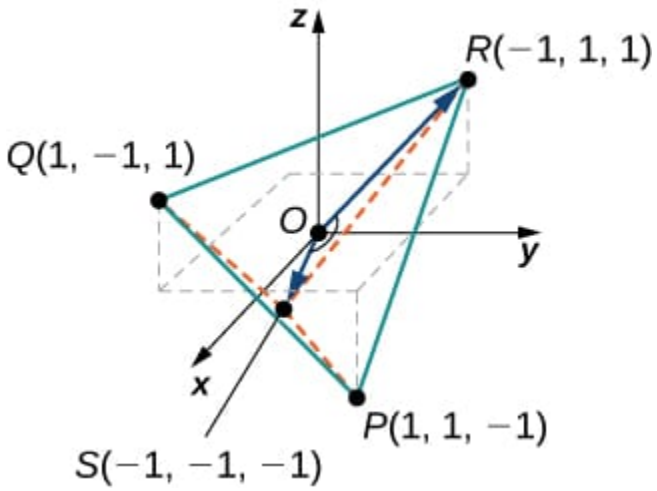


Figure $4.3E.3$: A tetrahedron in 3d-space with vertices located at points $P(1,1,-1), Q(1,-1,1), R(-1,1,1)$, and $S(-1,-1,-1)$. Two vectors are formed from point O. One points from O to point R, and another points from O to $S(-1,-1,-1)$.

Answer

a. $2\sqrt{2}$;

b. $109.47\,°$

52. Vector $\mathbf{p} = \langle 150, 225, 375 \rangle$ represents the price of certain models of bicycles sold by a bicycle shop. Vector $\mathbf{n} = \langle 10, 7, 9 \rangle$ represents the number of bicycles sold of each model, respectively. Compute the dot product $\mathbf{p} \cdot \mathbf{n}$ and state its meaning.

53. [T] Two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are represented by vectors with initial points that are at the origin. The first force has a magnitude of 20 lb and passes through the point $P(1, 1, 0)$. The second force has a magnitude of 40 lb and passes through the point $Q(0, 1, 1)$. Let $\mathbf{F}$ be the resultant force of forces $\mathbf{F}_1$ and $\mathbf{F}_2$.
   a. Find the magnitude of $\mathbf{F}$. (Round the answer to one decimal place.)
   b. Find the direction angles of $\mathbf{F}$. (Express the answer in degrees rounded to one decimal place.)

Answer

a. $\| \mathbf{F}_1 + \mathbf{F}_2 \| = 52.9$ lb;
b. The direction angles are $\alpha = 74.5\,°$, $\beta = 36.7\,°$, and $\gamma = 57.7\,°$.

54. [T] Consider $\mathbf{r}(t) = \langle \cos t, \sin t, 2t \rangle$ the position vector of a particle at time $t \in [0, 30]$, where the components of $\mathbf{r}$ are expressed in centimeters and time in seconds. Let $\vec{OP}$ be the position vector of the particle after 1 sec.
   a. Show that all vectors $\vec{PQ}$, where $Q(x, y, z)$ is an arbitrary point, orthogonal to the instantaneous velocity vector $\mathbf{v}(1)$ of the particle after 1 sec, can be expressed as $\vec{PQ} = \langle x - \cos 1, y - \sin 1, z - 2 \rangle$, where $x \sin 1 - y \cos 1 - 2z + 4 = 0$. The set of point Q describes a plane called the *normal plane* to the path of the particle at point P.
   b. Use a CAS to visualize the instantaneous velocity vector and the normal plane at point P along with the path of the particle.

Contributors

Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

Problems 39-46 and 53 & 54 were created by Paul Seeburger (Monroe Community College).
Solution to Problem 59 also was added by Paul Seeburger.

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4.4: The Cross Product

Learning Objectives

- Calculate the cross product of two given vectors.
- Use determinants to calculate a cross product.
- Find a vector orthogonal to two given vectors.
- Determine areas and volumes by using the cross product.
- Calculate the torque of a given force and position vector.

Imagine a mechanic turning a wrench to tighten a bolt. The mechanic applies a force at the end of the wrench. This creates rotation, or torque, which tightens the bolt. We can use vectors to represent the force applied by the mechanic, and the distance (radius) from the bolt to the end of the wrench. Then, we can represent torque by a vector oriented along the axis of rotation. Note that the torque vector is orthogonal to both the force vector and the radius vector.

In this section, we develop an operation called the **cross product**, which allows us to find a vector orthogonal to two given vectors. Calculating torque is an important application of cross products, and we examine torque in more detail later in the section.

The Cross Product and Its Properties

The dot product is a multiplication of two vectors that results in a scalar. In this section, we introduce a product of two vectors that generates a third vector orthogonal to the first two. Consider how we might find such a vector. Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be nonzero vectors. We want to find a vector $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ orthogonal to both $\mathbf{u}$ and $\mathbf{v}$—that is, we want to find $\mathbf{w}$ such that $\mathbf{u} \cdot \mathbf{w} = 0$ and $\mathbf{v} \cdot \mathbf{w} = 0$. Therefore, w_1, w_2, and w_3 must satisfy

$$u_1 w_1 + u_2 w_2 + u_3 w_3 = 0 \tag{4.4.1}$$

$$v_1 w_1 + v_2 w_2 + v_3 w_3 = 0. \tag{4.4.2}$$

If we multiply the top equation by v_3 and the bottom equation by u_3 and subtract, we can eliminate the variable w_3, which gives

$$(u_1 v_3 - v_1 u_3) w_1 + (u_2 v_3 - v_2 u_3) w_2 = 0.$$

If we select

$$w_1 = u_2 v_3 - u_3 v_2$$
$$w_2 = -(u_1 v_3 - u_3 v_1),$$

we get a possible solution vector. Substituting these values back into the original equations (Equations 4.4.1 and 4.4.2) gives

$$w_3 = u_1 v_2 - u_2 v_1.$$

That is, vector

$$\mathbf{w} = \langle u_2 v_3 - u_3 v_2, -(u_1 v_3 - u_3 v_1), u_1 v_2 - u_2 v_1 \rangle$$

is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$, which leads us to define the following operation, called the *cross product*.

Definition: Cross Product

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$. Then, the **cross product** $\mathbf{u} \times \mathbf{v}$ is vector

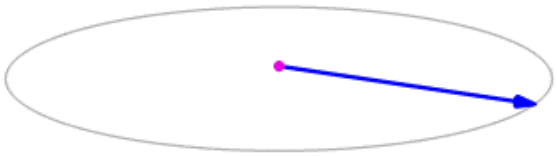
$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= (u_2 v_3 - u_3 v_2)\mathbf{i} - (u_1 v_3 - u_3 v_1)\mathbf{j} + (u_1 v_2 - u_2 v_1)\mathbf{k} \\ &= \langle u_2 v_3 - u_3 v_2, -(u_1 v_3 - u_3 v_1), u_1 v_2 - u_2 v_1 \rangle. \end{aligned} \tag{4.4.3}$$

From the way we have developed $\mathbf{u} \times \mathbf{v}$, it should be clear that the cross product is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. However, it never hurts to check. To show that $\mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{u}$, we calculate the dot product of $\mathbf{u}$ and $\mathbf{u} \times \mathbf{v}$.

$$\begin{aligned}\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) &= \langle u_1, u_2, u_3 \rangle \cdot \langle u_2 v_3 - u_3 v_2, -u_1 v_3 + u_3 v_1, u_1 v_2 - u_2 v_1 \rangle \\ &= u_1(u_2 v_3 - u_3 v_2) + u_2(-u_1 v_3 + u_3 v_1) + u_3(u_1 v_2 - u_2 v_1) \\ &= u_1 u_2 v_3 - u_1 u_3 v_2 - u_1 u_2 v_3 + u_2 u_3 v_1 + u_1 u_3 v_2 - u_2 u_3 v_1 \\ &= (u_1 u_2 v_3 - u_1 u_2 v_3) + (-u_1 u_3 v_2 + u_1 u_3 v_2) + (u_2 u_3 v_1 - u_2 u_3 v_1) \\ &= 0\end{aligned}$$

Interactive Element

In a similar manner, we can show that the cross product is also orthogonal to $\mathbf{v}$. The interactive app is unfortunately not accessible.



The applet shows the cross product $\mathbf{a} \times \mathbf{b}$ (vertical, in pink) changes as the angle between the vectors $\mathbf{a}$ (blue) and $\mathbf{b}$ (red) changes. The cross product (purple) is always perpendicular to both vectors, and has magnitude zero when the vectors are parallel and maximum magnitude $\|\mathbf{a}\|\|\mathbf{b}\|$ when they are perpendicular. (Public Domain work credited to Lucas Vieira, Wikimedia Commons).

Example 4.4.1: Finding a Cross Product

Let $\mathbf{p} = \langle -1, 2, 5 \rangle$ and $\mathbf{q} = \langle 4, 0, -3 \rangle$ (Figure 4.4.1). Find $\mathbf{p} \times \mathbf{q}$.

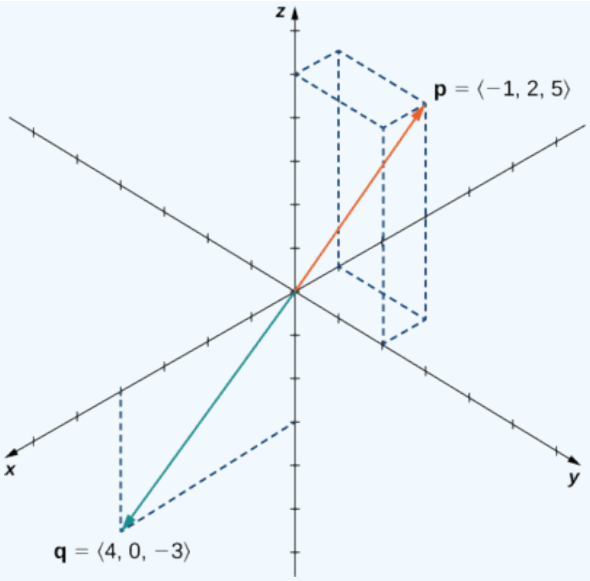


Figure 4.4.1: Two vectors in 3d-space system. The first vector is labeled “p = <-1, 2, 5>.” The second vector is labeled “q = <4, 0, -3>.”

Solution

Substitute the components of the vectors into Equation 4.4.3:

$$\begin{aligned}\mathbf{p}\times\mathbf{q} &= \langle -1,2,5\rangle \times \langle 4,0,-3\rangle \\ &= \langle p_2q_3 - p_3q_2, -(p_1q_3 - p_3q_1), p_1q_2 - p_2q_1\rangle \\ &= \langle 2(-3) - 5(0), -(-1)(-3) + 5(4), (-1)(0) - 2(4)\rangle \\ &= \langle -6, 17, -8\rangle.\end{aligned}$$

Exercise 4.4.1

Find $\mathbf{p}\times\mathbf{q}$ for $\mathbf{p} = \langle 5,1,2\rangle$ and $\mathbf{q} = \langle -2,0,1\rangle$. Express the answer using standard unit vectors.

Hint available at the end of this section.

Although it may not be obvious from Equation 4.4.3, the direction of $\mathbf{u}\times\mathbf{v}$ is given by the right-hand rule. If we hold the right hand out with the fingers pointing in the direction of $\mathbf{u}$, then curl the fingers toward vector $\mathbf{v}$, the thumb points in the direction of the cross product, as shown in Figure 4.4.2.

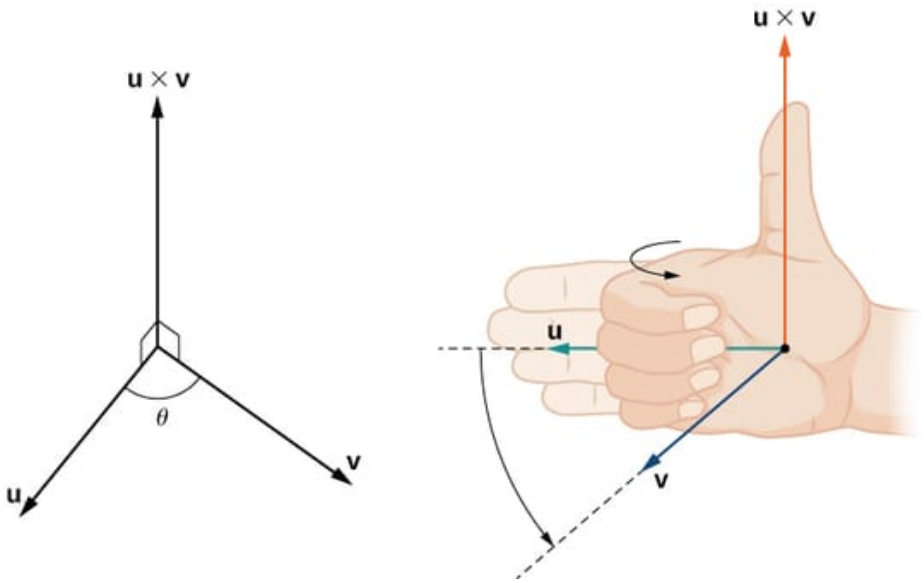


Figure 4.4.2: The left image shows two of the vectors $\mathbf{u}$ and $\mathbf{v}$ originates at the same point with an angle (\theta \) between them . The third vector is perpendicular to u and v, and it is $\mathbf{u} \times \mathbf{v}$. The second image has three vectors, $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$. Also, on the image of these three vectors is a right hand. The fingers are in the direction of $\mathbf{u}$. As the hand is closing, the direction of the closing fingers is the direction of $\mathbf{v}$. The thumb is up and in the direction of $\mathbf{u} \times \mathbf{v}$. The direction of $\mathbf{u} \times \mathbf{v}$ *is determined by the right-hand rule.*

Notice what this means for the direction of $\mathbf{v} \times \mathbf{u}$. If we apply the right-hand rule to $\mathbf{v} \times \mathbf{u}$, we start with our fingers pointed in the direction of $\mathbf{v}$, then curl our fingers toward the vector $\mathbf{u}$. In this case, the thumb points in the opposite direction of $\mathbf{u} \times \mathbf{v}$. (Try it!)

Example 4.4.2: Anti-commutativity of the Cross Product

Let $\mathbf{u} = \langle 0, 2, 1 \rangle$ and $\mathbf{v} = \langle 3, -1, 0 \rangle$. Calculate $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ and graph them.

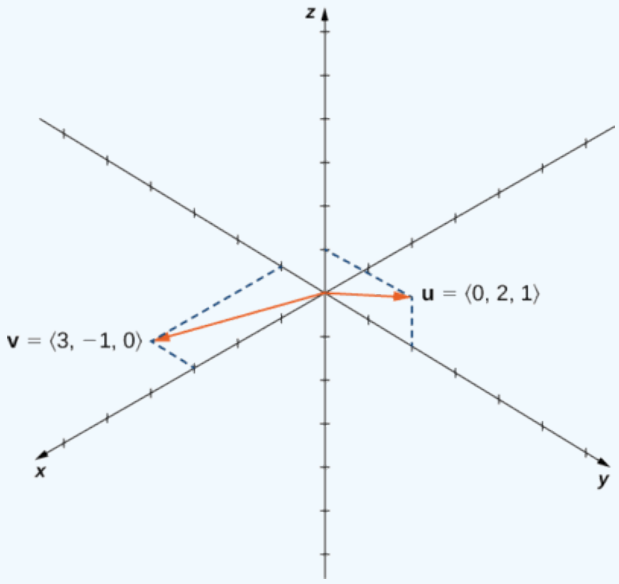


Figure 4.4.3: Two vectors in 3d-space originate at the origin, $\mathbf{u} = \langle 0, 2, 1 \rangle$ and $\mathbf{v} = \langle 3, -1, 0 \rangle$.

Solution

We have

$$\mathbf{u} \times \mathbf{v} = \langle (0 + 1), -(0 - 3), (0 - 6) \rangle = \langle 1, 3, -6 \rangle$$

$$\mathbf{v} \times \mathbf{u} = \langle (-1-0), -(3-0), (6-0) \rangle = \langle -1, -3, 6 \rangle.$$

We see that, in this case, $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$ (Figure 4.4.4). We prove this in general later in this section.

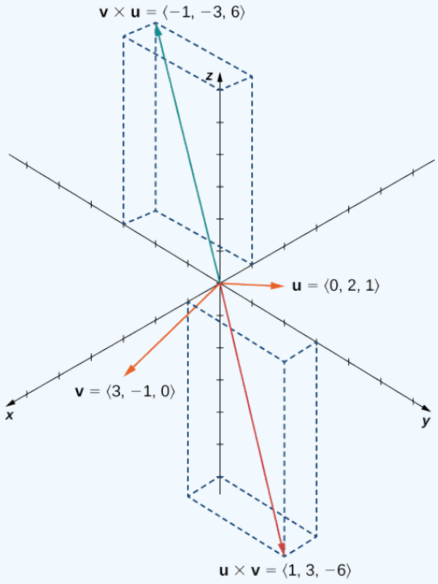


Figure 4.4.4: The cross products $\mathbf{u} \times \mathbf{v}$ *and* $\mathbf{v} \times \mathbf{u}$ *are both orthogonal to* $\mathbf{u}$ *and* $\mathbf{v}$*, but in opposite directions.*

Exercise 4.4.2

Suppose vectors $\mathbf{u}$ and $\mathbf{v}$ lie in the xy-plane (the z-component of each vector is zero). Now suppose the x- and y-components of $\mathbf{u}$ and the y-component of $\mathbf{v}$ are all positive, whereas the x-component of $\mathbf{v}$ is negative. Assuming the coordinate axes are oriented in the usual positions, in which direction does $\mathbf{u} \times \mathbf{v}$ point?

Hint available at the end of this section.

The cross products of the standard unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ can be useful for simplifying some calculations, so let's consider these cross products. A straightforward application of the definition shows that

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}. \tag{4.4.4}$$

(The cross product of two vectors is a vector, so each of these products results in the zero vector, not the scalar 0.) It's up to you to verify the calculations on your own.

Furthermore, because the cross product of two vectors is orthogonal to each of these vectors, we know that the cross product of $\mathbf{i}$ and $\mathbf{j}$ is parallel to $\mathbf{k}$. Similarly, the vector product of $\mathbf{i}$ and $\mathbf{k}$ is parallel to $\mathbf{j}$, and the vector product of $\mathbf{j}$ and $\mathbf{k}$ is parallel to $\mathbf{i}$.

We can use the right-hand rule to determine the direction of each product. Then we have

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}$$
$$\mathbf{j} \times \mathbf{i} = -\mathbf{k}$$

$$\mathbf{j} \times \mathbf{k} = \mathbf{i}$$
$$\mathbf{k} \times \mathbf{j} = -\mathbf{i}$$

$$\mathbf{k} \times \mathbf{i} = \mathbf{j}$$
$$\mathbf{i} \times \mathbf{k} = -\mathbf{j}.$$

These formulas come in handy later.

Example 4.4.3: Cross Product of Standard Unit Vectors

Find $\mathbf{i} \times (\mathbf{j} \times \mathbf{k})$.

Solution

We know that $\mathbf{j} \times \mathbf{k} = \mathbf{i}$. Therefore, $\mathbf{i} \times (\mathbf{j} \times \mathbf{k}) = \mathbf{i} \times \mathbf{i} = \mathbf{0}$.

Exercise 4.4.3

Find $(\mathbf{i} \times \mathbf{j}) \times (\mathbf{k} \times \mathbf{i})$.

Hint available at the end of this section.

As we have seen, the dot product is often called the *scalar product* because it results in a scalar. The cross product results in a vector, so it is sometimes called the **vector product.** These operations are both versions of vector multiplication, but they have very different properties and applications. Let's explore some properties of the cross product. We prove only a few of them. Proofs of the other properties are left as exercises.

Properties of the Cross Product

Let $\mathbf{u}, \mathbf{v},$ and $\mathbf{w}$ be vectors in space, and let c be a scalar.

i. Anticommutative property:

$$\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u}) \tag{4.4.5}$$

ii. Distributive property:

$$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w} \tag{4.4.6}$$

iii. Multiplication by a constant:

$$c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v}) \tag{4.4.7}$$

iv. Cross product of the zero vector:

$$\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0} \tag{4.4.8}$$

v. Cross product of a vector with itself:

$$\mathbf{v} \times \mathbf{v} = \mathbf{0} \tag{4.4.9}$$

vi. Scalar triple product:

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} \tag{4.4.10}$$

Proof

For property i, we want to show $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$. We have

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \langle u_1, u_2, u_3 \rangle \times \langle v_1, v_2, v_3 \rangle \\ &= \langle u_2 v_3 - u_3 v_2, -u_1 v_3 + u_3 v_1, u_1 v_2 - u_2 v_1 \rangle \\ &= -\langle u_3 v_2 - u_2 v_3, -u_3 v_1 + u_1 v_3, u_2 v_1 - u_1 v_2 \rangle \\ &= -\langle v_1, v_2, v_3 \rangle \times \langle u_1, u_2, u_3 \rangle \\ &= -(\mathbf{v} \times \mathbf{u}). \end{aligned}$$

Unlike most operations we've seen, the cross product is not commutative. This makes sense if we think about the right-hand rule.

For property $iv.$, this follows directly from the definition of the cross product. We have

$$\mathbf{u} \times \mathbf{0} = \langle u_2(0) - u_3(0), -(u_2(0) - u_3(0)), u1(0) - u_2(0) \rangle = \langle 0, 0, 0 \rangle = \mathbf{0}. \tag{4.4.11}$$

Then, by property i., $\mathbf{0} \times \mathbf{u} = \mathbf{0}$ as well. Remember that the dot product of a vector and the zero vector is the *scalar* 0, whereas the cross product of a vector with the zero vector is the *vector* $\mathbf{0}$.

Property $vi.$ looks like the associative property, but note the change in operations:

$$\begin{aligned}
\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) &= u \cdot \langle v_2 w_3 - v_3 w_2, -v_1 w_3 + v_3 w_1, v_1 w_2 - v_2 w_1 \rangle \\
&= u_1(v_2 w_3 - v_3 w_2) + u_2(-v_1 w_3 + v_3 w_1) + u_3(v_1 w_2 - v_2 w_1) \\
&= u_1 v_2 w_3 - u_1 v_3 w_2 - u_2 v_1 w_3 + u_2 v_3 w_1 + u_3 v_1 w_2 - u_3 v_2 w_1 \\
&= (u_2 v_3 - u_3 v_2) w_1 + (u_3 v_1 - u_1 v_3) w_2 + (u_1 v_2 - u_2 v_1) w_3 \\
&= \langle u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1 \rangle \cdot \langle w_1, w_2, w_3 \rangle = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}.
\end{aligned}$$

□

Example 4.4.4: Using the Properties of the Cross Product

Use the cross product properties to calculate $(2\mathbf{i} \times 3\mathbf{j}) \times \mathbf{j}$.

Solution

$$\begin{aligned}
(2\mathbf{i} \times 3\mathbf{j}) \times \mathbf{j} &= 2(\mathbf{i} \times 3\mathbf{j}) \times \mathbf{j} \\
&= 2(3)(\mathbf{i} \times \mathbf{j}) \times \mathbf{j} \\
&= (6\mathbf{k}) \times \mathbf{j} \\
&= 6(\mathbf{k} \times \mathbf{j}) \\
&= 6(-\mathbf{i}) = -6\mathbf{i}.
\end{aligned}$$

Exercise 4.4.4

Use the properties of the cross product to calculate $(\mathbf{i} \times \mathbf{k}) \times (\mathbf{k} \times \mathbf{j})$.

Hint available at the end of this section.

So far in this section, we have been concerned with the direction of the vector $\mathbf{u} \times \mathbf{v}$, but we have not discussed its magnitude. It turns out there is a simple expression for the magnitude of $\mathbf{u} \times \mathbf{v}$ involving the magnitudes of $\mathbf{u}$ and $\mathbf{v}$, and the sine of the angle between them.

Magnitude of the Cross Product

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors, and let θ be the angle between them. Then, $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \cdot \|\mathbf{v}\| \cdot \sin\theta$.

Proof

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be vectors, and let θ denote the angle between them. Then

$$\begin{aligned}\|\mathbf{u}\times\mathbf{v}\|^2 &= (u_2v_3-u_3v_2)^2+(u_3v_1-u_1v_3)^2+(u_1v_2-u_2v_1)^2\\ &= u_2^2v_3^2-2u_2u_3v_2v_3+u_3^2v_2^2+u_3^2v_1^2-2u_1u_3v_1v_3+u_1^2v_3^2+u_1^2v_2^2-2u_1u_2v_1v_2+u_2^2v_1^2\\ &= u_1^2v_1^2+u_1^2v_2^2+u_1^2v_3^2+u_2^2v_1^2+u_2^2v_2^2+u_2^2v_3^2+u_3^2v_1^2+u_3^2v_2^2+u_3^2v_3^2-(u_1^2v_1^2+u_2^2v_2^2+u_3^2v_3^2+2u_1u_2v_1v_2+2u_1u_3v_1v_3+2u_2u_3v_2v_3)\\ &= (u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2)-(u_1v_1+u_2v_2+u_3v_3)^2\\ &= \|\mathbf{u}\|^2\|\mathbf{v}\|^2-(\mathbf{u}\cdot\mathbf{v})^2\\ &= \|\mathbf{u}\|^2\|\mathbf{v}\|^2-\|\mathbf{u}\|^2\|\mathbf{v}\|^2\cos^2\theta\\ &= \|\mathbf{u}\|^2\|\mathbf{v}\|^2(1-\cos^2\theta)\\ &= \|\mathbf{u}\|^2\|\mathbf{v}\|^2(\sin^2\theta).\end{aligned}$$

Taking square roots and noting that $\sqrt{\sin^2\theta}=\sin\theta$ for $0\le\theta\le 180\,°$, we have the desired result:

$$\|\mathbf{u}\times\mathbf{v}\|=\|\mathbf{u}\|\|\mathbf{v}\|\sin\theta. \tag{4.4.12}$$

□

This definition of the cross product allows us to visualize or interpret the product geometrically. It is clear, for example, that the cross product is defined only for vectors in three dimensions, not for vectors in two dimensions. In two dimensions, it is impossible to generate a vector simultaneously orthogonal to two nonparallel vectors.

Example 4.4.5: Calculating the Cross Product

Find the magnitude of the cross product of $\mathbf{u}=\langle 0,4,0\rangle$ and $\mathbf{v}=\langle 0,0,-3\rangle$.

Solution

We have

$$\begin{aligned}\|\mathbf{u}\times\mathbf{v}\| &= \|\mathbf{u}\|\cdot\|\mathbf{v}\|\cdot\sin\theta\\ &= \sqrt{0^2+4^2+0^2}\cdot\sqrt{0^2+0^2+(-3)^2}\cdot\sin\frac{\pi}{2}\\ &= 4(3)(1)=12\end{aligned}$$

Exercise 4.4.5

Find the magnitude of $\mathbf{u}\times\mathbf{v}$, where $\mathbf{u}=\langle -8,0,0\rangle$ and $\mathbf{v}=\langle 0,2,0\rangle$.

Hint available at the end of this section.

Determinants and the Cross Product

Using Equation 4.4.3 to find the cross product of two vectors is straightforward, and it presents the cross product in the useful component form. The formula, however, is complicated and difficult to remember. Fortunately, we have an alternative. We can calculate the cross product of two vectors using determinant notation.

A 2×2 determinant is defined by

$$\begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix}=a_1b_2-b_1a_2. \tag{4.4.13}$$

For example,

$$\begin{vmatrix} 3 & -2 \\ 5 & 1 \end{vmatrix} = 3(1) - 5(-2) = 3 + 10 = 13. \tag{4.4.14}$$

A 3×3 determinant is defined in terms of 2×2 determinants as follows:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}. \tag{4.4.15}$$

Equation 4.4.15 is referred to as the expansion of the determinant along the first row. Notice that the multipliers of each of the 2×2 determinants on the right side of this expression are the entries in the first row of the 3×3 determinant. Furthermore, each of the 2×2 determinants contains the entries from the 3×3 determinant that would remain if you crossed out the row and column containing the multiplier. Thus, for the first term on the right, a_1 is the multiplier, and the 2×2 determinant contains the entries that remain if you cross out the first row and first column of the 3×3 determinant. Similarly, for the second term, the multiplier is a_2, and the 2×2 determinant contains the entries that remain if you cross out the first row and second column of the 3×3 determinant. Notice, however, that the coefficient of the second term is negative. The third term can be calculated in similar fashion.

Example 4.4.6: Using Expansion Along the First Row to Compute a 3×3 Determinant

Evaluate the determinant $\begin{vmatrix} 2 & 5 & -1 \\ -1 & 1 & 3 \\ -2 & 3 & 4 \end{vmatrix}$.

Solution

We have

$$\begin{aligned} \begin{vmatrix} 2 & 5 & -1 \\ -1 & 1 & 3 \\ -2 & 3 & 4 \end{vmatrix} &= 2 \begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix} - 5 \begin{vmatrix} -1 & 3 \\ -2 & 4 \end{vmatrix} - 1 \begin{vmatrix} -1 & 1 \\ -2 & 3 \end{vmatrix} \\ &= 2(4-9) - 5(-4+6) - 1(-3+2) \\ &= 2(-5) - 5(2) - 1(-1) = -10 - 10 + 1 \\ &= -19 \end{aligned}$$

Exercise 4.4.6

Evaluate the determinant $\begin{vmatrix} 1 & -2 & -1 \\ 3 & 2 & -3 \\ 1 & 5 & 4 \end{vmatrix}$.

Hint available at the end of this section.

Technically, determinants are defined only in terms of arrays of real numbers. However, the determinant notation provides a useful mnemonic device for the cross product formula.

Rule: Cross Product Calculated by a Determinant

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be vectors. Then the cross product $\mathbf{u} \times \mathbf{v}$ is given by

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}. \tag{4.4.16}$$

Example 4.4.7: Using Determinant Notation to find $\mathbf{p} \times \mathbf{q}$

Let $\mathbf{p} = \langle -1, 2, 5 \rangle$ and $\mathbf{q} = \langle 4, 0, -3 \rangle$. Find $\mathbf{p} \times \mathbf{q}$.

Solution

We set up our determinant by putting the standard unit vectors across the first row, the components of $\mathbf{u}$ in the second row, and the components of $\mathbf{v}$ in the third row. Then, we have

$$\begin{aligned}\mathbf{p} \times \mathbf{q} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 5 \\ 4 & 0 & -3 \end{vmatrix} = \begin{vmatrix} 2 & 5 \\ 0 & -3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & 5 \\ 4 & -3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & 2 \\ 4 & 0 \end{vmatrix} \mathbf{k} \\ &= (-6 - 0)\mathbf{i} - (3 - 20)\mathbf{j} + (0 - 8)\mathbf{k} \\ &= -6\mathbf{i} + 17\mathbf{j} - 8\mathbf{k}.\end{aligned}$$

Notice that this answer confirms the calculation of the cross product in Example 4.4.1.

Exercise 4.4.7

Use determinant notation to find $\mathbf{a} \times \mathbf{b}$, where $\mathbf{a} = \langle 8, 2, 3 \rangle$ and $\mathbf{b} = \langle -1, 0, 4 \rangle$.

Hint available at the end of this section.

Using the Cross Product

The cross product is very useful for several types of calculations, including finding a vector orthogonal to two given vectors, computing areas of triangles and parallelograms, and even determining the volume of the three-dimensional geometric shape made of parallelograms known as a *parallelepiped*. The following examples illustrate these calculations.

Example 4.4.8: Finding a Unit Vector Orthogonal to Two Given Vectors

Let $\mathbf{a} = \langle 5, 2, -1 \rangle$ and $\mathbf{b} = \langle 0, -1, 4 \rangle$. Find a unit vector orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

Solution

The cross product $\mathbf{a} \times \mathbf{b}$ is orthogonal to both vectors $\mathbf{a}$ and $\mathbf{b}$. We can calculate it with a determinant:

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 2 & -1 \\ 0 & -1 & 4 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -1 & 4 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 5 & -1 \\ 0 & 4 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 5 & 2 \\ 0 & -1 \end{vmatrix} \mathbf{k} \\ &= (8 - 1)\mathbf{i} - (20 - 0)\mathbf{j} + (-5 - 0)\mathbf{k} \\ &= 7\mathbf{i} - 20\mathbf{j} - 5\mathbf{k}.\end{aligned}$$

Normalize this vector to find a unit vector in the same direction:

$$\|\mathbf{a} \times \mathbf{b}\| = \sqrt{(7)^2 + (-20)^2 + (-5)^2} = \sqrt{474}.$$

Thus, $\left\langle \frac{7}{\sqrt{474}}, \frac{-20}{\sqrt{474}}, \frac{-5}{\sqrt{474}} \right\rangle$ is a unit vector orthogonal to $\mathbf{a}$ and $\mathbf{b}$.

Simplified, this vector becomes $\left\langle \frac{7\sqrt{474}}{474}, \frac{-10\sqrt{474}}{237}, \frac{-5\sqrt{474}}{474} \right\rangle$.

Exercise 4.4.8

Find a unit vector orthogonal to both $\mathbf{a}$ and $\mathbf{b}$, where $\mathbf{a} = \langle 4, 0, 3 \rangle$ and $\mathbf{b} = \langle 1, 1, 4 \rangle$.

Hint available at the end of this section.

To use the cross product for calculating areas, we state and prove the following theorem.

Area of a Parallelogram

If we locate vectors $\mathbf{u}$ and $\mathbf{v}$ such that they form adjacent sides of a parallelogram, then the area of the parallelogram is given by $\|\mathbf{u} \times \mathbf{v}\|$ (Figure 4.4.5).

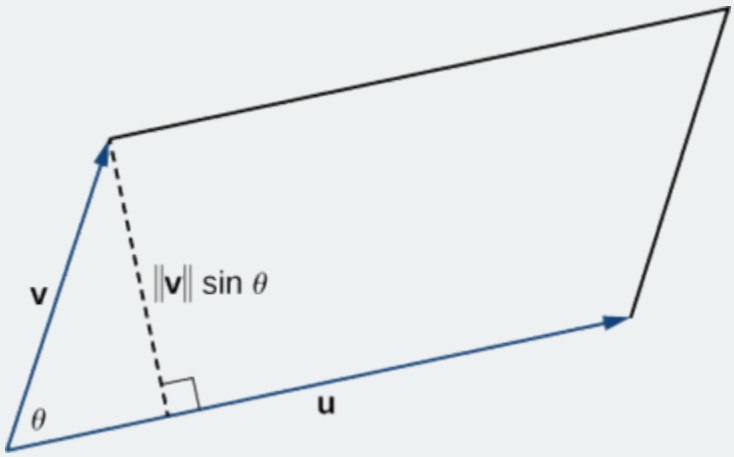


Figure 4.4.5: The parallelogram with adjacent sides $\mathbf{u}$ and $\mathbf{v}$ has base $\|\mathbf{u}\|$ and height $\|\mathbf{v}\| \sin \theta$.

Proof

We show that the magnitude of the cross product is equal to the base times height of the parallelogram.

$$\begin{aligned} \text{Area of a parallelogram} &= \text{base} \times \text{height} \\ &= \|\mathbf{u}\|(\|\mathbf{v}\| \sin \theta) \\ &= \|\mathbf{u} \times \mathbf{v}\| \end{aligned}$$

□

Example 4.4.9: Finding the Area of a Triangle

Let $P = (1, 0, 0), Q = (0, 1, 0)$, and $R = (0, 0, 1)$ be the vertices of a triangle (Figure 4.4.6). Find its area.

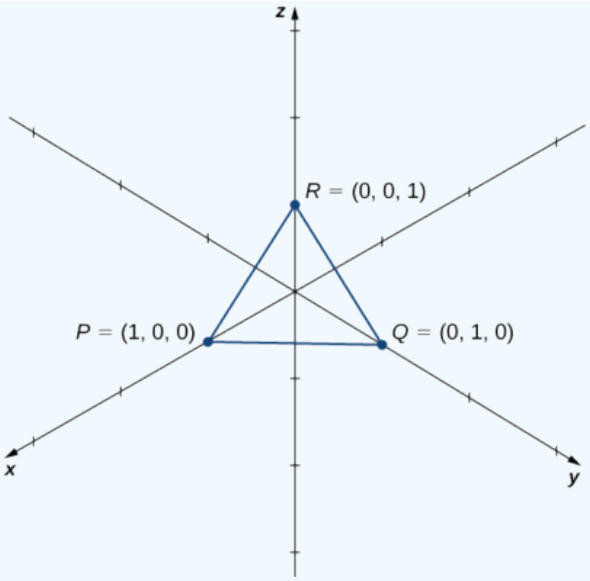


Figure 4.4.6: A 3D Cartesian coordinate system with three labeled points: $P = (1, 0, 0), Q = (0, 1, 0)$, and $R = (0, 0, 1)$. These points lie on the positive x-axis, y-axis, and z-axis, respectively. A triangle is formed by connecting these points with straight lines. The triangle is oriented in 3D space, with each vertex representing a unit distance along one of the coordinate axes.

Solution

We have $\vecd{PQ} = \langle 0 - 1, 1 - 0, 0 - 0\rangle = \langle -1, 1, 0\rangle$ and $\vecd{PR} = \langle 0 - 1, 0 - 0, 1 - 0\rangle = \langle -1, 0, 1\rangle$. The area of the parallelogram with adjacent sides $\vecd{PQ}$ and $\vecd{PR}$ is given by $\| \vecd{PQ} \times \vecd{PR} \|$:

$$\begin{aligned} \vecd{PQ} \times \vecd{PR} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} \\ &= (1-0)\mathbf{i} - (-1-0)\mathbf{j} + (0-(-1))\mathbf{k} \\ &= \mathbf{i} + \mathbf{j} + \mathbf{k} \end{aligned}$$

$$\begin{aligned} \| \vecd{PQ} \times \vecd{PR} \| &= \| \langle 1, 1, 1\rangle \| \\ &= \sqrt{1^2 + 1^2 + 1^2} \\ &= \sqrt{3}. \end{aligned}$$

The area of ΔPQR is half the area of the parallelogram or $\sqrt{3}/2$ units^2.

Example 4.4.10 Finding Area of a Parallelogram

Find the area of the parallelogram that has vertices $(0, -2, -1)$, $(2, 2, 0)$, $(1, 1, 1)$, and $(3, 5, 2)$.

Solution

- **Video length:** 4 minutes and 32 seconds
- **Context**: The video explains how to use the cross product to find the area of a parallelogram.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

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Exercise 4.4.9

Find the area of the parallelogram $PQRS$ with vertices $P(1,1,0)$, $Q(7,1,0)$, $R(9,4,2)$, and $S(3,4,2)$.

Hint available at the end of this section.

The Triple Scalar Product

Because the cross product of two vectors is a vector, it is possible to combine the dot product and the cross product. The dot product of a vector with the cross product of two other vectors is called the **triple scalar product** because the result is a scalar.

Definition: Triple Scalar Product

The **triple scalar product** of vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}). \tag{4.4.17}$$

Calculating a Triple Scalar Product

The triple scalar product of vectors

$$\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k} \tag{4.4.18}$$

$$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k} \tag{4.4.19}$$

and

$$\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k} \tag{4.4.20}$$

is the determinant of the 3×3 matrix formed by the components of the vectors:

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}. \tag{4.4.21}$$

Proof

The calculation is straightforward.

$$\begin{aligned} \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) &= \langle u_1, u_2, u_3 \rangle \cdot \langle v_2w_3 - v_3w_2, -v_1w_3 + v_3w_1, v_1w_2 - v_2w_1 \rangle \\ &= u_1(v_2w_3 - v_3w_2) + u_2(-v_1w_3 + v_3w_1) + u_3(v_1w_2 - v_2w_1) \\ &= u_1(v_2w_3 - v_3w_2) - u_2(v_1w_3 - v_3w_1) + u_3(v_1w_2 - v_2w_1) \\ &= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}. \end{aligned}$$

□

Example 4.4.11: Calculating the Triple Scalar Product

Let $\mathbf{u} = \langle 1, 3, 5 \rangle$, $\mathbf{v} = \langle 2, -1, 0 \rangle$ and $\mathbf{w} = \langle -3, 0, -1 \rangle$. Calculate the triple scalar product $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$.

Solution

Apply Equation 4.4.21 directly:

$$\begin{aligned}\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) &= \begin{vmatrix}1 & 3 & 5\\ 2 & -1 & 0\\ -3 & 0 & -1\end{vmatrix}\\ &= 1\begin{vmatrix}-1 & 0\\ 0 & -1\end{vmatrix} - 3\begin{vmatrix}2 & 0\\ -3 & -1\end{vmatrix} + 5\begin{vmatrix}2 & -1\\ -3 & 0\end{vmatrix}\\ &= (1-0) - 3(-2-0) + 5(0-3)\\ &= 1+6-15 = -8.\end{aligned}$$

Exercise 4.4.10

Calculate the triple scalar product $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$, where $\mathbf{a}=\langle 2,-4,1\rangle$, $\mathbf{b}=\langle 0,3,-1\rangle$, and $\mathbf{c}=\langle 5,-3,3\rangle$.

Hint available at the end of this section.

When we create a matrix from three vectors, we must be careful about the order in which we list the vectors. If we list them in a matrix in one order and then rearrange the rows, the absolute value of the determinant remains unchanged. However, each time two rows switch places, the determinant changes sign:

$$\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix} = d \qquad \begin{vmatrix}b_1 & b_2 & b_3\\ a_1 & a_2 & a_3\\ c_1 & c_2 & c_3\end{vmatrix} = -d \qquad \begin{vmatrix}b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\\ a_1 & a_2 & a_3\end{vmatrix} = d \qquad \begin{vmatrix}c_1 & c_2 & c_3\\ b_1 & b_2 & b_3\\ a_1 & a_2 & a_3\end{vmatrix} = -d$$

Verifying this fact is straightforward, but rather messy. Let's take a look at this with an example:

$$\begin{aligned}\begin{vmatrix}1 & 2 & 1\\ -2 & 0 & 3\\ 4 & 1 & -1\end{vmatrix} &= \begin{vmatrix}0 & 3\\ 1 & -1\end{vmatrix} - 2\begin{vmatrix}-2 & 3\\ 4 & -1\end{vmatrix} + \begin{vmatrix}-2 & 0\\ 4 & 1\end{vmatrix}\\ &= (0-3) - 2(2-12) + (-2-0)\\ &= -3+20-2 = 15.\end{aligned}$$

Switching the top two rows we have

$$\begin{aligned}\begin{vmatrix}-2 & 0 & 3\\ 1 & 2 & 1\\ 4 & 1 & -1\end{vmatrix} &= -2\begin{vmatrix}2 & 1\\ 1 & -1\end{vmatrix} + 3\begin{vmatrix}1 & 2\\ 4 & 1\end{vmatrix}\\ &= -2(-2-1) + 3(1-8)\\ &= 6-21 = -15.\end{aligned}$$

Rearranging vectors in the triple products is equivalent to reordering the rows in the matrix of the determinant. Let $\mathbf{u}=u_1\mathbf{i}+u_2\mathbf{j}+u_3\mathbf{k}$, $\mathbf{v}=v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k}$, and $\mathbf{w}=w_1\mathbf{i}+w_2\mathbf{j}+w_3\mathbf{k}$. Applying Note, we have

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix}u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\\ w_1 & w_2 & w_3\end{vmatrix} \tag{4.4.22}$$

and

$$\mathbf{u}\cdot(\mathbf{w}\times\mathbf{v}) = \begin{vmatrix}u_1 & u_2 & u_3\\ w_1 & w_2 & w_3\\ v_1 & v_2 & v_3\end{vmatrix}. \tag{4.4.23}$$

We can obtain the determinant for calculating $\mathbf{u} \cdot (\mathbf{w} \times \mathbf{v})$ by switching the bottom two rows of $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$. Therefore, $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = -\mathbf{u} \cdot (\mathbf{w} \times \mathbf{v})$.

Following this reasoning and exploring the different ways we can interchange variables in the triple scalar product lead to the following identities:

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = -\mathbf{u} \cdot (\mathbf{w} \times \mathbf{v}) \tag{4.4.24}$$

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}). \tag{4.4.25}$$

Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors in standard position. If $\mathbf{u}$ and $\mathbf{v}$ are not scalar multiples of each other, then these vectors form adjacent sides of a parallelogram. We saw in Note that the area of this parallelogram is $\|\mathbf{u} \times \mathbf{v}\|$. Now suppose we add a third vector $\mathbf{w}$ that does not lie in the same plane as $\mathbf{u}$ and $\mathbf{v}$ but still shares the same initial point. Then these vectors form three edges of a **parallelepiped**, a three-dimensional prism with six faces that are each parallelograms, as shown in Figure 4.4.7. The volume of this prism is the product of the figure's height and the area of its base. The triple scalar product of $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ provides a simple method for calculating the volume of the parallelepiped defined by these vectors.

Volume of a Parallelepiped

The volume of a parallelepiped with adjacent edges given by the vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ is the absolute value of the triple scalar product (Figure 4.4.7):

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|. \tag{4.4.26}$$

Note that, as the name indicates, the triple scalar product produces a scalar. The volume formula just presented uses the absolute value of a scalar quantity.

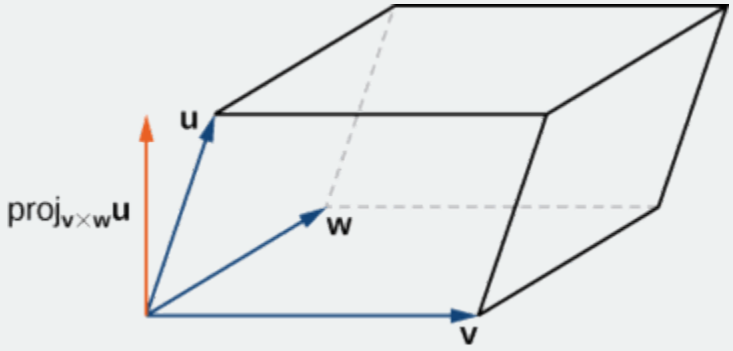


Figure 4.4.7: A parallelepiped formed by 3 vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ with height given by $\|\text{proj}_{\mathbf{v}\times\mathbf{w}}\mathbf{u}\|$.

Proof

The area of the base of the parallelepiped is given by $\|\mathbf{v} \times \mathbf{w}\|$. The height of the figure is given by $\|\text{proj}_{\mathbf{v}\times\mathbf{w}}\mathbf{u}\|$. The volume of the parallelepiped is the product of the height and the area of the base, so we have

$$\begin{aligned} V &= \| \text{ proj}_{\mathbf{v}\times\mathbf{w}}\mathbf{u} \| \, \|\mathbf{v} \times \mathbf{w}\| \\ &= || \, \frac{\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})}{\|\mathbf{v} \times \mathbf{w}\|} \, || \, \|\mathbf{v} \times \mathbf{w}\| \\ &= \|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})\|. \end{aligned}$$

□

Example 4.4.12: Calculating the Volume of a Parallelepiped

Let $\mathbf{u} = \langle -1, -2, 1 \rangle, \mathbf{v} = \langle 4, 3, 2 \rangle$, and $\mathbf{w} = \langle 0, -5, -2 \rangle$. Find the volume of the parallelepiped with adjacent edges $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ (Figure 4.4.8).

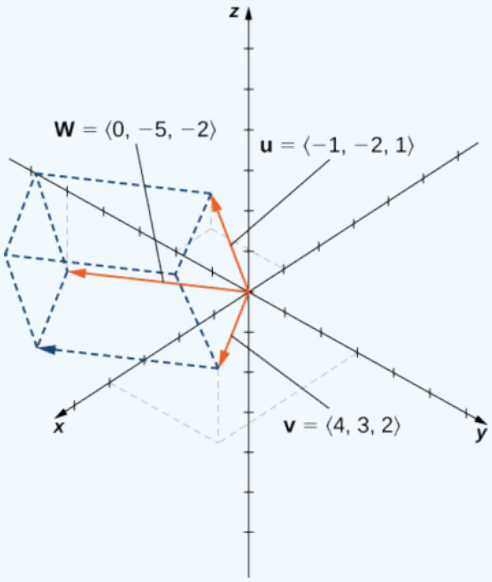


Figure 4.4.8: A parallelogram formed by 3 vectors $\mathbf{u} = \langle -1, -2, 1 \rangle, \mathbf{v} = \langle 4, 3, 2 \rangle$, and $\mathbf{w} = \langle 0, -5, -2 \rangle$.

Solution

We have

$$\begin{aligned} \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) &= \begin{vmatrix} -1 & -2 & 1 \\ 4 & 3 & 2 \\ 0 & -5 & -2 \end{vmatrix} \\ &= (-1) \begin{vmatrix} 3 & 2 \\ -5 & -2 \end{vmatrix} + 2 \begin{vmatrix} 4 & 2 \\ 0 & -2 \end{vmatrix} + \begin{vmatrix} 4 & 3 \\ 0 & -5 \end{vmatrix} \\ &= (-1)(-6 + 10) + 2(-8 - 0) + (-20 - 0) \\ &= -4 - 16 - 20 \\ &= -40. \end{aligned}$$

Thus, the volume of the parallelepiped is $|-40| = 40$ units^3

Watch the accompanying video for a visual demonstration of the concept.

- **Video length:** 2 minutes 28 seconds.
- **Context**: This video demonstrates how to take the determinant of a matrix to calculate the volume of a parallelepiped with edges $\mathbf{u} = 2\mathbf{i} - \mathbf{k}, \mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, and $\mathbf{w} = \mathbf{j} + 2\mathbf{k}$.

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Exercise 4.4.11

Find the volume of the parallelepiped formed by the vectors $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} - \mathbf{k}, \mathbf{b} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$, and $\mathbf{c} = 3\mathbf{j} + \mathbf{k}$.

Hint available at the end of this section.

Applications of the Cross Product

The cross product appears in many practical applications in mathematics, physics, and engineering. Let's examine some of these applications here, including the idea of torque, with which we began this section. Other applications show up in later chapters, particularly in our study of vector fields such as gravitational and electromagnetic fields (Introduction to Vector Calculus).

Example 4.4.13: Using the Triple Scalar Product

Use the triple scalar product to show that vectors $\mathbf{u} = \langle 2, 0, 5\rangle, \mathbf{v} = \langle 2, 2, 4\rangle$, and $\mathbf{w} = \langle 1, -1, 3\rangle$ are coplanar—that is, show that these vectors lie in the same plane.

Solution

Start by calculating the triple scalar product to find the volume of the parallelepiped defined by $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$:

$$\begin{aligned}\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) &= \begin{vmatrix} 2 & 0 & 5 \\ 2 & 2 & 4 \\ 1 & -1 & 3 \end{vmatrix} \\ &= [2(2)(3) + (0)(4)(1) + 5(2)(-1)] - [5(2)(1) + (2)(4)(-1) + (0)(2)(3)] \\ &= 2 - 2 = 0.\end{aligned}$$

The volume of the parallelepiped is 0 units^3, so one of the dimensions must be zero. Therefore, the three vectors all lie in the same plane.

Exercise 4.4.12

Are the vectors $\mathbf{a} = \mathbf{i} + \mathbf{j} - \mathbf{k}, \mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}$, and $\mathbf{c} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ coplanar?

Hint available at the end of this section.

Example 4.4.14: Finding an Orthogonal Vector

Only a single plane can pass through any set of three noncolinear points. Find a vector orthogonal to the plane containing points $P = (9, -3, -2), Q = (1, 3, 0)$, and $R = (-2, 5, 0)$.

Solution

The plane must contain vectors $\vecd{PQ}$ and $\vecd{QR}$:

$$\vecd{PQ} = \langle 1 - 9, 3 - (-3), 0 - (-2)\rangle = \langle -8, 6, 2\rangle$$

$$\vecd{QR} = \langle -2 - 1, 5 - 3, 0 - 0\rangle = \langle -3, 2, 0\rangle.$$

The cross product $\vecd{PQ} \times \vecd{QR}$ produces a vector orthogonal to both $\vecd{PQ}$ and $\vecd{QR}$. Therefore, the cross product is orthogonal to the plane that contains these two vectors:

$$\vecd{PQ} \times \vecd{QR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -8 & 6 & 2 \\ -3 & 2 & 0 \end{vmatrix}$$
$$= 0\mathbf{i} - 6\mathbf{j} - 16\mathbf{k} - (-18\mathbf{k} + 4\mathbf{i} + 0\mathbf{j})$$
$$= -4\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}.$$

We have seen how to use the triple scalar product and how to find a vector orthogonal to a plane. Now we apply the cross product to real-world situations.

Sometimes a force causes an object to rotate. For example, turning a screwdriver or a wrench creates this kind of rotational effect, called torque.

Definition: Torque

Torque, τ (the Greek letter tau), measures the tendency of a force to produce rotation about an axis of rotation. Let $\mathbf{r}$ be a vector with an initial point located on the axis of rotation and with a terminal point located at the point where the force is applied, and let vector $\mathbf{F}$ represent the force. Then torque is equal to the cross product of r and F:

$$\tau = \mathbf{r} \times \mathbf{F}. \tag{4.4.27}$$

See Figure 4.4.9.

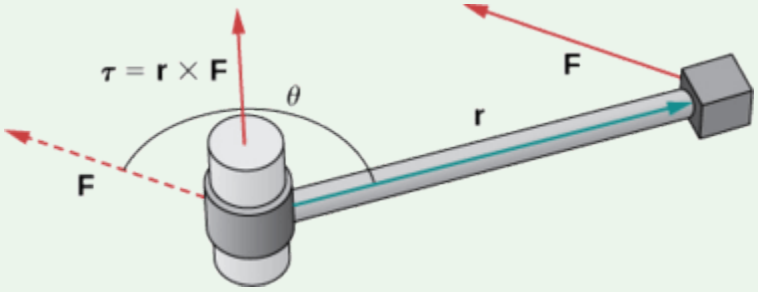


Figure 4.4.9: This image illustrates the concept of torque in physics. Torque τ is shown as the result of a force $\mathbf{F}$ applied at a distance $\mathbf{r}$ from a pivot point. The equation

$$\tau = \mathbf{r} \times \mathbf{F}. \tag{4.4.28}$$

highlights that torque is the cross product of the position vector $\mathbf{r}$ and the force vector $\mathbf{F}$. Torque measures how a force causes an object to rotate.

Think about using a wrench to tighten a bolt. The torque τ applied to the bolt depends on how hard we push the wrench (force) and how far up the handle we apply the force (distance). The torque increases with a greater force on the wrench at a greater distance from the bolt. Common units of torque are the newton-meter or foot-pound. Although torque is dimensionally equivalent to work (it has the same units), the two concepts are distinct. Torque is used specifically in the context of rotation, whereas work typically involves motion along a line.

Example 4.4.15: Evaluating Torque

A bolt is tightened by applying a force of 6 N to a 0.15-m wrench (Figure 4.4.10). The angle between the wrench and the force vector is $40\,°$. Find the magnitude of the torque about the center of the bolt. Round the answer to two decimal places.

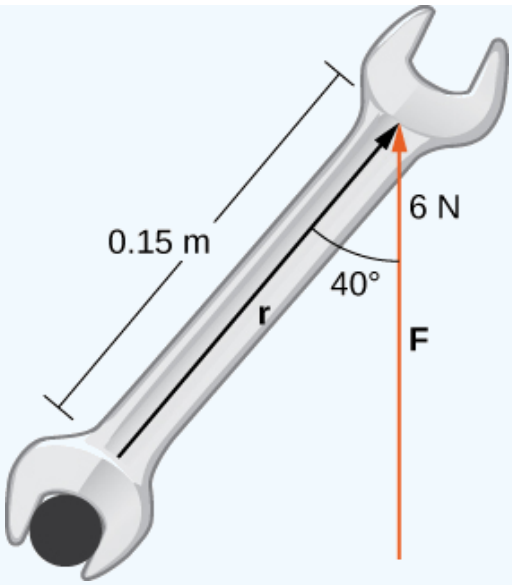


Figure 4.4.10: A wrench being used to demonstrate torque. The wrench is 0.15 meters long, measured as the distance r from the pivot point (the bolt) to where the force is applied. A force $\mathbf{F}$ of 6 newtons is applied at the end of the wrench, at an angle of 40° to the wrench's length. The force is represented as a vertical orange arrow, and the angle 40° is marked between the wrench and the force vector. This setup highlights how torque depends on the force, the distance from the pivot, and the angle of application.

Solution:

Substitute the given information into the equation defining torque:

$$\begin{aligned} \|\boldsymbol{\tau}\| &= \|\mathbf{r} \times \mathbf{F}\| \\ &= \|\mathbf{r}\| \, \| \, \mathbf{F} \, \| \sin\theta \\ &= (0.15\,\text{m})(6\,\text{N}) \sin 40^\circ \\ &\approx 0.58\,\text{N}\cdot\text{m}. \end{aligned}$$

Watch the accompanying video for a visual demonstration of the concept.

- **Video length:** 3 minutes 11 seconds.
- **Context**: This video demonstrates how to find the torque when a wrench is applied to a bolt.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Exercise 4.4.14

Calculate the force required to produce 15 N $\cdot$ m torque at an angle of 30° from a 150-cm rod.

Hint available at the end of this section.

Key Concepts

- The cross product $\mathbf{u} \times \mathbf{v}$ of two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is a vector orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. Its length is given by $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \cdot \|\mathbf{v}\| \cdot \sin\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$. Its direction is given by the right-hand rule.

- The algebraic formula for calculating the cross product of two vectors,

  $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, is

  $\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$.

- The cross product satisfies the following properties for vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$, and scalar c:

  $$\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$$

  $$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$$

  $$c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v})$$

  $$\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$$

  $$\mathbf{v} \times \mathbf{v} = \mathbf{0}$$

  $$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$$

- The cross product of vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is the determinant $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$
- If vectors $\mathbf{u}$ and $\mathbf{v}$ form adjacent sides of a parallelogram, then the area of the parallelogram is given by $\|\mathbf{u} \times \mathbf{v}\|$.
- The triple scalar product of vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ is $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$.
- The volume of a parallelepiped with adjacent edges given by vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ is $V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$.
- If the triple scalar product of vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ is zero, then the vectors are coplanar. The converse is also true: If the vectors are coplanar, then their triple scalar product is zero.
- The cross product can be used to identify a vector orthogonal to two given vectors or to a plane.
- Torque $\boldsymbol{\tau}$ measures the tendency of a force to produce rotation about an axis of rotation. If force $\mathbf{F}$ is acting at a distance (displacement) $\mathbf{r}$ from the axis, then torque is equal to the cross product of $\mathbf{r}$ and $\mathbf{F}$: $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$.

Key Equations

- **The cross product of two vectors in terms of the unit vectors**

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k} \tag{4.4.29}$$

Glossary

Cross Product

$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$, where $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$

Determinant

a real number associated with a square matrix

Parallelepiped

a three-dimensional prism with six faces that are parallelograms

Torque

the effect of a force that causes an object to rotate

Triple Scalar Product

the dot product of a vector with the cross product of two other vectors: $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$

Vector Product

the cross product of two vectors

Answers

Exercise 4.4.1

Use the formula $\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$.

$\mathbf{p} \times \mathbf{q} = \mathbf{i} - 9\mathbf{j} + 2\mathbf{k}$

Exercise 4.4.2

Remember the right-hand rule (Figure 4.4.2).

Up (the positive z-direction)

Exercise 4.4.3

Remember the right-hand rule (Figure 4.4.2).

$-\mathbf{i}$

Exercise 4.4.4

$\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$

$-\mathbf{k}$

Exercise 4.4.5

Vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

16

Exercise 4.4.6

Expand along the first row. Don't forget the second term is negative!

40

Exercise 4.4.7

Calculate the determinant $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 8 & 2 & 3 \\ -1 & 0 & 4 \end{vmatrix}$.

$\mathbf{a} \times \mathbf{b} = 8\mathbf{i} - 35\mathbf{j} + 2\mathbf{k}$

Exercise 4.4.8

Normalize the cross product.

$\left\langle \frac{-3}{\sqrt{194}}, \frac{-13}{\sqrt{194}}, \frac{4}{\sqrt{194}} \right\rangle$ or, simplified as $\left\langle \frac{-3\sqrt{194}}{194}, \frac{-13\sqrt{194}}{194}, \frac{2\sqrt{194}}{97} \right\rangle$

Exercise 4.4.9

Sketch the parallelogram and identify two vectors that form adjacent sides of the parallelogram.

$6\sqrt{13}$ units^2

Exercise 4.4.10

Place the vectors as the rows of a 3×3 matrix, then calculate the determinant.

17

Exercise 4.4.11

Calculate the triple scalar product by finding a determinant.

8 units^3

Exercise 4.4.12

Calculate the triple scalar product.

No, the triple scalar product is $-4 \neq 0$, so the three vectors form the adjacent edges of a parallelepiped. They are not coplanar.

Exercise 4.4.14

$\|\boldsymbol{\tau}\| = 15$ N $\cdot$ m and $\|\mathbf{r}\| = 1.5$ m

20 N

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4.4E: Exercises for Section 4.4

Exercises 1-4

The vectors u and v are given.

a. **Find the cross product $\mathrm{u} \times \mathrm{v}$ of the vectors u and v.**
b. **Express the answer in component form.Sketch the vectors u, v, and $\mathrm{u} \times \mathrm{v}$.**

1. $\mathbf{u} = \langle 2, 0, 0 \rangle, \quad \mathbf{v} = \langle 2, 2, 0 \rangle$

Answer

a. $\mathbf{u} \times \mathbf{v} = \langle 0, 0, 4 \rangle;$
b.

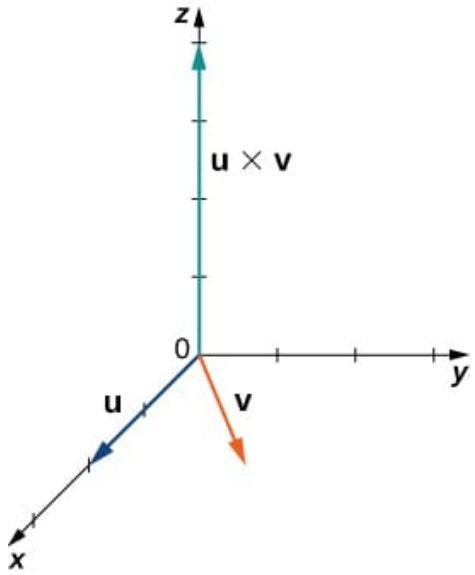


Figure $4.4E.1$: The cross product of two vectors \mathbf{u} (on the x-axis) and \mathbf{v} (in quadrant 1 of the xy-plane) forms a vector \mathbf{u}\times\mathbf{v} on the z-axis.

2. $\mathbf{u} = \langle 3, 2, -1 \rangle, \quad \mathbf{v} = \langle 1, 1, 0 \rangle$
3. $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j}, \quad \mathbf{v} = \mathbf{j} + 2\mathbf{k}$

Answer

a. $\mathbf{u} \times \mathbf{v} = \langle 6, -4, 2 \rangle;$
b.

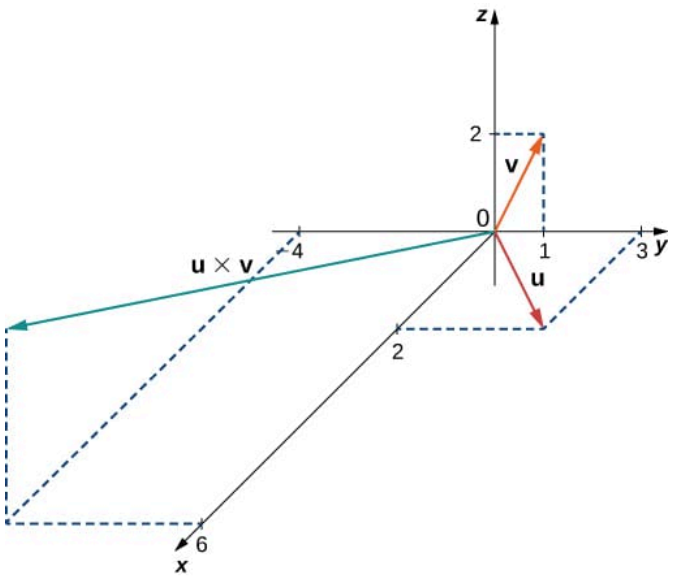


Figure $4.4E.2$: The cross product of two vectors \mathbf{u} (on the xy-plane) and \mathbf{v} (on the yz-plane) forms a vector \mathbf{u}\times\mathbf{v} that is perpendicular to both vectors \mathbf{u} and \mathbf{v}.

4. $\mathbf{u} = 2\mathbf{j} + 3\mathbf{k}, \quad \mathbf{v} = 3\mathbf{i} + \mathbf{k}$
5. Simplify $(\mathbf{i} \times \mathbf{i} - 2\mathbf{i} \times \mathbf{j} - 4\mathbf{i} \times \mathbf{k} + 3\mathbf{j} \times \mathbf{k}) \times \mathbf{i}$.

   Answer

   $-2\mathbf{j} - 4\mathbf{k}$

6. Simplify $\mathbf{j} \times (\mathbf{k} \times \mathbf{j} + 2\mathbf{j} \times \mathbf{i} - 3\mathbf{j} \times \mathbf{j} + 5\mathbf{i} \times \mathbf{k})$.

Exercises 7-10

Vectors u and v are given. Find unit vector w in the direction of the cross product vector $\mathrm{u} \times \mathrm{v}$. Express your answer using standard unit vectors.

7. $\mathbf{u} = \langle 3, -1, 2 \rangle, \quad \mathbf{v} = \langle -2, 0, 1 \rangle$

   Answer

   $\mathbf{w} = -\frac{\sqrt{6}}{18}\mathbf{i} - \frac{7\sqrt{6}}{18}\mathbf{j} - \frac{\sqrt{6}}{9}\mathbf{k}$

8. $\mathbf{u} = \langle 2, 6, 1 \rangle, \quad \mathbf{v} = \langle 3, 0, 1 \rangle$
9. $\mathbf{u} = \vec{AB}, \quad \mathbf{v} = \vec{AC}$, where $A(1, 0, 1)$, $B(1, -1, 3)$, and $C(0, 0, 5)$

   Answer

   $\mathbf{w} = -\frac{4\sqrt{21}}{21}\mathbf{i} - \frac{2\sqrt{21}}{21}\mathbf{j} - \frac{\sqrt{21}}{21}\mathbf{k}$

10. $\mathbf{u} = \vec{OP}, \quad \mathbf{v} = \vec{PQ}$, where $P(-1, 1, 0)$ and $Q(0, 2, 1)$
11. Determine the real number α such that $\mathbf{u} \times \mathbf{v}$ and $\mathbf{i}$ are orthogonal, where $\mathbf{u} = 3\mathbf{i} + \mathbf{j} - 5\mathbf{k}$ and $\mathbf{v} = 4\mathbf{i} - 2\mathbf{j} + \alpha\mathbf{k}$.

    Answer

    $\alpha = 10$

12. Show that $\mathbf{u} \times \mathbf{v}$ and $2\mathbf{i} - 14\mathbf{j} + 2\mathbf{k}$ cannot be orthogonal for any α real number, where $\mathbf{u} = \mathbf{i} + 7\mathbf{j} - \mathbf{k}$ and $\mathbf{v} = \alpha\mathbf{i} + 5\mathbf{j} + \mathbf{k}$.
13. Show that $\mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$, where $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors.
14. Show that $\mathbf{v} \times \mathbf{u}$ is orthogonal to $(\mathbf{u} \cdot \mathbf{v})(\mathbf{u} + \mathbf{v}) + \mathbf{u}$, where $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors.
15. Calculate the determinant $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 7 \\ 2 & 0 & 3 \end{vmatrix}$.

Answer

$$-3\mathbf{i}+11\mathbf{j}+2\mathbf{k}$$

16. Calculate the determinant $\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & 3 & -4\\ 1 & 6 & -1\end{vmatrix}$.

Exercises 17-18

The vectors $\mathbf{u}$ and $\mathbf{v}$ are given. Use determinant notation to find vector $\mathbf{w}$ orthogonal to vectors $\mathbf{u}$ and $\mathbf{v}$.

17. $\mathbf{u}=\langle -1,0,e^t\rangle, \quad \mathbf{v}=\langle 1,e^{-t},0\rangle$, where t is a real number

Answer

$$\mathbf{w}=\langle -1,e^t,-e^{-t}\rangle$$

18. $\mathbf{u}=\langle 1,0,x\rangle, \quad \mathbf{v}=\langle \frac{2}{x},1,0\rangle$, where x is a nonzero real number

19. Find vector $(\mathbf{a}-2\mathbf{b})\times\mathbf{c}$, where $\mathbf{a}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & -1 & 5\\ 0 & 1 & 8\end{vmatrix}, \mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & 1 & 1\\ 2 & -1 & -2\end{vmatrix}$, and $\mathbf{c}=\mathbf{i}+\mathbf{j}+\mathbf{k}$.

Answer

$$-26\mathbf{i}+17\mathbf{j}+9\mathbf{k}$$

20. Find vector $\mathbf{c}\times(\mathbf{a}+3\mathbf{b})$, where $\mathbf{a}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 5 & 0 & 9\\ 0 & 1 & 0\end{vmatrix}, \mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & -1 & 1\\ 7 & 1 & -1\end{vmatrix}$, and $\mathbf{c}=\mathbf{i}-\mathbf{k}$.

21. [T] Use the cross product $\mathbf{u}\times\mathbf{v}$ to find the acute angle between vectors $\mathbf{u}$ and $\mathbf{v}$, where $\mathbf{u}=\mathbf{i}+2\mathbf{j}$ and $\mathbf{v}=\mathbf{i}+\mathbf{k}$. Express the answer in degrees rounded to the nearest integer.

Answer

$$72\,^\circ$$

22. [T] Use the cross product $\mathbf{u}\times\mathbf{v}$ to find the obtuse angle between vectors $\mathbf{u}$ and $\mathbf{v}$, where $\mathbf{u}=-\mathbf{i}+3\mathbf{j}+\mathbf{k}$ and $\mathbf{v}=\mathbf{i}-2\mathbf{j}$. Express the answer in degrees rounded to the nearest integer.
23. Use the sine and cosine of the angle between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ to prove Lagrange's identity: $\|\mathbf{u}\times\mathbf{v}\|^2=\|\mathbf{u}\|^2\|\mathbf{v}\|^2-(\mathbf{u}\cdot\mathbf{v})^2$.
24. Verify Lagrange's identity $\|\mathbf{u}\times\mathbf{v}\|^2=\|\mathbf{u}\|^2\|\mathbf{v}\|^2-(\mathbf{u}\cdot\mathbf{v})^2$ for vectors $\mathbf{u}=-\mathbf{i}+\mathbf{j}-2\mathbf{k}$ and $\mathbf{v}=2\mathbf{i}-\mathbf{j}$.
25. Nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are called collinear if there exists a nonzero scalar α such that $\mathbf{v}=\alpha\mathbf{u}$. Show that $\mathbf{u}$ and $\mathbf{v}$ are collinear if and only if $\mathbf{u}\times\mathbf{v}=0$.
26. Nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are called *collinear* if there exists a nonzero scalar α such that $\mathbf{v}=\alpha\mathbf{u}$. Show that vectors $\vecd{AB}$ and $\vecd{AC}$ are collinear, where $A(4,1,0),\ B(6,5,-2)$, and $C(5,3,-1)$.
27. Find the area of the parallelogram with adjacent sides $\mathbf{u}=\langle 3,2,0\rangle$ and $\mathbf{v}=\langle 0,2,1\rangle$.

Answer

$$7$$

28. Find the area of the parallelogram with adjacent sides $\mathbf{u}=\mathbf{i}+\mathbf{j}$ and $\mathbf{v}=\mathbf{i}+\mathbf{k}$.
29. Consider points $A(3,-1,2),\ B(2,1,5)$, and $C(1,-2,-2)$.
    a. Find the area of parallelogram $ABCD$ with adjacent sides $\vecd{AB}$ and $\vecd{AC}$.
    b. Find the area of triangle ABC.
    c. Find the distance from point A to line BC.

Answer

a. $5\sqrt{6}$;
b. $\frac{5\sqrt{6}}{2}$;
c. $\frac{5\sqrt{6}}{\sqrt{59}}=\frac{5\sqrt{354}}{59}$

30. Consider points $A(2,-3,4)$, $B(0,1,2)$, and $C(-1,2,0)$.
    a. Find the area of parallelogram $ABCD$ with adjacent sides $\vecd{AB}$ and $\vecd{AC}$.
    b. Find the area of triangle ABC.
    c. Find the distance from point B to line AC.

Exercises 31-32

Vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are given.

a. **Find the triple scalar product $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$.**
b. **Find the volume of the parallelepiped with the adjacent edges $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.**

31. $\mathbf{u}=\mathbf{i}+\mathbf{j}$, $\mathbf{v}=\mathbf{j}+\mathbf{k}$, and $\mathbf{w}=\mathbf{i}+\mathbf{k}$

Answer

$a.2$; $b.2$ units^3

32. $\mathbf{u}=\langle -3,5,-1\rangle$, $\mathbf{v}=\langle 0,2,-2\rangle$, and $\mathbf{w}=\langle 3,1,1\rangle$
33. Calculate the triple scalar products $\mathbf{v}\cdot(\mathbf{u}\times\mathbf{w})$ and $\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})$, where $\mathbf{u}=\langle 1,1,1\rangle$, $\mathbf{v}=\langle 7,6,9\rangle$, and $\mathbf{w}=\langle 4,2,7\rangle$.

Answer

$\mathbf{v}\cdot(\mathbf{u}\times\mathbf{w})=-1$, $\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})=1$

34. Calculate the triple scalar products $\mathbf{w}\cdot(\mathbf{v}\times\mathbf{u})$ and $\mathbf{u}\cdot(\mathbf{w}\times\mathbf{v})$, where $\mathbf{u}=\langle 4,2,-1\rangle$, $\mathbf{v}=\langle 2,5,-3\rangle$, and $\mathbf{w}=\langle 9,5,-10\rangle$.
35. Find vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ with a triple scalar product given by the determinant $\begin{vmatrix}1&2&3\\0&2&5\\8&9&2\end{vmatrix}$. Determine their triple scalar product.

Answer

$\mathbf{a}=\langle 1,2,3\rangle$, $\mathbf{b}=\langle 0,2,5\rangle$, $\mathbf{c}=\langle 8,9,2\rangle$; $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=-9$

36. The triple scalar product of vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is given by the determinant $\begin{vmatrix}0&-2&1\\0&1&4\\1&-3&7\end{vmatrix}$. Find vector $\mathbf{a}-\mathbf{b}+\mathbf{c}$.
37. Consider the parallelepiped with edges \(OA,OB,\) and OC, where $A(2,1,0)$, $B(1,2,0)$, and $C(0,1,\alpha)$.
    a. Find the real number $\alpha>0$ such that the volume of the parallelepiped is 3 units^3.
    b. For $\alpha=1$, find the height h from vertex C of the parallelepiped. Sketch the parallelepiped.

Answer

a. $\alpha=1$;
b. h=1 unit,

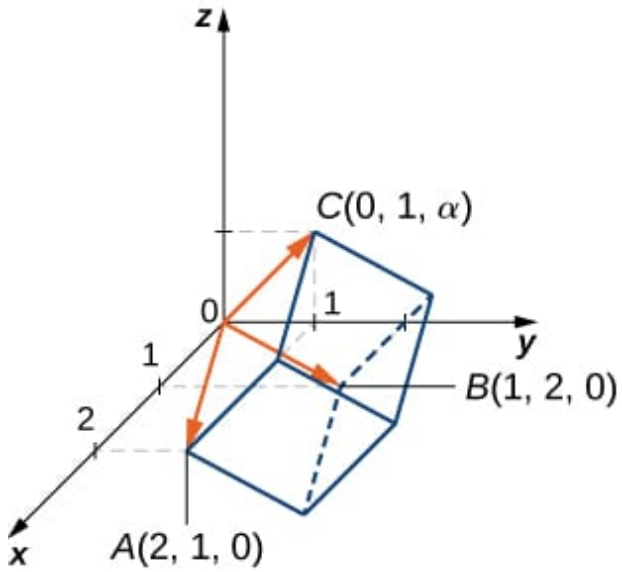


Figure $4.4E.3$: A box in 3d-space formed by three vectors pointing from the origin to the vertices $A(2,1,0)$, $B(1,2,0)$, and $C(0,1,\alpha)$.

38. Consider points $A(\alpha,0,0)$, $B(0,\beta,0)$, and $C(0,0,\gamma)$, with α, β, and γ positive real numbers.
    a. Determine the volume of the parallelepiped with adjacent sides $\vecd{OA}$, $\vecd{OB}$, and $\vecd{OC}$.
    b. Find the volume of the tetrahedron with vertices O, A, B, and C. (Hint: The volume of the tetrahedron is $1/6$ of the volume of the parallelepiped.)
    c. Find the distance from the origin to the plane determined by A, B, and C. Sketch the parallelepiped and tetrahedron.
39. Let u, v, and w be three-dimensional vectors and c be a real number. Prove the following properties of the cross product.
    a. $\mathbf{u}\times\mathbf{u}=\mathbf{0}$
    b. $\mathbf{u}\times(\mathbf{v}+\mathbf{w})=(\mathbf{u}\times\mathbf{v})+(\mathbf{u}\times\mathbf{w})$
    c. $c(\mathbf{u}\times\mathbf{v})=(c\mathbf{u})\times\mathbf{v}=\mathbf{u}\times(c\mathbf{v})$
    d. $\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v})=\mathbf{0}$
40. Show that vectors $\mathbf{u}=\langle 1,0,-8\rangle$, $\mathbf{v}=\langle 0,1,6\rangle$, and $\mathbf{w}=\langle -1,9,3\rangle$ satisfy the following properties of the cross product.
    a. $\mathbf{u}\times\mathbf{u}=\mathbf{0}$
    b. $\mathbf{u}\times(\mathbf{v}+\mathbf{w})=(\mathbf{u}\times\mathbf{v})+(\mathbf{u}\times\mathbf{w})$
    c. $c(\mathbf{u}\times\mathbf{v})=(c\mathbf{u})\times\mathbf{v}=\mathbf{u}\times(c\mathbf{v})$
    d. $\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v})=\mathbf{0}$
41. Nonzero vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are said to be linearly dependent if one of the vectors is a linear combination of the other two. For instance, there exist two nonzero real numbers α and β such that $\mathbf{w}=\alpha\mathbf{u}+\beta\mathbf{v}$. Otherwise, the vectors are called linearly independent. Show that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ could be placed on the same plane if and only if they are linear dependent.
42. Consider vectors $\mathbf{u}=\langle 1,4,-7\rangle$, $\mathbf{v}=\langle 2,-1,4\rangle$, $\mathbf{w}=\langle 0,-9,18\rangle$, and $\mathbf{p}=\langle 0,-9,17\rangle$.
    a. Show that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ can be placed on the same plane by using their triple scalar product
    b. Show that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ can be placed on the same plane by using the definition that there exist two nonzero real numbers α and β such that $w=\alpha u+\beta v$.
    c. Show that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{p}$ are linearly independent—that is, none of the vectors is a linear combination of the other two.
43. Consider points $A(0,0,2)$, $B(1,0,2)$, $C(1,1,2)$, and $D(0,1,2)$. Are vectors $\vecd{AB}$, $\vecd{AC}$, and $\vecd{AD}$ linearly dependent (that is, one of the vectors is a linear combination of the other two)?

    Answer

    Yes, $\vecd{AD}=\alpha\vecd{AB}+\beta\vecd{AC}$, where $\alpha=-1$ and $\beta=1$.

44. Show that vectors $\mathbf{i}+\mathbf{j}$, $\mathbf{i}-\mathbf{j}$, and $\mathbf{i}+\mathbf{j}+\mathbf{k}$ are linearly independent—that is, there exist two nonzero real numbers α and β such that $\mathbf{i}+\mathbf{j}+\mathbf{k}=\alpha(\mathbf{i}+\mathbf{j})+\beta(\mathbf{i}-\mathbf{j})$.
45. Let $\mathbf{u}=\langle u_1,u_2\rangle$ and $\mathbf{v}=\langle v_1,v_2\rangle$ be two-dimensional vectors. The cross product of vectors $\mathbf{u}$ and $\mathbf{v}$ is not defined. However, if the vectors are regarded as the three-dimensional vectors $\tilde{\mathbf{u}}=\langle u_1,u_2,0\rangle$ and $\tilde{\mathbf{v}}=\langle v_1,v_2,0\rangle$, respectively, then, in this case, we can define the cross product of $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{v}}$. In particular, in determinant notation, the cross product of $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{v}}$ is given by

$$\tilde{\mathbf{u}} \times \tilde{\mathbf{v}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & 0 \\ v_1 & v_2 & 0 \end{vmatrix}.$$

Use this result to compute $(\cos\theta\ \mathbf{i} + \sin\theta\ \mathbf{j}) \times (\sin\theta\ \mathbf{i} - \cos\theta\ \mathbf{j})$, where θ is a real number.

Answer

$-\mathbf{k}$

46. Consider points $P(2,1), Q(4,2)$, and $R(1,2)$.
    a. Find the area of triangle \(PQR\).
    b. Determine the distance from point R to the line passing through P and Q.
47. Determine a vector of magnitude 10 perpendicular to the plane passing through the x-axis and point $P(1,2,4)$.

Answer

$\langle 0, \pm 4\sqrt{5}, 2\sqrt{5} \rangle$

48. Determine a unit vector perpendicular to the plane passing through the *z*-axis and point $A(3,1,-2)$.
49. Consider $\mathbf{u}$ and $\mathbf{v}$ two three-dimensional vectors. If the magnitude of the cross product vector $\mathbf{u} \times \mathbf{v}$ is k times larger than the magnitude of vector $\mathbf{u}$, show that the magnitude of $\mathbf{v}$ is greater than or equal to k, where k is a natural number.
50. [T] Assume that the magnitudes of two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are known. The function $f(\theta) = \|\mathbf{u}\|\|\mathbf{v}\| \sin\theta$ defines the magnitude of the cross product vector $\mathbf{u} \times \mathbf{v}$, where $\theta \in [0, \pi]$ is the angle between $\mathbf{u}$ and $\mathbf{v}$.
    a. Graph the function f.
    b. Find the absolute minimum and maximum of function f. Interpret the results.
    c. If $\|\mathbf{u}\| = 5$ and $\|\mathbf{v}\| = 2$, find the angle between $\mathbf{u}$ and $\mathbf{v}$ if the magnitude of their cross product vector is equal to 9.
51. Find all vectors $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ that satisfy the equation $\langle 1,1,1 \rangle \times \mathbf{w} = \langle -1,-1,2 \rangle$. Hint: You should be able to write all components of $\mathbf{w}$ in terms of one of the constants w_1, w_2, or w_3.

Answer

Writing all components in terms of the constant w_3, one way to represent these vectors is: $\mathbf{w} = \langle w_3 - 1, w_3 + 1, w_3 \rangle$, where w_3 is any real number.
Note that we could use any parameter we wish here. We could set $w_3 = a$. Then $\mathbf{w} = \langle a - 1, a + 1, a \rangle$ would also represent these vectors.

52. Solve the equation $\mathbf{w} \times \langle 1,0,-1 \rangle = \langle 3,0,3 \rangle$, where $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ is a nonzero vector with a magnitude of 3.
53. [T] A mechanic uses a 12-in. wrench to turn a bolt. The wrench makes a $30°$ angle with the horizontal. If the mechanic applies a vertical force of 10 lb on the wrench handle, what is the magnitude of the torque at point P (see Figure $4.4E.4$)? Express the answer in foot-pounds rounded to two decimal places.

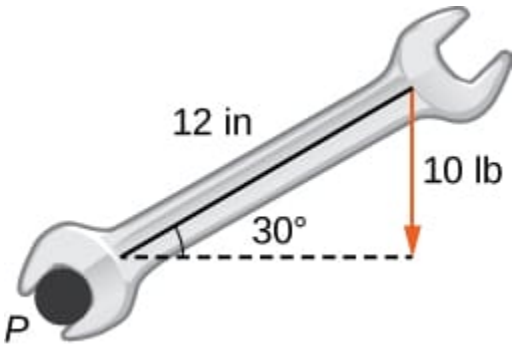


Figure $4.4E.4$: The wrench 12 inches long makes a $30°$ angle with the horizontal. A vertical force of 10 lb on the wrench handle at one end of the wrench.

Answer

8.66 ft-lb

54. [T] A boy applies the brakes on a bicycle by applying a downward force of 20 lb on the pedal when the 6-in. crank makes a $40°$ angle with the horizontal (see Figure $4.4E.5$). Find the torque at point P. Express your answer in foot-pounds rounded to two decimal places.

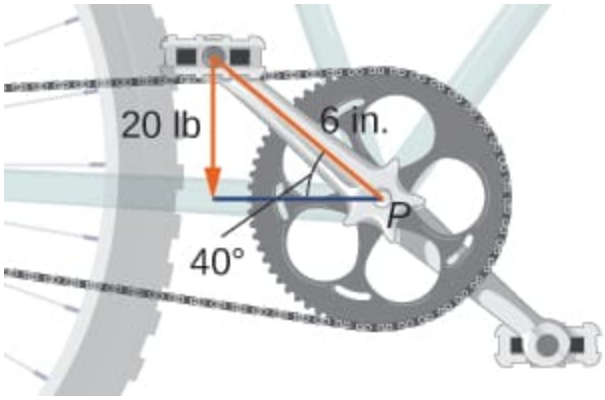


Figure $4.4E.5$: A force of 20 lb is applying downward on the pedal when the 6-in. crank makes a $40°$ angle with the horizontal.

55. [T] Find the magnitude of the force that needs to be applied to the end of a 20-cm wrench located on the positive direction of the y-axis if the force is applied in the direction $\langle 0, 1, -2 \rangle$ and it produces a 100 N·m torque to the bolt located at the origin.

Answer

$250\sqrt{5}$ N ≈ 559 N

56. [T] What is the magnitude of the force required to be applied to the end of a 1-ft wrench at an angle of $35°$ to produce a torque of 20 N·m?

57. [T] The force vector $\mathbf{F}$ acting on a proton with an electric charge of 1.6×10^{-19} C (in coulombs) moving in a magnetic field $\mathbf{B}$ where the velocity vector $\mathbf{v}$ is given by $\mathbf{F} = 1.6 \times 10^{-19}(\mathbf{v} \times \mathbf{B})$ (here, $\mathbf{v}$ is expressed in meters per second, $\mathbf{B}$ is in tesla [T], and $\mathbf{F}$ is in newtons [N]). Find the force that acts on a proton that moves in the xy-plane at velocity $\mathbf{v} = 10^5\mathbf{i} + 10^5\mathbf{j}$ (in meters per second) in a magnetic field given by $\mathbf{B} = 0.3\mathbf{j}$.

Answer

$\mathbf{F} = 4.8 \times 10^{-15}$ kN

58. [T] The force vector $\mathbf{F}$ acting on a proton with an electric charge of 1.6×10^{-19} C moving in a magnetic field $\mathbf{B}$ where the velocity vector $\mathbf{v}$ is given by $\mathbf{F} = 1.6 \times 10^{-19}(\mathbf{v} \times \mathbf{B})$ (here, $\mathbf{v}$ is expressed in meters per second, $\mathbf{B}$ in T, and $\mathbf{F}$ in N). If the magnitude of force $\mathbf{F}$ acting on a proton is 5.9×10^{-17} N and the proton is moving at the speed of 300 m/sec in magnetic field $\mathbf{B}$ of magnitude 2.4 T, find the angle between velocity vector $\mathbf{v}$ of the proton and magnetic field $\mathbf{B}$. Express the answer in degrees rounded to the nearest integer.

59. [T] Consider $\mathbf{r}(t) = \langle \cos t, \ \sin t, \ 2t \rangle$ the position vector of a particle at time $t \in [0, 30]$, where the components of $\mathbf{r}$ are expressed in centimeters and time in seconds. Let $\vec{OP}$ be the position vector of the particle after 1 sec.
    a. Determine unit vector $\mathbf{B}(t)$ (called the binormal unit vector) that has the direction of cross product vector $\mathbf{v}(t) \times \mathbf{a}(t)$, where $\mathbf{v}(t)$ and $\mathbf{a}(t)$ are the instantaneous velocity vector and, respectively, the acceleration vector of the particle after t seconds.
    b. Use a CAS to visualize vectors $\mathbf{v}(1), \ \mathbf{a}(1)$, and $\mathbf{B}(1)$ as vectors starting at point P along with the path of the particle.

Answer

a. $\mathbf{B}(t) = \langle \frac{2\sqrt{5}\sin t}{5}, -\frac{2\sqrt{5}\cos t}{5}, \frac{\sqrt{5}}{5} \rangle$;
b.

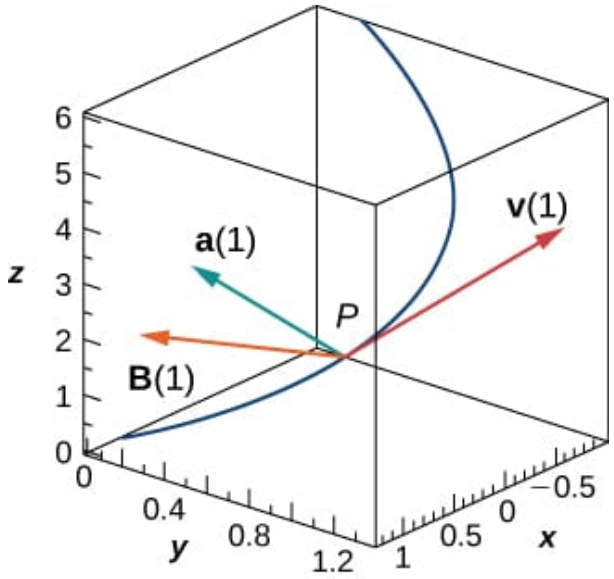


Figure $4.4E.6$: A three-dimensional parametric curve within a Cartesian coordinate system. At a specific point P on the curve, three key vectors are shown: $\mathbf{v}(1)$, the velocity vector (red), tangent to the curve at P; $\mathbf{a}(1)$, the acceleration vector (green), indicating the rate of change of velocity; and $\mathbf{B}(1)$, possibly representing another contextual vector such as a binormal or force vector (orange). The curve and vectors demonstrate the dynamics of motion at P, highlighting the relationship between position, velocity, and acceleration in 3D space.

50. A solar panel is mounted on the roof of a house. The panel may be regarded as positioned at the points of coordinates (in meters) $A(8,0,0), B(8,18,0), C(0,18,8)$, and $D(0,0,8)$ (see Figure $4.4E.7$).

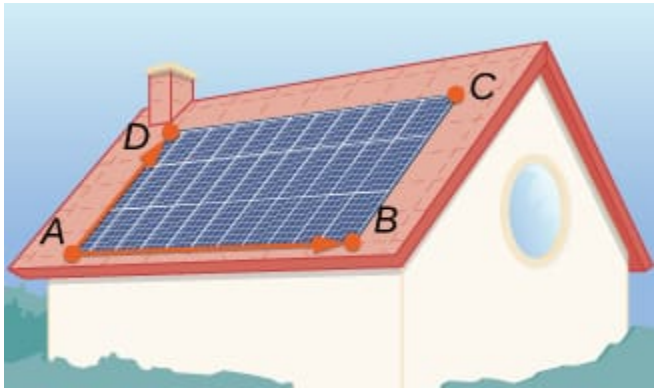


Figure $4.4E.7$: A house with a slanted roof equipped with solar panels. The solar panels are represented as a rectangular array covering a significant portion of one side of the roof. Key points labeled A, B, C, and D mark the corners of the solar panel array. The roof features a chimney near point D, and a circular window is visible on the house's wall below the roof.

a. Find vector $\mathbf{n} = \vecd{AB} \times \vecd{AD}$ perpendicular to the surface of the solar panels. Express the answer using standard unit vectors. Note that the magnitude of this vector should give us the area of rectangle $ABCD$.
b. Assume unit vector $\mathbf{s} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$ points toward the Sun at a particular time of the day and the flow of solar energy is $\mathbf{F} = 900\mathbf{s}$ (in watts per square meter $[W/m^2]$). Find the predicted amount of electrical power the panel can produce, which is given by the dot product of vectors $\mathbf{F}$ and $\mathbf{n}$ (expressed in watts).
c. Determine the angle of elevation of the Sun above the solar panel. Express the answer in degrees rounded to the nearest whole number. (Hint: The angle between vectors $\mathbf{n}$ and $\mathbf{s}$ and the angle of elevation are complementary.)

Contributors

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4.5: Equations of Lines and Planes in Space

Learning Objectives

- Write the vector, parametric, and symmetric of a line through a given point in a given direction, and a line through two given points.
- Find the distance from a point to a given line.
- Write the vector and scalar equations of a plane through a given point with a given normal.
- Find the distance from a point to a given plane.
- Find the angle between two planes.

By now, we are familiar with writing equations that describe a line in two dimensions. To write an equation for a line, we must know two points on the line, or we must know the direction of the line and at least one point through which the line passes. In two dimensions, we use the concept of slope to describe the orientation, or direction, of a line. In three dimensions, we describe the direction of a line using a vector parallel to the line. In this section, we examine how to use equations to describe lines and planes in space.

Equations for a Line in Space

Let's first explore what it means for two vectors to be parallel. Recall that parallel vectors must have the same or opposite directions. If two nonzero vectors, $\mathbf{u}$ and $\mathbf{v}$, are parallel, we claim there must be a scalar, k, such that $\mathbf{u} = k\mathbf{v}$. If $\mathbf{u}$ and $\mathbf{v}$ have the same direction, simply choose

$$k = \frac{\|\mathbf{u}\|}{\|\mathbf{v}\|}. \tag{4.5.1}$$

If $\mathbf{u}$ and $\mathbf{v}$ have opposite directions, choose

$$k = -\frac{\|\mathbf{u}\|}{\|\mathbf{v}\|}. \tag{4.5.2}$$

Note that the converse holds as well. If $\mathbf{u} = k\mathbf{v}$ for some scalar k, then either $\mathbf{u}$ and $\mathbf{v}$ have the same direction $(k > 0)$ or opposite directions $(k < 0)$, so $\mathbf{u}$ and $\mathbf{v}$ are parallel. Therefore, two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel if and only if $\mathbf{u} = k\mathbf{v}$ for some scalar k. By convention, the zero vector $\mathbf{0}$ is considered to be parallel to all vectors.

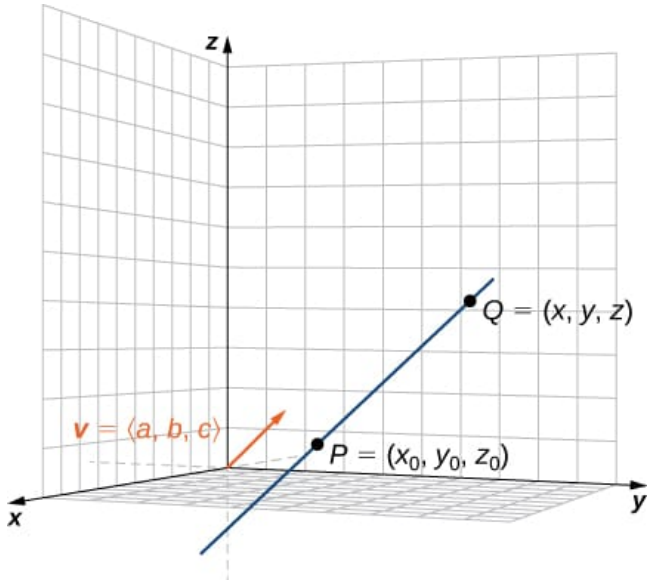


Figure 4.5.1: A three-dimensional Cartesian coordinate system with axes labeled x, y, and z. There is a line passing through two points in 3D space, labeled $P(x_0, y_0, z_0)$ and $Q(x, y, z)$. A vector $\mathbf{v} = \langle a, b, c \rangle$ is shown in orange, indicating the direction of the line.

As in two dimensions, we can describe a line in space using a point on the line and the direction of the line, or a parallel vector, which we call the **direction vector** (Figure 4.5.1). Let L be a line in space passing through point $P(x_0, y_0, z_0)$. Let $\mathbf{v} = \langle a, b, c \rangle$ be a vector parallel to L. Then, for any point on line $Q(x, y, z)$, we know that $\vecd{PQ}$ is parallel to $\mathbf{v}$. Thus, as we just discussed, there is a scalar, t, such that $\vecd{PQ} = t\mathbf{v}$, which gives

$$\begin{aligned} \vecd{PQ} &= t\mathbf{v} \\ \langle x - x_0, y - y_0, z - z_0 \rangle &= t\langle a, b, c \rangle \\ \langle x - x_0, y - y_0, z - z_0 \rangle &= \langle ta, tb, tc \rangle. \end{aligned} \tag{4.5.3}$$

Using vector operations, we can rewrite Equation 4.5.3

$$\begin{aligned} \langle x - x_0, y - y_0, z - z_0 \rangle &= \langle ta, tb, tc \rangle \\ \langle x, y, z \rangle - \langle x_0, y_0, z_0 \rangle &= t\langle a, b, c \rangle \\ \underbrace{\langle x, y, z \rangle}_{\mathbf{r}} &= \underbrace{\langle x_0, y_0, z_0 \rangle}_{\mathbf{r}_o} + t\underbrace{\langle a, b, c \rangle}_{\mathbf{v}}. \end{aligned}$$

Setting $\mathbf{r} = \langle x, y, z \rangle$ and $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$, we now have the **vector equation of a line:**

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}. \tag{4.5.4}$$

Equating components, Equation 4.5.4 shows that the following equations are simultaneously true: $x - x_0 = ta, y - y_0 = tb$, and $z - z_0 = tc$. If we solve each of these equations for the component variables x, y, and z, we get a set of equations in which each variable is defined in terms of the parameter t and that, together, describe the line. This set of three equations forms a set of **parametric equations of a line:**

$$x = x_0 + ta$$

$$y = y_0 + tb$$

$$z = z_0 + tc.$$

If we solve each of the equations for t assuming a, b, and c are nonzero, we get a different description of the same line:

$$\frac{x - x_0}{a} = t$$

$$\frac{y - y_0}{b} = t$$

$$\frac{z - z_0}{c} = t.$$

Because each expression equals t, they all have the same value. We can set them equal to each other to create **symmetric equations of a line:**

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

We summarize the results in the following theorem.

Parametric and Symmetric Equations of a Line

A line L parallel to vector $\mathbf{v} = \langle a, b, c \rangle$ and passing through point $P(x_0, y_0, z_0)$ can be described by the following parametric equations:

$$x = x_0 + ta, \quad y = y_0 + tb, \quad z = z_0 + tc \tag{4.5.5}$$

If the constants $a, b,$ and c are all nonzero, then L can be described by the symmetric equation of the line:

$$\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}. \tag{4.5.6}$$

The parametric equations of a line are not unique. Using a different parallel vector or a different point on the line leads to a different, equivalent representation. Each set of parametric equations leads to a related set of symmetric equations, so it follows that a symmetric equation of a line is not unique either.

Example 4.5.1: Equations of a Line in Space

Find parametric and symmetric equations of the line passing through points $(1,4,-2)$ and $(-3,5,0)$.

Solution

First, identify a vector parallel to the line:

$$\mathbf{v}=\langle -3-1,5-4,0-(-2)\rangle=\langle -4,1,2\rangle.$$

Use either of the given points on the line to complete the parametric equations:

$$x=1-4t$$
$$y=4+t,$$

and

$$z=-2+2t.$$

Solve each equation for t to create the symmetric equation of the line:

$$\frac{x-1}{-4}=y-4=\frac{z+2}{2}.$$

Exercise 4.5.1

Find parametric and symmetric equations of the line passing through points $(1,-3,2)$ and $(5,-2,8)$.

Hint: available at the end of this section.

Answer available at the end of this section.

Sometimes we don't want the equation of a whole line, just a line segment. In this case, we limit the values of our parameter t . For example, let $P(x_0,y_0,z_0)$ and $Q(x_1,y_1,z_1)$ be points on a line, and let $\mathbf{p}=\langle x_0,y_0,z_0\rangle$ and $\mathbf{q}=\langle x_1,y_1,z_1\rangle$ be the associated position vectors. In addition, let $\mathbf{r}=\langle x,y,z\rangle$. We want to find a vector equation for the line segment between P and Q. Using P as our known point on the line, and $\vecd{PQ}=\langle x_1-x_0,y_1-y_0,z_1-z_0\rangle$ as the direction vector equation, Equation 4.5.4 gives

$$\mathbf{r}=\mathbf{p}+t(\vecd{PQ}). \tag{4.5.7}$$

Equation 4.5.7 can be expanded using properties of vectors:

$$\begin{aligned}\mathbf{r}&=\mathbf{p}+t(\vecd{PQ})\\ &=\langle x_0,y_0,z_0\rangle+t\langle x_1-x_0,y_1-y_0,z_1-z_0\rangle\\ &=\langle x_0,y_0,z_0\rangle+t(\langle x_1,y_1,z_1\rangle-\langle x_0,y_0,z_0\rangle)\\ &=\langle x_0,y_0,z_0\rangle+t\langle x_1,y_1,z_1\rangle-t\langle x_0,y_0,z_0\rangle\\ &=(1-t)\langle x_0,y_0,z_0\rangle+t\langle x_1,y_1,z_1\rangle\\ &=(1-t)\mathbf{p}+t\mathbf{q}.\end{aligned}$$

Thus, the vector equation of the line passing through P and Q is

$$\mathbf{r} = (1-t)\mathbf{p} + t\mathbf{q}. \tag{4.5.8}$$

Remember that we did not want the equation of the whole line, just the line segment between P and Q. Notice that when $t = 0$, we have $r = p$, and when $t = 1$, we have $\mathbf{r} = \mathbf{q}$. Therefore, the vector equation of the line segment between P and Q is

$$\mathbf{r} = (1-t)\mathbf{p} + t\mathbf{q}, 0 \leq t \leq 1. \tag{4.5.9}$$

Going back to Equation 4.5.4, we can also find parametric equations for this line segment. We have

$$\begin{aligned} \mathbf{r} &= \mathbf{p} + t(\vec{PQ}) \\ \langle x, y, z\rangle &= \langle x_0, y_0, z_0\rangle + t\langle x_1 - x_0, y_1 - y_0, z_1 - z_0\rangle \\ &= \langle x_0 + t(x_1 - x_0), y_0 + t(y_1 - y_0), z_0 + t(z_1 - z_0)\rangle. \end{aligned}$$

Then, the parametric equations are

$$\begin{aligned} x &= x_0 + t(x_1 - x_0) \\ y &= y_0 + t(y_1 - y_0) \\ z &= z_0 + t(z_1 - z_0),\ 0 \leq t \leq 1. \end{aligned}$$

Example 4.5.2: Parametric Equations of a Line Segment

Find parametric equations of the line segment between the points $P(2, 1, 4)$ and $Q(3, -1, 3)$.

Solution

Start with the parametric equations for a line (Equations ???) and work with each component separately:

$$\begin{aligned} x &= x_0 + t(x_1 - x_0) \\ &= 2 + t(3 - 2) \\ &= 2 + t, \end{aligned}$$

$$\begin{aligned} y &= y_0 + t(y_1 - y_0) \\ &= 1 + t(-1 - 1) \\ &= 1 - 2t, \end{aligned}$$

and

$$\begin{aligned} z &= z_0 + t(z_1 - z_0) \\ &= 4 + t(3 - 4) \\ &= 4 - t. \end{aligned}$$

Therefore, the parametric equations for the line segment are

$$\begin{aligned} x &= 2 + t \\ y &= 1 - 2t \\ z &= 4 - t,\ 0 \leq t \leq 1. \end{aligned}$$

Exercise 4.5.2

Find parametric equations of the line segment between points $P(-1, 3, 6)$ and $Q(-8, 2, 4)$.

Answer available at the end of this section.

Distance between a Point and a Line

We already know how to calculate the distance between two points in space. We now expand this definition to describe the distance between a point and a line in space. Several real-world contexts exist when it is important to be able to calculate these distances. When building a home, for example, builders must consider “setback” requirements, when structures or fixtures have to be a certain distance from the property line. Air travel offers another example. Airlines are concerned about the distances between populated areas and proposed flight paths.

Let L be a line in the plane and let M be any point not on the line. Then, we define distance d from M to L as the length of line segment $\overline{MP}$, where P is a point on L such that $\overline{MP}$ is perpendicular to L (Figure $4.5.2$).

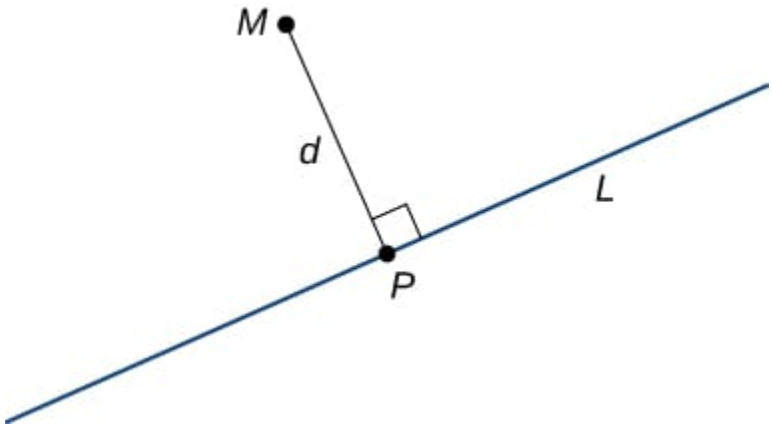


Figure $4.5.2$: The distance from point M to line L is the length of $\overline{MP}$ where $\overline{MP}$ is perpendicular to line L.

When we’re looking for the distance between a line and a point in space, Figure $4.5.2$ still applies. We still define the distance as the length of the perpendicular line segment connecting the point to the line. In space, however, there is no clear way to know which point on the line creates such a perpendicular line segment, so we select an arbitrary point on the line and use properties of vectors to calculate the distance. Therefore, let P be an arbitrary point on line L and let $\mathbf{v}$ be a direction vector for L (Figure $4.5.3$).

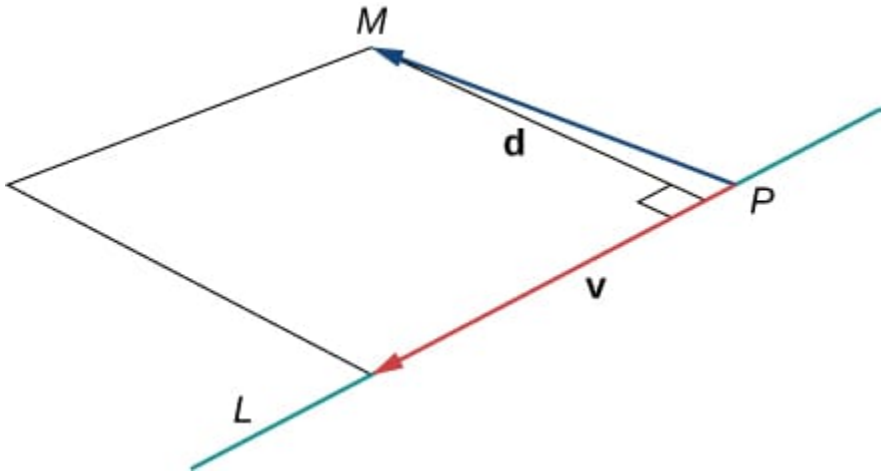


Figure $4.5.3$: Vectors $\vecd{PM}$ *and* $\mathbf{v}$ *form two sides of a parallelogram with base* $\|\mathbf{v}\|$ *and height* d*, which is the distance between a line and a point in space.*

Vectors $\vecd{PM}$ and $\mathbf{v}$ form two sides of a parallelogram with area $\|\vecd{PM} \times \mathbf{v}\|$. Using a formula from geometry, the area of this parallelogram can also be calculated as the product of its base and height:

$$\|\vecd{PM} \times \mathbf{v}\| = \|\mathbf{v}\| d. \tag{4.5.10}$$

We can use this formula to find a general formula for the distance between a line in space and any point not on the line.

Distance from a Point to a Line

Let L be a line in space passing through point P with direction vector $\mathbf{v}$. If M is any point not on L, then the distance from M to L is

$$d = \frac{\|\vecd{PM} \times \mathbf{v}\|}{\|\mathbf{v}\|}. \tag{4.5.11}$$

Example 4.5.3: Calculating the Distance from a Point to a Line

Find the distance between the point $M = (1, 1, 3)$ and line $\frac{x-3}{4} = \frac{y+1}{2} = z - 3$.

Solution:

From the symmetric equations of the line, we know that vector $\mathbf{v} = \langle 4, 2, 1\rangle$ is a direction vector for the line. Setting the symmetric equations of the line equal to zero, we see that point $P(3, -1, 3)$ lies on the line. Then,

$$\begin{aligned} \vecd{PM} &= \langle 1-3, 1-(-1), 3-3\rangle \\ &= \langle -2, 2, 0\rangle. \end{aligned}$$

To calculate the distance, we need to find $\vecd{PM} \times \mathbf{v}$:

$$\begin{aligned} \vecd{PM} \times \mathbf{v} &= \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 2 & 0 \\ 4 & 2 & 1 \end{bmatrix} \\ &= (2-0)\mathbf{i} - (-2-0)\mathbf{j} + (-4-8)\mathbf{k} \\ &= 2\mathbf{i} + 2\mathbf{j} - 12\mathbf{k}. \end{aligned}$$

Therefore, the distance between the point and the line is (Figure 4.5.4)

$$\begin{aligned} d &= \frac{\|\vecd{PM} \times \mathbf{v}\|}{\|\mathbf{v}\|} \\ &= \frac{\sqrt{2^2 + 2^2 + 12^2}}{\sqrt{4^2 + 2^2 + 1^2}} \\ &= \frac{2\sqrt{38}}{\sqrt{21}} \\ &= \frac{2\sqrt{798}}{21} \text{ units} \end{aligned}$$

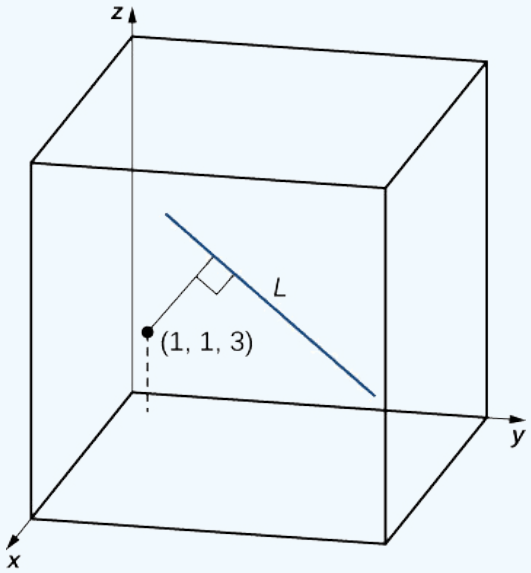


Figure 4.5.4: A 3D line L and a point $(1,1,3)$ with a perpendicular distance to the line marked by a right angle.Point $(1,1,3)$ is approximately 2.7 units from the line L with symmetric equations $\frac{x-3}{4}=\frac{y+1}{2}=z-3$.

Exercise 4.5.3

Find the distance between point $(0,3,6)$ and the line with parametric equations $x=1-t, y=1+2t, z=5+3t$.

Hint available at the end of this section.

Answer available at the end of this section.

Relationships between Lines

Given two lines in the two-dimensional plane, the lines are equal, they are parallel but not equal, or they intersect in a single point. In three dimensions, a fourth case is possible. If two lines in space are not parallel, but do not intersect, then the lines are said to be **skew lines** (Figure 4.5.5).

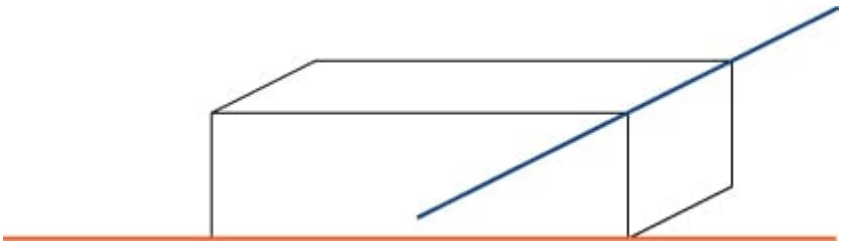


Figure 4.5.5: Two lines blue and orange with different directions may not cross in 3d-space.

To classify lines as parallel but not equal, equal, intersecting, or skew, we need to know two things: whether the direction vectors are parallel and whether the lines share a point (Figure 4.5.6).

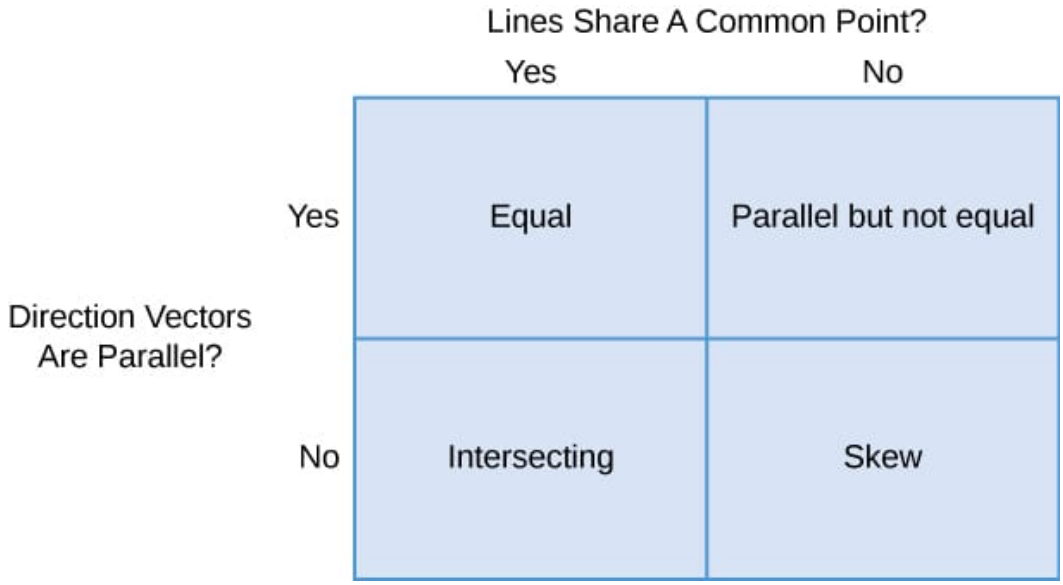


Figure 4.5.6: A table to determine the relationship between two lines based on whether their direction vectors are parallel and whether they share a point. **Equal**: Lines share a common point and have parallel direction vectors. **Parallel but not equal**: Lines do not share a common point but have parallel direction vectors. **Intersecting**: Lines share a common point but are not parallel. **Skew**: Lines do not share a common point and are not parallel.

Example 4.5.4: Classifying Lines in Space

For each pair of lines, determine whether the lines are equal, parallel but not equal, skew, or intersecting.

a.

- $L_1 : x = 2s - 1, y = s - 1, z = s - 4$
- $L_2 : x = t - 3, y = 3t + 8, z = 5 - 2t$

b.

- $L_1 : x = -y = z$
- $L_2 : \dfrac{x-3}{2} = y = z - 2$

c.

- $L_1 : x = 6s - 1, y = -2s, z = 3s + 1$
- $L_2 : \dfrac{x-4}{6} = \dfrac{y+3}{-2} = \dfrac{z-1}{3}$

Solution

a. Line L_1 has direction vector $\mathbf{v}_1 = \langle 2, 1, 1 \rangle$; line L_2 has direction vector $\mathbf{v}_2 = \langle 1, 3, -2 \rangle$. Because the direction vectors are not parallel vectors, the lines are either intersecting or skew. To determine whether the lines intersect, we see if there is a point, (x, y, z), that lies on both lines. To find this point, we use the parametric equations to create a system of equalities:

$$2s - 1 = t - 3; \tag{4.5.12}$$

$$s - 1 = 3t + 8; \tag{4.5.13}$$

$$s - 4 = 5 - 2t. \tag{4.5.14}$$

By the first equation, $t = 2s + 2$. Substituting into the second equation yields

$$s - 1 = 3(2s + 2) + 8$$

$$s - 1 = 6s + 6 + 8$$

$$5s = -15$$

$$s = -3.$$

Substitution into the third equation, however, yields a contradiction:

$$s - 4 = 5 - 2(2s + 2)$$

$$s - 4 = 5 - 4s - 4$$

$$5s = 5$$

$$s = 1.$$

There is no single point that satisfies the parametric equations for L_1 and L_2 simultaneously. These lines do not intersect, so they are skew (see Figure $4.5.7$).

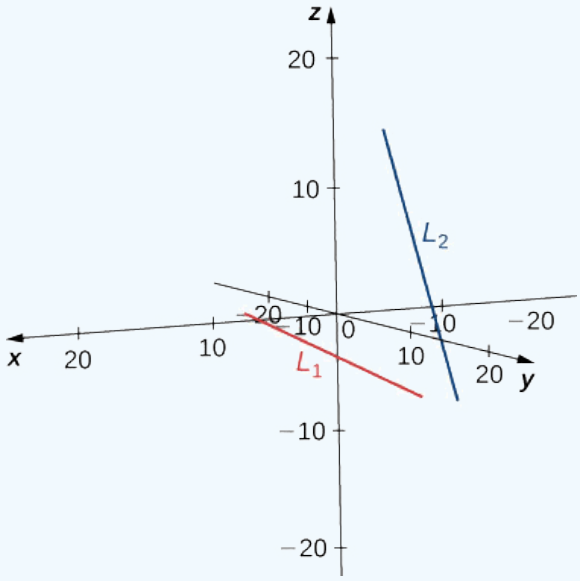


Figure $4.5.7$: Two non-intersecting lines L_1 red and L_2 blue in space.

b. Line L_1 has direction vector $\mathbf{v}_1 = \langle 1, -1, 1 \rangle$ and passes through the origin, $(0, 0, 0)$. Line L_2 has a different direction vector, $\mathbf{v}_2 = \langle 2, 1, 1 \rangle$, so these lines are not parallel or equal. Let r represent the parameter for line L_1 and let s represent the parameter for L_2:

$$x = r \; x = 2s + 3$$

$$y = -r \; y = s$$

$$z = r \; z = s + 2.$$

Solve the system of equations to find $r = 1$ and $s = -1$. If we need to find the point of intersection, we can substitute these parameters into the original equations to get $(1, -1, 1)$ (see Figure $4.5.8$).

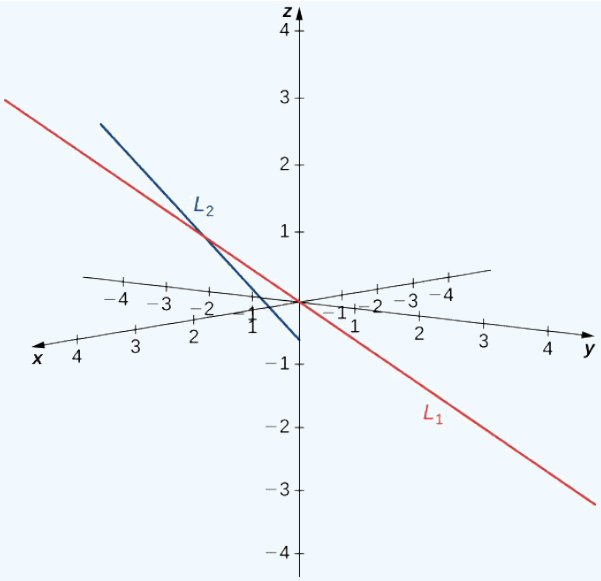


Figure 4.5.8: Two intersecting lines L_1 red and L_2 blue in space.

c. Lines L_1 and L_2 have equivalent direction vectors: $\mathbf{v} = \langle 6, -2, 3 \rangle$. These two lines are parallel (see Figure 4.5.9).

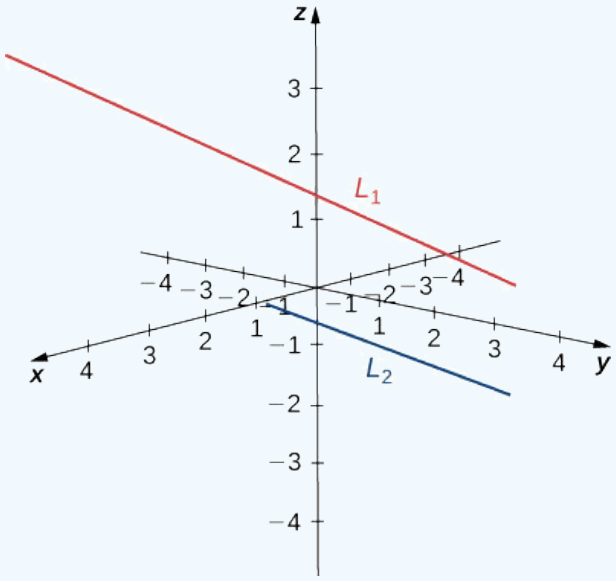


Figure 4.5.9: Two parallel lines L_1 red and L_2 blue in space.

Exercise 4.5.4

Describe the relationship between the lines with the following parametric equations:

$$x = 1 - 4t, y = 3 + t, z = 8 - 6t$$

$$x = 2 + 3s, y = 2s, z = -1 - 3s.$$

Hint available at the end of this section.

Answer available at the end of this section.

Equations for a Plane

We know that a line is determined by two points. In other words, for any two distinct points, there is exactly one line that passes through those points, whether in two dimensions or three. Similarly, given any three points that do not all lie on the same line, there is a unique plane that passes through these points. Just as a line is determined by two points, a plane is determined by three.

This may be the simplest way to characterize a plane, but we can use other descriptions as well. For example, given two distinct, intersecting lines, there is exactly one plane containing both lines. A plane is also determined by a line and any point that does not lie on the line. These characterizations arise naturally from the idea that a plane is determined by three points. Perhaps the most surprising characterization of a plane is actually the most useful.

Imagine a pair of orthogonal vectors that share an initial point. Visualize grabbing one of the vectors and twisting it. As you twist, the other vector spins around and sweeps out a plane. Here, we describe that concept mathematically. Let $\mathbf{n} = \langle a, b, c \rangle$ be a vector and $P = (x_0, y_0, z_0)$ be a point. Then the set of all points $Q = (x, y, z)$ such that $\vecd{PQ}$ is orthogonal to $\mathbf{n}$ forms a plane (Figure 4.5.10). We say that $\mathbf{n}$ is a **normal vector**, or perpendicular to the plane. Remember, the dot product of orthogonal vectors is zero. This fact generates the **vector equation of a plane**:

$$\mathbf{n} \cdot \vecd{PQ} = 0. \tag{4.5.15}$$

Rewriting this equation provides additional ways to describe the plane:

$$\begin{aligned} \mathbf{n} \cdot \vecd{PQ} &= 0 \\ \langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle &= 0 \\ a(x - x_0) + b(y - y_0) + c(z - z_0) &= 0. \end{aligned}$$

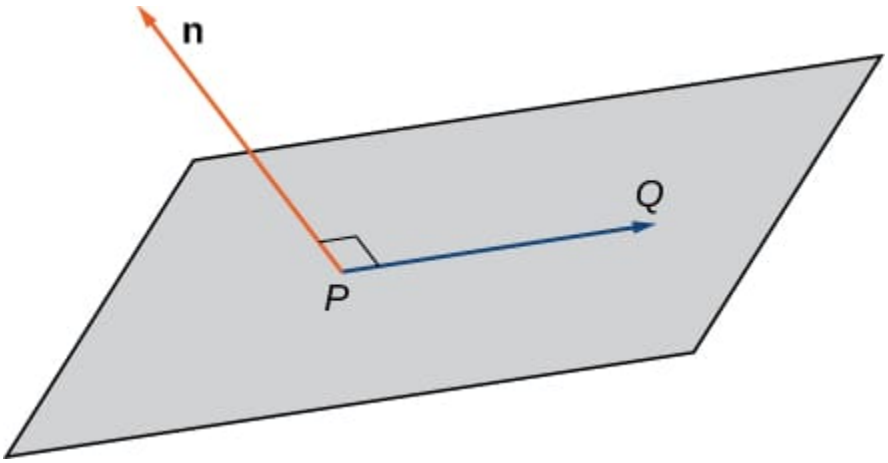


Figure 4.5.10: Given a point P and vector $\mathbf{n}$, *the set of all points* Q *with* $\vecd{PQ}$ *orthogonal to* $\mathbf{n}$ *forms a plane.*

Definition: Scalar Equation of a Plane

Given a point P and vector $\mathbf{n}$, the set of all points Q satisfying the equation $\mathbf{n} \cdot \vecd{PQ} = 0$ forms a plane. The equation

$$\mathbf{n} \cdot \vecd{PQ} = 0$$

is known as the *vector equation of a plane*.

The **scalar equation of a plane** containing point $P = (x_0, y_0, z_0)$ with normal vector $\vec{n} = \langle a, b, c \rangle$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

This equation can be expressed as $ax + by + cz + d = 0$, where $d = -ax_0 - by_0 - cz_0$. This form of the equation is sometimes called the **general form of the equation of a plane**.

As described earlier in this section, any three points that do not all lie on the same line determine a plane. Given three such points, we can find an equation for the plane containing these points.

Example 4.5.5: Writing an Equation of a Plane Given Three Points in the Plane

Write an equation for the plane containing points $P=(1,1,-2), Q=(0,2,1)$, and $R=(-1,-1,0)$ in both standard and general forms.

Solution

To write an equation for a plane, we must find a normal vector for the plane. We start by identifying two vectors in the plane:

$$\begin{aligned} \vecd{PQ} &= \langle 0-1, 2-1, 1-(-2)\rangle \\ &= \langle -1, 1, 3\rangle \\ \vecd{QR} &= \langle -1-0, -1-2, 0-1\rangle \\ &= \langle -1, -3, -1\rangle. \end{aligned}$$

The cross product $\vecd{PQ} \times \vecd{QR}$ is orthogonal to both $\vecd{PQ}$ and $\vecd{QR}$, so it is normal to the plane that contains these two vectors:

$$\begin{aligned} \mathbf{n} &= \vecd{PQ} \times \vecd{QR} \\ &= \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 3 \\ -1 & -3 & -1 \end{bmatrix} \\ &= (-1+9)\mathbf{i} - (1+3)\mathbf{j} + (3+1)\mathbf{k} \\ &= 8\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}. \end{aligned}$$

Thus, $n=\langle 8,-4,4\rangle$, and we can choose any of the three given points to write an equation of the plane:

$$\begin{aligned} 8(x-1) - 4(y-1) + 4(z+2) &= 0 \\ 8x - 4y + 4z + 4 &= 0. \end{aligned}$$

Watch the accompanying video for a visual demonstration of the concept.

- **Video length:** 5 minutes 10 seconds.
- **Context:** This video demonstrates how to find the equation of the plane that contains three points $(1,0,-2), (3,5,1)$, and $(4,2,-1)$

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

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The scalar equations of a plane vary depending on the normal vector and point chosen.

Example 4.5.6: Writing an Equation for a Plane Given a Point and a Line

Find an equation of the plane that passes through point $(1,4,3)$ and contains the line given by $x=\dfrac{y-1}{2}=z+1$.

Solution

Symmetric equations describe the line that passes through point $(0, 1, -1)$ parallel to vector $\mathbf{v}_1 = \langle 1, 2, 1 \rangle$ (see Figure 4.5.11). Use this point and the given point, $(1, 4, 3)$, to identify a second vector parallel to the plane:

$$\mathbf{v}_2 = \langle 1 - 0, 4 - 1, 3 - (-1) \rangle = \langle 1, 3, 4 \rangle.$$

Use the cross product of these vectors to identify a normal vector for the plane:

$$\begin{aligned} \mathbf{n} &= \mathbf{v}_1 \times \mathbf{v}_2 \\ &= \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 1 & 3 & 4 \end{bmatrix} \\ &= (8 - 3)\mathbf{i} - (4 - 1)\mathbf{j} + (3 - 2)\mathbf{k} \\ &= 5\mathbf{i} - 3\mathbf{j} + \mathbf{k}. \end{aligned}$$

The scalar equations for the plane are $5x - 3(y - 1) + (z + 1) = 0$ and $5x - 3y + z + 4 = 0$.

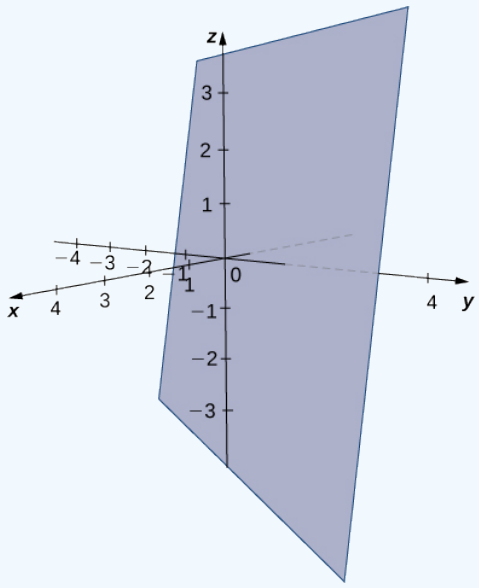


Figure 4.5.11: A plane passes through point $(0, 1, -1)$ and is parallel to vector $\mathbf{v}_1 = \langle 1, 2, 1 \rangle$.

Exercise 4.5.6

Find an equation of the plane containing the lines L_1 and L_2:

$$L_1 : x = -y = z$$

$$L_2 : \frac{x - 3}{2} = y = z - 2.$$

Hint available at the end of this section.

Now that we can write an equation for a plane, we can use the equation to find the distance d between a point P and the plane. It is defined as the shortest possible distance from P to a point on the plane.

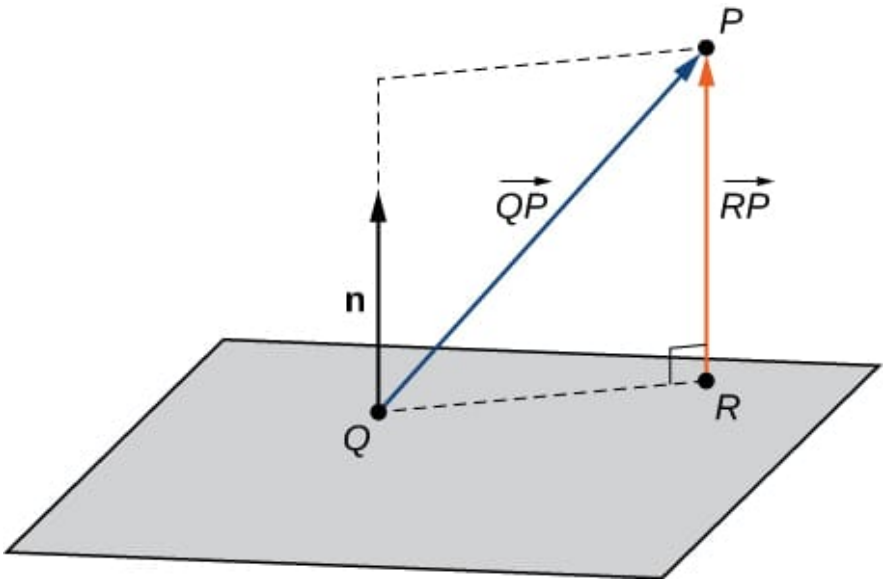


Figure 4.5.12: A point P above a plane, with its perpendicular projection R onto the plane, while the vectors $\vecd{QP}$, $\vecd{RP}$, and the normal vector $\vecd{n}$ illustrate the geometric relationships between the point P, the plane, and its normal. To find the shortest distance from point P to the plane containing points Q and R, we calculate the projection of vector $\vecd{QP}$ onto the normal vector $\vecd{n}$, which is equivalent to vector $\vecd{RP}$.

Just as we find the two-dimensional distance between a point and a line by calculating the length of a line segment perpendicular to the line, we find the three-dimensional distance between a point and a plane by calculating the length of a line segment perpendicular to the plane. Let R bet the point in the plane such that $\vecd{RP}$ is orthogonal to the plane, and let Q be an arbitrary point in the plane. Then the projection of vector $\vecd{QP}$ onto the normal vector describes vector $\vecd{RP}$, as shown in Figure 4.5.12.

The Distance between a Plane and a Point

Suppose a plane with normal vector $\mathbf{n}$ passes through point Q. The distance d from the plane to a point P not in the plane is given by

$$d = \|\text{proj}_{\mathbf{n}} \vecd{QP}\| = |\text{ comp}_{\mathbf{n}} \vecd{QP}\ | = \frac{|\ \vecd{QP} \cdot \mathbf{n}\ |}{\|\mathbf{n}\|}. \tag{4.5.16}$$

Example 4.5.7: Distance between a Point and a Plane

Find the distance between point $P = (3, 1, 2)$ and the plane given by $x - 2y + z = 5$ (Figure 4.5.13).

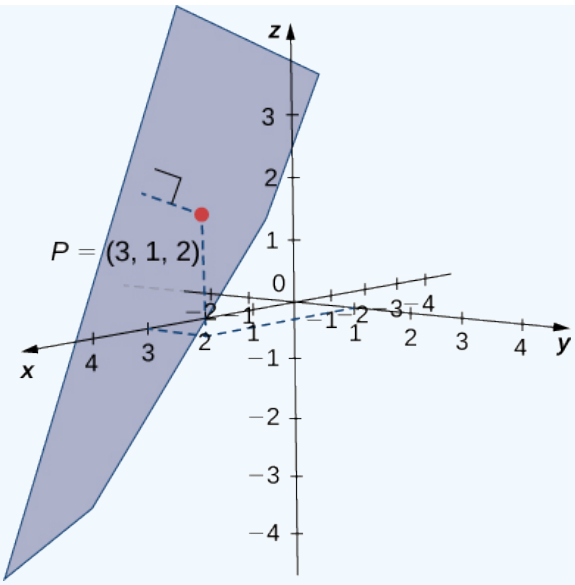


Figure 4.5.13: The distance between a point $P = (3, 1, 2)$ in a red dot and the plane $x - 2y + z = 5$.

Solution

The coefficients of the plane's equation provide a normal vector for the plane: $\mathbf{n} = \langle 1, -2, 1 \rangle$. To find vector $\vecd{QP}$, we need a point in the plane. Any point will work, so set $y = z = 0$ to see that point $Q = (5, 0, 0)$ lies in the plane. Find the component form of the vector from Q to P:

$$\vecd{QP} = \langle 3 - 5, 1 - 0, 2 - 0 \rangle = \langle -2, 1, 2 \rangle.$$

Apply the distance formula from Equation:

$$\begin{aligned} d &= \frac{|\vecd{QP} \cdot \mathbf{n}|}{\|\mathbf{n}\|} \\ &= \frac{|\langle -2, 1, 2 \rangle \cdot \langle 1, -2, 1 \rangle|}{\sqrt{1^2 + (-2)^2 + 1^2}} \\ &= \frac{|-2 - 2 + 2|}{\sqrt{6}} \\ &= \frac{2}{\sqrt{6}} = \frac{\sqrt{6}}{3} \text{ units.} \end{aligned}$$

Watch the accompanying video for a visual demonstration of the concept.

- **Video length:** 2 minutes 23 seconds.
- **Context**: This video demonstrates how to find the distance from a point $(3, 4, 1)$ to a plane $3x - 2y + 6z = 14$.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

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Exercise 4.5.7

Find the distance between point $P = (5, -1, 0)$ and the plane given by $4x + 2y - z = 3$.

Hint available at the end of this section.

Parallel and Intersecting Planes

We have discussed the various possible relationships between two lines in two dimensions and three dimensions. When we describe the relationship between two planes in space, we have only two possibilities: the two distinct planes are parallel or they intersect. When two planes are parallel, their normal vectors are parallel. When two planes intersect, the intersection is a line (Figure 4.5.14).

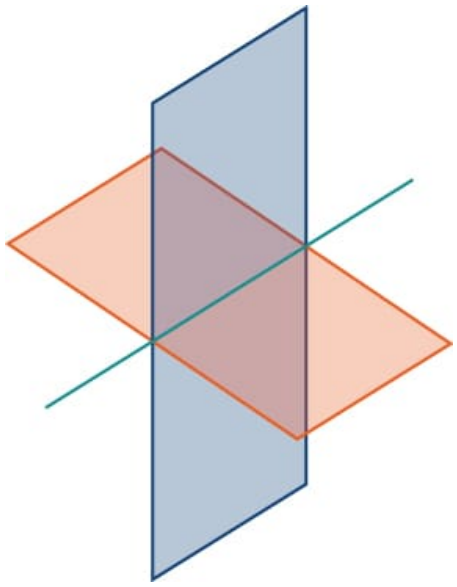


Figure 4.5.14: The intersection of two nonparallel planes blue and orange is a green line in space.

We can use the equations of the two planes to find parametric equations for the line of intersection.

Example 4.5.8: Finding the Line of Intersection for Two Planes

Find parametric and symmetric equations for the line formed by the intersection of the planes given by $x + y + z = 0$ and $2x - y + z = 0$ (see the following figure).

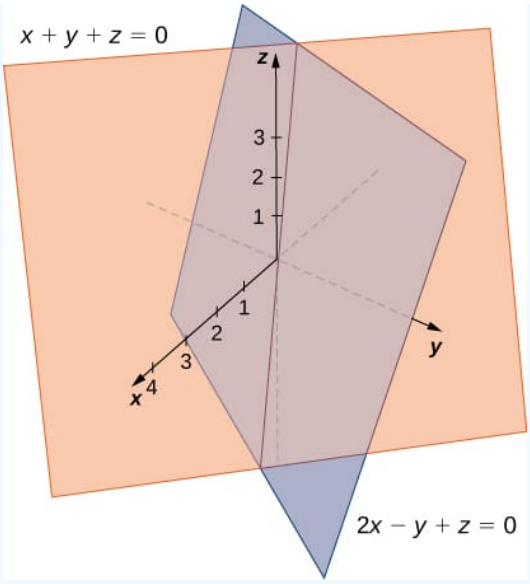


Figure 4.5.15: A line formed by the intersection of two planes $x + y + z = 0$ and $2x - y + z = 0$.

Solution

Note that the two planes have nonparallel normals, so the planes intersect. Further, the origin satisfies each equation, so we know the line of intersection passes through the origin. Add the plane equations so we can eliminate one of the variables, in this case, y:

$$x + y + z = 0$$

$$2x - y + z = 0$$

$$\text{---------------}$$

$$3x + 2z = 0.$$

This gives us $x = -\frac{2}{3}z$.We substitute this value into the first equation to express y in terms of z:

$$\begin{aligned} x + y + z &= 0 \\ -\frac{2}{3}z + y + z &= 0 \\ y + \frac{1}{3}z &= 0 \\ y &= -\frac{1}{3}z \end{aligned}$$

We now have the first two variables, x and y, in terms of the third variable, z. Now we define z in terms of t. To eliminate the need for fractions, we choose to define the parameter t as $t = -\frac{1}{3}z$. Then, $z = -3t$. Substituting the parametric representation of z back into the other two equations, we see that the parametric equations for the line of intersection are $x = 2t, y = t, z = -3t$. The symmetric equations for the line are $\frac{x}{2} = y = \frac{z}{-3}$.

Exercise 4.5.8

Find parametric equations for the line formed by the intersection of planes $x + y - z = 3$ and $3x - y + 3z = 5$.

Hint available at the end of this section.

In addition to finding the equation of the line of intersection between two planes, we may need to find the angle formed by the intersection of two planes. For example, builders constructing a house need to know the angle where different sections of the roof meet to know whether the roof will look good and drain properly. We can use normal vectors to calculate the angle between the two planes. We can do this because the angle between the normal vectors is the same as the angle between the planes. Figure 4.5.16 shows why this is true.

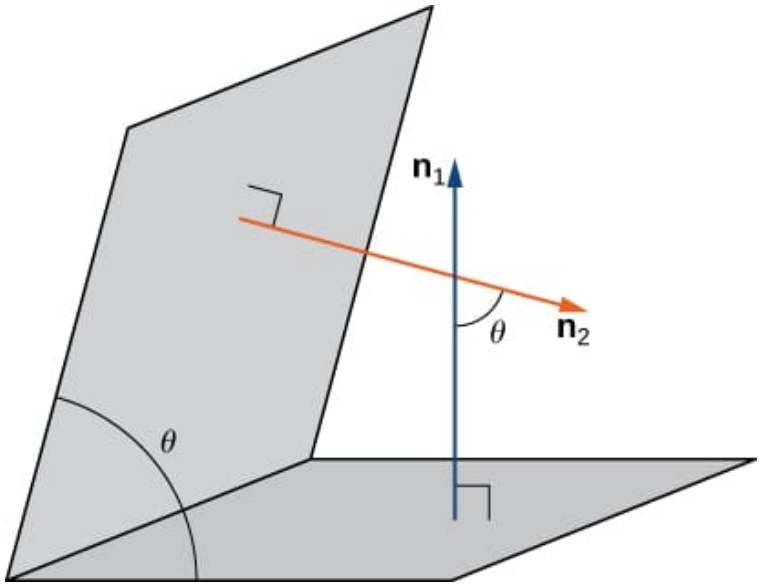


Figure 4.5.16: The angle between two planes has the same measure as the angle between the normal vectors for the planes.

We can find the measure of the angle θ between two intersecting planes by first finding the cosine of the angle, using the following equation:

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|}. \tag{4.5.17}$$

We can then use the angle to determine whether two planes are parallel or orthogonal or if they intersect at some other angle.

Example 4.5.9: Finding the Angle between Two Planes

Determine whether each pair of planes is parallel, orthogonal, or neither. If the planes are intersecting, but not orthogonal, find the measure of the angle between them. Give the answer in radians and round to two decimal places.

a. $x + 2y - z = 8$ and $2x + 4y - 2z = 10$
b. $2x - 3y + 2z = 3$ and $6x + 2y - 3z = 1$
c. $x + y + z = 4$ and $x - 3y + 5z = 1$

Solution:

a. The normal vectors for these planes are $\mathbf{n}_1 = \langle 1, 2, -1\rangle$ and $\mathbf{n}_2 = \langle 2, 4, -2\rangle$. These two vectors are scalar multiples of each other. The normal vectors are parallel, so the planes are parallel.
b. The normal vectors for these planes are $\mathbf{n}_1 = \langle 2, -3, 2\rangle$ and $\mathbf{n}_2 = \langle 6, 2, -3\rangle$. Taking the dot product of these vectors, we have

$$\begin{aligned}\mathbf{n}_1 \cdot \mathbf{n}_2 &= \langle 2, -3, 2\rangle \cdot \langle 6, 2, -3\rangle \\ &= 2(6) - 3(2) + 2(-3) = 0.\end{aligned}$$

The normal vectors are orthogonal, so the corresponding planes are orthogonal as well.
c. The normal vectors for these planes are $\mathbf{n}_1 = \langle 1, 1, 1\rangle$ and $\mathbf{n}_2 = \langle 1, -3, 5\rangle$:

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|}$$
$$= \frac{|\langle 1,1,1\rangle \cdot \langle 1,-3,5\rangle|}{\sqrt{1^2+1^2+1^2}\sqrt{1^2+(-3)^2+5^2}}$$
$$= \frac{3}{\sqrt{105}}$$

Then $\theta = \arccos\frac{3}{\sqrt{105}} \approx 1.27$ rad.
Thus the angle between the two planes is about 1.27 rad, or approximately 73 °.

Exercise 4.5.9

Find the measure of the angle between planes $x + y - z = 3$ and $3x - y + 3z = 5$. Give the answer in radians and round to two decimal places.

Hint available at the end of this section.

Answer available at the end of this section.

When we find that two planes are parallel, we may need to find the distance between them. To find this distance, we simply select a point in one of the planes. The distance from this point to the other plane is the distance between the planes.

Previously, we introduced the formula for calculating this distance in Equation 4.5.16:

$$d = \frac{\vecd{QP} \cdot \mathbf{n}}{\|\mathbf{n}\|}, \tag{4.5.18}$$

where Q is a point on the plane, P is a point not on the plane, and $\vec{n}$ is the normal vector that passes through point Q. Consider the distance from point (x_0, y_0, z_0) to plane $ax + by + cz + k = 0$. Let (x_1, y_1, z_1) be any point in the plane. Substituting into the formula yields

$$d = \frac{|a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1)|}{\sqrt{a^2+b^2+c^2}}$$
$$= \frac{|ax_0 + by_0 + cz_0 + k|}{\sqrt{a^2+b^2+c^2}}.$$

We state this result formally in the following theorem.

Distance from a Point to a Plane

Let $P(x_0, y_0, z_0)$ be a point. The distance from P to plane $ax + by + cz + k = 0$ is given by

$$d = \frac{|ax_0 + by_0 + cz_0 + k|}{\sqrt{a^2+b^2+c^2}}. \tag{4.5.19}$$

Example 4.5.10: Finding the Distance between Parallel Planes

Find the distance between the two parallel planes given by $2x + y - z = 2$ and $2x + y - z = 8$.

Solution

Point $(1, 0, 0)$ lies in the first plane. The desired distance, then, is

$$d = \frac{|ax_0 + by_0 + cz_0 + k|}{\sqrt{a^2 + b^2 + c^2}}$$
$$= \frac{|2(1) + 1(0) + (-1)(0) + (-8)|}{\sqrt{2^2 + 1^2 + (-1)^2}}$$
$$= \frac{6}{\sqrt{6}} = \sqrt{6} \text{ units}$$

Exercise 4.5.10:

Find the distance between parallel planes $5x - 2y + z = 6$ and $5x - 2y + z = -3$.

Hint available at the end of this section.

Answer available at the end of this section.

Distance between Two Skew Lines

Finding the distance from a point to a line or from a line to a plane seems like a pretty abstract procedure. But, if the lines represent pipes in a chemical plant or tubes in an oil refinery or roads at an intersection of highways, confirming that the distance between them meets specifications can be both important and awkward to measure. One way is to model the two pipes as lines, using the techniques in this chapter, and then calculate the distance between them. The calculation involves forming vectors along the directions of the lines and using both the cross product and the dot product.

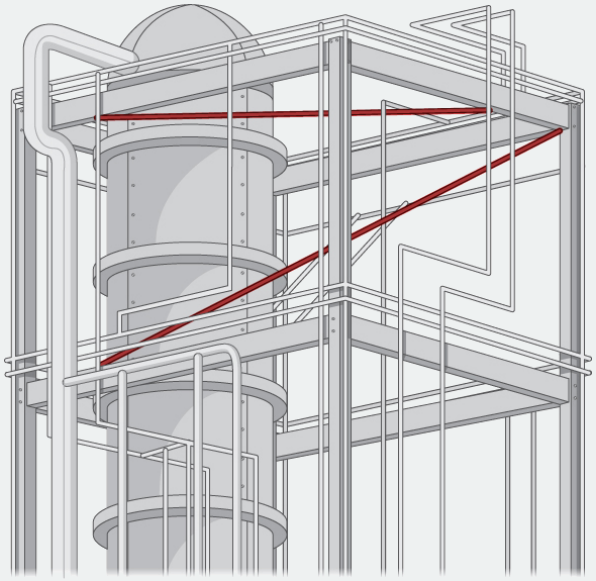


Figure 4.5.17: Industrial pipe installations often feature pipes running in different directions. How can we find the distance between two skew pipes (highlighted in red)?

The symmetric forms of two lines, L_1 and L_2, are

$$L_1 : \frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1} \tag{4.5.20}$$

$$L_2 : \frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}. \tag{4.5.21}$$

You are to develop a formula for the distance d between these two lines, in terms of the values $a_1, b_1, c_1; a_2, b_2, c_2; x_1, y_1, z_1;$ and x_2, y_2, z_2. The distance between two lines is usually taken to mean the minimum distance, so this is the length of a line segment or the length of a vector that is perpendicular to both lines and intersects both lines.

1. First, write down two vectors, $\mathbf{v}_1$ and $\mathbf{v}_2$, that lie along L_1 and L_2, respectively.
2. Find the cross product of these two vectors and call it $\mathbf{N}$. This vector is perpendicular to $\mathbf{v}_1$ and $\mathbf{v}_2$, and hence is perpendicular to both lines.
3. From vector $\mathbf{N}$, form a unit vector $\mathbf{n}$ in the same direction.
4. Use symmetric equations to find a convenient vector $\mathbf{v}_{12}$ that lies between any two points, one on each line. Again, this can be done directly from the symmetric equations.
5. The dot product of two vectors is the magnitude of the projection of one vector onto the other—that is, $\mathbf{A} \cdot \mathbf{B} = \|\mathbf{A}\|\|\mathbf{B}\| \cos\theta$, where θ is the angle between the vectors. Using the dot product, find the projection of vector $\mathbf{v}_{12}$ found in step 4 onto unit vector $\mathbf{n}$ found in step 3. This projection is perpendicular to both lines, and hence its length must be the perpendicular distance d between them. Note that the value of d may be negative, depending on your choice of vector $\mathbf{v}_{12}$ or the order of the cross product, so use absolute value signs around the numerator.
6. Check that your formula gives the correct distance of $|-25|/\sqrt{198} \approx 1.78$ between the following two lines:

$$L_1 : \frac{x-5}{2} = \frac{y-3}{4} = \frac{z-1}{3} \tag{4.5.22}$$

$$L_2 : \frac{x-6}{3} = \frac{y-1}{5} = \frac{z}{7}. \tag{4.5.23}$$

7. Is your general expression valid when the lines are parallel? If not, why not? (Hint: What do you know about the value of the cross product of two parallel vectors? Where would that result show up in your expression for d?)
8. Demonstrate that your expression for the distance is zero when the lines intersect. Recall that two lines intersect if they are not parallel and they are in the same plane. Hence, consider the direction of $\mathbf{n}$ and $\mathbf{v}_{12}$. What is the result of their dot product?
9. Consider the following application. Engineers at a refinery have determined they need to install support struts between many of the gas pipes to reduce damaging vibrations. To minimize cost, they plan to install these struts at the closest points between adjacent skewed pipes. Because they have detailed schematics of the structure, they are able to determine the correct lengths of the struts needed, and hence manufacture and distribute them to the installation crews without spending valuable time making measurements.

The rectangular frame structure has the dimensions $4.0 \times 15.0 \times 10.0$ m (height, width, and depth). One sector has a pipe entering the lower corner of the standard frame unit and exiting at the diametrically opposed corner (the one farthest away at the top); call this L_1. A second pipe enters and exits at the two different opposite lower corners; call this L_2 (Figure $4.5.18$).

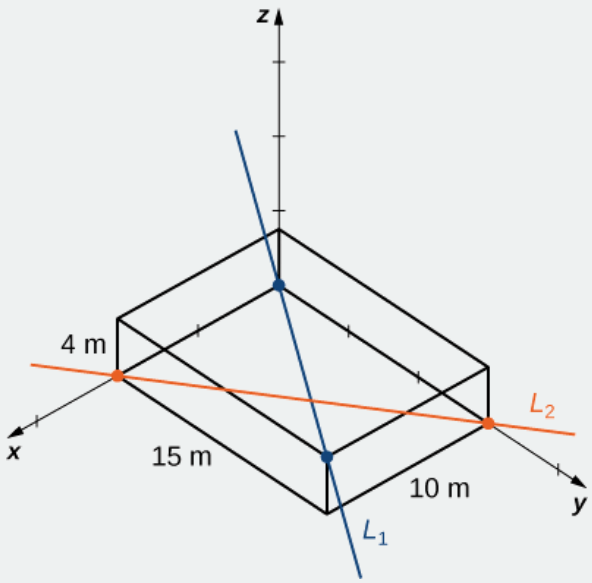


Figure $4.5.18$: Two lines L_1 blue and L_2 orange representing two pipes cross through a standard frame unit in 3d-space.

Write down the vectors along the lines representing those pipes, find the cross product between them from which to create the unit vector $\mathbf{n}$, define a vector that spans two points on each line, and finally determine the minimum distance between the lines. (Take the origin to be at the lower corner of the first pipe.) Similarly, you may also develop the symmetric equations for each line and substitute directly into your formula.

Key Concepts

- In three dimensions, the direction of a line is described by a direction vector. The vector equation of a line with direction vector $\mathbf{v} = \langle a, b, c \rangle$ passing through point $P = (x_0, y_0, z_0)$ is $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$, where $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$ is the position vector of point P. This equation can be rewritten to form the parametric equations of the line: $x = x_0 + ta, y = y_0 + tb$, and $z = z_0 + tc$. The line can also be described with the symmetric equations $\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$.
- Let L be a line in space passing through point P with direction vector $\mathbf{v}$. If Q is any point not on L, then the distance from Q to L is $d = \frac{\|\vecd{PQ} \times \mathbf{v}\|}{\|\mathbf{v}\|}$.
- In three dimensions, two lines may be parallel but not equal, equal, intersecting, or skew.
- Given a point P and vector $\mathbf{n}$, the set of all points Q satisfying equation $\mathbf{n} \cdot \vecd{PQ} = 0$ forms a plane. Equation $\mathbf{n} \cdot \vecd{PQ} = 0$ is known as the *vector equation of a plane*.
- The scalar equation of a plane containing point $P = (x_0, y_0, z_0)$ with normal vector $\mathbf{n} = \langle a, b, c \rangle$ is $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$. This equation can be expressed as $ax + by + cz + d = 0$, where $d = -ax_0 - by_0 - cz_0$. This form of the equation is sometimes called the general *form of the equation of a plane*.
- Suppose a plane with normal vector n passes through point Q. The distance D from the plane to point P not in the plane is given by

$$D = \|\text{proj}_{\mathbf{n}} \vecd{QP}\| = |\text{ comp}_{\mathbf{n}} \vec{QP}| = \frac{|\vec{QP} \cdot \mathbf{n}|}{\|\mathbf{n}\|.} \tag{4.5.24}$$

- The normal vectors of parallel planes are parallel. When two planes intersect, they form a line.
- The measure of the angle θ between two intersecting planes can be found using the equation: $\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|}$, where $\mathbf{n}_1$ and $\mathbf{n}_2$ are normal vectors to the planes.
- The distance D from point (x_0, y_0, z_0) to plane $ax + by + cz + d = 0$ is given by

$$D = \frac{|a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1)|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}} \tag{4.5.25}$$

.

Key Equations

- **Vector Equation of a Line**

  $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$

- **Parametric Equations of a Line**

  $x = x_0 + ta, y = y_0 + tb$, and $z = z_0 + tc$

- **Symmetric Equations of a Line**

  $\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$

- **Vector Equation of a Plane**

  $\mathbf{n} \cdot \vecd{PQ} = 0$

- **Scalar Equation of a Plane**

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

- **Distance between a Plane and a Point**

$$d=\|\text{proj}_{\mathbf{n}}\vec{QP}\|=|\text{ comp}_{\mathbf{n}}\vec{QP}\,|=\frac{|\,\vec{QP}\cdot\mathbf{n}\,|}{\|\mathbf{n}\|}$$

Glossary

Direction Vector
A vector parallel to a line that is used to describe the direction, or orientation, of the line in space.

General Form of the Equation of a Plane
An equation in the form $ax+by+cz+d=0$, where $\mathbf{n}=\langle a,b,c\rangle$ is a normal vector of the plane, $P=(x_0,y_0,z_0)$ is a point on the plane, and $d=-ax_0-by_0-cz_0$.

Normal Vector
A vector perpendicular to a plane.

Parametric Equations of a Line
The set of equations $x=x_0+ta, y=y_0+tb$, and $z=z_0+tc$ describing the line with direction vector $v=\langle a,b,c\rangle$ passing through point (x_0,y_0,z_0).

Scalar Equation of a Plane
The equation $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$ used to describe a plane containing point $P=(x_0,y_0,z_0)$ with normal vector $n=\langle a,b,c\rangle$ or its alternate form $ax+by+cz+d=0$, where $d=-ax_0-by_0-cz_0$.

Skew Lines
Two lines that are not parallel but do not intersect.

Symmetric Equations of a Line
The equations $\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$ describing the line with direction vector $v=\langle a,b,c\rangle$ passing through point (x_0,y_0,z_0).

Vector Equation of a Line
The equation $\mathbf{r}=\mathbf{r}_0+t\mathbf{v}$ used to describe a line with direction vector $\mathbf{v}=\langle a,b,c\rangle$ passing through point $P=(x_0,y_0,z_0)$, where $\mathbf{r}_0=\langle x_0,y_0,z_0\rangle$, is the position vector of point P.

Vector Equation of a Plane
The equation $\mathbf{n}\cdot\vec{PQ}=0$, where P is a given point in the plane, Q is any point in the plane, and $\mathbf{n}$ is a normal vector of the plane.

Answers

Exercise 4.5.1
Start by finding a vector parallel to the line.

Exercise 4.5.1
Possible set of parametric equations: $x=1+4t, y=-3+t, z=2+6t$; related set of symmetric equations:

$$\frac{x-1}{4}=y+3=\frac{z-2}{6}$$

Exercise 4.5.2

$$x=-1-7t, y=3-t, z=6-2t, 0\le t\le 1$$

Exercise 4.5.3
Find a vector with initial point $(0,3,6)$ and a terminal point on the line, and then find a direction vector for the line.

Exercise 4.5.3

$$\sqrt{\frac{10}{7}}=\frac{\sqrt{70}}{7}\text{ units}$$

Exercise 4.5.4
Start by identifying direction vectors for each line. Is one a multiple of the other?

Exercise 4.5.4

These lines are skew because their direction vectors are not parallel and there is no point (x, y, z) that lies on both lines.

Exercise 4.5.6

Hint: The cross product of the lines' direction vectors gives a normal vector for the plane.

$$-2(x-1)+(y+1)+3(z-1)=0$$

or

$$-2x+y+3z=0$$

Exercise 4.5.7

Point $(0, 0, -3)$ lies on the plane.

$$\frac{15}{\sqrt{21}}=\frac{5\sqrt{21}}{7}\text{ units} \tag{4.5.26}$$

Exercise 4.5.8

Add the two equations, then express z in terms of x. Then, express y in terms of x.

$$x=t, y=7-3t, z=4-2t$$

Exercise 4.5.9

Use the coefficients of the variables in each equation to find a normal vector for each plane.

Exercise 4.5.9

$$1.44\,\text{rad}$$

Exercise 4.5.10**:**

Set $x = y = 0$ to find a point on the first plane.

Exercise 4.5.10**:**

$$\frac{9}{\sqrt{30}}=\frac{3\sqrt{30}}{10}\text{ units}$$

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4.5E: Exercises for Section 4.5

Exercises 1 - 4

Points P and Q are given. Let L be the line passing through points P and Q.

a. **Find the vector equation of line L.**
b. **Find parametric equations of line L.**
c. **Find symmetric equations of line L.**
d. **Find parametric equations of the line segment determined by P and Q.**

1. $P(-3,5,9), \quad Q(4,-7,2)$

Answer

a. $\mathbf{r} = \langle -3,5,9\rangle + t\langle 7,-12,-7\rangle, \quad t \in R;$
b. $x = -3+7t, \quad y = 5-12t, \quad z = 9-7t, \quad t \in R;$
c. $\dfrac{x+3}{7} = \dfrac{y-5}{-12} = \dfrac{z-9}{-7};$
d. $x = -3+7t, \quad y = 5-12t, \quad z = 9-7t, \quad 0 \le t \le 1$

2. $P(4,0,5), \quad Q(2,3,1)$
3. $P(-1,0,5), \quad Q(4,0,3)$

Answer

a. $\mathbf{r} = \langle -1,0,5\rangle + t\langle 5,0,-2\rangle, \quad t \in R;$
b. $x = -1+5t, y = 0, z = 5-2t, \quad t \in R;$
c. $\dfrac{x+1}{5} = \dfrac{z-5}{-2}, \quad y = 0;$
d. $x = -1+5t, \quad y = 0, \quad z = 5-2t, \quad t \in [0,1]$

4. $P(7,-2,6), \quad Q(-3,0,6)$

Exercises 5 - 8

Point P and vector $\mathbf{v}$ are given. Let L be the line passing through point P with direction $\mathbf{v}$.

a. **Find parametric equations of line L.**
b. **Find symmetric equations of line L.**
c. **Find the intersection of the line with the xy-plane.**

5. $P(1,-2,3), \ \mathbf{v} = \langle 1,2,3\rangle$

Answer

a. $x = 1+t, \quad y = -2+2t, \quad z = 3+3t, \quad t \in R;$
b. $\dfrac{x-1}{1} = \dfrac{y+2}{2} = \dfrac{z-3}{3};$
c. $(0,-4,0)$

6. $P(3,1,5), \ \mathbf{v} = \langle 1,1,1\rangle$
7. $P(3,1,5), \ \mathbf{v} = \backslash\text{vec}dQR$, where $Q(2,2,3)$ and $R(3,2,3)$

Answer

a. $x = 3+t, \quad y = 1, \quad z = 5, \quad t \in R;$
b. $y = 1, \quad z = 5;$
c. The line does not intersect the xy-plane.

8. $P(2,3,0)$, $\mathbf{v} = \vecd{QR}$, where $Q(0,4,5)$ and $R(0,4,6)$

Exercises 9-10

Line L is given.

a. **Find a point P that belongs to the line and a direction vector $\mathbf{v}$ of the line. Express $\mathbf{v}$ in component form.**
b. **Find the distance from the origin to line L.**

9. $x = 1 + t, \quad y = 3 + t, \quad z = 5 + 4t, \quad t \in R$

Answer

a. A possible point and direction vector are $P(1,3,5)$ and $\mathbf{v} = \langle 1,1,4 \rangle$, but these answers are not unique.
b. $\sqrt{3}$ units

10. $-x = y + 1, \quad z = 2$
11. Find the distance between point $A(-3,1,1)$ and the line of symmetric equations

$x = -y = -z.$

Answer

$\frac{2\sqrt{2}}{\sqrt{3}} = \frac{2\sqrt{6}}{3}$ units

12. Find the distance between point $A(4,2,5)$ and the line of parametric equations

$x = -1 - t, \quad y = -t, \quad z = 2, \quad t \in R.$

Exercises 13 - 14

Lines L_1 and L_2 are given.

a. **Verify whether lines L_1 and L_2 are parallel.**
b. **If the lines L_1 and L_2 are parallel, then find the distance between them.**

13. $L_1 : x = 1 + t, \quad y = t, \quad z = 2 + t, \quad t \in R$ and $L_2 : x - 3 = y - 1 = z - 3$

Answer

a. Parallel;
b. $\frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{6}}{3}$ units

14. $L_1 : x = 2, \quad y = 1, \quad z = t, \quad t \in R$ and $L_2 : x = 1, \quad y = 1, \quad z = 2 - 3t, \quad t \in R$
15. Show that the line passing through points $P(3,1,0)$ and $Q(1,4,-3)$ is perpendicular to the line with equation $x = 3t, \quad y = -32 + 8t, \quad z = -9 + 6t, \quad t \in R.$

Answer

$\vecd{PQ} = \langle -2,3,-3 \rangle$ is the direction vector of the line through points P and Q, and the direction vector of the line defined by the parametric equations above is $\mathbf{v} = \langle 3,8,6 \rangle$.
Since $\mathbf{v} \cdot \vecd{PQ} = -6 + 24 - 18 = 0$, the two direction vectors are orthogonal.
Now all we need to show is that the two lines intersect.
The line through points $P(3,1,0)$ and $Q(1,4,-3)$ has parametric equations: $x = 3 - 2u$, $y = 1 + 3u$, and $z = -3u$.
Setting the x- and z-coordinates of the two lines equal, we obtain the system of equations:

$$3t = 3 - 2u \quad \text{and} \quad -9 + 6t = -3u$$

Solving this system using substitution gives us, $u = -3$ and $t = 3$. Plugging these values of t and u back into the parametric equations of these two lines gives us the intersection point with coordinates $(9,-8,9)$ on both

lines.
Therefore the lines intersect and the line through points P and Q with direction vector $\vecd{PQ}$ is perpendicular to the other line.

16. Are the lines of equations $x=-2+2t, \quad y=-6, \quad z=2+6t, \quad t\in R$ and $x=-1+t, \quad y=1+t, \quad z=t, \quad t\in R$, perpendicular to each other?
17. Find the point of intersection of the lines of equations $x=-2y=3z$ and $x=-5-t, \quad y=-1+t, \quad z=t-11, \quad t\in R$.

Answer

$(-12,6,-4)$

18. Find the intersection point of the x-axis with the line of parametric equations $x=10+t, \quad y=2-2t, \quad z=-3+3t, \quad t\in R$.

Exercises 19 - 22

Lines L_1 and L_2 are given. Determine whether the lines are *equal*, *parallel but not equal*, *skew*, or *intersecting*.

19. $L_1: x=y-1=-z$ and $L_2: x-2=-y=\dfrac{z}{2}$

Answer

The lines are skew.

20. $L_1: x=2t, \quad y=0, \quad z=3, \quad t\in R$ and $L_2: x=0, \quad y=8+s, \quad z=7+s, \quad s\in R$
21. $L_1: x=-1+2t, \quad y=1+3t, \quad z=7t, \quad t\in R$ and $L_2: x-1=\frac{2}{3}(y-4)=\frac{2}{7}z-2$

Answer

The lines are equal.

22. $L_1: 3x=y+1=2z$ and $L_2: x=6+2t, \quad y=17+6t, \quad z=9+3t, \quad t\in R$
23. Consider line L of symmetric equations $x-2=-y=\dfrac{z}{2}$ and point $A(1,1,1)$.
    a. Find parametric equations for a line parallel to L that passes through point A.
    b. Find symmetric equations of a line skew to L and that passes through point A.
    c. Find symmetric equations of a line that intersects L and passes through point A.

Answer

a. $x=1+t, \quad y=1-t, \quad z=1+2t, \quad t\in R$
b. For instance, the line passing through A with direction vector $j: x=1, z=1$
c. For instance, the line passing through A and point $(2,0,0)$ that belongs to L is a line that intersects; $L: \frac{x-1}{-1}=y-1=z-1$

24. Consider line L of parametric equations $x=t, \quad y=2t, \quad z=3, \quad t\in R$.
    a. Find parametric equations for a line parallel to L that passes through the origin.
    b. Find parametric equations of a line skew to L that passes through the origin.
    c. Find symmetric equations of a line that intersects L and passes through the origin.

Exercises 25 - 28

Point P and vector n are given.

a. **Find the scalar equation of the plane that passes through P and has normal vector n.**
b. **Find the general form of the equation of the plane that passes through P and has normal vector n.**

25. $P(0,0,0), \quad \mathbf{n}=3\mathbf{i}-2\mathbf{j}+4\mathbf{k}$

Answer

a. $3x-2y+4z=0$
b. $3x-2y+4z=0$

26. $P(3,2,2), \quad \mathbf{n} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$
27. $P(1,2,3), \quad \mathbf{n} = \langle 1,2,3 \rangle$

Answer

a. $(x-1) + 2(y-2) + 3(z-3) = 0$
b. $x + 2y + 3z - 14 = 0$

28. $P(0,0,0), \quad \mathbf{n} = \langle -3,2,-1 \rangle$

Exercises 29 - 32

The equation of a plane is given.

a. **Find normal vector $\mathbf{n}$ to the plane. Express $\mathbf{n}$ using standard unit vectors.**
b. **Find the intersections of the plane with each of the coordinate axes (its intercepts).**
c. **Sketch the plane.**

29. [T] $4x + 5y + 10z - 20 = 0$

Answer

a. $\mathbf{n} = 4\mathbf{i} + 5\mathbf{j} + 10\mathbf{k}$
b. $(5,0,0), \ (0,4,0)$, and $(0,0,2)$
c.

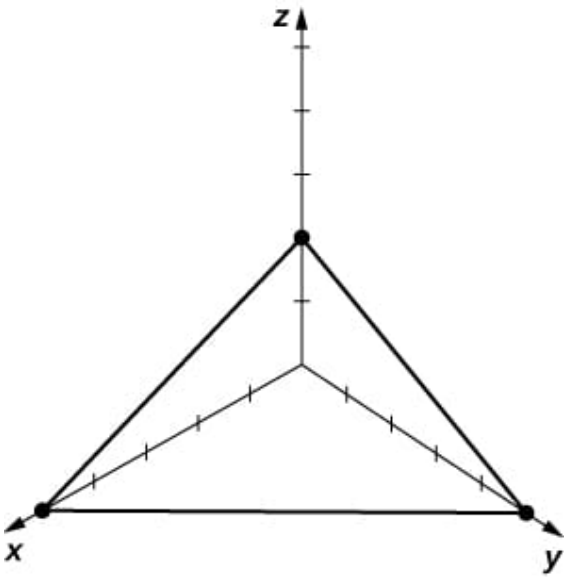


Figure $4.5E.1$: A triangular plane with vertices on xyz-coordinate space where x=5, y=5, and z=2.

30. $3x + 4y - 12 = 0$
31. $3x - 2y + 4z = 0$

Answer

a. $\mathbf{n} = 3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$
b. $(0,0,0)$

c.

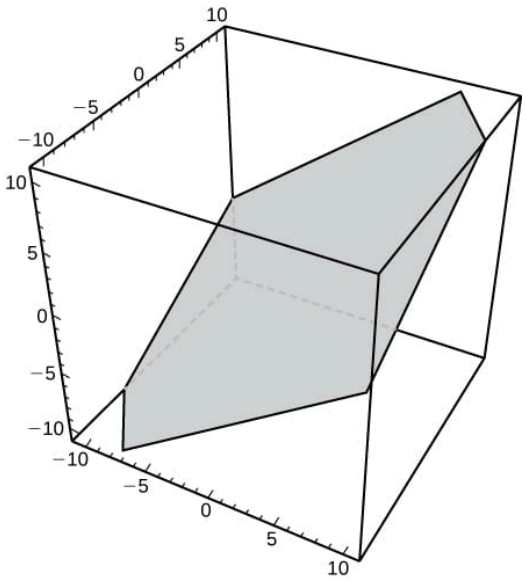


Figure $4.5E.2$: A plane lies inside the square box with x ∈ [-10,10], y ∈ [-10,10], and z ∈ [-10,10].

32. $x + z = 0$
33. Given point $P(1, 2, 3)$ and vector $\mathbf{n} = \mathbf{i} + \mathbf{j}$, find point Q on the x-axis such that $\vec{PQ}$ and $\mathbf{n}$ are orthogonal.

Answer

$(3, 0, 0)$

34. Show there is no plane perpendicular to $\mathbf{n} = \mathbf{i} + \mathbf{j}$ that passes through points $P(1, 2, 3)$ and $Q(2, 3, 4)$.
35. Find parametric equations of the line passing through point $P(-2, 1, 3)$ that is perpendicular to the plane of equation $2x - 3y + z = 7$.

Answer

$x = -2 + 2t, \quad y = 1 - 3t, \quad z = 3 + t, \quad t \in R$

36. Find symmetric equations of the line passing through point $P(2, 5, 4)$ that is perpendicular to the plane of equation $2x + 3y - 5z = 0$.
37. Show that line $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-2}{4}$ is parallel to plane $x - 2y + z = 6$.
38. Find the real number α such that the line of parametric equations $x = t, \quad y = 2 - t, \quad z = 3 + t, \quad t \in R$ is parallel to the plane of equation $\alpha x + 5y + z - 10 = 0$.

Exercises 39 - 42

The equations of two planes are given.

a. **Determine whether the planes are *parallel*, *orthogonal*, or *neither*.**
b. **If the planes are neither parallel nor orthogonal, then find the measure of the angle between the planes. Express the answer in degrees rounded to the nearest integer.**
c. **If the planes intersect, find the line of intersection of the planes, providing the parametric equations of this line.**

39. [T] $x + y + z = 0, \quad 2x - y + z - 7 = 0$

Answer

a. The planes are neither parallel nor orthogonal.
b. $62\,^\circ$
c. $x = -1 + 2t$
$y = -4 + t$
$z = 5 - 3t$

40. $5x - 3y + z = 4, \quad x + 4y + 7z = 1$
41. $x - 5y - z = 1, \quad 5x - 25y - 5z = -3$

Answer

a. The planes are parallel.

42. [T] $x - 3y + 6z = 4, \quad 5x + y - z = 4$

Exercises 43 - 46

Determine whether the given line intersects with the given plane. If they do intersect, state the point of intersection.

43. Plane: $2x + y - z = 11$ Line: $x = 1 + t, \quad y = 3 - 2t, \quad z = 2 + 4t$

Answer

They intersect at point $(-1, 7, -6)$.

44. Plane: $-x + 2y + z = 2$ Line: $x = 1 + 2t, \quad y = -2 + t, \quad z = 5 - 3t$

Answer

They intersect at point $\left(-\frac{1}{3}, -\frac{8}{3}, 7\right)$.

45. Plane: $x - 3y + 2z = 4$ Line: $x = 2 - t, \quad y = t, \quad z = 4 + 2t$

Answer

The line does not intersect with this plane.

46. Plane: $x - 3y + 2z = 10$ Line: $x = 2 - t, \quad y = t, \quad z = 4 + 2t$

Answer

The line is actually fully contained in this plane, so every point on the line is on the plane. For example, when $t = 0$ we have the point, $(2, 0, 4)$.

47. Show that the lines of equations $x = t, \quad y = 1 + t, \quad z = 2 + t, \quad t \in R$, and $\frac{x}{2} = \frac{y - 1}{3} = z - 3$ are skew, and find the distance between them.

Answer

$\frac{1}{\sqrt{6}} = \frac{\sqrt{6}}{6}$ units

48. Show that the lines of equations $x = -1 + t, \quad y = -2 + t, \quad z = 3t, \quad t \in R$, and $x = 5 + s, \quad y = -8 + 2s, \quad z = 7s, \quad s \in R$ are skew, and find the distance between them.
49. Consider point $C(-3, 2, 4)$ and the plane of equation $2x + 4y - 3z = 8$.
    a. Find the radius of the sphere with center C tangent to the given plane.
    b. Find point P of tangency.

Answer

a. $r = \frac{18}{\sqrt{29}} = \frac{18\sqrt{29}}{29}$
b. $P\left(-\frac{51}{29}, \frac{130}{29}, \frac{62}{29}\right)$

50. Consider the plane of equation $x - y - z - 8 = 0$.
    a. Find the equation of the sphere with center C at the origin that is tangent to the given plane.
    b. Find parametric equations of the line passing through the origin and the point of tangency.
51. Two children are playing with a ball. The girl throws the ball to the boy. The ball travels in the air, curves 3 ft to the right, and falls 5 ft away from the girl (see Figure $4.5E.3$). If the plane that contains the trajectory of the ball is perpendicular to the ground, find its equation.

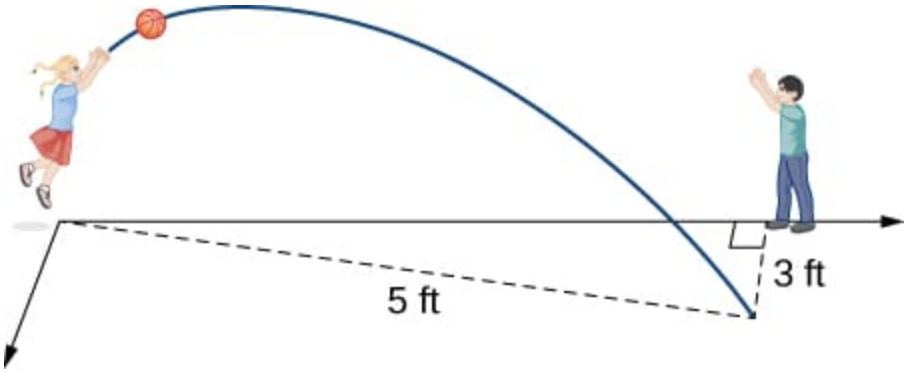


Figure $4.5E.3$: The girl throws the ball to the boy. The ball travels in the air, curves 3 ft to the right, and falls 5 ft away from the girl. The 3ft line segment, the 5ft line segment, and the line segment between the boy and the girl form a right triangle with the 5ft line segment as the hypotenuse.

Answer

$4x - 3y = 0$

52. [T] John allocates d dollars to consume monthly three goods of prices a, b, and c. In this context, the budget equation is defined as $ax + by + cz = d$, where $x \geq 0,\ y \geq 0$, and $z \geq 0$ represent the number of items bought from each of the goods. The budget set is given by $\{(x, y, z) \mid ax + by + cz \leq d,\ x \geq 0,\ y \geq 0,\ z \geq 0\}$, and the budget plane is the part of the plane of equation $ax + by + cz = d$ for which $x \geq 0,\ y \geq 0$, and $z \geq 0$. Consider $a = \$8,\ b = \$5,\ c = \$10$, and $d = \$500$.
    a. Use a CAS to graph the budget set and budget plane.
    b. For $z = 25$, find the new budget equation and graph the budget set in the same system of coordinates.
53. [T] Consider $\mathbf{r}(t) = \langle \sin t, \cos t, 2t \rangle$ the position vector of a particle at time $t \in [0, 3]$, where the components of $\mathbf{r}$ are expressed in centimeters and time is measured in seconds. Let $\vec{OP}$ be the position vector of the particle after 1 sec.
    a. Determine the velocity vector $\mathbf{v}(1)$ of the particle after 1 sec.
    b. Find the scalar equation of the plane that is perpendicular to $v(1)$ and passes through point P. This plane is called the normal plane to the path of the particle at point P.
    c. Use a CAS to visualize the path of the particle along with the velocity vector and normal plane at point P.

Answer

a. $\mathbf{v}(1) = \langle \cos 1, -\sin 1, 2 \rangle$
b. $(\cos 1)(x - \sin 1) - (\sin 1)(y - \cos 1) + 2(z - 2) = 0$
c.

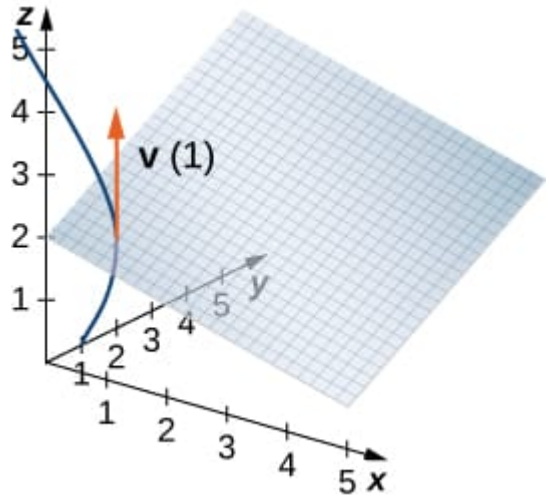


Figure $4.5E.4$: A three-dimensional Cartesian coordinate system with a parametric curve plotted in the xyz-space. A smooth blue curve represents the path of a particle or object in 3D space. An orange velocity vector $\mathbf{v}(1)$ is tangent to the curve at a specific point, indicating the direction and rate of motion at that point. A semi-transparent grid plane is shown in the background, likely used as a reference to provide perspective and a spatial framework for the curve.

54. [T] A solar panel is mounted on the roof of a house. The panel may be regarded as positioned at the points of coordinates (in meters) $A(8, 0, 0),\ B(8, 18, 0),\ C(0, 18, 8)$, and $D(0, 0, 8)$ (see Figure $4.5E.5$).

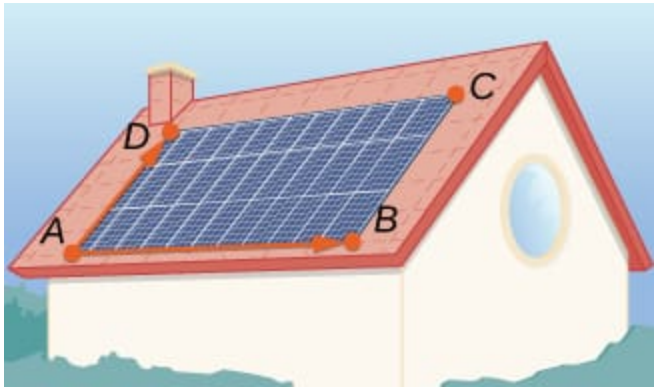


Figure $4.5E.5$: A house with a slanted roof equipped with solar panels. The solar panels are represented as a rectangular array covering a significant portion of one side of the roof. Key points labeled A, B, C, and D mark the corners of the solar panel array. The roof features a chimney near point D, and a circular window is visible on the house's wall below the roof.

a. Find the general form of the equation of the plane that contains the solar panel by using points $A,\ B$, and C, and show that its normal vector is equivalent to $\vecd{AB} \times \vecd{AD}$.
b. Find parametric equations of line L_1 that passes through the center of the solar panel and has direction vector $\mathbf{s} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$, which points toward the position of the Sun at a particular time of day.
c. Find symmetric equations of line L_2 that passes through the center of the solar panel and is perpendicular to it.
d. Determine the angle of elevation of the Sun above the solar panel by using the angle between lines L_1 and L_2.

Contributors

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Problems 15 and 43 - 46 created by Paul Seeburger

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4.6: Quadric Surfaces

Learning Objectives

- Identify a cylinder as a type of three-dimensional surface.
- Recognize the main features of ellipsoids, paraboloids, and hyperboloids.
- Use traces to draw the intersections of quadric surfaces with the coordinate planes.

We have been exploring vectors and vector operations in three-dimensional space, and we have developed equations to describe lines, planes, and spheres. In this section, we use our knowledge of planes and spheres, which are examples of three-dimensional figures called *surfaces*, to explore a variety of other surfaces that can be graphed in a three-dimensional coordinate system.

Identifying Cylinders

The first surface we'll examine is the cylinder. Although most people immediately think of a hollow pipe or a soda straw when they hear the word cylinder, here we use the broad mathematical meaning of the term. As we have seen, cylindrical surfaces don't have to be circular. A rectangular heating duct is a cylinder, as is a rolled-up yoga mat, the cross-section of which is a spiral shape.

In the two-dimensional coordinate plane, the equation $x^2 + y^2 = 9$ describes a circle centered at the origin with radius 3. In three-dimensional space, this same equation represents a surface. Imagine copies of a circle stacked on top of each other centered on the z-axis (Figure $4.6.1$), forming a hollow tube. We can then construct a cylinder from the set of lines parallel to the z-axis passing through the circle $x^2 + y^2 = 9$ in the xy-plane, as shown in the figure. In this way, any curve in one of the coordinate planes can be extended to become a surface.

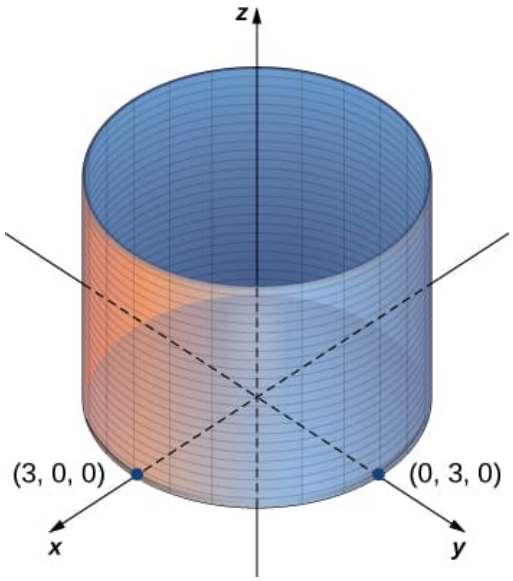


Figure $4.6.1$: In three-dimensional space, the graph of equation $x^2 + y^2 = 9$ is a cylinder with radius 3 centered on the z-axis. It continues indefinitely in the positive and negative directions.

Definition: Cylinders and Rulings

A set of lines parallel to a given line passing through a given curve is known as a cylindrical surface, or **cylinder.** The parallel lines are called **rulings.**

From this definition, we can see that we still have a cylinder in three-dimensional space, even if the curve is not a circle. Any

curve can form a cylinder, and the rulings that compose the cylinder may be parallel to any given line (Figure 4.6.2).

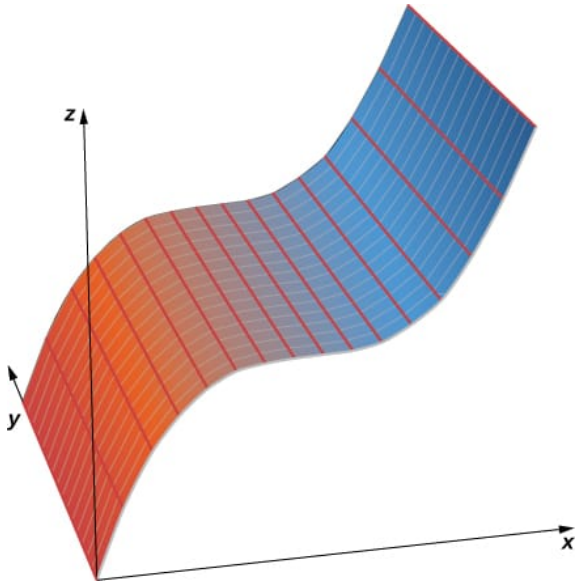


Figure 4.6.2: In three-dimensional space, the graph of equation $z = x^3$ is a cylindrical sheet with rulings parallel to the y-axis.

Interactive Element

Interactive View of a Cylindrical Surface

Below is the graph of the cylindrical surface $y + x^3 - x = 0$. You can move it around and also change its equation to see the effect on the graph. The interactive app is unfortunately not accessible.

Example 4.6.1: Graphing Cylindrical Surfaces

Sketch the graphs of the following cylindrical surfaces.

a. $x^2 + z^2 = 25$
b. $z = 2x^2 - y$
c. $y = \sin x$

Solution

a. The variable y can take on any value without limit. Therefore, the lines ruling this surface are parallel to the y-axis. The intersection of this surface with the xz-plane forms a circle centered at the origin with radius 5 (see Figure 4.6.3).

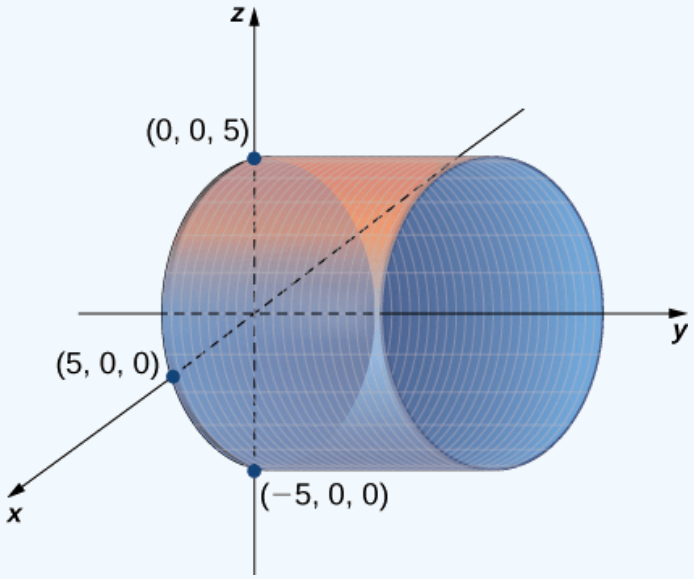


Figure 4.6.3: The graph of equation $x^2 + z^2 = 25$ is a cylinder with radius 5 centered on the y-axis.

b. In this case, the equation contains all three variables —x, y, and z— so none of the variables can vary arbitrarily. The easiest way to visualize this surface is to use a computer graphing utility (Figure 4.6.4).

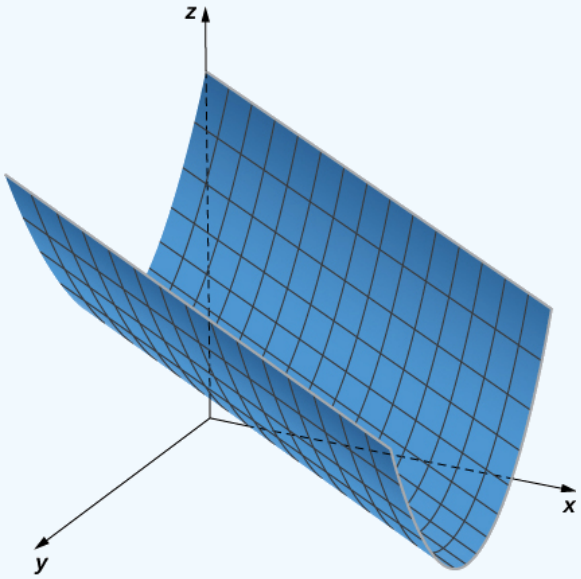


Figure 4.6.4: A graph of the surface of a parabolic cylinder using a computer graphing utility. The surface curving upward.

c. In this equation, the variable z can take on any value without limit. Therefore, the lines composing this surface are parallel to the z-axis. The intersection of this surface with the xy-plane outlines curve $y = \sin x$ (Figure 4.6.5).

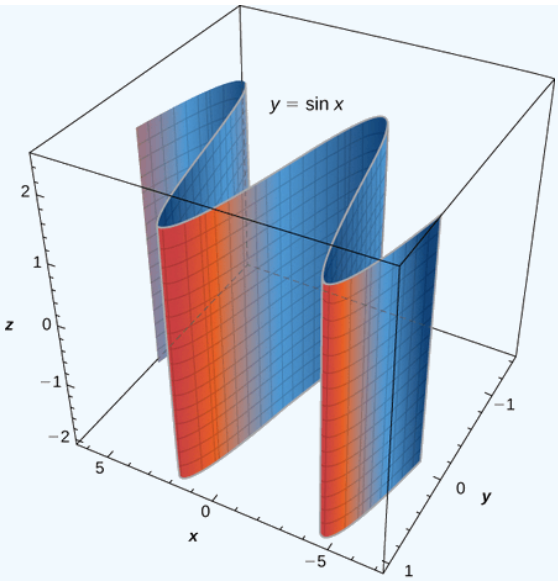


Figure 4.6.5: The graph of equation $y = \sin x$ is formed by a set of lines parallel to the z-axis passing through curve $y = \sin x$ in the xy -plane. The surface is a sinusoidal cylinder.

Watch the accompanying video for a visual demonstration of the concept.

- **Video length:** 2 minutes and 23 seconds
- **Context:** This video demonstrates how to sketch a parabolic cylinder $x^2 + z = 4$.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Exercise 4.6.1:

Sketch or use a graphing tool to view the graph of the cylindrical surface defined by equation $z = y^2$.

Hint available at the end of this section.

When sketching surfaces, we have seen that it is useful to sketch the intersection of the surface with a plane parallel to one of the coordinate planes. These curves are called traces. We can see them in the plot of the cylinder in Figure 4.6.6.

Definition: Traces

The **traces** of a surface are the cross-sections created when the surface intersects a plane parallel to one of the coordinate planes.

Traces are useful in sketching cylindrical surfaces. For a cylinder in three dimensions, though, only one set of traces is useful. Notice, in Figure 4.6.6, that the trace of the graph of $z = \sin x$ in the xz-plane is useful in constructing the graph. The trace in the xy-plane, though, is just a series of parallel lines, and the trace in the yz-plane is simply one line.

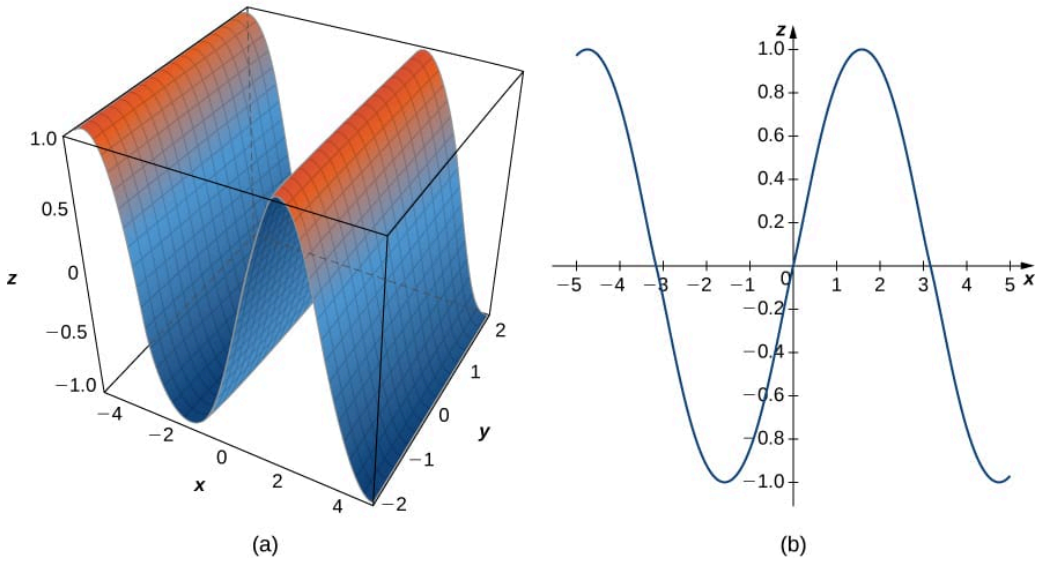


Figure 4.6.6: (a) A view of the graph of equation $z = \sin x$. (b) To find the trace of the graph in the xz-plane, set $y = 0$. The trace is simply a two-dimensional sine wave.

Cylindrical surfaces are formed by a set of parallel lines. Not all surfaces in three dimensions are constructed so simply, however. We now explore more complex surfaces, and traces are an important tool in this investigation.

Quadric Surfaces

We have learned about surfaces in three dimensions described by first-order equations; these are planes. Some other common types of surfaces can be described by second-order equations. We can view these surfaces as three-dimensional extensions of the conic sections we discussed earlier: the ellipse, the parabola, and the hyperbola. We call these graphs quadric surfaces

Definition: Quadric Surfaces and Conic Sections

Quadric surfaces are the graphs of equations that can be expressed in the form

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Jz + K = 0. \tag{4.6.1}$$

When a quadric surface intersects a coordinate plane, the trace is a conic section.

An *ellipsoid* is a surface described by an equation of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Set $x = 0$ to see the trace of the ellipsoid in the yz-plane. To see the traces in the xy- and xz-planes, set $z = 0$ and $y = 0$, respectively. Notice that, if $a = b$, the trace in the xy -plane is a circle. Similarly, if $a = c$, the trace in the xz-plane is a circle and, if $b = c$, then the trace in the yz-plane is a circle. A sphere, then, is an ellipsoid with $a = b = c$.

Example 4.6.2: Sketching an Ellipsoid

Sketch the ellipsoid

$$\frac{x^2}{2^2} + \frac{y^2}{3^2} + \frac{z^2}{5^2} = 1. \tag{4.6.2}$$

Solution

Start by sketching the traces. To find the trace in the xy-plane, set $z = 0$: $\frac{x^2}{2^2} + \frac{y^2}{3^2} = 1$ (Figure $4.6.7$). To find the other traces, first set $y = 0$ and then set $x = 0$.

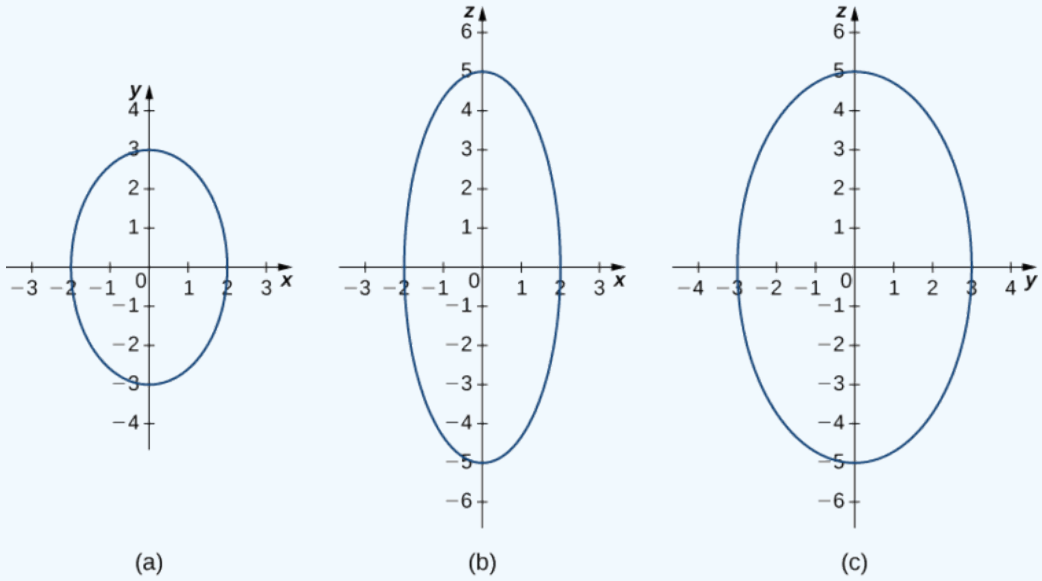


Figure $4.6.7$: (a) This graph represents the trace of equation $\frac{x^2}{2^2} + \frac{y^2}{3^2} + \frac{z^2}{5^2} = 1$ in the xy-plane, when we set $z = 0$. (b) When we set $y = 0$, we get the trace of the ellipsoid in the xz-plane, which is an ellipse. (c) When we set $x = 0$, we get the trace of the ellipsoid in the yz-plane, which is also an ellipse.

Now that we know what traces of this solid look like, we can sketch the surface in three dimensions (Figure $4.6.8$).

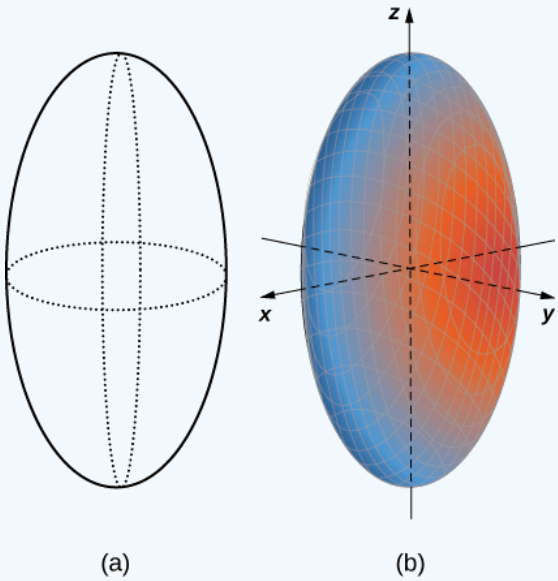


Figure $4.6.8$: (a) The traces provide a framework for the surface of an ellipsoid. (b) The surface of an ellipsoid with center at the origin.

The trace of an ellipsoid is an ellipse in each of the coordinate planes. However, this does not have to be the case for all quadric surfaces. Many quadric surfaces have traces that are different kinds of conic sections, and this is usually indicated by the name of the surface. For example, if a surface can be described by an equation of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c} \tag{4.6.3}$$

then we call that surface an **elliptic paraboloid**. The trace in the xy-plane is an ellipse, but the traces in the xz-plane and yz -plane are parabolas (Figure $4.6.9$). Other elliptic paraboloids can have other orientations simply by interchanging the variables to

give us a different variable in the linear term of the equation $\frac{x^2}{a^2}+\frac{z^2}{c^2}=\frac{y}{b}$ or $\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{x}{a}$.

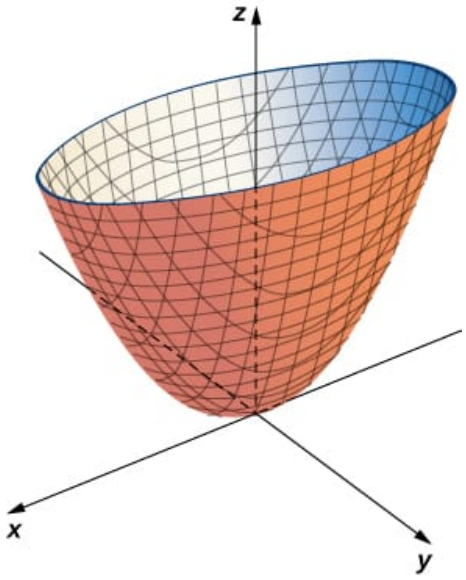


Figure 4.6.9: This quadric surface is called an elliptic paraboloid which looks like a parabolic surface resembling an upward-facing bowl.

Interactive Element

Quadric Surface Interactive Exploration

Move the sliders to change the coefficients of $ax^2+by^2+cz^2=d$. You can also drag and rotate the axes to change your perspective. The interactive app is unfortunately not accessible.

Example 4.6.3: Identifying Traces of Quadric Surfaces

Describe the traces of the elliptic paraboloid $x^2+\frac{y^2}{2^2}=\frac{z}{5}$.

Solution

To find the trace in the xy-plane, set $z=0: x^2+\frac{y^2}{2^2}=0$. The trace in the plane $z=0$ is simply one point, the origin. Since a single point does not tell us what the shape is, we can move up the z-axis to an arbitrary plane to find the shape of other traces of the figure.

The trace in plane $z=5$ is the graph of equation $x^2+\frac{y^2}{2^2}=1$, which is an ellipse. In the xz-plane, the equation becomes $z=5x^2$. The trace is a parabola in this plane and in any plane with the equation $y=b$.

In planes parallel to the yz-plane, the traces are also parabolas, as we can see in Figure 4.6.10.

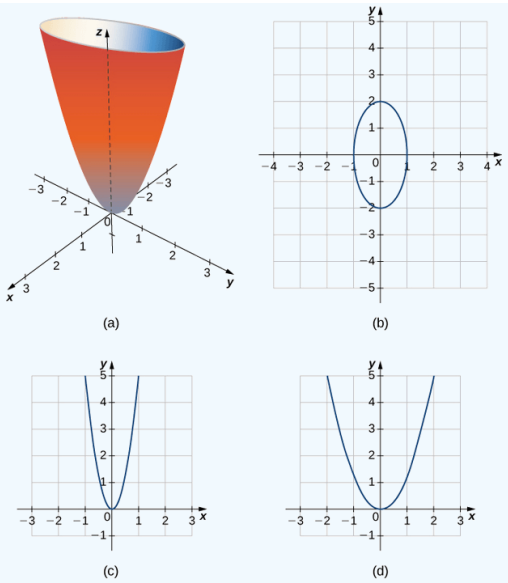


Figure 4.6.10: (a) Sketch of the paraboloid $x^2 + \frac{y^2}{2^2} = \frac{z}{5}$. (b) The trace in plane $z = 5$ is an ellipse. (c) The trace in the xz-plane is a parabola. (d) The trace in the yz-plane is a parabola.

Exercise 4.6.2:

A hyperboloid of one sheet is any surface that can be described with an equation of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$. Describe the traces of the hyperboloid of one sheet given by equation $\frac{x^2}{3^2} + \frac{y^2}{2^2} - \frac{z^2}{5^2} = 1$.

Hint available at the end of this section.

Hyperboloids of one sheet have some fascinating properties. For example, they can be constructed using straight lines, such as in the sculpture in Figure 4.6.12a. In fact, cooling towers for nuclear power plants are often constructed in the shape of a hyperboloid. The builders are able to use straight steel beams in the construction, which makes the towers very strong while using relatively little material (Figure 4.6.12b).

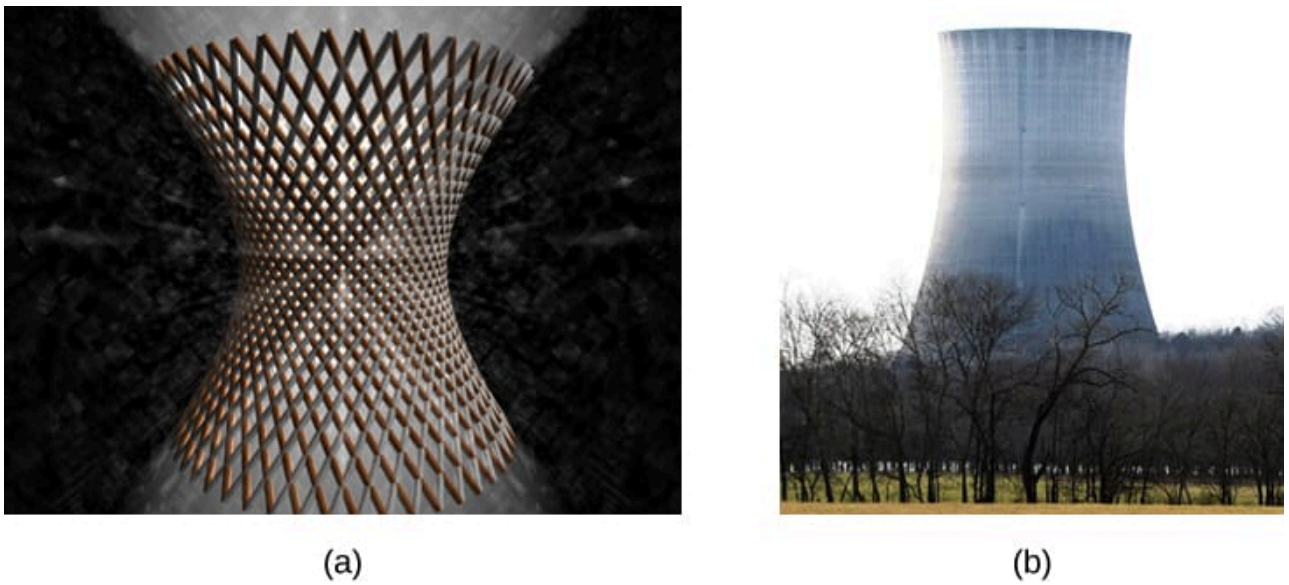


Figure 4.6.12: (a) A sculpture in the shape of a hyperboloid can be constructed of straight lines. (b) Cooling towers for nuclear power plants are often built in the shape of a hyperboloid.(Credit: CC-BY-NC-SA ; Wikimedia Commons)

Example 4.6.4: Chapter Opener: Finding the Focus of a Parabolic Reflector

Energy hitting the surface of a parabolic reflector is concentrated at the focal point of the reflector (Figure 4.6.13). If the surface of a parabolic reflector is described by equation $\frac{x^2}{100} + \frac{y^2}{100} = \frac{z}{4}$, where is the focal point of the reflector?

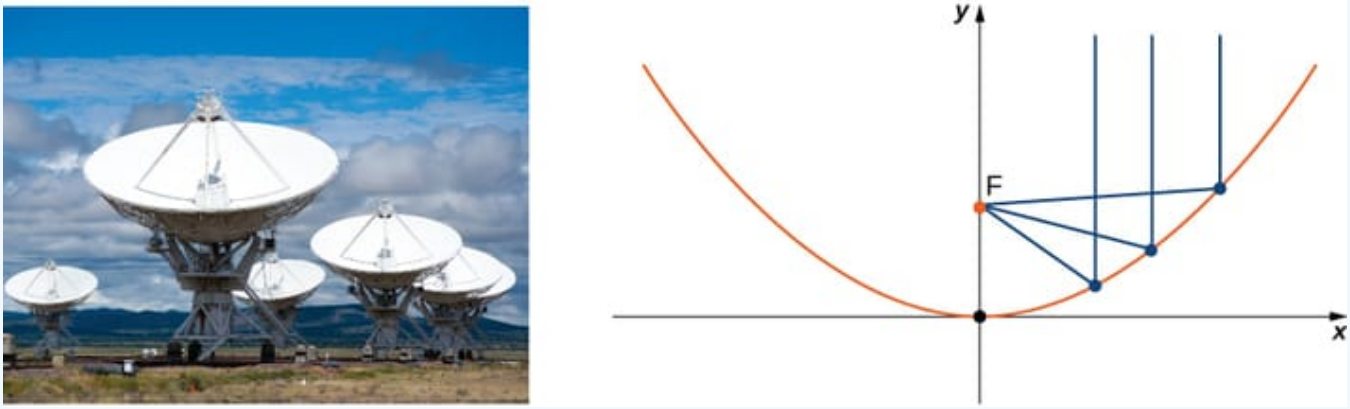


Figure 4.6.13: Energy reflects off of the parabolic reflector and is collected at the focal point. (CC-BY-2.0 modification of work by CGP Grey, Wikimedia Commons)

Solution

Since *z* is the first-power variable, the axis of the reflector corresponds to the z-axis. The coefficients of x^2 and y^2 are equal, so the cross-section of the paraboloid perpendicular to the z-axis is a circle. We can consider a trace in the xz-plane or the yz-plane; the result is the same. Setting $y = 0$, the trace is a parabola opening up along the z-axis, with standard equation $x^2 = 4pz$, where p is the focal length of the parabola. In this case, this equation becomes $x^2 = 100 \cdot \frac{z}{4} = 4pz$ or $25 = 4p$. So p is 6.25 m, which tells us that the focus of the paraboloid is 6.25 m up the axis from the vertex. Because the vertex of this surface is the origin, the focal point is $(0, 0, 6.25)$.

Seventeen standard quadric surfaces can be derived from the general equation

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Jz + K = 0. \tag{4.6.4}$$

The following figures summarizes the most important ones.

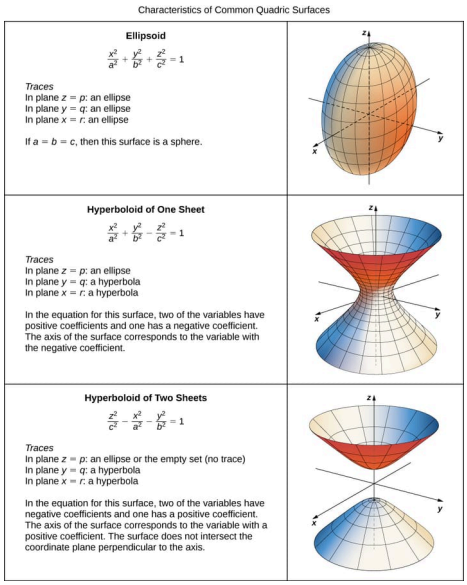


Figure 4.6.14: Characteristics of Common Quadratic Surfaces: Ellipsoid, Hyperboloid of One Sheet, Hyperboloid of Two Sheets is shown in a table with two columns and three rows. The three rows represent the first 3 quadric surfaces: ellipsoid, hyperboloid of one sheet, and hyperboloid of two sheets. The equations and traces are in the first column. The second column has the graphs of the surfaces. The ellipsoid graph is a vertical oblong round shape. The hyperboloid of one sheet is circular on the top and the bottom and narrow in the middle. The hyperboloid in two sheets has two parabolic domes opposite of each other.

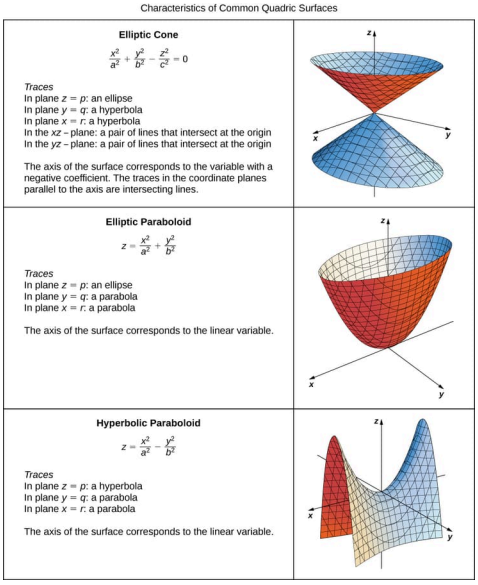


Figure 4.6.15: Characteristics of Common Quadratic Surfaces: Elliptic Cone, Elliptic Paraboloid, Hyperbolic Paraboloid is shown in a table with two columns and three rows. The three rows represent the second 6 quadric surfaces: elliptic cone, elliptic paraboloid, and hyperbolic paraboloid. The equations and traces are in the first column. The second column has the graphs of the surfaces. The elliptic cone has two cones touching at the points. The elliptic paraboloid is similar to a cone but oblong. The hyperbolic paraboloid has a bend in the middle similar to a saddle.

Example 4.6.5: Identifying Equations of Quadric Surfaces

Identify the surfaces represented by the given equations.

a. $16x^2+9y^2+16z^2=144$
b. $9x^2-18x+4y^2+16y-36z+25=0$

Solution

a. The x, y, and z terms are all squared, and are all positive, so this is probably an ellipsoid. However, let's put the equation into the standard form for an ellipsoid just to be sure. We have

$$16x^2 + 9y^2 + 16z^2 = 144.$$

Dividing through by 144 gives

$$\frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{9} = 1.$$

So, this is, in fact, an ellipsoid, centered at the origin.

b. We first notice that the z term is raised only to the first power, so this is either an elliptic paraboloid or a hyperbolic paraboloid. We also note there are x terms and y terms that are not squared, so this quadric surface is not centered at the origin. We need to complete the square to put this equation in one of the standard forms. We have

$$\begin{aligned} 9x^2 - 18x + 4y^2 + 16y - 36z + 25 &= 0 \\ 9x^2 - 18x + 4y^2 + 16y + 25 &= 36z \\ 9(x^2 - 2x) + 4(y^2 + 4y) + 25 &= 36z \\ 9(x^2 - 2x + 1 - 1) + 4(y^2 + 4y + 4 - 4) + 25 &= 36z \\ 9(x-1)^2 - 9 + 4(y+2)^2 - 16 + 25 &= 36z \\ 9(x-1)^2 + 4(y+2)^2 &= 36z \\ \frac{(x-1)^2}{4} + \frac{(y-2)^2}{9} &= z. \end{aligned}$$

This is an elliptic paraboloid centered at $(1, 2, 0)$.

Watch the accompanying video for a visual demonstration of the concept.

- **Video length**: 2 minutes 47 seconds.
- **Context:** This video demonstrates how to identify and sketching an ellipsoid $4x^2 + y^2 + 16z^2 = 16$

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Watch the accompanying video for another visual demonstration of the concept.

- **Video length:** 1 minute 32 seconds.
- **Context:** This video demonstrates how to identify and sketch a paraboloid $x^2 + y^2 - z = 1$

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Exercise 4.6.3

Identify the surface represented by equation $9x^2 + y^2 - z^2 + 2z - 10 = 0$.

Hint available at the end of this section.

Key Concepts

- A set of lines parallel to a given line passing through a given curve is called a *cylinder,* or a cylindrical surface. The parallel lines are called *rulings*.
- The intersection of a three-dimensional surface and a plane is called a trace. To find the trace in the xy-, yz-, or xz-planes, set $z = 0, x = 0$, or $y = 0$, respectively.
- Quadric surfaces are three-dimensional surfaces with traces composed of conic sections. Every quadric surface can be expressed with an equation of the form

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Jz + K = 0.$$

- To sketch the graph of a quadric surface, start by sketching the traces to understand the framework of the surface.
- Important quadric surfaces are summarized in Figures $4.6.13$ and $4.6.14$.

Glossary

Cylinder
A set of lines parallel to a given line passing through a given curve.

Ellipsoid
A three-dimensional surface described by an equation of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$; all traces of this surface are ellipses.

Elliptic Cone
A three-dimensional surface described by an equation of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$; traces of this surface include ellipses and intersecting lines.

Elliptic Paraboloid
A three-dimensional surface described by an equation of the form $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$; traces of this surface include ellipses and parabolas.

Hyperboloid of One Sheet
A three-dimensional surface described by an equation of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$; traces of this surface include ellipses and hyperbolas.

Hyperboloid of Two Sheets
A three-dimensional surface described by an equation of the form $\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$; traces of this surface include ellipses and hyperbolas.

Quadric Surfaces
Surfaces in three dimensions having the property that the traces of the surface are conic sections (ellipses, hyperbolas, and parabolas).

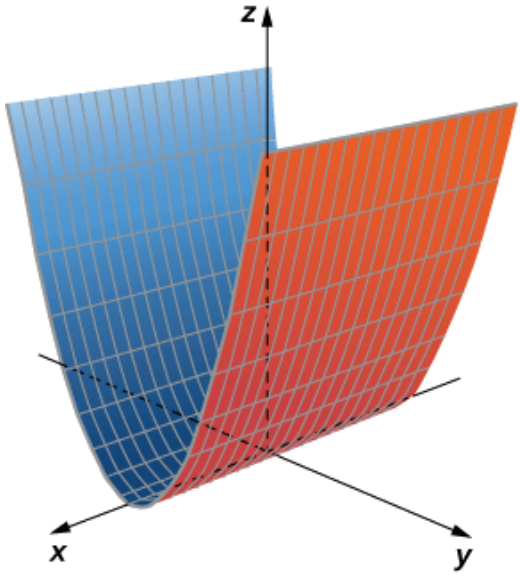
Rulings
Parallel lines that make up a cylindrical surface.

Trace
The intersection of a three-dimensional surface with a coordinate plane.

Answers

Exercise $4.6.1$**:**

The variable x can take on any value without limit.



Exercise $4.6.2$**:**

To find the traces in the coordinate planes, set each variable to zero individually.

The traces parallel to the xy-plane are ellipses and the traces parallel to the xz- and yz-planes are hyperbolas. Specifically, the trace in the xy-plane is ellipse $\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$, the trace in the xz-plane is hyperbola $\frac{x^2}{3^2} - \frac{z^2}{5^2} = 1$, and the trace in the yz-plane is hyperbola $\frac{y^2}{2^2} - \frac{z^2}{5^2} = 1$ (see Figure $4.6.11$).

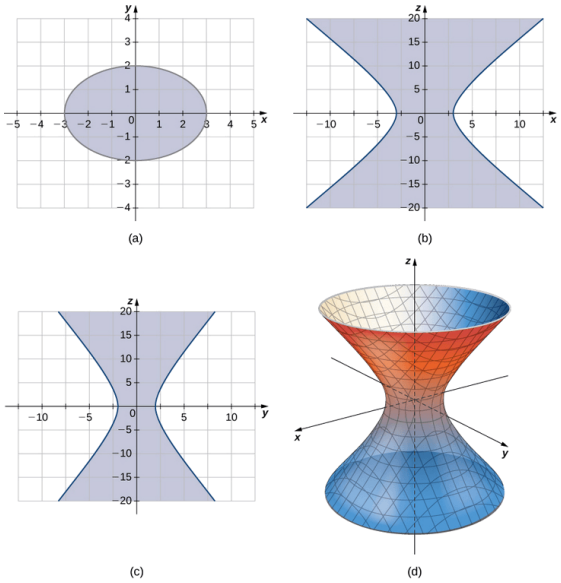


Figure $4.6.11$: (a) Trace in the xz-plane is an ellipse. (b)The trace in the xz-plane is a hyperboloid. (c) The trace in xz-plane is a hyperboloid. (d) Sketch of the hyperboloid of one sheet given by equation $\frac{x^2}{3^2} + \frac{y^2}{2^2} - \frac{z^2}{5^2} = 1$.

Exercise $4.6.3$

Look at the signs and powers of the x, y, and z terms

Hyperboloid of one sheet, centered at $(0, 0, 1)$.

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4.6E: Exercises for Section 4.6

Exercises 1 - 6

Sketch and describe the cylindrical surface of the given equation.

1. [T] $x^2 + z^2 = 1$

Answer

The surface is a cylinder with the rulings parallel to the y-axis.

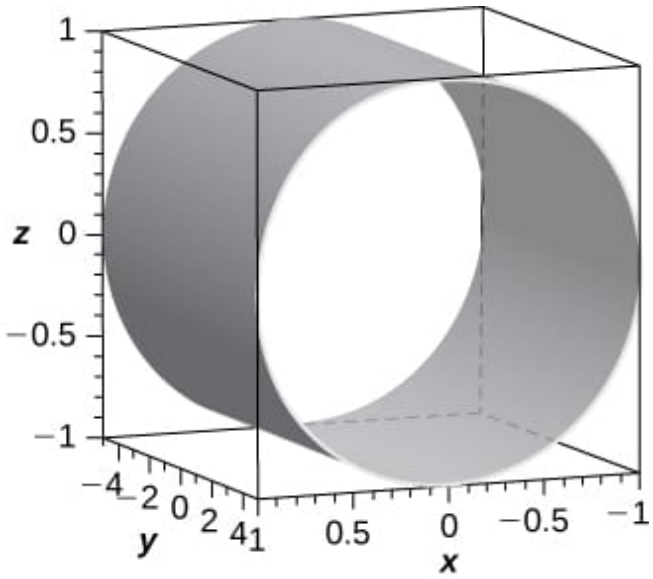


Figure $4.6E.1$: A cylinder with the rulings parallel to the y-axis. A cylinder is inside the box with x ∈ [-1,1], y ∈ [-4,4], and z ∈ [-1,1].

2. [T] $x^2 + y^2 = 9$
3. [T] $z = \cos\left(\frac{\pi}{2} + x\right)$

Answer

The surface is a cylinder with rulings parallel to the y-axis.

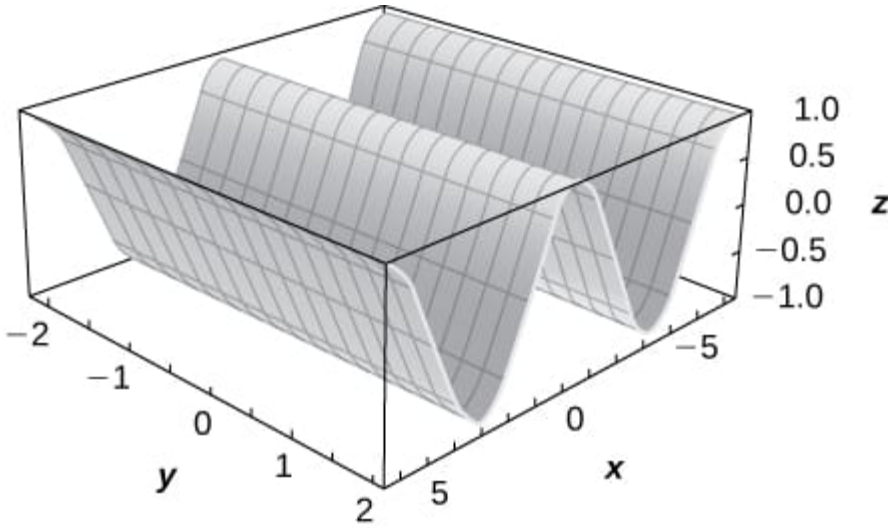


Figure $4.6E.2$: A cylinder sheet with the rulings parallel to the y-axis. A cylinder is inside the box with x ∈ [-5,5], y ∈ [-2,2], and z ∈ [-1,1].

4. [T] $z = e^x$
5. [T] $z = 9 - y^2$

Answer

The surface is a cylinder with rulings parallel to the x-axis.

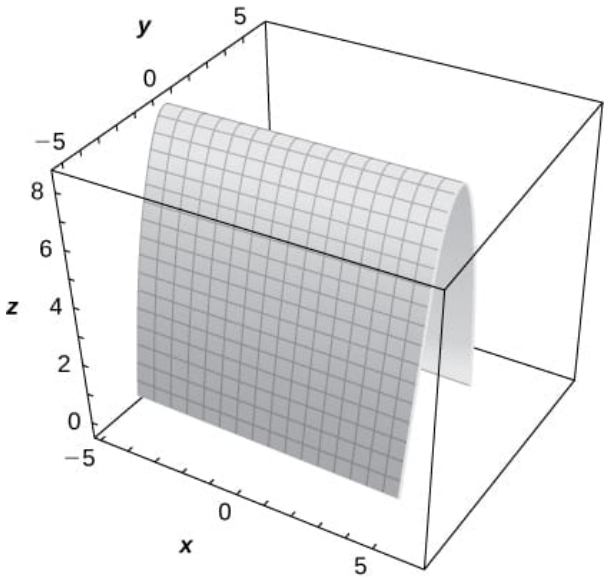


Figure $4.6E.3$: A cylinder sheet with the rulings parallel to the x-axis. A cylinder is inside the box with x ∈ [-5,5], y ∈ [-5,5], and z ∈ [0,8].

6. [T] $z = \ln x$

Exercises 7 - 10

The graph of a quadric surface is given.

a. **Specify the name of the quadric surface.**
b. **Determine the axis of symmetry of the quadric surface.**

7.

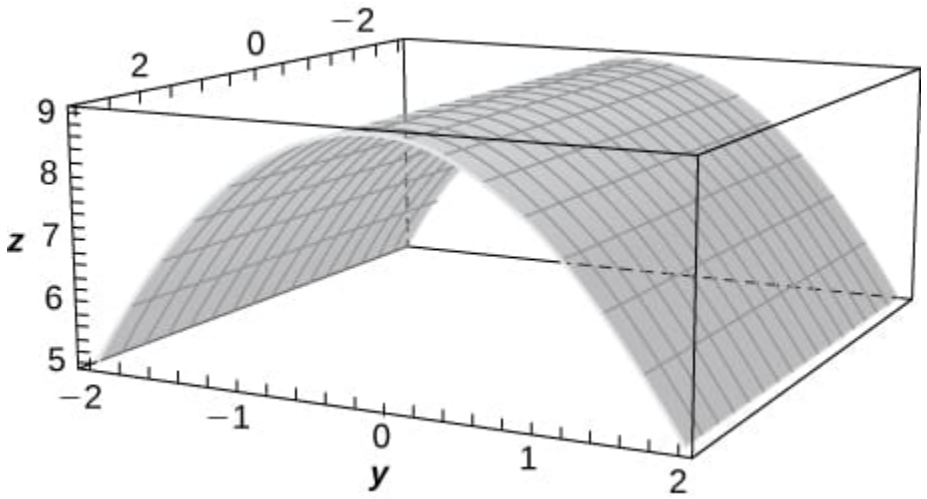


Figure $4.6E.4$: A cylinder sheet with the rulings parallel to the x-axis. A cylinder is inside the box with x ∈ [-2,2], y ∈ [-2,2], and z ∈ [5,9].

Answer

a. Cylinder;
b. The x-axis

8.

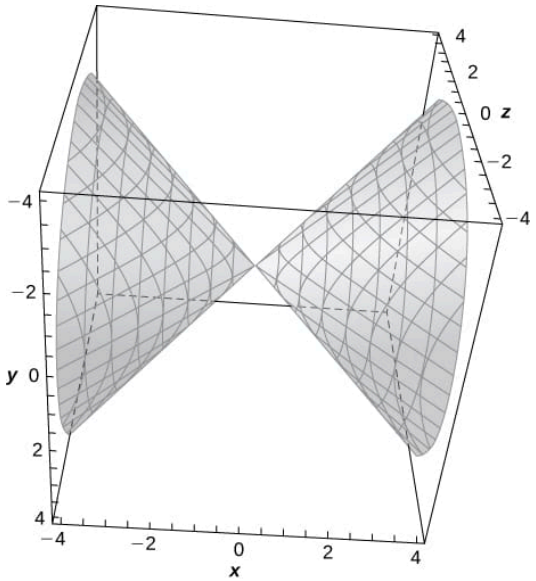


Figure $4.6E.5$: An elliptic cone of two sheets with x-axis as the axis of symmetry inside the box with x ∈ [-4,4], y ∈ [-4,4], and z ∈ [-4,4].

9.

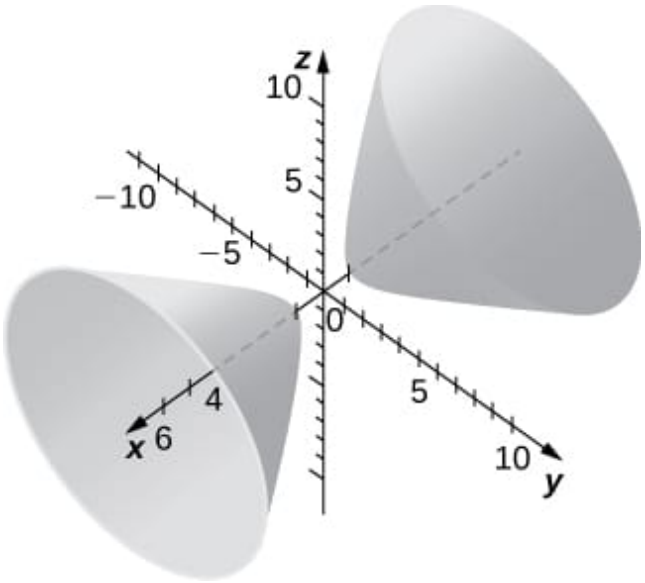


Figure $4.6E.6$: A hyperboloid of two sheets with x-axis as the axis of symmetry in 3d-space.

Answer

a. Hyperboloid of two sheets;
b. The x-axis

10.

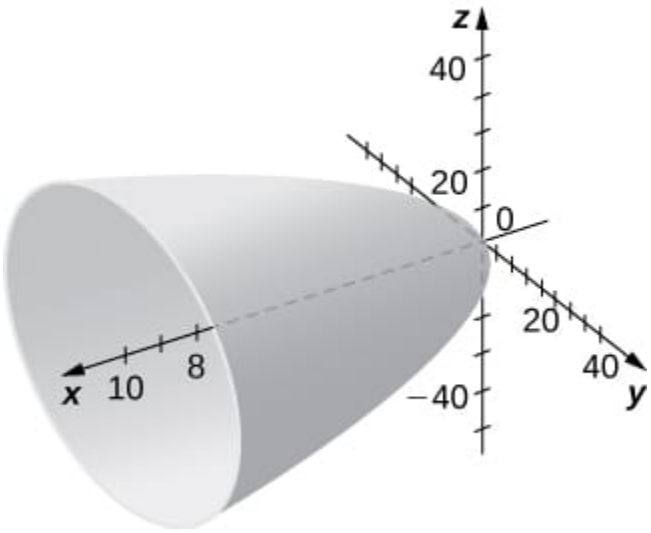


Figure $4.6E.7$: A paraboloid with x-axis as the axis of symmetry in 3d-space.

Exercises 11 - 16

Match the given quadric surface with its corresponding equation in standard form.

a. $\frac{x^2}{4}+\frac{y^2}{9}-\frac{z^2}{12}=1$
b. $\frac{x^2}{4}-\frac{y^2}{9}-\frac{z^2}{12}=1$
c. $\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{12}=1$

d. $z^2 = 4x^2 + 3y^2$
e. $z = 4x^2 - y^2$
f. $4x^2 + y^2 - z^2 = 0$

11. Hyperboloid of two sheets

Answer

b.

12. Ellipsoid
13. Elliptic paraboloid

Answer

d.

14. Hyperbolic paraboloid
15. Hyperboloid of one sheet

Answer

a.

16. Elliptic cone

Exercises 17 - 28

Rewrite the given equation of the quadric surface in standard form. Identify the surface.

17. $-x^2 + 36y^2 + 36z^2 = 9$

Answer

$-\frac{x^2}{9} + \frac{y^2}{\frac{1}{4}} + \frac{z^2}{\frac{1}{4}} = 1$, hyperboloid of one sheet with the x-axis as its axis of symmetry

18. $-4x^2 + 25y^2 + z^2 = 100$
19. $-3x^2 + 5y^2 - z^2 = 10$

Answer

$-\frac{x^2}{\frac{10}{3}} + \frac{y^2}{2} - \frac{z^2}{10} = 1$, hyperboloid of two sheets with the y-axis as its axis of symmetry

20. $3x^2 - y^2 - 6z^2 = 18$
21. $5y = x^2 - z^2$

Answer

$y = -\frac{z^2}{5} + \frac{x^2}{5}$, hyperbolic paraboloid with the y-axis as its axis of symmetry

22. $8x^2 - 5y^2 - 10z = 0$
23. $x^2 + 5y^2 + 3z^2 - 15 = 0$

Answer

$\frac{x^2}{15} + \frac{y^2}{3} + \frac{z^2}{5} = 1$, ellipsoid

24. $63x^2 + 7y^2 + 9z^2 - 63 = 0$
25. $x^2 + 5y^2 - 8z^2 = 0$

Answer

$\frac{x^2}{40} + \frac{y^2}{8} - \frac{z^2}{5} = 0$, elliptic cone with the z-axis as its axis of symmetry

26. $5x^2 - 4y^2 + 20z^2 = 0$
27. $6x = 3y^2 + 2z^2$

Answer

$x = \frac{y^2}{2} + \frac{z^2}{3}$, elliptic paraboloid with the x-axis as its axis of symmetry

28. $49y = x^2 + 7z^2$

Exercises 29 - 34

Find the trace of the given quadric surface in the specified plane of coordinates and sketch it.

29. [T] $x^2 + z^2 + 4y = 0, \quad z = 0$

Answer

Parabola $y = -\frac{x^2}{4}$,

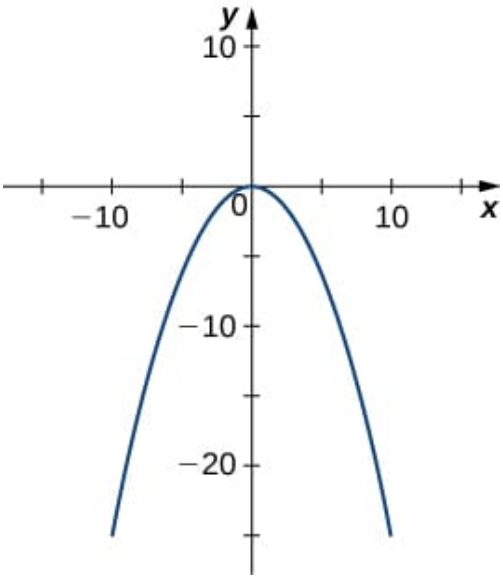


Figure $4.6E.8$: A downward parabola with vertex at origin on the xy-plane.

30. [T] $x^2 + z^2 + 4y = 0, \quad x = 0$
31. [T] $-4x^2 + 25y^2 + z^2 = 100, \quad x = 0$

Answer

Ellipse $\frac{y^2}{4} + \frac{z^2}{100} = 1$,

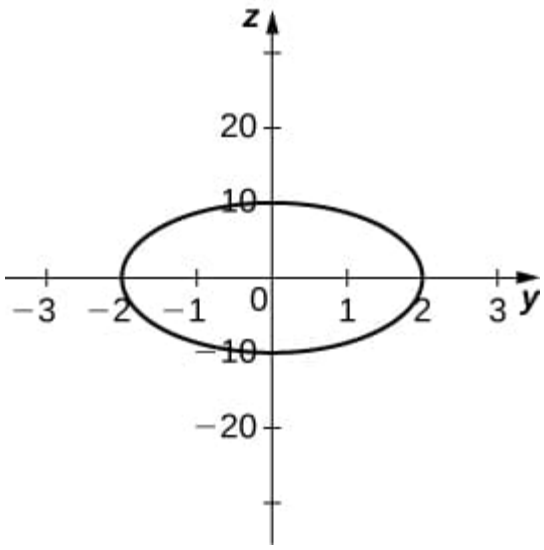


Figure $4.6E.9$: An ellipse centered at origin with vertices (2,0), (-2,0), (0,10), and (0,-10) on the yz-plane.

32. [T] $-4x^2 + 25y^2 + z^2 = 100, \quad y = 0$
33. [T] $x^2 + \frac{y^2}{4} + \frac{z^2}{100} = 1, \quad x = 0$

Answer

Ellipse $\frac{y^2}{4} + \frac{z^2}{100} = 1,$

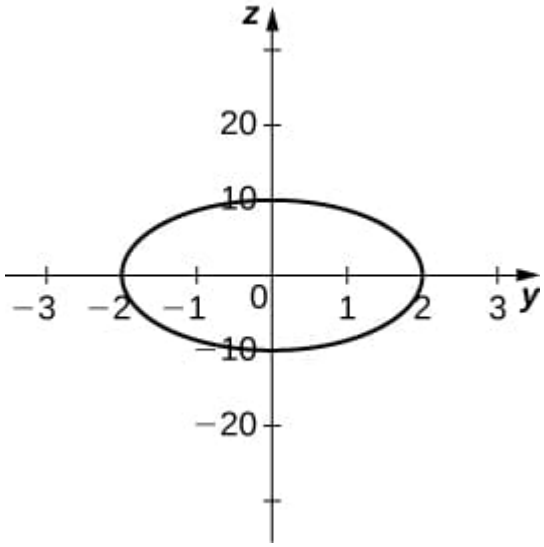


Figure $4.6E.10$: An ellipse centered at origin with vertices (2,0), (-2,0), (0,10), and (0,-10) on the yz-plane.

34. [T] $x^2 - y - z^2 = 1, \quad y = 0$
35. Use Figure $4.6E.11$ to answer the questions.

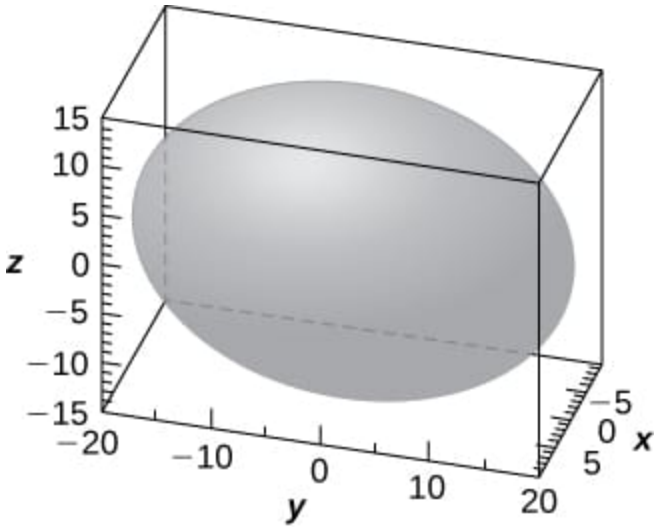


Figure $4.6E.11$: An ellipsoid with y-axis as the axis of symmetry. It is inside a box with x ∈ [-5,5], y ∈ [-20,20], and z ∈ [-15,15].

a. Specify the name of the quadric surface.
b. Which of the equations—$16x^2 + 9y^2 + 36z^2 = 3600$, $9x^2 + 36y^2 + 16z^2 = 3600$, or $36x^2 + 9y^2 + 16z^2 = 3600$ —corresponds to the graph?
c. Use b. to write the equation of the quadric surface in standard form.

Answer

a. Ellipsoid
b. The third equation
c. $\frac{x^2}{100} + \frac{y^2}{400} + \frac{z^2}{225} = 1$

36. Use Figure $4.6E.12$ to answer the questions.

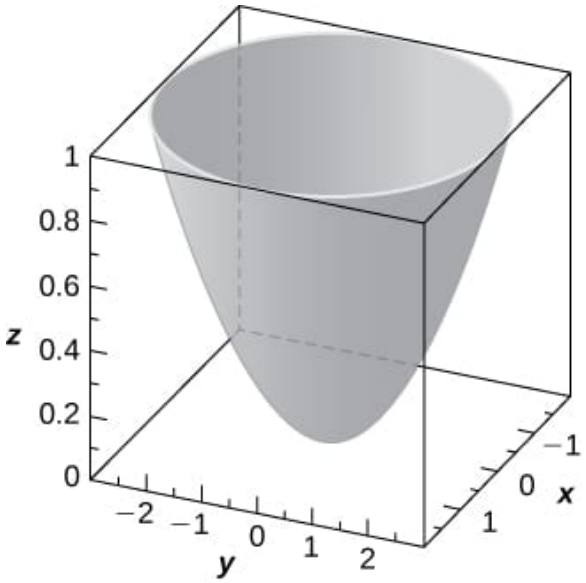


Figure $4.6E.12$: A paraboloid with z-axis as the axis of symmetry. It is inside a box with x ∈ [-1,1], y ∈ [-2,2], and z ∈ [0,1].

a. Specify the name of the quadric surface.
b. Which of the equations—$36z = 9x^2 + y^2$, $9x^2 + 4y^2 = 36z$, or $-36z = -81x^2 + 4y^2$ —corresponds to the graph above?
c. Use b. to write the equation of the quadric surface in standard form.

Exercises 37 - 42

The equation of a quadric surface is given.

a. **Use the method of completing the square to write the equation in standard form.**
b. **Identify the surface.**

37. $x^2 + 2z^2 + 6x - 8z + 1 = 0$

Answer

a. $\dfrac{(x+3)^2}{16} + \dfrac{(z-2)^2}{8} = 1$
b. Cylinder centered at $(-3, 2)$ with rulings parallel to the y-axis

38. $4x^2 - y^2 + z^2 - 8x + 2y + 2z + 3 = 0$
39. $x^2 + 4y^2 - 4z^2 - 6x - 16y - 16z + 5 = 0$

Answer

a. $\dfrac{(x-3)^2}{4} + (y-2)^2 - (z+2)^2 = 1$
b. Hyperboloid of one sheet centered at $(3, 2, -2)$, with the z-axis as its axis of symmetry

40. $x^2 + z^2 - 4y + 4 = 0$
41. $x^2 + \dfrac{y^2}{4} - \dfrac{z^2}{3} + 6x + 9 = 0$

Answer

a. $(x+3)^2 + \dfrac{y^2}{4} - \dfrac{z^2}{3} = 0$
b. Elliptic cone centered at $(-3, 0, 0)$, with the z-axis as its axis of symmetry

42. $x^2 - y^2 + z^2 - 12z + 2x + 37 = 0$
43. Write the standard form of the equation of the ellipsoid centered at the origin that passes through points $A(2, 0, 0)$, $B(0, 0, 1)$, and $C\left(12, \sqrt{11}, \frac{1}{2}\right)$.

Answer

$\dfrac{x^2}{4} + \dfrac{y^2}{16} + z^2 = 1$

44. Write the standard form of the equation of the ellipsoid centered at point $P(1, 1, 0)$ that passes through points $A(6, 1, 0)$, $B(4, 2, 0)$ and $C(1, 2, 1)$.
45. Determine the intersection points of elliptic cone $x^2 - y^2 - z^2 = 0$ with the line of symmetric equations $\dfrac{x-1}{2} = \dfrac{y+1}{3} = z$.

Answer

$(1, -1, 0)$ and $\left(\frac{13}{3}, 4, \frac{5}{3}\right)$

46. Determine the intersection points of parabolic hyperboloid $z = 3x^2 - 2y^2$ with the line of parametric equations $x = 3t$, $y = 2t$, $z = 19t$, where $t \in R$.
47. Find the equation of the quadric surface with points $P(x, y, z)$ that are equidistant from point $Q(0, -1, 0)$ and plane of equation $y = 1$. Identify the surface.

Answer

$x^2 + z^2 + 4y = 0$, elliptic paraboloid

48. Find the equation of the quadric surface with points $P(x, y, z)$ that are equidistant from point $Q(0, 2, 0)$ and plane of equation $y = -2$. Identify the surface.
49. If the surface of a parabolic reflector is described by equation $400z = x^2 + y^2$, find the focal point of the reflector.

Answer

$(0, 0, 100)$

50. Consider the parabolic reflector described by equation $z = 20x^2 + 20y^2$. Find its focal point.
51. Show that quadric surface $x^2 + y^2 + z^2 + 2xy + 2xz + 2yz + x + y + z = 0$ reduces to two parallel planes.
52. Show that quadric surface $x^2 + y^2 + z^2 - 2xy - 2xz + 2yz - 1 = 0$ reduces to two parallel planes passing.
53. [T] The intersection between cylinder $(x-1)^2 + y^2 = 1$ and sphere $x^2 + y^2 + z^2 = 4$ is called a *Viviani curve*.

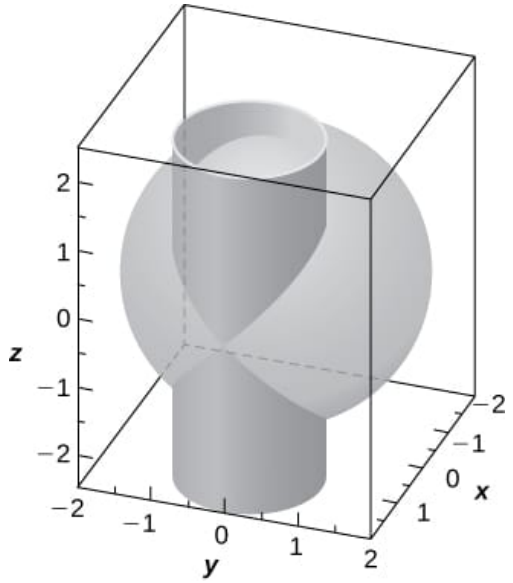


Figure $4.6E.13$: The intersection between cylinder $(x-1)^2 + y^2 = 1$ and sphere $x^2 + y^2 + z^2 = 4$ inside the box with x ∈ [-2,2], y ∈ [-2,2], and z ∈ [-2,2].

a. Solve the system consisting of the equations of the surfaces to find the equation of the intersection curve. (Hint: Find x and y in terms of z.)
b. Use a computer algebra system (CAS) or CalcPlot3D to visualize the intersection curve on sphere $x^2 + y^2 + z^2 = 4$.

Answer

a. $x = 2 - \dfrac{z^2}{2}, \quad y = \pm\dfrac{z}{2}\sqrt{4 - z^2}$, where $z \in [-2, 2]$;

b.

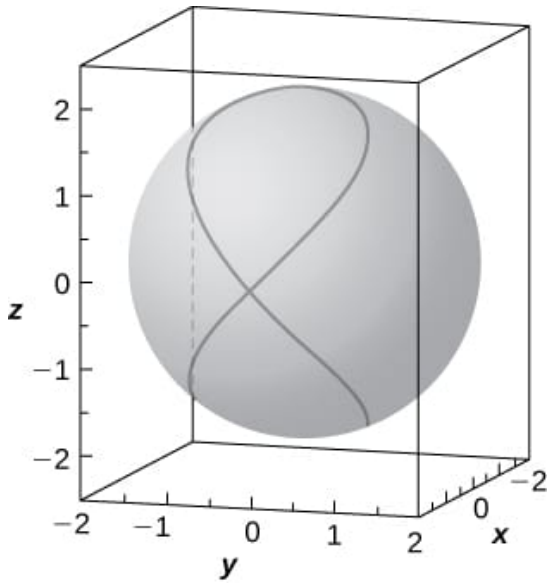


Figure $4.6E.14$: The intersection between cylinder $(x-1)^2 + y^2 = 1$ and sphere $x^2 + y^2 + z^2 = 4$ inside the box with x ∈ [-2,2], y ∈ [-2,2], and z ∈ [-2,2]. The curve of intersection looks like the number 8 written on the surface of a sphere centered at origin with radius of 2.

54. Hyperboloid of one sheet $25x^2 + 25y^2 - z^2 = 25$ and elliptic cone $-25x^2 + 75y^2 + z^2 = 0$ are represented in Figure $4.6E.15$

along with their intersection curves. Identify the intersection curves and find their equations (Hint: Find y from the system consisting of the equations of the surfaces.)

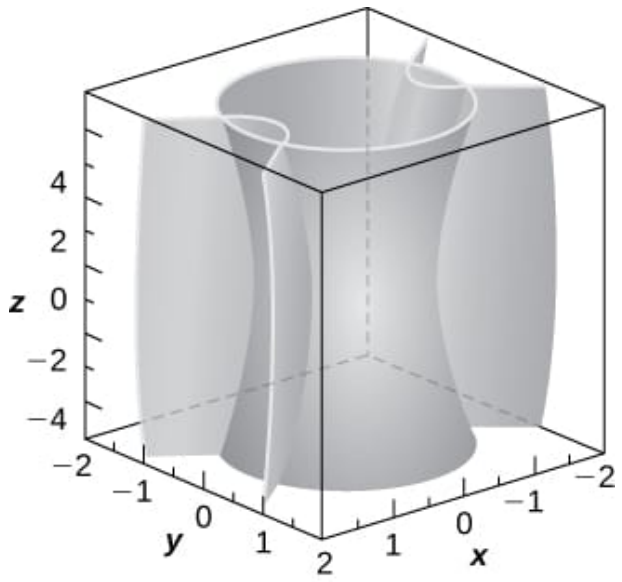


Figure $4.6E.15$: The intersection of the hyperboloid of one sheet $25x^2 + 25y^2 - z^2 = 25$ and elliptic cone $-25x^2 + 75y^2 + z^2 = 0$ inside the box with x ∈ [-2,2], y ∈ [-2,2], and z ∈ [-4,4].

55. [T] Use a CAS or CalcPlot3D to create the intersection between cylinder $9x^2 + 4y^2 = 18$ and ellipsoid $36x^2 + 16y^2 + 9z^2 = 144$, and find the equations of the intersection curves.

Answer

two ellipses of equations $\frac{x^2}{2} + \frac{y^2}{\frac{9}{2}} = 1$ in planes $z = \pm 2\sqrt{2}$

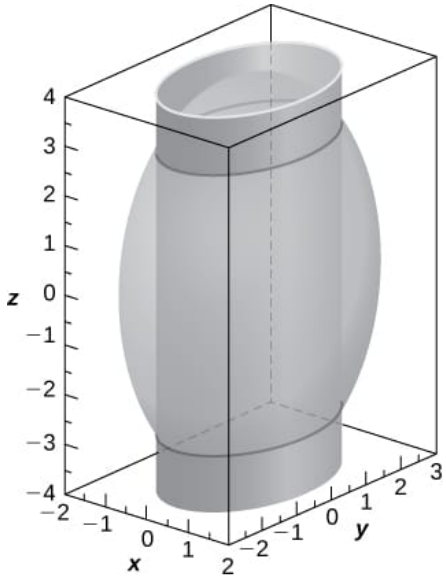


Figure $4.6E.16$: the intersection between cylinder $9x^2 + 4y^2 = 18$ and ellipsoid $36x^2 + 16y^2 + 9z^2 = 144$ inside the box with x ∈ [-3,3], y ∈ [-2,2], and z ∈ [-4,4].

56. [T] A spheroid is an ellipsoid with two equal semiaxes. For instance, the equation of a spheroid with the z-axis as its axis of symmetry is given by $\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = 1$, where a and c are positive real numbers. The spheroid is called oblate if \(ca\).

a. The eye cornea is approximated as a prolate spheroid with an axis that is the eye, where $a = 8.7mm$ and $c = 9.6mm$.Write the equation of the spheroid that models the cornea and sketch the surface.
b. Give two examples of objects with prolate spheroid shapes.

57. [T] In cartography, Earth is approximated by an oblate spheroid rather than a sphere. The radii at the equator and poles are approximately 3963mi and 3950mi, respectively.
a. Write the equation in standard form of the ellipsoid that represents the shape of Earth. Assume the center of Earth is at the origin and that the trace formed by plane $z = 0$ corresponds to the equator.
b. Sketch the graph.
c. Find the equation of the intersection curve of the surface with plane $z = 1000$ that is parallel to the xy-plane. The intersection curve is called a *parallel*.
d. Find the equation of the intersection curve of the surface with plane $x + y = 0$ that passes through the z-axis. The intersection curve is called a meridian.

Answer

a. $\frac{x^2}{3963^2} + \frac{y^2}{3963^2} + \frac{z^2}{3950^2} = 1$

b.

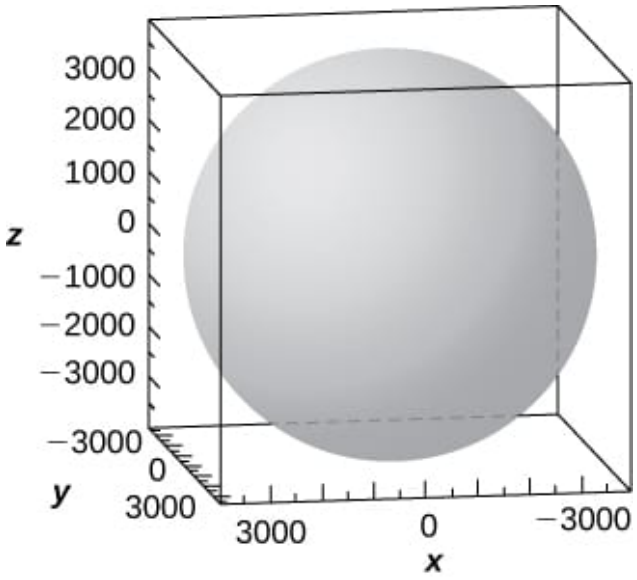


Figure $4.6E.17$: An sphere tilted a little bit to the x-axis inside the box with x ∈ [-3000,3000], y ∈ [-3000,3000], and z ∈ [-3000,3000].

c. The intersection curve is the ellipse of equation $\frac{x^2}{3963^2} + \frac{y^2}{3963^2} = \frac{(2950)(4950)}{3950^2}$, and the intersection is an ellipse.
d. The intersection curve is the ellipse of equation $\frac{2y^2}{3963^2} + \frac{z^2}{3950^2} = 1.$

58. [T] A set of buzzing stunt magnets (or “rattlesnake eggs”) includes two sparkling, polished, superstrong spheroid-shaped magnets well-known for children’s entertainment. Each magnet is 1.625 in. long and 0.5 in. wide at the middle. While tossing them into the air, they create a buzzing sound as they attract each other.
a. Write the equation of the prolate spheroid centered at the origin that describes the shape of one of the magnets.
b. Write the equations of the prolate spheroids that model the shape of the buzzing stunt magnets. Use a CAS or CalcPlot3D to create the graphs.

59. [T] A heart-shaped surface is given by equation $(x^2 + \frac{9}{4}y^2 + z^2 - 1)^3 - x^2z^3 - \frac{9}{80}y^2z^3 = 0.$
a. Use a CAS or CalcPlot3D to graph the surface that models this shape.
b. Determine and sketch the trace of the heart-shaped surface on the xz-plane.

Answer

a.

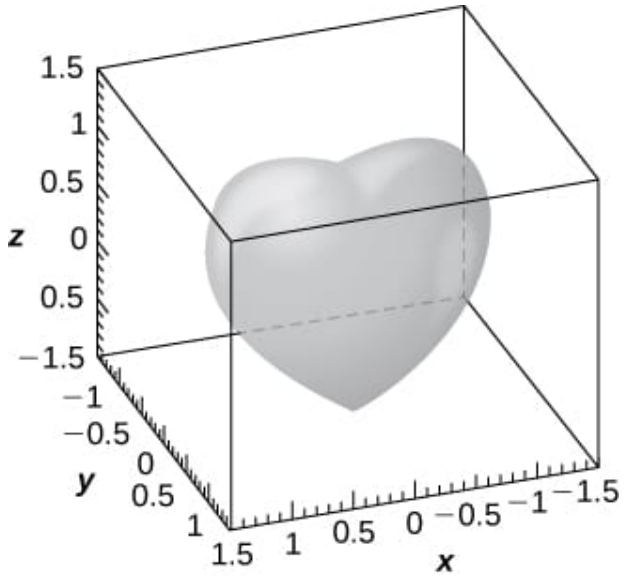


Figure $4.6E.18$: A heart-shaped surface is given by equation $(x^2+\frac{9}{4}y^2+z^2-1)^3-x^2z^3-\frac{9}{80}y^2z^3=0$. inside the box with x ∈ [-1.5,1.5], y ∈ [-1.5,1.5], and z ∈ [-1.5,1.5].

b. The intersection curve is $(x^2+z^2-1)^3-x^2z^3=0$.

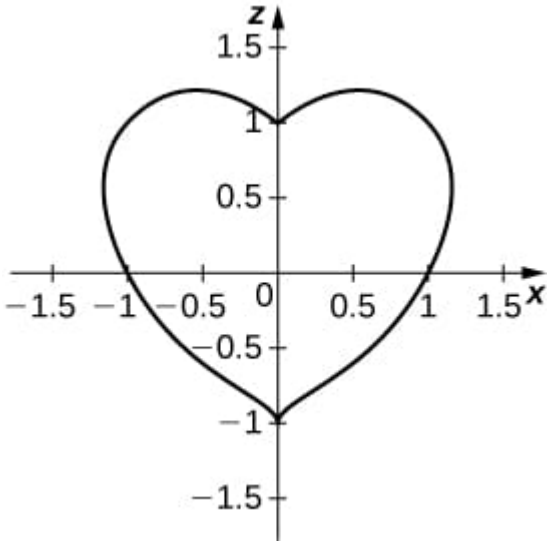


Figure $4.6E.19$: The intersection curve $(x^2+z^2-1)^3-x^2z^3=0$ on the xy-plane. It looks like the shape of a heart centered at origin.

30. [T] The ring torus symmetric about the z-axis is a special type of surface in topology and its equation is given by $(x^2+y^2+z^2+R^2-r^2)^2=4R^2(x^2+y^2)$, where $R>r>0$. The numbers R and r are called are the major and minor radii, respectively, of the surface. Figure $4.6E.20$ shows a ring torus for which $R=2$ and $r=1$.

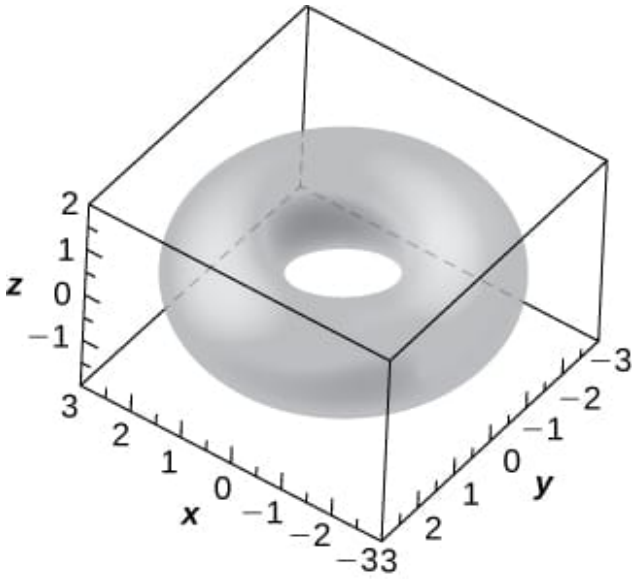


Figure $4.6E.20$: A torus (which looks like a donut) inside the box with x ∈ [-3,3], y ∈ [-3,3], and z ∈ [-2,2].

a. Write the equation of the ring torus with $R = 2$ and $r = 1$, and use a CAS or CalcPlot3D to graph the surface. Compare the graph with the figure given.
b. Determine the equation and sketch the trace of the ring torus from a. on the xy-plane.
c. Give two examples of objects with ring torus shapes.

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4.7: Chapter 4 Review Exercises

Exercises 1 - 4

Determine whether the statement is *true* or *false*. Justify the answer with a proof or a counterexample.

1. For vectors $\mathbf{a}$ and $\mathbf{b}$ and any given scalar c, $c(\mathbf{a} \cdot \mathbf{b}) = (c\mathbf{a}) \cdot \mathbf{b}$.

   Answer

   True; See proof in Section 11.3

2. For vectors $\mathbf{a}$ and $\mathbf{b}$ and any given scalar c, $c(\mathbf{a} \times \mathbf{b}) = (c\mathbf{a}) \times \mathbf{b}$.
3. The symmetric equation for the line of intersection between two planes $x + y + z = 2$ and $x + 2y - 4z = 5$ is given by $-\frac{x-1}{6} = \frac{y-1}{5} = z$.

   Answer

   False, Converting the symmetric equations above to the parametric equations of the line, we get:
   $x = 1 - 6t$
   $y = 1 + 5t$
   $z = t$

   If this line lies on each plane, we should get an identity (like 5 = 5) when we substitute each expression of t in the each plane's equation.

   Substituting into the first plane equation, we get: $(1 - 6t) + (1 + 5t) + t = 2$ ✓

   Thus we know that this line does lie on the first plane.

   But when we substitute into the second plane equation, we obtain: $(1 - 6t) + 2(1 + 5t) - 4(t) = 1 - 6t + 2 + 10t - 4t = 3 \neq 5$

   Since we do not get an identity, we know this line is **not** on the second plane and can therefore not be the line of intersection of the two planes.

4. If $\mathbf{a} \cdot \mathbf{b} = 0$, then $\mathbf{a}$ is perpendicular to $\mathbf{b}$.

   Answer

   False, since $\mathbf{a}$ or $\mathbf{b}$ could also be the zero vector.

Exercises 5-6

Use the given vectors to find the quantities.

5. $\mathbf{a} = 9\mathbf{i} - 2\mathbf{j}, \quad \mathbf{b} = -3\mathbf{i} + \mathbf{k}$
   a. $3\mathbf{a} + \mathbf{b}$
   b. $\|\mathbf{a}\|$
   c. $\mathbf{a} \times \|\mathbf{b} \times \mathbf{a}\|$
   d. $\|\mathbf{b} \times \mathbf{a}\|$

   Answer

   a. $\langle 24, -6, 1 \rangle$
   b. $\sqrt{85}$
   c. Can't cross a vector with a scalar
   d. 11

6. $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - 9\mathbf{k}, \quad \mathbf{b} = -\mathbf{i} + 2\mathbf{k}, \quad \mathbf{c} = 4\mathbf{i} - 2\mathbf{j} + \mathbf{k}$
   a. $2\mathbf{a} - \mathbf{b}$
   b. $\|\mathbf{b} \times \mathbf{c}\|$
   c. $\mathbf{b} \times (\mathbf{b} \times \mathbf{c})$
   d. $\mathbf{c} \times \|\mathbf{b} \times \mathbf{a}\|$
   e. $\text{Proj}_{\mathbf{a}}\mathbf{b}$
7. Find the values of a such that vectors $\langle 2, 4, a\rangle$ and $\langle 0, -1, a\rangle$ are orthogonal.

Answer

$a = \pm 2$

Exercises 8-9

Find the unit vectors.

8. Find the unit vector that has the same direction as vector $\mathbf{v}$ that begins at $(0, -3)$ and ends at $(4, 10)$.
9. Find the unit vector that has the same direction as vector $\mathbf{v}$ that begins at $(1, 4, 10)$ and ends at $(3, 0, 4)$.

Answer

$\langle \frac{1}{\sqrt{14}}, -\frac{2}{\sqrt{14}}, -\frac{3}{\sqrt{14}}\rangle = \langle \frac{\sqrt{14}}{14}, -\frac{\sqrt{14}}{7}, -\frac{3\sqrt{14}}{14}\rangle$

Exercises 10-11

Find the area or volume of the given shapes.

10. The parallelogram spanned by vectors $\mathbf{a} = \langle 1, 13\rangle$ and $\mathbf{b} = \langle 3, 21\rangle$
11. The parallelepiped formed by $\mathbf{a} = \langle 1, 4, 1\rangle$ and $\mathbf{b} = \langle 3, 6, 2\rangle$, and $\mathbf{c} = \langle -2, 1, -5\rangle$

Answer

27 units^2

Exercises 12-13

Find the parametric equations and the vector equation of the line with the given properties.

12. The line that passes through point $(2, -3, 7)$ that is parallel to vector $\langle 1, 3, -2\rangle$
13. The line that passes through points $(1, 3, 5)$ and $(-2, 6, -3)$

Answer

$x = 1 - 3t, y = 3 + 3t, z = 5 - 8t, \quad \mathbf{r}(t) = (1 - 3t)\mathbf{i} + 3(1 + t)\mathbf{j} + (5 - 8t)\mathbf{k}$

Exercises 14-15

Find the equation of the plane with the given properties.

14. The plane that passes through point $(4, 7, -1)$ and has normal vector $\mathbf{n} = \langle 3, 4, 2\rangle$
15. The plane that passes through points $(0, 1, 5), (2, -1, 6)$, and $(3, 2, 5)$.

Answer

$-x + 3y + 8z = 43$

Exercises 16-17

Find the traces for the surfaces in planes $x = k$, $y = k$, and $z = k$. Then, describe and draw the surfaces.

16. $9x^2 + 4y^2 - 16y + 36z^2 = 20$
17. $x^2 = y^2 + z^2$

Answer

$x = k$ trace: $k^2 = y^2 + z^2$ is a circle, $y = k$ trace: $x^2 - z^2 = k^2$ is a hyperbola (or a pair of lines if $k = 0$), $z = k$ trace: $x^2 - y^2 = k^2$ is a hyperbola (or a pair of lines if $k = 0$). The surface is a cone.

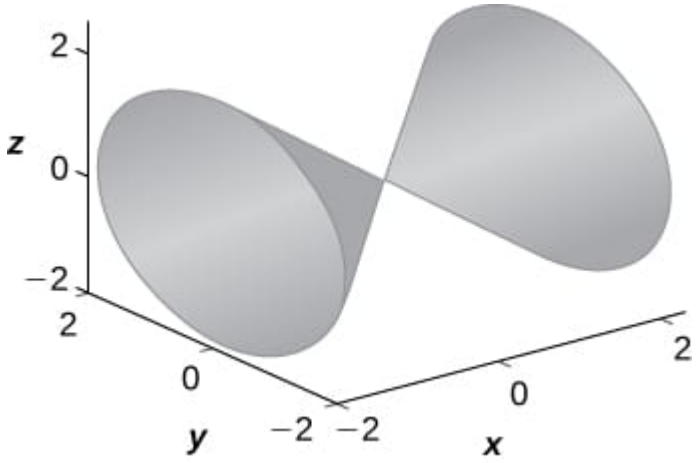


Figure 4.7.1: A cone of two sheets with x-axis as the axis of symmetry in 3d-space.

Exercises 18-19

Write the given equation in cylindrical coordinates and spherical coordinates.

18. $x^2 + y^2 + z^2 = 144$
19. $z = x^2 + y^2 - 1$

Answer

Cylindrical: $z = r^2 - 1$, spherical: $\cos\varphi = \rho\sin^2\varphi - \frac{1}{\rho}$

Exercises 20-21

Convert the given equations from cylindrical or spherical coordinates to rectangular coordinates. Identify the given surface.

20. $\rho^2(\sin^2(\varphi) - \cos^2(\varphi)) = 1$
21. $r^2 - 2r\cos(\theta) + z^2 = 1$

Answer

$x^2 - 2x + y^2 + z^2 = 1$, sphere

Exercises 22-23

Consider a small boat crossing a river.

22. If the boat velocity is 5 km/h due north in still water and the water has a current of 2 km/h due west (see Figure $4.7.2$), what is the velocity of the boat relative to shore? What is the angle θ that the boat is actually traveling?

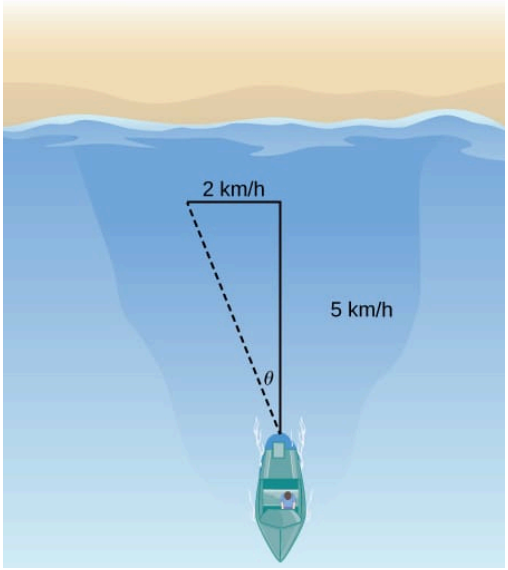


Figure 4.7.2: A boat traveling on water. A horizontal component of 2 km/h is the current velocity. A vertical component of 5 km/h is the boat velocity. The angle θ between the horizontal and vertical lines is marked.

23. When the boat reaches the shore, two ropes are thrown to people to help pull the boat ashore. One rope is at an angle of $25\,^\circ$ and the other is at $35\,^\circ$. If the boat must be pulled straight and at a force of 500 N, find the magnitude of force for each rope (see Figure $4.7.3$).

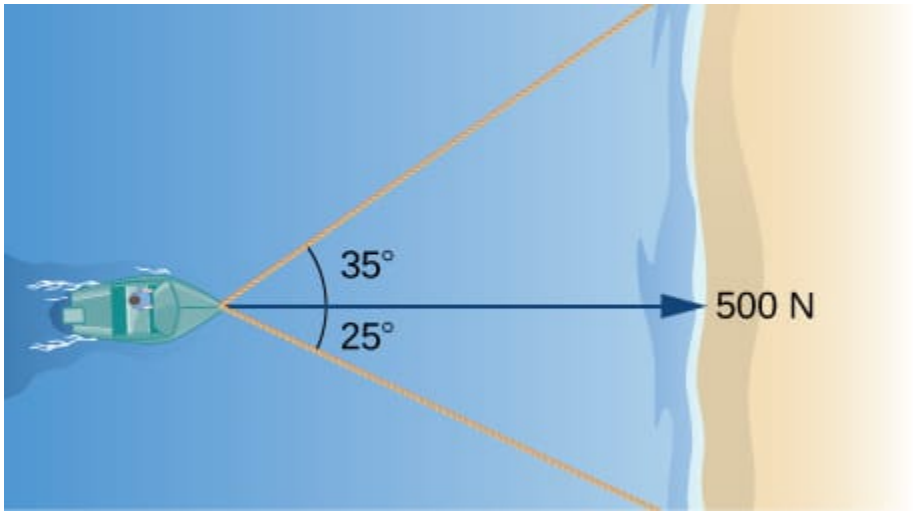


Figure 4.7.3: Two ropes are thrown to help pull the boat ashore. One rope is at an angle of $25\,^\circ$ and the other is at $35\,^\circ$. The boat is straight to the shore and it is pulled with a combined force of 500 N.

Answer

331 N and 244 N

24. An airplane is flying in the direction of 52° east of north with a speed of 450 mph. A strong wind has a bearing 33° east of north with a speed of 50 mph. What is the resultant ground speed and bearing of the airplane?
25. Calculate the work done by moving a particle from position $(1, 2, 0)$ to $(8, 4, 5)$ along a straight line with a force $\mathbf{F} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$.

Answer

15 J

Exercises 26-27

Consider your unsuccessful attempt to take the tire off your car using a wrench to loosen the bolts. Assume the wrench is 0.3 m long and you are able to apply a 200-N force.

26. Because your tire is flat, you are only able to apply your force at a $60\,^{\circ}$ angle. What is the torque at the center of the bolt? Assume this force is not enough to loosen the bolt.
27. Someone lends you a tire jack and you are now able to apply a 200-N force at an $80\,^{\circ}$ angle. Is your resulting torque going to be more or less? What is the new resulting torque at the center of the bolt? Assume this force is not enough to loosen the bolt.

Answer

More, 59.09 J

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5: VECTOR-VALUED FUNCTIONS

A vector-valued function, also referred to as a vector function, is a mathematical function of one or more variables whose range is a set of multidimensional vectors or infinite-dimensional vectors. The input of a vector-valued function could be a scalar or a vector. Vector-valued functions provide a useful method for studying various curves both in the plane and in three-dimensional space. We can apply this concept to calculate the velocity, acceleration, arc length, and curvature of an object's trajectory. In this chapter, we examine these methods and show how they are used.

In 1705, using Sir Isaac Newton's new laws of motion, the astronomer Edmond Halley made a prediction. He stated that comets that had appeared in 1531, 1607, and 1682 were actually the same comet and that it would reappear in 1758. Halley was proved to be correct, although he did not live to see it. However, the comet was later named in his honor. Halley's Comet follows an elliptical path through the solar system, with the Sun appearing at one focus of the ellipse. This motion is predicted by Johannes Kepler's first law of planetary motion, which we mentioned briefly previously. Kepler's third law of planetary motion can be used with the calculus of vector-valued functions to find the average distance of Halley's Comet from the Sun.



Figure 5.1: Halley's Comet appeared in view of Earth in 1986 and will appear again in 2061.(CC-BY-2.0, Wikimedia Commons)

Vector-valued functions provide a useful method for studying various curves both in the plane and in three-dimensional space. We can apply this concept to calculate the velocity, acceleration, arc length, and curvature of an object's trajectory. In this chapter, we examine these methods and show how they are used.

5.1: Vector-Valued Functions and Space Curves

5.1E: Exercises for Section 5.1

5.2: Calculus of Vector-Valued Functions

5.2E: Exercises for Section 5.2

5.3: Arc Length and Curvature

5.3E: Exercises for Section 5.3

5.4: Motion in Space

5.4E: Exercises for Section 5.4

5.5: Chapter 5 Review Exercises

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5.1: Vector-Valued Functions and Space Curves

Learning Objectives

- Write the general equation of a vector-valued function in component form and unit-vector form.
- Recognize parametric equations for a space curve.
- Describe the shape of a helix and write its equation.
- Define the limit of a vector-valued function.

Our study of vector-valued functions combines ideas from our earlier examination of single-variable calculus with our description of vectors in three dimensions from the preceding chapter. In this section, we extend concepts from earlier chapters and also examine new ideas concerning curves in three-dimensional space. These definitions and theorems support the presentation of material in the rest of this chapter and also in the remaining chapters of the text.

Definition of a Vector-Valued Function

Our first step in studying the calculus of vector-valued functions is to define what exactly a vector-valued function is. We can then look at graphs of vector-valued functions and see how they define curves in both two and three dimensions.

Definition: Vector-Valued Functions

A **vector-valued function** is a function of the form

$$\mathbf{r(t)} = f(t)\mathbf{i} + g(t)\mathbf{j} \text{ or } \mathbf{r}(t) = f(t)\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}, \tag{5.1.1}$$

where the component functions f, g, and h, are real-valued functions of the parameter t. Vector-valued functions are also written in the form

$$\mathbf{r}(t) = \langle f(t),\, g(t)\rangle \text{ or } \mathbf{r}(t) = \langle f(t),\, g(t),\, h(t)\rangle. \tag{5.1.2}$$

In both cases, the first form of the function defines a two-dimensional vector-valued function; the second form describes a three-dimensional vector-valued function.

The parameter t can lie between two real numbers: $a \le t \le b$. Another possibility is that the value of t might take on all real numbers. Last, the component functions themselves may have domain restrictions that enforce restrictions on the value of t. We often use t as a parameter because t can represent time.

Example 5.1.1: Finding the domain of a vector-valued function

State the domain of the vector-valued function $\mathbf{r}(t) = \sqrt{2-t}\ \mathbf{i} + \ln(t+3)\ \mathbf{j} + e^t\ \mathbf{k}$.

Solution

We first consider the natural domain of each component function. Note that we list the domains in both **set-builder notation** and **interval notation**.

Function:	Domain:
$\sqrt{2-t}$	$\{t \mid t \le 2\}$ or $(-\infty, 2]$
$\ln(t+3)$	$\{t \mid t > -3\}$ or $(-3, \infty)$
e^t	$(-\infty, \infty)$

The domain of $\mathbf{r}$ is the intersection of these domains, so it must contain all values of t that work in all three, but no value of t that does not work in any one of these functions.

Hence, the domain of $\mathbf{r}$ is: $\mathrm{D_r} : \{\ t\ |\ -3 < t \leq 2\}$ or $(-3, 2]$.

Note that only one form of the domain of $\mathbf{r}$ need be given. The first, $\{\ t\ |\ -3 < t \leq 2\}$, is in set-builder notation, while the second, $(-3, 2]$, is in interval notation.

Example 5.1.2: Evaluating Vector-Valued Functions and Determining Domains

For each of the following vector-valued functions, evaluate $\mathbf{r}(0)$, $\mathbf{r}(\frac{\pi}{2})$, and $\mathbf{r}(\frac{2\pi}{3})$. Do any of these functions have domain restrictions?

1. $\mathbf{r}(t) = 4\cos t\ \mathbf{i} + 3\sin t\ \mathbf{j}$
2. $\mathbf{r}(t) = 3\tan t\ \mathbf{i} + 4\sec t\ \mathbf{j} + 5t\ \mathbf{k}$

Solution

1. To calculate each of the function values, substitute the appropriate value of t into the function:

$$\begin{aligned}
\mathbf{r}(0) &= 4\cos(0)\mathbf{i} + 3\sin(0)\mathbf{j} \\
&= 4\mathbf{i} + 0\mathbf{j} = 4\mathbf{i} \\
\mathbf{r}\left(\frac{\pi}{2}\right) &= 4\cos\left(\frac{\pi}{2}\right)\mathbf{i} + 3\sin\left(\frac{\pi}{2}\right)\mathbf{j} \\
&= 0\mathbf{i} + 3\mathbf{j} = 3\mathbf{j} \\
\mathbf{r}\left(\frac{2\pi}{3}\right) &= 4\cos\left(\frac{2\pi}{3}\right)\mathbf{i} + 3\sin\left(\frac{2\pi}{3}\right)\mathbf{j} \\
&= 4\left(-\tfrac{1}{2}\right)\mathbf{i} + 3\left(\tfrac{\sqrt{3}}{2}\right)\mathbf{j} = -2\mathbf{i} + \tfrac{3\sqrt{3}}{2}\mathbf{j}
\end{aligned}$$

   To determine whether this function has any domain restrictions, consider the component functions separately. The first component function is $f(t) = 4\cos t$ and the second component function is $g(t) = 3\sin t$. Neither of these functions has a domain restriction, so the domain of $\mathbf{r}(t) = 4\cos t\ \mathbf{i} + 3\sin t\ \mathbf{j}$ is all real numbers.
2. To calculate each of the function values, substitute the appropriate value of *t* into the function:

$$\begin{aligned}
\mathbf{r}(0) &= 3\tan(0)\mathbf{i} + 4\sec(0)\mathbf{j} + 5(0)\mathbf{k} \\
&= 0\mathbf{i} + 4j + 0\mathbf{k} = 4\mathbf{j} \\
\mathbf{r}\left(\frac{\pi}{2}\right) &= 3\tan\left(\frac{\pi}{2}\right)\mathbf{i} + 4\sec\left(\frac{\pi}{2}\right)\mathbf{j} + 5\left(\frac{\pi}{2}\right)\mathbf{k}, \text{ which does not exist} \\
\mathbf{r}\left(\frac{2\pi}{3}\right) &= 3\tan\left(\frac{2\pi}{3}\right)\mathbf{i} + 4\sec\left(\frac{2\pi}{3}\right)\mathbf{j} + 5\left(\frac{2\pi}{3}\right)\mathbf{k} \\
&= 3(-\sqrt{3})\mathbf{i} + 4(-2)\mathbf{j} + \frac{10\pi}{3}\mathbf{k} \\
&= (-3\sqrt{3})\mathbf{i} - 8\mathbf{j} + \frac{10\pi}{3}\mathbf{k}
\end{aligned}$$

   To determine whether this function has any domain restrictions, consider the component functions separately. The first component function is $f(t) = 3\tan t$, the second component function is $g(t) = 4\sec t$, and the third component function is $h(t) = 5t$. The first two functions are not defined for odd multiples of $\frac{\pi}{2}$, so the function is not defined for odd multiples of $\frac{\pi}{2}$. Therefore,

$$\mathrm{D_r} = \left\{t \,|\, t \neq \frac{(2n+1)\pi}{2}\right\},$$

   where n is any integer.

Watch the accompanying video for a visual demonstration of this concept.

- **Video Length**: 1 minute 52 seconds.
- **Context**: This video demonstrates how to find the domain of a vector-valued function for the function $\mathbf{r}(t) = \frac{1}{t-3}\mathbf{i} - t^2\mathbf{j} + \sqrt{t+1}\mathbf{k}$

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Exercise 5.1.1

For the vector-valued function $\mathbf{r}(t) = (t^2 - 3t)\ \mathbf{i} + (4t + 1)\ \mathbf{j}$, evaluate $\mathbf{r}(0),\ \mathbf{r}(1)$, and $\mathbf{r}(-4)$. Does this function have any domain restrictions?

Hint available at the end of this section.

Example 5.1.1 illustrates an important concept. The domain of a vector-valued function consists of real numbers. The domain can be all real numbers or a subset of the real numbers. The range of a vector-valued function consists of vectors. Each real number in the domain of a vector-valued function is mapped to either a two- or a three-dimensional vector.

Graphing Vector-Valued Functions

Recall that a plane vector consists of two quantities: direction and magnitude. Given any point in the plane (the *initial point*), if we move in a specific direction for a specific distance, we arrive at a second point. This represents the *terminal point* of the vector. We calculate the components of the vector by subtracting the coordinates of the initial point from the coordinates of the terminal point.

A vector is considered to be in *standard position* if the initial point is located at the origin. When graphing a vector-valued function, we typically graph the vectors in the domain of the function in standard position, because doing so guarantees the uniqueness of the graph. This convention applies to the graphs of three-dimensional vector-valued functions as well. The graph of a vector-valued function of the form

$$\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j}$$

consists of the set of all points $(f(t),\ g(t))$, and the path it traces is called a **plane curve**. The graph of a vector-valued function of the form

$$\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$$

consists of the set of all points $(f(t),\ g(t),\ h(t))$, and the path it traces is called a **space curve**. Any representation of a plane curve or space curve using a vector-valued function is called a **vector parameterization** of the curve.

Each plane curve and space curve has an **orientation**, indicated by arrows drawn in on the curve, that shows the direction of motion along the curve as the value of the parameter t increases.

Example 5.1.3 : Graphing a Vector-Valued Function

Create a graph of each of the following vector-valued functions:

a. The plane curve represented by $\mathbf{r}(t) = 4\cos t\ \mathbf{i} + 3\sin t\ \mathbf{j}$, $0 \le t \le 2\pi$
b. The plane curve represented by $\mathbf{r}(t) = 4\cos(t^3)\ \mathbf{i} + 3\sin(t^3)\ \mathbf{j}$, $0 \le t \le \sqrt[3]{2\pi}$
c. The space curve represented by $\mathbf{r}(t) = 4\cos t\ \mathbf{i} + 4\sin t\ \mathbf{j} + t\ \mathbf{k}$, $0 \le t \le 4\pi$

Solution

a. As with any graph, we start with a table of values. We then graph each of the vectors in the second column of the table in standard position and connect the terminal points of each vector to form a curve (Figure 5.1.1). This curve turns out to be an ellipse centered at the origin.

Table 5.1.1: Table of Values for $\mathbf{r}(t) = 4\cos t\ \mathbf{i} + 3\sin t\ \mathbf{j},\ 0 \le t \le 2\pi$

t	$\mathbf{r}(t)$	t	$\mathbf{r}(t)$
0	$4\mathbf{i}$	π	$-4\mathbf{i}$
$\frac{\pi}{4}$	$2\sqrt{2}\mathbf{i} + \frac{3\sqrt{2}}{2}\mathbf{j}$	$\frac{5\pi}{4}$	$-2\sqrt{2}\mathbf{i} - \frac{3\sqrt{2}}{2}\mathbf{j}$
$\frac{\pi}{2}$	$3\mathbf{j}$	$\frac{3\pi}{2}$	$-3\mathbf{j}$
$\frac{3\pi}{4}$	$-2\sqrt{2}\mathbf{i} + \frac{3\sqrt{2}}{2}\mathbf{j}$	$\frac{7\pi}{4}$	$2\sqrt{2}\mathbf{i} - \frac{3\sqrt{2}}{2}\mathbf{j}$
2π	$4\mathbf{i}$		

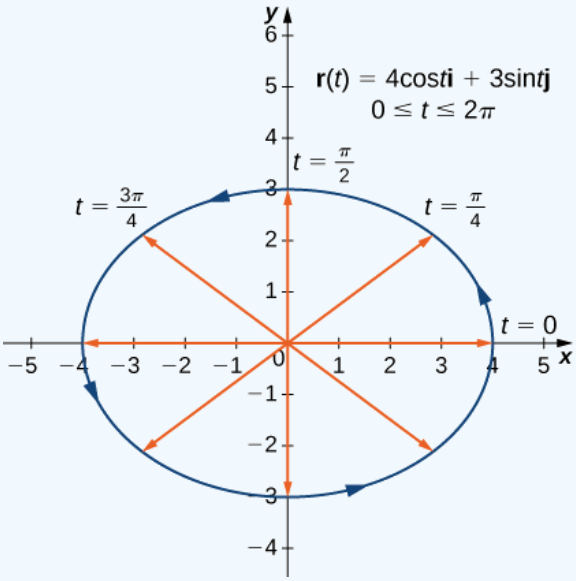


Figure 5.1.1: This is the graph of an ellipse centered at the origin. The graph is the vector-valued function $\mathbf{r}(t) = 4\cos t\mathbf{i} + 3\sin t\mathbf{j}$ over $0 \le t \le 2\pi$. The x values range over $[-4, 4]$ and the y values range over $[-3, 3]$. The ellipse has arrows on the curve representing counter-clockwise orientation. There are also line segments inside of the ellipse to the curve at different increments of t. The increments are $t = 0, t = \pi/4, t = \pi/2, t = 3\pi/4$.

b. The table of values for $\mathbf{r}(t) = 4\cos(t^3)\ \mathbf{i} + 3\sin(t^3)\ \mathbf{j},\ 0 \le t \le \sqrt[3]{2\pi}$ is as follows:

Table 5.1.2: Table of Values for $\mathbf{r}(t) = 4\cos(t^3)\ \mathbf{i} + 3\sin(t^3)\ \mathbf{j},\ 0 \le t \le \sqrt[3]{2\pi}$

t	$\mathbf{r}(t)$	t	$\mathbf{r}(t)$
0	$4\mathbf{i}$	$\sqrt[3]{\pi}$	$-4\mathbf{i}$
$\sqrt[3]{\frac{\pi}{4}}$	$2\sqrt{2}\mathbf{i} + \frac{3\sqrt{2}}{2}\mathbf{j}$	$\sqrt[3]{\frac{5\pi}{4}}$	$-2\sqrt{2}\mathbf{i} - \frac{3\sqrt{2}}{2}\mathbf{j}$
$\sqrt[3]{\frac{\pi}{2}}$	$3\mathbf{j}$	$\sqrt[3]{\frac{3\pi}{2}}$	$-3\mathbf{j}$
$\sqrt[3]{\frac{3\pi}{4}}$	$-2\sqrt{2}\mathbf{i} + \frac{3\sqrt{2}}{2}\mathbf{j}$	$\sqrt[3]{\frac{7\pi}{4}}$	$2\sqrt{2}\mathbf{i} - \frac{3\sqrt{2}}{2}\mathbf{j}$

t	$\mathbf{r}(t)$	t	$\mathbf{r}(t)$
$\sqrt[3]{2\pi}$	$4\mathbf{i}$		

The graph of this curve is also an ellipse centered at the origin.

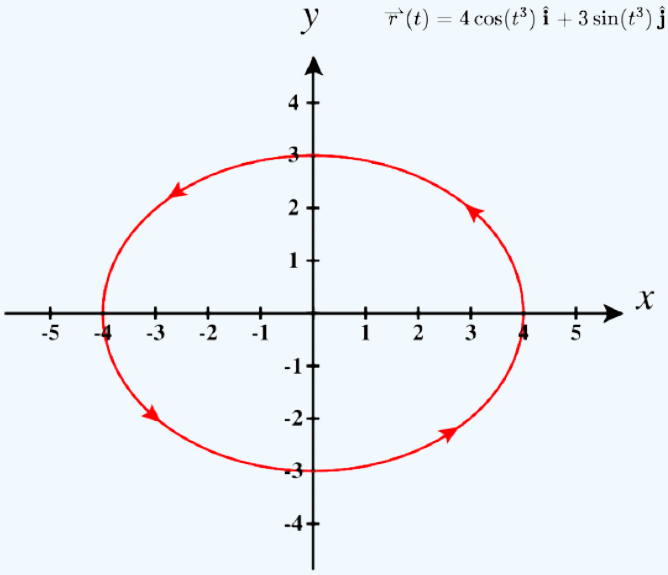


Figure 5.1.2: This is the graph of an ellipse centered at the origin. The graph is the vector-valued function $\mathbf{r}(t) = 4\cos(t^3)\mathbf{i} + 3\sin(t^3)\mathbf{j}$ over $0 \le t \le \sqrt[3]{2\pi}$. The x values range over $[-4, 4]$ and the y values range over $[-3, 3]$. The ellipse has arrows on the curve representing counter-clockwise orientation.

c. We go through the same procedure for a three-dimensional vector function.

Table 5.1.3*: Table of Values for* $\mathrm{r(t)} = 4\cos\mathrm{t}\mathbf{i} + 4\sin\mathrm{t}\mathbf{j} + \mathrm{t}\mathbf{k}$, $0 \le \mathrm{t} \le 4\pi$

t	$\mathbf{r}(t)$	t	$\mathbf{r}(t)$
0	$4\mathbf{i}$	π	$-4\mathbf{i} + \pi\mathbf{k}$
$\frac{\pi}{4}$	$2\sqrt{2}\mathbf{i} + 2\sqrt{2}\mathbf{j} + \frac{\pi}{4}\mathbf{k}$	$\frac{5\pi}{4}$	$-2\sqrt{2}\mathbf{i} - 2\sqrt{2}\mathbf{j} + \frac{5\pi}{4}\mathbf{k}$
$\frac{\pi}{2}$	$4\mathbf{j} + \frac{\pi}{2}\mathbf{k}$	$\frac{3\pi}{2}$	$-4\mathbf{j} + \frac{3\pi}{2}\mathbf{k}$
$\frac{3\pi}{4}$	$-2\sqrt{2}\mathbf{i} + 2\sqrt{2}\mathbf{j} + \frac{3\pi}{4}\mathbf{k}$	$\frac{7\pi}{4}$	$2\sqrt{2}\mathbf{i} - 2\sqrt{2}\mathbf{j} + \frac{7\pi}{4}\mathbf{k}$
2π	$4\mathbf{j} + 2\pi\mathbf{k}$		

The values then repeat themselves, except for the fact that the coefficient of $\mathbf{k}$ is always increasing (5.1.3). This curve is called a helix. Notice that if the $\mathbf{k}$ component is eliminated, then the function becomes $\mathbf{r}(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j}$, which is a circle of radius 4 centered at the origin.

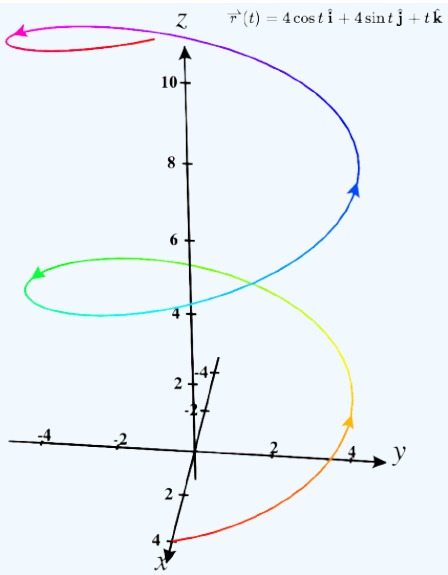


Figure 5.1.3: This is the graph of a helix centered about the z-axis. The graph is the vector-valued function $\mathbf{r}(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j} + t\mathbf{k}$ over $0 \leq t \leq 4\pi$. The x values range over $[-4, 4]$, the y values range over $[-4, 4]$ and the z values range over $[0, 4\pi]$. The helix begins at point $(4, 0, 0)$, wraps around the z-axis at a constant distance of 4 from it, makes 2 full revolutions, and stops at $(4, 0, 4\pi)$. The curve has arrows along it showing the orientation.

Watch the accompanying video for a visual demonstration of this concept.

- **Video Length**: 1 minute 48 seconds
- **Context**: This video demonstrates how to sketch the graph of a helix presented as a vector-valued function: $\mathbf{r}(t) = 3\cos t\mathbf{i} + 3t\mathbf{j} + 3\sin t\mathbf{k}$

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You may notice that the graphs in parts a. and b. are identical. This happens because the function describing curve b is a so-called reparameterization of the function describing curve a. In fact, any curve has an infinite number of reparameterizations; for example, we can replace t with $2t$ in any of the three previous curves without changing the shape of the curve. The interval over which t is defined may change, but that is all. We return to this idea later in this chapter when we study arc-length parameterization. As mentioned, the name of the shape of the curve of the graph in 5.1.3 is a *helix*. The curve resembles a spring, with a circular cross-section looking down along the z-axis. It is possible for a helix to be elliptical in cross-section as well. For example, the vector-valued function $\mathbf{r}(t) = 4\cos t\ \mathbf{i} + 3\sin t\ \mathbf{j} + t\ \mathbf{k}$ describes an elliptical helix. The projection of this helix into the xy-plane is an ellipse. Last, the arrows in the graph of this helix indicate the orientation of the curve as t progresses from 0 to 4π.

Exercise 5.1.2

Create a graph of the vector-valued function $\mathbf{r}(t) = (t^2 - 1)\mathbf{i} + (2t - 3)\mathbf{j}$, $0 \leq t \leq 3$.

Hint available at the end of this section.

Answer available at the end of this section.

The video example below compares the concepts of *intersection* versus *collision* of the paths two vector functions. Two vector functions' graphs intersect if the corresponding graphs share a common point. However, they collide if there exists a shared point in the intersection of the two graphs that each graph attains at the same value t. In other words, if we think of t as time, and the

position of two different objects given by the vector functions, then the two objects collide if they are ever in the same position at the *same* time.

Example 5.1.4

Two particles travel along the curves $\mathbf{r}(t)$ and $\mathbf{w}(t)$. Determine if their paths intersect and if they collide.

$\mathbf{r}(t) = 2t\mathbf{i} + (t-1)\mathbf{j} + (t^2+1)\mathbf{k}$

$\mathbf{w}(t) = (t^2+2)\mathbf{i} + (t^2-2)\mathbf{j} + (t^2+6)\mathbf{k}$

Solution

- **Video length**: 4 minutes 47 seconds
- **Context**: This video demonstrates how to determine if two parametric curves in 3-dimensional space intersect and/or collide. Question: Two particles travel along the curves $\mathbf{r}(t) = 2t\mathbf{i} + (t-1)\mathbf{j} + (t^2+1)\mathbf{k}$ and $\mathbf{w}(t) = (t^2+2)\mathbf{i} + (t^2-2)\mathbf{j} + (t^2+6)\mathbf{k}$. Determine if their paths intersect and if they collide.

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At this point, you may notice a similarity between vector-valued functions and parameterized curves. Indeed, given a vector-valued function $\mathbf{r}(t) = f(t)\ \mathbf{i} + g(t)\ \mathbf{j}$ we can define $x = f(t)$ and $y = g(t)$. If a restriction exists on the values of t (for example, t is restricted to the interval $[a, b]$ for some constants \(a

Finding a Vector-Valued Function to Trace out the Graph of a Function $y = f(x)$

As you can see in the examples above, a vector-valued function traces out a curve in the plane or in space. What if we wish to write a vector-valued function that traces out the graph of a particular curve in the xy-plane?

What function's graph is traced out by the vector-valued function in Exercise $5.1.2$ above: $\mathbf{r}(t) = t\ \mathbf{i} + t^3\ \mathbf{j}$? It looks like the graph of $y = x^3$, doesn't it?

Remembering what was just said about the components of the vector-valued function corresponding to the parametic equations of a parameterized curve, we see that here we have:

$$x = t$$
$$y = t^3$$

Since $x = t$, we can replace t in the equation $y = t^3$ with x, giving us the function: $y = x^3$.

So we were correct in our guess.

How could we write a vector-valued function to trace out the graph of a function, $y = f(x)$?

Well, there are two orientations to consider: **left-to-right** and **right-to-left**.

Note

To trace out the graph of $y = f(x)$ from left-to-right, use: $\mathbf{r}(t) = t\ \mathbf{i} + f(t)\ \mathbf{j}$

Note that what's important here is to have the x component be an increasing function. *Any increasing function will work.* We could use $x = t^3$, for example. But then we would need to remember to replace x in the function $f(x)$ with this expression t^3, giving us $y = f(t^3)$. This means the function $y = f(x)$ could also be parameterized from left-to-right by the vector-valued function: $\mathbf{r}(t) = t^3\ \mathbf{i} + f(t^3)\ \mathbf{j}$

Note

To trace out the graph of $y = f(x)$ from right-to-left, use: $\mathbf{r}(t) = -t\ \mathbf{i} + f(-t)\ \mathbf{j}$

Again note that we could use any decreasing function of t for the x component and obtain a vector-valued function that traces out the graph of $y = f(x)$ from right-to-left. Using $x = -t$ is just the simplest decreasing function we can choose.

Example 5.1.5: Finding a vector-valued function to trace the graph of $y = f(x)$

Determine a vector-valued function that will trace out the graph of $y = \cos x$ from left-to-right, and another one to trace it out from right-to-left.

Solution

Left-to-right: $\mathbf{r}(t) = t\ \mathbf{i} + \cos t\ \mathbf{j}$

Right-to-left: $\mathbf{r}(t) = -t\ \mathbf{i} + \cos(-t)\ \mathbf{j}$

Finding a Vector-Valued Function to Trace out the Graph of an Equation in x and y and Vice Versa

What if we wish to find a vector-valued function to trace out the graph of a circle, an ellipse, or a hyperbola, given its implicit equation?

Well, note that in Example 5.1.3, the vector-valued function $\mathbf{r}(t) = 4\cos t\ \mathbf{i} + 3\sin t\ \mathbf{j}$ traced out the graph of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

In this vector-valued function we see that:

$$x = 4\cos t \quad \text{and} \quad y = 3\sin t \tag{5.1.3}$$

What we need now is a way to convert this to an implicit equation involving x and y. To accomplish this, remember the Pythagorean identity,

$$\cos^2 t + \sin^2 t = 1 \tag{5.1.4}$$

.

Now all we need to do is solve the above equations for $\cos t$ and $\sin t$ and we can substitute into this identity to obtain an equation in x and y.

So:

$$\cos t = \frac{x}{4} \quad \text{and} \quad \sin t = \frac{y}{3} \tag{5.1.5}$$

Substituting into the identity gives us:

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = 1 \tag{5.1.6}$$

Simplifying this implicit equation gives us the implicit equation of the ellipse in Example 5.1.3 that we wrote above:

$$\frac{x^2}{16} + \frac{y^2}{9} = 1 \tag{5.1.7}$$

To go the other way and find a vector-valued function that traces out an ellipse requires us to simply take these steps in the opposite direction!

Example 5.1.6: Writing a vector-valued function for a given circle, ellipse, or hyperbola

Write a vector-valued function that traces out each of the following implicit curves:

a. The ellipse: $\frac{x^2}{16} + \frac{y^2}{9} = 1$
b. The circle: $x^2 + y^2 = 4$
c. The hyperbola: $\frac{x^2}{25} - \frac{y^2}{16} = 1$

Solution

a. Let's just use the process shown above in reverse. First, let's rewrite the implicit equation so it shows a sum of quantities squared equals one.

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$

Now we need the identity we used above, $\cos^2 t + \sin^2 t = 1$.

Equating the parts being squared (note we actually have a choice here about which to make $\cos t$ and which to make $\sin t$), we get:

$$\frac{x}{4} = \cos t \quad \text{and} \quad \frac{y}{3} = \sin t$$

Now we just need to solve for x and y.

$$x = 4\cos t \quad \text{and} \quad y = 3\sin t$$

We can now write a vector-valued function that traces out this ellipse: $\mathbf{r}(t) = 4\cos t\ \mathbf{i} + 3\sin t\ \mathbf{j}$

Note that we could also have written $\mathbf{r}(t) = 4\sin t\ \mathbf{i} + 3\cos t\ \mathbf{j}$, since we could have chosen to switch the $\sin t$ and $\cos t$ above. It will trace out the same ellipse, but with the opposite orientation.

b. Tracing out a circle is fairly straightforward, not really needing the process we showed above, although it still may be helpful at first. Remember that all the vectors on the unit circle can be represented in the form: $\mathbf{v} = \cos\theta\ \mathbf{i} + \sin\theta\ \mathbf{j}$.

So the vector-valued function, $\mathbf{r}(t) = \cos t\ \mathbf{i} + \sin t\ \mathbf{j}$, will trace out the unit circle with equation, $x^2 + y^2 = 1$.

To obtain a circle of radius 2 centered at the origin (which is the graph of $x^2 + y^2 = 4$), we just need to multiply through this vector-valued function by a scalar factor of 2.

Thus, a vector-valued function that will trace out this circle is: $\mathbf{r}(t) = 2\cos t\ \mathbf{i} + 2\sin t\ \mathbf{j}$.

Note again that another possibility is: $\mathbf{r}(t) = 2\sin t\ \mathbf{i} + 2\cos t\ \mathbf{j}$. It will trace out the same circle, but with the opposite orientation.

To use the technique above, you start by dividing each term in the equation by the square of the radius, here 4, thus putting the circle equation in "ellipse form". The rest of the steps follow the pattern shown in part a.

c. To trace out a hyperbola of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ or $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, we need to locate a trigonometric identity showing the difference of two squares equals 1. If you don't already have such an identity memorized, we can obtain one from the Pythagorean identity used above. That is,

$$\cos^2 t + \sin^2 t = 1$$

Dividing through each term by $\cos^2 t$

$$\frac{\cos^2 t}{\cos^2 t} + \frac{\sin^2 t}{\cos^2 t} = \frac{1}{\cos^2 t}$$

yields,

$$1 + \tan^2 t = \sec^2 t$$

Rewriting this equation gives us the identity we need:

$$\sec^2 t - \tan^2 t = 1$$

Now, the equation of this hyperbola is:

$$\frac{x^2}{25} - \frac{y^2}{16} = 1$$

Rewriting the left side to show the quantities that are squared:

$$\left(\frac{x}{5}\right)^2 - \left(\frac{y}{4}\right)^2 = 1$$

We can then equate the squared expressions corresponding terms:

$$\frac{x}{5} = \sec t \quad \text{and} \quad \frac{y}{4} = \tan t$$

Solving for x and y, we have:

$$x = 5\sec t \quad \text{and} \quad y = 4\tan t$$

So a vector-valued function that will trace out the hyperbola

$$\frac{x^2}{25} - \frac{y^2}{16} = 1 \tag{5.1.8}$$

is

$$\mathbf{r}(t) = 5\sec t\,\mathbf{i} + 4\tan t\,\mathbf{j}.$$

Parameterizing a Piecewise Path

There are times when it is necessary to parameterize a path made up of pieces of different curves. This piecewise path may be open or form the boundary of a closed region as does the example shown in Figure 5.1.4. In addition to determining a vector-valued function to trace out each piece separately, with the indicated orientation, we also need to determine a suitable range of values for the parameter t.

Note that there are many ways to parameterize any one piece, so there are many correct ways to parameterize a path in this way.

Example 5.1.7: Parameterizing a piecewise path

Determine a piecewise parameterization of the path shown in Figure 5.1.4, starting with $t = 0$ and continuing on through each piece.

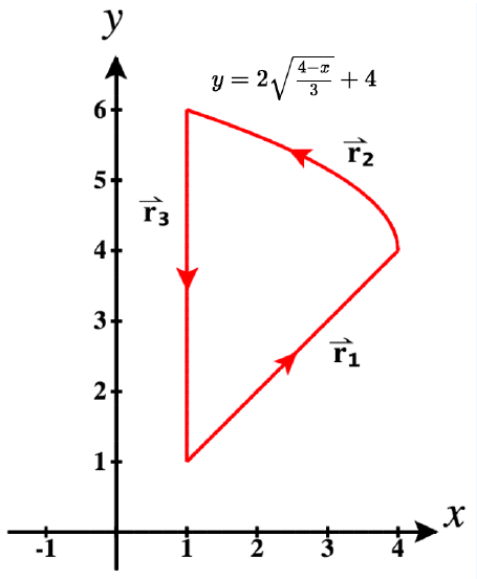


Figure 5.1.1: A closed curve in the xy-plane with three segments, labeled with directional vectors $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$. $\mathbf{r}_1$ is a straight line segment from $(1,1)$ to $(4,4)$. $\mathbf{r}_2$ is a curved segment from $(4,4)$ to $(1,6)$ along the curve given by $y = 2\sqrt{\frac{4-x}{3}} + 4$. $\mathbf{r}_3$ is a straight line segment from $(1,6)$ to $(1,1)$.

Solution

Our first task is to identify the three pieces in this piecewise path.

Note how we labeled these sequentially as $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$. Now we need to identify the function for each and write the corresponding vector-valued function with the correct orientation (left-to-right or right-to-left).

Determining $\mathbf{r}_1$: The equation of the linear function in this piece is $y = x$.

Since it is oriented from left-to-right between $t = 1$ and $t = 4$, we can write:

$$\mathbf{r}_{1a}(t) = t\,\mathbf{i} + t\,\mathbf{j} \quad \text{for} \quad 1 \le t \le 4$$

If we wish to begin this piece at $t = 0$, we just need to shift the value of t one unit to the left. One way to do this is to write $\mathbf{r}_{1a}$ in terms of t_1 instead of t to make the translation easier to see.

Thus, we have $\mathbf{r}_{1a}(t_1) = t_1\ \mathbf{i} + t_1\ \mathbf{j}$ for $1 \le t_1 \le 4$.

Subtracting 1 from each part of this range of parameter values, we have: $0 \le t_1 - 1 \le 3$.

Now we let $t = t_1 - 1$. Solving for t_1, we obtain: $t_1 = t + 1$.

Replacing t_1 with the expression $t + 1$ will effectively shift the range of parameter values one unit to the left.

So, starting with $t = 0$, we have:

$$\mathbf{r}_1(t) = (t+1)\,\mathbf{i} + (t+1)\,\mathbf{j} \quad \text{for} \quad 0 \le t \le 3$$

Double-check that this vector-valued function will trace out this segment in the correct direction before going on to r_2.

Determining $\mathbf{r}_2$: This piece has a label showing the function whose graph it traces along. If it were oriented from left-to-right, we would have:

$$\text{Left-to-right:} \quad \mathbf{r}_{2a}(t) = t\,\mathbf{i} + \left(2\sqrt{\frac{4-t}{3}} + 4\right)\mathbf{j} \quad \text{for} \quad 1 \le t \le 4$$

But since we need it to be oriented from right-to-left, we need to replace t with $-t$ in the function and we need to divide through the range inequality by -1 to obtain the corresponding range. Thus we obtain:

$$\mathbf{r}_{2b}(t) = -t\,\mathbf{i} + \left(2\sqrt{\frac{4-(-t)}{3}} + 4\right)\mathbf{j} \quad \text{for} \quad -4 \le t \le -1$$

Check that it works!

Now we wish to have this piece start at $t = 3$ just after the first one finishes. Again let's make this easier to see by writing r_{2b} in terms on t_2.

$$\mathbf{r}_{2b}(t_2) = -t_2\,\mathbf{i} + \left(2\sqrt{\frac{4-(-t_2)}{3}} + 4\right)\mathbf{j} \quad \text{for} \quad -4 \le t_2 \le -1$$

To force r_2 to start with $t = 3$ instead of $t = -4$, we need to add 7 to each part of the inequality. This yields: $3 \le t_2 + 7 \le 6$.

Let $t = t_2 + 7$. Then solving for $-t_2$ (since this is what we need to replace in r_{2b}), we have: $-t_2 = 7 - t$.

Replacing $-t_2$ with $(7 - t)$ in $\mathbf{r}_{2b}$, we obtain:

$$\mathbf{r}_2(t) = (7-t)\,\mathbf{i} + \left(2\sqrt{\frac{4-(7-t)}{3}} + 4\right)\mathbf{j} \quad \text{for} \quad 3 \le t \le 6$$

This can be combined with our earlier result for r_1 to write a piecewise-defined vector-valued function that traces out the first two pieces, starting at $t = 0$:

$$\mathbf{r}(t) = \begin{cases} (t+1)\,\mathbf{i} + (t+1)\,\mathbf{j}, & 0 \le t \le 3 \\ (7-t)\,\mathbf{i} + \left(2\sqrt{\frac{t-3}{3}} + 4\right)\mathbf{j}, & 3 < t \le 6 \end{cases}$$

Note that one small modification was made to the second range so that when $t = 3$, there is no confusion about which piece to evaluate.

Determining $\mathbf{r}_3$: To determine this last piece we need to think a little differently. This is because it is a vertical segment, which cannot be represented with a function of the form, $y = f(x)$. Note that it could be represented by a function of the form $x = f(y)$. Letting $y = t$, we can write $x = f(t)$ and writing a parameterization in increasing y values (bottom-to-top), we'd get: $\mathbf{r}(t) = f(t)\ \mathbf{i} + t\ \mathbf{j}$.

The equation of this line is $x = 1$. Thus, if we wished to parameterize this segment with upward orientation (increasing values of y), we have:

$$\mathbf{r}_{3a}(t) = 1\,\mathbf{i} + t\,\mathbf{j} \quad \text{for} \quad 1 \le t \le 6$$

But since we wish to use a downward orientation (decreasing values of y), we need to use a decreasing function of t for y. As before, the simplest case is to use $y = -t$. Then, in the general case, we'd trace a function $x = f(y)$ in a downwards orientation with $\mathbf{r}(t) = f(-t)\ \mathbf{i} - t\ \mathbf{j}$.

In the case of r_3, this gives us:

$$\mathbf{r}_{3b}(t) = 1\,\mathbf{i} - t\,\mathbf{j} \quad \text{for} \quad -6 \le t \le -1$$

Note that since $x = 1,\ \ f(-t) = 1$, that is, it did not change the first component since it was constant and not a variable function of the parameter t.

Also note that since we negated t, we also had to negate the range, dividing it through by -1.

As above, to facilitate the translation, we'll replace t with t_3, giving us:

$$\mathbf{r}_{3b}(t) = 1\,\mathbf{i} - t_3\,\mathbf{j} \quad \text{for} \quad -6 \le t_3 \le -1$$

Now, we wish this final piece to start at $t = 6$ where the second piece we formed above leaves off. We see that we need to add 12 to the range of paramater t to accomplish this, giving us a new range of $6 \leq t_3 + 12 \leq 11$.

Let $t = t_3 + 12$. Then solving for $-t_3$ (since this is what we need to replace in r_{3b}), we have: $-t_3 = 12 - t$.

Replacing $-t_3$ with $(12 - t)$ in $\mathbf{r}_{3b}$, we obtain:

$$\mathbf{r}_3(t) = 1\,\mathbf{i} + (12 - t)\,\mathbf{j} \quad \text{for} \quad 6 \leq t \leq 11$$

Check that this still traces out this vertical segment from top-to-bottom.

We can now state the final answer as a single piecewise-defined vector-valued function that traces out this entire path, starting when $t = 0$.

$$\mathbf{r}(t) = \begin{cases} (t+1)\,\mathbf{i} + (t+1)\,\mathbf{j}, & 0 \leq t \leq 3 \\ (7-t)\,\mathbf{i} + \left(2\sqrt{\frac{t-3}{3}} + 4\right)\,\mathbf{j}, & 3 < t \leq 6 \\ 1\,\mathbf{i} + (12-t)\,\mathbf{j} & 6 < t \leq 11 \end{cases}$$

Be sure to verify that this single vector-valued function does indeed trace out the entire path!

Limits and Continuity of a Vector-Valued Function

We now take a look at the limit of a vector-valued function. This is important to understand to study the calculus of vector-valued functions.

Definition: Limit of a Vector-Valued Function

A vector-valued function $\mathbf{r}$ approaches the **limit** $\mathbf{L}$ as t approaches a, written

$$\lim_{t \to a} \mathbf{r}(t) = \mathbf{L}, \tag{5.1.9}$$

provided

$$\lim_{t \to a} \|\mathbf{r}(t) - \mathbf{L}\| = 0. \tag{5.1.10}$$

This is a rigorous definition of the limit of a vector-valued function. In practice, we use the following theorem:

Theorem: Limit of a Vector-Valued Function

Let f, g, and h be functions of t. Then the limit of the vector-valued function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$ as t approaches a is given by

$$\lim_{t \to a} \mathbf{r}(t) = [\lim_{t \to a} f(t)]\mathbf{i} + [\lim_{t \to a} g(t)]\mathbf{j}, \tag{5.1.11}$$

provided the limits $\lim_{t \to a} f(t)$ and $\lim_{t \to a} g(t)$ exist.

Similarly, the limit of the vector-valued function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ as t approaches a is given by

$$\lim_{t \to a} \mathbf{r}(t) = [\lim_{t \to a} f(t)]\mathbf{i} + [\lim_{t \to a} g(t)]\mathbf{j} + [\lim_{t \to a} h(t)]\mathbf{k}, \tag{5.1.12}$$

provided the limits $\lim_{t \to a} f(t)$, $\lim_{t \to a} g(t)$ and $\lim_{t \to a} h(t)$ exist.

In the following example, we show how to calculate the limit of a vector-valued function.

Example 5.1.8: Evaluating the Limit of a Vector-Valued Function

For each of the following vector-valued functions, calculate $\lim\limits_{t\to 3}\mathbf{r}(t)$ for

a. $\mathbf{r}(t)=(t^2-3t+4)\mathbf{i}+(4t+3)\mathbf{j}$
b. $\mathbf{r}(t)=\frac{2t-4}{t+1}\mathbf{i}+\frac{t}{t^2+1}\mathbf{j}+(4t-3)\mathbf{k}$

Solution

a. Use Equation 5.1.11 and substitute the value $t=3$ into the two component expressions:

$$\begin{aligned}\lim_{t\to 3}\mathbf{r}(t) &= \lim_{t\to 3}\left[(t^2-3t+4)\mathbf{i}+(4t+3)\mathbf{j}\right]\\ &= \left[\lim_{t\to 3}(t^2-3t+4)\right]\mathbf{i}+\left[\lim_{t\to 3}(4t+3)\right]\mathbf{j}\\ &= 4\mathbf{i}+15\mathbf{j}\end{aligned}$$

b. Use Equation 5.1.12 and substitute the value $t=3$ into the three component expressions:

$$\begin{aligned}\lim_{t\to 3}\mathbf{r}(t) &= \lim_{t\to 3}\left(\frac{2t-4}{t+1}\mathbf{i}+\frac{t}{t^2+1}\mathbf{j}+(4t-3)\mathbf{k}\right)\\ &= \left[\lim_{t\to 3}\left(\frac{2t-4}{t+1}\right)\right]\mathbf{i}+\left[\lim_{t\to 3}\left(\frac{t}{t^2+1}\right)\right]\mathbf{j}+\left[\lim_{t\to 3}(4t-3)\right]\mathbf{k}\\ &= \tfrac{1}{2}\mathbf{i}+\tfrac{3}{10}\mathbf{j}+9\mathbf{k}\end{aligned}$$

Exercise 5.1.3

Calculate $\lim\limits_{t\to 2}\mathbf{r}(t)$ for the function $\mathbf{r}(t)=\sqrt{t^2+3t-1}\ \mathbf{i}-(4t-3)\mathbf{j}-\sin\frac{(t+1)\pi}{2}\mathbf{k}$

Hint available at the end of this section.

Answer available at the end of this section.

Now that we know how to calculate the limit of a vector-valued function, we can define **continuity at a point** for such a function.

Definition: Continuity at a Point

Let f, g, and h be functions of t.

The vector-valued function $\mathbf{r}(t)=f(t)\mathbf{i}+g(t)\mathbf{j}$ is **continuous** at $t=a$ if the following three conditions hold:

i. $\mathbf{r}(a)$ is defined
ii. $\lim\limits_{t\to a}\mathbf{r}(t)$ exists
iii. $\lim\limits_{t\to a}\mathbf{r}(t)=\mathbf{r}(a)$

Similarly, the vector-valued function $\mathbf{r}(t)=f(t)\mathbf{i}+g(t)\mathbf{j}+h(t)\mathbf{k}$ is **continuous** at $t=a$ if the following three conditions hold:

i. $\mathbf{r}(a)$ is defined
ii. $\lim\limits_{t\to a}\mathbf{r}(t)$ exists
iii. $\lim\limits_{t\to a}\mathbf{r}(t)=\mathbf{r}(a)$

Summary

- A vector-valued function is a function of the form $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$ or $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where the component functions f, g, and h are real-valued functions of the parameter t.
- The graph of a vector-valued function of the form $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$ is called a *plane curve*. The graph of a vector-valued function of the form $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ is called a *space curve*.
- It is possible to represent an arbitrary plane curve by a vector-valued function.
- To calculate the limit of a vector-valued function, calculate the limits of the component functions separately.

Key Equations

- **Vector-valued function**
  $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$ or $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$,or $\mathbf{r}(t) = \langle f(t), g(t)\rangle$ or $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$
- **Limit of a vector-valued function**
  $\lim\limits_{t\to a}\mathbf{r}(t) = [\lim\limits_{t\to a} f(t)]\mathbf{i} + [\lim\limits_{t\to a} g(t)]\mathbf{j}$ or $\lim\limits_{t\to a}\mathbf{r}(t) = [\lim\limits_{t\to a} f(t)]\mathbf{i} + [\lim\limits_{t\to a} g(t)]\mathbf{j} + [\lim\limits_{t\to a} h(t)]\mathbf{k}$

Glossary

Component Functions
The component functions of the vector-valued function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$ are $f(t)$ and $g(t)$, and the component functions of the vector-valued function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ are $f(t)$, $g(t)$ and $h(t)$.

Helix
A three-dimensional curve in the shape of a spiral.

Limit of a Vector-Valued Function
A vector-valued function $\mathbf{r}(t)$ has a limit $\mathbf{L}$ as t approaches a if $\lim t \to a\, |\mathbf{r}(t) - \mathbf{L}| = 0$.

Plane Curve
The set of ordered pairs $(f(t), g(t))$ together with their defining parametric equations $x = f(t)$ and $y = g(t)$.

Reparameterization
An alternative parameterization of a given vector-valued function.

Space Curve
The set of ordered triples $(f(t), g(t), h(t))$ together with their defining parametric equations $x = f(t)$, $y = g(t)$ and $z = h(t)$.

Vector Parameterization
Any representation of a plane or space curve using a vector-valued function.

Vector-Valued Function
A function of the form $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$ or $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$,where the component functions f, g, and h are real-valued functions of the parameter t.

Answers

Exercise 5.1.1

Substitute the appropriate values of t into the function.

$\mathbf{r}(0) = \mathbf{j},\ \mathbf{r}(1) = -2\mathbf{i} + 5\mathbf{j},\ \mathbf{r}(-4) = 28\mathbf{i} - 15\mathbf{j}$

The domain of $\mathbf{r}(t) = (t^2 - 3t)\mathbf{i} + (4t + 1)\mathbf{j}$ is all real numbers.

Exercise 5.1.2

Start by making a table of values, then graph the vectors for each value of t.

Exercise 5.1.2

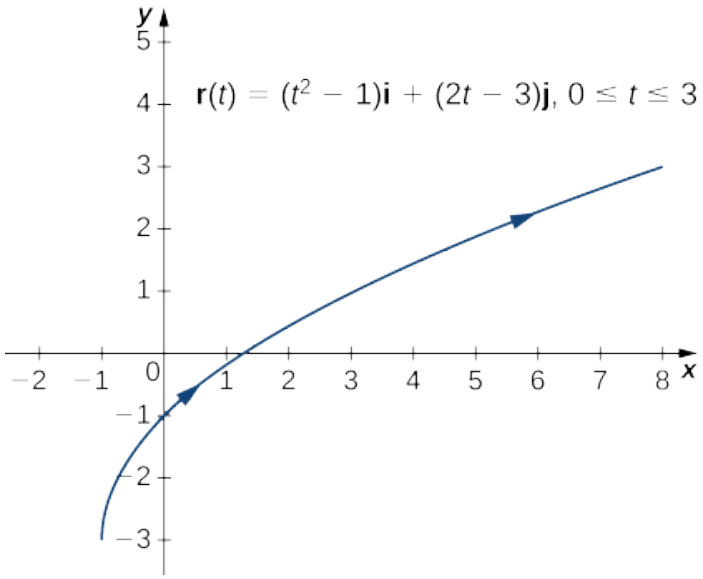


Figure 5.1.4: This figure is a graph of the function r(t) = (t^2-1)i + (2t-3)j, for the values of t from 0 to 3. The curve begins in the 3rd quadrant at the ordered pair (-1,-3) and increases up through the 1st quadrant. It is increasing and has arrows on the curve representing orientation to the right.

Exercise 5.1.3

Use Equation 5.1.12 from the preceding theorem.

Exercise 5.1.3

$$\lim_{t\to 2} \mathbf{r}(t) = 3\mathbf{i} - 5\mathbf{j} + \mathbf{k} \tag{5.1.13}$$

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5.1E: Exercises for Section 5.1

Exercises 1 - 4 Introduction to Vector-Valued Functions

1. Give the component functions $x = f(t)$ and $y = g(t)$ for the vector-valued function $\mathbf{r}(t) = 3\sec t\ \mathbf{i} + 2\tan t\ \mathbf{j}$.

   Answer

   Here we can say that $f(t) = 3\sec t, \quad g(t) = 2\tan t$

   so we have $x(t) = 3\sec t, \quad y(t) = 2\tan t.$

2. Given $\mathbf{r}(t) = 3\sec t\mathbf{i} + 2\tan t\mathbf{j}$, find the following values (if possible).

   a. $\mathbf{r}(\frac{\pi}{4})$
   b. $\mathbf{r}(\pi)$
   c. $\mathbf{r}(\frac{\pi}{2})$

3. Sketch the curve of the vector-valued function $\mathbf{r}(t) = 3\sec t\mathbf{i} + 2\tan t\mathbf{j}$ and give the orientation of the curve. Sketch asymptotes as a guide to the graph.

   Answer

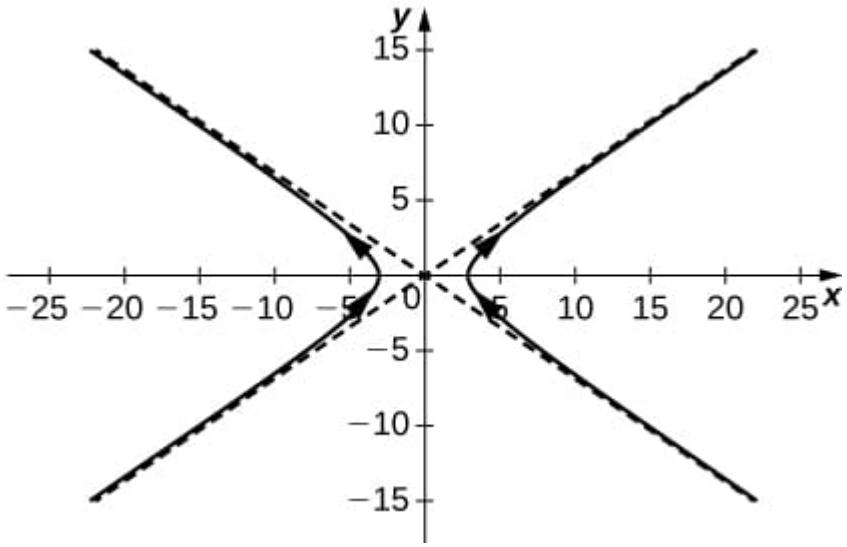


Figure $5.1E.1$: The graph in the xy-plane of a hyperbola centered at the origin. The hyperbola passes through the points $\pm 3, 0)$ has two branches that are mirror images, opening towards positive x-axis and negative x-axis. The asymptotes of the hyperbola, $y = \pm\frac{2}{3}$ are represented by dashed diagonal lines crossing through the origin, guiding the direction of the hyperbola branches.

Exercises 4 - 7 Limits of Vector-Valued Functions

4. Evaluate $\lim\limits_{t\to 0}\left(e^t\mathbf{i} + \frac{\sin t}{t}\mathbf{j} + e^{-t}\mathbf{k}\right)$
5. Given the vector-valued function $\mathbf{r}(t) = \langle\cos t, \sin t\rangle$ find the following values:

   a. $\lim\limits_{t\to\frac{\pi}{4}}\mathbf{r}(t)$
   b. $\mathbf{r}(\frac{\pi}{3})$
   c. Is $\mathbf{r}(t)$ continuous at $t = \frac{\pi}{3}$?
   d. Graph $\mathbf{r}(t)$.

Answer

a. $\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$,
b. $\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle$,
c. Yes, the limit as t approaches $\frac{\pi}{3}$ is equal to $\mathrm{r}(\frac{\pi}{3})$,
d.

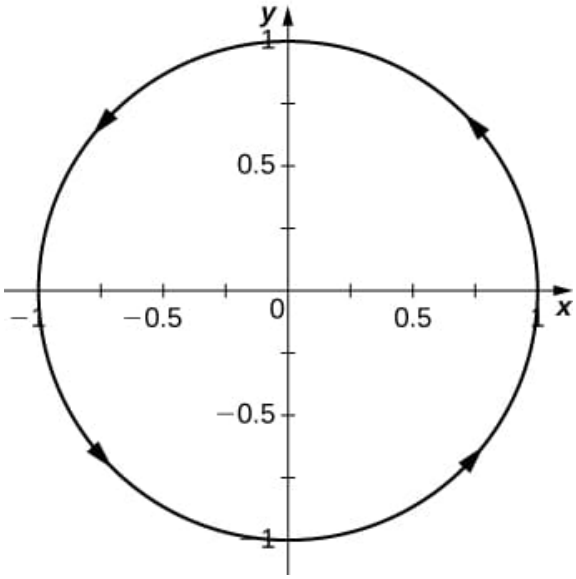


Figure $5.1E.2$: The graph in the xy-plane of circle of radius 1 with counter-clockwise orientation.

6. Given the vector-valued function $\mathbf{r}(t) = \langle t, t^2 + 1 \rangle$, find the following values:

   a. $\lim\limits_{t \to -3} \mathbf{r}(t)$
   b. $\mathbf{r}(-3)$
   c. Is $\mathbf{r}(t)$ continuous at $x = -3$?
   d. $\mathbf{r}(t + 2) - \mathbf{r}(t)$

7. Let $\mathbf{r}(t) = e^t \mathbf{i} + \sin t \mathbf{j} + \ln t \mathbf{k}$. Find the following values:

   a. $\mathbf{r}(\frac{\pi}{4})$
   b. $\lim\limits_{t \to \frac{\pi}{4}} \mathbf{r}(t)$
   c. Is $\mathbf{r}(t)$ continuous at $t = \frac{\pi}{4}$?

   Answer

   a. $\langle e^{\frac{\pi}{4}}, \frac{\sqrt{2}}{2}, \ln(\frac{\pi}{4}) \rangle$;
   b. $\langle e^{\frac{\pi}{4}}, \frac{\sqrt{2}}{2}, \ln(\frac{\pi}{4}) \rangle$;
   c. Yes

Exercises 8 - 13

Find the limit of the following vector-valued functions at the indicated value of t.

8. $\lim\limits_{t \to 4} \langle \sqrt{t - 3}, \frac{\sqrt{t} - 2}{t - 4}, \tan(\frac{\pi}{t}) \rangle$
9. $\lim\limits_{t \to \frac{\pi}{2}} \mathbf{r}(t)$ for $\mathbf{r}(t) = e^t \mathbf{i} + \sin t \mathbf{j} + \ln t \mathbf{k}$

   Answer

   $\langle e^{\frac{\pi}{2}}, 1, \ln(\frac{\pi}{2}) \rangle$

10. $\lim\limits_{t \to \infty} \langle e^{-2t}, \frac{2t + 3}{3t - 1}, \arctan(2t) \rangle$

11. $\lim_{t\to e^2}\langle t\ln(t), \frac{\ln t}{t^2}, \sqrt{\ln(t^2)}\rangle$

Answer

$2e^2\mathbf{i} + \frac{2}{e^4}\mathbf{j} + 2\mathbf{k}$

12. $\lim_{t\to\frac{\pi}{6}}\langle\cos 2t, \sin 2t, 1\rangle$
13. $\lim_{t\to\infty}\mathbf{r}(t)$ for $\mathbf{r}(t) = 2e^{-t}\mathbf{i} + e^{-t}\mathbf{j} + \ln(t-1)\mathbf{k}$

Answer

The limit does not exist because the limit of $\ln(t-1)$ as t approaches infinity does not exist.

Exercise 14 - 17 Domain of a Vector-Valued Function

Find the domain of the vector-valued functions.

14. Domain: $\mathbf{r}(t) = \langle t^2, t, \sin t\rangle$
15. Domain: $\mathbf{r}(t) = \langle t^2, \tan t, \ln t\rangle$

Answer

$\mathrm{D}_\mathbf{r} = \{t \mid t > 0, t \neq (2k+1)\frac{\pi}{2}, \text{ where } k \text{ is any integer}\}$

16. Domain: $\mathbf{r}(t) = \langle t^2, \sqrt{t-3}, \frac{3}{2t+1}\rangle$
17. Domain: $\mathbf{r}(t) = \langle \csc(t), \frac{1}{\sqrt{t-3}}, \ln(t-2)\rangle$

Answer

$\mathrm{D}_\mathbf{r} = \{t \mid t > 3, t \neq n\pi, \text{ where } n \text{ is any integer}\}$

18.

a. Find the domain of $\mathbf{r}(t) = 2e^{-t}\mathbf{i} + e^{-t}\mathbf{j} + \ln(t-1)\mathbf{k}$.
b. For what values of t is $\mathbf{r}(t) = 2e^{-t}\mathbf{i} + e^{-t}\mathbf{j} + \ln(t-1)\mathbf{k}$ continuous?

Answer

a. $\mathrm{D}_\mathbf{r} : (1, \infty)$
b. All t such that $t \in (1, \infty)$

19. Domain: $\mathbf{r}(t) = (\arccos t)\ \mathbf{i} + \sqrt{2t-1}\ \mathbf{j} + \ln(t)\ \mathbf{k}$

Answer

$\mathrm{D}_\mathbf{r} : \left[\frac{1}{2}, 1\right]$

Exercises 20 - 22 Visualizing Vector-Valued Functions

20. Describe the curve defined by the vector-valued function $\mathbf{r}(t) = (1+t)\mathbf{i} + (2+5t)\mathbf{j} + (-1+6t)\mathbf{k}$.
21. Let $\mathbf{r}(t) = \langle\cos t, t, \sin t\rangle$ and use it to answer the following questions.

a. For what values of t is $\mathbf{r}(t)$ continuous?
b. Sketch the graph of $\mathbf{r}(t)$.

Answer

a. $\mathbf{r}$ is continuous for all real numbers, i.e., for $t \in \mathbb{R}$.
b. Note that there should be a z on the vertical axis in the cross-section in image (a) below instead of the y.

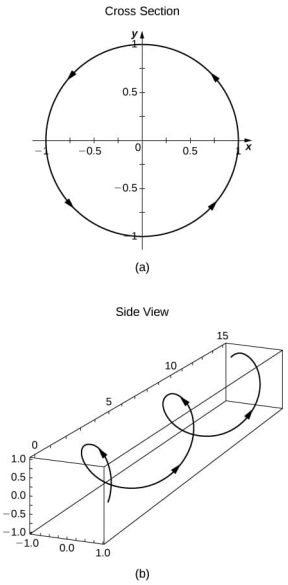


Figure $5.1E.3$: Top image, (a), shows counterclockwise oriented path on the unit circle in the xy-plane. Bottom image, (b), shows a helix - corkscrew path - in the xyz-space with z-coordinate varying as the circular motion continues as in image (a).

22. Produce a careful sketch of the graph of $\mathbf{r}(t) = t^2\ \mathbf{i} + t\ \mathbf{j}$.

Exercises 23 - 25

Use a graphing utility to sketch each of the vector-valued functions:

23. [T] $\mathbf{r}(t) = 2\cos^2 t\mathbf{i} + (2 - \sqrt{t})\mathbf{j}$

Answer

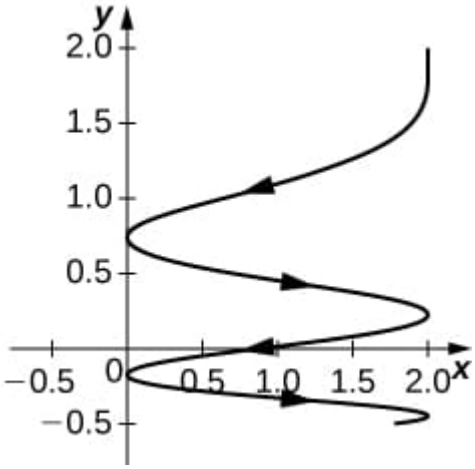
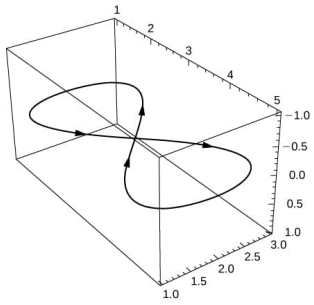


Figure $5.1E.4$: The graph in the xy-plane of a curve shows a spiraling curve passing from the origin (0,0). The curve makes multiple oscillations as it extends vertically, like sideways sinusoidal curve, but with diminishing frequency. The x values range over $[0, 2]$ and the y values range over $[-0.5, 2]$ in the graph.

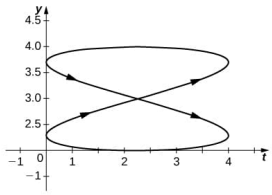
24. [T] $\mathbf{r}(t) = \langle e^{\cos(3t)}, e^{-\sin(t)} \rangle$
25. [T] $\mathbf{r}(t) = \langle 2 - \sin(2t), 3 + 2\cos t \rangle$

Answer



(a)

View in the *yt*-plane



(b)

Figure $5.1E.5$: The first of two curves, (a), shows a graph in the xyz-space with a looped curve. The curve loops twice, like a "figure 8", with arrows indicating directionality. The second curve, (b), is in the yt-plane. In the curve, t ranges over $[0, 4]$ and y ranges over $[0, 4]$. It features a similar looped curve. The curve in plot (b) also has arrows indicating direction.

Exercises 26 - 33 Finding Equations in x and y for the Path Traced out by Vector-Valued Functions

Eliminate the parameter t, write the equation in Cartesian coordinates, then sketch the graph of the vector-valued functions.

26. $\mathbf{r}(t) = 2t\mathbf{i} + t^2\mathbf{j}$

(*Hint:* Let $x = 2t$ and $y = t^2$. Solve the first equation for t in terms of x and substitute this result into the second equation.)

27. $\mathbf{r}(t) = t^3\mathbf{i} + 2t\mathbf{j}$

Answer

$y = 2\sqrt[3]{x}$, a variation of the cube-root function

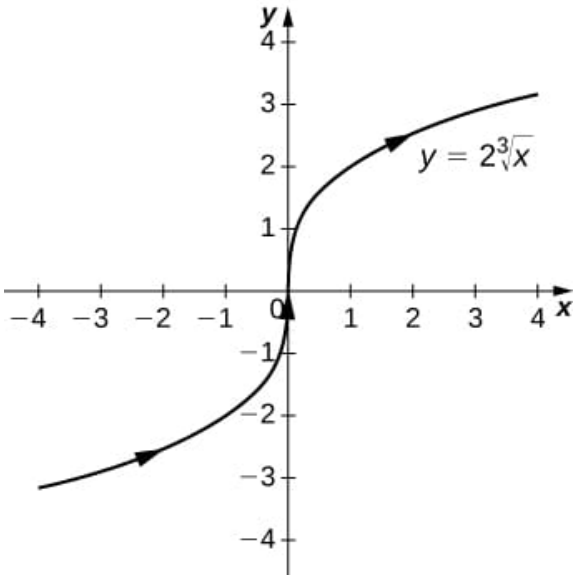


Figure $5.1E.6$: The graph of $y = 2\sqrt[3]{x}$. It's an increasing function that starts in the third quadrant where the graph is concave up, passes through the origin, and moves into the first quadrant where the graph is concave down. Two arrows on the curve indicate the direction, which is from left to right.

28. $\mathbf{r}(t) = \sin t\ \mathbf{i} + \cos t\ \mathbf{j}$
29. $\mathbf{r}(t) = 3\cos t\ \mathbf{i} + 3\sin t\ \mathbf{j}$

Answer

$x^2 + y^2 = 9$, a circle centered at $(0, 0)$ with radius 3, and a counterclockwise orientation

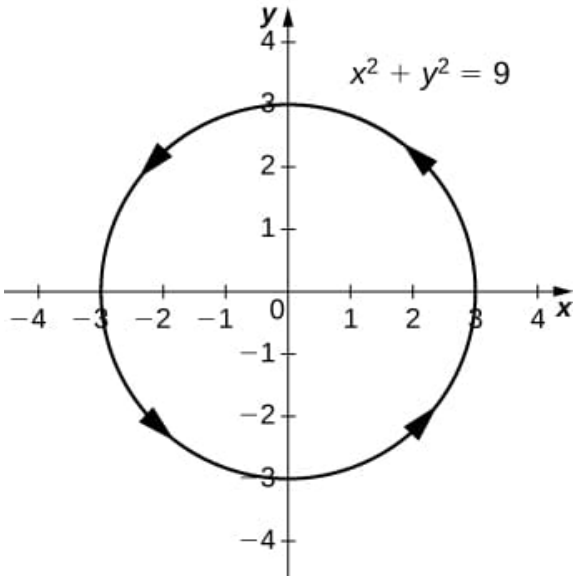


Figure $5.1E.7$: The graph of $x^2 + y^2 = 9$ in the xy-plane of a circle or radius 3 oriented counterclockwise.

30. $\mathbf{r}(t) = \langle \sin t, 4\cos t \rangle$
31. $\mathbf{r}(t) = 2\sin t\ \mathbf{i} - 3\cos t\ \mathbf{j}$

Answer

$\frac{x^2}{4} + \frac{y^2}{9} = 1$, an ellipse centered at $(0, 0)$ with intercepts at $x = \pm 2$ and $y = \pm 3$, and a clockwise orientation

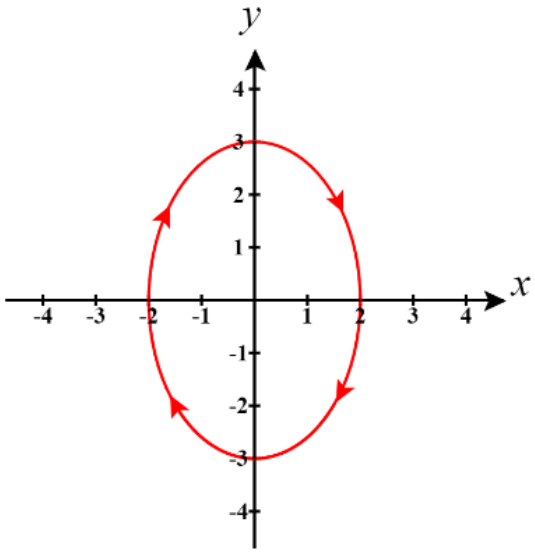


Figure $5.1E.8$: The graph in the xy-plane of an ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$ with clockwise orientation.

32. $\mathbf{r}(t)=\tan t\ \mathbf{i}-2\sec t\ \mathbf{j}$

33. $\mathbf{r}(t)=3\sec t\ \mathbf{i}+4\tan t\ \mathbf{j}$

Answer

$\frac{x^2}{9}-\frac{y^2}{16}=1$, a hyperbola centered at $(0,0)$ with x-intercepts $(3,0)$ and $(-3,0)$, with orientation shown

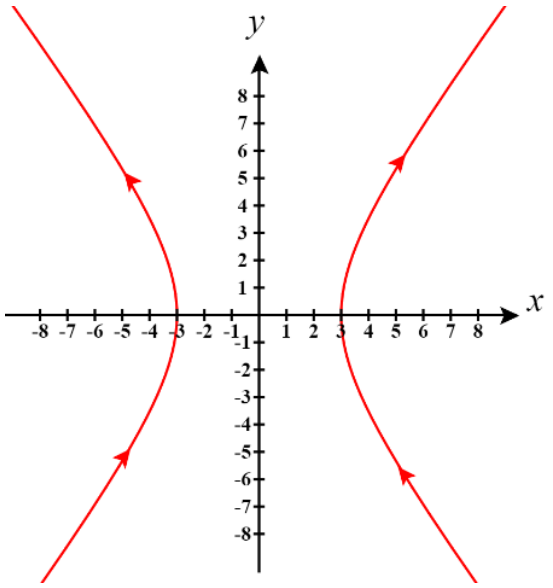


Figure $5.1E.9$: The graph in the xy-plane of the hyperbola $\frac{x^2}{9}-\frac{y^2}{16}=1$. The orientation for both sections of the curve is upward - in the direction of increasing y.

Exercises 34 - 40 Finding a Vector-Valued Function to Trace out the Graph of an Equation in x and y

Find a vector-valued function that traces out the given curve in the indicated direction.

34. $4x^2+9y^2=36$; clockwise and counterclockwise

35. $y = x^2$; from left to right

Answer

$\mathbf{r}(t) = \langle t, t^2 \rangle$, where t increases

36. The line through P and Q where P is $(1, 4, -2)$ and Q is $(3, 9, 6)$
37. The circle, $x^2 + y^2 = 36$, oriented clockwise, with position $(-6, 0)$ at time $t = 0$.

Answer

$\mathbf{r}(t) = -6\cos t\ \mathbf{i} + 6\sin t\ \mathbf{j}$

38. The ellipse, $x^2 + \dfrac{y^2}{36} = 1$, oriented counterclockwise

Answer

$\mathbf{r}(t) = \cos t\ \mathbf{i} + 6\sin t\ \mathbf{j}$

39. The hyperbola, $\dfrac{y^2}{36} - x^2 = 1$, top piece is oriented from left-to-right

Answer

$\mathbf{r}(t) = \tan t\ \mathbf{i} + 6\sec t\ \mathbf{j}$

40. The hyperbola, $\dfrac{x^2}{49} - \dfrac{y^2}{64} = 1$, right piece is oriented from bottom-to-top

Answer

$\mathbf{r}(t) = 7\sec t\ \mathbf{i} + 8\tan t\ \mathbf{j}$

Exercises 41 - 44 Parameterizing a Piecewise Path

Provide a parameterization for each piecewise path. Try to write a parameterization that starts with $t = 0$ and progresses on through values of t as you move from one piece to another.

41.

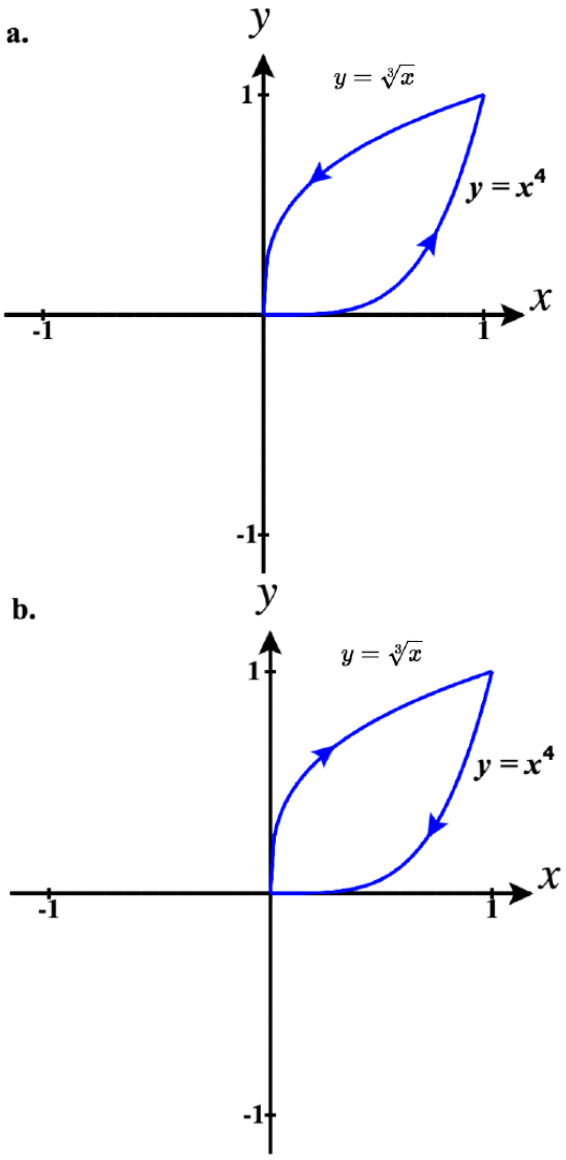


Figure $5.1E.10$: Figure (a) shows a graph in the xy-plane. The graph features a closed, asymmetric loop shaped like a petal. Both x and y values ranging from -1 to 1. The curve includes the origin and extends into the first quadrant. The curve is composed of two distinct functions marked as $y = \sqrt[3]{x}$ and $y = x^4$. Arrows on the curve indicate the direction of the loop, moving counter-clockwise. There are tick marks at -1 and 1 on both axes. Figure (b) shows the same curve as (a), but is the orientation is clockwise.

Answer

a. $\mathbf{r}_1(t) = t\ \mathbf{i} + t^4\ \mathbf{j}$ for $0 \le t \le 1$

$\mathbf{r}_2(t) = -t\ \mathbf{i} + \sqrt[3]{-t}\ \mathbf{j}$ for $-1 \le t \le 0$

So a piecewise parameterization of this path is:

$$\mathbf{r}(t) = \begin{cases} t\,\mathbf{i} + t^4\,\mathbf{j}, & 0 \le t \le 1 \\ (2-t)\ \mathbf{i} + \sqrt[3]{2-t}\,\mathbf{j}, & 1 < t \le 2 \end{cases}$$

b. $\mathbf{r}_1(t) = t\ \mathbf{i} + \sqrt[3]{t}\ \mathbf{j}$ for $0 \le t \le 1$

$\mathbf{r}_2(t) = -t\ \mathbf{i} + (-t)^4\ \mathbf{j}$ for $-1 \le t \le 0$

So a piecewise parameterization of this path is:

$$\mathbf{r}(t) = \begin{cases} t\,\mathbf{i} + \sqrt[3]{t}\,\mathbf{j}, & 0 \le t \le 1 \\ (2-t)\ \mathbf{i} + (2-t)^4\ \mathbf{j}, & 1 < t \le 2 \end{cases}$$

42.

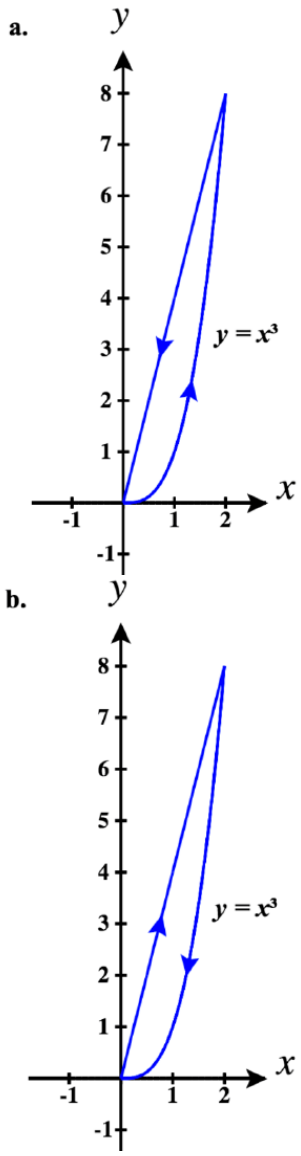


Figure $5.1E.11$: Figure (a) shows a graph in the xy-plane. The graph is over the x values 0 to 2, and the y values 0 to 8. The graph features a closed loop made up of the function $y = x^3$, and the line segment from the point $(0, 0)$ to$(2, 8)$. Arrows on the curve indicate the direction of the loop, moving counter-clockwise. Figure (b) shows the same graph as (a), but is the orientation is clockwise.

13.

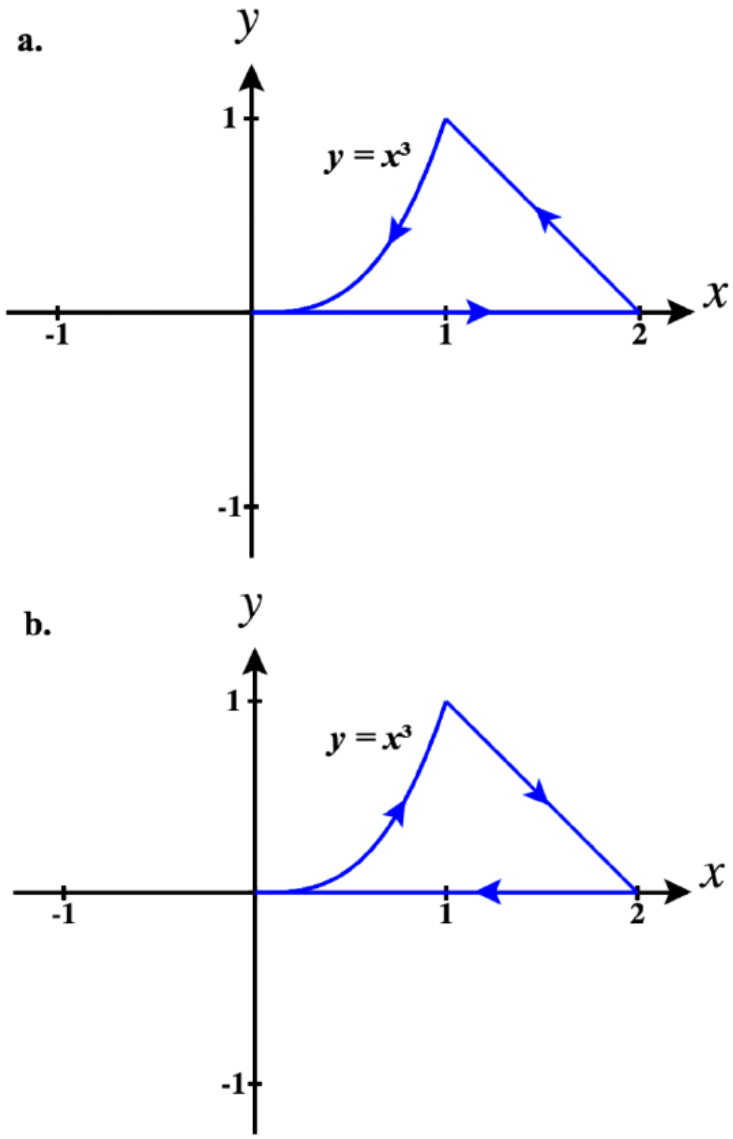


Figure $5.1E.12$: Figure (a) shows a graph in the xy-plane. The graph is over the x values 0 to 2, and the y values 0 to 1. The graph features a closed loop made up of three sections: the function $y = x^3$, the line segment from the point $(1, 1)$ to $(2, 0)$, and the line segment from the point $(2, 0)$ to $(0, 0)$. Arrows on the curve indicate the direction of the loop, moving counter-clockwise. Figure (b) shows the same graph as (a), but is the orientation is clockwise.

Answer

a. $\mathbf{r}_1(t) = t\ \mathbf{i} + 0\ \mathbf{j}$ for $0 \le t \le 2$

$\mathbf{r}_2(t) = -t\ \mathbf{i} + (2 + t)\ \mathbf{j}$ for $-2 \le t \le -1$
$\mathbf{r}_3(t) = -t\ \mathbf{i} + (-t)^3\ \mathbf{j}$ for $-1 \le t \le 0$

So a piecewise parameterization of this path is:

$$\mathbf{r}(t) = \begin{cases} t\,\mathbf{i}, & 0 \le t \le 2 \\ (4 - t)\ \mathbf{i} + (t - 2)\ \mathbf{j}, & 2 < t \le 3 \\ (4 - t)\ \mathbf{i} + (4 - t)^3\ \mathbf{j}, & 3 < t \le 4 \end{cases}$$

b. $\mathbf{r}_1(t) = t\ \mathbf{i} + t^3\ \mathbf{j}$ for $0 \le t \le 1$

$\mathbf{r}_2(t) = t\ \mathbf{i} + (2 - t)\ \mathbf{j}$ for $1 \le t \le 2$
$\mathbf{r}_3(t) = -t\ \mathbf{i} + 0\ \mathbf{j}$ for $-2 \le t \le 0$

So a piecewise parameterization of this path is:

$$\mathbf{r}(t) = \begin{cases} t\,\mathbf{i} + t^3\,\mathbf{j}, & 0 \le t \le 1 \\ t\,\mathbf{i} + (2-t)\,\mathbf{j}, & 1 < t \le 2 \\ (4-t)\,\mathbf{i}, & 2 < t \le 4 \end{cases}$$

14.

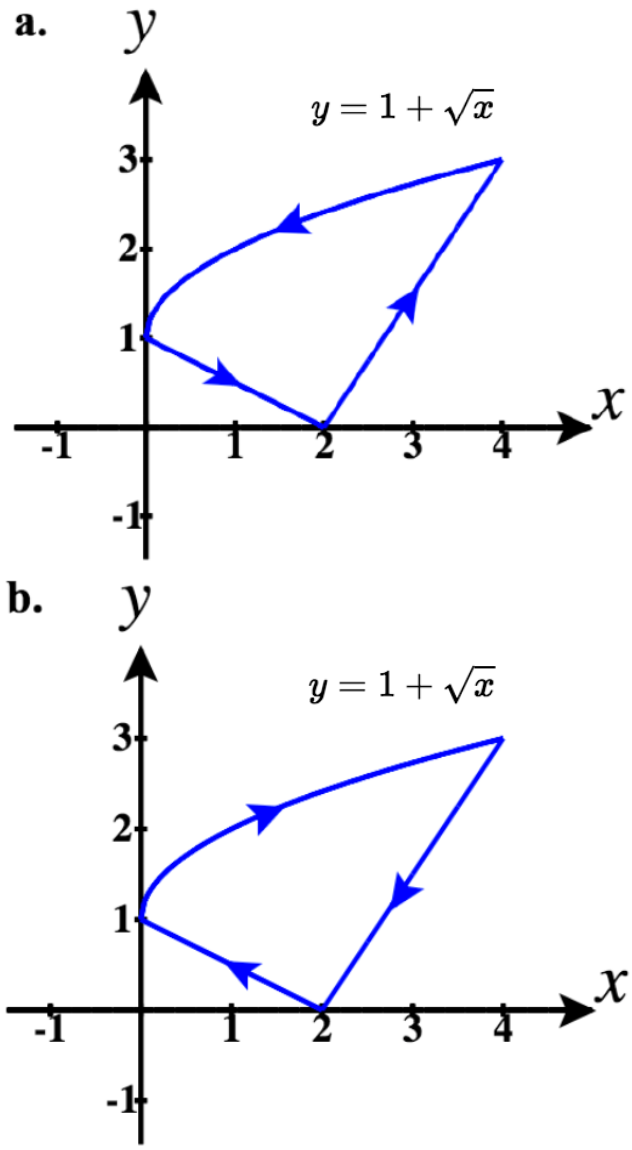


Figure $5.1E.13$: Figure (a) shows a graph in the xy-plane. The graph is over the x values 0 to 4, and the y values 0 to 3. The graph features a closed loop made up of three sections: the function $y = 1 + \sqrt{x}$ from point $(0, 1)$ to $(4, 3)$, the line segment from the point $(4, 3)$ to $(2, 0)$, and the line segment from the point $(2, 0)$ to $(0, 1)$. Arrows on the curve indicate the direction of the loop, moving counter-clockwise. Figure (b) shows the same graph as (a), but is the orientation is clockwise.

Exercises 45 - 48 Additional Vector-Valued Function Questions

Consider the curve described by the vector-valued function $\mathbf{r}(t) = (50e^{-t}\cos t)\mathbf{i} + (50e^{-t}\sin t)\mathbf{j} + (5 - 5e^{-t})\mathbf{k}$.

45) What is the initial point of the path corresponding to $\mathbf{r}(0)$?

Answer

$(50, 0, 0)$

46. What is $\lim_{t\to\infty} \mathbf{r}(t)$?

47. [T] Use technology to sketch the curve.

Answer

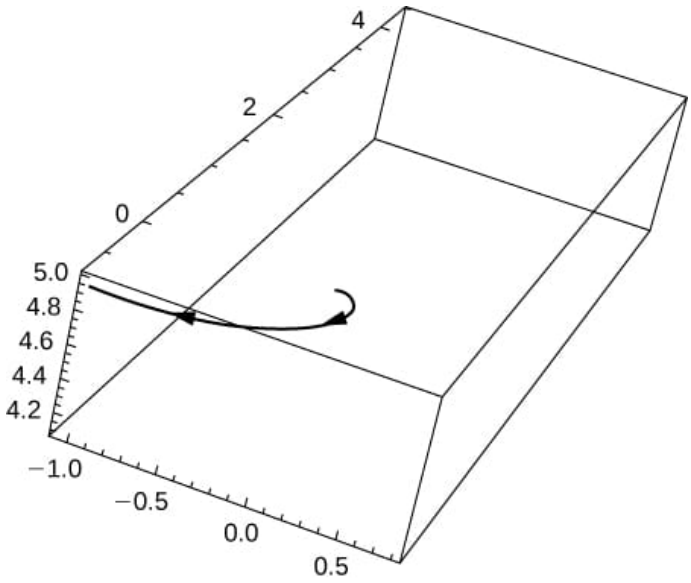


Figure $5.1E.14$: This three-dimensional plot is enclosed within a rectangular box-like frame representing the three axes of the coordinate system. The axes are labeled with numerical ticks: the horizontal axis ranges approximately from -0.5 to 0.5, the vertical axis extends upwards from 4.4 to 5, and the depth axis extends from -0.5 to about 4 in. This three-dimensional graph starts at a point in the center of the frame and curves upward and forward as the direction of the arrow indicates.

48. Eliminate the parameter *t* to show that $z = 5 - \dfrac{r}{10}$ where $r^2 = x^2 + y^2$.

49. [T] Let $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + 0.3\sin(2t)\mathbf{k}$. Use technology to graph the curve (called the *roller-coaster curve*) over the interval $[0, 2\pi)$. Choose at least two views to determine the peaks and valleys.

Answer

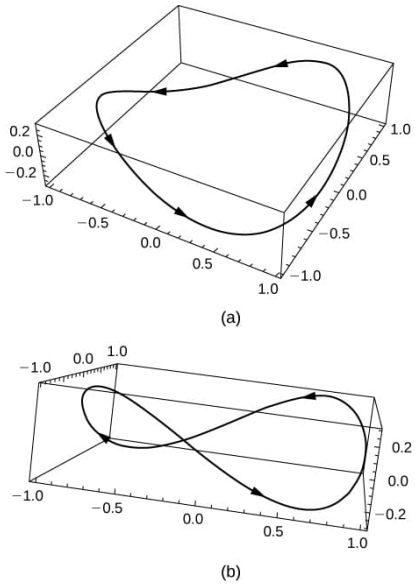


Figure $5.1E.15$: Two 3D plots labeled (a) and (b) illustrating different views of a curve in three-dimensional space. In plot (a), the curve forms a loop within a cubic frame. The axes are labeled from -1.0 to 1.0 in both the horizontal and depth directions, with the vertical axis ranging from -0.2 to 0.2. The curve includes arrows along its length, indicating the direction of motion through the loop, shaped like the boundary of a circular saddle. In plot (b), the same curve is presented from a different perspective. The axes are labeled identically, with the horizontal axis extending from -1.0 to 1.0, the vertical axis from -0.2 to 0.2, and the depth axis from -1.0 to 1.0.The curve still forms a loop but appears more elongated due to the new angle of view. It looks like a "figure 8".

50. [T] Use the result of the preceding problem to construct an equation of a roller coaster with a steep drop from the peak and steep incline from the “valley.” Then, use technology to graph the equation.
51. Use the results of the preceding two problems to construct an equation of a path of a roller coaster with more than two turning points (peaks and valleys).

Answer

One possibility is $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + \sin(4t)\mathbf{k}$. By increasing the coefficient of t in the third component, the number of turning points will increase.

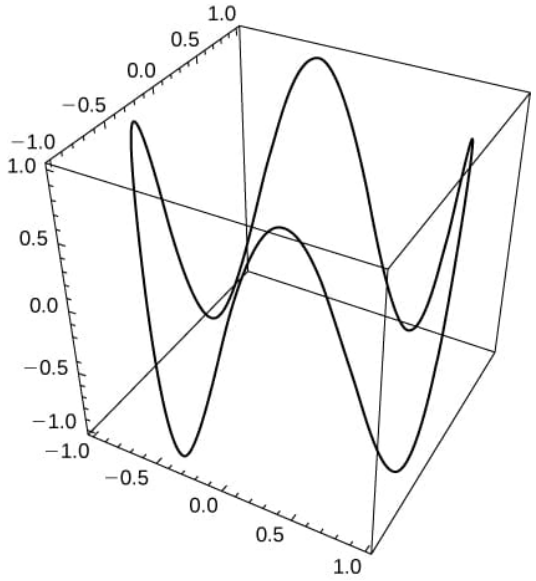


Figure $5.1E.1$: A three-dimensional plot of a sinusoidal curve within a cube-shaped frame, representing a periodic function in 3D space. The curve oscillates along all three axes, forming wave-like patterns that pass through the cube's center. The axes are all labeled with tick marks ranging from -1.0 to 1.0 along all three dimensions (horizontal, vertical, and depth). The curve exhibits multiple peaks and valleys, crossing the axes at various points, falling on the surface of a cylinder.

52. Complete the following investigation.

a. Graph the curve $\mathbf{r}(t) = (4 + \cos(18t))\cos(t)\mathbf{i} + (4 + \cos(18t)\sin(t))\mathbf{j} + 0.3\sin(18t)\mathbf{k}$ using two viewing angles of your choice to see the overall shape of the curve.
b. Does the curve resemble a “slinky”?
c. What changes to the equation should be made to increase the number of coils of the slinky?

Contributors

- Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

- Paul Seeburger (Monroe Community College) created problems 12, 14, 19, 22, 30-33, 37- 44.

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5.2: Calculus of Vector-Valued Functions

Learning Objectives

- Write an expression for the derivative of a vector-valued function.
- Find the tangent vector at a point for a given position vector.
- Determine where a vector-valued function is smooth.
- Find the unit tangent vector at a point for a given position vector and explain its significance.
- Calculate the definite integral of a vector-valued function.

To study the calculus of vector-valued functions, we follow a similar path to the one we took in studying real-valued functions. First, we define the derivative, then we examine applications of the derivative, then we move on to defining integrals. However, we will find some interesting new ideas along the way as a result of the vector nature of these functions and the properties of space curves.

Derivatives of Vector-Valued Functions

Now that we have seen what a vector-valued function is and how to take its limit, the next step is to learn how to differentiate a vector-valued function. The definition of the derivative of a vector-valued function is nearly identical to the definition of a real-valued function of one variable. However, because the range of a vector-valued function consists of vectors, the same is true for the range of the derivative of a vector-valued function.

Definition: Derivative of Vector-Valued Functions

The **derivative** of a vector-valued function $\mathbf{r}(t)$ is

$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t} \tag{5.2.1}$$

provided the limit exists. If $\mathbf{r}'(t)$ exists, then $\mathbf{r}(t)$ is differentiable at t. If $\mathbf{r}\,'(t)$ exists for all t in an open interval (a, b) then $\mathbf{r}(t)$ is differentiable over the interval (a, b). For the function to be differentiable over the closed interval $[a, b]$, the following two limits must exist as well:

$$\mathbf{r}'(a) = \lim_{\Delta t \to 0^+} \frac{\mathbf{r}(a+\Delta t) - \mathbf{r}(a)}{\Delta t} \tag{5.2.2}$$

and

$$\mathbf{r}'(b) = \lim_{\Delta t \to 0^-} \frac{\mathbf{r}(b+\Delta t) - \mathbf{r}(b)}{\Delta t} \tag{5.2.3}$$

Graphical Interpretation of the Derivative: Recall that the derivative of a real-valued function can be interpreted as the slope of a tangent line or the instantaneous rate of change of the function. The derivative of a vector-valued function can be understood to be an instantaneous rate of change as well; for example, when the function represents the position of an object at a given point in time, the derivative represents its velocity at that same point in time.

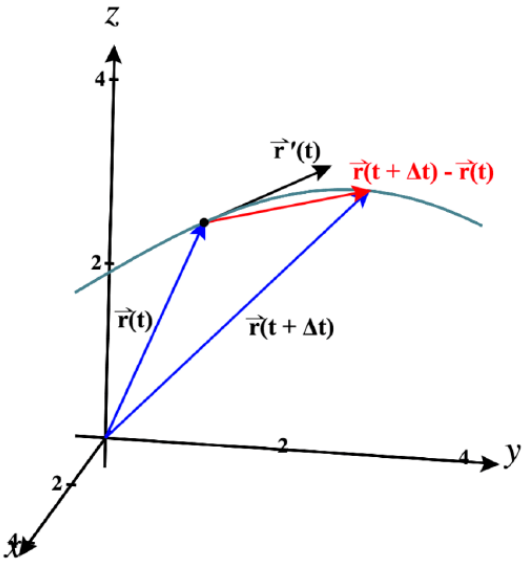


Figure 5.2.1 This is a three-dimensional graph in the xyz-space. A curved path in green shows the trajectory or a vector-valued function $\mathbf{r}(t)$of time t. Two vectors, $\mathbf{r}(t)$ and $\mathbf{r}(t+\Delta t)$, are shown as position vectors. $\mathbf{r}(t)$ extends to a black dot on the curve, while $\mathbf{r}(t+\Delta t)$ reaches a point further along the curve. A red arrow represents the vector $\mathbf{r}(t+\Delta t)-\mathbf{r}(t)$. An additional black arrow labeled $\mathbf{r}'(t)$ is depicted as the tangent to the curve at the black dot. Diagram shows how the direction of the average velocity vector (red arrow) approaches the direction of velocity vector (black arrow) representing the derivative.

To understand what this derivative represents geometrically, it helps to consider the diagram shown in Figure 5.2.1. For the purposes of this illustration, consider $\mathbf{r}(t)$ to the represent position of an object.

We see vectors $\mathbf{r}(t)$ and $\mathbf{r}(t+\Delta t)$, where Δt is a small change in time producing a second vector that is farther along in the direction of motion.

The difference of these two vectors gives us the **displacement vector,** $\mathbf{r}(t+\Delta t)-\mathbf{r}(t)$, shown here in red.

As $\Delta t \to 0$, $\mathbf{r}(t+\Delta t)$ will approach $\mathbf{r}(t)$, and the displacement vector $\mathbf{r}(t+\Delta t)-\mathbf{r}(t)$ will become closer and closer to being tangent to the curve at the terminal point of $\mathbf{r}(t)$.

Although the length of the displacement vector is approaching 0, we are actually taking the limit not of the displacement vector itself, but of the average velocity over the time Δt, found by dividing the displacement vector by the scalar Δt.

$$\text{Average Velocity} \quad = \quad \frac{\mathbf{r}(t+\Delta t)-\mathbf{r}(t)}{\Delta t} \tag{5.2.4}$$

Since Δt is also approaching 0, dividing by it will tend to lengthen the average velocity vector (as long as $\Delta t < 1$).

The limit gives us the vector $\mathbf{r}\,'(t)$, the velocity of the motion along the curve, which is tangent to the curve at this point.

Many of the rules for calculating derivatives of real-valued functions can be applied to calculating the derivatives of vector-valued functions as well. Recall that the derivative of a real-valued function can be interpreted as the slope of a tangent line or the instantaneous rate of change of the function. The derivative of a vector-valued function can be understood to be an instantaneous rate of change as well; for example, when the function represents the position of an object at a given point in time, the derivative represents its velocity at that same point in time.

We now demonstrate taking the derivative of a vector-valued function.

Example 5.2.1: Finding the Derivative of a Vector-Valued Function

Use the definition to calculate the derivative of the function

$$\mathbf{r}(t) = (3t+4)\,\mathbf{i} + (t^2 - 4t + 3)\,\mathbf{j}.$$

Solution

Let's use Equation 5.2.1:

$$\begin{aligned}
\mathbf{r}'(t) &= \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{[(3(t+\Delta t)+4)\,\mathbf{i} + ((t+\Delta t)^2 - 4(t+\Delta t)+3)\,\mathbf{j}] - [(3t+4)\,\mathbf{i} + (t^2-4t+3)\,\mathbf{j}]}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{(3t+3\Delta t+4)\,\mathbf{i} - (3t+4)\,\mathbf{i} + (t^2+2t\Delta t+(\Delta t)^2-4t-4\Delta t+3)\,\mathbf{j} - (t^2-4t+3)\,\mathbf{j}}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{(3\Delta t)\,\mathbf{i} + (2t\Delta t+(\Delta t)^2-4\Delta t)\,\mathbf{j}}{\Delta t} \\
&= \lim_{\Delta t \to 0} (3\,\mathbf{i} + (2t+\Delta t-4)\,\mathbf{j}) \\
&= 3\,\mathbf{i} + (2t-4)\,\mathbf{j}
\end{aligned}$$

Exercise 5.2.1

Use the definition to calculate the derivative of the function $\mathbf{r}(t) = (2t^2+3)\ \mathbf{i} + (5t-6)\ \mathbf{j}$.

Hint available at the end of this section.

Answer available at the end of this section.

Notice that in the calculations in Example 5.2.1, we could also obtain the answer by first calculating the derivative of each component function, then putting these derivatives back into the vector-valued function. This is always true for calculating the derivative of a vector-valued function, whether it is in two or three dimensions. We state this in the following theorem. The proof of this theorem follows directly from the definitions of the limit of a vector-valued function and the derivative of a vector-valued function.

Theorem: Differentiation of Vector-Valued Functions

Let f, g, and h be differentiable functions of t.

1. If $\mathbf{r}(t) = f(t)\ \mathbf{i} + g(t)\ \mathbf{j}$ then
$$\mathbf{r}'(t) = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j}. \tag{5.2.5}$$
2. If $\mathbf{r}(t) = f(t)\ \mathbf{i} + g(t)\ \mathbf{j} + h(t)\ \mathbf{k}$ then
$$\mathbf{r}'(t) = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k}. \tag{5.2.6}$$

Example 5.2.2: Calculating the Derivative of Vector-Valued Functions

Calculate the derivative of each of the following functions.

a. $\mathbf{r}(t) = (6t+8)\ \mathbf{i} + (4t^2+2t-3)\ \mathbf{j}$
b. $\mathbf{r}(t) = 3\cos t\ \mathbf{i} + 4\sin t\ \mathbf{j}$
c. $\mathbf{r}(t) = e^t \sin t\ \mathbf{i} + e^t \cos t\ \mathbf{j} - e^{2t}\ \mathbf{k}$

Solution

We use the above theorem on differentiation of vector-valued functions to take the derivative.

a. The first component of

$$\mathbf{r}(t) = (6t+8)\,\mathbf{i} + (4t^2+2t-3)\,\mathbf{j}$$

is $f(t) = 6t+8$. The second component is $g(t) = 4t^2+2t-3$. We have $f\,'(t) = 6$ and $g\,'(t) = 8t+2$, so the Theorem 5.2.1 gives $\mathbf{r}\,'(t) = 6\ \mathbf{i} + (8t+2)\ \mathbf{j}$.

b. The first component is $f(t) = 3\cos t$ and the second component is $g(t) = 4\sin t$. We have $f\,'(t) = -3\sin t$ and $g\,'(t) = 4\cos t$, so we obtain $\mathbf{r}\,'(t) = -3\sin t\ \mathbf{i} + 4\cos t\ \mathbf{j}$.

c. The first component of $\mathbf{r}(t) = e^t\sin t\ \mathbf{i} + e^t\cos t\ \mathbf{j} - e^{2t}\ \mathbf{k}$ is $f(t) = e^t\sin t$, the second component is $g(t) = e^t\cos t$, and the third component is $h(t) = -e^{2t}$. We have $f\,'(t) = e^t(\sin t + \cos t)$, $g\,'(t) = e^t(\cos t - \sin t)$, and $h\,'(t) = -2e^{2t}$, so the theorem gives $\mathbf{r}\,'(t) = e^t(\sin t + \cos t)\ \mathbf{i} + e^t(\cos t - \sin t)\ \mathbf{j} - 2e^{2t}\ \mathbf{k}$.

Exercise 5.2.2

Calculate the derivative of the function

$$\mathbf{r}(t) = (t\ln t)\,\mathbf{i} + (5e^t)\,\mathbf{j} + (\cos t - \sin t)\,\mathbf{k}.$$

Hint available at the end of this section.

Answer available at the end of this section.

Describing Motion with Vector-Valued Functions

As has been mentioned already in this section, vector-valued functions are often used to describe motion in the plane or in space. Now that we understand the derivative of a vector-valued function, let's make the relationships clear.

First, since the components of a vector-valued function represent the coordinates of a point on the corresponding curve, it may help to write

$$\mathbf{r}(t) = x(t)\ \mathbf{i} + y(t)\ \mathbf{j} + z(t)\ \mathbf{k} \tag{5.2.7}$$

Now we can define how to calculate the velocity, speed, acceleration, and jerk of the object's motion.

Definition: Velocity, Speed, Acceleration and Jerk

Let $\mathbf{r}(t)$ be a twice-differentiable vector-valued function of the parameter t that represents the position of an object as a function of time.

The **velocity vector** $\mathbf{v}(t)$ of the object is given by

$$\text{Velocity } = \mathbf{v}(t) = \mathbf{r}'(t). \tag{5.2.8}$$

The **speed** is defined to be

$$\text{Speed } = \|\mathbf{v}(t)\| = \|\mathbf{r}'(t)\| = \frac{ds}{dt}. \tag{5.2.9}$$

The **acceleration vector** $\mathbf{a}(t)$ is defined to be

$$\text{Acceleration } = \mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t). \tag{5.2.10}$$

If $\mathbf{r}(t)$ has a third derivative, we can also define the **jerk vector** $\backslash\text{vecd}\text{jerk}(t)$ to be

$$\text{Jerk } = \backslash\text{vecd}\text{jerk}(t) = \mathbf{a}'(t) = \mathbf{r}'''(t). \tag{5.2.11}$$

Now let's see how this looks in the context of a particular position function.

Example 5.2.3: Finding velocity, speed, acceleration, and jerk

Consider the position function,

$$\mathbf{r}(t) = (1 - t^2)\,\mathbf{i} + t\,\mathbf{j}, \text{ where time is in seconds} \tag{5.2.12}$$

a. Determine the velocity, speed, acceleration, and jerk as functions of time for this position function.
b. Evaluate each of these functions, including the position function, at times $t = 0$ sec and $t = 2$ sec.
c. Sketch a graph of this position function and draw in each of the non-zero vectors from part b on the graph of $\mathbf{r}$, starting at the location of the object on the curve at the corresponding time.

Solution

a. Since $\mathbf{r}(t) = (1 - t^2)\ \mathbf{i} + t\ \mathbf{j}$, we have

$$\begin{aligned}
\mathbf{v}(t) &= \mathbf{r}'(t) = -2t\,\mathbf{i} + 1\,\mathbf{j}, \\
\text{speed}(t) &= \|v(t)\| = \sqrt{(-2t)^2 + (1)^2} = \sqrt{4t^2 + 1}, \\
\mathbf{a}(t) &= \mathbf{v}'(t) = -2\,\mathbf{i}, \\
\backslash\text{vecd}\text{jerk}(t) &= \mathbf{0}
\end{aligned}$$

b. Evaluating these functions at time $t = 0$, we get:

$$\begin{aligned}
\mathbf{r}(0) &= (1 - 0^2)\,\mathbf{i} + 0\,\mathbf{j} = \mathbf{i} \\
\mathbf{v}(0) &= -2(0)\,\mathbf{i} + 1\,\mathbf{j} = \mathbf{j} \\
\text{speed}(0) &= \sqrt{4(0)^2 + 1} = \sqrt{1} = 1\,\text{unit/sec} \\
\mathbf{a}(0) &= -2\,\mathbf{i}, \\
\backslash\text{vecd}\text{jerk}(0) &= \mathbf{0}
\end{aligned}$$

And for time $t = 2$, we have

$$\begin{aligned}
\mathbf{r}(2) &= (1 - 2^2)\,\mathbf{i} + 2\,\mathbf{j} = -3\,\mathbf{i} + 2\,\mathbf{j} \\
\mathbf{v}(2) &= -2(2)\,\mathbf{i} + 1\,\mathbf{j} = -4\,\mathbf{i} + \mathbf{j} \\
\text{speed}(2) &= \sqrt{4(2)^2 + 1} = \sqrt{17}\,\text{units/sec} \\
\mathbf{a}(0) &= -2\,\mathbf{i}, \\
\backslash\text{vecd}\text{jerk}(0) &= \mathbf{0}
\end{aligned}$$

c. The graph of this curve is shown below in Figure 5.2.2. We then drew in and labeled the vectors $\mathbf{v}(0)$, $\mathbf{a}(0)$, $\mathbf{v}(2)$, and $\mathbf{a}(2)$.

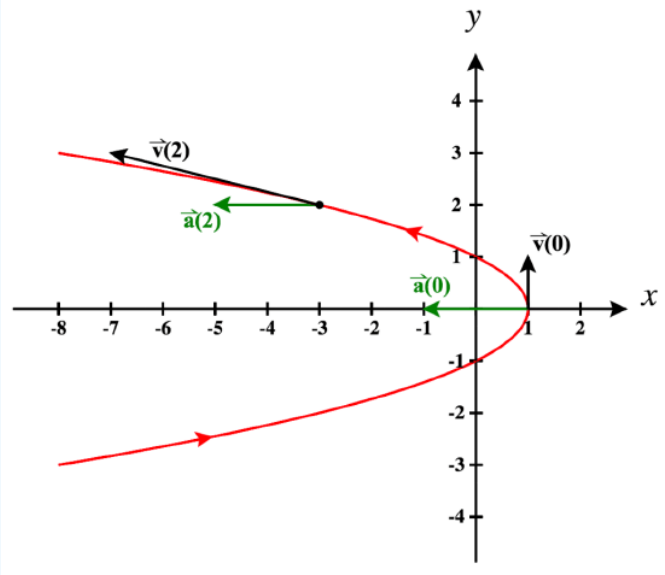


Figure 5.2.2: The graph of a vector-valued function depicting the position of an object is shown with a red curve. The x values range from -8 to 2, and the y values range from -4 to 4. The red curve starts in the third quadrant at about $(-8,-3)$, passes through the point $(1,0)$ and curves towards the point $(-8,3)$ in the second quadrant. The curve is concave left. There are vectors illustrated with arrows along the curve. At the point $(1,0)$, a black arrow labeled $\mathbf{v}(0)$ points upward indicating the velocity at that point, and green arrow labeled $\mathbf{a}(0)$ points leftward, indicating the acceleration. At the point $(-3,2)$, a black arrow labeled $\mathbf{v}(2)$ points tangent to the curve indicating the velocity at that point, and green arrow labeled $\mathbf{a}(2)$ points leftward, indicating the acceleration. In each case, the velocity vector is tangent to the curve and the acceleration vector points in the "inside" of the curve.

Smooth Vector-Valued Functions

Definition: Smoothness

Let $\mathbf{r}(t) = f(t)\ \mathbf{i} + g(t)\ \mathbf{j} + h(t)\ \mathbf{k}$ be the parameterization of a curve that is differentiable on an open interval I.

Then $\mathbf{r}(t)$ is **smooth** on the open interval I, if

$$\mathbf{r}'(t) \neq \mathbf{0}, \text{ for any value of } t \text{ in the interval } I.$$

To put this another way, $\mathbf{r}(t)$ is **smooth** on the open interval I if:

1. $\mathbf{r}(t)$ is defined and continuous on I, and
2. $f',\ g'$ and h' are all continuous on I, that is, $\mathbf{r}'(t)$ is defined and continuous on I, and
3. $\mathbf{r}'(t) \neq \mathbf{0}$ for all values of t in the interval I.

The concept of smoothness is very closely connected to the application of vector-valued functions to motion. If you have ever ridden rides at an amusement park, you can probably identify some where the ride felt smooth, and others where there are moments during the ride where it was not smooth. These moments often correspond to points in the motion where the relevant parameterization is not smooth by the definition above.

To identify open intervals in the domain of $\mathbf{r}$ on which $\mathbf{r}$ is smooth, we usually identify values of t for which $\mathbf{r}$ is smooth.

In addition to using the definition above to identify these points, it is often apparent from the graph of the vector-valued function $\mathbf{r}(t)$. A **cusp**, i.e., a sharp turning point, is a common occurrence at points where a curve is not smooth. In Figure 5.2.3 below, you can quickly identify points at which the parameterization fails to be smooth by locating cusps in the graph.

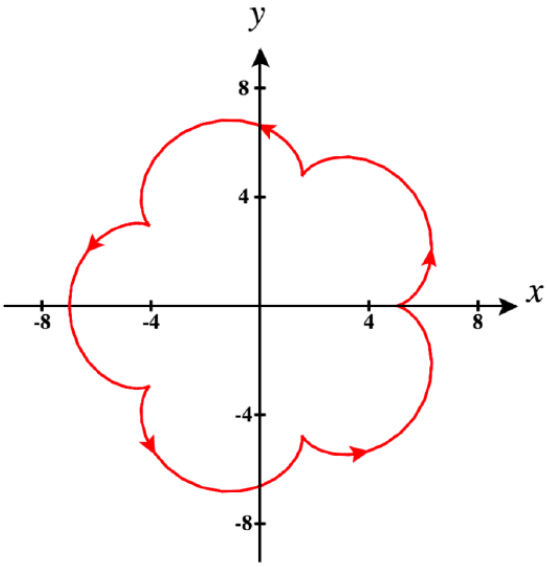


Figure 5.2.3: This epicycloid is a curve resembling a 5-petaled flower, symmetrically distributed around the origin. Each petal curves smoothly outward, reaching peak at a distance of about 7 from the origin, and forming a circular path back towards the origin, reaching a distance about 5 from the origin. The orientation is counterclockwise. The epicycloid is not smooth at the points between its petals. There are five such points on this graph.

When we consider the motion described by $\mathbf{r}(t)$, there are at least two additional events that can occur at points where the parameterization fails to be smooth: a reversal of motion or a momentary pause in the motion. To summarize these events:

If a vector-valued function $\mathbf{r}(t)$ is *not* smooth at time t, we will observe that:

1. There is a cusp at the associated point on the graph of $\mathbf{r}(t)$, or
2. The motion reverses itself at the associated point, causing the motion to travel back along the same path in the opposite direction, or
3. The motion actually stops and starts up again, with no visual cue, that is, where the curve appears smooth.

Let's consider a couple examples to see how this can look.

Example 5.2.4: Determining open intervals on which a vector-valued function is smooth

Determine the open intervals on which each of the following vector-valued functions is smooth:

a. $\mathbf{r}(t) = (2t^3 - 3t^2) \quad \mathbf{i} + (t^2 - 2t) \quad \mathbf{j} + 2 \quad \mathbf{k}$
b. $\mathbf{r}(t) = t^3 \quad \mathbf{i} + \sqrt{t^2 - 1}\ \mathbf{j}$

Solution

a. First we need to consider the domain of $\mathbf{r}$. In this case, $\mathrm{D_r} : (-\infty, \infty)$.

Then we need to find $\mathbf{r}'(t)$. Since $\mathbf{r}(t) = (2t^3 - 3t^2) \quad \mathbf{i} + (t^2 - 2t) \quad \mathbf{j} + 2 \quad \mathbf{k}$,

$$\mathbf{r}'(t) = \left(6t^2 - 6t\right)\ \mathbf{i} + (2t - 2)\ \mathbf{j} \tag{5.2.13}$$

Next we need to consider the domain of $\mathbf{r}'$. Here, $\mathrm{D_{r'}} : (-\infty, \infty)$.

Finally we need to determine if there are any values of t in $\mathrm{D_r}$ where $\mathbf{r}'(t) = \mathbf{0}$.

To do this, we set each component of $\mathbf{r}'(t)$ equal to 0 and solve for t.

$$t^2 - 6t = 0 \quad \text{and} \quad 2t - 2 = 0 \tag{5.2.14}$$
$$6t(t-1) = 0 \qquad 2(t-1) = 0 \tag{5.2.15}$$
$$t = 0 \quad \text{or} \quad t = 1 \qquad t = 1 \tag{5.2.16}$$

Since $t = 1$ makes both components of $\mathbf{r}'$ equal $\mathbf{0}$, we have $\mathbf{r}'(1) = \mathbf{0}$, so we know that $\mathbf{r}$ is *not* smooth at $t = 1$.

Note however that $\mathbf{r}$ is smooth at $t = 0$, because although one of the components of $\mathbf{r}'$ is 0, the other one is not.

Therefore we can conclude that $\mathbf{r}$ is smooth on the open intervals:

$$(-\infty, 1) \quad \text{and} \quad (1, \infty) \tag{5.2.17}$$

Note in the Figure $5.2.4$, there is a **cusp** in the curve when $t = 1$.

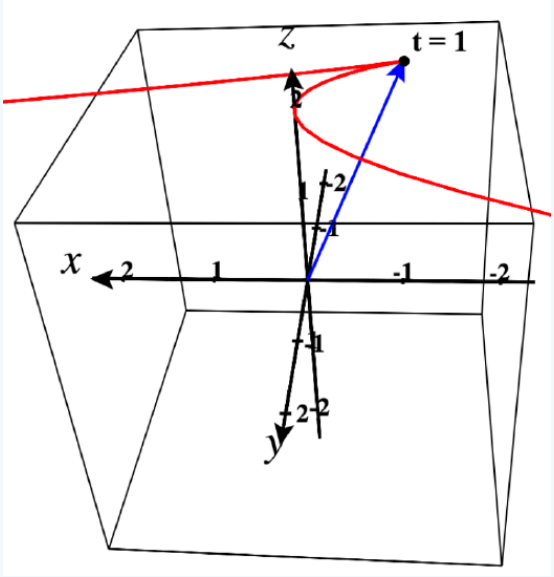


Figure $5.2.4$: Plot of $\mathbf{r}(t) = (2t^3 - 3t^2) \quad \mathbf{i} + (t^2 - 2t) \quad \mathbf{j} + 2 \quad \mathbf{k}$. The curve is shown in red curve navigates through the space. There is a blue arrow pointing to the position on the curve at $t = 1$. At that position, the red curve has a cusp, meaning that it approaches the point and turns in the opposite direction, indicating that $\mathbf{r}'(1) = \vec{0}$.

b. The domain of $\mathbf{r}$ is $\mathrm{D_r} : (-\infty, -1] \cup [1, \infty)$, since $t^2 - 1$ is less than 0 between -1 and 1.

Now we calculate $\mathbf{r}'(t)$. Since $\mathbf{r}(t) = t^3 \quad \mathbf{i} + \sqrt{t^2 - 1}\ \mathbf{j}$, we have

$$\mathbf{r}'(t) = 3t^2\ \mathbf{i} + \frac{t}{\sqrt{t^2 - 1}}\ \mathbf{j} \tag{5.2.18}$$

Then, since we cannot have zero in the denominator, $t = -1$ and $t = 1$ are not in the domain of $\mathbf{r}'$. Thus, $\mathrm{D_{r'}} : (-\infty, -1) \cup (1, \infty)$.

Lastly we determine if there are any values of t in $\mathrm{D_r}$ where $\mathbf{r}'(t) = \mathbf{0}$.

Setting each component of $\mathbf{r}'(t)$ equal to 0 and solve for t we get:

$$t^2 = 0 \quad \text{and} \quad \frac{t}{\sqrt{t^2 - 1}} = 0 \tag{5.2.19}$$
$$t = 0 \qquad t = 0 \tag{5.2.20}$$

If $t = 0$ were in the domain of $\mathbf{r}'$, then $\mathbf{r}$ would not be smooth there, but since it is not in the domain, we have nothing to remove, and we can conclude that $\mathbf{r}$ is smooth on the entire domain of its derivative $\mathbf{r}'$, that is,

$$\mathbf{r} \text{ is smooth on the open intervals:} \quad (-\infty, -1) \quad \text{and} \quad (1, \infty). \tag{5.2.21}$$

Observe the two separate smooth pieces of this function in the graph below.

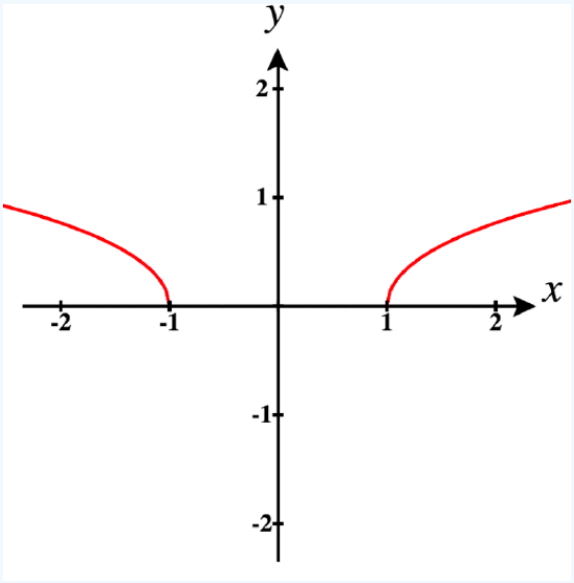


Figure 5.2.5: Graph of $\mathbf{r}(t) = t^3 \quad \mathbf{i} + \sqrt{t^2 - 1}\ \mathbf{j}$. The curve is mapped in two segments that are symmetric across the y-axis. The right segment passes through the point $(1, 0)$ and increases with the graph concave down over $x \geq 1$. The left segment is the mirror image of that.

Watch the accompanying video for a visual demonstration of this concept.

- **Video Length**: 3 minute 13 seconds.
- **Context**: This video demonstrates how to determine where a vector-valued function is smooth for $\mathbf{r}(t) = (t + \frac{1}{t}) \quad \mathbf{i} + (t^2 - 2t)\ \mathbf{j}$

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

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Exercise 5.2.2

The epicycloid in Figure 5.2.3 is given by $\mathbf{r}(t) = (6\cos t - \cos(6t)) \quad \mathbf{i} + (6\sin t - \sin(6t)) \quad \mathbf{j}$, for $0 \leq t \leq 2\pi$. At what values of t do the cusps occur?

Answer available at the end of this section.

We can extend to vector-valued functions the properties of the derivative that we presented previously. In particular, the constant multiple rule, the sum and difference rules, the product rule, and the chain rule all extend to vector-valued functions. However, in the case of the product rule, there are actually three extensions:

1. for a real-valued function multiplied by a vector-valued function,
2. for the dot product of two vector-valued functions, and
3. for the cross product of two vector-valued functions.

Theorem: Properties of the Derivative of Vector-Valued Functions

Let $\mathbf{r}$ and $\mathbf{u}$ be differentiable vector-valued functions of t, let f be a differentiable real-valued function of t, and let c be a scalar.

i.	$\frac{d}{dt}[c\mathbf{r}(t)]$	$=$	$c\mathbf{r}'(t)$	Scalar multiple
ii.	$\frac{d}{dt}[\mathbf{r}(t) \pm \mathbf{u}(t)]$	$=$	$\mathbf{r}'(t) \pm \mathbf{u}'(t)$	Sum and difference
iii.	$\frac{d}{dt}[f(t)\mathbf{u}(t)]$	$=$	$f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$	Scalar product
iv.	$\frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{u}(t)]$	$=$	$\mathbf{r}'(t) \cdot \mathbf{u}(t) + \mathbf{r}(t) \cdot \mathbf{u}'(t)$	Dot product
v.	$\frac{d}{dt}[\mathbf{r}(t) \times \mathbf{u}(t)]$	$=$	$\mathbf{r}'(t) \times \mathbf{u}(t) + \mathbf{r}(t) \times \mathbf{u}'(t)$	Cross product
vi.	$\frac{d}{dt}[\mathbf{r}(f(t))]$	$=$	$\mathbf{r}'(f(t)) \cdot f'(t)$	Chain rule
vii.	If $\mathbf{r}(t) \cdot \mathbf{r}(t)$	$=$	c, then $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$.	

Proof

The proofs of the first two properties follow directly from the definition of the derivative of a vector-valued function. The third property can be derived from the first two properties, along with the product rule. Let $\mathbf{u}(t) = g(t)\ \mathbf{i} + h(t)\ \mathbf{j}$. Then

$$\begin{aligned}\frac{d}{dt}[f(t)\mathbf{u}(t)] &= \frac{d}{dt}[f(t)(g(t)\,\mathbf{i} + h(t)\,\mathbf{j})] \\ &= \frac{d}{dt}[f(t)g(t)\,\mathbf{i} + f(t)h(t)\,\mathbf{j}] \\ &= \frac{d}{dt}[f(t)g(t)]\,\mathbf{i} + \frac{d}{dt}[f(t)h(t)]\,\mathbf{j} \\ &= (f'(t)g(t) + f(t)g'(t))\,\mathbf{i} + (f'(t)h(t) + f(t)h'(t))\,\mathbf{j} \\ &= f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t).\end{aligned}$$

To prove property iv. let $\mathbf{r}(t) = f_1(t)\ \mathbf{i} + g_1(t)\ \mathbf{j}$ and $\mathbf{u}(t) = f_2(t)\ \mathbf{i} + g_2(t)\ \mathbf{j}$. Then

$$\begin{aligned}\frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{u}(t)] &= \frac{d}{dt}[f_1(t)f_2(t) + g_1(t)g_2(t)] \\ &= f_1{}'(t)f_2(t) + f_1(t)f_2{}'(t) + g_1{}'(t)g_2(t) + g_1(t)g_2{}'(t) = f_1{}'(t)f_2(t) + g_1{}'(t)g_2(t) + f_1(t)f_2{}'(t) + g_1(t)g_2{}'(t) \\ &= (f_1{}'\,\mathbf{i} + g_1{}'\,\mathbf{j}) \cdot (f_2\,\mathbf{i} + g_2\,\mathbf{j}) + (f_1\,\mathbf{i} + g_1\,\mathbf{j}) \cdot (f_2{}'\,\mathbf{i} + g_2{}'\,\mathbf{j}) \\ &= \mathbf{r}'(t) \cdot \mathbf{u}(t) + \mathbf{r}(t) \cdot \mathbf{u}'(t).\end{aligned}$$

The proof of property v. is similar to that of property iv. Property vi. can be proved using the chain rule. Last, property vii. follows from property iv:

$$\begin{aligned}\frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{r}(t)] &= \frac{d}{dt}[c] \\ \mathbf{r}'(t) \cdot \mathbf{r}(t) + \mathbf{r}(t) \cdot \mathbf{r}'(t) &= 0 \\ 2\mathbf{r}(t) \cdot \mathbf{r}'(t) &= 0 \\ \mathbf{r}(t) \cdot \mathbf{r}'(t) &= 0\end{aligned}$$

Example 5.2.5

Show that $\frac{d}{dt}[\mathbf{r}(f(t))] = \mathbf{r}\,'(f(t)) \cdot f\,'(t)$

Solution

Video length: 2 minutes 53 seconds

Context: This video demonstrates how to prove the chain rule for the derivative of a vector valued function composed with a function.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Now for some examples using these properties.

Example 5.2.6: Using the Properties of Derivatives of Vector-Valued Functions

Given the vector-valued functions

$$\mathbf{r}(t) = (6t+8)\,\mathbf{i} + (4t^2+2t-3)\,\mathbf{j} + 5t\,\mathbf{k}$$

and

$$\mathbf{u}(t) = (t^2-3)\,\mathbf{i} + (2t+4)\,\mathbf{j} + (t^3-3t)\,\mathbf{k},$$

calculate each of the following derivatives using the properties of the derivative of vector-valued functions.

a. $\dfrac{d}{dt}[\mathbf{r}(t)\cdot\mathbf{u}(t)]$
b. $\dfrac{d}{dt}[\mathbf{u}(t)\times\mathbf{u}\,'(t)]$

Solution

We have $\mathbf{r}\,'(t) = 6\ \mathbf{i} + (8t+2)\ \mathbf{j} + 5\ \mathbf{k}$ and $\mathbf{u}\,'(t) = 2t\ \mathbf{i} + 2\ \mathbf{j} + (3t^2-3)\ \mathbf{k}$. Therefore, according to **property iv**:

a.

$$\begin{aligned}\frac{d}{dt}[\mathbf{r}(t)\cdot\mathbf{u}(t)] &= \mathbf{r}'(t)\cdot\mathbf{u}(t) + \mathbf{r}(t)\cdot\mathbf{u}'(t)\\ &= (6\,\mathbf{i}+(8t+2)\,\mathbf{j}+5\,\mathbf{k})\cdot((t^2-3)\,\mathbf{i}+(2t+4)\,\mathbf{j}+(t^3-3t)\,\mathbf{k})\\ &+((6t+8)\,\mathbf{i}+(4t^2+2t-3)\,\mathbf{j}+5t\,\mathbf{k})\cdot(2t\,\mathbf{i}+2\,\mathbf{j}+(3t^2-3)\,\mathbf{k})\\ &= 6(t^2-3)+(8t+2)(2t+4)+5(t^3-3t)\\ &+2t(6t+8)+2(4t^2+2t-3)+5t(3t^2-3)\\ &= 20t^3+42t^2+26t-16.\end{aligned}$$

b. First, we need to adapt **property v** for this problem:

$$\frac{d}{dt}[\mathbf{u}(t)\times\mathbf{u}'(t)] = \mathbf{u}'(t)\times\mathbf{u}'(t) + \mathbf{u}(t)\times\mathbf{u}''(t).$$

Recall that the cross product of any vector with itself is zero. Furthermore,$\mathbf{u}\,''(t)$ represents the second derivative of $\mathbf{u}(t)$:

$$\mathbf{u}''(t) = \frac{d}{dt}[\mathbf{u}'(t)] = \frac{d}{dt}[2t\,\mathbf{i}+2\,\mathbf{j}+(3t^2-3)\,\mathbf{k}] = 2\,\mathbf{i}+6t\,\mathbf{k}.$$

Therefore,

$$\begin{aligned}\frac{d}{dt}[\mathbf{u}(t)\times\mathbf{u}'(t)] &= 0+((t^2-3)\,\mathbf{i}+(2t+4)\,\mathbf{j}+(t^3-3t)\,\mathbf{k})\times(2\,\mathbf{i}+6t\,\mathbf{k})\\ &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ t^2-3 & 2t+4 & t^3-3t\\ 2 & 0 & 6t\end{vmatrix}\\ &= 6t(2t+4)\,\mathbf{i}-(6t(t^2-3)-2(t^3-3t))\,\mathbf{j}-2(2t+4)\,\mathbf{k}\\ &= (12t^2+24t)\,\mathbf{i}+(12t-4t^3)\,\mathbf{j}-(4t+8)\,\mathbf{k}.\end{aligned}$$

Exercise 5.2.3

Calculate $\frac{d}{dt}[\mathbf{r}(t)\cdot\mathbf{r}\,'(t)]$ and $\frac{d}{dt}[\mathbf{u}(t)\times\mathbf{r}(t)]$ for the vector-valued functions:

- $\mathbf{r}(t)=\cos t\ \mathbf{i}+\sin t\ \mathbf{j}-e^{2t}\ \mathbf{k}$
- $\mathbf{u}(t)=t\ \mathbf{i}+\sin t\ \mathbf{j}+\cos t\ \mathbf{k}$,

Hint available at the end of this section.

Answer available at the end of this section.

Tangent Vectors and Unit Tangent Vectors

Recall that the derivative at a point can be interpreted as the slope of the tangent line to the graph at that point. In the case of a vector-valued function, the derivative provides a tangent vector to the curve represented by the function. Consider the vector-valued function

$$\mathbf{r}(t)=\cos t\,\mathbf{i}+\sin t\,\mathbf{j} \tag{5.2.22}$$

The derivative of this function is

$$\mathbf{r}'(t)=-\sin t\,\mathbf{i}+\cos t\,\mathbf{j}$$

If we substitute the value $t=\pi/6$ into both functions we get

$$\mathbf{r}\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}\,\mathbf{i}+\frac{1}{2}\,\mathbf{j}$$

and

$$\mathbf{r}'\left(\frac{\pi}{6}\right)=-\frac{1}{2}\,\mathbf{i}+\frac{\sqrt{3}}{2}\,\mathbf{j}.$$

The graph of this function appears in Figure 5.2.1, along with the vectors $\mathbf{r}\left(\frac{\pi}{6}\right)$ and $\mathbf{r}'\left(\frac{\pi}{6}\right)$.

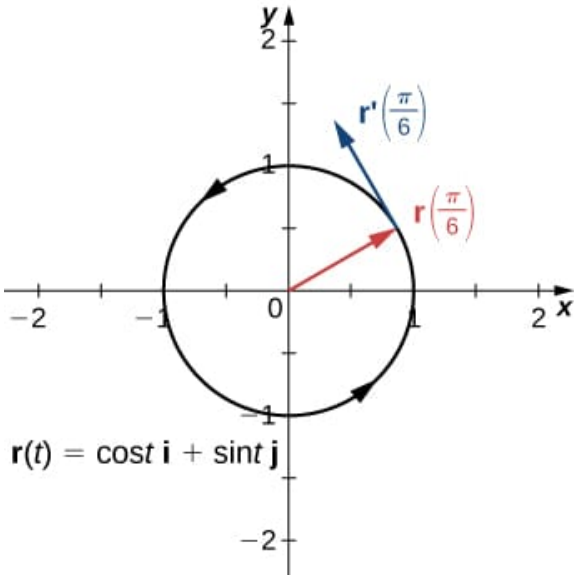


Figure 5.2.6: This is the graph of a circle represented by the vector-valued function $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$. It is a circle centered at the origin with radius of 1, and counter-clockwise orientation. It has a vector from the origin pointing to the curve and labeled $\mathbf{r}(\pi/6)$. At the same point on the circle there is a tangent vector labeled $\mathbf{r}'(\pi/6)$.

Notice that the vector $\mathbf{r}\,'\left(\frac{\pi}{6}\right)$ is tangent to the circle at the point corresponding to $t = \frac{\pi}{6}$. This is an example of a tangent vector to the plane curve defined by Equation 5.2.22.

Definition: Principal Unit Tangent Vector

Let C be a curve defined by a vector-valued function $\mathbf{r}$, and assume that $\mathbf{r}\,'(t)$ exists when $\mathrm{t} = \mathrm{t}_0$ A tangent vector $\mathbf{r}$ at $t = t_0$ is any vector such that, when the tail of the vector is placed at point $\mathbf{r}(t_0)$ on the graph, vector $\mathbf{r}$ is tangent to curve C. Vector $\mathbf{r}\,'(t_0)$ is an example of a tangent vector at point $t = t_0$. Furthermore, assume that $\mathbf{r}\,'(t) \neq 0$. The **principal unit tangent** vector at t is defined to be

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}, \tag{5.2.23}$$

provided $\|\mathbf{r}\,'(t)\| \neq 0$.

The unit tangent vector is exactly what it sounds like: a unit vector that is tangent to the curve. To calculate a unit tangent vector, first find the derivative $\mathbf{r}\,'(t)$. Second, calculate the magnitude of the derivative. The third step is to divide the derivative by its magnitude.

Example 5.2.7: Finding a Unit Tangent Vector

Find the unit tangent vector for each of the following vector-valued functions:

a. $\mathbf{r}(t) = \cos t\ \mathbf{i} + \sin t\ \mathbf{j}$
b. $\mathbf{u}(t) = (3t^2 + 2t)\ \mathbf{i} + (2 - 4t^3)\ \mathbf{j} + (6t + 5)\ \mathbf{k}$

Solution

a.

$$\begin{aligned}
\text{First step:} \quad \mathbf{r}'(t) &= -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} \\
\text{Second step:} \quad \|\mathbf{r}'(t)\| &= \sqrt{(-\sin t)^2 + (\cos t)^2} = 1 \\
\text{Third step:} \quad \mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{-\sin t\,\mathbf{i} + \cos t\,\mathbf{j}}{1} = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}
\end{aligned}$$

b.

$$\begin{aligned}
\text{First step:} \quad \mathbf{r}'(t) &= (6t+2)\,\mathbf{i} - 12t^2\,\mathbf{j} + 6\,\mathbf{k} \\
\text{Second step:} \quad \|\mathbf{r}'(t)\| &= \sqrt{(6t+2)^2 + (-12t^2)^2 + 6^2} \\
&= \sqrt{144t^4 + 36t^2 + 24t + 40} \\
&= 2\sqrt{36t^4 + 9t^2 + 6t + 10} \\
\text{Third step:} \quad \mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{(6t+2)\,\mathbf{i} - 12t^2\,\mathbf{j} + 6\,\mathbf{k}}{2\sqrt{36t^4 + 9t^2 + 6t + 10}} \\
&= \frac{3t+1}{\sqrt{36t^4 + 9t^2 + 6t + 10}}\,\mathbf{i} - \frac{6t^2}{\sqrt{36t^4 + 9t^2 + 6t + 10}}\,\mathbf{j} + \frac{3}{\sqrt{36t^4 + 9t^2 + 6t + 10}}\,\mathbf{k}
\end{aligned}$$

Exercise 5.2.4

Find the unit tangent vector for the vector-valued function

$$\mathbf{r}(t) = (t^2 - 3)\,\mathbf{i} + (2t+1)\,\mathbf{j} + (t-2)\,\mathbf{k}.$$

Hint available at the end of this section.

Answer available at the end of this section.

Interactive Element

Observing the Tangent Vectors of a Vectors Valued Function

Below shows the graph of a vector valued function, but a vector values function is more than just a static graph. If you hit the Animate button, you will see the tangent vector move and change as time, t progresses. You can also choose to observe the unit tangent vector.

\(\vecs{r}(t)=, -3

\(\vecs{r}(t)=<\frac{5\cos(t)}{t}, \frac{5\sin(t)}{t}>, 1

\(\vecs{r}(t)=<3\cos(5t)\cos(t), 3\cos(5t)\sin(t)>, 0

\(\vecs{r}(t)=<2\sin(t), 4\sin^2(t)>, 0

\(\vecs{r}(t)=, -3

\(\vecs{r}(t)=<3\cos(-t), 4\sin(-t)>, 0

\(\vecs{r}(t)=, 0

\(\vecs{r}(t)=, -2

\(\vecs{r}(t)=, -2

\(\vecs{r}(t)=<\frac{10}{1+t^2}, \frac{10}{1+t^4}>, -1

Tangent Vector Unit Tangent Vector

Animate Next

Integrals of Vector-Valued Functions

We introduced antiderivatives of real-valued functions in Antiderivatives and definite integrals of real-valued functions in The Definite Integral. Each of these concepts can be extended to vector-valued functions. Also, just as we can calculate the derivative of a vector-valued function by differentiating the component functions separately, we can calculate the antiderivative in the same manner. Furthermore, the Fundamental Theorem of Calculus applies to vector-valued functions as well.

The antiderivative of a vector-valued function appears in applications. For example, if a vector-valued function represents the velocity of an object at time *t*, then its antiderivative represents position. Or, if the function represents the acceleration of the object at a given time, then the antiderivative represents its velocity.

Definition: Definite and Indefinite Integrals of Vector-Valued Functions

Let f, g, and h be integrable real-valued functions over the closed interval $[a, b]$.

1. The **indefinite integral** of a vector-valued function $\mathbf{r}(t) = f(t)\ \mathbf{i} + g(t)\ \mathbf{j}$ is

$$\int [f(t)\,\mathbf{i} + g(t)\,\mathbf{j}]\,dt = \left[\int f(t)\,dt\right]\,\mathbf{i} + \left[\int g(t)\,dt\right]\,\mathbf{j}. \tag{5.2.24}$$

   The **definite integral** of a vector-valued function is

$$\int_a^b [f(t)\,\mathbf{i} + g(t)\,\mathbf{j}]\,dt = \left[\int_a^b f(t)\,dt\right]\,\mathbf{i} + \left[\int_a^b g(t)\,dt\right]\,\mathbf{j}. \tag{5.2.25}$$

2. The **indefinite integral** of a vector-valued function $\mathbf{r}(t) = f(t)\ \mathbf{i} + g(t)\ \mathbf{j} + h(t)\ \mathbf{k}$ is

$$\int [f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}]\,dt = \left[\int f(t)\,dt\right]\,\mathbf{i} + \left[\int g(t)\,dt\right]\,\mathbf{j} + \left[\int h(t)\,dt\right]\,\mathbf{k}. \tag{5.2.26}$$

   The **definite integral** of the vector-valued function is

$$\int_a^b [f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}]\,dt = \left[\int_a^b f(t)\,dt\right]\,\mathbf{i} + \left[\int_a^b g(t)\,dt\right]\,\mathbf{j} + \left[\int_a^b h(t)\,dt\right]\,\mathbf{k}. \tag{5.2.27}$$

Since the indefinite integral of a vector-valued function involves indefinite integrals of the component functions, each of these component integrals contains an integration constant. They can all be different. For example, in the two-dimensional case, we can have

$$\int f(t)\,dt = F(t) + C_1 \text{ and } \int g(t)\,dt = G(t) + C_2,$$

where F and G are antiderivatives of f and g, respectively. Then

$$\begin{aligned}\int [f(t)\,\mathbf{i} + g(t)\,\mathbf{j}]\,dt &= \left[\int f(t)\,dt\right]\,\mathbf{i} + \left[\int g(t)\,dt\right]\,\mathbf{j} \\ &= (F(t) + C_1)\,\mathbf{i} + (G(t) + C_2)\,\mathbf{j} \\ &= F(t)\,\mathbf{i} + G(t)\,\mathbf{j} + C_1\,\mathbf{i} + C_2\,\mathbf{j} \\ &= F(t)\,\mathbf{i} + G(t)\,\mathbf{j} + \mathbf{C}\end{aligned}$$

where $\mathbf{C} = C_1\ \mathbf{i} + C_2\ \mathbf{j}$. Therefore, the *integration constants* becomes a *constant vector*.

Example 5.2.8: Integrating Vector-Valued Functions

Calculate each of the following integrals:

a. $\int [(3t^2 + 2t)\ \mathbf{i} + (3t - 6)\ \mathbf{j} + (6t^3 + 5t^2 - 4)\ \mathbf{k}]\ dt$

b. $\int [\langle t, t^2, t^3\rangle \times \langle t^3, t^2, t\rangle]\ dt$

c. $\int_0^{\frac{\pi}{3}} [\sin 2t\ \mathbf{i} + \tan t\ \mathbf{j} + e^{-2t}\ \mathbf{k}]\ dt$

Solution

a. We use the first part of the definition of the integral of a space curve:

$$\begin{aligned}\int [(3t^2 + 2t)\,\mathbf{i} + (3t - 6)\,\mathbf{j} + (6t^3 + 5t^2 - 4)\,\mathbf{k}]\,dt &= \left[\int 3t^2 + 2t\,dt\right]\,\mathbf{i} + \left[\int 3t - 6\,dt\right]\,\mathbf{j} + \left[\int 6t^3 + 5t^2 - 4\,dt\right]\,\mathbf{k} \\ &= (t^3 + t^2)\,\mathbf{i} + \left(\frac{3}{2}t^2 - 6t\right)\,\mathbf{j} + \left(\frac{3}{2}t^4 + \frac{5}{3}t^3 - 4t\right)\,\mathbf{k} + \mathbf{C}.\end{aligned}$$

b. First calculate $\langle t, t^2, t^3\rangle \times \langle t^3, t^2, t\rangle$:

$$\begin{aligned}\langle t, t^2, t^3\rangle \times \langle t^3, t^2, t\rangle &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & t^2 & t^3 \\ t^3 & t^2 & t \end{vmatrix} \\ &= (t^2(t) - t^3(t^2))\,\mathbf{i} - (t^2 - t^3(t^3))\,\mathbf{j} + (t(t^2) - t^2(t^3))\,\mathbf{k} \\ &= (t^3 - t^5)\,\mathbf{i} + (t^6 - t^2)\,\mathbf{j} + (t^3 - t^5)\,\mathbf{k}.\end{aligned}$$

Next, substitute this back into the integral and integrate:

$$\begin{aligned}\int [\langle t, t^2, t^3\rangle \times \langle t^3, t^2, t\rangle]\,dt &= \int (t^3 - t^5)\,\mathbf{i} + (t^6 - t^2)\,\mathbf{j} + (t^3 - t^5)\,\mathbf{k}\,dt \\ &= \left(\frac{t^4}{4} - \frac{t^6}{6}\right)\,\mathbf{i} + \left(\frac{t^7}{7} - \frac{t^3}{3}\right)\,\mathbf{j} + \left(\frac{t^4}{4} - \frac{t^6}{6}\right)\,\mathbf{k} + \mathbf{C}.\end{aligned}$$

c. Use the second part of the definition of the integral of a space curve:

$$\begin{aligned}\int_0^{\frac{\pi}{3}} [\sin 2t\,\mathbf{i} + \tan t\,\mathbf{j} + e^{-2t}\,\mathbf{k}]\,dt &= \left[\int_0^{\frac{\pi}{3}} \sin 2t\,dt\right]\,\mathbf{i} + \left[\int_0^{\frac{\pi}{3}} \tan t\,dt\right]\,\mathbf{j} + \left[\int_0^{\frac{\pi}{3}} e^{-2t}\,dt\right]\,\mathbf{k} \\ &= \left(-\tfrac{1}{2}\cos 2t\right)\Big|_0^{\pi/3}\,\mathbf{i} - \left(\ln|\cos t|\right)\Big|_0^{\pi/3}\,\mathbf{j} - \left(\tfrac{1}{2}e^{-2t}\right)\Big|_0^{\pi/3}\,\mathbf{k} \\ &= \left(-\tfrac{1}{2}\cos\tfrac{2\pi}{3} + \tfrac{1}{2}\cos 0\right)\,\mathbf{i} - \left(\ln\left(\cos\tfrac{\pi}{3}\right) - \ln(\cos 0)\right)\,\mathbf{j} - \left(\tfrac{1}{2}e^{-2\pi/3} - \tfrac{1}{2}e^{-2(0)}\right)\,\mathbf{k} \\ &= \left(\tfrac{1}{4} + \tfrac{1}{2}\right)\,\mathbf{i} - (-\ln 2)\,\mathbf{j} - \left(\tfrac{1}{2}e^{-2\pi/3} - \tfrac{1}{2}\right)\,\mathbf{k} \\ &= \tfrac{3}{4}\,\mathbf{i} + (\ln 2)\,\mathbf{j} + \left(\tfrac{1}{2} - \tfrac{1}{2}e^{-2\pi/3}\right)\,\mathbf{k}.\end{aligned}$$

Watch the accompanying video for a visual demonstration of this concept.

- **Video Length**: 1 minute 37 seconds.
- **Context**: This video demonstrates how to integrate a vector-valued function: Find the indefinite integral $\int [\cos t\ \mathbf{i} - \sin t\ \mathbf{j} + e^t\ \mathbf{k}]\ dt$

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Exercise 5.2.5

Calculate the following integral:

$$\int_1^3 [(2t+4)\,\mathbf{i} + (3t^2 - 4t)\,\mathbf{j}]\,dt$$

Hint available at the end of this section.

Answer available at the end of this section.

Example 5.2.9

Find $\mathbf{r}(t)$ given that $\mathbf{r}''(t) = 2\cos t\mathbf{i} - \sin t\mathbf{k}$, $\mathbf{r}'(0) = 2\mathbf{j}$, $\mathbf{r}(0) = \mathbf{j} - \mathbf{k}$.

Solution

Video Length: 5 minutes 30 seconds

Context: This video demonstrates how to solve an initial value problem for a vector differential equation.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Integration in the Context of Motion

Above, we defined velocity as the derivative of position and acceleration as the derivative of velocity. Integration allows us to go the other way! As you learned in first semester calculus, integration allows us to generate an object's velocity as a function of time given its acceleration and an initial velocity. We can also find the object's position function by integrating the velocity and using an initial position.

Let's consider an example.

Example 5.2.10: Finding an object's velocity from its acceleration

An object has an acceleration over time given by $\mathbf{a}(t) = \sin t\ \mathbf{i} - 2t\ \mathbf{j}$, and it's initial velocity was $\mathbf{v}(0) = 2\ \mathbf{i} + \ \mathbf{j}$.

a. Find the object's velocity as a function of time.
b. Assuming the object was located at the point (2, 3, -1) when time $t = 0$, determine the object's position function and find its location at time $t = 3$ sec.

Solution

a. First, we find the velocity as the antiderivative of the acceleration.

$$\begin{aligned}\mathbf{v}(t) &= \int \mathbf{a}(t)\ dt = \int (\sin t\,\mathbf{i} - 2t\,\mathbf{j})\ dt \\ &= -\cos t\,\mathbf{i} - t^2\,\mathbf{j} + \mathbf{C}_1\end{aligned}$$

Now we use the initial velocity to determine $\mathbf{C}_1$.

$$\begin{aligned}\mathbf{v}(0) = -\cos 0\,\mathbf{i} - (0)^2\,\mathbf{j} + \mathbf{C}_1 &= 2\,\mathbf{i} + \mathbf{j} \\ -\mathbf{i} + \mathbf{C}_1 &= 2\,\mathbf{i} + \mathbf{j}\end{aligned}$$

$$\text{And so} \quad \mathbf{C}_1 = 3\,\mathbf{i} + \mathbf{j}$$

Incorporating this constant vector into our velocity function from above, we obtain the velocity describing this object's motion over time:

$$\mathbf{v}(t) = (3 - \cos t)\ \mathbf{i} + \left(1 - t^2\right)\ \mathbf{j} \tag{5.2.28}$$

b. Since the object's position is at the point (2, 3, -1) when time $t = 0$, we know $\mathbf{r}(0) = 2\ \mathbf{i} + 3\ \mathbf{j} - \ \mathbf{k}$.

Now, to determine the object's position function, we integrate it's velocity.

$$\begin{aligned}\mathbf{r}(t) = \int \mathbf{v}(t)\ dt &= \int \left[(3 - \cos t)\ \mathbf{i} + \left(1 - t^2\right)\ \mathbf{j}\right] dt \\ &= (3t - \sin t)\ \mathbf{i} + \left(t - \frac{t^3}{3}\right)\ \mathbf{j} + \mathbf{C}_2\end{aligned}$$

Now we use the initial position to determine $\mathbf{C}_2$.

$$\mathbf{r}(0) = (3(0) - \sin 0)\ \mathbf{i} + \left(0 - \frac{(0)^3}{3}\right)\ \mathbf{j} + \mathbf{C}_2 = 2\,\mathbf{i} + 3\,\mathbf{j} - \ \mathbf{k}$$

$$\text{And so} \quad \mathbf{C}_2 = 2\,\mathbf{i} + 3\,\mathbf{j} - \ \mathbf{k}$$

Incorporating this constant vector into our position function from above, we obtain:

$$\mathbf{r}(t) = (2 + 3t - \sin t)\ \mathbf{i} + \left(3 + t - \frac{t^3}{3}\right)\ \mathbf{j} - \ \mathbf{k} \tag{5.2.29}$$

To find the object's position at time $t = 3$ seconds, we just evaluate this position function at $t = 3$.

$$\begin{aligned}\mathbf{r}(3) &= (2 + 3(3) - \sin 3)\ \mathbf{i} + \left(3 + 3 - \frac{(3)^3}{3}\right)\ \mathbf{j} - \ \mathbf{k} \\ &= (11 - \sin 3)\ \mathbf{i} - 3\,\mathbf{j} - \ \mathbf{k}\end{aligned}$$

This position vector indicates that the object will be located at the point $(11 - \sin 3, -3, -1)$ at time $t = 3$ seconds.

The Meaning of Definite Integrals of Vector-Valued Functions

Now let's turn our attention to the meaning of a definite integral of a vector-valued function. The context in which this will make the most sense is where the function we integrate is a velocity function. That is,

$$\int_a^b \mathbf{v}(t)\ dt \tag{5.2.30}$$

We know that the antiderivative of velocity is position and that this definite integral gives us the change in position over the time interval, $a \le t \le b$. In other words,

$$\int_a^b \mathbf{v}(t)\ dt = \mathbf{r}(b) - \mathbf{r}(a) \tag{5.2.31}$$

Thus, the definite integral of velocity over a time interval $a \le t \le b$ gives us the **displacement vector** that indicates the change in position over this time interval.

In general, the definite integral

$$\int_a^b \mathbf{r}(t) \quad dt = \mathbf{q}(b) - \mathbf{q}(a), \text{ where } \mathbf{q}(t) \text{ is the antiderivative of } \mathbf{r}(t),$$

gives us a change in the antiderivative of our vector-valued function over the given inteval $[a, b]$. This will always be a constant vector that would fit from tip to tip of the vectors given by the antiderivative function at $t = a$ and $t = b$, respectively (assuming the vectors were placed in standard position).

Summary

- To calculate the derivative of a vector-valued function, calculate the derivatives of the component functions, then put them back into a new vector-valued function.
- Many of the properties of differentiation of scalar functions also apply to vector-valued functions.
- The derivative of a vector-valued function $\mathbf{r}(t)$ is also a tangent vector to the curve. The unit tangent vector $\mathbf{T}(t)$ is calculated by dividing the derivative of a vector-valued function by its magnitude.
- The antiderivative of a vector-valued function is found by finding the antiderivatives of the component functions, then putting them back together in a vector-valued function.
- The definite integral of a vector-valued function is found by finding the definite integrals of the component functions, then putting them back together in a vector-valued function.

Key Equations

- **Derivative of a vector-valued function**

$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$

- **Principal unit tangent vector**

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$

- **Indefinite integral of a vector-valued function**

$$\int [f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}]\,dt = \left[\int f(t)\,dt\right]\,\mathbf{i} + \left[\int g(t)\,dt\right]\,\mathbf{j} + \left[\int h(t)\,dt\right]\,\mathbf{k}$$

- **Definite integral of a vector-valued function**

$$\int_a^b [f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}]\,dt = \left[\int_a^b f(t)\,dt\right]\,\mathbf{i} + \left[\int_a^b g(t)\,dt\right]\,\mathbf{j} + \left[\int_a^b h(t)\,dt\right]\,\mathbf{k}$$

Glossary

Definite Integral of a Vector-Valued Function
The vector obtained by calculating the definite integral of each of the component functions of a given vector-valued function, then using the results as the components of the resulting function.

Derivative of a Vector-Valued Function
The derivative of a vector-valued function $\mathbf{r}(t)$ is $\mathbf{r}\,'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}$, provided the limit exists.

Indefinite Integral of a Vector-Valued Function
A vector-valued function with a derivative that is equal to a given vector-valued function.

Principal Unit Tangent Vector
A unit vector tangent to a curve *C*.

Tangent Vector
To $\mathbf{r}(t)$ at $t = t_0$ any vector $\mathbf{v}$ such that, when the tail of the vector is placed at point $\mathbf{r}(t_0)$ on the graph, vector $\mathbf{v}$ is tangent to curve *C*.

Answers

Exercise 5.2.1
Use Equation 5.2.1.

Exercise 5.2.1

$$\mathbf{r}'(t) = 4t\,\mathbf{i} + 5\,\mathbf{j}$$

Exercise 5.2.2
Identify the component functions and use theorem on differentiation of vector-valued functions.

Exercise 5.2.2

$$\mathbf{r}'(t) = (1 + \ln t)\,\mathbf{i} + 5e^t\,\mathbf{j} - (\sin t + \cos t)\,\mathbf{k}$$

Exercise 5.2.2
At $t = 0, 2\pi/5, 4\pi/5, 6\pi/5, 8\pi/5, 2\pi$ although the point corresponding to $t = 0$ and $t = 2\pi$ is the same, namely $(5, 0)$.

Exercise 5.2.3
Follow the same steps as in Example 5.2.6.

Exercise 5.2.3
$\frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{r}\,'(t)] = 8e^{4t}$
$\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{r}(t)] = -(e^{2t}(\cos t + 2\sin t) + \cos 2t)\ \mathbf{i} + (e^{2t}(2t + 1) - \sin 2t)\ \mathbf{j} + (t\cos t + \sin t - \cos 2t)\ \mathbf{k}$

Exercise 5.2.4
Follow the same steps as in Example 5.2.7.

Exercise 5.2.4

$$\mathbf{T}(t) = \frac{2t}{\sqrt{4t^2 + 5}}\,\mathbf{i} + \frac{2}{\sqrt{4t^2 + 5}}\,\mathbf{j} + \frac{1}{\sqrt{4t^2 + 5}}\,\mathbf{k}$$

Exercise 5.2.5
Use the definition of the definite integral of a plane curve.

Exercise 5.2.5

$$\int_1^3 [(2t + 4)\,\mathbf{i} + (3t^2 - 4t)\,\mathbf{j}]\,dt = 16\,\mathbf{i} + 10\,\mathbf{j}$$

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5.2E: Exercises for Section 5.2

Exercises 1 - 10 Taking Derivatives of Vector-Valued Functions

Compute the derivative of each vector-valued function.

1. $\mathbf{r}(t) = t^3\ \mathbf{i} + 3t^2\ \mathbf{j} + \dfrac{t^3}{6}\ \mathbf{k}$

   Answer

   $$\mathbf{r}'(t) = 3t^2\ \mathbf{i} + 6t\ \mathbf{j} + \tfrac{1}{2}t^2\ \mathbf{k}$$

2. $\mathbf{r}(t) = \sin(t)\ \mathbf{i} + \cos(t)\ \mathbf{j} + e^t\ \mathbf{k}$
3. $\mathbf{r}(t) = e^{-t}\ \mathbf{i} + \sin(3t)\ \mathbf{j} + 10\sqrt{t}\ \mathbf{k}$. A sketch of the graph is shown here. Notice the varying periodic nature of the graph.

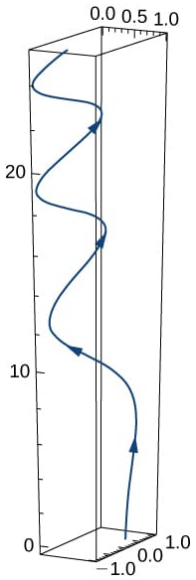


Figure $5.2E.1$: Graph of $\mathbf{r}(t) = e^{-t}\mathbf{i} + \sin(3t)\mathbf{j} + 10\sqrt{t}\mathbf{k}$. The curve is shown in a box frame with horizontal axis values ranging over $[0, 1]$, vertical axis values ranging over $[0, 26]$, and depth axis values ranging over $[-1, 1]$. The curve is shaped like a corkscrew starting from the bottom-back of the frame, moving up and forward with increasing osciallation frequency.

   Answer

   $$\mathbf{r}'(t) = -e^{-t}\ \mathbf{i} + 3\cos(3t)\ \mathbf{j} + \frac{5}{\sqrt{t}}\ \mathbf{k}$$

4. $\mathbf{r}(t) = e^t\ \mathbf{i} + 2e^t\ \mathbf{j} +\ \mathbf{k}$
5. $\mathbf{r}(t) =\ \mathbf{i} +\ \mathbf{j} +\ \mathbf{k}$

   Answer

   $$\mathbf{r}'(t) = \langle 0, 0, 0\rangle = \mathbf{0}$$

6. $\mathbf{r}(t) = te^t\ \mathbf{i} + t\ln(t)\ \mathbf{j} + \sin(3t)\ \mathbf{k}$
7. $\mathbf{r}(t) = \left\langle \dfrac{1}{t+1}, \arctan(t), \ln t^3 \right\rangle$

   Answer

   $$\mathbf{r}'(t) = \langle \frac{-1}{(t+1)^2}, \frac{1}{1+t^2}, \frac{3}{t}\rangle$$

8. $\mathbf{r}(t) = \langle \tan 2t,\ \sec 2t,\ \sin^2 t\rangle$

9. $\mathbf{r}(t) = \langle 3,\ 4\sin(3t),\ t\cos(t)\rangle$

Answer

$$\mathbf{r}'(t) = \langle 0, 12\cos(3t), \cos t - t\sin t\rangle$$

10. $\mathbf{r}(t) = t^2\ \mathbf{i} + te^{-2t}\ \mathbf{j} - 5e^{-4t}\ \mathbf{k}$
11.

a. Describe and sketch the curve represented by the vector-valued function $\mathbf{r}(t) = \langle 6t, 6t - t^2\rangle$.
b. Locate the highest point on the curve $\mathbf{r}(t) = \langle 6t, 6t - t^2\rangle$ and give the value of the function at this point.

Answer

b. $\mathbf{r}(t) = \langle 18, 9\rangle$ at $t = 3$

12. Find the parametric equations of the tangent line to the curve $\mathbf{r}(t) = \langle t, t^2, t\rangle$ at $t = 2$.
13. Find the parametric equations of the tangent line to the curve $\mathbf{r}(t) = \langle e^t, e^{-t}, 0\rangle$ at $t = 0$.

Answer

$$x = 1 + t, \quad y = 1 - t, \quad z = 0$$

14. Compute the first, second, and third derivatives of $\mathbf{r}(t) = 3t\ \mathbf{i} + 6\ln(t)\ \mathbf{j} + 5e^{-3t}\ \mathbf{k}$.

Exercises 15 - 17 Describing Motion with Vector-Valued Functions

Find the velocity and acceleration at the given times, plot the graph of the position function, and draw in the velocity and acceleration vectors at the corresponding locations on the curve.

15. $\mathbf{r}(t) = t\ \mathbf{i} + t^3\ \mathbf{j}$, at $t = 0$ and at $t = 1$

Answer

	$t = 0:$	$t = 1:$
$\mathbf{r}(t) = t\,\mathbf{i} + t^3\,\mathbf{j},$	$\mathbf{r}(0) = \mathbf{0},$	$\mathbf{r}(1) = \mathbf{i} + \mathbf{j}$
$\mathbf{v}(t) = \mathbf{i} + 3t^2\,\mathbf{j},$	$\mathbf{v}(0) = \mathbf{i},$	$\mathbf{v}(1) = \mathbf{i} + 3\,\mathbf{j}$
$\mathbf{a}(t) = 6t\,\mathbf{j},$	$\mathbf{a}(0) = \mathbf{0},$	$\mathbf{a}(1) = 6\,\mathbf{j}$

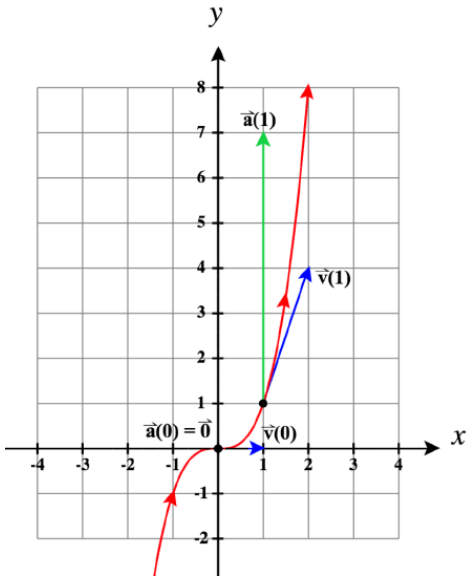


Figure $5.2E.2$: Path along graph of $y = x^3$ from left-to-right. Position $(0, 0)$ is labeled with a velocity vector $\mathbf{v}(0) = \langle 1, 0\rangle$ and acceleration vector $\mathbf{a}(0) = \mathbf{0}$. Position $(1, 1)$ is labeled with a velocity vector $\mathbf{v}(1) = \langle 1, 3\rangle$ and acceleration vector $\mathbf{a}(1) = \langle 1, 7\rangle$.

16. $\mathbf{r}(t) = \cos t\ \mathbf{i} + \sin 3t\ \mathbf{j}$, at $t = 0$, at $t = \frac{\pi}{4}$, and at $t = \frac{\pi}{2}$

17. $\mathbf{r}(t) = \ln t\ \mathbf{i} + (t-2)\ \mathbf{j}$, at $t = 1$ and at $t = 2$

Answer

	$t = 1:$	$t = 2:$
$\mathbf{r}(t) = (\ln t)\,\mathbf{i} + (t-2)\,\mathbf{j},$	$\mathbf{r}(1) = -\mathbf{j},$	$\mathbf{r}(2) = (\ln 2)\,\mathbf{i}$
$\mathbf{v}(t) = \dfrac{1}{t}\,\mathbf{i} + \mathbf{j},$	$\mathbf{v}(1) = \mathbf{i} + \mathbf{j},$	$\mathbf{v}(2) = \frac{1}{2}\mathbf{i} + \mathbf{j}$
$\mathbf{a}(t) = -\dfrac{1}{t^2}\,\mathbf{i},$	$\mathbf{a}(1) = -\mathbf{i},$	$\mathbf{a}(2) = -\frac{1}{4}\,\mathbf{i}$

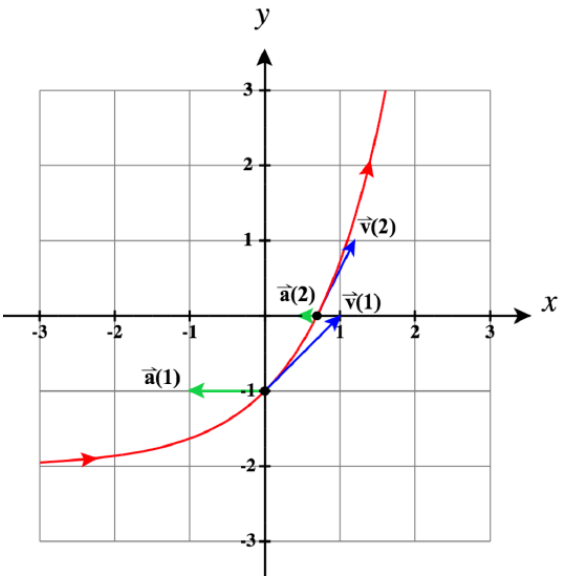


Figure $5.2E.3$: Path along graph of $\mathbf{r}(t) = \ln(t)\mathbf{i} + (t-2)\mathbf{j}$ with direction arrow showing direction from left to right. Position $(0, -1)$ is labeled with a velocity vector $\mathbf{v}(1) = \langle 1, 1\rangle$ and acceleration vector $\mathbf{a}(1) = \langle -1, 0\rangle$. Position $(\ln(2), 0)$ is labeled with a velocity vector $\mathbf{v}(2) = \langle \frac{1}{2}, 1\rangle$ and acceleration vector $\mathbf{a}(1) = \langle -\frac{1}{4}, 0\rangle$.

Exercises 18 - 24

Find the velocity, speed, and acceleration of a particle with the given position function. Remember that the speed is the magnitude of the velocity represented by $\|\mathbf{v}(t)\|$ or $\|\mathbf{r}\,'(t)\|$.

18. $\mathbf{r}(t) = e^{2t}\ \mathbf{i} + \sin t\ \mathbf{j}$
19. $\mathbf{r}(t) = \cos t^3\ \mathbf{i} + \sin t^3\ \mathbf{j}$

Answer

$\mathbf{v}(t) = -3t^2 \sin t^3\ \mathbf{i} + 3t^2 \cos t^3\ \mathbf{j}$, $\text{speed}(t) = \|\mathbf{v}(t)\| = 3t^2$, $\mathbf{a}(t) = (-6t \sin t^3 - 9t^4 \cos t^3)\ \mathbf{i} + (6t \cos t^3 - 9t^4 \sin t^3)\ \mathbf{j}$

20. $\mathbf{r}(t) = \langle e^t, e^{-t}, 0\rangle$
21. $\mathbf{r}(t) = \langle t + \cos t, t - \sin t\rangle$

Answer

$\mathbf{v}(t) = \langle 1 - \sin t, 1 - \cos t\rangle$, $\text{speed}(t) = \|\mathbf{v}(t)\| = \sqrt{3 - 2(\sin t + \cos t)}$, $\mathbf{a}(t) = \langle -\cos t, \sin t\rangle$

22. $\mathbf{r}(t) = \dfrac{2t-1}{2t+1}\ \mathbf{i} + \ln(1 - 4t^2)\ \mathbf{j}$
23. $\mathbf{r}(t) = \cos 3t\ \mathbf{i} + \sin 3t\ \mathbf{j} + 0.5t\ \mathbf{k}$

Answer

$\mathbf{v}(t) = -3 \sin 3t\ \mathbf{i} + 3 \cos 3t\ \mathbf{j} + 0.5\ \mathbf{k}$, $\text{speed}(t) = \|\mathbf{v}(t)\| = \sqrt{9.25}$ units/sec, $\mathbf{a}(t) = -9 \cos 3t\ \mathbf{i} - 9 \sin 3t\ \mathbf{j}$

24. $\mathbf{r}(t) = e^{-t}\ \mathbf{i} + (\ln t)\ \mathbf{j} + (\sin 7t)\ \mathbf{k}$
25. Consider the position vector for a particle to be $\mathbf{r}(t) = t\ \mathbf{i} + t^2\ \mathbf{j} + t^3\ \mathbf{k}$. The graph is shown here:

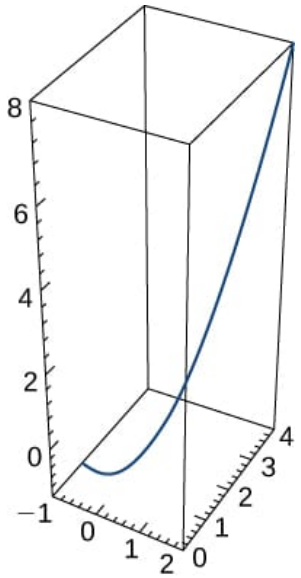


Figure $5.2E.4$: The curve is shown in a box frame with horizontal axis values ranging over $[-1, 2]$, vertical axis values ranging over $[-0.5, 8]$, and depth axis values ranging over $[0, 4]$. The curve goes from approximately the left corner of the base of the frame to the top right back corner.

a. Find the velocity vector at any time.
b. Find the speed of the particle at time $t = 2$ sec.
c. Find the acceleration at time $t = 2$ sec.

Answer

a. $\mathbf{v}(t) = \mathbf{i} + 2t\ \mathbf{j} + 3t^2\ \mathbf{k}$
b. $\sqrt{161}$ units/sec
Note that $\text{speed}(t) = \|\mathbf{v}(t)\| = \sqrt{1^2 + (2t)^2 + (3t^2)^2} = \sqrt{1 + 4t^2 + 9t^4}$.
Hence, $\text{speed}(2) = \sqrt{1 + 16 + 9(16)} = \sqrt{161}$ units/sec.
c. Since $\mathbf{a}(t) = 2\ \mathbf{j} + 6t\ \mathbf{k}$, $\mathbf{a}(2) = 2\ \mathbf{j} + 12\ \mathbf{k}$

26. A particle travels along the path of an ellipse with the equation $\mathbf{r}(t) = \cos t\ \mathbf{i} + 2\sin t\ \mathbf{j} + 0\ \mathbf{k}$. Find the following:

a. Velocity of the particle
b. Speed of the particle at $t = \frac{\pi}{4}$
c. Acceleration of the particle at $t = \frac{\pi}{4}$

Answer

a. $\mathbf{v}(t) = \langle -\sin t, 2\cos t, 0\rangle$

c. $\mathbf{a}(t) = \langle -\frac{\sqrt{2}}{2}, -\sqrt{2}, 0\rangle$

27. Show that if the speed of a particle traveling along a curve represented by a vector-valued function is constant, then the velocity function is always perpendicular to the acceleration function.

Answer

$$\begin{aligned} \|\mathbf{v}(t)\| &= k \\ \mathbf{v}(t)\cdot\mathbf{v}(t) &= k^2 \\ \frac{d}{dt}\Big(\mathbf{v}(t)\cdot\mathbf{v}(t)\Big) &= \frac{d}{dt}\Big(k^2\Big) = 0 \\ \mathbf{v}(t)\cdot\mathbf{v}'(t) + \mathbf{v}'(t)\cdot\mathbf{v}(t) &= 0 \\ 2\mathbf{v}(t)\cdot\mathbf{v}'(t) &= 0 \\ \mathbf{v}(t)\cdot\mathbf{v}'(t) &= 0 \end{aligned}$$

The last statement implies that the velocity and acceleration are perpendicular or orthogonal.

29. Given the vector-valued function $\mathbf{r}(t) = \langle \tan t, \sec t, 0\rangle$ (graph is shown here), find the following:

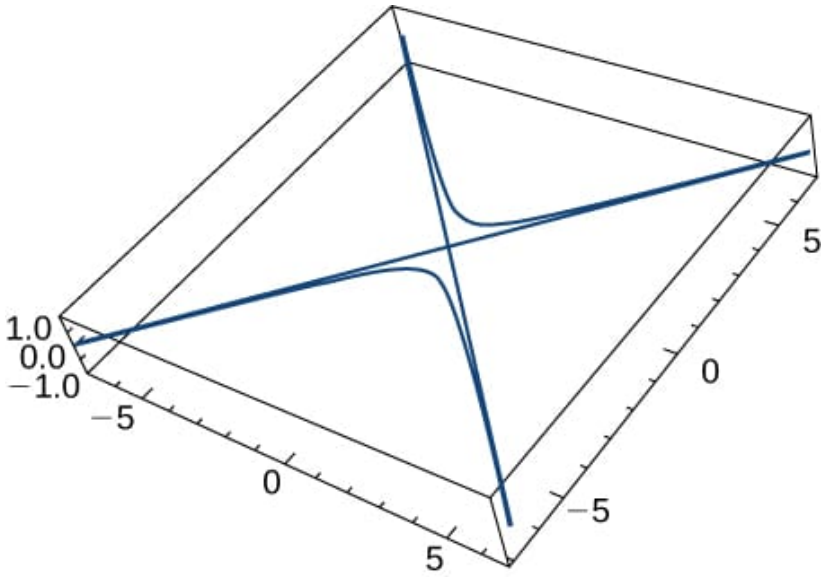


Figure $5.2E.5$: The curve is shown in a box frame with horizontal axis values ranging over $[-7, 7]$, vertical axis values ranging over $[-1, 1]$, and depth axis values ranging over $[-7, 7]$. The curve is symmetric about the center of the horizontal axis and the center of the depth axis, and is shaped like a hyperbola.

a. Velocity
b. Speed
c. Acceleration

Answer

a. $\mathbf{v}(t) = \langle \sec^2 t,\ \sec t \tan t \rangle$
b. $\|\mathbf{v}(t)\| = \sqrt{\sec^4 t + \sec^2 t \tan^2 t} = \sqrt{(\sec^2 t)(\sec^2 t + \tan^2 t)}$
c. $\mathbf{a}(t) = \langle 2 \sec^2 t \tan t,\ \sec t \tan^2 t + \sec^3 t \rangle$

29. Find the minimum speed of a particle traveling along the curve $\mathbf{r}(t) = \langle t + \cos t, t - \sin t \rangle$, where $t \in [0, 2\pi)$. Then also find its maximum speed on this interval.

Answer

Min. speed is $\sqrt{3 - 2\sqrt{2}} \approx 0.41421$ when $t = \frac{\pi}{4}$.
Max. speed is $\sqrt{3 + 2\sqrt{2}} \approx 2.41421$ when $t = \frac{5\pi}{4}$.

Exercises 30 - 31

Consider a particle that moves on a circular path of radius b according to the function $\mathbf{r}(t) = b\cos(\omega t)\ \mathbf{i} + b\sin(\omega t)\ \mathbf{j}$, where ω is the angular velocity, $\frac{d\theta}{dt}$.

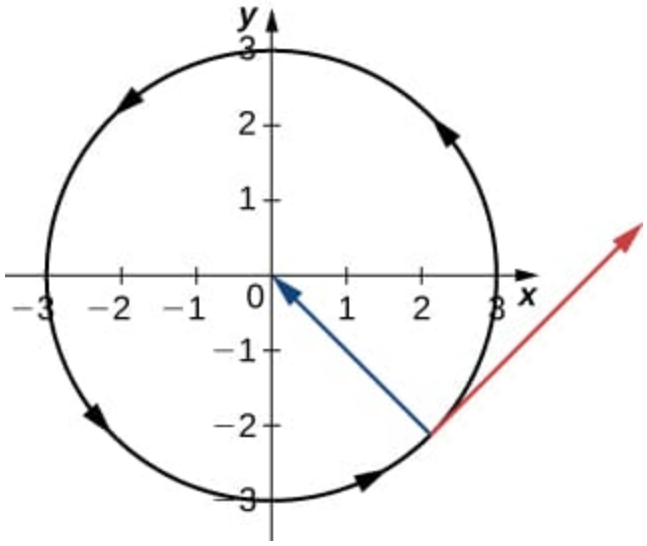


Figure $5.2E.6$: The graph of a circle with radius 3 oriented in the counterclockwise direction. The velocity vector is shown at a point on the circle in the fourth quadrant in red and is tangent to the circle. The acceleration vector is shown at the same point in blue and points to the center of the circle.

30. Show that the speed of the particle is proportional to the angular velocity.
31. Find the velocity function and show that $\mathbf{v}(t)$ is always orthogonal to $\mathbf{r}(t)$.

Answer

$\mathbf{r}'(t) = -b\omega \sin(\omega t)\ \mathbf{i} + b\omega \cos(\omega t)\ \mathbf{j}$. To show orthogonality, note that $\mathbf{r}'(t) \cdot \mathbf{r}(t) = 0$.

Exercises 32 - 40 Smoothness of Vector-Valued Functions

a. **Determine any values of t at which $\mathbf{r}$ is not smooth.**
b. **Determine the open intervals on which $\mathbf{r}$ is smooth.**
c. **Graph the vector-valued function and describe its behavior at the points where it is not smooth.**

32. $\mathbf{r}(t) = \langle 3t, 5t^2 - 1 \rangle$
33. $\mathbf{r}(t) = t^3\ \mathbf{i} + 5t^2\ \mathbf{j}$

Answer

a. $\mathbf{r}$ is not smooth at $t = 0$, since $\mathbf{r}'(0) = \mathbf{0}$.

b. $\mathbf{r}$ is smooth on the open intervals $(-\infty, 0)$ and $(0, \infty)$.

c. There is a cusp when $t = 0$.

34. $\mathbf{r}(t) = \langle 5,\ 2\sin(t),\ \cos(t) \rangle$
35. $\mathbf{r}(t) = \langle t^3 - 3t^2, 7 \rangle$

Answer

a. $\mathbf{r}$ is not smooth at $t = 0$ and $t = 2$, since $\mathbf{r}'(0) = \mathbf{0}$ and $\mathbf{r}'(2) = \mathbf{0}$.

b. $\mathbf{r}$ is smooth on the open intervals $(-\infty, 0)$, $(0, 2)$, and $(2, \infty)$.

c. The motion on the curve reverses along the same path at both $t = 0$ and $t = 2$.

36. $\mathbf{r}(t) = t^2\ \mathbf{i} + t^3\ \mathbf{j} - 5e^{-4t}\ \mathbf{k}$
37. $\mathbf{r}(t) = \left\langle \ln(t^2 + 4t + 5),\ \dfrac{t^3}{3} - 4t,\ 5 \right\rangle$

Answer

a. $\mathbf{r}$ is not smooth at $t = -2$, since $\mathbf{r}'(-2) = \mathbf{0}$. Since the domain of $\mathbf{r}$ is $(-\infty, \infty)$, this is all we have to remove.
b. $\mathbf{r}$ is smooth on the open intervals $(-\infty, -2)$ and $(-2, \infty)$.
c. There is a cusp when $t = -2$.

38. $\mathbf{r}(t) = (5\cos t - \cos 5t)\ \mathbf{i} + (5\sin t - \sin 5t)\ \mathbf{j}$, for $0 \le t \le 2\pi$
39. $\mathbf{r}(t) = \sqrt{t^3 + 9t^2}\ \mathbf{i} + (t^2 + 12t)\ \mathbf{j} + 7\ \mathbf{k}$

Answer

a. The domain of $\mathbf{r}$ is $[-9, \infty)$.

And $\mathbf{r}$ is not smooth at $t = -6$, since $\mathbf{r}'(-6) = \mathbf{0}$.
The domain of $\mathbf{r}'$ is $(-9, \infty)$, since $\mathbf{r}'$ is undefined at $t = -9$.

b. $\mathbf{r}$ is smooth on the open intervals $(-9, -6)$ and $(-6, \infty)$.

c. There is a cusp when $t = -6$.

40. $\mathbf{r}(t) = \cos^3 t\ \mathbf{i} + \sin t\ \mathbf{j}$, for $0 \le t \le 2\pi$

Answer

a. The domain of $\mathbf{r}$ is $(-\infty, \infty)$.

$\mathbf{r}'(t) = -3(\cos^2 t)(\sin t)\ \mathbf{i} + \cos t\ \mathbf{j}$. It's domain is also $(-\infty, \infty)$.
But note that both components have a factor of $\cos t$, so both components will be 0 when $\cos t = 0$.
Therefore, $\mathbf{r}$ is not smooth at $t = \frac{\pi}{2}$ and at $t = \frac{3\pi}{2}$, since $\mathbf{r}'\left(\frac{\pi}{2}\right) = \mathbf{0}$ and $\mathbf{r}'\left(\frac{3\pi}{2}\right) = \mathbf{0}$. Note then that $\mathbf{r}$ is not smooth for any odd multiple of $\frac{\pi}{2}$, that is for $t = \frac{(2n+1)\pi}{2}$, for any integer value n.

b. $\mathbf{r}$ is smooth on the open intervals $\left(\frac{(2n-1)\pi}{2}, \frac{(2n+1)\pi}{2}\right)$, for any integer value n.

c. There is a cusp when $t = \frac{(2n+1)\pi}{2}$, for any integer value n.

Exercises 41 - 43 Properties of the Derivative

Evaluate each expression given that $\mathbf{r}(t) = t\ \mathbf{i} + t^2\ \mathbf{j} - t^4\ \mathbf{k}$ and $\mathbf{s}(t) = \sin(t)\ \mathbf{i} + e^t\ \mathbf{j} + \cos(t)\ \mathbf{k}$

41. $\frac{d}{dt}\left[\mathbf{r}(t^2)\right]$

Answer

$$\frac{d}{dt}\left[\mathbf{r}(t^2)\right] = \langle 2t, 4t^3, -8t^7 \rangle$$

42. $\frac{d}{dt}\left[t^2 \cdot \mathbf{s}(t)\right]$
43. $\frac{d}{dt}\left[\mathbf{r}(t) \cdot \mathbf{s}(t)\right]$

Answer

$$\frac{d}{dt}\left[\mathbf{r}(t) \cdot \mathbf{s}(t)\right] = \sin t + 2te^t - 4t^3\cos t + t\cos t + t^2e^t + t^4\sin t$$

44. Find $\mathbf{r}'(t) \cdot \mathbf{r}''(t)$ *for* $\mathbf{r}(t) = -3t^5\ \mathbf{i} + 5t\ \mathbf{j} + 2t^2\ \mathbf{k}$.

Answer

$$\mathbf{r}'(t) \cdot \mathbf{r}''(t) = 900t^7 + 16t$$

45. Given $\mathbf{r}(t) = t\ \mathbf{i} + 3t\ \mathbf{j} + t^2\ \mathbf{k}$ and $\mathbf{u}(t) = 4t\ \mathbf{i} + t^2\ \mathbf{j} + t^3\ \mathbf{k}$, find $\frac{d}{dt}(\mathbf{r}(t) \times \mathbf{u}(t))$.

46. Evaluate $\frac{d}{dt}\left[\mathbf{u}(t) \times \mathbf{u}\,'(t)\right]$ given $\mathbf{u}(t) = t^2\ \mathbf{i} - 2t\ \mathbf{j} + \ \mathbf{k}$.

Answer

$$\frac{d}{dt}\left[\mathbf{u}(t) \times \mathbf{u}\,'(t)\right] = 0\ \mathbf{i} + 2\ \mathbf{j} + 4t\ \mathbf{k}$$

47. Given $\mathbf{r}(t) = t\ \mathbf{i} + 2\sin t\ \mathbf{j} + 2\cos t\ \mathbf{k}$ and $\mathbf{u}(t) = \frac{1}{t}\ \mathbf{i} + 2\sin t\ \mathbf{j} + 2\cos t\ \mathbf{k}$, find the following:

a. $\mathbf{r}(t) \times \mathbf{u}(t)$
b. $\frac{d}{dt}\big(\mathbf{r}(t) \times \mathbf{u}(t)\big)$
c. Now, use the product rule for the derivative of the cross product of two vectors and show this result is the same as the answer for the preceding problem.

Answer

a. $\mathbf{r}(t) \times \mathbf{u}(t) = \left\langle 0,\ 2(\cos t)\left(\frac{1}{t} - t\right),\ 2(\sin t)\left(t - \frac{1}{t}\right)\right\rangle$
b. $\frac{d}{dt}\big(\mathbf{r}(t) \times \mathbf{u}(t)\big) = \left\langle 0,\ 2(\sin t)\left(t - \frac{1}{t}\right) - 2(\cos t)\left(1 + \frac{1}{t^2}\right),\ 2(\sin t)\left(1 + \frac{1}{t^2}\right) + 2(\cos t)\left(t - \frac{1}{t}\right)\right\rangle$

Exercises 48 - 51 Unit Tangent Vectors

Find a *unit tangent vector* at the indicated value of t.

48. $\mathbf{r}(t) = 3t^3\ \mathbf{i} + 2t^2\ \mathbf{j} + \dfrac{1}{t}\ \mathbf{k}; \quad t = 1$
49. $\mathbf{r}(t) = t\ \mathbf{i} + \sin(2t)\ \mathbf{j} + \cos(3t)\ \mathbf{k}; \quad t = \frac{\pi}{3}$

Answer

$\mathbf{r}'\left(\frac{\pi}{3}\right) = \langle 1,\ -1, 0\rangle$ is a tangent vector, so a unit tangent vector would be:
$\frac{1}{\sqrt{2}}\langle 1, -1, 0\rangle \quad = \quad \langle \frac{\sqrt{2}}{2},\ -\frac{\sqrt{2}}{2},\ 0\rangle$

50. $\mathbf{r}(t) = \cos(2t)\ \mathbf{i} + 2\sin t\ \mathbf{j} + t^2\ \mathbf{k}; \quad t = \frac{\pi}{2}$
51. $\mathbf{r}(t) = 3e^t\ \mathbf{i} + 2e^{-3t}\ \mathbf{j} + 4e^{2t}\ \mathbf{k}; \quad t = \ln(2)$

Answer

$\mathbf{r}'(\ln(2)) = \langle 6, -\frac{3}{4}, 32\rangle$ is a tangent vector, so a unit tangent vector would be:
$$\frac{1}{\sqrt{1060.5625}}\left\langle 6, -\tfrac{3}{4}, 32\right\rangle \quad = \quad \left\langle \frac{24\sqrt{16969}}{16969}, -\frac{12\sqrt{16969}}{67876}, \frac{128\sqrt{16969}}{16969}\right\rangle$$

Exercises 52 - 58

Find the unit tangent vector $\mathbf{T}(t)$ for the following parameterized curves.

52. $\mathbf{r}(t) = t\ \mathbf{i} + 3t\ \mathbf{j} + t^2\ \mathbf{k}$
53. $\mathbf{r}(t) = 6\ \mathbf{i} + \cos(3t)\ \mathbf{j} + 3\sin(4t)\ \mathbf{k}, \quad 0 \le t < 2\pi$

Answer

$$\mathbf{T}(t) = \frac{1}{\sqrt{9\sin^2(3t) + 144\cos^2(4t)}}\langle 0, -3\sin(3t), 12\cos(4t)\rangle$$

54. $\mathbf{r}(t) = \langle t\cos t, t\sin t\rangle$
55. $\mathbf{r}(t) = \langle t + 1, 2t + 1, 2t + 2\rangle$

Answer

$\mathbf{T}(t) = \frac{1}{3}\langle 1, 2, 2\rangle$

56. $\mathbf{r}(t) = \cos t\ \mathbf{i} + \sin t\ \mathbf{j} + \sin t\ \mathbf{k}, \quad 0 \le t < 2\pi$. Two views of this curve are presented here:

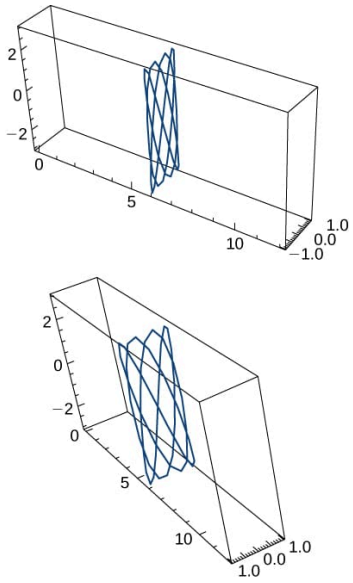


Figure $5.2E.7$:

Two 3D plots showing a helical trajectory, represented by a tightly coiled curve within a rectangular frame. The top plot presents the helical curve viewed from the side. The curve oscillates sinusoidally along the vertical axis while progressing horizontally, creating a series of loops. The horizontal axis ranges from 0 to 12, the vertical axis ranges from -2.5 to 2.5, and the depth axis ranges from -1.0 to 1.0. The bottom plot shows the same helical curve from a tilted perspective, providing a clearer view of the looping structure. The axes are labeled similarly, with the same ranges. The tilted perspective highlights the helical motion as the curve twists through the 3D space, emphasizing its periodic and sinusoidal nature.

57. $\mathbf{r}(t) = \left\langle t, \frac{1}{t} \right\rangle$. The graph is shown here:

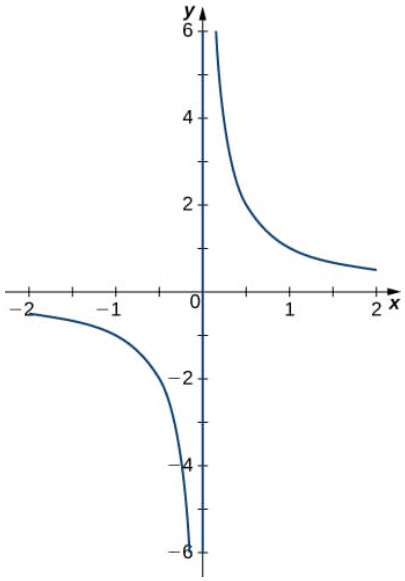


Figure $5.2E.1$: A 2D Cartesian plot illustrating the graph of a rational function, with two distinct branches. The graph features vertical and horizontal asymptotes. The $x)\ values shown on the horizontal axis range from -2 to 2, while the \ y$ values shown on the vertical axis range from -6 to 6. The curve has two parts: one branch in the first quadrant, which approaches the horizontal asymptote $y = 0$ as $x)\ increases and approaches the \ y$ axis as x approaches 0 from the right. The other branch in the third quadrant, which approaches the horizontal asymptote $y = 0$ as $x)\ decreases and approaches the \ y$ axis as x approaches 0 from the left.

Answer

$$\mathbf{T}(t) = \left\langle \frac{t^2}{\sqrt{t^4+1}}, \frac{-1}{\sqrt{t^4+1}} \right\rangle$$

58. $\mathbf{r}(t) = 3\cos(4t)\ \mathbf{i} + 3\sin(4t)\ \mathbf{j} + 5t\ \mathbf{k}, \quad 1 \le t \le 2$

Answer

$$\mathbf{T}(t) = -\tfrac{12}{13}\sin(4t)\ \mathbf{i} + \tfrac{12}{13}\cos(4t)\ \mathbf{j} + \tfrac{5}{13}\ \mathbf{k}$$

59. A particle travels along the path of a helix with the equation $\mathbf{r}(t) = \cos(t)\ \mathbf{i} + \sin(t)\ \mathbf{j} + t\ \mathbf{k}$. See the graph presented here:

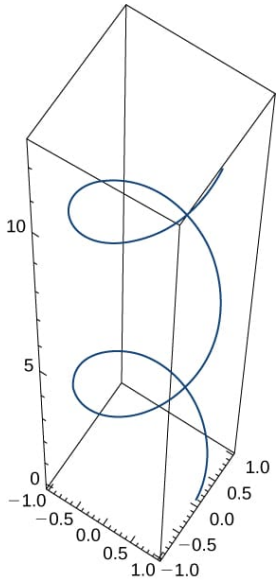


Figure $5.2E.8$: A 3D plot illustrating a single helical trajectory, represented by a smooth, spiraling curve within a tall rectangular box. The curve starts near the bottom, spirals upward in a circular pattern, and extends to the top of the frame, maintaining a consistent radius. The vertical axis ranges from 0 to 13, representing the height of the helix. The horizontal and depth axes range from -1.0 to 1.0, showing the radial spread of the helix. The plot highlights the periodic and rotational nature of the helical motion, with the curve continuously winding as it ascends in the vertical direction.

Find the following:

a. Velocity of the particle at any time
b. Speed of the particle at any time
c. Acceleration of the particle at any time
d. Find the unit tangent vector for the helix

Answer

a. $\mathbf{v}(t) = \langle -\sin t, \cos t, 1 \rangle$

c. $\mathbf{a}(t) = -\cos t\ \mathbf{i} - \sin t\ \mathbf{j} + 0\ \mathbf{k}$

Exercises 60 - 65 Integration of Vector-Valued Functions

60. Evaluate $\displaystyle\int \left(e^t\,\mathbf{i} + \sin t\,\mathbf{j} + \frac{1}{2t-1}\,\mathbf{k} \right)\ dt$

61. Evaluate $\displaystyle\int_0^1 \mathbf{r}(t)\ dt$, where $\mathbf{r}(t) = \left\langle \sqrt[3]{t}, \frac{1}{t+1}, e^{-t} \right\rangle$

Answer

$$\tfrac{3}{4}\ \mathbf{i} + \ln(2)\ \mathbf{j} + (1 - \tfrac{1}{e})\ \mathbf{k}$$

62. Evaluate $\displaystyle\int_0^3 \|t\ \mathbf{i} + t^2\ \mathbf{j}\| dt$.

Answer

$$\tfrac{1}{3}(10^{\frac{3}{2}} - 1)$$

53. The acceleration function, initial velocity, and initial position of a particle are

$$\begin{aligned}\mathbf{a}(t) &= -5\cos t\,\mathbf{i} - 5\sin t\,\mathbf{j},\\ \mathbf{v}(0) &= 9\,\mathbf{i} + 2\,\mathbf{j}, \quad \text{and}\\ \mathbf{r}(0) &= 5\,\mathbf{i}\end{aligned}$$

Find $\mathbf{v}(t)$ and $\mathbf{r}(t)$.

Answer

$$\begin{aligned}\mathbf{v}(t) &= (9 - 5\sin t)\ \mathbf{i} + (-3 + 5\cos t)\ \mathbf{j}\\ \mathbf{r}(t) &= (9t + 5\cos t)\ \mathbf{i} + (-3t + 5\sin t)\ \mathbf{j}\end{aligned}$$

54. Find the antiderivative of $\mathbf{r}'(t) = \cos(2t)\ \mathbf{i} - 2\sin t\ \mathbf{j} + \dfrac{1}{1+t^2}\ \mathbf{k}$ that satisfies the initial condition $\mathbf{r}(0) = 3\ \mathbf{i} - 2\ \mathbf{j} + \ \mathbf{k}$.

55. An object starts from rest at point $P(1, 2, 0)$ and moves with an acceleration of $\mathbf{a}(t) = \ \mathbf{j} + 2\ \mathbf{k}$, where $\|\mathbf{a}(t)\|$ is measured in feet per second per second. Find the location of the object after $t = 2$ sec.

Contributors:

- Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.
- Paul Seeburger (Monroe Community College) created questions 11 - 19.

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5.3: Arc Length and Curvature

Learning Objectives

- Determine the length of a particle's path in space by using the arc-length function.
- Explain the meaning of the curvature of a curve in space and state its formula.
- Describe the meaning of the normal and binormal vectors of a curve in space.

In this section, we study formulas related to curves in both two and three dimensions, and see how they are related to various properties of the same curve. For example, suppose a vector-valued function describes the motion of a particle in space. We would like to determine how far the particle has traveled over a given time interval, which can be described by the arc length of the path it follows. Or, suppose that the vector-valued function describes a road we are building and we want to determine how sharply the road curves at a given point. This is described by the curvature of the function at that point. We explore each of these concepts in this section.

Arc Length for Vector Functions

We have seen how a vector-valued function describes a curve in either two or three dimensions. Recall that the formula for the arc length of a curve defined by the parametric functions $x = x(t), y = y(t), t_1 \leq t \leq t_2$ is given by

$$s = \int_{t_1}^{t_2} \sqrt{(x'(t))^2 + (y'(t))^2}dt.$$

In a similar fashion, if we define a smooth curve using a vector-valued function $\mathbf{r}(t) = f(t)\ \mathbf{i} + g(t)\ \mathbf{j}$, where $a \leq t \leq b$, the arc length is given by the formula

$$s = \int_{a}^{b} \sqrt{(f'(t))^2 + (g'(t))^2}dt.$$

In three dimensions, if the vector-valued function is described by $\mathbf{r}(t) = f(t)\ \mathbf{i} + g(t)\ \mathbf{j} + h(t)\ \mathbf{k}$ over the same interval $a \leq t \leq b$, the arc length is given by

$$s = \int_{a}^{b} \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}dt.$$

Theorem: Arc-Length Formulas for Plane and Space curves

Plane curve: Given a smooth curve C defined by the function $\mathbf{r}(t) = f(t)\ \mathbf{i} + g(t)\ \mathbf{j}$, where t lies within the interval $[a, b]$, the arc length of C over the interval is

$$s = \int_{a}^{b} \sqrt{[f'(t)]^2 + [g'(t)]^2}dt \tag{5.3.1}$$

$$= \int_{a}^{b} \|\mathbf{r}'(t)\|dt. \tag{5.3.2}$$

Space curve: Given a smooth curve C defined by the function $\mathbf{r}(t) = f(t)\ \mathbf{i} + g(t)\ \mathbf{j} + h(t)\ \mathbf{k}$, where t lies within the interval $[a, b]$, the arc length of C over the interval is

$$s = \int_{a}^{b} \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}dt \tag{5.3.3}$$

$$= \int_{a}^{b} \|\mathbf{r}'(t)\|dt. \tag{5.3.4}$$

The two formulas are very similar; they differ only in the fact that a space curve has three component functions instead of two. Note that the formulas are defined for smooth curves: curves where the vector-valued function $\mathbf{r}(t)$ is differentiable with a non-zero derivative. The smoothness condition guarantees that the curve has no cusps (or corners) that could make the formula problematic.

Example 5.3.1: Finding the Arc Length

Calculate the arc length for each of the following vector-valued functions:

a. $\mathbf{r}(t) = (3t - 2)\ \mathbf{i} + (4t + 5)\ \mathbf{j}, \quad 1 \leq t \leq 5$
b. $\mathbf{r}(t) = \langle t\cos t, t\sin t, 2t\rangle, 0 \leq t \leq 2\pi$

Solution

a. Using Equation 5.3.2, $\mathbf{r}\,'(t) = 3\ \mathbf{i} + 4\ \mathbf{j}$, so

$$\begin{aligned} s &= \int_a^b \|\mathbf{r}'(t)\|dt \\ &= \int_1^5 \sqrt{3^2 + 4^2}dt \\ &= \int_1^5 5dt = 5t\big|_1^5 = 20. \end{aligned}$$

b. Using Equation 5.3.4, $\mathbf{r}\,'(t) = \langle \cos t - t\sin t, \sin t + t\cos t, 2\rangle$, so

$$\begin{aligned} s &= \int_a^b \|\mathbf{r}'(t)\|dt \\ &= \int_0^{2\pi} \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 2^2}dt \\ &= \int_0^{2\pi} \sqrt{(\cos^2 t - 2t\sin t\cos t + t^2\sin^2 t) + (\sin^2 t + 2t\sin t\cos t + t^2\cos^2 t) + 4}dt \\ &= \int_0^{2\pi} \sqrt{\cos^2 t + \sin^2 t + t^2(\cos^2 t + \sin^2 t) + 4}dt \\ &= \int_0^{2\pi} \sqrt{t^2 + 5}dt \end{aligned}$$

Here we can use a table integration formula

$$\int \sqrt{u^2 + a^2}du = \frac{u}{2}\sqrt{u^2 + a^2} + \frac{a^2}{2}\ln\left|u + \sqrt{u^2 + a^2}\right| + C,$$

so we obtain

$$\begin{aligned} \int_0^{2\pi} \sqrt{t^2 + 5}dt &= \frac{1}{2}\left(t\sqrt{t^2 + 5} + 5\ln\left|t + \sqrt{t^2 + 5}\right|\right)_0^{2\pi} \\ &= \frac{1}{2}\left(2\pi\sqrt{4\pi^2 + 5} + 5\ln\left(2\pi + \sqrt{4\pi^2 + 5}\right)\right) - \frac{5}{2}\ln\sqrt{5} \\ &\approx 25.343 \text{ units}. \end{aligned}$$

Watch the accompanying video for a visual demonstration of this concept.

- **Video Length**: 1 minute 52 seconds.
- **Context**: This video demonstrates how to find the arc length of the parameterized curve $\mathbf{r}(t) = t^2\ \mathbf{i} + \sin t\ \mathbf{j} - 2\cos t\ \mathbf{k}$.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

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Exercise 5.3.1

Calculate the arc length of the parameterized curve

$$\mathbf{r}(t) = \langle 2t^2 + 1, 2t^2 - 1, t^3 \rangle, \quad 0 \leq t \leq 3.$$

Hint available at the end of this section.

We now return to the helix introduced earlier in this chapter. A vector-valued function that describes a helix can be written in the form

$$\mathbf{r}(t) = R\cos\left(\frac{2\pi Nt}{h}\right)\mathbf{i} + R\sin\left(\frac{2\pi Nt}{h}\right)\mathbf{j} + t\,\mathbf{k}, 0 \leq t \leq h,$$

where R represents the radius of the helix, h represents the height (distance between two consecutive turns), and the helix completes N turns. Let's derive a formula for the arc length of this helix using Equation 5.3.4. First of all,

$$\mathbf{r}'(t) = -\frac{2\pi NR}{h}\sin\left(\frac{2\pi Nt}{h}\right)\mathbf{i} + \frac{2\pi NR}{h}\cos\left(\frac{2\pi Nt}{h}\right)\mathbf{j} + \mathbf{k}.$$

Therefore,

$$\begin{aligned}
s &= \int_a^b \|\mathbf{r}'(t)\|dt \\
&= \int_0^h \sqrt{\left(-\frac{2\pi NR}{h}\sin\left(\frac{2\pi Nt}{h}\right)\right)^2 + \left(\frac{2\pi NR}{h}\cos\left(\frac{2\pi Nt}{h}\right)\right)^2 + 1^2}dt \\
&= \int_0^h \sqrt{\frac{4\pi^2N^2R^2}{h^2}\left(\sin^2\left(\frac{2\pi Nt}{h}\right) + \cos^2\left(\frac{2\pi Nt}{h}\right)\right) + 1}dt \\
&= \int_0^h \sqrt{\frac{4\pi^2N^2R^2}{h^2} + 1}dt \\
&= \left[t\sqrt{\frac{4\pi^2N^2R^2}{h^2} + 1}\right]_0^h \\
&= h\sqrt{\frac{4\pi^2N^2R^2 + h^2}{h^2}} \\
&= \sqrt{4\pi^2N^2R^2 + h^2}.
\end{aligned}$$

This gives a formula for the length of a wire needed to form a helix with N turns that has radius R and height h.

Arc-Length Parameterization

We now have a formula for the arc length of a curve defined by a vector-valued function. Let's take this one step further and examine what an **arc-length function** is.

If a vector-valued function represents the position of a particle in space as a function of time, then the arc-length function measures how far that particle travels as a function of time. The formula for the arc-length function follows directly from the

formula for arc length:

$$s = \int_a^t \sqrt{(f'(u))^2 + (g'(u))^2 + (h'(u))^2}du. \tag{5.3.5}$$

If the curve is in two dimensions, then only two terms appear under the square root inside the integral. The reason for using the independent variable *u* is to distinguish between time and the variable of integration. Since $s(t)$ measures distance traveled as a function of time, $s\,'(t)$ measures the speed of the particle at any given time. Since we have a formula for $s(t)$ in Equation 5.3.5, we can differentiate both sides of the equation:

$$\begin{aligned} s'(t) &= \frac{d}{dt}\left[\int_a^t \sqrt{(f'(u))^2 + (g'(u))^2 + (h'(u))^2}du\right] \\ &= \frac{d}{dt}\left[\int_a^t \|\mathbf{r}'(u)\|du\right] \\ &= \|\mathbf{r}'(t)\|. \end{aligned}$$

If we assume that $\mathbf{r}(t)$ defines a smooth curve, then the arc length is always increasing, so $s\,'(t) > 0$ for $t > a$. Last, if $\mathbf{r}(t)$ is a curve on which $\|\mathbf{r}\,'(t)\| = 1$ for all t, then

$$s(t) = \int_a^t \|\mathbf{r}'(u)\|\,du = \int_a^t 1\,du = t - a,$$

which means that t represents the arc length as long as $a = 0$.

Theorem: Arc-Length Function

Let $\mathbf{r}(t)$ describe a smooth curve for $t \geq a$. Then the arc-length function is given by

$$s(t) = \int_a^t \|\mathbf{r}'(u)\|\,du \tag{5.3.6}$$

Furthermore,

$$\frac{ds}{dt} = \|\mathbf{r}'(t)\| > 0.$$

If $\|\mathbf{r}\,'(t)\| = 1$ for all $t \geq a$, then the parameter t represents the arc length from the starting point at $t = a$.

A useful application of this theorem is to find an alternative parameterization of a given curve, called an **arc-length parameterization**. Recall that any vector-valued function can be reparameterized via a change of variables. For example, if we have a function $\mathbf{r}(t) = \langle 3\cos t, 3\sin t\rangle, 0 \leq t \leq 2\pi$ that parameterizes a circle of radius 3, we can change the parameter from t to $4t$, obtaining a new parameterization $\mathbf{r}(t) = \langle 3\cos 4t, 3\sin 4t\rangle$. The new parameterization still defines a circle of radius 3, but now we need only use the values $0 \leq t \leq \pi/2$ to traverse the circle once.

Suppose that we find the arc-length function $s(t)$ and are able to solve this function for t as a function of s. We can then reparameterize the original function $\mathbf{r}(t)$ by substituting the expression for t back into $\mathbf{r}(t)$. The vector-valued function is now written in terms of the parameter s. Since the variable s represents the arc length, we call this an *arc-length parameterization* of the original function $\mathbf{r}(t)$. One advantage of finding the arc-length parameterization is that the distance traveled along the curve starting from $s = 0$ is now equal to the parameter s. The arc-length parameterization also appears in the context of curvature (which we examine later in this section) and line integrals.

Example 5.3.2: Finding an Arc-Length Parameterization

Find the arc-length parameterization for each of the following curves:

a. $\mathbf{r}(t) = 4\cos t\ \mathbf{i} + 4\sin t\ \mathbf{j}, \quad t \geq 0$

b. $\mathbf{r}(t) = \langle t+3, 2t-4, 2t\rangle, \quad t \geq 3$

Solution

a. First we find the arc-length function using Equation 5.3.5:

$$\begin{aligned} s(t) &= \int_a^t \|\mathbf{r}'(u)\|\, du \\ &= \int_0^t \|\langle -4\sin u, 4\cos u\rangle\|\, du \\ &= \int_0^t \sqrt{(-4\sin u)^2 + (4\cos u)^2}\, du \\ &= \int_0^t \sqrt{16\sin^2 u + 16\cos^2 u}\, du \\ &= \int_0^t 4\, du = 4t, \end{aligned}$$

which gives the relationship between the arc length s and the parameter t as $s = 4t$; so, $t = s/4$. Next we replace the variable t in the original function $\mathbf{r}(t) = 4\cos t\ \mathbf{i} + 4\sin t\ \mathbf{j}$ with the expression $s/4$ to obtain

$$\mathbf{r}(s) = 4\cos\left(\frac{s}{4}\right)\mathbf{i} + 4\sin\left(\frac{s}{4}\right)\mathbf{j}.$$

This is the arc-length parameterization of $\mathbf{r}(t)$. Since the original restriction on t was given by $t \geq 0$, the restriction on s becomes $s/4 \geq 0$, or $s \geq 0$.

b. The arc-length function is given by Equation 5.3.5:

$$\begin{aligned} s(t) &= \int_a^t \|\mathbf{r}'(u)\|\, du \\ &= \int_3^t \|\langle 1, 2, 2\rangle\|\, du \\ &= \int_3^t \sqrt{1^2 + 2^2 + 2^2}\, du \\ &= \int_3^t 3\, du \\ &= 3t - 9. \end{aligned}$$

Therefore, the relationship between the arc length s and the parameter t is $s = 3t - 9$, so $t = \frac{s}{3} + 3$. Substituting this into the original function $\mathbf{r}(t) = \langle t+3, 2t-4, 2t\rangle$ yields

$$\mathbf{r}(s) = \langle\left(\frac{s}{3}+3\right)+3,\ 2\left(\frac{s}{3}+3\right)-4,\ 2\left(\frac{s}{3}+3\right)\rangle = \langle\frac{s}{3}+6, \frac{2s}{3}+2, \frac{2s}{3}+6\rangle.$$

This is an arc-length parameterization of $\mathbf{r}(t)$. The original restriction on the parameter t was $t \geq 3$, so the restriction on s is $(s/3) + 3 \geq 3$, or $s \geq 0$.

Watch the accompanying video for a visual demonstration of this concept.

- **Video Length**: 2 minute 32 seconds.
- **Context**: This video demonstrates how to parameterize by the arc length for the curve $\mathbf{r}(t) = 3t\ \mathbf{i} - 2\sin t\ \mathbf{j} + 2\cos t\ \mathbf{k}$.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Exercise 5.3.2

Find the arc-length function for the helix

$$\mathbf{r}(t) = \langle 3\cos t, 3\sin t, 4t\rangle, \quad t \geq 0.$$

Then, use the relationship between the arc length and the parameter t to find an arc-length parameterization of $\mathbf{r}(t)$.

Hint available at the end of this section.

Curvature

An important topic related to arc length is curvature. The concept of curvature provides a way to measure how sharply a smooth curve turns. A circle has constant curvature. The smaller the radius of the circle, the greater the curvature.

Think of driving down a road. Suppose the road lies on an arc of a large circle. In this case you would barely have to turn the wheel to stay on the road. Now suppose the radius is smaller. In this case you would need to turn more sharply to stay on the road. In the case of a curve other than a circle, it is often useful first to inscribe a circle to the curve at a given point so that it is tangent to the curve at that point and "hugs" the curve as closely as possible in a neighborhood of the point (Figure 5.3.1). The curvature of the graph at that point is then defined to be the same as the curvature of the inscribed circle.

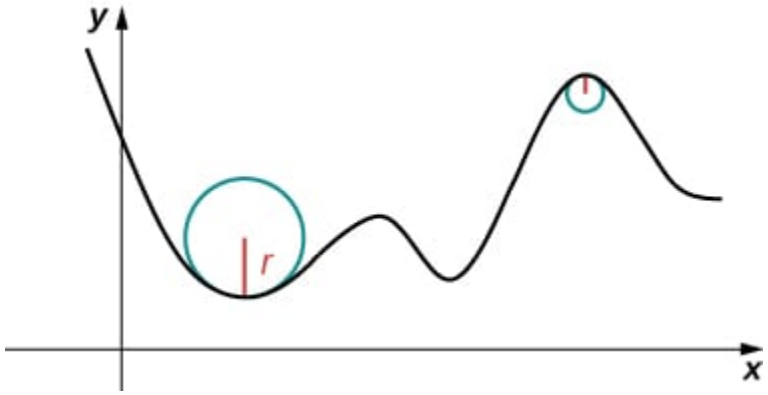


Figure 5.3.1: The graph represents the curvature of a function $y = f(x)$. The sharper the turn in the graph, the greater the curvature, and the smaller the radius of the inscribed circle.

Definition: Curvature

Let C be a smooth curve in the plane or in space given by $\mathbf{r}(s)$, where s is the arc-length parameter. The **curvature** κ at s is

$$\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\| = \|\mathbf{T}'(s)\|. \tag{5.3.7}$$

Interactive Element

View the Circle of Curvature

Below you can see the graph of a function, its curvature (κ), its radius of curvature (r), and its circle of curvature. You can move the slider to see how the circle of curvature changes as you move along the curve. Click "Next" to see a new curve.

$f(x) = \frac{x^3}{4} - x^2 - x + 1$

$f(x) = \sin(x)$

$f(x) = \ln(x)$

$f(x) = \cos(3x)e^{x/3}$

$f(x) = 2\sin(x) + 2\cos(3x)$

$f(x) = \sin(2x) + \ln(x+2)$

$f(x) = 3x + \frac{1}{5x} - x^2 + 1$

$f(x) = \frac{\sin(4x)}{x}$

$f(x) = \ln(x)\cos(3x)$

$f(x) = \ln(x+2) + \sin(2x) - e^{-x}$

2

x = 3

Next

The formula in the definition of curvature is not always very useful in terms of calculation. In particular, recall that $\mathbf{T}(t)$ represents the unit tangent vector to a given vector-valued function $\mathbf{r}(t)$, and the formula for $\mathbf{T}(t)$ is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\| \mathbf{r}'(t) \|}. \tag{5.3.8}$$

To use the formula for curvature, it is first necessary to express $\mathbf{r}(t)$ in terms of the arc-length parameter s, then find the unit tangent vector $\mathbf{T}(s)$ for the function $\mathbf{r}(s)$, then take the derivative of $\mathbf{T}(s)$ with respect to s. This is a tedious process. Fortunately, there are equivalent formulas for curvature.

Theorem: Alternate Formulas of Curvature

If C is a smooth curve given by $\mathbf{r}(t)$, then the curvature κ of C at t is given by

$$\kappa = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}. \tag{5.3.9}$$

If C is a three-dimensional curve, then the curvature can be given by the formula

$$\kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}. \tag{5.3.10}$$

If C is the graph of a function $y = f(x)$ and both y' and y'' exist, then the curvature κ at point (x, y) is given by

$$\kappa = \frac{|y''|}{[1 + (y')^2]^{3/2}}. \tag{5.3.11}$$

Proof

The first formula follows directly from the chain rule:

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt},$$

where s is the arc length along the curve C. Dividing both sides by ds/dt, and taking the magnitude of both sides gives

$$\left\| \frac{d\mathbf{T}}{ds} \right\| = \left\| \frac{\mathbf{T}'(t)}{\frac{ds}{dt}} \right\|.$$

Since $ds/dt = \|\mathbf{r}\,'(t)\|$, this gives the formula for the curvature κ of a curve C in terms of any parameterization of C:

$$\kappa = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}.$$

In the case of a three-dimensional curve, we start with the formulas $\mathbf{T}(t) = (\mathbf{r}\,'(t))/\|\mathbf{r}\,'(t)\|$ and $ds/dt = \|\mathbf{r}\,'(t)\|$. Therefore, $\mathbf{r}\,'(t) = (ds/dt)\mathbf{T}(t)$. We can take the derivative of this function using the scalar product formula:

$$\mathbf{r}''(t) = \frac{d^2s}{dt^2}\mathbf{T}(t) + \frac{ds}{dt}\mathbf{T}'(t).$$

Using these last two equations we get

$$\begin{aligned}\mathbf{r}'(t) \times \mathbf{r}''(t) &= \frac{ds}{dt}\mathbf{T}(t) \times \left(\frac{d^2s}{dt^2}\mathbf{T}(t) + \frac{ds}{dt}\mathbf{T}'(t)\right) \\ &= \frac{ds}{dt}\frac{d^2s}{dt^2}\mathbf{T}(t) \times \mathbf{T}(t) + (\frac{ds}{dt})^2\mathbf{T}(t) \times \mathbf{T}'(t).\end{aligned}$$

Since $\mathbf{T}(t) \times \mathbf{T}(t) = 0$, this reduces to

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \left(\frac{ds}{dt}\right)^2 \mathbf{T}(t) \times \mathbf{T}'(t).$$

Since $\mathbf{T}\,'$ is parallel to $\mathbf{N}$, and $\mathbf{T}$ is orthogonal to $\mathbf{N}$, it follows that $\mathbf{T}$ and $\mathbf{T}\,'$ are orthogonal. This means that $\|\mathbf{T} \times \mathbf{T}\,'\| = \|\mathbf{T}\|\|\mathbf{T}\,'\|\sin(\pi/2) = \|\mathbf{T}\,'\|$, so

$$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = \left(\frac{ds}{dt}\right)^2 \|\mathbf{T}'(t)\|.$$

Now we solve this equation for $\|\mathbf{T}\,'(t)\|$ and use the fact that $ds/dt = \|\mathbf{r}\,'(t)\|$:

$$\|\mathbf{T}'(t)\| = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^2}.$$

Then, we divide both sides by $\|\mathbf{r}\,'(t)\|$. This gives

$$\kappa = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}.$$

This proves 5.3.10. To prove 5.3.11, we start with the assumption that curve C is defined by the function $y = f(x)$. Then, we can define $\mathbf{r}(t) = x\ \mathbf{i} + f(x)\ \mathbf{j} + 0\ \mathbf{k}$. Using the previous formula for curvature:

$$\mathbf{r}'(t) = \mathbf{i} + f'(x)\,\mathbf{j}$$
$$\mathbf{r}''(t) = f''(x)\,\mathbf{j}$$
$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & f'(x) & 0 \\ 0 & f''(x) & 0 \end{vmatrix} = f''(x)\,\mathbf{k}.$$

Therefore,

$$\kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{|f''(x)|}{(1 + [f'(x)]^2)^{3/2}}$$

Example 5.3.3: Finding Curvature

Find the curvature for each of the following curves at the given point:

a. $\mathbf{r}(t) = 4\cos t\ \mathbf{i} + 4\sin t\ \mathbf{j} + 3t\ \mathbf{k}, \quad t = \dfrac{4\pi}{3}$
b. $f(x) = \sqrt{4x - x^2}, \quad x = 2$

Solution

a. This function describes a helix.

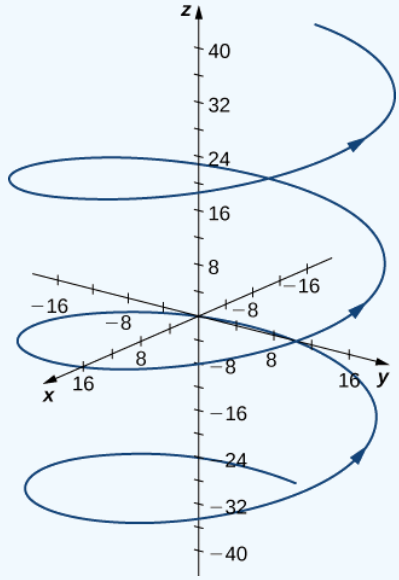


Figure 5.3.2: The curve shown here is a helix that spirals around the z-axis. It begins below the xy-plane and spirals up with upward orientation.

The curvature of the helix at $t = (4\pi)/3$ can be found by using 5.3.9. First, calculate $\mathbf{T}(t)$:

$$\begin{aligned} \mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \\ &= \frac{\langle -4\sin t, 4\cos t, 3\rangle}{\sqrt{(-4\sin t)^2 + (4\cos t)^2 + 3^2}} \\ &= \langle -\frac{4}{5}\sin t, \frac{4}{5}\cos t, \frac{3}{5}\rangle. \end{aligned}$$

Next, calculate $\mathbf{T}'(t)$:

$$\mathbf{T}'(t) = \langle -\frac{4}{5}\cos t, -\frac{4}{5}\sin t, 0\rangle.$$

Last, apply 5.3.9 :

$$\begin{aligned}\kappa &= \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\|\langle -\frac{4}{5}\cos t, -\frac{4}{5}\sin t, 0\rangle\|}{\|\langle -4\sin t, 4\cos t, 3\rangle\|} \\ &= \frac{\sqrt{(-\frac{4}{5}\cos t)^2 + (-\frac{4}{5}\sin t)^2 + 0^2}}{\sqrt{(-4\sin t)^2 + (4\cos t)^2 + 3^2}} \\ &= \frac{4/5}{5} = \frac{4}{25}.\end{aligned}$$

The curvature of this helix is constant at all points on the helix.

b. This function describes a semicircle.

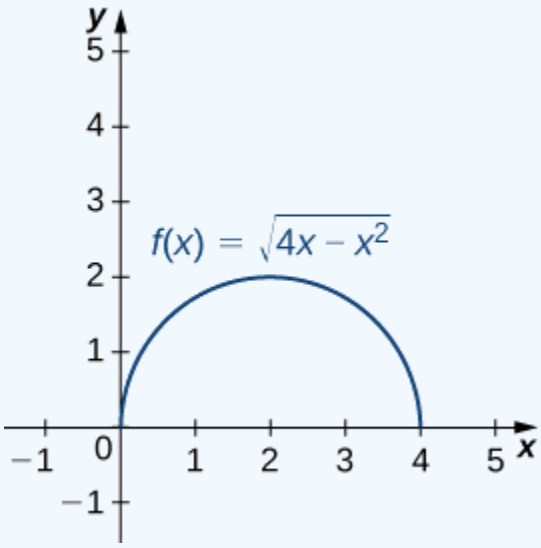


Figure 5.3.3: This figure is the graph of a semicircle. It is in the first quadrant. The semicircle begins at the origin and stops at the point $4, 0$ on the x-axis. The semicircle represents the function $f(x) = \sqrt{4x - x^2}$.

To find the curvature of this graph, we must use 5.3.11. First, we calculate y' and y'' :

$$\begin{aligned}y &= \sqrt{4x - x^2} = (4x - x^2)^{1/2} \\ y' &= \frac{1}{2}(4x - x^2)^{-1/2}(4 - 2x) = (2 - x)(4x - x^2)^{-1/2} \\ y'' &= -(4x - x^2)^{-1/2} + (2 - x)(-\frac{1}{2})(4x - x^2)^{-3/2}(4 - 2x) \\ &= -\frac{4x - x^2}{(4x - x^2)^{3/2}} - \frac{(2 - x)^2}{(4x - x^2)^{3/2}} \\ &= \frac{x^2 - 4x - (4 - 4x + x^2)}{(4x - x^2)^{3/2}} \\ &= -\frac{4}{(4x - x^2)^{3/2}}.\end{aligned}$$

Then, we apply 5.3.11:

$$\begin{aligned}\kappa &= \frac{|y''|}{[1+(y')^2]^{3/2}} \\ &= \frac{\left|-\frac{4}{(4x-x^2)^{3/2}}\right|}{\left[1+((2-x)(4x-x^2)^{-1/2})^2\right]^{3/2}} = \frac{\left|\frac{4}{(4x-x^2)^{3/2}}\right|}{\left[1+\frac{(2-x)^2}{4x-x^2}\right]^{3/2}} \\ &= \frac{\left|\frac{4}{(4x-x^2)^{3/2}}\right|}{\left[\frac{4x-x^2+x^2-4x+4}{4x-x^2}\right]^{3/2}} = \left|\frac{4}{(4x-x^2)^{3/2}}\right| \cdot \frac{(4x-x^2)^{3/2}}{8} \\ &= \frac{1}{2}.\end{aligned}$$

The curvature of this circle is equal to the reciprocal of its radius. There is a minor issue with the absolute value in 5.3.11 ; however, a closer look at the calculation reveals that the denominator is positive for any value of x.

Watch the accompanying video for a visual demonstration of this concept.

- **Video Length**: 4 minute 24 seconds.
- **Context**: This video demonstrates how to find the curvature of the parameterized curve $\mathbf{r}(t) = t\ \mathbf{i} + (t^2-1)\ \mathbf{j} + \frac{1}{3}t^3\ \mathbf{k}$ at the point $P = \left(2, 3, \frac{8}{3}\right)$.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Below is a video example in which we maximize the curvature for a curve for which y is given as a function of x.

Example 5.3.4

Find the point on the curve where at which the curvature is maximum: $y = e^x$

Solution

- **Video length**: 5 minutes 15 seconds.
- **Context**: This video demonstrates how to maximize curvature.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

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Exercise 5.3.3

Find the curvature of the curve defined by the function

$$y = 3x^2 - 2x + 4$$

at the point $x = 2$.

Hint available at the end of this section.

The next video example shows how to compute the curvature of a curve if it's given using the arc length parameter. This actually cuts down the work a bit, allowing us to use the formula in the definition in (5.3.7) directly rather than formulas we developed afterwards.

Example 5.3.5

Find the curvature κ of the curve where s is the arc length parameter.

$\mathbf{r}(s) = \frac{3}{5}s\mathbf{i} - 2\sin(\frac{2s}{5})\mathbf{j} + 2\cos(\frac{2s}{5})\mathbf{k}$

Solution

- **Video length:** 3 minutes 22 seconds.
- **Context**: This video demonstrates how to find the curvature of a curve that is parameterized by arc length.

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The Normal and Binormal Vectors

We have seen that the derivative $\mathbf{r}\,'(t)$ of a vector-valued function is a tangent vector to the curve defined by $\mathbf{r}(t)$, and the unit tangent vector $\mathbf{T}(t)$ can be calculated by dividing $\mathbf{r}\,'(t)$ by its magnitude. When studying motion in three dimensions, two other vectors are useful in describing the motion of a particle along a path in space: the principal unit normal vector and the binormal vector.

Definition: Principal Unit Normal and Binormal Vector

Let C be a three-dimensional smooth curve represented by $\mathbf{r}$ over an open interval I. If $\mathbf{T}\,'(t) \neq \mathbf{0}$, then the **principal unit normal vector** at t is defined to be

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}. \tag{5.3.12}$$

The **binormal vector** at t is defined as

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t), \tag{5.3.13}$$

where $\mathbf{T}(t)$ is the unit tangent vector.

Note that, by definition, the binormal vector is orthogonal to both the unit tangent vector and the normal vector. Furthermore, $\mathbf{B}(t)$ is always a unit vector. This can be shown using the formula for the magnitude of a cross product.

$$\|\mathbf{B}(t)\| = \|\mathbf{T}(t) \times \mathbf{N}(t)\| = \|\mathbf{T}(t)\|\|\mathbf{N}(t)\| \sin\theta, \tag{5.3.14}$$

where θ is the angle between $\mathbf{T}(t)$ and $\mathbf{N}(t)$. Since $\mathbf{N}(t)$ is the derivative of a unit vector, property (vii) of the derivative of a vector-valued function tells us that $\mathbf{T}(t)$ and $\mathbf{N}(t)$ are orthogonal to each other, so $\theta = \pi/2$. Furthermore, they are both unit vectors, so

their magnitude is 1. Therefore, $\|\mathbf{T}(t)\|\|\mathbf{N}(t)\|\sin\theta = (1)(1)\sin(\pi/2) = 1$ and $\mathbf{B}(t)$ is a unit vector.

The principal unit normal vector can be challenging to calculate because the unit tangent vector involves a quotient, and this quotient often has a square root in the denominator. In the three-dimensional case, finding the cross product of the unit tangent vector and the unit normal vector can be even more cumbersome. Fortunately, we have alternative formulas for finding these two vectors, and they are presented in Motion in Space.

Example 5.3.6: Finding the Principal Unit Normal Vector and Binormal Vector

For each of the following vector-valued functions, find the principal unit normal vector. Then, if possible, find the binormal vector.

1. $\mathbf{r}(t) = 4\cos t\ \mathbf{i} - 4\sin t\ \mathbf{j}$
2. $\mathbf{r}(t) = (6t+2)\ \mathbf{i} + 5t^2\ \mathbf{j} - 8t\ \mathbf{k}$

Solution

1. This function describes a circle.

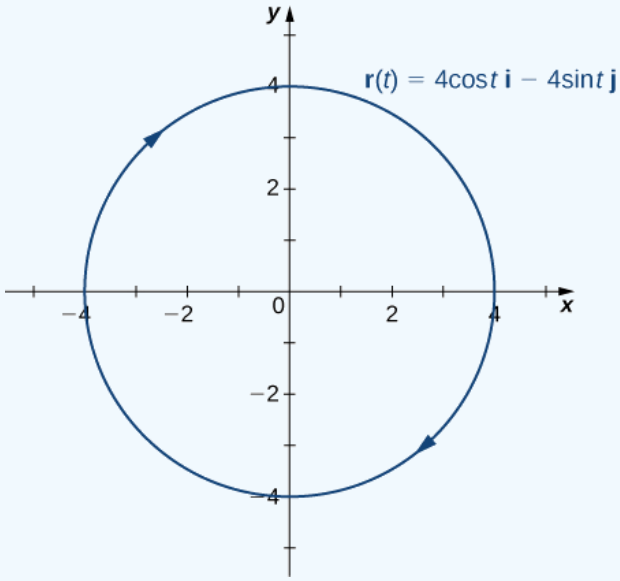


Figure 5.3.4: This figure is the graph of a circle centered at the origin with radius of 2. The orientation of the circle is clockwise. It represents the vector-valued function $\mathbf{r}(t) = 4\cos t\ \mathbf{i} - 4\sin t\ \mathbf{j}$.

To find the principal unit normal vector, we first must find the unit tangent vector $\mathbf{T}(t)$:

$$\begin{aligned}\mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}\\ &= \frac{-4\sin t\,\mathbf{i} - 4\cos t\,\mathbf{j}}{\sqrt{(-4\sin t)^2 + (-4\cos t)^2}}\\ &= \frac{-4\sin t\,\mathbf{i} - 4\cos t\,\mathbf{j}}{\sqrt{16\sin^2 t + 16\cos^2 t}}\\ &= \frac{-4\sin t\,\mathbf{i} - 4\cos t\,\mathbf{j}}{\sqrt{16(\sin^2 t + \cos^2 t)}}\\ &= \frac{-4\sin t\,\mathbf{i} - 4\cos t\,\mathbf{j}}{4}\\ &= -\sin t\,\mathbf{i} - \cos t\,\mathbf{j}.\end{aligned}$$

Next, we use 5.3.12 :

$$\begin{aligned}\mathbf{N}(t) &= \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} \\ &= \frac{-\cos t\,\mathbf{i} + \sin t\,\mathbf{j}}{\sqrt{(-\cos t)^2 + (\sin t)^2}} \\ &= \frac{-\cos t\,\mathbf{i} + \sin t\,\mathbf{j}}{\sqrt{\cos^2 t + \sin^2 t}} \\ &= -\cos t\,\mathbf{i} + \sin t\,\mathbf{j}.\end{aligned}$$

Notice that the unit tangent vector and the principal unit normal vector are orthogonal to each other for all values of t:

$$\begin{aligned}\mathbf{T}(t) \cdot \mathbf{N}(t) &= \langle -\sin t, -\cos t\rangle \cdot \langle -\cos t, \sin t\rangle \\ &= \sin t \cos t - \cos t \sin t \\ &= 0.\end{aligned}$$

Furthermore, the principal unit normal vector points toward the center of the circle from every point on the circle. Since $\mathbf{r}(t)$ defines a curve in two dimensions, we cannot calculate the binormal vector.

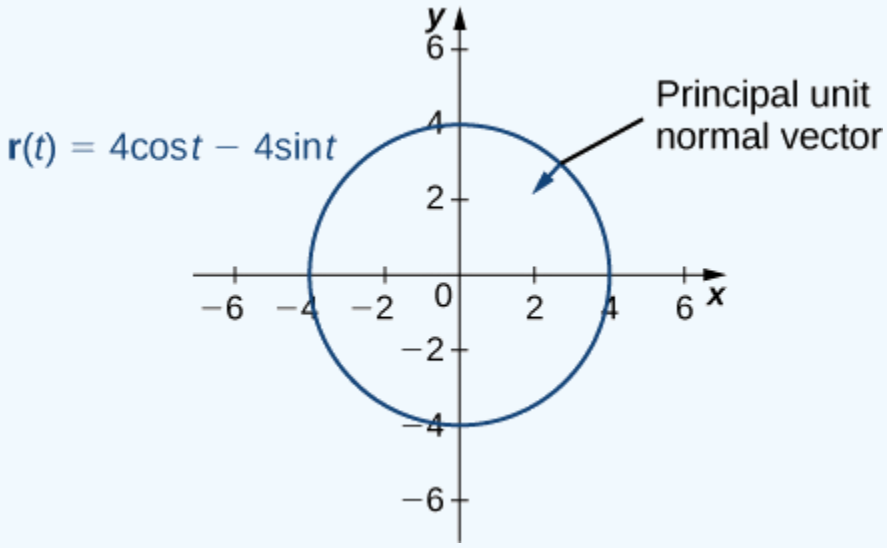


Figure 5.3.5: The same circle from Figure 5.3.4 is graphed. On the circle in the first quadrant is a vector pointing inward, labeled “principal unit normal vector”.

2. This function looks like this:

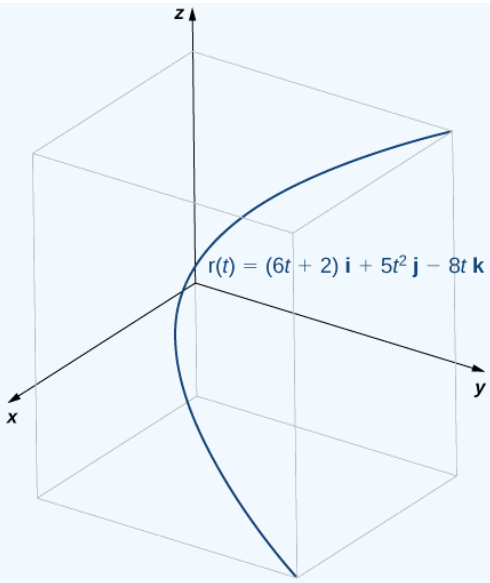


Figure 5.3.6: This figure is a curve in 3 dimensions. It is inside of a box. The box represents the first octant. The curve starts at the bottom right of the box and curves through the box in a parabolic curve to the top.

To find the principal unit normal vector, we first find the unit tangent vector $\mathbf{T}(t)$:

$$\begin{aligned}
\mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \\
&= \frac{6\,\mathbf{i} + 10t\,\mathbf{j} - 8\,\mathbf{k}}{\sqrt{6^2 + (10t)^2 + (-8)^2}} \\
&= \frac{6\,\mathbf{i} + 10t\,\mathbf{j} - 8\,\mathbf{k}}{\sqrt{36 + 100t^2 + 64}} \\
&= \frac{6\,\mathbf{i} + 10t\,\mathbf{j} - 8\,\mathbf{k}}{\sqrt{100(t^2 + 1)}} \\
&= \frac{3\,\mathbf{i} - 5t\,\mathbf{j} - 4\,\mathbf{k}}{5\sqrt{t^2 + 1}} \\
&= \frac{3}{5}(t^2 + 1)^{-1/2}\,\mathbf{i} - t(t^2 + 1)^{-1/2}\,\mathbf{j} - \frac{4}{5}(t^2 + 1)^{-1/2}\,\mathbf{k}.
\end{aligned}$$

Next, we calculate $\mathbf{T}\,'(t)$ and $\|\mathbf{T}\,'(t)\|$:

$$\begin{aligned}\mathbf{T}'(t) &= \frac{3}{5}(-\frac{1}{2})(t^2+1)^{-3/2}(2t)\,\mathbf{i} - ((t^2+1)^{-1/2} - t(\frac{1}{2})(t^2+1)^{-3/2}(2t))\,\mathbf{j} - \frac{4}{5}(-\frac{1}{2})(t^2+1)^{-3/2}(2t)\,\mathbf{k} \\ &= -\frac{3t}{5(t^2+1)^{3/2}}\,\mathbf{i} - \frac{1}{(t^2+1)^{3/2}}\,\mathbf{j} + \frac{4t}{5(t^2+1)^{3/2}}\,\mathbf{k} \\ \|\mathbf{T}'(t)\| &= \sqrt{\Big(-\frac{3t}{5(t^2+1)^{3/2}}\Big)^2 + \Big(-\frac{1}{(t^2+1)^{3/2}}\Big)^2 + \Big(\frac{4t}{5(t^2+1)^{3/2}}\Big)^2} \\ &= \sqrt{\frac{9t^2}{25(t^2+1)^3} + \frac{1}{(t^2+1)^3} + \frac{16t^2}{25(t^2+1)^3}} \\ &= \sqrt{\frac{25t^2+25}{25(t^2+1)^3}} \\ &= \sqrt{\frac{1}{(t^2+1)^2}} \\ &= \frac{1}{t^2+1}.\end{aligned}$$

Therefore, according to 5.3.12 :

$$\begin{aligned}\mathbf{N}(t) &= \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} \\ &= \Big(-\frac{3t}{5(t^2+1)^{3/2}}\,\mathbf{i} - \frac{1}{(t^2+1)^{3/2}}\,\mathbf{j} + \frac{4t}{5(t^2+1)^{3/2}}\,\mathbf{k}\Big)(t^2+1) \\ &= -\frac{3t}{5(t^2+1)^{1/2}}\,\mathbf{i} - \frac{5}{5(t^2+1)^{1/2}}\,\mathbf{j} + \frac{4t}{5(t^2+1)^{1/2}}\,\mathbf{k} \\ &= -\frac{3t\,\mathbf{i} + 5\,\mathbf{j} - 4t\,\mathbf{k}}{5\sqrt{t^2+1}}.\end{aligned}$$

Once again, the unit tangent vector and the principal unit normal vector are orthogonal to each other for all values of t:

$$\begin{aligned}\mathbf{T}(t)\cdot\mathbf{N}(t) &= \Big(\frac{3\,\mathbf{i} - 5t\,\mathbf{j} - 4\,\mathbf{k}}{5\sqrt{t^2+1}}\Big)\cdot\Big(-\frac{3t\,\mathbf{i} + 5\,\mathbf{j} - 4t\,\mathbf{k}}{5\sqrt{t^2+1}}\Big) \\ &= \frac{3(-3t) - 5t(-5) - 4(4t)}{25(t^2+1)} \\ &= \frac{-9t + 25t - 16t}{25(t^2+1)} \\ &= 0.\end{aligned}$$

Last, since $\mathbf{r}(t)$ represents a three-dimensional curve, we can calculate the binormal vector using 5.3.13 :

$$
\begin{aligned}
\mathbf{B}(t) &= \mathbf{T}(t) \times \mathbf{N}(t) \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{3}{5\sqrt{t^2+1}} & -\dfrac{5t}{5\sqrt{t^2+1}} & -\dfrac{4}{5\sqrt{t^2+1}} \\ -\dfrac{3t}{5\sqrt{t^2+1}} & -\dfrac{5}{5\sqrt{t^2+1}} & \dfrac{4t}{5\sqrt{t^2+1}} \end{vmatrix} \\
&= \left(\left(-\frac{5t}{5\sqrt{t^2+1}}\right)\left(\frac{4t}{5\sqrt{t^2+1}}\right) - \left(-\frac{4}{5\sqrt{t^2+1}}\right)\left(-\frac{5}{5\sqrt{t^2+1}}\right)\right)\mathbf{i} \\
&\quad - \left(\left(\frac{3}{5\sqrt{t^2+1}}\right)\left(\frac{4t}{5\sqrt{t^2+1}}\right) - \left(-\frac{4}{5\sqrt{t^2+1}}\right)\left(-\frac{3t}{5\sqrt{t^2+1}}\right)\right)\mathbf{j} \\
&\quad + \left(\left(\frac{3}{5\sqrt{t^2+1}}\right)\left(-\frac{5}{5\sqrt{t^2+1}}\right) - \left(-\frac{5t}{5\sqrt{t^2+1}}\right)\left(-\frac{3t}{5\sqrt{t^2+1}}\right)\right)\mathbf{k} \\
&= \left(\frac{-20t^2-20}{25(t^2+1)}\right)\mathbf{i} + \left(\frac{-15-15t^2}{25(t^2+1)}\right)\mathbf{k} \\
&= -20\left(\frac{t^2+1}{25(t^2+1)}\right)\mathbf{i} - 15\left(\frac{t^2+1}{25(t^2+1)}\right)\mathbf{k} \\
&= -\frac{4}{5}\,\mathbf{i} - \frac{3}{5}\,\mathbf{k}.
\end{aligned}
$$

Exercise 5.3.4

Find the unit normal vector for the vector-valued function $\mathbf{r}(t) = (t^2 - 3t)\ \mathbf{i} + (4t + 1)\ \mathbf{j}$ and evaluate it at $t = 2$.

Hint available at the end of this section.

For any smooth curve in three dimensions that is defined by a vector-valued function, we now have formulas for the unit tangent vector $\mathbf{T}$, the unit normal vector $\mathbf{N}$, and the binormal vector $\mathbf{B}$. The unit normal vector and the binormal vector form a plane that is perpendicular to the curve at any point on the curve, called the normal plane. In addition, these three vectors form a frame of reference in three-dimensional space called the **Frenet frame of reference** (also called the **TNB** frame) (Figure 5.3.7). Last, the plane determined by the vectors $\mathbf{T}$ and $\mathbf{N}$ forms the **osculating plane** of C at any point P on the curve.

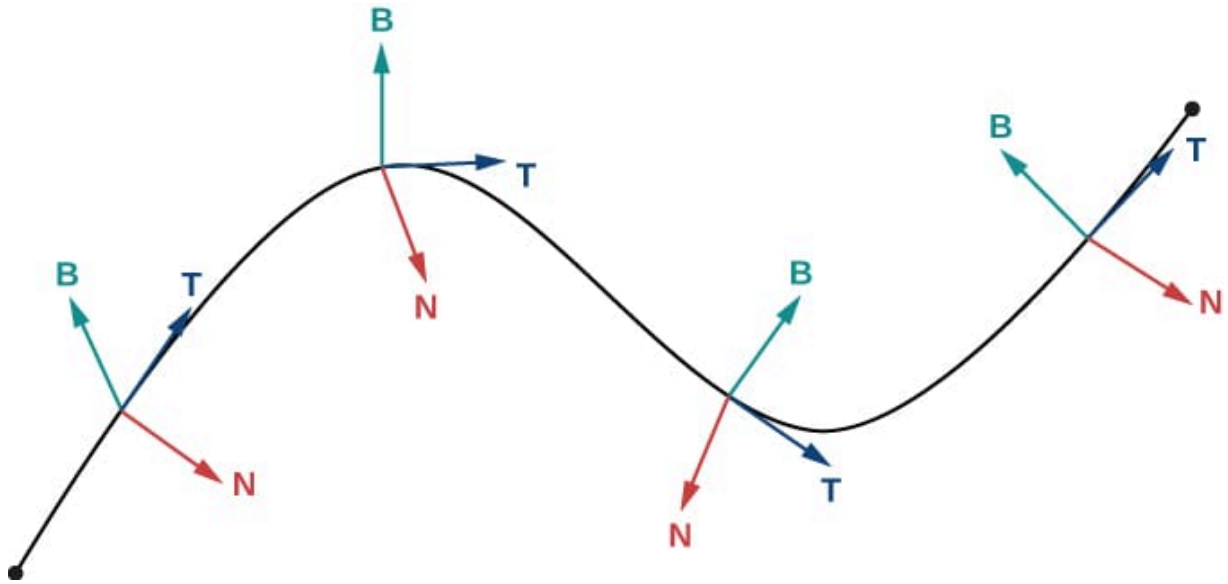


Figure 5.3.7: This figure depicts a Frenet frame of reference. At every point P on a three-dimensional curve, the unit tangent, unit normal, and binormal vectors form a three-dimensional frame of reference.

Suppose we form a circle in the osculating plane of C at point P on the curve. Assume that the circle has the same curvature as

the curve does at point P and let the circle have radius r. Then, the curvature of the circle is given by $\frac{1}{r}$. We call r the radius of curvature of the curve, and it is equal to the reciprocal of the curvature. If this circle lies on the concave side of the curve and is tangent to the curve at point P, then this circle is called the **osculating circle** of C at P, as shown in Figure 5.3.8.

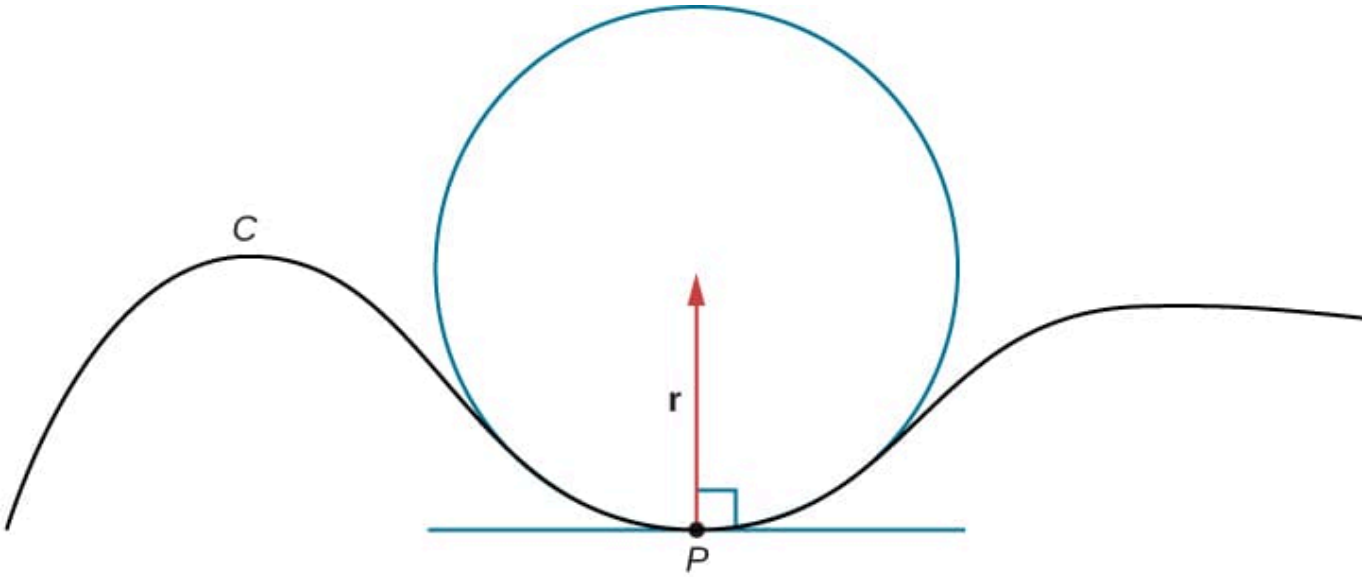


Figure 5.3.8: In this osculating circle, the circle is tangent to curve C at point P and shares the same curvature.

For more information on osculating circles, see this demonstration on curvature and torsion, this article on osculating circles, and this discussion of Serret formulas.

To find the equation of an osculating circle in two dimensions, we need find only the center and radius of the circle.

Example 5.3.7: Finding the Equation of an Osculating Circle

Find the equation of the osculating circle of the curve defined by the function $y = x^3 - 3x + 1$ at $x = 1$.

Solution

Figure 5.3.9 shows the graph of $y = x^3 - 3x + 1$.

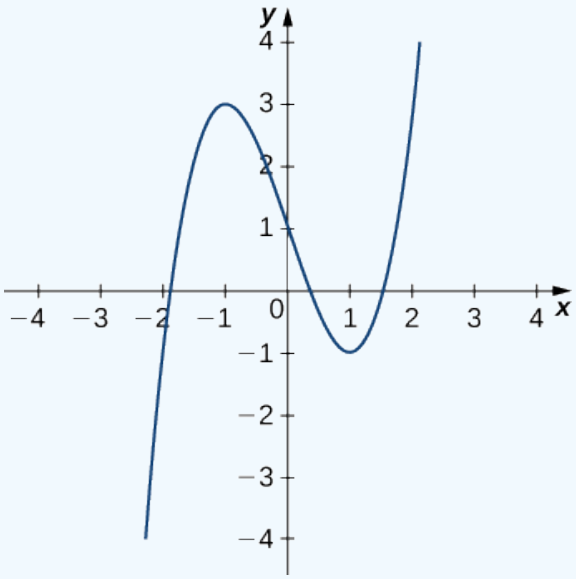


Figure 5.3.9: This is the graph of a cubic function $y = x^3 - 3x + 1$. The curve increases, reaches a maximum value of 3 at $x = -1$, decreases passing through the y-axis at 1, then reaching a minimum value of -1 at $x = 1$ before increasing again. We want to find the osculating circle of this graph at the point where $x = 1$.

First, let's calculate the curvature at $x = 1$:

$$\kappa = \frac{|f''(x)|}{\left(1 + [f'(x)]^2\right)^{3/2}} = \frac{|6x|}{(1 + [3x^2 - 3]^2)^{3/2}}. \tag{5.3.15}$$

This gives $\kappa = 6$. Therefore, the radius of the osculating circle is given by $R = \frac{1}{\kappa} = \frac{1}{6}$. Next, we then calculate the coordinates of the center of the circle. When $x = 1$, the slope of the tangent line is zero. Therefore, the center of the osculating circle is directly above the point on the graph with coordinates $(1, -1)$. The center is located at $(1, -\frac{5}{6})$. The formula for a circle with radius r and center (h, k) is given by $(x - h)^2 + (y - k)^2 = r^2$. Therefore, the equation of the osculating circle is $(x - 1)^2 + (y + \frac{5}{6})^2 = \frac{1}{36}$. The graph and its osculating circle appears in the following graph (Figure 5.3.10).

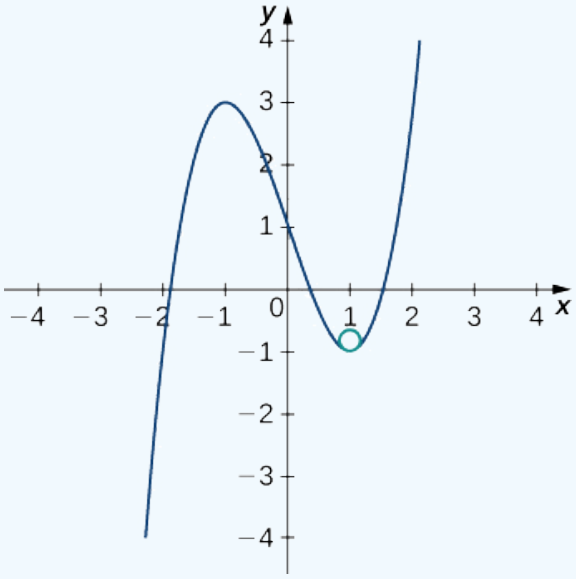


Figure 5.3.10: This is the graph of the cubic function described in Figure 5.3.9. There is a small circle inside of the bend of the curve at $x = 1$. The osculating circle has radius $R = \frac{1}{6}$.

Exercise 5.3.5

Find the equation of the osculating circle of the curve defined by the vector-valued function $y = 2x^2 - 4x + 5$ at $x = 1$.

Hint available at the end of this section.

Answer available at the end of this section.

Key Concepts

- The arc-length function for a vector-valued function is calculated using the integral formula $s(t) = \int_a^b \|\mathbf{r}\,'(t)\|\ dt$. This formula is valid in both two and three dimensions.
- The curvature of a curve at a point in either two or three dimensions is defined to be the curvature of the inscribed circle at that point. The arc-length parameterization is used in the definition of curvature.
- There are several different formulas for curvature. The curvature of a circle is equal to the reciprocal of its radius.
- The principal unit normal vector at t is defined to be

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}.$$

- The binormal vector at t is defined as $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$, where $\mathbf{T}(t)$ is the unit tangent vector.
- The Frenet frame of reference is formed by the unit tangent vector, the principal unit normal vector, and the binormal vector.
- The osculating circle is tangent to a curve at a point and has the same curvature as the tangent curve at that point.

Key Equations

- **Arc length of space curve**
  $s = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\ dt = \int_a^b \|\mathbf{r}\,'(t)\|\ dt$
- **Arc-length function**
  $s(t) = \int_a^t \sqrt{f'(u))^2 + (g'(u))^2 + (h'(u))^2}\ du \ \ or \ \ s(t) = \int_a^t \|\mathbf{r}\,'(u)\|\ du$
- $\kappa = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} \ \ or \ \ \kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} \ \ or \ \ \kappa = \frac{|y''|}{[1+(y')^2]^{3/2}}$
- **Principal unit normal vector**
  $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$
- **Binormal vector**
  $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$

Glossary

Arc-Length Function
A function $s(t)$ that describes the arc length of curve C as a function of t.

Arc-Length Parameterization
A reparameterization of a vector-valued function in which the parameter is equal to the arc length.

Binormal Vector
A unit vector orthogonal to the unit tangent vector and the unit normal vector.

Curvature
The derivative of the unit tangent vector with respect to the arc-length parameter.

Frenet Frame of Reference
(TNB frame) a frame of reference in three-dimensional space formed by the unit tangent vector, the unit normal vector, and the binormal vector.

Normal Plane
A plane that is perpendicular to a curve at any point on the curve.

Osculating Circle
A circle that is tangent to a curve C at a point P and that shares the same curvature.

Osculating Plane
The plane determined by the unit tangent and the unit normal vector.

Principal Unit Normal Vector
A vector orthogonal to the unit tangent vector, given by the formula $\frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$.

Radius of Curvature
The reciprocal of the curvature.

Smooth
Curves where the vector-valued function $\mathbf{r}(t)$ is differentiable with a non-zero derivative.

Answers

Exercise 5.3.1

Use Equation 5.3.4.

$\mathbf{r}\,'(t) = \langle 4t, 4t, 3t^2 \rangle$, so $s = \frac{1}{27}(113^{3/2} - 32^{3/2}) \approx 37.785$ units

Exercise 5.3.2

Start by finding the arc-length function.

$s = 5t$, or $t = s/5$. Substituting this into $\mathbf{r}(t) = \langle 3\cos t, 3\sin t, 4t \rangle$ gives

$$\mathbf{r}(s) = \langle 3\cos\left(\frac{s}{5}\right), 3\sin\left(\frac{s}{5}\right), \frac{4s}{5} \rangle, \quad s \geq 0$$

Exercise 5.3.3

Use 5.3.11.

$\kappa = \frac{6}{101^{3/2}} \approx 0.0059$

First, find $\mathbf{T}(t)$, then use 5.3.12.

$\mathbf{N}(2) = \frac{\sqrt{2}}{2}(\mathbf{i} - \mathbf{j})$

Exercise 5.3.5

Use 5.3.11 to find the curvature of the graph, then draw a graph of the function around $x = 1$ to help visualize the circle in relation to the graph.

Exercise 5.3.5

$\kappa = \frac{4}{[1+(4x-4)^2]^{3/2}}$

At the point $x = 1$, the curvature is equal to 4. Therefore, the radius of the osculating circle is $\frac{1}{4}$.

A graph of this function appears next in Figure 5.3.11:

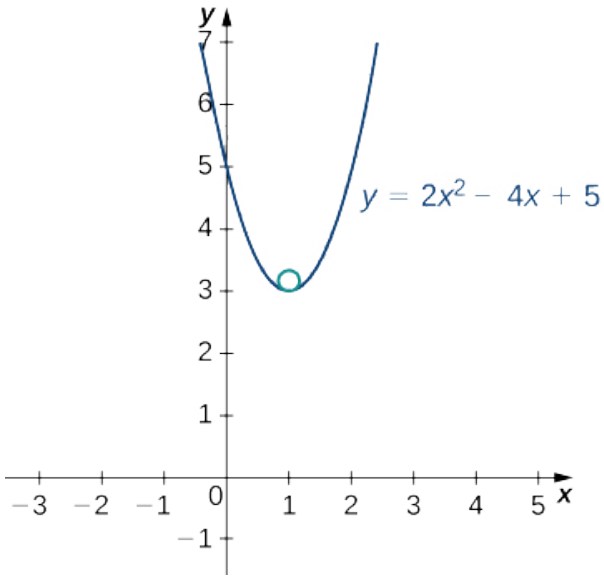


Figure 5.3.11: This figure is the graph of the function $y = 2x^2 - 4x + 5$. The curve is a parabola opening up with vertex at $(1, 3)$. The osculating circle is shown at the vertex.

The vertex of this parabola is located at the point $(1, 3)$. Furthermore, the center of the osculating circle is directly above the vertex. Therefore, the coordinates of the center are $(1, \frac{13}{4})$. The equation of the osculating circle is
$(x-1)^2 + (y - \frac{13}{4})^2 = \frac{1}{16}$.

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5.3E: Exercises for Section 5.3

Exercises 1 - 5 Determining Arc Length

Find the arc length of the curve on the given interval.

1. $\mathbf{r}(t) = t^2\ \mathbf{i} + (2t^2 + 1)\ \mathbf{j}, \quad 1 \le t \le 3$

   Answer

   $8\sqrt{5}$ units

2. $\mathbf{r}(t) = t^2\ \mathbf{i} + 14t\ \mathbf{j}, \quad 0 \le t \le 7$. This portion of the graph is shown here:

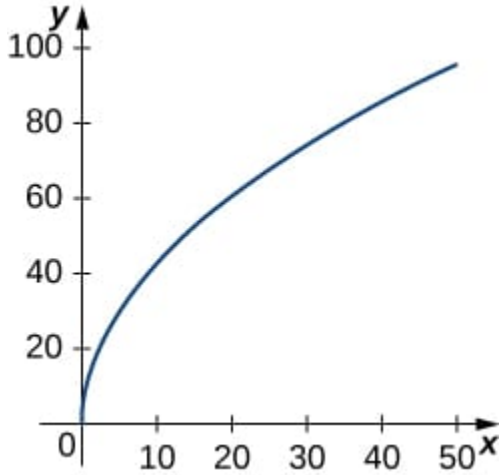


Figure $5.3E.1$: The graph of the given vector function is a smooth planar curve. The x values along the horizontal axis ranges from 0 to 50, and the y values along the vertical axis ranges from 0 to 100. The curve starts at the origin and increases in a concave down way.

3. $\mathbf{r}(t) = \langle t^2 + 1, 4t^3 + 3 \rangle, \quad -1 \le t \le 0$

   Answer

   $\frac{1}{54}(37^{3/2} - 1)$ units

4. $\mathbf{r}(t) = \langle 2\sin t, 5t, 2\cos t \rangle, \quad 0 \le t \le \pi$. This portion of the graph is shown here:

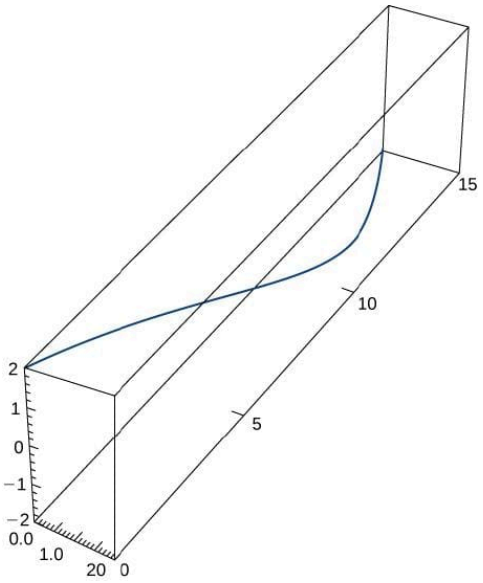


Figure $5.3E.2$: The graph of the given vector function is a smooth space curve. The plot is enclosed within a rectangular box with the horizontal axis ranging from 0 to 2, the vertical axis ranging from -2 to 2, and the depth axis ranging from 0 to 15. The curve goes from the bottom-left back corner, gradually ascending diagonally along a sinusoidal curve through the space, and toward the far-left top corner of the box.

5. $\mathbf{r}(t) = \langle e^{-t}\cos t, e^{-t}\sin t\rangle$ over the interval $[0, \frac{\pi}{2}]$. Here is the portion of the graph on the indicated interval:

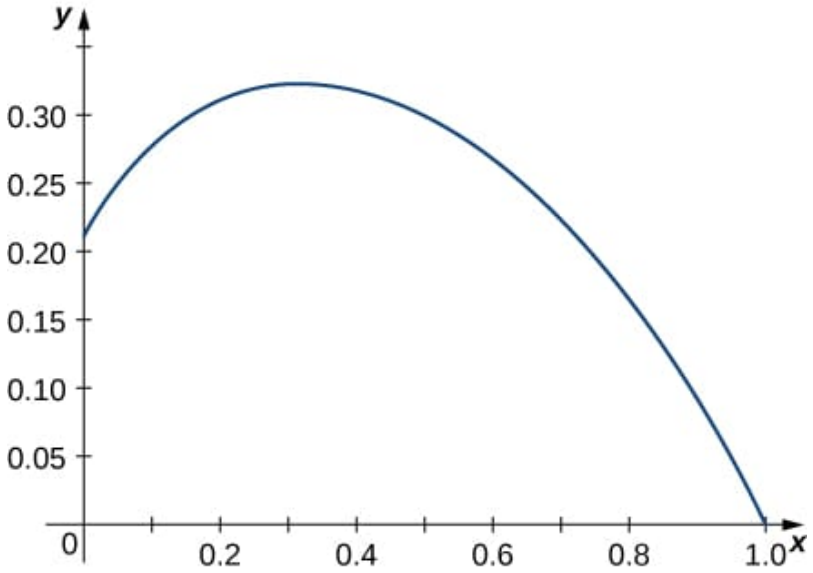


Figure $5.3E.3$: The graph of the given vector function is a smooth planar curve. The x values along the horizontal axis ranges from 0 to approximately 1, and the y values along the vertical axis ranges from 0 to approximately 0.35. The curve is in the first quadrant, goes from the point $(1, 0)$ to approximately $(0, 0.21)$. The shape of the curve is concave down and it reaches a local maximum at about $(0.32, 0.32)$.

6. Set up an integral to represent the arc length from $t = 0$ to $t = 2$ along the curve traced out by $\mathbf{r}(t) = \langle t,\ t^4\rangle$. Then use technology to approximate this length to the nearest thousandth of a unit.
7. Find the length of one turn of the helix given by $\mathbf{r}(t) = \frac{1}{2}\cos t\ \mathbf{i} + \frac{1}{2}\sin t\ \mathbf{j} + \frac{\sqrt{3}}{2}\ t\ \mathbf{k}$.

Answer

Length $= 2\pi$ units

8. Find the arc length of the vector-valued function $\mathbf{r}(t) = -t\ \mathbf{i} + 4t\ \mathbf{j} + 3t\ \mathbf{k}$ over $[0, 1]$.
9. A particle travels in a circle with the equation of motion $\mathbf{r}(t) = 3\cos t\ \mathbf{i} + 3\sin t\ \mathbf{j} + 0\ \mathbf{k}$. Find the distance traveled around the circle by the particle.

Answer

6π units

10. Set up an integral to find the circumference of the ellipse with the equation $\mathbf{r}(t) = \cos t\ \mathbf{i} + 2\sin t\ \mathbf{j} + 0\ \mathbf{k}$.
11. Find the length of the curve $\mathbf{r}(t) = \langle \sqrt{2}t,\ e^t,\ e^{-t} \rangle$ over the interval $0 \le t \le 1$. The graph is shown here:

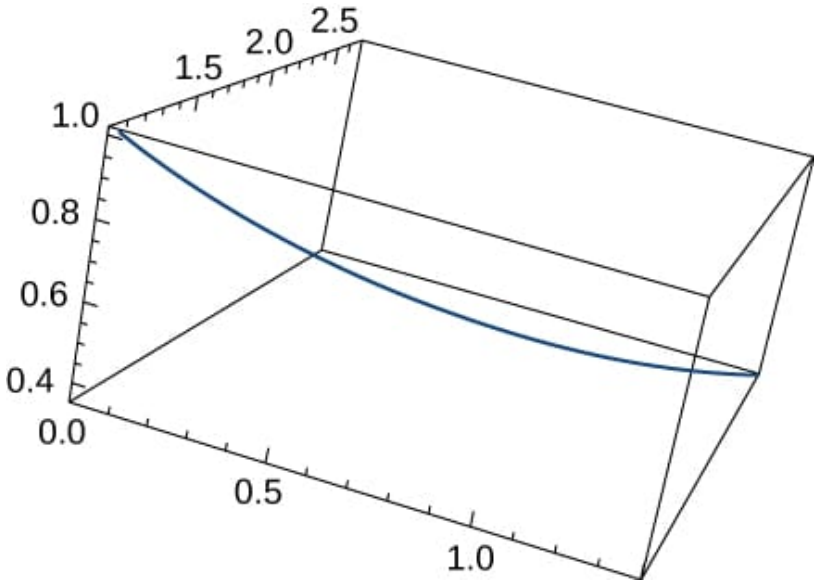


Figure $5.3E.4$: The graph of the given vector function is a smooth space curve. The plot is enclosed within a rectangular box with the horizontal axis ranging from 0 to 1.5, the vertical axis ranging from 0.4 to 1, and the depth axis ranging from 1 to 2.5. The curve goes from the top-left front corner, quickly descending diagonally, and ending at the bottom-right back corner.

Answer

$\left(e - \frac{1}{e}\right)$ units

12. Find the length of the curve $\mathbf{r}(t) = \langle 2\sin t,\ 5t,\ 2\cos t \rangle$ for $t \in [-10, 10]$.

Exercises 13 - 26 Unit Tangent Vectors and Unit Normal Vectors

13. The position function for a particle is $\mathbf{r}(t) = a\cos(\omega t)\ \mathbf{i} + b\sin(\omega t)\ \mathbf{j}$. Find the unit tangent vector and the unit normal vector at $t = 0$.

Solution:

$$\mathbf{r}'(t) = -a\omega\sin(\omega t)\,\mathbf{i} + b\omega\cos(\omega t)\,\mathbf{j}$$

$$\|\mathbf{r}'(t)\| = \sqrt{a^2\omega^2\sin^2(\omega t) + b^2\omega^2\cos^2(\omega t)}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{-a\omega\sin(\omega t)\,\mathbf{i} + b\omega\cos(\omega t)\,\mathbf{j}}{\sqrt{a^2\omega^2\sin^2(\omega t) + b^2\omega^2\cos^2(\omega t)}}$$

$$\mathbf{T}(0) = \frac{b\omega\,\mathbf{j}}{\sqrt{(b\omega)^2}} = \frac{b\omega\,\mathbf{j}}{|b\omega|}$$

If $b\omega > 0,\ \mathbf{T}(0) = \mathbf{j}$, and if $b\omega < 0,\ T(0) = -\mathbf{j}$

Answer

If $b\omega > 0,\ \mathbf{T}(0) = \mathbf{j}$, and if $b\omega < 0,\ \mathbf{T}(0) = -\mathbf{j}$

If $a > 0,\ \mathbf{N}(0) = -\mathbf{i}$, and if $a < 0,\ \mathbf{N}(0) = \mathbf{i}$

14. Given $\mathbf{r}(t) = a\cos(\omega t)\ \mathbf{i} + b\sin(\omega t)\ \mathbf{j}$, find the binormal vector $\mathbf{B}(0)$.
15. Given $\mathbf{r}(t) = \langle 2e^t, e^t\cos t, e^t\sin t \rangle$, determine the unit tangent vector $\mathbf{T}(t)$.

Answer

$$\mathbf{T}(t) = \left\langle \frac{2}{\sqrt{6}}, \frac{\cos t - \sin t}{\sqrt{6}}, \frac{\cos t + \sin t}{\sqrt{6}} \right\rangle$$
$$= \left\langle \frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{6}(\cos t - \sin t), \frac{\sqrt{6}}{6}(\cos t + \sin t) \right\rangle$$

16. Given $\mathbf{r}(t) = \langle 2e^t,\ e^t \cos t,\ e^t \sin t \rangle$, find the unit tangent vector $\mathbf{T}(t)$ evaluated at $t = 0$, $\mathbf{T}(0)$.
17. Given $\mathbf{r}(t) = \langle 2e^t,\ e^t \cos t,\ e^t \sin t \rangle$, determine the unit normal vector $\mathbf{N}(t)$.

Answer

$$\mathbf{N}(t) = \langle 0,\ -\tfrac{\sqrt{2}}{2}(\sin t + \cos t),\ \tfrac{\sqrt{2}}{2}(\cos t - \sin t) \rangle$$

18. Given $\mathbf{r}(t) = \langle 2e^t,\ e^t \cos t,\ e^t \sin t \rangle$, find the unit normal vector $\mathbf{N}(t)$ evaluated at $t = 0$, $\mathbf{N}(0)$.

Answer

$$\mathbf{N}(0) = \langle 0,\ -\tfrac{\sqrt{2}}{2},\ \tfrac{\sqrt{2}}{2} \rangle$$

19. Given $\mathbf{r}(t) = t\ \mathbf{i} + t^2\ \mathbf{j} + t\ \mathbf{k}$, find the unit tangent vector $\mathbf{T}(t)$. The graph is shown here:

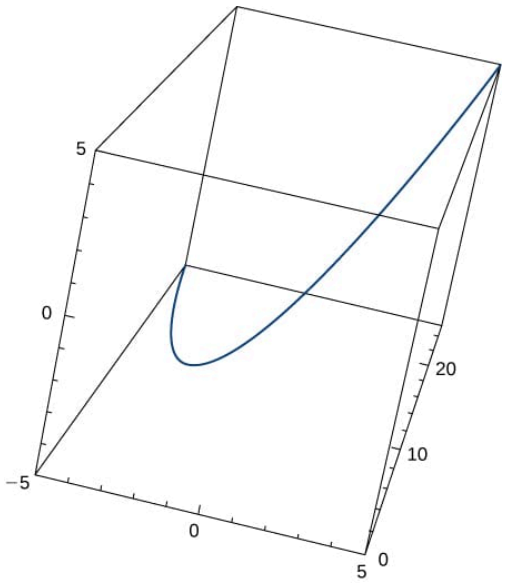


Figure $5.3E.5$: The graph of the given vector function is a smooth space curve. The plot is enclosed within a rectangular box with the horizontal axis ranging from -5 to 5, the vertical axis ranging from -5 to 5, and the depth axis ranging from 0 to 26. The curve goes from the bottom-left back corner, moving forward and up, then back and up, ending at the top-right back corner.

Answer

$$\mathbf{T}(t) = \frac{1}{\sqrt{4t^2 + 2}} < 1, 2t, 1 >$$

20) Find the unit tangent vector $\mathbf{T}(t)$ and unit normal vector $\mathbf{N}(t)$ at $t = 0$ for the plane curve $\mathbf{r}(t) = \langle t^3 - 4t, 5t^2 - 2 \rangle$. The graph is shown here:

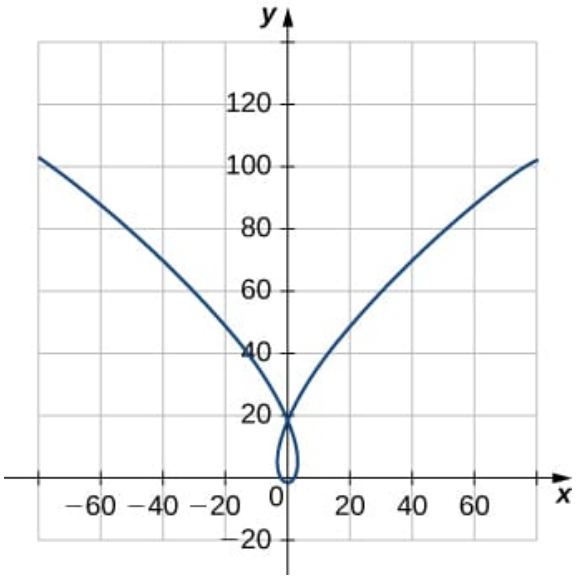


Figure $5.3E.6$: The graph of the given vector function is a smooth planar curve. The x values along the horizontal axis ranges from -80 to approximately 80, and the y values along the vertical axis ranges from -20 to approximately 140. The curve is symmetric across the y-axis, with one narrow region near the origin where the curve intersects itself. As x moves farther from 0 in both positive and negative directions, the curve widens and rises symmetrically along the y-axis.

21. Find the unit tangent vector $\mathbf{T}(t)$ for $\mathbf{r}(t) = 3t\ \mathbf{i} + 5t^2\ \mathbf{j} + 2t\ \mathbf{k}$.

Answer

$$\mathbf{T}(t) = \frac{1}{\sqrt{100t^2 + 13}}(3\ \mathbf{i} + 10t\ \mathbf{j} + 2\ \mathbf{k})$$

22. Find the principal normal vector to the curve $\mathbf{r}(t) = \langle 6\cos t, 6\sin t\rangle$ at the point determined by $t = \frac{\pi}{3}$.
23. Find $\mathbf{T}(t)$ for the curve $\mathbf{r}(t) = (t^3 - 4t)\ \mathbf{i} + (5t^2 - 2)\ \mathbf{j}$.

Answer

$$\mathbf{T}(t) = \frac{1}{\sqrt{9t^4 + 76t^2 + 16}}\left((3t^2 - 4)\ \mathbf{i} + 10t\ \mathbf{j}\right)$$

24. Find $\mathbf{N}(t)$ for the curve $\mathbf{r}(t) = (t^3 - 4t)\ \mathbf{i} + (5t^2 - 2)\ \mathbf{j}$.
25. Find the unit tangent vector $\mathbf{T}(t)$ for $\mathbf{r}(t) = \langle 2\sin t,\ 5t,\ 2\cos t\rangle$.

Answer

$$\mathbf{T}(t) = \langle \tfrac{2\sqrt{29}}{29}\cos t,\ \tfrac{5\sqrt{29}}{29},\ -\tfrac{2\sqrt{29}}{29}\sin t\rangle$$

26. Find the unit normal vector $\mathbf{N}(t)$ for $\mathbf{r}(t) = \langle 2\sin t,\ 5t,\ 2\cos t\rangle$.

Answer

$$\mathbf{N}(t) = \langle -\sin t,\ 0,\ -\cos t\rangle$$

Exercises 27 - 29 Arc Length Parameterizations

27. Find the arc-length function $\mathbf{s}(t)$ for the line segment given by $\mathbf{r}(t) = \langle 3 - 3t,\ 4t\rangle$. Then write the arc-length parameterization of r with s as the parameter.

Answer

Arc-length function: $s(t) = 5t$; The arc-length parameterization of $\mathbf{r}(t)$: $\mathbf{r}(s) = \left(3 - \frac{3s}{5}\right)\ \mathbf{i} + \frac{4s}{5}\ \mathbf{j}$

28. Parameterize the helix $\mathbf{r}(t) = \cos t\ \mathbf{i} + \sin t\ \mathbf{j} + t\ \mathbf{k}$ using the arc-length parameter s, from $t = 0$.

29. Parameterize the curve using the arc-length parameter s, at the point at which $t = 0$ for $\mathbf{r}(t) = e^t \sin t\ \mathbf{i} + e^t \cos t\ \mathbf{j}$

Answer

$$\mathbf{r}(s) = \left(1 + \frac{s}{\sqrt{2}}\right) \sin\left(\ln\left(1 + \frac{s}{\sqrt{2}}\right)\right)\ \mathbf{i} + \left(1 + \frac{s}{\sqrt{2}}\right) \cos\left(\ln\left(1 + \frac{s}{\sqrt{2}}\right)\right)\ \mathbf{j}$$

Exercises 30 - 50 Curvature and the Osculating Circle

30. Find the curvature of the curve $\mathbf{r}(t) = 5\cos t\ \mathbf{i} + 4\sin t\ \mathbf{j}$ at $t = \pi/3$. (*Note:* The graph is an ellipse.)

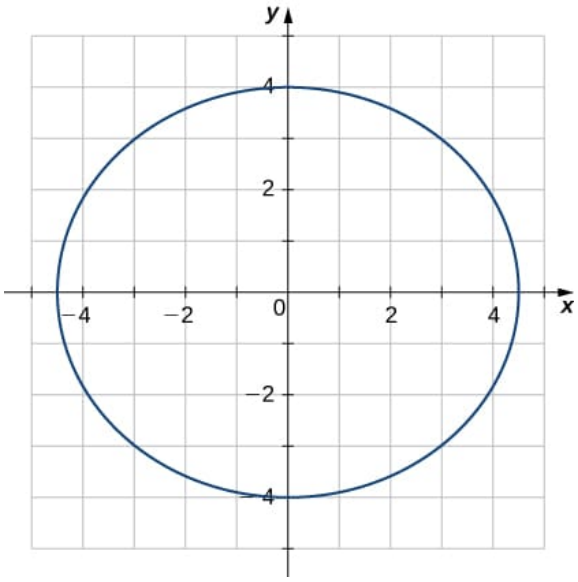


Figure $5.3E.7$: The graph of the given vector function is an ellipse centered at the origin, with x values ranging from -3.5 to 3.5, and y values ranging from -4 to 4.

31. Find the x-coordinate at which the curvature of the curve $y = 1/x$ is a maximum value.

Answer

The maximum value of the curvature occurs at $x = 1$.

32. Find the curvature of the curve $\mathbf{r}(t) = 5\cos t\ \mathbf{i} + 5\sin t\ \mathbf{j}$. Does the curvature depend upon the parameter t?
33. Find the curvature κ for the curve $y = x - \frac{1}{4}x^2$ at the point $x = 2$.

Answer

$\frac{1}{2}$

34. Find the curvature κ for the curve $y = \frac{1}{3}x^3$ at the point $x = 1$.
35. Find the curvature κ of the curve $\mathbf{r}(t) = t\ \mathbf{i} + 6t^2\ \mathbf{j} + 4t\ \mathbf{k}$. The graph is shown here:

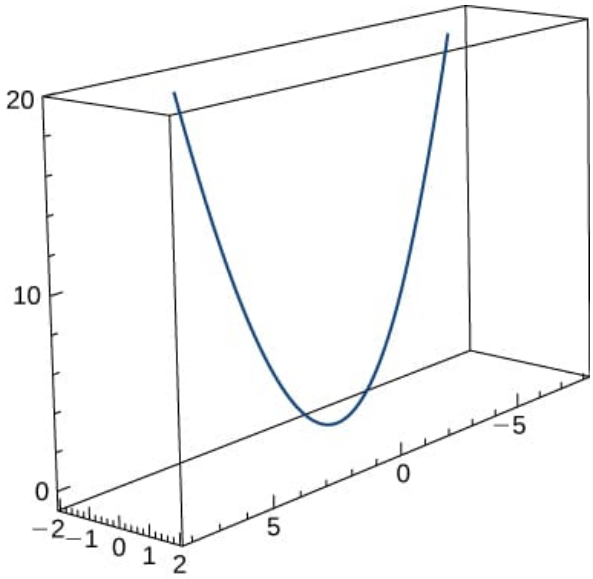


Figure $5.3E.8$: The graph of the given vector function has a parabolic shape. The curve opens upward and appears symmetric, with its lowest point near the center at the origin. The plot is enclosed within a rectangular box with the horizontal axis ranging from -2 to 2, the vertical axis ranging from 0 to 20, and the depth axis ranging from 8 to -8.

Answer

$$\kappa \approx \frac{49.477}{(17+144t^2)^{3/2}}$$

36. Find the curvature of $\mathbf{r}(t) = \langle 2\sin t, 5t, 2\cos t\rangle$.
37. Find the curvature of $\mathbf{r}(t) = \sqrt{2}t\ \mathbf{i} + e^t\ \mathbf{j} + e^{-t}\ \mathbf{k}$ at point $P(0,1,1)$.

Answer

$\frac{1}{2\sqrt{2}}$

38. At what point does the curve $y = e^x$ have maximum curvature?
39. What happens to the curvature as $x \to \infty$ for the curve $y = e^x$?

Answer

The curvature approaches zero.

40. Find the point of maximum curvature on the curve $y = \ln x$.
41. Find the equations of the normal plane and the osculating plane of the curve $\mathbf{r}(t) = \langle 2\sin(3t), t, 2\cos(3t)\rangle$ at point $(0, \pi, -2)$.

Answer

$y = 6x + \pi$ and $x + 6y = 6\pi$

42. Find equations of the osculating circles of the ellipse $4y^2 + 9x^2 = 36$ at the points $(2,0)$ and $(0,3)$.
43. Find the equation for the osculating plane at point $t = \pi/4$ on the curve $\mathbf{r}(t) = \cos(2t)\ \mathbf{i} + \sin(2t)\ \mathbf{j} + t\ \mathbf{k}$.

Answer

$x + 2z = \frac{\pi}{2}$

44. Find the radius of curvature of $6y = x^3$ at the point $(2, \frac{4}{3})$.
45. Find the curvature at each point (x,y) on the hyperbola $\mathbf{r}(t) = \langle a\cosh(t), b\sinh(t)\rangle$.

Answer

$$\frac{a^4b^4}{(b^4x^2 + a^4y^2)^{3/2}}$$

46. Calculate the curvature of the circular helix $\mathbf{r}(t) = r\sin(t)\ \mathbf{i} + r\cos(t)\ \mathbf{j} + t\ \mathbf{k}$.
47. Find the radius of curvature of $y = \ln(x+1)$ at point $(2, \ln 3)$.

Answer

$\frac{10\sqrt{10}}{3}$

18. Find the radius of curvature of the hyperbola $xy = 1$ at point $(1, 1)$.

Exercises 49 - 51

A particle moves along the plane curve C described by $\mathbf{r}(t) = t\ \mathbf{i} + t^2\ \mathbf{j}$. Use this parameterization to answer the following questions.

19. Find the length of the curve over the interval $[0, 2]$.

Answer

$\frac{1}{4}\left[4\sqrt{17} + \ln\left(4 + \sqrt{17}\right)\right]$ units ≈ 4.64678 units

20. Find the curvature of the plane curve at $t = 0, 1, 2$.
21. Describe the curvature as t increases from $t = 0$ to $t = 2$.

Answer

The curvature is decreasing over this interval.

Exercises 52 - 54

The surface of a large cup is formed by revolving the graph of the function $y = 0.25x^{1.6}$ from $x = 0$ to $x = 5$ about the y-axis (measured in centimeters).

22. **[T]** Use technology to graph the surface.
23. Find the curvature κ of the generating curve as a function of x.

Answer

$$\kappa = \frac{30}{x^{2/5}\left(25 + 4x^{6/5}\right)^{3/2}}$$

Note that initially your answer may be:

$$\frac{6}{25x^{2/5}\left(1 + \frac{4}{25}x^{6/5}\right)^{3/2}}$$

We can simplify it as follows:

$$\begin{aligned}\frac{6}{25x^{2/5}\left(1 + \frac{4}{25}x^{6/5}\right)^{3/2}} &= \frac{6}{25x^{2/5}\left[\frac{1}{25}\left(25 + 4x^{6/5}\right)\right]^{3/2}} \\ &= \frac{6}{25x^{2/5}\left(\frac{1}{25}\right)^{3/2}\left[25 + 4x^{6/5}\right]^{3/2}} \\ &= \frac{6}{\frac{25}{125}x^{2/5}\left[25 + 4x^{6/5}\right]^{3/2}} \\ &= \frac{30}{x^{2/5}\left(25 + 4x^{6/5}\right)^{3/2}}\end{aligned}$$

24. **[T]** Use technology to graph the curvature function.

Contributors:

- Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

- Paul Seeburger (Monroe Community College) created question 6.

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5.4: Motion in Space

Learning Objectives

- Describe the velocity and acceleration vectors of a particle moving in space.
- Explain the tangential and normal components of acceleration.
- State Kepler's laws of planetary motion.

We have now seen how to describe curves in the plane and in space, and how to determine their properties, such as arc length and curvature. All of this leads to the main goal of this chapter, which is the description of motion along plane curves and space curves. We now have all the tools we need; in this section, we put these ideas together and look at how to use them.

Motion Vectors in the Plane and in Space

Our starting point is using vector-valued functions to represent the position of an object as a function of time. All of the following material can be applied either to curves in the plane or to space curves. For example, when we look at the orbit of the planets, the curves defining these orbits all lie in a plane because they are elliptical. However, a particle traveling along a helix moves on a curve in three dimensions.

Definition: Velocity, Acceleration, and Speed

Let $\mathbf{r}(t)$ be a twice-differentiable vector-valued function of the parameter t that represents the position of an object as a function of time.

The **velocity** vector $\mathbf{v}(t)$ of the object is given by

$$\text{Velocity } = \mathbf{v}(t) = \mathbf{r}'(t). \tag{5.4.1}$$

The **acceleration** vector $\mathbf{a}(t)$ is defined to be

$$\text{Acceleration } = \mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t). \tag{5.4.2}$$

The **speed** is defined to be

$$\text{Speed } = v(t) = \|\mathbf{v}(t)\| = \|\mathbf{r}'(t)\| = \frac{ds}{dt}. \tag{5.4.3}$$

Since $\mathbf{r}(t)$ can be in either two or three dimensions, these vector-valued functions can have either two or three components. In two dimensions, we define $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ and in three dimensions $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Then the velocity, acceleration, and speed can be written as shown in the following table.

Table 5.4.1: Formulas for Position, Velocity, Acceleration, and Speed

Quantity	Two Dimensions	Three Dimensions
Position	$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$	$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$
Velocity	$\mathbf{v}(t) = x\,'(t)\mathbf{i} + y\,'(t)\mathbf{j}$	$\mathbf{v}(t) = x\,'(t)\mathbf{i} + y\,'(t)\mathbf{j} + z\,'(t)\mathbf{k}$
Acceleration	$\mathbf{a}(t) = x\,''(t)\mathbf{i} + y\,''(t)\mathbf{j}$	$\mathbf{a}(t) = x\,''(t)\mathbf{i} + y\,''(t)\mathbf{j} + z\,''(t)\mathbf{k}$
Speed	$\|\mathbf{v}(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2}$	$\|\mathbf{v}(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$

Example 5.4.1: Studying Motion Along a Parabola

A particle moves in a parabolic path defined by the vector-valued function $\mathbf{r}(t) = t^2\mathbf{i} + \sqrt{5 - t^2}\mathbf{j}$, where t measures time in seconds.

a. Find the velocity, acceleration, and speed as functions of time.
b. Sketch the curve along with the velocity vector at time $t = 1$.

Solution

a. We use Equations 5.4.1, 5.4.2, and 5.4.3:

$$\begin{aligned}
\mathbf{v}(t) = \mathbf{r}'(t) &= 2t\mathbf{i} - \frac{t}{\sqrt{5 - t^2}}\mathbf{j} \\
\mathbf{a}(t) = \mathbf{v}'(t) &= 2\mathbf{i} - 5(5 - t^2)^{-\frac{3}{2}}\mathbf{j} \\
||\mathbf{v}(t)|| &= ||\mathbf{r}'(t)|| \\
&= \sqrt{(2t)^2 + \left(-t \Big/ \sqrt{5 - t^2}\right)^2} \\
&= \sqrt{4t^2 + \frac{t^2}{5 - t^2}} \\
&= \sqrt{\frac{21t^2 - 4t^4}{5 - t^2}}.
\end{aligned}$$

b. The graph of $\mathbf{r}(t) = t^2\mathbf{i} + \sqrt{5 - t^2}\mathbf{j}$ is a portion of a parabola (Figure 5.4.1).

When $t = 1$, $\mathbf{r}(1) = (1)^2\mathbf{i} + \sqrt{5 - (1)^2}\mathbf{j} \quad = \quad \mathbf{i} + \sqrt{4}\mathbf{j} \quad = \quad \mathbf{i} + 2\mathbf{j}$.

Thus the particle would be located at the point $(1, 2)$ when $t = 1$.

The velocity vector at $t = 1$ is

$$\begin{aligned}
\mathbf{v}(1) = \mathbf{r}'(1) &= 2(1)\mathbf{i} - \frac{1}{\sqrt{5 - 1^2}}\mathbf{j} \\
&= \quad 2\mathbf{i} - \frac{1}{2}\mathbf{j}
\end{aligned}$$

and the acceleration vector at $t = 1$ is

$$\mathbf{a}(1) = \mathbf{v}'(1) = 2\mathbf{i} - 5(5 - 1^2)^{-3/2}\mathbf{j} \quad = \quad 2\mathbf{i} - \frac{5}{8}\mathbf{j}. \tag{5.4.4}$$

Notice that the velocity vector is tangent to the path, as is always the case.

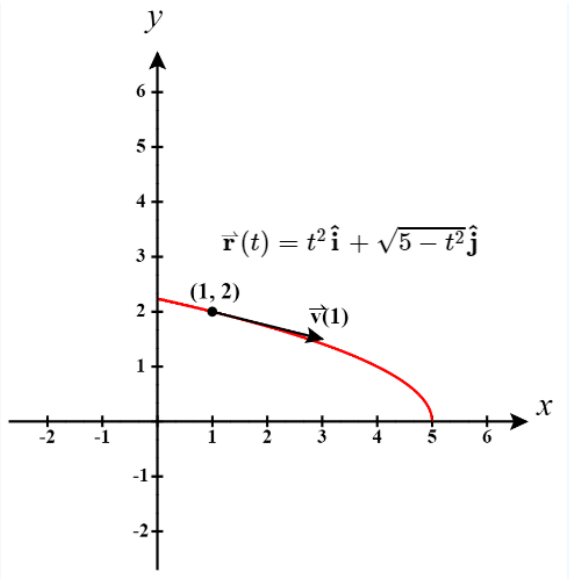


Figure 5.4.1: This graph depicts the velocity vector at time $t = 1$ for a particle moving in a parabolic path. The given plane curve $\mathbf{r}(t)$ is graphed in the first quadrant. It's a parabola that opens to the left and passes through $(0, \sqrt{5})$ and $(5, 0)$, with latter being the vertex of the parabola. A vector is sketched with its tail at $(1, 2)$ and tip at $(3, 1.5)$. The vector is tangent the parabola at $(1, 2)$.

Watch the accompanying video for a visual demonstration of this concept.

- **Video Length**: 4 minute 26 seconds.
- **Context**: This video demonstrates how to find the velocity, acceleration, unit tangent, and unit normal vectors for $\mathbf{r}(t) = 2t\mathbf{i} + t^2\mathbf{j}$.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Exercise 5.4.1

A particle moves in a path defined by the vector-valued function $\mathbf{r}(t) = (t^2 - 3t)\ \mathbf{i} + (2t - 4)\ \mathbf{j} + (t + 2)\ \mathbf{k}$, where t measures time in seconds and where distance is measured in feet. Find the velocity, acceleration, and speed as functions of time.

Hint available at the end of this section.

To gain a better understanding of the velocity and acceleration vectors, imagine you are driving along a curvy road. If you do not turn the steering wheel, you would continue in a straight line and run off the road. The speed at which you are traveling when you run off the road, coupled with the direction, gives a vector representing your velocity, as illustrated in Figure 5.4.2.

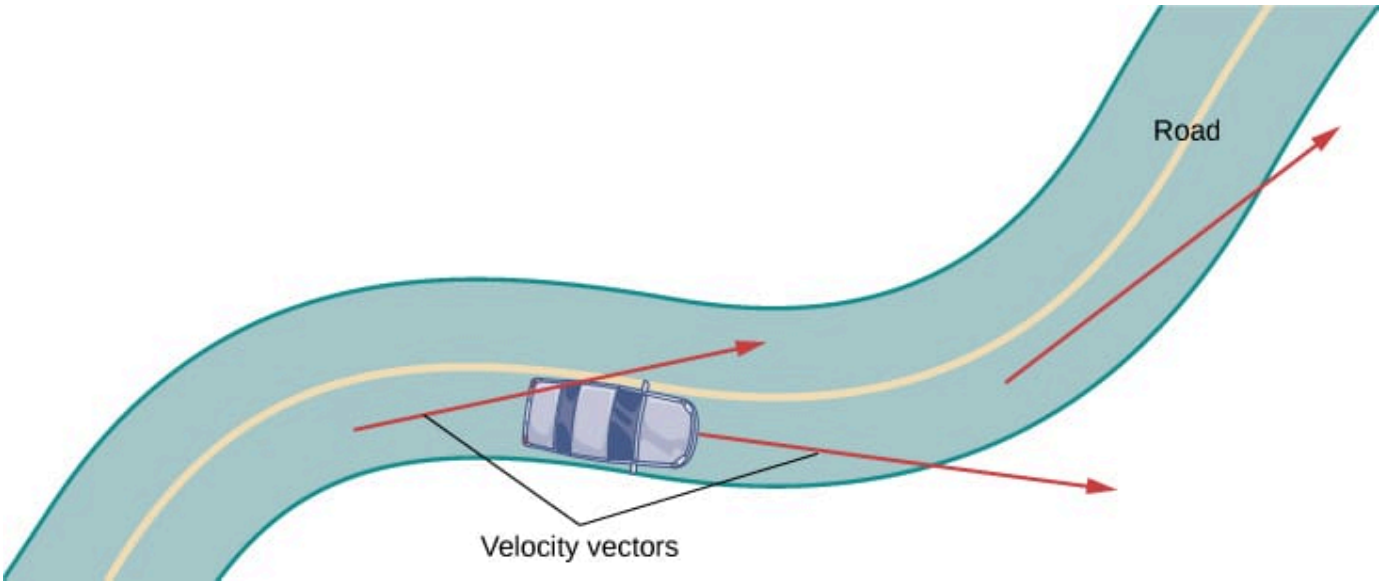


Figure 5.4.2: At each point along a road traveled by a car, the velocity vector of the car is tangent to the path traveled by the car.

However, the fact that you must turn the steering wheel to stay on the road indicates that your velocity is always changing (even if your speed is not) because your *direction* is constantly changing to keep you on the road. As you turn to the right, your acceleration vector also points to the right. As you turn to the left, your acceleration vector points to the left. This indicates that your velocity and acceleration vectors are constantly changing, regardless of whether your actual speed varies (Figure 5.4.3).

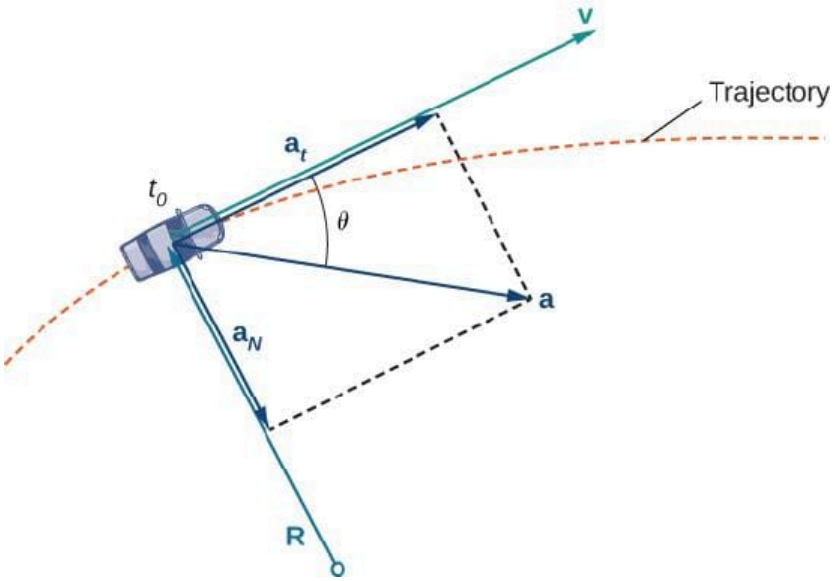


Figure 5.4.3: The path the car is traveling is curve represented by a dotted line. The position of the car is labeled t_0 on the curve. At this point there are two pairwise orthogonal vector shown: $\mathbf{a}_T$ is tangent to the path of the car and $\mathbf{a}_N$ points in the normal direction on the "inside" of the path of the car. These two vectors are the tangent and normal components of the acceleration vector $\mathbf{a}$ shown "between" the first two vectors. has angle between vector $\mathbf{a}$ and $\mathbf{a}_T$ is marked as θ. The acceleration vector points toward the "inside" of the turn at all times.

Components of the Acceleration Vector

We can combine some of the concepts discussed in Arc Length and Curvature with the acceleration vector to gain a deeper understanding of how this vector relates to motion in the plane and in space. Recall that the unit tangent vector $\mathbf{T}$ and the unit normal vector $\mathbf{N}$ form an osculating plane at any point P on the curve defined by a vector-valued function $\mathbf{r}(t)$. The following theorem shows that the acceleration vector $\mathbf{a}(t)$ lies in the osculating plane and can be written as a linear combination of the unit tangent and the unit normal vectors.

Theorem 5.4.1: The Plane of the Acceleration Vector

The acceleration vector $\mathbf{a}(t)$ of an object moving along a curve traced out by a twice-differentiable function $\mathbf{r}(t)$ lies in the plane formed by the unit tangent vector $\mathbf{T}(t)$ and the principal unit normal vector $\mathbf{N}(t)$ to C. Furthermore,

$$\mathbf{a}(t) = v'(t)\mathbf{T}(t) + [v(t)]^2\kappa\mathbf{N}(t) \tag{5.4.5}$$

Here, $v(t) = \|\mathbf{v}(t)\|$ is the speed of the object and κ is the curvature of C traced out by $\mathbf{r}(t)$.

Proof

Because $\mathbf{v}(t) = \mathbf{r}\,'(t)$ and $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||}$, we have $\mathbf{v}(t) = ||\mathbf{r}\,'(t)||\mathbf{T}(t) = v(t)\mathbf{T}(t)$.

Now we differentiate this equation:

$$\mathbf{a}(t) = \mathbf{v}'(t) = \frac{d}{dt}\left(v(t)\mathbf{T}(t)\right) = v'(t)\mathbf{T}(t) + v(t)\mathbf{T}'(t)$$

Since $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{||\mathbf{T}'(t)||}$, we know $\mathbf{T}\,'(t) = ||\mathbf{T}\,'(t)||\mathbf{N}(t)$, so

$$\mathbf{a}(t) = v'(t)\mathbf{T}(t) + v(t)||\mathbf{T}'(t)||\mathbf{N}(t).$$

A formula for curvature is $\kappa = \dfrac{||\mathbf{T}'(t)||}{||\mathbf{r}'(t)||}$, so $\mathbf{T}'(t) = \kappa||\mathbf{r}'(t)|| = \kappa v(t)$.

This gives $\mathbf{a}(t) = v\,'(t)\mathbf{T}(t) + \kappa(v(t))^2\mathbf{N}(t)$.

□

The coefficients of $\mathbf{T}(t)$ and $\mathbf{N}(t)$ are referred to as the **tangential component of acceleration** and the **normal component of acceleration**, respectively. We write $a_{\mathbf{T}}$ to denote the tangential component and $a_{\mathbf{N}}$ to denote the normal component.

Theorem 5.4.2: Tangential and Normal Components of Acceleration

Let $\mathbf{r}(t)$ be a vector-valued function that denotes the position of an object as a function of time. Then $\mathbf{a}(t) = \mathbf{r}\,''(t)$ is the acceleration vector. The tangential and normal components of acceleration $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$ are given by the formulas

$$a_{\mathbf{T}} = \mathbf{a}\cdot\mathbf{T} = \frac{\mathbf{v}\cdot\mathbf{a}}{||\mathbf{v}||} \tag{5.4.6}$$

and

$$a_{\mathbf{N}} = \mathbf{a}\cdot\mathbf{N} = \frac{||\mathbf{v}\times\mathbf{a}||}{||\mathbf{v}||} = \sqrt{||\mathbf{a}||^2 - (a_{\mathbf{T}})^2}. \tag{5.4.7}$$

These components are related by the formula

$$\mathbf{a}(t) = a_{\mathbf{T}}\mathbf{T}(t) + a_{\mathbf{N}}\mathbf{N}(t). \tag{5.4.8}$$

Here $\mathbf{T}(t)$ is the unit tangent vector to the curve defined by $\mathbf{r}(t)$, and $\mathbf{N}(t)$ is the unit normal vector to the curve defined by $\mathbf{r}(t)$.

The normal component of acceleration is also called the *centripetal component of acceleration* or sometimes the *radial component of acceleration*. To understand centripetal acceleration, suppose you are traveling in a car on a circular track at a constant speed. Then, as we saw earlier, the acceleration vector points toward the center of the track at all times. As a rider in

the car, you feel a pull toward the *outside* of the track because you are constantly turning. This sensation acts in the opposite direction of centripetal acceleration. The same holds true for non-circular paths. The reason is that your body tends to travel in a straight line and resists the force resulting from acceleration that push it toward the side. Note that at point B in Figure $5.4.4$ the acceleration vector is pointing backward. This is because the car is decelerating as it goes into the curve.

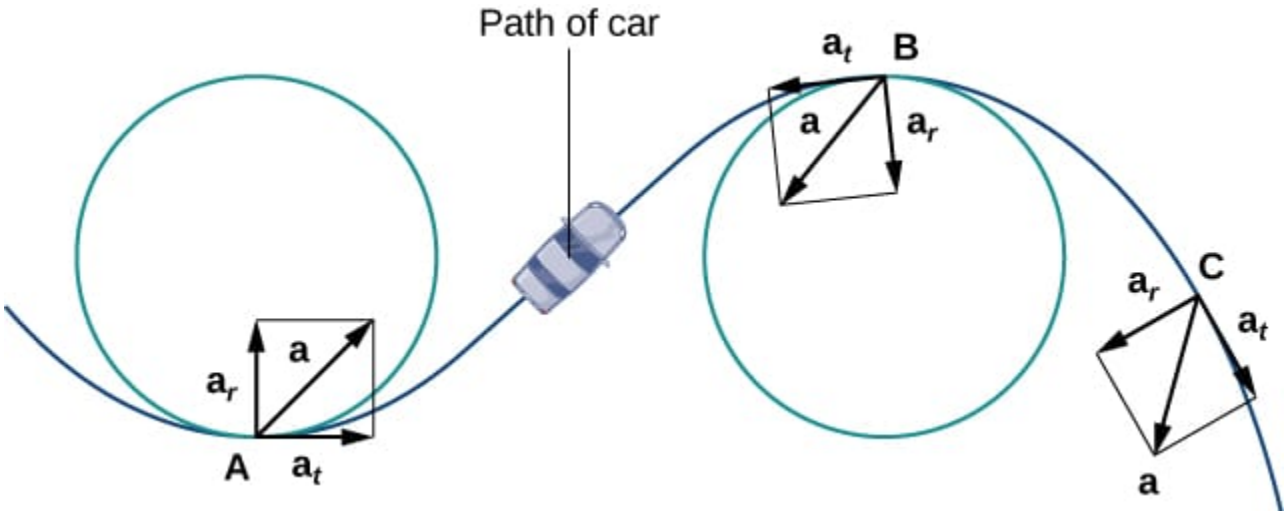


Figure $5.4.4$: The tangential and normal components of acceleration can be used to describe the acceleration vector. This figure has a curve representing the path of a car. The curve decreases and increases. There are two circles along the path The first circle has point A where the curve meets the circle. At point A there are three vectors. The first vector is $\mathbf{a}_T$ and is tangent to the curve at A. The second vector is $\mathbf{a}_N$ and is orthogonal to vector $\mathbf{a}_T$. In between these vectors is vector $\mathbf{a}$. The second circle has point B where the curve where there's a similar circle showing the decomposition of $\mathbf{a}$ into $\mathbf{a}_T$ and $\mathbf{a}_N$. At a third point C, there's no circle shown, but a similar decomposition of the acceleration vector $\mathbf{a}$ into tangential and normal components is shown. In each case, $\mathbf{a}_T$ points in the tangent direction, whereas $\mathbf{a}_N$ points on the "inside" of the curve (the direction in which the curve is turning.

The tangential and normal unit vectors at any given point on the curve provide a frame of reference at that point. The tangential and normal components of acceleration are the projections of the acceleration vector onto $\mathbf{T}$ and $\mathbf{N}$, respectively.

Example $5.4.2$: Finding Components of Acceleration

A particle moves in a path defined by the vector-valued function $\mathbf{r}(t) = t^2\ \mathbf{i} + (2t-3)\ \mathbf{j} + (3t^2 - 3t)\ \mathbf{k}$, where t measures time in seconds and distance is measured in feet.

a. Find $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$ as functions of t.
b. Find $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$ at time $t = 2$.

Solution

a. Let's start deriving the velocityand acceleration functions:

$$\begin{aligned}\mathbf{v}(t) &= \mathbf{r}'(t) \\ &= 2t\,\mathbf{i} + 2\,\mathbf{j} + (6t-3)\,\mathbf{k} \\ \mathbf{a}(t) &= \mathbf{v}'(t) \\ &= 2\,\mathbf{i} + 6\,\mathbf{k}\end{aligned}$$

Now we apply Equation $5.4.6$:

$$\begin{aligned} a_{\mathbf{T}} &= \frac{\mathbf{v}\cdot\mathbf{a}}{||\mathbf{v}||} \\ &= \frac{(2t\,\mathbf{i}+2\,\mathbf{j}+(6t-3)\,\mathbf{k})\cdot(2\,\mathbf{i}+6\,\mathbf{k})}{||2t\,\mathbf{i}+2\,\mathbf{j}+(6t-3)\,\mathbf{k}||} \\ &= \frac{4t+6(6t-3)}{\sqrt{(2t)^2+2^2+(6t-3)^2}} \\ &= \frac{40t-18}{40t^2-36t+13} \end{aligned}$$

Now we can apply Equation 5.4.7:

$$\begin{aligned} a_{\mathbf{N}} &= \sqrt{||\mathbf{a}||^2 - a_{\mathbf{T}}} \\ &= \sqrt{||2\,\mathbf{i}+6\,\mathbf{k}||^2 - \left(\frac{40t-18}{\sqrt{40t^2-36+13}}\right)^2} \\ &= \sqrt{4+36-\frac{(40t-18)^2}{40t^2-36t+13}} \\ &= \sqrt{\frac{40(40t^2-36t+13)-(1600t^2-1440t+324)}{40t^2-36t+13}} \\ &= \sqrt{\frac{196}{40t^2-36t+13}} \\ &= \frac{14}{\sqrt{40t^2-36t+13}} \end{aligned}$$

b. We must evaluate each of the answers from part a at $t=2$:

$$\begin{aligned} a_{\mathbf{T}}(2) &= \frac{40(2)-18}{\sqrt{40(2)^2-36(2)+13}} \\ &= \frac{80-18}{\sqrt{160-72+13}} \\ &= \frac{62}{\sqrt{101}} \\ a_{\mathbf{N}}(2) &= \frac{14}{\sqrt{40(2)^2-36(2)+13}} \\ &= \frac{14}{\sqrt{160-72+13}} = \frac{14}{\sqrt{101}}. \end{aligned}$$

The units of acceleration are feet per second squared, as are the units of the normal and tangential components of acceleration.

Watch the accompanying video for a visual demonstration of this concept.

- **Video Length**: 5 minutes 3 seconds.
- **Context**: This video demonstrates how to find the unit tangent and normal vectors, and the components of acceleration for $\mathbf{r}(t) = 2t\mathbf{i} + \cos t\mathbf{j} - \sin t\mathbf{k}$.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

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Exercise 5.4.2

An object moves in a path defined by the vector-valued function $\mathbf{r}(t) = 4t\ \mathbf{i} + t^2\ \mathbf{j}$, where t measures time in seconds.

a. Find a_{T} and a_{N} as functions of t.
b. Find a_{T} and a_{N} at time $t = -3$.

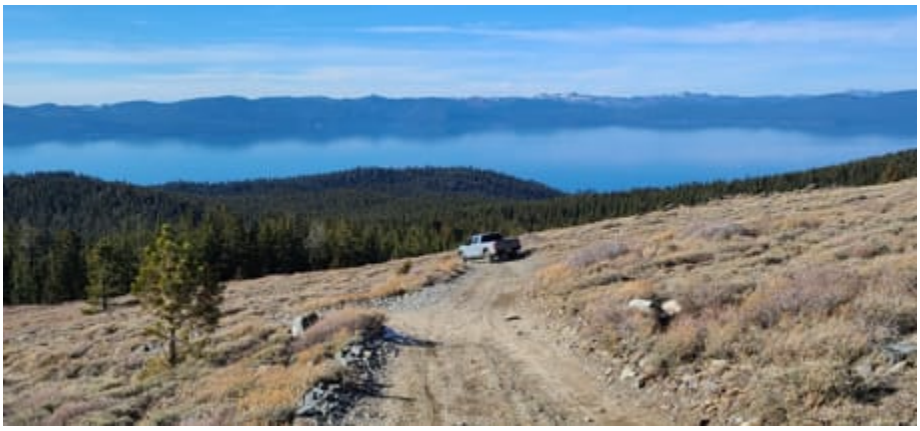
Hint available at the end of this section.

Answer available at the end of this section.

Interactive Element

Finding the Components of Acceleration of a Truck on a Road

A self driving truck will be driving on a remote dirt road and it's possible position functions have already been determined and validated in several tests. Your job will be to validate the code that will determine the components of acceleration once it is written. You will be given the position function below and you must determine $a_{\vecs{T}}$ and $a_{\vecs{N}}$ at the given value of t rounded to four decimal places.



$\vecs{r}(t) = (t+1)\hat{i} + (1-t^2)\hat{j} + e^{-t}\hat{k}$ at $t = 1$

$\vecs{r}(t) = \cos(t)\hat{i} + \sin(t)\hat{j} + \sqrt{t}\hat{k}$ at $t = 4$

$\vecs{r}(t) = (2t-1)\hat{i} + \cos(t)\hat{j} + \sin(2t)\hat{k}$ at $t = 2$

$\vecs{r}(t) = \frac{1}{t+1}\hat{i} + \frac{1}{t+1}\hat{j} + \frac{1}{t+3}\hat{k}$ at $t = 1$

$\vecs{r}(t) = \sin(t)\hat{i} + \sin(2t)\hat{j} + \sin(3t)\hat{k}$ at $t = 2$

$\vecs{r}(t) = \frac{1}{t+1}\hat{i} + \frac{1}{(t+1)^2}\hat{j} + \frac{1}{(t+1)^3}\hat{k}$ at $t = 1$

$\vecs{r}(t) = e^{-t}\hat{i} + e^{-2t}\hat{j} + e^{-3t}\hat{k}$ at $t = 0.5$

$\vecs{r}(t) = (t + \sin(t))\hat{i} + t^2\hat{j} + e^{t}\hat{k}$ at $t = 0.8$

$\vecs{r}(t) = (t + \frac{1}{t+2})\hat{i} + (t^2 - t - 1)\hat{j} + (t^3 - 3t^2)\hat{k}$ at $t = 3$

$\vecs{r}(t) = e^{-t}\hat{i} + \cos(2t)\hat{j} + \ln(t+1)\hat{k}$ at $t = 2$

$a_{\vec{T}}(t)$ = $a_{\vec{N}}(t)$ =

Start

Click on the Start button to begin.

Projectile Motion

Now let's look at an application of vector functions. In particular, let's consider the effect of gravity on the motion of an object as it travels through the air, and how it determines the resulting trajectory of that object. In the following, we ignore the effect of air resistance. This situation, with an object moving with an initial velocity but with no forces acting on it other than gravity, is known as projectile motion. It describes the motion of objects from golf balls to baseballs, and from arrows to cannonballs.

First we need to choose a coordinate system. If we are standing at the origin of this coordinate system, then we choose the positive y-axis to be up, the negative y-axis to be down, and the positive x-axis to be forward (i.e., away from the thrower of the object). The effect of gravity is in a downward direction, so Newton's second law tells us that the force on the object resulting from gravity is equal to the mass of the object times the acceleration resulting from gravity, or $\mathbf{F}_g = m\mathbf{a}$, where $\mathbf{F}_g$ represents the force from gravity and $\mathbf{a} = -g\ \mathbf{j}$ represents the acceleration resulting from gravity at Earth's surface. The value of g in the English system of measurement is approximately 32 ft/sec^2 and it is approximately 9.8 m/sec^2 in the metric system. This is the only force acting on the object. Since gravity acts in a downward direction, we can write the force resulting from gravity in the form $\mathbf{F}_g = -mg\ \mathbf{j}$, as shown in Figure 5.4.5.

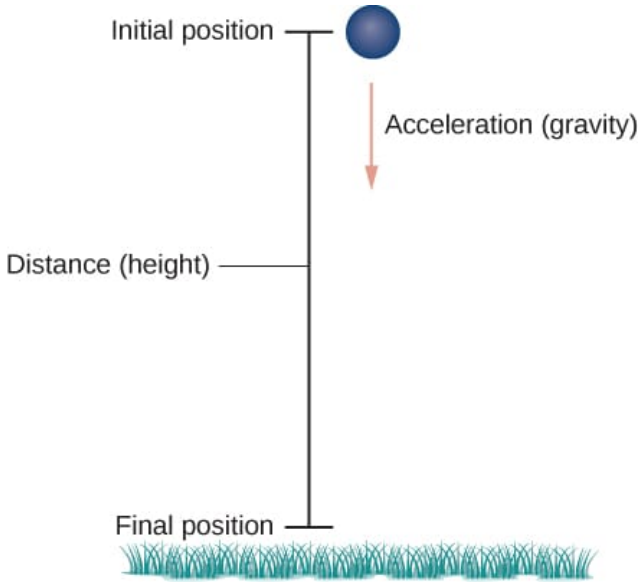


Figure 5.4.5: This figure is a ball falling in a vertical path. The ball is at the top at the initial position. From the ball a vector is drawn vertically downward labeled "acceleration". The vertical line is labeled "distance". At the bottom of the line it is labeled "final position". There is also grass at the bottom of the figure.

Newton's second law also tells us that $F = m\mathbf{a}$, where $\mathbf{a}$ represents the acceleration vector of the object. This force must be equal to the force of gravity at all times, so we therefore know that

$$\begin{aligned} \mathbf{F} &= \mathbf{F}_g \\ m\mathbf{a} &= -mg\mathbf{j} \\ \mathbf{a} &= -g\mathbf{j}. \end{aligned}$$

Now we use the fact that the acceleration vector is the first derivative of the velocity vector. Therefore, we can rewrite the last equation in the form

$$\mathbf{v}'(t) = -g\mathbf{j} \tag{5.4.9}$$

By taking the antiderivative of each side of this equation we obtain

$$\mathbf{v}(t) = \int -g\,\mathbf{j}\,dt = -gt\,\mathbf{j} + \mathbf{C}_1 \tag{5.4.10}$$

for some constant vector $\mathbf{C}_1$. To determine the value of this vector, we can use the velocity of the object at a fixed time, say at time $t = 0$. We call this velocity the *initial velocity*: $\mathbf{v}(0) = \mathbf{v}_0$. Therefore, $\mathbf{v}(0) = -g(0)\ \mathbf{j} + \mathbf{C}_1 = \mathbf{v}_0$ and $\mathbf{C}_1 = \mathbf{v}_0$. This gives the velocity vector as $\mathbf{v}(t) = -gt\ \mathbf{j} + \mathbf{v}_0$.

Next we use the fact that velocity $\mathbf{v}(t)$ is the derivative of position $\mathbf{s}(t)$. This gives the equation

$$\mathbf{s}'(t) = -gt\,\mathbf{j} + \mathbf{v}_0. \tag{5.4.11}$$

Taking the antiderivative of both sides of this equation leads to

$$\begin{aligned}\mathbf{s}(t) &= \int -gt\,\mathbf{j} + \mathbf{v}_0\,dt \\ &= -\frac{1}{2}gt^2\,\mathbf{j} + \mathbf{v}_0 t + \mathbf{C}_2\end{aligned}$$

with another unknown constant vector $\mathbf{C}_2$. To determine the value of $\mathbf{C}_2$, we can use the position of the object at a given time, say at time $t = 0$. We call this position the *initial position*: $\mathbf{s}(0) = \mathbf{s}_0$. Therefore, $\mathbf{s}(0) = -(1/2)g(0)^2\ \mathbf{j} + \mathbf{v}_0(0) + \mathbf{C}_2 = \mathbf{s}_0$. This gives the position of the object at any time as

$$\mathbf{s}(t) = -12gt^2\,\mathbf{j} + \mathbf{v}_0 t + \mathbf{s}_0. \tag{5.4.12}$$

Let's take a closer look at the initial velocity and initial position. In particular, suppose the object is thrown upward from the origin at an angle θ to the horizontal, with initial speed $\mathbf{v}_0$. How can we modify the previous result to reflect this scenario? First, we can assume it is thrown from the origin. If not, then we can move the origin to the point from where it is thrown. Therefore, $\mathbf{s}_0 = \mathbf{0}$, as shown in Figure 5.4.6.

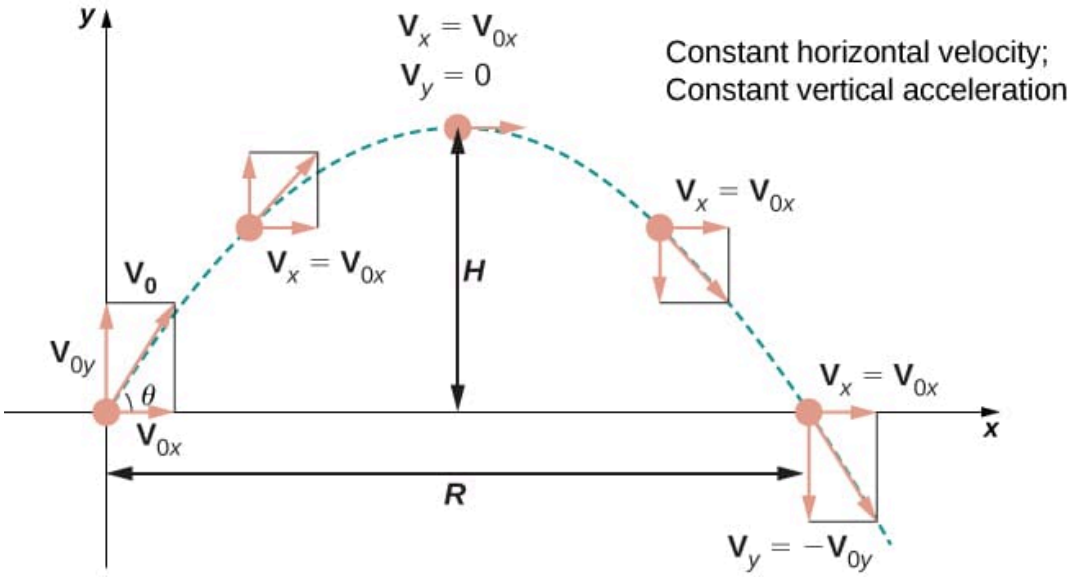


Figure 5.4.6: This figure has an upside down parabolic curve representing projectile motion. The figure is labeled "constant horizontal velocity; constant vertical acceleration". The curve is in the first quadrant beginning and ending on the x-axis. The height of the curve is labeled "H". The distance on the x-axis is labeled "R". Angle theta represents the direction of the projectile at the origin. Five points are labeled on the graph with vectors. The vectors are labeled "v" with subscripts representing directions. At each of the five points, $\mathbf{v}_x$ is the same. However, at the first point, $\mathbf{v}_y$ points up with the longest vector. At the second point, $\mathbf{v}_y$ points up but with a shorter vector. At the third point, the projectile is at the highest point, and $\mathbf{v}_y$ is the 0 vector. At the fourth point, $\mathbf{v}_y$ points down with a shorter vector. At the final point where the projectile hits the ground, $\mathbf{v}_y$ points down with the longest vector (same length as initially).

We can rewrite the initial velocity vector in the form $\mathbf{v}_0 = v_0 \cos\theta\ \mathbf{i} + v_0 \sin\theta\ \mathbf{j}$. Then the equation for the position function $\mathbf{s}(t)$ becomes

$$\begin{aligned}\mathbf{s}(t) &= -\frac{1}{2}gt^2\,\mathbf{j} + v_0 t\cos\theta\,\mathbf{i} + v_0 t\sin\theta\,\mathbf{j}\\ &= v_0 t\cos\theta\,\mathbf{i} + v_0 t\sin\theta\,\mathbf{j} - \frac{1}{2}gt^2\,\mathbf{j}\\ &= v_0 t\cos\theta\,\mathbf{i} + \left(v_0 t\sin\theta - \frac{1}{2}gt^2\right)\mathbf{j}.\end{aligned}$$

The coefficient of $\mathbf{i}$ represents the horizontal component of $\mathbf{s}(t)$ and is the horizontal distance of the object from the origin at time t. The maximum value of the horizontal distance (measured at the same initial and final altitude) is called the range R. The coefficient of $\mathbf{j}$ represents the vertical component of $\mathbf{s}(t)$ and is the altitude of the object at time t. The maximum value of the vertical distance is the height H.

Example 5.4.3: Motion of a Cannonball

During an Independence Day celebration, a cannonball is fired from a cannon on a cliff toward the water. The cannon is aimed at an angle of 30° above horizontal and the initial speed of the cannonball is 600 ft/sec. The cliff is 100 ft above the water (Figure 5.4.7).

a. Find the maximum height of the cannonball.
b. How long will it take for the cannonball to splash into the sea?
c. How far out to sea will the cannonball hit the water?

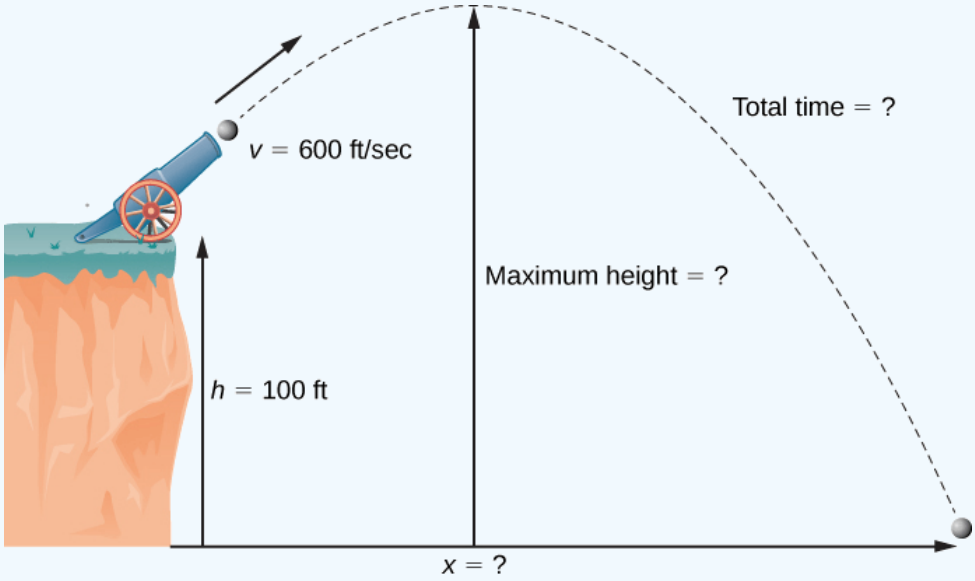


Figure 5.4.7: This figure has a cannon at the edge of a cliff aimed upwards. There is a cannonball coming out of the cannon. The path of the cannonball is an downward opening parabola represented with a broken line. The maximum height is labeled with "?". The height of the cliff is 100 feet.

Solution

We use the equation

$$\mathbf{s}(t) = v_0 t\cos\theta\,\mathbf{i} + \left(v_0 t\sin\theta - \frac{1}{2}gt^2\right)\mathbf{j}$$

with $\theta = 30^\circ$, $g = 32\frac{\text{ft}}{\text{sec}^2}$, and $v_0 = 600\frac{\text{ft}}{\text{sec}^2}$. Then the position equation becomes

$$\begin{aligned}\mathbf{s}(t) &= 600t(\cos 30^\circ)\,\mathbf{i} + \left(600t\sin 30^\circ - \frac{1}{2}(32)t^2\right)\mathbf{j}\\ &= 300t\sqrt{3}\,\mathbf{i} + \left(300t - 16t^2\right)\mathbf{j}\end{aligned}$$

a. The cannonball reaches its maximum height when the vertical component of its velocity is zero, because the cannonball is neither rising nor falling at that point. The velocity vector is

$$\begin{aligned}\mathbf{v}(t) &= \mathbf{s}'(t)\\ &= 300\sqrt{3}\,\mathbf{i} + (300 - 32t)\,\mathbf{j}\end{aligned}$$

Therefore, the vertical component of velocity is given by the expression $300 - 32t$. Setting this expression equal to zero and solving for t gives $t = 9.375$ sec. The height of the cannonball at this time is given by the vertical component of the position vector, evaluated at $t = 9.375$.

$$\begin{aligned}\mathbf{s}(9.375) &= 300(9.375)\sqrt{3}\,\mathbf{i} + (300(9.375) - 16(9.375)^2)\,\mathbf{j}\\ &= 4871.39\,\mathbf{i} + 1406.25\,\mathbf{j}\end{aligned}$$

Therefore, the maximum height of the cannonball is 1406.39 ft above the cannon, or 1506.39 ft above sea level.

b. When the cannonball lands in the water, it is 100 ft below the cannon. Therefore, the vertical component of the position vector is equal to –100. Setting the vertical component of $\mathbf{s}(t)$ equal to –100 and solving, we obtain

$$\begin{aligned}300t - 16t^2 &= -100\\ 16t^2 - 300t - 100 &= 0\\ 4t^2 - 75 - 25 &= 0\\ t &= \frac{75 \pm \sqrt{(-75)^2 - 4(4)(-25)}}{2(4)}\\ &= \frac{75 \pm \sqrt{6025}}{8}\\ &= \frac{75 \pm 5\sqrt{241}}{8}\end{aligned}$$

The positive value of t that solves this equation is approximately 19.08. Therefore, the cannonball hits the water after approximately 19.08 sec.

c. To find the distance out to sea, we simply substitute the answer from part (b) into $\mathbf{s}(t)$:

$$\begin{aligned}\mathbf{s}(19.08) &= 300(19.08)\sqrt{3}\,\mathbf{i} + \left(300(19.08) - 16(19.08)^2\right)\,\mathbf{j}\\ &= 9914.26\,\mathbf{i} - 100.7424\,\mathbf{j}\end{aligned}$$

Therefore, the ball hits the water about 9914.26 ft away from the base of the cliff. Notice that the vertical component of the position vector is very close to –100, which tells us that the ball just hit the water. Note that 9914.26 feet is not the true range of the cannon since the cannonball lands in the ocean at a location below the cannon. The range of the cannon would be determined by finding how far out the cannonball is when its height is 100 ft above the water (the same as the altitude of the cannon).

Watch the accompanying video for a visual demonstration of this concept.

- **Video Length**: 6 minutes 20 seconds
- **Context**: This video demonstrates how to solve the following motion problem using the tools of this section: A projectile is fired with an initial angle of 20° from the horizontal in order to hit a target at ground level 400 meters away. Find the initial speed the projectile should be fired and find its maximal height.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

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Exercise 5.4.3

An archer fires an arrow at an angle of 40° above the horizontal with an initial speed of 98 m/sec. The height of the archer is 171.5 cm. Find the horizontal distance the arrow travels before it hits the ground.

Hint available at the end of this section.

Answer available at the end of this section.

One final question remains: In general, what is the maximum distance a projectile can travel, given its initial speed? To determine this distance, we assume the projectile is fired from ground level and we wish it to return to ground level. In other words, we want to determine an equation for the range. In this case, the equation of projectile motion is

$$\mathbf{s} = v_0 t \cos\theta\,\mathbf{i} + \left(v_0 t \sin\theta - \frac{1}{2}gt^2\right)\mathbf{j}. \tag{5.4.13}$$

Setting the second component equal to zero and solving for t yields

$$\begin{aligned} v_0 t \sin\theta - \frac{1}{2}gt^2 &= 0 \\ t\left(v_0 \sin\theta - \frac{1}{2}gt\right) &= 0 \end{aligned}$$

Therefore, either $t = 0$ or $t = \dfrac{2v_0 \sin\theta}{g}$. We are interested in the second value of t, so we substitute this into $\mathbf{s}(t)$, which gives

$$\begin{aligned} \mathbf{s}\left(\frac{2v_0\sin\theta}{g}\right) &= v_0\left(\frac{2v_0\sin\theta}{g}\right)\cos\theta\,\mathbf{i} + \left(v_0\left(\frac{2v_0\sin\theta}{g}\right)\sin\theta - \frac{1}{2}g\left(\frac{2v_0\sin\theta}{g}\right)^2\right)\mathbf{j} \\ &= \left(\frac{2v_0^2\sin\theta\cos\theta}{g}\right)\mathbf{i} \\ &= \frac{v_0^2\sin 2\theta}{g}\,\mathbf{i}. \end{aligned}$$

Thus, the expression for the range of a projectile fired at an angle θ is

$$R = \frac{v_0^2\sin 2\theta}{g}\,\mathbf{i}. \tag{5.4.14}$$

The only variable in this expression is θ. To maximize the distance traveled, take the derivative of the coefficient of $\mathbf{i}$ with respect to θ and set it equal to zero:

$$\begin{aligned} \frac{d}{d\theta}\left(\frac{v_0^2\sin 2\theta}{g}\right) &= 0 \\ \frac{2v_0^2\cos 2\theta}{g} &= 0 \\ \theta &= 45^\circ \end{aligned}$$

This value of θ) is the smallest positive value that makes the derivative equal to zero. Therefore, in the absence of air resistance, the best angle to fire a projectile (to maximize the range) is at a 45° angle. The distance it travels is given by

$$\mathbf{s}\left(\frac{2v_0\sin 45^\circ}{g}\right) = \frac{v_0^2\sin 90^\circ}{g}\,\mathbf{i} = \frac{v_0^2}{g}\,\mathbf{i} \tag{5.4.15}$$

Therefore, the range for an angle of 45° is $\frac{v_0^2}{g}$ units.

Example 5.4.4

A baseball is hit at 150 ft/sec from a height of 3 ft towards an 8-ft fence that is 400 ft from the home plate. Find the minimum angle required for a home run. Assume gravity is the only force working on the ball.

Solution

- **Video Length**: 6 minutes 47 seconds
- **Context**: This video demonstrates how to find the maximum angle to hit a homerun: A baseball is hit at 150 ft/sec from a height of 3 ft towards an 8 ft fence that is 400 ft from home plate. Find the minimum angle required for a home run. Assume gravity is the only force acting on the ball.

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Kepler's Laws

During the early 1600s, Johannes Kepler was able to use the amazingly accurate data from his mentor Tycho Brahe to formulate his three laws of planetary motion, now known as Kepler's laws of planetary motion. These laws also apply to other objects in the solar system in orbit around the Sun, such as comets (e.g., Halley's comet) and asteroids. Variations of these laws apply to satellites in orbit around Earth.

Theorem 5.4.2: Kepler's Laws of Planetary Motion

1. The path of any planet about the Sun is elliptical in shape, with the center of the Sun located at one focus of the ellipse (the law of ellipses).
2. A line drawn from the center of the Sun to the center of a planet sweeps out equal areas in equal time intervals (the law of equal areas) (Figure 5.4.8).
3. The ratio of the squares of the periods of any two planets is equal to the ratio of the cubes of the lengths of their semimajor orbital axes (the Law of Harmonies).

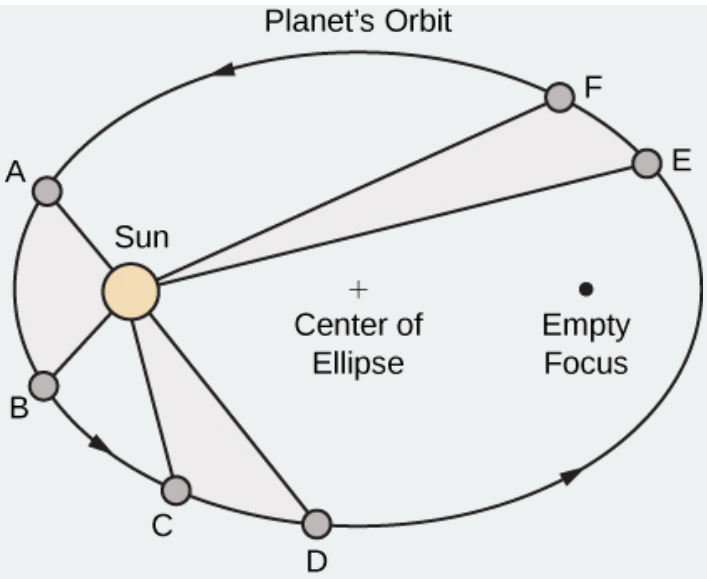


Figure 5.4.8: This figure is an elliptical curve labeled "planets orbit". The sun is represented towards the left inside the ellipse, at a focal point. Along the ellipse there are points A, B, C, D, E, F. There are line segments from the sun to each point. The Sun is located at a focus of the elliptical orbit of any planet. The area enclosed between the Sun, A, B is shaded. The area enclosed between the Sun, C, D is shaded. The area enclosed between the Sun, E, F is shaded. The shaded areas are all equal, assuming that the amount of time measured as the planet moves is the same for each region.

Kepler's third law is especially useful when using appropriate units. In particular, *1 astronomical unit* is defined to be the average distance from Earth to the Sun, and is now recognized to be 149,597,870,700 m or, approximately 93,000,000 mi. We therefore write 1 A.U. = 93,000,000 mi. Since the time it takes for Earth to orbit the Sun is 1 year, we use Earth years for units of time. Then, substituting 1 year for the period of Earth and 1 A.U. for the average distance to the Sun, Kepler's third law can be written as

$$T_p^2 = D_p^3$$

for any planet in the solar system, where T_P is the period of that planet measured in Earth years and D_P is the average distance from that planet to the Sun measured in astronomical units. Therefore, if we know the average distance from a planet to the Sun (in astronomical units), we can then calculate the length of its year (in Earth years), and vice versa.

Kepler's laws were formulated based on observations from Brahe; however, they were not proved formally until Sir Isaac Newton was able to apply calculus. Furthermore, Newton was able to generalize Kepler's third law to other orbital systems, such as a moon orbiting around a planet. Kepler's original third law only applies to objects orbiting the Sun.

Proof

Let's now prove Kepler's first law using the calculus of vector-valued functions. First we need a coordinate system. Let's place the Sun at the origin of the coordinate system and let the vector-valued function $\mathbf{r}(t)$ represent the location of a planet as a function of time. Newton proved Kepler's law using his second law of motion and his law of universal gravitation. Newton's second law of motion can be written as $\mathbf{F} = m\mathbf{a}$, where $\mathbf{F}$ represents the net force acting on the planet. His law of universal gravitation can be written in the form $\mathbf{F} = -\dfrac{GmM}{||\mathbf{r}||^2} \cdot \dfrac{\mathbf{r}}{||\mathbf{r}||}$, which indicates that the force resulting from the gravitational attraction of the Sun points back toward the Sun, and has magnitude $\dfrac{GmM}{||\mathbf{r}||^2}$ (Figure 5.4.9).

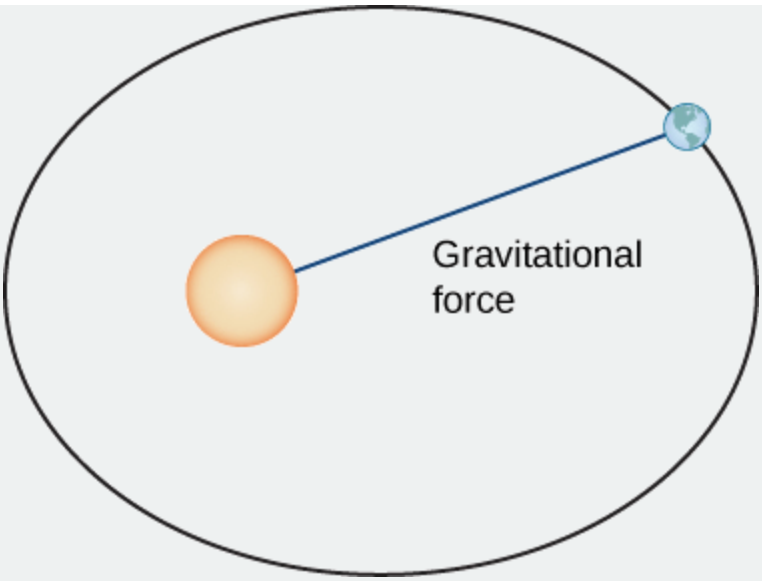


Figure 5.4.9: This figure is an ellipse with a circle to the left on the inside at a focal point. The circle represents the sun. On the ellipse is a smaller circle representing Earth. The line segment drawn between the circles is labeled “gravitational force”. The gravitational force between Earth and the Sun is equal to the mass of the earth times its acceleration.

Setting these two forces equal to each other, and using the fact that $\mathbf{a}(t) = \mathbf{v}\,'(t)$, we obtain

$$m\mathbf{v}'(t) = -\frac{GmM}{\|\mathbf{r}\|^2} \cdot \frac{\mathbf{r}}{\|\mathbf{r}\|},$$

which can be rewritten as

$$\frac{d\mathbf{v}}{dt} = -\frac{GM}{||\mathbf{r}||^3}\mathbf{r}.$$

This equation shows that the vectors $d\mathbf{v}/dt$ and $\mathbf{r}$ are parallel to each other, so $d\mathbf{v}/dt \times \mathbf{r} = \mathbf{0}$. Next, let’s differentiate $\mathbf{r} \times \mathbf{v}$ with respect to time:

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{v}) = \frac{d\mathbf{r}}{dt} \times \mathbf{v} + \mathbf{r} \times \frac{d\mathbf{v}}{dt} = \mathbf{v} \times \mathbf{v} + \mathbf{0} = \mathbf{0}. \tag{5.4.16}$$

This proves that $\mathbf{r} \times \mathbf{v}$ is a constant vector, which we call $\mathbf{C}$. Since $\mathbf{r}$ and $\mathbf{v}$ are both perpendicular to $\mathbf{C}$ for all values of t, they must lie in a plane perpendicular to $\mathbf{C}$. Therefore, the motion of the planet lies in a plane.

Next we calculate the expression $d\mathbf{v}/dt \times \mathbf{C}$:

$$\frac{d\mathbf{v}}{dt} \times \mathbf{C} = -\frac{GM}{||\mathbf{r}||^3}\mathbf{r} \times (\mathbf{r} \times \mathbf{v}) = -\frac{GM}{||\mathbf{r}||^3}[(\mathbf{r} \cdot \mathbf{v})\mathbf{r} - (\mathbf{r} \cdot \mathbf{r})\mathbf{v}]. \tag{5.4.17}$$

The last equality in Equation 5.4.16 is from the triple cross product formula (Introduction to Vectors in Space). We need an expression for $\mathbf{r} \cdot \mathbf{v}$. To calculate this, we differentiate $\mathbf{r} \cdot \mathbf{r}$ with respect to time:

$$\frac{d}{dt}(\mathbf{r} \cdot \mathbf{r}) = \frac{d\mathbf{r}}{dt} \cdot \mathbf{r} + \mathbf{r} \cdot \frac{d\mathbf{r}}{dt} = 2\mathbf{r} \cdot \frac{d\mathbf{r}}{dt} = 2\mathbf{r} \cdot \mathbf{v}. \tag{5.4.18}$$

Since $\mathbf{r} \cdot \mathbf{r} = ||\mathbf{r}||^2$, we also have

$$\frac{d}{dt}(\mathbf{r} \cdot \mathbf{r}) = \frac{d}{dt}||\mathbf{r}||^2 = 2||\mathbf{r}||\frac{d}{dt}||\mathbf{r}||. \tag{5.4.19}$$

Combining Equation 5.4.18 and Equation 5.4.19, we get

$$2\mathbf{r}\cdot\mathbf{v} = 2||\mathbf{r}||\frac{d}{dt}||\mathbf{r}||$$
$$\mathbf{r}\cdot\mathbf{v} = ||\mathbf{r}\|\frac{d}{dt}||\mathbf{r}||.$$

Substituting this into Equation 5.4.17 gives us

$$\begin{aligned}\frac{d\mathbf{v}}{dt}\times\mathbf{C} &= -\frac{GM}{\|\mathbf{r}\|^3}\left[(\mathbf{r}\cdot\mathbf{v})\,\mathbf{r} - \|\mathbf{r}\|^2\,\mathbf{v}\right]\\ &= -\frac{GM}{\|\mathbf{r}\|^3}\left[\|\mathbf{r}\|\frac{d}{dt}\|\mathbf{r}\|\,\mathbf{r} - \|\mathbf{r}\|^2\,\mathbf{v}\right]\\ &= -GM\left[\frac{1}{\|\mathbf{r}\|^2}\frac{d}{dt}\|\mathbf{r}\|\,\mathbf{r} - \frac{1}{\|\mathbf{r}\|}\,\mathbf{v}\right]\\ &= GM\left[\frac{\mathbf{v}}{\|\mathbf{r}\|} - \frac{\mathbf{r}}{\|\mathbf{r}\|^2}\frac{d}{dt}\|\mathbf{r}\|\right].\end{aligned} \tag{5.4.20}$$

However,

$$\begin{aligned}\frac{d}{dt}\frac{\mathbf{r}}{||\mathbf{r}||} &= \frac{\frac{d}{dt}(\mathbf{r})||\mathbf{r}|| - \mathbf{r}\frac{d}{dt}||\mathbf{r}||}{||\mathbf{r}||^2}\\ &= \frac{\frac{d\mathbf{r}}{dt}}{||\mathbf{r}||} - \frac{\mathbf{r}}{||\mathbf{r}||^2}\frac{d}{dt}||\mathbf{r}||\\ &= \frac{\mathbf{v}}{||\mathbf{r}||} - \frac{\mathbf{r}}{||\mathbf{r}||^2}\frac{d}{dt}||\mathbf{r}||.\end{aligned}$$

Therefore, Equation 5.4.20 becomes

$$\frac{d\mathbf{v}}{dt}\times\mathbf{C} = GM\left(\frac{d}{dt}\frac{\mathbf{r}}{||\mathbf{r}||}\right).$$

Since $\mathbf{C}$ is a constant vector, we can integrate both sides and obtain

$$\mathbf{v}\times\mathbf{C} = GM\frac{\mathbf{r}}{||\mathbf{r}||} + \mathbf{D},$$

where $\mathbf{D}$ is a constant vector. Our goal is to solve for $||\mathbf{r}||$. Let's start by calculating $\mathbf{r}\cdot(\mathbf{v}\times\mathbf{C}$:

$$\mathbf{r}\cdot(\mathbf{v}\times\mathbf{C} = GM\frac{||\mathbf{r}||^2}{||\mathbf{r}||} + \mathbf{r}\cdot\mathbf{D} = GM||\mathbf{r}|| + \mathbf{r}\cdot\mathbf{D}.$$

However, $\mathbf{r}\cdot(\mathbf{v}\times\mathbf{C}) = (\mathbf{r}\times\mathbf{v})\cdot\mathbf{C}$, so

$$(\mathbf{r}\times\mathbf{v})\cdot\mathbf{C} = GM||\mathbf{r}|| + \mathbf{r}\cdot\mathbf{D}.$$

Since $\mathbf{r}\times\mathbf{v} = \mathbf{C}$, we have

$$||\mathbf{C}||^2 = GM||\mathbf{r}|| + \mathbf{r}\cdot\mathbf{D}.$$

Note that $\mathbf{r}\cdot\mathbf{D} = ||\mathbf{r}||||\mathbf{D}||\cos\theta$, where θ is the angle between $\mathbf{r}$ and $\mathbf{D}$. Therefore,

$$||\mathbf{C}||^2 = GM||\mathbf{r}|| + ||\mathbf{r}||||\mathbf{D}||\cos\theta$$

Solving for $||\mathbf{r}||$,

$$||\mathbf{r}|| = \frac{||\mathbf{C}||^2}{GM + ||\mathbf{D}||\cos\theta} = \frac{||\mathbf{C}||^2}{GM}\left(\frac{1}{1+e\cos\theta}\right).$$

where $e = ||\mathbf{D}||/GM$. This is the polar equation of a conic with a focus at the origin, which we set up to be the Sun. It is a hyperbola if $e > 1$, a parabola if $e = 1$, or an ellipse if $e < 1$. Since planets have closed orbits, the only possibility is an ellipse. However, at this point it should be mentioned that hyperbolic comets do exist. These are objects that are merely passing through the solar system at speeds too great to be trapped into orbit around the Sun. As they pass close enough to the Sun, the gravitational field of the Sun deflects the trajectory enough so the path becomes hyperbolic.

□

Kepler's third law of planetary motion can be modified to the case of one object in orbit around an object other than the Sun, such as the Moon around the Earth. In this case, Kepler's third law becomes

$$P^2 = \frac{4\pi^2 a^3}{G(m+M)}, \tag{5.4.21}$$

where *m* is the mass of the Moon and *M* is the mass of Earth, *a* represents the length of the major axis of the elliptical orbit, and *P* represents the period.

Example 5.4.5: Using Kepler's Third Law for Nonheliocentric Orbits

Given that the mass of the Moon is 7.35×10^{22} kg, the mass of Earth is 5.97×10^{24} kg, $G = 6.67 \times 10^{-11}\text{m/kg} \cdot \sec^2$, and the period of the moon is 27.3 days, let's find the length of the major axis of the orbit of the Moon around Earth.

Solution

It is important to be consistent with units. Since the universal gravitational constant contains seconds in the units, we need to use seconds for the period of the Moon as well:

$$27.3 \text{ days} \times \frac{24 \text{ hr}}{1 \text{ day}} \times \frac{3600 \text{ esc}}{1 \text{ hour}} = 2,358,720 \text{ sec}$$

Substitute all the data into Equation 5.4.21 and solve for a:

$$\begin{aligned}
(2,358,720sec)^2 &= \frac{4\pi^2 a^3}{\left(6.67 \times 10^{-11} \frac{m}{\text{kg} \times \text{sec}^2}\right)(7.35 \times 10^{22}\text{kg} + 5.97 \times 10^{24}\text{kg})} \\
5.563 \times 10^{12} &= \frac{4\pi^2 a^3}{(6.67 \times 10^{-11}\text{m}^3)(6.04 \times 10^{24})} \\
(5.563 \times 10^{12})(6.67 \times 10^{-11}\text{m}^3)(6.04 \times 10^{24}) &= 4\pi^2 a^3 \\
a^3 &= \frac{2.241 \times 10^{27}}{4\pi^2}\text{m}^3 \\
a &= 3.84 \times 10^8 \text{m} \\
&\approx 384,000\,\text{km}.
\end{aligned}$$

Analysis

According to solarsystem.nasa.gov, the actual average distance from the Moon to Earth is 384,400 km. This is calculated using reflectors left on the Moon by Apollo astronauts back in the 1960s.

Exercise 5.4.4

Titan is the largest moon of Saturn. The mass of Titan is approximately $1.35 \times 10^{23} kg$. The mass of Saturn is approximately 5.68×10^{26} kg. Titan takes approximately 16 days to orbit Saturn. Use this information, along with the universal gravitation constant $G = 6.67 \times 10^{-11}\text{m/kg} \cdot \sec^2$ to estimate the distance from Titan to Saturn.

Hint available at the end of this section.

Answer available at the end of this section.

Example 5.4.6: Halley's Comet

We now return to the chapter opener, which discusses the motion of Halley's comet around the Sun. Kepler's first law states that Halley's comet follows an elliptical path around the Sun, with the Sun as one focus of the ellipse. The period of Halley's comet is approximately 76.1 years, depending on how closely it passes by Jupiter and Saturn as it passes through the outer solar system. Let's use $T = 76.1$ years. What is the average distance of Halley's comet from the Sun?



Figure 5.4.10: This is a picture of Halley's Comet. It is a bright ball of light towards the right of the picture with a tail of trailing light. There are also stars throughout the picture.(CC-BY-2.0, Wikimedia Commons)

Solution

Using the equation $T^2 = D^3$ with $T = 76.1$, we obtain $D^3 = 5791.21$, so $D \approx 17.96$ A.U. This comes out to approximately 1.67×10^9 mi.

A natural question to ask is: What are the maximum (aphelion) and minimum (perihelion) distances from Halley's Comet to the Sun? The eccentricity of the orbit of Halley's Comet is 0.967 (Check the source for more information). Recall that the formula for the eccentricity of an ellipse is $e = c/a$, where *a* is the length of the semimajor axis and *c* is the distance from the center to either focus. Therefore, $0.967 = c/17.96$ and $c \approx 17.37$ A.U. Subtracting this from *a* gives the perihelion distance $p = a - c = 17.96 - 17.37 = 0.59$ A.U. According to the National Space Science Data Center (Check the Source for more information), the perihelion distance for Halley's comet is 0.587 A.U. To calculate the aphelion distance, we add

$$P = a + c = 17.96 + 17.37 = 35.33 \text{ A.U.}$$

This is approximately 3.3×10^9 mi. The average distance from Pluto to the Sun is 39.5 A.U. (Check the Source for more information), so it would appear that Halley's Comet stays just within the orbit of Pluto.

Navigating a Banked Turn

How fast can a racecar travel through a circular turn without skidding and hitting the wall? The answer could depend on several factors:

- The weight of the car;
- The friction between the tires and the road;
- The radius of the circle;
- The "steepness" of the turn.

In this project we investigate this question for NASCAR racecars at the Bristol Motor Speedway in Tennessee. Before considering this track in particular, we use vector functions to develop the mathematics and physics necessary for answering questions such as this.

A car of mass m moves with constant angular speed ω around a circular curve of radius R (Figure 5.4.9). The curve is banked at an angle θ. If the height of the car off the ground is h, then the position of the car at time t is given by the function $\mathbf{r}(t) =< R\cos(\omega t), R\sin(\omega t), h >$.

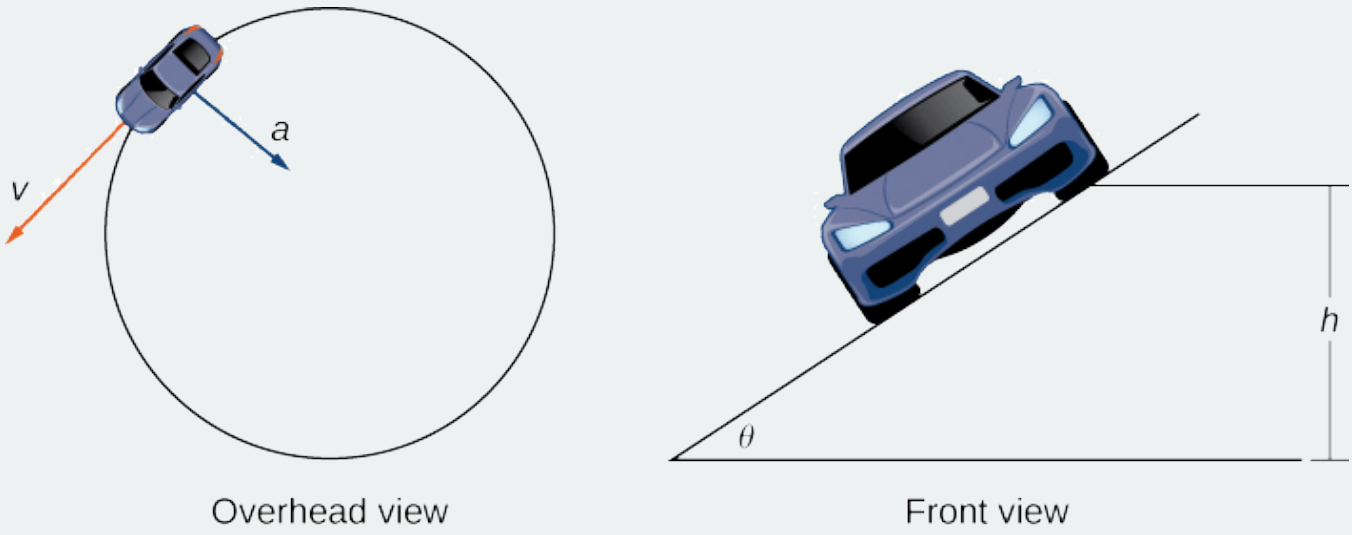


Figure 5.4.11: This figure has two graphics. The first is a circle with a car on the circle. The circle is labeled “overhead view”. From the car there is a vector labeled $\mathbf{v}$ tangent to the circle. There is also a vector towards the center from the car labeled $\mathbf{a}$. The second graphic is labeled “front view”. It is the car at an angle. The angle is labeled θ. The height of the cars tilt is labeled “h”.

1. Find the velocity function $\mathbf{v}(t)$ of the car. Show that $\mathbf{v}$ is tangent to the circular curve. This means that, without a force to keep the car on the curve, the car will shoot off of it.
2. Show that the speed of the car is ωR. Use this to show that $(2\pi 4)/\|\mathbf{v}\| = (2\pi)/\omega$.
3. Find the acceleration $\mathbf{a}$. Show that this vector points toward the center of the circle and that $\|\mathbf{a}\| = R\omega^2$.
4. The force required to produce this circular motion is called the *centripetal force*, and it is denoted $\mathbf{F}_{cent}$. This force points toward the center of the circle (not toward the ground). Show that $\|\mathbf{F}_{cent}\| = \left(m|\mathbf{v}|^2\right)/R$.

As the car moves around the curve, three forces act on it: gravity, the force exerted by the road (this force is perpendicular to the ground), and the friction force (Figure 5.4.10). Because describing the frictional force generated by the tires and the road is complex, we use a standard approximation for the frictional force. Assume that $\mathbf{f} = \mu\mathbf{N}$ for some positive constant μ. The constant μ is called the *coefficient of friction*.

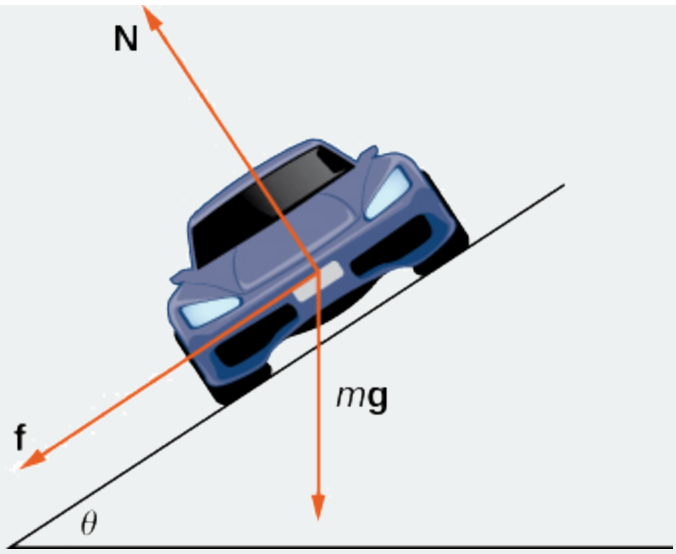


Figure 5.4.12: The car has three forces acting on it: gravity (denoted by mg), the friction force $\mathbf{f}$, and the force exerted by the road $\mathbf{N}$.

Let v_{max} denote the maximum speed the car can attain through the curve without skidding. In other words, v_{max} is the fastest speed at which the car can navigate the turn. When the car is traveling at this speed, the magnitude of the centripetal force is

$$\|\mathbf{F}_{cent}\| = \frac{m(v_{max})^2}{R}. \tag{5.4.22}$$

The next three questions deal with developing a formula that relates the speed v_{max} to the banking angle θ.

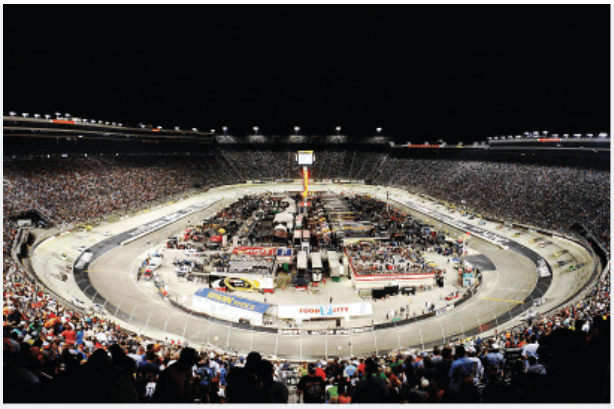
5. Show that $\mathbf{N}\cos\theta = m\mathbf{g} + \mathbf{f}\sin\theta$. Conclude that $\mathbf{N} = (m\mathbf{g})/(\cos\theta - \mu\sin\theta)$.
6. The centripetal force is the sum of the forces in the horizontal direction, since the centripetal force points toward the center of the circular curve. Show that

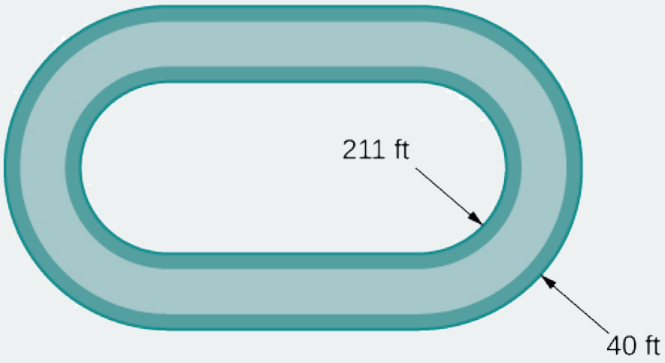
   $$\mathbf{F}_{cent} = \mathbf{N}\sin\theta + \mathbf{f}\cos\theta. \tag{5.4.23}$$

   Conclude that

   $$\mathbf{F}_{cent} = \frac{\sin\theta + \mu\cos\theta}{cos\theta - \mu\sin\theta} m\mathbf{g}. \tag{5.4.24}$$

7. Show that $(v_{\max})^2 = ((\sin\theta + \mu\ cos\theta)/(\cos\theta - \mu\sin\theta))gR$. Conclude that the maximum speed does not actually depend on the mass of the car.
   Now that we have a formula relating the maximum speed of the car and the banking angle, we are in a position to answer the questions like the one posed at the beginning of the project.
   The Bristol Motor Speedway is a NASCAR short track in Bristol, Tennessee. The track has the approximate shape shown in Figure 5.4.11. Each end of the track is approximately semicircular, so when cars make turns they are traveling along an approximately circular curve. If a car takes the inside track and speeds along the bottom of turn 1, the car travels along a semicircle of radius approximately 211 ft with a banking angle of 24°. If the car decides to take the outside track and speeds along the top of turn 1, then the car travels along a semicircle with a banking angle of 28°. (The track has variable angle banking.)





(a) (b)

Figure 5.4.13: This figure has two graphics. The first is a picture of a raceway. In image (a), there are cars on the track and fans in the stands. In the second graphic, (b), there is an oval drawing of a raceway. The inner radius of a curve is labeled “211 ft” and the width of the track is labeled “40 ft”. At the Bristol Motor Speedway, Bristol, Tennessee (a), the turns have an inner radius of about 211 ft and a width of 40 ft (b). (credit: part (a) photo by Raniel Diaz, Flickr)

The coefficient of friction for a normal tire in dry conditions is approximately 0.7. Therefore, we assume the coefficient for a NASCAR tire in dry conditions is approximately 0.98.

Before answering the following questions, note that it is easier to do computations in terms of feet and seconds, and then convert the answers to miles per hour as a final step.

8. In dry conditions, how fast can the car travel through the bottom of the turn without skidding?
9. In dry conditions, how fast can the car travel through the top of the turn without skidding?
10. In wet conditions, the coefficient of friction can become as low as 0.1. If this is the case, how fast can the car travel through the bottom of the turn without skidding?
11. Suppose the measured speed of a car going along the outside edge of the turn is 105 mph. Estimate the coefficient of friction for the car’s tires.

Key Concepts

- If $\mathbf{r}(t)$ represents the position of an object at time *t*, then $\mathbf{r}'(t)$ represents the velocity and $\mathbf{r}\,''(t)$ represents the acceleration of the object at time *t*. The magnitude of the velocity vector is speed.
- The acceleration vector always points toward the concave side of the curve defined by $\mathbf{r}(t)$. The tangential and normal components of acceleration a_{T} and a_{N} are the projections of the acceleration vector onto the unit tangent and unit normal vectors to the curve.
- Kepler’s three laws of planetary motion describe the motion of objects in orbit around the Sun. His third law can be modified to describe motion of objects in orbit around other celestial objects as well.
- Newton was able to use his law of universal gravitation in conjunction with his second law of motion and calculus to prove Kepler’s three laws.

Key Equations

- **Velocity**

$$\mathbf{v}(t) = \mathbf{r}'(t)$$

- **Acceleration**

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$$

- **Speed**

$$v(t) = ||\mathbf{v}(t)|| = ||\mathbf{r}'(t)|| = \frac{ds}{dt}$$

- **Tangential component of acceleration**

$$a_{\mathrm{T}} = \mathbf{a} \cdot \mathbf{T} = \frac{\mathbf{v} \cdot \mathbf{a}}{||\mathbf{v}||}$$

- **Normal component of acceleration**

$$a_{\mathrm{N}} = \mathbf{a} \cdot \mathbf{N} = \frac{||\mathbf{v} \times \mathbf{a}||}{||\mathbf{v}||} = \sqrt{||\mathbf{a}||^2 - a_{\mathrm{T}}}$$

Glossary

Acceleration Vector
The second derivative of the position vector.

Kepler's Laws of Planetary Motion
Three laws governing the motion of planets, asteroids, and comets in orbit around the Sun.

Normal Component of Acceleration
The coefficient of the unit normal vector $\mathbf{N}$ when the acceleration vector is written as a linear combination of $\mathbf{T}$ and $\mathbf{N}$.

Projectile Motion
Motion of an object with an initial velocity but no force acting on it other than gravity.

Tangential Component of Acceleration
The coefficient of the unit tangent vector $\mathbf{T}$ when the acceleration vector is written as a linear combination of $\mathbf{T}$ and $\mathbf{N}$.

Velocity Vector
The derivative of the position vector.

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- Edited by Paul Seeburger
  Paul Seeburger added finding point $(1, 2)$ when $t = 1$ and added part 3 (finding the equations of the tangent line) in Example $5.4.1$.
  He also created Figure $5.4.1$.

Answers

Exercise $5.4.1$

Use Equations $5.4.1$, $5.4.2$, and $5.4.3$

Exercise $5.4.2$

Use Equations $5.4.6$ and $5.4.7$

Exercise $5.4.2$

a.

$$\begin{aligned} a_{\mathrm{T}} &= \frac{\mathbf{v}(t) \cdot \mathbf{a}(t)}{||\mathbf{v}(t)||} = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{||\mathbf{r}'(t)||} \\ &= \frac{(4\,\mathbf{i} + 2t\,\mathbf{j}) \cdot (2\,\mathbf{j})}{||4\,\mathbf{i} + 2t\,\mathbf{j}||} \\ &= \frac{4t}{\sqrt{4^2 + (2t)^2}} \\ &= \frac{2t}{\sqrt{2 + t^2}} \end{aligned}$$

$$
\begin{aligned}
a_{\mathrm{N}} &= \sqrt{||\mathbf{a}||^2 - a_{\mathrm{T}}^2} \\
&= \sqrt{||2\,\mathbf{j}||^2 - \left(\frac{2t}{\sqrt{2+t^2}}\right)^2} \\
&= \sqrt{4 - \frac{4t^2}{2+t^2}}
\end{aligned}
$$

b.
$$
\begin{aligned}
a_{\mathrm{T}}(-3) &= \frac{2(-3)}{\sqrt{2+(-3)^2}} \\
&= \frac{-6}{\sqrt{11}}
\end{aligned}
$$

$$
\begin{aligned}
a_{\mathrm{N}}(-3) &= \sqrt{4 - \frac{4(-3)^2}{2+(-3)^2}} \\
&= \sqrt{4 - \frac{36}{11}} \\
&= \sqrt{\frac{8}{11}} \\
&= \frac{2\sqrt{2}}{\sqrt{11}}
\end{aligned}
$$

Exercise 5.4.3

The equation for the position vector needs to account for the height of the archer in meters.

Exercise 5.4.3

967.15 m

Exercise 5.4.4

Make sure your units agree, then use Equation 5.4.21.

Exercise 5.4.4

$$
a \approx 1.224 \times 10^9 \mathrm{m} = 1,224,000 \mathrm{km} \tag{5.4.25}
$$

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5.4E: Exercises for Section 5.4

Exercise 1

1. Given $\mathbf{r}(t) = (3t^2 - 2)\ \mathbf{i} + (2t - \sin t)\ \mathbf{j}$:

    a. Find the velocity of a particle moving along this curve.
    b. Find the acceleration of a particle moving along this curve.

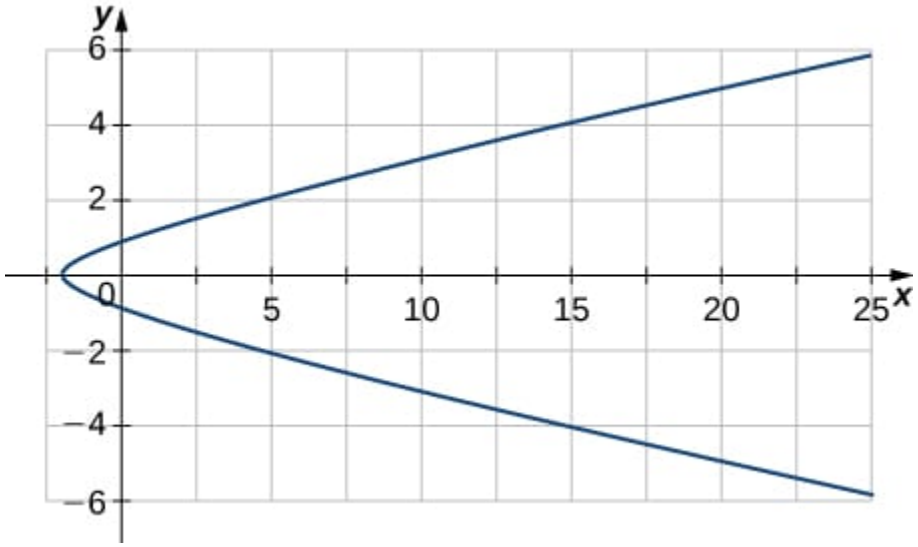


Figure $5.4E.1$: This is the graph of the given vector function representing the position. It's a curve in the xy-plane that widens symmetrically about the x-axis, as it moves to the right along the x-axis. The curve begins at the origin $(-2, 0)$ and expands to the right, forming a shape like a narrow flap of an envelope.

Answer

a. $\mathbf{v}(t) = 6t\ \mathbf{i} + (2 - \cos t)\ \mathbf{i}$
b. $\mathbf{a}(t) = 6\ \mathbf{i} + \sin t\ \mathbf{i}$

Exercises 2 - 5

Given the position function, find the velocity, acceleration, and speed in terms of the parameter t.

2. $\mathbf{r}(t) = e^{-t}\ \mathbf{i} + t^2\ \mathbf{j} + \tan t\ \mathbf{k}$
3. $\mathbf{r}(t) = \langle 3\cos t,\ 3\sin t,\ t^2 \rangle$

Answer

$$\mathbf{v}(t) = -3\sin t\ \mathbf{i} + 3\cos t\ \mathbf{j} + 2t\ \mathbf{k}$$
$$\mathbf{a}(t) = -3\cos t\ \mathbf{i} - 3\sin t\ \mathbf{j} + 2\ \mathbf{k}$$
$$\text{Speed}(t) = \|\mathbf{v}(t)\| = \sqrt{9 + 4t^2}$$

4. $\mathbf{r}(t) = t^5\ \mathbf{i} + (3t^2 + 2t - 5)\ \mathbf{j} + (3t - 1)\ \mathbf{k}$
5. $\mathbf{r}(t) = 2\cos t\ \mathbf{j} + 3\sin t\ \mathbf{k}$. The graph is shown here:

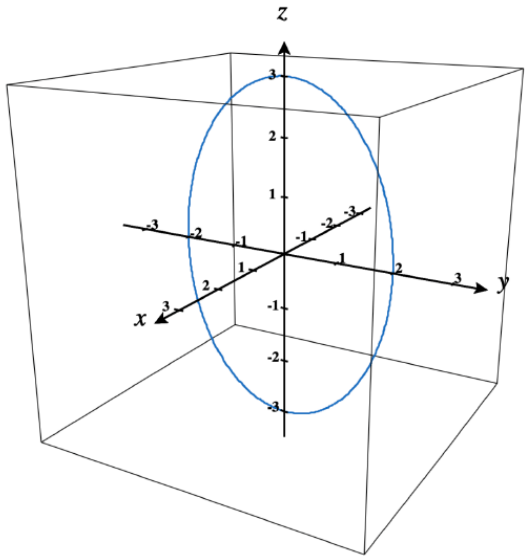


Figure $5.4E.2$: The graph of the given vector function is an ellipse graphed in the three-dimensional space. The ellipse is contained in tye yz -plane, with its major axis along the z-axis and minor axis along the y-axis.

Answer

$$\mathbf{v}(t) = -2\sin t\ \mathbf{j} + 3\cos t\ \mathbf{k}$$
$$\mathbf{a}(t) = -2\cos t\ \mathbf{j} - 3\sin t\ \mathbf{k}$$
$$\text{Speed}(t) = \|\mathbf{v}(t)\| = \sqrt{4\sin^2 t + 9\cos^2 t} = \sqrt{4 + 5\cos^2 t}$$

Exercises 6 - 8

Find the velocity, acceleration, and speed of a particle with the given position function.

6. $\mathbf{r}(t) = \langle t^2 - 1, t\rangle$
7. $\mathbf{r}(t) = \langle e^t, e^{-t}\rangle$

Answer

$$\mathbf{v}(t) = \langle e^t, -e^{-t}\rangle,$$
$$\mathbf{a}(t) = \langle e^t, e^{-t}\rangle,$$
$$\|\mathbf{v}(t)\| = \sqrt{e^{2t} + e^{-2t}}$$

8. $\mathbf{r}(t) = \langle \sin t, t, \cos t\rangle$. The graph is shown here:

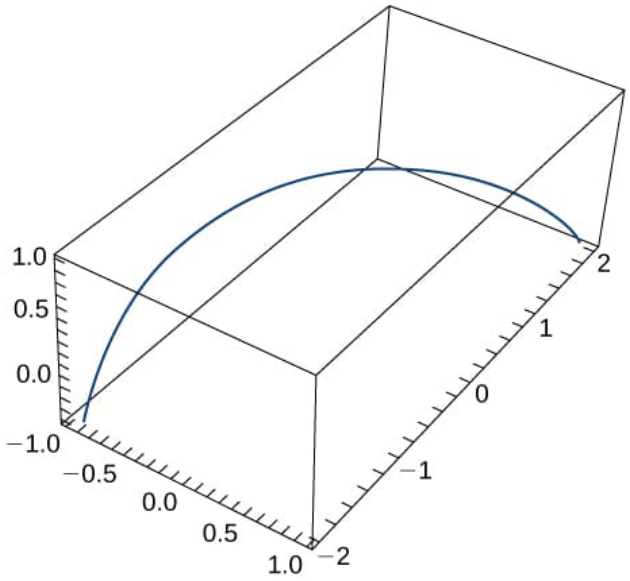


Figure $5.4E.3$: The graph of the given vector function is a curve that forms a smooth arc. The plot is enclosed within a rectangular box with the horizontal axis ranging from -1 to 1, the vertical axis ranging from -0.5 to 1, and the depth axis ranging from -2 to 2. The curve goes from lower-left front, rises to a peak near the center, and then descends as it moves toward the lower-right back.

9. The position function of an object is given by $\mathbf{r}(t) = \langle t^2, 5t, t^2 - 16t \rangle$. At what time is the speed a minimum?

Answer

$t = 4$

10. Let $\mathbf{r}(t) = r\cosh(\omega t)\ \mathbf{i} + r\sinh(\omega t)\ \mathbf{j}$. Find the velocity and acceleration vectors and show that the acceleration is proportional to $\mathbf{r}(t)$.
11. Consider the motion of a point on the circumference of a rolling circle. As the circle rolls, it generates the cycloid $\mathbf{r}(t) = (\omega t - \sin(\omega t))\ \mathbf{i} + (1 - \cos(\omega t))\ \mathbf{j}$, where ω is the angular velocity of the circle and b is the radius of the circle:

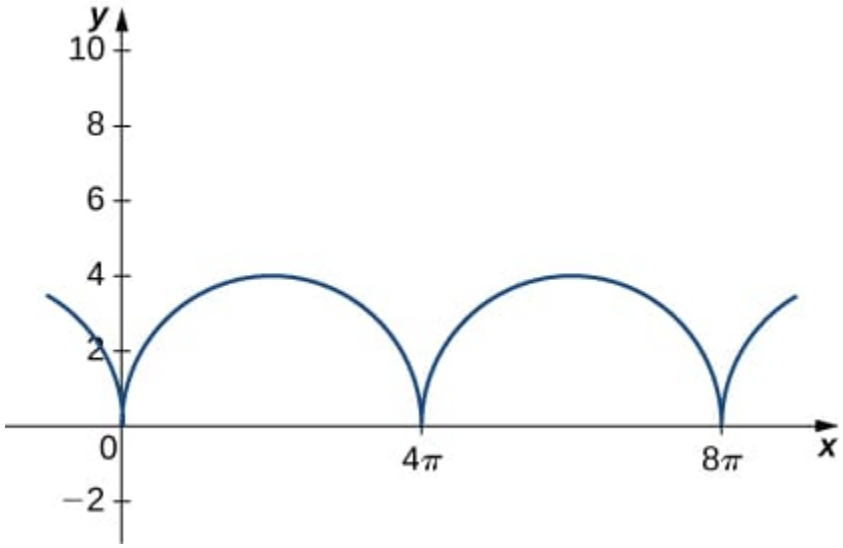


Figure $5.4E.4$: This is the graph of the given cycloid in the Cartesian plane. The curve alternates between peaks and valleys, with peaks occurring at $y = 4$ and valleys touching the x-axis at regular intervals, $x = 4n\pi$, with n an integer.

Find the equations for the velocity, acceleration, and speed of the particle at any time.

Answer

$\mathbf{v}(t) = (\omega - \omega\cos(\omega t))\ \mathbf{i} + (\omega\sin(\omega t))\ \mathbf{j}$
$\mathbf{a}(t) = (\omega^2\sin(\omega t))\ \mathbf{i} + (\omega^2\cos(\omega t))\ \mathbf{j}$

$$\begin{aligned}\text{speed}(t) &= \sqrt{(\omega - \omega\cos(\omega t))^2 + (\omega\sin(\omega t))^2} \\ &= \sqrt{\omega^2 - 2\omega^2\cos(\omega t) + \omega^2\cos^2(\omega t) + \omega^2\sin^2(\omega t)} \\ &= \sqrt{2\omega^2(1-\cos(\omega t))}\end{aligned}$$

12. A person on a hang glider is spiraling upward as a result of the rapidly rising air on a path having position vector $\mathbf{r}(t) = (3\cos t)\ \mathbf{i} + (3\sin t)\ \mathbf{j} + t^2\ \mathbf{k}$. The path is similar to that of a helix, although it is not a helix. The graph is shown here:

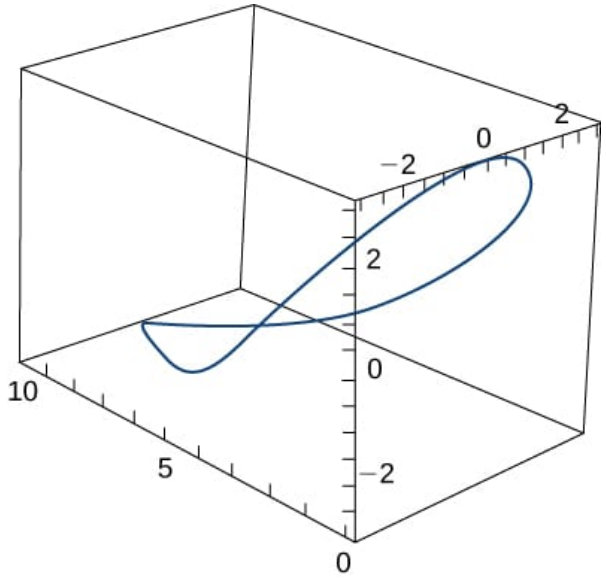


Figure $5.4E.5$: The graph of the given vector function is a curve that forms a smooth arc. The plot is enclosed within a rectangular box with the horizontal axis ranging from 10 to 0, the vertical axis ranging from -3 to 3, and the depth axis ranging from -3 to 3. The curve is a closed loop. The curve begins near the bottom-left corner of the graph, rises as it moves to the right, turns around, and then descends back to the starting point, forming a shape the like boundary of a "potato chip".

Find the following quantities:

a. The velocity and acceleration vectors
b. The glider's speed at any time
c. The times, if any, at which the glider's acceleration is orthogonal to its velocity

Answer

b. $\|\mathbf{v}(t)\| = \sqrt{9 + 4t^2}$

13. Given that $\mathbf{r}(t) = \langle e^{-5t}\sin t,\ e^{-5t}\cos t,\ 4e^{-5t}\rangle$ is the position vector of a moving particle, find the following quantities:

a. The velocity of the particle
b. The speed of the particle
c. The acceleration of the particle

Answer

a. $\mathbf{v}(t) = \langle e^{-5t}(\cos t - 5\sin t),\ -e^{-5t}(\sin t + 5\cos t),\ -20e^{-5t}\rangle$
c. $\mathbf{a}(t) = \langle e^{-5t}(-\sin t - 5\cos t) - 5e^{-5t}(\cos t - 5\sin t),\ -e^{-5t}(\cos t - 5\sin t) + 5e^{-5t}(\sin t + 5\cos t),\ 100e^{-5t}\rangle$

14. Find the maximum speed of a point on the circumference of an automobile tire of radius 1 ft when the automobile is traveling at 55 mph.
15. Find the position vector-valued function $\mathbf{r}(t)$, given that $\mathbf{a}(t) = \mathbf{i} + e^t\ \mathbf{j}$, $\mathbf{v}(0) = 2\ \mathbf{j}$, and $\mathbf{r}(0) = 2\ \mathbf{i}$.
16. Find $\mathbf{r}(t)$ given that $\mathbf{a}(t) = -32\ \mathbf{j}, \mathbf{v}(0) = 600\sqrt{3}\ \mathbf{i} + 600\ \mathbf{j}$, and $\mathbf{r}(0) = \mathbf{0}$.
17. The acceleration of an object is given by $\mathbf{a}(t) = t\ \mathbf{j} + t\ \mathbf{k}$. The velocity at $t = 1$ sec is $\mathbf{v}(1) = 5\ \mathbf{j}$ and the position of the object at $t = 1$ sec is $\mathbf{r}(1) = 0\ \mathbf{i} + 0\ \mathbf{j} + 0\ \mathbf{k}$. Find the object's position at any time.

Answer

$$\mathbf{r}(t) = 0\ \mathbf{i} + \left(\tfrac{1}{6}t^3 + 4.5t - \tfrac{14}{3}\right)\ \mathbf{j} + \left(\tfrac{1}{6}t^3 - \tfrac{1}{2}t + \tfrac{1}{3}\right)\ \mathbf{k}$$

Exercises 18 - 22

Projectile Motion

18. A projectile is shot in the air from ground level with an initial velocity of 500 m/sec at an angle of 60° with the horizontal.

    a. At what time does the projectile reach maximum height?
    b. What is the approximate maximum height of the projectile?
    c. At what time is the maximum range of the projectile attained?
    d. What is the maximum range?
    e. What is the total flight time of the projectile?

    Answer

    a. 44.185 sec
    c. $t = 88.37$ sec
    e. $t = 88.37$ sec

19. A projectile is fired at a height of 1.5 m above the ground with an initial velocity of 100 m/sec and at an angle of 30° above the horizontal. Use this information to answer the following questions:
    a. Determine the maximum height of the projectile.
    b. Determine the range of the projectile.

    Answer

    The range is approximately 886.29 m.

20. A golf ball is hit in a horizontal direction off the top edge of a building that is 100 ft tall. How fast must the ball be launched to land 450 ft away?
21. A projectile is fired from ground level at an angle of 8° with the horizontal. The projectile is to have a range of 50 m. Find the minimum velocity (speed) necessary to achieve this range.

    Answer

    $v = 42.16$ m/sec

22. Prove that an object moving in a straight line at a constant speed has an acceleration of zero.

Exercises 23 - 31

Find the tangential and normal components of acceleration.

23. $\mathbf{r}(t) = t^2\ \mathbf{i} + 2t\ \mathbf{j}$ when $t = 1$.

    Answer

    $$a_{\mathbf{T}} = \sqrt{2}, \quad a_{\mathbf{N}} = \sqrt{2}$$

24. $\mathbf{r}(t) = \langle \cos(2t),\ \sin(2t), 1 \rangle$
25. $\mathbf{r}(t) = \langle e^t \cos t,\ e^t \sin t,\ e^t \rangle$. The graph is shown here:

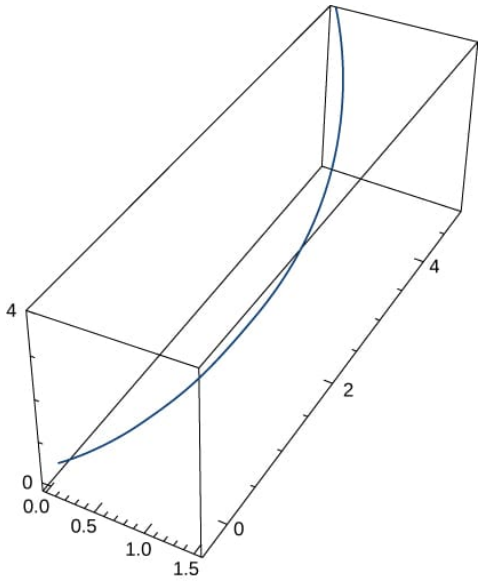


Figure $5.4E.6$: The graph of the given vector function is a curve that forms a smooth arc. The plot is enclosed within a rectangular box with the horizontal axis ranging from 0 to 1.5, the vertical axis ranging from 0 to 4, and the depth axis ranging from -0.5 to 5. The curve goes from lower-left front, rises slowly as it moves to the right, then left, ending in the upper-left back.

Answer

$$a_{\mathbf{T}} = \sqrt{3}e^t, \quad a_{\mathbf{N}} = \sqrt{2}e^t$$

26. $\mathbf{r}(t) = \langle \frac{2}{3}(1+t)^{3/2},\ \frac{2}{3}(1-t)^{3/2},\ \sqrt{2}t \rangle$
27. $\mathbf{r}(t) = \left\langle 2t,\ t^2,\ \dfrac{t^3}{3} \right\rangle$

Answer

$$a_{\mathbf{T}} = 2t, \quad a_{\mathbf{N}} = 2$$

28. $\mathbf{r}(t) = t^2\ \mathbf{i} + t^2\ \mathbf{j} + t^3\ \mathbf{k}$
29. $\mathbf{r}(t) = \langle 6t,\ 3t^2,\ 2t^3 \rangle$

Answer

$$a_{\mathbf{T}} = \frac{6t + 12t^3}{\sqrt{1 + t^2 + t^4}}, \quad a_{\mathbf{N}} = 6\sqrt{\frac{1 + 4t^2 + t^4}{1 + t^2 + t^4}}$$

30. $\mathbf{r}(t) = 3\cos(2\pi t)\ \mathbf{i} + 3\sin(2\pi t)\ \mathbf{j}$

Answer

$$a_{\mathbf{T}} = 0, \quad a_{\mathbf{N}} = 12\pi^2$$

31. $\mathbf{r}(t) = a\cos(\omega t)\ \mathbf{i} + b\sin(\omega t)\ \mathbf{j}$ at $t = 0$.

Exercises 32 - 36

Answer

$$a_{\mathbf{T}} = 0, \quad a_{\mathbf{N}} = a\omega^2$$

32. Suppose that the position function for an object in three dimensions is given by the equation $\mathbf{r}(t) = t\cos(t)\ \mathbf{i} + t\sin(t)\ \mathbf{j} + 3t\ \mathbf{k}$.

    a. Show that the particle moves on a circular cone.
    b. Find the angle between the velocity and acceleration vectors when $t = 1.5$.
    c. Find the tangential and normal components of acceleration when $t = 1.5$.

Answer

c. $a_{\mathrm{T}} = 0.43 \text{ m/sec}^2, \quad a_{\mathrm{N}} = 2.46 \text{ m/sec}^2$

33. The force on a particle is given by $\mathbf{f}(t) = (\cos t)\ \mathbf{i} + (\sin t)\ \mathbf{j}$. The particle is located at point $(c, 0)$ at $t = 0$. The initial velocity of the particle is given by $\mathbf{v}(0) = v_0\ \mathbf{j}$. Find the path of the particle of mass m. (Recall, $\mathbf{F} = m\mathbf{a}$.)

Answer

$$\mathbf{r}(t) = \left(\frac{-\cos t}{m} + c + \tfrac{1}{m}\right)\ \mathbf{i} + \left(\frac{-\sin t}{m} + \left(v_0 + \tfrac{1}{m}\right) t\right)\ \mathbf{j}$$

34. An automobile that weighs 2700 lb makes a turn on a flat road while traveling at 56 ft/sec. If the radius of the turn is 70 ft, what is the required frictional force to keep the car from skidding?

35. Using Kepler's laws, it can be shown that $v_0 = \sqrt{\dfrac{2GM}{r_0}}$ is the minimum speed needed when $\theta = 0$ so that an object will escape from the pull of a central force resulting from mass M. Use this result to find the minimum speed when $\theta = 0$ for a space capsule to escape from the gravitational pull of Earth if the probe is at an altitude of 300 km above Earth's surface.

Answer

10.94 km/sec

36. Find the time in years it takes the dwarf planet Pluto to make one orbit about the Sun given that $a = 39.5$ A.U.

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5.5: Chapter 5 Review Exercises

Exercises 1 - 4

***True or False*? Justify your answer with a proof or a counterexample.**

1. A parametric equation that passes through points P and Q can be given by $\mathbf{r}(t) = \langle t^2,\ 3t+1,\ t-2\rangle$, where $P(1,4,-1)$ and $Q(16,11,2)$.
2. $\dfrac{d}{dt}\Big[\mathbf{u}(t)\times\mathbf{u}(t)\Big] = 2\mathbf{u}\,'(t)\times\mathbf{u}(t)$

   Answer

   False, $\dfrac{d}{dt}\Big[\mathbf{u}(t)\times\mathbf{u}(t)\Big] = \mathbf{0}$.
3. The curvature of a circle of radius r is constant everywhere. Furthermore, the curvature is equal to $1/r$.
4. The speed of a particle with a position function $\mathbf{r}(t)$ is $\dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$.

   Answer

   False, it is $\|\mathbf{r}\,'(t)\|$

Exercises 5 - 6

Find the domains of the vector-valued functions.

5. $\mathbf{r}(t) = \langle \sin(t),\ \ln(t),\ \sqrt{t}\rangle$
6. $\mathbf{r}(t) = \left\langle e^t,\ \dfrac{1}{\sqrt{4-t}},\ \sec t\right\rangle$

   Answer

   $t<4,\ \ t\neq\dfrac{n\pi}{2}$

Exercises 7 - 9

Sketch the curves for the following vector equations. Use a calculator if needed.

7. **[T]** $\mathbf{r}(t) = \langle t^2,\ t^3\rangle$
8. **[T]** $\mathbf{r}(t) = \langle \sin(20t)e^{-t},\ \cos(20t)e^{-t},\ e^{-t}\rangle$

Answer

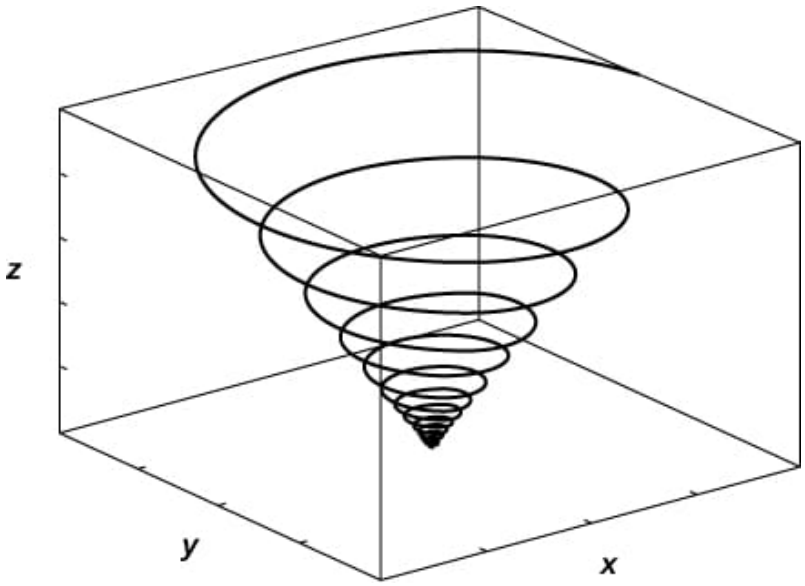


Figure 5.5.1: This illustrates a three-dimensional spiral that tapers downward, forming a cone-like shape. The spiral starts at a wide radius near the top of the plot and progressively narrows as it approaches the apex at the bottom. The curve follows a helical pattern, descending smoothly along the z-axis.

Exercises 9 - 10

Find a vector function that describes the following curves.

9. Intersection of the cylinder $x^2 + y^2 = 4$ with the plane $x + z = 6$
10. Intersection of the cone $z = \sqrt{x^2 + y^2}$ and plane $z = y - 4$

Answer

$$\mathbf{r}(t) = \left\langle t,\ 2 - \tfrac{t^2}{8},\ -2 - \tfrac{t^2}{8} \right\rangle$$

Exercises 11 - 12

Find the derivatives of $\mathbf{u}(t),\ \mathbf{u}\,'(t),\ \mathbf{u}\,'(t) \times \mathbf{u}(t),\ \mathbf{u}(t) \times \mathbf{u}\,'(t)$, and $\mathbf{u}(t) \cdot \mathbf{u}\,'(t)$. Find the unit tangent vector.

11. $\mathbf{u}(t) = \langle e^t,\ e^{-t} \rangle$
12. $\mathbf{u}(t) = \langle t^2,\ 2t + 6,\ 4t^5 - 12 \rangle$

Answer

$\mathbf{u}\,'(t) = \langle 2t,\ 2,\ 20t^4 \rangle$,
$\mathbf{u}\,''(t) = \langle 2,\ 0,\ 80t^3 \rangle$,
$\dfrac{d}{dt}\left[\mathbf{u}\,'(t) \times \mathbf{u}(t)\right] = \langle -480t^3 - 160t^4,\ 24 + 75t^2,\ 12 + 4t \rangle$,
$\dfrac{d}{dt}\left[\mathbf{u}(t) \times \mathbf{u}\,'(t)\right] = \langle 480t^3 + 160t^4,\ -24 - 75t^2,\ -12 - 4t \rangle$,
$\dfrac{d}{dt}\left[\mathbf{u}(t) \cdot \mathbf{u}\,'(t)\right] = 720t^8 - 9600t^3 + 6t^2 + 4$,
unit tangent vector: $\mathbf{T}(t) = \dfrac{2t}{\sqrt{400t^8 + 4t^2 + 4}}\ \mathbf{i} + \dfrac{2}{\sqrt{400t^8 + 4t^2 + 4}}\ \mathbf{j} + \dfrac{20t^4}{\sqrt{400t^8 + 4t^2 + 4}}\ \mathbf{k}$

Exercises 13 - 14

Evaluate the following integrals.

13. $\int \left(\tan(t)\sec(t)\,\mathbf{i} - te^{3t}\,\mathbf{j}\right)\ dt$
14. $\int_1^4 \mathbf{u}(t)\ dt$, with $\mathbf{u}(t) = \left\langle \frac{\ln t}{t},\ \frac{1}{\sqrt{t}},\ \sin\left(\frac{t\pi}{4}\right)\right\rangle$

Answer

$$\frac{\ln(4^2)}{2}\ \mathbf{i} + 2\ \mathbf{j} + \frac{2(2+\sqrt{2})}{\pi}\ \mathbf{k}$$

Exercises 15 - 16

Find the length for the following curves.

15. $\mathbf{r}(t) = \langle 3t,\ 4\cos t,\ 4\sin t\rangle$ for $1 \le t \le 4$
16. $\mathbf{r}(t) = 2\ \mathbf{i} + t\ \mathbf{j} + 3t^2\ \mathbf{k}$ for $0 \le t \le 1$

Answer

$$\frac{\sqrt{37}}{2} + \tfrac{1}{12}\sinh^{-1} 6$$

Exercises 17 - 18

Reparameterize the following functions with respect to their arc length measured from $t = 0$ in direction of increasing t.

17. $\mathbf{r}(t) = 2t\ \mathbf{i} + (4t-5)\ \mathbf{j} + (1-3t)\ \mathbf{k}$
18. $\mathbf{r}(t) = \cos(2t)\ \mathbf{i} + 8t\ \mathbf{j} - \sin(2t)\ \mathbf{k}$

Answer

$$\mathbf{r}(t(s)) = \cos\left(\tfrac{2s}{\sqrt{65}}\right)\ \mathbf{i} + \tfrac{8s}{\sqrt{65}}\ \mathbf{j} - \sin\left(\tfrac{2s}{\sqrt{65}}\right)\ \mathbf{k}$$

Exercises 19 - 20

Find the curvature for the following vector functions.

19. $\mathbf{r}(t) = (2\sin t)\ \mathbf{i} - 4t\ \mathbf{j} + (2\cos t)\ \mathbf{k}$
20. $\mathbf{r}(t) = \sqrt{2}e^t\ \mathbf{i} + \sqrt{2}e^{-t}\ \mathbf{j} + 2t\ \mathbf{k}$

Exercises 21 - 24

Answer

$$\frac{e^{2t}}{\left(e^{2t}+1\right)^2}$$

21. Find the unit tangent vector, the unit normal vector, and the binormal vector for $\mathbf{r}(t) = 2\cos t\ \mathbf{i} + 3t\ \mathbf{j} + 2sint\ \mathbf{k}$.
22. Find the tangential and normal acceleration components with the position vector $\mathbf{r}(t) = \langle \cos t,\ \sin t,\ e^t\rangle$.

Answer

$$a_T = \frac{e^{2t}}{1+e^{2t}},$$

$$a_N = \frac{\sqrt{2e^{2t} + 4e^{2t}\sin t\cos t + 1}}{1+e^{2t}}$$

23. A Ferris wheel car is moving at a constant speed v and has a constant radius r. Find the tangential and normal acceleration of the Ferris wheel car.
24. The position of a particle is given by $\mathbf{r}(t) = \langle t^2,\ \ln t,\ \sin(\pi t)\rangle$, where t is measured in seconds and r is measured in meters. Find the velocity, acceleration, and speed functions. What are the position, velocity, speed, and acceleration of the particle at

1 sec?

Answer

$\mathbf{v}(t) = \langle 2t,\ \frac{1}{t},\ \pi\cos(\pi t)\rangle\ \text{m/sec},$
$\mathbf{a}(t) = \langle 2,\ -\frac{1}{t^2},\ -\pi^2\sin(\pi t)\rangle\ \text{m/sec}^2,$
$\text{speed}(t) = \sqrt{4t^2 + \frac{1}{t^2} + \pi^2\cos^2(\pi t)}\ \text{m/sec};$
At $t = 1$, $\mathbf{r}(1) = \langle 1, 0, 0\rangle$ m, $\mathbf{v}(1) = \langle 2, -1, \pi\rangle$ m/sec, $\mathbf{a}(1) = \langle 2, -1, 0\rangle$ m/sec^2, and $\text{speed}(1) = \sqrt{5+\pi^2}$ m/sec

Exercises 25 - 26

The following problems consider launching a cannonball out of a cannon. The cannonball is shot out of the cannon with an angle θ and initial velocity $\mathbf{v}_0$. The only force acting on the cannonball is gravity, so we begin with a constant acceleration $\mathbf{a}(t) = -g\ \mathbf{j}$.

25. Find the velocity vector function $\mathbf{v}(t)$.
26. Find the position vector $\mathbf{r}(t)$ and the parametric representation for the position.

Answer

$\mathbf{r}(t) = \mathbf{v}_0 t - \frac{gt^2}{2}\ \mathbf{j},$
$\mathbf{r}(t) = \langle v_0(\cos\theta)t,\ \ v_0(\sin\theta)t, -\frac{gt^2}{2}\rangle$ where $v_0 = \|\mathbf{v}_0\|.$

27. At what angle do you need to fire the cannonball for the horizontal distance to be greatest? What is the total distance it would travel?

Contributors

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6: APPENDICES

6.1: A- Table of Derivatives
6.2: B- Table of Integrals

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6.1: A- Table of Derivatives

General Formulas

1. $\frac{d}{dx}(c) = 0$
2. $\frac{d}{dx}(f(x) + g(x)) = f\,'(x) + g\,'(x)$
3. $\frac{d}{dx}(f(x)g(x)) = f\,'(x)g(x) + f(x)g\,'(x)$
4. $\frac{d}{dx}(x^n) = nx^{n-1}, \quad$ for real numbers n
5. $\frac{d}{dx}(cf(x)) = cf\,'(x)$
6. $\frac{d}{dx}(f(x) - g(x)) = f\,'(x) - g\,'(x)$
7. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$
8. $\frac{d}{dx}[f(g(x))] = f\,'(g(x)) \cdot g\,'(x)$

Trigonometric Functions

9. $\frac{d}{dx}(\sin x) = \cos x$
10. $\frac{d}{dx}(\tan x) = \sec^2 x$
11. $\frac{d}{dx}(\sec x) = \sec x \tan x$
12. $\frac{d}{dx}(\cos x) = -\sin x$
13. $\frac{d}{dx}(\cot x) = -\csc^2 x$
14. $\frac{d}{dx}(\csc x) = -\csc x \cot x$

Inverse Trigonometric Functions

15. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
16. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
17. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$

18. $\dfrac{d}{dx}\left(\cos^{-1}x\right)=\dfrac{-1}{\sqrt{1-x^2}}$

19. $\dfrac{d}{dx}\left(\cot^{-1}x\right)=\dfrac{-1}{1+x^2}$

20. $\dfrac{d}{dx}\left(\csc^{-1}x\right)=\dfrac{-1}{|x|\sqrt{x^2-1}}$

Exponential and Logarithmic Functions

21. $\dfrac{d}{dx}\left(e^x\right)=e^x$

22. $\dfrac{d}{dx}\left(\ln|x|\right)=\dfrac{1}{x}$

23. $\dfrac{d}{dx}\left(b^x\right)=b^x\ln b$

24. $\dfrac{d}{dx}\left(\log_b x\right)=\dfrac{1}{x\ln b}$

Hyperbolic Functions

25. $\dfrac{d}{dx}\left(\sinh x\right)=\cosh x$

26. $\dfrac{d}{dx}\left(\tanh x\right)=\text{sech}^2\ x$

27. $\dfrac{d}{dx}\left(\text{sech}x\right)=-\text{sech}\ x\tanh x$

28. $\dfrac{d}{dx}\left(\cosh x\right)=\sinh x$

29. $\dfrac{d}{dx}\left(\coth x\right)=-\text{csch}^2\ x$

30. $\dfrac{d}{dx}\left(\text{csch}\,x\right)=-\text{csch}x\coth x$

Inverse Hyperbolic Functions

31. $\dfrac{d}{dx}\left(\sinh^{-1}x\right)=\dfrac{1}{\sqrt{x^2+1}}$

32. $\dfrac{d}{dx}\left(\tanh^{-1}x\right)=\dfrac{1}{1-x^2}\quad(|x|<1)$

33. \(\quad \dfrac{d}{dx}\left(\text{sech}^{-1}\,x\right)=\dfrac{-1}{x\sqrt{1-x^2}}\quad (0

34. $\dfrac{d}{dx}\left(\cosh^{-1}x\right)=\dfrac{1}{\sqrt{x^2-1}}\quad(x>1)$

35. $\dfrac{d}{dx}\left(\coth^{-1}x\right)=\dfrac{1}{1-x^2}\quad(|x|>1)$

36. $\dfrac{d}{dx}\left(\text{csch}^{-1}x\right)=\dfrac{-1}{|x|\sqrt{1+x^2}}\quad(x\neq 0)$

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6.2: B- Table of Integrals

Basic Integrals

1. $\int u^n \, du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$
2. $\int \frac{du}{u} = \ln |u| + C$
3. $\int e^u \, du = e^u + C$
4. $\int a^u \, du = \frac{a^u}{\ln a} + C$
5. $\int \sin u \, du = -\cos u + C$
6. $\int \cos u \, du = \sin u + C$
7. $\int \sec^2 u \, du = \tan u + C$
8. $\int \csc^2 u \, du = -\cot u + C$
9. $\int \sec u \tan u \, du = \sec u + C$
10. $\int \csc u \cot u \, du = -\csc u + C$
11. $\int \tan u \, du = \ln |\sec u| + C$
12. $\int \cot u \, du = \ln |\sin u| + C$
13. $\int \sec u \, du = \ln |\sec u + \tan u| + C$
14. $\int \csc u \, du = \ln |\csc u - \cot u| + C$
15. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$
16. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$
17. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{|u|}{a} + C$

Trigonometric Integrals

18. $\int \sin^2 u \, du = \frac{1}{2}u - \frac{1}{4}\sin 2u + C$

19. $\int \cos^2 u \ du = \frac{1}{2}u + \frac{1}{4}\sin 2u + C$

20. $\int \tan^2 u \ du = \tan u - u + C$

21. $\int \cot^2 u \ du = -\cot u - u + C$

22. $\int \sin^3 u \ du = -\frac{1}{3}(2 + \sin^2 u)\cos u + C$

23. $\int \cos^3 u \ du = \frac{1}{3}(2 + \cos^2 u)\sin u + C$

24. $\int \tan^3 u \ du = \frac{1}{2}\tan^2 u + \ln|\cos u| + C$

25. $\int \cot^3 u \ du = -\frac{1}{2}\cot^2 u - \ln|\sin u| + C$

26. $\int \sec^3 u \ du = \frac{1}{2}\sec u \tan u + \frac{1}{2}\ln|\sec u + \tan u| + C$

27. $\int \csc^3 u \ du = -\frac{1}{2}\csc u \cot u + \frac{1}{2}\ln|\csc u - \cot u| + C$

28. $\int \sin^n u \ du = \frac{-1}{n}\sin^{n-1} u \cos u + \frac{n-1}{n}\int \sin^{n-2} u \ du$

29. $\int \cos^n u \ du = \frac{1}{n}\cos^{n-1} u \sin u + \frac{n-1}{n}\int \cos^{n-2} u \ du$

30. $\int \tan^n u \ du = \frac{1}{n-1}\tan^{n-1} u - \int \tan^{n-2} u \ du$

31. $\int \cot^n u \ du = \frac{-1}{n-1}\cot^{n-1} u - \int \cot^{n-2} u \ du$

32. $\int \sec^n u \ du = \frac{1}{n-1}\tan u \sec^{n-2} u + \frac{n-2}{n-1}\int \sec^{n-2} u \ du$

33. $\int \csc^n u \ du = \frac{-1}{n-1}\cot u \csc^{n-2} u + \frac{n-2}{n-1}\int \csc^{n-2} u \ du$

34. $\int \sin au \sin bu \ du = \frac{\sin(a-b)u}{2(a-b)} - \frac{\sin(a+b)u}{2(a+b)} + C$

35. $\int \cos au \cos bu \ du = \frac{\sin(a-b)u}{2(a-b)} + \frac{\sin(a+b)u}{2(a+b)} + C$

36. $\int \sin au \cos bu \ du = -\frac{\cos(a-b)u}{2(a-b)} - \frac{\cos(a+b)u}{2(a+b)} + C$

37. $\int u \sin u \ du = \sin u - u \cos u + C$

38. $\int u \cos u \ du = \cos u + u \sin u + C$

39. $\int u^n \sin u \ du = -u^n \cos u + n\int u^{n-1} \cos u \ du$

40. $\int u^n \cos u \ du = u^n \sin u - n\int u^{n-1} \sin u \ du$

41. Erroneous nesting of equation structures

Exponential and Logarithmic Integrals

42. $\int ue^{au}\ du = \frac{1}{a^2}(au-1)e^{au} + C$

43. $\int u^n e^{au}\ du = \frac{1}{a}u^n e^{au} - \frac{n}{a}\int u^{n-1}e^{au}\ du$

44. $\int e^{au}\sin bu\ du = \frac{e^{au}}{a^2+b^2}(a\sin bu - b\cos bu) + C$

45. $\int e^{au}\cos bu\ du = \frac{e^{au}}{a^2+b^2}(a\cos bu + b\sin bu) + C$

46. $\int \ln u\ du = u\ln u - u + C$

47. $\int u^n \ln u\ du = \frac{u^{n+1}}{(n+1)^2}[(n+1)\ln u - 1] + C$

48. $\int \frac{1}{u\ln u}\ du = \ln|\ln u| + C$

Hyperbolic Integrals

49. $\int \sinh u\ du = \cosh u + C$

50. $\int \cosh u\ du = \sinh u + C$

51. $\int \tanh u\ du = \ln\cosh u + C$

52. $\int \coth u\ du = \ln|\sinh u| + C$

53. $\int \operatorname{sech}\ u\ du = \tan^{-1}|\sinh u| + C$

54. $\int \operatorname{csch}\ u\ du = \ln|\tanh\frac{1}{2}u| + C$

55. $\int \operatorname{sech}^2 u\ du = \tanh\ u + C$

56. $\int \operatorname{csch}^2 u\ du = -\coth\ u + C$

57. $\int \operatorname{sech}\ u\tanh u\ du = -\operatorname{sech}\ u + C$

58. $\int \operatorname{csch}\ u\coth u\ du = -\operatorname{csch}\ u + C$

Inverse Trigonometric Integrals

59. $\int \sin^{-1} u\ du = u\sin^{-1} u + \sqrt{1-u^2} + C$

60. $\int \cos^{-1} u \; du = u\cos^{-1} u - \sqrt{1-u^2} + C$

61. $\int \tan^{-1} u \; du = u\tan^{-1} u - \frac{1}{2}\ln(1+u^2) + C$

62. $\int u\sin^{-1} u \; du = \frac{2u^2-1}{4}\sin^{-1} u + \frac{u\sqrt{1-u^2}}{4} + C$

63. $\int u\cos^{-1} u \; du = \frac{2u^2-1}{4}\cos^{-1} u - \frac{u\sqrt{1-u^2}}{4} + C$

64. $\int u\tan^{-1} u \; du = \frac{u^2+1}{2}\tan^{-1} u - \frac{u}{2} + C$

65. $\int u^n \sin^{-1} u \; du = \frac{1}{n+1}\left[u^{n+1}\sin^{-1} u - \int \frac{u^{n+1}\,du}{\sqrt{1-u^2}}\right], \quad n \neq -1$

66. $\int u^n \cos^{-1} u \; du = \frac{1}{n+1}\left[u^{n+1}\cos^{-1} u + \int \frac{u^{n+1}\,du}{\sqrt{1-u^2}}\right], \quad n \neq -1$

67. $\int u^n \tan^{-1} u \; du = \frac{1}{n+1}\left[u^{n+1}\tan^{-1} u - \int \frac{u^{n+1}\,du}{1+u^2}\right], \quad n \neq -1$

Integrals Involving $a^2 + u^2$, $a > 0$

68. $\int \sqrt{a^2+u^2} \; du = \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln\left(u+\sqrt{a^2+u^2}\right) + C$

69. $\int u^2\sqrt{a^2+u^2} \; du = \frac{u}{8}(a^2+2u^2)\sqrt{a^2+u^2} - \frac{a^4}{8}\ln\left(u+\sqrt{a^2+u^2}\right) + C$

70. $\int \frac{\sqrt{a^2+u^2}}{u} \; du = \sqrt{a^2+u^2} - a\ln\left|\frac{a+\sqrt{a^2+u^2}}{u}\right| + C$

71. $\int \frac{\sqrt{a^2+u^2}}{u^2} \; du = -\frac{\sqrt{a^2+u^2}}{u} + \ln\left(u+\sqrt{a^2+u^2}\right) + C$

72. $\int \frac{du}{\sqrt{a^2+u^2}} = \ln\left(u+\sqrt{a^2+u^2}\right) + C$

73. $\int \frac{u^2}{\sqrt{a^2+u^2}} \; du = \frac{u}{2}\left(\sqrt{a^2+u^2}\right) - \frac{a^2}{2}\ln\left(u+\sqrt{a^2+u^2}\right) + C$

74. $\int \frac{du}{u\sqrt{a^2+u^2}} = \frac{-1}{a}\ln\left|\frac{\sqrt{a^2+u^2}+a}{u}\right| + C$

75. $\int \frac{du}{u^2\sqrt{a^2+u^2}} = -\frac{\sqrt{a^2+u^2}}{a^2u} + C$

76. $\int \frac{du}{(a^2+u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2+u^2}} + C$

Integrals Involving $u^2 - a^2$, $a > 0$

77. $\int \sqrt{u^2-a^2} \; du = \frac{u}{2}\sqrt{u^2-a^2} - \frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right| + C$

78. $$\int u^2\sqrt{u^2-a^2}\,du=\frac{u}{8}(2u^2-a^2)\sqrt{u^2-a^2}-\frac{a^4}{8}\ln\left|u+\sqrt{u^2-a^2}\right|+C$$

79. $$\int \frac{\sqrt{u^2-a^2}}{u}\,du=\sqrt{u^2-a^2}-a\cos^{-1}\frac{a}{|u|}+C$$

80. $$\int \frac{\sqrt{u^2-a^2}}{u^2}\,du=-\frac{\sqrt{u^2-a^2}}{u}+\ln\left|u+\sqrt{u^2-a^2}\right|+C$$

81. $$\int \frac{du}{\sqrt{u^2-a^2}}=\ln\left|u+\sqrt{u^2-a^2}\right|+C$$

82. $$\int \frac{u^2}{\sqrt{u^2-a^2}}\,du=\frac{u}{2}\sqrt{u^2-a^2}+\frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right|+C$$

83. $$\int \frac{du}{u^2\sqrt{u^2-a^2}}=\frac{\sqrt{u^2-a^2}}{a^2u}+C$$

84. $$\int \frac{du}{(u^2-a^2)^{3/2}}=-\frac{u}{a^2\sqrt{u^2-a^2}}+C$$

Integrals Involving $a^2 - u^2$, $a > 0$

85. $$\int \sqrt{a^2-u^2}\,du=\frac{u}{2}\sqrt{a^2-u^2}+\frac{a^2}{2}\sin^{-1}\frac{u}{a}+C$$

86. $$\int u^2\sqrt{a^2-u^2}\,du=\frac{u}{8}(2u^2-a^2)\sqrt{a^2-u^2}+\frac{a^4}{8}\sin^{-1}\frac{u}{a}+C$$

87. $$\int \frac{\sqrt{a^2-u^2}}{u}\,du=\sqrt{a^2-u^2}-a\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right|+C$$

88. $$\int \frac{\sqrt{a^2-u^2}}{u^2}\,du=\frac{-1}{u}\sqrt{a^2-u^2}-\sin^{-1}\frac{u}{a}+C$$

89. $$\int \frac{u^2}{\sqrt{a^2-u^2}}\,du=\frac{1}{2}\left(-u\sqrt{a^2-u^2}+a^2\sin^{-1}\frac{u}{a}\right)+C$$

90. $$\int \frac{du}{u\sqrt{a^2-u^2}}=-\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right|+C$$

91. $$\int \frac{du}{u^2\sqrt{a^2-u^2}}=-\frac{1}{a^2u}\sqrt{a^2-u^2}+C$$

92. $$\int \left(a^2-u^2\right)^{3/2}\,du=-\frac{u}{8}\left(2u^2-5a^2\right)\sqrt{a^2-u^2}+\frac{3a^4}{8}\sin^{-1}\frac{u}{a}+C$$

93. $$\int \frac{du}{(a^2-u^2)^{3/2}}=-\frac{u}{a^2\sqrt{a^2-u^2}}+C$$

Integrals Involving $2au - u^2$, $a > 0$

94. $$\int \sqrt{2au-u^2}\,du=\frac{u-a}{2}\sqrt{2au-u^2}+\frac{a^2}{2}\cos^{-1}\left(\frac{a-u}{a}\right)+C$$

95. $$\int \frac{du}{\sqrt{2au-u^2}}=\cos^{-1}\left(\frac{a-u}{a}\right)+C$$

96. $$\int u\sqrt{2au-u^2}\,du = \frac{2u^2-au-3a^2}{6}\sqrt{2au-u^2} + \frac{a^3}{2}\cos^{-1}\left(\frac{a-u}{a}\right) + C$$

97. $$\int \frac{du}{u\sqrt{2au-u^2}} = -\frac{\sqrt{2au-u^2}}{au} + C$$

Integrals Involving $a + bu$, $a \neq 0$

98. $$\int \frac{u}{a+bu}\,du = \frac{1}{b^2}(a+bu-a\ln|a+bu|) + C$$

99. $$\int \frac{u^2}{a+bu}\,du = \frac{1}{2b^3}\left[(a+bu)^2 - 4a(a+bu) + 2a^2\ln|a+bu|\right] + C$$

100. $$\int \frac{du}{u(a+bu)} = \frac{1}{a}\ln\left|\frac{u}{a+bu}\right| + C$$

101. $$\int \frac{du}{u^2(a+bu)} = -\frac{1}{au} + \frac{b}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$$

102. $$\int \frac{u}{(a+bu)^2}\,du = \frac{a}{b^2(a+bu)} + \frac{1}{b^2}\ln|a+bu| + C$$

103. $$\int \frac{u}{u(a+bu)^2}\,du = \frac{1}{a(a+bu)} - \frac{1}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$$

104. $$\int \frac{u^2}{(a+bu)^2}\,du = \frac{1}{b^3}\left(a+bu - \frac{a^2}{a+bu} - 2a\ln|a+bu|\right) + C$$

105. $$\int u\sqrt{a+bu}\,du = \frac{2}{15b^2}(3bu-2a)(a+bu)^{3/2} + C$$

106. $$\int \frac{u}{\sqrt{a+bu}}\,du = \frac{2}{3b^2}(bu-2a)\sqrt{a+bu} + C$$

107. $$\int \frac{u^2}{\sqrt{a+bu}}\,du = \frac{2}{15b^3}(8a^2+3b^2u^2-4abu)\sqrt{a+bu} + C$$

108. $$\int \frac{du}{u\sqrt{a+bu}} = \begin{cases} \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}\right| + C, & \text{if } a > 0 \\ \frac{\sqrt{2}}{\sqrt{-a}}\tan^{-1}\sqrt{\frac{a+bu}{-a}} + C, & \text{if } a < 0 \end{cases}$$

109. $$\int \frac{\sqrt{a+bu}}{u}\,du = 2\sqrt{a+bu} + a\int \frac{du}{u\sqrt{a+bu}}$$

110. $$\int \frac{\sqrt{a+bu}}{u^2}\,du = -\frac{\sqrt{a+bu}}{u} + \frac{b}{2}\int \frac{du}{u\sqrt{a+bu}}$$

111. $$\int u^n\sqrt{a+bu}\,du = \frac{2}{b(2n+3)}\left[u^n(a+bu)^{3/2} - na\int u^{n-1}\sqrt{a+bu}\,du\right]$$

112. $$\int \frac{u^n}{\sqrt{a+bu}}\,du = \frac{2u^n\sqrt{a+bu}}{b(2n+1)} - \frac{2na}{b(2n+1)}\int \frac{u^{n-1}}{\sqrt{a+bu}}\,du$$

113. $$\int \frac{du}{u^n\sqrt{a+bu}} = -\frac{\sqrt{a+bu}}{a(n-1)u^{n-1}} - \frac{b(2n-3)}{2a(n-1)}\int \frac{du}{u^{n-1}\sqrt{a+bu}}$$

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