MATHEMATICS FOR LIBERAL ARTS



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Mathematics for Liberal Arts

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Detailed Licensing

Licensing

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CHAPTER OVERVIEW

1: Sets and Venn Diagrams

1.1: Basics of Sets

1.2: Union, Intersection, and Complement

1.3: Venn Diagrams

1.4: Cardinality

1.5: Exercises

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1.1: Basics of Sets

An art collector might own a collection of paintings, while a music lover might keep a collection of CDs. Any collection of items can form a **set**.

Set

A **set** is a collection of distinct objects, called **elements** of the set

A set can be defined by describing the contents, or by listing the elements of the set, enclosed in curly brackets.

Example 1

Some examples of sets defined by describing the contents:

a. The set of all even numbers
b. The set of all books written about travel to Chile

Some examples of sets defined by listing the elements of the set:

a. {1, 3, 9, 12}
b. {red, orange, yellow, green, blue, indigo, purple}

A set simply specifies the contents; order is not important. The set represented by {1, 2, 3} is equivalent to the set {3, 1, 2}.

Notation

Commonly, we will use a variable to represent a set, to make it easier to refer to that set later.

The symbol $\in$ means "is an element of".

A set that contains no elements, {}, is called the **empty set** and is notated $\emptyset$

Example 2

Let $A = \{1, 2, 3, 4\}$

To notate that 2 is element of the set, we'd write $2 \in A$

Sometimes a collection might not contain all the elements of a set. For example, Chris owns three Madonna albums. While Chris's collection is a set, we can also say it is a **subset** of the larger set of all Madonna albums.

Subset

A **subset** of a set A is another set that contains only elements from the set A, but may not contain all the elements of A.

If B is a subset of A, we write $B \subseteq A$

A **proper subset** is a subset that is not identical to the original set – it contains fewer elements.

If B is a proper subset of A, we write $B \subset A$

Example 3

Consider these three sets

$A =$ the set of all even numbers $\quad B = \{2, 4, 6\} \quad C = \{2, 3, 4, 6\}$

Here $B \subset A$ since every element of B is also an even number, so is an element of A.

More formally, we could say $B \subset A$ since if $x \in B$, then $x \in A$

It is also true that $B \subset C$.

C is not a subset of A, since C contains an element, 3 , that is not contained in A

Example 4

Suppose a set contains the plays “Much Ado About Nothing”, “MacBeth”, and “A Midsummer’s Night Dream”. What is a larger set this might be a subset of?

Solution

There are many possible answers here. One would be the set of plays by Shakespeare. This is also a subset of the set of all plays ever written. It is also a subset of all British literature.

Try it Now 1

The set $A = \{1, 3, 5\}$. What is a larger set this might be a subset of?

Answer

There are several answers: The set of all odd numbers less than 10. The set of all odd numbers. The set of all integers. The set of all real numbers.

Every set is a subset of itself.

Every set can be written as a subset of itself.

For any set, call it X, $X \subseteq X$.

The Empty set is a subset of every set.

The Empty set is a subset of every set, including itself.

Even if we don't see it listed in the set, the empty set is a subset of every set, including the empty set!

Notation: The empty set is a subset of every set.

For any set, call in X,

$\{\} \subseteq X$

Or

$\emptyset \subseteq X$

Notation: The empty set is a subset of the empty Set.

$\{\} \subseteq \{\}$

Or

$\emptyset \subseteq \emptyset$

Example 5

List all of the subsets of set $A = \{blue, red, yellow\}$.

Solution

Any one of the three elements can form a subset.

- $\{blue\}, \{red\}, \{yellow\}$

Now any pair of elements will form a subset.

- $\{blue, red\}, \{blue, yellow\}, \{red, yellow\}$

Now every set is a subset of itself.

- $\{blue, red, yellow\}$

Don't forget that the empty set is a subset of every set (even though we don't see it listed).

- $\{\}$

So the set $A = \{blue, red, yellow\}$ has 8 subsets total!

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1.2: Union, Intersection, and Complement

Commonly sets interact. For example, you and a new roommate decide to have a house party, and you both invite your circle of friends. At this party, two sets are being combined, though it might turn out that there are some friends that were in both sets.

Union, Interection, and Complement

The **union** of two sets contains all the elements contained in either set (or both sets).

The union is notated $A \cup B$

More formally, $x \in A \cup B$ if $x \in A$ or $x \in B$ (or both)

The **intersection** of two sets contains only the elements that are in both sets.

The intersection is notated $A \cap B$

More formally, $x \in A \cap B$ if $x \in A$ and $x \in B$

The **complement** of a set A contains everything that is *not* in the set A.

The complement is notated A, or A^c, or sometimes $\sim A$.

Example 5

Consider the sets:

$A = \{ \text{red, green, blue} \}$ $B = \{ \text{red, yellow, orange} \}$ $C = \{ \text{red, orange, yellow, green, blue, purple} \}$

a. Find $A \cup B$
b. Find $A \cap B$
c. Find $A^c \cap C$

Solution

a) The union contains all the elements in either set: $A \cup B = \{ \text{red, green, blue, yellow, orange} \}$

Notice we only list red once.

b) The intersection contains all the elements in both sets: $A \cap B = \{ \text{red} \}$

c) Here we're looking for all the elements that are not in set A and are also in C.

$A^c \cap C = \{ \text{orange, yellow, purple} \}$

Try it Now 2

Using the sets from the previous example, find $A \cup C$ and $B^c \cap A$

Answer

$A \cup C = \{ \text{red, orange, yellow, green, blue purple} \}$

$B^c \cap A = \{ \text{green, blue} \}$

Notice that in the example above, it would be hard to just ask for A^c, since everything from the color fuchsia to puppies and peanut butter are included in the complement of the set. For this reason, complements are usually only used with intersections, or when we have a universal set in place.

Universal Set

A **universal set** is a set that contains all the elements we are interested in. This would have to be defined by the context.

A complement is relative to the universal set, so A^C contains all the elements in the universal set that are not in A.

Example 6

a. If we were discussing searching for books, the universal set might be all the books in the library.
b. If we were grouping your Facebook friends, the universal set would be all your Facebook friends.
c. If you were working with sets of numbers, the universal set might be all whole numbers, all integers, or all real numbers

Example 7

Suppose the universal set is $U =$ all whole numbers from 1 to 9. If $A = \{1, 2, 4\}$, then

$A^c = \{3, 5, 6, 7, 8, 9\}$

As we saw earlier with the expression $A^c \cap C$, set operations can be grouped together. Grouping symbols can be used like they are with arithmetic - to force an order of operations.

Example 8

Suppose

$H = \{$ cat, dog, rabbit, mouse $\}, F = \{$ dog, cow, duck, pig, rabbit $\}$ $W = \{$ duck, rabbit, deer, frog, mouse $\}$

a. Find $(H \cap P) \cup W$
b. Find $H \cap (F \cup W)$
c. Find $(H \cap P) \cap W$

Solution

a) We start with the intersection: $H \cap F = \{$ dog, rabbit $\}$

Now we union that result with W : $(H \cap F) \cup W = \{$dog, duck, rabbit, deer, frog, mouse $\}$

b) We start with the union: $F \cup W = \{$dog, cow, rabbit, duck, pig, deer, frog, mouse $\}$

Now we intersect that result with H : $H \cap (F \cup W) = \{$ dog, rabbit, mouse $\}$

c) We start with the intersection: $H \cap F = \{$dog, rabbit $\}$

Now we want to find the elements of W that are not in $\mathrm{H} \cap F$

$(H \cap P)^c \cap W = \{$ duck, deer, frog, mouse $\}$

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1.3: Venn Diagrams

To visualize the interaction of sets, John Venn in 1880 thought to use overlapping circles, building on a similar idea used by Leonhard Euler in the 18th century. These illustrations now called **Venn Diagrams**.

Venn Diagram

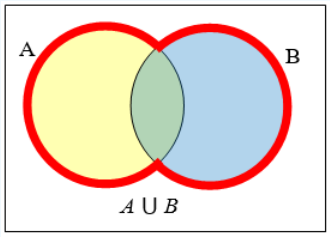
A Venn diagram represents each set by a circle, usually drawn inside of a containing box representing the universal set. Overlapping areas indicate elements common to both sets.

Basic Venn diagrams can illustrate the interaction of two or three sets.

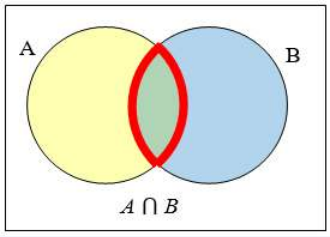
Example 9

Create Venn diagrams to illustrate $A \cup B$, $A \cap B$, and $A^c \cap B$

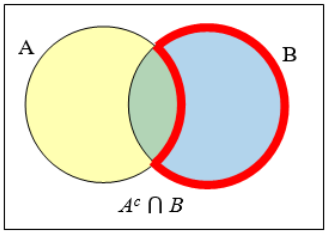
$A \cup B$ contains all elements in either set.



$A \cup B$ contains all elements in either set.



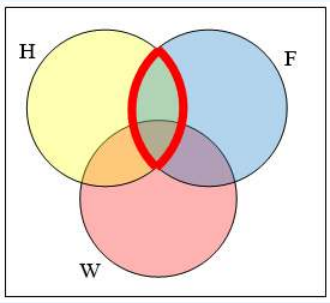
$A \cap B$ contains only those elements in both sets - in the overlap of the circles.



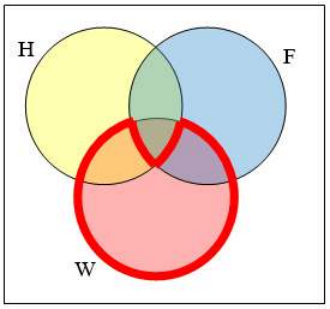
Example 10

Use a Venn diagram to illustrate $(H \cap P)^c \cap W$

We'll start by identifying everything in the set $H \cap P$



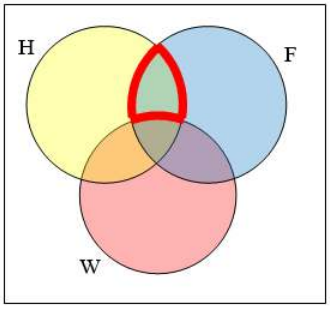
Now, $(H \cap P)^c \cap W$ will contain everything not in the set identified above that is also in set W



Example 11

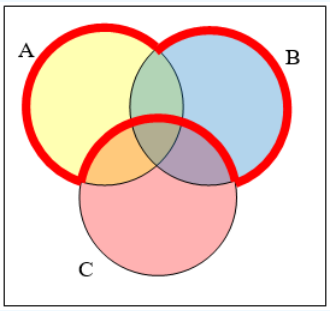
Create an expression to represent the outlined part of the Venn diagram shown.

The elements in the outlined set are in sets H and F, but are not in set W. So we could represent this set as $H \cap F \cap W$



Try it Now 3

Create an expression to represent the outlined portion of the Venn diagram shown



Answer

$A \cup B \cap C^c$

Page Error: The P in example 10 above should be an F to match the image of the Venn Diagrams.

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1.4: Cardinality

Often times we are interested in the number of items in a set or subset. This is called the cardinality of the set.

Cardinality

The number of elements in a set is the cardinality of that set.

The cardinality of the set A is often notated as $|A|$ or $n(A)$

Example 12

Let $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{2, 4, 6, 8\}$

What is the cardinality of B? $A \cup B$, $A \cap B$?

Solution

The cardinality of B is 4, since there are 4 elements in the set.

The cardinality of $A \cup B$ is 7, since $A \cup B = \{1, 2, 3, 4, 5, 6, 8\}$, which contains 7 elements.

The cardinality of $A \cap B$ is 3 , since $A \cap B = \{2, 4, 6\}$, which contains 3 elements.

Example 13

What is the cardinality of $P =$ the set of English names for the months of the year?

Solution

The cardinality of this set is 12, since there are 12 months in the year.

Sometimes we may be interested in the cardinality of the union or intersection of sets, but not know the actual elements of each set. This is common in surveying.

Example 14

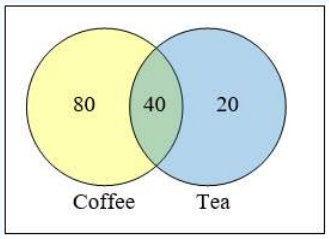
A survey asks 200 people "What beverage do you drink in the morning", and offers choices:

- Tea only
- Coffee only
- Both coffee and tea

Suppose 20 report tea only, 80 report coffee only, 40 report both. How many people drink tea in the morning? How many people drink neither tea or coffee?

Solution

This question can most easily be answered by creating a Venn diagram. We can see that we can find the people who drink tea by adding those who drink only tea to those who drink both: 60 people.



We can also see that those who drink neither are those not contained in the any of the three other groupings, so we can count those by subtracting from the cardinality of the universal set, 200.

$200 - 20 - 80 - 40 = 60$ people who drink neither.

Example 15

A survey asks: Which online services have you used in the last month:

- Twitter
- Facebook
- Have used both

The results show 40% of those surveyed have used Twitter, 70% have used Facebook, and 20% have used both. How many people have used neither Twitter or Facebook?

Solution

Let T be the set of all people who have used Twitter, and F be the set of all people who have used Facebook. Notice that while the cardinality of F is 70% and the cardinality of T is 40%, the cardinality of $F \cup T$ is not simply $70\% + 40\%$, since that would count those who use both services twice. To find the cardinality of $F \cup T$, we can add the cardinality of F and the cardinality of T, then subtract those in intersection that we've counted twice. In symbols,

$\mathrm{n}(F \cup T) = \mathrm{n}(F) + \mathrm{n}(T) - \mathrm{n}(F \cap T)$

$\mathrm{n}(F \cup T) = 70\% + 40\% - 20\% = 90\%$

Now, to find how many people have not used either service, we're looking for the cardinality of $(F \cup T)^c$. since the universal set contains 100% of people and the cardinality of $F \cup T = 90\%$, the cardinality of $(F \cup 7)^c$ must be the other 10%

The previous example illustrated two important properties

Cardinality properties

$\mathrm{n}(A \cup B) = \mathrm{n}(A) + \mathrm{n}(B) - \mathrm{n}(A \cap B)$

$n\left(A^{\circ}\right) = n(U) - n(A)$

Notice that the first property can also be written in an equivalent form by solving for the cardinality of the intersection:

$\mathrm{n}(A \cap B) = \mathrm{n}(A) + \mathrm{n}(B) - \mathrm{n}(A \cup B)$

Example 16

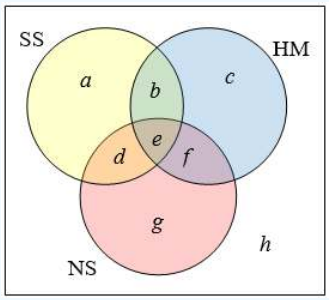
Add text here.Fifty students were surveyed, and asked if they were taking a social science (SS), humanities (HM) or a natural science (NS) course the next quarter.

21 were taking a SS course	26 were taking a HM course
19 were taking a NS course	9 were taking SS and HM
7 were taking SS and NS	10 were taking HM and NS
3 were taking all three	7 were taking none

How many students are only taking a SS course?

Solution

It might help to look at a Venn diagram.



From the given data, we know that there are 3 students in region e and 7 students in region h

since 7 students were taking a SS and NS course, we know that $n(d) + n(e) = 7$. since we know there are 3 students in region 3 , there must be
$7 - 3 = 4$ students in region d

Similarly, since there are 10 students taking $\mathbf{HM}$ and $\mathbf{NS}$, which includes regions e and f, there must be

$10 - 3 = 7$ students in region f

Since 9 students were taking **SS** and **HM**, there must be $9-3=6$ students in region b

Now, we know that 21 students were taking a SS course. This includes students from regions a, b, d, and e. since we know the number of students in all but region a, we can determine that $21-6-4-3=8$ students are in region a

8 students are taking only a SS course.

Try it Now 4

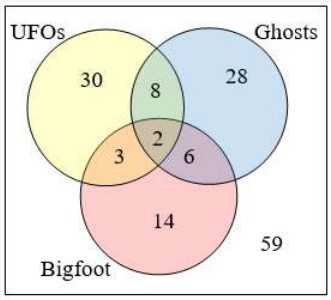
One hundred fifty people were surveyed and asked if they believed in UFOs, ghosts, and Bigfoot.

43 believed in UFOs
44 believed in ghosts
25 believed in Bigfoot
10 believed in UFOs and ghosts
8 believed in ghosts and Bigfoot
5 believed in UFOs and Bigfoot
2 believed in all three

How many people surveyed believed in at least one of these things?

Answer

Starting with the intersection of all three circles, we work our way out. since 10 people believe in UFOs and Ghosts, and 2 believe in all three, that leaves 8 that believe in only UFOs and Ghosts. We work our way out, filling in all the regions. Once we have, we can add up all those regions, getting 91 people in the union of all three sets. This leaves $150-91=59$ who believe in none.



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1.5: Exercises

1. List out the elements of the set “The letters of the word Mississipi”

2. List out the elements of the set “Months of the year”

3. Write a verbal description of the set $\{3, 6, 9\}$

4. Write a verbal description of the set $\{a, i, e, 0, u\}$

5. Is $\{1, 3, 5\}$ a subset of the set of odd integers?

6. Is $\{A, B, C\}$a subset of the set of letters of the alphabet?

For problems 7-12, consider the sets below, and indicate if each statement is true or false.

$A = \{1, 2, 3, 4, 5\}$ $B = \{1, 3, 5\}$ $C = \{4, 6\}$ $U = \{$ numbers from 0 to 10$\}$

7. $3 \in B$

8. $5 \in C$

9. $B \subset A$

10. $C \subset A$

11. $C \subset B$

12. $C \subset U$

Using the sets from above, and treating U as the Universal set, find each of the following:

13. $A \cup B$

14. $A \cup C$

15. $A \cap C$

16. $B \cap C$

17. A^c

18. B^c

Let $D = \{b, a, c, k\}$, $E = \{t, a, s, k\}$, $F = \{b, a, t, h\}$. Using these sets, find the following:

19. $D^c \cap E$

20. $F^c \cap D$

21. $(D \cap E) \cup F$

22. $D \cap (E \cup P)$

23. $(F \cap E)^c \cap D$

24. $(D \cup E)^c \cap F$

Create a Venn diagram to illustrate each of the following:

25. $(F \cap E) \cup D$

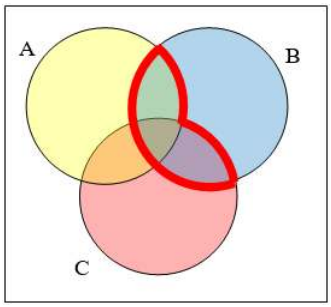
26. $(D \cup E)^c \cap F$

27. $(F^c \cap E') \cap D$

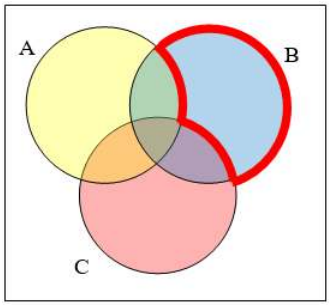
28. $(D \cup E) \cup F$

Write an expression for the shaded region.

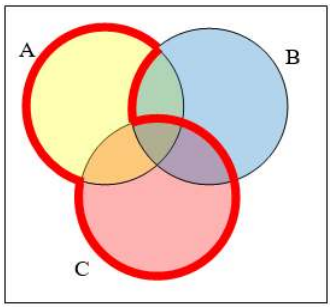
29.



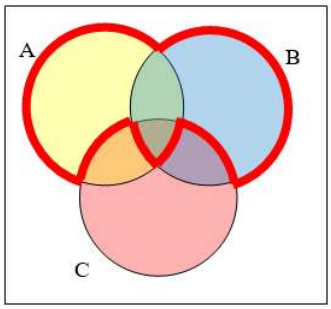
30.



31.



32.



Let $A = \{1, 2, 3, 4, 5\}$ $B = \{1, 3, 5\}$ $C = \{4, 6\}$. Find the cardinality of the given set.

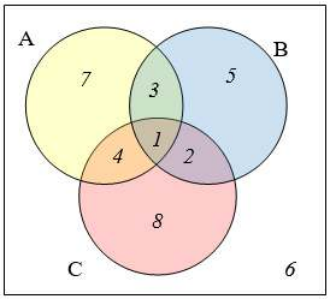
33. $\mathrm{n}(A)$

34. $\mathrm{n}(B)$

35. $\mathrm{n}(A \cup C)$

36. $\mathrm{n}(A \cap C)$

The Venn diagram here shows the cardinality of each set. Use this in 37-40 to find the cardinality of given set.



37. $\mathbf{n}(A \cap C)$

38. $\mathbf{n}(B \cup C)$

39. $\mathbf{n}\left(A \cap B \cap C^2\right)$

40. $\mathbf{n}\left(A \cap B^c \cap C\right)$

41. If $n(G) = 20, n(H) = 30, n(G \cap H) = 5$, find $n(G \cup H)$

42. If $n(G) = 5, n(H) = 8, n(G \cap H) = 4$, find $n(G \cup H)$

43. A survey was given asking whether they watch movies at home from Netflix, Redbox, or a video store. Use the results to determine how many people use Redbox.

52 only use Netflix	**62 only use Redbox**
24 only use a video store	**16 use only a video store and Redbox**
48 use only Netflix and Redbox	**30 use only a video store and Netflix**
10 use all three	**25 use none of these**

44. A survey asked buyers whether color, size, or brand influenced their choice of cell phone. The results are below. How many people were influenced by brand?

5 only said color	**8 only said size**
16 only said brand	**20 said only color and size**
42 said only color and brand	**53 said only size and brand**
102 said all three	**20 said none of these**

45. Use the given information to complete a Venn diagram, then determine: a) how many students have seen exactly one of these movies, and b) how many had seen only *Star Wars*.

18 had seen The Matrix (M)	**24 had seen Star Wars (SW)**
20 had seen Lord of the Rings (LotR)	**10 had seen M and SW**
14 had seen LotR and SW	**12 had seen M and LotR**
6 had seen all three	

46. A survey asked people what alternative transportation modes they use. Using the data to complete a Venn diagram, then determine: a) what percent of people only ride the bus, and b) how many people don't use any alternate transportation.

30 **20**
25 **5**
10 **12**
2

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CHAPTER OVERVIEW

2: Logic

2.1: Statements- Compound, Conditional and Biconditional

2.2: Truth Tables- Conjunction (and), Disjunction (or), Negation (not)

2.3: Truth Tables- Conditional, Biconditional

2.4: Verifying Arguments

2.4.1: Statements, Connectives, and Quantifiers

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2.1: Statements- Compound, Conditional and Biconditional

Statements

Definition: Statement

A **statement** is in logic is a declarative sentence that is either true or false. We represent statements by lowercase letters such as *p*, *q*, or *r*.

Examples of statements:

1. Guillermo has the flu.
2. Either George Washington or Abraham Lincoln was the first president of the United States
3. If Paul eats pizza, then Paul will drink soda.

Examples of Non-statements:

1. Go clean your room. (This is not declaring a truth or false)
2. Why are you here? (This is a question)
3. This sentence is false. (This is a paradox. It cannot be either true or false. If we assume that this sentence is true, then we must conclude that it is false. If we assume that it is false, then we must conclude that it is true.)

Definition: Connective and Compound statement

A **connective** on a statement is a word or combination of words that combines one or more statements to make a new mathematical statement.

A **compound statement** is a statement that contains one or more connectives. Because these connectives are used so frequently in logic, we give them names and use special symbols to represent them.

Definition: Connectives, Logical Operators

A **negation** expresses the word "not" and uses the symbol $\sim$: "not p" is notated $\sim p$.

> *Law of double negation* ~(~A)=A
>
> Two negations result in the elimination of the ~ symbol.
>
> It is sort of like in mathematics when we see two negatives, 2-(-2)=2+2=4.
>
> Example in words: "I'm **not not** going to the party" means "I am going to the party."

A **conjunction** expresses the word "and" between two statements and uses the symbol $\wedge$: p and q is notated $p \wedge q$.

A **disjunction** expresses the word "or" between two statements and uses the symbol $\vee$: p or q is notated $p \vee q$.

A **conditional** statement is of the form "If ... then ..." and uses the symbol $\rightarrow$ $\rightarrow$: If p, then q is notated $p \rightarrow q$.

A **biconditional** statement expresses "if and only if" between two statements and uses the symbol $\leftrightarrow$: p if and only if q is notated $p \leftrightarrow q$.

You can remember the first two symbols by relating them to the shapes for the union and intersection. $A \wedge B$ would be the elements that exist in both sets, as in in $A \cap B$. Likewise, $A \vee B$ would be the elements that exist in either set (either or), like in $A \cup B$ "A union B". When we are working with sets, we use the rounded version of the symbols; when we are working with statements, we use the pointy version.

Example

Translate each statement into symbolic notation. Let P represent "I like Pepsi" and let C represent "I like Coke."

1. I like Pepsi or I like Coke.
2. I like Pepsi and I like Coke.
3. I do not like Pepsi.
4. It is not the case that I like Pepsi or Coke.
5. I like Pepsi and I do not like Coke.

Solution

1. $P \vee C$
2. $P \wedge C$
3. $\sim P$
4. $\sim (P \vee C)$
5. $P \wedge \sim C$

As you can see, we can use parentheses to organize more complicated statements.

Exercise: You Try

Translate “We have carrots or we will not make soup” into symbols. Let C represent “we have carrots” and let S represent “we will make soup”.

Answer

$C \vee \sim S$

More About Conditional Statements.

Let's now focus on **conditional statements**. Conditional Statements use "If..., then..."

The "If" is associated with the **antecedent** (the first part or condition part of the statement), and the "then" part is called the **consequent** (the part that follows).

- If it is raining, then I have an umbrella.
- If I am cold, then I wear a jacket.
- If the sky is red, then I eat pizza :)

Definition: Inverse, Converse, Contapositive

For the conditional statement "if p, then q" notated $p \rightarrow q$:

The **inverse** of a conditional statement is the negation of both the antecedent and the consequent.

The inverse of "if p, then q" is "if not p, then not q".

Notation for the inverse of a conditional statement: $\sim p \rightarrow \sim q$.

The **converse** of a conditional statement switches the order of the antecedent and the consequent.

The converse of " if p, then q" is "if q, then p."

Notation for the converse is $q \rightarrow p$.

The **contrapositive** of a conditional statement both negates and switches the antecedent and consequent.

The contrapositive of "if p, then q" is "if not q, then not p."

The notation for the contrapositive is $\sim q \rightarrow \sim p$.

Example

Write the inverse, converse and contrapositive for the statement below.

If it is raining, then the grass is wet.

Solution

Inverse: If it is *not* raining, then the grass is *not* wet.

Converse: If the grass is wet, then it is raining.

Contrapositive: If the grass is not wet, then it is not raining.

Example: You try

If the cherries are red, then they are ready to be picked.

Solution

Inverse: If the cherries are not red, then they are not ready to be picked.

Converse: If the cherries are ready to be picked, then the cherries are red.

Contrapositive: If the cherries are not ready to be picked, then the cherries are not red.

Definition: Biconditional Statement

A **biconditional** statement reads "p if and only if q," denoted $p \iff q$. It is the combination of conditional states in two directions, "p then q" and "q then p."

$p \to q$ AND $q \to p$ results in $p \iff q$.

Negation of Compound Statements

Demorgan's Law: Negation of compound statements

We use **DeMorgan's Law** to negate a compound statement, one with "and" or "or."

Negate a conjunction (and).

$\sim (p \wedge q) \equiv \sim p \vee \sim q$

Negate a disjunction (or).

$\sim (p \vee q) \equiv \sim p \wedge \sim q$

Example: DeMorgan's Law

For Valentine's Day, you did not get your sweetie flowers or candy: Which of the following statements is logically equivalent?

1. You did not get them flowers or did not get them candy.
2. You did not get them flowers and did not get them candy.
3. You got them flowers or got them candy.

Solution

1. This statement does not go far enough; it leaves open the possibility that you got them one of the two things.
2. This statement is equivalent to the original; $\sim (f \vee c)$ is equivalent to $\sim f \wedge \sim c$.
3. This statement says that you got them something, but we know that you did not.

Exercise: You try

Negate the statement "John went to dinner and a concert."

Answer

Let's call "John went to dinner" d and "John went to a concert" c.

So the statement above can be written d∧c.

The negation is ~(d∧c) ≡ ~d∨~c which reads "**John did not go to dinner or he did not go to the concert**." This statement is the negation of "John went to dinner and a concert."

Negation of the Conditional

To negate a conditional statement we use the following equivalence.

The negation of a conditional statement is logically equivalent to a conjunction of the antecedent and the negation of the consequent.

$\sim(p \rightarrow q)$ is equivalent to $p \wedge \sim q$

Example : Negation of conditional

Which of the following statements is equivalent to the negation of “If you don’t grease the pan, then the food will stick to it” ?

a. I didn’t grease the pan and the food didn’t stick to it.

b. I didn’t grease the pan and the food stuck to it.

c. I greased the pan and the food didn’t stick to it.

Solution

a. This is correct; it is the conjunction of the antecedent and the negation of the consequent. To disprove that not greasing the pan will cause the food to stick, I have to not grease the pan and have the food not stick.

b. This is essentially the original statement with no negation; the “if…then” has been replaced by “and”.

c. This essentially agrees with the original statement and cannot disprove it.

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2.2: Truth Tables- Conjunction (and), Disjunction (or), Negation (not)

Because complex statements can get tricky to think about, we can create a **truth table** to keep track of what truth values for the simple statements make the complex statement true and false.

Truth Table

A table showing what the resulting truth value of a complex statement is for all the possible truth values for the simple statements.

Example 1

Suppose you're picking out a new couch, and your significant other says "get a sectional *or* something with a chaise".

This is a complex statement made of two simpler conditions: "is a sectional", and "has a chaise". For simplicity, let's use p to designate "is a sectional", and q to designate "has a chaise".

A truth table for this situation would look like this:

p	q	p or q
T	T	T
T	F	T
F	T	T
F	F	F

In the table, T is used for true, and F for false. In the first row, if p is true and q is also true, then the complex statement "p or *q*" is true. This would be a sectional that also has a chaise, which meets our desire. (Remember that *or* in logic is not exclusive; if the couch has both features, it meets the condition.)

In the previous example about the couch, the truth table was really just summarizing what we already know about how the *or* statement work. The truth tables for the basic *and*, *or*, and *not* statements are shown below.

Basic Truth Tables

Negation - Expresses "not" which means the opposite truth value.

p	$\sim p$
T	F
F	T

Conjunction - Expresses "and" which means both p and q must be true.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction - Expresses "or" which means either p or q can be true or both.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Truth tables really become useful when we analyze more complex statements.

Example 2

Create a truth table for the statement $p \vee \sim q$

Solution

When we create the truth table, we need to list all the possible truth value combinations for p and q. Notice how the first column contains 2 Ts followed by 2 Fs, and the second column alternates T, F, T, F. This pattern ensures that all 4 combinations are considered.

p	q
T	T
T	F
F	T
F	F

After creating columns with those initial values, we create a third column for the expression $\sim q$. Now we will temporarily ignore the column for p and write the truth values for $\sim q$

p	q	$\sim q$
T	T	F
T	F	T
F	T	F
F	F	T

Next we can find the truth values of $p \vee \sim q$, using the first and third columns.

p	q	$\sim q$	$p \vee \sim q$
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

The truth table shows that $p \vee \sim q$ is true in three cases and false in one case. If you're wondering what the point of this is, suppose it is the last day of the baseball season and two teams, who are not playing each other, are competing for the final playoff spot. Anaheim will make the playoffs if it wins its game or if Boston does not win its game. (Anaheim owns the tie-breaker; if both teams win, or if both teams lose, then Anaheim gets the playoff spot.) If $p =$ Anaheim wins its game and $q =$ Boston wins its game, then $p \vee \sim q$ represents the situation "Anaheim wins its game or Boston does not win its game". The truth table shows us the different scenarios related to Anaheim making the playoffs. In the first row, Anaheim wins its game and Boston wins its game, so it is true that Anaheim makes the playoffs. In the second row, Anaheim wins and Boston does not win, so it is true that Anaheim makes the playoffs. In the third row, Anaheim does not win its game and Boston wins its game, so it is false that Anaheim makes the playoffs. In the fourth row, Anaheim does not win and Boston does not win, so it is true that Anaheim makes the playoffs.

Try it Now 1

Create a truth table for this statement: $\sim p \wedge q$

Answer

p	q	$\sim p$	$\sim p \wedge q$
T	T	F	F
T	F	F	F
F	T	T	T
F	F	T	F

Example 3

Create a truth table for the statement $p \wedge \sim (q \vee r)$

Solution

It helps to work from the inside out when creating a truth table, and to create columns in the table for intermediate operations. We start by listing all the possible truth value combinations for p, q, and r. Notice how the first column contains 4 Ts followed by **4Fs**, the second column contains **2Ts, 2Fs**, then repeats, and the last column alternates **T, F, T, F** ... This pattern ensures that all 8 combinations are considered. After creating columns with those initial values, we create a fourth column for the innermost expression, $q \vee r$. Now we will temporarily ignore the column for p and focus on q and r, writing the truth values for $q \vee r$

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

p	q	r	$q \vee r$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	F

Next we can find the negation of $q \vee r$, working off the $q \vee r$ Column we just created. (Ignore the first three columns and simply negate the values in the $q \vee r$ column.)

p	q	r	$q \vee r$	$\sim (q \vee r)$
T	T	T	T	F
T	T	F	T	F
T	F	T	T	F
T	F	F	F	T
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	T

Finally, we find the values of p and $\sim (q \vee r)$. (Ignore the second, third, and fourth columns.)

p	q	r	$q \vee r$	$\sim (q \vee r)$	$p \wedge \sim (q \vee r)$
T	T	T	T	F	F
T	T	F	T	F	F
T	F	T	T	F	F
T	F	F	F	T	T
F	T	T	T	F	F
F	T	F	T	F	F
F	F	T	T	F	F
F	F	F	F	T	F

It turns out that this complex expression is true in only one case: when p is true, q is false, and r is false. To illustrate this situation, suppose that Anaheim will make the playoffs if: (1) Anaheim wins, and (2) neither Boston nor Cleveland wins. TFF is the only scenario in which Anaheim will make the playoffs.

Try it Now 2

Create a truth table for this statement: $(\sim p \wedge q) \vee \sim q$

Answer

p	q	$\sim p$	$\sim p \wedge q$	$\sim q$	$(\sim p \wedge q) \vee \sim q$
T	T	F	F	F	F
T	F	F	F	T	T
F	T	T	T	F	T
F	F	T	F	T	T

You have probably noticed that different truth tables have different numbers of rows, depending on how many variables (or simple statements) that we have. When there is only one simple statement (such as our truth table for negation), we have two rows, one for true and another for false. When we have two simple statements, p and q, there are four rows in the truth table. For Example 3, we had three simple statements, p, q and r, with eight rows in the truth table. So for each new simple statement, the number of rows doubles. This pattern continues, so we can generalize it as follows.

Definition: The Number of Lines in a Truth Table

A statement with k variables - or simple statements - will have a truth table with 2^k rows.

Logically Equivalent

Two statements are **logically equivalent** if they have the same simple statements and when their truth tables are computed, the final columns in the tables are identical.

DeMorgan's Laws

The negation of a conjunction is logically equivalent to the disjunction of the negation of the statements making up the conjunction. To negate an “and” statement, negate each part and change the “and” to “or”.

$$\sim (p \wedge q) \text{ is logically equivalent to } \sim p \vee \sim q$$

The negation of a disjunction is logically equivalent to the conjunction of the negation of the statements making up the disjunction. To negate an “or” statement, negate each part and change the “or” to “and”.

$$\sim (p \vee q) \text{ is logically equivalent to } \sim p \wedge \sim q$$

Example 4

For Valentine’s Day, you did not get your sweetie flowers or candy: Which of the following statements is logically equivalent?

a. You did not get them flowers or did not get them candy.
b. You did not get them flowers and did not get them candy.
c. You got them flowers or got them candy.

Solution

a. This statement does not go far enough; it leaves open the possibility that you got them one of the two things.
b. This statement is equivalent to the original; $\sim (f \vee c)$ is equivalent to $\sim f \wedge \sim c$
c. This statement says that you got them something, but we know that you did not.

? Try it Now 3

To serve as the President of the US, a person must have been born in the US, must be at least 35 years old, and must have lived in the US for at least 14 years. What minimum set of conditions would disqualify someone from serving as President?

Answer

Failing to meet just one of the three conditions is all it takes to be disqualified. A person is disqualified if they were not born in the US, or are not at least 35 years old, or have not lived in the US for at least 14 years. The key word here is “or” instead of “and”.

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2.3: Truth Tables- Conditional, Biconditional

Conditional

A **conditional** is a logical compound statement in which a statement p, called the hypothesis, implies a statement q, called the conclusion.

A conditional is written as $p \rightarrow q$ and is translated as "if p, then q".

Example 1

The English statement “If it is raining, then there are clouds is the sky” is a conditional statement. It makes sense because if the hypothesis “it is raining” is true, then the conclusion “there are clouds in the sky” must also be true.

Notice that the statement tells us nothing of what to expect if it is not raining; there might be clouds in the sky, or there might not. If the hypothesis is false, then the conclusion becomes irrelevant.

Example 2

Suppose you order a team jersey online on Tuesday and want to receive it by Friday so you can wear it to Saturday’s game. The website says that if you pay for expedited shipping, you will receive the jersey by Friday. In what situation is the website telling a lie?

There are four possible outcomes:

1. You pay for expedited shipping and receive the jersey by Friday
2. You pay for expedited shipping and don’t receive the jersey by Friday
3. You don’t pay for expedited shipping and receive the jersey by Friday
4. You don’t pay for expedited shipping and don’t receive the jersey by Friday

Only one of these outcomes proves that the website was lying: the second outcome in which you pay for expedited shipping but don’t receive the jersey by Friday. The first outcome is exactly what was promised, so there’s no problem with that. The third outcome is not a lie because the website never said what would happen if you didn’t pay for expedited shipping; maybe the jersey would arrive by Friday whether you paid for expedited shipping or not. The fourth outcome is not a lie because, again, the website didn’t make any promises about when the jersey would arrive if you didn’t pay for expedited shipping.

It may seem strange that the third outcome in the previous example, in which the first part is false but the second part is true, is not a lie. Remember, though, that if the hypothesis is false, we cannot make any judgment about the conclusion. The website never said that paying for expedited shipping was the *only* way to receive the jersey by Friday.

Example 3

A friend tells you “If you upload that picture to Facebook, you’ll lose your job.” Under what conditions can you say that your friend was wrong?

There are four possible outcomes:

1. You upload the picture and lose your job
2. You upload the picture and don’t lose your job
3. You don’t upload the picture and lose your job
4. You don’t upload the picture and don’t lose your job

There is only one possible case in which you can say your friend was wrong: the second outcome in which you upload the picture but still keep your job. In the last two cases, your friend didn’t say anything about what would happen if you didn’t upload the picture, so you can’t say that their statement was wrong. Even if you didn’t upload the picture and lost your job

anyway, your friend never said that you were guaranteed to keep your job if you didn't upload the picture; you might lose your job for missing a shift or punching your boss instead.

Truth Table for the Conditional

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Again, if the hypothesis p is false, we cannot prove that the statement is a lie, so the result of the third and fourth rows is true.

Example 4

Construct a truth table for the statement $(m \wedge \sim p) \rightarrow r$

Solution

We start by constructing a truth table with 8 rows to cover all possible scenarios. Next, we can focus on the hypothesis, $m \wedge \sim p$.

m	p	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

m	p	r	$\sim p$
T	T	T	F
T	T	F	F
T	F	T	T
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	T

m	p	r	$\sim p$	$m \wedge \sim p$
T	T	T	F	F
T	T	F	F	F
T	F	T	T	T
T	F	F	T	T
F	T	T	F	F
F	T	F	F	F
F	F	T	T	F
F	F	F	T	F

Now we can create a column for the conditional. Because it can be confusing to keep track of all the Ts and **F**s, why don't we copy the column for r to the right of the column for $m \wedge \sim p$? This makes it a lot easier to read the conditional from left to right.

m	p	r	$\sim p$	$m \wedge \sim p$	r	$(m \wedge \sim p) \rightarrow r$
T	T	T	F	F	T	T
T	T	F	F	F	F	T
T	F	T	T	T	T	T
T	F	F	T	T	F	F
F	T	T	F	F	T	T
F	T	F	F	F	F	T
F	F	T	T	F	T	T
F	F	F	T	F	F	T

When m is true, p is false, and r is false- -the fourth row of the table-then the hypothesis $m \wedge \sim p$ will be true but the conclusion false, resulting in an invalid conditional; every other case gives a valid conditional.

If you want a real-life situation that could be modeled by $(m \wedge \sim p) \rightarrow r$, consider this: let $m =$ we order meatballs, $p =$ we order pasta, and $r =$ Rob is happy. The statement $(m \wedge \sim p) \rightarrow r$ is "if we order meatballs and don't order pasta, then Rob is happy". If m is true (we order meatballs), p is false (we don't order pasta), and r is false (Rob is not happy), then the statement is false, because we satisfied the hypothesis but Rob did not satisfy the conclusion.

For any conditional, there are three related statements, the converse, the inverse, and the contrapositive.

Derived Forms of a Conditional

The original conditional is "if p, then q" $p \rightarrow q$

The converse is "if q, then p" $q \rightarrow p$

The inverse is "if not p, then not q" $\sim p \rightarrow \sim q$

The contrapositive is "if not q, then not p" $\sim q \rightarrow \sim p$

Example 5

Consider again the conditional “If it is raining, then there are clouds in the sky.” It seems reasonable to assume that this is true.

The converse would be “If there are clouds in the sky, then it is raining.” This is not always true.

The inverse would be “If it is not raining, then there are not clouds in the sky.” Likewise, this is not always true.

The contrapositive would be “If there are not clouds in the sky, then it is not raining.” This statement is true, and is equivalent to the original conditional.

Looking at truth tables, we can see that the original conditional and the contrapositive are logically equivalent, and that the converse and inverse are logically equivalent.

		Conditional	Converse	Inverse	Contrapositive
p	q	$p \rightarrow q$	$q \rightarrow p$	$\sim p \rightarrow \sim q$	$\sim q \rightarrow \sim p$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

Equivalent

Equivalence

A conditional statement and its contrapositive are logically equivalent.

The converse and inverse of a conditional statement are logically equivalent.

In other words, the original statement and the contrapositive must agree with each other; they must both be true, or they must both be false. Similarly, the converse and the inverse must agree with each other; they must both be true, or they must both be false.

We typically represent the conditional using the words, "if ..., then ...," but there are other ways this logical connective can be represented in English. Consider the conditional from Example 5: "If it is raining, then there are clouds in the sky." We could equivalently write, "It is raining *only if* there are clouds in the sky." You can probably imagine how these two statements are saying the same thing - whenever it's raining outside, it is a safe conclusion there are clouds in the sky as well. Some other wordings that communicate the same information use either "sufficient" or "necessary." For example, "Raining is a sufficient condition for it to be cloudy," and "Being cloudy is a necessary condition for it to be raining." Here is a table summarizing the different wordings.

Different Wordings of the Conditional

The following statements are equivalent:

- If p, then q.
- q only if p.
- p is sufficient for q.
- q is necessary for p.

In everyday life, we often have a stronger meaning in mind when we use a conditional statement. Consider “If you submit your hours today, then you will be paid next Friday.” What the payroll rep really means is “If you submit your hours today, then you will be paid next Friday, and if you don’t submit your hours today, then you won’t be paid next Friday.” The conditional statement if *t*, then *p* also includes the inverse of the statement: if not *t*, then not *p*. A more compact way to express this statement is “You will be paid next Friday *if and only if* you submit your timesheet today.” A statement of this form is called a **biconditional**.

Biconditional

A biconditional is a logical conditional statement in which the hypothesis and conclusion are interchangeable.

A biconditional is written as $p \leftrightarrow q$ and is translated as " p if and only if q".

Because a biconditional statement $p \leftrightarrow q$ is equivalent to $(p \rightarrow q) \wedge (q \rightarrow p)$, we may think of it as a conditional statement combined with its converse: if p, then q and if q, then p. The double-headed arrow shows that the conditional statement goes from left to right and from right to left. A biconditional is considered true as long as the hypothesis and the conclusion have the same truth value; that is, they are either both true or both false.

Truth Table for the Biconditional

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Notice that the fourth row, where both components are false, is true; if you don’t submit your timesheet and you don’t get paid, the person from payroll told you the truth.

Example 6

Suppose this statement is true: “The garbage truck comes down my street if and only if it is Thursday morning.” Which of the following statements could be true?

a. It is noon on Thursday and the garbage truck did not come down my street this morning.
b. It is Monday and the garbage truck is coming down my street.
c. It is Wednesday at 11:59PM and the garbage truck did not come down my street today.

Solution

a. This cannot be true. This is like the second row of the truth table; it is true that I just experienced Thursday morning, but it is false that the garbage truck came.
b. This cannot be true. This is like the third row of the truth table; it is false that it is Thursday, but it is true that the garbage truck came.
c. This could be true. This is like the fourth row of the truth table; it is false that it is Thursday, but it is also false that the garbage truck came, so everything worked out like it should.

Try it Now 1

Suppose this statement is true: "I wear my running shoes if and only if I am exercising." Determine whether each of the following statements must be true or false.

a. I am exercising and I am not wearing my running shoes.
b. I am wearing my running shoes and I am not exercising.
c. I am not exercising and I am not wearing my running shoes.

Answer

Choices a & b are false; c is true.

Example 7

Create a truth table for the statement $(A \vee B) \leftrightarrow \sim C$

Solution

Whenever we have three component statements, we start by listing all the possible truth value combinations for A, B, and C. After creating those three columns, we can create a fourth column for the hypothesis, $A \vee B$. Now we will temporarily ignore the column for C and focus on A and B, writing the truth values for $A \vee B$.

A	B	C
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

A	B	C	$A \vee B$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	T
F	F	T	F
F	F	F	F

Next we can create a column for the negation of C. (Ignore the $A \vee B$ column and simply negate the values in the C column.)

A	B	C	$A \vee B$	$\sim C$
T	T	T	T	F
T	T	F	T	T
T	F	T	T	F
T	F	F	T	T
F	T	T	T	F
F	T	F	T	T
F	F	T	F	F
F	F	F	F	T

Finally, we find the truth values of $(A \vee B) \leftrightarrow \sim C$. Remember, a biconditional is true when the truth value of the two parts match, but it is false when the truth values do not match.

A	B	C	$A \vee B$	$\sim C$	$(A \vee B) \leftrightarrow \sim C$
T	T	T	T	F	F
T	T	F	T	T	T
T	F	T	T	F	F
T	F	F	T	T	T
F	T	T	T	F	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	F	T	F

To illustrate this situation, suppose your boss needs you to do either project A or project B (or both, if you have the time). If you do one of the projects, you will not get a crummy review (C is for crummy). So $(A \vee B) \leftrightarrow \sim C$ means "You will not get a crummy review if and only if you do project A or project B." Looking at a few of the rows of the truth table, we can see how this works out. In the first row, $A, B,$ and C are all true: you did both projects and got a crummy review, which is not what your boss told you would happen! That is why the final result of the first row is false. In the fourth row, A is true, B is false, and C is false: you did project A and did not get a crummy review. This is what your boss said would happen, so the final result of this row is true. And in the eighth row, $A, B,$ and C are all false: you didn't do either project and did not get a crummy review. This is not what your boss said would happen, so the final result of this row is false. (Even though you may be happy that your boss didn't follow through on the threat, the truth table shows that your boss lied about what would happen.)

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2.4: Verifying Arguments

Arguments

An **argument** is a series of statements called **premises** followed by a single statement called the **conclusion**.

An argument is **valid** if whenever all the premises are true, then the conclusion must also be true. Otherwise the argument is **invalid**.

Arguments can also be analyzed using truth tables, although this can be a lot of work.

Analyzing Arguments Using Truth Tables

To analyze an argument with a truth table:

1. Make a truth table with a seperate column for each premise and the conclusion.
2. Examine *only* the lines in the table which all of the premises are true.
3. If the conclusion is also true for the lines you examined in step 2, the argument is valid.
4. If the conclusion is false for even one of the lines you examined in step 2, the argument is invalid.

Example 1

Consider the argument

Premise: **If you bought bread, then you went to the store.**
Premise: **You bought bread.**
Conclusion: **You went to the store.**

Solution

While this example is fairly obviously a valid argument, we can analyze it using a truth table by representing each of the premises symbolically.

We'll let b represent "you bought bread" and s represent "you went to the store". Then the argument becomes:

Premise: $b \to s$
Premise: b
Conclusion: s

To test the validity, we create a truth table and a separate column for both premises and the conclusion:

b	s	$b \to s$	b	s
T	T	T	T	T
T	F	F	T	
F	T	T	F	
F	F	T	F	

We only examine the first line since that is the only time both premises are true. We see that the conclusion is also true. This argument is valid.

Example 2

Determine whether the argument is valid:

Premise: **If I have a shovel, I can dig a hole.**
Premise: **I dug a hole.**
Conclusion: **Therefore, I had a shovel.**

Solution

Let $S =$ have a shovel, $D =$ **dig** a hole. The argument becomes:

Premise: $S \to D$
Premise: D
Conclusion: S

We can construct a truth table with a seperate column for each premise and the conclusion:

S	D	$S \to D$	D	S
T	T	T	T	T
T	F	F	F	
F	T	T	T	F
F	F	T	F	

We need to examine both Line 1 and Line 3 since both premises are true in both cases. Since the conclusion is false in Line 3, the argument is invalid.

Rather than making a truth table for every argument, we may be able to recognize certain common forms of arguments that are valid (or invalid). If we can determine that an argument fits one of the common forms, we can immediately state whether it is valid or invalid.

The Law of Detachment

The law of detachment is:

Premise: $p \to q$
Premise: p
Conclusion: q

Example 3

Recall this argument from an earlier example:

Premise: **If you bought bread, then you went to the store.**
Premise: **You bought bread.**
Conclusion: **You went to the store.**

In symbolic form:

Premise: $b \to s$
Premise: b
Conclusion: s

This argument has the structure described by the law of detachment. (The second premise and the conclusion are simply the two parts of the first premise detached from each other.) Instead of making a truth table, we can say that this argument is valid by stating that it satisfies the law of detachment.

The Law of Contraposition

The law of contraposition is:

Premise: $p \to q$
Premise: $\sim q$
Conclusion: $\sim p$

Notice that the second premise and the conclusion look like the contrapositive of the first premise, $\sim q \to \sim p$, but they have been detached. You can think of the law of contraposition as a combination of the law of detachment and the fact that the contrapositive is logically equivalent to the original statement.

Example 4

Premise: **If I drop my phone into the swimming pool, my phone will be ruined.**
Premise: **My phone isn't ruined.**
Conclusion: **I didn't drop my phone into the swimming pool.**

If we let $d =$ I drop the phone in the pool and $r =$ the phone is ruined, then we can represent the argument this way:

Premise: $d \to r$
Premise: $\sim r$
Conclusion: $\sim d$

The form of this argument matches what we need to invoke the law of contraposition, so it is a valid argument.

Try it Now 1

Is this argument valid?

Premise: **If you brushed your teeth before bed, then your toothbrush will be wet.**
Premise: **Your toothbrush is dry.**
Conclusion: **You didn't brush your teeth before bed.**

Answer

Let $b =$ brushed teeth and $w =$ toothbrush is wet.

Premise: $b \to w$
Premise: $\sim w$
Conclusion: $\sim b$

This argument is valid by the Law of Contraposition.

The Law of Syllogism

The Law of Syllogism is:

Premise: $p \to q$
Premise: $q \to r$
Conclusion: $p \to r$

The earlier example about buying a shirt at the mall is an example illustrating the transitive property. It describes a chain reaction: if the first thing happens, then the second thing happens, and if the second thing happens, then the third thing happens. Therefore, if we want to ignore the second thing, we can say that if the first thing happens, then we know the third thing will happen. We don't have to mention the part about buying jeans; we can simply say that the first event leads to the final event. We could even have more than two premises; as long as they form a chain reaction, the transitive property will give us a valid argument.

Example 5

Premise: **If a soccer player commits a reckless foul, she will receive a yellow card.**
Premise: **If Hayley receives a yellow card, she will be suspended for the next match.**
Conclusion: **If Hayley commits a reckless foul, she will be suspended for the next match.**

If we let $r =$ committing a reckless foul, $y =$ receiving a yellow card, and $s =$ being suspended, then our argument looks like this:

Premise: $r \to y$
Premise: $y \to s$
Conclusion: $r \to s$

This argument has the exact structure required to use the transitive property, so it is a valid argument.

Try it Now 2

Is this argument valid?

Premise: If the old lady swallows a fly, she will swallow a spider.
Premise: If the old lady swallows a spider, she will swallow a bird.
Premise: Premise: If the old lady swallows a bird, she will swallow a cat.
Premise: If the old lady swallows a cat, she will swallow a dog.
Premise: If the old lady swallows a dog, she will swallow a goat.
Premise: If the old lady swallows a goat, she will swallow a cow.
Premise: If the old lady swallows a cow, she will swallow a horse.
Premise: If the old lady swallows a horse, she will die, of course.
Conclusion: If the old lady swallows a fly, she will die, of course.

Answer

This argument is valid by the Law of Syllogism, which can involve more than two premises, as long as they continue the chain reaction. The premises $f \to s, s \to b, b \to c, c \to d\ d \to g, g \to w, w \to h, h \to x$ can be reduced to $f \to x$. (Because we had already used c and d we decided to use w for cow and x for death. If the old lady swallows the fly, she will eventually eat a horse and die.

Disjunctive Syllogism

Disjunctive syllogism is:

Premise: $p \vee q$
Premise: $\sim p$
Conclusion: q

The order of the two parts of the disjunction isn't important. In other words, we could have the premises $p \vee q$ and $\sim q$, and the conclusion p

Example 6

Premise: I can either drive or take the train.
Premise: I refuse to drive.
Conclusion: I will take the train.

If we let $d = I$ drive and $t = I$ take the train, then the symbolic representation of the argument is:

Premise: $d \vee t$
Premise: $\sim d$
Conclusion: t

This argument is valid because it has the form of a disjunctive syllogism. I have two choices, and one of them is not going to happen, so the other one must happen.

Try it Now 3

Is this argument valid?

Premise: Alison was required to write a 10-page paper or give a 5-minute speech.
Premise: Alison did not give a 5-minute speech.
Conclusion: Alison wrote a 10-page paper.

Answer

Let $p =$ wrote a paper and $s =$ gave a speech.

Premise: $p \vee s$
Premise: $\neg s$
Conclusion: p

This argument is valid by Disjunctive Syllogism. Alison had to do one or the other; she didn't choose the speech, so she must have chosen the paper.

Keep in mind that, when you are determining the validity of an argument, you must assume that the premises are true. If you don't agree with one of the premises, you need to keep your personal opinion out of it. Your job is to pretend that the premises are true and then determine whether they force you to accept the conclusion. You may attack the premises in a court of law or a political discussion, of course, but here we are focusing on the structure of the arguments, not the truth of what they actually say.

We have just looked at four forms of valid arguments; there are two common forms that represent *invalid* arguments, which are also called *fallacies*.

The Fallacy of the Converse

The fallacy of the converse is:

Premise: $p \to q$
Premise: q
Conclusion: p

Notice that the second premise and the conclusion look like the converse of the first premise, $q \to p$, but they have been detached. The fallacy of the converse incorrectly tries to assert that the converse of a statement is equivalent to that statement.

Example 7

Premise: **If I drink coffee after noon, then I have a hard time falling asleep that night.**
Premise: **I had a hard time falling asleep last night.**
Conclusion: **I drank coffee after noon yesterday.**

If we let $c =$ **I** drink coffee after noon and $h =$ **I** have a hard time falling asleep, then our argument looks like this:

Premise: $c \to h$
Premise: h
Conclusion: c

This argument uses converse reasoning, so it is an invalid argument. There could be plenty of other reasons why I couldn't fall asleep: I could be worried about money, my neighbors might have been setting off fireworks, …

Try it Now 4

Is this argument valid?

Premise: **If you pull that fire alarm, you will get in big trouble.**
Premise: **You got in big trouble.**
Conclusion: **You must have pulled the fire alarm.**

Answer

Let $f =$ pulled fire alarm and $t =$ got in big trouble.

Premise: $f \to t$
Premise: t
Conclusion: f

The Fallacy of the Inverse

The fallacy of the inverse is:

Premise: $p \to q$
Premise: $\sim p$
Conclusion: $\sim q$

Again, notice that the second premise and the conclusion look like the inverse of the first premise, $\sim p \to \sim q$, but they have been detached. The fallacy of the inverse incorrectly tries to assert that the inverse of a statement is equivalent to that statement.

Example 8

Premise: **If you listen to the Grateful Dead, then you are a hippie.**
Premise: **Sky doesn't listen to the Grateful Dead.**
Conclusion: **Sky is not a hippie.**

If we let $g =$ listen to the Grateful Dead and $h =$ is a hippie, then this is the argument:

Premise: $g \to h$
Premise: $\sim g$
Conclusion: $\sim h$

This argument is invalid because it uses inverse reasoning. The first premise does not imply that all hippies listen to the Grateful Dead; there could be some hippies who listen to Phish instead.

Try it Now 5

Is this argument valid?

Premise: **If a hockey player trips an opponent, he will be assessed a 2-minute penalty.**
Premise: **Alexei did not trip an opponent.**
Conclusion: **Alexei will not be assessed a 2-minute penalty.**

Answer

Let $t =$ tripped and $p =$ got a penalty.

Premise: $t \to p$
Premise: $\sim t$
Conclusion: $\sim p$

This argument is invalid because it has the form of the Fallacy of the Inverse. Alexei may have gotten a penalty for an infraction other than tripping.

Of course, arguments are not limited to these six basic forms; some arguments have more premises, or premises that need to be rearranged before you can see what is really happening. There are plenty of other forms of arguments that are valid or invalid. If an argument doesn't seem to fit the pattern of any of these common forms, though, you may want to use a truth table instead.

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2.4.1: Statements, Connectives, and Quantifiers

Statements

A **statement** in logic is a declarative sentence that is either true or false. We represent statements by lowercase letters such as *p*, *q*, or *r*.

Examples of statements:

1. Guillermo has the flu.
2. Either George Washington or Abraham Lincoln was the first president of the United States
3. If Paul eats pizza, then Paul will drink soda.

Examples of Non-statements:

1. Go clean your room. (This is not declaring a truth or false)
2. Why are you here? (This is a question)
3. This sentence is false. (This is a paradox. It cannot be either true or false. If we assume that this sentence is true, then we must conclude that it is false. If we assume that it is false, then we must conclude that it is true.)

It is possible to form new statements from existing statements by connecting the statements with words such as “and” and “or” or by negating the statement.

Compound Statements and Connectives

A **connective** on a statement is a word or combination of words that combines one or more statements to make a new mathematical statement.

A **compound statement** is a statement that contains one or more connectives. Because these connectives are used so frequently in logic, we give them names and use special symbols to represent them.

Connectives

A **negation** expresses the word "not" and uses the symbol $\sim$: not p is notated $\sim p$

A **conjunction** expresses the word "and" between two statements and uses the symbol $\wedge$: p and q is notated $p \wedge q$

A **disjunction** expresses the word "or" between two statements and uses the symbol $\vee$: p or q is notated $p \vee q$

A **conditional** statement is of the form "If ... then ..." and uses the symbol $\rightarrow$: If p, then q is notated $p \rightarrow q$

A **biconditional** statement expresses "if and only if" between two statements and uses the symbol $\leftrightarrow$: p if and only if q is notated $p \leftrightarrow q$

You can remember the first two symbols by relating them to the shapes for the union and intersection. $A \wedge B$ would be the elements that exist in both sets, in $A \cap B$. Likewise, $A \vee B$ would be the elements that exist in either set, in $A \cup B$. When we are working with sets, we use the rounded version of the symbols; when we are working with statements, we use the pointy version.

Example 1

Translate each statement into symbolic notation. Let P represent "I like Pepsi" and let C represent "I like Coke."

a. I like Pepsi or I like Coke.
b. I like Pepsi and I like Coke.
c. I do not like Pepsi.
d. It is not the case that I like Pepsi or Coke.
e. I like Pepsi and I do not like Coke.

Solution

a. $P \vee C$
b. $P \wedge C$
c. $\sim P$
d. $\sim (P \vee C)$
e. $P \wedge \sim C$

As you can see, we can use parentheses to organize more complicated statements.

Try it Now 1

Translate "We have carrots or we will not make soup" into symbols. Let C represent "we have carrots" and let S represent "we will make soup".

Answer

$C \vee \sim S$

Quantifiers

A **universal quantifier** states that an entire set of things share a characteristic. (all, every, or none)

An **existential quantifier** states that a set contains at least one element. (some, many, or at least one)

Something interesting happens when we **negate** – or state the opposite of – a quantified statement.

Example 2

Suppose your friend says "Everybody cheats on their taxes." What is the minimum amount of evidence you would need to prove your friend wrong?

To show that it is not true that everybody cheats on their taxes, all you need is one person who does not cheat on their taxes. It would be perfectly fine to produce more people who do not cheat, but one counterexample is all you need.

It is important to note that you do not need to show that absolutely nobody cheats on their taxes.

Example 3

Suppose your friend says "One of these six cartons of milk is leaking." What is the minimum amount of evidence you would need to prove your friend wrong?

Solution

In this case, you would need to check all six cartons and show that none of them is leaking. You cannot disprove your friend's statement by checking only one of the cartons.

When we negate a statement with a universal quantifier, we get a statement with an existential quantifier, and vice-versa.

Negating a Quantified Statement

The negation of "all A are B" is "at least one A is not B".

The negation of "no A are B" is "at least one A is B".

The negation of "at least one A is B" is "no A are B".

The negation of "at least one A is not B" is "all A are B".

Example 4

"Somebody brought a flashlight." Write the negation of this statement.

The negation is "Nobody brought a flashlight."

Example 5

"There are no prime numbers that are even." Write the negation of this statement.

The negation is "At least one prime number is even."

Try it Now 2

Write the negation of "All Icelandic children learn English in school."

Answer

At least one Icelandic child did not learn English in school.

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CHAPTER OVERVIEW

3: Probability

3.1: Introduction to Probability
3.2: Basic Concepts
3.3: Working with Events
3.4: Bayes Theorem
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3.1: Introduction to Probability

The probability of a specified event is the chance or likelihood that it will occur. There are several ways of viewing probability. One would be **experimental** in nature, where we repeatedly conduct an experiment. Suppose we flipped a coin over and over and over again and it came up heads about half of the time; we would expect that in the future whenever we flipped the coin it would turn up heads about half of the time. When a weather reporter says “there is a 10% chance of rain tomorrow,” she is basing that on prior evidence; that out of all days with similar weather patterns, it has rained on 1 out of 10 of those days.

Another view would be **subjective** in nature, in other words an educated guess. If someone asked you the probability that the Seattle Mariners would win their next baseball game, it would be impossible to conduct an experiment where the same two teams played each other repeatedly, each time with the same starting lineup and starting pitchers, each starting at the same time of day on the same field under the precisely the same conditions. Since there are so many variables to take into account, someone familiar with baseball and with the two teams involved might make an educated guess that there is a 75% chance they will win the game; that is, *if* the same two teams were to play each other repeatedly under identical conditions, the Mariners would win about three out of every four games. But this is just a guess, with no way to verify its accuracy, and depending upon how educated the educated guesser is, a subjective probability may not be worth very much.

We will return to the experimental and subjective probabilities from time to time, but in this course we will mostly be concerned with **theoretical** probability, which is defined as follows: Suppose there is a situation with n equally likely possible outcomes and that m of those n outcomes correspond to a particular event; then the **probability** of that event is defined as $\frac{m}{n}$.

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3.2: Basic Concepts

If you roll a die, pick a card from deck of playing cards, or randomly select a person and observe their hair color, we are executing an experiment or procedure. In probability, we look at the likelihood of different outcomes. We begin with some terminology.

Events and Outcomes

The result of an experiment is called an **outcome**.

An **event** is any particular outcome or group of outcomes.

A **simple event** is an event that cannot be broken down further

The **sample space** is the set of all possible simple events.

Example 1

If we roll a standard 6-sided die, describe the sample space and some simple events.

The sample space is the set of all possible simple events: $\{1, 2, 3, 4, 5, 6\}$

Some examples of simple events:

We roll a 1

We roll a 5

Some compound events:

We roll a number bigger than 4

We roll an even number



Two dice



One die

Basic Probability

Given that all outcomes are equally likely, we can compute the probability of an event E using this formula:

$$P(E) = \frac{\text{Number of outcomes corresponding to the event E}}{\text{Total number of equally - likely outcomes}}$$

Example 2

If we roll a 6-sided die, calculate

a. P(rolling a 1)
b. P(rolling a number bigger than 4)

Solution

Recall that the sample space is $\{1, 2, 3, 4, 5, 6\}$

a. There is one outcome corresponding to “rolling a 1”, so the probability is $\frac{1}{6}$
b. There are two outcomes bigger than a 4, so the probability is $\frac{2}{6} = \frac{1}{3}$

Probabilities are essentially fractions, and can be reduced to lower terms like fractions.

Example 3

Let's say you have a bag with 20 cherries, 14 sweet and 6 sour. If you pick a cherry at random, what is the probability that it will be sweet?

Solution

There are 20 possible cherries that could be picked, so the number of possible outcomes is 20. Of these 20 possible outcomes, 14 are favorable (sweet), so the probability that the cherry will be sweet is $\frac{14}{20} = \frac{7}{10}$.

There is one potential complication to this example, however. It must be assumed that the probability of picking any of the cherries is the same as the probability of picking any other. This wouldn't be true if (let us imagine) the sweet cherries are smaller than the sour ones. (The sour cherries would come to hand more readily when you sampled from the bag.) Let us keep in mind, therefore, that when we assess probabilities in terms of the ratio of favorable to all potential cases, we rely heavily on the assumption of equal probability for all outcomes.

? Try it Now 1

At some random moment, you look at your clock and note the minutes reading.

a. What is probability the minutes reading is 15?
b. What is the probability the minutes reading is 15 or less?

Answer

There are 60 possible readings, from 00 to 59.

a. $\frac{1}{60}$
b. $\frac{16}{60}$ (counting 00 through 15)

Cards

A standard deck of 52 playing cards consists of four **suits** (hearts, spades, diamonds and clubs). Spades and clubs are black while hearts and diamonds are red. Each suit contains 13 cards, each of a different **rank**: an Ace (which in many games functions as both a low card and a high card), cards numbered 2 through 10, a Jack, a Queen and a King.

✓ Example 4

Compute the probability of randomly drawing one card from a deck and getting an Ace.

Solution

There are 52 cards in the deck and 4 Aces so

$$P(Ace) = \frac{4}{52} = \frac{1}{13} \approx 0.0769$$

We can also think of probabilities as percents: There is a 7.69% chance that a randomly selected card will be an Ace.

Notice that the smallest possible probability is 0 – if there are no outcomes that correspond with the event. The largest possible probability is 1 – if all possible outcomes correspond with the event.

Certain and Impossible events

An **impossible event** has a probability of 0.

A **certain event** has a probability of 1.

The probability of any event must be $0 \le P(E) \le 1$.

In the course of this chapter, *if you compute a probability and get an answer that is negative or greater than 1, you have made a mistake and should check your work.*

Example 5

A fair coin is flipped 3 times. What is the probability that we get exactly two heads?

Solution

Let's first consider the sample space for this experiment. Below are the possible outcomes for flipping a fair coin 3 times.

{HHH, HTH, HHT, THH, TTH, HTH, HTT, TTT}

The number of outcomes with exactly two heads (H) is 3, and the total number of outcomes is 8.

Recall that basic probability is calculated as $P(E) = \frac{\text{frequency of event}}{\text{total number of outcomes}}$ So we have $P(\text{exactly two H}) = \frac{\text{number of exactly 2 head outcomes}}{\text{total number of outcomes}} = \frac{3}{8}$.

If we need to convert this number into a decimal or percent, we simply divide the numerator by the denominator (for the decimal) then multiply by 100 (for the percent). Calculating, we get $3 \div 8 = 0.375$ and $0.375 \times 100 = 37.5\%$. Notice that as is appropriate for all probability examples, the decimal number is between 0 and 1.

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3.3: Working with Events

Complementary Events

Now let us examine the probability that an event does **not** happen. As in the previous section, consider the situation of rolling a six-sided die and first compute the probability of rolling a six: the answer is $P(\text{six}) = \frac{1}{6}$. Now consider the probability that we do *not* roll a six: there are 5 outcomes that are not a six, so the answer is $P(\text{not a six}) = \frac{5}{6}$. Notice that

$$P(\text{six}) + P(\text{not a six}) = \frac{1}{6} + \frac{5}{6} = \frac{6}{6} = 1$$

This is not a coincidence. Consider a generic situation with n possible outcomes and an event E that corresponds to m of these outcomes. Then the remaining $n - m$ outcomes correspond to E not happening, thus

$$P(\text{not } E) = \frac{n-m}{n} = \frac{n}{n} - \frac{m}{n} = 1 - \frac{m}{n} = 1 - P(E)$$

Complement of an Event

The **complement** of an event is the event "E doesn't happen"

The notation $\bar{E}$ is used for the complement of event E.

We can compute the probability of the complement using $P(\bar{E}) = 1 - P(E)$

Notice also that $P(E) = 1 - P(\bar{E})$

Example 5

If you pull a random card from a deck of playing cards, what is the probability it is not a heart?

Solution

There are 13 hearts in the deck, so

$$P(\text{heart}) = \frac{13}{52} = \frac{1}{4}.$$

The probability of *not* drawing a heart is the complement:

$$P(\text{ not heart }) = 1 - P(\text{ heart }) = 1 - \frac{1}{4} = \frac{3}{4}$$

Probability of two independent events

Example 6

Suppose we flipped a coin and rolled a die, and wanted to know the probability of getting a head on the coin and a 6 on the die.

Solution

We could list all possible outcomes: $\{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}$.

Notice there are $2 \cdot 6 = 12$ total outcomes. Out of these, only 1 is the desired outcome, so the probability is $\frac{1}{12}$.

The prior example was looking at two independent events.

Independent Events

Events A and B are **independent events** if the probability of Event B occurring is the same whether or not Event A occurs.

Example 7

Are these events independent?

a. A fair coin is tossed two times. The two events are (1) first toss is a head and (2) second toss is a head.
b. The two events (1) "It will rain tomorrow in Houston" and (2) "It will rain tomorrow in Galveston" (a city near Houston).
c. You draw a card from a deck, then draw a second card without replacing the first.

Solution

a. The probability that a head comes up on the second toss is $\frac{1}{2}$ regardless of whether or not a head came up on the first toss, so these events are independent.
b. These events are not independent because it is more likely that it will rain in Galveston on days it rains in Houston than on days it does not.
c. The probability of the second card being red depends on whether the first card is red or not, so these events are not independent.

When two events are independent, the probability of both occurring is the product of the probabilities of the individual events.

$P(A \text{ and } B)$ for Independent Events

If events A and B are independent, then the probability of both A and B occurring is

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

where $P(A \text{ and } B)$ is the probability of events A and B both occurring, $P(A)$ is the probability of event A occurring, and $P(B)$ is the probability of event B occurring

If you look back at the coin and die example from earlier, you can see how the number of outcomes of the first event multiplied by the number of outcomes in the second event multiplied to equal the total number of possible outcomes in the combined event.

Example 8

In your drawer you have 10 pairs of socks, 6 of which are white, and 7 tee shirts, 3 of which are white. If you randomly reach in and pull out a pair of socks and a tee shirt, what is the probability both are white?

Solution

The probability of choosing a white pair of socks is $\frac{6}{10}$.

The probability of choosing a white tee shirt is $\frac{3}{7}$.

The probability of both being white is $\frac{6}{10} \cdot \frac{3}{7} = \frac{18}{70} = \frac{9}{35}$

Try it Now 2

A card is pulled a deck of cards and noted. The card is then replaced, the deck is shuffled, and a second card is removed and noted. What is the probability that both cards are Aces?

The previous examples looked at the probability of *both* events occurring. Now we will look at the probability of *either* event occurring.

Answer

since the second draw is made after replacing the first card, these events are independent The probability of an ace on each draw is $\frac{4}{52} = \frac{1}{13}$, so the probability of an Ace on both **draws** is $\frac{1}{13} \cdot \frac{1}{13} = \frac{1}{169}$

Example 9

Suppose we flipped a coin and rolled a die, and wanted to know the probability of getting a head on the coin *or* a 6 on the die.

Solution

Here, there are still 12 possible outcomes:

$$\{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}$$

By simply counting, we can see that 7 of the outcomes have a head on the coin *or* a 6 on the die *or* both – we use *or* inclusively here (these 7 outcomes are $H1, H2, H3, H4, H5, H6, T6$), so the probability is $\frac{7}{12}$. How could we have found this from the individual probabilities?

As we would expect, $\frac{1}{2}$ of these outcomes have a head, and $\frac{1}{6}$ of these outcomes have a 6 on the die. If we add these, $\frac{1}{2} + \frac{1}{6} = \frac{6}{12} + \frac{2}{12} = \frac{8}{12}$, which is not the correct probability. Looking at the outcomes we can see why: the outcome H6 would have been counted twice, since it contains both a head and a 6; the probability of both a head *and* rolling a 6 is $\frac{1}{12}$.

If we subtract out this double count, we have the correct probability: $\frac{8}{12} - \frac{1}{12} = \frac{7}{12}$.

$P(A \text{ or } B)$

The probability of either A or B occurring (or both) is

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Example 10

Suppose we draw one card from a standard deck. What is the probability that we get a Queen or a King?

Solution

There are 4 Queens and 4 Kings in the deck, hence 8 outcomes corresponding to a Queen or King out of 52 possible outcomes. Thus the probability of drawing a Queen or a King is:

$$P(\text{ King or Queen}) = \frac{8}{52}$$

Note that in this case, there are no cards that are both a Queen and a King, so $P(\text{King and Queen}) = 0$. Using our probability rule, we could have said:

$$P(\text{ King or Queen}) = P(\text{ King }) + P(\text{ Queen}) - P(\text{ King and Queen}) = \frac{4}{52} + \frac{4}{52} - 0 = \frac{8}{52}$$

In the last example, the events were **mutually exclusive**, so $P(A \text{ or } B) = P(A) + P(B)$.

Example 11

Suppose we draw one card from a standard deck. What is the probability that we get a red card or a King?

Solution

- Half the cards are red, so P(\text{red})=\frac{26}{52}\)
- There are four kings, so P(\text{King})=\frac{4}{52}\)
- There are two red kings, so $P(\text{Red and King}) = \frac{2}{52}$\)

We can then calculate

$$P(\text{Red or King}) = P(\text{Red}) + P(\text{King}) - P(\text{Red and King}) = \frac{26}{52} + \frac{4}{52} - \frac{2}{52} = \frac{28}{52}$$

Try it Now 3

In your drawer you have 10 pairs of socks, 6 of which are white, and 7 tee shirts, 3 of which are white. If you reach in and randomly grab a pair of socks and a tee shirt, what the probability at least one is white?

Answer

$P(\text{ white sock and white tee }) = \frac{6}{10} \cdot \frac{3}{7} = \frac{9}{35}$

$P(\text{ white sock or white tee }) = \frac{6}{10} + \frac{3}{7} - \frac{9}{35} = \frac{27}{35}$

Example 12

The table below shows the number of survey subjects who have received and not received a speeding ticket in the last year, and the color of their car. Find the probability that a randomly chosen person:

a. Has a red car *and* got a speeding ticket
b. Has a red car *or* got a speeding ticket.

	Speeding ticket	No speeding ticket	Total
Red car	15	135	150
Not red car	45	470	515
Total	60	605	665

Solution

We can see that 15 people of the 665 surveyed had both a red car and got a speeding ticket, so the probability is $\frac{15}{665} \approx 0.0226$.

Notice that having a red car and getting a speeding ticket are not independent events, so the probability of both of them occurring is not simply the product of probabilities of each one occurring.

We could answer this question by simply adding up the numbers: 15 people with red cars and speeding tickets + 135 with red cars but no ticket + 45 with a ticket but no red car = 195 people. So the probability is $\frac{195}{665} \approx 0.2932$.

We also could have found this probability by:

$$\begin{aligned} &\text{P(had a red car)} + \text{P(got a speeding ticket)} - \text{P(had a red car and got a speeding ticket)} \\ &= \frac{150}{665} + \frac{60}{665} - \frac{15}{665} = \frac{195}{665} \end{aligned}$$

Conditional Probability

Often it is required to compute the probability of an event given that another event has occurred.

Example 13

What is the probability that two cards drawn at random from a deck of playing cards will both be aces?

Solution

It might seem that you could use the formula for the probability of two independent events and simply multiply $\frac{4}{52} \cdot \frac{4}{52} = \frac{1}{169}$. This would be incorrect, however, because the two events are not independent. If the first card drawn is an ace, then the probability that the second card is also an ace would be lower because there would only be three aces left in the deck.

Once the first card chosen is an ace, the probability that the second card chosen is also an ace is called the **conditional probability** of drawing an ace. In this case the "condition" is that the first card is an ace. Symbolically, we write this as:

P(ace on second draw | an ace on the first draw).

The vertical bar "|" is read as "given," so the above expression is short for "The probability that an ace is drawn on the second draw given that an ace was drawn on the first draw." What is this probability? After an ace is drawn on the first draw, there are

3 aces out of 51 total cards left. This means that the conditional probability of drawing an ace after one ace has already been drawn is $\frac{3}{51} = \frac{1}{17}$.

Thus, the probability of both cards being aces is $\frac{4}{52} \cdot \frac{3}{51} = \frac{12}{2652} = \frac{1}{221}$.

Conditional Probability

The probability the event B occurs, given that event A has happened, is represented as

$P(B|A)$

This is read as "the probability of B given A"

Example 14

Find the probability that a die rolled shows a 6, given that a flipped coin shows a head.

Solution

These are two independent events, so the probability of the die rolling a 6 is $\frac{1}{6}$, regardless of the result of the coin flip.

Example 15

The table below shows the number of survey subjects who have received and not received a speeding ticket in the last year, and the color of their car. Find the probability that a randomly chosen person:

a) Has a speeding ticket *given* they have a red car

b) Has a red car *given* they have a speeding ticket

	Speeding ticket	No speeding ticket	Total
Red car	15	135	150
Not red car	45	470	515
Total	60	605	665

Solution

a) Since we know the person has a red car, we are only considering the 150 people in the first row of the table. Of those, 15 have a speeding ticket, so

P(ticket | red car) = $\frac{15}{150} = \frac{1}{10} = 0.1$

b) Since we know the person has a speeding ticket, we are only considering the 60 people in the first column of the table. Of those, 15 have a red car, so

P(red car | ticket) = $\frac{15}{60} = \frac{1}{4} = 0.25$.

Notice from the last example that $P(B|A)$ is **not** equal to $P(A|B)$.

These kinds of conditional probabilities are what insurance companies use to determine your insurance rates. They look at the conditional probability of you having accident, given your age, your car, your car color, your driving history, etc., and price your policy based on that likelihood.

Conditional Probability Formula

If Events A and B are not independent, then

$P(A \text{ and } B) = P(A) \cdot P(B|A)$

Example 16

If you pull 2 cards out of a deck, what is the probability that both are spades?

Solution

The probability that the first card is a spade is $\frac{13}{52}$.

The probability that the second card is a spade, given the first was a spade, is $\frac{12}{51}$, since there is one less spade in the deck, and one less total cards.

The probability that both cards are spades is $\frac{13}{52} \cdot \frac{12}{51} = \frac{156}{2652} \approx 0.0588$

Example 17

If you draw two cards from a deck, what is the probability that you will get the Ace of Diamonds and a black card?

Solution

You can satisfy this condition by having Case A or Case B, as follows:

Case A) you can get the Ace of Diamonds first and then a black card or

Case B) you can get a black card first and then the Ace of Diamonds.

Let's calculate the probability of Case A. The probability that the first card is the Ace of Diamonds is $\frac{1}{52}$. The probability that the second card is black given that the first card is the Ace of Diamonds is $\frac{26}{51}$ because 26 of the remaining 51 cards are black. The probability is therefore $\frac{1}{52} \cdot \frac{26}{51} = \frac{1}{102}$.

Now for Case B: the probability that the first card is black is $\frac{26}{52} = \frac{1}{2}$. The probability that the second card is the Ace of Diamonds given that the first card is black is $\frac{1}{51}$. The probability of Case B is therefore $\frac{1}{2} \cdot \frac{1}{51} = \frac{1}{102}$, the same as the probability of Case 1.

Recall that the probability of A or B is $P(\text{A}) + P(\text{B}) - P(\text{A and B})$. In this problem, $P(\text{A and B}) = 0$ since the first card cannot be the Ace of Diamonds and be a black card. Therefore, the probability of Case A or Case B is $\frac{1}{102} + \frac{1}{102} = \frac{2}{102} = \frac{1}{51}$. The probability that you will get the Ace of Diamonds and a black card when drawing two cards from a deck is $\frac{1}{51}$.

Try it Now 4

In your drawer you have 10 pairs of socks, 6 of which are white. If you reach in and randomly grab two pairs of socks, what is the probability that both are white?

Answer

$\frac{6}{10} \cdot \frac{5}{9} = \frac{30}{90} = \frac{1}{3}$

Example 18

A home pregnancy test was given to women, then pregnancy was verified through blood tests. The following table shows the home pregnancy test results. Find

a) P(not pregnant | positive test result)

b) P(positive test result | not pregnant)

	Positive test	Negative test	Total
Pregnant	70	4	74
Not Pregnant	5	14	19
Total	75	18	93

Solution

a) Since we know the test result was positive, we're limited to the 75 women in the first column, of which 5 were not pregnant. P(not pregnant | positive test result) = $\frac{5}{75} \approx 0.067$.

b) Since we know the woman is not pregnant, we are limited to the 19 women in the second row, of which 5 had a positive test. P(positive test result | not pregnant) = $\frac{5}{19} \approx 0.263$.

The second result is what is usually called a false positive: A positive result when the woman is not actually pregnant.

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3.4: Bayes Theorem

In this section we concentrate on the more complex conditional probability problems we began looking at in the last section.

Example 19

Suppose a certain disease has an incidence rate of 0.1% (that is, it afflicts 0.1% of the population). A test has been devised to detect this disease. The test does not produce false negatives (that is, anyone who has the disease will test positive for it), but the false positive rate is 5% (that is, about 5% of people who take the test will test positive, even though they do not have the disease). Suppose a randomly selected person takes the test and tests positive. What is the probability that this person actually has the disease?

Solution

There are two ways to approach the solution to this problem. One involves an important result in probability theory called Bayes' theorem. We will discuss this theorem a bit later, but for now we will use an alternative and, we hope, much more intuitive approach.

Let's break down the information in the problem piece by piece.

Suppose a certain disease has an incidence rate of 0.1% (that is, it afflicts 0.1% of the population). The percentage 0.1% can be converted to a decimal number by moving the decimal place two places to the left, to get 0.001. In turn, 0.001 can be rewritten as a fraction: 1/1000. This tells us that about 1 in every 1000 people has the disease. (If we wanted we could write P(disease)=0.001.)

A test has been devised to detect this disease. The test does not produce false negatives (that is, anyone who has the disease will test positive for it). This part is fairly straightforward: everyone who has the disease will test positive, or alternatively everyone who tests negative does not have the disease. (We could also say P(positive | disease)=1.)

The false positive rate is 5% (that is, about 5% of people who take the test will test positive, even though they do not have the disease). This is even more straightforward. Another way of looking at it is that of every 100 people who are tested and do not have the disease, 5 will test positive even though they do not have the disease. (We could also say that P(positive | no disease)=0.05.)

Suppose a randomly selected person takes the test and tests positive. What is the probability that this person actually has the disease? Here we want to compute P(disease|positive). We already know that P(positive|disease)=1, but remember that conditional probabilities are not equal if the conditions are switched.

Rather than thinking in terms of all these probabilities we have developed, let's create a hypothetical situation and apply the facts as set out above. First, suppose we randomly select 1000 people and administer the test. How many do we expect to have the disease? Since about 1/1000 of all people are afflicted with the disease, $\frac{1}{1000}$ of 1000 people is 1. (Now you know why we chose 1000.) Only 1 of 1000 test subjects actually has the disease; the other 999 do not.

We also know that 5% of all people who do not have the disease will test positive. There are 999 disease-free people, so we would expect $(0.05)(999) = 49.95$ (so, about 50) people to test positive who do not have the disease.

Now back to the original question, computing P(disease|positive). There are 51 people who test positive in our example (the one unfortunate person who actually has the disease, plus the 50 people who tested positive but don't). Only one of these people has the disease, so

P(disease | positive) $\approx \frac{1}{51} \approx 0.0196$

or less than 2%. Does this surprise you? This means that of all people who test positive, over 98% *do not have the disease*.

The answer we got was slightly approximate, since we rounded 49.95 to 50. We could redo the problem with 100,000 test subjects, 100 of whom would have the disease and $(0.05)(99,900) = 4995$ test positive but do not have the disease, so the exact probability of having the disease if you test positive is

P(disease | positive) $\approx \frac{100}{5095} \approx 0.0196$

which is pretty much the same answer.

But back to the surprising result. *Of all people who test positive, over 98% do not have the disease.* If your guess for the probability a person who tests positive has the disease was wildly different from the right answer (2%), don't feel bad. The exact same problem was posed to doctors and medical students at the Harvard Medical School 25 years ago and the results revealed in a 1978 *New England Journal of Medicine* article. Only about 18% of the participants got the right answer. Most of the rest thought the answer was closer to 95% (perhaps they were misled by the false positive rate of 5%).

So at least you should feel a little better that a bunch of doctors didn't get the right answer either (assuming you thought the answer was much higher). But the significance of this finding and similar results from other studies in the intervening years lies not in making math students feel better but in the possibly catastrophic consequences it might have for patient care. If a doctor thinks the chances that a positive test result nearly guarantees that a patient has a disease, they might begin an unnecessary and possibly harmful treatment regimen on a healthy patient. Or worse, as in the early days of the AIDS crisis when being HIV-positive was often equated with a death sentence, the patient might take a drastic action and commit suicide.

As we have seen in this hypothetical example, the most responsible course of action for treating a patient who tests positive would be to counsel the patient that they most likely do *not* have the disease and to order further, more reliable, tests to verify the diagnosis.

One of the reasons that the doctors and medical students in the study did so poorly is that such problems, when presented in the types of statistics courses that medical students often take, are solved by use of Bayes' theorem, which is stated as follows:

Bayes' Theorem

$$P(A|B) = \frac{P(A)P(B|A)}{P(A)P(B|A)+P(\bar{A})P(B|\bar{A})}$$

In our earlier example, this translates to

$$P(\text{ disease } | \text{ positive }) = \frac{P(\text{ disease })P(\text{ positive } | \text{ disease })}{P(\text{ disease })P(\text{ positive } | \text{ disease })+P(\text{ no disease })P(\text{ positive } | \text{ no disease })}$$

Plugging in the numbers gives

$$P(\text{ disease } | \text{ positive }) = \frac{(0.001)(1)}{(0.001)(1)+(0.999)(0.05)} \approx 0.0196$$

which is exactly the same answer as our original solution.

The problem is that you (or the typical medical student, or even the typical math professor) are much more likely to be able to remember the original solution than to remember Bayes' theorem. Psychologists, such as Gerd Gigerenzer, author of *Calculated Risks: How to Know When Numbers Deceive You*, have advocated that the method involved in the original solution (which Gigerenzer calls the method of "natural frequencies") be employed in place of Bayes' Theorem. Gigerenzer performed a study and found that those educated in the natural frequency method were able to recall it far longer than those who were taught Bayes' theorem. When one considers the possible life-and-death consequences associated with such calculations it seems wise to heed his advice.

Example 20

A certain disease has an incidence rate of 2%. If the false negative rate is 10% and the false positive rate is 1%, compute the probability that a person who tests positive actually has the disease.

Solution

Imagine 10,000 people who are tested. Of these 10,000, 200 will have the disease; 10% of them, or 20, will test negative and the remaining 180 will test positive. Of the 9800 who do not have the disease, 98 will test positive. So of the 278 total people who test positive, 180 will have the disease. Thus

$$P(\text{ disease } | \text{ positive }) = \frac{180}{278} \approx 0.647$$

so about 65% of the people who test positive will have the disease.

Using Bayes theorem directly would give the same result:

$$P(\text{ disease } | \text{ positive }) = \frac{(0.02)(0.90)}{(0.02)(0.90)+(0.98)(0.01)} = \frac{0.018}{0.0278} \approx 0.647$$

Try it Now 5

A certain disease has an incidence rate of 0.5%. If there are no false negatives and if the false positive rate is 3%, compute the probability that a person who tests positive actually has the disease.

Answer

Out of 100,000 people, 500 would have the disease. Of those, all 500 would test positive. Of the 99,500 without the disease, 2,985 would falsely test positive and the other 96,515 would test negative.

$$\text{P}(\text{ disease } | \text{ positive }) = \frac{500}{500+2985} = \frac{500}{3485} \approx 14.3\%$$

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3.5: Counting

Counting? You already know how to count or you wouldn't be taking a college-level math class, right? Well yes, but what we'll really be investigating here are ways of counting *efficiently*. When we get to the probability situations a bit later in this chapter we will need to count some *very* large numbers, like the number of possible winning lottery tickets. One way to do this would be to write down every possible set of numbers that might show up on a lottery ticket, but believe me: you don't want to do this.

Basic Counting

We will start, however, with some more reasonable sorts of counting problems in order to develop the ideas that we will soon need.

Example 21

Suppose at a particular restaurant you have three choices for an appetizer (soup, salad or breadsticks) and five choices for a main course (hamburger, sandwich, quiche, fajita or pizza). If you are allowed to choose exactly one item from each category for your meal, how many different meal options do you have?

Solution

Solution 1: One way to solve this problem would be to systematically list each possible meal:

soup + hamburger	soup + sandwich	soup + quiche
soup + fajita	soup + pizza	salad + hamburger
salad + sandwich	salad + quiche	salad + fajita
salad + pizza	breadsticks + hamburger	breadsticks + sandwich
breadsticks + quiche	breadsticks + fajita	breadsticks + pizza

Assuming that we did this systematically and that we neither missed any possibilities nor listed any possibility more than once, the answer would be 15. Thus you could go to the restaurant 15 nights in a row and have a different meal each night.

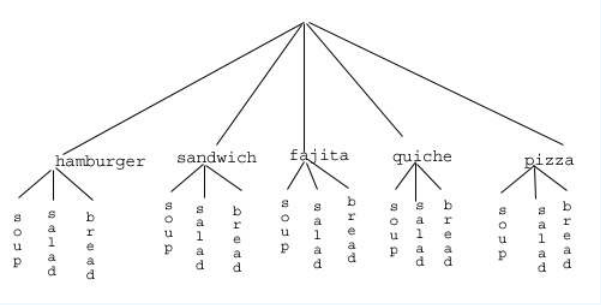
Solution 2: Another way to solve this problem would be to list all the possibilities in a table:

	hamburger	**sandwich**	**quiche**	**fajita**	**pizza**
soup	Soup + burger				
salad	Salad + burger				
bread	*etc.*				

In each of the cells in the table we could list the corresponding meal: soup + hamburger in the upper left corner, salad + hamburger below it, etc. But if we didn't really care *what* the possible meals are, only *how many* possible meals there are, we could just count the number of cells and arrive at an answer of 15, which matches our answer from the first solution. (It's always good when you solve a problem two different ways and get the same answer!)

Solution 3: We already have two perfectly good solutions. Why do we need a third? The first method was not very systematic, and we might easily have made an omission. The second method was better, but suppose that in addition to the appetizer and the main course we further complicated the problem by adding desserts to the menu: we've used the rows of the table for the appetizers and the columns for the main courses—where will the desserts go? We would need a third dimension, and since drawing 3-D tables on a 2-D page or computer screen isn't terribly easy, we need a better way in case we have three categories to choose form instead of just two.

So, back to the problem in the example. What else can we do? Let's draw a **tree diagram**:



This is called a "tree" diagram because at each stage we branch out, like the branches on a tree. In this case, we first drew five branches (one for each main course) and then for each of those branches we drew three more branches (one for each appetizer). We count the number of branches at the final level and get (surprise, surprise!) 15.

If we wanted, we could instead draw three branches at the first stage for the three appetizers and then five branches (one for each main course) branching out of each of those three branches.

OK, so now we know how to count possibilities using tables and tree diagrams. These methods will continue to be useful in certain cases, but imagine a game where you have two decks of cards (with 52 cards in each deck) and you select one card from each deck. Would you really want to draw a table or tree diagram to determine the number of outcomes of this game?

Let's go back to the previous example that involved selecting a meal from three appetizers and five main courses, and look at the second solution that used a table. Notice that one way to count the number of possible meals is simply to number each of the appropriate cells in the table, as we have done above. But another way to count the number of cells in the table would be multiply the number of rows (3) by the number of columns (5) to get 15. Notice that we could have arrived at the same result without making a table at all by simply multiplying the number of choices for the appetizer (3) by the number of choices for the main course (5). We generalize this technique as the *basic counting rule*:

Basic Counting Rule

If we are asked to choose one item from each of two separate categories where there are m items in the first category and n items in the second category, then the total number of available choices is $m \cdot n$

This is sometimes called the multiplication rule for probabilities.

Example 22

There are 21 novels and 18 volumes of poetry on a reading list for a college English course. How many different ways can a student select one novel and one volume of poetry to read during the quarter?

Solution

There are 21 choices from the first category and 18 for the second, so there are $21 \cdot 18 = 378$ possibilities.

The Basic Counting Rule can be extended when there are more than two categories by applying it repeatedly, as we see in the next example.

Example 23

Suppose at a particular restaurant you have three choices for an appetizer (soup, salad or breadsticks), five choices for a main course (hamburger, sandwich, quiche, fajita or pasta) and two choices for dessert (pie or ice cream). If you are allowed to choose exactly one item from each category for your meal, how many different meal options do you have?

Solution

There are 3 choices for an appetizer, 5 for the main course and 2 for dessert, so there are $3 \cdot 5 \cdot 2 = 30$ possibilities.

Example 24

A quiz consists of 3 true-or-false questions. In how many ways can a student answer the quiz?

Solution

There are 3 questions. Each question has 2 possible answers (true or false), so the quiz may be answered in $2 \cdot 2 \cdot 2 = 8$ different ways. Recall that another way to write $2 \cdot 2 \cdot 2$ is 2^3 which is much more compact.

Try it Now 6

Suppose at a particular restaurant you have eight choices for an appetizer, eleven choices for a main course and five choices for dessert. If you are allowed to choose exactly one item from each category for your meal, how many different meal options do you have?

Answer

$8 \cdot 11 \cdot 5 = 440$ menu combinations

Permutations

In this section we will develop an even faster way to solve some of the problems we have already learned to solve by other means. Let's start with a couple examples.

Example 25

How many different ways can the letters of the word MATH be rearranged to form a four-letter code word?

Solution

This problem is a bit different. Instead of choosing one item from each of several different categories, we are repeatedly choosing items from the *same* category (the category is: the letters of the word MATH) and each time we choose an item we *do not replace* it, so there is one fewer choice at the next stage: we have 4 choices for the first letter (say we choose A), then 3 choices for the second (M, T and H; say we choose H), then 2 choices for the next letter (M and T; say we choose M) and only one choice at the last stage (T). Thus there are $4 \cdot 3 \cdot 2 \cdot 1 = 24$ ways to spell a code worth with the letters MATH.

In this example, we needed to calculate $n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$. This calculation shows up often in mathematics, and is called the **factorial**, and is notated $n!$

Factorial

$n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$

Example 26

How many ways can five different door prizes be distributed among five people?

Solution

There are 5 choices of prize for the first person, 4 choices for the second, and so on. The number of ways the prizes can be distributed will be $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ ways.

Now we will consider some slightly different examples.

Example 27

A charity benefit is attended by 25 people and three gift certificates are given away as door prizes: one gift certificate is in the amount of $100, the second is worth $25 and the third is worth $10. Assuming that no person receives more than one prize, how many different ways can the three gift certificates be awarded?

Solution

Using the Basic Counting Rule, there are 25 choices for the person who receives the $\$100$ certificate, 24 remaining choices for the $\$25$ certificate and 23 choices for the $\$10$ certificate, so there are $25 \cdot 24 \cdot 23 = 13,800$ ways in which the prizes can be awarded.

Example 28

Eight sprinters have made it to the Olympic finals in the 100-meter race. In how many different ways can the gold, silver and bronze medals be awarded?

Solution

Using the Basic Counting Rule, there are 8 choices for the gold medal winner, 7 remaining choices for the silver, and 6 for the bronze, so there are $8 \cdot 7 \cdot 6 = 336$ ways the three medals can be awarded to the 8 runners.

Note that in these preceding examples, the gift certificates and the Olympic medals were awarded *without replacement*; that is, once we have chosen a winner of the first door prize or the gold medal, they are not eligible for the other prizes. Thus, at each succeeding stage of the solution there is one fewer choice (25, then 24, then 23 in the first example; 8, then 7, then 6 in the second). Contrast this with the situation of a multiple choice test, where there might be five possible answers — A, B, C, D or E — for each question on the test.

Note also that *the order of selection was important* in each example: for the three door prizes, being chosen first means that you receive substantially more money; in the Olympics example, coming in first means that you get the gold medal instead of the silver or bronze. In each case, if we had chosen the same three people in a different order there might have been a different person who received the $100 prize, or a different goldmedalist. (Contrast this with the situation where we might draw three names out of a hat to each receive a $10 gift certificate; in this case the order of selection is *not* important since each of the three people receive the same prize. Situations where the order is *not* important will be discussed in the next section.)

We can generalize the situation in the two examples above to any problem *without replacement* where the *order of selection is important*. If we are arranging in order r items out of n possibilities (instead of 3 out of 25 or 3 out of 8 as in the previous examples), the number of possible arrangements will be given by

$$n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)$$

If you don't see why $(n-r+1)$ is the right number to use for the last factor, just think back to the first example in this section, where we calculated $25 \cdot 24 \cdot 23$ to get $13,800$. In this case $n = 25$ and $r = 3$, so $n - r + 1 = 25 - 3 + 1 = 23$, which is exactly the right number for the final factor.

Now, why would we want to use this complicated formula when it's actually easier to use the Basic Counting Rule, as we did in the first two examples? Well, we won't actually use this formula all that often, we only developed it so that we could attach a special notation and a special definition to this situation where we are choosing r items out of n possibilities *without replacement* and where the *order of selection is important*. In this situation we write:

Permutations

$${}_nP_r = n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)$$

We say that there are ${}_nP_r$ **permutations** of size r that may be selected from among n choices *without replacement* when *order matters*.

It turns out that we can express this result more simply using factorials.

$${}_nP_r = \frac{n!}{(n-r)!}$$

In practicality, we usually use technology rather than factorials or repeated multiplication to compute permutations.

Example 29

I have nine paintings and have room to display only four of them at a time on my wall. How many different ways could I do this?

Solution

Since we are choosing 4 paintings out of 9 *without replacement* where the *order of selection is important* there are ${}_9P_4 = 9 \cdot 8 \cdot 7 \cdot 6 = 3,024$ permutations.

Example 30

How many ways can a four-person executive committee (president, vice-president, secretary, treasurer) be selected from a 16-member board of directors of a non-profit organization?

Solution

We want to choose 4 people out of 16 without replacement and where the order of selection is important. So the answer is ${}_{16}P_4 = 16 \cdot 15 \cdot 14 \cdot 13 = 43,680.$

Try it Now 7

How many 5 character passwords can be made using the letters A through Z

a. if repeats are allowed
b. if no repeats are allowed

Answer

There are 26 characters.

a. $26^5 = 11,881,376.$
b. ${}_{26}P_5 = 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 = 7,893,600$

Combinations

In the previous section we considered the situation where we chose r items out of n possibilities *without replacement* and where the *order of selection was important*. We now consider a similar situation in which the order of selection is *not* important.

Example 31

A charity benefit is attended by 25 people at which three $50 gift certificates are given away as door prizes. Assuming no person receives more than one prize, how many different ways can the gift certificates be awarded?

Solution

Using the Basic Counting Rule, there are 25 choices for the first person, 24 remaining choices for the second person and 23 for the third, so there are $25 \cdot 24 \cdot 23 = 13,800$ ways to choose three people. Suppose for a moment that Abe is chosen first, Bea second and Cindy third; this is one of the 13,800 possible outcomes. Another way to award the prizes would be to choose Abe first, Cindy second and Bea third; this is another of the 13,800 possible outcomes. But either way Abe, Bea and Cindy each get $50, so it doesn't really matter the order in which we select them. In how many different orders can Abe, Bea and Cindy be selected? It turns out there are 6:

ABC ACB BAC BCA CAB CBA

How can we be sure that we have counted them all? We are really just choosing 3 people out of 3, so there are $3 \cdot 2 \cdot 1 = 6$ ways to do this; we didn't really need to list them all, we can just use permutations!

So, out of the 13,800 ways to select 3 people out of 25, six of them involve Abe, Bea and Cindy. The same argument works for any other group of three people (say Abe, Bea and David or Frank, Gloria and Hildy) so each three-person group is counted *six times*. Thus the 13,800 figure is six times too big. The number of distinct three-person groups will be $\frac{13,800}{6} = 2300$.

We can generalize the situation in this example above to any problem of choosing a collection of items *without replacement* where the *order of selection is* ***not*** *important*. If we are choosing r items out of n possibilities (instead of 3 out of 25 as in the previous examples), the number of possible choices will be given by , and we could use this formula for computation. However this situation arises so frequently that we attach a special notation and a special definition to this situation where we are choosing r items out of n possibilities *without replacement* where the *order of selection is* ***not*** *important*.

Combinations

$_nC_r = \frac{_nP_r}{rP_r}$

We say that there are $_nC_r$ **combinations** of size r that may be selected from among n choices *without replacement* where *order doesn't matter*.

We can also write the combinations formula in terms of factorials:

$_nC_r = \frac{n!}{(n-r)!r!}$

Example 32

A group of four students is to be chosen from a 35-member class to represent the class on the student council. How many ways can this be done?

Solution

Since we are choosing 4 people out of 35 *without replacement* where the *order of selection is* ***not*** *important* there are = $_{35}C_4 = \frac{35 \cdot 34 \cdot 33 \cdot 32}{4 \cdot 3 \cdot 2 \cdot 1} = 52,360$ combinations.

Try it Now 8

The United States Senate Appropriations Committee consists of 29 members; the Defense Subcommittee of the Appropriations Committee consists of 19 members. Disregarding party affiliation or any special seats on the Subcommittee, how many different 19-member subcommittees may be chosen from among the 29 Senators on the Appropriations Committee?

Answer

Order does not matter. $_{29}C_{19} = 20,030,010$ possible subcommittees

In the preceding Try it Now problem we assumed that the 19 members of the Defense Subcommittee were chosen without regard to party affiliation. In reality this would never happen: if Republicans are in the majority they would never let a majority of Democrats sit on (and thus control) any subcommittee. (The same of course would be true if the Democrats were in control.) So let's consider the problem again, in a slightly more complicated form:

Example 33

The United States Senate Appropriations Committee consists of 29 members, 15 Republicans and 14 Democrats. The Defense Subcommittee consists of 19 members, 10 Republicans and 9 Democrats. How many different ways can the members of the Defense Subcommittee be chosen from among the 29 Senators on the Appropriations Committee?

Solution

In this case we need to choose 10 of the 15 Republicans and 9 of the 14 Democrats. There are $_{15}C_{10} = 3003$ ways to choose the 10 Republicans and $_{14}C_9 = 2002$ ways to choose the 9 Democrats. But now what? How do we finish the problem?

Suppose we listed all of the possible 10-member Republican groups on 3003 slips of red paper and all of the possible 9-member Democratic groups on 2002 slips of blue paper. How many ways can we choose one red slip and one blue slip? This is a job for the Basic Counting Rule! We are simply making one choice from the first category and one choice from the second category, just like in the restaurant menu problems from earlier.

There must be $3003 \cdot 2002 = 6,012,006$ possible ways of selecting the members of the Defense Subcommittee.

Probability using Permutations and Combinations

We can use permutations and combinations to help us answer more complex probability questions

Example 34

A 4 digit PIN number is selected. What is the probability that there are no repeated digits?

Solution

There are 10 possible values for each digit of the PIN (namely: $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$), so there are $10 \cdot 10 \cdot 10 \cdot 10 = 10^4 = 10000$ total possible PIN numbers.

To have no repeated digits, all four digits would have to be different, which is selecting without replacement. We could either compute 10 \cdot 9 \cdot 8 \cdot 7, or notice that this is the same as the permutation $_{10}P_4 = 5040$.

The probability of no repeated digits is the number of 4 digit PIN numbers with no repeated digits divided by the total number of 4 digit PIN numbers. This probability is $\frac{_{10}P_4}{10^4} = \frac{5040}{10000} = 0.504$

Example 35

In a certain state's lottery, 48 balls numbered 1 through 48 are placed in a machine and six of them are drawn at random. If the six numbers drawn match the numbers that a player had chosen, the player wins $1,000,000. In this lottery, the order the numbers are drawn in doesn't matter. Compute the probability that you win the million-dollar prize if you purchase a single lottery ticket.

Solution

In order to compute the probability, we need to count the total number of ways six numbers can be drawn, and the number of ways the six numbers on the player's ticket could match the six numbers drawn from the machine. Since there is no stipulation that the numbers be in any particular order, the number of possible outcomes of the lottery drawing is $_{48}C_6 = 12,271,512$. Of these possible outcomes, only one would match all six numbers on the player's ticket, so the probability of winning the grand prize is:

$\frac{_6C_6}{_{48}C_6} = \frac{1}{12271512} \approx 0.0000000815$

Example 36

In the state lottery from the previous example, if five of the six numbers drawn match the numbers that a player has chosen, the player wins a second prize of $1,000. Compute the probability that you win the second prize if you purchase a single lottery ticket.

Solution

As above, the number of possible outcomes of the lottery drawing is $_{48}C_6 = 12,271,512$. In order to win the second prize, five of the six numbers on the ticket must match five of the six winning numbers; in other words, we must have chosen five of the six winning numbers and one of the 42 losing numbers. The number of ways to choose 5 out of the 6 winning numbers is given by $_6C_5 = 6$ and the number of ways to choose 1 out of the 42 losing numbers is given by $_{42}C_1 = 42$. Thus the number of

favorable outcomes is then given by the Basic Counting Rule: ${}_6C_5 \cdot {}_{42}C_1 = 6 \cdot 42 = 252$. So the probability of winning the second prize is.

$\frac{({}_6C_5)({}_{42}C_1)}{{}_{48}C_6} = \frac{252}{12271512} \approx 0.0000205$

Try it Now 9

A multiple-choice question on an economics quiz contains 10 questions with five possible answers each. Compute the probability of randomly guessing the answers and getting 9 questions correct.

Answer

There are $5^{10} = 9,765,625$ different ways the exam can be answered. There are 10 possible locations for the one missed question, and in each of those locations there are 4 wrong answers, so there are 40 ways the test could be answered with one wrong answer.

$\mathrm{P}(\text{9 answers correct}) = \frac{40}{5^{10}} \approx 0.0000041$ chance

Example 37

Compute the probability of randomly drawing five cards from a deck and getting exactly one Ace.

Solution

In many card games (such as poker) the order in which the cards are drawn is not important (since the player may rearrange the cards in his hand any way he chooses); in the problems that follow, we will assume that this is the case unless otherwise stated. Thus we use combinations to compute the possible number of 5-card hands, ${}_{52}C_5$. This number will go in the denominator of our probability formula, since it is the number of possible outcomes.

For the numerator, we need the number of ways to draw one Ace and four other cards (none of them Aces) from the deck. Since there are four Aces and we want exactly one of them, there will be ${}_4C_1$ ways to select one Ace; since there are 48 non-Aces and we want 4 of them, there will be ${}_{48}C_4$\) ways to select the four non-Aces. Now we use the Basic Counting Rule to calculate that there will be ${}_4C_1 \cdot {}_{48}C_4$ ways to choose one ace and four non-Aces.

Putting this all together, we have

$P(\text{one Ace}) = \frac{({}_4C_1)({}_{48}C_4)}{{}_{52}C_5} = \frac{778320}{2598960} \approx 0.299$

Example 38

Compute the probability of randomly drawing five cards from a deck and getting exactly two Aces.

Solution

The solution is similar to the previous example, except now we are choosing 2 Aces out of 4 and 3 non-Aces out of 48; the denominator remains the same:

$P(\text{two Aces}) = \frac{({}_4C_2)({}_{48}C_3)}{{}_{52}C_5} = \frac{103776}{2598960} \approx 0.0399$

It is useful to note that these card problems are remarkably similar to the lottery problems discussed earlier.

Try it Now 10

Compute the probability of randomly drawing five cards from a deck of cards and getting three Aces and two Kings.

Answer

$P(\text{three Aces and two Kings}) = \frac{({}_4C_3)({}_4C_2)}{{}_{52}C_5} = \frac{24}{2598960} \approx 0.0000092$

Birthday Problem

Let's take a pause to consider a famous problem in probability theory:

Suppose you have a room full of 30 people. What is the probability that there is at least one shared birthday?

Take a guess at the answer to the above problem. Was your guess fairly low, like around 10%? That seems to be the intuitive answer ($\frac{30}{365}$, perhaps?). Let's see if we should listen to our intuition. Let's start with a simpler problem, however.

Example 39

Suppose three people are in a room. What is the probability that there is at least one shared birthday among these three people?

Solution

There are a lot of ways there could be at least one shared birthday. Fortunately there is an easier way. We ask ourselves "What is the alternative to having at least one shared birthday?" In this case, the alternative is that there are **no** shared birthdays. In other words, the alternative to "at least one" is having **none**. In other words, since this is a complementary event,

$$P(\text{at least one}) = 1 - P(\text{none})$$

We will start, then, by computing the probability that there is no shared birthday. Let's imagine that you are one of these three people. Your birthday can be anything without conflict, so there are 365 choices out of 365 for your birthday. What is the probability that the second person does not share your birthday? There are 365 days in the year (let's ignore leap years) and removing your birthday from contention, there are 364 choices that will guarantee that you do not share a birthday with this person, so the probability that the second person does not share your birthday is $\frac{364}{365}$. Now we move to the third person. What is the probability that this third person does not have the same birthday as either you or the second person? There are 363 days that will not duplicate your birthday or the second person's, so the probability that the third person does not share a birthday with the first two is $\frac{363}{365}$.

We want the second person not to share a birthday with you *and* the third person not to share a birthday with the first two people, so we use the multiplication rule:

$$P(\text{no shared birthday}) = \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \approx 0.9918$$

and then subtract from 1 to get

$$P(\text{shared birthday}) = 1 - P(\text{no shared birthday}) = 1 - 0.9918 = 0.0082.$$

This is a pretty small number, so maybe it makes sense that the answer to our original problem will be small. Let's make our group a bit bigger.

Example 40

Suppose five people are in a room. What is the probability that there is at least one shared birthday among these five people?

Solution

Continuing the pattern of the previous example, the answer should be

$$P(\text{shared birthday}) = 1 - \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdot \frac{362}{365} \cdot \frac{361}{365} \approx 0.0271$$

Note that we could rewrite this more compactly as

$$P(\text{shared birthday}) = 1 - \frac{{}_{365}P_5}{365^5} \approx 0.0271$$

which makes it a bit easier to type into a calculator or computer, and which suggests a nice formula as we continue to expand the population of our group.

Example 41

Suppose 30 people are in a room. What is the probability that there is at least one shared birthday among these 30 people?

Solution

Here we can calculate

$$P(\text{ shared birthday }) = 1 - \frac{{}_{365}P_{30}}{365^{30}} \approx 0.706$$

which gives us the surprising result that when you are in a room with 30 people there is a 70% chance that there will be at least one shared birthday!

If you like to bet, and if you can convince 30 people to reveal their birthdays, you might be able to win some money by betting a friend that there will be at least two people with the same birthday in the room anytime you are in a room of 30 or more people. (Of course, you would need to make sure your friend hasn't studied probability!) You wouldn't be guaranteed to win, but you should win more than half the time.

This is one of many results in probability theory that is counterintuitive; that is, it goes against our gut instincts. If you still don't believe the math, you can carry out a simulation. Just so you won't have to go around rounding up groups of 30 people, someone has kindly developed a Java applet so that you can conduct a computer simulation. Go to this web page: http://statweb.stanford.edu/~susan/surprise/Birthday.html, and once the applet has loaded, select 30 birthdays and then keep clicking Start and Reset. If you keep track of the number of times that there is a repeated birthday, you should get a repeated birthday about 7 out of every 10 times you run the simulation.

Try it Now 11

Suppose 10 people are in a room. What is the probability that there is at least one shared birthday among these 10 people?

Answer

$$P(\text{shared birthday}) = 1 - \frac{{}_{365}P_{10}}{365^{10}} \approx 0.117$$

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3.6: Expected Value

Expected value is perhaps the most useful probability concept we will discuss. It has many applications, from insurance policies to making financial decisions, and it's one thing that the casinos and government agencies that run gambling operations and lotteries hope most people never learn about.

Example 42

[1]In the casino game roulette, a wheel with 38 spaces (18 red, 18 black, and 2 green) is spun. In one possible bet, the player bets \$1 on a single number. If that number is spun on the wheel, then they receive \$36 (their original \$1 + \$35). Otherwise, they lose their \$1. On average, how much money should a player expect to win or lose if they play this game repeatedly?



Solution

Suppose you bet \$1 on each of the 38 spaces on the wheel, for a total of \$38 bet. When the winning number is spun, you are paid \$36 on that number. While you won on that one number, overall you've lost \$2. On a per-space basis, you have "won" $-\$2/\$38 \approx -\$0.053$. In other words, on average you lose 5.3 cents per space you bet on.

We call this average gain or loss the expected value of playing roulette. Notice that no one ever loses exactly 5.3 cents: most people (in fact, about 37 out of every 38) lose \$1 and a very few people (about 1 person out of every 38) gain \$35 (the \$36 they win minus the \$1 they spent to play the game).

There is another way to compute expected value without imagining what would happen if we play every possible space. There are 38 possible outcomes when the wheel spins, so the probability of winning is $\frac{1}{38}$. The complement, the probability of losing, is $\frac{37}{38}$.

Summarizing these along with the values, we get this table:

Outcome	Probability of outcome
\$35	$\frac{1}{38}$
$-\$1$	$\frac{37}{38}$

Notice that if we multiply each outcome by its corresponding probability we get $\$35 \cdot \frac{1}{38} = 0.9211$ and $-\$1 \cdot \frac{37}{38} = -0.9737$, and if we add these numbers we get $0.9211 + (-0.9737) \approx -0.053$, which is the expected value we computed above.

Expected Value

Expected Value is the average gain or loss of an event if the procedure is repeated many times.

We can compute the expected value by multiplying each outcome by the probability of that outcome, then adding up the products.

Try it Now 12

You purchase a raffle ticket to help out a charity. The raffle ticket costs \$5. The charity is selling 2000 tickets. One of them will be drawn and the person holding the ticket will be given a prize worth \$4000. Compute the expected value for this raffle.

Answer

$$(\$3,995) \cdot \frac{1}{2000} + (-\$5) \cdot \frac{1999}{2000} \approx -\$3.00$$

Example 43

In a certain state's lottery, 48 balls numbered 1 through 48 are placed in a machine and six of them are drawn at random. If the six numbers drawn match the numbers that a player had chosen, the player wins \$1,000,000. If they match 5 numbers, then win \$1,000. It costs \$1 to buy a ticket. Find the expected value.

Solution

Earlier, we calculated the probability of matching all 6 numbers and the probability of matching 5 numbers:

$\frac{{}_6C_6}{{}_{48}C_6} = \frac{1}{12271512} \approx 0.0000000815$ for all 6 numbers,

$\frac{({}_6C_5)({}_{42}C_1)}{{}_{48}C_6} = \frac{252}{12271512} \approx 0.0000205$ for 5 numbers.

Our probabilities and outcome values are:

Outcome	Probability of outcome
$\$999,999$	$\frac{1}{12271512}$
$\$999$	$\frac{252}{12271512}$
$-\$1$	$1-\frac{253}{12271512}=\frac{12271259}{12271512}$

The expected value, then is:

$(\$999,999)\cdot\frac{1}{12271512}+(\$999)\cdot\frac{252}{12271512}+(-\$1)\cdot\frac{12271259}{12271512}\approx-\$0.898$

On average, one can expect to lose about 90 cents on a lottery ticket. Of course, most players will lose $1.

In general, if the expected value of a game is negative, it is not a good idea to play the game, since on average you will lose money. It would be better to play a game with a positive expected value (good luck trying to find one!), although keep in mind that even if the *average* winnings are positive it could be the case that most people lose money and one very fortunate individual wins a great deal of money. If the expected value of a game is 0, we call it a **fair game**, since neither side has an advantage.

Not surprisingly, the expected value for casino games is negative for the player, which is positive for the casino. It must be positive or they would go out of business. Players just need to keep in mind that when they play a game repeatedly, their expected value is negative. That is fine so long as you enjoy playing the game and think it is worth the cost. But it would be wrong to expect to come out ahead.

? Try it Now 13

A friend offers to play a game, in which you roll 3 standard 6-sided dice. If all the dice roll different values, you give him $1. If any two dice match values, you get $2. What is the expected value of this game? Would you play?

Answer

Suppose you roll the first die. The probability the second will be different is $\frac{5}{6}$. The probability that the third roll is different than the previous two is $\frac{4}{6}$, so the probability that the three dice are different is $\frac{5}{6}\cdot\frac{4}{6}=\frac{20}{36}$. The probability that two dice will match is the complement, $1-\frac{20}{36}=\frac{16}{36}$.

The expected value is: $(\$2)\cdot\frac{16}{36}+(-\$1)\cdot\frac{20}{36}=\frac{12}{36}\approx\$0.33$. Yes, it is in your advantage to play. On average, you'd win $0.33 per play.

Expected value also has applications outside of gambling. Expected value is very common in making insurance decisions.

✓ Example 44

A 40-year-old man in the U.S. has a 0.242% risk of dying during the next year[2]. An insurance company charges $275 for a life-insurance policy that pays a $100,000 death benefit. What is the expected value for the person buying the insurance?

Solution

The probabilities and outcomes are

Outcome	Probability of outcome
$\$100,000-\$275=\$99,725$	0.00242
$-\$275$	$1-0.00242=0.99758$

The expected value is $(\$99,725)(0.00242) + (-\$275)(0.99758) = -\$33$.

Not surprisingly, the expected value is negative; the insurance company can only afford to offer policies if they, on average, make money on each policy. They can afford to pay out the occasional benefit because they offer enough policies that those benefit payouts are balanced by the rest of the insured people.

For people buying the insurance, there is a negative expected value, but there is a security that comes from insurance that is worth that cost.

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[1] Photo CC-BY-SA http://www.flickr.com/photos/stoneflower/

[2] According to the estimator at www.numericalexample.com/inde...=article&id=91

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3.7: Exercises

1. A ball is drawn randomly from a jar that contains 6 red balls, 2 white balls, and 5 yellow balls. Find the probability of the given event.

a. A red ball is drawn
b. A white ball is drawn

2. Suppose you write each letter of the alphabet on a different slip of paper and put the slips into a hat. What is the probability of drawing one slip of paper from the hat at random and getting:

a. A consonant
b. A vowel

3. A group of people were asked if they had run a red light in the last year. 150 responded "yes", and 185 responded "no". Find the probability that if a person is chosen at random, they have run a red light in the last year.

4. In a survey, 205 people indicated they prefer cats, 160 indicated they prefer dots, and 40 indicated they don't enjoy either pet. Find the probability that if a person is chosen at random, they prefer cats.

5. Compute the probability of tossing a six-sided die (with sides numbered 1 through 6) and getting a 5.

6. Compute the probability of tossing a six-sided die and getting a 7.

7. Giving a test to a group of students, the grades and gender are summarized below. If one student was chosen at random, find the probability that the student was female.

	A	B	C	Total
Male	8	18	13	39
Female	10	4	12	26
Total	18	22	25	65

8. The table below shows the number of credit cards owned by a group of individuals. If one person was chosen at random, find the probability that the person had no credit cards.

	Zero	One	Two or more	Total
Male	9	5	19	33
Female	18	10	20	48
Total	27	15	39	81

9. Compute the probability of tossing a six-sided die and getting an even number.

10. Compute the probability of tossing a six-sided die and getting a number less than 3.

11. If you pick one card at random from a standard deck of cards, what is the probability it will be a King?

12. If you pick one card at random from a standard deck of cards, what is the probability it will be a Diamond?

13. Compute the probability of rolling a 12-sided die and getting a number other than 8.

14. If you pick one card at random from a standard deck of cards, what is the probability it is not the Ace of Spades?

15. Referring to the grade table from question #7, what is the probability that a student chosen at random did NOT earn a C?

16. Referring to the credit card table from question #8, what is the probability that a person chosen at random has at least one credit card?

17. A six-sided die is rolled twice. What is the probability of showing a 6 on both rolls?

18. A fair coin is flipped twice. What is the probability of showing heads on both flips?

19. A die is rolled twice. What is the probability of showing a 5 on the first roll and an even number on the second roll?

20. Suppose that 21% of people own dogs. If you pick two people at random, what is the probability that they both own a dog?

21. Suppose a jar contains 17 red marbles and 32 blue marbles. If you reach in the jar and pull out 2 marbles at random, find the probability that both are red.

22. Suppose you write each letter of the alphabet on a different slip of paper and put the slips into a hat. If you pull out two slips at random, find the probability that both are vowels.

23. Bert and Ernie each have a well-shuffled standard deck of 52 cards. They each draw one card from their own deck. Compute the probability that:

a. Bert and Ernie both draw an Ace.
b. Bert draws an Ace but Ernie does not.
c. neither Bert nor Ernie draws an Ace.
d. Bert and Ernie both draw a heart.
e. Bert gets a card that is not a Jack and Ernie draws a card that is not a heart.

24. Bert has a well-shuffled standard deck of 52 cards, from which he draws one card; Ernie has a 12-sided die, which he rolls at the same time Bert draws a card. Compute the probability that:

a. Bert gets a Jack and Ernie rolls a five.
b. Bert gets a heart and Ernie rolls a number less than six.
c. Bert gets a face card (Jack, Queen or King) and Ernie rolls an even number.
d. Bert gets a red card and Ernie rolls a fifteen.
e. Bert gets a card that is not a Jack and Ernie rolls a number that is not twelve.

25. Compute the probability of drawing a King from a deck of cards and then drawing a Queen.

26. Compute the probability of drawing two spades from a deck of cards.

27. A math class consists of 25 students, 14 female and 11 male. Two students are selected at random to participate in a probability experiment. Compute the probability that

a. a male is selected, then a female.
b. a female is selected, then a male.
c. two males are selected.
d. two females are selected.
e. no males are selected.

28. A math class consists of 25 students, 14 female and 11 male. Three students are selected at random to participate in a probability experiment. Compute the probability that

a. a male is selected, then two females.
b. a female is selected, then two males.
c. two females are selected, then one male.
d. three males are selected.
e. three females are selected.

29. Giving a test to a group of students, the grades and gender are summarized below. If one student was chosen at random, find the probability that the student was female and earned an A.

	A	B	C	Total
Male	8	18	13	39
Female	10	4	12	26
Total	18	22	25	65

30. The table below shows the number of credit cards owned by a group of individuals. If one person was chosen at random, find the probability that the person was male and had two or more credit cards.

	Zero	One	Two or more	Total
Male	9	5	19	33
Female	18	10	20	48
Total	27	15	39	81

31.A jar contains 6 red marbles numbered 1 to 6 and 8 blue marbles numbered 1 to 8. A marble is drawn at random from the jar. Find the probability the marble is red or odd-numbered.

32. A jar contains 4 red marbles numbered 1 to 4 and 10 blue marbles numbered 1 to 10. A marble is drawn at random from the jar. Find the probability the marble is blue or even-numbered.

33. Referring to the table from #29, find the probability that a student chosen at random is female or earned a B.

34. Referring to the table from #30, find the probability that a person chosen at random is male or has no credit cards.

35. Compute the probability of drawing the King of hearts or a Queen from a deck of cards.

36. Compute the probability of drawing a King or a heart from a deck of cards.

37. A jar contains 5 red marbles numbered 1 to 5 and 8 blue marbles numbered 1 to 8. A marble is drawn at random from the jar. Find the probability the marble is

a. Even-numbered given that the marble is red.
b. Red given that the marble is even-numbered.

38. A jar contains 4 red marbles numbered 1 to 4 and 8 blue marbles numbered 1 to 8. A marble is drawn at random from the jar. Find the probability the marble is

a. Odd-numbered given that the marble is blue.
b. Blue given that the marble is odd-numbered.

39. Compute the probability of flipping a coin and getting heads, given that the previous flip was tails.

40. Find the probability of rolling a “1” on a fair die, given that the last 3 rolls were all ones.

41. Suppose a math class contains 25 students, 14 females (three of whom speak French) and 11 males (two of whom speak French). Compute the probability that a randomly selected student speaks French, given that the student is female.

42. Suppose a math class contains 25 students, 14 females (three of whom speak French) and 11 males (two of whom speak French). Compute the probability that a randomly selected student is male, given that the student speaks French.

43. A certain virus infects one in every 400 people. A test used to detect the virus in a person is positive 90% of the time if the person has the virus and 10% of the time if the person does not have the virus. Let A be the event "the person is infected" and B be the event "the person tests positive".

a. Find the probability that a person has the virus given that they have tested positive, i.e. find $P(A|B)$.
b. Find the probability that a person does not have the virus given that they test negative, i.e. find $P(\text{ not } A|\text{ not } B)$.

44. A certain virus infects one in every 2000 people. A test used to detect the virus in a person is positive 96% of the time if the person has the virus and 4% of the time if the person does not have the virus. Let A be the event "the person is infected" and B be the event "the person tests positive".

a. Find the probability that a person has the virus given that they have tested positive, i.e. find $P(A|B)$.
b. Find the probability that a person does not have the virus given that they test negative, i.e. find $P(\text{ not } A|\text{ not } B)$.

45. A certain disease has an incidence rate of 0.3%. If the false negative rate is 6% and the false positive rate is 4%, compute the probability that a person who tests positive actually has the disease.

46. A certain disease has an incidence rate of 0.1%. If the false negative rate is 8% and the false positive rate is 3%, compute the probability that a person who tests positive actually has the disease.

47. A certain group of symptom-free women between the ages of 40 and 50 are randomly selected to participate in mammography screening. The incidence rate of breast cancer among such women is 0.8%. The false negative rate for the mammogram is 10%.

The false positive rate is 7%. If a the mammogram results for a particular woman are positive (indicating that she has breast cancer), what is the probability that she actually has breast cancer?

48. About 0.01% of men with no known risk behavior are infected with HIV. The false negative rate for the standard HIV test 0.01% and the false positive rate is also 0.01%. If a randomly selected man with no known risk behavior tests positive for HIV, what is the probability that he is actually infected with HIV?

49. A boy owns 2 pairs of pants, 3 shirts, 8 ties, and 2 jackets. How many different outfits can he wear to school if he must wear one of each item?

50. At a restaurant you can choose from 3 appetizers, 8 entrees, and 2 desserts. How many different three-course meals can you have?

51. How many three-letter "words" can be made from 4 letters "FGHI" if

a. repetition of letters is allowed
b. repetition of letters is not allowed

52. How many four-letter "words" can be made from 6 letters "AEBWDP" if

a. repetition of letters is allowed
b. repetition of letters is not allowed

53. All of the license plates in a particular state feature three letters followed by three digits (e.g. ABC 123). How many different license plate numbers are available to the state's Department of Motor Vehicles?

54. A computer password must be eight characters long. How many passwords are possible if only the 26 letters of the alphabet are allowed?

55. A pianist plans to play 4 pieces at a recital. In how many ways can she arrange these pieces in the program?

56. In how many ways can first, second, and third prizes be awarded in a contest with 210 contestants?

57. Seven Olympic sprinters are eligible to compete in the 4 x 100 m relay race for the USA Olympic team. How many four-person relay teams can be selected from among the seven athletes?

58. A computer user has downloaded 25 songs using an online file-sharing program and wants to create a CD-R with ten songs to use in his portable CD player. If the order that the songs are placed on the CD-R is important to him, how many different CD-Rs could he make from the 25 songs available to him?

59. In western music, an octave is divided into 12 pitches. For the film *Close Encounters of the Third Kind*, director Steven Spielberg asked composer John Williams to write a five-note theme, which aliens would use to communicate with people on Earth. Disregarding rhythm and octave changes, how many five-note themes are possible if no note is repeated?

60. In the early twentieth century, proponents of the Second Viennese School of musical composition (including Arnold Schönberg, Anton Webern and Alban Berg) devised the twelve-tone technique, which utilized a tone row consisting of all 12 pitches from the chromatic scale in any order, but with not pitches repeated in the row. Disregarding rhythm and octave changes, how many tone rows are possible?

61. In how many ways can 4 pizza toppings be chosen from 12 available toppings?

62. At a baby shower 17 guests are in attendance and 5 of them are randomly selected to receive a door prize. If all 5 prizes are identical, in how many ways can the prizes be awarded?

63. In the 6/50 lottery game, a player picks six numbers from 1 to 50. How many different choices does the player have if order doesn’t matter?

64. In a lottery daily game, a player picks three numbers from 0 to 9. How many different choices does the player have if order doesn’t matter?

65. A jury pool consists of 27 people. How many different ways can 11 people be chosen to serve on a jury and one additional person be chosen to serve as the jury foreman?

66. The United States Senate Committee on Commerce, Science, and Transportation consists of 23 members, 12 Republicans and 11 Democrats. The Surface Transportation and Merchant Marine Subcommittee consists of 8 Republicans and 7 Democrats. How many ways can members of the Subcommittee be chosen from the Committee?

67. You own 16 CDs. You want to randomly arrange 5 of them in a CD rack. What is the probability that the rack ends up in alphabetical order?

68. A jury pool consists of 27 people, 14 men and 13 women. Compute the probability that a randomly selected jury of 12 people is all male.

69. In a lottery game, a player picks six numbers from 1 to 48. If 5 of the 6 numbers match those drawn, they player wins second prize. What is the probability of winning this prize?

70. In a lottery game, a player picks six numbers from 1 to 48. If 4 of the 6 numbers match those drawn, they player wins third prize. What is the probability of winning this prize?

71. Compute the probability that a 5-card poker hand is dealt to you that contains all hearts.

72. Compute the probability that a 5-card poker hand is dealt to you that contains four Aces.

73. A bag contains 3 gold marbles, 6 silver marbles, and 28 black marbles. Someone offers to play this game: You randomly select on marble from the bag. If it is gold, you win $3. If it is silver, you win $2. If it is black, you lose $1. What is your expected value if you play this game?

74. A friend devises a game that is played by rolling a single six-sided die once. If you roll a 6, he pays you $3; if you roll a 5, he pays you nothing; if you roll a number less than 5, you pay him $1. Compute the expected value for this game. Should you play this game?

75. In a lottery game, a player picks six numbers from 1 to 23. If the player matches all six numbers, they win 30,000 dollars. Otherwise, they lose $1. Find the expected value of this game.

76. A game is played by picking two cards from a deck. If they are the same value, then you win $5, otherwise you lose $1. What is the expected value of this game?

77. A company estimates that 0.7% of their products will fail after the original warranty period but within 2 years of the purchase, with a replacement cost of $350. If they offer a 2 year extended warranty for $48, what is the company's expected value of each warranty sold?

78. An insurance company estimates the probability of an earthquake in the next year to be 0.0013. The average damage done by an earthquake it estimates to be $60,000. If the company offers earthquake insurance for $100, what is their expected value of the policy?

Exploration

Some of these questions were adapted from puzzles at mindyourdecisions.com.

79. A small college has been accused of gender bias in its admissions to graduate programs.

a. Out of 500 men who applied, 255 were accepted. Out of 700 women who applied, 240 were accepted. Find the acceptance rate for each gender. Does this suggest bias?
b. The college then looked at each of the two departments with graduate programs, and found the data below. Compute the acceptance rate within each department by gender. Does this suggest bias?
c. Looking at our results from Parts *a* and *b*, what can you conclude? Is there gender bias in this college's admissions? If so, in which direction?

80. A bet on "black" in Roulette has a probability of 18/38 of winning. If you win, you double your money. You can bet anywhere from $1 to $100 on each spin.

a. Suppose you have $10, and are going to play until you go broke or have $20. What is your best strategy for playing?
b. Suppose you have $10, and are going to play until you go broke or have $30. What is your best strategy for playing?

81. Your friend proposes a game: You flip a coin. If it's heads, you win $1. If it's tails, you lose $1. However, you are worried the coin might not be fair coin. How could you change the game to make the game fair, without replacing the coin?

82. Fifty people are in a line. The first person in the line to have a birthday matching someone in front of them will win a prize. Of course, this means the first person in the line has no chance of winning. Which person has the highest likelihood of winning?

83. Three people put their names in a hat, then each draw a name, as part of a randomized gift exchange. What is the probability that no one draws their own name? What about with four people?

84. How many different "words" can be formed by using all the letters of each of the following words exactly once?

a. "ALICE"
b. "APPLE"

85. How many different "words" can be formed by using all the letters of each of the following words exactly once?

a. "TRUMPS"
b. "TEETER"

86. The *Monty Hall problem* is named for the host of the game show *Let's make a Deal*. In this game, there would be three doors, behind one of which there was a prize. The contestant was asked to choose one of the doors. Monty Hall would then open one of the other doors to show there was no prize there. The contestant was then asked if they wanted to stay with their original door, or switch to the other unopened door. Is it better to stay or switch, or does it matter?

87. Suppose you have two coins, where one is a fair coin, and the other coin comes up heads 70% of the time. What is the probability you have the fair coin given each of the following outcomes from a series of flips?

a. 5 Heads and 0 Tails
b. 8 Heads and 3 Tails
c. 10 Heads and 10 Tails
d. 3 Heads and 8 Tails

88. Suppose you have six coins, where five are fair coins, and one coin comes up heads 80% of the time. What is the probability you have a fair coin given each of the following outcomes from a series of flips?

a. 5 Heads and 0 Tails
b. 8 Heads and 3 Tails
c. 10 Heads and 10 Tails
d. 3 Heads and 8 Tails

89. In this problem, we will explore probabilities from a series of events.

a. If you flip 20 coins, how many would you *expect* to come up "heads", on average? Would you expect *every* flip of 20 coins to come up with exactly that many heads?
b. If you were to flip 20 coins, what would you consider a "usual" result? An "unusual" result?
c. Flip 20 coins (or one coin 20 times) and record how many come up "heads". Repeat this experiment 9 more times. Collect the data from the entire class.
d. When flipping 20 coins, what is the theoretic probability of flipping 20 heads?
e. Based on the class's experimental data, what appears to be the probability of flipping 10 heads out of 20 coins?
f. The formula ${}_nC_x p^x (1-p)^{n-x}$ will compute the probability of an event with probability p occurring x times out of n, such as flipping x heads out of n coins where the probability of heads is $p = \frac{1}{2}$. Use this to compute the theoretic probability of flipping 10 heads out of 20 coins.
g. If you were to flip 20 coins, based on the class's experimental data, what range of values would you consider a "usual" result? What is the combined probability of these results? What would you consider an "unusual" result? What is the combined probability of these results?
h. We'll now consider a simplification of a case from the 1960s. In the area, about 26% of the jury eligible population was black. In the court case, there were 100 men on the juror panel, of which 8 were black. Does this provide evidence of racial bias in jury selection?

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CHAPTER OVERVIEW

4: Descriptive Statistics

Statistics are often presented in an effort to add credibility to an argument or advice. You can see this by paying attention to television advertisements. Many of the numbers thrown about in this way do not represent careful statistical analysis. They can be misleading, and push you into decisions that you might find cause to regret. These chapters will help you learn statistical essentials. It will make you into an intelligent consumer of statistical claims.

4.1: Numerical Summaries of Data

4.2: Measures of Central Tendency

4.3: Measures of Variation

4.4: Exercises

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4.1: Numerical Summaries of Data

It is often desirable to use a few numbers to summarize a distribution. One important aspect of a distribution is where its center is located. Measures of central tendency are discussed first. A second aspect of a distribution is how spread out it is. In other words, how much the data in the distribution vary from one another. The second section describes measures of variability.

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4.2: Measures of Central Tendency

Let's begin by trying to find the most "typical" value of a data set.

Note that we just used the word "typical" although in many cases you might think of using the word "average." We need to be careful with the word "average" as it means different things to different people in different contexts. One of the most common uses of the word "average" is what mathematicians and statisticians call the **arithmetic mean**, or just plain old **mean** for short. "Arithmetic mean" sounds rather fancy, but you have likely calculated a mean many times without realizing it; the mean is what most people think of when they use the word "average".

Mean

The **mean** of a set of data is the sum of the data values divided by the number of values.

Example 14

Marci's exam scores for her last math class were: 79, 86, 82, 94. The mean of these values would be:

Solution

$$\frac{79+86+82+94}{4} = 85.25.$$

Typically we round means to one more decimal place than the original data had. In this case, we would round 85.25 to 85.3.

Example 15

The number of touchdown (TD) passes thrown by each of the 31 teams in the National Football League in the 2000 season are shown below.

37 33 33 32 29 28 28 23 22 22 22 21 21 21 20

20 19 19 18 18 18 18 16 15 14 14 14 12 12 9 6

Solution

Adding these values, we get 634 total TDs. Dividing by 31, the number of data values, we get $\frac{634}{31} = 20.4516$. It would be appropriate to round this to 20.5.

It would be most correct for us to report that "The mean number of touchdown passes thrown in the NFL in the 2000 season was 20.5 passes," but it is not uncommon to see the more casual word "average" used in place of "mean.

Try it Now 4

The price of a jar of peanut butter at 5 stores was: $3.29, $3.59, $3.79, $3.75, and $3.99. Find the mean price.

Answer

Adding the prices and dividing by 5 we get the mean price: $3.682.

Example 16

The one hundred families in a particular neighborhood are asked their annual household income, to the nearest $5 thousand dollars. The results are summarized in a frequency table below.

Income (thousands of dollars)	Frequency
15	6
20	8
25	11
30	17
35	19
40	20
45	12
50	7

Solution

Calculating the mean by hand could get tricky if we try to type in all 100 values:

$$\frac{15+\cdots+15+20+\cdots+20+25+\cdots+25+\cdots}{100}$$

We could calculate this more easily by noticing that adding 15 to itself six times is the same as $15 \cdot 6 = 90$. Using this simplification, we get

$$\frac{15 \cdot 6+20 \cdot 8+25 \cdot 11+30 \cdot 17+35 \cdot 19+40 \cdot 20+45 \cdot 12+50 \cdot 7}{100}=\frac{3390}{100}=33.9$$

The mean household income of our sample is 33.9 thousand dollars ($33,900).

Example 17

Extending off the last example, suppose a new family moves into the neighborhood example that has a household income of $5 million ($5000 thousand). Adding this to our sample, our mean is now:

Solution

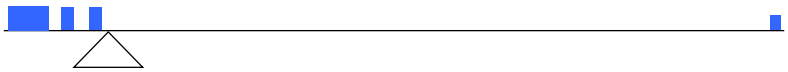
$$\frac{15 \cdot 6+20 \cdot 8+25 \cdot 11+30 \cdot 17+35 \cdot 19+40 \cdot 20+45 \cdot 12+50 \cdot 7+5000 \cdot 1}{101}$$
$$=\frac{8390}{101}=83.069$$

While 83.1 thousand dollars ($83,069) is the correct mean household income, it no longer represents a “typical” value.

Imagine the data values on a see-saw or balance scale. The mean is the value that keeps the data in balance, like in the picture below.



If we graph our household data, the $5 million data value is so far out to the right that the mean has to adjust up to keep things in balance



For this reason, when working with data that have **outliers** – values far outside the primary grouping – it is common to use a different measure of center, the **median**.

Median

To find the median, begin by listing the data in order from smallest to largest, or largest to smallest.

If the number of data values, N, is odd, then the median is the middle data value. This value can be found by rounding $\frac{N}{2}$ up to the next whole number.

If the number of data values is even, there is no one middle value, so we find the mean of the two middle values (values $\frac{N}{2}$ and $\frac{N}{2}+1$)

Example 18

Returning to the football touchdown data, we would start by listing the data in order. Luckily, it was already in decreasing order, so we can work with it without needing to reorder it first.

37 33 33 32 29 28 28 23 22 22 22 21 21 21 20

20 19 19 18 18 18 18 16 15 14 14 14 12 12 9 6

Solution

Since there are 31 data values, an odd number, the median will be the middle number, the 16th data value (\frac{31}{2} = 15.5\), round up to 16, leaving 15 values below and 15 above). The 16th data value is 20, so the median number of touchdown passes in the 2000 season was 20 passes. Notice that for this data, the median is fairly close to the mean we calculated earlier, 20.5.

Example 19

Find the median of these quiz scores: 5 10 8 6 4 8 2 5 7 7

We start by listing the data in order: 2 4 5 5 6 7 7 8 8 10

Solution

Since there are 10 data values, an even number, there is no one middle number. So we find the mean of the two middle numbers, 6 and 7, and get $\frac{6+7}{2} = 6.5$.

The median quiz score was 6.5.

Try it Now 5

The price of a jar of peanut butter at 5 stores were: $3.29, $3.59, $3.79, $3.75, and $3.99. Find the median price.

Answer

First we put the data in order: $3.29, $3.59, $3.75, $3.79, $3.99. Since there are an odd number of data, the median will be the middle value, $3.75.

Example 20

Let us return now to our original household income data

Income (thousands of dollars)	Frequency
15	6
20	8
25	11
30	17
35	19
40	20
45	12
50	7

Solution

Here we have 100 data values. If we didn't already know that, we could find it by adding the frequencies. Since 100 is an even number, we need to find the mean of the middle two data values - the 50th and 51st data values. To find these, we start counting

up from the bottom:

There are 6 data values of \$15, so	Values 1 to 6 are \$15 thousand
The next 8 data values are \$20, so	Values 7 to $(6+8)=14$ are \$20 thousand
The next 11 data values are \$25, so	Values 15 to $(14+11)=25$ are \$25 thousand
The next 17 data values are \$30, so	Values 26 to $(25+17)=42$ are \$30 thousand
The next 19 data values are \$35, so	Values 43 to $(42+19)=61$ are \$35 thousand

From this we can tell that values 50 and 51 will be \$35 thousand, and the mean of these two values is \$35 thousand. The median income in this neighborhood is \$35 thousand.

Example 21

If we add in the new neighbor with a \$5 million household income, then there will be 101 data values, and the 51^{st} value will be the median. As we discovered in the last example, the 51^{st} value is \$35 thousand. Notice that the new neighbor did not affect the median in this case. The median is not swayed as much by outliers as the mean is.

In addition to the mean and the median, there is one other common measurement of the "typical" value of a data set: the **mode**.

Mode

The **mode** is the element of the data set that occurs most frequently.

The mode is fairly useless with data like weights or heights where there are a large number of possible values. The mode is most commonly used for categorical data, for which median and mean cannot be computed.

Example 22

In our vehicle color survey, we collected the data

Color	Frequency
Blue	3
Green	5
Red	4
White	3
Black	2
Grey	3

For this data, Green is the mode, since it is the data value that occurred the most frequently.

It is possible for a data set to have more than one mode if several categories have the same frequency, or no modes if each every category occurs only once.

Try it Now 6

Reviewers were asked to rate a product on a scale of 1 to 5. Find

a. The mean rating
b. The median rating
c. The mode rating

Rating	Frequency
1	4
2	8
3	7
4	3
5	1

Answer

a. The mean is $\frac{1 \cdot 4 + 2 \cdot 8 + 3 \cdot 7 + 4 \cdot 3 + 5 \cdot 1}{23} \approx 2.5$

b. There are 23 data values, so the median will be the 12^{th} data value. Ratings of 1 are the first 4 values, while a rating of 2 are the next 8 values, so the 12^{th} value will be a rating of 2. The median is 2.

c. The mode is the most frequent rating. The mode rating is 2.

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4.3: Measures of Variation

Consider these three sets of quiz scores:

- Section A: 5 5 5 5 5 5 5 5 5 5
- Section B: 0 0 0 0 0 10 10 10 10 10
- Section C: 4 4 4 5 5 5 5 6 6 6

All three of these sets of data have a mean of 5 and median of 5, yet the sets of scores are clearly quite different. In section A, everyone had the same score; in section B half the class got no points and the other half got a perfect score, assuming this was a 10-point quiz. Section C was not as consistent as section A, but not as widely varied as section B.

In addition to the mean and median, which are measures of the "typical" or "middle" value, we also need a measure of how "spread out" or varied each data set is.

There are several ways to measure this "spread" of the data. The first is the simplest and is called the **range**.

Range

The range is the difference between the maximum value and the minimum value of the data set.

Example 23

Using the quiz scores from above,

For section A, the range is 0 since both maximum and minimum are 5 and $5–5 = 0$

For section B, the range is 10 since $10–0 = 10$

For section C, the range is 2 since $6–4 = 2$

In the last example, the range seems to be revealing how spread out the data is. However, suppose we add a fourth section, Section D, with scores 0 5 5 5 5 5 5 5 5 10.

This section also has a mean and median of 5. The range is 10, yet this data set is quite different than Section B. To better illuminate the differences, we'll have to turn to more sophisticated measures of variation.

Standard deviation

The standard deviation is a measure of variation based on measuring how far each data value deviates, or is different, from the mean. A few important characteristics:

- Standard deviation is always positive. Standard deviation will be zero if all the data values are equal, and will get larger as the data spreads out.
- Standard deviation has the same units as the original data.
- Standard deviation, like the mean, can be highly influenced by outliers.

Using the data from section D, we could compute for each data value the difference between the data value and the mean:

data value	deviation: data value - mean
0	$0-5=-5$
5	$5-5=0$
5	$5-5=0$
5	$5-5=0$
5	$5-5=0$
5	$5-5=0$
5	$5-5=0$
5	$5-5=0$
5	$5-5=0$
10	$10-5=5$

We would like to get an idea of the "average" deviation from the mean, but if we find the average of the values in the second column the negative and positive values cancel each other out (this will always happen), so to prevent this we square every value in the second column:

We then add the squared deviations up to get $25+0+0+0+0+0+0+0+0+25=50$. Ordinarily we would then divide by the number of scores, n, (in this case, 10) to find the mean of the deviations. But we only do this if the data set represents a population; if the data set represents a sample (as it almost always does), we instead divide by $n-1$ (in this case, $10-1=9$).[1]

So in our example, we would have $\frac{50}{10}=5$ if section D represents a population and $\frac{50}{9}=$ about 5.56 if section D represents a sample. These values (5 and 5.56) are called, respectively, the **population variance** and the **sample variance** for section D.

Variance can be a useful statistical concept, but note that the units of variance in this instance would be points-squared since we squared all of the deviations. What are points-squared? Good question. We would rather deal with the units we started with (points in this case), so to convert back we take the square root and get:

$$\text{population standard deviation}=\sqrt{\frac{50}{10}}=\sqrt{5}\approx 2.2$$

or

$$\text{sample standard deviation}=\sqrt{\frac{50}{9}}\approx 2.4$$

If we are unsure whether the data set is a sample or a population, we will usually assume it is a sample, and we will round answers to one more decimal place than the original data, as we have done above.

To compute standard deviation:

1. Find the deviation of each data from the mean. In other words, subtract the mean from the data value.
2. Square each deviation.
3. Add the squared deviations.
4. Divide by n, the number of data values, if the data represents a whole population; divide by n–1 if the data is from a sample.
5. Compute the square root of the result.

Example 24

Computing the standard deviation for Section B above, we first calculate that the mean is 5. Using a table can help keep track of your computations for the standard deviation:

data value	deviation: data value - mean	deviation squared
0	$0-5=-5$	$(-5)^2=25$
0	$0-5=-5$	$(-5)^2=25$
0	$0-5=-5$	$(-5)^2=25$
0	$0-5=-5$	$(-5)^2=25$
0	$0-5=-5$	$(-5)^2=25$
10	$10-5=5$	$(5)^2=25$
10	$10-5=5$	$(5)^2=25$
10	$10-5=5$	$(5)^2=25$
10	$10-5=5$	$(5)^2=25$
10	$10-5=5$	$(5)^2=25$

Assuming this data represents a population, we will add the squared deviations, divide by 10, the number of data values, and compute the square root:

$$\sqrt{\frac{25+25+25+25+25+25+25+25+25+25}{10}}=\sqrt{\frac{250}{10}}=5$$

Notice that the standard deviation of this data set is much larger than that of section D since the data in this set is more spread out.

For comparison, the standard deviations of all four sections are:

Section A: 5 5 5 5 5 5 5 5 5 5	Standard deviation: 0
Section B: 0 0 0 0 0 10 10 10 10 10	Standard deviation: 5
Section C: 4 4 4 5 5 5 5 6 6 6	Standard deviation: 0.8
Section D: 0 5 5 5 5 5 5 5 5 10	Standard deviation: 2.2

Try it Now 7

The price of a jar of peanut butter at 5 stores were: \$3.29, \$3.59, \$3.79, \$3.75, and \$3.99. Find the standard deviation of the prices.

Answer

Earlier we found the mean of the data was \$3.682.

data value	deviation: data value - mean	deviation squared
3.29	$3.29-3.682=-0.391$	0.153664
3.59	$3.59-3.682=-0.092$	0.008464
3.79	$3.79-3.682=0.108$	0.011664
3.75	$3.75-3.682=0.068$	0.004624
3.99	$3.99-3.682=0.308$	0.094864

This data is from a sample, so we will add the squared deviations, divide by 4, the number of data values minus 1, and compute the square root:

$\sqrt{\frac{0.153664+0.008464+0.011664+0.004624+0.094864}{4}}\approx \$0.261$

Where standard deviation is a measure of variation based on the mean, **quartiles** are based on the median.

Quartiles

Quartiles are values that divide the data in quarters.

The first quartile (Q_1) is the value so that 25% of the data values are below it; the third quartile (Q_3) is the value so that 75% of the data values are below it. You may have guessed that the second quartile is the same as the median, since the median is the value so that 50% of the data values are below it.

This divides the data into quarters; 25% of the data is between the minimum and Q_1, 25% is between Q_1 and the median, 25% is between the median and Q_3, and 25% is between Q_3 and the maximum value

While quartiles are not a 1-number summary of variation like standard deviation, the quartiles are used with the median, minimum, and maximum values to form a **5 number summary** of the data.

Five number summary

The five number summary takes this form:

Minimum, Q_1, Median, Q_3, Maximum

To find the first quartile, we need to find the data value so that 25% of the data is below it. If n is the number of data values, we compute a locator by finding 25% of n. If this locator is a decimal value, we round up, and find the data value in that position. If the locator is a whole number, we find the mean of the data value in that position and the next data value. This is identical to the process we used to find the median, except we use 25% of the data values rather than half the data values as the locator.

To find the first quartile, Q_1

Begin by ordering the data from smallest to largest

Compute the locator: $L = 0.25n$

If L is a decimal value:

Round up to $L+$

Use the data value in the $L+^{\text{th}}$ position

If L is a whole number:

Find the mean of the data values in the L^{th} and $L+1^{\text{th}}$ positions.

To find the third quartile, Q_3

Use the same procedure as for Q_1, but with locator: $L = 0.75n$

Examples should help make this clearer.

Example 25

Suppose we have measured 9 females and their heights (in inches), sorted from smallest to largest are:

59 60 62 64 66 67 69 70 72

To find the first quartile we first compute the locator: 25% of 9 is $L = 0.25(9) = 2.25$. Since this value is not a whole number, we round up to 3. The first quartile will be the third data value: 62 inches.

To find the third quartile, we again compute the locator: 75% of 9 is $0.75(9) = 6.75$. Since this value is not a whole number, we round up to 7. The third quartile will be the seventh data value: 69 inches.

Example 26

Suppose we had measured 8 females and their heights (in inches), sorted from smallest to largest are:

59 60 62 64 66 67 69 70

To find the first quartile we first compute the locator: 25% of 8 is $L = 0.25(8) = 2$. Since this value is a whole number, we will find the mean of the 2nd and 3rd data values: (60+62)/2 = 61, so the first quartile is 61 inches.

The third quartile is computed similarly, using 75% instead of 25%. $L = 0.75(8) = 6$. This is a whole number, so we will find the mean of the 6th and 7th data values: $\frac{67+69}{2} = 68$, so Q_3 is 68.

Note that the median could be computed the same way, using 50%.

The 5-number summary combines the first and third quartile with the minimum, median, and maximum values.

Example 27

For the 9 female sample, the median is 66, the minimum is 59, and the maximum is 72. The 5 number summary is: 59, 62, 66, 69, 72.

For the 8 female sample, the median is 65, the minimum is 59, and the maximum is 70, so the 5 number summary would be: 59, 61, 65, 68, 70.

Example 28

Returning to our quiz score data. In each case, the first quartile locator is $0.25(10) = 2.5$, so the first quartile will be the 3rd data value, and the third quartile will be the 8th data value. Creating the five-number summaries:

Section and data	5-number summary
Section A: 5 5 5 5 5 5 5 5 5 5	5, 5, 5, 5, 5
Section B: 0 0 0 0 0 10 10 10 10 10	0, 0, 5, 10, 10
Section C: 4 4 4 5 5 5 5 6 6 6	4, 4, 5, 6, 6
Section D: 0 5 5 5 5 5 5 5 5 10	0, 5, 5, 5, 10

Of course, with a relatively small data set, finding a five-number summary is a bit silly, since the summary contains almost as many values as the original data.

Try it Now 8

The total cost of textbooks for the term was collected from 36 students. Find the 5 number summary of this data.

$140 $160 $160 $165 $180 $220 $235 $240 $250 $260 $280 $285

$285 $285 $290 $300 $300 $305 $310 $310 $315 $315 $320 $320

$330 $340 $345 $350 $355 $360 $360 $380 $395 $420 $460 $460

Answer

The data is already in order, so we don't need to sort it first.

The minimum value is $140 and the maximum is $460.

There are 36 data values so $n = 36. n/2 = 18$, which is a whole number, so the median is the mean of the 18^{th} and 19^{th} data values, $\$305$ and $\$310$. The median is $\$307.50$

To find the first quartile, we calculate the locator, $L = 0.25(36) = 9$. since this is a whole number, we know Q_1 is the mean of the 9^{th} and 10^{th} data values, $\$250$ and $\$260$. $Q_1 = \$255$

To find the third quartile, we calculate the locator, $L = 0.75(36) = 27$. since this is a whole number, we know Q_3 is the mean of the 27^{th} and 28^{th} data values, $\$345$ and $\$350$.
$Q_3 = \$347.50$

The 5 number summary of this data is: $140, $255, $307.50, $347.50, $460

Example 29

Returning to the household income data from earlier, create the five-number summary.

Income (thousands of dollars)	Frequency
15	6
20	8
25	11
30	17
35	19
40	20
45	12
50	7

Solution

By adding the frequencies, we can see there are 100 data values represented in the table. In Example 20, we found the median was \$35 thousand. We can see in the table that the minimum income is \$15 thousand, and the maximum is \$50 thousand.

To find Q_1, we calculate the locator: L = 0.25(100) = 25. This is a whole number, so Q_1 will be the mean of the 25th and 26th data values.

Counting up in the data as we did before,

There are 6 data values of \$15, so	Values 1 to 6 are \$15 thousand
The next 8 data values are \$20, so	Values 7 to (6+8)=14 are \$20 thousand
The next 11 data values are \$25, so	Values 15 to (14+11)=25 are \$25 thousand
The next 17 data values are \$30, so	Values 26 to (25+17)=42 are \$30 thousand

The 25th data value is \$25 thousand, and the 26th data value is \$30 thousand, so Q_1 will be the mean of these: $(25 + 30)/2 = \$27.5$ thousand.

To find Q_3, we calculate the locator: $L = 0.75(100) = 75$. This is a whole number, so Q_3 will be the mean of the 75th and 76th data values. Continuing our counting from earlier,

Math mode is not properly terminated

Both the 75th and 76th data values lie in this group, so Q_3 will be \$40 thousand.

Putting these values together into a five-number summary, we get: 15, 27.5, 35, 40, 50

Note that the 5 number summary divides the data into four intervals, each of which will contain about 25% of the data. In the previous example, that means about 25% of households have income between \$40 thousand and \$50 thousand.

For visualizing data, there is a graphical representation of a 5-number summary called a **box plot**, or box and whisker graph.

Box plot

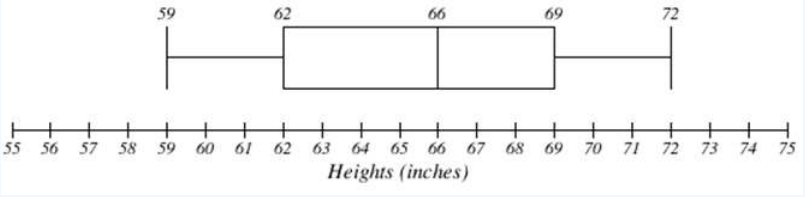
A **box plot** is a graphical representation of a five-number summary.

To create a box plot, a number line is first drawn. A box is drawn from the first quartile to the third quartile, and a line is drawn through the box at the median. “Whiskers” are extended out to the minimum and maximum values.

Example 30

The box plot below is based on the 9 female height data with 5 number summary:

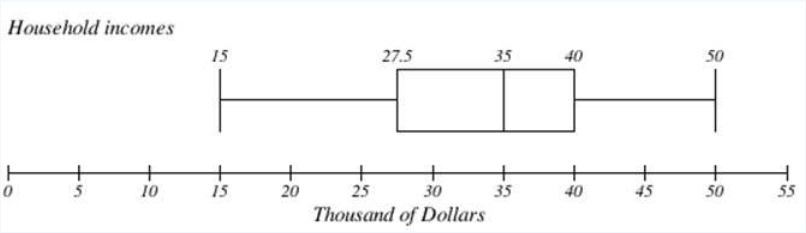
59, 62, 66, 69, 72.



Example 31

The box plot below is based on the household income data with 5 number summary:

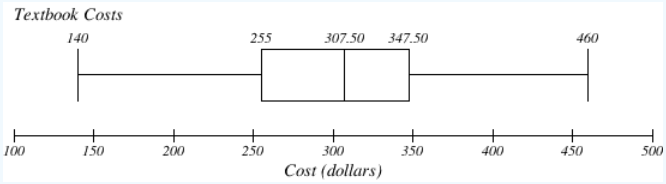
15, 27.5, 35, 40, 50



Try it Now 9

Create a boxplot based on the textbook price data from the last Try it Now.

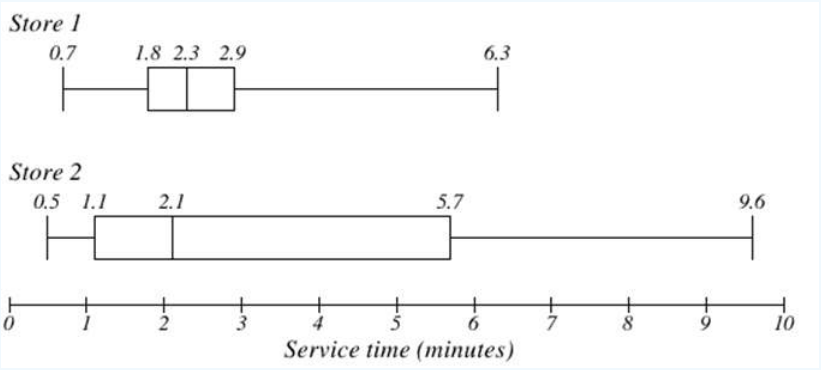
Answer



Box plots are particularly useful for comparing data from two populations.

Example 32

The box plot of service times for two fast-food restaurants is shown below.



While store 2 had a slightly shorter median service time (2.1 minutes vs. 2.3 minutes), store 2 is less consistent, with a wider spread of the data.

At store 1, 75% of customers were served within 2.9 minutes, while at store 2, 75% of customers were served within 5.7 minutes.

Which store should you go to in a hurry? That depends upon your opinions about luck – 25% of customers at store 2 had to wait between 5.7 and 9.6 minutes.

Example 33

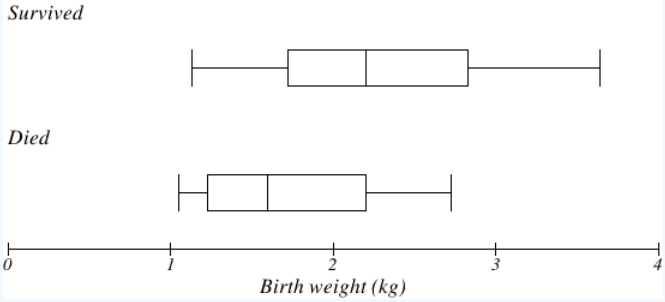
The boxplot below is based on the birth weights of infants with severe idiopathic respiratory distress syndrome (SIRDS)[2]. The boxplot is separated to show the birth weights of infants who survived and those that did not.

Comparing the two groups, the boxplot reveals that the birth weights of the infants that died appear to be, overall, smaller than the weights of infants that survived. In fact, we can see that the median birth weight of infants that survived is the same as the third quartile of the infants that died.

Similarly, we can see that the first quartile of the survivors is larger than the median weight of those that died, meaning that over 75% of the survivors had a birth weight larger than the median birth weight of those that died.

Looking at the maximum value for those that died and the third quartile of the survivors, we can see that over 25% of the survivors had birth weights higher than the heaviest infant that died.

The box plot gives us a quick, albeit informal, way to determine that birth weight is quite likely linked to survival of infants with SIRDS.



[1] The reason we do this is highly technical, but we can see how it might be useful by considering the case of a small sample from a population that contains an outlier, which would increase the average deviation: the outlier very likely won't be included in the sample, so the mean deviation of the sample would underestimate the mean deviation of the population; thus we divide by a slightly smaller number to get a slightly bigger average deviation.

[2] van Vliet, P.K. and Gupta, J.M. (1973) Sodium bicarbonate in idiopathic respiratory distress syndrome. *Arch. Disease in Childhood*, **48**, 249–255. As quoted on http://openlearn.open.ac.uk/mod/ouco...&section=1.1.3

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4.4: Exercises

Skills

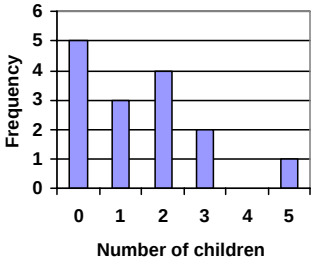
1. The table below shows scores on a Math test.
   a. Complete the frequency table for the Math test scores
   b. Construct a histogram of the data
   c. Construct a pie chart of the data

80	50	50	90	70	70	100	60	70	80	70	50
90	100	80	70	30	80	80	70	100	60	60	50

2. A group of adults where asked how many cars they had in their household
   a. Complete the frequency table for the car number data
   b. Construct a histogram of the data
   c. Construct a pie chart of the data

1	4	2	2	1	2	3	3	1	4	2	2
1	2	1	3	2	2	1	2	1	1	1	2

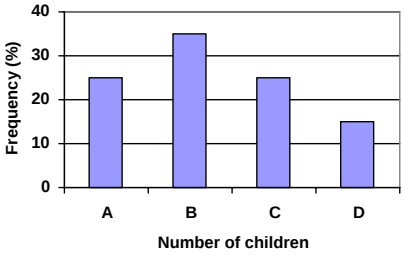
3. A group of adults were asked how many children they have in their families. The bar graph to the right shows the number of adults who indicated each number of children.
   a. How many adults where questioned?
   b. What percentage of the adults questioned had 0 children?



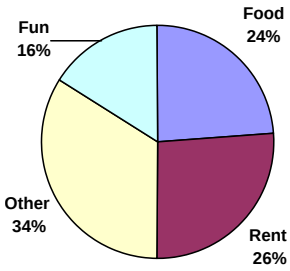
4. Jasmine was interested in how many days it would take an order from Netflix to arrive at her door. The graph below shows the data she collected.
   a. How many movies did she order?
   b. What percentage of the movies arrived in one day?



5. The bar graph below shows the *percentage* of students who received each letter grade on their last English paper. The class contains 20 students. What number of students earned an A on their paper?



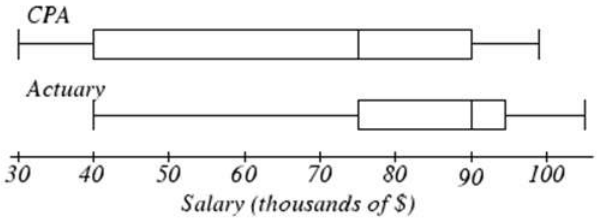
6. Kori categorized her spending for this month into four categories: Rent, Food, Fun, and Other. The percents she spent in each category are pictured here. If she spent a total of $2600 this month, how much did she spend on rent?



7. A group of diners were asked how much they would pay for a meal. Their responses were: $7.50, $8.25, $9.00, $8.00, $7.25, $7.50, $8.00, $7.00.
   a. Find the mean
   b. Find the median
   c. Write the 5-number summary for this data
   d. Find the standard deviation of this data
8. You recorded the time in seconds it took for 8 participants to solve a puzzle. The times were: 15.2, 18.8, 19.3, 19.7, 20.2, 21.8, 22.1, 29.4.
   a. Find the mean
   b. Find the median
   c. Write the 5-number summary for this data
   d. Find the standard deviation of this data
9. Refer back to the histogram from question #3.
   a. Compute the mean number of children for the group surveyed
   b. Compute the median number of children for the group surveyed
   c. Write the 5-number summary for this data.
   d. Create box plot.
10. Refer back to the histogram from question #4.
    a. Compute the mean number of shipping days
    b. Compute the median number of shipping days
    c. Write the 5-number summary for this data.
    d. Create box plot.

Concepts

11. The box plot below shows salaries for Actuaries and CPAs. Kendra makes the median salary for an Actuary. Kelsey makes the first quartile salary for a CPA. Who makes more money? How much more?



12. Referring to the boxplot above, what percentage of actuaries makes more than the median salary of a CPA?

Exploration

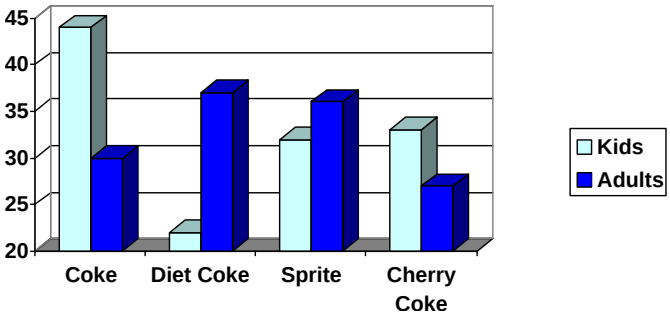
13. Studies are often done by pharmaceutical companies to determine the effectiveness of a treatment program. Suppose that a new AIDS antibody drug is currently under study. It is given to patients once the AIDS symptoms have revealed themselves. Of interest is the average length of time in months patients live once starting the treatment. Two researchers each follow a different set of 40 AIDS patients from the start of treatment until their deaths. The following data (in months) are collected.

Researcher 1: 3; 4; 11; 15; 16; 17; 22; 44; 37; 16; 14; 24; 25; 15; 26; 27; 33; 29; 35; 44; 13; 21; 22; 10; 12; 8; 40; 32; 26; 27; 31; 34; 29; 17; 8; 24; 18; 47; 33; 34

Researcher 2: 3; 14; 11; 5; 16; 17; 28; 41; 31; 18; 14; 14; 26; 25; 21; 22; 31; 2; 35; 44; 23; 21; 21; 16; 12; 18; 41; 22; 16; 25; 33; 34; 29; 13; 18; 24; 23; 42; 33; 29

a. Create comparative histograms of the data
b. Create comparative boxplots of the data

14. A graph appears below showing the number of adults and children who prefer each type of soda. There were 130 adults and kids surveyed. Discuss some ways in which the graph below could be improved



15. Make up three data sets with 5 numbers each that have:
    a. the same mean but different standard deviations.
    b. the same mean but different medians.
    c. the same median but different means.

16. A sample of 30 distance scores measured in yards has a mean of 7, a variance of 16, and a standard deviation of 4.
    a. You want to convert all your distances from yards to feet, so you multiply each score in the sample by 3. What are the new mean, median, variance, and standard deviation?
    b. You then decide that you only want to look at the distance past a certain point. Thus, after multiplying the original scores by 3, you decide to subtract 4 feet from each of the scores. Now what are the new mean, median, variance, and standard deviation?

17. In your class, design a poll on a topic of interest to you and give it to the class.
    a. Summarize the data, computing the mean and five-number summary.
    b. Create a graphical representation of the data.
    c. Write several sentences about the topic, using your computed statistics as evidence in your writing.

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CHAPTER OVERVIEW

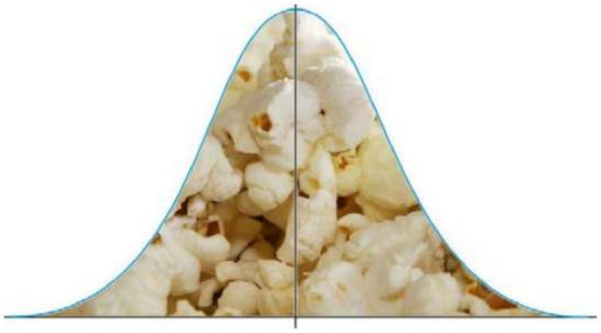
5: Normal Distribution

Introduction

Most high schools have a set amount of time in-between classes during which students must get to their next class. If you were to stand at the door of your statistics class and watch the students coming in, think about how the students would enter. Usually, one or two students enter early, then more students come in, then a large group of students enter, and finally, the number of students entering decreases again, with one or two students barely making it on time, or perhaps even coming in late!

Now consider this. Have you ever popped popcorn in a microwave? Think about what happens in terms of the rate at which the kernels pop. For the first few minutes, nothing happens, and then, after a while, a few kernels start popping. This rate increases to the point at which you hear most of the kernels popping, and then it gradually decreases again until just a kernel or two pops.

Here's something else to think about. Try measuring the height, shoe size, or the width of the hands of the students in your class. In most situations, you will probably find that there are a couple of students with very low measurements and a couple with very high measurements, with the majority of students centered on a particular value.



All of these examples show a typical pattern that seems to be a part of many real-life phenomena. In statistics, because this pattern is so pervasive, it seems to fit to call it normal, or more formally, the normal distribution. The normal distribution is an extremely important concept, because it occurs so often in the data we collect from the natural world, as well as in many of the more theoretical ideas that are the foundation of statistics. This chapter explores the details of the normal distribution.

5.1: The Standard Normal Probability Distribution

5.2: The Density Curve of a Normal Distribution

5.3: Application of Normal Distributions

5.4: The Central Limit Theorem

5.5: Exercises

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5.1: The Standard Normal Probability Distribution

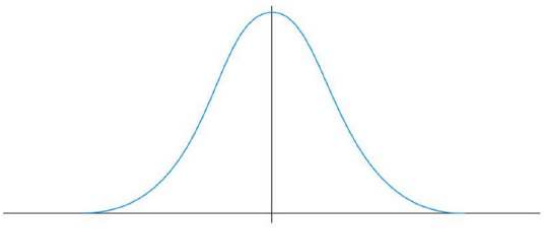
Learning Objectives

1. Identify the characteristics of a normal distribution.
2. Identify and use the Empirical Rule (68-95-99.7 Rule) for normal distributions.
3. Calculate a z-score and relate it to probability.
4. Determine if a data set corresponds to a normal distribution.

The Characteristics of a Normal Distribution

Shape

When graphing the data from each of the examples in the introduction, the distributions from each of these situations would be mound-shaped and mostly symmetric. A normal distribution is a perfectly symmetric, mound-shaped distribution. It is commonly referred to the as a normal curve, or bell curve.

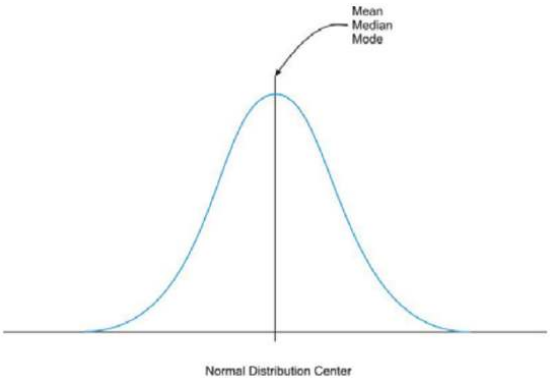


The Normal Distribution

Because so many real data sets closely approximate a normal distribution, we can use the idealized normal curve to learn a great deal about such data. With a practical data collection, the distribution will never be exactly symmetric, so just like situations involving probability, a true normal distribution only results from an infinite collection of data. Also, it is important to note that the normal distribution describes a continuous random variable.

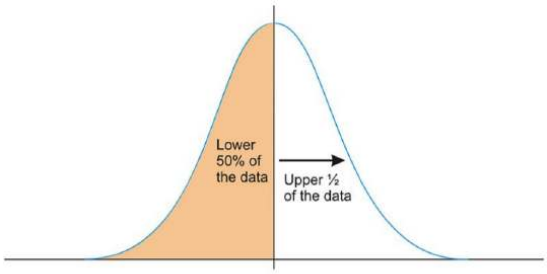
Center

Due to the exact symmetry of a normal curve, the center of a normal distribution, or a data set that approximates a normal distribution, is located at the highest point of the distribution, and all the statistical measures of center we have already studied (the mean, median, and mode) are equal.



Normal Distribution Center

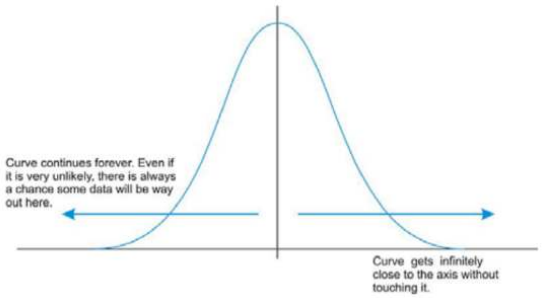
It is also important to realize that this center peak divides the data into two equal parts.



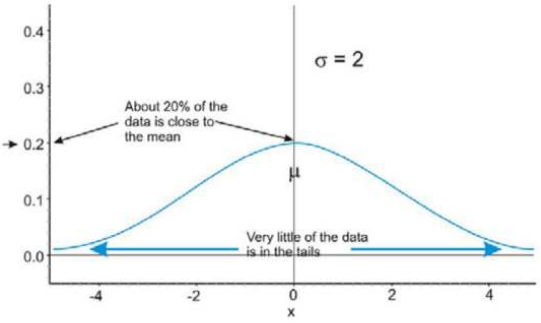
Normal Distribution divided into equal halves by the center

Spread

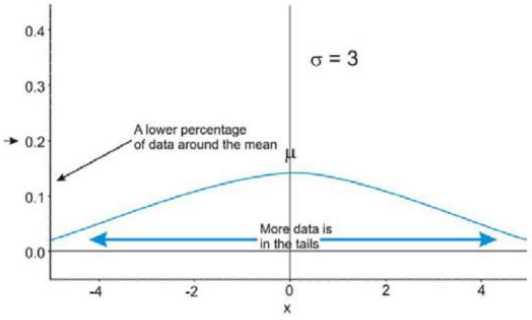
Let's go back to our popcorn example. The bag advertises a certain time, beyond which you risk burning the popcorn. From experience, the manufacturers know when most of the popcorn will stop popping, but there is still a chance that there are those rare kernels that will require more (or less) time to pop than the time advertised by the manufacturer. The directions usually tell you to stop when the time between popping is a few seconds, but aren't you tempted to keep going so you don't end up with a bag full of un-popped kernels? Because this is a real, and not theoretical, situation, there will be a time when the popcorn will stop popping and start burning, but there is always a chance, no matter how small, that one more kernel will pop if you keep the microwave going. In an idealized normal distribution of a continuous random variable, the distribution continues infinitely in both directions.



Because of this infinite spread, the range would not be a useful statistical measure of spread. The most common way to measure the spread of a normal distribution is with the standard deviation, or the typical distance away from the mean. Because of the symmetry of a normal distribution, the standard deviation indicates how far away from the maximum peak the data will be. Here are two normal distributions with the same center (mean):



normal distribution, standard deviation = 2

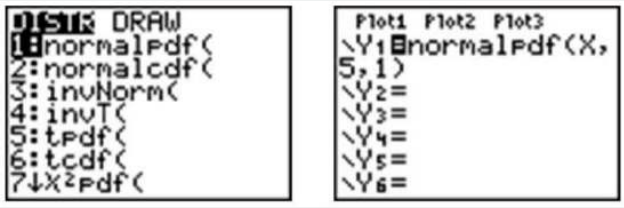


normal distribution, standard deviation = 3

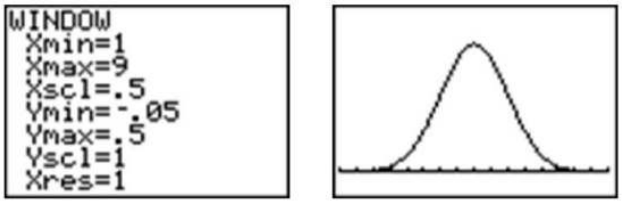
The first distribution pictured above has a smaller standard deviation, and so more of the data are heavily concentrated around the mean than in the second distribution. Also, in the first distribution, there are fewer data values at the extremes than in the second distribution. Because the second distribution has a larger standard deviation, the data are spread farther from the mean value, with more of the data appearing in the tails.

Technology Note: Investigating the Normal Distribution on a TI-83/84 Graphing Calculator

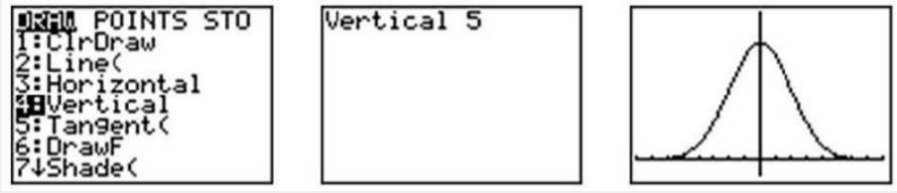
We can graph a normal curve for a probability distribution on the TI-83/84 calculator. To do so, first press [Y=]. To create a normal distribution, we will draw an idealized curve using something called a density function. The command is called 'normalpdf(', and it is found by pressing [2nd][DISTR][1]. Enter an X to represent the random variable, followed by the mean and the standard deviation, all separated by commas. For this example, choose a mean of 5 and a standard deviation of 1.



Adjust your window to match the following settings and press [GRAPH].



Press [2ND][QUIT] to go to the home screen. We can draw a vertical line at the mean to show it is in the center of the distribution by pressing [2ND][DRAW] and choosing 'Vertical'. Enter the mean, which is 5, and press [ENTER].

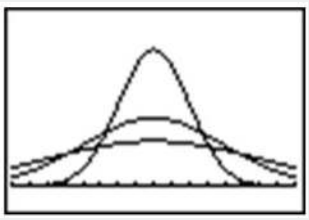


Remember that even though the graph appears to touch the x-axis, it is actually just very close to it.

In your Y= Menu, enter the following to graph 3 different normal distributions, each with a different standard deviation:

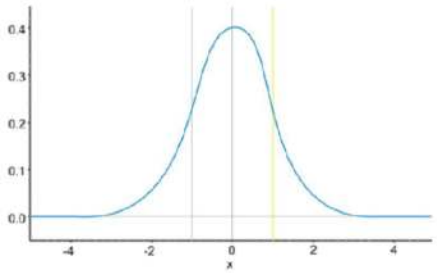


This makes it easy to see the change in spread when the standard deviation changes.



The Empirical Rule for Normal Distributions

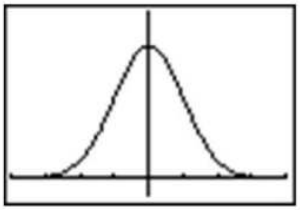
Because of the similar shape of all normal distributions, we can measure the percentage of data that is a certain distance from the mean no matter what the standard deviation of the data set is. The following graph shows a normal distribution with $\mu = 0$ and $\sigma = 1$. This curve is called a *standard normal curve.* In this case, the values of x represent the number of standard deviations away from the mean.



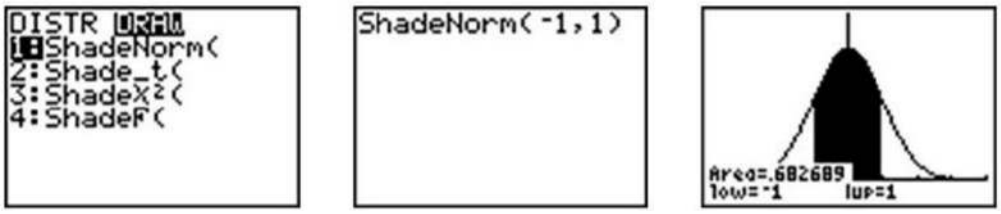
The Standard Normal Distribution

Notice that vertical lines are drawn at points that are exactly one standard deviation to the left and right of the mean. We have consistently described standard deviation as a measure of the typical distance away from the mean. How much of the data is actually within one standard deviation of the mean? To answer this question, think about the space, or area, under the curve. The entire data set, or 100% of it, is contained under the whole curve. What percentage would you estimate is between the two lines? To help estimate the answer, we can use a graphing calculator. Graph a standard normal distribution over an appropriate window.

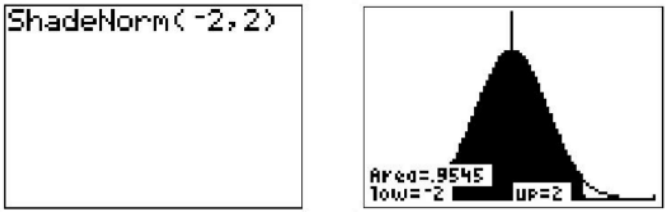




Now press [2ND][DISTR], go to the DRAW menu, and choose ‘ShadeNorm(’. Insert ‘−1, 1’ after the ‘Shade-Norm(’ command and press [ENTER]. It will shade the area within one standard deviation of the mean.



The calculator also gives a very accurate estimate of the area. We can see from the rightmost screenshot above that approximately 68% of the area is within one standard deviation of the mean. If we venture to 2 standard deviations away from the mean, how much of the data should we expect to capture? Make the following changes to the 'ShadeNorm(' command to find out:



Notice from the shading that almost all of the distribution is shaded, and the percentage of data is close to 95%. If you were to venture to 3 standard deviations from the mean, 99.7%, or virtually all of the data, is captured, which tells us that very little of the data in a normal distribution is more than 3 standard deviations from the mean.

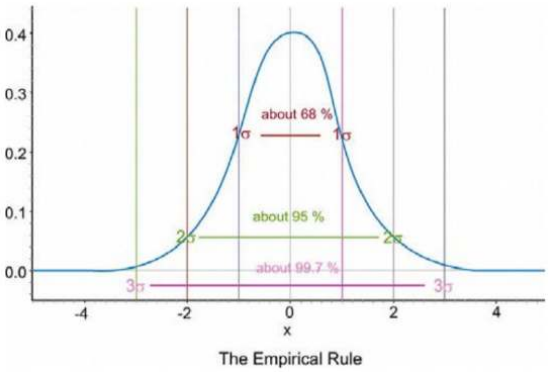


Notice that the calculator actually makes it look like the entire distribution is shaded because of the limitations of the screen resolution, but as we have already discovered, there is still some area under the curve further out than that. These three approximate percentages, 68%, 95%, and 99.7%, are extremely important and are part of what is called the *Empirical Rule*.

The Empirical Rule states that the percentages of data in a normal distribution within 1, 2, and 3 standard deviations of the mean are approximately 68%, 95%, and 99.7%, respectively.

On the Web

http://tinyurl.com/2ue78u Explore the Empirical Rule.



The Empirical Rule

Calculating and Interpreting z-Scores

A **z-score** is a measure of the number of standard deviations a particular data point is away from the mean. For example, let's say the mean score on a test for your statistics class was an 82, with a standard deviation of 7 points. If your score was an 89, it is exactly one standard deviation to the right of the mean; therefore, your z-score would be 1. If, on the other hand, you scored a 75, your score would be exactly one standard deviation below the mean, and your z-score would be −1. All values that are below the mean have negative z-scores, while all values that are above the mean have positive z-scores. A z-score of −2 would represent a value that is exactly 2 standard deviations below the mean, so in this case, the value would be $82 - 14 = 68$.

To calculate a *z-score* for which the numbers are not so obvious, you take the deviation and divide it by the standard deviation.

$$z = \frac{\text{deviation}}{\text{standard deviation}}$$

You may recall that deviation is the mean value of the variable subtracted from the observed value, so in symbolic terms, the z-score would be:

$$z = \frac{x - \mu}{\sigma}$$

As previously stated, since σ is always positive, z will be positive when x is greater than μ and negative when x is less than μ. A z-score of zero means that the term has the same value as the mean. The value of z represents the number of standard deviations the given value of x is above or below the mean.

Example 5.1.1

What is the z-score for an A on the test described above, which has a mean score of 82? (Assume that an A is a 93.)

Solution

The z-score can be calculated as follows:

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{93 - 82}{7}$$
$$z \approx 1.57$$

If we know that the test scores from the last example are distributed normally, then a z-score can tell us something about how our test score relates to the rest of the class. From the Empirical Rule, we know that about 68% of the students would have scored between a z-score of −1 and 1, or between a 75 and an 89, on the test. If 68% of the data is between these two values, then that leaves the remaining 32% in the tail areas. Because of symmetry, half of this, or 16%, would be in each individual tail.

Example 5.1.2

On a nationwide math test, the mean was 65 and the standard deviation was 10. If Robert scored 81, what was his z-score?

Solution

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{81 - 65}{10}$$
$$z \approx 1.60$$

Example 5.1.3

On a college entrance exam, the mean was 70, and the standard deviation was 8. If Helen's zscore was −1.5, what was her exam score?

Solution

Since $z = \frac{x - \mu}{\sigma}$, then we can rewrite this formula solving for x:

$$x = \mu + z\sigma$$

Now, we can obtain Helen's exam score with the given parameters:

$$x = \mu + z\sigma$$

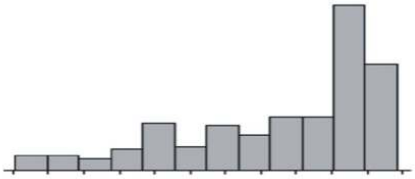
$$x = 70 + (-1.5)(8)$$

$$x = 58$$

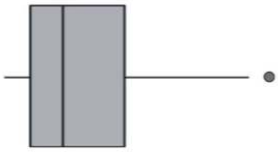
Thus, Helen's exam score was 58; notice a score of 58 is below the mean and this makes sense since her z-score was negative.

Assessing Normality

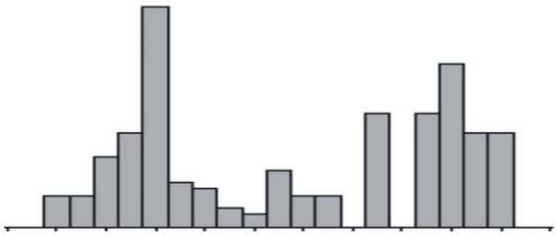
The best way to determine if a data set approximates a normal distribution is to look at a visual representation. Histograms and box plots can be useful indicators of normality, but they are not always definitive. It is often easier to tell if a data set is not normal from these plots.



Skewed left distribution



Skewed right distribution with outliers



Bimodal Distribution

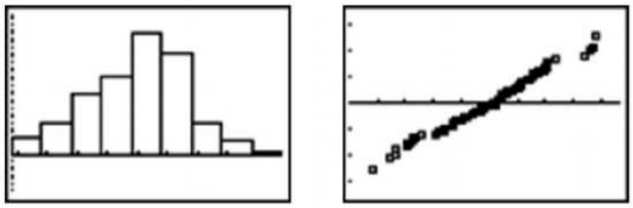
If a data set is skewed right, it means that the right tail is significantly longer than the left. Similarly, skewed left means the left tail has more weight than the right. A bimodal distribution, on the other hand, has two modes, or peaks. For instance, with a histogram of the heights of American 30-yearold adults, you will see a bimodal distribution−one mode for males and one mode for females.

There is a plot we can use to determine if a distribution is normal called a **normal probability plot** or **normal quantile plot**. To make this plot by hand, first order your data from smallest to largest. Then, determine the quantile of each data point. Finally, using a table of standard normal probabilities, determine the closest z-score for each quantile. Plot these z-scores against the actual data values. To make a normal probability plot using your calculator, enter your data into a list, then use the last type of graph in the STAT PLOT menu, as shown below:



If the data set is normal, then this plot will be perfectly linear. The closer to being linear the normal probability plot is, the more closely the data set approximates a normal distribution.

Look below at the histogram and the normal probability plot for the same data.



The histogram is fairly symmetric and mound-shaped and appears to display the characteristics of a normal distribution. When the z-scores of the quantiles of the data are plotted against the actual data values, the normal probability plot appears strongly linear, indicating that the data set closely approximates a normal distribution. The following example will allow you to see how a normal probability plot is made in more detail.

Example 5.1.4

The following data set tracked high school seniors' involvement in traffic accidents. The participants were asked the following question: "During the last 12 months, how many accidents have you had while you were driving (whether or not you were responsible)?"

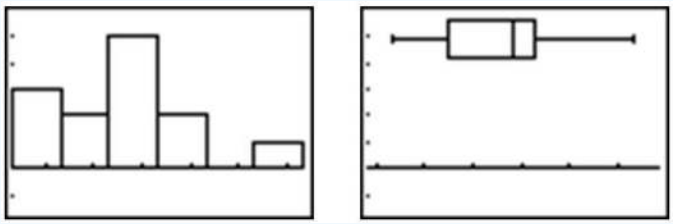
Solution

Year	Percentage of high school seniors who said they were involved in no traffic accidents
1991	75.7
1992	76.9
1993	76.1
1994	75.7
1995	75.3
1996	74.1
1997	74.4
1998	74.4
1999	75.1
2000	75.1
2001	75.5
2002	75.5
2003	75.8

Figure: Percentage of high school seniors who said they were involved in no traffic accidents.

Source: Sourcebook of Criminal Justice Statistics: http://www.albany.edu/sourcebook/pdf/t352.pdf

Here is a histogram and a box plot of this data:

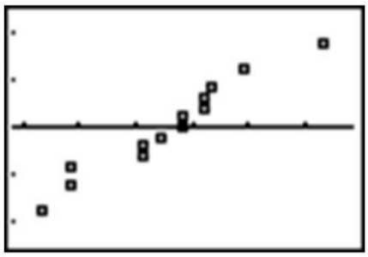


The histogram appears to show a roughly mound-shaped and symmetric distribution. The box plot does not appear to be significantly skewed, but the various sections of the plot also do not appear to be overly symmetric, either. In the following chart, the data has been reordered from smallest to largest, the quantiles have been determined, and the closest corresponding z-scores have been found using a table of standard normal probabilities.

Table of Quantiles and Corresponding z-scores for Senior No-accident Data.

Year	Percentage of high school seniors who said they were involved in no traffic accidents	Quantiles	z-score
1996	74.1	$\frac{1}{13} = 0.078$	-1.42
1997	74.4	$\frac{2}{13} = 0.154$	-1.02
1998	74.4	$\frac{3}{13} = 0.231$	-0.74
1999	75.1	$\frac{4}{13} = 0.286$	-0.56
2000	75.1	$\frac{5}{13} = 0.385$	-0.29
1995	75.3	$\frac{6}{13} = 0.462$	-0.09
2001	75.5	$\frac{7}{13} = 0.538$	0.1
2002	75.5	$\frac{8}{13} = 0.615$	0.29
1991	75.7	$\frac{9}{13} = 0.692$	0.59
1994	75.7	$\frac{10}{13} = 0.769$	0.74
2003	75.8	$\frac{11}{13} = 0.846$	1.02
1993	76.1	$\frac{12}{13} = 0.923$	1.43
1992	76.9	$\frac{13}{13} = 1$	3.49

Here is a plot of the percentages versus the z-scores of their quantiles, or the normal probability plot:



Remember that you can simplify this process by simply entering the percentages into a L1 in your calculator and selecting the normal probability plot option (the last type of plot) in STAT PLOT.

While not perfectly linear, this plot does have a strong linear pattern, and we would, therefore, conclude that the distribution is reasonably normal.

Exercises

1. Which of the following data sets is most likely to be normally distributed? For the other choices, explain why you believe they would not follow a normal distribution.

a) The hand span (measured from the tip of the thumb to the tip of the extended 5^{th} finger) of a random sample of high school seniors

b) The annual salaries of all employees of a large shipping company

c) The annual salaries of a random sample of 50 CEOs of major companies, 25 women and 25 men

d) The dates of 100 pennies taken from a cash drawer in a convenience store

2. The grades on a statistics mid-term for a high school are normally distributed, with $\mu = 81$ and $\sigma = 6.3$. Calculate the z-scores for each of the following exam grades. Draw and label a sketch for each example. 65, 83, 93, 100

3. Assume that the mean weight of 1-year-old girls in the USA is normally distributed, with a mean of about 9.5 kilograms and a standard deviation of approximately 1.1 kilograms. Without using a calculator, estimate the percentage of 1-year-old girls who meet the following conditions. Draw a sketch and shade the proper region for each problem.

a) Less than 8.4 kg

b) Between 7.3 kg and 11.7 kg

c) More than 12.8 kg

4. For a standard normal distribution, place the following in order from smallest to largest.

a) The percentage of data below 1

b) The percentage of data below −1

c) The mean

d) The standard deviation

e) The percentage of data above 2

5. The 2007 AP Statistics examination scores were not normally distributed, with $\mu = 2.8$ and $\sigma = 1.34$ [1]. What is the approximate z-score that corresponds to an exam score of 5? (The scores range from 1 to 5.)

a) 0.786

b) 1.46

c) 1.64

d) 2.20

e) A z-score cannot be calculated because the distribution is not normal.

6. The heights of 5^{th}-grade boys in the USA is approximately normally distributed, with a mean height of 143.5 cm and a standard deviation of about 7.1 cm. What is the probability that a randomly chosen 5^{th}-grade boy would be taller than 157.7 cm?

7. A statistics class bought some sprinkle (or jimmies) doughnuts for a treat and noticed that the number of sprinkles seemed to vary from doughnut to doughnut, so they counted the sprinkles on each doughnut.

Here are the results: 241, 282, 258, 223, 133, 335, 322, 323, 354, 194, 332, 274, 233, 147, 213, 262, 227, and 366.

Create a histogram, dot plot, or box plot for this data. Comment on the shape, center, and spread of the distribution.

Notes:

1. Data available on the College Board Website: http://professionals.collegeboard.com/data-reportsresearch/ap/archi ved/2007

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5.2: The Density Curve of a Normal Distribution

Learning Objectives

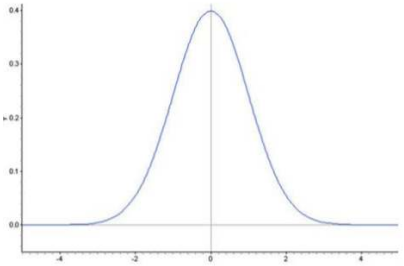
1. Identify the properties of a normal density curve and the relationship between concavity and standard deviation.
2. Convert between z-scores and areas under a normal probability curve.
3. Calculate probabilities that correspond to left, right, and middle areas from a z-score table.
4. Calculate probabilities that correspond to left, right, and middle areas using a graphing calculator.
5. Calculate for unknown values other than the z-score and area.

Introduction

In this section, we will continue our investigation of normal distributions to include density curves and learn various methods for calculating probabilities from the normal density curve.

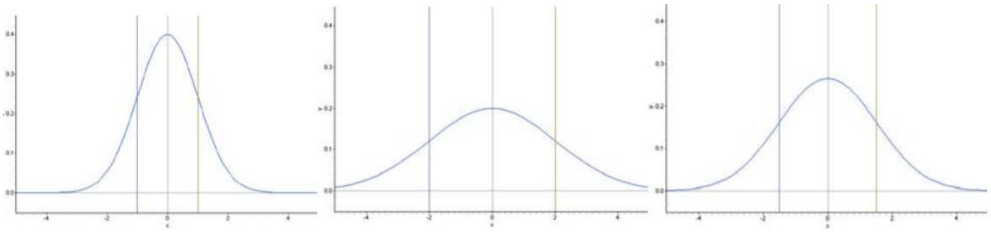
Density Curves

A **density curve** is an idealized representation of a distribution in which the area under the curve is defined to be 1. Density curves need not be normal, but the normal density curve will be the most useful to us.



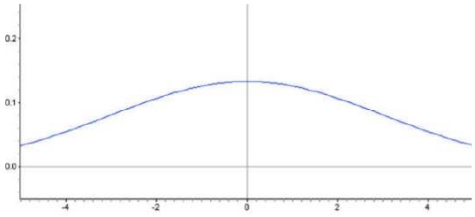
Inflection Points on a Normal Density Curve

We already know from the Empirical Rule that approximately $\frac{2}{3}$ of the data in a normal distribution lies within 1 standard deviation of the mean. With a normal density curve, this means that about 68% of the total area under the curve is within z-scores of ± 1. Look at the following three density curves:



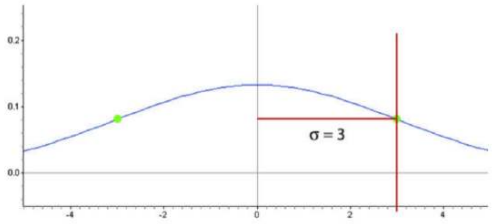
Notice that the curves are spread increasingly wider. Lines have been drawn to show the points that are one standard deviation on either side of the mean. Look at where this happens on each density curve.

Here is a normal distribution with an even larger standard deviation.



Is it possible to predict the standard deviation of this distribution by estimating the x-coordinate of a point on the density curve? Read on to find out!

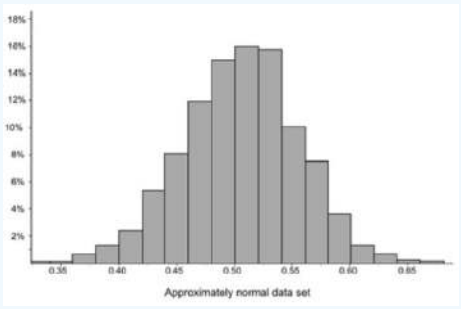
You may have noticed that the density curve changes shape at two points in each of our examples. These are the points where the curve changes concavity. Starting from the mean and heading outward to the left and right, the curve is **concave down**. (It looks like a mountain, or 'n' shape.) After passing these points, the curve is **concave up**. (It looks like a valley, or 'u' shape.) The points at which the curve changes from being concave up to being concave down are called the **inflection points**. On a normal density curve, these inflection points are always exactly one standard deviation away from the mean.



In this example, the standard deviation is 3 units. We can use this concept to estimate the standard deviation of a normally distributed data set.

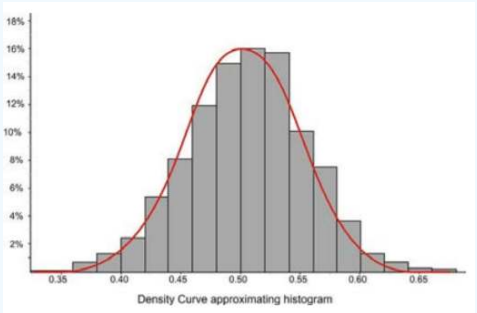
Example 5.2.1

Estimate the standard deviation of the distribution represented by the following histogram.

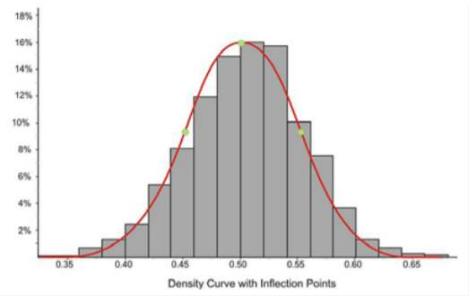


Solution

This distribution is fairly normal, so we could draw a density curve to approximate it as follows:



Now estimate the inflection points as shown below:



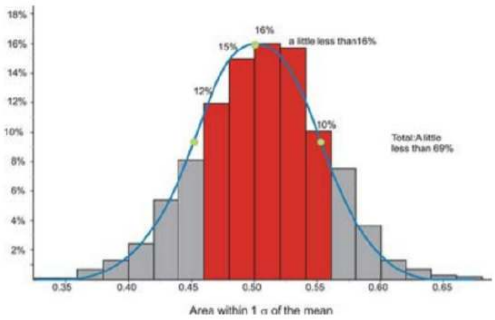
It appears that the mean is about 0.5 and that the x-coordinates of the inflection points are about 0.45 and 0.55, respectively. This would lead to an estimate of about 0.05 for the standard deviation.

The actual statistics for this distribution are as follows:

$$s \approx 0.04988$$

$$\bar{x} \approx 0.4997$$

We can verify these figures by using the expectations from the Empirical Rule. In the following graph, we have highlighted the bins that are contained within one standard deviation of the mean.



If you estimate the relative frequencies from each bin, their total is remarkably close to 68%. Make sure to divide the relative frequencies from the bins on the ends by 2 when performing your calculation.

Convert Between z-scores and Areas

While it is convenient to estimate areas under a normal curve using the Empirical Rule, we often need more precise methods to calculate these areas. Luckily, we can use formulas or technology to help us with the calculations.

z-scores

All normal distributions have the same basic shape, and therefore, rescaling and re-centering can be implemented to change any normal distributions to one with a mean of 0 and a standard deviation of 1. This configuration is referred to as a **standard normal distribution**. In a standard normal distribution, the variable along the horizontal axis is the z-score. This score is another measure of the performance of an individual score in a population. To review, the z-score measures how many standard deviations a score is away from the mean. The z-score of the term x in a population distribution whose mean is μ and whose standard deviation is σ is given by: $\frac{x-\mu}{\sigma}$. Since σ is always positive, z will be positive when x is greater than μ and negative when x is less than μ. A z-score of 0 means that the term has the same value as the mean. The value of z is the number of standard deviations the given value of x is above or below the mean.

Example 5.2.2

On a nationwide math test, the mean was 65 and the standard deviation was 10. If Robert scored 81, what was his z-score?

Solution

$$z = \frac{x-\mu}{\sigma}$$

$$z = \frac{81 - 65}{1.6}$$

$$z = 10$$

Example 5.2.3

On a college entrance exam, the mean was 70 and the standard deviation was 8. If Helen's z-score was −1.5, what was her exam score?

Solution

Recall, the equation to obtain x is

$$x = \mu + z\sigma$$

Using this equation, we can find Helen's score:

$$x = \mu + z\sigma$$

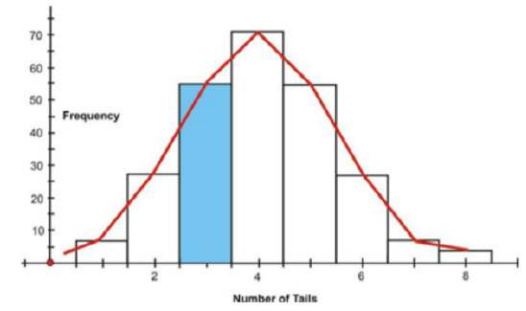
$$x = 70 + (-1.5)(8)$$

$$x = 58$$

Now you will see how z-scores are used to determine the probability of an event.

Example 5.2.4

Suppose you were to toss 8 coins 256 times. The following figure shows the histogram and the approximating normal curve for the experiment. The random variable represents the number of tails obtained.



The blue section of the graph represents the probability that exactly 3 of the coins turned up tails. Geometrically, this probability represents the area of the blue shaded bar divided by the total area of the bars. The area of the blue shaded bar is approximately equal to the area under the normal curve from 2.5 to 3.5.

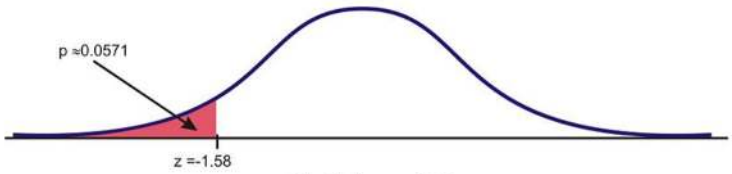
Since areas under normal curves correspond to the probability of an event occurring, a special normal distribution table is used to calculate the probabilities. *This table can be found at the end of this section where the area is given from the mean.* The following is an example of a table of z-scores and a brief explanation of how it works: http://tinyurl.com/2ce9ogv.

The values inside the given table represent the areas under the standard normal curve for values between 0 and the relative z-score. For example, to determine the area under the curve between zscores of 0 and 2.36, look in the intersecting cell for the row labeled 2.3 and the column labeled 0.06. The area under the curve is 0.4909. To determine the area between 0 and a negative value, look in the intersecting cell of the row and column which sums to the absolute value of the number in question. For example, the area under the curve between −1.3 and 0 is equal to the area under the curve between 1.3 and 0, so look at the cell that is the intersection of the 1.3 row and the 0.00 column. (The area is 0.4032.)

Calculate Probabilities that Correspond to z-scores and Areas

It is extremely important, especially when you first start with these calculations, that you get in the habit of relating it to the normal distribution by drawing a sketch of the situation. In this case, simply draw a sketch of a standard normal curve with the appropriate

region shaded and labeled.



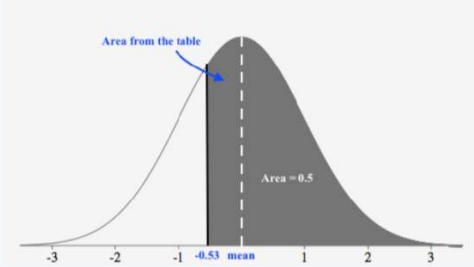
Sketch for z ≤ -1.58

Example 5.2.5

Find the probability of choosing a value that is greater than $z = -0.528$, or $P(z > -0.528)$.

Solution

Before even using the table, first draw a figure with the shaded region. This z-score is just below the mean, so the answer should be more than 0.5.



Next, read the table to find the correct probability for the data below this z-score. We must first round this z-score to −0.53, so this will slightly under-estimate the probability, but it is the best we can do using the table. Looking up a z-score of −0.53, we see

z	0.00	0.01	0.02	0.03
0.00	0.00000	0.00399	0.00798	0.01197
0.10	0.03983	0.04380	0.04776	0.05172
0.20	0.07926	0.08317	0.08706	0.09095
0.30	0.11791	0.12712	0.12552	0.12930
0.40	0.15542	0.15910	0.16276	0.16640
0.50	0.19146	0.19497	0.19847	**0.20194**

The table returns an area of 0.20194. Since the area from the mean to $z = -0.53$ is 0.20194 and the area on the right of the mean is 0.5, then the area of the shaded region is

$$0.5 + 0.20194 = 0.70194$$

Thus, the probability of choosing a value that is greater than $z = -0.528$ is 0.7019.

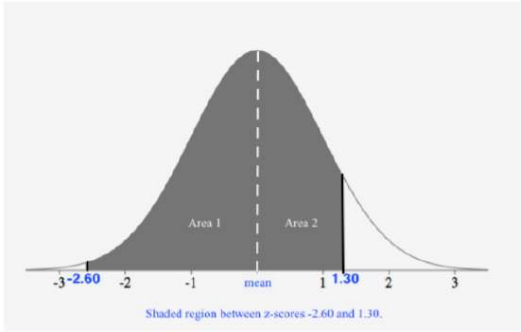
What about values between two z-scores? While it is an interesting and worthwhile exercise to do this using a table, we can also use statistical software or a graphing calculator.

Example 5.2.6

Find $P(-2.60 < z < 1.30)$.

Solution

First, we draw a figure with the shaded region:



Since the table gives us the area from the mean to a z-score, we can see that we will add the areas, Area 1 + Area 2, to obtain the area of the shaded region, resulting in the probability. Let's look up the z-scores on the table to find the area from the mean to each z-score:

z	0.00
1.30	0.40320
2.60	0.49534

Area 1 is 0.49534 and Area 2 is 0.40320. Adding these two together, we get

$$P(-2.60 < z < 1.30) = \text{ Area } 1 + \text{ Area } 2 = 0.49534 + 0.40320 = 0.89854$$

Thus, the probability $P(-2.60 < z < 1.30) = 0.89854$.

The probability can also be found using the TI-83/84 calculator. Use the 'normalcdf(−2.60, 1.30, 0, 1)' command, and the calculator will return the result 0.898538. The syntax for this command is 'normalcdf(min, max, μ, σ)'. When using this command, you do not need to first standardize. You can use the mean and standard deviation of the given distribution.

Technology Note: The 'normalcdf(' Command on TI-83/84 Calculator

Your graphing calculator has already been programmed to calculate probabilities for a normal density curve using what is called a **cumulative density function**. The command you will use is found in the DISTR menu, which you can bring up by pressing [2ND][DISTR].



Press [2] to select the 'normalcdf(' command, which has a syntax of 'normalcdf(lower bound, upper bound, mean, standard deviation)'.

The command has been programmed so that if you do not specify a mean and standard deviation, it will default to the standard normal curve, with $\mu = 0$ and $\sigma = 1$.

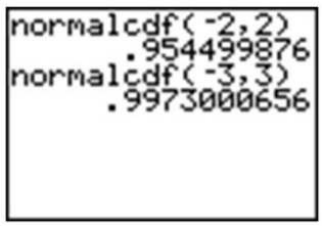
For example, entering 'normalcdf(−1, 1)' will specify the area within one standard deviation of the mean, which we already know to be approximately 0.68.



Try verifying the other values from the Empirical Rule.

Summary:

'Normalcdf (a, b, μ, σ)' gives values of the cumulative normal density function. In other words, it gives the probability of an event occurring between $x = a$ and $x = b$, or the area under the probability density curve between the vertical lines $x = a$ and $x = b$, where the normal distribution has a mean of μ and a standard deviation of σ. If μ and σ are not specified, it is assumed that $\mu = 0$ and $\sigma = 1$.

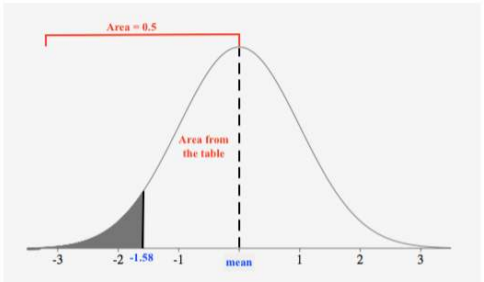


Example 5.2.7

Find the probability $P(z < -1.58)$.

Solution

First, we draw a figure with the shaded region:



Since the table gives us the area from the mean to a z-score and the total area to the left of the mean is 0.5, we can see that we will subtract the area given in the table from 0.5 to obtain the area of the shaded region, resulting in the probability. Let's look up the z-score on the table to find the area from the mean to the z-score:

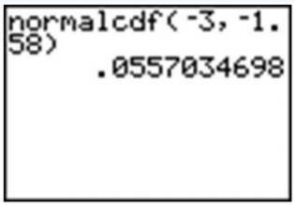
z	0.08
1.50	0.44295

The area from the mean to $z = -1.58$ is 0.44295. Subtracting this from 0.5, we get

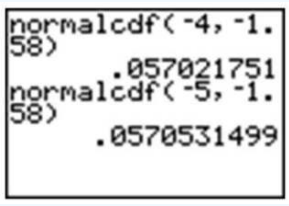
$$P(z < -1.58) = 0.5 - 0.44295 = 0.05705$$

Doing this on the calculator, we must have both an upper and lower bound. Technically, though, the density curve does not have a lower bound, as it continues infinitely in both directions. We do know, however, that a very small percentage of the data

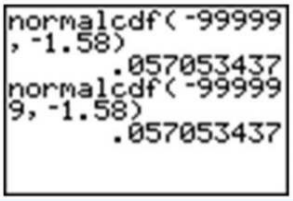
is below 3 standard deviations to the left of the mean. Use −3 as the lower bound and see what answer you get.



The answer is fairly accurate, but you must remember that there is really still some area under the probability density curve, even though it is just a little, that we are leaving out if we stop at −3. If you look at the z-table, you can see that we are, in fact, leaving out about $0.5 - 0.4987 = 0.0013$. Next, try going out to −4 and −5.



Once we get to −5, the answer is quite accurate. Since we cannot really capture all the data, entering a sufficiently small value should be enough for any reasonable degree of accuracy. A quick and easy way to handle this is to enter −99999 (or “a bunch of nines”). It really doesn’t matter exactly how many nines you enter. The difference between five and six nines will be beyond the accuracy that even your calculator can display.

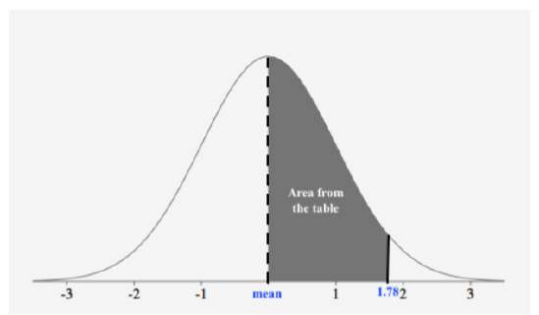


Example 5.2.8

Find the probability $P(0 < z < 1.78)$.

Solution

First, we draw a figure with the shaded region:



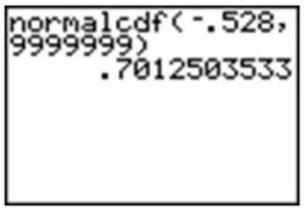
Since the table gives us the area from the mean to a z-score, we can see that whatever area is given from the table results in the probability. Let’s look up the z-score on the table to find the area from the mean to the z-score.

z	0.08
1.70	0.46246

The area from the mean to $z = 1.78$ is 0.46246. Thus,

$$P(z < 1.78) = 0.46246$$

We are at an advantage using the calculator because we do not have to round off the z-score in this example. Let's try this example with the calculator. Enter the 'normalcdf(' command, using −0.528 to "a bunch of nines." The nines represent a ridiculously large upper bound that will insure that the unaccounted-for probability will be so small that it will be virtually undetectable.



Remember that because of rounding, our answer from the table was slightly too small, so when we subtracted it from 1, our final answer was slightly too large. The calculator answer of about 0.70125 is a more accurate approximation than the answer arrived at by using the table.

Standardizing

In most practical problems involving normal distributions, the curve will not be as we have seen so far, with $\mu = 0$ and $\sigma = 1$. When using a z-table, you will first have to standardize the distribution by calculating the z-score(s).

Example 5.2.9

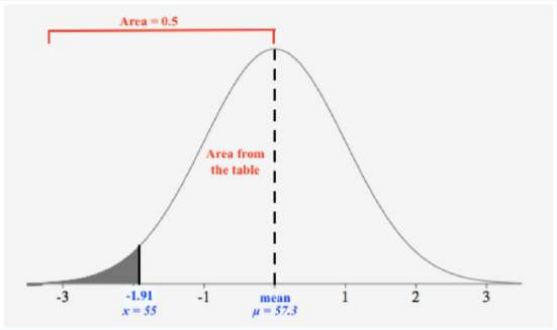
A candy company sells small bags of candy and attempts to keep the number of pieces in each bag the same, though small differences due to random variation in the packaging process lead to different amounts in individual packages. A quality control expert from the company has determined that the mean number of pieces in each bag is normally distributed, with a mean of 57.3 and a standard deviation of 1.2. Endy opened a bag of candy and felt he was cheated. His bag contained only 55 candies. Does Endy have reason to complain?

Solution

To determine if Endy was cheated, we need to find the probability of selecting a bag of candy with 55 or fewer candies, i.e., we let $x = 55$. Let's calculate the z-score for 55:

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{55 - 57.3}{1.2}$$
$$z \approx -1.91$$

Next, we can draw a figure to see the shaded region:

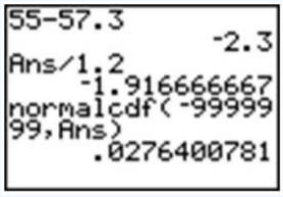


Using a table, we obtain a value 0.47193. This is the area from the mean to $z = -1.91$. We can subtract this value from 0.5 since the area on the left of the mean is 0.5:

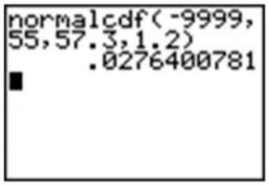
$$0.5 - 0.47193 = 0.02807$$

Hence, there is about a 3% chance that he would get a bag of candy with 55 or fewer pieces, so Endy should feel cheated because the chances of getting a bag with 55 or fewer candies is so low.

Using a graphing calculator, the results would look as follows (the 'Ans' function has been used to avoid rounding off the z-score):



However, one of the advantages of using a calculator is that it is unnecessary to standardize. We can simply enter the mean and standard deviation from the original population distribution of candy, avoiding the z-score calculation completely.



Unknown Value Problems

If you understand the relationship between the area under a density curve and mean, standard deviation, and z-scores, you should be able to solve problems in which you are provided all but one of these values and are asked to calculate the remaining value. In the last lesson, we found the probability that a variable is within a particular range, or the area under a density curve within that range. What if you are asked to find a value that gives a particular probability? We rewrite the z-score formula $z = \frac{x - \mu}{\sigma}$ as

$$x = \mu + z \cdot \sigma \quad \text{or} \quad \mu = x - z \cdot \sigma$$

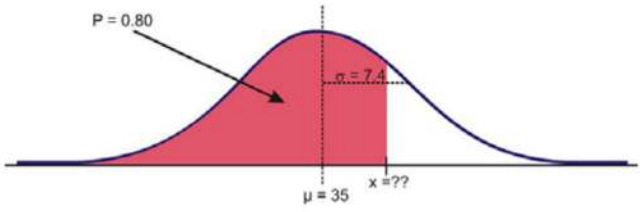
Unknown Original Value, x

Example 5.2.10

Given the normally-distributed random variable x, with $\mu = 35$ and $\sigma = 7.4$, what is the value of x where the probability of experiencing a value less than it is 80%?

Solution

As suggested before, it is important and helpful to sketch the distribution.



Sketch of distribution

We need to find a z-score from the table that corresponds to the area from the mean. Since the area on the left of the mean is 0.5, we see that the area from the mean to x is 0.30, i.e.,

$$P(z < x) = 0.8$$

and this implies that

$$P(0 < z < x) = 0.8 - 0.5 = 0.3$$

We need to find, somewhere in the areas given in the table, an area of 0.3 (or the closest to it) and its corresponding z-score. Let's take a look:

z	0.04	0.05
0.80	0.29955	0.30234

We see the closest area to 0.3, given in the table, is 0.29955, which has a corresponding z-score of 0.84. Hence, we use $z = 0.84$ for the z-score in the formula to obtain x. Given $\mu = 35$ and $\sigma = 7.4$, we get

$$x = \mu + z \cdot \sigma$$

$$x = 35 + 0.84(7.4)$$

$$x = 41.216$$

Thus, the value of x where the probability of experiencing a value less than it is 80% is 41.216. In general, when we want to obtain an x value from a given probability, we find the z-score first, then plug-n-chug this into the rewritten z-score formula.

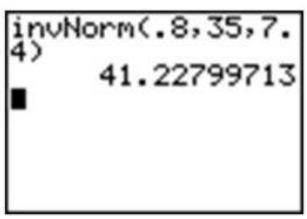
When we were given a value of the variable and were asked to find the percentage or probability, the 'normalcdf(' command on a graphing calculator. But how do we find a value given the percentage? Graphing calculators and computer software are much more convenient and accurate. The command on the TI-83/84 calculator is 'invNorm('. You may have seen it already in the DISTR menu.



The syntax for this command is as follows:

'InvNorm(percentage or probability to the left, mean, standard deviation)'

Make sure to enter the values in the correct order, such as in the example below:



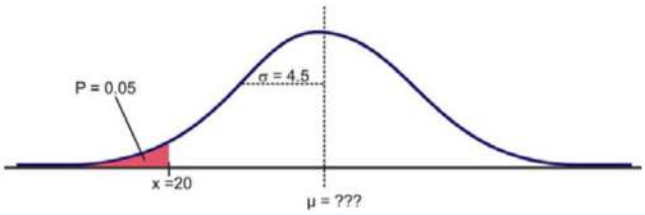
Unknown Mean or Standard Deviation

Example 5.2.11

For a normally distributed random variable, $\sigma = 4.5$, $x = 20$, and $P = 0.05$, find .

Solution

To solve this problem, first draw a sketch:



We need to find a z-score from the table that corresponds to the area from the mean. Since the area on the left of $x = 20$ is 0.05, we see that the area from the mean to x is 0.45, i.e.,

$$P(z < x) = 0.05$$

and this implies that

$$P(x < z < 0) = 0.5 - 0.05 = 0.45$$

We need to find, somewhere in the areas given in the table, an area of 0.45 (or the closest to it) and its corresponding negative z-score, since the x value lies below the mean. Let's take a look:

z	0.04	0.05
1.60	0.44950	0.45953

We see the closest area to 0.45, given in the table, is 0.44950, which has a corresponding z-score of -1.64. Recall, the z-score is negative because the x value lies below the mean. Hence, we use $z = -1.64$ for the z-score in the formula to obtain x. Given $\sigma = 4.5$ and $x = 20$, we get

$$\mu = x - z \cdot \sigma$$

$$\mu = 20 - (-1.64)(4.5)$$

$$\mu = 41.216$$

Thus, the mean is 27.38.

We could also use the 'invNorm(' command on the calculator. The result, -1.645, confirms the prediction that the value is less than 2 standard deviations from the mean.

Now, plug in the known quantities into the z-score formula and solve for μ as follows:

$$z = \frac{x - \mu}{\sigma}$$

$$\mu = x - z\sigma$$

$$\mu \approx 20 - (-1.645)(4.5)$$

$$z \approx 27.402$$

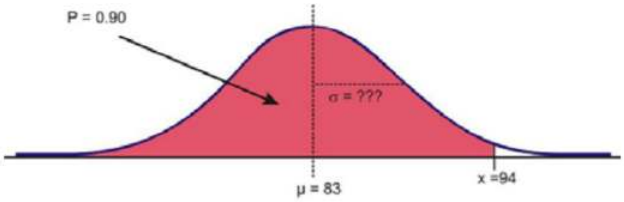
We can see there was little discrepancy from using the table and using the calculator. However, since we were eye-balling from the table, the calculator gives more accurate results.

Example 5.2.12

For a normally-distributed random variable, $\mu = 83$, $x = 94$, and $P = 0.90$. Find σ.

Solution

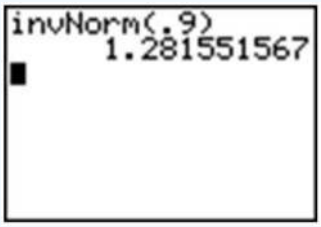
Again, let's first look at a sketch of the distribution.



Sketch of Distribution

Since about 97.5% of the data is below 2 standard deviations, it seems reasonable to estimate that the x value is less than two standard deviations away from the mean and that σ might be around 7 or 8.

Again, the first step to see if our prediction is right is to use 'invNorm(' to calculate the z-score. Remember that since we are not entering a mean or standard deviation, the result is based on the assumption that $\mu = 0$ and $\sigma = 1$.



Now, use the z-score formula and solve for σ as follows:

$$z = \frac{x - \mu}{\sigma}$$

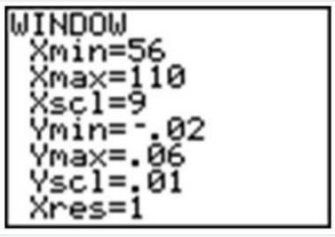
$$\sigma = \frac{x - \mu}{z}$$

$$\sigma = \frac{94 - 83}{1.282}$$

$$\sigma \approx 8.583$$

Technology Note: Drawing a Distribution on the TI-83/84 Calculator

The TI-83/84 calculator will draw a distribution for you, but before doing so, we need to set an appropriate window (see screen below) and delete or turn off any functions or plots. Let's use the last example and draw the shaded region below 94 under a normal curve with $\mu = 83$ and $\sigma = 8.583$. Remember from the Empirical Rule that we probably want to show about 3 standard deviations away from 83 in either direction. If we use 9 as an estimate for σ, then we should open our window 27 units above and below 83. The y settings can be a bit tricky, but with a little practice, you will get used to determining the maximum percentage of area near the mean.

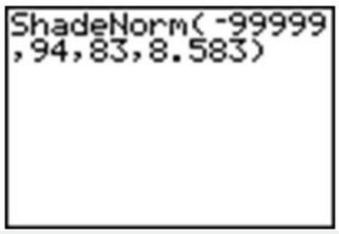


The reason that we went below the x-axis is to leave room for the text, as you will see.

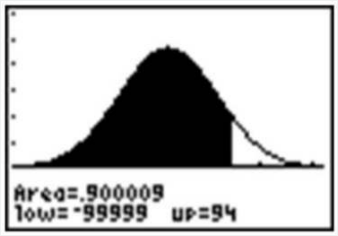
Now, press [2ND][DISTR] and arrow over to the DRAW menu.

Choose the 'ShadeNorm(' command. With this command, you enter the values just as if you were doing a 'normal cdf(' calculation. The syntax for the 'ShadeNorm(' command is as follows: 'ShadeNorm(lower bound, upper bound, mean, standard deviation)'

Enter the values shown in the following screenshot:



Next, press [ENTER] to see the result. It should appear as follows:

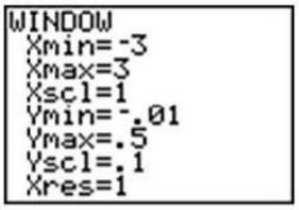


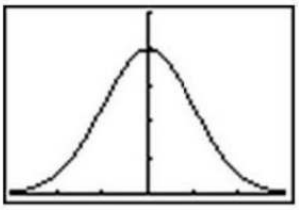
Technology Note: The 'normalpdf(' Command on TI-83/84 Calculator

You may have noticed that the first option in the DISTR menu is 'normalpdf(', which stands for a normal probability density function. It is the option you used in Lesson 5.1 to draw the graph of a normal distribution. Many students wonder what this function is for and occasionally even use it by mistake to calculate what they think are cumulative probabilities, but this function is actually the mathematical formula for drawing a normal distribution. You can find this formula in the resources at the end of the lesson if you are interested. The numbers this function returns are not really useful to us statistically. The primary purpose for this function is to draw the normal curve.

To do this, first be sure to turn off any plots and clear out any functions. Then press [Y=], insert 'normalpdf(', enter 'X', and close the parentheses as shown. Because we did not specify a mean and standard deviation, the standard normal curve will be drawn. Finally, enter the following window settings, which are necessary to fit most of the curve on the screen (think about the Empirical Rule when deciding on settings), and press [GRAPH]. The normal curve below should appear on your screen.





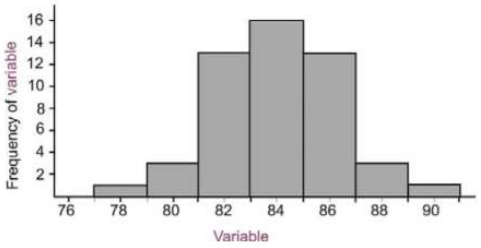


Important links: Tables and Calculators

- This link leads you to a z-score table and an explanation of how to use it: http://tinyurl.com/2ce9ogv
- Here's a normal distribution calculator. Use this calculator to verify your answers and drawing a figure: http://tinyurl.com/n6uwo5m

Exercises

1. Estimate the standard deviation of the following distribution.



2. Calculate the following probabilities using only the z-table. Show all your work.

a) $P(z \geq -0.79)$

b) $P(-1 \leq z \leq 1)$ Show all work.

c) $P(-1.56 < z < 0.32)$

3. Brielle's statistics class took a quiz, and the results were normally distributed, with a mean of 85 and a standard deviation of 7. She wanted to calculate the percentage of the class that got a *B* (between 80 and 90). She used her calculator and was puzzled by the result. Here is a screen shot of her calculator:



Explain her mistake and the resulting answer on the calculator, and then calculate the correct answer.

4. Which grade is better: A 78 on a test whose mean is 72 and standard deviation is 6.5, or an 83 on a test whose mean is 77 and standard deviation is 8.4. Justify your answer and draw sketches of each distribution.

5. Teachers A and B have final exam scores that are approximately normally distributed, with the mean for Teacher A equal to 72 and the mean for Teacher B equal to 82. The standard deviation of Teacher A's scores is 10, and the standard deviation of Teacher B's scores is 5.

a) With which teacher is a score of 90 more impressive? Support your answer with appropriate probability calculations and with a sketch.

b) With which teacher is a score of 60 more discouraging? Again, support your answer with appropriate probability calculations and with a sketch.

6. For each of the following problems, X is a continuous random variable with a normal distribution and the given mean and standard deviation. P is the probability of a value of the distribution being less than x. Find the missing value and sketch and shade the distribution.

Mean	Standard Deviation	x	P, probability
85	4.5		0.68
	1	16	0.05
73		85	0.91
93	5		0.90

7. What is the z-score for the lower quartile in a standard normal distribution?

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5.3: Application of Normal Distributions

Learning Objectives

- Apply the characteristics of a normal distribution to solving applications.

Introduction

The normal distribution is the foundation for statistical inference and will be an essential part of many of those topics in later chapters. In the meantime, this section will cover some of the types of questions that can be answered using the properties of a normal distribution. The first examples deal with more theoretical questions that will help you master basic understandings and computational skills, while the later problems will provide examples with real data, or at least a real context.

Normal Distributions with Real Data

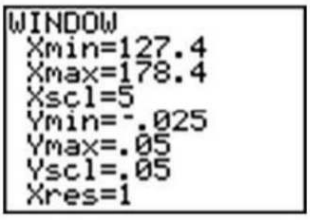
The foundation of performing experiments by collecting surveys and samples is most often based on the normal distribution, as you will learn in greater detail in later chapters. Here are two examples to get you started.

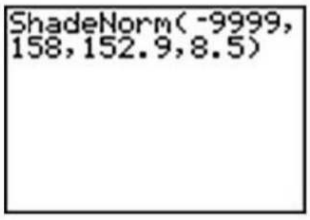
Example 5.3.1

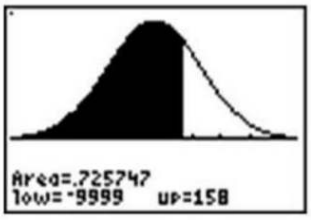
The Information Centre of the National Health Service in Britain collects and publishes a great deal of information and statistics on health issues affecting the population. One such comprehensive data set tracks information about the health of children [1]. According to its statistics, in 2006, the mean height of 12-year-old boys was 152.9 cm, with a standard deviation estimate of approximately 8.5 cm. (These are not the exact figures for the population, and in later chapters, we will learn how they are calculated and how accurate they may be, but for now, we will assume that they are a reasonable estimate of the true parameters.) If 12-year-old Cecil is 158 cm, approximately what percentage of all 12-year-old boys in Britain is he taller than?

Solution

We first must assume that the height of 12-year-old boys in Britain is normally distributed, and this seems like a reasonable assumption to make. As always, draw a sketch and estimate a reasonable answer prior to calculating the percentage. In this case, let's use the calculator to sketch the distribution and the shading. First, decide on an appropriate window that includes about 3 standard deviations on either side of the mean. In this case, 3 standard deviations is about 25.5 cm, so add and subtract this value to/from the mean to find the horizontal extremes. Then enter the appropriate 'ShadeNorm(' command as shown:



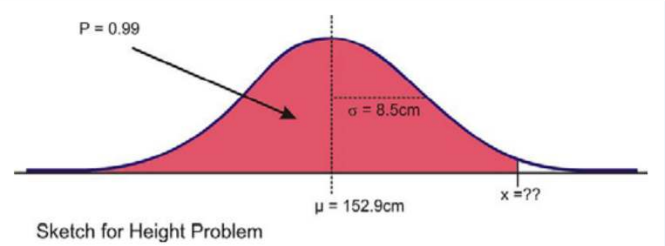




From this data, we would estimate that Cecil is taller than about 73% of 12-year-old boys. We could also phrase our assumption this way: the probability of a randomly selected British 12- year-old boy being shorter than Cecil is about 0.73. Often with data like this, we use percentiles. We would say that Cecil is in the 73rd percentile for height among 12-year-old boys in Britain.

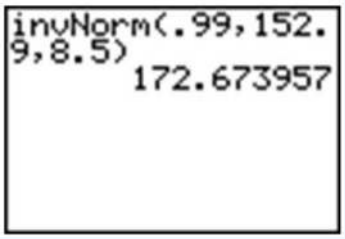
How tall would Cecil need to be in order to be in the top 1% of all 12-year-old boys in Britain?

Here is a sketch:



Sketch for Height Problem

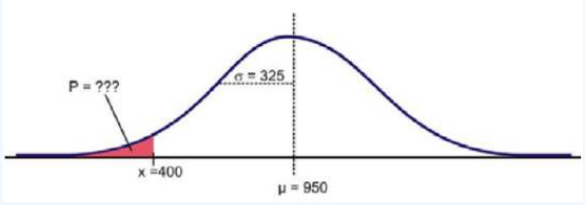
In this case, we are given the percentage, so we need to use the 'invNorm(' command as shown.



Our results indicate that Cecil would need to be about 173 cm tall to be in the top 1% of 12-year-old boys in Britain.

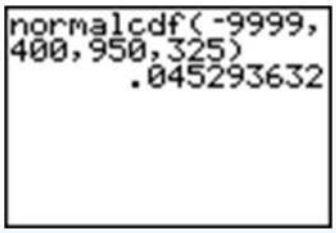
Example 5.3.2

Suppose that the distribution of the masses of female marine iguanas in Puerto Villamil in the Galapagos Islands is approximately normal, with a mean mass of 950 g and a standard deviation of 325 g. There are very few young marine iguanas in the populated areas of the islands, because feral cats tend to kill them. How rare is it that we would find a female marine iguana with a mass less than 400 g in this area?



Solution

Using a graphing calculator, we can approximate the probability of a female marine iguana being less than 400 grams as follows:



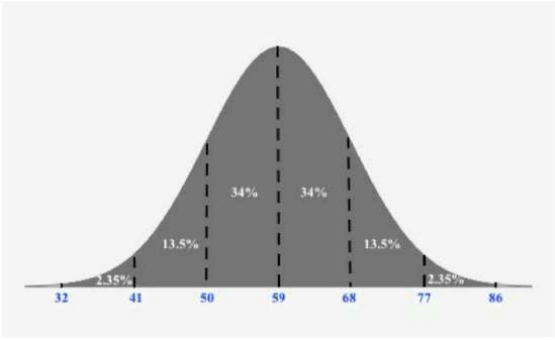
With a probability of approximately 0.045, or only about 5%, we could say it is rather unlikely that we would find an iguana this small.

Example 5.3.3

The physical plant at the main campus of a large state university receives daily requests to replace florescent lightbulbs. The distribution of the number of daily requests is bell-shaped and has a mean of 59 and a standard deviation of 9. Using the Empirical rule, what is the approximate percentage of lightbulb replacement requests numbering between 59 and 77?

Solution

Since we want to use the Empirical Rule, we should draw a figure reflecting the Empirical Rule given the mean is 59 and the standard deviation is 9. Recall, 1 standard deviation from the mean is 59 ± 9, two standard deviations from the mean is $59 \pm 2 \cdot 9$, and 3 standard deviations from the mean is $59 \pm 3 \cdot 9$.



Once we make this figure, we can easily the percentage of lightbulb replacement requests numbering between 59 and 77:

$$34\% + 13.5\% = 47.5\%$$

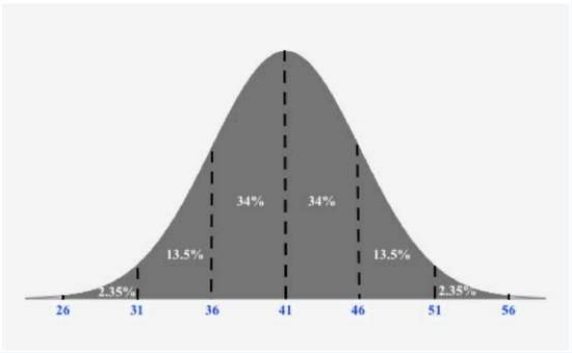
Thus, 47.5% of lightbulb replacement requests numbering between 59 and 77.

Example 5.3.4

A company has a policy of retiring company cars; this policy looks at number of miles driven, purpose of trips, style of car and other features. The distribution of the number of months in service for the fleet of cars is bell-shaped and has a mean of 41 months and a standard deviation of 5 months. Using the Empirical Rule, what is the approximate percentage of cars that remain in service between 46 and 56 months?

Solution

Since we want to use the Empirical Rule, we should draw a figure reflecting the Empirical Rule given the mean is 41 and the standard deviation is 5:



Once we make this figure, we can easily the approximate percentage of cars that remain in service between 46 and 56 months:

$$13.5\% + 2.35\% = 15.85\%$$

Thus, 15.85% of cars remain in service between 46 and 56 months.

Exercises

1. Which of the following intervals contains the middle 95% of the data in a standard normal distribution?

   a) $z < 2$

   b) $z \leq 1.645$

c) $z \leq 1.96$

d) $-1.645 \leq z \leq 1.645$

e) $-1.96 \leq z \leq 1.96$

2. The manufacturing process at a metal-parts factory produces some slight variation in the diameter of metal ball bearings. The quality control experts claim that the bearings produced have a mean diameter of 1.4 cm. If the diameter is more than 0.0035 cm too wide or too narrow, they will not work properly. In order to maintain its reliable reputation, the company wishes to ensure that no more than one-tenth of 1% of the bearings that are made are ineffective. What would the standard deviation of the manufactured bearings need to be in order to meet this goal?

3. Suppose that the wrapper of a certain candy bar lists its weight as 2.13 ounces. Naturally, the weights of individual bars vary somewhat. Suppose that the weights of these candy bars vary according to a normal distribution, with $\mu = 2.2$ ounces and $\sigma = 0.04$ ounces.

a) What proportion of the candy bars weigh less than the advertised weight?

b) What proportion of the candy bars weight between 2.2 and 2.3 ounces?

c) A candy bar of what weight would be heavier than all but 1% of the candy bars out there?

d) If the manufacturer wants to adjust the production process so that no more than 1 candy bar in 1000 weighs less than the advertised weight, what would the mean of the actual weights need to be? (Assume the standard deviation remains the same.)

e) If the manufacturer wants to adjust the production process so that the mean remains at 2.2 ounces and no more than 1 candy bar in 1000 weighs less than the advertised weight, how small does the standard deviation of the weights need to be?

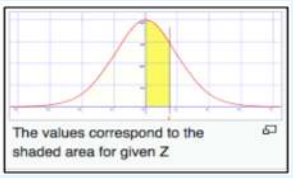
4. The Acme Company manufactures widgets. The distribution of widget weights is bell-shaped. The widget weights have a mean of 51 ounces and a standard deviation of 4 ounces. Use the Empirical Rule to answer the following questions.

a) 99.7% of the widget weights lie between what two weights?

b) What percentage of the widget weights lie between 43 and 63 ounces?

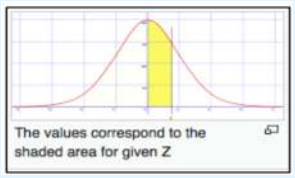
c) What percentage of the widget weights lie above 47?

Standard Normal Table Area under the Normal Curve from 0 to z



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.00	0.00000	0.00399	0.00798	0.01197	0.01595	0.01994	0.02392	0.02790	0.03188	0.03586
0.10	0.03983	0.04380	0.04776	0.05172	0.05567	0.05962	0.06356	0.06749	0.07142	0.07535
0.20	0.07926	0.08317	0.08706	0.09095	0.09483	0.09871	0.10257	0.10642	0.11026	0.11409
0.30	0.11791	0.12172	0.12552	0.12930	0.13307	0.13683	0.14058	0.14431	0.14803	0.15173
0.40	0.15542	0.15910	0.16276	0.16640	0.17003	0.17364	0.17724	0.18082	0.18439	0.18793
0.50	0.19146	0.19497	0.19847	0.20194	0.20540	0.20884	0.21226	0.21566	0.21904	0.22240
0.60	0.22575	0.22907	0.23237	0.23565	0.23891	0.24215	0.24537	0.24857	0.25175	0.25490
0.70	0.25804	0.26115	0.26424	0.26730	0.27035	0.27337	0.27637	0.27935	0.28230	0.28524
0.80	0.28814	0.29103	0.29389	0.29673	0.29955	0.30234	0.30511	0.30785	0.31057	0.31327
0.90	0.31594	0.31859	0.32121	0.32381	0.32639	0.32894	0.33147	0.33398	0.33646	0.33891
1.00	0.34134	0.34375	0.34614	0.34849	0.35083	0.35314	0.35543	0.35769	0.35993	0.36214
1.10	0.36433	0.36650	0.36864	0.37076	0.37286	0.37493	0.37698	0.37900	0.38100	0.38298
1.20	0.38493	0.38686	0.38877	0.39065	0.39251	0.39435	0.39617	0.39796	0.39973	0.40147
1.30	0.40320	0.40490	0.40658	0.40824	0.40988	0.41149	0.41308	0.41466	0.41621	0.41774
1.40	0.41924	0.42073	0.42220	0.42364	0.42507	0.42647	0.42785	0.42922	0.43056	0.43189
1.50	0.43319	0.43448	0.43574	0.43699	0.43822	0.43943	0.44062	0.44179	0.44295	0.44408
1.60	0.44520	0.44630	0.44738	0.44845	0.44950	0.45053	0.45154	0.45254	0.45352	0.45449
1.70	0.45543	0.45637	0.45728	0.45818	0.45907	0.45994	0.46080	0.46164	0.46246	0.46327
1.80	0.46407	0.46485	0.46562	0.46638	0.46712	0.46784	0.46856	0.46926	0.46995	0.47062
1.90	0.47128	0.47193	0.47257	0.47320	0.47381	0.47441	0.47500	0.47558	0.47615	0.47670
2.00	0.47725	0.47778	0.47831	0.47882	0.47932	0.47982	0.48030	0.48077	0.48124	0.48169
2.10	0.48214	0.48257	0.48300	0.48341	0.48382	0.48422	0.48461	0.48500	0.48537	0.48574
2.20	0.48610	0.48645	0.48679	0.48713	0.48745	0.48778	0.48809	0.48840	0.48870	0.48899
2.30	0.48928	0.48956	0.48983	0.49010	0.49036	0.49061	0.49086	0.49111	0.49134	0.49158
2.40	0.49180	0.49202	0.49224	0.49245	0.49266	0.49286	0.49305	0.49324	0.49343	0.49361
2.50	0.49379	0.49396	0.49413	0.49430	0.49446	0.49461	0.49477	0.49492	0.49506	0.49520
2.60	0.49534	0.49547	0.49560	0.49573	0.49585	0.49598	0.49609	0.49621	0.49632	0.49643
2.70	0.49653	0.49664	0.49674	0.49683	0.49693	0.49702	0.49711	0.49720	0.49728	0.49736

Standard Normal Table Area under the Normal Curve from 0 to z



2.80	0.49744	0.49752	0.49760	0.49767	0.49774	0.49781	0.49788	0.49795	0.49801	0.49807
2.90	0.49813	0.49819	0.49825	0.49831	0.49836	0.49841	0.49846	0.49851	0.49856	0.49861
3.00	0.49865	0.49869	0.49874	0.49878	0.49882	0.49886	0.49889	0.49893	0.49896	0.49900
3.10	0.49903	0.49906	0.49910	0.49913	0.49916	0.49918	0.49921	0.49924	0.49926	0.49929
3.20	0.49931	0.49934	0.49936	0.49938	0.49940	0.49942	0.49944	0.49946	0.49948	0.49950
3.30	0.49952	0.49953	0.49955	0.49957	0.49958	0.49960	0.49961	0.49962	0.49964	0.49965
3.40	0.49966	0.49968	0.49969	0.49970	0.49971	0.49972	0.49973	0.49974	0.49975	0.49976
3.50	0.49977	0.49978	0.49978	0.49979	0.49980	0.49981	0.49981	0.49982	0.49983	0.49983
3.60	0.49984	0.49985	0.49985	0.49986	0.49986	0.49987	0.49987	0.49988	0.49988	0.49989
3.70	0.49989	0.49990	0.49990	0.49990	0.49991	0.49991	0.49992	0.49992	0.49992	0.49992
3.80	0.49993	0.49993	0.49993	0.49994	0.49994	0.49994	0.49994	0.49995	0.49995	0.49995
3.90	0.49995	0.49995	0.49996	0.49996	0.49996	0.49996	0.49996	0.49996	0.49997	0.49997
4.00	0.49997	0.49997	0.49997	0.49997	0.49997	0.49997	0.49998	0.49998	0.49998	0.49998

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5.4: The Central Limit Theorem

Suppose X is a random variable with a distribution that may be known or unknown (it can be any distribution). Using a subscript that matches the random variable, suppose:

a. $\mu_x =$ the mean of X
b. $\sigma_x =$ the standard deviation of X

If you draw random samples of size n, then as n increases, the random variable $\bar{X}$ which consists of sample means, tends to be normally distributed and

$$\bar{X} \sim N\left(\mu_x, \frac{\sigma_x}{\sqrt{n}}\right). \tag{5.4.1}$$

The central limit theorem for sample means says that if you keep drawing larger and larger samples (such as rolling one, two, five, and finally, ten dice) and calculating their means, the sample means form their own *normal distribution* (the sampling distribution). The normal distribution has the same mean as the original distribution and a variance that equals the original variance divided by, the sample size. The variable n is the number of values that are averaged together, not the number of times the experiment is done.

To put it more formally, if you draw random samples of size n, the distribution of the random variable $\bar{X}$, which consists of sample means, is called the *sampling distribution of the mean*. The sampling distribution of the mean approaches a normal distribution as n, the sample size, increases.

The random variable $\bar{X}$ has a different z-score associated with it from that of the random variable X. The mean $\bar{x}$ is the value of $\bar{X}$ in one sample.

$$z = \frac{\bar{x} - \mu_x}{\left(\frac{\sigma_x}{\sqrt{n}}\right)} \tag{5.4.2}$$

- μ_x is the average of both X and $\bar{X}$.
- $\sigma\bar{x} = \frac{\sigma_x}{\sqrt{n}} =$ standard deviation of $\bar{X}$ and is called the standard error of the mean.

Howto: Find probabilities for means on the calculator

2^{nd} DISTR

2:normalcdf

$$\text{normalcdf}\left(\text{lower value of the area, upper value of the area, mean, } \frac{\text{standard deviation}}{\sqrt{\text{sample size}}}\right)$$

where:

- *mean* is the mean of the original distribution
- *standard deviation* is the standard deviation of the original distribution
- *sample size* $= n$

Example 5.4.1

An unknown distribution has a mean of 90 and a standard deviation of 15. Samples of size $n = 25$ are drawn randomly from the population.

a. Find the probability that the sample mean is between 85 and 92.
b. Find the value that is two standard deviations above the expected value, 90, of the sample mean.

Answer

a.

Let $X =$ one value from the original unknown population. The probability question asks you to find a probability for the **sample mean**.

Let $\bar{X} =$ the mean of a sample of size 25. Since $\mu_x = 90, \sigma_x = 15$, and $n = 25$,

$$\bar{X} \sim N(90, \frac{15}{\sqrt{25}}).$$

Find $P(85 < x < 92)$. Draw a graph.

$$P(85 < x < 92) = 0.6997$$

The probability that the sample mean is between 85 and 92 is 0.6997.

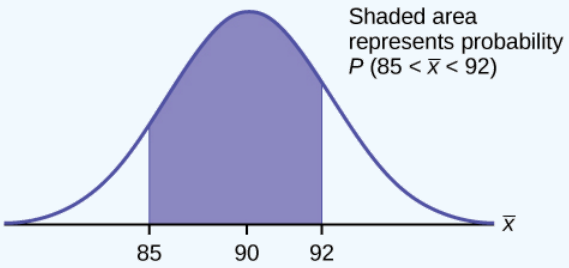


Figure 5.4.1.

normalcdf (lower value, upper value, mean, standard error of the mean)

The parameter list is abbreviated (lower value, upper value, μ, $\frac{\sigma}{\sqrt{n}}$)

normalcdf $(85, 92, 90, \frac{15}{\sqrt{25}}) = 0.6997$

b.

To find the value that is two standard deviations above the expected value 90, use the formula:

$$\text{value} = \mu_x + (\#\text{ofTSDEVs})\left(\frac{\sigma_x}{\sqrt{n}}\right)$$

$$= 90 + 2\left(\frac{15}{\sqrt{25}}\right) = 96$$

The value that is two standard deviations above the expected value is 96.

The standard error of the mean is

$$\frac{\sigma_x}{\sqrt{n}} = \frac{15}{\sqrt{25}} = 3.$$

Recall that the standard error of the mean is a description of how far (on average) that the sample mean will be from the population mean in repeated simple random samples of size n.

? Exercise 5.4.1

An unknown distribution has a mean of 45 and a standard deviation of eight. Samples of size n = 30 are drawn randomly from the population. Find the probability that the sample mean is between 42 and 50.

Answer

$$P(42 < \bar{x} < 50) = \left(42, 50, 45, \frac{8}{\sqrt{30}}\right) = 0.9797$$

Example 5.4.2

The length of time, in hours, it takes an "over 40" group of people to play one soccer match is normally distributed with a **mean of two hours** and a **standard deviation of 0.5 hours**. A **sample of size** $n = 50$ is drawn randomly from the population. Find the probability that the **sample mean** is between 1.8 hours and 2.3 hours.

Answer

Let $X =$ the time, in hours, it takes to play one soccer match.

The probability question asks you to find a probability for the **sample mean time, in hours**, it takes to play one soccer match.

Let $\bar{X} =$ the mean time, in hours, it takes to play one soccer match.

If $\mu_x =$ _____, $\sigma_x =$ _____, and $n =$ _____, then $X \sim N$(_____, _____) by the central limit theorem for means.

$\mu_x = 2, \sigma_x = 0.5, n = 50$, and $X \sim N\left(2, \frac{0.5}{\sqrt{50}}\right)$

Find $P(1.8 < \bar{x} < 2.3)$. Draw a graph.

$P(1.8 < \bar{x} < 2.3) = 0.9977$

normalcdf $\left(1.8, 2.3, 2, \frac{.5}{\sqrt{50}}\right) = 0.9977$

The probability that the mean time is between 1.8 hours and 2.3 hours is 0.9977.

Exercise 5.4.2

The length of time taken on the SAT for a group of students is normally distributed with a mean of 2.5 hours and a standard deviation of 0.25 hours. A sample size of $n = 60$ is drawn randomly from the population. Find the probability that the sample mean is between two hours and three hours.

Answer

$$P(2 < \bar{x} < 3) = \text{normalcdf}\left(2, 3, 2.5, \frac{0.25}{\sqrt{60}}\right) = 1$$

Calculator SKills

To find percentiles for means on the calculator, follow these steps.

- 2nd DIStR
- 3:invNorm

$$k = \text{invNorm}\left(\text{area to the left of} k, \text{mean}, \frac{\text{standard deviation}}{\sqrt{samplesize}}\right)$$

where:

- k = the k^{th} percentile
- *mean* is the mean of the original distribution
- *standard deviation* is the standard deviation of the original distribution
- *sample size* = n

Example 5.4.3

In a recent study reported Oct. 29, 2012 on the Flurry Blog, the mean age of tablet users is 34 years. Suppose the standard deviation is 15 years. Take a sample of size $n = 100$.

a. What are the mean and standard deviation for the sample mean ages of tablet users?
b. What does the distribution look like?
c. Find the probability that the sample mean age is more than 30 years (the reported mean age of tablet users in this particular study).
d. Find the 95th percentile for the sample mean age (to one decimal place).

Answer

a. Since the sample mean tends to target the population mean, we have $\mu_{\bar{x}} = \mu = 34$. The sample standard deviation is given by:

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{15}{\sqrt{100}} = \frac{15}{10} = 1.5$$

b. The central limit theorem states that for large sample sizes (n), the sampling distribution will be approximately normal.
c. The probability that the sample mean age is more than 30 is given by:

$$P(\bar{X} > 30) = \text{normalcdf}(30, E99, 34, 1.5) = 0.9962$$

d. Let k = the 95th percentile.

$$k = \text{invNorm}\left(0.95, 34, \frac{15}{\sqrt{100}}\right) = 36.5$$

Exercise 5.4.3

In an article on Flurry Blog, a gaming marketing gap for men between the ages of 30 and 40 is identified. You are researching a startup game targeted at the 35-year-old demographic. Your idea is to develop a strategy game that can be played by men from their late 20s through their late 30s. Based on the article's data, industry research shows that the average strategy player is 28 years old with a standard deviation of 4.8 years. You take a sample of 100 randomly selected gamers. If your target market is 29- to 35-year-olds, should you continue with your development strategy?

Answer

You need to determine the probability for men whose mean age is between 29 and 35 years of age wanting to play a strategy game.

$$P(29 < \bar{x} < 35) = \text{normalcdf}\left(29, 35, 28, \frac{4.8}{\sqrt{100}}\right) = 0.0186 \quad (5.4.3)$$

You can conclude there is approximately a 1.9% chance that your game will be played by men whose mean age is between 29 and 35.

Example 5.4.4

The mean number of minutes for app engagement by a tablet user is 8.2 minutes. Suppose the standard deviation is one minute. Take a sample of 60.

a. What are the mean and standard deviation for the sample mean number of app engagement by a tablet user?
b. What is the standard error of the mean?
c. Find the 90th percentile for the sample mean time for app engagement for a tablet user. Interpret this value in a complete sentence.
d. Find the probability that the sample mean is between eight minutes and 8.5 minutes.

Answer

a. $\mu = \mu = 8.2 \sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{1}{\sqrt{60}} = 0.13$

b. This allows us to calculate the probability of sample means of a particular distance from the mean, in repeated samples of size 60.

c. Let k = the 90th percentile

$k = \text{invNorm}\left(0.90, 8.2, \frac{1}{\sqrt{60}}\right) = 8.37$. This values indicates that 90 percent of the average app engagement time for table users is less than 8.37 minutes.

d. $P(8 < \bar{x} < 8.5) = \text{normalcdf}\left(8, 8.5, 8.2, \frac{1}{\sqrt{60}}\right) = 0.9293$

? Exercise 5.4.4

Cans of a cola beverage claim to contain 16 ounces. The amounts in a sample are measured and the statistics are $n = 34$, $\bar{x} = 16.01$ ounces. If the cans are filled so that $\mu = 16.00$ ounces (as labeled) and $\sigma = 0.143$ ounces, find the probability that a sample of 34 cans will have an average amount greater than 16.01 ounces. Do the results suggest that cans are filled with an amount greater than 16 ounces?

Answer

We have $P(\bar{x} > 16.01) = \text{normalcdf}\left(16.01, E99, 16, \frac{0.143}{\sqrt{34}}\right) = 0.3417$. Since there is a 34.17% probability that the average sample weight is greater than 16.01 ounces, we should be skeptical of the company's claimed volume. If I am a consumer, I should be glad that I am probably receiving free cola. If I am the manufacturer, I need to determine if my bottling processes are outside of acceptable limits.

Summary

In a population whose distribution may be known or unknown, if the size (n) of samples is sufficiently large, the distribution of the sample means will be approximately normal. The mean of the sample means will equal the population mean. The standard deviation of the distribution of the sample means, called the standard error of the mean, is equal to the population standard deviation divided by the square root of the sample size (n).

Formula Review

- The Central Limit Theorem for Sample Means:

$$\bar{X} \sim N\left(\mu_x, \frac{\sigma_x}{\sqrt{n}}\right)$$

- The Mean $\bar{X}$: σ_x
- Central Limit Theorem for Sample Means z-score and standard error of the mean:

$$z = \frac{\bar{x} - \mu_x}{\left(\frac{\sigma_x}{\sqrt{n}}\right)}$$

- Standard Error of the Mean (Standard Deviation ($\bar{X}$)):

$$\frac{\sigma_x}{\sqrt{n}}$$

Glossary

Average

a number that describes the central tendency of the data; there are a number of specialized averages, including the arithmetic mean, weighted mean, median, mode, and geometric mean.

Central Limit Theorem

Given a random variable (RV) with known mean μ and known standard deviation, σ, we are sampling with size n, and we are interested in two new RVs: the sample mean, $\bar{X}$, and the sample sum, $\sum X$. If the size (n) of the sample is sufficiently large, then $\bar{X} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$ and $\sum X \sim N(n\mu, (\sqrt{n})(\sigma))$. If the size ($n$) of the sample is sufficiently large, then the distribution of the sample means and the distribution of the sample sums will approximate a normal distributions regardless of the shape of the population. The mean of the sample means will equal the population mean, and the mean of the sample sums will equal n times the population mean. The standard deviation of the distribution of the sample means, $\frac{\sigma}{\sqrt{n}}$, is called the standard error of the mean.

Normal Distribution

a continuous random variable (RV) with pdf $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$, where μ is the mean of the distribution and σ is the standard deviation; notation: $X \sim N(\mu, \sigma)$. If $\mu = 0$ and $\sigma = 1$, the RV is called a **standard normal distribution**.

Standard Error of the Mean

the standard deviation of the distribution of the sample means, or $\frac{\sigma}{\sqrt{n}}$.

References

1. Baran, Daya. “20 Percent of Americans Have Never Used Email.”WebGuild, 2010. Available online at www.webguild.org/20080519/20-...ver-used-email (accessed May 17, 2013).
2. Data from The Flurry Blog, 2013. Available online at blog.flurry.com (accessed May 17, 2013).
3. Data from the United States Department of Agriculture.

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5.5: Exercises

1. The physical plant at the main campus of a large state university receives daily requests to replace fluorescent light bulbs. The distribution of the number of daily requests is bell-shaped and has a mean of 56 and a standard deviation of 4. Using the Empirical Rule, what is the approximate percentage of light bulb replacement requests numbering between 56 and 68?

2. A company has a policy of retiring company cars; this policy looks at the number of miles driven, the purpose of trips, style of car and other features. The distribution of the number of months in service for the fleet of cars is bell-shaped and has a mean of 65 months and a standard deviation of 4 months. Using the Empirical Rule, what is the approximate percentage of cars that remain in service between 57 and 61 months?

3. The Acme Company manufactures widgets. The distribution of widget weights is bell-shaped. The widget weights have a mean of 48 ounces and a standard deviation of 11 ounces. Suggestion: sketch the distribution in order to answer these questions.

   a. 99.7% of the widget weights lie between _____ and _____.

   b. What percentage of the widget weights lie between 26 and 81 ounces?

   c. What percentage of the widget weights lie above 37?

4. For a standard normal distribution, find the following probabilities:

   a. $P(z < 1.42)$

   b. $P(z > -2.52)$

   c. $P(-2.06 < z < 2.81)$

5. For a standard normal distribution, given $P(z < c) = 0.7055$, find c.

6. For a standard normal distribution, given $P(z > c) = 0.7109$, find c.

7. On a nationwide math test, the mean was 72 and the standard deviation was 10. If Roberto scored 70, what was his z-score?

8. On a nationwide math test, the mean was 66 and the standard deviation was 4. If Roberto scored 75, what was his z-score?

9. On a nationwide math test, the mean was 57 and the standard deviation was 4. If Roberto scored 85, what was his z-score?

10. A quick survey of peanut butter prices had a standard deviation and mean of $0.26 and $3.68, respectively. Compute the area for a peanut butter jar costing less than $3.50.

11. A quick survey of peanut butter prices had a standard deviation and mean of $0.26 and $3.68, respectively. Compute the area for a peanut butter jar costing more than $4.25.

12. A quick survey of peanut butter prices had a standard deviation and mean of $0.26 and $3.68, respectively. Compute the area for a peanut butter jar costing between $3.50 and $4.25.

13. A quick survey of peanut butter prices had a standard deviation and mean of $0.81 and $3.22, respectively. Compute the price for a peanut butter jar costing given the area from the mean is 0.48422.

14. A quick survey of peanut butter prices had a standard deviation and mean of $1.53 and $2.22, respectively. Compute the price for a peanut butter jar costing given the area from the mean is 0.13683.

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CHAPTER OVERVIEW

6: Geometry

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6.1: Lines

Geometry (from Greek words meaning earth-measure) originally developed as a means of surveying land areas, In its simplest form, it is a study of figures that can be drawn on a perfectly smooth flat surface, or **plane**. It is this **plane geometry** which we will study in this bock and which serves as a foundation for trigonometry, solid and analytic geometry, and calculus.

The simplest figures that can be drawn on a plane are the point and the line. By a line we will always mean a **straight line**. Through two distinct points one and only one (straight) line can be drawn. The line through points A and B will be denoted by $\overleftrightarrow{AB}$ (Figure 6.1.1). The arrows indicate that the line extends indefinitely in each direction, The **line segment** from A to B consists of A, B and that part of $\overleftrightarrow{AB}$ between A and B, It is denoted by AB (some textbooks use the notation $\overline{AB}$ for line segment). The ray $\overrightarrow{AB}$ is the part of $\overleftrightarrow{AB}$ which begins at A and extends indefinitely in the direction of B.

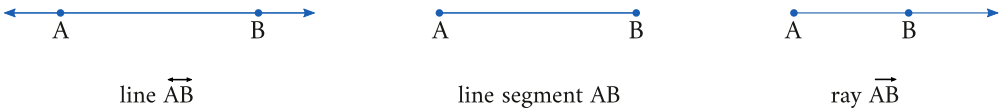


Figure 6.1.1: Line $\overleftrightarrow{AB}$, line segment $\overline{AB}$, and ray $\overrightarrow{AB}$. (CC BY-NC 4.0; Ümit Kaya via LibreTexts)

We assume everyone is familiar with the notion of length of a line segment and how it can be measured in inches, or feet, or meters, etc, The distance between two points A and B is the same as the length of AB.

Two line segments are equal if they have the same length, e.g., in Figure 6.1.2, $AB = CD$,



Figure 6.1.2: $AB = CD$. (CC BY-NC 4.0; Ümit Kaya via LibreTexts)

We often indicate two line segments are equal by marking them in the same way, e.g., in Figure 6.1.3, $AB = CD$ and $EF = GH$.



Figure 6.1.3: $AB = CD$ and $EF = GH$. (CC BY-NC 4.0; Ümit Kaya via LibreTexts)

Example 6.1.1

Find x if $AB = CD$:

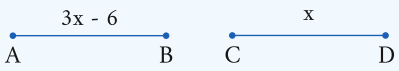


Figure 6.1.E1: (CC BY-NC 4.0; Ümit Kaya via LibreTexts)

Solution

$$\begin{aligned} AB &= CD \\ 3x - 6 &= x \\ 3x - x &= 6 \\ 2x &= 6 \\ x &= 3 \end{aligned}$$

Check:

AB	=	CD
3x - 6		x
3(3) - 6		3
9 - 6		
3		

Answer: $x = 3$.

Notice that in Example **6.1.1** we have not indicated the unit of measurement. Strictly speaking, we should specify that $AB = 3x - 6$ inches (or feet or meters) and that $BC = x$ inches. However since the answer would still be $x = 3$ we will usually omit this information to save space.

We say that B is the midpoint of AC if B is A point on AC and $AB = BC$ (Figure **6.1.4**).

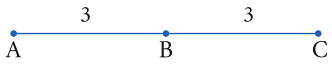


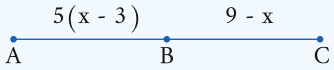
Figure **6.1.4**: B is the midpoint of AC. (CC BY-NC 4.0; Ümit Kaya via LibreTexts)

Example 6.1.2

Find x and AC if B is the midpoint of AC and $AB = 5(x - 3)$ and $BC = 9 - x$,

Solution

We first draw a picture to help visualize the given information:



Since B is a midpoint,

$$\begin{aligned} AB &= BC \\ 5(x-3) &= 9-x \\ 5x-15 &= 9-x \\ 5x+x &= 9+15 \\ 6x &= 24 \\ x &= 4 \end{aligned}$$

Check:

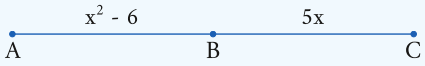
AB	=	BC
5(x - 3)		9 - x
5(4 - 3)		9 - 4
5(1)		5
5		

We obtain $AC = AB + BC = 5 + 5 = 10$.

Answer: $x = 4$, $AC = 10$.

Example 6.1.3

Find AB if B is the midpoint of AC:



Solution

$$\begin{aligned} AB &= BC \\ x^2-6 &= 5x \\ x^2-5x-6 &= 0 \\ (x-6)(x+1) &= 0 \end{aligned}$$

$$\begin{aligned} x-6 &= 0 \\ x &= 6 \end{aligned} \qquad \begin{aligned} x+1 &= 0 \\ x &= -1 \end{aligned}$$

If $x=6$ then $AB=x^2-6=6^2-6=36-6=30$.

If $x=-1$ then $AB=(-1)^2-6=1-6=-5$.

We reject the answer $x=-1$ and $AB=-5$ because the length of a line segment is always positive. Therefore $x=6$ and $AB=30$.

Check:

AB	=	BC
x^2 - 6		5x
6^2 - 6		5(6)
36 - 6		30
30		

Answer: $AB=30$.

Three points are collinear if they lie on the same line.



Figure 6.1.5: A, B, and C are collinear $AB=5$, $BC=3$, and $AC=8$ (CC BY-NC 4.0; Ümit Kaya via LibreTexts)

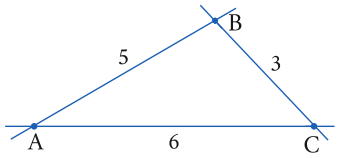


Figure 6.1.6: A, B, and C are not collinear. $AB=5$, $BC=3$, $AC=6$. (CC BY-NC 4.0; Ümit Kaya via LibreTexts)

A, B, and C are collinear if and only if $AB+BC=AC$.

Example 6.1.4

If A, B, and C are collinear and $AC=7$, find x:



Solution

$$\begin{aligned} AB + BC &= AC \\ 8 - 2x + x + 1 &= 7 \\ 9 - x &= 7 \\ 2 &= x \end{aligned}$$

Check:

AB + BC	=	AC
8 - 2x + x + 1		7
8 - 2(2) + 2 + 1		
8 - 4 + 3		
4 + 3		
7		

Answer: $x = 2$

Historical Note

Geometry originated in the solution of practical problems, The architectural remains of Babylon, Egypt, and other ancient civilizations show a knowledge of simple geometric relationships, The digging of canals, erection of buildings, and the laying out of cities required computations of lengths, areas, and volumes, Surveying is said to have developed in Egypt so that tracts of land could be relocated after the annual overflow of the Nile, Geometry was also utilized by ancient civilizations in their astronomical observations and the construction of their calendars.

The Greeks transformed the practical geometry of the Babylonians and Egyptians into an organized body of knowledge. Thales (c, 636 - c. 546 B,C.), one of the "seven wise men" of antiquity, is credited with being the first to obtain geometrical results by logical reasoning, instead of just by intuition and experiment. Pythagoras (c. 582 - c. 507 B,C.) continued the work of Thales, He founded the Pythagorean school, a mystical society devoted to the unified study of philosophy, mathematics, and science, About 300 B,C., Euclid, a Greek teacher of mathematics at the university at Alexandria, wrote a systematic exposition of elementary geometry called the **Elements**, In his **Elements**, Euclid used a few simple principles, called *axioms* or *postulates*, to derive most of the mathematics known at the time, For over 2000 years, Euclid's Elements has been accepted as the standard textbook of geometry and is the basis for most other elementary texts, including this one.

Problems

1. Find x if $AB = CD$:



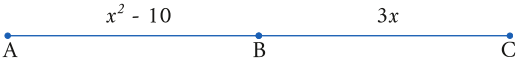
2. Find x if $AB = CD$:



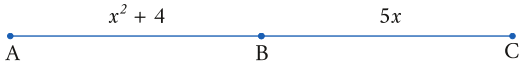
3. Find x and AC if B is the midpoint of AC and $AB = 3(x - 5)$ and $BC = x + 3$.

4. Find x and AC if B is the midpoint of AC and $AB = 2x + 9$ and $BC = 5(x - 9)$,

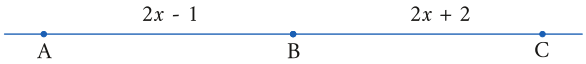
5. Find AB if B is the midpoint of AC:



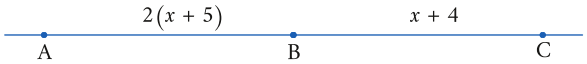
6, Find AB if B is the midpoint of AC:



7. If A, B, and C are collinear and $AC = 13$ find x:



8. If A, B, and C are collinear and $AC = 26$ find x:



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6.2: Angle Classifications

Angles are classified according to their measures as follow:

- An **acute angle** is an angle whose measure is between $0°$ and $90°$.
- A **right angle** is an angle whose measure is $90°$. We often use a little square to indicate a right angle.
- An **obtuse angle** is an angle whose measure is between $90°$ and $180°$.
- A **straight angle** is an angle whose measure is $180°$. A straight angle is just a straight line with one of its points designated as the vertex.
- A **reflex angle** is an angle whose measure is greater than $180°$.

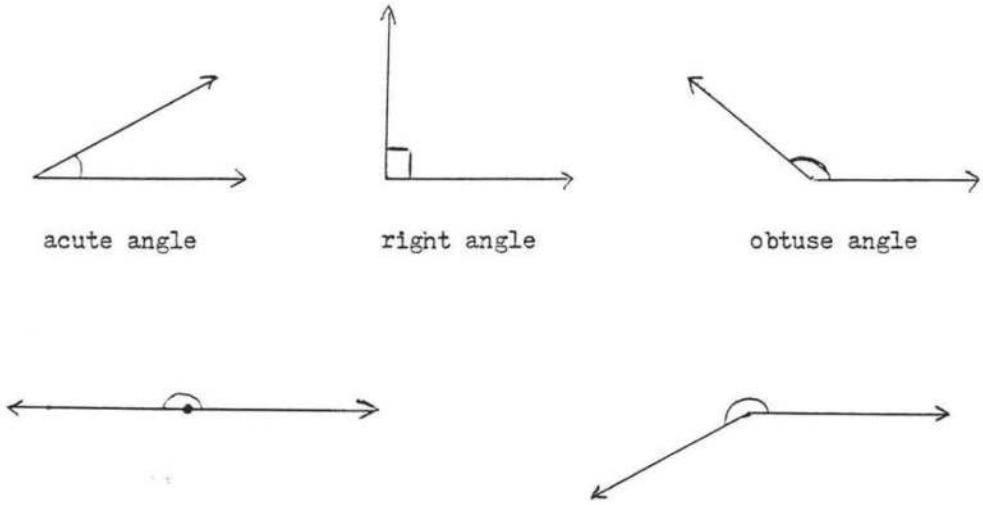


Figure 6.2.1: Angles classified according to their measures.

Notice that an angle can be measured in two ways. In Figure 6.2.2, $\angle ABC$ is a reflex of $240°$ or an obtuse angle of $120°$ depending on how it is measured. Unless otherwise indicated, we will always assume the angle has measure less than $180°$.

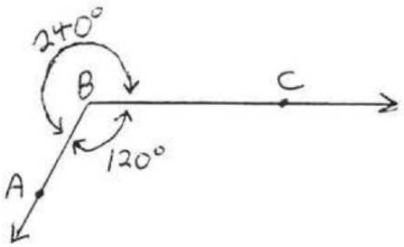


Figure 6.2.2: $\angle ABC$ can be measured in two different ways.

The lines are **perpendicular** if they meet to form right angles. In Figure 6.2.3, $\overleftrightarrow{AB}$ is perpendicular to $\overleftrightarrow{CD}$. The symbol for perpendicular is $\perp$ and we write $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$.

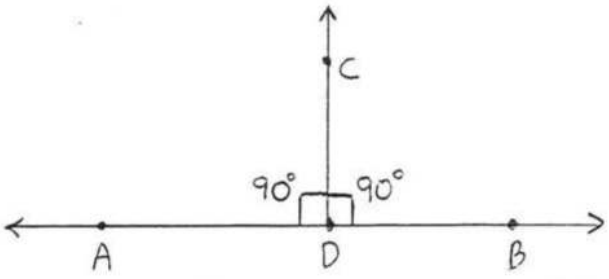


Figure 6.2.3: $\overleftrightarrow{AB}$ is perpendicular to $\overleftrightarrow{CD}$.

The **perpendicular bisector** of a line segment is a line perpendicular to the line segment at its midpoint, In Figure **6.2.4**, $\overleftrightarrow{CD}$ is a perpendicular bisector of AB.

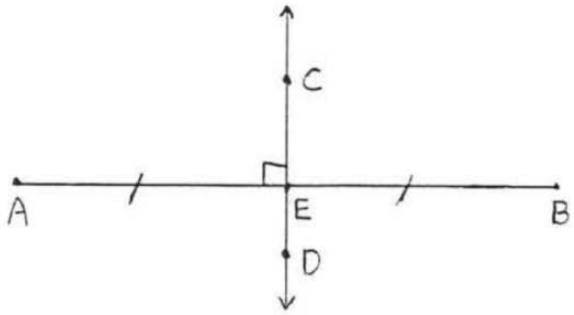


Figure **6.2.4**: $\overleftrightarrow{CD}$ is a perpendicular bisector of AB.

Two angles are called complementary if the sum of their measures is $90°$. Each angle is called the complement of the other. For example, angles of $60°$ and $30°$ are complementary.

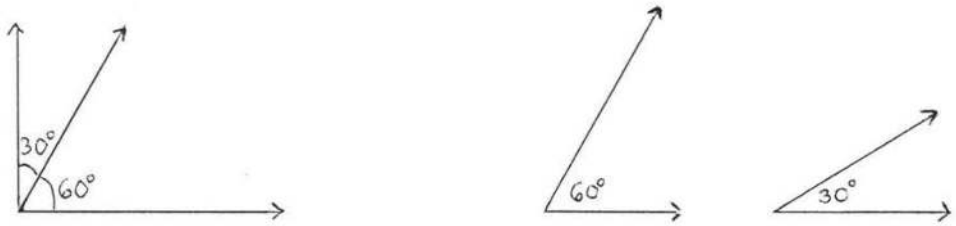


Figure **6.2.5**: Complementary angles.

Example 6.2.1

Find the complement of a $40°$ angle.

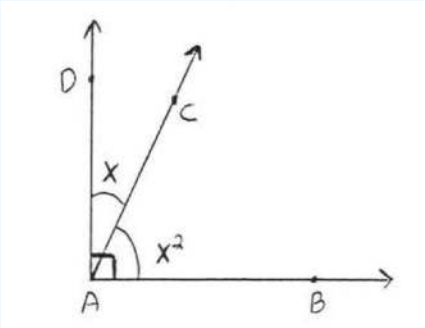
Solution

$90° - 40° = 50°$.

Answer: $50°$.

Example 6.2.2

Find x and the complementary angles:



Solution

Since $\angle BAD = 90°$,

$$\begin{aligned} x^2+x &= 90^\circ \\ x^2+x-90 &= 0 \\ (x-9)(x+10) &= 0 \end{aligned} \tag{6.2.1}$$

$$\begin{aligned} x-9 &= 0 \\ x &= 9 \end{aligned} \qquad \begin{aligned} x+10 &= 0 \\ x &= -10 \end{aligned} \tag{6.2.2}$$

$\angle CAD = x = 90^\circ$. $\angle CAD = x = -10^\circ$.

$\angle BAC = x^2 = 9^2 = 81^\circ$.

$\angle BAC = \angle CAD = 81^\circ + 9^\circ = 90^\circ$.

We reject the answer $x = -10$ because the measure of an angle is always positive.(In trigonometry, when directed angles are introduced, angles can have negative measure. In this book, however, all angles will be thought of as having positive measure,)

Check, $x = 9$:

$x^2 + x = 90^\circ$

$9^2 + 9$

$81 + 9$

90°

Answer: $x = 9$, $\angle CAD = 9^\circ$, $\angle BAC = 81^\circ$.

Two angles are called supplementary if the sum of their measures is 180°. Each angle is called the supplement of the other. For example, angle of 150° and 30° are supplementary.



Figure 6.2.6: Supplementary angles.

Example 6.2.3

Find the supplement of an angle of 40°.

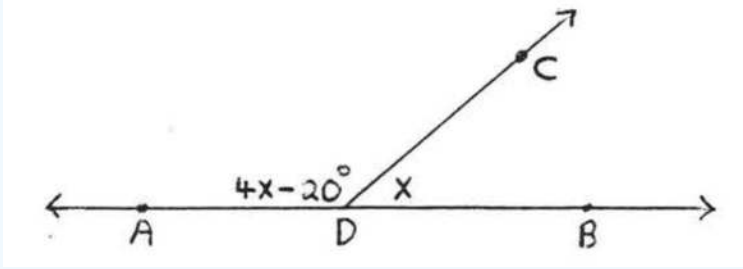
Solution

$180^\circ - 40^\circ = 140^\circ$.

Answer: 140°.

Example 6.2.4

Find x and the supplementary angles:



Solution

Since $\angle ADB = 180°$,

$$\begin{aligned} 4x - 20 + x &= 180° \\ 5x &= 180 + 20 \\ 5x &= 200 \\ x &= 40 \end{aligned} \tag{6.2.3}$$

$\angle ADC = 4x - 20 = 4(40) - 20 = 160 - 20 = 140°$

$\angle BDC = x = 40°$,

$\angle ADC + \angle BDC = 140° + 40° = 180°$.

Check:

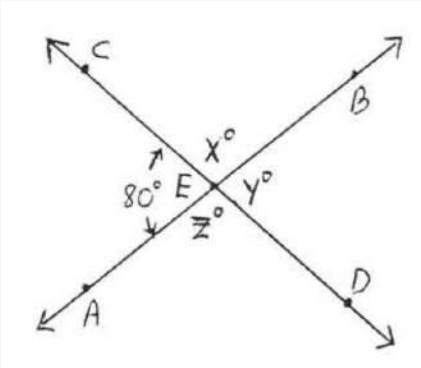
$$\begin{aligned} 4x - 20 + x &= 180^\circ \\ 4(40) - 20 + 40 & \\ 160 - 20 + 40 & \\ 140 + 40 & \\ 180^\circ & \end{aligned}$$

Answer

$x = 40, \angle ADC = 140°, \angle BDC = 40°$.

Example 6.2.5

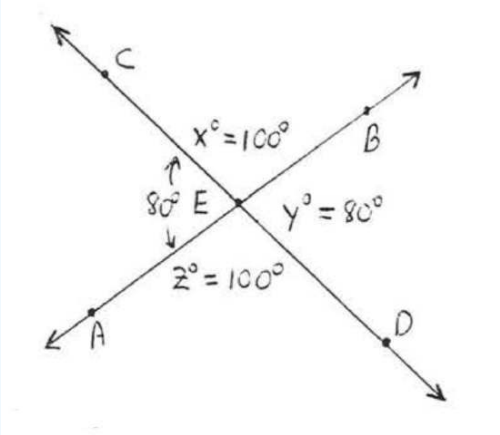
Find x, y, z:



Solution

$x° = 180° - 80° = 100°$ because $x°$ and $80°$ are the measures of supplementary angles.

$$y^\circ = 180^\circ - x^\circ = 180^\circ - 100^\circ = 80^\circ.$$
$$z^\circ = 180^\circ - 80^\circ = 100^\circ. \tag{6.2.4}$$



Answer: $x = 100$, $y = 80$, $z = 100$.

When two lines intersect as in EXAMPLE E, they form two pairs of angles that are opposite to each other called vertical angles, In Figure 6.2.7, $\angle x$ and $\angle x'$ are one pair of vertical angles. $\angle y$ and $\angle y'$ a.re the other pair of vertical angles, As suggested by Example 6.2.5, $\angle x = \angle x'$ and $\angle y = \angle y'$. To see this in general, we can reason as follows: $\angle x$ is the supplement of $\angle y$ so $\angle x = 180^\circ - \angle y$. $\angle x'$ is also the supplement of $\angle y$ so $\angle x' = 180 - \angle y$. Therefore $\angle x = \angle x'$. Similarly, we can show $\angle y = \angle y'$. Therefore vertical angles are always equal.

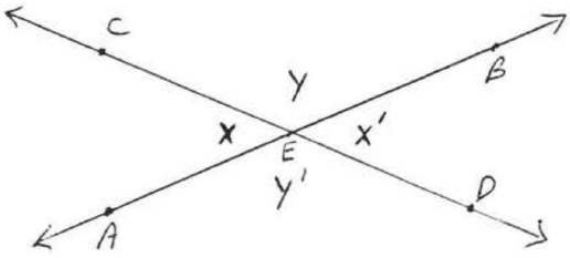
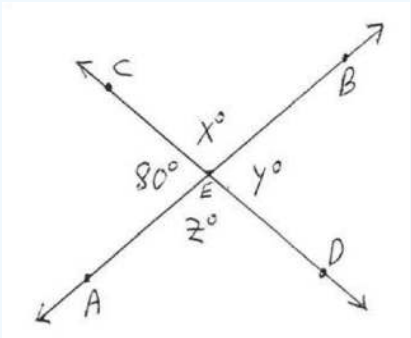


Figure 6.2.7: $\angle x$, $\angle x'$ and $\angle y$, $\angle y'$ are pairs of vertical angles.

We can now use "vertical angles are equal" in solving problems:

Example 6.2.5 (repeated)

Find x, y, and z:



Solution

$\angle x = 180^\circ - 80^\circ = 100^\circ$ because $\angle x$ is the supplement of 80°.

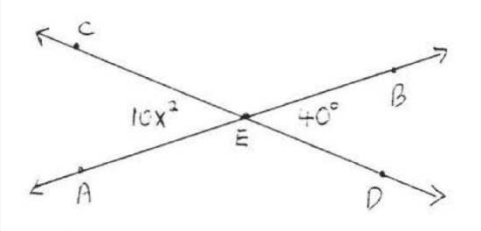
$\angle y = 80°$ because vertical angles are equal.

$\angle z = \angle x = 100°$ because vertical angles are equal.

Answer: $x = 100$, $y = 80$, $z = 100$.

Example 6.2.6

Find x:



Solution

Since vertical angles are equal, $10x^2 = 40°$.

$$\begin{array}{rcl} \text{Method 1:} \quad 10x^2 & = & 40 \\ 10x^2 - 40 & = & 0 \\ (10)(x^2 - 4) & = & 0 \\ x^2 - 4 & = & 0 \\ (x+2)(x-2) & = & 0 \end{array}$$

$$\begin{array}{rcl} x + 2 & = & 0 \\ x & = & -2 \end{array} \qquad \begin{array}{rcl} x - 2 & = & 0 \\ x & = & 2 \end{array}$$

$$\begin{array}{rcl} \text{Method 2:} \quad 10x^2 & = & 40 \\ \frac{10x^2}{10} & = & \frac{40}{10} \\ x^2 & = & 4 \\ x & = & \pm 2 \end{array}$$

If $x = 2$ then $\angle AEC = 10x^2 = 10(2)^2 = 10(4) = 40°$.

If $x = -2$ then $\angle AEC = 10x^2 = 10(-2)^2 = 10(4) = 40°$.

We accept the solution $x = -2$ even though x is negative because the value of the angle $10x^2$ is still positive.

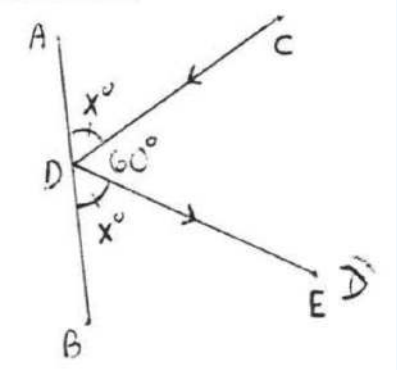
Check:



Answer: $x = 2$ or $x = -2$.

Example 6.2.7

In the diagram, AB represents a mirror, CD represents a ray of light approaching the mirror from C, and E represents the eye of a person observing the ray as it is reflected from the mirror at D. According to a law of physics, $\angle CDA$, called the angle of incidence, equals $\angle EDB$, called the angle of reflection. If $\angle CDE = 60°$, how much is the angle of incidence?



Solution

Let $\backslash x^{\circ} = \angle CDA = \angle EDB$.

$$\begin{aligned} x + x + 60 &= 180 \\ 2x + 60 &= 180 \\ 2x &= 120 \\ x &= 60 \end{aligned} \tag{6.2.5}$$

Answer: 60°

Note on Theorems and Postulates

The statement "vertical angles are always equal" is an example of a **theorem**. A theorem is a statement which we can prove to be true, A proof is a process of reasoning which uses statements already known to be true to show the truth of a new statement, A example of a proof is the discussion preceding the statement "vertical angles are always equal." We used facts about supplementary angles that were already known to establish the new statement, that "vertical angles are always equal."

Ideally we would like to prove all statements in mathematics which we think are true. However before we can begin proving anything we need some true statements with which to start. Such statements should be so self-evident as not to require proofs themselves, A statement of this kind, which we assume to be true without proof, is called a **postulate** or an **axiom**. An example of a postulate is the assumption that all angles can be measured in degrees. This was used without actually being stated in our proof that "vertical angles are always equal,"

Theorems, proofs, and postulates constitute the heart of mathematics and we will encounter many more of them as we continue our study of geometry.

Problems

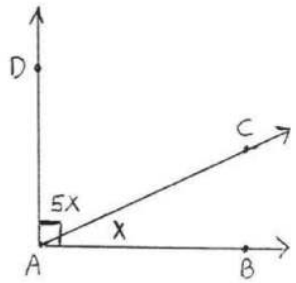
1. Find the complement of an angle of

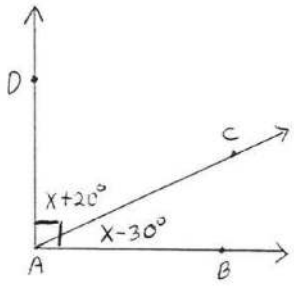
a. 37°
b. 45°
c. 53°
d. 60°

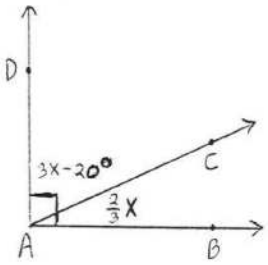
2. Find the complement of an angle of

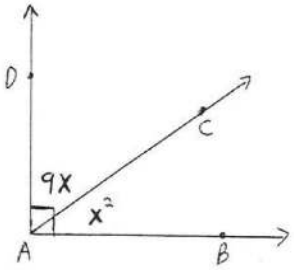
a. 30°
b. 40°
c. 50°
d. 81°

3 - 6. Find x and the complementary angles:

3. 

4. 

5. 

6. 

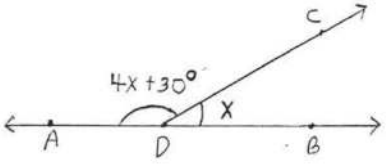
7. Find the supplement of an angle of

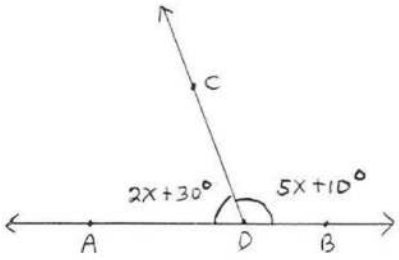
a. $30°$
b. $37°$
c. $90°$
d. $120°$

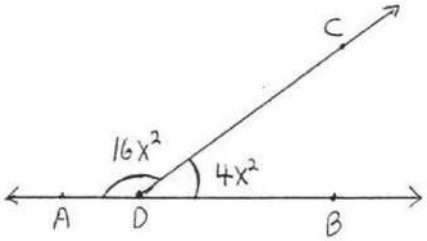
8. Find the supplement of an angle of

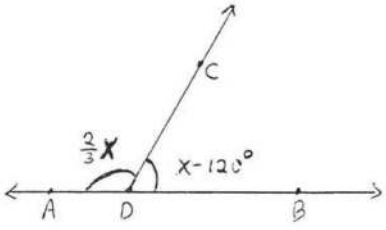
a. $45°$
b. $52°$
c. $85°$
d. $135°$

9 - 14. Find x and the supplementary angles:

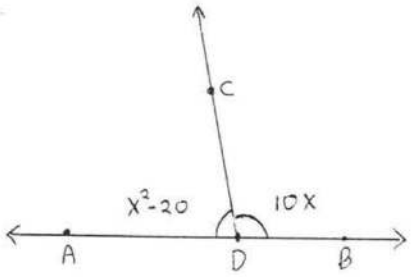
9. 

10. 

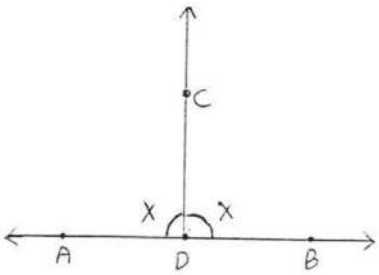
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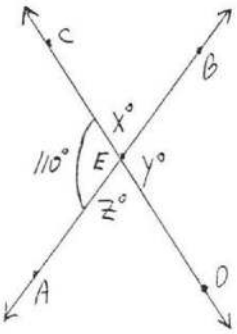


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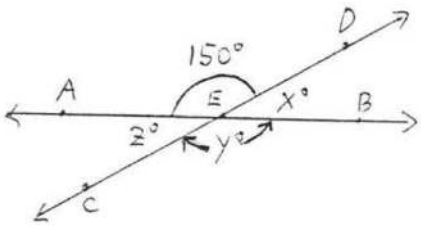


15 - 22. Find x, y, and z:

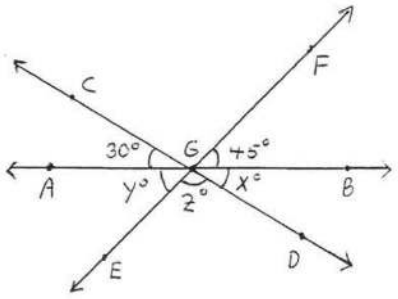
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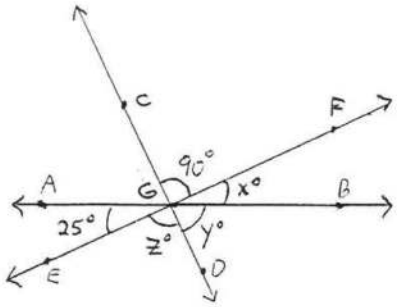
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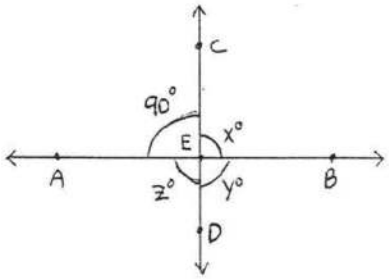
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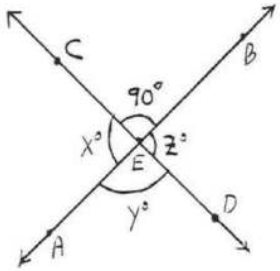
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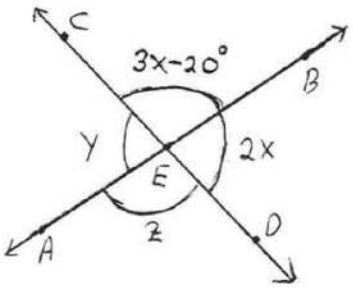
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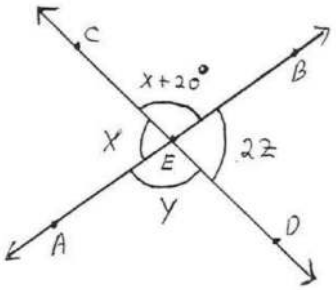
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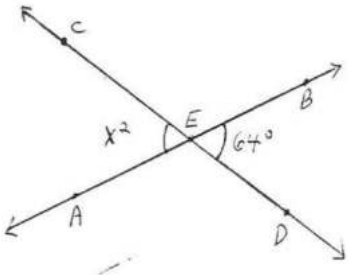


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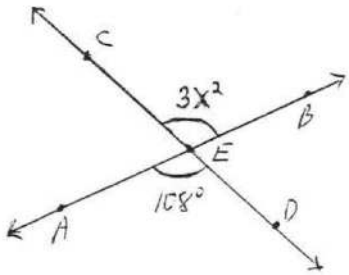


23 - 26. Find x:

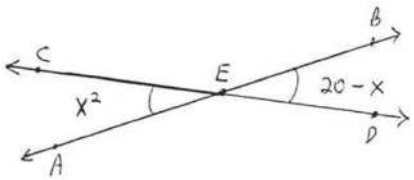
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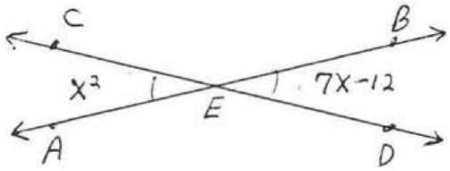
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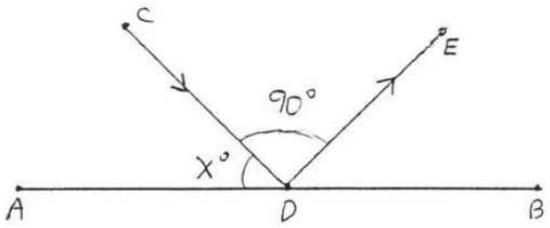
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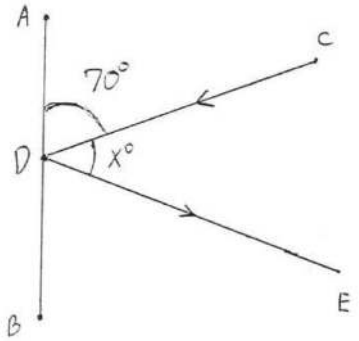
26.



27. Find the angle of incidence, $\angle CDA$:



28. Find x if the angle of incidence is $40°$:



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6.3: Parallel Lines

Two lines are **parallel** if they do not meet, no matter how far they are extended. The symbol for parallel is ||. In Figure 6.3.1, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$. The arrow marks are used to indicate the lines are parallel.

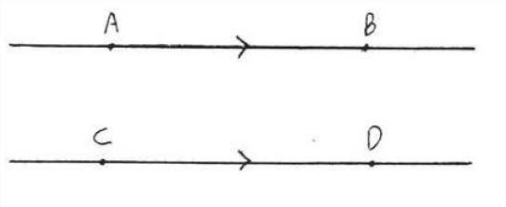


Figure 6.3.1: $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel.They do not meet no matter how far they are extended.

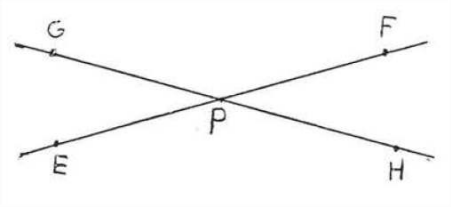


Figure 6.3.1: $\overleftrightarrow{EF}$ and $\overleftrightarrow{GH}$ are not parallel. They meet at point P.

We make the following assumption about parallel lines, called the **parallel postulate.**

Theorem 6.3.1: Parallel Postulate

The probabilities assigned to events by a distribution function on a sample space are given by

Through a point not on a given line one and only one line can be drawn parallel to the given line. So in Figure 6.3.3, there is exactly one line that can be drawn through C that is parallel to $\overleftrightarrow{AB}$.

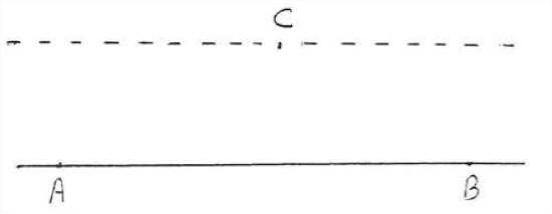


Figure 6.3.3: There is exactly one line that can be drawn through C parallel to $\overrightarrow{AB}$.

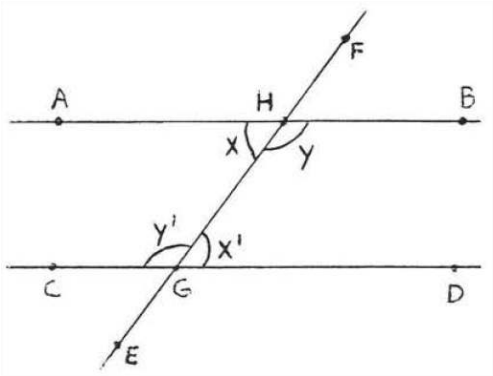


Figure 6.3.4: $\overleftrightarrow{EF}$ is a transversal.

A **transversal** is a line that intersects two other lines at two distinct points. In Figure 6.3.4, $\overleftrightarrow{EF}$ is a transversal. $\angle x$ and $\angle x'$ are called **alternate interior angles** of lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$. The word "alternate," here, means that the angles are on different sides of the transversal, one angle formed with $\overleftrightarrow{AB}$ and the other formed with $\overleftrightarrow{CD}$. The word "interior" means that they are between the two lines. Notice that they form the letter "Z." (Figure 6.3.5). $\angle y$ and $\angle y'$ are also alternate interior angles. They also form a "Z" though It is stretched out and backwards. Viewed from the side, the letter "Z" may also look like an "N."

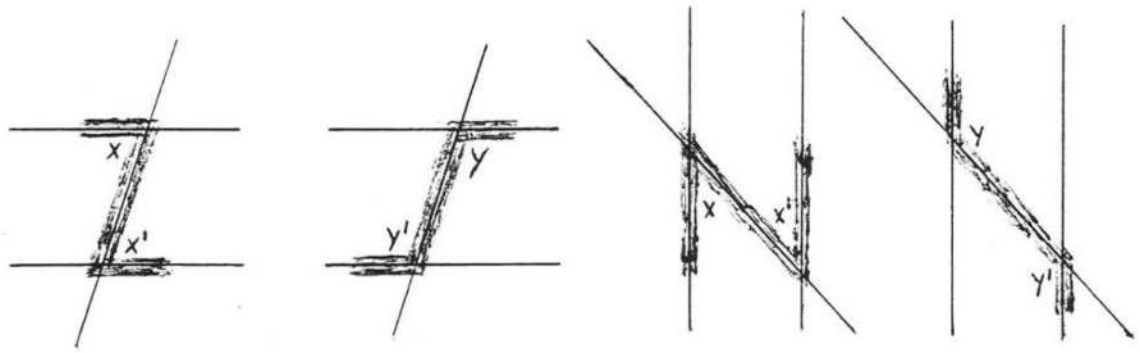


Figure 6.3.5: Alternate interior form the letters "Z" or "N". The letter may be stretched out or backwards.

Alternate interior angles are important because of the following theorem:

Theorem 6.3.1 The "Z" Theorem

If two lines are parallel then their alternate interior angles are equal, If the alternate interior angles of two lines are equal then the lines must 'oe parallel,

In Figure 6.3.6, $\overleftrightarrow{AB}$ must be parallel to $\overleftrightarrow{CD}$ because the alternate interior angles are both $30°$. Notice that the other pair of alternate interior angles, $\angle y$ and $\angle y'$, are also equal. They are both $150°$. In Figure 6.3.7, the lines are not parallel and none of the alternate interior angles are equal.

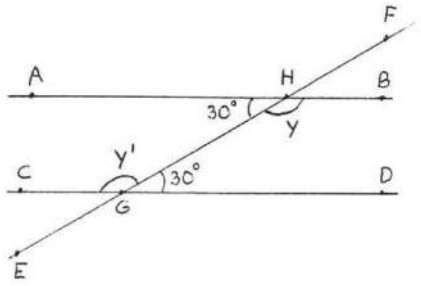


Figure 6.3.6: The lines are parallel and their alternate interior angles are equal.

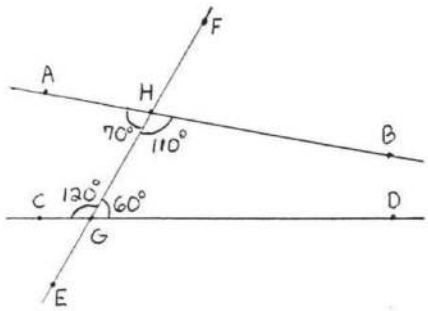
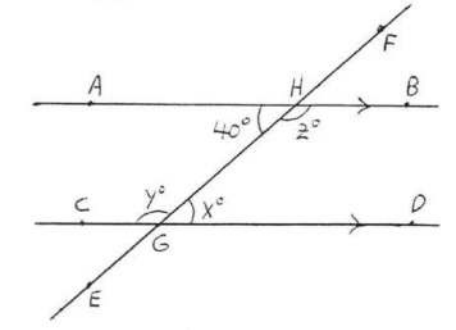


Figure 6.3.7: The lines are not parallel and their alternate interior angles are not equal.

The Proof of Theorem 6.3.1 is complicated and will be deferred to the appendix.

Example 6.3.1

Find x, y and z:



Solution

$\overleftrightarrow{AB} \| \overleftrightarrow{CD}$ since the arrows indicate parallel lines. $x° = 40°$ because alternate interior angles of parallel lines are equal. $y° = z° = 180° - 40° = 140°$.

Answer: $x = 40, y = 140, z = 140$.

Corresponding angles of two lines are two angles which are on the same side of the two lines and the same side of the transversal, In Figure 6.3.8, $\angle w$ and $\angle w'$ are corresponding angles of lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$. They form the letter "F." $\angle x$ and $\angle x'$, $\angle y$ and $\angle y'$, and $\angle z$ and $\angle z'$ are other pairs of corresponding angles of $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$. They all form the letter "F"', though it might be a backwards or upside down "F" (Figure 6.3.9).

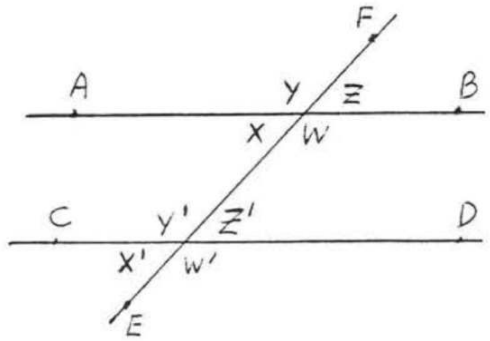


Figure 6.3.8: Four pairs of corresponding angles are illustrated.

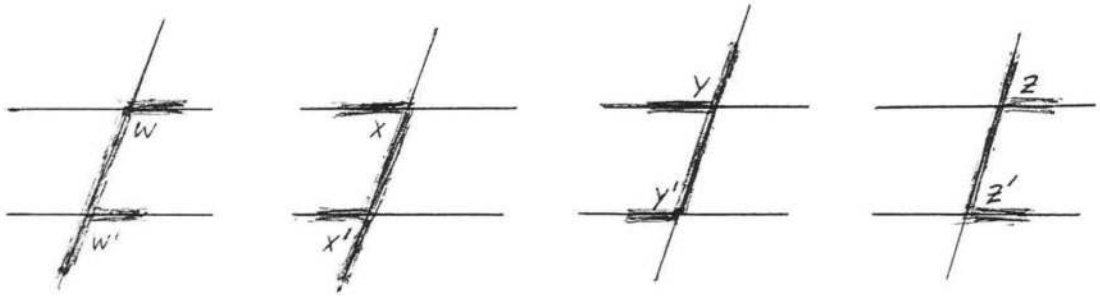


Figure 6.3.9: Corresponding angles form the letter "F," though it may be a backwards or upside down "F."

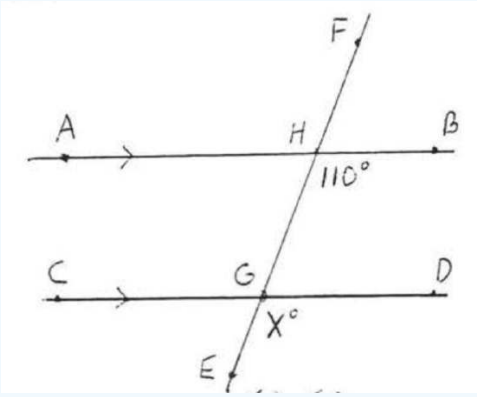
Corresponding angles are important because of the following theorem:

Theorem 6.3.2: The "F" Theorem

If two lines are parallel then their corresponding angles are equal. If the corresponding angles of two lines are equal then the lines must be parallel.

Example 6.3.2

Find x:



Solution

The arrow indicate $\overleftrightarrow{AB} \| \overleftrightarrow{CD}$. Therefore $x° = 110°$ because $x°$ and $110°$ are the measures of corresponding angles of the parallel lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$.

Answer: $x = 110$.

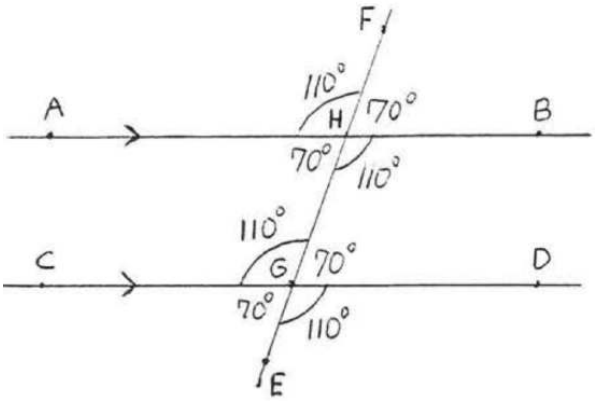


Figure $PageIndex10$: Each pair of corresponding angles is equal.

Notice that we can now find all the other angles in Example $PageIndex2$. Each one is either supplementary to one of the $110°$ angles or forms equal vertical angles with one of them (Figure $PageIndex10$). Therefore **all** the corresponding angles are equal, Also each pair of alternate interior angles is equal. It is not hard to see that if just one pair of corresponding angles or one pair of alternate interior angles are equal then so are all other pairs of corresponding and alternate interior angles.

Proof of Theorem 6.3.2: The corresponding angles will be equal if the alternate interior angles are equal and vice versa. Therefore Theorem 6.3.2 follows directly from Therorem 6.3.1.

In Figure 6.3.11, $\angle x$ and $\angle x'$ are called **interior angles on the same side of the transversal**.(In some textbooks, interior angles on the same sdie of the transversal are called cointerior angles.) $\angle y$ and $\angle y'$ are also interior angles on the same side of the transversal, Notice that each pair of angles forms the letter "C." Compare Figure 6.3.11 with Figure 10 and also with Example 6.3.1, The following theorem is then apparent:

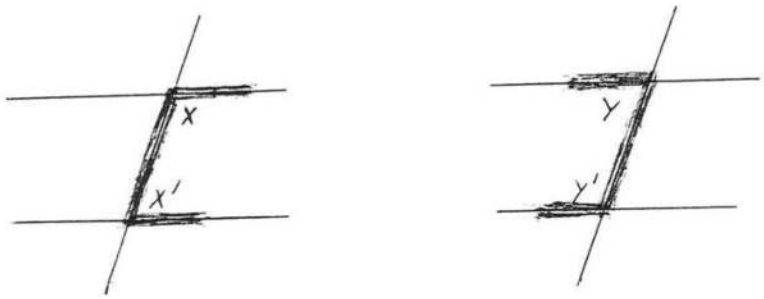


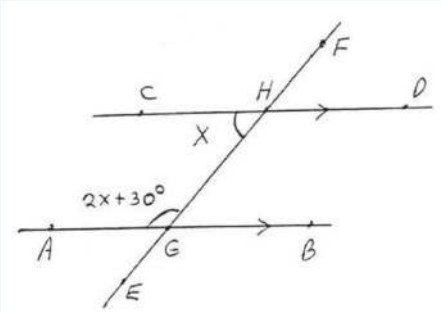
Figure **6.3.11**: Interior angles on the same side of the transversal form the letter "*C*". It may also be a backwards "*C*."

Theorem 6.3.3:The "C" Theorem

If two lines are parallel then the interior angles on the same side of the transversal are supplementary (they add uP to **180°**). If the interior angles of two lines on the same side of the transversal are supplementary then the lines must be parallel.

Example 6.3.3

Find x and the marked angles:



Solution

The lines are parallel so by Theorem **6.3.3** the two labelled angles must be supplementary.

$$\begin{aligned} x+2x+30 &= 180 \\ 3x+30 &= 180 \\ 3x &= 180-30 \\ 3x &= 150 \\ x &= 50 \end{aligned} \tag{6.3.1}$$

$\angle CHG = x = 50°$

$\angle AGH = 2x+30 = 2(50)+30 = 100+30 = 130°$.

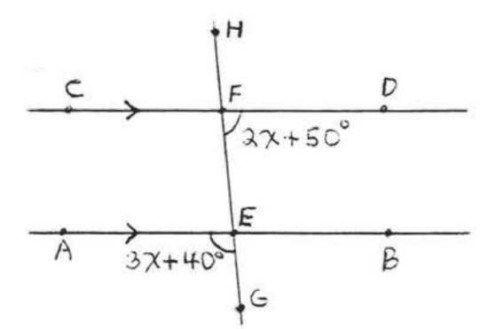
Check:

$$\begin{aligned} x + 2x + 3x &= 180^{\circ} \\ 50 + 2(50) + 30 & \\ 50 + 130 & \\ 180 & \end{aligned}$$

Answer: $x = 50$, $\angle CHG = 50°$, $\angle AGHa = 130°$.

Example 6.3.4

Find x and the marked angles:



Solution

$\angle BEF = 3x + 40°$ because vertical angles are equal. $\angle BEF$ and $\angle DFE$ are interior angles on the same side of the transversal, and therefore are supplementary because the lines are parallel.

$$\begin{aligned} 3x + 40 + 2x + 50 &= 180 \\ 5x + 90 &= 180 \\ 5x &= 180 - 90 \\ 5x &= 90 \\ x &= 18 \end{aligned} \tag{6.3.2}$$

$\angle AEC = 3x + 40 = 3(18) + 40 = 54 + 40 = 94°$

$\angle DFE = 2x + 50 = 2(18) + 50 = 36 + 50 = 86°$

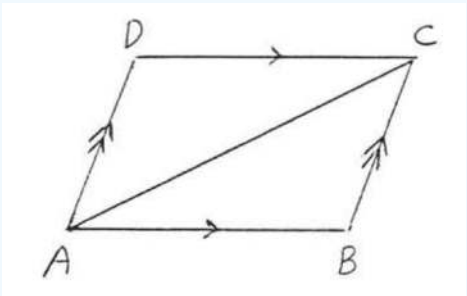
Check:

$$\begin{aligned} 3x + 40 + 2x + 50 &= 180 \\ 3(18) + 40 + 2(18) + 50 & \\ 54 + 40 + 36 + 50 & \\ 94 + 86 & \\ 180 & \end{aligned}$$

Answer: $x = 18$, $\angle AEG = 94°$, $\angle DFE = 86°$.

Example 6.3.5

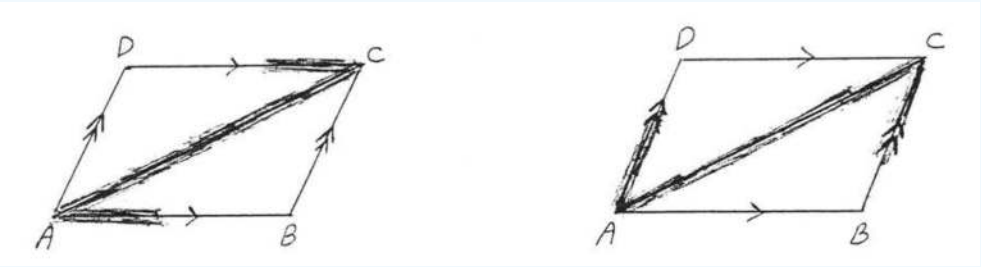
List all pairs of alternate interior angles in the diagram, (The single arrow indicates $\overleftrightarrow{AB}$ is parallel to $\overleftrightarrow{CD}$ and the double arrow indicates $\overleftrightarrow{AD}$ is parallel to $\overleftrightarrow{BC}$.



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Solution

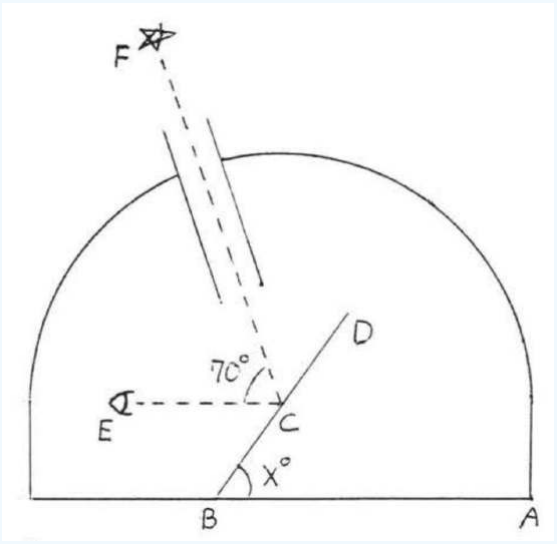
We see if a letter Z or N can be formed using the line segments in the diagram (Figure 6.3.12),



Answer: $\angle DCA$ and $\angle CAB$ are alternate interior angles of lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$. $\angle DAC$ and $\angle ACB$ are alternate interior angles of lines $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$

Example 6.3.6

A telescope is pointed at a star $70°$ above the horizon, What angle $x°$ must the mirror BD make with the horizontal so that the star can be seen in the eyepiece E?

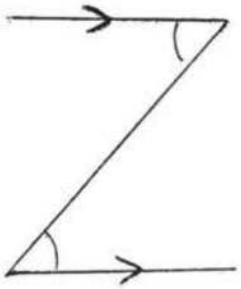


Solution

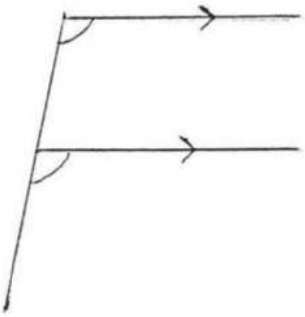
$x° = \angle BCE$ because they are alternate interior angles of parallel lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CE}$. $\angle DCF = \angle BCE = x°$ because the angle of incidence is equal to the angle of reflection. Therefore

$$\begin{aligned} x + 70 + x &= 180 \\ 2x + 70 &= 180 \\ 2x &= 110 \\ x &= 55 \end{aligned} \tag{6.3.3}$$

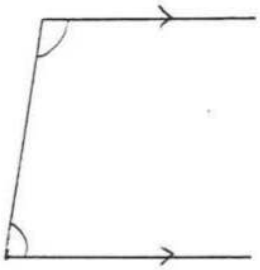
Answer: $55°$



Alternate interior angles of paralle lines are equal. They form the letter "Z."



Corresponding angles of parallel lines are equal. They form the letter "F."



Interior angles on the same sides of the transversal of parallel lines are supplementary. They form the letter "G."

Historical Note

The parallel postulate given earlier in this section is the equivalent of the fifth postulate of Euclid's Elements. Euclid was correct in assuming it as a postulate rather than trying to prove it as a theorem, However this did not become clear to the mathematical world until the nineteenth century, 2200 years later, In the interim, scores of prominent mathematicians attempted unsuccessfully to give a satisfactory proof of the parallel postulate. They felt that it was not as self-evident as a postulate should be, and that it required some formal justification,

In 1826, N, I, Lobachevsky, a Russian mathematician, presented a system of geometry based on the assumption that through a given point more than one straight line can be drawn parallel to a given line (Figure **6.3.13**). In 1854, the German mathematician Georg Bernhard Riemann proFosed a system of geometriJ in which there are no parallel lines at all, A gecmetry in which the parallel postulate has been replaced by some other postulate is called a non-Euclidean geometry. The existence of these geometries shows that the parallel postulate need not necessarily be true. Indeed Einstein used the geometry of Riemann as the basis for his theory of relativity.

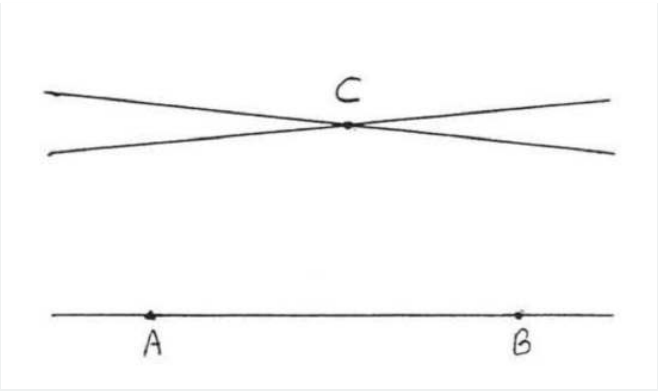


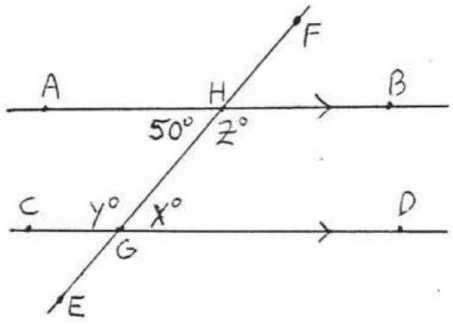
Figure *PageIndex13*: In the geometry of Lobachevsky, more than one line can be drawn through C parallel to AB.

Of course our original parallel postulate makes the most sense for ordinary applications, and we use it throughout this book, However, for applications where great distances are involved, such as in astronomy, it may well be that a non-Euclidean geometry gives a better approximation of physical reality.

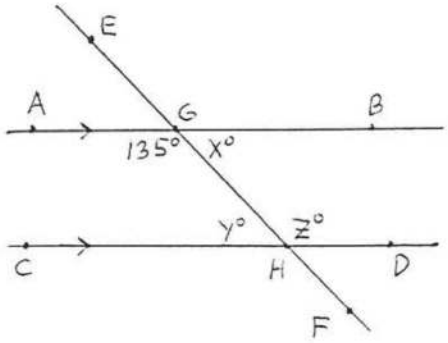
Problems

For each of the following, state the theorem(s) used in obtaining your answer (for example, "alternate interior angles of parallel lines are equal"). Lines marked with the same arrow are assumed to be parallel,

1 - 2. Find x, y, and z:

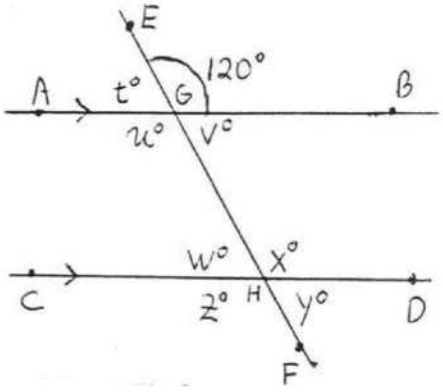


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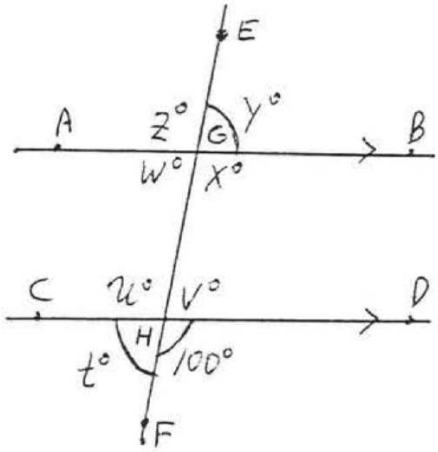


2.

3 - 4. Find t, u, v, w, x, y, and z:

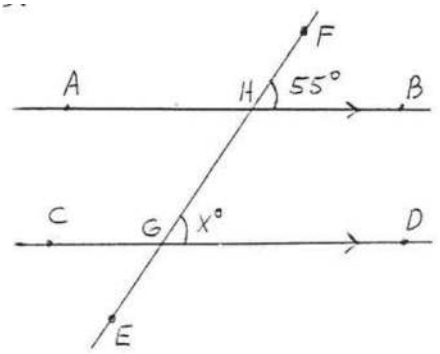


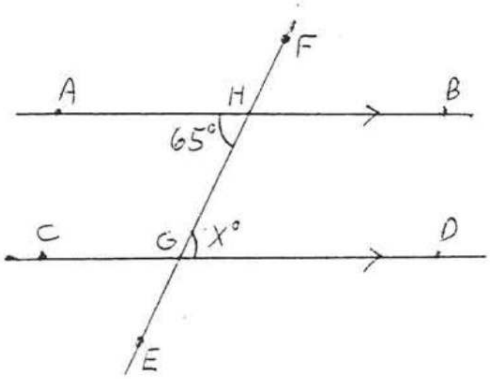
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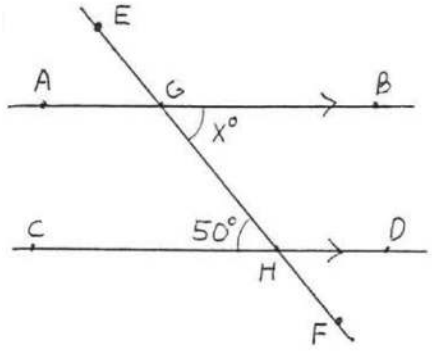


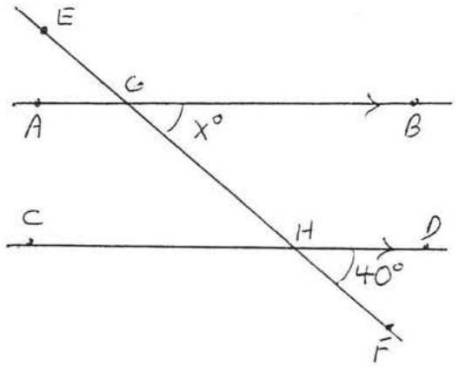
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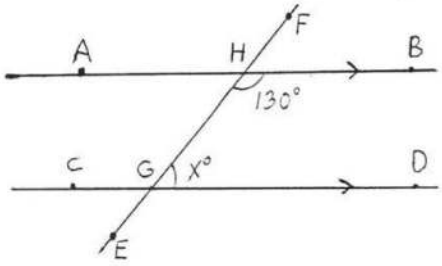
5 - 10. Find x:

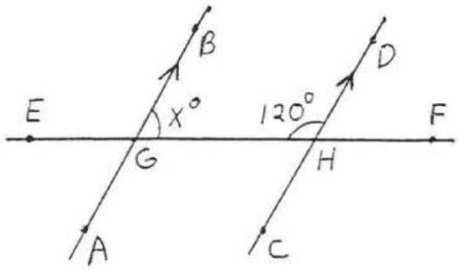
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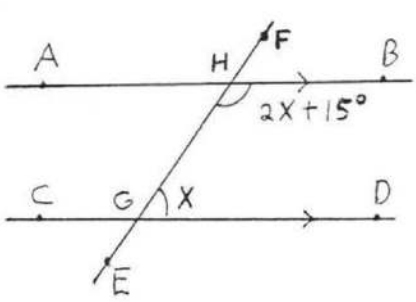
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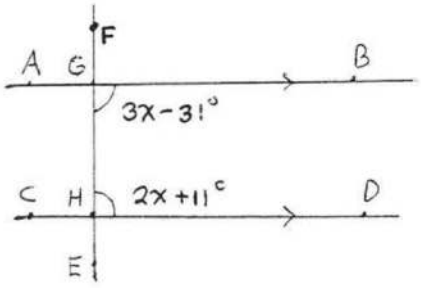
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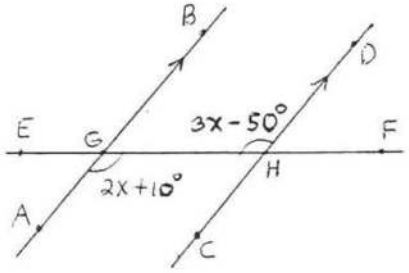
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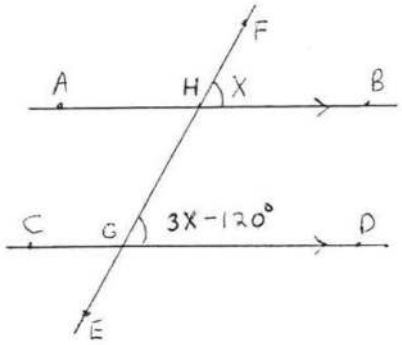
11 - 18. Find x and the marked angles:

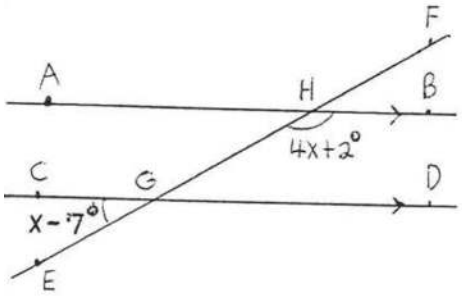
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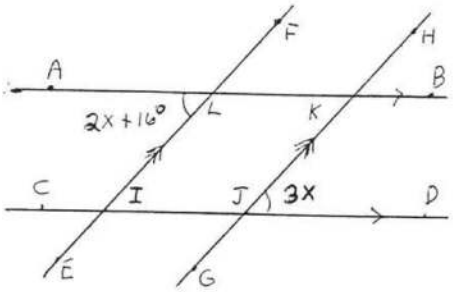
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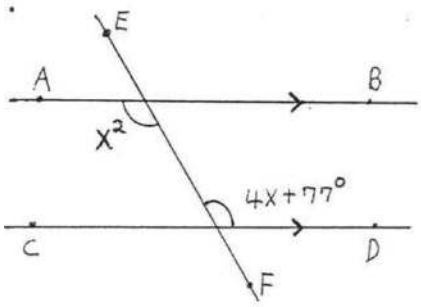
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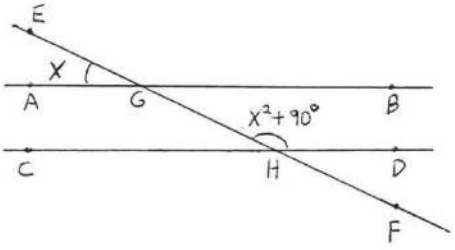
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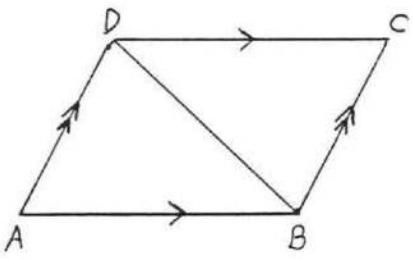
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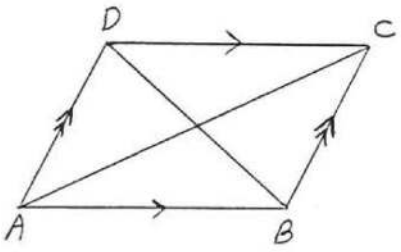
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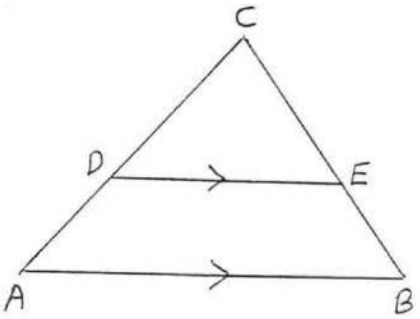
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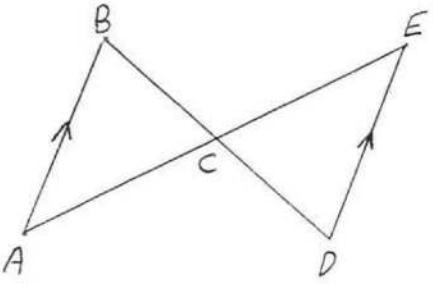
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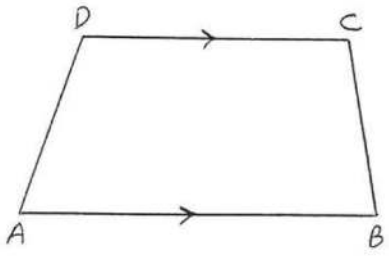
19 - 26. For each of the following, list all pairs of alternate interior angles and corresponding angles, If there are none, then list all pairs of interior angles on the same side of the transversal. Indicate the parallel lines which form each pair of angles.

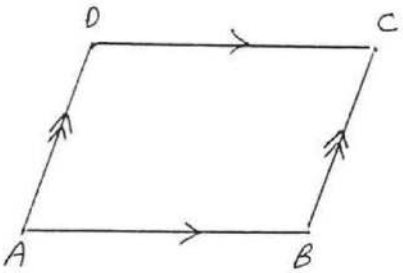
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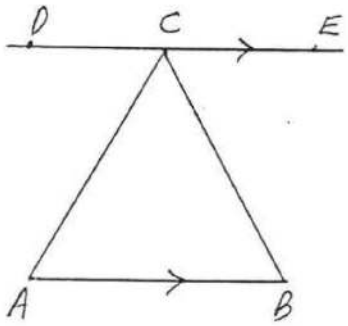
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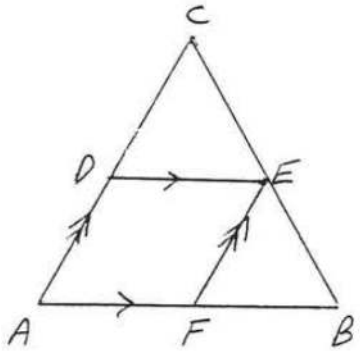
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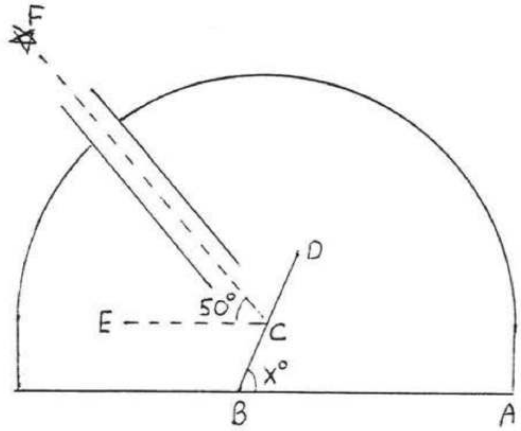
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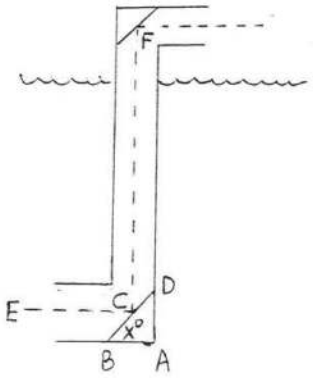
25. 

26. 

27. A telescope is pointed at a star $50°$ above the horizon. What angle $x°$ must the mirror BD make wiht the horizontal so that the star can be seen in the eyepiece E?



28. A periscope is used by sailors in a submarine to see objects on the surface of the water, If $\angle ECF = 90°$, what angle $x°$ does the mirror BD make with the horizontal?



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6.4: Proportions

In our discussion of similar triangles the idea of a proportion will play an important role. In this section we will review the important properties of proportions.

A **proportion** is an equation which states that two fractions are equal. For example, $\frac{2}{6}=\frac{4}{12}$ is a **proportion**. We sometimes say "2 is to 6 as 4 is to 12." This is also written $2:6=4:12$. The **extremes** of this proportion are the numbers 2 and 12 and the **means** are the numbers 6 and 4. Notice that the product of the means $6\times 4=24$ is the same as the product of the extremes $2\times 12=24$.

Theorem 6.4.1

If $\frac{a}{b}=\frac{c}{d}$ then $ad=be$. Conversely, if $ad=bc$ then $\frac{a}{b}=\frac{c}{d}$. (The product of the means is equal to the product of the extremes).

EXAMPLES:

- $\frac{2}{6}=\frac{4}{12}$ and $2\times 12=6\times 4$ are both true.
- $\frac{2}{3}=\frac{6}{9}$ and $2\times 9=3\times 6$ are both true.
- $\frac{1}{4}=\frac{4}{12}$ and $1\times 12=4\times 4$ are both false.

Proof of Theorem 1: If $\frac{a}{b}=\frac{c}{d}$, multiply both sides of the equation by bd:

$$\frac{a}{\cancel{b}}(\cancel{b}d)=\frac{c}{\cancel{d}}(b\cancel{d}) \tag{6.4.1}$$

We obtain $ad=bc$.

Conversely, if $ad=bc$, divide both sides of the equation by bd:

$$\frac{d\cancel{d}}{b\cancel{d}}=\frac{\cancel{b}c}{\cancel{b}d} \tag{6.4.2}$$

The result is $\frac{a}{b}=\frac{c}{d}$.

The following theorem shows that we can interchange the means or the extremes or both of them simultaneously and still have a valid proportion:

Theorem 6.4.2

If one of the following is true then they are all true:

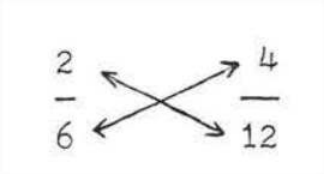
1. $\frac{a}{b}=\frac{c}{d}$
2. $\frac{a}{c}=\frac{b}{d}$
3. $\frac{d}{b}=\frac{c}{a}$
4. $\frac{d}{c}=\frac{b}{a}$

Proof

If any one of these proportions is true when $ad=bc$ by Theorem 6.4.1. The remaining proportions can then be obtained from $ad=bc$ by division, as in Theorem 6.4.1.

EXAMPLE: $\frac{2}{6}=\frac{4}{12},\frac{2}{4}=\frac{6}{12},\frac{12}{6}=\frac{4}{2},\frac{12}{4}=\frac{6}{2}$ are all true because $2\times 12=6\times 4$.

The process of converting a proportion $\frac{2}{6}=\frac{4}{12}$ to the equivalent equation $2\times 12=6\times 4$ is sometimes called **cross multiplication**. The idea is conveyed by the following notation:



Example 6.4.1

Find x : $\frac{3}{x}=\frac{4}{20}$

Solution

By "cross multiplication,"

$$\begin{aligned} 3(20) &= x(4) \\ 60 &= 4x \\ 15 &= x \end{aligned} \tag{6.4.3}$$

Check:

$\frac{3}{x}=\frac{3}{15}=\frac{1}{5}$. $\frac{4}{20}=\frac{1}{5}$.

Answer: $x=15$.

Example 6.4.2

Find x: $\frac{x-1}{x-3}=\frac{2x+2}{x+1}$

Solution

$$\begin{aligned} (x-1)(x+1) &= (x-3)(2x+2) \\ x^2-1 &= 2x^2-4x-6 \\ 0 &= x^2-4x-5 \\ 0 &= (x-5)(x+1) \\ 0 &= x-5 \qquad 0=x+1 \\ 5 &= x \qquad -1=x \end{aligned} \tag{6.4.4}$$

Check, $x=5$:

$\frac{x-1}{x-3}=\frac{5-1}{5-3}=\frac{4}{2}=2$. $\frac{2x+2}{x+1}=\frac{2(5)+2}{5+1}=\frac{12}{6}=2$.

Check, $x=-1$:

$\frac{x-1}{x-3}=\frac{-1-1}{-1-3}=\frac{-2}{-4}=\frac{1}{2}$. $\frac{2x+2}{x+1}=\frac{2(-1)+2}{-1+1}=\frac{-2+2}{0}=\frac{0}{0}$.

Since $\frac{0}{0}$ is undefined, we reject this answer.

Answer: $x=5$.

Problems

1 - 12. Find x:

1. $\frac{6}{x}=\frac{18}{3}$
2. $\frac{4}{x}=\frac{2}{6}$
3. $\frac{7}{1}=\frac{x}{3}$

4. $\frac{x}{8} = \frac{9}{6}$
5. $\frac{7}{1} = \frac{x}{3}$
6. $\frac{10}{2} = \frac{25}{x}$
7. $\frac{x+5}{x} = \frac{5}{4}$
8. $\frac{x-6}{4} = \frac{5}{10}$
9. $\frac{3+x}{x} = \frac{3}{2}$
10. $\frac{x}{x+3} = \frac{4}{x}$
11. $\frac{3x-3}{2x+6} = \frac{x-1}{x}$
12. $\frac{3x-6}{x-2} = \frac{2x+2}{x-1}$

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SECTION OVERVIEW

6.4: Similar Triangles

6.5.1: Proportions

6.5.2: Similar Triangles

6.5.3: Transversals to Three Parallel Lines

6.5.4: Pythagorean Theorem

6.5.5: Special Right Triangles

6.5.6: Distance from a Point to a Line

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6.5.1: Proportions

In our discussion of similar triangles the idea of a proportion will play an important role. In this section we will review the important properties of proportions.

A **proportion** is an equation which states that two fractions are equal. For example, $\frac{2}{6}=\frac{4}{12}$ is a **proportion**. We sometimes say "2 is to 6 as 4 is to 12." This is also written $2:6=4:12$. The **extremes** of this proportion are the numbers 2 and 12 and the **means** are the numbers 6 and 4. Notice that the product of the means $6\times 4=24$ is the same as the product of the extremes $2\times 12=24$.

Theorem 6.5.1.1

If $\frac{a}{b}=\frac{c}{d}$ then $ad=be$. Conversely, if $ad=bc$ then $\frac{a}{b}=\frac{c}{d}$. (The product of the means is equal to the product of the extremes).

EXAMPLES:

- $\frac{2}{6}=\frac{4}{12}$ and $2\times 12=6\times 4$ are both true.
- $\frac{2}{3}=\frac{6}{9}$ and $2\times 9=3\times 6$ are both true.
- $\frac{1}{4}=\frac{4}{12}$ and $1\times 12=4\times 4$ are both false.

Proof of Theorem 1: If $\frac{a}{b}=\frac{c}{d}$, multiply both sides of the equation by bd:

$$\frac{a}{\cancel{b}}(\cancel{b}d)=\frac{c}{\cancel{d}}(b\cancel{d}) \tag{6.5.1.1}$$

We obtain $ad=bc$.

Conversely, if $ad=bc$, divide both sides of the equation by bd:

$$\frac{d\cancel{d}}{b\cancel{d}}=\frac{\cancel{b}c}{\cancel{b}d} \tag{6.5.1.2}$$

The result is $\frac{a}{b}=\frac{c}{d}$.

The following theorem shows that we can interchange the means or the extremes or both of them simultaneously and still have a valid proportion:

Theorem 6.5.1.2

If one of the following is true then they are all true:

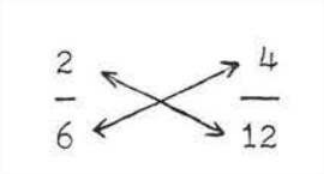
1. $\frac{a}{b}=\frac{c}{d}$
2. $\frac{a}{c}=\frac{b}{d}$
3. $\frac{d}{b}=\frac{c}{a}$
4. $\frac{d}{c}=\frac{b}{a}$

Proof

If any one of these proportions is true when $ad=bc$ by Theorem 6.5.1.1. The remaining proportions can then be obtained from $ad=bc$ by division, as in Theorem 6.5.1.1.

EXAMPLE: $\frac{2}{6}=\frac{4}{12},\frac{2}{4}=\frac{6}{12},\frac{12}{6}=\frac{4}{2},\frac{12}{4}=\frac{6}{2}$ are all true because $2\times 12=6\times 4$.

The process of converting a proportion $\frac{2}{6}=\frac{4}{12}$ to the equivalent equation $2\times 12=6\times 4$ is sometimes called **cross multiplication**. The idea is conveyed by the following notation:



Example 6.5.1.1

Find x : $\frac{3}{x}=\frac{4}{20}$

Solution

By "cross multiplication,"

$$\begin{aligned} 3(20) &= x(4) \\ 60 &= 4x \\ 15 &= x \end{aligned} \tag{6.5.1.3}$$

Check:

$\frac{3}{x}=\frac{3}{15}=\frac{1}{5}$. $\frac{4}{20}=\frac{1}{5}$.

Answer: $x=15$.

Example 6.5.1.2

Find x: $\frac{x-1}{x-3}=\frac{2x+2}{x+1}$

Solution

$$\begin{aligned} (x-1)(x+1) &= (x-3)(2x+2) \\ x^2-1 &= 2x^2-4x-6 \\ 0 &= x^2-4x-5 \\ 0 &= (x-5)(x+1) \\ 0 &= x-5 \qquad 0=x+1 \\ 5 &= x \qquad -1=x \end{aligned} \tag{6.5.1.4}$$

Check, $x=5$:

$\frac{x-1}{x-3}=\frac{5-1}{5-3}=\frac{4}{2}=2$. $\frac{2x+2}{x+1}=\frac{2(5)+2}{5+1}=\frac{12}{6}=2$.

Check, $x=-1$:

$\frac{x-1}{x-3}=\frac{-1-1}{-1-3}=\frac{-2}{-4}=\frac{1}{2}$. $\frac{2x+2}{x+1}=\frac{2(-1)+2}{-1+1}=\frac{-2+2}{0}=\frac{0}{0}$.

Since $\frac{0}{0}$ is undefined, we reject this answer.

Answer: $x=5$.

Problems

1 - 12. Find x:

1. $\frac{6}{x}=\frac{18}{3}$
2. $\frac{4}{x}=\frac{2}{6}$
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4. $\frac{x}{8} = \frac{9}{6}$
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6. $\frac{10}{2} = \frac{25}{x}$
7. $\frac{x+5}{x} = \frac{5}{4}$
8. $\frac{x-6}{4} = \frac{5}{10}$
9. $\frac{3+x}{x} = \frac{3}{2}$
10. $\frac{x}{x+3} = \frac{4}{x}$
11. $\frac{3x-3}{2x+6} = \frac{x-1}{x}$
12. $\frac{3x-6}{x-2} = \frac{2x+2}{x-1}$

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6.5.2: Similar Triangles

Two triangles are said to be similar if they have equal sets of angles. In Figure **6.5.2.1**, $\triangle ABC$ is similar to $\triangle DEF$. The angles which are equal are called **corresponding angles**. In Figure **6.5.2.1**, $\angle A$ corresponds to $\angle D$, $\angle B$ corresponds to $\angle E$, and $\angle C$ corresponds to $\angle F$. The sides joining corresponding vertices are called **corresponding sides**. In Figure **6.5.2.1**, AB corresponds to DE, BC corresponds to EF, and AC corresponds to DF. The symbol for similar is $\sim$. The similarity statement $\triangle ABC \sim \triangle DEF$ will always be written so that corresponding vertices appear in the same order.

For the triangles in Figure **6.5.2.1**, we could also write $\triangle BAC \sim \triangle BDF$ or $\triangle ACB \sim \triangle DFE$ but never $\triangle ABC \sim \triangle EDF$ nor $\triangle ACB \sim \triangle DEF$.

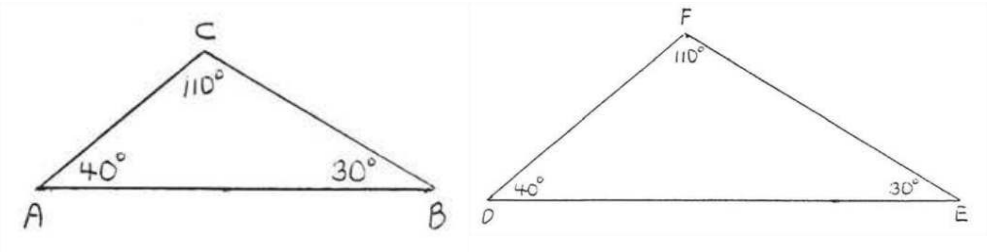
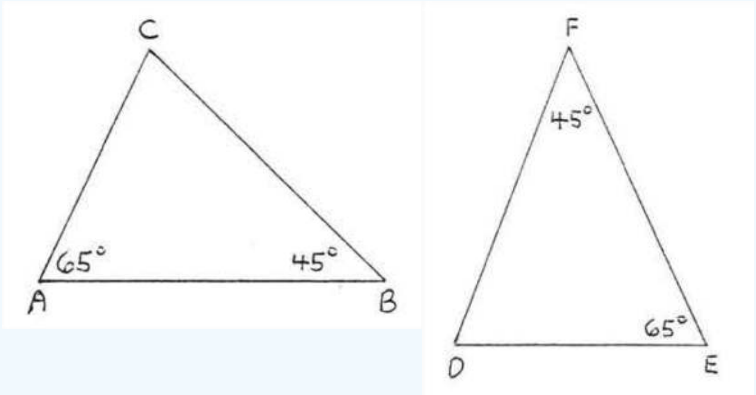


Figure **6.5.2.1**: $\triangle ABC$ is similar to $\triangle DEF$.

We can tell which sides correspond from the similarity statement. For example, if $\triangle ABC \sim \triangle DEF$, then side AB corresponds to side DE because both are the first two letters. BC corresponds to EF because both are the last two letters, AC corresponds to DF because both consist of the first and last letters.

Example 6.5.2.1

Determine if the triangles are similar, and if so, write the similarity statement:



Solution

$$\angle C = 180° - (65° + 45°) = 180° - 110° = 70°$$

$$\angle D = 180° - (65° + 45°) = 180° - 110° = 70°$$

Therefore both triangles have the same angles and $\triangle ABC \sim \triangle EFD$.

Answer: $\triangle ABC \sim \triangle EFD$.

Example A suggests that to prove similarity it is only necessary to know that **two** of the corresponding angles are equal:

Theorem 6.5.2.1

Two triangles are similar if two angles of one equal two angles of the other ($AA = AA$).

In Figure 6.5.2.2, $\triangle ABC \sim \triangle DEF$ because $\angle A = \angle D$ and $\angle B = \angle E$.

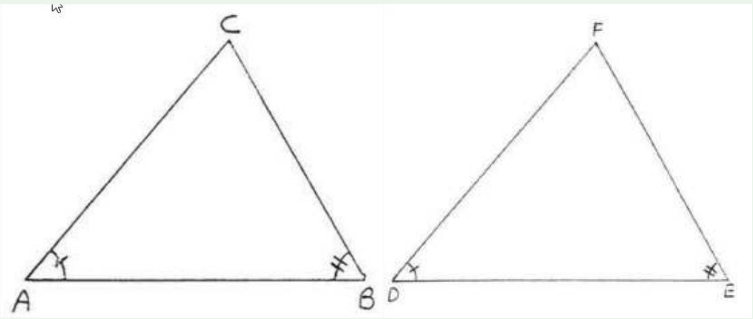


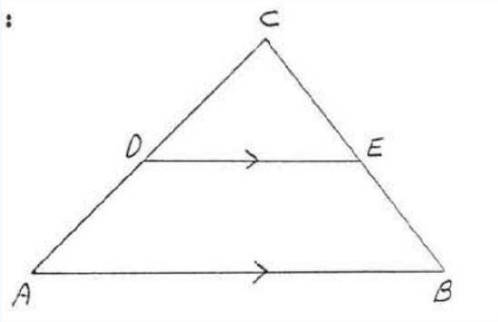
Figure 6.5.2.2. $\triangle ABC \sim \triangle DEF$ because $AA = AA$.

Proof

$\triangle C = 180° - (\angle A + \angle B) = 180° - (\angle D + \angle E) = \angle F$.

Example 6.5.2.2

Determine which triangles are similar and write a similarity statement:



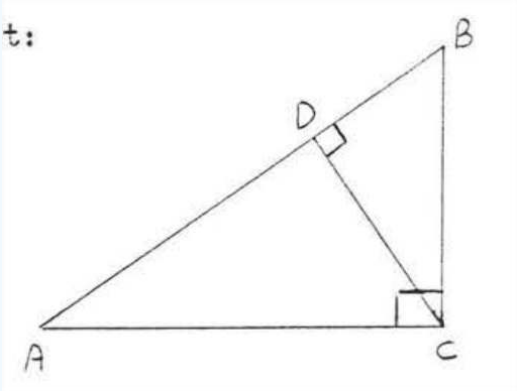
Solution

$\angle A = \angle CDE$ because they are corresponding angles of parallel lines. $\angle C = \angle C$ because of identity. Therefore $\triangle ABC \sim \triangle DEC$ by $AA = AA$.

Answer: $\triangle ABC \sim \triangle DEC$.

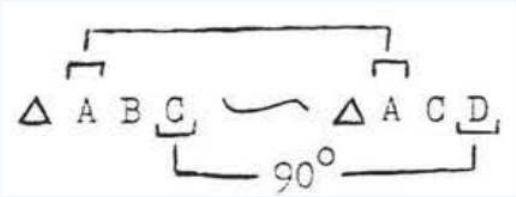
Example 6.5.2.3

Determine which triangles are similar and write a similarity statement:

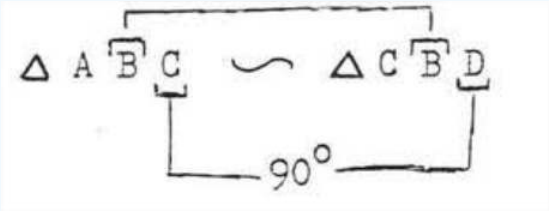


Solution

$\angle A = \angle A$ identity. $\angle ACB = \angle ADC = 90°$. Therefore



Also $\angle B = \angle B$, identity, $\angle BDC = \angle BCA = 90°$. Therefore



Answer: $\triangle ABC \sim \triangle ACD \sim \triangle CBD$.

Similar triangIes are important because of the following theorem:

Theorem 6.5.2.2

The corresponding sides of similar triangles are proportional. This means that if $\triangle ABC \sim \triangle DEF$ then

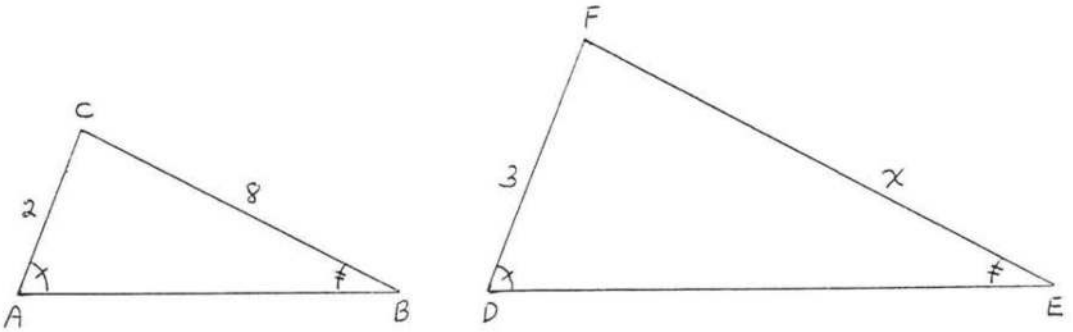
$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}.$$

That is, the first two letters of $\triangle ABC$ are to the first two letters of $\triangle DEF$ as the last two letters of $\triangle ABC$ are to the last two letters of $\triangle DEF$ as the first and last letters of $\triangle ABC$ are to the first and last letters of $\triangle DEF$.

Before attempting to prove Theorem 6.5.2.2, we will give several examples of how it is used:

Example 6.5.2.4

Find x:



Solution

$\angle A = \angle D$ and $\angle B = \angle E$ so $\triangle ABC \sim \triangle DEF$. By Theorem **6.5.2.2**,

$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$.

We will ignore $\frac{AB}{DE}$ here since we do not know and do not have to find either AB or DE.

$$\begin{aligned} \frac{BC}{EF} &= \frac{AC}{DF} \\ \frac{8}{x} &= \frac{2}{3} \\ 24 &= 2x \\ 12 &= x \end{aligned} \qquad (6.5.2.1)$$

Check:

$$\frac{BC}{EF} = \frac{AC}{DF}$$

$$\frac{8}{x} \quad \Big| \quad \frac{2}{3}$$

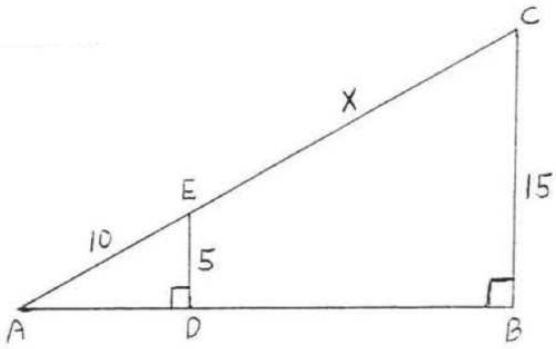
$$\frac{8}{12}$$

$$\frac{2}{3}$$

Answer: $x = 12$.

✔ Example **6.5.2.5**

Find x:



Solution

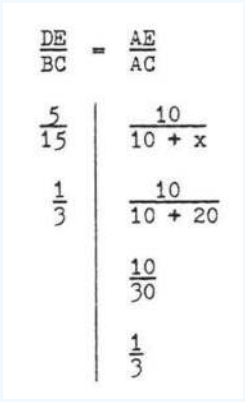
$\angle A = \angle A, \angle ADE = \angle ABC$, so $\triangle ADE \sim \triangle ABC$ by $AA = AA$.

$$\frac{AD}{AB} = \frac{DE}{BC} = \frac{AE}{AC}.$$

We ignore $\frac{AD}{AB}$.

$$\begin{aligned} \frac{DE}{BC} &= \frac{AE}{AC} \\ \frac{5}{15} &= \frac{10}{10+x} \\ 5(10+x) &= 15(10) \\ 50+5x &= 150 \\ 5x &= 150-50 \\ 5x &= 100 \\ x &= 20 \end{aligned} \tag{6.5.2.2}$$

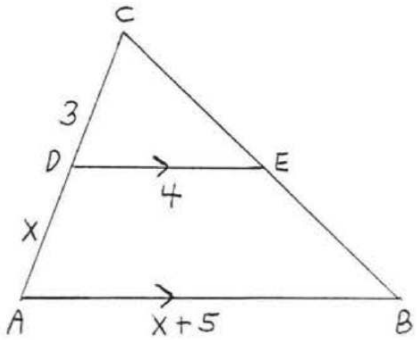
Check:



Answer: $x = 20$.

Example 6.5.2.6

Find x:



Solution

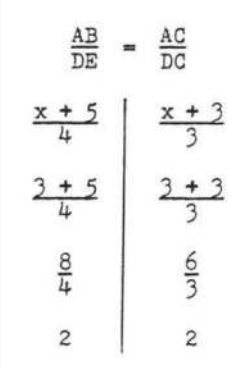
$\angle A = \angle CDE$ because they are corresponding angles of parallel lines. $\angle C = \angle C$ because of identity. Therefore $\triangle ABC \sim \triangle DEC$ by $AA = AA$.

$$\frac{AB}{DE} = \frac{BC}{EC} = \frac{AC}{DC}$$

We ignore $\frac{BC}{EC}$:

$$\begin{aligned} \frac{AB}{DE} &= \frac{AC}{DC} \\ \frac{x+5}{4} &= \frac{x+3}{3} \\ (x+5)(3) &= (4)(x+3) \\ 3x+15 &= 4x+12 \\ 15-12 &= 4x-3x \\ 3 &= x \end{aligned} \tag{6.5.2.3}$$

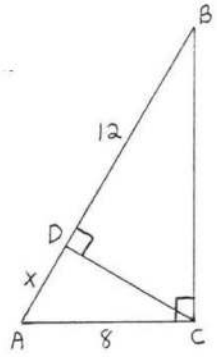
Check:



Answer: $x = 3$.

Example 6.5.2.7

Find x:



Solution

$\angle A = \angle A$, $\angle ACB = \angle ADC = 90°$, $\triangle ABC \sim \triangle ACD$.

$$\begin{aligned} \frac{AB}{AC} &= \frac{AC}{AD} \\ \frac{x+12}{8} &= \frac{8}{x} \\ (x+12)(x) &= (8)(8) \\ x^2+12x &= 64 \\ x^2+12x-64 &= 0 \\ (x-4)(x+16) &= 0 \\ x=4 \qquad x &= -16 \end{aligned} \tag{6.5.2.4}$$

We reject the answer $x = -16$ because $AD = x$ cannot be negative.

Check, $x = 4$

$$\frac{AB}{AC} = \frac{AC}{AD}$$

$$\frac{x+12}{8} \quad \frac{8}{x}$$

$$\frac{4+12}{8} \quad \frac{8}{4}$$

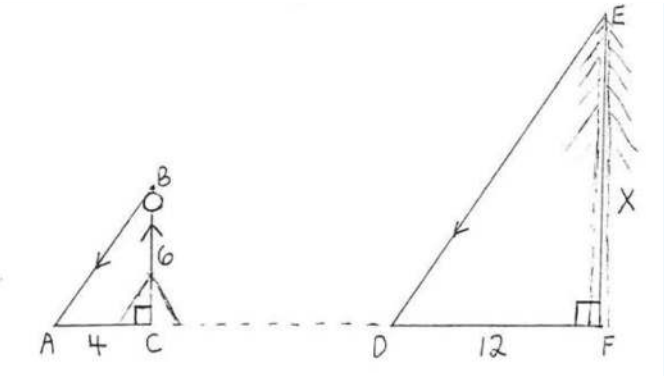
$$\frac{16}{8} \quad 2$$

$$2$$

Answer: $x = 4$.

Example 6.5.2.8

A tree casts a shadow 12 feet long at the same time a 6 foot man casts a shadow 4 feet long. What is the height of the tree?



Solution

In the diagram AB and DE are parallel rays of the sun. Therefore $\angle A = \angle D$ because they are corresponding angles of parallel lines with respect to the transversal AF. Since also $\angle C = \angle F = 90°$, we have $\triangle ABC \sim \triangle DEF$ by $AA = AA$.

$$\begin{aligned} \frac{AC}{DF} &= \frac{BC}{EF} \\ \frac{4}{12} &= \frac{6}{x} \\ 4x &= 72 \\ x &= 18 \end{aligned} \tag{6.5.2.5}$$

Answer: $x = 18$ feet.

Proof of Theorem **6.5.2.2** ("The corresponding sides of similar triangles are proportional"):

We illustrate the proof using the triangles of Example **6.5.2.4** (Figure **6.5.2.3**). The proof for other similar triangles follows the same pattern. Here we will prove that $x = 12$ so that $\frac{2}{3} = \frac{8}{x}$.

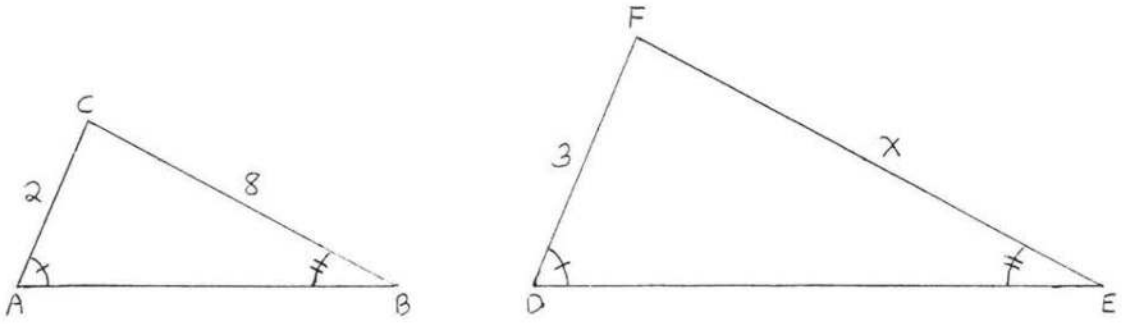


Figure **6.5.2.3**. The triangles of Example **6.5.2.4**.

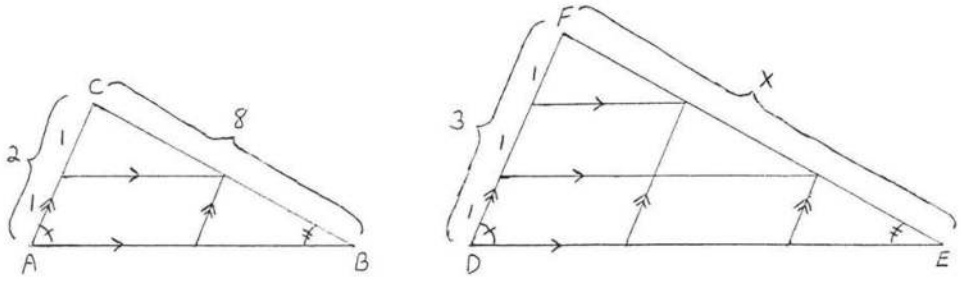


Figure **6.5.2.4**: Draw lines parallel to the sides of $\triangle ABC$ and $\triangle DEF$.

First draw lines parallel to the sides of $\triangle ABC$ and $\triangle DEF$ as shown in Figure **6.5.2.4**. The corresponding angles of these parallel lines are equal and each of the parallelograms with a side equal to 1 has its opposite side equal to 1 as well, Therefore all of the

small triangles with a side equal to 1 are congruent by $AAS = AAS$. The corresponding sides of these triangles form side $BC = 8$ of $\triangle ABC$ (see Figure 6.5.2.5). Therefore each of these sides must equal 4 and $x = EF = 4 + 4 + 4 = 12$ (Figure 6.5.2.6).

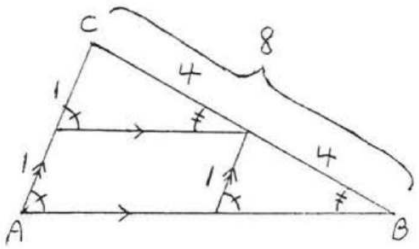


Figure 6.5.2.5. The small triangles are congruent hence the corresponding sides lying on BC must each be equal to 4.

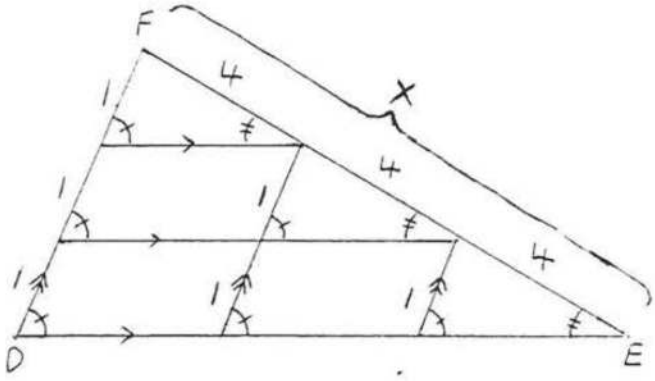


Figure 6.5.2.6. The small triangles of $\triangle DEF$ are congruent to the small triangles of $\triangle ABC$ hence $x = EF = 4 + 4 + 4 = 12$.

(Note to instructor: This proof can be carried out whenever the lengths of the sides of the triangles are rational numbers. However, since irrational numbers can be approximated as closely as necessary by rationals, the proof extends to that case as well.)

Historical Note

Thales (c. 600 B.C.) used the proportionality of sides of similar triangles to measure the heights of the pyramids in Egypt. His method was much like the one we used in Example 6.5.2.8 to measure the height of trees.

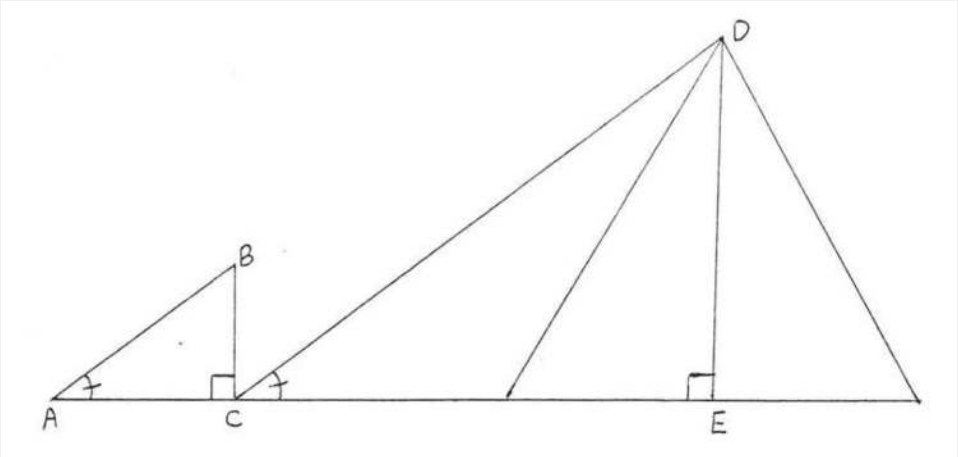


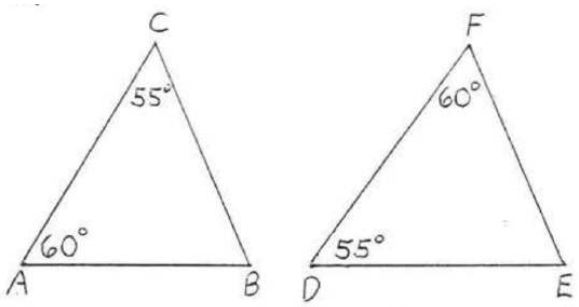
Figure 6.5.2.7. Using similar triangles to measure the height of a pyramid.

In Figure 6.5.2.7, DE represent the height of the pyramid and CE is the length of its shadow. BC represents a vertical stick and AC is the length of its shadow. We have $\triangle ABC \sim \triangle CDE$. Thales was able to measure directly the lengths AC, BC, and CE. Substituting these values in the proportion $\frac{BC}{DE} = \frac{AC}{CE}$, he was able to find the height DE.

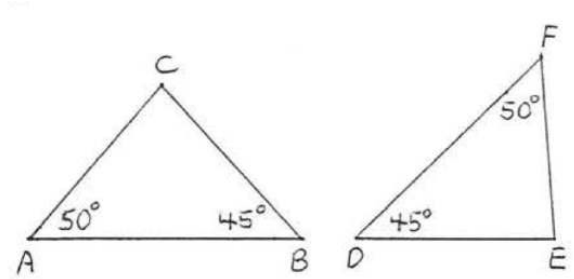
Problems

1 - 6. Determine which triangles are similar and write the similarity statement:

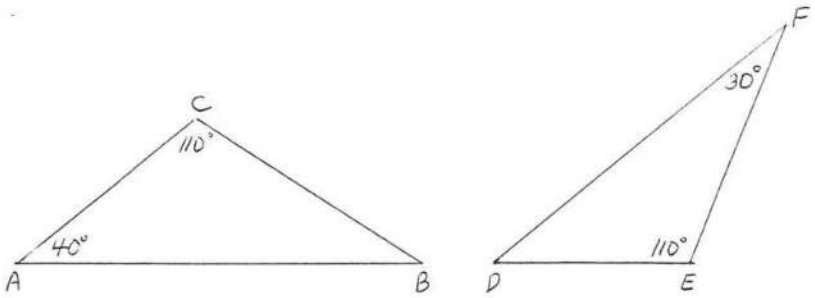
1.



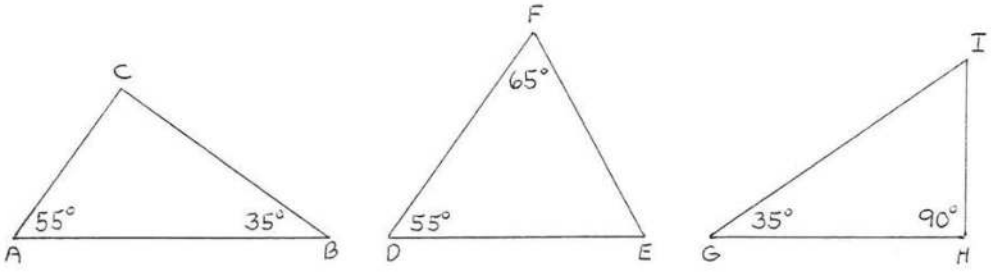
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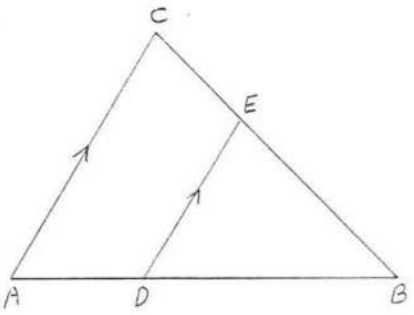
3.



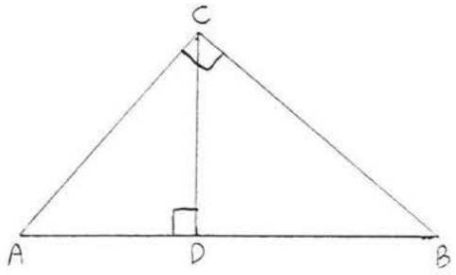
4.



5.



6.



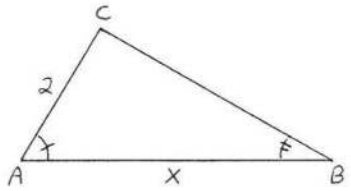
7 - 22. For each of the following

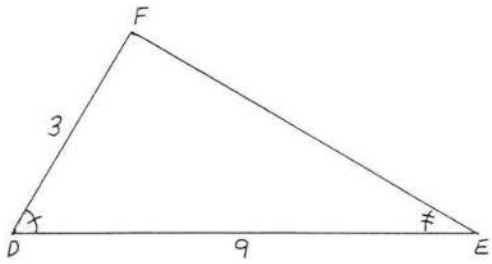
(1) write the similarity statement

(2) write the proportion between the corresponding sides

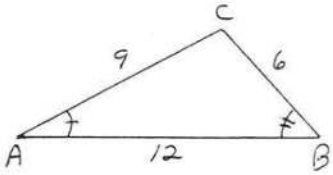
(3) solve for x or x and y.

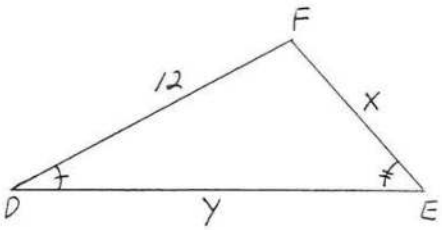
7.



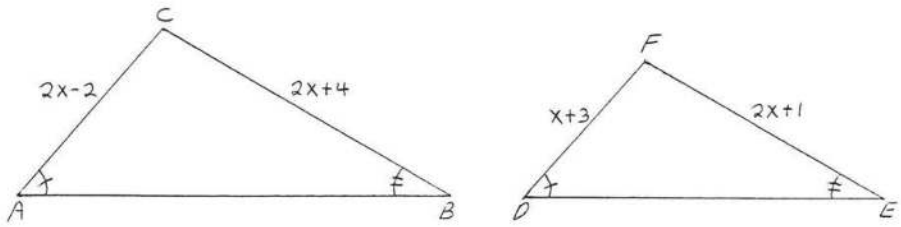


8.

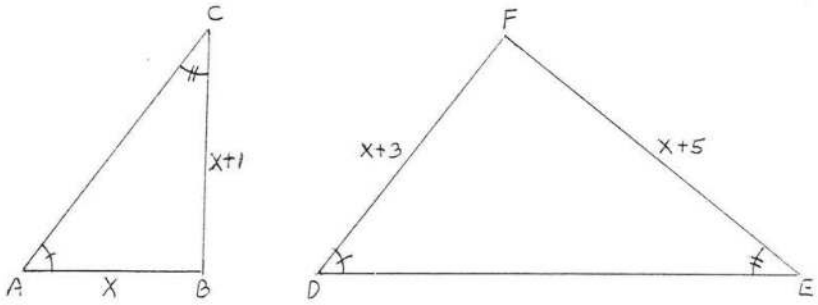




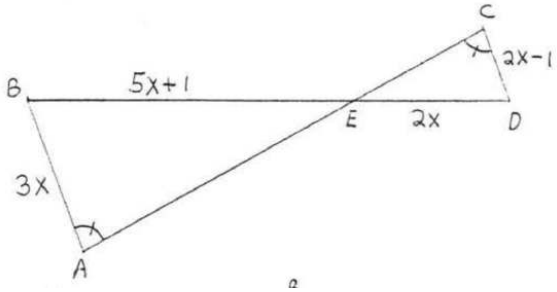
9.


C
2x-2
2x+4
A
B
F
x+3
2x+1
D
E

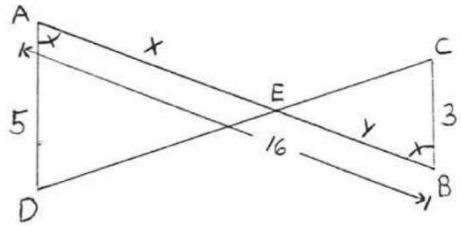
10.


C
x+1
A
X
B
F
x+3
x+5
D
E

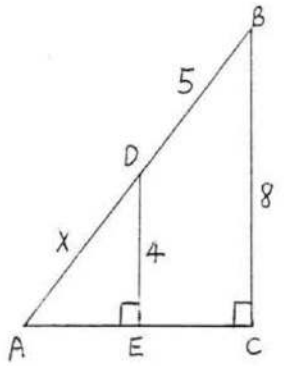
11.


C
2x-1
B
5x+1
E
2x
D
3x
A

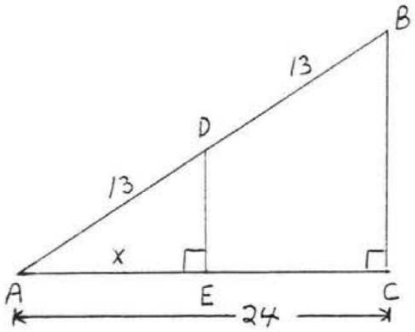
12.


A
x
C
E
5
3
y
16
B
D

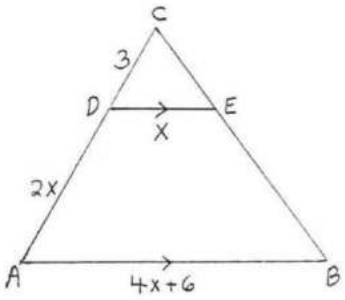
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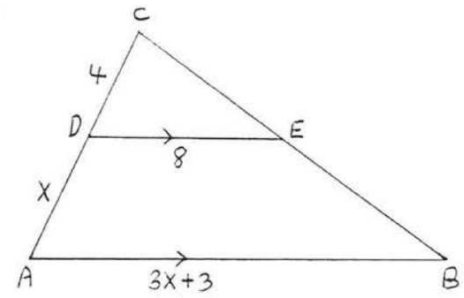
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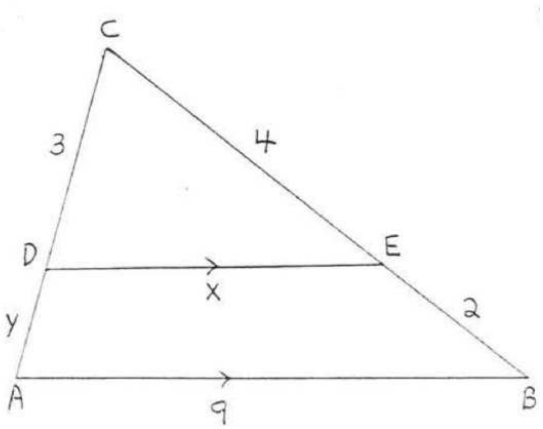
15.



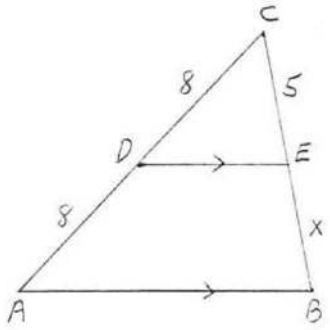
16.



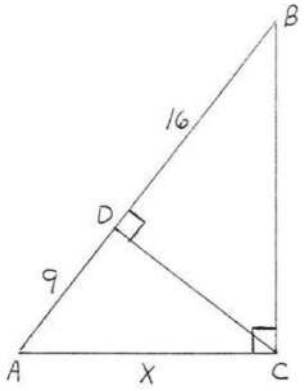
17.



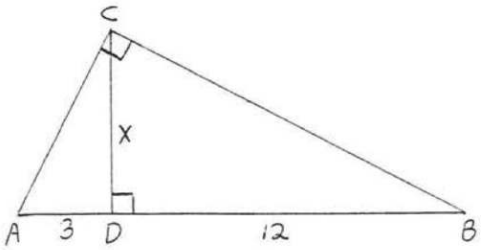
18.



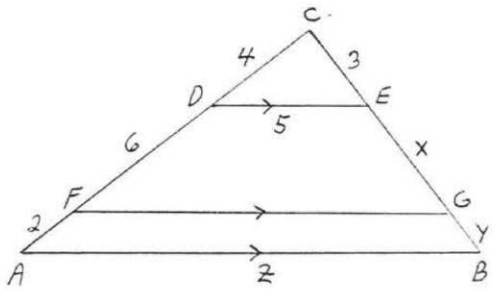
19.



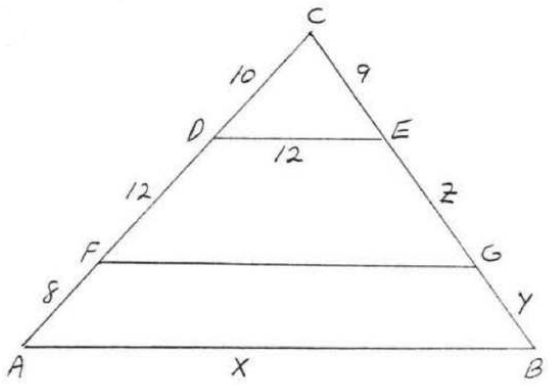
20.



21.

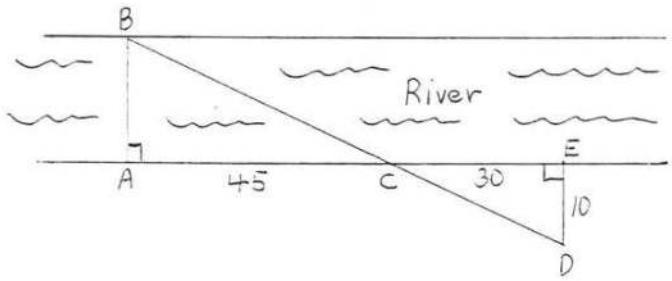


22.



23. A flagpole casts a shadow 80 feet long at the same time a 5 foot boy casts a shadow 4 feet long. How tall is the flagpole?

24. Find the width ***AB*** of the river:



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6.5.3: Transversals to Three Parallel Lines

In Chapter 1 we defined a transversal to be a line which intersects two other lines, We will now extend the definition to a line which intersects three other lines. In Figure **6.5.3.1**, AB is a transversal to three lines.

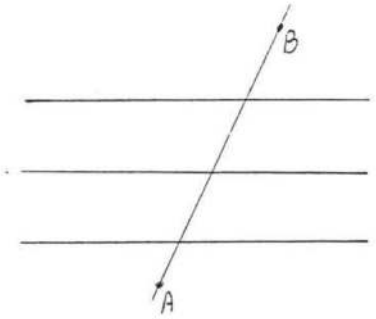


Figure **6.5.3.1**: AB is a transversal to three lines.

If the three lines are parallel and we have two such transversals we may state the following theorem:

Theorem 6.5.3.1

The line segments formed by two transversals crossing three parallel lines are proportional.

In Figure **6.5.3.2**, $\frac{a}{b} = \frac{c}{d}$.

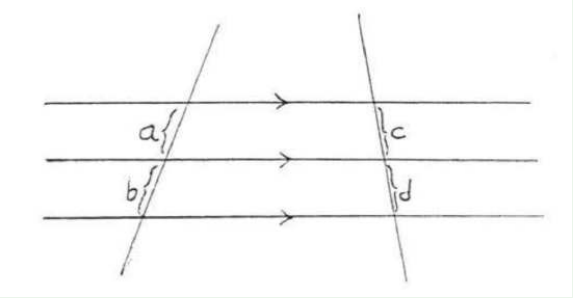
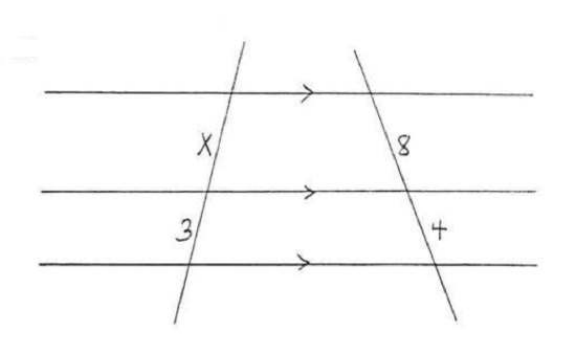


Figure **6.5.3.2**: The line segments formed by the transversals are proportional.

Example 6.5.3.1

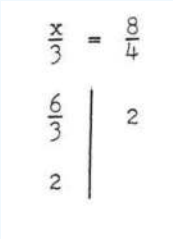
Find x:



Solution

$$\begin{aligned} \frac{x}{3} &= \frac{8}{4} \\ 4x &= 24 \\ x &= 6 \end{aligned} \tag{6.5.3.1}$$

Check:



Answer: $x = 6$.

Proof of Theorem 6.5.3.1

Draw GB and HC parallel to DF (Figure 6.5.3.3). The corresponding angles of the parallel lines are equal and so $\triangle BCH \sim \triangle ABG$. Therefore

$$\frac{BC}{AB} = \frac{CH}{BG}. \tag{6.5.3.2}$$

Now $CH = FE = c$ and $BG = ED = d$ because they are the opposite sides of a parallelogram. Substituting, we obtain

$$\frac{a}{b} = \frac{c}{d}. \tag{6.5.3.3}$$

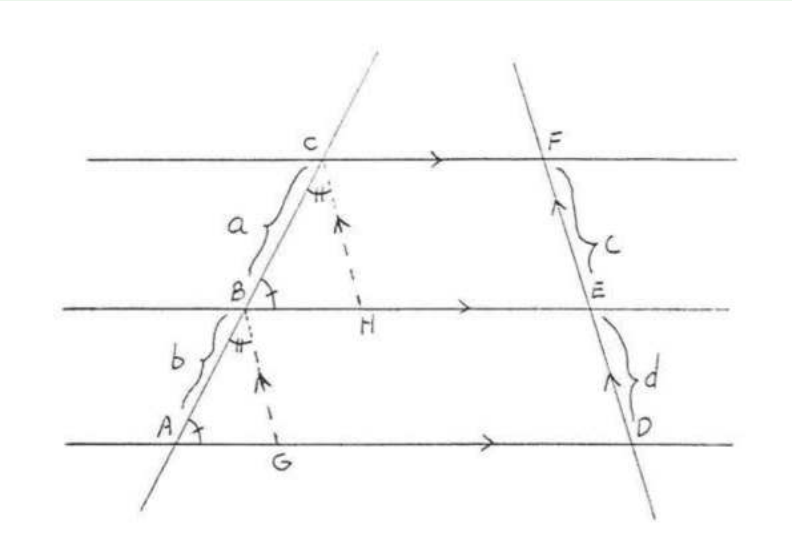
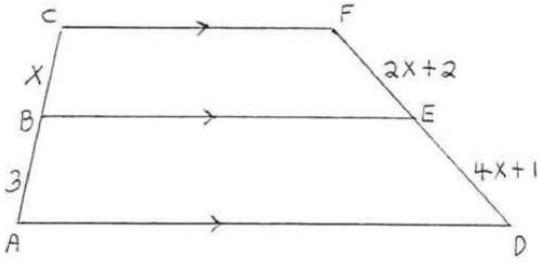


Figure 6.5.3.3: Draw GB and HC parallel to DF.

Example 6.5.3.2

Find x:



Solution

$$\begin{aligned} \frac{x}{3} &= \frac{2x+2}{4x+1} \\ (x)(4x+1) &= (3)(2x+2) \\ 4x^2+x &= 6x+6 \\ 4x^2-5x-6 &= 0 \\ (x-2)(4x+3) &= 0 \end{aligned}$$

$$\begin{aligned} x-2 &= 0 \\ x &= -2 \end{aligned} \quad \text{or} \quad \begin{aligned} 4x+3 &= 0 \\ 4x &= -3 \\ x &= -\frac{3}{4} \end{aligned}$$

We reject $x = -\frac{3}{4}$ because $BC = x$ cannot be negative.

Check, $x = 2$:

$$\frac{x}{3} = \frac{2x+2}{4x+1}$$

$$\frac{2}{3} \quad \frac{2(2)+2}{4(2)+1}$$

$$\frac{4+2}{8+1}$$

$$\frac{6}{9}$$

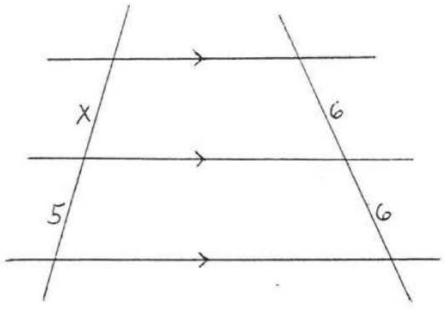
$$\frac{2}{3}$$

Answer: $x = 2$.

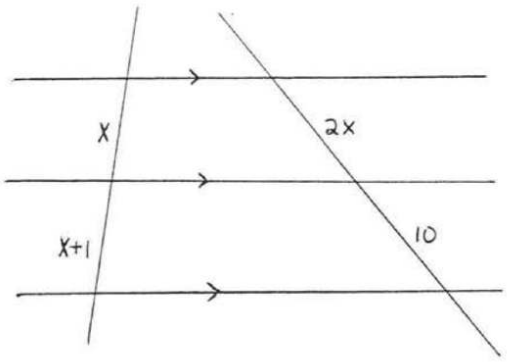
Problems

1 - 6. Find x:

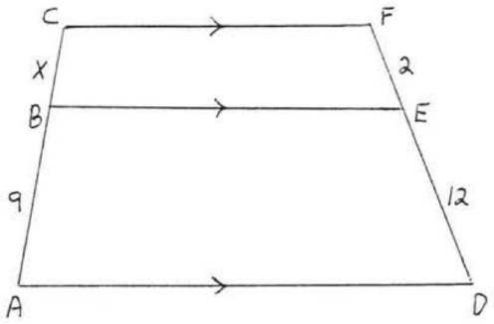
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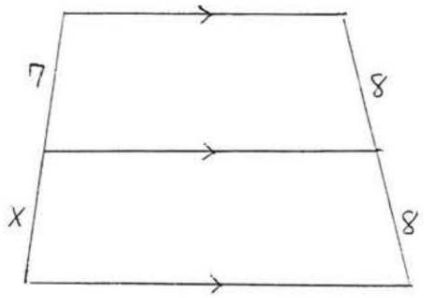
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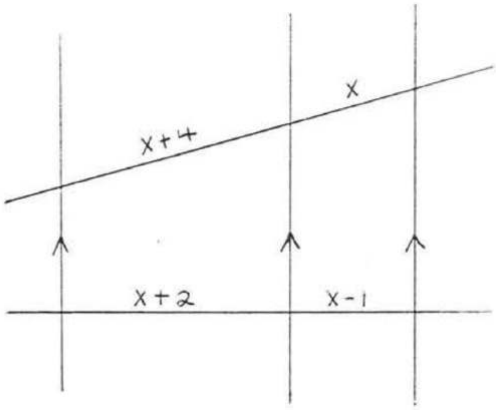


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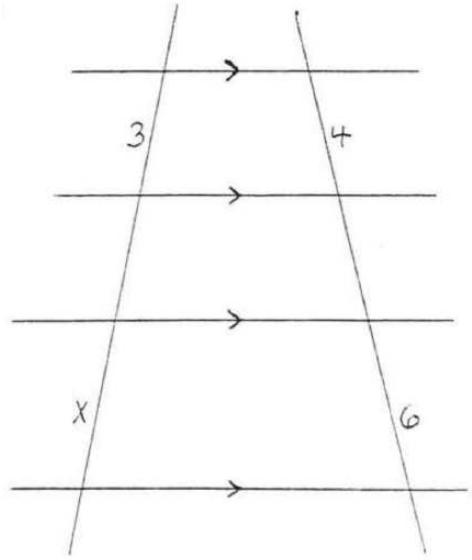


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6.



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6.5.4: Pythagorean Theorem

In a right triangle, the sides of the right angle are called the **legs** of the triangle and the remaining side is called the **hypotenuse**. In Figure 6.5.4.1, side AC and BC are the legs and side AB is the hypotenuse.

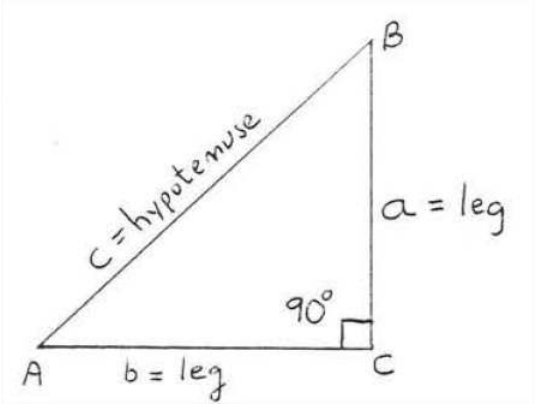


Figure 6.5.4.1: A right triangle.

The following is one of the most famous theorems in mathematics.

Theorem 6.5.4.1: Pythagorean Theorem

In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the legs. That is,

$$\text{leg}^2 + \text{leg}^2 = \text{hypotenuse}^2 \tag{6.5.4.1}$$

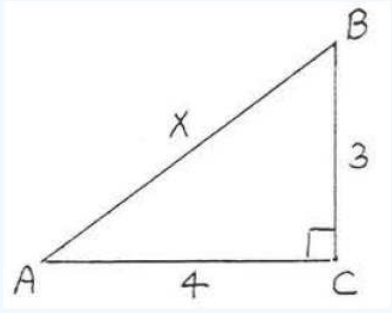
Thus, for the sides of the triangle in Figure 6.5.4.1,

$$a^2 + b^2 = c^2$$

Before we prove Theorem 6.5.4.1, we will give several examples.

Example 6.5.4.1

Find x



Solution

$$\begin{aligned} \text{leg}^2 + \text{leg}^2 &= \text{hyp}^2 \\ 3^2 + 4^2 &= x^2 \\ 9 + 16 &= x^2 \\ 25 &= x^2 \\ 5 &= x \end{aligned}$$

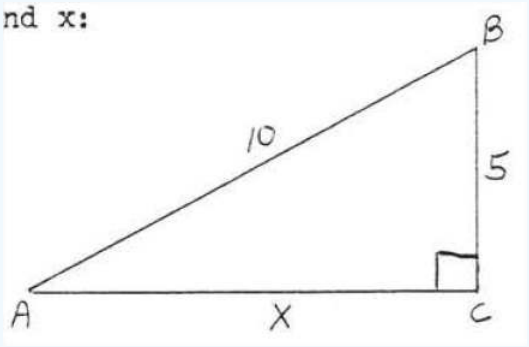
Check:

$$\begin{array}{rcl} \text{leg}^2 + \text{leg}^2 & = & \text{hyp}^2 \\ 3^2 + 4^2 & & x^2 \\ 9 + 16 & & 5^2 \\ 25 & & 25 \end{array}$$

Answer: $x = 5$.

Example 6.5.4.2

Find x:



Solution

$$\begin{aligned} \text{leg}^2 + \text{leg}^2 &= \text{hyp}^2 \\ 5^2 + x^2 &= 10^2 \\ 25 + x^2 &= 100 \\ x^2 &= 75 \\ x &= \sqrt{75} = \sqrt{25}\sqrt{3} = 5\sqrt{3} \end{aligned}$$

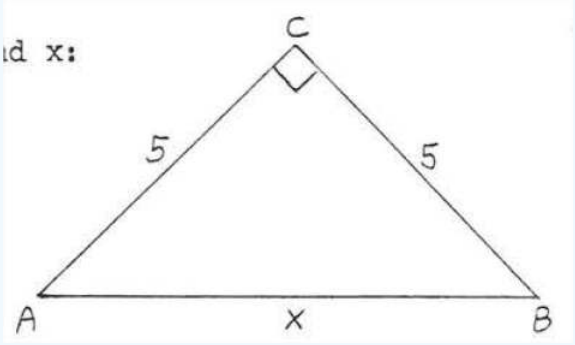
Check:

$$\begin{array}{rcl} \text{leg}^2 + \text{leg}^2 & = & \text{hyp}^2 \\ 5^2 + x^2 & & 10^2 \\ 25 + (5\sqrt{3})^2 & & 100 \\ 25 + 25\sqrt{9} & & \\ 25 + 25(3) & & \\ 25 + 75 & & \\ 100 & & \end{array}$$

Answer: $x = 5\sqrt{3}$.

Example 6.5.4.3

Find x:



Solution

$$\begin{aligned} \text{leg}^2 + \text{leg}^2 &= \text{hyp}^2 \\ 5^2 + 5^2 &= x^2 \\ 25 + 25 &= x^2 \\ 50 &= x^2 \\ x &= \sqrt{50} = \sqrt{25}\sqrt{2} = 5\sqrt{2} \end{aligned}$$

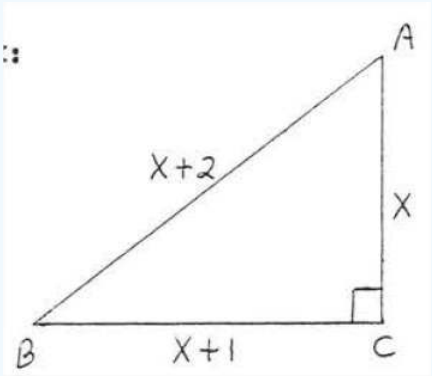
Check:

$$\begin{array}{rcl} \text{leg}^2 + \text{leg}^2 & = & \text{hyp}^2 \\ 5^2 + 5^2 & & x^2 \\ 25 + 25 & & (5\sqrt{2})^2 \\ 50 & & 25.\sqrt{4} \\ & & 25(2) \\ & & 50 \end{array}$$

Answer: $x = 5\sqrt{2}$.

Example 6.5.4.4

Find x



Solution

$$\begin{aligned} \text{leg}^2 + \text{leg}^2 &= \text{hyp}^2 \\ x^2 + (x+1)^2 &= (x+2)^2 \\ x^2 + x^2 + 2x + 1 &= x^2 + 4x + 4 \\ x^2 + x^2 + 2x + 1 - x^2 - 4x - 4 &= 0 \\ x^2 - 2x - 3 &= 0 \\ (x-3)(x+1) &= 0 \end{aligned}$$

$$\begin{aligned} x - 3 &= 0 \quad & x + 1 &= 0 \\ x &= 3 & x &= -1 \end{aligned}$$

We reject $x = -1$ because $AC = x$ cannot be negative.

Check, $x = 3$:

$$\begin{array}{r|l} \text{leg}^2 + \text{leg}^2 = & \text{hyp}^2 \\ x^2 + (x+1)^2 & (x+2)^2 \\ 3^2 + (3+1)^2 & (3+2)^2 \\ 9 + 4^2 & 5^2 \\ 9 + 16 & 25 \\ 25 & \end{array}$$

Answer: $x = 3$.

We will now restate and prove Theorem **6.5.4.1**:

Theorem 6.5.4.1 Pythagorean Theorem

In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the legs. That is,

$\text{leg}^2 + \text{leg}^2 = \text{hypotenuse}^2$.

In Figure **6.5.4.1**,

$a^2 + b^2 = c^2$.

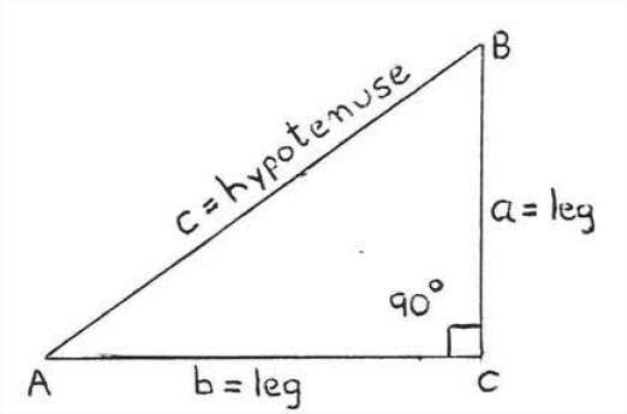


Figure **6.5.4.1**. A right triangle.

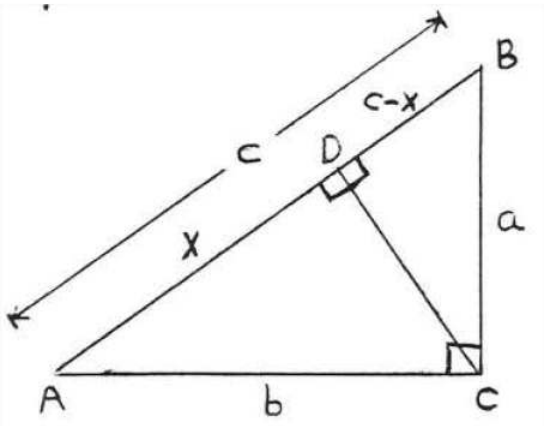


Figure 6.5.4.2. Draw CD perpendicular to AB.

Proof

In Figure 6.5.4.1, draw CD perpendicular to AB. Let $x = AD$. Then $BD = c - x$ (Figure 6.5.4.2). As in Example 6.5.4.3, section 4.2, $\triangle ABC \sim \triangle ACD$ and $\triangle ABC \sim \triangle CBD$. From these two similarities we obtain two proportions:

$\triangle ABC \backsim \triangle ACD$

$$\frac{AB}{AC} = \frac{AC}{AD}$$

$$\frac{c}{b} = \frac{b}{x}$$

$$cx = b^2$$

$\triangle ABC \backsim \triangle CBD$

$$\frac{AB}{CB} = \frac{BC}{BD}$$

$$\frac{c}{a} = \frac{a}{c - x}$$

$$c(c - x) = a^2$$

$$c^2 - cx = a^2$$

$$c^2 - b^2 = a^2$$

$$c^2 = a^2 + b^2$$

The converse of the Pythagorean Theorem also holds:

Theorem 6.5.4.2 (converse of the Pythagorean Theorem).

In a triangle, if the square of one side is equal to the sun of the squares of the other two sides then the triangle is a right triangle.

In Figure 6.5.4.3, if $c^2 = a^2 + b^2$ then $\triangle ABC$ is a right triangle with $\angle C = 90°$.

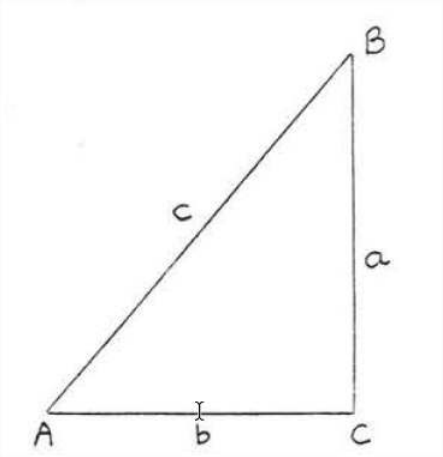


Figure 6.5.4.3: If $c^2 = a^2 + b^2$ then $\angle C = 90°$.

Proof

Draw a new triangle, $\triangle DEF$, so that $\angle F = 90°$, $d = a$, and $e = b$ (Figure 6.5.4.4). $\triangle DEF$ is a right triangle, so by Theorem 6.5.4.1, $f^2 = d^2 + e^2$. We have $f^2 = d^2 + e^2 = a^2 + b^2 = c^2$ and therefore $f = c$. Therefore $\triangle ABC \cong \triangle DEF$ because $SSS = SSS$. Therefore, $\angle C + \angle F = 90°$.

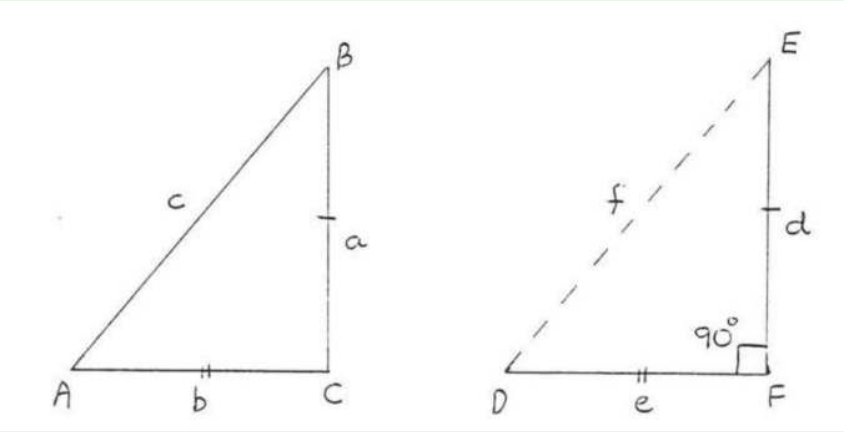
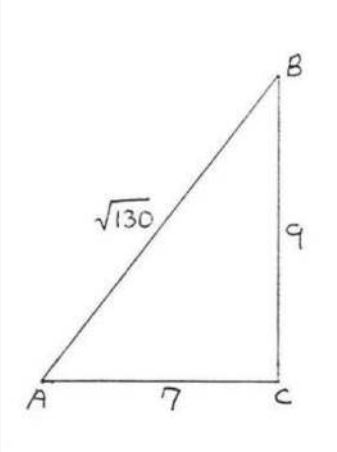


Figure 6.5.4.4: Given $\triangle ABC$, draw $\triangle DEF$ so that $\angle F = 90°$, $d = a$ and $e = b$.

Example 6.5.4.5

Is $\triangle ABC$ a right triangle?



Solution

$AC^2 = 7^2 = 49$

$BC^2 = 9^2 = 81$

$AB^2 = (\sqrt{130})^2 = 130$

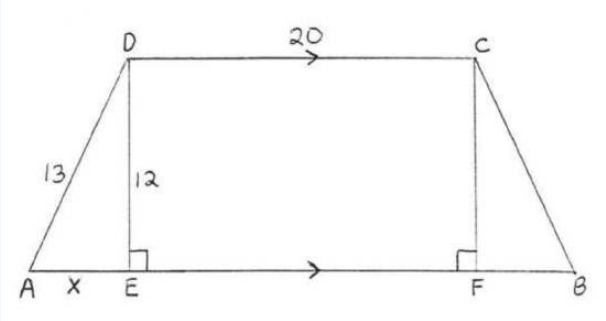
$49 + 81 = 130$.

so by Theorem 6.5.4.2, $\triangle ABC$ is a right triangle.

Answer: yes.

Example 6.5.4.6

Find x and AB:



Solution

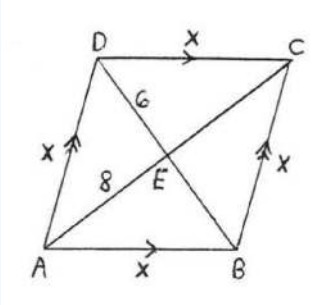
$$\begin{aligned} x^2 + 12^2 &= 13^2 \\ x^2 + 144 &= 169 \\ x^2 &= 169 - 144 \\ x^2 &= 25 \\ x &= 5 \end{aligned}$$

$CDEF$ is a rectangle so $EF = CD = 20$ and $CF = DE = 12$. Therefore $FB = 5$ and $AB = AE + EF + FB = 5 + 20 + 5 = 30$.

Answer: $x = 5$, $AB = 30$.

Example 6.5.4.7

Find x, AC and BD:



Solution

$ABCD$ is a rhombus. The diagonals of a rhombus are perpendicular and bisect each other.

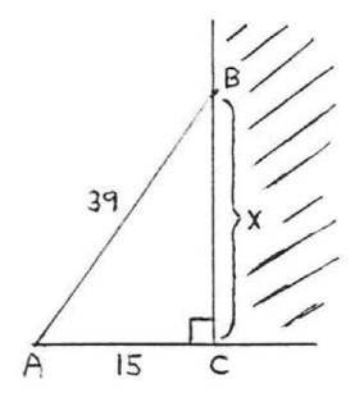
$$\begin{aligned} 6^2+8^2 &= x^2 \\ 36+64 &= x^2 \\ 100 &= x^2 \\ 10 &= x \end{aligned}$$

$AC = 8+8 = 16, BD = 6+6 = 12.$

Answer: $x = 10, AC = 16, BD = 12.$

Example 6.5.4.8

A ladder 39 feet long leans against a building, How far up the side of the building does the ladder reach if the foot of the ladder is 15 feet from the building?



Solution

$$\begin{aligned} \text{leg}^2+\text{leg}^2 &= \text{hyp}^2 \\ x^2+15^2 &= 39^2 \\ x^2+225 &= 1521 \\ x^2 &= 1521-225 \\ x^2 &= 1296 \\ x &= \sqrt{1296} = 36 \end{aligned}$$

Answer: 36 feet.

Historical Note

Pythagoras (c. 582 - 507 B.C.) was not the first to discover the theorem which bears his name. It was known long before his time by the Chinese, the Babylonians, and perhaps also the Egyptians and the Hindus, According to tradition, Pythagoras was the first to give a nroof of the theorem, His proof probably made use of areas, like the one suggested. In Figure **6.5.4.5** below, (each square contains four congruent right triangles with sides of lengths a, b, and c, In addition the square on the left contains a square with side a and a square with side b while the one on the right contains a square with side c.)

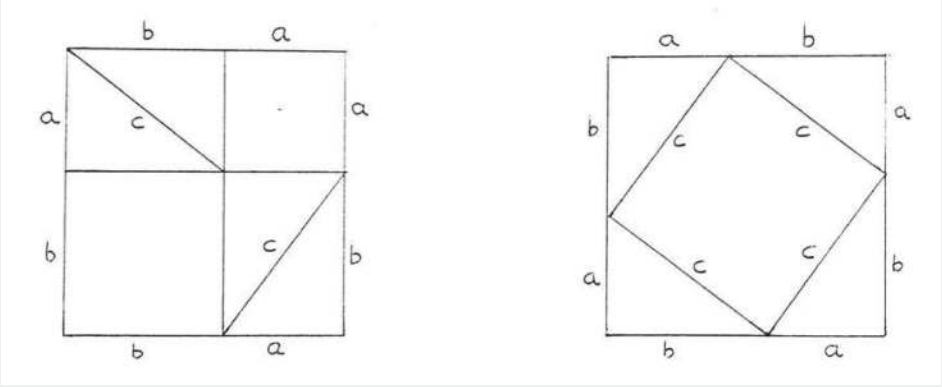


Figure **6.5.4.5**: Pythagoras may have proved $a^2 + b^2 = c^2$ in this way.

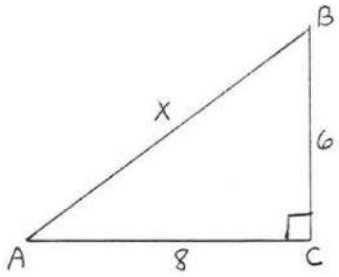
Since the time of Pythagoras, at least several hundred different proofs of the Pythagorean Theorem have been proposed, Pythagoras was the founder of the Pythagorean school, a secret religious society devoted to the study of philosophy, mathematics, and science. Its membership was a select group, which tended to keep the discoveries and practices of the society secret from outsiders. The Pythagoreans believed that numbers were the ultimate components of the universe and that all physical relationships could be expressed with whole numbers, This belief was prompted in part by their discovery that the notes of the musical scale were related by numerical ratios. The Pythagoreans made important contributions to medicine, physics, and astronomy, In geometry, they are credited with the angle s

um theorem for triangles, the properties of parallel lines, and the theory of similar triangles and proportions.

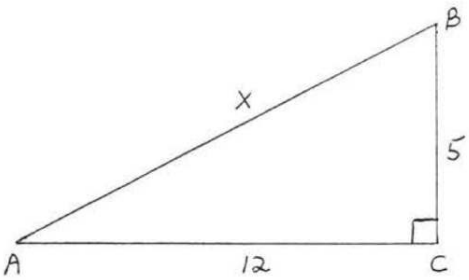
Problems

1 - 10. Find x. Leave answers in simplest radical form.

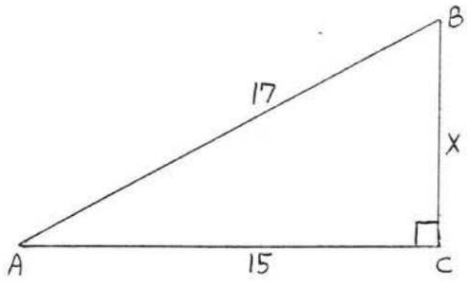
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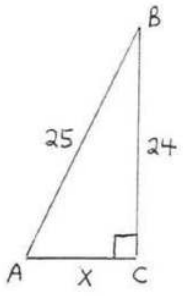
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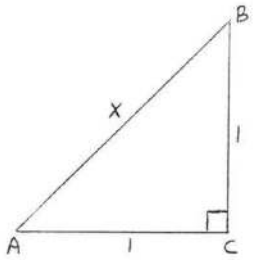
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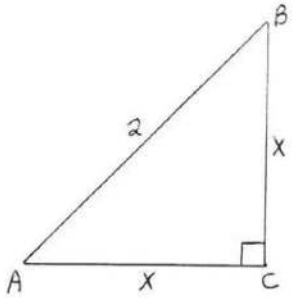
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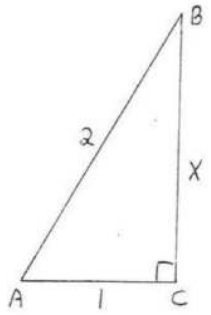
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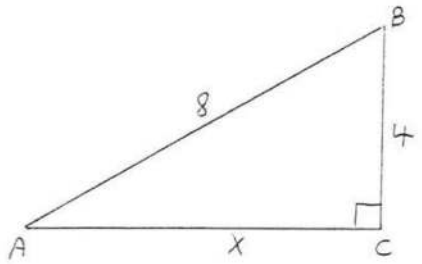
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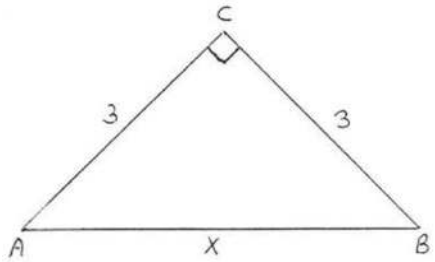
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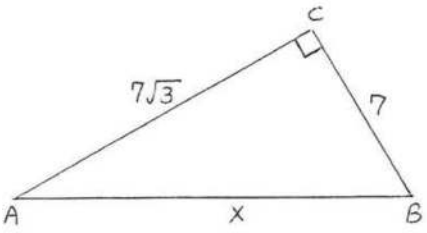
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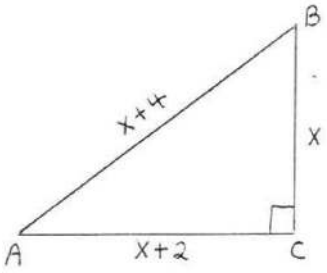


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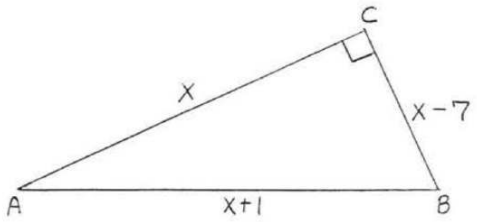


11 - 14. Find x and all sides of the triangle:

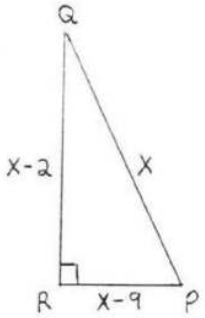
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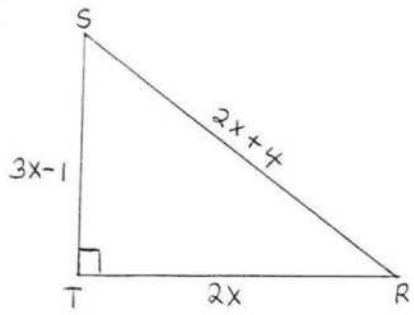
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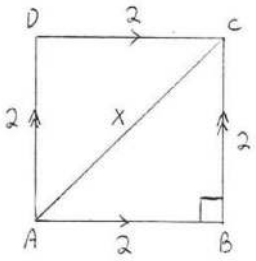


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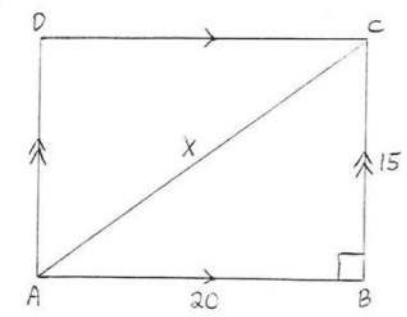


15 - 16. Find x:

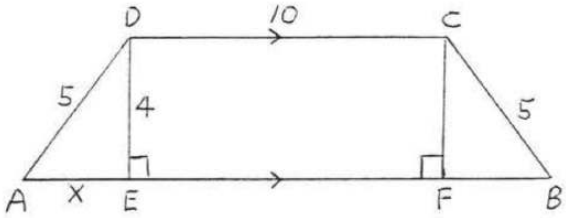
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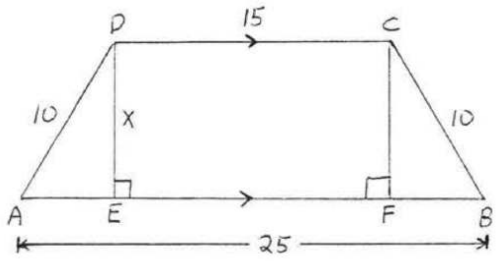
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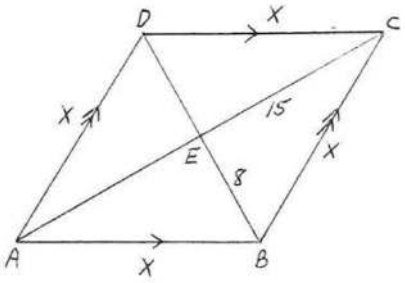
17. Find x and AB.



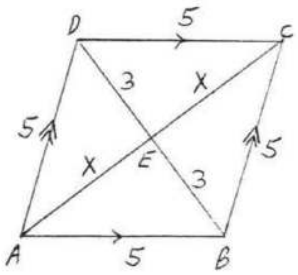
18. Find x:



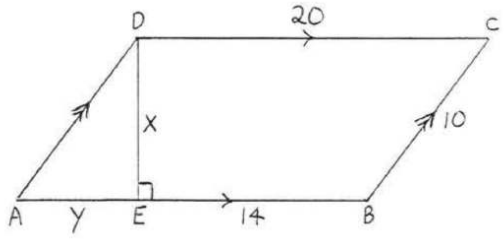
19. Find x, AC and BD:



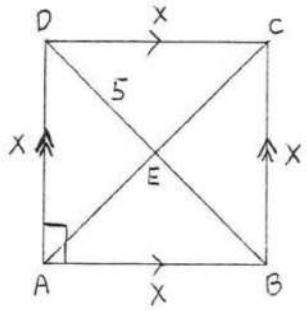
20. Find x, AC and BD:



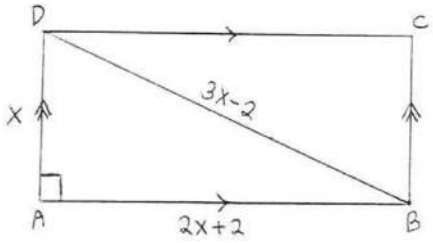
21. Find x and y:



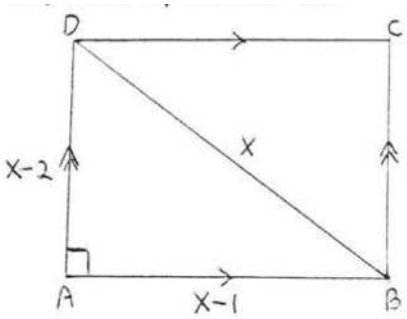
22. Find x, AC and BD:



23. Find x, AB and BD:

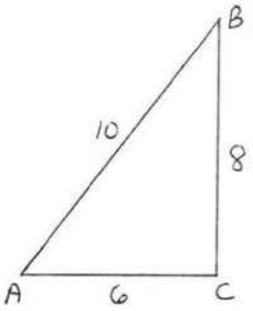


24. Find x, AB and AD:

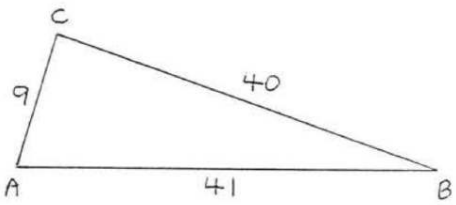


25 - 30. Is $\triangle ABC$ a right triangle?

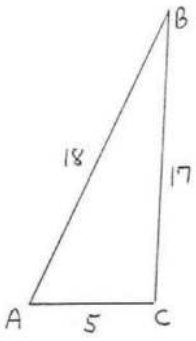
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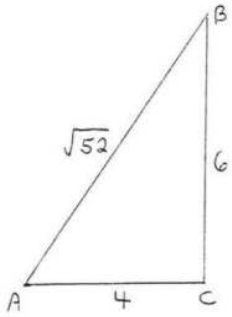
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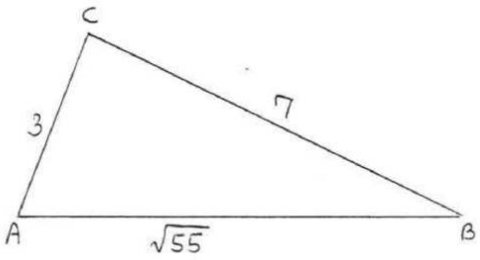
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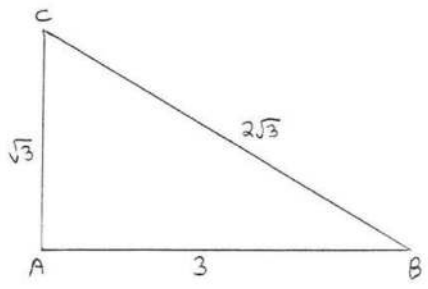
28.



29.



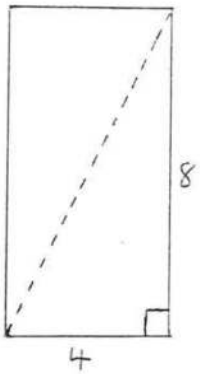
30.



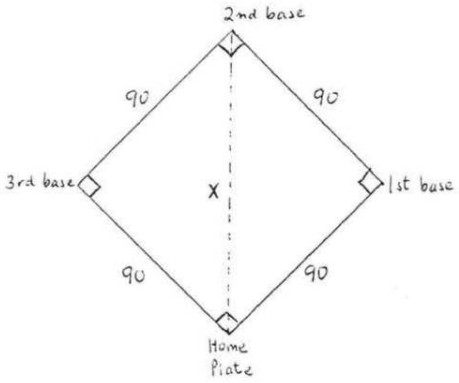
31. A ladder 25 feet long leans against a building, How far up the side of the building does the ladder reach if the foot of the ladder is 7 feet from the building?

32. A man travels 24 miles east and then 10 miles north. At the end of his journey how far is he from his starting point?

33. Can a table 9 feet wide (with its legs folded) fit through a rectangular doorway 4 feet by 8 feet?



34. A baseball diamond is a square 90 feet on each side, Find the distance from home plate to second base (leave answer in simplest radical form).



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6.5.5: Special Right Triangles

There are two kinds of right triangle which deserve special attention: the $30° - 60° - 90°$ right triangle and the $45° - 45° - 90°$ right triangle.

30°−60°−90° Right Triangles

A triangle whose angles are $30°, 60°,$ and $90°$ is called a $30° - 60° - 90°$ triangle. $\triangle ABC$ in Figure 6.5.5.1 is a $30° - 60° - 90°$ triangle with side $AC = 1$.

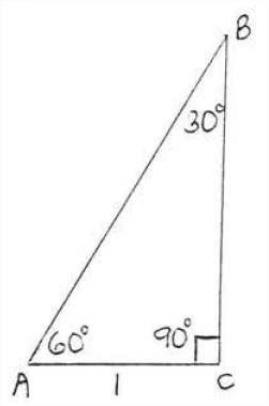


Figure 6.5.5.1: A $30° - 60° - 90°$ triangle.

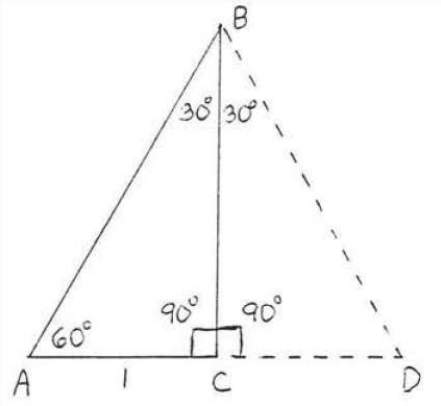


Figure 6.5.5.2: Draw BD and CD.

To learn more about this triangle let us draw lines BD and CD as in Figure 6.5.5.2. $\triangle ABC \cong \triangle DBC$ by $ASA = ASA$ so $AC = DC = 1$. $\triangle ABD$ is an equiangular triangle so all the sides must be equal to 2.. Therefore $AB = 2$ (Figure 6.5.5.3).

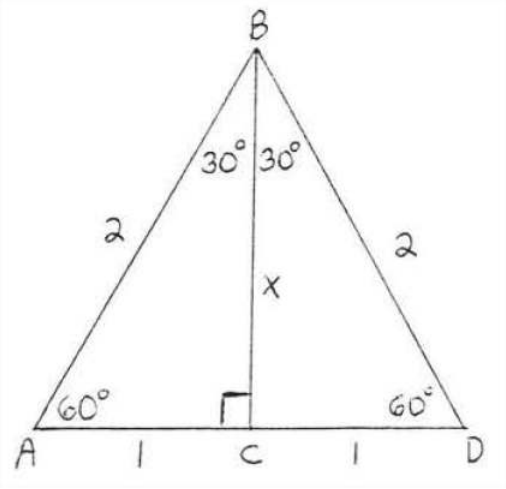


Figure 6.5.5.3: $\triangle ABD$ is equiangular with all sides equal to 2 .

Let $x = BC$. Let us find x. Applying the Pythagorean Theorem to $\triangle ABC$,

$$\begin{aligned} \text{leg}^2 + \text{leg}^2 &= \text{hyp}^2 \\ 1^2 + x^2 &= x^2 \\ 1 + x^2 &= 4 \\ x^2 &= 3 \\ x &= \sqrt{3} \end{aligned}$$

Now suppose we are given another $30° - 60° - 90°$ triangle $\triangle DEF$, with side $DF = 8$ (Figure 6.5.5.4). $\triangle DEF$ is similar to $\triangle ABC$ of Figure 6.5.5.3 Therefore

$$\frac{DF}{AC} = \frac{DE}{AB} \quad \text{and} \quad \frac{DF}{AC} = \frac{EF}{BC}$$
$$\frac{8}{1} = \frac{DE}{2} \qquad \frac{8}{1} = \frac{EF}{\sqrt{3}}$$
$$16 = DE \qquad 8\sqrt{3} = EF$$

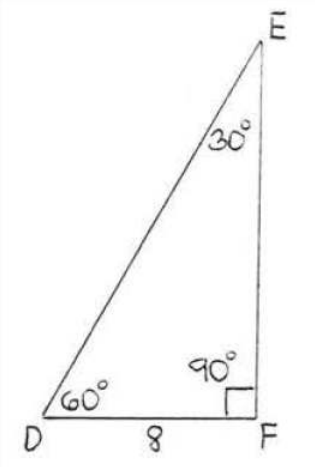


Figure 6.5.5.3:

Our conclusions about triangles ABC and DEF suggest the following theorem:

Theorem 6.5.5.1

In the $30° - 60° - 90°$ triangle the hypotenuse is always twice as large as the leg opposite the $30°$ angle (the shorter leg). The leg opposite the $60°$ angle (the longer leg) is always equal to the shorter leg times $\sqrt{3}$.

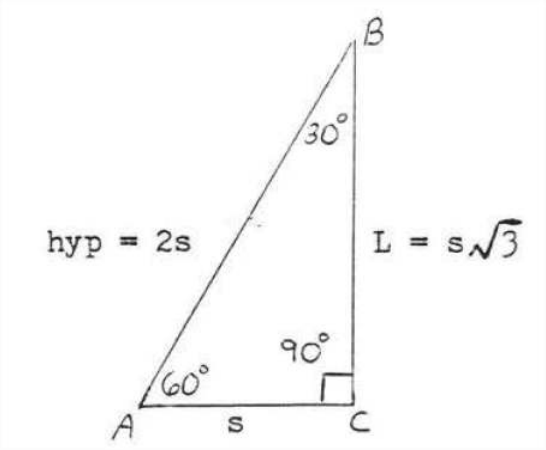


Figure 6.5.5.5: The hypotenuse is twice the shorter leg and the longer leg is equal to the shorter leg times the $\sqrt{3}$.

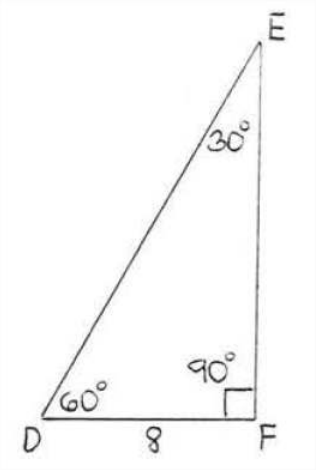
In Figure 6.5.5.5, $s =$ shorter leg, $L =$ longer leg, and hyp = hypotenuse. Theorem 6.5.5.1 says that

Erroneous nesting of equation structures

Note that the longer leg is always the leg opposite (furthest away from) the $60°$ angle and the shorter leg is always the leg opposite (furthest away from) the $30°$ angle.

Example 6.5.5.1

Find x and y:



Solution

$\angle B = 180° - (60° + 90°) = 180° - 150° = 30°$, so $\triangle ABC$ is a $30° - 60° - 90°$ triangle. By theorem 6.5.5.1,

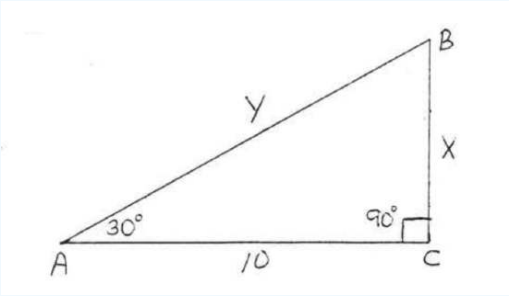
$$\begin{aligned} \text{hyp} &= 2s & L &= s\sqrt{3} \\ y &= 2(7) = 14. & x &= 7\sqrt{3} \end{aligned}$$

Answer

$x = 7\sqrt{3}$, $y = 14$.

Example 6.5.5.2

Find x and y:



Solution

$\angle B = 60°$ so $\triangle ABC$ is a $30° - 60° - 90°$ triangle. By Theorem 6.5.5.1,

$$\begin{aligned} L &= s\sqrt{3} \\ 10 &= x\sqrt{3} \\ \frac{10}{\sqrt{3}} &= \frac{x\cancel{\sqrt{3}}}{\cancel{\sqrt{3}}} \\ x &= \frac{10}{\sqrt{3}} = \frac{10}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{10\sqrt{3}}{3}. \\ \text{hyp} &= 2s \\ y &= 2x = 2\left(\frac{10\sqrt{3}}{3}\right) = \frac{20\sqrt{3}}{3} \end{aligned}$$

Answer

$x = \frac{10\sqrt{3}}{3}, y = \frac{20\sqrt{3}}{3}$.

45°−45°−90° Right Triangles

The second special triangle we will consider is the $45° - 45° - 90°$ triangle. A triangle whose angles are $45°$, $45°$, and $90°$ is called a $45° - 45° - 90°$ triangle or an isosceles right triangle. $\triangle ABC$ in Figure 6.5.5.6 is a $45° - 45° - 90°$ triangle with side $AC = 1$.

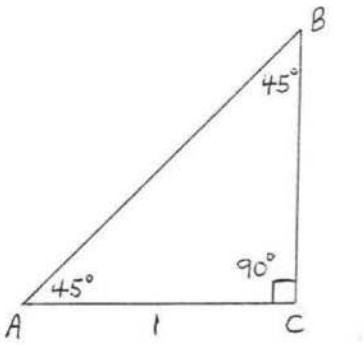


Figure 6.5.5.6: A $45° - 45° - 90°$ triangle.

Since $\angle A = \angle B = 45°$, the sides opposite these angles must be equal (Theorem 2.5.2, Section 2.5). Therefore $AC = BC = 1$.

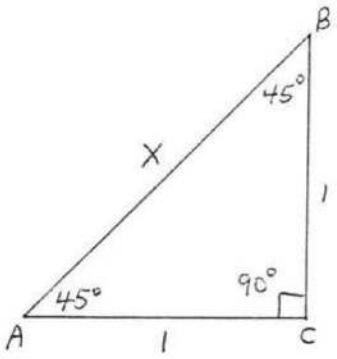


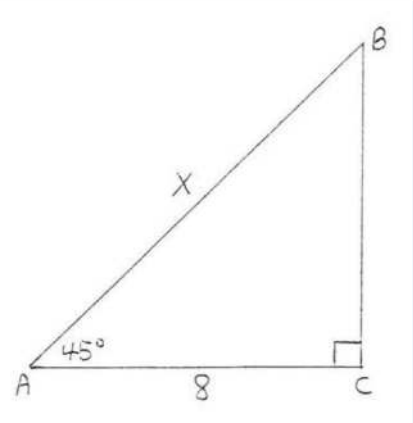
Figure 6.5.5.7: The legs of the $45° - 45° - 90°$ triangle are equal.

Let $x = AB$ (Figure 6.5.5.7). By the Pythagorean Theorem,

$$\begin{aligned} \text{leg}^2 + \text{leg}^2 &= \text{hyp}^2 \\ 1^2 + 1^2 &= x^2 \\ 1 + 1 &= x^2 \\ 2 &= x^2 \\ \sqrt{2} &= x \end{aligned}$$

Example 6.5.5.3

Find x:



Solution

$\angle B = 180° - (45° + 90°) = 180° - 135° = 45°$. So $\triangle ABC$ is a $45° - 45° - 90°$ triangle. $AC = BC = 8$ because these sides are opposite equal angles. By the Pythagorean Theorem,

$$\begin{aligned} \text{leg}^2 + \text{leg}^2 &= \text{hyp}^2 \\ 8^2 + 8^2 &= x^2 \\ 64 + 64 &= x^2 \\ 128 &= x^2 \\ x &= \sqrt{128} = \sqrt{64}\sqrt{2} = 8\sqrt{2} \end{aligned}$$

Answer

$x = 8\sqrt{2}$.

The triangles of Figure 6.5.5.6 and Example 6.5.5.3 suggest the following theorem:

Theorem 6.5.5.2

In the $45° - 45° - 90°$ triangle the legs are equal and the hypotenuse is equal to either leg times $\sqrt{2}$.

In Figure 6.5.5.8, hyp is the hypotenuse and L is the length of each leg. Theorem 6.5.5.2 says that

$$\boxed{\text{hyp} = \sqrt{2}L} \tag{6.5.5.1}$$

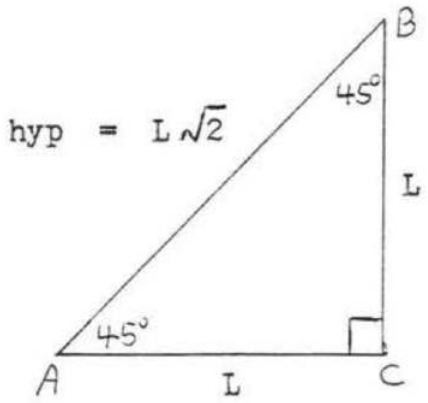
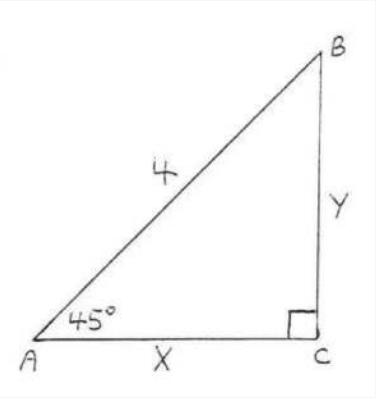


Figure 6.5.5.8: The legs are equal and the hypotenuse is equal to either leg times $\sqrt{2}$.

https://math.libretexts.org/@go/page/208692

Example 6.5.5.4

Find x and y:



Solution

$\angle B = 45°$. So $\triangle ABC$ is an isosceles right triangle and $x = y$.

$$\begin{aligned} x^2 + y^2 &= 4^2 \\ x^2 + x^2 &= 16 \\ 2x^2 &= 16 \\ x^2 &= 8 \\ x &= \sqrt{8} = \sqrt{4}\sqrt{2} = 2\sqrt{2} \end{aligned}$$

Answer: $x = y = 2\sqrt{2}$.

Another method:

$\triangle ABC$ is a $45° - 45° - 90°$ triangle. Hence by Theorem 6.5.5.2,

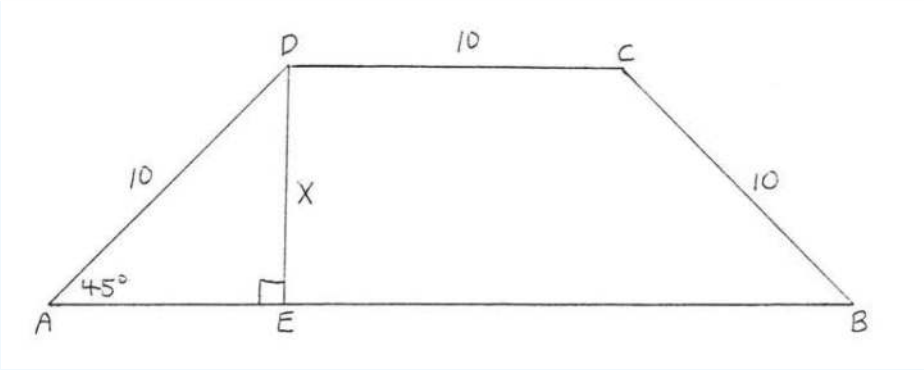
$$\begin{aligned} \text{hyp} &= L\sqrt{2} \\ 4 &= x\sqrt{2} \\ \frac{4}{\sqrt{2}} &= \frac{x\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}} \\ x &= \frac{4}{\sqrt{2}} = \frac{4}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{4\sqrt{2}}{2} = 2\sqrt{2} \end{aligned}$$

Answer

$x = y = 2\sqrt{2}$.

Example 6.5.5.5

Find AB:



Solution

$\triangle ADE$ is a $45° - 45° - 90°$ triangle. Hence

$$\begin{aligned} \text{hyp} &= L\sqrt{2} \\ 10 &= x\sqrt{2} \\ \frac{10}{\sqrt{2}} &= \frac{x\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}} \\ x &= \frac{10}{\sqrt{2}} = \frac{10}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{10\sqrt{2}}{2} = 5\sqrt{2} \\ AE = x &= 5\sqrt{2} \end{aligned}$$

Now draw CF perpendicular to AB (Figure 6.5.5.9). $\angle B = 45°$ since $ABCD$ is an isosceles trapezoid (Theorem 3.2.4, Section 3.2).

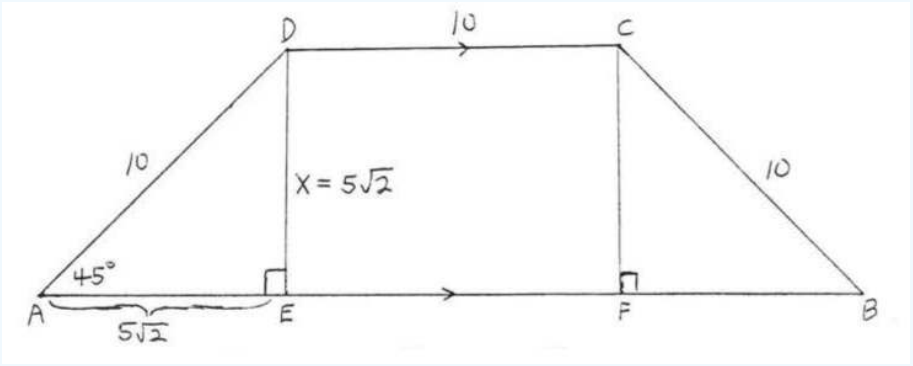


Figure 6.5.5.9: Draw CF perpendicular to AB.

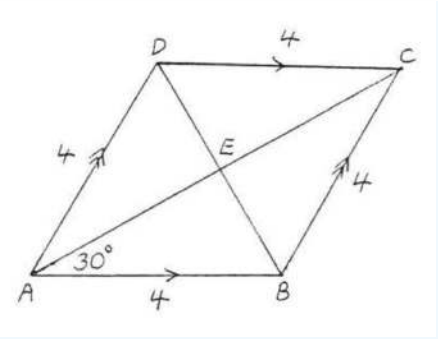
So $\triangle BCF$ is a $45° - 45° - 90°$ triangle congruent to $\triangle ADE$ and therefore $BF = 5\sqrt{2}$. $CDEF$ is a rectangle and therefore $EF = 10$. We have $AB = AE + EF + FB = 5\sqrt{2} + 10 + 5\sqrt{2} = 10\sqrt{2} + 10$.

Answer

$AB = 10\sqrt{2} + 10$.

Example 6.5.5.6

Find AC and BD:



Solution

$ABCD$ is a rhombus. The diagonals AC and BD are perpendicular and bisect each other. $\angle AEB = 90°$ and $\angle ABE = 180° - (90° - 30°) = 60°$. So $\triangle AEB$ is a $30° - 60° - 90°$ triangle.

$$\begin{aligned} \text{hyp} &= 2s & L &= s\sqrt{3} \\ 4 &= 2(BE) & AE &= 2\sqrt{3} \\ 2 &= BE & AC &= 2\sqrt{3} + 2\sqrt{3} \\ bd &= 2 + 2 = 4 & AC &= 4\sqrt{3} \end{aligned}$$

Answer

$AC = 4\sqrt{3}, BD = 4$.

Historical Note: Irrational Numbers

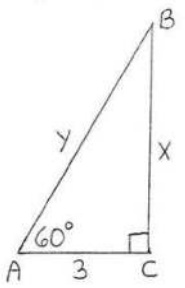
The Pythagoreans believed that all physical relation ships could be expressed with whole numbers. However the sides of the special triangles described in this section are related by **irrational** numbers, $\sqrt{2}$ and $\sqrt{3}$. An irrational number is a number which can be approximated, but not expressed exactly, by a ratio of whole numbers. For example $\sqrt{2}$ can be approximated with increasing accuracy by such ratios as $1.4 = \frac{14}{10}$, $1.41 = \frac{141}{100}$, $1.414 = \frac{1414}{1000}$, etc., but there is no fraction of whole numbers which is **exactly** equal to $\sqrt{2}$. (For more details and a proof, see the book by Richardson listed in the References). The Pythagoreans discovered that $\sqrt{2}$ was irrational in about the 5th century B.C. It was a tremendous shock to them that not all triangles could be measured "exactly." They may have even tried to keep this discovery a secret for fear of the damage it would do to their philosophical credibility.

The inability of the Pythagoreans to accept irrational numbers had unfortunate consequences for the development of mathematics. Later Greek mathematicians avoided giving numerical values to lengths of line segments. Problems whose algebraic solutions might be irrational numbers, such as those involving quadratic equations, were instead stated and solved geometrically. The result was that geometry flourished at the expense of algebra. It was left for the Hindus and the Arabs to resurrect the study of algebra in the Middle Ages. And it was not until the 19th century that irrational numbers were placed in the kind of logical framework that the Greeks had given to geometry 2000 years before.

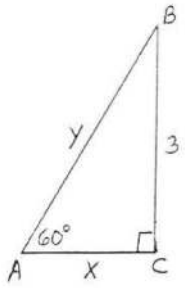
Problems

1 - 10. Find x and y:

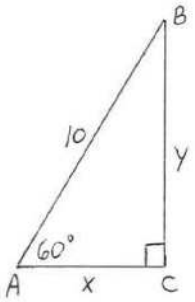
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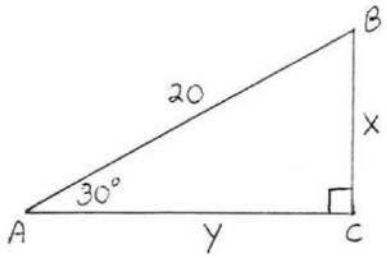
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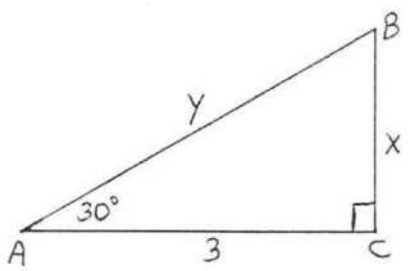


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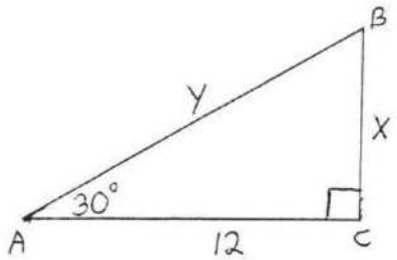


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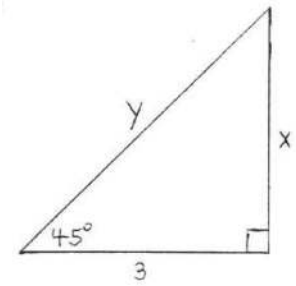
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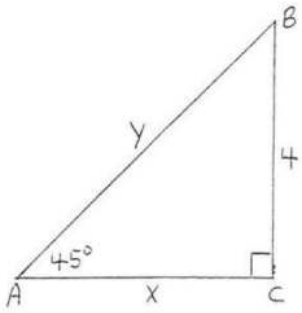
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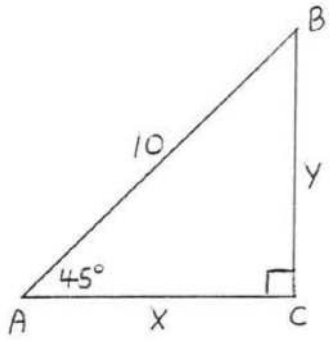
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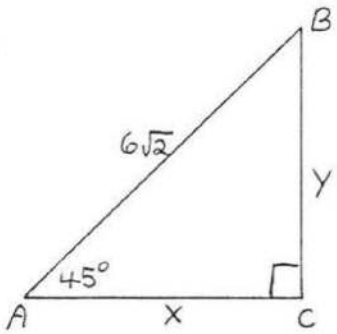
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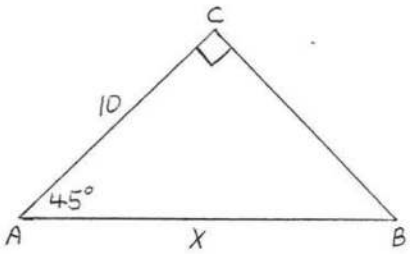


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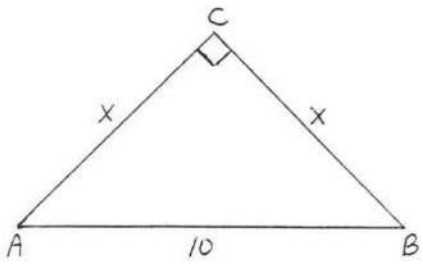


11 - 14. Find x:

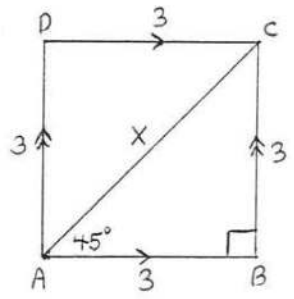
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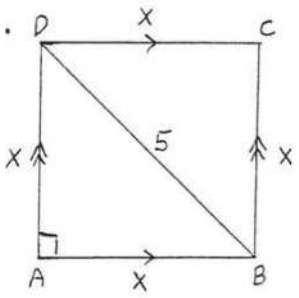
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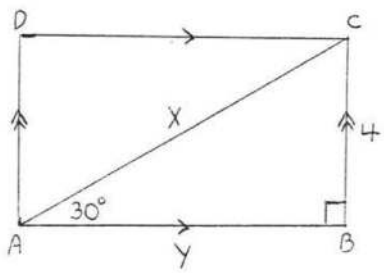


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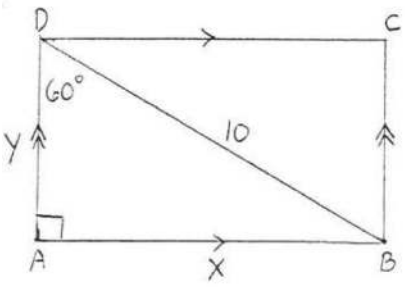


15 - 20. Find x and y:

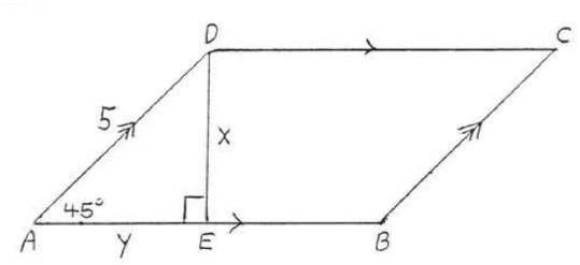
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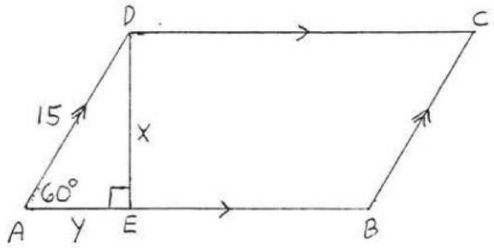
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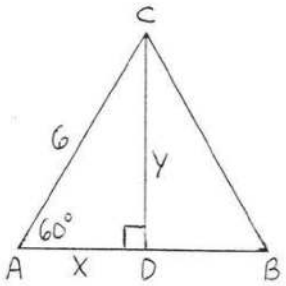
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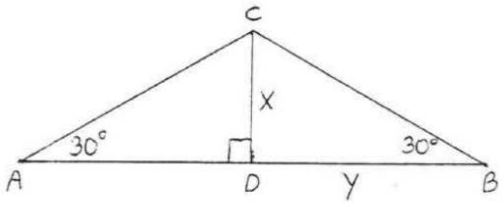
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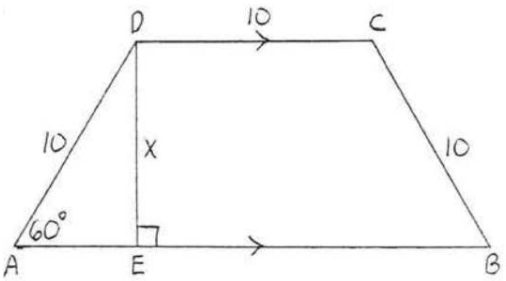


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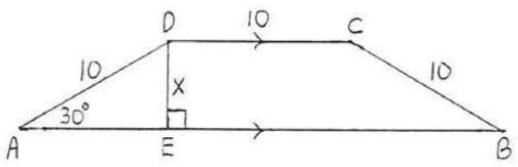


21 - 22. Find x and AB:

21.

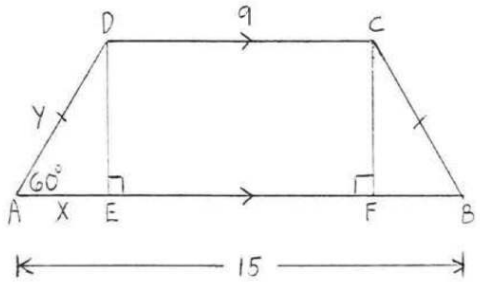


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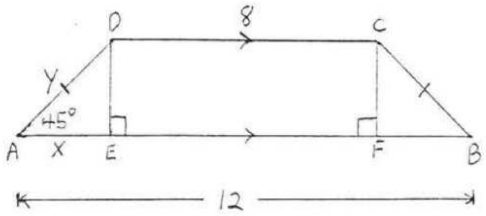


23 - 24. Find x and y:

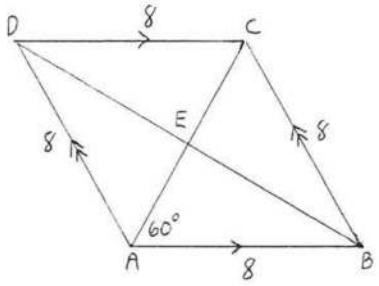
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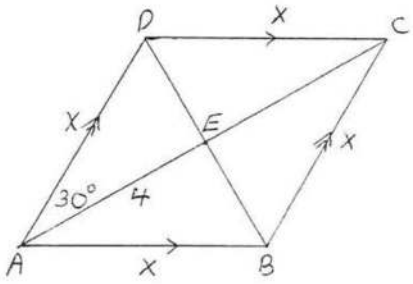
24.



25. Find AC and BD:



26. Find x, AC and BD:



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6.5.6: Distance from a Point to a Line

Suppose we are given a point P and a line $\overleftrightarrow{AB}$ as in Figure 6.5.6.1. We would like to find the shortest line segment that can be drawn from P to $\overleftrightarrow{AB}$.

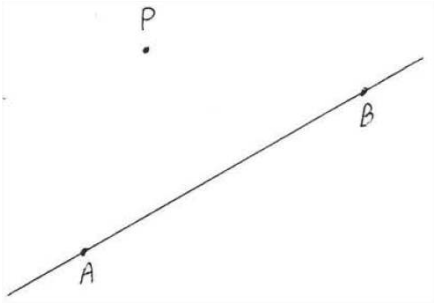


Figure 6.5.6.1: Point P and Line $\overleftrightarrow{AB}$.

First we will prove a theorem:

Theorem 6.5.6.1

In a right triangle, the hypotenuse is larger than either leg. In Figure 6.5.6.1, $c > a$ and $c > b$. (The symbol ">" means "is greater than.")

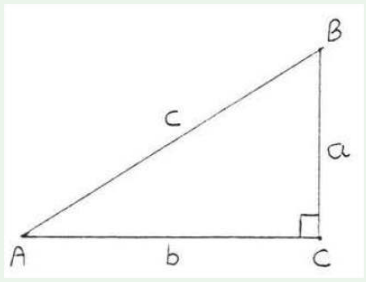


Figure 6.5.6.2: c is larger than either a or b.

Proof

By the Pythagorean Theorem,

$$c = \sqrt{a^2 + b^2} > \sqrt{a^2} = a.$$

$c = \sqrt{a^2 + b^2} > \sqrt{b^2} = b.$

Now we can give the answer to our question:

Theorem 6.5.6.2

The perpendicular is the shortest line segment that can be drawn from a point to a straight line.

In Figure 6.5.6.3 the shortest line segment from P to $\overleftrightarrow{AB}$ is PD. Any other line segment, such as PC, must be longer.

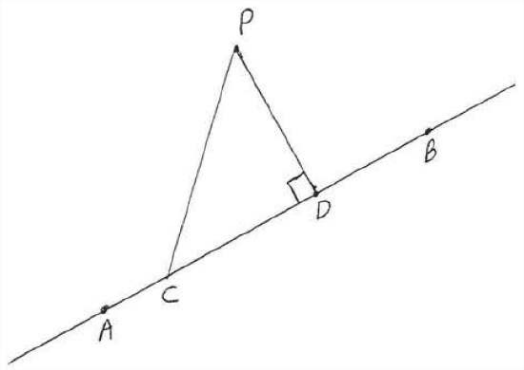


Figure 6.5.6.3: PD is the shortest line segment from P to $\overleftrightarrow{AB}$.

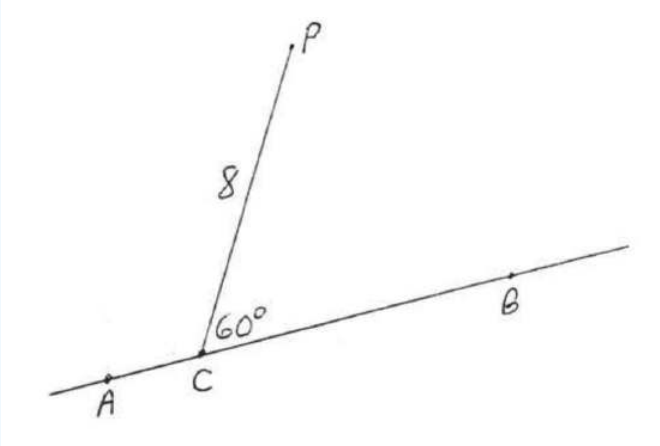
Proof

PC is the hypotenuse of right triangle PCD. Therefore by Theorem 6.5.6.1, $PC > PD$.

We define the distance from a point to a line to be the length of the perpendicular.

Example 6.5.6.1

Find the distance from P to $\overleftrightarrow{AB}$:



Solution

Draw PD perpendicular to $\overleftrightarrow{AB}$ (Figure 6.5.6.4). $\triangle PCD$ is a $30° - 60° - 90°$ triangle.

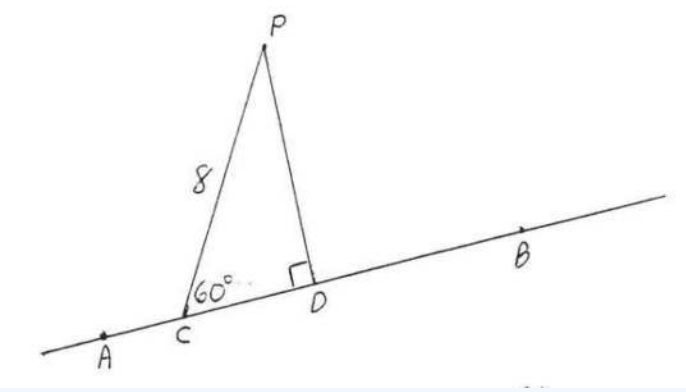


Figure 6.5.6.4: Draw PD perpendicular to $\overleftrightarrow{AB}$.

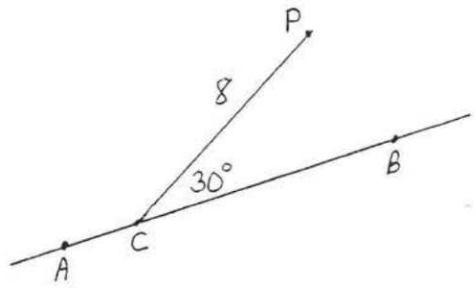
$$\begin{aligned} \text{hyp} &= 2s \\ 8 &= 2(CD) \\ 4 &= CD \\ L &= s\sqrt{3} \\ PD &= 4\sqrt{3} \end{aligned}$$

Answer: $4\sqrt{3}$

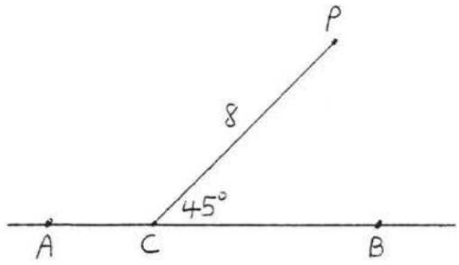
Problems

1 - 6. Find the distance from P to $\overleftrightarrow{AB}$:

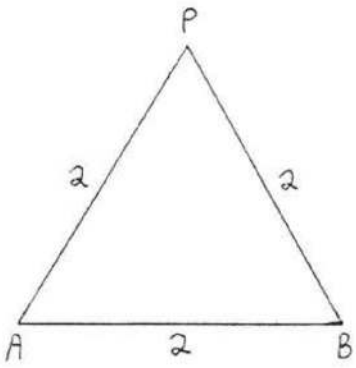
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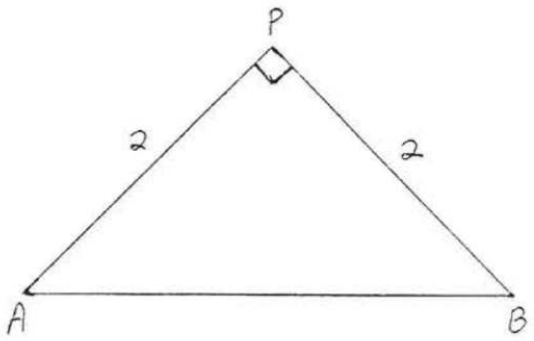
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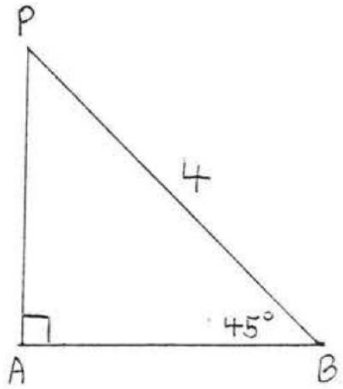
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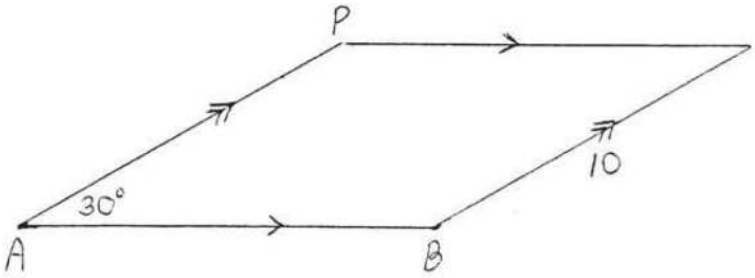
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6.6: Similar Triangles

Two triangles are said to be similar if they have equal sets of angles. In Figure **6.6.1**, $\triangle ABC$ is similar to $\triangle DEF$. The angles which are equal are called **corresponding angles**. In Figure **6.6.1**, $\angle A$ corresponds to $\angle D$, $\angle B$ corresponds to $\angle E$, and $\angle C$ corresponds to $\angle F$. The sides joining corresponding vertices are called **corresponding sides**. In Figure **6.6.1**, AB corresponds to DE, BC corresponds to EF, and AC corresponds to DF. The symbol for similar is $\sim$. The similarity statement $\triangle ABC \sim \triangle DEF$ will always be written so that corresponding vertices appear in the same order.

For the triangles in Figure **6.6.1**, we could also write $\triangle BAC \sim \triangle BDF$ or $\triangle ACB \sim \triangle DFE$ but never $\triangle ABC \sim \triangle EDF$ nor $\triangle ACB \sim \triangle DEF$.

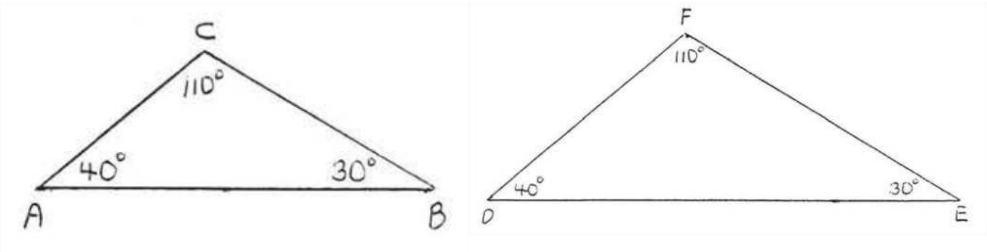
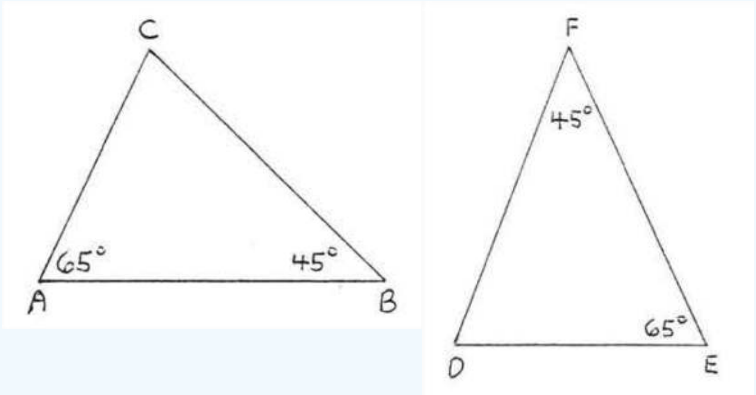


Figure **6.6.1**: $\triangle ABC$ is similar to $\triangle DEF$.

We can tell which sides correspond from the similarity statement. For example, if $\triangle ABC \sim \triangle DEF$, then side AB corresponds to side DE because both are the first two letters. BC corresponds to EF because both are the last two letters, AC corresponds to DF because both consist of the first and last letters.

Example 6.6.1

Determine if the triangles are similar, and if so, write the similarity statement:



Solution

$$\angle C = 180° - (65° + 45°) = 180° - 110° = 70°$$

$$\angle D = 180° - (65° + 45°) = 180° - 110° = 70°$$

Therefore both triangles have the same angles and $\triangle ABC \sim \triangle EFD$.

Answer: $\triangle ABC \sim \triangle EFD$.

Example A suggests that to prove similarity it is only necessary to know that **two** of the corresponding angles are equal:

Theorem 6.6.1

Two triangles are similar if two angles of one equal two angles of the other ($AA = AA$).

In Figure 6.6.2, $\triangle ABC \sim \triangle DEF$ because $\angle A = \angle D$ and $\angle B = \angle E$.

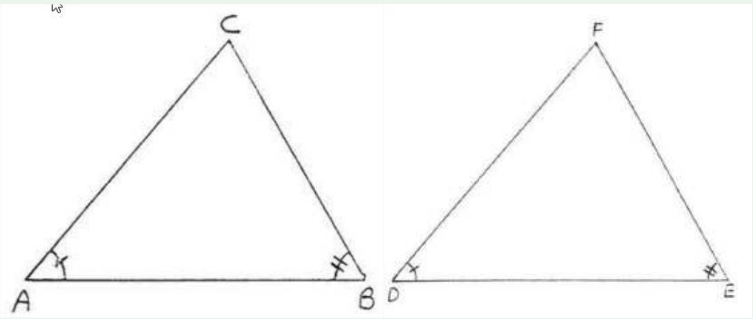


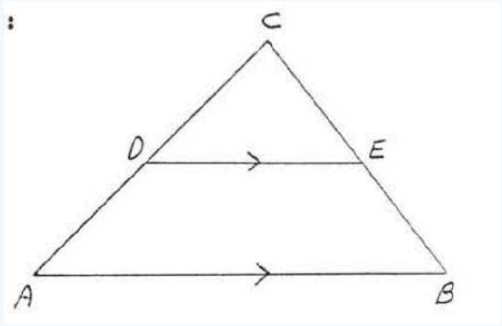
Figure 6.6.2. $\triangle ABC \sim \triangle DEF$ because $AA = AA$.

Proof

$\triangle C = 180° - (\angle A + \angle B) = 180° - (\angle D + \angle E) = \angle F$.

Example 6.6.2

Determine which triangles are similar and write a similarity statement:



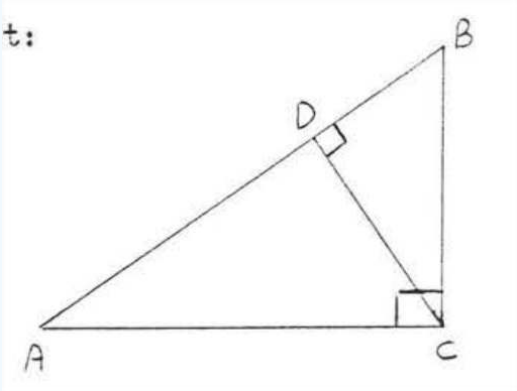
Solution

$\angle A = \angle CDE$ because they are corresponding angles of parallel lines. $\angle C = \angle C$ because of identity. Therefore $\triangle ABC \sim \triangle DEC$ by $AA = AA$.

Answer: $\triangle ABC \sim \triangle DEC$.

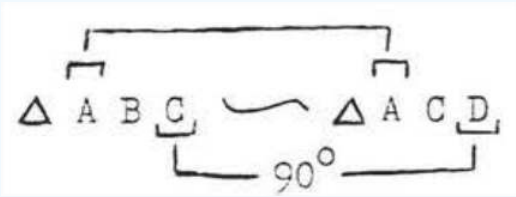
Example 6.6.3

Determine which triangles are similar and write a similarity statement:

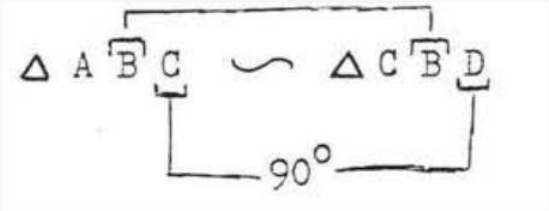


Solution

$\angle A = \angle A$ identity. $\angle ACB = \angle ADC = 90°$. Therefore



Also $\angle B = \angle B$, identity, $\angle BDC = \angle BCA = 90°$. Therefore



Answer: $\triangle ABC \sim \triangle ACD \sim \triangle CBD$.

Similar triangIes are important because of the following theorem:

Theorem 6.6.2

The corresponding sides of similar triangles are proportional. This means that if $\triangle ABC \sim \triangle DEF$ then

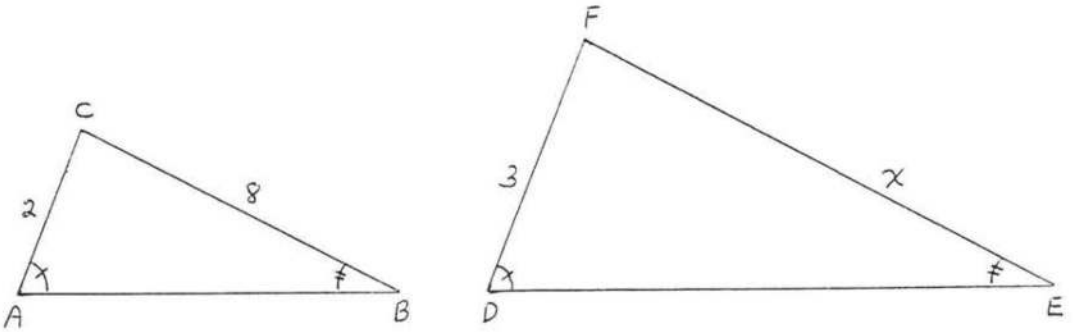
$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}.$$

That is, the first two letters of $\triangle ABC$ are to the first two letters of $\triangle DEF$ as the last two letters of $\triangle ABC$ are to the last two letters of $\triangle DEF$ as the first and last letters of $\triangle ABC$ are to the first and last letters of $\triangle DEF$.

Before attempting to prove Theorem 6.6.2, we will give several examples of how it is used:

Example 6.6.4

Find x:



Solution

$\angle A = \angle D$ and $\angle B = \angle E$ so $\triangle ABC \sim \triangle DEF$. By Theorem **6.6.2**,

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}.$$

We will ignore $\frac{AB}{DE}$ here since we do not know and do not have to find either AB or DE.

$$\begin{aligned} \frac{BC}{EF} &= \frac{AC}{DF} \\ \frac{8}{x} &= \frac{2}{3} \\ 24 &= 2x \\ 12 &= x \end{aligned} \tag{6.6.1}$$

Check:

$$\frac{BC}{EF} = \frac{AC}{DF}$$

$$\frac{8}{x} \quad \Big| \quad \frac{2}{3}$$

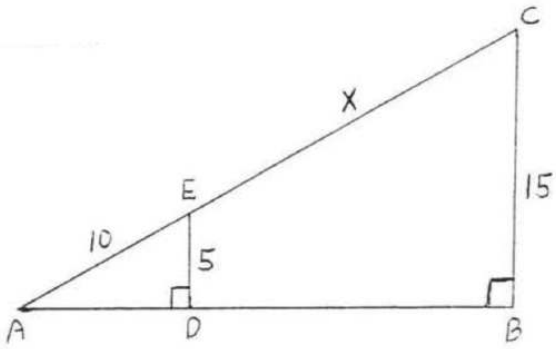
$$\frac{8}{12}$$

$$\frac{2}{3}$$

Answer: $x = 12$.

Example 6.6.5

Find x:



Solution

$\angle A = \angle A, \angle ADE = \angle ABC$, so $\triangle ADE \sim \triangle ABC$ by $AA = AA$.

$$\frac{AD}{AB} = \frac{DE}{BC} = \frac{AE}{AC}.$$

We ignore $\frac{AD}{AB}$.

$$\begin{aligned} \frac{DE}{BC} &= \frac{AE}{AC} \\ \frac{5}{15} &= \frac{10}{10+x} \\ 5(10+x) &= 15(10) \\ 50+5x &= 150 \\ 5x &= 150-50 \\ 5x &= 100 \\ x &= 20 \end{aligned} \tag{6.6.2}$$

Check:

$$\frac{DE}{BC} = \frac{AE}{AC}$$

$$\frac{5}{15} \quad \frac{10}{10+x}$$

$$\frac{1}{3} \quad \frac{10}{10+20}$$

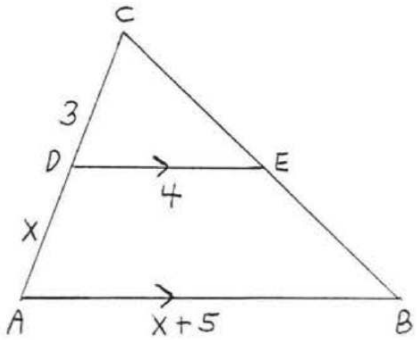
$$\frac{10}{30}$$

$$\frac{1}{3}$$

Answer: $x = 20$.

Example 6.6.6

Find x:



Solution

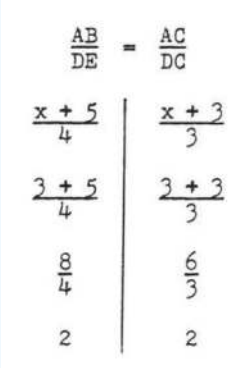
$\angle A = \angle CDE$ because they are corresponding angles of parallel lines. $\angle C = \angle C$ because of identity. Therefore $\triangle ABC \sim \triangle DEC$ by $AA = AA$.

$$\frac{AB}{DE} = \frac{BC}{EC} = \frac{AC}{DC}$$

We ignore $\frac{BC}{EC}$:

$$\begin{aligned} \frac{AB}{DE} &= \frac{AC}{DC} \\ \frac{x+5}{4} &= \frac{x+3}{3} \\ (x+5)(3) &= (4)(x+3) \\ 3x+15 &= 4x+12 \\ 15-12 &= 4x-3x \\ 3 &= x \end{aligned} \tag{6.6.3}$$

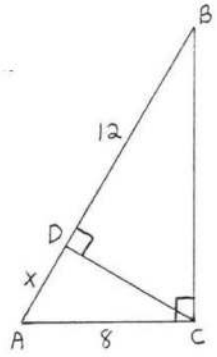
Check:



Answer: $x = 3$.

Example 6.6.7

Find x:



Solution

$\angle A = \angle A$, $\angle ACB = \angle ADC = 90°$, $\triangle ABC \sim \triangle ACD$.

$$\begin{aligned} \frac{AB}{AC} &= \frac{AC}{AD} \\ \frac{x+12}{8} &= \frac{8}{x} \\ (x+12)(x) &= (8)(8) \\ x^2+12x &= 64 \\ x^2+12x-64 &= 0 \\ (x-4)(x+16) &= 0 \\ x=4 \qquad x &= -16 \end{aligned} \tag{6.6.4}$$

We reject the answer $x = -16$ because $AD = x$ cannot be negative.

Check, $x = 4$

$$\frac{AB}{AC} = \frac{AC}{AD}$$

$$\frac{x+12}{8} \quad \frac{8}{x}$$

$$\frac{4+12}{8} \quad \frac{8}{4}$$

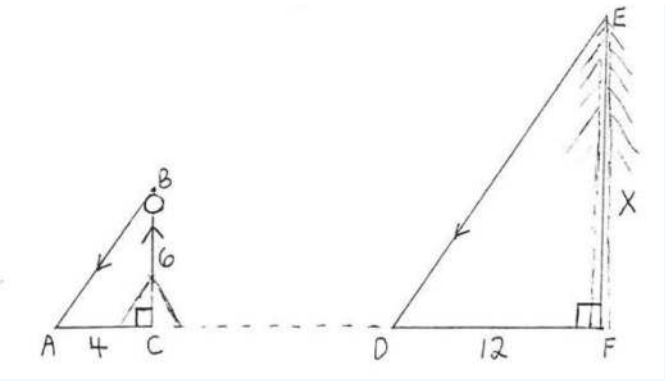
$$\frac{16}{8} \quad 2$$

$$2$$

Answer: $x = 4$.

Example 6.6.8

A tree casts a shadow 12 feet long at the same time a 6 foot man casts a shadow 4 feet long. What is the height of the tree?



Solution

In the diagram AB and DE are parallel rays of the sun. Therefore $\angle A = \angle D$ because they are corresponding angles of parallel lines with respect to the transversal AF. Since also $\angle C = \angle F = 90°$, we have $\triangle ABC \sim \triangle DEF$ by $AA = AA$.

$$\begin{aligned} \frac{AC}{DF} &= \frac{BC}{EF} \\ \frac{4}{12} &= \frac{6}{x} \\ 4x &= 72 \\ x &= 18 \end{aligned} \tag{6.6.5}$$

Answer: $x = 18$ feet.

Proof of Theorem **6.6.2** ("The corresponding sides of similar triangles are proportional"):

We illustrate the proof using the triangles of Example **6.6.4** (Figure **6.6.3**). The proof for other similar triangles follows the same pattern. Here we will prove that $x = 12$ so that $\frac{2}{3} = \frac{8}{x}$.

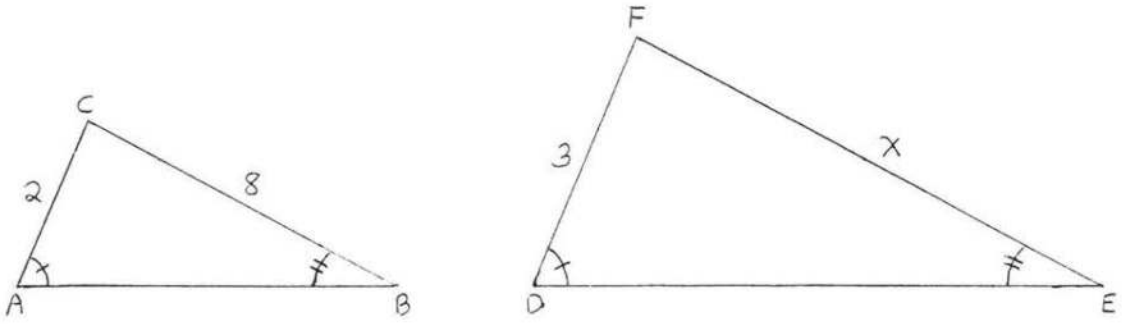


Figure **6.6.3**. The triangles of Example **6.6.4**.

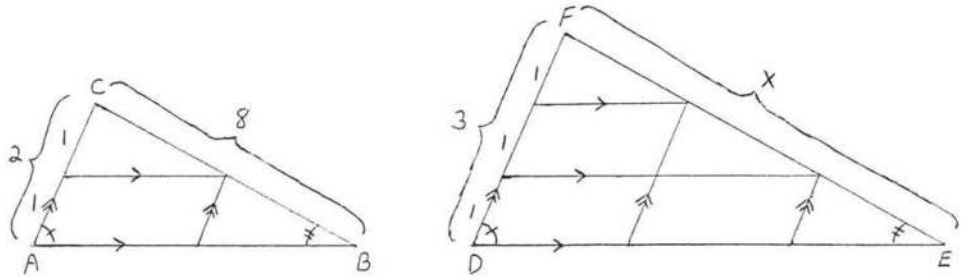


Figure **6.6.4**: Draw lines parallel to the sides of $\triangle ABC$ and $\triangle DEF$.

First draw lines parallel to the sides of $\triangle ABC$ and $\triangle DEF$ as shown in Figure **6.6.4**. The corresponding angles of these parallel lines are equal and each of the parallelograms with a side equal to 1 has its opposite side equal to 1 as well, Therefore all of the small

triangles with a side equal to 1 are congruent by $AAS = AAS$. The corresponding sides of these triangles form side $BC = 8$ of $\triangle ABC$ (see Figure 6.6.5). Therefore each of these sides must equal 4 and $x = EF = 4 + 4 + 4 = 12$ (Figure 6.6.6).

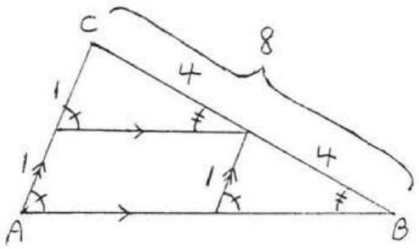


Figure 6.6.5. The small triangles are congruent hence the corresponding sides lying on BC must each be equal to 4.

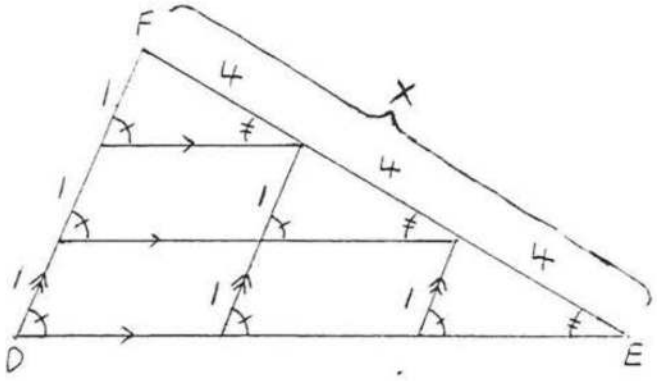


Figure 6.6.6. The small triangles of $\triangle DEF$ are congruent to the small triangles of $\triangle ABC$ hence $x = EF = 4 + 4 + 4 = 12$.

(Note to instructor: This proof can be carried out whenever the lengths of the sides of the triangles are rational numbers. However, since irrational numbers can be approximated as closely as necessary by rationals, the proof extends to that case as well.)

Historical Note

Thales (c. 600 B.C.) used the proportionality of sides of similar triangles to measure the heights of the pyramids in Egypt. His method was much like the one we used in Example 6.6.8 to measure the height of trees.

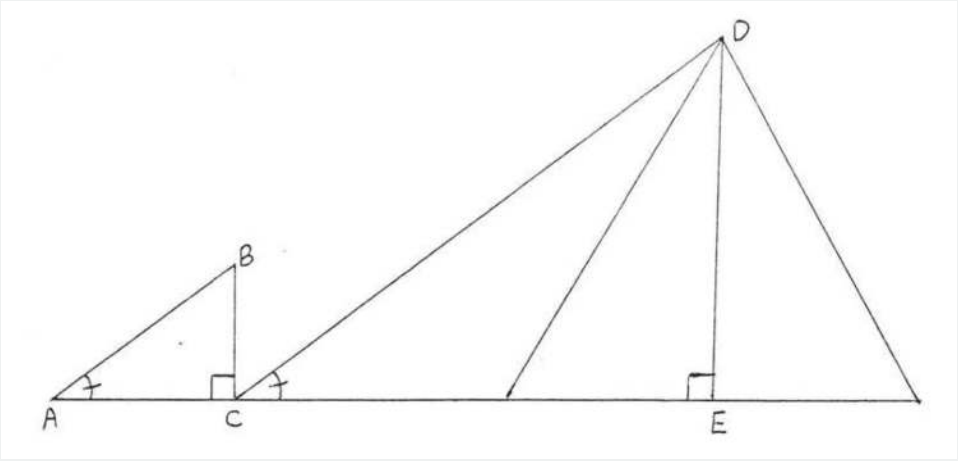


Figure 6.6.7. Using similar triangles to measure the height of a pyramid.

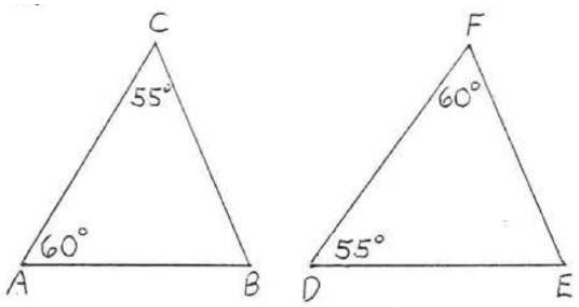
In Figure 6.6.7, DE represent the height of the pyramid and CE is the length of its shadow. BC represents a vertical stick and AC is the length of its shadow. We have $\triangle ABC \sim \triangle CDE$. Thales was able to measure directly the lengths AC, BC, and CE. Substituting these values in the proportion $\frac{BC}{DE} = \frac{AC}{CE}$, he was able to find the height DE.

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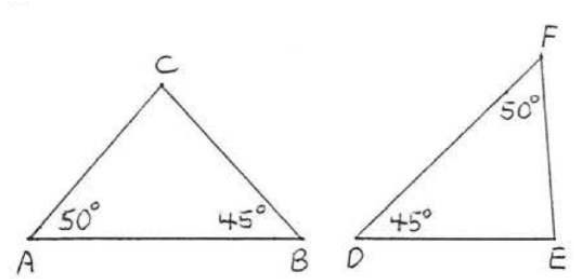
Problems

1 - 6. Determine which triangles are similar and write the similarity statement:

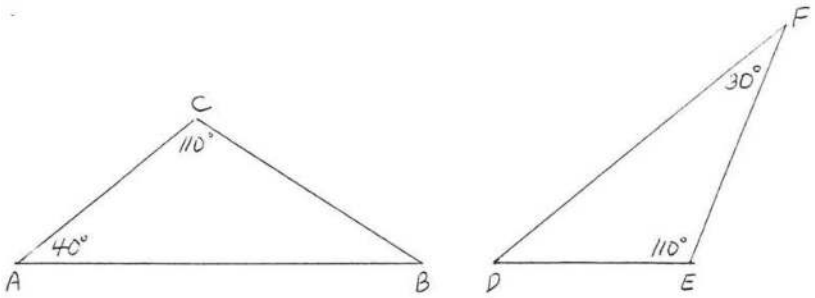
1.



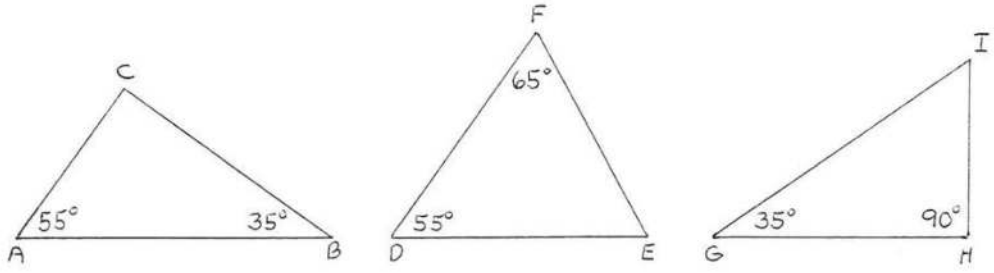
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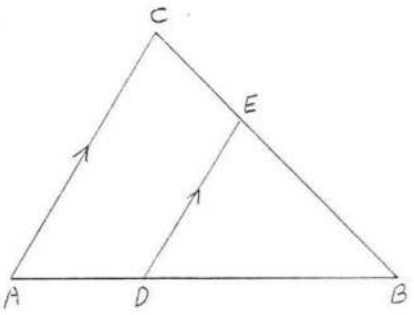
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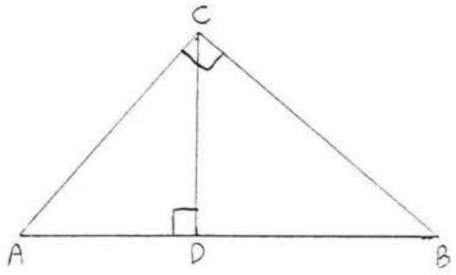
4.



5.



6.



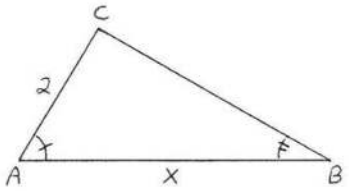
7 - 22. For each of the following

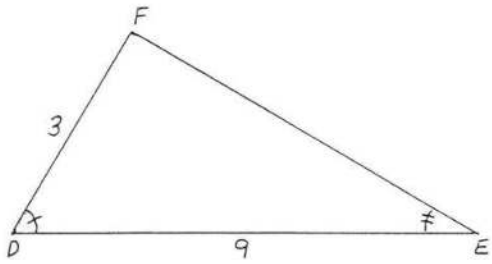
(1) write the similarity statement

(2) write the proportion between the corresponding sides

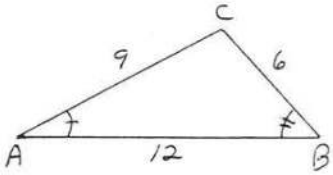
(3) solve for x or x and y.

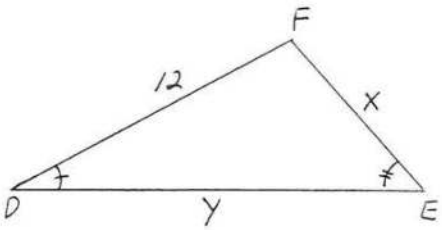
7.



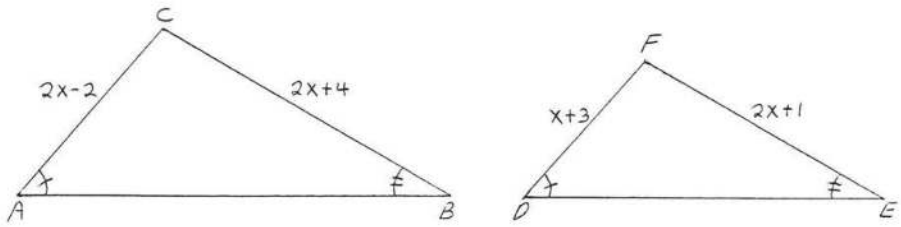


8.

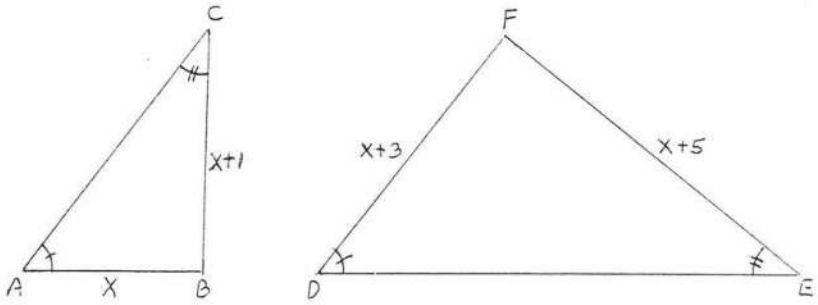




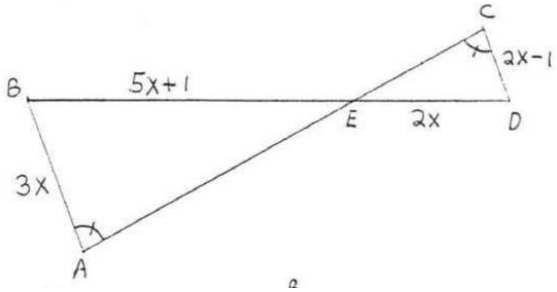
9.


C
2x-2
2x+4
A
B
F
x+3
2x+1
D
E

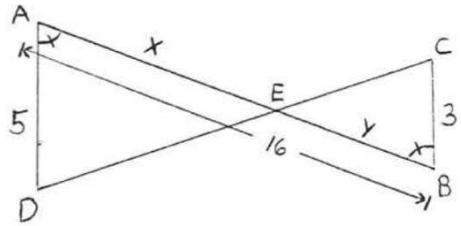
10.


C
x+1
A
X
B
F
x+3
x+5
D
E

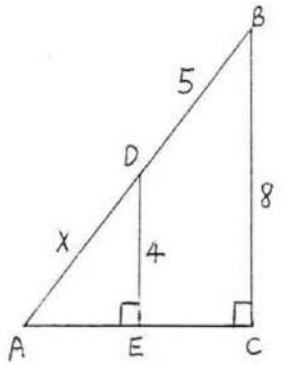
11.


C
2x-1
B
5x+1
E
2x
D
3x
A

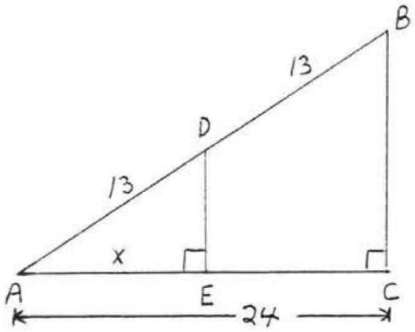
12.


A
x
C
E
5
3
y
16
D
B

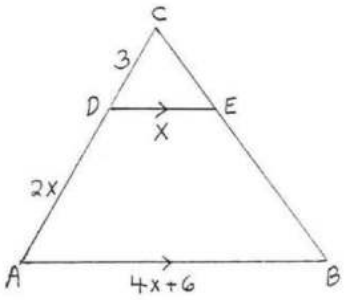
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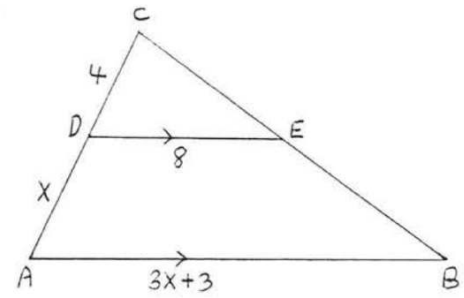
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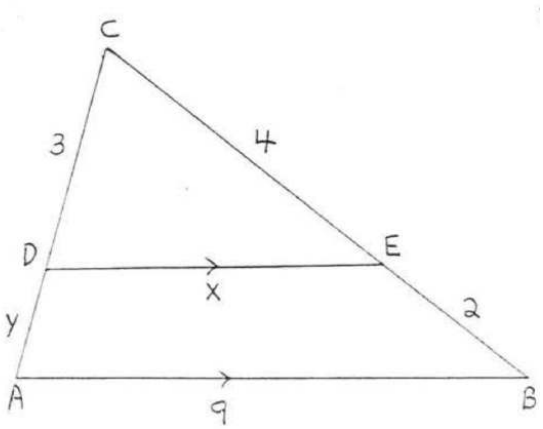
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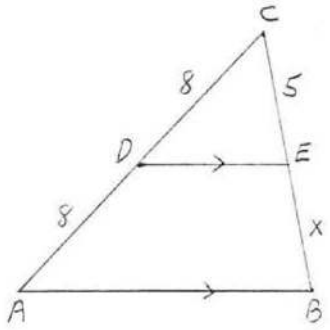
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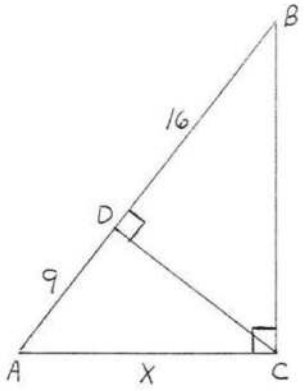
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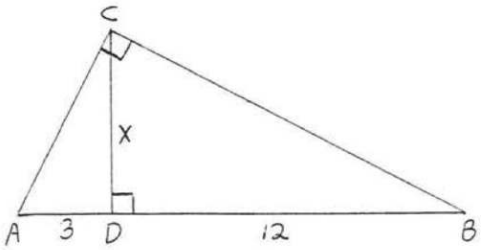
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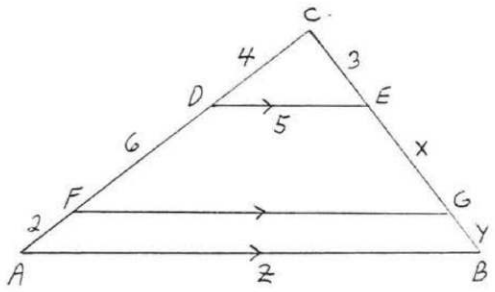
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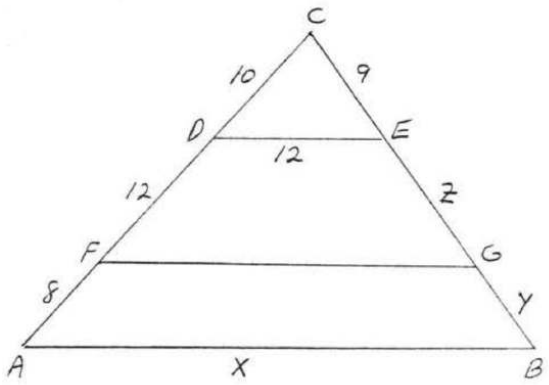
20.



21.

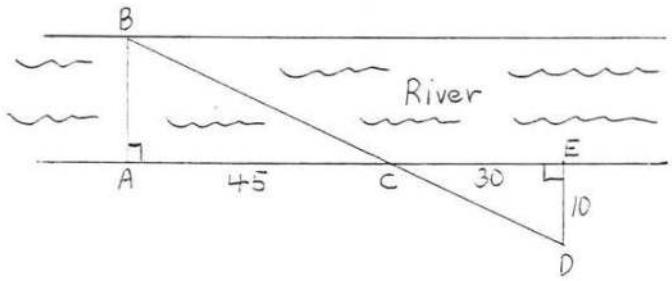


22.



23. A flagpole casts a shadow 80 feet long at the same time a 5 foot boy casts a shadow 4 feet long. How tall is the flagpole?

24. Find the width ***AB*** of the river:



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6.7: The SSS Theorem

We now consider the case where the side of two triangles are known to be of the same length.

Theorem 6.7.1: Side-Side-Side (SSS) Theorem

Two triangles are congruent if three sides of one are equal respectively to three sides of the other ($SSS = SSS$).

Theorem 6.7.1 is demonstrated in Figure 6.7.1: if $a = d, b = e$, and $c = f$ then $\triangle ABC \cong \triangle DEF$

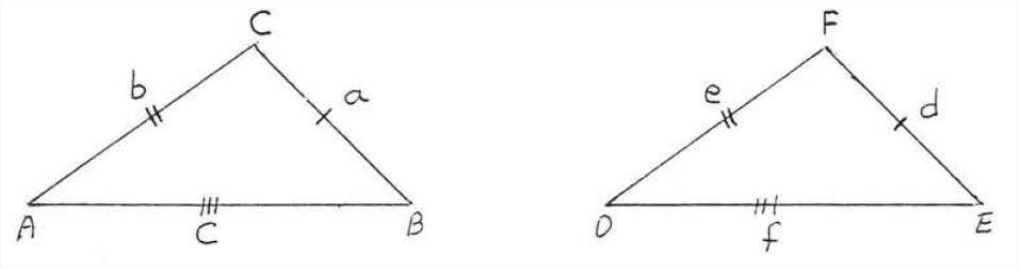
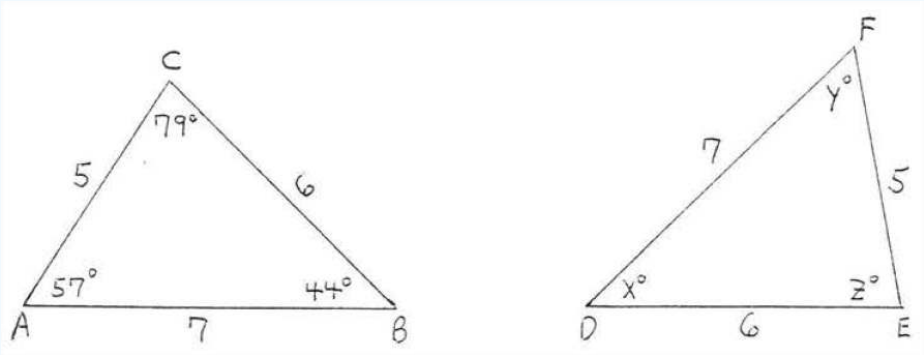


Figure 6.7.1: $\triangle ABC \cong \triangle DEF$ because $SSS = SSS$.

Example 6.7.1

Find x, y, z :



Solution

$AB = 7 = DF$. Therefore $\angle C$, the angle opposite AB must correspond to $\angle E$, the angle opposite DF. In the same way $\angle A$ corresponds to $\angle F$ and $\angle B$ corresponds to $\angle D$. We have $\triangle ABC \cong \triangle FDE$ by $SSS = SSS$, so

$x° = \angle D = \angle B = 44°$

$y° = \angle F = \angle A = 57°$

$z° = \angle E = \angle C = 79°$

Answer: $x = 44, y = 57, z = 79$

Proof of Theorem 6.7.1

In Figure 6.7.1, place $\triangle ABC$ and $\triangle DEF$ so that their longest sides coincide, in this case AB and DE. This can be done because $AB = c = r = DE$. Now draw CF, forming angles 1, 2, 3, and 4 (Figure 6.7.2). The rest of the proof will be presented in double-column form:

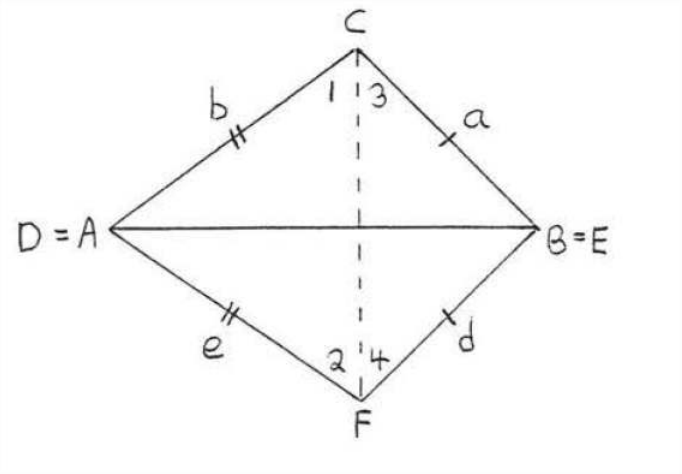
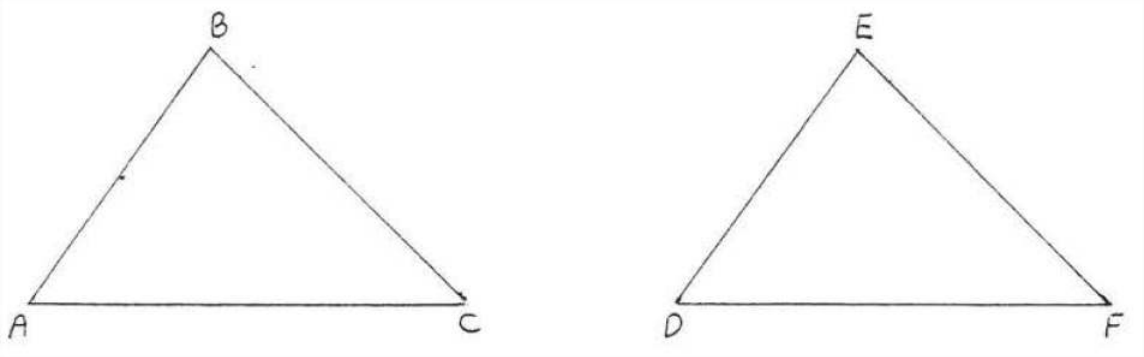


Figure **6.7.2**: Place $\triangle ABC$ and $\triangle DEF$ so that AB and DE coincide and draw CF.

Statement	Reasons
1. $\angle 1 = \angle 2$.	1. The base angles of isosceles triangle CAF are equal (Theorem **6.7.1**, section 2.5).
2. $\angle 3 = \angle 4$.	2. The base angles of isosceles triangle CBF are equal.
3. $\angle C = \angle F$.	3. $\angle C = \angle 1 + \angle 3 = \angle 2 + \angle 4 = \angle F$.
4. $AC = DF$.	4. Given, $AC = b = e = DF$.
5. $BC = EF$.	5. Given, $BC = a = d = EF$.
6. $\triangle ABC \cong \triangle DEF$.	6. $SAS = SAS$: $AC, \angle C, BC$ of $\triangle ABC = DF, \angle F, EF$ of $\triangle DEF$.

Example 6.7.2

Given $AB = DE, BC = EF$, and $AC = DF$. Prove $\angle C = \angle F$



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Solution

Statements	Reasons
1. $AB = DE$.	1. Given.
2. $BC = EF$.	2. Given.
3. $AC = DF$.	3. Given.
4. $\triangle ABC \cong \triangle DEF$.	4. $SSS = SSS$: AB, BC, AC of $\triangle ABC = DE, EF, DF$ of $\triangle DEF$.
5. $\angle C = \angle F$.	5. Corresponding angles of congruent triangles are equal.

Application: Triangular Bracing

The SSS Theorem is the basis of an important principle of construction engineering called **triangular bracing**. Imagine the line segments in Figure 6.7.3 to be beans of wood or steel joined at the endpoints by nails or screws. If pressure is applied to one of the sides, $ABCD$ will collapse and look like $A'B'C'D'$.

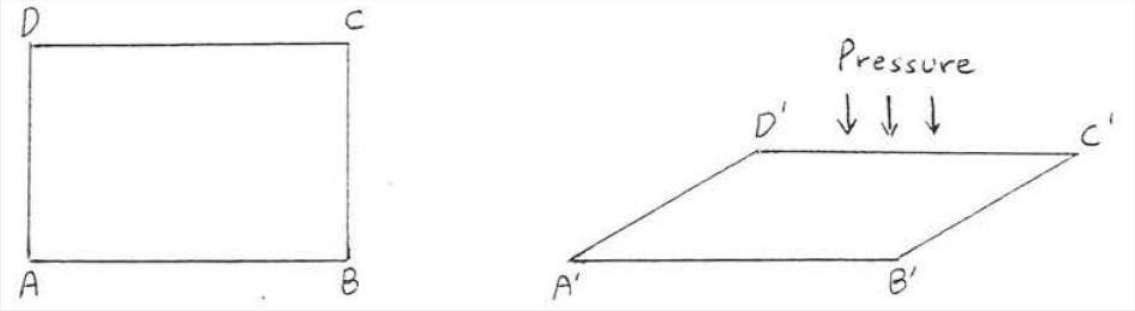


Figure 6.7.3: $ABCD$ collapses into $A'B'C'D'$, when pressure is applied.

Now suppose points A and C are joined by a new beam, called a **brace** (Figure 6.7.4). The structure will not collapse as long as the beans remain unbroken and joined together. It is impossible to deform $ABCD$ into any other shape $A'B'C'D'$ because if $AB = A'B'$, $BC = B'C'$, and $AC = A'C'$ then $\triangle ABC$ would be congruent to $\triangle A'B'C'$ by $SSS = SSS$.

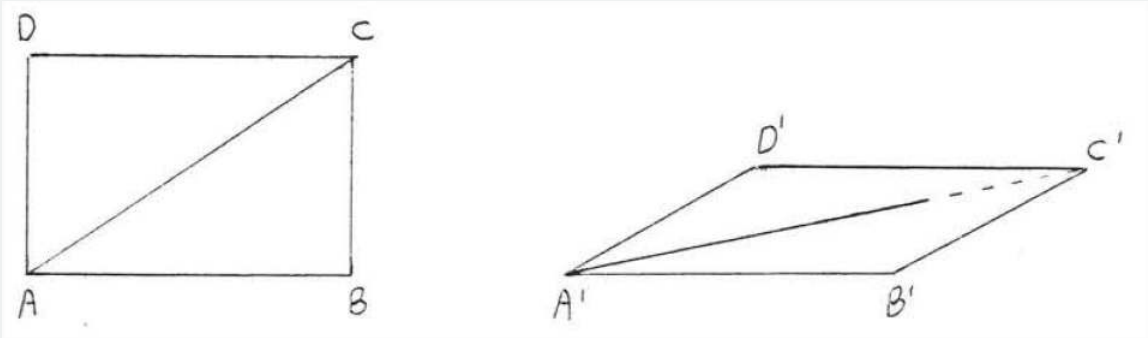


Figure 6.7.4: $ABCD$ cannot collapse into $A'B'C'D'$ as long as the beams remain unbroken and Jotned together.

We sometimes say that a triangle is a **rigid** figure; once the sides of a triangle are fixed the angles cannot be changed. Thus in Figure 6.7.4, the shape of $\triangle ABC$ cannot be changed as long as the lengths of its sides remain the same.

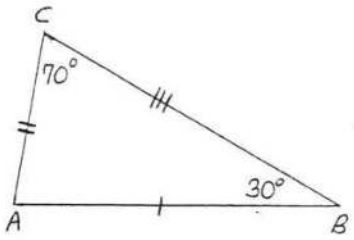
Problems

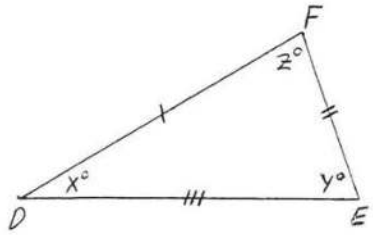
1 - 8. For each of the following (1) write the congruence statement,

(2) given the reason for (1) (SAS, ASA, AAS, or SSS Theorem), and

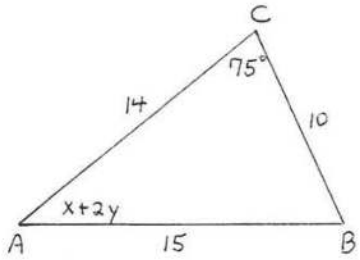
(3) find x, or x and y, or x, y, and z.

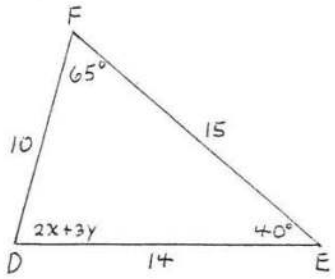
1.


C
70°
30°
A
B

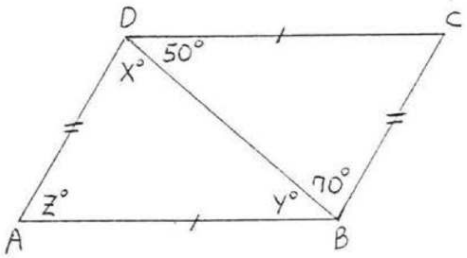

F
z°
x°
y°
D
E

2.

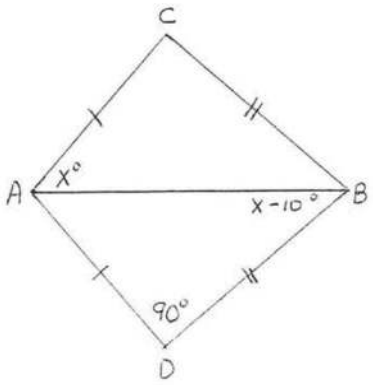

C
75°
14
10
x+2y
A
15
B


F
65°
10
15
2x+3y
40°
D
14
E

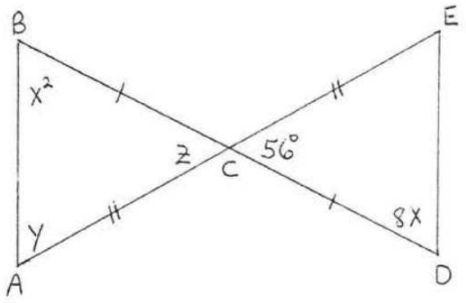
3.


D
C
50°
x°
z°
y°
70°
A
B

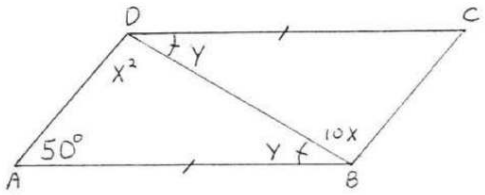
4.


C
A
x°
B
x-10°
90°
D

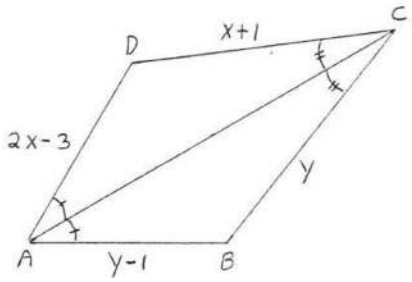
5.



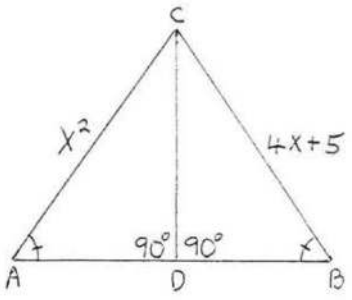
6.



7.

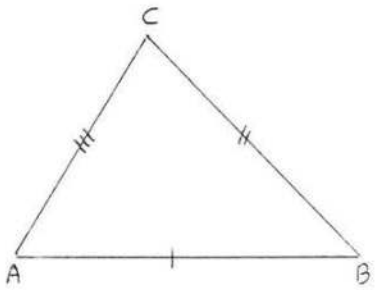


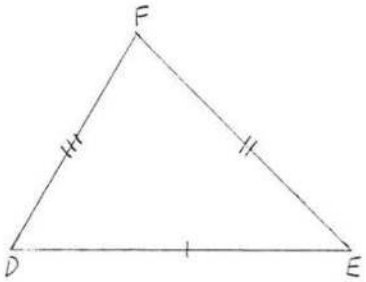
8.



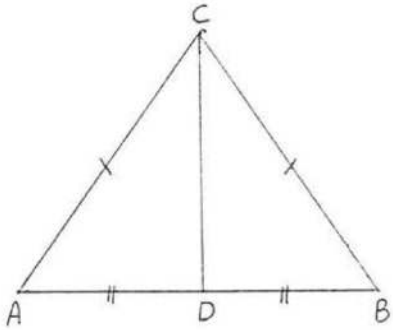
9. Given $AB = DE$, $BC = EF$, and $AC = DF$. Prove $\angle A = \angle D$.

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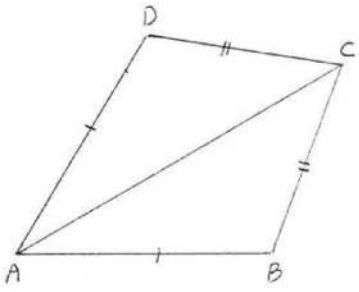




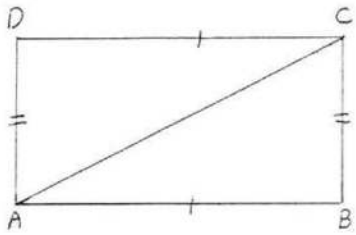
10. Given $AC = BC$. $AD = BD$. Prove $\angle ADC = \angle BDC$.



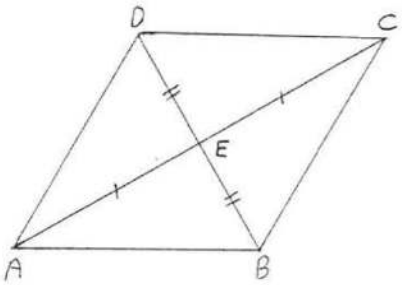
11. Given $AB = AD$, $BC = DC$. Prove $\angle BAC = \angle CAD$.



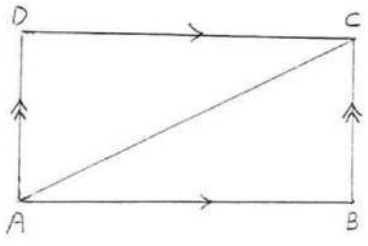
12. Given $AB = CD$, $BC = DA$. Prove $\angle BAC = \angle DCA$.



13. Given $AE = CE$, $BE = ED$. Prove $AB = CD$.



14. Given $AB||CD$, $AD||BC$. Prove $AB = CD$.



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6.8: The SAS Theorem

We have said that two triangles are congruent if all their correspond ing sides and angles are equal, However in some cases, it is possible to conclude that two triangles are congruent, with only partial information about their sides and angles.

Suppose we are told that $\triangle ABC$ has $\angle A = 53°$, $AB = 5$ inches, and $AC = 3$ inches. Let us attenpt to sketch $\triangle ABC$. We first draw an angle of $53°$ with a protractor and label it $\angle A$. Using a ruler, we find the point 5 inches from the vertex on one side of the angle and label it B, On the other side of the angle, we find the point 3 inches from the vertex and label it C, See Figure 6.8.1, There is now only one way for us to complete our sketch of $\triangle ABC$, and that is to connect points B and C with a line segment, We could now measure BC, $\angle B$, and $\angle C$ to find the remaining parts of the triangle.

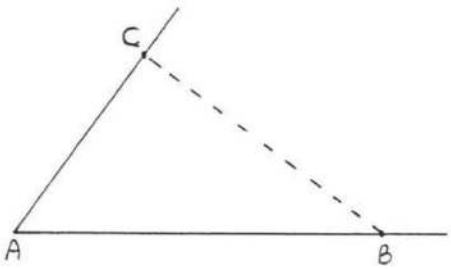


Figure 6.8.1: Sketching $\triangle ABC$.

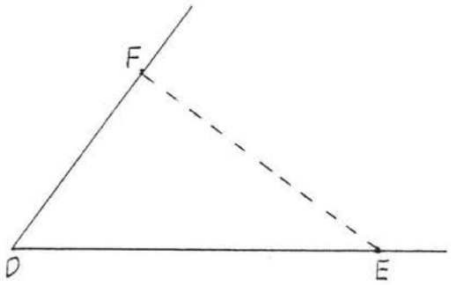


Figure 6.8.2: Sketching $\triangle DEF$.

Suppose now $\triangle DEF$ were another triangle, with $\angle D = 53°$, $DE = 5$ inches, and $DF = 3$ inches. We could sketch $\triangle DEF$ just as we did $\triangle ABC$, and then measure EF, $\angle E$, and $\angle F$ (Figure 6.8.2). It is clear that we must have $BC = EF$, $\angle B = \angle E$, and $\angle C = \angle F$ because both triangles were drawn in exactly the same way. Therefore $\triangle ABC \cong \triangle DEF$.

- In $\triangle ABC$, we say that $\angle A$ is the angle included between sides AB and AC.
- In $\triangle DEF$, we say that $\angle D$ is the angle included between sides DE and DF.

Our discussion suggests the following theorem:

Theorem 6.8.1 (SAS or Side-Angle-Side Theorem)

Two triangles are congruent if two sides and the included angle of one are equal respectively to two sides and the included angle of the other,

In Figure 6.8.1 and 6.8.2, $\triangle ABC \cong \triangle DEF$ because AB, AC, and $\angle A$ are equal respectively to DE, DF and $\angle D$.

We sometimes abbreviate Theorem 6.8.1 by simply writing $SAS = SAS$.

Example 6.8.1

In $\triangle PQR$ name the angle included between sides

1. PQ and QR,
2. PQ and PR,

3. PR and QR,

Solution

Note that the included angle is named by the letter that is common to both sides, For (1), the letter "Q" is common to PQ and QR and so $\angle Q$ is included between sides PQ and QR. Similarly for (2) and (3).

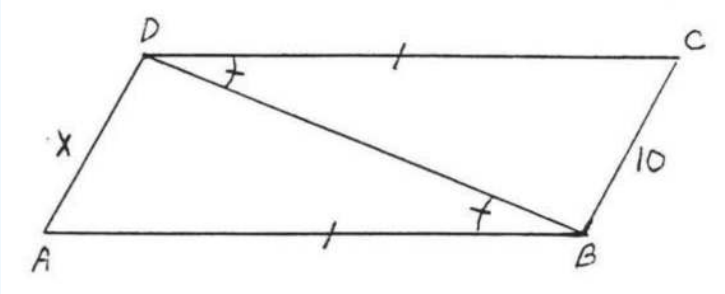
Answer: (1) $\angle Q$, (2) $\angle P$, (3) $\angle R$.

Example 6.8.2

For the two triangles in the diagram

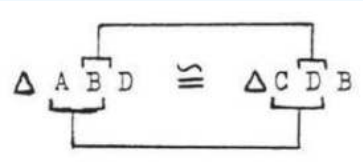
1. list two sides and an included angle of each triangle that are respectively equal, using the infonnation given in the diagram,
2. write the congruence statement,

and (3) find x by identifying a pair of corresponding sides of the congruent triangles.



Solution

(1) The angles and sides that are marked the same way in the diagram are assumed to be equal, So $\angle B$ in $\triangle ABD$ is equal to $\angle D$ in $\triangle BCD$. Therefore, "B" corresponds to "D." We also have $AB = CD$. Therefore "A" must corresponds to "C". Thus, if the triangles are congruent, the correspondence must be



Finally, BD (the same as DB) is a side common to both triangles, Summaryzing,

	$\underline{\triangle ABD}$		$\underline{\triangle CDB}$	
Side	AB	$=$	CD	(marked = in diagram)
Included Angle	$\angle B$	$=$	$\angle D$	(marked = in diagram)
Side	BD	$=$	DB	(common side)

(2) $\triangle ABD \cong \triangle CDB$ because of the SAS Theorem ($SAS = SAS$).

(3) $x = AD = CB = 10$ because AD and CB are corresponding sides (first and third letters in the congruence statement) a.~d corresponding sides of congruent triangles are equal.

Answer:

(1) AB, $\angle B$, BD of $\triangle ABD = CD$, $\angle D$, DB of $\triangle CDB$.

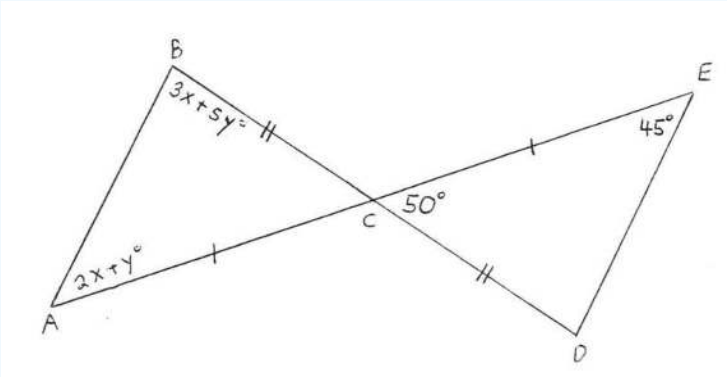
(2) $\triangle ABD \cong \triangle CDB$.

(3) $x = AD = CB = 10$.

Example 6.8.3

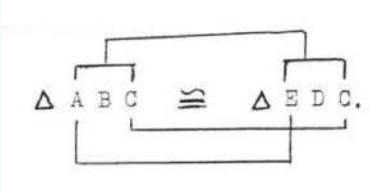
For the two triangles in the diagram

1. list two sides and an included angle of each triangle that are respectively equal, using the information given in the diagram.
2. write the congruence statement, and
3. find x and y by identifying a pair of corresponding sides of the congruent triangles.



Solution

(1) $AC = CE$ and $BC = CD$ because they are mar!,;:ed the same way. We also know that $\angle ACB = \angle ECD = 50°$ because vertical angles are equal. Therefore "C" in $\triangle ABC$ corresponds to "C" in $\triangle CDE$. Since $AC = CE$, we must have that "A" in $\triangle ABC$ corresponds to "E" in $\triangle CDE$. Thus, if the triangles are congruent, the correspondence must be



We summarize:

	$\triangle ABC$		$\triangle EDC$	
Side	AC	=	EC	(marked = in diagram)
Included Angle	$\angle ACB$	=	$\angle ECD$	(vertical angles are =)
Side	BC	=	DC	(marked = in diagram)

(2) $\triangle ABC \cong \triangle EDC$ because of the SAS theorem. ($SAS = SAS$)

(3) $\angle A = \angle E$ and $\angle B = \angle D$ because they are corres:9onding angles of the congruent triangles. $\angle D = 85°$ because the sum of the angles of $\triangle EDC$ must be $180°$. ($\angle D = 180° - (50° + 45°) = 180° - 95° = 85°$). We obtain a system of two equations in the two unknowns x and y:

$$\angle A = \angle E \longrightarrow 2x + y = 45 \xrightarrow{-5} -10x - 5y = -225$$
$$\angle B = \angle D \longrightarrow 3x + 5y = 85 \longrightarrow 3x + 5y = 85$$
$$-7x = -140$$
$$x = 20$$

Substituting for x in the first original equation,

$$\begin{aligned} 2x+y &= 45 \\ 2(20)+y &= 45 \\ 40+y &= 45 \\ y &= 45-40 \\ y &= 5 \end{aligned} \tag{6.8.1}$$

Check:

$\angle A$	$\angle E$
$2x + y$	45°
$2(20) + 5$	
$40 + 5$	
45°	

$\angle B$	$\angle D$
$3x + 5y$	85°
$3(20) + 5(5)$	
$60 + 25$	
85°	

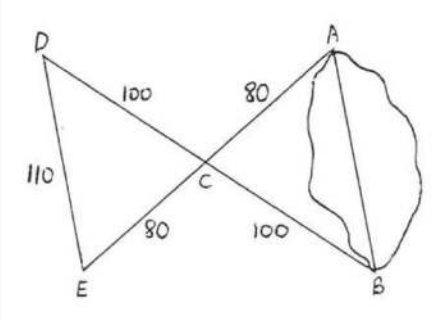
Answer:

1. AC, $\angle ACB$, BC of $\triangle ABC$ = $EC, \angle ECD, DC$ of $\triangle EDC$.
2. $\triangle ABC \cong \triangle EDC$.
3. $x = 20, y = 5$.

Example 6.8.4

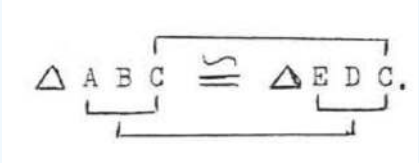
The following procedure was used to measure the d.istance AB across a pond: From a point C, AC and BC were measured and found to be 80 and 100 feet respectively. Then AC was extended to E so that $AC = CE$ and BC was extended to D so that $BC = CD$. Finally, DE we found to be 110 feet.

1. Write the congruence statement.
2. Give a reason for (1).
3. Find AB.



Solution

(1) $\angle ACB = \angle ECD$ because vertical angles are equal. Therefore the "C's" correspond, $AC = EC$ so A must correspond to E. We have



(2) $SAS = SAS$. Sides AC, BC, and included angle C of ABC are equal respectively to EC, DC, and included angle C of $\angle EDC$.

(3) $AB = ED$ ecause they are corresponding sides of congruent triangles, Since $ED = 110$, $AB = 110$.

Answer

(1) $\triangle ABC \cong \triangle EDC$.

(2) $SAS = SAS$: AC, $\angle C$, BC of $\triangle ABC = EC$, $\angle C$, DC of $\triangle EDC$.

(3) $AB = 110 feet$.

Historical Note

The SAS Theorem is Proposition 4 in Euclid's **Elements**, Both our discussion and Suclit's proof of the SAS Theoremimplicitly use the following principle: If a geometric construction is repeated in a different location (or what amounts to the same thing is "moved" to a different location) then the size and shape of the figure remain the same, There is evidence that Euclid used this principle reluctantly, and many mathematicians have since questioned its use in formal proofs, They feel that it makes too strong an assumption about the nature of physical space and is an inferior form of geometric reasoning. Bertrand Russell (1872 - 1970), for example, has suggested that we would be better off assuming the SAS Theorem as a **postulate**, This is in fact done in a system of axioms for Euclidean geometry devised by David Hilbert (1862 - 1943), a system that has gained much favor with modern mathematicians. Hilbert was the leading exponent of the "formalist school," which sought to discover exactly what assumptions underlie each branch of mathematics and to remove all logical ambiguities, Hilbert's system, however, is too formal for an introductory course in geometry,

Problems

1 - 4. For each of the following (1) draw the triangle with the two sides and the included angle and (2) measure the remaining side and angles:

1. $AB = 2$ inches, $AC = 1$ inch, $\angle A = 60°$.

2. $DE = 2$ inches, $DF = 1$ inch, $\angle D = 60°$.

3. $AB = 2$ inches, $AC = 3$ inches, $\angle A = 40°$.

4. $DE = 2$ inches, $DF = 3$ inches, $\angle D = 40°$.

5 - 8. Name the angle included between sides

5. AB and BC in $\triangle ABC$.

6. XY and YZ in $\triangle XYZ$.

7. DE and DF in $\triangle DEF$.

8. RS and TS in $\triangle RST$.

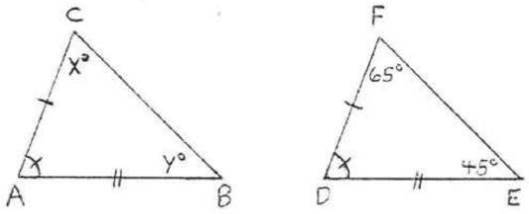
9 - 22. For each of the following.

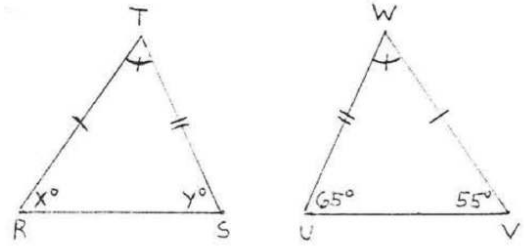
(1) list two sides and an included angle of each triangle that are respectively equal, using the information given in the diagram,

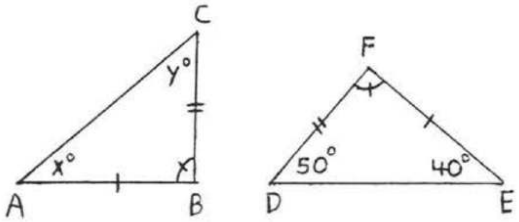
(2) write the congruence statement,

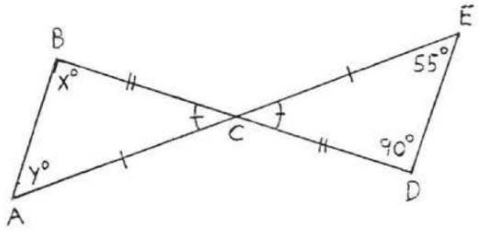
(3) find x, or x and y.

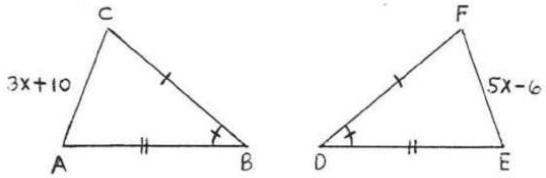
Assume that angles or sides marked in the same way are eQual.

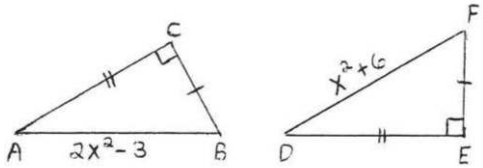
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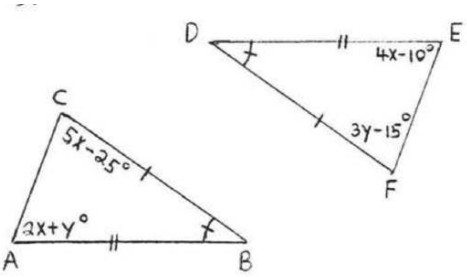
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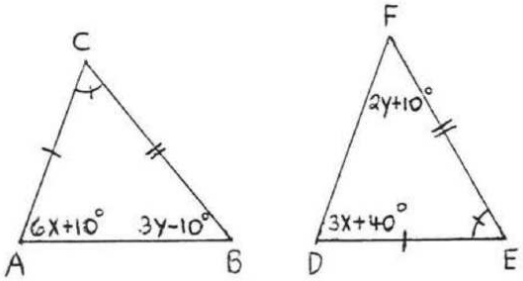
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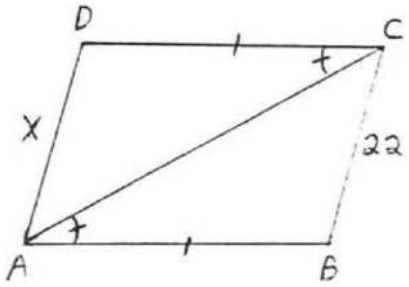
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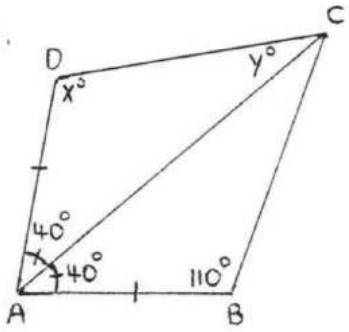
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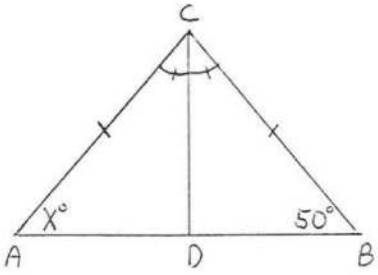
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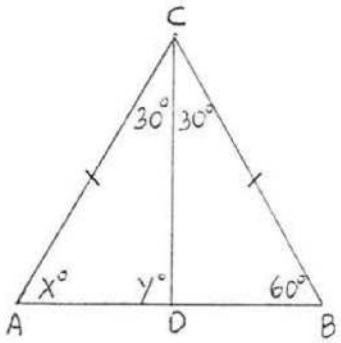
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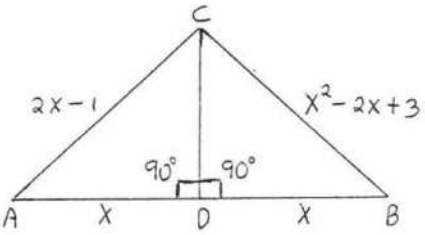
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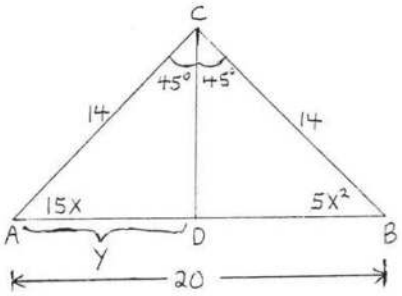
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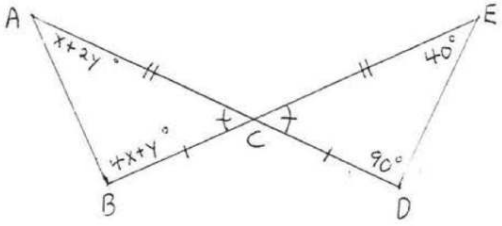
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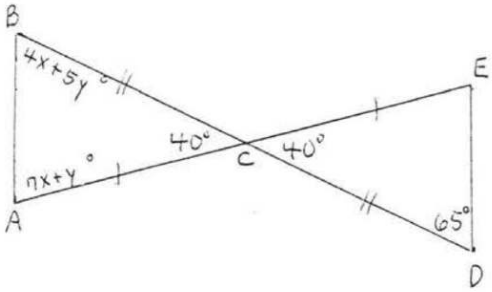
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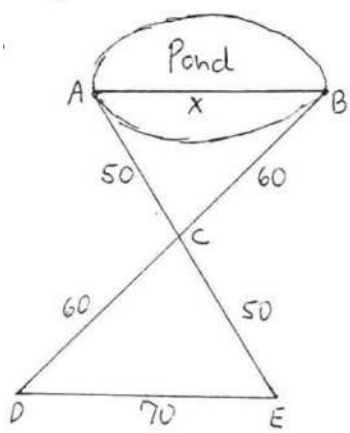
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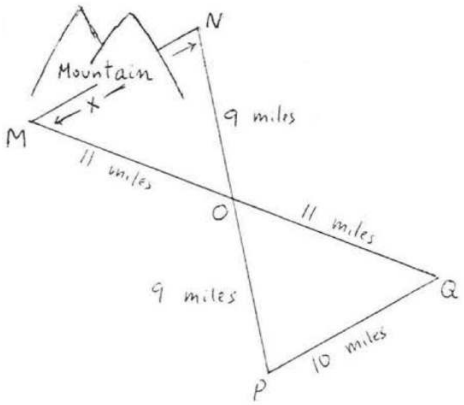
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25.



26.



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6.9: The SSS Theorem

We now consider the case where the side of two triangles are known to be of the same length.

Theorem 6.9.1: Side-Side-Side (SSS) Theorem

Two triangles are congruent if three sides of one are equal respectively to three sides of the other ($SSS = SSS$).

Theorem 6.9.1 is demonstrated in Figure 6.9.1: if $a = d, b = e$, and $c = f$ then $\triangle ABC \cong \triangle DEF$

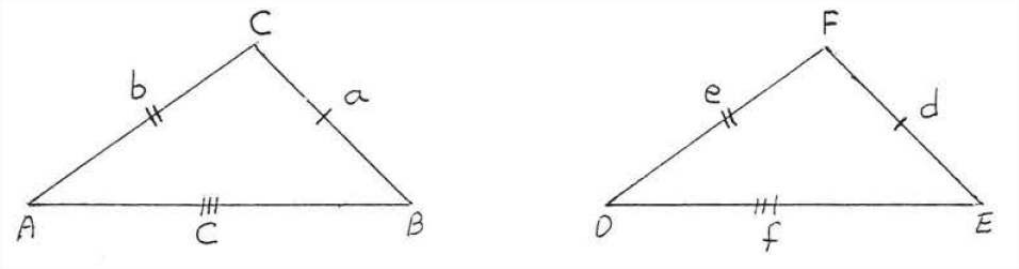
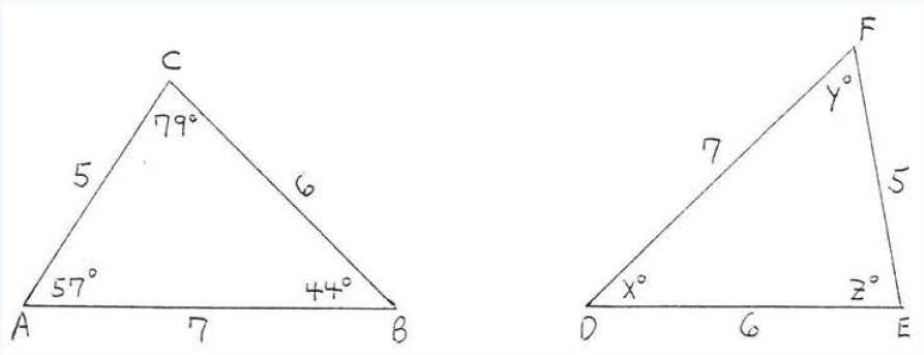


Figure 6.9.1: $\triangle ABC \cong \triangle DEF$ because $SSS = SSS$.

Example 6.9.1

Find x, y, z :



Solution

$AB = 7 = DF$. Therefore $\angle C$, the angle opposite AB must correspond to $\angle E$, the angle opposite DF. In the same way $\angle A$ corresponds to $\angle F$ and $\angle B$ corresponds to $\angle D$. We have $\triangle ABC \cong \triangle FDE$ by $SSS = SSS$, so

$x° = \angle D = \angle B = 44°$

$y° = \angle F = \angle A = 57°$

$z° = \angle E = \angle C = 79°$

Answer: $x = 44, y = 57, z = 79$

Proof of Theorem 6.9.1

In Figure 6.9.1, place $\triangle ABC$ and $\triangle DEF$ so that their longest sides coincide, in this case AB and DE. This can be done because $AB = c = r = DE$. Now draw CF, forming angles 1, 2, 3, and 4 (Figure 6.9.2). The rest of the proof will be presented in double-column form:

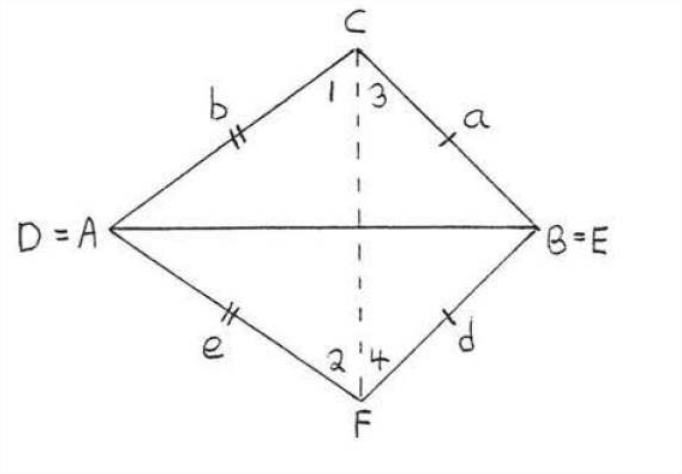
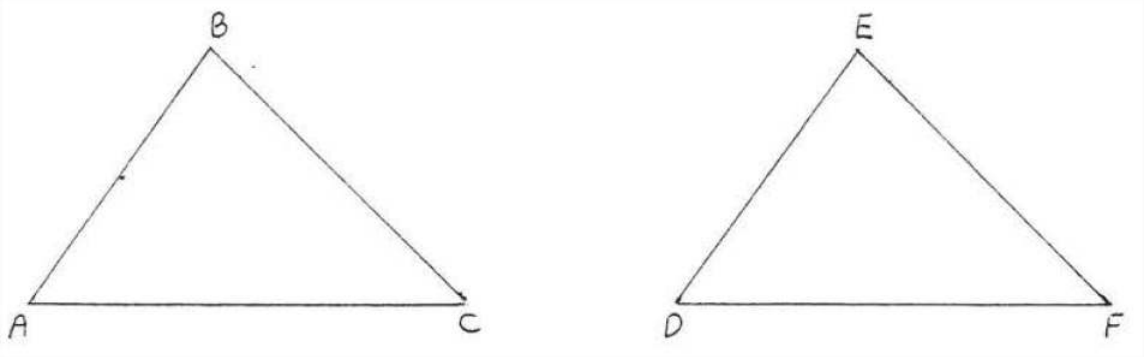


Figure **6.9.2**: Place $\triangle ABC$ and $\triangle DEF$ so that AB and DE coincide and draw CF.

Statement	Reasons
1. $\angle 1 = \angle 2$.	1. The base angles of isosceles triangle CAF are equal (Theorem **6.9.1**, section 2.5).
2. $\angle 3 = \angle 4$.	2. The base angles of isosceles triangle CBF are equal.
3. $\angle C = \angle F$.	3. $\angle C = \angle 1 + \angle 3 = \angle 2 + \angle 4 = \angle F$.
4. $AC = DF$.	4. Given, $AC = b = e = DF$.
5. $BC = EF$.	5. Given, $BC = a = d = EF$.
6. $\triangle ABC \cong \triangle DEF$.	6. $SAS = SAS$: $AC, \angle C, BC$ of $\triangle ABC = DF, \angle F, EF$ of $\triangle DEF$.

Example 6.9.2

Given $AB = DE, BC = EF$, and $AC = DF$. Prove $\angle C = \angle F$



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Solution

Statements	Reasons
1. $AB = DE$.	1. Given.
2. $BC = EF$.	2. Given.
3. $AC = DF$.	3. Given.
4. $\triangle ABC \cong \triangle DEF$.	4. $SSS = SSS$: AB, BC, AC of $\triangle ABC = DE, EF, DF$ of $\triangle DEF$.
5. $\angle C = \angle F$.	5. Corresponding angles of congruent triangles are equal.

Application: Triangular Bracing

The SSS Theorem is the basis of an important principle of construction engineering called **triangular bracing**. Imagine the line segments in Figure 6.9.3 to be beans of wood or steel joined at the endpoints by nails or screws. If pressure is applied to one of the sides, $ABCD$ will collapse and look like $A'B'C'D'$.

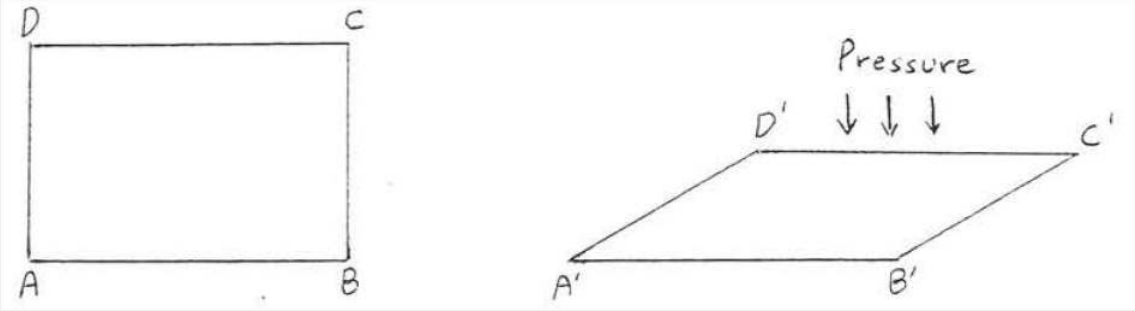


Figure 6.9.3: $ABCD$ collapses into $A'B'C'D'$, when pressure is applied.

Now suppose points A and C are joined by a new beam, called a **brace** (Figure 6.9.4). The structure will not collapse as long as the beans remain unbroken and joined together. It is impossible to deform $ABCD$ into any other shape $A'B'C'D'$ because if $AB = A'B'$, $BC = B'C'$, and $AC = A'C'$ then $\triangle ABC$ would be congruent to $\triangle A'B'C'$ by $SSS = SSS$.

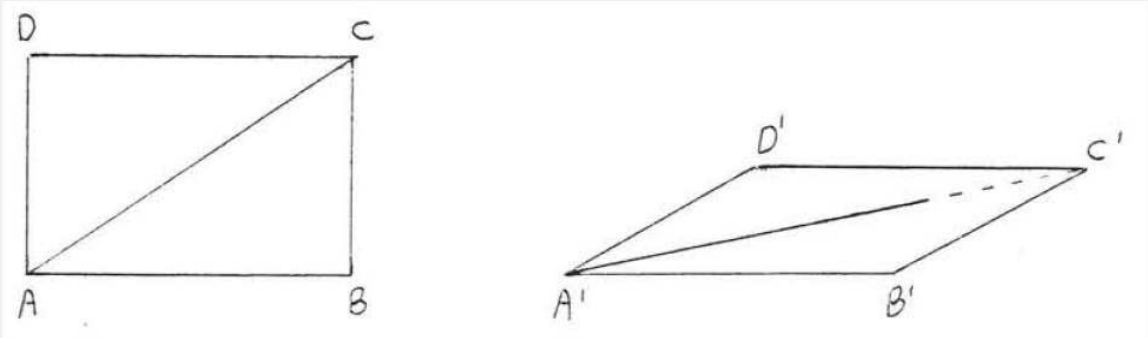


Figure 6.9.4: $ABCD$ cannot collapse into $A'B'C'D'$ as long as the beams remain unbroken and Jotned together.

We sometimes say that a triangle is a **rigid** figure; once the sides of a triangle are fixed the angles cannot be changed. Thus in Figure 6.9.4, the shape of $\triangle ABC$ cannot be changed as long as the lengths of its sides remain the same.

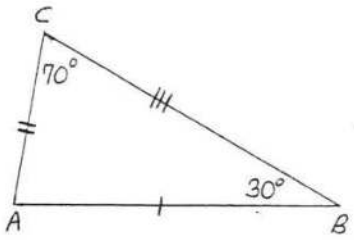
Problems

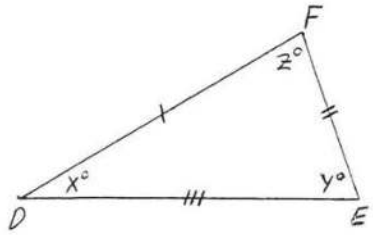
1 - 8. For each of the following (1) write the congruence statement,

(2) given the reason for (1) (SAS, ASA, AAS, or SSS Theorem), and

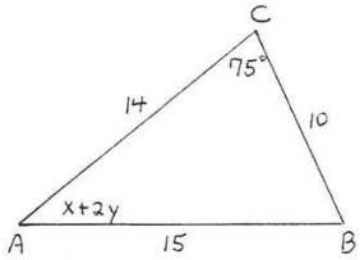
(3) find x, or x and y, or x, y, and z.

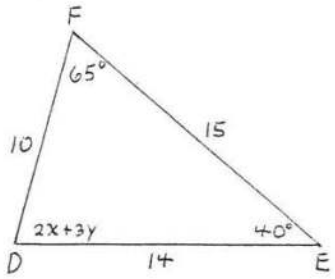
1.


C
70°
30°
A
B

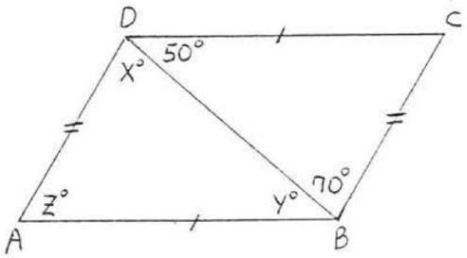

F
z°
x°
y°
D
E

2.

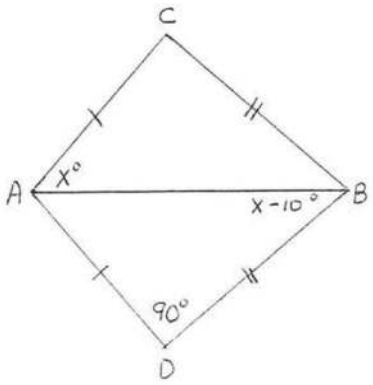

C
75°
14
10
x+2y
A
15
B


F
65°
10
15
2x+3y
40°
D
14
E

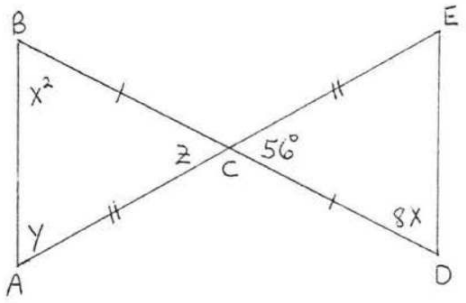
3.


D
C
50°
x°
z°
y°
70°
A
B

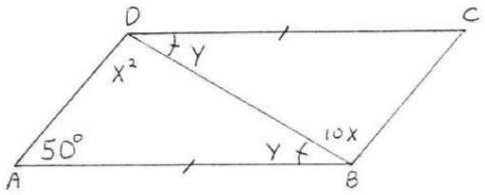
4.


C
A
x°
B
x-10°
90°
D

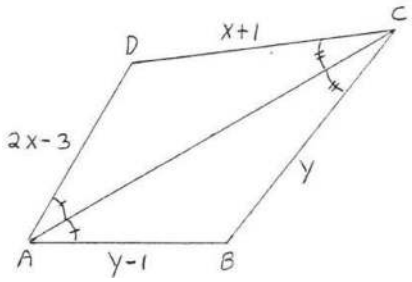
5.



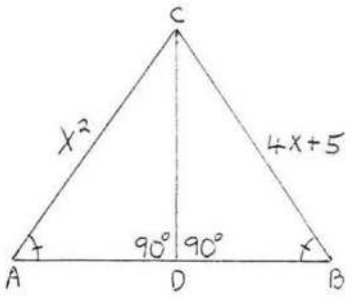
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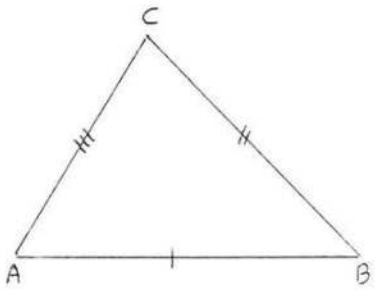


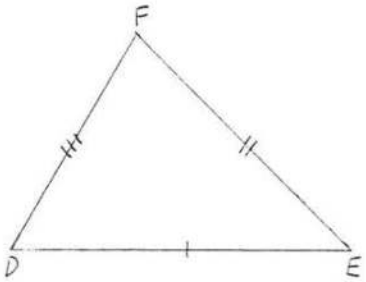
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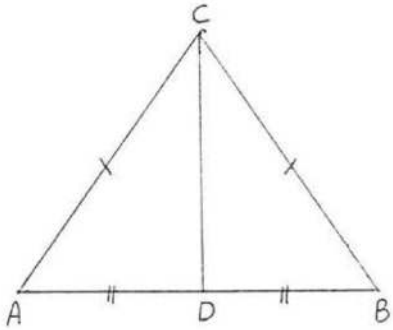
9. Given $AB = DE$, $BC = EF$, and $AC = DF$. Prove $\angle A = \angle D$.

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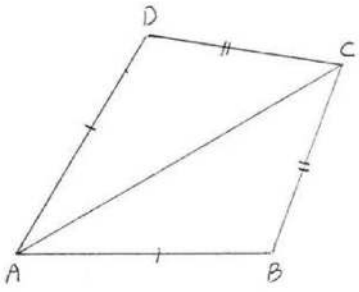




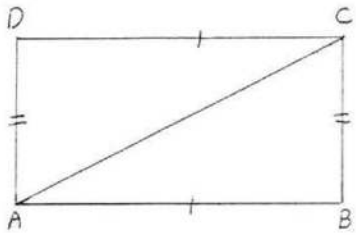
10. Given $AC = BC$. $AD = BD$. Prove $\angle ADC = \angle BDC$.



11. Given $AB = AD$, $BC = DC$. Prove $\angle BAC = \angle CAD$.

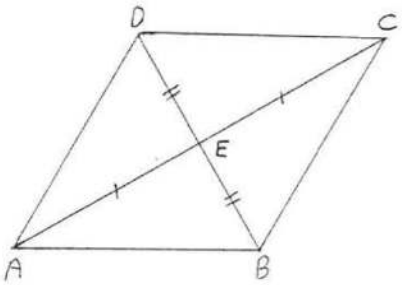


12. Given $AB = CD$, $BC = DA$. Prove $\angle BAC = \angle DCA$.

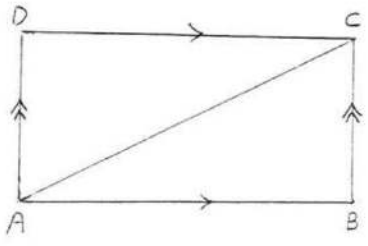


13. Given $AE = CE$, $BE = ED$. Prove $AB = CD$.

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14. Given $AB||CD$, $AD||BC$. Prove $AB = CD$.



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6.10: Pythagorean Theorem

In a right triangle, the sides of the right angle are called the **legs** of the triangle and the remaining side is called the **hypotenuse**. In Figure 6.10.1, side AC and BC are the legs and side AB is the hypotenuse.

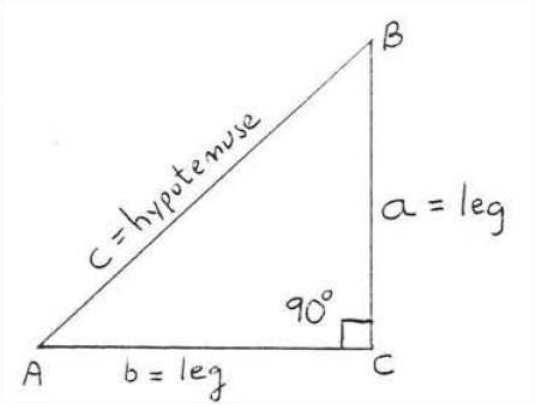


Figure 6.10.1: A right triangle.

The following is one of the most famous theorems in mathematics.

Theorem 6.10.1: Pythagorean Theorem

In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the legs. That is,

$$\text{leg}^2 + \text{leg}^2 = \text{hypotenuse}^2 \tag{6.10.1}$$

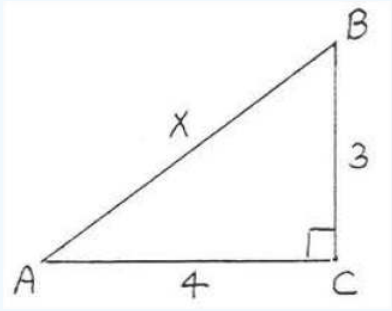
Thus, for the sides of the triangle in Figure 6.10.1,

$$a^2 + b^2 = c^2$$

Before we prove Theorem 6.10.1, we will give several examples.

Example 6.10.1

Find x



Solution

$$\begin{aligned} \text{leg}^2 + \text{leg}^2 &= \text{hyp}^2 \\ 3^2 + 4^2 &= x^2 \\ 9 + 16 &= x^2 \\ 25 &= x^2 \\ 5 &= x \end{aligned}$$

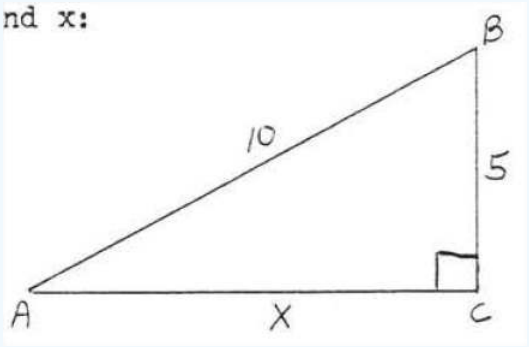
Check:

$$\begin{array}{rcl} \text{leg}^2 + \text{leg}^2 & = & \text{hyp}^2 \\ 3^2 + 4^2 & & x^2 \\ 9 + 16 & & 5^2 \\ 25 & & 25 \end{array}$$

Answer: $x = 5$.

Example 6.10.2

Find x:



Solution

$$\begin{aligned} \text{leg}^2 + \text{leg}^2 &= \text{hyp}^2 \\ 5^2 + x^2 &= 10^2 \\ 25 + x^2 &= 100 \\ x^2 &= 75 \\ x &= \sqrt{75} = \sqrt{25}\sqrt{3} = 5\sqrt{3} \end{aligned}$$

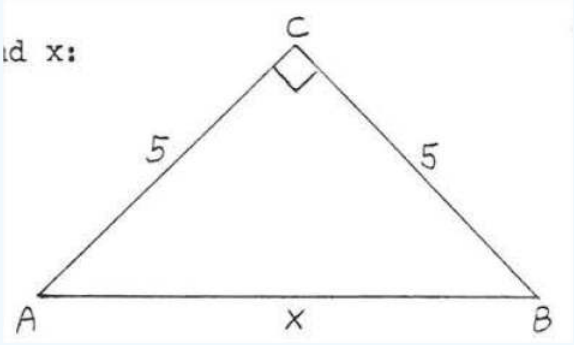
Check:

$$\begin{array}{rcl} \text{leg}^2 + \text{leg}^2 & = & \text{hyp}^2 \\ 5^2 + x^2 & & 10^2 \\ 25 + (5\sqrt{3})^2 & & 100 \\ 25 + 25\sqrt{9} & & \\ 25 + 25(3) & & \\ 25 + 75 & & \\ 100 & & \end{array}$$

Answer: $x = 5\sqrt{3}$.

Example 6.10.3

Find x:



Solution

$$\begin{aligned} \text{leg}^2 + \text{leg}^2 &= \text{hyp}^2 \\ 5^2 + 5^2 &= x^2 \\ 25 + 25 &= x^2 \\ 50 &= x^2 \\ x &= \sqrt{50} = \sqrt{25}\sqrt{2} = 5\sqrt{2} \end{aligned}$$

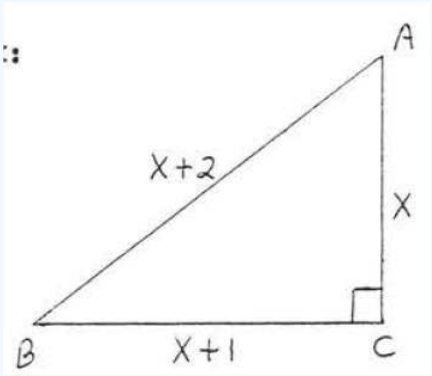
Check:

$$\begin{array}{rcl} \text{leg}^2 + \text{leg}^2 & = & \text{hyp}^2 \\ 5^2 + 5^2 & & x^2 \\ 25 + 25 & & (5\sqrt{2})^2 \\ 50 & & 25.\sqrt{4} \\ & & 25(2) \\ & & 50 \end{array}$$

Answer: $x = 5\sqrt{2}$.

Example 6.10.4

Find x



Solution

$$\begin{aligned} \text{leg}^2 + \text{leg}^2 &= \text{hyp}^2 \\ x^2 + (x+1)^2 &= (x+2)^2 \\ x^2 + x^2 + 2x + 1 &= x^2 + 4x + 4 \\ x^2 + x^2 + 2x + 1 - x^2 - 4x - 4 &= 0 \\ x^2 - 2x - 3 &= 0 \\ (x-3)(x+1) &= 0 \end{aligned}$$

$$\begin{aligned} x - 3 &= 0 \quad & x + 1 &= 0 \\ x &= 3 & x &= -1 \end{aligned}$$

We reject $x = -1$ because $AC = x$ cannot be negative.

Check, $x = 3$:

$$\begin{array}{r|l} \text{leg}^2 + \text{leg}^2 = & \text{hyp}^2 \\ x^2 + (x+1)^2 & (x+2)^2 \\ 3^2 + (3+1)^2 & (3+2)^2 \\ 9 + 4^2 & 5^2 \\ 9 + 16 & 25 \\ 25 & \end{array}$$

Answer: $x = 3$.

We will now restate and prove Theorem **6.10.1**:

Theorem 6.10.1 Pythagorean Theorem

In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the legs. That is,

$\text{leg}^2 + \text{leg}^2 = \text{hypotenuse}^2$.

In Figure **6.10.1**,

$a^2 + b^2 = c^2$.

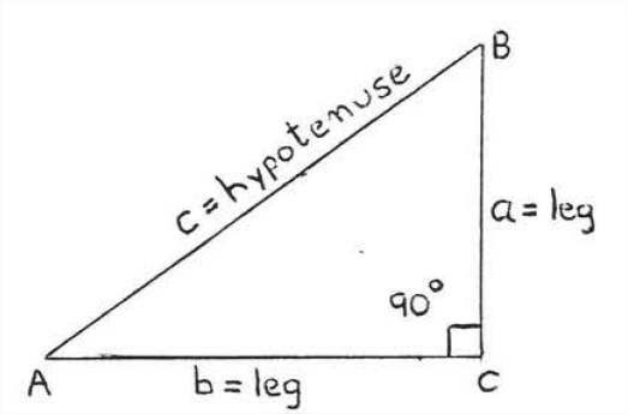


Figure **6.10.1**. A right triangle.

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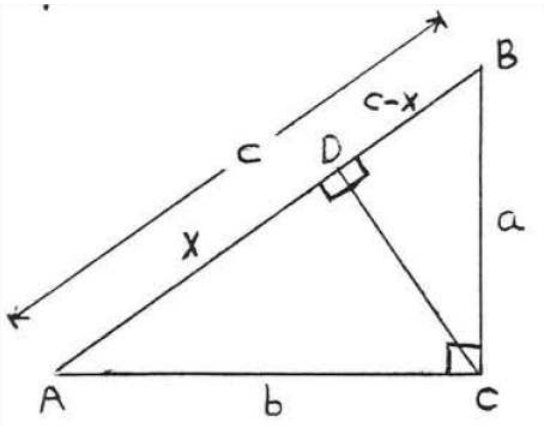


Figure 6.10.2. Draw CD perpendicular to AB.

Proof

In Figure 6.10.1, draw CD perpendicular to AB. Let $x = AD$. Then $BD = c - x$ (Figure 6.10.2). As in Example 6.10.3, section 4.2, $\triangle ABC \sim \triangle ACD$ and $\triangle ABC \sim \triangle CBD$. From these two similarities we obtain two proportions:

$\triangle ABC \sim \triangle ACD$

$$\frac{AB}{AC} = \frac{AC}{AD}$$

$$\frac{c}{b} = \frac{b}{x}$$

$$cx = b^2$$

$\triangle ABC \sim \triangle CBD$

$$\frac{AB}{CB} = \frac{BC}{BD}$$

$$\frac{c}{a} = \frac{a}{c - x}$$

$$c(c - x) = a^2$$

$$c^2 - cx = a^2$$

$$c^2 - b^2 = a^2$$

$$c^2 = a^2 + b^2$$

The converse of the Pythagorean Theorem also holds:

Theorem 6.10.2 (converse of the Pythagorean Theorem).

In a triangle, if the square of one side is equal to the sun of the squares of the other two sides then the triangle is a right triangle.

In Figure 6.10.3, if $c^2 = a^2 + b^2$ then $\triangle ABC$ is a right triangle with $\angle C = 90°$.

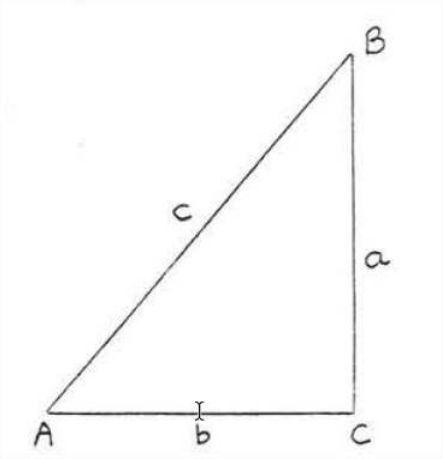


Figure 6.10.3: If $c^2 = a^2 + b^2$ then $\angle C = 90°$.

Proof

Draw a new triangle, $\triangle DEF$, so that $\angle F = 90°$, $d = a$, and $e = b$ (Figure 6.10.4). $\triangle DEF$ is a right triangle, so by Theorem 6.10.1, $f^2 = d^2 + e^2$. We have $f^2 = d^2 + e^2 = a^2 + b^2 = c^2$ and therefore $f = c$. Therefore $\triangle ABC \cong \triangle DEF$ because $SSS = SSS$. Therefore, $\angle C + \angle F = 90°$.

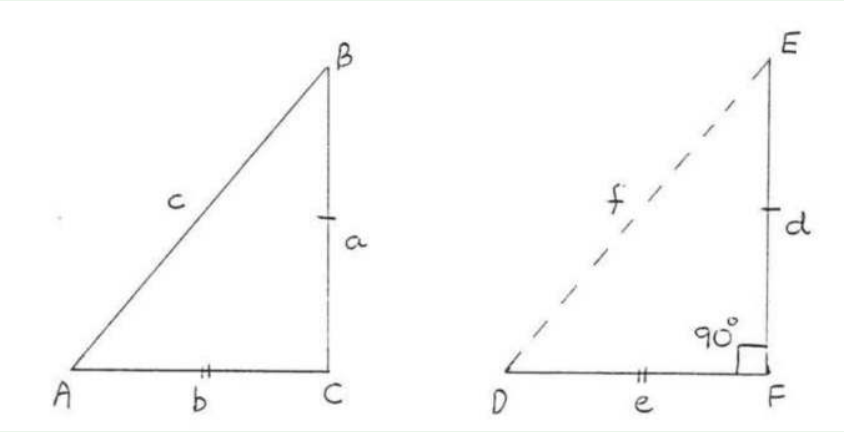
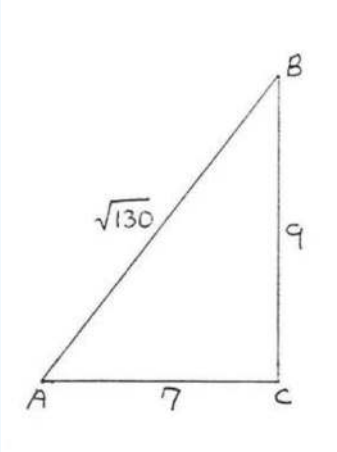


Figure 6.10.4: Given $\triangle ABC$, draw $\triangle DEF$ so that $\angle F = 90°$, $d = a$ and $e = b$.

Example 6.10.5

Is $\triangle ABC$ a right triangle?



Solution

$AC^2 = 7^2 = 49$

$BC^2 = 9^2 = 81$

$AB^2 = (\sqrt{130})^2 = 130$

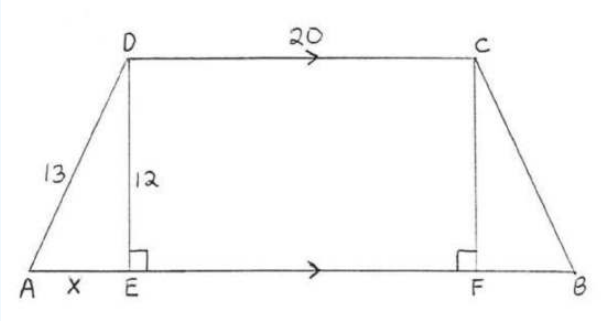
$49 + 81 = 130$.

so by Theorem 6.10.2, $\triangle ABC$ is a right triangle.

Answer: yes.

Example 6.10.6

Find x and AB:



Solution

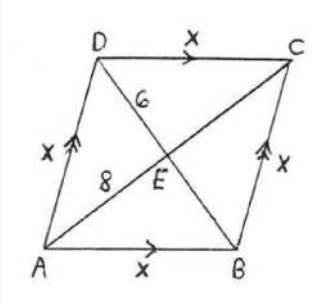
$$\begin{aligned} x^2 + 12^2 &= 13^2 \\ x^2 + 144 &= 169 \\ x^2 &= 169 - 144 \\ x^2 &= 25 \\ x &= 5 \end{aligned}$$

$CDEF$ is a rectangle so $EF = CD = 20$ and $CF = DE = 12$. Therefore $FB = 5$ and $AB = AE + EF + FB = 5 + 20 + 5 = 30$.

Answer: $x = 5$, $AB = 30$.

Example 6.10.7

Find x, AC and BD:



Solution

$ABCD$ is a rhombus. The diagonals of a rhombus are perpendicular and bisect each other.

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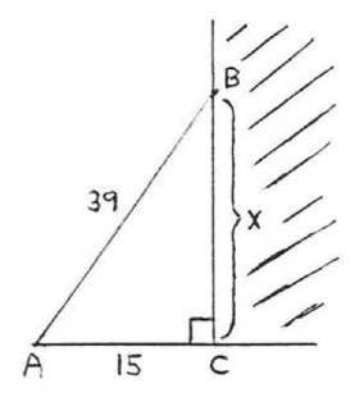
$$\begin{aligned} 6^2+8^2 &= x^2 \\ 36+64 &= x^2 \\ 100 &= x^2 \\ 10 &= x \end{aligned}$$

$AC = 8+8 = 16, BD = 6+6 = 12.$

Answer: $x = 10, AC = 16, BD = 12.$

Example 6.10.8

A ladder 39 feet long leans against a building, How far up the side of the building does the ladder reach if the foot of the ladder is 15 feet from the building?



Solution

$$\begin{aligned} \text{leg}^2+\text{leg}^2 &= \text{hyp}^2 \\ x^2+15^2 &= 39^2 \\ x^2+225 &= 1521 \\ x^2 &= 1521-225 \\ x^2 &= 1296 \\ x &= \sqrt{1296} = 36 \end{aligned}$$

Answer: 36 feet.

Historical Note

Pythagoras (c. 582 - 507 B.C.) was not the first to discover the theorem which bears his name. It was known long before his time by the Chinese, the Babylonians, and perhaps also the Egyptians and the Hindus, According to tradition, Pythagoras was the first to give a nroof of the theorem, His proof probably made use of areas, like the one suggested. In Figure **6.10.5** below, (each square contains four congruent right triangles with sides of lengths a, b, and c, In addition the square on the left contains a square with side a and a square with side b while the one on the right contains a square with side c.)

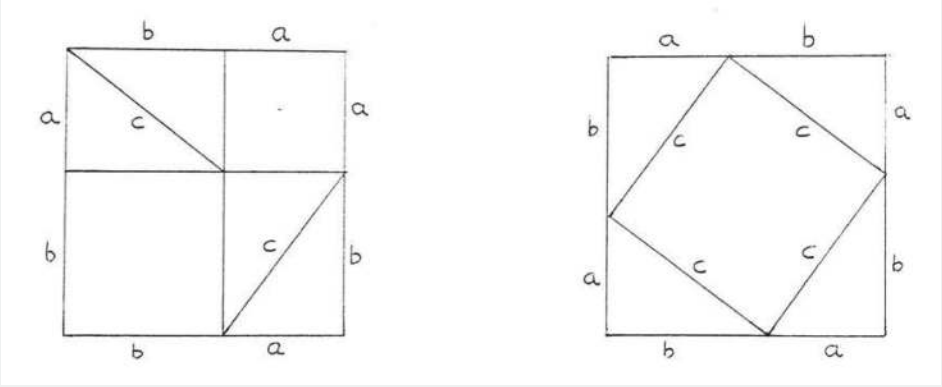


Figure 6.10.5: Pythagoras may have proved $a^2 + b^2 = c^2$ in this way.

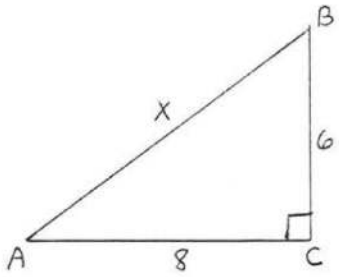
Since the time of Pythagoras, at least several hundred different proofs of the Pythagorean Theorem have been proposed, Pythagoras was the founder of the Pythagorean school, a secret religious society devoted to the study of philosophy, mathematics, and science. Its membership was a select group, which tended to keep the discoveries and practices of the society secret from outsiders. The Pythagoreans believed that numbers were the ultimate components of the universe and that all physical relationships could be expressed with whole numbers, This belief was prompted in part by their discovery that the notes of the musical scale were related by numerical ratios. The Pythagoreans made important contributions to medicine, physics, and astronomy, In geometry, they are credited with the angle s

um theorem for triangles, the properties of parallel lines, and the theory of similar triangles and proportions.

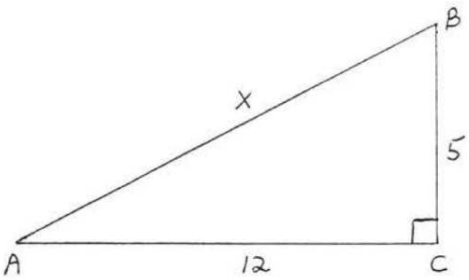
Problems

1 - 10. Find x. Leave answers in simplest radical form.

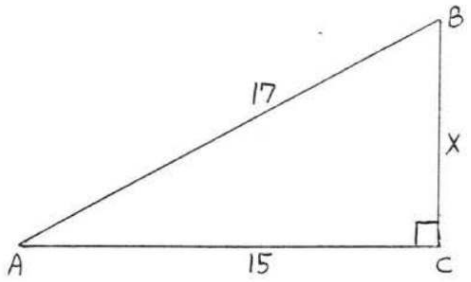
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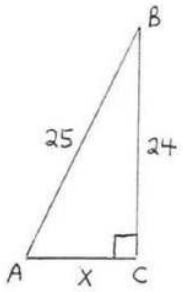
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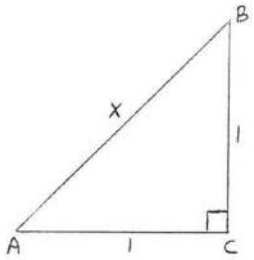
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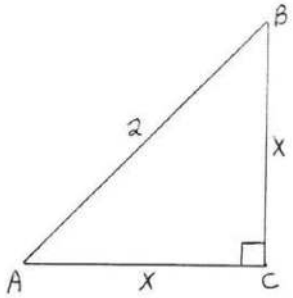
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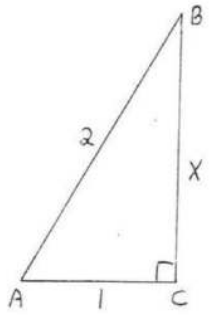
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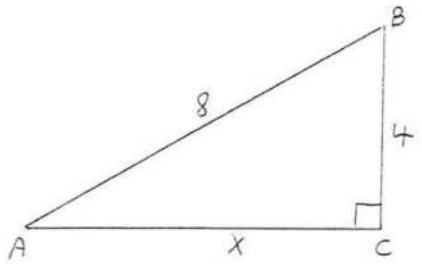
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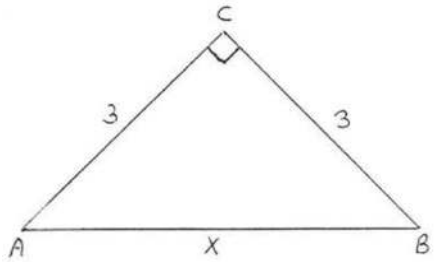
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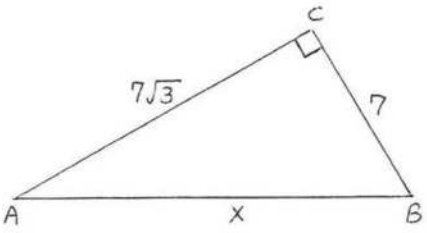
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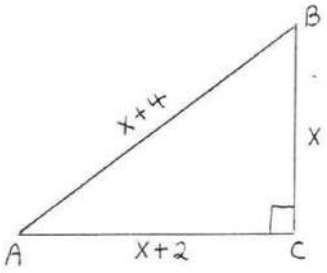


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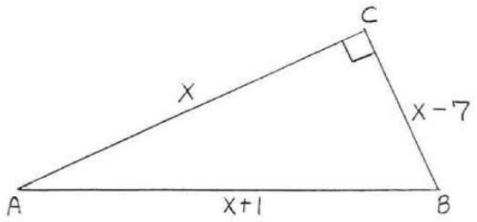


11 - 14. Find x and all sides of the triangle:

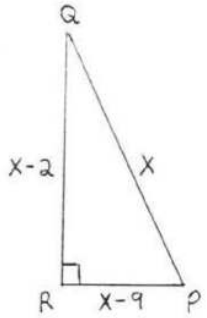
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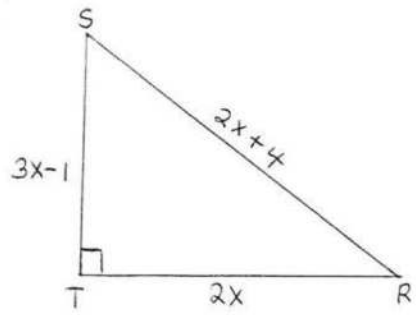
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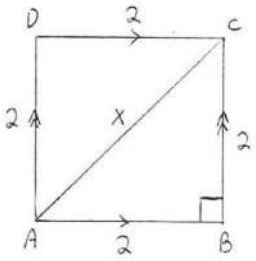


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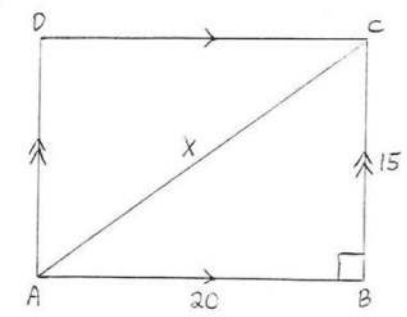


15 - 16. Find x:

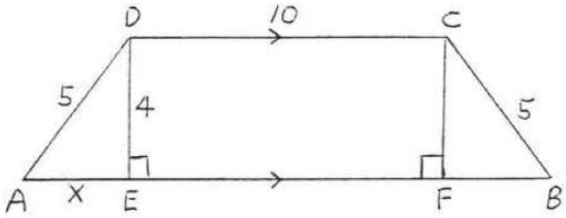
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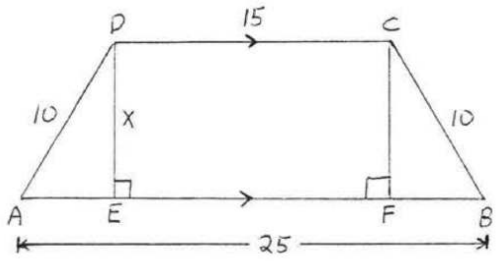
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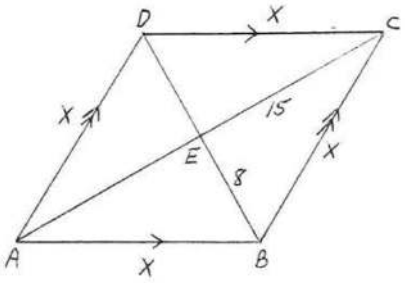
17. Find x and AB.



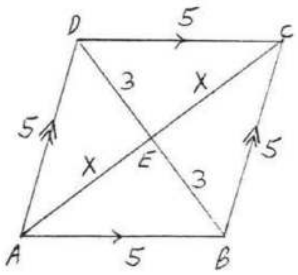
18. Find x:



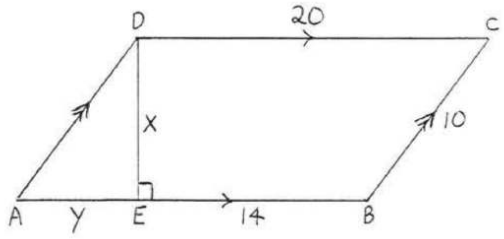
19. Find x, AC and BD:



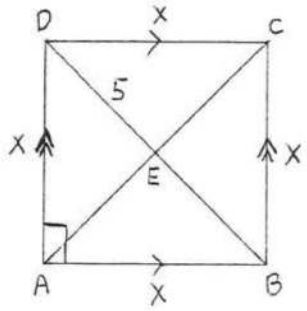
20. Find x, AC and BD:



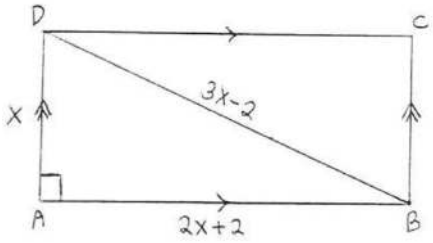
21. Find x and y:



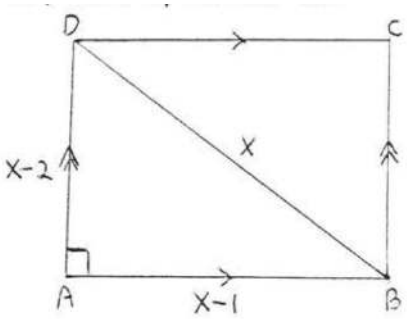
22. Find x, AC and BD:



23. Find x, AB and BD:

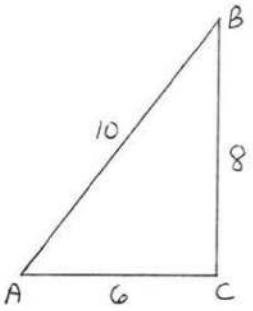


24. Find x, AB and AD:

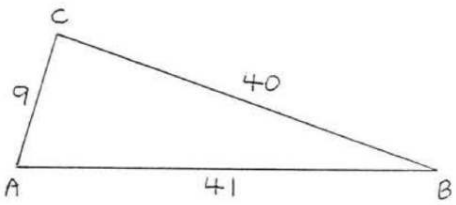


25 - 30. Is $\triangle ABC$ a right triangle?

25.

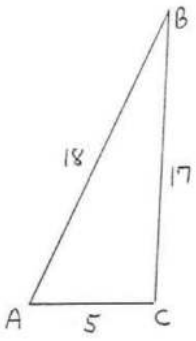


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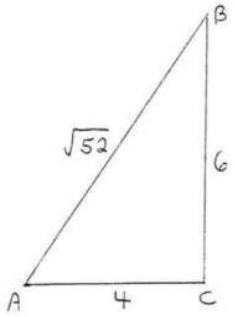


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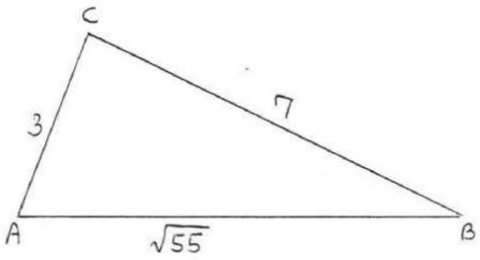
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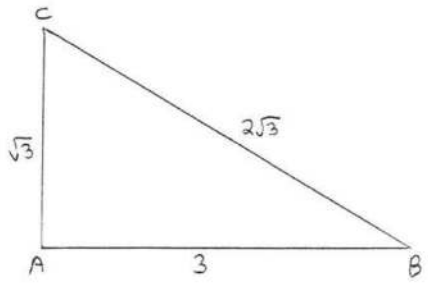
28.



29.



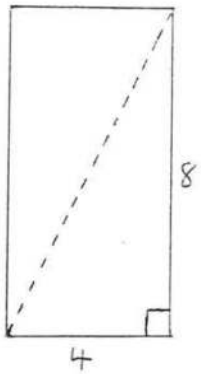
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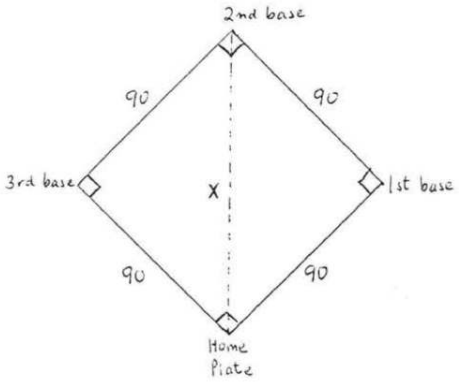
31. A ladder 25 feet long leans against a building, How far up the side of the building does the ladder reach if the foot of the ladder is 7 feet from the building?

32. A man travels 24 miles east and then 10 miles north. At the end of his journey how far is he from his starting point?

33. Can a table 9 feet wide (with its legs folded) fit through a rectangular doorway 4 feet by 8 feet?



34. A baseball diamond is a square 90 feet on each side, Find the distance from home plate to second base (leave answer in simplest radical form).



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CHAPTER OVERVIEW

7: Graph Theory

7.1: Introduction
7.2: Graphs
7.3: Shortest Path
7.4: Euler Circuits and the Chinese Postman Problem
7.5: Eulerization and the Chinese Postman Problem
7.6: Hamiltonian Circuits and the Traveling Salesman Problem
7.7: Spanning Trees
7.8: Exercise

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7.1: Introduction

In the modern world, planning efficient routes is essential for business and industry, with applications as varied as product distribution, laying new fiber optic lines for broadband internet, and suggesting new friends within social network websites like Facebook.

This field of mathematics started nearly 300 years ago as a look into a mathematical puzzle (we'll look at it in a bit). The field has exploded in importance in the last century, both because of the growing complexity of business in a global economy and because of the computational power that computers have provided us.

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7.2: Graphs

Drawing Graphs

Example 1

Here is a portion of a housing development from Missoula, Montana[1]. As part of her job, the development's lawn inspector has to walk down every street in the development making sure homeowners' landscaping conforms to the community requirements.

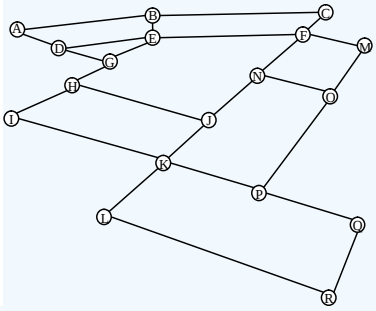


Solution

Naturally, she wants to minimize the amount of walking she has to do. Is it possible for her to walk down every street in this development without having to do any backtracking? While you might be able to answer that question just by looking at the picture for a while, it would be ideal to be able to answer the question for any picture regardless of its complexity.

To do that, we first need to simplify the picture into a form that is easier to work with. We can do that by drawing a simple line for each street. Where streets intersect, we will place a dot.



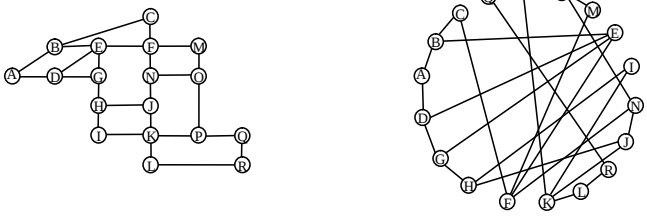


This type of simplified picture is called a **graph**.

Graphs, Vertices, and Edges

A **graph** consists of a set of dots, called **vertices**, and a set of **edges** connecting pairs of vertices.

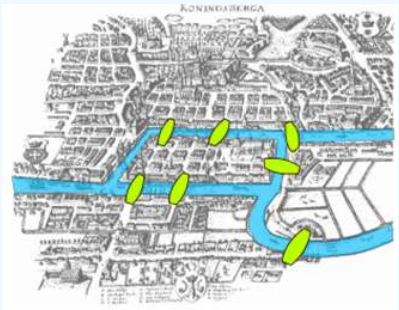
While we drew our original graph to correspond with the picture we had, there is nothing particularly important about the layout when we analyze a graph. Both of the graphs below are equivalent to the one drawn above since they show the same edge connections between the same vertices as the original graph.



You probably already noticed that we are using the term *graph* differently than you may have used the term in the past to describe the graph of a mathematical function.

Example 2

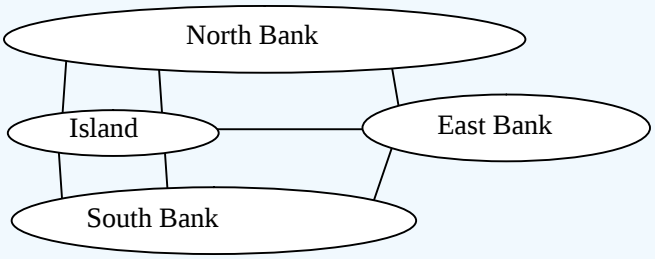
Back in the 18th century in the Prussian city of Königsberg, a river ran through the city and seven bridges crossed the forks of the river. The river and the bridges are highlighted in the picture to the right[2].



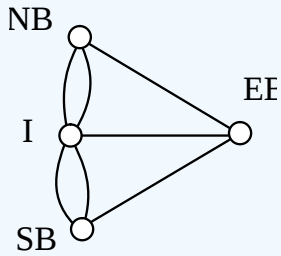
As a weekend amusement, townsfolk would see if they could find a route that would take them across every bridge once and return them to where they started.

Solution

Leonard Euler (pronounced OY-lur), one of the most prolific mathematicians ever, looked at this problem in 1735, laying the foundation for graph theory as a field in mathematics. To analyze this problem, Euler introduced edges representing the bridges:



Since the size of each land mass it is not relevant to the question of bridge crossings, each can be shrunk down to a vertex representing the location:



Notice that in this graph there are *two* edges connecting the north bank and island, corresponding to the two bridges in the original drawing. Depending upon the interpretation of edges and vertices appropriate to a scenario, it is entirely possible and reasonable to have more than one edge connecting two vertices.

While we haven't answered the actual question yet of whether or not there is a route which crosses every bridge once and returns to the starting location, the graph provides the foundation for exploring this question.

Definitions

While we loosely defined some terminology earlier, we now will try to be more specific.

Vertex

A vertex is a dot in the graph that could represent an intersection of streets, a land mass, or a general location, like “work” or “school”. Vertices are often connected by edges. Note that vertices only occur when a dot is explicitly placed, not whenever two edges cross. Imagine a freeway overpass – the freeway and side street cross, but it is not possible to change from the side street to the freeway at that point, so there is no intersection and no vertex would be placed.

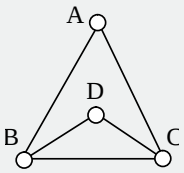
Edges

Edges connect pairs of vertices. An edge can represent a physical connection between locations, like a street, or simply that a route connecting the two locations exists, like an airline flight.

Adjacency Table

An alternate way to represent a graph is with an **adjacency table**. This lists the vertices on the row and column headers of a table, and has a value in each cell if the vertices are connected with an edge. Later the value might represent something like the length or cost of the edge, but for now we will just put a 1 where there is an edge.

For example, this graph



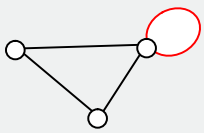
could be represented with the adjacency table

Adjacency table

	A	B	C	D
A		1	1	
B	1		1	1
C	1	1		1
D		1	1	

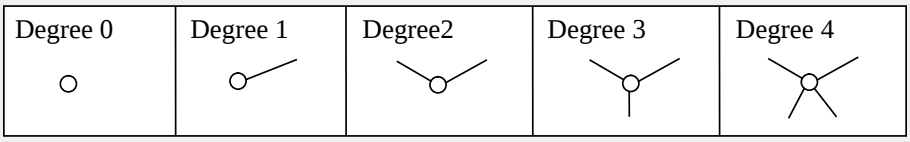
Loop

A loop is a special type of edge that connects a vertex to itself. Loops are not used much in street network graphs.



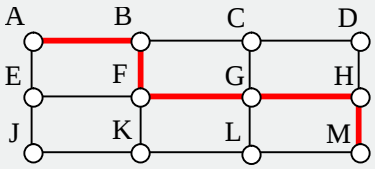
Degree of a vertex

The degree of a vertex is the number of edges meeting at that vertex. It is possible for a vertex to have a degree of zero or larger.



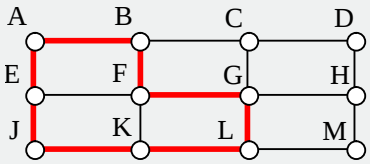
Path

A path is a sequence of vertices using the edges. Usually we are interested in a path between two vertices. For example, a path from vertex A to vertex M is shown below. It is one of many possible paths in this graph.



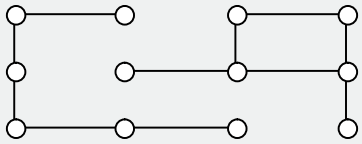
Circuit

A circuit is a path that begins and ends at the same vertex. A circuit starting and ending at vertex A is shown below.



Connected

A graph is connected if there is a path from any vertex to any other vertex. Every graph drawn so far has been connected. The graph below is **disconnected**; there is no way to get from the vertices on the left to the vertices on the right.



Weights

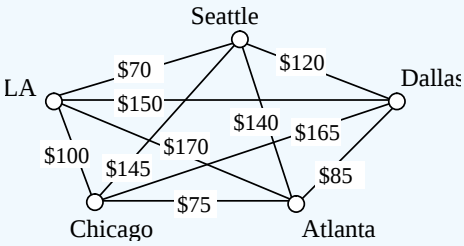
Depending upon the problem being solved, sometimes weights are assigned to the edges. The weights could represent the distance between two locations, the travel time, or the travel cost. It is important to note that the distance between vertices in a graph does not necessarily correspond to the weight of an edge.

Try it Now 1

The graph below shows 5 cities. The weights on the edges represent the airfare for a one-way flight between the cities.

a. How many vertices and edges does the graph have?
b. Is the graph connected?
c. What is the degree of the vertex representing LA?
d. If you fly from Seattle to Dallas to Atlanta, is that a path or a circuit?

e. If you fly from LA to Chicago to Dallas to LA, is that a path or a circuit?



▼ Adjacency Table

Adjacency table with cities as vertices and edges representing flight cost

	Seattle	Dallas	Atlanta	Chicago	LA
Seattle	*	$120	$140	$145	$70
Dallas	$120	*	$85	$165	$150
Atlanta	$140	$85	*	$75	$170
Chicago	$145	$165	$75	*	$100
LA	$70	$150	$170	$100	*

Answer

a. 5 vertices, 10 edges

b. Yes, it is connected

c. The vertex is degree 4

d. A path

e. A circuit

[1] Sam Beebe. http://www.flickr.com/photos/sbeebe/2850476641/, CC-BY

[2] Bogdan Giuşcă. http://en.Wikipedia.org/wiki/File:Ko...rg_bridges.png

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7.3: Shortest Path

When you visit a website like Google Maps or use your Smartphone to ask for directions from home to your Aunt's house in Pasadena, you are usually looking for a shortest path between the two locations. These computer applications use representations of the street maps as graphs, with estimated driving times as edge weights.

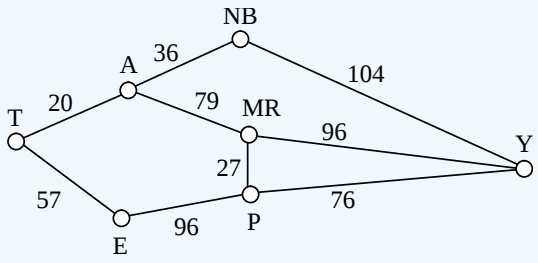
While often it is possible to find a shortest path on a small graph by guess-and-check, our goal in this chapter is to develop methods to solve complex problems in a systematic way by following **algorithms**. An algorithm is a step-by-step procedure for solving a problem. Dijkstra's (pronounced dike-stra) algorithm will find the shortest path between two vertices.

Dijkstra's Algorithm

1. Mark the ending vertex with a distance of zero. Designate this vertex as current.
2. Find all vertices leading to the current vertex. Calculate their distances to the end. Since we already know the distance the current vertex is from the end, this will just require adding the most recent edge. Don't record this distance if it is longer than a previously recorded distance.
3. Mark the current vertex as visited. We will never look at this vertex again.
4. Mark the vertex with the smallest distance as current, and repeat from step 2.

Example 3

Suppose you need to travel from Tacoma, WA (vertex T) to Yakima, WA (vertex Y). Looking at a map, it looks like driving through Auburn (A) then Mount Rainier (MR) might be shortest, but it's not totally clear since that road is probably slower than taking the major highway through North Bend (NB). A graph with travel times in minutes is shown below. An alternate route through Eatonville (E) and Packwood (P) is also shown.



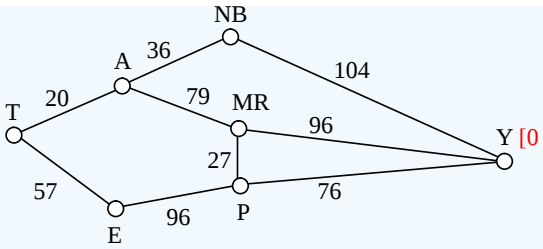
▼ Adjacency Table

Adjacency table with travel time in minutes between cities

	T	A	E	NB	MR	P	Y
T		20	57				
A	20			36	79		
E	57					96	
NB		36					104
MR		79				27	96
P			96		27		76
Y				104	96	76	

Solution

Step 1: Mark the ending vertex with a distance of zero. The distances will be recorded in [brackets] after the vertex name.



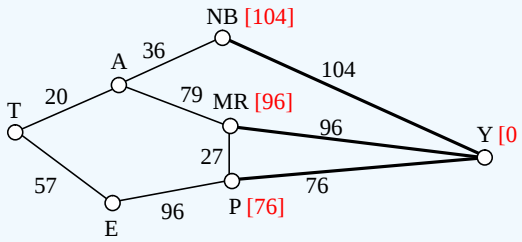
▼ Adjacency Table

In the adjacency table, we will record the distances in the column headers

Adjacency table with travel time in minutes between cities

	T	A	E	NB	MR	P	Y [0]
T		20	57				
A	20			36	79		
E	57					96	
NB		36					104
MR		79				27	96
P			96		27		76
Y				104	96	76	

Step 2: For each vertex leading to Y, we calculate the distance to the end. For example, NB is a distance of 104 from the end, and MR is 96 from the end. Remember that distances in this case refer to the travel time in minutes.



▼ Adjacency Table

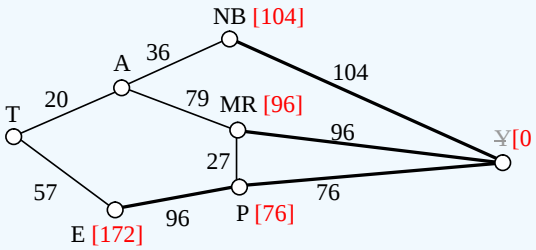
Adjacency table with travel time in minutes between cities

	T	A	E	NB [104]	MR [96]	P [76]	Y [0]
T		20	57				
A	20			36	79		
E	57					96	
NB		36					104
MR		79				27	96
P			96		27		76
Y				104	96	76	

Step 3 & 4: We mark Y as visited, and mark the vertex with the smallest recorded distance as current. At this point, P will be designated current. Back to step 2.

Step 2 (#2): For each vertex leading to P (and not leading to a visited vertex) we find the distance from the end. Since E is 96 minutes from P, and we've already calculated P is 76 minutes from Y, we can compute that E is 96+76 = 172 minutes from Y.

If we make the same computation for MR, we'd calculate 76+27 = 103. Since this is larger than the previously recorded distance from Y to MR, we will *not* replace it.



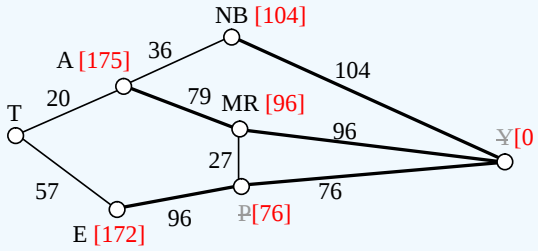
▼ Adjacency Table

Adjacency table with travel time in minutes between cities

	T	A	E [172]	NB [104]	MR [96]	P [76] current	Y [0] visited
T		20	57				
A	20			36	79		
E	57					96	
NB		36					104
MR		79				27	96
P			96		27		76
Y				104	96	76	

Step 3 & 4 (#2): We mark P as visited, and designate the vertex with the smallest recorded distance as current: MR. Back to step 2.

Step 2 (#3): For each vertex leading to MR (and not leading to a visited vertex) we find the distance to the end. The only vertex to be considered is A, since we've already visited Y and P. Adding MR's distance 96 to the length from A to MR gives the distance 96+79 = 175 minutes from A to Y.



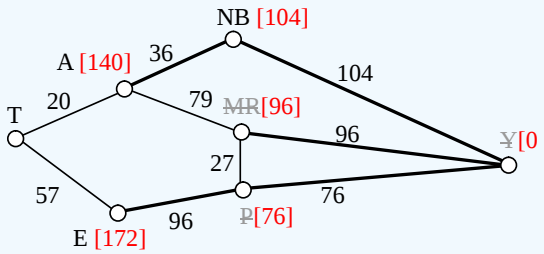
▼ Adjacency Table

Adjacency table with travel time in minutes between cities

	T	A [175]	E [172]	NB [104]	MR [96] current	P [76] visited	Y [0] visited
T		20	57				
A	20			36	79		
E	57					96	
NB		36					104
MR		79				27	96
P			96		27		76
Y				104	96	76	

Step 3 & 4 (#3): We mark MR as visited, and designate the vertex with smallest recorded distance as current: NB. Back to step 2.

Step 2 (#4): For each vertex leading to NB, we find the distance to the end. We know the shortest distance from NB to Y is 104 and the distance from A to NB is 36, so the distance from A to Y through NB is 104+36 = 140. Since this distance *is* shorter than the previously calculated distance from Y to A through MR, we replace it.



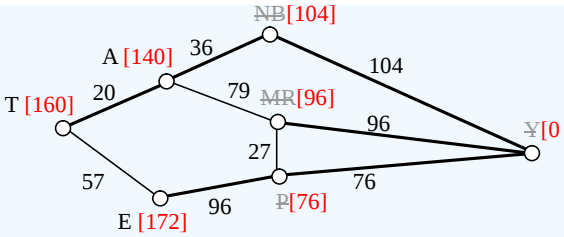
▼ Adjacency Table

Adjacency table with travel time in minutes between cities

	T	A [140]	E [172]	NB [104] current	MR [96] visited	P [76] visited	Y [0] visited
T		20	57				
A	20			36	79		
E	57					96	
NB		36					104
MR		79				27	96
P			96		27		76
Y				104	96	76	

Step 3 & 4 (#4): We mark NB as visited, and designate A as current, since it now has the shortest distance.

Step 2 (#5): T is the only non-visited vertex leading to A, so we calculate the distance from T to Y through A: 20+140 = 160 minutes.



▼ Adjacency Table

Adjacency table with travel time in minutes between cities

	T [160]	A [140] current	E [172]	NB [104] visited	MR [96] visited	P [76] visited	Y [0] visited
T		20	57				
A	20			36	79		
E	57					96	
NB		36					104
MR		79				27	96
P			96		27		76
Y				104	96	76	

Step 3 & 4 (#5): We mark A as visited, and designate E as current.

Step 2 (#6): The only non-visited vertex leading to E is T. Calculating the distance from T to Y through E, we compute $172 + 57 = 229$ minutes. Since this is longer than the existing marked time, we do not replace it.

Step 3 (#6): We mark E as visited. Since all vertices have been visited, we are done.

From this, we know that the shortest path from Tacoma to Yakima will take 160 minutes. Tracking which sequence of edges yielded 160 minutes, we see the shortest path is T-A-NB-Y.

Dijkstra's algorithm is an **optimal algorithm**, meaning that it always produces the actual shortest path, not just a path that is pretty short, provided one exists. This algorithm is also **efficient**, meaning that it can be implemented in a reasonable amount of time. Dijkstra's algorithm takes around V^2 calculations, where V is the number of vertices in a graph[1]. A graph with 100 vertices would take around 10,000 calculations. While that would be a lot to do by hand, it is not a lot for computer to handle. It is because of this efficiency that your car's GPS unit can compute driving directions in only a few seconds.

In contrast, an **inefficient** algorithm might try to list all possible paths then compute the length of each path. Trying to list all possible paths could easily take 10^{25} calculations to compute the shortest path with only 25 vertices; that's a 1 with 25 zeros after it! To put that in perspective, the fastest computer in the world would still spend over 1000 years analyzing all those paths.

✔ Example 4

A shipping company needs to route a package from Washington, D.C. to San Diego, CA. To minimize costs, the package will first be sent to their processing center in Baltimore, MD then sent as part of mass shipments between their various processing centers, ending up in their processing center in Bakersfield, CA. From there it will be delivered in a small truck to San Diego.

The travel times, in hours, between their processing centers are shown in the table below. Three hours has been added to each travel time for processing.

Travel times adjacency table

	Baltimore	Denver	Dallas	Chicago	Atlanta	Bakersfield
Baltimore	*			15	14	
Denver		*		18	24	19
Dallas			*	18	15	25
Chicago	15	18	18	*	14	
Atlanta	14	24	15	14	*	
Bakersfield		19	25			*

Find the shortest path from Baltimore to Bakersfield.

Solution

While we could draw a graph, we can also work directly from the table.

Step 1: The ending vertex, Bakersfield, is marked as current.

Step 2: All cities connected to Bakersfield, in this case Denver and Dallas, have their distances calculated; we'll mark those distances in the column headers.

Step 3 & 4: Mark Bakersfield as visited. Here, we are labeling it as such in the column header and putting a line through the corresponding row and column of the table as a quick visual indicator. We mark Denver as current, labeled as such in the column header, since it is the vertex with the shortest distance.

Travel times adjacency table

	Baltimore	**Denver** [19] current	Dallas [25]	Chicago	Atlanta	~~Bakersfield [0] visited~~
Baltimore	*			15	14	
Denver		*		18	24	~~19~~
Dallas			*	18	15	~~25~~
Chicago	15	18	18	*	14	
Atlanta	14	24	15	14	*	
~~Bakersfield~~		~~19~~	~~25~~			~~*~~

Step 2 (#2): For cities connected to Denver, calculate distance to the end. For example, Chicago is 18 hours from Denver, and Denver is 19 hours from the end, the distance for Chicago to the end is 18+19 = 37 (Chicago to Denver to Bakersfield). Atlanta is 24 hours from Denver, so the distance to the end is 24+19 = 43 (Atlanta to Denver to Bakersfield).

Step 3 & 4 (#2): We mark Denver as visited and mark Dallas as current.

Travel times adjacency table

	Baltimore	~~Denver~~ ~~[19] visited~~	**Dallas** [25] current	Chicago [37]	Atlanta [43]	~~Bakersfield~~ ~~[0] visited~~
Baltimore	*			15	14	
~~Denver~~		~~*~~		~~18~~	~~24~~	~~19~~
Dallas			*	18	15	~~25~~
Chicago	15	~~18~~	18	*	14	
Atlanta	14	~~24~~	15	14	*	
~~Bakersfield~~		~~19~~	~~25~~			~~*~~

Step 2 (#3): For cities connected to Dallas, calculate the distance to the end. For Chicago, the distance from Chicago to Dallas is 18 and from Dallas to the end is 25, so the distance from Chicago to the end through Dallas would be 18+25 = 43. Since this is longer than the currently marked distance for Chicago, we do not replace it. For Atlanta, we calculate 15+25 = 40. Since this is shorter than the currently marked distance for Atlanta, we replace the existing distance.

Step 3 & 4 (#3): We mark Dallas as visited, and mark Chicago as current.

Travel times adjacency table

	Baltimore	~~Denver~~ ~~[19] visited~~	~~Dallas~~ ~~[25] visited~~	**Chicago** [37] current	Atlanta [40]	~~Bakersfield~~ ~~[0] visited~~
Baltimore	*			15	14	
~~Denver~~		~~*~~		~~18~~	~~24~~	~~19~~
~~Dallas~~			~~*~~	~~18~~	~~15~~	~~25~~
Chicago	15	~~18~~	~~18~~	*	14	
Atlanta	14	~~24~~	~~15~~	14	*	
~~Bakersfield~~		~~19~~	~~25~~			~~*~~

Step 2 (#4): Baltimore and Atlanta are the only non-visited cities connected to Chicago. For Baltimore, we calculate 15+37 = 52 and mark that distance. For Atlanta, we calculate 14+37 = 51. Since this is longer than the existing distance of 40 for Atlanta, we do not replace that distance.

Step 3 & 4 (#4): Mark Chicago as visited and Atlanta as current.

Travel times adjacency table

	Baltimore [52]	~~Denver~~ ~~[19] visited~~	~~Dallas~~ ~~[25] visited~~	~~Chicago~~ ~~[37] visited~~	**Atlanta** [40] current	~~Bakersfield~~ ~~[0] visited~~
Baltimore	*			~~15~~	14	
~~Denver~~		~~*~~		~~18~~	~~24~~	~~19~~
~~Dallas~~			~~*~~	~~18~~	~~15~~	~~25~~
~~Chicago~~	~~15~~	~~18~~	~~18~~	~~*~~	~~14~~	
Atlanta	14	~~24~~	~~15~~	~~14~~	*	
~~Bakersfield~~		~~19~~	~~25~~			~~*~~

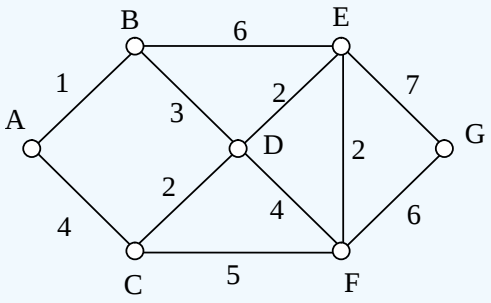
Step 2 (#5): The distance from Atlanta to Baltimore is 14. Adding that to the distance already calculated for Atlanta gives a total distance of 14+40 = 54 hours from Baltimore to Bakersfield through Atlanta. Since this is larger than the currently calculated distance, we do not replace the distance for Baltimore.

Step 3 & 4 (#5): We mark Atlanta as visited. All cities have been visited and we are done.

The shortest route from Baltimore to Bakersfield will take 52 hours, and will route through Chicago and Denver.

Try it Now 2

Find the shortest path between vertices A and G in the graph below.



▼ Adjacency Table

Adjacency table with edge weights

	A	B	C	D	E	F	G
A	*	1	4				
B	1	*		4	6		
C	4		*	2		5	
D		4	2	*	2	4	
E		6		2	*	2	7
F			5	4	2	*	6
G					7	6	*

Answer

The shortest path is ABDEG, with length 13

[1] It can be made to run faster through various optimizations to the implementation.

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7.4: Euler Circuits and the Chinese Postman Problem

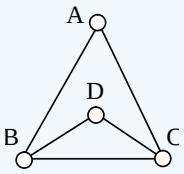
In the first section, we created a graph of the Königsberg bridges and asked whether it was possible to walk across every bridge once. Because Euler first studied this question, these types of paths are named after him.

Euler Path

An **Euler path** is a path that uses every edge in a graph with no repeats. Being a path, it does not have to return to the starting vertex.

Example 5

In the graph shown below, there are several Euler paths.



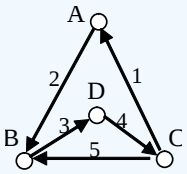
▼ Adjacency Table

Adjacency table

	A	B	C	D
A	*	1	1	
B	1	*	1	1
C	1	1	*	1
D		1	1	*

Solution

One such path is CABDCB. The path is shown in arrows to the right, with the order of edges numbered.

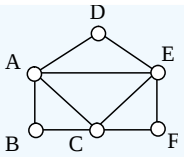


Euler Circuit

An **Euler circuit** is a circuit that uses every edge in a graph with no repeats. Being a circuit, it must start and end at the same vertex.

Example 6

The graph below has several possible Euler circuits.



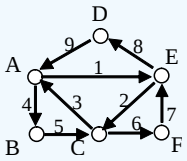
▼ Adjacency Table

Adjacency table

	A	B	C	D	E	F
A	*	1	1	1	1	
B	1	*	1			
C	1	1	*		1	1
D	1			*	1	
E	1		1	1	*	1
F			1		1	*

Solution

Here's a couple, starting and ending at vertex A: ADEACEFCBA and AECABCFEDA. The second is shown in arrows.



Look back at the example used for Euler paths – does that graph have an Euler circuit? A few tries will tell you no; that graph does not have an Euler circuit. When we were working with shortest paths, we were interested in the optimal path. With Euler paths and circuits, we're primarily interested in whether an Euler path or circuit *exists*.

Why do we care if an Euler circuit exists? Think back to our housing development lawn inspector from the beginning of the chapter. The lawn inspector is interested in walking as little as possible. The ideal situation would be a circuit that covers every street with no repeats. That's an Euler circuit! Luckily, Euler solved the question of whether or not an Euler path or circuit will exist.

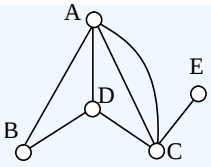
Euler's Path and Circuit Theorems

A graph will contain an Euler path if it contains at most two vertices of odd degree.

A graph will contain an Euler circuit if all vertices have even degree

Example 7

In the graph below, vertices A and C have degree 4, since there are 4 edges leading into each vertex. B is degree 2, D is degree 3, and E is degree 1. This graph contains two vertices with odd degree (D and E) and three vertices with even degree (A, B, and C), so Euler's theorems tell us this graph has an Euler path, but not an Euler circuit.



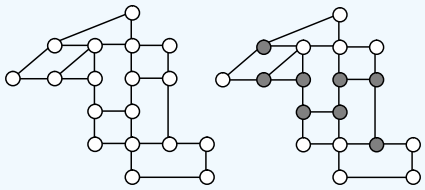
▼ Adjacency Table

Adjacency table, with values showing the number of edges connecting each pair

	A	B	C	D	E
A	*	1	2	1	
B	1	*		1	
C	2		*	1	1
D	1	1	1	*	
E			1		*

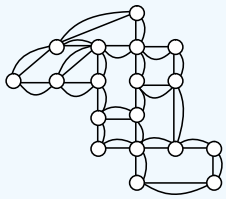
Example 8

Is there an Euler circuit on the housing development lawn inspector graph we created earlier in the chapter? All the highlighted vertices have odd degree. Since there are more than two vertices with odd degree, there are no Euler paths or Euler circuits on this graph. Unfortunately our lawn inspector will need to do some backtracking.



Example 9

When it snows in the same housing development, the snowplow has to plow both sides of every street. For simplicity, we'll assume the plow is out early enough that it can ignore traffic laws and drive down either side of the street in either direction. This can be visualized in the graph by drawing two edges for each street, representing the two sides of the street.



Notice that every vertex in this graph has even degree, so this graph does have an Euler circuit.

Now we know how to determine if a graph has an Euler circuit, but if it does, how do we find one? While it usually is possible to find an Euler circuit just by pulling out your pencil and trying to find one, the more formal method is **Fleury's algorithm.**

Fleury's Algorithm

1. Start at any vertex if finding an Euler circuit. If finding an Euler path, start at one of the two vertices with odd degree.
2. Choose any edge leaving your current vertex, provided deleting that edge will not separate the graph into two disconnected sets of edges.
3. Add that edge to your circuit, and delete it from the graph.
4. Continue until you're done.

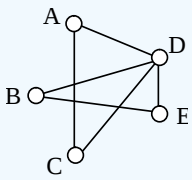
Example 10

Let's find an Euler Circuit on this graph using Fleury's algorithm, starting at vertex A.

Solution

Step 1:

Original Graph. Choosing edge AD.



▼ Adjacency Table

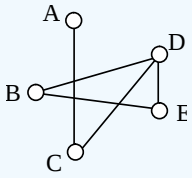
Adjacency table

	A	B	C	D	E
A	*		1	1	
B		*		1	1
C	1		*	1	
D	1	1	1	*	1
E		1		1	*

Circuit so far: AD

Step 2:

AD deleted. D is current. Can't choose DC since that would disconnect graph. Choosing DE



▼ Adjacency Table

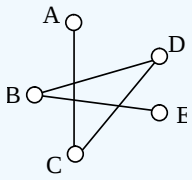
Adjacency table

	A	B	C	D	E
A	*		1	deleted	
B		*		1	1
C	1		*	1	
D	deleted	1	1	*	1
E		1		1	*

Circuit so far: ADE

Step 3:

E is current. From here, there is only one option, so the rest of the circuit is determined.



▼ Adjacency Table

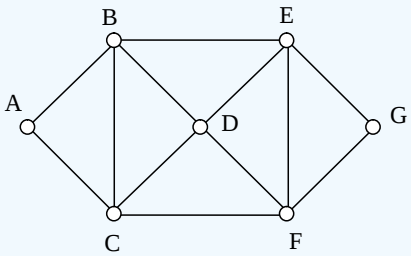
Adjacency table

	A	B	C	D	E
A	*		1		
B		*		1	1
C	1		*	1	
D		1	1	*	deleted
E		1		deleted	*

Circuit: ADEBDCA

Try it Now 3

Does the graph below have an Euler Circuit? If so, find one.



▼ Adjacency Table

Adjacency table

	A	B	C	D	E
A	*		1	1	
B		*		1	1
C	1		*	1	
D	1	1	1	*	1
E		1		1	*

Answer

Yes, all vertices have even degree so this graph has an Euler Circuit. There are several possibilities. One is: ABEGFCDFEDBCA

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7.5: Eulerization and the Chinese Postman Problem

Not every graph has an Euler path or circuit, yet our lawn inspector still needs to do her inspections. Her goal is to minimize the amount of walking she has to do. In order to do that, she will have to duplicate some edges in the graph until an Euler circuit exists.

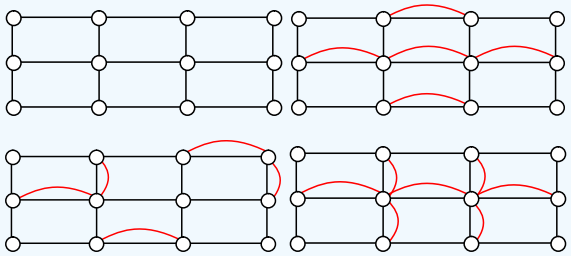
Eulerization

Eulerization is the process of adding edges to a graph to create an Euler circuit on a graph. To eulerize a graph, edges are duplicated to connect pairs of vertices with odd degree. Connecting two odd degree vertices increases the degree of each, giving them both even degree. When two odd degree vertices are not directly connected, we can duplicate all edges in a path connecting the two.

Note that we can only duplicate edges, not create edges where there wasn't one before. Duplicating edges would mean walking or driving down a road twice, while creating an edge where there wasn't one before is akin to installing a new road!

Example 11

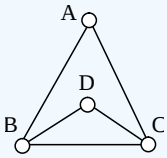
For the rectangular graph shown, three possible eulerizations are shown. Notice in each of these cases the vertices that started with odd degrees have even degrees after eulerization, allowing for an Euler circuit.



In the example above, you'll notice that the last eulerization required duplicating seven edges, while the first two only required duplicating five edges. If we were eulerizing the graph to find a walking path, we would want the eulerization with minimal duplications. If the edges had weights representing distances or costs, then we would want to select the eulerization with the minimal total added weight.

Try it Now 4

Eulerize the graph shown, then find an Euler circuit on the eulerized graph.



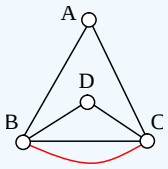
▼ Adjacency Table

Adjacency Table

	A	B	C	D
A	*	1	1	
B	1	*	1	1
C	1	1	*	1
D		1	1	*

Answer

This graph can be eulerized by duplicating the edge BC, as shown. One possible Euler circuit on the eulerized graph is ACDBCBA



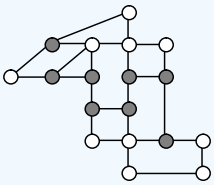
▼ Adjacency Table

Adjacency Table

	A	B	C	D
A	*	1	1	
B	1	*	2	1
C	1	2	*	1
D		1	1	*

Example 12

Looking again at the graph for our lawn inspector from Examples 1 and 8, the vertices with odd degree are shown highlighted. With eight vertices, we will always have to duplicate at least four edges.



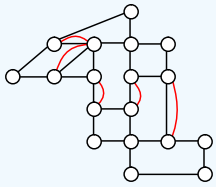
▼ Adjacency Table

Adjacency Table, odd vertices indicated in row headers

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R
A		1		1														
B odd	1		1		1													
C		1				1												
D odd	1				1		1											
E		1		1		1	1											
F			1		1								1	1				
G odd				1	1			1										
H odd							1		1	1								
I								1			1							
J odd								1			1			1				
K									1	1		1				1		
L											1							1
M						1									1			
N odd						1				1					1			
O odd													1	1		1		
P odd											1				1		1	
Q																1		1
R												1					1	

Solution

In this case, we need to duplicate five edges since two odd degree vertices are not directly connected. Without weights we can't be certain this is the eulerization that minimizes walking distance, but it looks pretty good.



▼ Adjacency Table

Adjacency Table after Eulerization

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R
A		1		1														
B	1		1		2													
C		1				1												
D	1				2		1											
E		2		2		1	1											
F			1		1								1	1				
G				1	1			2										
H							2		1	1								
I								1			1							
J								1			1			2				
K									1	1		1				1		
L											1							1
M						1									1			
N						1				2					1			
O													1	1		2		
P											1				2		1	
Q																1		1
R												1					1	

The problem of finding the optimal eulerization is called the Chinese Postman Problem, a name given by an American in honor of the Chinese mathematician Mei-Ko Kwan who first studied the problem in 1962 while trying to find optimal delivery routes for postal carriers. This problem is important in determining efficient routes for garbage trucks, school buses, parking meter checkers, street sweepers, and more.

Unfortunately, algorithms to solve this problem are fairly complex. Some simpler cases are considered in the exercises.

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7.6: Hamiltonian Circuits and the Traveling Salesman Problem

In the last section, we considered optimizing a walking route for a postal carrier. How is this different than the requirements of a package delivery driver? While the postal carrier needed to walk down every street (edge) to deliver the mail, the package delivery driver instead needs to visit every one of a set of delivery locations. Instead of looking for a circuit that covers every edge once, the package deliverer is interested in a circuit that visits every vertex once.

Hamiltonian Circuits and Paths

A **Hamiltonian circuit** is a circuit that visits every vertex once with no repeats. Being a circuit, it must start and end at the same vertex. A **Hamiltonian path** also visits every vertex once with no repeats, but does not have to start and end at the same vertex.

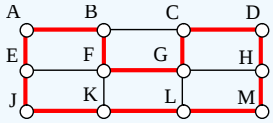
Hamiltonian circuits are named for William Rowan Hamilton who studied them in the 1800's.

Example 13

One Hamiltonian circuit is shown on the graph below. There are several other Hamiltonian circuits possible on this graph. Notice that the circuit only has to visit every vertex once; it does not need to use every edge.

Solution

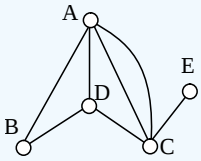
This circuit could be notated by the sequence of vertices visited, starting and ending at the same vertex: ABFGCDHMLKJEA. Notice that the same circuit could be written in reverse order, or starting and ending at a different vertex.



Unlike with Euler circuits, there is no nice theorem that allows us to instantly determine whether or not a Hamiltonian circuit exists for all graphs.[1]

Example 14

Does a Hamiltonian path or circuit exist on the graph below?



▼ Adjacency Table

Adjacency table, with values showing the number of edges connecting each pair

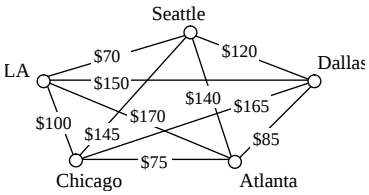
	A	B	C	D	E
A	*	1	2	1	
B	1	*		1	
C	2		*	1	1
D	1	1	1	*	
E			1		*

Solution

We can see that once we travel to vertex E there is no way to leave without returning to C, so there is no possibility of a Hamiltonian circuit. If we start at vertex E we can find several Hamiltonian paths, such as ECDAB and ECABD.

With Hamiltonian circuits, our focus will not be on existence, but on the question of optimization; given a graph where the edges have weights, can we find the optimal Hamiltonian circuit; the one with lowest total weight.

This problem is called the **Traveling salesman problem** (TSP) because the question can be framed like this: Suppose a salesman needs to give sales pitches in four cities. He looks up the airfares between each city, and puts the costs in a graph. In what order should he travel to visit each city once then return home with the lowest cost?



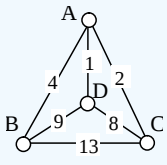
To answer this question of how to find the lowest cost Hamiltonian circuit, we will consider some possible approaches. The first option that might come to mind is to just try all different possible circuits.

Brute Force Algorithm (a.k.a. exhaustive search)

1. List all possible Hamiltonian circuits
2. Find the length of each circuit by adding the edge weights
3. Select the circuit with minimal total weight.

Example 15

Apply the Brute force algorithm to find the minimum cost Hamiltonian circuit on the graph below.



▼ Adjacency Table

Adjacency table cost
edge costs

	A	B	C	D
A	*	4	2	1
B	4	*	13	9
C	2	13	*	8
D	1	9	8	*

Solution

To apply the Brute force algorithm, we list all possible Hamiltonian circuits and calculate their weight:

Circuit	Weight
ABCDA	$4+13+8+1+26$
ABDCA	$4+9+8+2=23$
ACBDA	$2+13+9+1=25$

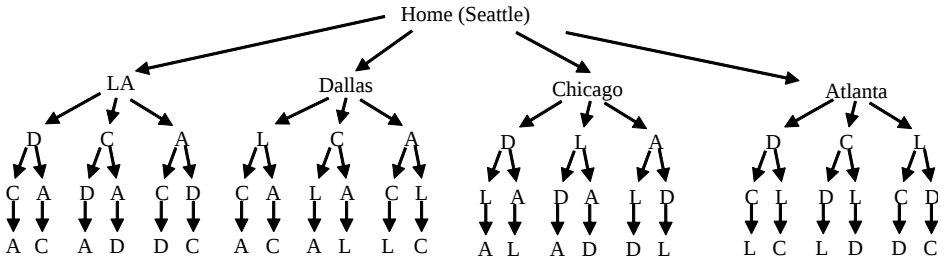
Note: These are the unique circuits on this graph. All other possible circuits are the reverse of the listed ones or start at a different vertex, but result in the same weights.

From this we can see that the second circuit, ABDCA, is the optimal circuit.

The Brute force algorithm is optimal; it will always produce the Hamiltonian circuit with minimum weight. Is it efficient? To answer that question, we need to consider how many Hamiltonian circuits a graph could have. For simplicity, let's look at the worst-case possibility, where every vertex is connected to every other vertex. This is called a **complete graph.**

Suppose we had a complete graph with five vertices like the air travel graph above. From Seattle there are four cities we can visit first. From each of those, there are three choices. From each of those cities, there are two possible cities to visit next. There is then only one choice for the last city before returning home.

This can be shown visually:



Counting the number of routes, we can see there are $4\cdot3\cdot2\cdot1=24$ routes. For six cities there would be $5\cdot4\cdot3\cdot2\cdot1=120$ routes.

Number of Possible Circuits

For n vertices in a complete graph, there will be $(n-1)!=(n-1)(n-2)(n-3)\cdots3\cdot2\cdot1$ routes. Half of these are duplicates in reverse order, so there are $\frac{(n-1)!}{2}$ unique circuits.

The exclamation symbol, !, is read "factorial" and is shorthand for the product shown.

Example 16

How many circuits would a complete graph with 8 vertices have?

Solution

A complete graph with 8 vertices would have $(8-1)! = 7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$ possible Hamiltonian circuits. Half of the circuits are duplicates of other circuits but in reverse order, leaving 2520 unique routes.

While this is a lot, it doesn't seem unreasonably huge. But consider what happens as the number of cities increase:

Citices	Unique Hamiltonian Circuits
9	8 ! / 2 = 20,160
10	9 ! / 2 = 181,440
11	10 ! / 2 = 1,814,400
15	14 ! / 2 = 43,589,145,600
20	19 ! / 2=60,822,550,204,416,000

As you can see the number of circuits is growing extremely quickly. If a computer looked at one billion circuits a second, it would still take almost two years to examine all the possible circuits with only 20 cities! Certainly Brute Force is **not** an efficient algorithm.

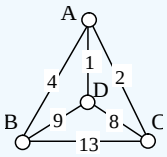
Unfortunately, no one has yet found an efficient *and* optimal algorithm to solve the TSP, and it is very unlikely anyone ever will. Since it is not practical to use brute force to solve the problem, we turn instead to **heuristic algorithms**; efficient algorithms that give approximate solutions. In other words, heuristic algorithms are fast, but may or may not produce the optimal circuit.

Nearest Neighbor Algorithm (NNA)

1. Select a starting point.
2. Move to the nearest unvisited vertex (the edge with smallest weight).
3. Repeat until the circuit is complete.

Example 17

Consider our earlier graph, shown here.



▼ Adjacency Table

Adjacency table cost edge costs

	A	B	C	D
A	*	4	2	1
B	4	*	13	9
C	2	13	*	8
D	1	9	8	*

Starting at vertex A, the nearest neighbor is vertex D with a weight of 1.

From D, the nearest neighbor is C, with a weight of 8.

From C, our only option is to move to vertex B, the only unvisited vertex, with a cost of 13.

From B we return to A with a weight of 4.

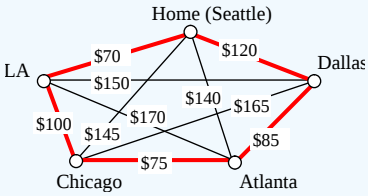
Solution

The resulting circuit is ADCBA with a total weight of $1 + 8 + 13 + 4 = 26$.

We ended up finding the worst circuit in the graph! What happened? Unfortunately, while it is very easy to implement, the NNA is a **greedy** algorithm, meaning it only looks at the immediate decision without considering the consequences in the future. In this case, following the edge AD forced us to use the very expensive edge BC later.

Example 18

Consider again our salesman needing to fly between 5 cities and return home.



▼ Adjacency Table

Solution

- Starting in Seattle, the nearest neighbor (cheapest flight) is to LA, at a cost of $70. From there:
- LA to Chicago: $100
- Chicago to Atlanta: $75
- Atlanta to Dallas: $85
- Dallas to Seattle: $120

Total cost: $450

In this case, nearest neighbor did find the optimal circuit.

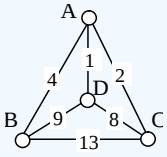
Going back to our first example, how could we improve the outcome? One option would be to redo the nearest neighbor algorithm with a different starting point to see if the result changed. Since nearest neighbor is so fast, doing it several times isn't a big deal.

Repeated Nearest Neighbor Algorithm (RNNA)

1. Do the Nearest Neighbor Algorithm starting at each vertex
2. Choose the circuit produced with minimal total weight

Example 19

We will revisit the graph from Example 17.



▼ Adjacency Table

Adjacency table cost
edge costs

	A	B	C	D
A	*	4	2	1
B	4	*	13	9
C	2	13	*	8
D	1	9	8	*

Starting at vertex A resulted in a circuit with weight 26.

Starting at vertex B, the nearest neighbor circuit is BADCB with a weight of 4+1+8+13 = 26. This is the same circuit we found starting at vertex A. No better.

Starting at vertex C, the nearest neighbor circuit is CADBC with a weight of 2+1+9+13 = 25. Better!

Starting at vertex D, the nearest neighbor circuit is DACBA. Notice that this is actually the same circuit we found starting at C, just written with a different starting vertex.

Solution

The RNNA was able to produce a slightly better circuit with a weight of 25, but still not the optimal circuit in this case. Notice that even though we found the circuit by starting at vertex C, we could still write the circuit starting at A: ADBCA or ACBDA.

Try it Now 5

The table below shows the time, in milliseconds, it takes to send a packet of data between computers on a network. If data needed to be sent in sequence to each computer, then notification needed to come back to the original computer, we would be solving the TSP. The computers are labeled A-F for convenience.

Time in milliseconds to send data between computers

	A	B	C	D	E	F
A	-	44	34	12	40	41
B	44	-	31	43	24	50
C	34	31	-	20	39	27
D	12	43	20	-	11	17
E	40	24	39	11	-	42
F	41	50	27	17	42	-

a. Find the circuit generated by the NNA starting at vertex B.

b. Find the circuit generated by the RNNA.

Answer

At each step, we look for the nearest location we haven't already visited.

From B the nearest computer is E with time 24.

From E, the nearest computer is D with time 11.

From D the nearest is A with time 12.

From A the nearest is C with time 34.

From C, the only computer we haven't visited is F with time 27

From F, we return back to B with time 50.

The NNA circuit from B is BEDACFB with time 158 milliseconds.

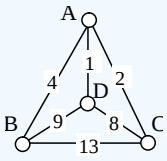
While certainly better than the basic NNA, unfortunately, the RNNA is still greedy and will produce very bad results for some graphs. As an alternative, our next approach will step back and look at the "big picture" – it will select first the edges that are shortest, and then fill in the gaps.

Sorted Edges Algorithm (a.k.a. Cheapest Link Algorithm)

1. Select the cheapest unused edge in the graph.
2. Repeat step 1, adding the cheapest unused edge to the circuit, unless:
    a. adding the edge would create a circuit that doesn't contain all vertices, or
    b. adding the edge would give a vertex degree 3.
3. Repeat until a circuit containing all vertices is formed.

Example 20

Using the four vertex graph from earlier, we can use the Sorted Edges algorithm.

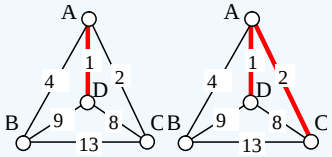


▼ Adjacency Table

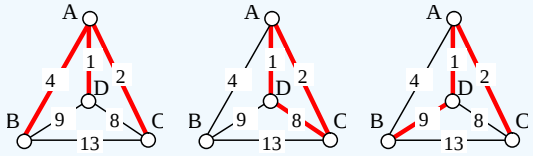
Solution

The cheapest edge is AD, with a cost of 1. We highlight that edge to mark it selected.

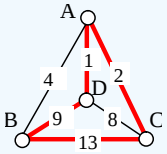
The next shortest edge is AC, with a weight of 2, so we highlight that edge.



For the third edge, we'd like to add AB, but that would give vertex A degree 3, which is not allowed in a Hamiltonian circuit. The next shortest edge is CD, but that edge would create a circuit ACDA that does not include vertex B, so we reject that edge. The next shortest edge is BD, so we add that edge to the graph.



We then add the last edge to complete the circuit: ACBDA with weight 25.



Notice that the algorithm did not produce the optimal circuit in this case; the optimal circuit is ACDBA with weight 23.

While the Sorted Edge algorithm overcomes some of the shortcomings of NNA, it is still only a heuristic algorithm, and does not guarantee the optimal circuit.

Example 21

Your teacher's band, *Derivative Work*, is doing a bar tour in Oregon. The driving distances are shown below. Plan an efficient route for your teacher to visit all the cities and return to the starting location. Use NNA starting at Portland, and then use Sorted Edges.

Driving distances between cities

	Ashland	Astoria	Bend	Corvallis	Crater Lake	Eugene	Newport	Portland	Salem	Seaside
Ashland	-	374	200	223	108	178	252	285	240	356
Astoria	374	-	255	166	433	199	135	95	136	17
Bend	200	255	-	128	277	128	180	160	131	247
Corvallis	223	166	128	-	430	47	52	84	40	155
Crater Lake	108	433	277	430	-	453	478	344	389	423
Eugene	178	199	128	47	453	-	91	110	64	181
Newport	252	135	180	52	478	91	-	114	83	117
Portland	285	95	160	84	344	110	114	-	47	78
Salem	240	136	131	40	389	64	83	47	-	118
Seaside	356	17	247	155	423	181	117	78	118	-

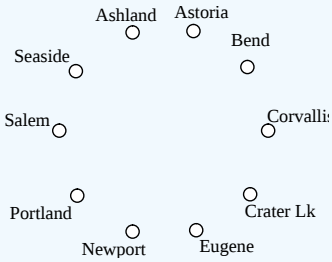
Solution

Using NNA with a large number of cities, you might find it helpful to mark off the cities as they're visited to keep from accidently visiting them again. Looking in the row for Portland, the smallest distance is 47, to Salem. Following that idea, our circuit will be:

Nearest Neighbor Route

Edge	Distance
Portland to Salem	47
Salem to Corvallis	40
Corvallis to Eugene	47
Eugene to Newport	91
Newport to Seaside	117
Seaside to Astoria	17
Astoria to Bend	255
Bend to Ashland	200
Ashland to Crater Lake	108
Crater Lake to Portland	344
Total trip length:	1266 miles

Using Sorted Edges, you might find it helpful to draw an empty graph, perhaps by drawing vertices in a circular pattern.

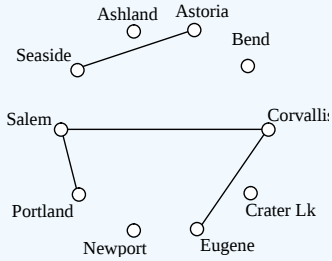


Adding edges to the graph as you select them will help you visualize any circuits or vertices with degree 3. If using adjacency tables, you could instead start with a blank table and add the edges to the table as you go.

We start adding the shortest edges:

- Seaside to Astoria: 17 miles
- Corvallis to Salem: 40 miles
- Portland to Salem: 47 miles
- Corvallis to Eugene: 47 miles

The graph after adding these edges is shown below.



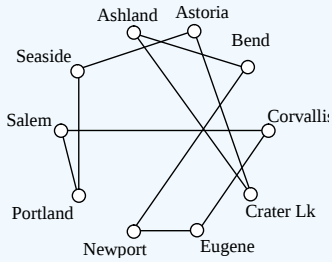
The next shortest edge is from Corvallis to Newport at 52 miles, but adding that edge would give Corvallis degree 3.

Continuing on, we can skip over any edge pair that contains Salem or Corvallis, since they both already have degree 2.

- Portland to Seaside: 78 miles

- Eugene to Newport: 91 miles
- Portland to Astoria: reject – closes circuit
- Ashland to Crater Lake: 108 miles

The graph after adding these edges is shown below.



At this point, we can skip over any edge pair that contains Salem, Seaside, Eugene, Portland, or Corvallis since they already have degree 2.

- Newport to Astoria: reject – closes circuit
- Newport to Bend: 180 miles
- Bend to Ashland: 200 miles

At this point the only way to complete the circuit is to add:

- Crater Lake to Astoria: 433 miles

The final circuit, written to start at Portland, is:

Portland, Salem, Corvallis, Eugene, Newport, Bend, Ashland, Crater Lake, Astoria, Seaside, Portland.

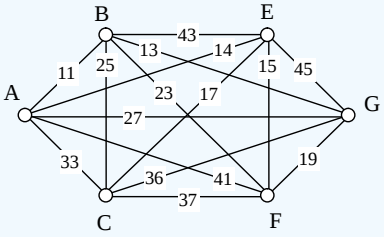
Total trip length: 1241 miles.

While better than the NNA route, neither algorithm produced the optimal route. The following route can make the tour in 1069 miles:

Portland, Astoria, Seaside, Newport, Corvallis, Eugene, Ashland, Crater Lake, Bend, Salem, Portland

Try it Now 6

Find the circuit produced by the Sorted Edges algorithm using the graph below.



▼ Adjacency Table

Adjacency table with weights

	A	B	C	E	F	G
A	-	11	33	14	41	27
B	11	-	25	43	23	13
C	33	25	-	17	37	36
E	14	43	17	-	15	45
F	41	23	37	15	-	19
G	27	13	36	45	19	-

Answer

AB: Add, cost 11

BG: Add, cost 13

AE: Add, cost 14

EF: Add, cost 15

EC: Skip (degree 3 at E)

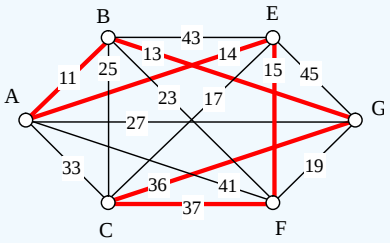
FG: Skip (would create a circuit not including C)

BF, BC, AG, AC: Skip (would cause a vertex to have degree 3)

GC: Add, cost 36

CF: Add, cost 37, completes the circuit

Final circuit: ABGCFEA



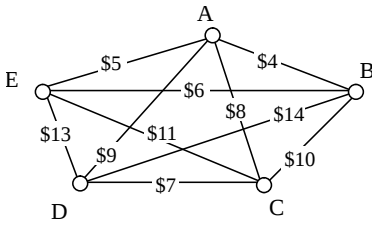
[1] There are some theorems that can be used in specific circumstances, such as Dirac's theorem, which says that a Hamiltonian circuit must exist on a graph with n vertices if each vertex has degree $n/2$ or greater.

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7.7: Spanning Trees

A company requires reliable internet and phone connectivity between their five offices (named A, B, C, D, and E for simplicity) in New York, so they decide to lease dedicated lines from the phone company. The phone company will charge for each link made. The costs, in thousands of dollars per year, are shown in the graph.



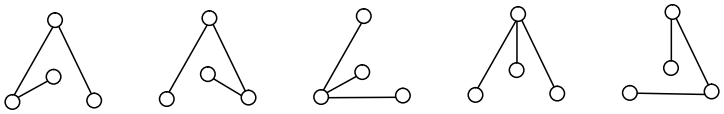
In this case, we don't need to find a circuit, or even a specific path; all we need to do is make sure we can make a call from any office to any other. In other words, we need to be sure there is a path from any vertex to any other vertex.

Spanning Tree

A **spanning tree** is a connected graph using all vertices in which there are no circuits.

In other words, there is a path from any vertex to any other vertex, but no circuits.

Some examples of spanning trees are shown below. Notice there are no circuits in the trees, and it is fine to have vertices with degree higher than two.



Usually we have a starting graph to work from, like in the phone example above. In this case, we form our spanning tree by finding a **subgraph** – a new graph formed using all the vertices but only some of the edges from the original graph. No edges will be created where they didn't already exist.

Of course, any random spanning tree isn't really what we want. We want the **minimum cost spanning tree (MCST)**.

Minimum Cost Spanning Tree (MCST)

The minimum cost spanning tree is the spanning tree with the smallest total edge weight.

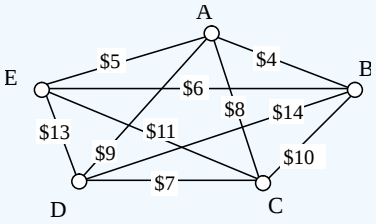
A nearest neighbor style approach doesn't make as much sense here since we don't need a circuit, so instead we will take an approach similar to sorted edges.

Kruskal's Algorithm

1. Select the cheapest unused edge in the graph.
2. Repeat step 1, adding the cheapest unused edge, unless:
   a. adding the edge would create a circuit
3. Repeat until a spanning tree is formed

Example 22

Using our phone line graph



▼ Adjacency Table

- Adjacency Table

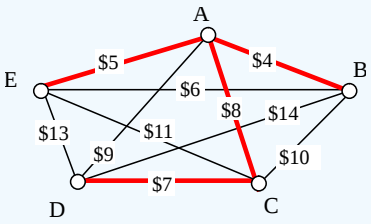
Adjacency table showing cost to connect offices, in thousands

	A	B	C	D	E
A	-	4	8	9	5
B	4	-	10	14	6
C	8	10	-	7	11
D	9	14	7	-	13
E	5	6	11	13	-

Begin adding edges:

- AB: $4 OK
- AE: $5 OK
- BE: $6 reject - closes circuit ABEA
- DC: $7 OK
- AC: $8 OK

At this point we stop – every vertex is now connected, so we have formed a spanning tree with cost $4 + $5 + $7 + $8 = $24 thousand a year.



Remarkably, Kruskal's algorithm is both optimal and efficient; we are guaranteed to always produce the optimal MCST.

Example 23

The power company needs to lay updated distribution lines connecting the ten Oregon cities below to the power grid. How can they minimize the amount of new line to lay?

Distances between cities

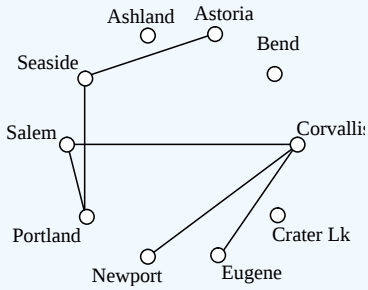
	Ashland	Astoria	Bend	Corvallis	Crater Lake	Eugene	Newport	Portland	Salem	Seaside
Ashland	-	374	200	223	108	178	252	285	240	356
Astoria	374	-	255	166	433	199	135	95	136	17
Bend	200	255	-	128	277	128	180	160	131	247
Corvallis	223	166	128	-	430	47	52	84	40	155
Crater Lake	108	433	277	430	-	453	478	344	389	423
Eugene	178	199	128	47	453	-	91	110	64	181
Newport	252	135	180	52	478	91	-	114	83	117
Portland	285	95	160	84	344	110	114	-	47	78
Salem	240	136	131	40	389	64	83	47	-	118
Seaside	356	17	247	155	423	181	117	78	118	-

Solution

Using Kruskal's algorithm, we add edges from cheapest to most expensive, rejecting any that close a circuit. We stop when the graph is connected.

- Seaside to Astoria: 17 miles
- Corvallis to Salem: 40 miles
- Portland to Salem: 47 miles
- Corvallis to Eugene: 47 miles
- Corvallis to Newport: 52 miles
- Salem to Eugene: reject - closes circuit Corvallis-Salem-Eugene-Corvallis
- Portland to Seaside: 78 miles

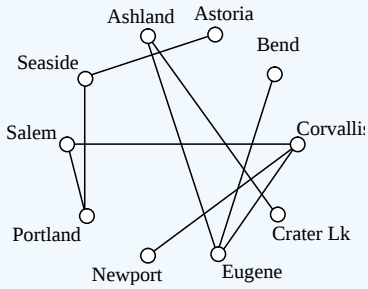
The graph up to this point is below.



Continuing,

- Newport to Salem: reject - closes circuit
- Corvallis to Portland: reject
- Eugene to Newport: reject
- Portland to Astoria: reject
- Ashland to Crater Lake: 108 miles
- Eugene to Portland: reject
- Newport to Portland: reject
- Newport to Seaside: reject
- Salem to Seaside: reject
- Bend to Eugene: 128 miles

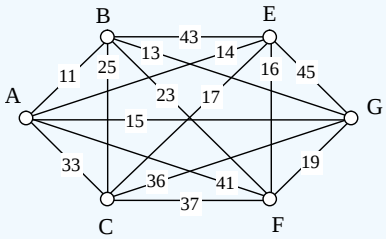
- Bend to Salem: reject
- Astoria to Newport: reject
- Salem to Astoria: reject
- Corvallis to Seaside: reject
- Portland to Bend: reject
- Astoria to Corvallis: reject
- Eugene to Ashland: 178 miles



This connects the graph. The total length of cable to lay would be 695 miles.

Try it Now 7

Find a minimum cost spanning tree on the graph below using Kruskal's algorithm.



▼ Adjacency Table

Adjacency table with weights

	A	B	C	E	F	G
A	-	11	33	14	41	27
B	11	-	25	43	23	13
C	33	25	-	17	37	36
E	14	43	17	-	15	45
F	41	23	37	15	-	19
G	27	13	36	45	19	-

Answer

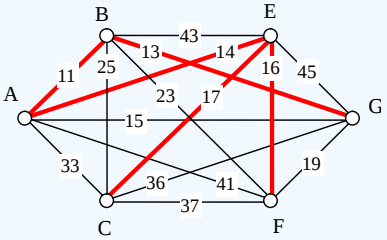
AB: Add, cost 11

BG: Add, cost 13

AE: Add, cost 14

AG: Skip, would create circuit ABGA

EF: Add, cost 16



EC: Add, cost 17

This completes the spanning tree

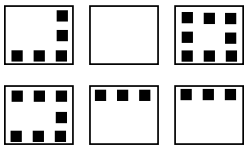
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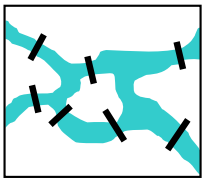
7.8: Exercise

Skills

1. To deliver mail in a particular neighborhood, the postal carrier needs to walk along each of the streets with houses (the dots). Create a graph with edges showing where the carrier must walk to deliver the mail.



2. Suppose that a town has 7 bridges as pictured below. Create a graph that could be used to determine if there is a path that crosses all bridges once.



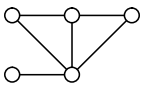
3. The table below shows approximate driving times (in minutes, without traffic) between five cities in the Dallas area. Create a weighted graph representing this data.

	Plano	Mesquite	Arlington	Denton
Fort Worth	54	52	19	42
Plano		38	53	41
Mesquite			43	56
Arlington				50

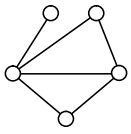
4. Shown in the table below are the one-way airfares between 5 cities[1]. Create a graph showing this data.

	Honolulu	London	Moscow	Cairo
Seattle	$159	$370	$654	$684
Honolulu		$830	$854	$801
London			$245	$323
Moscow				$329

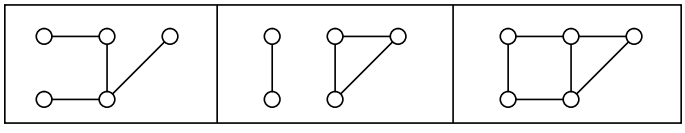
5. Find the degree of each vertex in the graph below.



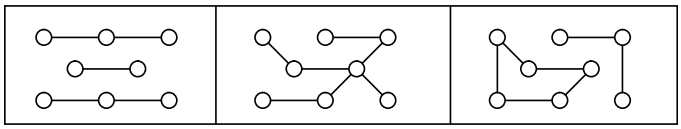
6. Find the degree of each vertex in the graph below.



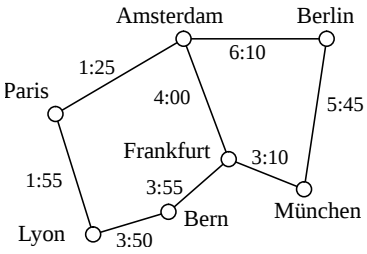
7. Which of these graphs are connected?



8. Which of these graphs are connected?

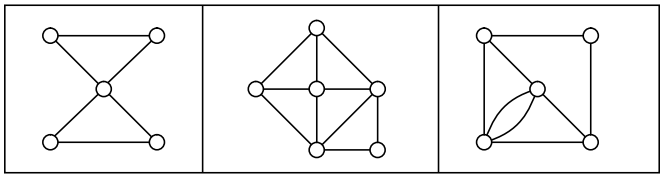


9. Travel times by rail for a segment of the Eurail system is shown below with travel times in hours and minutes[2]. Find path with shortest travel time from Bern to Berlin by applying Dijkstra's algorithm.

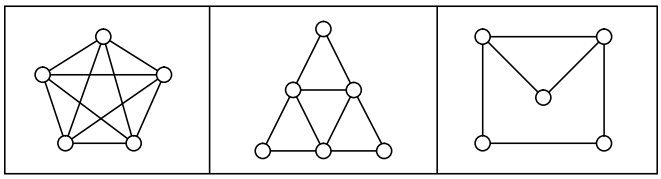


10. Using the graph from the previous problem, find the path with shortest travel time from Paris to München.

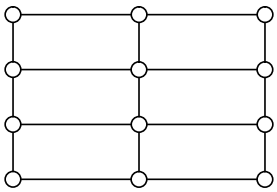
11. Does each of these graphs have an Euler circuit? If so, find it.



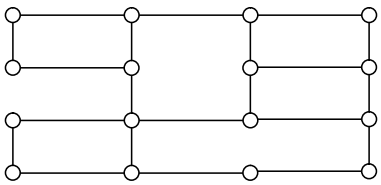
12. Does each of these graphs have an Euler circuit? If so, find it.



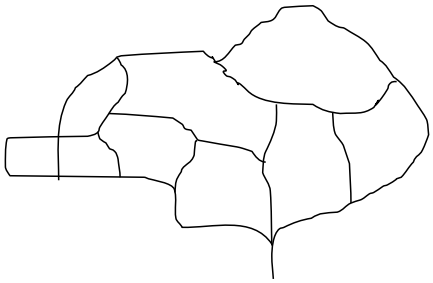
13. Eulerize this graph using as few edge duplications as possible. Then, find an Euler circuit.



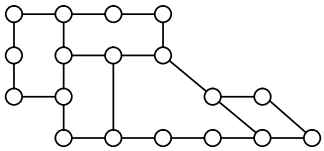
14. Eulerize this graph using as few edge duplications as possible. Then, find an Euler circuit.



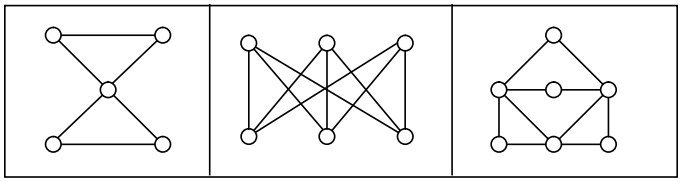
15. The maintenance staff at an amusement park need to patrol the major walkways, shown in the graph below, collecting litter. Find an efficient patrol route by finding an Euler circuit. If necessary, eulerize the graph in an efficient way.



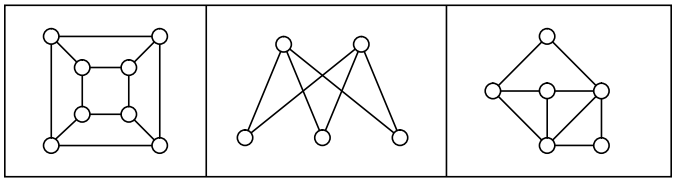
16. After a storm, the city crew inspects for trees or brush blocking the road. Find an efficient route for the neighborhood below by finding an Euler circuit. If necessary, eulerize the graph in an efficient way.



17. Does each of these graphs have at least one Hamiltonian circuit? If so, find one.



18. Does each of these graphs have at least one Hamiltonian circuit? If so, find one.



19. A company needs to deliver product to each of their 5 stores around the Dallas, TX area. Driving distances between the stores are shown below. Find a route for the driver to follow, returning to the distribution center in Fort Worth:

a. Using Nearest Neighbor starting in Fort Worth
b. Using Repeated Nearest Neighbor
c. Using Sorted Edges

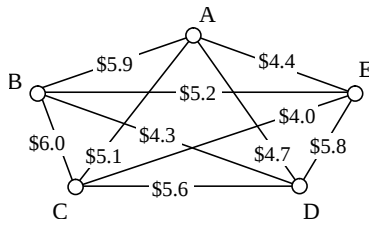
	Plano	Mesquite	Arlington	Denton
Fort Worth	54	52	19	42
Plano		38	53	41
Mesquite			43	56
Arlington				50

20. A salesperson needs to travel from Seattle to Honolulu, London, Moscow, and Cairo. Use the table of flight costs from problem #4 to find a route for this person to follow:

a. Using Nearest Neighbor starting in Seattle
b. Using Repeated Nearest Neighbor
c. Using Sorted Edges

21. When installing fiber optics, some companies will install a sonet ring; a full loop of cable connecting multiple locations. This is used so that if any part of the cable is damaged it does not interrupt service, since there is a second connection to the hub. A company has 5 buildings. Costs (in thousands of dollars) to lay cables between pairs of buildings are shown below. Find the circuit that will minimize cost:

a. Using Nearest Neighbor starting at building A
b. Using Repeated Nearest Neighbor
c. Using Sorted Edges



22. A tourist wants to visit 7 cities in Israel. Driving distances, in kilometers, between the cities are shown below[3]. Find a route for the person to follow, returning to the starting city:

a. Using Nearest Neighbor starting in Jerusalem
b. Using Repeated Nearest Neighbor
c. Using Sorted Edges

	Jerusalem	Tel Aviv	Haifa	Tiberias	Beer Sheba	Eilat
Jerusalem	_					
Tel Aviv	58	_				
Haifa	151	95	_			
Tiberias	152	134	69	_		
Beer Sheba	81	105	197	233	_	
Eilat	309	346	438	405	241	_
Nazareth	131	102	35	29	207	488

23. Find a minimum cost spanning tree for the graph you created in problem #3

24. Find a minimum cost spanning tree for the graph you created in problem #22

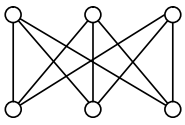
25. Find a minimum cost spanning tree for the graph from problem #21

Concepts

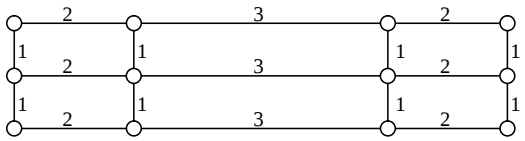
26. Can a graph have one vertex with odd degree? If not, are there other values that are not possible? Why?

27. A complete graph is one in which there is an edge connecting every vertex to every other vertex. For what values of n does complete graph with n vertices have an Euler circuit? A Hamiltonian circuit?

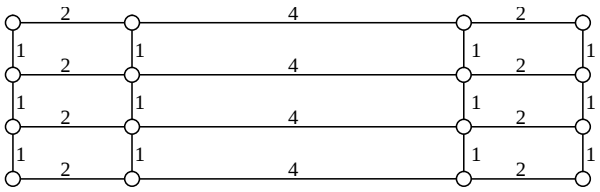
28. Create a graph by drawing n vertices in a row, then another n vertices below those. Draw an edge from each vertex in the top row to every vertex in the bottom row. An example when n=3 is shown below. For what values of n will a graph created this way have an Euler circuit? A Hamiltonian circuit?



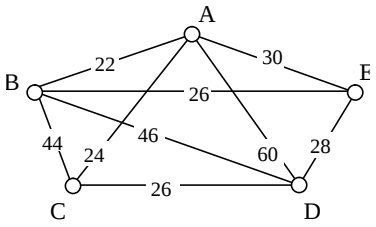
29. Eulerize this graph in the most efficient way possible, considering the weights of the edges.



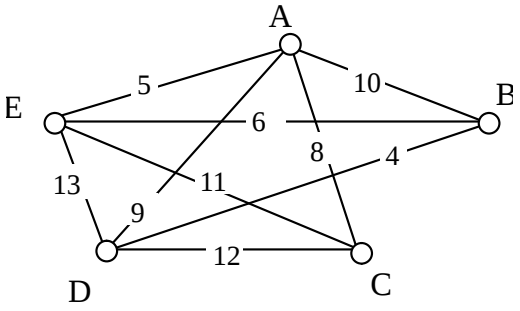
30. Eulerize this graph in the most efficient way possible, considering the weights of the edges.



31. Eulerize this graph in the most efficient way possible, considering the weights of the edges.



32. Eulerize this graph in the most efficient way possible, considering the weights of the edges.



Explorations

33. Social networks such as Facebook and LinkedIn can be represented using graphs in which vertices represent people and edges are drawn between two vertices when those people are “friends.” The table below shows a friendship table, where an X shows that two people are friends.

34.

	A	B	C	D	E	F	G	H	I
A		X	X			X	X		
B			X		X				
C					X				
D					X				X
E							X		X
F								X	X
G								X	
H									X

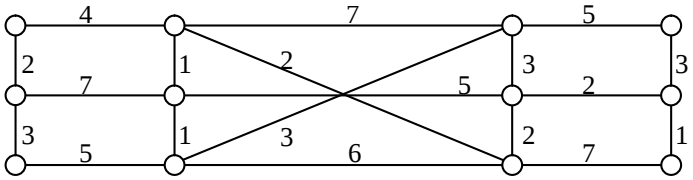
a. Create a graph of this friendship table
b. Find the shortest path from A to D. The length of this path is often called the “degrees of separation” of the two people.
c. Extension: Split into groups. Each group will pick 10 or more movies, and look up their major actors (www.imdb.com is a good source). Create a graph with each actor as a vertex, and edges connecting two actors in the same movie (note the movie name on the edge). Find interesting paths between actors, and quiz the other groups to see if they can guess the connections.

35. A spell checker in a word processing program makes suggestions when it finds a word not in the dictionary. To determine what words to suggest, it tries to find similar words. One measure of word similarity is the Levenshtein distance, which measures the number of substitutions, additions, or deletions that are required to change one word into another. For example, the words spit and spot are a distance of 1 apart; changing spit to spot requires one substitution (i for o). Likewise, spit is distance 1 from pit since the change requires one deletion (the s). The word spite is also distance 1 from spit since it requires one addition (the e). The word soot is distance 2 from spit since two substitutions would be required.

a. Create a graph using words as vertices, and edges connecting words with a Levenshtein distance of 1. Use the misspelled word “moke” as the center, and try to find at least 10 connected dictionary words. How might a spell checker use this graph?
b. Improve the method from above by assigning a weight to each edge based on the likelihood of making the substitution, addition, or deletion. You can base the weights on any reasonable approach: proximity of keys on a keyboard, common language errors, etc. Use Dijkstra’s algorithm to find the length of the shortest path from each word to “moke”. How might a spell checker use these values?

36. The graph below contains two vertices of odd degree. To eulerize this graph, it is necessary to duplicate edges connecting those two vertices.

1. Use Dijkstra’s algorithm to find the shortest path between the two vertices with odd degree. Does this produce the most efficient eulerization and solve the Chinese Postman Problem for this graph?



2. Suppose a graph has n odd vertices. Using the approach from part a, how many shortest paths would need to be considered? Is this approach going to be efficient?

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[1] Cheapest fares found when retrieved Sept 1, 2009 for travel Sept 22, 2009

[2] From www.eurail.com/eurail-railway-map

[3] From http://www.ddtravel-acc.com/Israel-c...s-distance.htm

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CHAPTER OVERVIEW

8: Measurement

Measurement is a number that describes the size or amount of something. You can measure many things like length, area, capacity, weight, temperature and time. In the United States, two main systems of measurement are used: the metric system and the U.S. customary measurement system.

The material in this chapter is from Developmental Math by NROC. Edited by Nancy Ikeda and Mareike Claassen

8.1: U.S. Customary Measurement System
8.2: Metric Units of Measurement
8.3: Temperature and Time
8.4: Conversion Between the Metric and US Customary Systems of Measurement
8.5: Chapter Review and Glossary
8.6: Exercises

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8.1: U.S. Customary Measurement System

Learning Objectives

1. Define units of length, weight and capacity.
2. Convert from one unit to another.
3. Perform arithmetic calculations on units of length, weight and capacity.
4. Solve application problems involving units of length, weight and capacity.

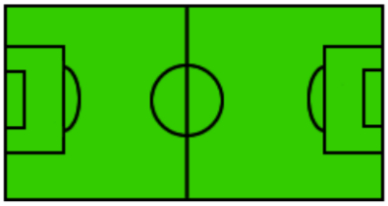
Introduction

Measurement is a number that describes the size or amount of something. You can measure many things like length, area, capacity, weight, temperature and time. In the United States, two main systems of measurement are used: the **metric system** and the **U.S. customary measurement system**. This section addresses the measurements of length, weight and capacity using the U.S. customary measurement system.

Suppose you want to purchase tubing for a project, and you see two signs in a hardware store: *$1.88 for 2 feet* of tubing and *$5.49 for 3 yards* of tubing. If both types of tubing will work equally well for your project, which is the better price? You need to know about two **units of measurement**, yards and feet, in order to determine the answer.

Units of Length

Length is the distance from one end of an object to the other end, or from one object to another. For example, the length of a letter-sized piece of paper is 11 inches. The system for measuring length in the United States is based on the four customary units of length: **inch, foot, yard,** and **mile**. Below are examples to show measurement in each of these units.

Unit	Description	Image
Inch/Inches	Some people donate their hair to be made into wigs for cancer patients who have lost hair as a result of treatment. One company requires hair donations to be at least 8 inches long.	
	Frame size of a bike: the distance from the center of the crank to the top of the seat tube. Frame size is usually measured in inches. This frame is 16 inches.	
Foot/Feet	Rugs are typically sold in standard lengths. One typical size is a rug that is 8 feet wide and 11 feet long. This is often described as an 8 by 11 rug.	
Yard/Yards	Soccer fields vary some in their size. An official field can be any length between 100 and 130 yards.	
Mile/Miles	A marathon is 26.2 miles long. One marathon route is shown in the map to the right.	

You can use any of these four U.S. customary measurement units to describe the length of something, but it makes more sense to use certain units for certain objects depending on their general size. For example, it makes more sense to describe the length of a rug in feet rather than miles, and to describe a marathon in miles rather than inches.

You may need to convert between units of measurement. For example, you might want to express your height using feet and inches (5 feet 4 inches) or using only inches (64 inches). You need to know the unit equivalents in order to make these conversions

between units.

The table below shows equivalents and conversion factors for the four customary units of measurement of length.

Unit Equivalents	Conversion Factors (longer to shorter units of measurement)	Conversion Factors (shorter to longer units of measurement)
1 foot = 12 inches	$\frac{12 \text{ inches}}{1 \text{ foot}}$	$\frac{1 \text{ foot}}{12 \text{ inches}}$
1 yard = 3 feet	$\frac{3 \text{ feet}}{1 \text{ yard}}$	$\frac{1 \text{ yard}}{3 \text{ feet}}$
1 mile = 5280 feet	$\frac{5280 \text{ feet}}{1 \text{ mile}}$	$\frac{1 \text{ mile}}{5280 \text{ feet}}$

Note that each of these conversion factors is a ratio of equal values, also called a unit fraction, so each conversion factor equals 1. Recall that multiplying a number by 1 does not change the number. Multiplying a measurement by a conversion factor does not change the size of the measurement at all since it is the same as multiplying by 1; it just changes the units that you are using to measure.

Converting Between Units of Length

You can use the conversion factors to convert a measurement, such as feet, to another type of measurement, such as inches.

Note that there are many more inches for a measurement than there are feet for the same measurement, as feet is a longer unit of measurement. You could use the conversion factor $\frac{12 \text{ inches}}{1 \text{ foot}}$.

If a length is measured in feet, and you'd like to convert the length to yards, you can think, "I am converting from a shorter unit to a longer one, so the length in yards will be less than the length in feet." You could use the conversion factor $\frac{1 \text{ yard}}{3 \text{ feet}}$.

If a distance is measured in miles, and you want to know how many feet it is, you can think, "I am converting from a longer unit of measurement to a shorter one, so the number of feet would be greater than the number of miles." You could use the conversion factor $\frac{5280 \text{ feet}}{1 \text{ mile}}$.

Another way to determine which conversion factor, or unit fraction to use, is to check which unit will need to be cancelled. I.e. Converting inches to feet, you want the denominator of the unit fraction to cancel out the original units of inches. Hence the appropriate conversion factors has to be in $\frac{\text{feet}}{\text{inches}}$. Then you just need to fill in the appropriate numbers. Since 1 foot = 12 inches, the conversion factor is $\frac{1 \text{ feet}}{12 \text{ inches}}$.

You can use the **factor label method (dimensional analysis)** to convert a length from one unit of measure to another using the conversion factors. In the factor label method, you multiply by **unit fractions** to convert a measurement from one unit to another. Study the example below to see how the factor label method can be used to convert $3\frac{1}{2}$ feet into an equivalent number of inches.

Example 8.1.1

How many inches are in $3\frac{1}{2}$ feet?

Solution

Begin by reasoning about your answer. Since a foot is longer than an inch, this means the answer would be greater than $3\frac{1}{2}$.

$$3\frac{1}{2} \text{ feet} = \_\_\_\_\_ \text{ inches}$$

Find the conversion factor that compares inches and feet, with "inches" in the numerator, and "feet" in the denominator. Then multiply

$$3\frac{1}{2} \text{ feet} \cdot \frac{12 \text{ inches}}{1 \text{ foot}} = \_\_\_\_\_ \text{ inches}$$

Rewrite the mixed number as an improper fraction before multiplying.

$$\frac{7 \text{ feet}}{2} \cdot \frac{12 \text{ inches}}{1 \text{ foot}} = \_\_\_\_\_ \text{ inches}$$

You can cancel the same units when they appear in the numerator *and* the denominator. Here “feet” and “foot” describe the same unit just in the singular and plural form. This eliminates the unit of "feet" from the problem and leaves the unit of "inches".

$$\frac{7 \cancel{\text{ feet}}}{2} \cdot \frac{12 \text{ inches}}{1 \cancel{\text{ foot}}} = \_\_\_\_\_ \text{ inches}$$

$$\frac{7}{2} \cdot \frac{12 \text{ inches}}{1} = \_\_\_\_\_ \text{ inches}$$

Rewrite as multiplication of numerators and denominators.

$$\frac{7 \cdot 12 \text{ inches}}{2 \cdot 1} = \_\_\_\_\_ \text{ inches}$$

Multiply.

$$\frac{84 \text{ inches}}{2} = \_\_\_\_\_ \text{ inches}$$

Divide.

$$\frac{84 \text{ inches}}{2} = 42 \text{ inches}$$

Answer: There are 42 inches in $3\frac{1}{2}$ feet.

Notice that by using the factor label method you can cancel the units out of the problem, just as if they were numbers. You can only cancel if the unit being canceled is in both the numerator and denominator of the fractions you are multiplying.

In the problem above, you canceled feet and foot leaving you with inches, which is what you were trying to find.

$$\frac{7 \cancel{\text{ feet}}}{2} \cdot \frac{12 \text{ inches}}{1 \cancel{\text{ foot}}} = \_\_\_\_\_ \text{ inches}$$

What if you had used the wrong conversion factor?

$$\frac{7 \text{ feet}}{2} \cdot \frac{1 \text{ foot}}{12 \text{ inches}} = \_\_\_\_\_ \text{ inches}$$

You could not cancel the feet because the unit is not the same in both the numerator and the denominator. So if you complete the computation, you would still have both feet and inches in the answer and no conversion would take place.

Here is another example of a length conversion using the factor label method.

✔ Example 8.1.2

How many yards is 7 feet?

Solution

Start by reasoning about the size of your answer. Since a yard is longer than a foot, there will be fewer yards. So your answer will be less than 7.

$$7 \text{ feet} = \_\_\_\_\_ \text{ yards}$$

Find the conversion factor that compares feet and yards, with yards in the numerator.

$$7 \text{ feet} \cdot \frac{1 \text{ yard}}{3 \text{ feet}} = \_\_\_\_\_ \text{ yards}$$

Rewrite the whole number as a fraction in order to multiply.

$$\frac{7 \text{ feet}}{1} \cdot \frac{1 \text{ yard}}{3 \text{ feet}} = \_\_\_\_\_ \text{ yards}$$

Cancel the similar units “feet” and “feet” leaving only yards.

$$\frac{7 \text{ feet}}{1} \cdot \frac{1 \text{ yard}}{3 \text{ feet}} = \_\_\_\_\_ \text{ yards}$$

$$\frac{7}{1} \cdot \frac{1 \text{ yard}}{3} = \_\_\_\_\_ \text{ yards}$$

Multiply.

$$\frac{7 \cdot 1 \text{ yard}}{1 \cdot 3} = \_\_\_\_\_ \text{ yards}$$

Divide.

$$\frac{7 \text{ yards}}{3} = 2\frac{1}{3} \text{ yards}$$

Answer: 7 feet equals $2\frac{1}{3}$ yards.

Note that if the units do not cancel to give you the answer you are trying to find, you may not have used the correct conversion factor.

Try It 8.1.1

How many feet are in $2\frac{1}{2}$ miles?

Answer

There are 5280 feet in a mile, so multiply $2\frac{1}{2}$ by $\frac{5280\text{feet}}{1\text{mile}}$ to get 13,200 feet.

Applying Unit Conversions

There are times when you will need to perform computations on measurements that are given in different units. For example, consider the tubing problem given earlier. You must decide which of the two options is a better price, and you have to compare prices given in different unit measurements.

In order to compare, you need to convert the measurements into one single, common unit of measurement. To be sure you have made the computation accurately, think about whether the unit you are converting to is smaller or larger than the number you have. Its relative size will tell you whether the number you are trying to find is greater or lesser than the given number.

Example 8.1.3

An interior decorator needs border trim for a home she is wallpapering. She needs 15 feet of border trim for the living room, 30 feet of border trim for the bedroom, and 26 feet of border trim for the dining room. How many yards of border trim does she need?

Solution

You need to find the total length of border trim that is needed for all three rooms in the house. You can only add or subtract numbers with the same units. Since the measurements for each room are given in feet, you can add the numbers.

$$15 \text{ feet} + 30 \text{ feet} + 26 \text{ feet} = 71 \text{ feet}$$

How many yards is 71 feet? Reason about the size of your answer. Since a yard is longer than a foot, there will be fewer yards. Expect your answer to be less than 71.

$$71 \text{ feet} = \_\_\_\_\_ \text{ yards}$$

Use the conversion factor $\frac{1 \text{ yard}}{3 \text{ feet}}$.

$$\frac{71 \text{ feet}}{1} \cdot \frac{1 \text{ yard}}{3 \text{ feet}} = \_\_\_\_\_ \text{ yards}$$

Since “feet” is in the numerator and denominator, you can cancel this unit.

$$\frac{71 \text{ feet}}{1} \cdot \frac{1 \text{ yard}}{3 \text{ feet}} = \_\_\_\_\_ \text{ yards}$$

$$\frac{71}{1} \cdot \frac{1 \text{ yard}}{3} = \_\_\_\_\_ \text{ yards}$$

Multiply.

$$\frac{71 \cdot 1 \text{ yards}}{1 \cdot 3} \quad \_\_\_\_\_ \text{ yards}$$
$$\frac{71 \text{ yards}}{3} = \_\_\_\_\_ \text{ yards}$$

Divide, and write as a mixed number.

$$\frac{71 \text{ yards}}{3} = 23\frac{2}{3} \text{ yards}$$

Answer: The interior decorator needs $23\frac{2}{3}$ yards of border trim.

The next example uses the factor label method to solve a problem that requires converting from miles to feet.

Example 8.1.4

Two runners were comparing how much they had trained earlier that day. Jo said, "According to my pedometer, I ran 8.3 miles." Alex said, "That's a little more than what I ran. I ran 8.1 miles." How many more feet did Jo run than Alex?

Solution

You need to find the difference between the distance Jo ran and the distance Alex ran. Since both distances are given in the same unit, you can subtract and keep the unit the same.

$$8.3 \text{ miles} – 8.1 \text{ miles} = 0.2 \text{ mile}$$
$$0.2 \text{ mile} = \frac{2}{10} \text{ miles}$$

Since the problem asks for the difference in *feet*, you must convert from miles to feet. How many feet is 0.2 mile? Reason about the size of your answer. Since a mile is longer than a foot, the distance when expressed as feet will be a number greater than 0.2.

$$\frac{2}{10} \text{ miles} = \_\_\_\_\_ \text{feet}$$

Use the conversion factor $\frac{5280 \text{ feet}}{1 \text{ mile}}$.

$$\frac{2 \text{ miles}}{10} \cdot \frac{5280 \text{ feet}}{1 \text{ mile}} = \_\_\_\_\_ \text{feet}$$
$$\frac{2}{10} \cdot \frac{5280 \text{ feet}}{1} = \_\_\_\_\_ \text{feet}$$

Multiply.

$$\frac{2 \cdot 5280 \text{ feet}}{10 \cdot 1} = \_\_\_\_\_ \text{feet}$$
$$\frac{10,560 \text{ feet}}{10} = \_\_\_\_\_ \text{feet}$$

Divide.

$$\frac{10,560 \text{ feet}}{10} = 1056 \text{ feet}$$

Answer: Jo ran 1056 feet more than Alex.

Now let's revisit the question from earlier.

Example 8.1.5

You are walking through a hardware store and notice two sales on tubing.

- 3 yards of Tubing A costs $5.49.
- Tubing B sells for $1.88 for 2 feet.

Either tubing is acceptable for your project. Which tubing is less expensive?

Solution

Find the unit price for each tubing. This will make it easier to compare.

Tubing A: 3 yards = $5.49

Find the cost per yard of Tubing A by dividing the cost of 3 yards of the tubing by 3.

$$\frac{\$5.49 \div 3}{3 \text{ yards} \div 3} = \frac{\$1.83}{1 \text{ yard}}$$

Tubing B is sold by the foot. Find the cost per foot by dividing $1.88 by 2 feet.

Tubing B: 2 feet = $1.88

$$\frac{\$1.88 \div 2}{2 \text{ feet} \div 2} = \frac{\$0.94}{1 \text{ foot}}$$

To compare the prices, you need to have the same unit of measure.

$$\frac{\$0.94}{1 \text{ foot}} \cdot \frac{3 \text{ feet}}{1 \text{ yard}} = \frac{\$}{\text{yard}}$$

Use the conversion factor $\frac{3 \text{ feet}}{1 \text{ yard}}$, cancel and multiply

$$\frac{\$0.94}{1} \cdot \frac{3}{1 \text{ yard}} = \frac{\$2.82}{1 \text{ yard}}$$

$$\therefore \$2.82 \text{ per yard}$$

Compare prices for 1 yard of each tubing.

Tubing A: $1.83 per yard

Tubing B: $2.82 per yard

Answer: Tubing A is less expensive than Tubing B.

In the problem above, you could also have found the price per foot for each kind of tubing and compared the unit prices of each per foot.

Try It 8.1.2

A fence company is measuring a rectangular area in order to install a fence around its perimeter. If the length of the rectangular area is 130 yards and the width is 75 feet, what is the total length of the distance to be fenced?

Answer

130 yards is equivalent to 390 feet. To find the perimeter, add length + length + width + width: 390 feet + 390 feet + 75 feet + 75 feet = 930 feet.

Summary

The four basic units of measurement that are used in the U.S. customary measurement system are: inch, foot, yard, and mile. Typically, people use yards, miles, and sometimes feet to describe long distances. Measurement in inches is common for shorter objects or lengths.

You need to convert from one unit of measure to another if you are solving problems that include measurements involving more than one type of measurement. Each of the units can be converted to one of the other units using the table of equivalents, the conversion factors, and/or the factor label method shown in this topic.

Weight

Introduction

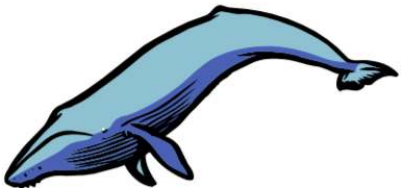
When you mention how heavy or light an object is, you are referring to its weight. In the U.S. customary system of measurement, weight is measured in ounces, pounds, and tons. Like other units of measurement that describe the same kind of quantity, you can

convert between these units and you sometimes need to do this to solve problems.

In 2010, the post office charged $0.44 to mail an item that weighed an ounce or less. The post office charged $0.17 for each additional ounce, or fraction of an ounce, of weight. How much did it cost to mail a package that weighed two pounds three ounces? To answer this question, you need to understand the relationship between ounces and pounds.

Units of Weight

You often use the word **weight** to describe how heavy or light an object or person is. Weight is measured in the U.S. customary system using three units: ounces, pounds, and tons. An **ounce** is the smallest unit for measuring weight, a **pound** is a larger unit, and a **ton** is the largest unit.

Whales are some of the largest animals in the world. Some species can reach weights of up to 200 tons- that's equal to 400,000 pounds	
Meat is a product that is typically sold by the pound. One pound of ground beef makes about four hamburger patties.	
Ounces are used to measure lighter objects. A stack of 11 pennies is equal to about one ounce.	

You can use any of the customary measurement units to describe the weight of something, but it makes more sense to use certain units depending on how heavy an object is. For example, it makes more sense to describe the weight of a human being in pounds rather than tons. It makes more sense to describe the weight of a car in tons rather than ounces.

1 pound = 16 ounces

1 ton = 2000 pounds

Converting Between Units of Weight

Four ounces is a typical serving size of meat. Since meat is sold by the pound, you might want to convert the weight of a package of meat from pounds to ounces in order to determine how many servings are contained in a package of meat.

The weight capacity of a truck is often provided in tons. You might need to convert pounds into tons if you are trying to determine whether a truck can safely transport a big shipment of heavy materials.

The table below shows the unit conversions and conversion factors that are used to make conversions between customary units of weight.

Unit Equivalents	Conversion Factors (heavier to lighter units of measurement)	Conversion Factors (lighter to heavier units of measurement)
1 pound = 16 ounces	$\frac{16 \text{ ounces}}{1 \text{ pound}}$	$\frac{1 \text{ pound}}{16 \text{ ounces}}$
1 ton = 2000 pounds	$\frac{2000 \text{ pounds}}{1 \text{ ton}}$	$\frac{1 \text{ ton}}{2000 \text{ pounds}}$

You can use the *factor label method* to convert one customary unit of weight to another customary unit of weight. This method uses unit fractions conversion factor, which allow you to "cancel" units to end up with your desired unit of measurement.

Two examples illustrating the factor label method are shown below.

Example 8.1.6

How many ounces are in $2\frac{1}{4}$ pounds?

Solution

Begin by reasoning about your answer. Since a pound is heavier than an ounce, expect your answer to be a number greater than $2\frac{1}{4}$.

$$2\frac{1}{4} \text{ pounds} = \_\_\_\_\_ \text{ounces}$$

Multiply by the conversion factor that relates ounces and pounds: $\frac{16 \text{ ounces}}{1 \text{ pound}}$.

$$2\frac{1}{4} \text{ pounds} \cdot \frac{16 \text{ ounces}}{1 \text{ pound}} = \_\_\_\_\_ \text{ounces}$$

Write the mixed number as an improper fraction.

$$\frac{9 \text{ pounds}}{4} \cdot \frac{16 \text{ ounces}}{1 \text{ pound}} = \_\_\_\_\_ \text{ounces}$$

The common unit "pound" can be canceled because it appears in both the numerator and denominator.

$$\frac{9}{4} \cdot \frac{16 \text{ ounces}}{1} = \_\_\_\_\_ \text{ounces}$$

Multiply and simplify.

$$\frac{9 \cdot 16 \text{ ounces}}{4 \cdot 1} = \_\_\_\_\_ \text{ounces}$$
$$\frac{144 \text{ ounces}}{4} = \_\_\_\_\_ \text{ounces}$$
$$\frac{144 \text{ ounces}}{4} = 36 \text{ ounces}$$

Answer: There are 36 ounces in $2\frac{1}{4}$ pounds.

Example 8.1.7

How many tons is 6500 pounds?

Solution

Begin by reasoning about your answer. Since a ton is heavier than a pound, expect your answer to be a number less than 6500.

$$6500 \text{ pounds} = \_\_\_\_\_ \text{tons}$$

Multiply by the conversion factor that relates tons to pounds: $\frac{1 \text{ ton}}{2000 \text{ pounds}}$.

$$6500 \text{ pounds} \cdot \frac{1 \text{ ton}}{2000 \text{ pounds}} = \_\_\_\_\_ \text{tons}$$

$$\frac{6500 \text{ pounds}}{1} \cdot \frac{1 \text{ ton}}{2000 \text{ pounds}} = \_\_\_\_\_ \text{tons}$$

Apply the Factor Label method.

$$\frac{6500}{1} \cdot \frac{1 \text{ ton}}{2000} = \_\_\_\_\_ \text{tons}$$

Multiply and simplify.

$$\frac{6500 \text{ tons}}{2000} = \_\_\_\_\_ \text{tons}$$

$$\frac{6500 \text{ tons}}{2000} = 3\frac{1}{4} \text{ tons}$$

Answer: There are 6500 pounds in $3\frac{1}{4}$ tons.

Try It 8.1.3

How many pounds is 72 ounces?

Answer

There are 16 ounces in one pound, so $72 \text{ ounces} \cdot \frac{1 \text{ pound}}{16 \text{ ounces}} = 4\frac{1}{2} \text{ pounds}$.

Applying Unit Conversions

There are times when you need to perform calculations on measurements that are given in different units. To solve these problems, you need to convert one of the measurements to the same unit of measurement as the other measurement.

Think about whether the unit you are converting to is smaller or larger than the unit you are converting from. This will help you be sure that you are making the right computation. You can use the factor label method to make the conversion from one unit to another.

Here is an example of a problem that requires converting between units.

Example 8.1.8

A municipal trash facility allows a person to throw away a maximum of 30 pounds of trash per week. Last week, 140 people threw away the maximum allowable trash. How many tons of trash did this equal?

Solution

Determine the total trash for the week expressed in pounds. If 140 people each throw away 30 pounds, you can find the total by multiplying.

$$140 \cdot 30 \text{ pounds} = 4200 \text{ pounds}$$

Then convert 4200 pounds to tons. Reason about your answer. Since a ton is heavier than a pound, expect your answer to be a number less than 4200.

$$4200 \text{ pounds} = \_\_\_\_\_ \text{ tons}$$

Find the conversion factor appropriate for the situation: $\frac{1 \text{ ton}}{2000 \text{ pounds}}$.

$$\frac{4200 \text{ pounds}}{1} \cdot \frac{1 \text{ ton}}{2000 \text{ pounds}} = \_\_\_\_\_ \text{tons}$$

$$\frac{4200}{1} \cdot \frac{1 \text{ ton}}{2000} = \_\_\_\_\_ \text{tons}$$

Multiply and simplify.

$$\frac{4200 \cdot 1 \text{ ton}}{1 \cdot 2000} = \_\_\_\_\_ \text{tons}$$

$$\frac{4200 \text{ tons}}{2000} = \_\_\_\_\_ \text{tons}$$

$$\frac{4200 \text{ tons}}{2000} = 2\frac{1}{10} \text{ tons}$$

Answer: The total amount of trash generated is $2\frac{1}{10}$ tons.

Let's revisit the post office problem that was posed earlier. We can use unit conversion to solve this problem.

Example 8.1.9

The post office charges $0.44 to mail something that weighs an ounce or less. The charge for each additional ounce, or fraction of an ounce, of weight is $0.17. At this rate, how much will it cost to mail a package that weighs 2 pounds 3 ounces?

Solution

Since the pricing is for ounces, convert the weight of the package from pounds and ounces into just ounces.

$$2 \text{ pounds } 3 \text{ ounces} = \_\_\_\_\_ \text{ounces}$$

First use the factor label method to convert 2 pounds to ounces.

$$\frac{2 \text{ pounds}}{1} \cdot \frac{16 \text{ ounces}}{\text{pound}} = \_\_\_\_\_ \text{ounces}$$

$$\frac{2}{1} \cdot \frac{16 \text{ ounces}}{1} = 32 \text{ ounces}$$

$$\therefore 2 \text{ pounds} = 32 \text{ ounces}$$

Add the additional 3 ounces to find the weight of the package. The package weighs 35 ounces.

$$32 \text{ ounces} + 3 \text{ ounces} = 35 \text{ ounces}$$

There are 34 additional ounces, since 35 – 1 = 34.

Apply the pricing formula. $0.44 for the first ounce and $0.17 for each additional ounce.

$$\$0.44(1) + \$0.17(34)$$

$$\$0.44 + \$5.78$$

$$\$0.44 + \$5.78 = \$6.22$$

Answer: It will cost $6.22 to mail a package that weighs 2 pounds 3 ounces.

Try It 8.1.4

The average weight of a northern Bluefin tuna is 1800 pounds. The average weight of a great white shark is $2\frac{1}{2}$ tons. On average, how much more does a great white shark weigh, in pounds, than a northern Bluefin tuna?

Answer

$2\frac{1}{2}$ tons = 5000 pounds. 5000 pounds – 1800 pounds = 3200 pounds.

Summary

In the U.S. customary system of measurement, weight is measured in three units: ounces, pounds, and tons. A pound is equivalent to 16 ounces, and a ton is equivalent to 2000 pounds. While an object's weight can be described using any of these units, it is typical to describe very heavy objects using tons and very light objects using an ounce. Pounds are used to describe the weight of many objects and people. Often, in order to compare the weights of two objects or people or to solve problems involving weight, you must convert from one unit of measurement to another unit of measurement. Using conversion factors with the factor label method is an effective strategy for converting units and solving problems.

Capacity

Introduction

Capacity is the amount of liquid (or other pourable substance) that an object can hold when it's full. When a liquid, such as milk, is being described in gallons or quarts, this is a measure of capacity.

Understanding units of capacity can help you solve problems like this: Sven and Johanna were hosting a potluck dinner. They did not ask their guests to tell them what they would be bringing, and three people ended up bringing soup. Erin brought 1 quart, Richard brought 3 pints, and LeVar brought 9 cups. How many cups of soup did they have all together?

Units of Capacity

There are five main units for measuring capacity in the U.S. customary measurement system. The smallest unit of measurement is a **fluid ounce**. "Ounce" is also used as a measure of weight, so it is important to use the word "fluid" with ounce when you are talking about capacity. Sometimes the prefix "fluid" is not used when it is clear from the context that the measurement is capacity, not weight.

The other units of capacity in the customary system are the **cup**, **pint**, **quart**, and **gallon**. The table below describes each unit of capacity and provides an example to illustrate the size of the unit of measurement.

Fluid Ounce A unit of capacity equal to $\frac{1}{8}$ of a cup. One fluid ounce of water at 62°F weighs about one ounce. The amount of liquid medicine is often measured in fluid ounces.	
Cup A unit equal to 8 fluid ounces. The capacity of a standard measuring cup is one cup.	
Pint A unit equal to 16 fluid ounces, or 2 cups. The capacity of a carton of ice cream is often measured in pints.	
Quart A unit equal to 32 fluid ounces, or 4 cups. You often see quarts of milk being sold in the supermarket.	
Gallon A unit equal to 4 quarts, or 128 fluid ounces. When you fill up your car with gasoline, the price of gas is often listed in dollars per gallon.	

You can use any of these five measurement units to describe the capacity of an object, but it makes more sense to use certain units for certain purposes. For example, it makes more sense to describe the capacity of a swimming pool in gallons and the capacity of an expensive perfume in fluid ounces. However, unlike the units for length and weight, the units of capacity are not that far apart and often multiple units would be appropriate for one situation.

Sometimes you will need to convert between units of measurement. For example, you might want to express 5 gallons of lemonade in cups if you are trying to determine how many 8-fluid ounce servings the amount of lemonade would yield.

The table below shows some of the most common equivalents and conversion factors for the five customary units of measurement of capacity.

Unit Equivalents	Conversion Factors (heavier to lighter units of measurement)	Conversion Factors (lighter to heavier units of measurement)
1 cup = 8 fluid ounces	$\frac{1 \text{ cup}}{8 \text{ fluid ounces}}$	$\frac{8 \text{ fluid ounces}}{1 \text{ cup}}$
1 pint = 2 cups	$\frac{1 \text{ pint}}{2 \text{ cups}}$	$\frac{2 \text{ cups}}{1 \text{ pint}}$
1 quart = 2 pints	$\frac{1 \text{ quart}}{2 \text{ pints}}$	$\frac{2 \text{ pints}}{1 \text{ quart}}$
1 quart = 4 cups	$\frac{1 \text{ quart}}{4 \text{ cups}}$	$\frac{4 \text{ cups}}{1 \text{ quart}}$
1 gallon = 4 quarts	$\frac{1 \text{ gallon}}{4 \text{ quarts}}$	$\frac{4 \text{ quarts}}{1 \text{ gallon}}$
1 gallon = 16 cups	$\frac{1 \text{ gallon}}{16 \text{ cups}}$	$\frac{16 \text{ cups}}{1 \text{ gallon}}$

Converting Between Units of Capacity

As with converting units of length and weight, you can use the factor label method to convert from one unit of capacity to another. An example of this method is shown below.

Example 8.1.10

How many pints is $2\frac{3}{4}$ gallons?

Solution

Begin by reasoning about your answer. Since a gallon is larger than a pint, expect the answer in pints to be a number greater than $2\frac{3}{4}$.

$$2\frac{3}{4} \text{ gallons} = \_\_\_\_\_ \text{ pints}$$

The table above does not contain a conversion factor for gallons and pints, so you cannot convert it in one step. However, you can use quarts as an intermediate unit, as shown here. Set up the equation so that two sets of labels cancel—gallons and quarts.

$$\frac{11 \text{ gallons}}{4} \cdot \frac{4 \text{ quarts}}{1 \text{ gallon}} \cdot \frac{2 \text{ pints}}{1 \text{ quart}} = \_\_\_\_\_ \text{ pints}$$

$$\frac{11}{4} \cdot \frac{4}{1} \cdot \frac{2 \text{ pints}}{1} = \_\_\_\_\_ \text{ pints}$$

Multiply and simplify.

$$\frac{11 \cdot 4 \cdot 2 \text{ pints}}{4 \cdot 1 \cdot 1} = \_\_\_\_\_ \text{ pints}$$

$$\frac{88 \text{ pints}}{4} = 22 \text{ pints}$$

Answer: $2\frac{3}{4}$ gallons is 22 pints.

Example 8.1.11

How many gallons is 32 fluid ounces?

Solution

Begin by reasoning about your answer. Since gallons is a larger unit than fluid ounces, expect the answer to be less than 32.

$$32 \text{ fluid ounces} = \_\_\_\_\_ \text{ gallons}$$

The table above does not contain a conversion factor for gallons and fluid ounces, so you cannot convert it in one step. Use a series of intermediate units, as shown here.

$$\frac{32 \text{ fluid ounces}}{1} \cdot \frac{1 \text{ cup}}{8 \text{ fluid ounces}} \cdot \frac{1 \text{ pint}}{2 \text{ cups}} \cdot \frac{1 \text{ quart}}{2 \text{ pints}} \cdot \frac{1 \text{ gallon}}{4 \text{ quarts}} \quad \_\_\_\_\_ \text{ gallons}$$

Cancel units that appear in both the numerator and denominator.

$$\frac{32}{1} \cdot \frac{1}{8} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1 \text{ gallon}}{4} = \_\_\_\_\_ \text{ gallons}$$

Multiply and simplify.

$$\frac{32 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \text{ gallon}}{1 \cdot 8 \cdot 2 \cdot 2 \cdot 4} = \_\_\_\_\_ \text{ gallons}$$

$$\frac{32 \text{ gallons}}{128} = \frac{1}{4} \text{ gallons}$$

Answer: 32 fluid ounces is the same as $\frac{1}{4}$ gallon.

Try It 8.1.5

Find the sum of 4 gallons and 2 pints. Express your answer in cups.

Answer

Each gallon has 16 cups, so $4 \cdot 16 = 64$ will give you the number of cups in 4 gallons. Each pint has 2 cups, so $2 \cdot 2 = 4$ will give you the number of cups in 2 pints. $64 + 4 = 68$ cups.

Applying Unit Conversions

There are times when you will need to combine measurements that are given in different units. In order to do this, you need to convert first so that the units are the same.

Consider the situation posed earlier in this topic.

Example 8.1.12

Sven and Johanna were hosting a potluck dinner. They did not ask their guests to tell them what they would be bringing, and three people ended up bringing soup. Erin brought 1 quart, Richard brought 3 pints, and LeVar brought 9 cups. How much soup did they have total?

Solution

Since the problem asks for the total amount of soup, you must add the three quantities. Before adding, you must convert the quantities to the same unit.

$$1 \text{ quart} + 3 \text{ pints} + 9 \text{ cups}$$

The problem does not require a particular unit, so you can choose. Cups might be the easiest computation because going to a smaller unit will multiply the numbers and not divide and yield fractions.

Converting 1 quart to cups is given in the table of equivalents.

$$1 \text{ quart} = 4 \text{ cups}$$

Use the factor label method to convert pints to cups.

$$\frac{3 \text{ pints}}{1} \cdot \frac{2 \text{ cups}}{1 \text{ pint}} = \_\_\_\_\_ \text{cups}$$

$$\frac{3}{1} \cdot \frac{2 \text{ cups}}{1} = 6 \text{ cups}$$

Add the 3 quantities.

$$4 \text{ cups} + 6 \text{ cups} + 9 \text{ cups} = 19 \text{ cups}$$

Answer: There are 19 cups of soup for the dinner.

Example 8.1.13

Natasha is making lemonade to bring to the beach. She has two containers. One holds one gallon and the other holds 2 quarts. If she fills both containers, how many cups of lemonade will she have?

Solution

This problem requires you to find the sum of the capacity of each container and then convert that sum to cups.

$$1 \text{ gallon} + 2 \text{ quarts} = \_\_\_\_\_ \text{ cups}$$

First, find the sum in quarts. 1 gallon is equal to 4 quarts.

$$4 \text{ quarts} + 2 \text{ quarts} = 6 \text{ quarts}$$

Since the problem asks for the capacity in cups, convert 6 quarts to cups.

$$\frac{6 \text{ quarts}}{1} \cdot \frac{2 \text{ pints}}{1 \text{ quart}} \cdot \frac{2 \text{ cups}}{1 \text{ pint}} = \_\_\_\_\_ \text{ cups}$$

Cancel units that appear in both the numerator and denominator.

$$\frac{6}{1} \cdot \frac{2}{1} \cdot \frac{2 \text{ cups}}{1} = \_\_\_\_\_ \text{ cups}$$

Multiply.

$$6 \cdot 2 \cdot 2 = 24 \text{ cups}$$

Answer: Natasha will have 24 cups of lemonade.

Another way to work the problem above would be to first change 1 gallon to 16 cups and change 2 quarts to 8 cups. Then add: 16 + 8 = 24 cups.

Try It 8.1.6

Alan is making chili. He is using a recipe that makes 24 cups of chili. He has a 5-quart pot and a 2- gallon pot and is trying to determine whether the chili will all fit in one of these pots. Which of the pots will fit the chili?

Answer

The chili will fit into the 2-gallon pot only. $5 \text{ quarts} = 5 \cdot 4 \text{ cups} = 20 \text{ cups}$, so 24 cups of chili will not fit into the 5-quart pot. $2 \text{ gallons} = 32 \text{ cups}$, so 24 cups of chili will fit in this pot.

Summary

There are five basic units for measuring capacity in the U.S. customary measurement system. These are the fluid ounce, cup, pint, quart, and gallon. These measurement units are related to one another, and capacity can be described using any of the units. Typically, people use gallons to describe larger quantities and fluid ounces, cups, pints, or quarts to describe smaller quantities. Often, in order to compare or to solve problems involving the amount of liquid in a container, you need to convert from one unit of measurement to another.

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8.2: Metric Units of Measurement

Learning Objectives

1. Describe the general relationship between the U.S. customary units and metric units of length, weight/mass, and capacity.
2. Define the metric prefixes and use them to perform basic conversions among metric units.
3. Perform arithmetic calculations on metric units of length, mass, and capacity.
4. Solve application problems involving metric units of length, mass, and volume.

Introduction

In the United States, both the **U.S. customary measurement system** and the **metric system** are used, especially in medical, scientific, and technical fields. In most other countries, the metric system is the primary system of measurement. If you travel to other countries, you will see that road signs list distances in kilometers and milk is sold in liters. People in many countries use words like “kilometer,” “liter,” and “milligram” to measure the length, capacity, and weight of different objects. These measurement units are part of the metric system.

Unlike the U.S. customary system of measurement, the metric system is based on 10s. For example, a liter is 10 times larger than a deciliter, and a centigram is 10 times larger than a milligram. This idea of “10” is not present in the U.S. customary system—there are 12 inches in a foot, and 3 feet in a yard…and 5280 feet in a mile! The benefit of the metric system is that the conversion works identically for any unit, i.e. converting grams to kilograms is the same as converting meters to kilometers. This system is not limited to units describing length, capacity, and weight, but also applies to units measuring time (seconds), current (amperes), power (watts) and many more.

So, what if you have to find out how many milligrams are in a decigram? Or, what if you want to convert meters to kilometers? Understanding how the metric system works is a good start.

What is Metric?

The metric system uses units such as meter, liter, and gram to measure length, capacity, and mass, just as the U.S. customary system uses feet, quarts, and ounces to measure these.

In addition to the difference in the basic units, the metric system is based on factors of 10, and different measures for length include kilometer, meter, decimeter, centimeter, and millimeter. Notice that the word “meter” is part of all of these units. The prefix determines the "size" of the unit and indicates a fixed power of 10. For example "kilo" represents $10^3 = 1000$.

The metric system also applies the idea that units within the system get larger or smaller by a power of 10. This means that a meter is 100 times larger than a centimeter, and a kilogram is 1000 times heavier than a gram. You will explore this idea a bit later. For now, notice how this idea of “getting bigger or smaller by 10” is very different than the relationship between units in the U.S. customary system, where 3 feet equals 1 yard, and 16 ounces equals 1 pound.

Length, Mass, and Capacity

The table below shows the basic units of the metric system. Note that the names of all metric units follow from these three basic units.

Length	Mass	Capacity
	basic units	
meter	gram	liter
	other units you may see	
kilometer	kilogram	kiloliter
centimeter	centigram	centiliter
millimeter	milligram	milliliter

In the metric system, the basic unit of length is the meter. A meter is slightly larger than a yardstick, or just over three feet.

The basic metric unit of mass is the gram. A regular-sized paperclip has a mass of about 1 gram.

Among scientists, one gram is defined as the mass of water that would fill a 1-centimeter cube. You may notice that the word "mass" is used here instead of "weight." In the sciences and technical fields, a distinction is made between weight and mass. Weight is a measure of the pull of gravity on an object. For this reason, an object's weight would be different if it was weighed on Earth or on the moon because of the difference in the gravitational forces. However, the object's mass would remain the same in both places because mass measures the amount of substance in an object. As long as you are planning on only measuring objects on Earth, you can use mass/weight fairly interchangeably—but it is worth noting that there is a difference!

Finally, the basic metric unit of capacity is the liter. A liter is slightly larger than a quart.

		
The handle of a shovel is about 1 meter.	A paperclip weighs about 1 gram.	A medium-sized container of milk is about 1 liter.

Though it is rarely necessary to convert between the customary and metric systems, sometimes it helps to have a mental image of how large or small some units are. The table below shows the relationship between some common units in both systems.

	Common Measurements in Customary and Metric Systems
Length	1 centimeter is a little less than half an inch.
	1.6 kilometers is about 1 mile.
	1 meter is about 3 inches longer than 1 yard.
Mass	1 kilogram is a little more than 2 pounds
	28 grams is about the same as 1 ounce.
Capacity	1 liter is a little more than 1 quart.
	4 liters is a little more than 1 gallon.

Prefixes in the Metric System

The metric system is a base-10 system. This means that each successive unit is 10 times larger than the previous one.

The names of metric units are formed by adding a prefix to the basic unit of measurement. To tell how large or small a unit is, you look at the prefix. To tell whether the unit is measuring length, mass, or capacity, you look at the base.

Prefixes in the Metric System						
kilo-	*hecto-*	*deka-*	*meter/gram/liter*	*deci-*	*centi-*	*milli-*
1000 times **larger** than base unit	100 times **larger** than base unit	10 times **larger** than base unit	base units	10 times **smaller** than base unit	100 times **smaller** than base unit	1000 times **smaller** than base unit

Using this table as a reference, you can see the following:

- A kilogram is 1000 times larger than one gram (so 1 kilogram = 1000 grams).

- A centimeter is 100 times smaller than one meter (so 1 meter = 100 centimeters).
- A dekaliter is 10 times larger than one liter (so 1 dekaliter = 10 liters).

Here is a similar table that just shows the metric units of measurement for mass, along with their size relative to 1 gram (the base unit). The common abbreviations for these metric units have been included as well.

Prefixes in the Metric System						
kilogram (kg)	hectogram (hg)	dekagram (dag)	gram (g)	decigram (dg)	centigram (cg)	milligram (mg)
1000 gram	100 gram	10 gram	gram	0.1 gram	0.01 gram	0.001 gram

Since the prefixes remain constant through the metric system, you could create similar charts for length and capacity. The prefixes have the same meanings whether they are attached to the units of length (meter), mass (gram), or capacity (liter).

Try It 8.2.1

Which of the following sets of three units are all metric measurements of length?

A. inch, foot, yard
B. kilometer, centimeter, millimeter
C. kilogram, gram, centigram
D. kilometer, foot, decimeter

Answer

A. Incorrect. Although these units do measure length, they are all units of measurement from the U.S. customary system.
B. Correct. All of these measurements are from the metric system. You can tell they are measurements of length because they all contain the word “meter.”
C. Incorrect. These measurements are from the metric system, but they are measurements of mass, not length.
D. Incorrect. Kilometer and decimeter are metric units of length, but foot is not.

Converting Units Up and Down the Metric Scale

Converting between metric units of measure requires knowledge of the metric prefixes and an understanding of the decimal system —that’s about it.

For instance, you can figure out how many centigrams are in one dekagram by using the table above. One dekagram is larger than one centigram, so you expect that one dekagram will equal many centigrams.

In the table, each unit is 10 times larger than the one to its immediate right. This means that 1 dekagram = 10 grams; 10 grams = 100 decigrams; and 100 decigrams = 1000 centigrams. So, 1 dekagram = 1000 centigrams.

Example 8.2.1

How many milligrams are in one decigram?

Solution

Identify locations of milligrams and decigrams.

kg	hg	dag	g	**dg**	cg	**mg**

Decigrams (dg) are larger than milligrams (mg), so you expect there to be many mg in one dg.

dg is 10 times larger than a cg, and a cg is 10 times larger than a mg.

kg	hg	dag	g	**dg**	cg	**mg**
				→	→ → →	→
					x10	x10

Since you are going from a larger unit to a smaller unit, multiply.

Multiply: $1 \cdot 10 \cdot 10$, to find the number of milligrams in one decigram.

$$1 \text{ dg} \cdot 10 \cdot 10 = 100 \text{ mg}$$

Answer: There are 100 milligrams (mg) in 1 decigram (dg).

Example 8.2.2

Convert 1 centimeter to kilometers.

Solution

Identify locations of kilometers and centimeters.

km	hm	dam	m	dm	**cm**	mm

Kilometers (km) are larger than centimeters (cm), so you expect there to be less than one km in a cm.

cm is 10 times smaller than a dm; a dm is 10 times smaller than a m, etc.

km	hm	dam	m	dm	**cm**	mm
←	← ← ←	← ← ←	← ← ←	← ← ←	←	
÷10	÷10	÷10	÷10	÷10		

Since you are going from a smaller unit to a larger unit, divide.

Divide: $1 \div 10 \div 10 \div 10 \div 10 \div 10$, to find the number of kilometers in one centimeter.

$$1 \text{ cm} \div 10 \div 10 \div 10 \div 10 \div 10 = 0.00001 \text{ km}$$

Answer: 1 centimeter (cm) = 0.00001 kilometers (km).

Once you begin to understand the metric system, you can use a shortcut to convert among different metric units. The size of metric units increases tenfold as you go up the metric scale. The decimal system works the same way: a tenth is 10 times larger than a hundredth; a hundredth is 10 times larger than a thousandth, etc. By applying what you know about decimals to the metric system, converting among units is as simple as moving decimal points.

Here is the first problem from above: How many milligrams are in one decigram? You can recreate the order of the metric units as shown below:

kg hg dag g dg cg mg

This question asks you to start with 1 decigram and convert that to milligrams. As shown above, milligrams is two places to the right of decigrams. You can just move the decimal point two places to the right to convert decigrams to milligrams: $1.\text{ dg} = 100.\text{ mg}$ (Note the location of the decimal points).

The same method works when you are converting from a smaller to a larger unit, as in the problem: Convert 1 centimeter to kilometers.

km hm dam m dm cm mm

Note that instead of moving to the right, you are now moving to the left—so the decimal point must do the same: $1.\ \text{cm} = 0.00001\ \text{km}$.

Try It 8.2.2

How many milliliters are in 1 liter?

Answer

There are 10 milliliters in a centiliter, 10 centiliters in a deciliter, and 10 deciliters in a liter. Multiply: $10 \cdot 10 \cdot 10$, to find the number of milliliters in a liter, which comes out 1000 milliliters in a liter.

Summary

The metric system is an alternative system of measurement used in most countries, as well as in the United States. The metric system is based on joining one of a series of prefixes, including kilo- , hecto-, deka-, deci-, centi-, and milli-, with a base unit of measurement, such as meter, liter, or gram. Units in the metric system are all related by a power of 10, which means that each successive unit is 10 times larger than the previous one. This makes converting one metric measurement to another a straightforward process, and is often as simple as moving a decimal point. It is always important, though, to consider the direction of the conversion. If you are converting a smaller unit to a larger unit, then the decimal point has to move to the left (making your number smaller); if you are converting a larger unit to a smaller unit, then the decimal point has to move to the right (making your number larger).

Converting within the Metric System

Introduction

While knowing the different units used in the metric system is important, the real purpose behind learning the metric system is for you to be able to use these measurement units to calculate the size, mass, or capacity of different objects. In practice, it is often necessary to convert one metric measurement to another unit—this happens frequently in the medical, scientific, and technical fields, where the metric system is commonly used.

If you have a prescription for 5000 mg of medicine, and upon getting it filled, the dosage reads 5 g of medicine, did the pharmacist make a mistake?

For a moment, imagine that you are a pharmacist. You receive three prescriptions for liquid amoxicillin: one calls for 2.5 centiliters, one calls for 0.3 deciliters, and one calls for 450 milliliters. Amoxicillin is stored in the refrigerator in 1 liter, 1 deciliter, and 1 centiliter containers. Which container should you use to ensure you are not wasting any of the unused drug?

To solve this problem, you need to know how to convert from one measurement to another as well as how to add different quantities together. Let's take a look at how to do this.

Converting from Larger to Smaller Units

Converting between measurements in the metric system is simply a matter of identifying the unit that you have, the unit that you want to convert to, and then counting the number of units between them. A basic example of this is shown below.

Example 8.2.3

Convert 1 kilometer to decimeters

Solution

Identify locations of kilometers and decimeters.

km	hm	dam	m	**dm**	cm	mm

Kilometers (km) are larger than decimeters (dm), so you expect there to be more than one dm in a km.

km	hm	dam	m	**dm**	cm	mm
→	→ → →	→ → →	→ → →	→		
	x10	x10	x10	x10		

Count the intermediate units, multiplying by 10 as you go. (Since you are going from a larger unit to a smaller unit, you multiply.)

Multiply to find the number of decimeters in one kilometer.

$$1 \text{ km} \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 10000 \text{ dm}$$

Answer: 1 kilometer = 10,000 decimeters

This problem is straightforward because you are converting 1 kilometer to another unit. The example below shows how you would solve this problem if you were asked to convert 8.2 kilometers to decimeters. Notice that most steps are the same; the critical difference is that you multiply by 8.2 in the final step.

Example 8.2.4

Convert 8.2 kilometers to decimeters

Solution

Identify locations of kilometers and decimeters.

km	hm	dam	m	**dm**	cm	mm

Kilometers (km) are larger than decimeters (dm), so you expect there to be more than one dm in a km.

km	hm	dam	m	**dm**	cm	mm
→	→ → →	→ → →	→ → →	→		
	x10	x10	x10	x10		

Count the intermediate units, multiplying by 10 as you go. Since you are going from a larger unit to a smaller unit, multiply.

Multiply to find the number of decimeters in 8.2 kilometers.

$$8.2 \text{ km} \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 82,000 \text{ dm}$$

Answer: 8.2 kilometer = 82,000 decimeters

You can also apply the rules of base 10 to use the “move the decimal” shortcut method in this example. Notice how decimeters (dm) is four places to the right of kilometers (km); similarly, you move the decimal point four places to the right when converting 8.2 kilometers to decimeters.

km hm dam m dm cm mm

8.2000 km = 82000. dm

8.2 km = 82000 dm

Example 8.2.5

Convert 0.55 liters to centiliters.

Solution

Count two places from liters to centiliters.

kl hl dal l dl cl ml

In 0.55 l, move the decimal point two places to the right.

$$0.55 \text{ l} = 055.\text{cl}$$

$$0.55 \text{ l} = 55 \text{ cl}$$

Answer: 0.55 liters = 55 centiliters

Try It 8.2.3

How many dekaliters are in 0.5 deciliters?

Answer

One deciliter is 100 times smaller than a dekaliter, so you move the decimal point two places to the left to convert 0.5 deciliters to 0.005 dekaliters.

Converting from Smaller to Larger Units

You can use similar processes when converting from smaller to larger units. When converting a larger unit to a smaller one, you multiply; when you convert a smaller unit to a larger one, you divide. Here is an example.

Example 8.2.6

Convert 739 centigrams to grams.

Solution

Identify locations of centigrams and grams.

kg	hg	dag	**g**	dg	**cg**	mg

Centigrams (cg) are smaller than grams (g), so you expect there to be less than 739 g in 739 cg.

Count the intermediate units, dividing by 10 as you go. Since you are going from a smaller unit to a larger unit, divide.

kg	hg	dag	**g**	dg	**cg**	mg
			←	← ← ←	←	
			÷10	÷10		

Since you are going from a smaller unit to a larger unit, divide.

Divide to find the number of grams in 739 centigrams.

$$\mathbf{739 \div 10 \div 10 = 7.39 \text{ g}}$$

Answer: 739 centigrams = 7.39 grams

Notice that the shortcut method of counting prefixes and moving the decimal the same number of places also works here. Just make sure you are moving the decimal point in the correct direction for the conversion.

Example 8.2.7

Convert 205.5 milliliters to kiloliters.

Solution

Count six places from milliliters to kiloliters.

kl hl dal l dl cl ml

Milliliter is smaller than kiloliter, so you expect the number 205.5 to get smaller as you move up the metric chart.

In 205.5 ml, move the decimal point six places to the left.

$$205.5 \text{ ml} = 0.0002055\text{kl}$$

Answer: 205.5 milliliters = 0.0002055 kiloliters

Try It 8.2.4

Convert 3085 milligrams to grams.

Answer

One gram is 1000 times larger than a milligram, so you can move the decimal point in 3085 three places to the left to get 3.085 grams.

Factor Label Method

There is yet another method that you can use to convert metric measurements—the **factor label method (dimensional analysis)**. You used this method when you were converting measurement units within the U.S. customary system.

The factor label method works the same in the metric system; it relies on the use of unit fractions and the reducing of intermediate units. The table below shows some of the **unit equivalents** and **unit fractions** for length in the metric system. (You should notice that all of the unit fractions contain a factor of 10. Remember that the metric system is based on the notion that each unit is 10 times larger than the one that came before it.)

Also, notice that two new prefixes have been added here: mega- (which is very big) and micro- (which is very small). Mega is abbbreviated with the prefix capital "M" (since lower case "m" is milli} and micro is represented with the prefix of the greek letter "mu" written with the symbol μ

Unit Equivalents	Conversion Factors	
1 meter = 1,000,000 micrometers	$\frac{1\text{ m}}{1,000,000\ \mu\text{m}}$	$\frac{1,000,000\ \mu\text{m}}{1\text{ m}}$
1 meter = 1000 millimeters	$\frac{1\text{ m}}{1000\text{ mm}}$	$\frac{1000\text{ mm}}{1\text{ m}}$
1 meter = 100 centimeters	$\frac{1\text{ m}}{100\text{ cm}}$	$\frac{100\text{ cm}}{1\text{ m}}$
1 meter = 10 decimeters	$\frac{1\text{ m}}{10\text{ dm}}$	$\frac{10\text{ dm}}{1\text{ m}}$
1 dekameter = 10 meters	$\frac{1\text{ dam}}{10\text{ m}}$	$\frac{10\text{ m}}{1\text{ dam}}$
1 hectometer = 100 meters	$\frac{1\text{ hm}}{100\text{ m}}$	$\frac{100\text{ m}}{1\text{ hm}}$
1 kilometer = 1000 meters	$\frac{1\text{ km}}{1000\text{ m}}$	$\frac{1000\text{ m}}{1\text{ km}}$
1 megameter = 1,000,000 meters	$\frac{1\text{ Mm}}{1,000,000\text{ m}}$	$\frac{1,000,000\text{ m}}{1\text{ Mm}}$

When applying the factor label method in the metric system, be sure to check that you are not skipping over any intermediate units of measurement.

Example 8.2.8

Convert 7225 centimeters to meters.

Solution

Meters is larger than centimeters, so you expect your answer to be less than 7225.

$$7225 \text{ cm} = \_\_\_\_\_ \text{m}$$

Using the factor label method, write 7225 cm as a fraction and use unit fractions to convert it to m.

$$\frac{7225 \text{ cm}}{1} \cdot \frac{1 \text{ m}}{100 \text{ cm}} = \_\_\_\_\_ \text{m}$$

Cancel similar units, multiply, and simplify.

$$\frac{7225}{1} \cdot \frac{1 \text{ m}}{100} = \frac{7225}{100} \text{ m}$$
$$\frac{7225 \text{ m}}{100} = 72.25 \text{ m}$$

Answer: 7225 centimeters = 72.25 meters

Try It 8.2.5

Using whichever method you prefer, convert 32.5 kilometers to meters.

Answer

To find the number of meters in 32.5 kilometers, you can set up the following conversion: $\frac{32.5 \text{ kilometers}}{1} \cdot \frac{1000 \text{ meters}}{1 \text{ kilometer}} = \frac{32,500 \text{ meters}}{1}$. The kilometer units cancel, leaving the answer as 32,500 meters.

Now that you have seen how to convert among metric measurements in multiple ways, let's revisit the problem posed earlier.

Example 8.2.9

If you have a prescription for 5000 mg of medicine, and upon getting it filled, the dosage reads 5 g of medicine, did the pharmacist make a mistake?

Solution

Need to convert mg to g.

$$5000 \text{ mg} = \_\_\_\_\_ \text{g?}$$

$$\frac{5000 \text{ mg}}{1} \cdot \frac{1 \text{ g}}{1000 \text{ mg}} = \_\_\_\_\_ \text{g}$$

'_' allowed only in math mode

$$\frac{5000 \cdot 1 \text{ g}}{1 \cdot 1000} = \frac{5000 \text{ g}}{1000}$$
$$\frac{5000 \text{ g}}{1000} = 5 \text{ g}$$

Answer: 5 g = 5000 mg, so the pharmacist did not make a mistake.

Summary

To convert among units in the metric system, identify the unit that you have, the unit that you want to convert to, and then count the number of units between them. If you are going from a larger unit to a smaller unit, you multiply by 10 successively. If you are

going from a smaller unit to a larger unit, you divide by 10 successively. The factor label method can also be applied to conversions within the metric system. To use the factor label method, you multiply the original measurement by unit fractions; this allows you to represent the original measurement in a different measurement unit.

Using Metric Conversions to Solve Problems

Introduction

Learning how to solve real-world problems using metric conversions is as important as learning how to do the conversions themselves. Mathematicians, scientists, nurses, and even athletes are often confronted with situations where they are presented with information using metric measurements, and must then make informed decisions based on that data.

To solve these problems effectively, you need to understand the context of a problem, perform conversions, and then check the reasonableness of your answer. Do all three of these steps and you will succeed in whatever measurement system you find yourself using.

Understanding Context and Performing Conversions

The first step in solving any real-world problem is to understand its context. This will help you figure out what kinds of solutions are reasonable (and the problem itself may give you clues about what types of conversions are necessary). Here is an example.

Example 8.2.10

In the Summer Olympic Games, athletes compete in races of the following lengths: 100 meters, 200 meters, 400 meters, 800 meters, 1500 meters, 5000 meters and 10,000 meters. If a runner were to run in all these races, how many kilometers would he run?

Solution

To figure out how many kilometers he would run, you need to first add all of the lengths of the races together and *then* convert that measurement to kilometers.

$$10000\ \text{m} + 5000\ \text{m} + 1500\ \text{m} + 800\ \text{m} + 400\ \text{m} + 200\ \text{m} + 100\ \text{m} = 18000\ \text{m}$$

Use the factor label method and unit fractions to convert from meters to kilometers.

$$\frac{18000\ \text{m}}{1} \cdot \frac{1\ \text{km}}{1000\ \text{m}} = \_\_\_\_\_\ \text{km}$$

Cancel, multiply, and solve.

$$\frac{18000}{1} \cdot \frac{1\ \text{km}}{1000} = \frac{18000\ \text{km}}{1000}$$

$$\frac{18000\ \text{km}}{1000} = 18\ \text{km}$$

Answer: The runner would run 18 kilometers.

This may not be likely to happen (a runner would have to be quite an athlete to compete in all of these races) but it is an interesting question to consider. The problem required you to find the total distance that the runner would run (in kilometers). The example showed how to add the distances, in meters, and then convert that number to kilometers.

An example with a different context, but still requiring conversions, is shown below.

Example 8.2.11

One bottle holds 295 dl while another one holds 28,000 ml. What is the difference in capacity between the two bottles?

Solution

The two measurements are in different units. You can convert both units to liters and then compare them

$$295\ \text{dl} = \_\_\_\_\_\ \text{l}$$

$$28000\ \text{ml} = \_\_\_\_\_\ \text{l}$$

Convert dl to liters.

$$\frac{295 \text{ dl}}{1} \cdot \frac{1 \text{ l}}{10 \text{ dl}} = \_\_\_\_\_ \text{l}$$

Cancel common units and multiply.

$$\frac{295}{1} \cdot \frac{1 \text{ l}}{10} = \_\_\_\_\_ \text{l}$$

295 dl = 29.5 liters.

$$\frac{295 \text{ l}}{10} = 29.5 \text{ l}$$

Convert ml to liters.

$$\frac{28000 \text{ ml}}{1} \cdot \frac{1 \text{ l}}{1000 \text{ ml}} = \_\_\_\_\_ \text{l}$$

$$\frac{28000}{1} \cdot \frac{1 \text{ l}}{1000} = \frac{28000 \text{ l}}{1000}$$

28,000 ml = 28 liters

$$\frac{28000 \text{ l}}{1000} = 28 \text{ l}$$

The question asks for "difference in capacity" between the bottles.

$$29.5 \text{ liters} – 28 \text{ liters} = 1.5 \text{ liters}$$

Answer: There is a difference in capacity of 1.5 liters between the two bottles.

This problem asked for the difference between two quantities. The easiest way to find this is to convert one quantity so that both quantities are measured in the same unit, and then subtract one from the other. Here you could have also chosen to convert both numbers to deciliters or both numbers to milliliters instead of liters.

Try It 8.2.6

One boxer weighs in at 85 kg. He is 80 dag heavier than his opponent. How much does his opponent weigh?

Answer

His opponent weights 84.2 kg since $80 \text{ dekagrams} = 0.8 \text{ kilograms}$, and $85 - 0.8 = 84.2$.

Checking your Conversions

Sometimes it is a good idea to check your conversions using a second method. This usually helps you catch any errors that you may make, such as using the wrong unit fractions or moving the decimal point the wrong way.

Example 8.2.12

A two-liter bottle contains 87 centiliters of oil and 4.1 deciliters of water. How much more liquid is needed to fill the bottle?

Solution

You are looking for the amount of liquid needed to fill the bottle. Convert both measurements to liters and then solve the problem.

$$87 \text{ cl} + 4.1 \text{ dl} + \_\_\_\_\_ = 2 \text{ l}$$

Convert 87 cl to liters.

$$87 \text{ cl} = \_\_\_\_\_ \text{l}$$

$$\frac{87 \text{ cl}}{1} \cdot \frac{1 \text{ l}}{100 \text{ cl}} = \_\_\_\_\_ \text{l}$$

$$\frac{87}{1} \cdot \frac{1 \text{ l}}{100} = \frac{87 \text{ l}}{100}$$

$$\frac{87\ \text{l}}{100} = 0.87\ \text{l}$$

Convert 4.1 dl to liters

$$4.1\ \text{dl} = \_\_\_\_\_\text{l}$$

$$\frac{4.1\ \text{dl}}{1} \cdot \frac{1\ \text{l}}{10\ \text{dl}} = \_\_\_\_\_\text{l}$$

$$\frac{4.1}{1} \cdot \frac{1\ \text{l}}{10} = \frac{4.1\ \text{l}}{10}$$

$$\frac{4.1\ \text{l}}{10} = 0.41\ \text{l}$$

Subtract to find how much more liquid is needed to fill the bottle.

$$87\ \text{cl} + 4.1\ \text{dl} + \_\_\_\_\_ = 2\ \text{l}$$

$$0.87\ \text{liter} + 0.41\ \text{liter} + \_\_\_\_\_ = 2\ \text{liters}$$

$$2\ \text{liters} - 0.87\ \text{liter} - 0.41\ \text{liter} = 0.72\ \text{liter}$$

Answer: The amount of liquid needed to fill the bottle is 0.72 liter.

Having come up with the answer, you could also check your conversions using the quicker "move the decimal" method, shown below.

Example 8.2.13

A two-liter bottle contains 87 centiliters of oil and 4.1 deciliters of water. How much more liquid is needed to fill the bottle?

Solution

You are looking for the amount of liquid needed to fill the bottle. Convert both measurements to liters and then solve the problem.

$$87\ \text{cl} + 4.1\ \text{dl} + \_\_\_\_\_ = 2\ \text{l}$$

Convert 87 cl to liters.

$$87\ \text{cl} = \_\_\_\_\_\text{l}$$

On the chart, l is two places to the left of cl.

kl hl dal **l** dl **cl** ml

Move the decimal point two places to the left in 87 cl.

.87. cl

87 cl = 0.87 l

Convert 4.1 dl to liters.

$$4.1\ \text{dl} = \_\_\_\_\_\text{l}$$

On the chart, l is one place to the left of dl.

kl hl dal **l** **dl** cl ml

Move the decimal point one place to the left in 4.1 dl.

.4.1 dl

4.1 dl= 0.41 l

Subtract to find how much more liquid is needed to fill the bottle.

$$87\ \text{cl} + 4.1\ \text{dl} + \_\_\_\_\_ = 2\ \text{l}$$

$$0.87 \text{ liter} + 0.41 \text{ liter} + \_\_\_\_\_ = 2 \text{ liters}$$

$$2 \text{ liters} - 0.87 \text{ liter} - 0.41 \text{ liter} = 0.72 \text{ liter}$$

Answer: The amount of liquid needed to fill the bottle is 0.72 liter.

The initial answer checks out—0.72 liter of liquid is needed to fill the bottle. Checking one conversion with another method is a good practice for catching any errors in scale.

Summary

Understanding the context of real-life application problems is important. Look for words within the problem that help you identify what operations are needed, and then apply the correct unit conversions. Checking your final answer by using another conversion method (such as the "move the decimal" method, if you have used the factor label method to solve the problem) can cut down on errors in your calculations.

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8.3: Temperature and Time

Learning Objectives

1. State the freezing and boiling points of water on the Celsius and Fahrenheit temperature scales.
2. Convert temperatures and times using conversion formulas.
3. Perform arithmetic calculations on temperature and time.
4. Solve application problems using temperature and time.

Introduction

Turn on the television any morning and you will see meteorologists talking about the day's weather forecast. In addition to telling you what the weather conditions will be like (sunny, cloudy, rainy, muggy), they also tell you the day's forecast for high and low temperatures. A hot summer day may reach 100° in Philadelphia, while a cool spring day may have a low of 40° in Seattle.

If you have been to other countries, though, you may notice that meteorologists measure heat and cold differently outside of the United States. For example, a TV weatherman in San Diego may forecast a high of 89°, but a similar forecaster in Tijuana, Mexico —which is only 20 miles south— may look at the same weather pattern and say that the day's high temperature is going to be 32°. What's going on here?

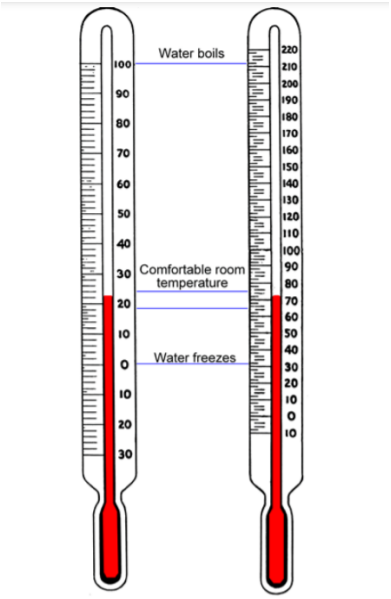
The difference is that the two countries use different temperature scales. In the United States, temperatures are usually measured using the **Fahrenheit** scale, while most countries that use the metric system use the **Celsius** scale to record temperatures. Learning about the different scales— including how to convert between them—will help you figure out what the weather is going to be like, no matter which country you find yourself in.

Measuring Temperature on Two Scales

Temperature

Temperature is used to describe the level of heat or cold of an object or environment.

Fahrenheit and Celsius are two different scales for measuring temperature.

A thermometer measuring a temperature of 22° Celsius is shown here.		A thermometer measuring a temperature of 72° Fahrenheit is shown here.
On the Celsius scale, water freezes at 0° and boils at 100°.		On the Fahrenheit scale, water freezes at 32° and boils at 212°.
If the United States were to adopt the Celsius scale, forecast temperatures would rarely go below -30° or above 45°. (A temperature of -18° may be forecast for a cold winter day in Michigan, while a temperature of 43° may be predicted for a hot summer day in Arizona.)		In the United States, forecast temperatures measured in Fahrenheit rarely go below -20° or above 120°. (A temperature of 0° may be forecast for a cold winter day in Michigan, while a temperature of 110° may be predicted for a hot summer day in Arizona.)
Most office buildings maintain an indoor temperature between 18°C and 24°C to keep employees comfortable.		Most office buildings maintain an indoor temperature between 65°F and 75°F to keep employees comfortable.

Try It 8.3.1

A cook puts a thermometer into a pot of water to see how hot it is. The thermometer reads 132°, but the water is not boiling yet. Which temperature scale is the thermometer measuring?

Answer

The thermometer is using Fahrenheit. Water boils at 212° on the Fahrenheit scale, so a measurement of 132° on a Fahrenheit scale is legitimate for hot (but non-boiling) water.

Converting Between the Scales

By looking at the two thermometers shown, you can make some general comparisons between the scales. For example, many people tend to be comfortable in outdoor temperatures between 50°F and 80°F (or between 10°C and 25°C). If a meteorologist predicts an average temperature of 0°C (or 32°F), then it is a safe bet that you will need a winter jacket.

Sometimes, it is necessary to convert a Celsius measurement to its exact Fahrenheit measurement or vice versa. For example, what if you want to know the temperature of your child in Fahrenheit, and the only thermometer you have measures temperature in Celsius measurement? Converting temperature between the systems is a straightforward process as long as you use the formulas provided below.

Temperature Conversion Formulas

To convert a Fahrenheit measurement to a Celsius measurement, use this formula.

$$C = \frac{5}{9}(F - 32) \tag{8.3.1}$$

To convert a Celsius measurement to a Fahrenheit measurement, use this formula.

$$F = \frac{9}{5}C + 32 \tag{8.3.2}$$

How were these formulas developed? They came from comparing the two scales. Since the freezing point is 0° in the Celsius scale and 32° on the Fahrenheit scale, we subtract 32 when converting from Fahrenheit to Celsius, and add 32 when converting from Celsius to Fahrenheit.

There is a reason for the fractions $\frac{5}{9}$ and $\frac{9}{5}$, also. There are 100 degrees between the freezing (0°) and boiling points (100°) of water on the Celsius scale and 180 degrees between the similar points (32° and 212°) on the Fahrenheit scale. Writing these two scales as a ratio, $\frac{F°}{C°}$, gives $\frac{180°}{100°} = \frac{180° \div 20}{100° \div 20} = \frac{9}{5}$. If you flip the ratio to be $\frac{C°}{F°}$, you get $\frac{100°}{180°} = \frac{100° \div 20}{180° \div 20} = \frac{5}{9}$. Notice how these fractions are used in the conversion formulas.

The example below illustrates the conversion of Celsius temperature to Fahrenheit temperature, using the boiling point of water, which is 100° C.

Example 8.3.1

The boiling point of water is 100°C. What temperature does water boil at in the Fahrenheit scale?

Solution

A Celsius temperature is given. To convert it to the Fahrenheit scale, use Equation 8.3.2.

$$F = \frac{9}{5}C + 32$$

Substitute 100 for C and multiply.

$$F = \frac{9}{5}(100) + 32$$

$$F = \frac{900}{5} + 32$$

Simplify $\frac{900}{5}$ by dividing numerator and denominator by 5.

$$F = \frac{900 \div 5}{5 \div 5} + 32$$

$$F = \frac{180}{1} + 32$$

Add 180 + 32.

$$F = 212$$

Answer: The boiling point of water is 212°F.

Example 8.3.2

Water freezes at 32°F. On the Celsius scale, what temperature is this?

Solution

A Fahrenheit temperature is given. To convert it to the Celsius scale, use Equation 8.3.1.

$$C = \frac{5}{9}(F - 32)$$

Substitute 32 for F and subtract.

$$C = \frac{5}{9}(32 - 32)$$

Any number multiplied by 0 is 0

$$C = \frac{5}{9}(0)$$

$$C = 0$$

Answer: The freezing point of water is 0°C.

The two previous problems used the conversion formulas to verify some temperature conversions that were discussed earlier—the boiling and freezing points of water. The next example shows how these formulas can be used to solve a real-world problem using different temperature scales.

Example 8.3.3

Two scientists are doing an experiment designed to identify the boiling point of an unknown liquid (remember different liquids have different boiling point). One scientist gets a result of 120°C; the other gets a result of 250°F. Which temperature is higher and by how much?

Solution

One temperature is given in °C, and the other is given in °F. To find the difference between them, we need to measure them on the same scale.

What is the difference between 120°C and 250°F?

Use the conversion formula to convert 120°C to °F. (You could convert 250°F to °C instead; this is explained in the text after this example.)

$$F = \frac{9}{5}C + 32$$

Substitute 120 for C.

$$F = \frac{9}{5}(120) + 32$$

Multiply.

$$F = \frac{1080}{5} + 32$$

Simplify $\frac{1080}{5}$ by dividing numerator and denominator by 5.

$$F = \frac{1080 \div 5}{5 \div 5} + 32$$

Add 216 + 32.

$$F = \frac{216}{1} + 32$$

You have found that 120°C = 248°F.

$$F = 248$$

To find the difference between 248°F and 250°F, subtract.

$$250°F – 248°F = 2°F$$

Answer: 250°F is the higher temperature by 2°F

You could have converted 250°F to °C instead, and then found the difference in the two measurements. (Had you done it this way, you would have found that 250°F = 121.1°C, and that 121.1°C is 1.1°C higher than 120°C.) Whichever way you choose, it is important to compare the temperature measurements within the same scale, and to apply the conversion formulas accurately.

Try It 8.3.2

Tatiana is researching vacation destinations, and she sees that the average summer temperature in Barcelona, Spain is around 26°C. What is the average temperature in degrees Fahrenheit?

Answer

Tatiana can find the Fahrenheit equivalent by solving the equation $F = \frac{9}{5}(26) + 32$. The result is 78.8° Fahrenheit, which rounds to 79° Fahrenheit.

Time

Time

Timeis used to describe the amount of existence between events.

Time is universal--everyone uses the same measurements of time. Some of the common units (along with their abbreviations) for time are listed in the following table.

Unit Conversion Table
1 minute (min) = 60 seconds (sec)
1 hour (hr) = 60 minutes
1 day (da) = 24 hours
1 week (wk) = 7 days
1 year (yr)= 52 weeks

Example 8.3.4

Convert 2,016 hr to weeks.

Solution

Looking in the unit conversion table under *time*, we see that $\mathbf{1\ wk = 7\ da}$ and that $\mathbf{1\ da = 24\ hr}$. To convert from hours to weeks, we must first convert from hours to days and then from days to weeks. We need two unit fractions.

The unit fraction needed for converting from hours to days is $\frac{\mathbf{1\ da}}{\mathbf{24\ hr}}$. The unit fraction needed for converting from days to weeks is $\frac{\mathbf{1\ wk}}{\mathbf{7\ da}}$.

$$\begin{aligned} \mathbf{2{,}016\ hr} &= \frac{\mathbf{2{,}016\ hr}}{\mathbf{1}} \cdot \frac{\mathbf{1\ da}}{\mathbf{24\ hr}} \cdot \frac{\mathbf{1\ wk}}{\mathbf{7\ da}} && \textbf{Divide out common units.} \\ &= \frac{\mathbf{2{,}016\ \cancel{hr}}}{\mathbf{1}} \cdot \frac{\mathbf{1\ \cancel{da}}}{\mathbf{24\ \cancel{hr}}} \cdot \frac{\mathbf{1\ wk}}{\mathbf{7\ \cancel{da}}} \\ &= \frac{\mathbf{2{,}016 \cdot 1\ wk}}{\mathbf{24 \cdot 7}} && \textbf{Simplify} \\ &= \mathbf{12\ wk} \end{aligned}$$

Thus, $\mathbf{2{,}016\ hr = 12\ wk}$.

Try It 8.3.3

Your employer tracks your accumulated vacation time in hours. You want to take a two week vacation and have accumulated 412 hours of paid time off. Do you have enough hours for your vacation?

Answer

To solve this we need to covert 412 hours to weeks.

The unit fraction needed for converting from hours to days is $\frac{\mathbf{1\ da}}{\mathbf{24\ hr}}$. The unit fraction needed for converting from days to weeks is $\frac{\mathbf{1\ wk}}{\mathbf{7\ da}}$.

$$\begin{aligned} \mathbf{412\ hr} &= \frac{\mathbf{412\ hr}}{\mathbf{1}} \cdot \frac{\mathbf{1\ da}}{\mathbf{24\ hr}} \cdot \frac{\mathbf{1\ wk}}{\mathbf{7\ da}} && \textbf{Divide out common units.} \\ &= \frac{\mathbf{412\ \cancel{hr}}}{\mathbf{1}} \cdot \frac{\mathbf{1\ \cancel{da}}}{\mathbf{24\ \cancel{hr}}} \cdot \frac{\mathbf{1\ wk}}{\mathbf{7\ \cancel{da}}} \\ &= \frac{\mathbf{412 \cdot 1\ wk}}{\mathbf{24 \cdot 7}} && \textbf{Simplify} \\ &= \mathbf{2.5\ wk} \end{aligned}$$

Thus, $412 \text{ hr} = 2.5 \text{ wk}$.

Yes, you have enough time for a 2.5 week vacation.

Try It 8.3.4

According to the Global Web Index Survey, the average US resident between 16-64 years of age spent 144 minutes per day on social media in 2021. How many days is that per year? Assume the year has 365 days.

Answer

To solve this we need to multiply 144 minutes per day by 365 days to get the total minutes per year.

$144 * 365 = 52,560$ minutes for the whole year.

The unit fraction needed for converting from minutes to hours is $\frac{1 \text{ hr}}{60 \text{ min}}$. The unit fraction needed for converting from hours to days is $\frac{1 \text{ da}}{24 \text{ hr}}$.

$$\begin{aligned} 52,560 \text{ min} &= \frac{52,560 \text{ min}}{1} \cdot \frac{1 \text{ hr}}{60 \text{ min}} \cdot \frac{1 \text{ da}}{24 \text{ hr}} && \text{Divide out common units.} \\ &= \frac{52,560 \cancel{\text{min}}}{1} \cdot \frac{1 \cancel{\text{hr}}}{60 \cancel{\text{min}}} \cdot \frac{1 \text{ da}}{24 \cancel{\text{hr}}} \\ &= \frac{52,560 \cdot 1 \text{ da}}{60 \cdot 24} && \text{Simplify} \\ &= 36.5 \text{ wk} \end{aligned}$$

Thus, $52{,}560 \text{ min} = 36.5 \text{ da}$.

The average American spends 36.5 days a year on social media.

Summary

Temperature is often measured in one of two scales: the Celsius scale and the Fahrenheit scale. A Celsius thermometer will measure the boiling point of water at 100° and its freezing point at 0°; a Fahrenheit thermometer will measure the same events at 212° for the boiling point of water and 32° as its freezing point. You can use conversion formulas to convert a measurement made in one scale to the other scale. Time is universal and is measured in seconds, minutes, hours, days, weeks and years.

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8.4: Conversion Between the Metric and US Customary Systems of Measurement

Learning Objectives

1. Estimate equivalence between metric units and US customary units.
2. Convert metric units to US customary units.
3. Convert US customary units to metric units.

Estimate equivalence between the Metric Units and US Customary Units

Before converting metric units to US customary units and vice versa, you need to familiarize the approximate metric equivalents of common customary units. The table below summarizes the commonly used metric and US customary units and their approximate equivalence. The notations enclosed are the abbreviations of each unit.

Metric Units	**US Customary Units**
Lengths	
1 meter (m)	3.28 feet (ft or ’)
1 meter (m)	39.4 inches (in)
1 kilometer (km)	0.62 mile (mi)
2.54 centimeter (cm)	1 inch (in)
0.9144 meter (m)	1 yard (yd)
Weight	
1 kilogram (kg)	2.2 pounds (lb)
28.3 grams (g)	1 ounce (oz)
Fluid Capacity	
1 milliliter (ml)	0.203 teaspoon (tsp)
29.57 milliliters (ml)	1 fluid ounce (fl oz)
1 liter (l)	4.23 cups (c)
3.79 liters (l)	1 gallon (gal)
473.176 milliliters (ml)	1 pint (pt)
0.95 liters (l)	1 quart (qt)

Additional Common Conversions

Metric Units	US Customary Units
Area	
6.45 square centimeters (cm^2)	1 square inch (in^2)
1 square meter (m^2)	1.196 square yard (yd^2)
4046.873 square meters (m^2)	1 acre (ac)
Volume and Capacity	
16.39 milliliter (ml)	1 cubic inch (in^3)
28.32 liters (l)	1 cubic foot (ft^3)
764.55 liters (l)	1 cubic yard (yd^3)

Convert Metric Units to US Customary Units

You have familiarized the commonly used equivalence between metric and US customary units. At this point, you will learn to convert metric units to US customary units.

To convert metric units to US customary units, use their estimate equivalence as the conversion factor in dimensional analysis. Write the equivalence as a fraction, and multiply it to the given measurement (in metric) to convert. Be sure that the units of measure to convert cancels out. For instance, to convert liters to gallons, be sure to cancel out liters and retain gallons as the final unit.

Now, let us consider some examples.

Example 8.4.1

The official regulators of a shipment company classify a parcel weighing more than 51 pounds (lb) as a heavy shipment. Andrea packed 24 terracotta plant containers to be shipped to her customers. If each plant container weighs 1.3 kilograms (kg), will her package be classified as a heavy shipment?

Solution

To determine whether the parcel is a heavy shipment or not, you need to convert the total weight of the plant containers to pounds.

Compute for the total weight.

$$\text{Total Weight } = 24 \times 1.3 \text{ kg} = 31.2 \text{ kg}$$

Convert the total weight to pounds. Use the metric to US customary units' equivalence: 1 kg = 2.2 lbs.

$$31.2 \cancel{\text{kg}} \cdot \frac{2.2 \text{ lb}}{1 \cancel{\text{kg}}} = 68.64 \text{ lb}$$

Since the total weight is more than 51 lb, the shipment is considered heavy.

Try It 8.4.1

Angelica who lives in Las Vegas, Nevada, visited her friend in Canada. She knows that Canada uses the metric system, so their speed limits are posted in kilometers per hour. On her way, she sees a road sign that says, "max 60 km/h". She knows the sign means "a maximum speed of 60 kilometers per hour" but she is unsure if this is higher than the speed limit in Las Vegas, Nevada, which is 65 miles per hour (mph). How can Angelica convert 60 km/h to miles per hour? Use the metric to US customary units' equivalence: 1 kilometer (km) = 0.62 mile.

Answer

$$\frac{60\ \cancel{\text{km}}}{1\ \text{hr}} \cdot \frac{0.62\ \text{mi}}{1\ \cancel{\text{km}}} = 37.2\ \text{mi / hr}$$

In Las Vegas, the limit is 65 miles per hour (mph). Since 37.2 mph is below this limit, 60km/h is not over the speed limit.

Example 8.4.2

Half of a meter of ribbon is required for wrapping presents enclosed in a rectangular box. How many presents using the same box size can be wrapped with 10 yards of ribbon?

Solution

Convert half a meter (0.5 m) to yards. Use the metric to US customary units' equivalence: 1 yard = 0.9144 meter.

$$0.5\ \cancel{\text{m}} \cdot \frac{1\ \text{yd}}{0.9144\ \cancel{\text{m}}} = 0.5468\ \text{yd}$$

The result means that one present requires 0.5468 yards of ribbon. Calculate the number of presents by dividing 10 yards by 0.5468 yards.

$$10\ \text{yd} \div \frac{0.5468\ \text{yd}}{1\ \text{present}} = 10\ \cancel{\text{yd}} \times \frac{1\ \text{present}}{0.5468\ \cancel{\text{yd}}} = 18.29\ \text{presents}$$

Approximately, 18 presents can be wrapped with 10 yards of ribbons.

Try It 8.4.2

Crystal and her friends are planning for a road trip, which they have determined to be 2000 kilometers (km) long. They want to rent a van that can travel 100 km using 9 liters of gas. Suppose the gas price is $3.74 a gallon (gal), how many gallons of gas they will need? How much money will the group of friends need to save for gas? Use the metric to US customary units' equivalence: 3.79 liters (l)= 1 gallon (gal).

Answer

They will need $\frac{2000\ \cancel{\text{km}}}{1} \cdot \frac{9\ l}{100\ \cancel{\text{km}}} = 180$ liters of gas.

Converting to gallons: $\frac{180\ \cancel{l}}{1} \cdot \frac{1\ \text{gal}}{3.79\ \cancel{l}} = 47.5$ gallons of gas

Calculating the price: $47.5 \times \$3.74 = \$177.63$

The team needs 47.5 gallons of gas for the trip, which will cost them $177.63.

Try It 8.4.3

Christine plans to put up a short-course swimming pool at her house. A short-course swimming pool usually measures 22.86 meters in length, at least 18.29 meters in width, and 2.5 meters in depth. Convert the dimensions of the pool to inches. Use the metric to US customary units' equivalence: 1 centimeter (cm) = 0.39 inch (in.) and 1 centimeter (cm) = 0.01 meter (m).

Answer

Length: $\frac{22.86\ \cancel{\text{m}}}{1} \cdot \frac{1\ \cancel{\text{cm}}}{0.01\ \cancel{\text{m}}} \cdot \frac{0.39\ \text{in}}{1\ \cancel{\text{cm}}} = 891.54$ inches

Width: $\frac{18.29\ \cancel{\text{m}}}{1} \cdot \frac{1\ \cancel{\text{cm}}}{0.01\ \cancel{\text{m}}} \cdot \frac{0.39\ \text{in}}{1\ \cancel{\text{cm}}} = 713.31$ inches

Depth: $\frac{2.5\ \cancel{\text{m}}}{1} \cdot \frac{1\ \cancel{\text{cm}}}{0.01\ \cancel{\text{m}}} \cdot \frac{0.39\ \text{in}}{1\ \cancel{\text{cm}}} = 97.5$ inches

Approximately, the pool measures 891.54 in by 713.31 in by 97.5 in.

Convert US Customary Units to Metric Units

You have learned how to convert metric units to US customary units. Now, you will learn to convert US customary units to metric units.

Example 8.4.3

Daniel drove 40 miles in two hours. What distance (in meters) did he cover in one minute? Use the US customary to metric units' equivalence: 0.62 mile (mi) = 1 kilometer (km), 1 km is equal to 1000 meters (m), and 1 hour = 60 minutes.

Solution

Convert 40 miles in two hours to kilometers per hour.

$$\frac{40 \cancel{\text{ mi}}}{2 \text{ hrs}} \cdot \frac{1 \text{ km}}{0.62 \cancel{\text{ mi}}} = 32.2581 \text{ km/hr}$$

Convert 32.2581 kilometers per hour to meters per hour.

$$\frac{32.2581 \cancel{\text{ km}}}{1 \text{ hr}} \cdot \frac{1000 \text{ m}}{1 \cancel{\text{ km}}} = 32,258.1 \text{ m/hr}$$

Convert 32,258.1 meters per hour to meters per minute.

$$\frac{32,258.1 \text{ m}}{1 \cancel{\text{ hr}}} \cdot \frac{1 \cancel{\text{ hr}}}{60 \text{ min}} = 537.635 \text{ m/min}$$

Daniel has covered 537.635 meters in one minute.

Try It 8.4.4

A supermarket has a car parking spot that is 9 feet wide and 18 feet long. Its ground-level parking has 50 parallel parking spots (side by side) of the same sizes. What is the area of the parking lot in square centimeters? Use the US customary units' equivalence: 1 foot (ft) = 12 inches (in) and 1 square inch (in^2) = 6.45 square centimeters (cm^2). (Hint: You will need to use the formula to find the area of a rectangle)

Answer

Convert 9 feet to inches: $\frac{9 \cancel{\text{ ft}}}{1} \cdot \frac{12 \text{ in}}{1 \cancel{\text{ ft}}} = 108 \text{ in}$

Convert 18 feet to inches: $\frac{18 \cancel{\text{ ft}}}{1} \cdot \frac{12 \text{ in}}{1 \cancel{\text{ ft}}} = 216 \text{ in}$

Find the area of one parking spot: 108 in x 216 in = 23,328 in^2

Find the area of the whole parking lot: 23,328 in^2 x 50 = 1,166,400 in^2

Convert to cm^2: $\frac{1,166,400 \cancel{\text{ in}^2}}{1} \cdot \frac{6.45 \text{ cm}^2}{1 \cancel{\text{ in}^2}} = 7,523,280 \text{ cm}^2$

The area of the parking lot is 7,523,280 square centimeters.

Example 8.4.4

Melody's garage is 45 feet long while David's garage is 10 yards and 10 feet long. Which house has a longer garage? How much is the difference (in meters)? Use the US customary to metric equivalence: 3.28 feet (ft) = 1 meter (m) and 1 foot (ft) = 0.3333 yard (yd)

Solution

Convert 10 yards to feet: $\frac{10\ \cancel{yd}}{1} \cdot \frac{1\ \text{ft}}{0.3333\ \cancel{yd}} = 30\ \text{ft}$

David's garage is 30 ft + 10 ft = 40 ft long. His garage is 5 feet shorter than Melody's garage.

Convert 5 feet to meters: $\frac{5\ \cancel{ft}}{1} \cdot \frac{1\ \text{m}}{3.28\ \cancel{ft}} = 1.52\ \text{m}$

Melody's garage is longer by 1.52 meters.

Try It 8.4.5

How many liters of water are needed to fill an Olympic swimming pool with a total volume of 3300 cubic yards? Assume that the water level is as deep as the pool. Use the US customary to metric units' equivalence: 1 cubic yard (yd^3) = 746.56 liters (l).

Answer

Convert cubic yards to liters: $\frac{3300\ \cancel{yd^3}}{1} \cdot \frac{746.56\ l}{1\ \cancel{yd^3}} = 2,463,648$ liters

At most 2,463,648 liters of water are needed to fill the swimming pool.

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8.5: Chapter Review and Glossary

Chapter Review

1.1: Length

The four basic units of measurement that are used for length in the U.S. customary measurement system are: inch, foot, yard, and mile. Typically, people use yards, miles, and sometimes feet to describe long distances. Measurement in inches is common for shorter objects or lengths. You need to convert from one unit of measure to another if you are solving problems that include measurements involving more than one type of measurement. Each of the units can be converted to one of the other units using the table of equivalents, the conversion factors, and/or the factor label method shown in this topic.

1.1: Weight

In the U.S. customary system of measurement, weight is measured in three units: ounces, pounds, and tons. A pound is equivalent to 16 ounces, and a ton is equivalent to 2,000 pounds. While an object's weight can be described using any of these units, it is typical to describe very heavy objects using tons and very light objects using an ounce. Pounds are used to describe the weight of many objects and people. Often, in order to compare the weights of two objects or people or to solve problems involving weight, you must convert from one unit of measurement to another unit of measurement. Using conversion factors with the factor label method is an effective strategy for converting units and solving problems.

1.1: Capacity

There are five basic units for measuring capacity in the U.S. customary measurement system. These are the fluid ounce, cup, pint, quart, and gallon. These measurement units are related to one another, and capacity can be described using any of the units. Typically, people use gallons to describe larger quantities and fluid ounces, cups, pints, or quarts to describe smaller quantities. Often, in order to compare or to solve problems involving the amount of liquid in a container, you need to convert from one unit of measurement to another.

1.2: The Metric System

The metric system is an alternative system of measurement used in most countries, as well as in the United States. The metric system is based on joining one of a series of prefixes, including kilo- , hecto-, deka-, deci-, centi-, and milli-, with a base unit of measurement, such as meter, liter, or gram. Units in the metric system are all related by a power of 10, which means that each successive unit is 10 times larger than the previous one. This makes converting one metric measurement to another a straightforward process, and is often as simple as moving a decimal point. It is always important, though, to consider the direction of the conversion. If you are converting a smaller unit to a larger unit, then the decimal point has to move to the left (making your number smaller); if you are converting a larger unit to a smaller unit, then the decimal point has to move to the right (making your number larger).

1.2: Converting within the Metric System

To convert among units in the metric system, identify the unit that you have, the unit that you want to convert to, and then count the number of units to move between them. If you are going from a larger unit to a smaller unit, you multiply by 10 successively. If you are going from a smaller unit to a larger unit, you divide by 10 successively. You can also move the decimal point of your number the same direction and same number of spots that you need to move from the original unit to the new unit. The factor label method can also be applied to conversions within the metric system. To use the factor label method, you multiply the original measurement by unit fractions; this allows you to represent the original measurement in a different measurement unit.

1.2: Using Metric Conversion to Solve Problems

Understanding the context of real-life application problems is important. Look for words within the problem that help you identify what operations are needed, and then apply the correct unit conversions. Checking your final answer by using another conversion method (such as the "move the decimal" method, if you have used the factor label method to solve the problem) can cut down on errors in your calculations.

1.3: Temperature and time

Temperature is often measured in one of two scales: the Celsius scale and the Fahrenheit scale. A Celsius thermometer will measure the boiling point of water at 100° and its freezing point at 0°; a Fahrenheit thermometer will measure the same events at 212° for the boiling point of water and 32° as its freezing point. You can use conversion formulas to convert a measurement made in one scale to the other scale. Time is universal and is measured in seconds, minutes, hours, days, weeks and years.

1.4: Conversion Between the Metric and US Customary Systems of Measurement

To convert metric units to US customary units or vice versa, use their estimate equivalence as the conversion factor in dimensional analysis. Write the equivalence as a fraction, and multiply it to the given measurement to convert. Be sure that the units of measure to convert cancels out.

Glossary:

capacity	The amount of liquid (or other pourable substance) that an object can hold when it's full.
Celsius	A measure of temperature commonly used in countries that use the metric system. On the Celsius scale, water freezes at 0° and boils at 100°.
cup	A unit of capacity equal to 8 fluid ounces.
factor label method	One method of converting a measurement from one unit of measurement to another unit of measurement. In this method, you multiply the original measurement by unit fractions containing different units of measurement to obtain the new unit of measurement.
Fahrenheit	A measure of temperature commonly used in the United States. On the Fahrenheit scale, water freezes at 32° F and boils at 212° F.
fluid ounce	A unit of capacity equal to $\frac{1}{8}$ of a cup. One fluid ounce of water at 62°F weighs about one ounce.
foot	A unit for measuring length in the U.S. customary measurement system. 1 foot = 12 inches
gallon	A unit equal to 4 quarts, or 128 fluid ounces.
gram	The base unit of mass in the metric system.
inch	A unit for measuring length in the U.S. customary measurement system. 1 foot = 12 inches
length	The distance from one end to the other or the distance from one point to another.
liter	The base unit of capacity in the metric system.
measurement	The use of standard units to find out the size or quantity of items such as length, width, height, mass, weight, volume, temperature or time.
meter	The base unit of length in the metric system.
metric system	A widely-used system of measurement that is based on the decimal system and multiples of 10.
mile	A unit for measuring length in the U.S. customary measurement system. 1 mile = 5280 feet or 1760 yards.
ounce	A unit for measuring weight in the U.S. customary measurement system. 16 ounces = 1 pound.
pint	A unit of capacity equal to 16 fluid ounces, or 2 cups.
pound	A unit for measuring weight in the U.S. customary measurement system. 16 ounces = 1 pound.
prefix	A short set of letters that denote the size of measurement units in the metric system. Metric prefixes include kilo-, hecto-, deka-, deci-, centi-, and milli-
quart	A unit of capacity equal to 32 fluid ounces, or 4 cups.
ton	A unit for measuring the weight of heavier items in the U.S. customary measurement system. 1 ton = 2000 pounds.
U.S. customary measurement system	The most common system of measurement used in the United States. It is based on English measurement systems of the 18^{th} century.
unit equivalents	Statements of equivalence between measurement units within a system or in comparison to another system of units. For example, 1 foot = 12 inches or 1 inch = 2.54 centimeters are both examples of unit equivalents.
unit fractions	A fraction where the numerator and denominator are equal amounts, as in $\frac{1\text{ kg}}{1000\text{ g}}$ or $\frac{12\text{ inches}}{1\text{ foot}}$. Unit fractions serve to help with conversions in the factor label method (dimensional analysis).
unit of measurement	A standard amount or quantity. For example, an inch is a unit of measurement.
weight	A mathematical description of how heavy an object is.

yard	A unit for measuring length in the U.S. customary measurement system. 1 yard = 3 feet or 36 inches.

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8.6: Exercises

For 1 - 26, convert the measurements. *Round your answer up to 3 decimal places, if needed.*

Exploration

Section 1.1 U.S. Customary Measurement System:

1. 5 yards = _____ feet
2. 6 yards = _____ inches
3. 108 inches = _____ yards
4. 257 miles = _____ feet
5. 253 miles = _____ feet
6. 8 pounds = _____ ounces
7. 370 ounces = _____ ton
8. 417 ounces = _____ ton
9. 112 cups = _____ gallons
10. $8\frac{1}{2}$ gallons = _____ cups
11. 7 quarts = _____ fluid ounces
12. 9 fluid ounces = _____ quarts
13. Carolyn is building a pen for her dog. The perimeter of the pen is 47 ft. The fencing material is sold by the yard. It costs $8.60 /yd. What is the cost of the material before taxes?

Section 1.2 Metric Units of Measurement

1. 8 meters = _____ centimeters
2. 4580 milliliters = _____ liters
3. 3520 milliliters = _____ liters
4. 2 millimeters = _____ meters
5. 171 centimeters = _____ meters
6. 5180 milliliters = _____ liters
7. 2610 grams = _____ kilograms
8. 2 kilograms = _____ grams
9. 58.3 millimeters = _____ centimeters
10. 0.3429 kilograms = _____ milligrams
11. Apollo Spas services 281 hot tubs. If each hot tub needs 105 milliliters of muriatic acid, how many liters of acid are needed for all of the hot tubs?
12. The photo sharing site Flickr had 2.7 billion photos in June 2012. Create a comparison to understand this number by assuming each picture is about 2 megabytes in size, and comparing to the data stored on other media like DVDs, iPods, or flash drives.
13. In June 2012, Twitter was reporting 400 million tweets per day. Each tweet can consist of up to 140 characters (letter, numbers, etc.). Create a comparison to help understand the amount of tweets in a year by imagining each character was a drop of water and comparing to filling something up.

Section 1.3 Temperature and Time

1. $14°C$ = _____ $°F$
2. $41°F$ = _____ $°C$
3. $12°C$ = _____ $°F$
4. $99°F$ = _____ $°C$
5. What unit of measurement do these abbreviations belong to?
   a. da
   b. wk
   c. F

d. C
e. yr
f. sec
g. min
h. hr

6. Convert the following:
   a. 21°C to °F
   b. 81°F to °C
   c. 235,000 sec to yr
   d. 67 hr to da
   e. 19 wk to hr
   f. 4 da to min
7. Australia uses Celsius to measure temperature. Your travel agent said it will be between 23° C and 27° C during your visit. What is the temperature range in Fahrenheit?
8. You want to binge watch a TV show that has 11 episodes that are 45 minutes each. How many hours do you need to set aside this weekend to watch the entire series?
9. A friend from Switzerland is coming to visit the U.S. in December. You tell them it will be about 35° when they're here. You forgot to tell them that the measurement was in degrees Fahrenheit. They are used to Celsius and assume that is the temperature you told them. Are they planning for cold or hot weather? What Celsius temperature, should you have told them?
10. Your aunt always tells you her new child's age in weeks. The last time you saw them she said her child was 93 weeks old. How much is that in years?
11. To tell your colleague in Europe the temperature you are currently experiencing, you need to tell it to them in Celsius. If it is currently 64°F, what would you tell them?
12. During the landing of the Mars Science Laboratory *Curiosity*, it was reported that the signal from the rover would take 14 minutes to reach earth. Radio signals travel at the speed of light, about 186,000 miles per second. How far was Mars from Earth when *Curiosity* landed?
13. It is estimated that a driver takes, on average, 1.5 seconds from seeing an obstacle to reacting by applying the brake or swerving. How far will a car traveling at 60 miles per hour travel (in feet) before the driver reacts to an obstacle?

Section 1.4 Conversion Between the Metric and US Customary Systems of Measurement

1. Convert the following between U.S. and Metric measurement systems.
   a. 52 inches to centimeters
   b. 2 miles to meters
   c. 11 feet to decimeters
   d. 47 ounces to grams
   e. 4580 milliliters to cups
   f. 22 millimeters to inches
   g. 2.3 tons to kilograms
   h. 2,610 grams to pounds
   i. 7 gallons to liters
   j. 10 cups to quarts
   k. 3.6 quarts to cups
   l. 29 cups to gallons
   m. 3 pints to milliliters
   n. 8 meters to yards
2. If you drove 100 km/hour on the freeway, were you exceeding the speed limit of 65mi/hr?
3. Would you expect to pay more for 10 gallons (40 quarts) of gasoline or for 40 liters of gasoline?
4. Which is the better deal: apples priced at $1.69 per pound or the same apples priced $3.72 per kilogram?
5. If you ran 100 yards in the same time as I ran 100 meters, who ran faster?

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