MATHEMATICS CONCEPTS AND SKILLS FOR ELEMENTARY SCHOOL TEACHERS



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Mathematics Concepts and Skills for Elementary School Teachers

Martin Romero

Course Student Learning Outcomes

Math 204 Learning and Process Goals

Mathematical Mindset Goals: Throughout this class....

- MM 1.0 Students will apply mathematical thinking processes (e.g., visualizing, conjecturing, justifying, and proving) to formulate structured solutions and clearly document problem-solving steps.
- MM 2.0 Students will actively engage in mathematics by using effective strategies, persisting through challenges, reflecting on their problem-solving process,
- MM 3.0 Students will demonstrate a growth mindset by valuing utility, diligence and effort in mathematics, understanding that mathematical skills are developed through perseverance, practice, and learning from mistakes.

What is a Student Learning Outcome (SLO), and Why Does It Matter?

A **Student Learning Outcome**, or **SLO**, is a clear statement that describes what you should be able to **know**, **do**, or **understand** after completing a class, assignment, or course.

It helps answer this question:
"What will I walk away with after learning this?"

- SLO 1.1 Students will **apply** and **evaluate** multiple problem-solving strategies, including Polya's 4-step process, mathematical thinking, and algebraic reasoning to mathematical problems and real-world scenarios.
- SLO 1.2 Students will **analyze**, t**ranslate**, **solve,** and **interpret** mathematical problems and real-world scenarios.
- SLO 1.3 Students will **create** a mathematical problem-solving scenario inspired by real-world observations and **analyze** it using strategies such as counting, measuring, identifying patterns, or quantifying spatial relationships.
- SLO 1.4 Students will be able to **analyze** and use mathematical language, symbols, and structures to communicate mathematical ideas effectively, demonstrating an understanding of precision and logic in problem-solving.
- SLO 2.1 Students will **demonstrate** an understanding of foundational set theory by defining and manipulating sets and using set operations (e.g., union, intersection, complement) to construct and analyze new sets.
- SLO 2.2 Students will **analyze** and **apply** the properties and functions of whole numbers, distinguishing between their various uses (cardinal, ordinal, and identification) and explaining relationships such as greater than and less than through set-based representations.
- SLO 2.3 Students will **compare** Egyptian, Roman, and Hindu-Arabic numeration systems, identifying differences in place value, positionality, and arithmetic operations, and evaluate each system's advantages and limitations.
- SLO 2.4 Students will **demonstrate** proficiency with the Hindu-Arabic numeration system by converting between bases, expanding numbers, and articulating the system's structure, base, and numeral naming conventions.
- SLO 3.1 Students will demonstrate a comprehensive understanding of whole number operations, including addition, subtraction, multiplication, and division, by modeling each operation with set and measurement models and analyzing their properties.
- SLO 3.2 Students will apply strategies for mental computation in addition, subtraction, and multiplication across different bases and demonstrate mastery of basic facts using commutativity, associativity, and distributive properties.
- SLO 3.3 Students will analyze and apply the properties of whole numbers in relation to inequality, including transitivity,

addition, and multiplication properties of "less than," as well as use addition to establish comparative relationships.

- SLO 3.4 Students will demonstrate an understanding of whole number exponents by defining exponents through repeated multiplication and proving fundamental exponent rules.
- SLO 3.5 Students will evaluate numerical expressions involving whole numbers using the conventional order of operations (including parentheses, exponents, multiplication, division, addition, and subtraction), to arrive at a unique and correct solution.
- SLO 4.1 Students will apply mental math and computational estimation techniques, such as using compatible numbers, properties, and rounding strategies, to solve mathematical problems efficiently and accurately.
- SLO 4.2 Students will compare how basic and scientific calculators interpret and solve expressions, distinguishing between arithmetic logic (step-by-step entry) and algebraic logic (order of operations) while using calculator functions effectively (e.g., memory, exponent, and parentheses keys) to computer accurate results.
- SLO 4.3 Students will model and justify standard written algorithms for basic operations (addition, subtraction, multiplication, division) using concrete models, place value, and mathematical properties.
- SLO 4.4 Students will compute sums, differences, products, and quotients in various bases (2–12) using standard algorithms and nontraditional methods, such as lattice and scaffold techniques.
- SLO 5.1 Students will analyze and apply properties of numbers, including divisibility rules and the Fundamental Theorem of Arithmetic, to determine the prime or composite nature of numbers and represent them using prime factorization
- SLO 5.2 Students will demonstrate proficiency in identifying prime numbers using methods such as the Sieve of Eratosthenes and analyzing the relationship between prime factors and number properties
- SLO 5.3 Students will compute the greatest common factor (GCF) and least common multiple (LCM) of pairs of numbers using various methods, including prime factorization, set intersection, and algorithms
- SLO 5.4 Students will evaluate the relationships between the GCF, LCM, and the product of two numbers, demonstrating a deeper understanding of number relationships and their practical applications.
- SLO 6.1 Students will model and compare fractions using visual, verbal, and symbolic representations, and explain how these models support conceptual understanding and simplification strategies.
- SLO 6.2 Students will analyze and compute sums and differences of fractions, applying addition and subtraction properties and using estimation techniques to verify the reasonableness of results of real-world problems.
- SLO 6.3 Students will use visual models and real-life contexts to represent and compute multiplication and division of fractions, justifying the methods they choose and interpreting the results.
- SLO 6.4 Students will evaluate the application of mental math and estimation techniques for operations with fractions, including rounding, range estimation, and using compatible numbers to simplify calculations.
- SLO 7.1 Students will demonstrate understanding of decimal numbers by representing them visually (using number lines, hundreds squares, and grids), translating between word names and expanded forms, and ordering them based on place value and fraction equivalence.
- SLO 7.2 Students will compute with decimals and justify operations, including multiplying or dividing by powers of 10, using estimation techniques to evaluate sums, differences, products, and quotients effectively.
- SLO 7.3 Students will analyze and convert between fractions, decimals, and percentages, distinguishing between terminating, repeating, and nonterminating decimal representations and classifying their corresponding fractions.
- SLO 7.4 Students will solve problems involving ratios, rates, and proportions, comparing part-to-part and part-to-whole quantities, interpreting units, and applying strategies such as cross multiplication and scaling.
- SLO 7.5 Students will apply percentages to solve real-world problems by representing n% on grids, converting between

percentages and fractions or decimals, and solving percent problems using equations, proportions, and calculators.

- SLO 7.6 Students will use scientific notation to represent decimal numbers efficiently and interpret scientific notation results in mathematical and scientific contexts.
- SLO 8.1 Students will represent integers using models such as the integer number line and colored chips, and demonstrate addition, subtraction, and ordering of integers using these tools.
- SLO 8.2 Students will analyze and apply the properties of integer operations, including addition, subtraction, multiplication, and division, to solve mathematical problems and verify relationships such as closure, commutativity, and distributivity.
- SLO 8.3 Students will explain integer multiplication and division through repeated addition, patterns, and missing-factor approaches, and compute using negative exponents and scientific notation.
- SLO 8.4 Students will evaluate relationships between integers, applying the properties of inequalities (e.g., transitivity, multiplication by a positive or negative number) and theorems such as additive cancellation and opposite of the opposite

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1: MATHEMATICAL THINKING, PROBLEM SOLVING AND MATH AS A LANGUAGE

Thinking Mathematically to Solve Problems

Elementary school teachers should understand the interplay between thinking mathematically and problem-solving because these skills are foundational to students' success in mathematics and beyond. Mathematical thinking involves understanding why concepts work, making connections, and applying knowledge, while problem-solving is the active process of using these skills in new and challenging situations. Together, they help students move beyond memorization and toward a deeper understanding of math. (Devlin, 2012; NCTM 2020)

Developing problem-solving skills in young children is essential for their long-term success in mathematics and beyond. Early exposure predicts later achievement, as deep mathematical thinking builds a strong foundation for advanced learning. It also fosters positive mathematical identities, helping all children, regardless of background, see themselves as capable problem solvers. Rather than rote memorization, problem-solving promotes conceptual understanding by encouraging reasoning, pattern recognition, and connection-making. Engaging, thought-provoking problems increase student motivation and enjoyment, leading to higher achievement, especially when children collaborate and explore multiple strategies. (Feikes, Schwingendorf, and Gregg, 2018; Seda and Brown, 2021). These skills align with key mathematical practices, such as reasoning, explaining, and making connections, as emphasized in the Common Core Standards (CDE, 2013). Ultimately, mathematical thinking extends beyond numbers, equipping students with critical thinking and perseverance, essential for lifelong success.

Teachers play a vital role in nurturing these abilities. By creating supportive learning environments, designing rich mathematical tasks, and facilitating meaningful discussions, teachers can help young learners become confident, capable problem solvers who see math as a tool to explore and understand the world around them. Chapter 1 will be the jumping off point of our journey to learn to be better mathematical thinkers, problem solvers, and ultimately better teachers of mathematics.

Chapter Learning Outcomes

Essential Questions: How do we approach complex problems in everyday life? What strategies can help us become better problem solvers? How can understanding mathematics as a language help us communicate complex ideas across different disciplines and real-world contexts?

Learning that Transfers: You will engage with problem-solving strategies and mathematical thinking to develop flexible, adaptable skills. These skills extend beyond solving mathematical problems and equip you to tackle complex, real-world situations and be empowered to innovate and think critically in diverse contexts.

Student Learning Objectives

- SLO 1.1 Students will **apply** and **evaluate** multiple problem-solving strategies, including Polya's 4-step process, mathematical thinking, and algebraic reasoning to mathematical problems and real-world scenarios.
- SLO 1.2 Students will **analyze**, t**ranslate**, **solve,** and **interpret** mathematical problems and real-world scenarios.
- SLO 1.3 Students will **create** a mathematical problem-solving scenario inspired by real-world observations and **analyze** it using strategies such as counting, measuring, identifying patterns, or quantifying spatial relationships.
- SLO 1.4 Students will be able to **analyze** and use mathematical language, symbols, and structures to communicate mathematical ideas effectively, demonstrating an understanding of precision and logic in problem-solving.

Puddles and Paths: A Lesson with Grandpa

Interactive Content

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

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Morgan and Grandpa, a fine pair to see,

Were headed to the park, just as happy as could be.

Morgan skipped and she twirled, her heart full of glee,

And she called out, “Grandpa, oh Grandpa, look at me!”

They walked down the lane when they came to a stop.

A puddle so giant, it looked like a swamp!

“What do we do?” Morgan asked with a frown.

“We can’t jump over or walk around town.”

“Let’s think,” said Grandpa, his eyes all aglow. “Problem-solving will guide where we go.

We’ll start with step one: just look and define.

What’s the problem, dear Morgan? What clues can you find?”

“The puddle is wide,” Morgan said with a grin. “But a path of small rocks runs right through the middle and in!”

“Step two,” Grandpa said, “is to think up a way.

Could we hop on the rocks to keep our shoes out of the spray?”

“Let’s try it!” said Morgan, as brave as could be.

She leapt to the first rock, then cried, “Grandpa, look at me!”

Step three was in action as Grandpa came too,

And soon they were over, their path good as new.

“What did we learn?” Grandpa asked with a cheer. “Step four is to check: was the solution clear?”

“It worked!” shouted Morgan. “The rocks kept us dry.

Problem-solving is fun when we give it a try!”

Story written by ChatGPT with thoughtful prompt engineering

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1.1: Thinking Mathematically

Thinking Like Mathematicians

Mathematical thinking is more than just memorizing formulas or getting the right answer—it's about exploring ideas, understanding why things work, and making connections. As future elementary school teachers, you'll help young children learn what it means to think mathematically, and that goes beyond simply following steps. You'll help them discover that math is about curiosity, patterns, problem-solving, and creativity (CDE, 2013; NCTM, 2020; Seda and Brown, 2021).

Imagine your students asking, "What is math really about?" or "What does it mean to be good at math?" Mathematical thinking involves reasoning, making sense of numbers, and understanding the relationships between different ideas. It's about encouraging children to explore, take risks, and see mistakes as opportunities to learn—all while focusing on the *why* behind each concept (Boaler, 2022)

As teachers, your role will be to foster this kind of thinking in your classrooms. This means shifting away from just teaching procedures to helping students develop a deep understanding of concepts—what we call moving from procedural to conceptual mathematics. By encouraging young learners to engage deeply, to make sense of math, and to see it as a tool for understanding the world around them, you're not only teaching math—you're teaching them how to think, play, wonder, and explore.

To embody these ideas in our class, you need to be the kind of student you want to have. Cultivate the math mindset that is needed not only to be successful in learning math but also in teaching it. Embrace curiosity, be open to making and learning from mistakes, and engage deeply with the concepts we explore. By modeling a positive attitude toward math and a willingness to take risks, you are preparing yourselves to inspire the same mindset in your future students.

Think About and Discuss

As we dive into this class, I encourage you to consider what your mathematical experiences have been. What emotions or memories surface when you think about learning math as a child? Were you encouraged to ask questions, to make mistakes, to explore? Or were you expected to memorize and follow procedures?

What can we do to make math feel less like a set of rigid rules and more like a powerful tool for understanding and interacting with the world?

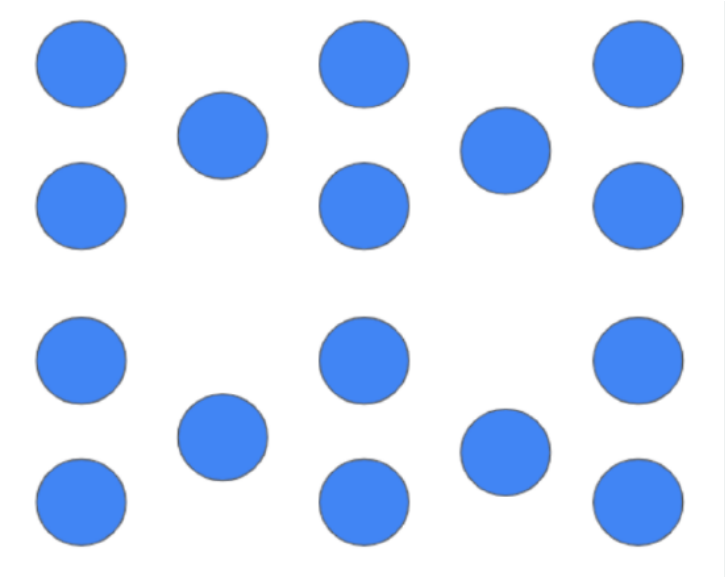
- How did your past experiences shape your current beliefs about what it means to be "good at math"?
- In what ways can you help your future students develop a growth mindset toward mathematics, seeing struggle and mistakes as natural and essential parts of learning?
- What does it mean to create an equitable math classroom—where every student feels seen, heard, and capable?
- How might mathematical thinking help students make sense of the world beyond the classroom?

Let's start this journey not just by thinking about *how* we'll teach math, but *why*—and for *whom*.

Before reading what the key aspects of mathematical thinking are, let's do some math.

Wonder, Play, Grow

Count the dots and be prepared to share how you counted them.



🧠**Let's Listen to Learn**❤️

Audio

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

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Now Add 17 + 18, how did you do it?

Take a look at the Video below from Math Celebrity Howie Hua. Compare your ideas with his and see how all of this connects to what we mean by mathematical thinking.

YouTube Video

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Mathematical Thinking

Now that you have had a chance to play with mathematics in a flexible way, here are some key aspects of what it means to think mathematically:

1. **Reasoning and Sense-Making**

   Mathematical thinking involves making sense of problems and persevering in solving them. This means being able to:

- Analyze problems to understand what is being asked.
- Explore different strategies for solving a problem.
- Justify and explain why a solution makes sense.
- Question assumptions and evaluate the reasonableness of results.

For example, if a student encounters a word problem, thinking mathematically means first understanding the relationships in the problem (not just looking for key words to "grab numbers") (CDE, 2013).

2. Recognizing Patterns and Structures

Mathematical thinking involves identifying patterns, structures, and relationships in numbers, shapes, and data. It includes:

- Seeing connections between ideas (e.g., recognizing that division and fractions are related).
- Noticing repeated reasoning (e.g., understanding why the distributive property works across many examples).
- Using structure to simplify problems (e.g., factoring expressions or breaking numbers into smaller, more manageable parts for mental math).

This habit of seeing structure allows mathematicians to generalize and extend their understanding to new situations (Boaler, 2022)

3. Abstract and Quantitative Thinking

Mathematical thinking requires students to move between concrete examples and abstract ideas. It involves:

- Representing situations symbolically, such as using equations, graphs, or diagrams.
- Understanding what quantities mean in context and interpreting their relationships (quantitative reasoning).
- Making sense of abstract ideas such as infinity, variables, or probability.

For example, when analyzing a real-world situation, students should be able to translate it into a mathematical model and interpret their results back in context (CDE, 2013).

4. Problem-Solving and Creativity

Mathematical thinking is inherently creative. It involves:

- Exploring multiple strategies to solve a problem.
- Trying new approaches if the first attempt doesn't work.
- Applying knowledge in novel or unfamiliar situations.

Students should feel empowered to take risks, make mistakes, and learn from them as part of the process of developing deeper understanding (Boaler, 2022).

5. Communication and Justification

A crucial aspect of mathematical thinking is being able to articulate your reasoning. This includes:

- Explaining why a particular solution is correct.
- Sharing and discussing ideas with peers.
- Writing arguments using precise mathematical language and notation.

For example, a student solving an equation should not only find the solution but also explain why their process works (NCTM, 2020)

6. Applying Mathematics to the Real World

Thinking mathematically includes applying math concepts to interpret and solve real-world problems. This involves:

- Modeling situations using appropriate tools (e.g., graphs, equations, simulations).
- Making decisions based on data and quantitative reasoning.

- Understanding the relevance of math in fields like science, technology, and finance.

For instance, understanding percentages and proportional reasoning helps students analyze topics like interest rates, population growth, or voting statistics (CDE, 2013).

7. Embracing Productive Struggle

Mathematical thinking requires a growth mindset—believing that ability in math develops through effort and learning. It involves:

- Persevering when problems seem challenging.
- Viewing mistakes as opportunities for learning.
- Building confidence in tackling complex ideas over time.

Students who think mathematically recognize that struggle is part of the learning process and a key to growth. (CDE, 2013; NCTM, 2020)

8. Using Tools and Technology Strategically

Mathematical thinking also includes selecting and using tools appropriately, such as calculators, graphs, manipulatives, or software, to support understanding and problem-solving (CDE, 2013) This means:

- Deciding when to use a tool and interpreting the results meaningfully.
- Using tools to experiment and explore concepts.

Wonder, Play, Grow

Use any method you can think of to solve this problem.

Math 1.1.1

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Think About and Discuss

Now that you have a process to solve the problem, discuss with your peers how they solved it. Then, think about how a 5th grader might approach the problem. Once you're ready, play the video below to see how it went and explore the aspects of mathematical thinking involved in her solution.

5th Grader Solving the Hearts Problem

https://vimeo.com/user7358057/review...961/a5004a5990

pw: cmlvideos

How to Foster Mathematical Thinking

As we can see, students have the extraordinary ability to think mathematically. As future teachers and caregivers for children, what can we do to help students develop this mathematical mindset? How can we foster productive mathematical thinking so

that students begin to develop a positive identity about their ability to do math mathematics.

Imagine what this would look like, sound like, and feel like in a classroom. The California Mathematics Framework emphasizes creating classrooms where students are encouraged to:

- Ask questions like "Why?" and "How do you know?"
- Share their thinking and learn from peers.
- Engage in rich tasks that require reasoning and creativity.
- Build connections between different mathematical ideas and real-world contexts.

Mathematical thinking is not just about getting the "right answer"; it's about cultivating habits of exploration, reasoning, and reflection that enable students to approach challenges with curiosity and confidence (CDE, 2013)

Doing Mathematics - Standards of Mathematical Practice

Future public school teachers need guidance as to what productive mathematical thinking looks like and what it means for students TO DO mathematics. This guidance comes in the form of The Standards for Mathematical Practice. This guidance describes varieties of expertise that mathematics educators at all levels should seek to develop in their students. Take a look below. But do not be overwhelmed, we will highlight these practices as we go in our class together. The goal is for you to recognize when you are doing these things and start to imagine what this looks like when you have your own students.

Math 1.1.2

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1.1.1: Math Mindset Written Homework

ARTICLE - Unlocking Student's Math Potential

Write a short response for each part (Circle, Square, Triangle), addressing the following prompts with the article content in mind.

Task 1: Circle - Square - Triangle Reflection (Use "*about*" 100 words for each)

- **Circle:**
  What's still rolling around in your head? What do you still not understand or feel uncertain about? Explain why you might have some cognitive dissonance with these ideas. Pose one or two thoughtful questions about teaching and learning that you'd like to discuss in class.

- **Square:**
  What ideas from the article that really "squares" with what you know or have experienced in your life or education? Explain how these ideas connect to you or your future teaching.

- **Triangle:**
  What is one POINTED idea from the article that you could use in your personal or teaching/learning life? Why does this matter to you, teachers, and/or students?

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1.1.2: Online Math Mindset HW

Online Homework

ADAPT 1.1.2.1

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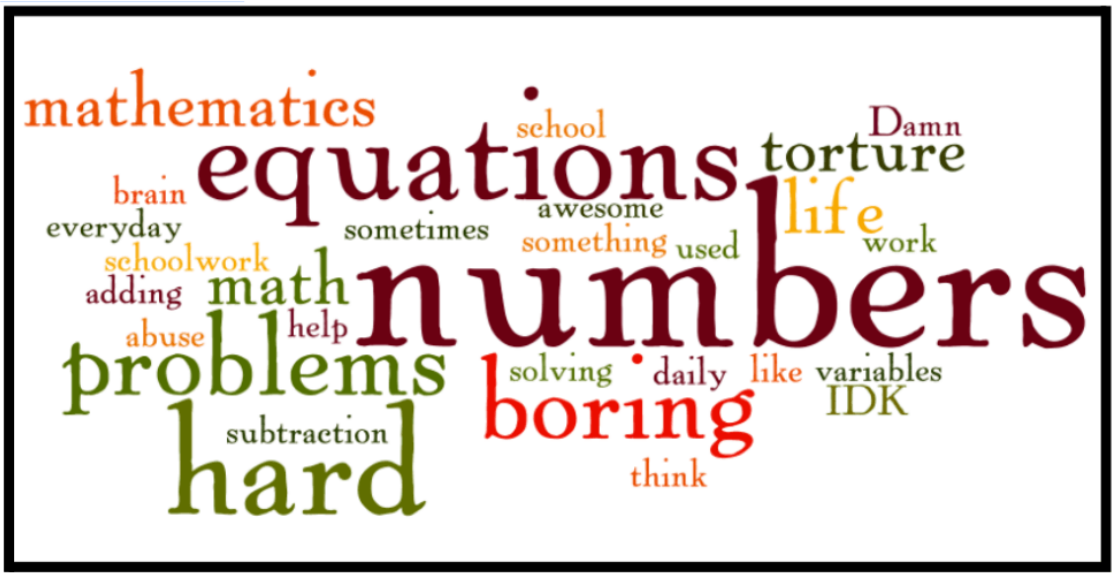
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1.2: What is Mathematics?

What is Mathematics?

A common question students often ask in one form or another about mathematics is, “When will I ever use this?” or “Why do we need to study math anyway?” To better answer this question for your students, it's important to reflect on your own answer to the question: *What is mathematics?* Below is a word cloud created from responses given by middle school students to that very question.[1] What do you see in the responses?



Think About and Discuss

Think about (or even write) what you think mathematics is. Then you will partner up with those near to you share your ideas.

As future elementary school teachers, your understanding of mathematics goes beyond solving equations or memorizing formulas; it involves recognizing mathematics as a way of thinking, reasoning, and problem-solving that is deeply connected to the world around us. By exploring the nature of mathematics, you can help your students see its relevance in everyday life and cultivate their curiosity and confidence in the subject. Below are some responses from mathematicians.

- **Keith Devlin** characterizes mathematics as "the science of patterns," emphasizing its role in identifying and analyzing patterns—whether real or abstract, visual or mental, from the natural world or human cognition.
- In *A Mathematician's Lament*, **Paul Lockhart** describes mathematics as "an art of human expression," akin to painting or music. He emphasizes that mathematics is a creative and imaginative pursuit, involving the invention of new ideas and the exploration of abstract concepts for their intrinsic beauty.
- **William Thurston**, a Fields Medal-winning mathematician, said, "Mathematics is not about numbers, equations, computations, or algorithms: it is about understanding."
- **Einstein** had insightful views on mathematics. He stated, "Pure mathematics is, in its way, the poetry of logical ideas." For Einstein, mathematics was a form of art that combined logic and creativity.
- **Francis Su** advocates for mathematics as a means to achieve human flourishing. In his book *Mathematics for Human Flourishing*, he presents an inclusive vision of mathematics, exploring its beauty and its capacity to develop virtues such as persistence, creativity, and critical thinking. Su's perspective underscores the humanistic aspects of mathematics, suggesting it contributes to personal growth and well-being.

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Audio

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Think About and Discuss

As a future elementary teacher, which of these descriptions of mathematics most aligns with how you hope your students will experience math in your classroom?

- What about the description resonates with your personal experience or beliefs about math?
- How might this perspective influence the way you teach or design learning experiences for your students?
- How could you help your students see mathematics in this way?

Understanding mathematics as a multifaceted discipline with deep connections to creativity, logic, and human flourishing opens the door to various perspectives. Each viewpoint—from Lockhart's emphasis on artistic expression to Su's focus on personal growth—offers valuable insights into how mathematics shapes our lives. However, for this course, we will center our exploration on Keith Devlin's definition of mathematics as "the science of patterns." This approach provides a foundational lens to understand mathematical ideas and their application, aligning with the idea that patterns, whether in nature, art, or abstract reasoning, are at the heart of mathematical thought. While the other interpretations are interconnected and essential, focusing on patterns allows us to frame our curriculum in a way that is both accessible and powerful for future educators.

Example 1.2.1

Complete the table following the same rule or pattern from the first rows.

Input	**Output**
15	5
18	8
21	11
22	?
?	13

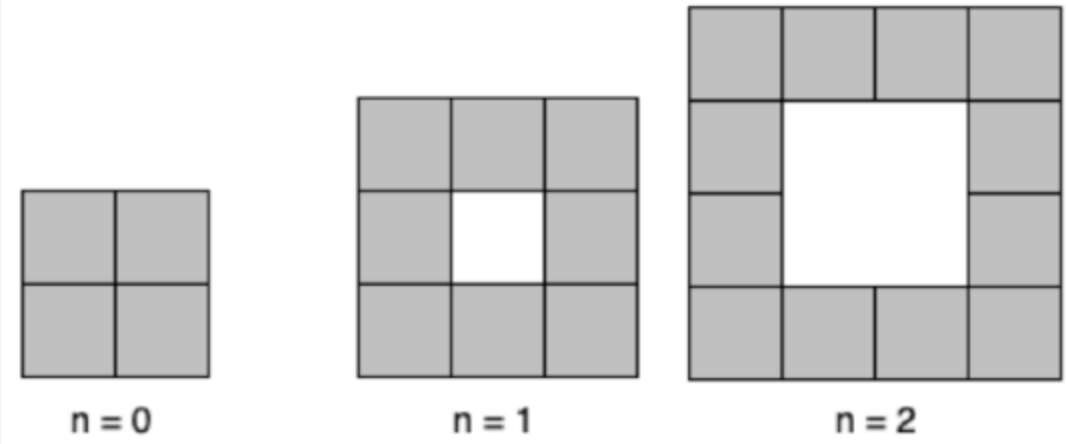
Solution available at the end of this section.

Wonder, Play, Grow

Now you try it on your own using the interactive below. Keep playing until you feel comfortable.

Example 1.2.2

Find the number of tiles around square pools of size n=3, 4, 8, and 10.



Solution available at the end of this section.

If you want to further Wonder, Play, Grow with Pool Tile Problem, then use this Growing Patterns LINK for an interactive experience.

Answers

Example 1.2.1

The pattern follows an sequence where both input and output increase by 3, but at 22, the increase is only by 1 instead of 3. Using this rule, the missing values are 22→12 and 23→13. How did you think about it?

Example 1.2.2

For n=0 the pool is a 2×2, so the total number of shaded tiles is **4**. **For n=1 t**he pool is a 3×3 grid with the center tile removed, so the number of shaded tiles is **8**. **For n=2 t**he pool is a 4×4 grid with a 2×2 center removed, so the number of shaded tiles is **12**.

Pool of size 3 → 16 shaded tiles **Pool of size 4** → 20 shaded tiles **Pool of size 8** → 36 shaded tiles **Pool of size 10** → 44 shaded tiles.

Can you come up with a pattern to find the number of tiles for any size pool of length n?

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1.2.1: Discussion Board Math Mindset HW

Assignment: Exploring the Essence of Mathematics

What Is Mathematics and What Does It Mean to "Do" It?

Task 1: Choose and Engage with a Resource

- Select **one** of the following resources to explore the essence of mathematics. These resources emphasize the creativity, logic, and relevance of mathematics in our world

- Watch the video or read the article.
- Take notes on key ideas or quotes that resonate with you.

Resources

YouTube Video

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Task 2: Personal Reflection POST 1

Write a short paragraph (6–8 sentences) addressing the following: POST THIS TO THE DISCUSSION BOARD

1. What perspective on mathematics was presented and How does this perspective compare to your previous understanding of mathematics?
2. How did the resource impact your understanding of what mathematics is and what it means to "do" mathematics?
3. Share one surprising or thought-provoking idea from the resource.

Task 3: Family Engagement POST 2

- Share one idea from the resource you explored with a family member or loved one from outside of class. You can also include anything else from chapter 1 regarding what you have experienced.

- Work with them to find patterns in their everyday surroundings. These could be patterns in nature (leaves, flowers), architecture (tiles, windows), or routines (daily schedules). Take a photo, sketch, or describe at least ONE pattern you found together.
- Ask them about how they use math in their job or daily tasks. Encourage them to share examples that aren't immediately obvious.
- Write a summary of your conversation (5–7 sentences). Include:
  1. Who you talked to.
  2. The pattern.
  3. What they discussed about using math in their jobs or daily tasks.
  4. Something you learned in the experience with your family member or loved one.

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1.3: Introduction to Problem Solving

Introduction: Problem Solving as the Heart of Mathematics

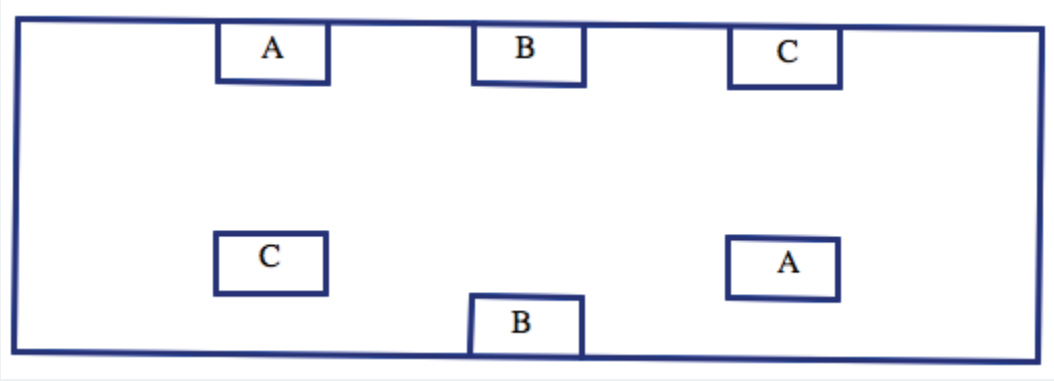
The ultimate goal of our course is to empower you to help your future students achieve meaningful and lasting mathematical understanding. We begin our mathematical activity with problem solving because it is the cornerstone of mathematics learning and teaching. Problem solving is considered the primary reason for studying mathematics. It is not just about learning procedures but also about developing a deep understanding of why and how mathematics works (Feikes, Schwingendorf, and Gregg, 2018). Learning through problem solving is an effective method for grasping mathematical concepts (Finan, 2006). It allows students to engage actively in the learning process, constructing their own understanding rather than passively receiving information (CDE, 2013) Real understanding in mathematics comes from solving problems, turning it into a sense-making activity (Feikes, Schwingendorf, and Gregg, 2018).

Engaging with problems fosters not only the ability to solve specific mathematical tasks but also the development of broader problem-solving strategies that are applicable in everyday life and other disciplines. Problem solving encourages creativity, critical thinking, and perseverance. It allows students to explore, test, and refine their ideas, leading to a deeper understanding of mathematical concepts. George Pólya's four-step process—understanding the problem, devising a plan, carrying out the plan, and looking back—is a powerful framework that you will use throughout this course to approach and solve problems systematically. This method not only provides a clear structure for tackling mathematical challenges but also cultivates habits of mind that are essential for effective teaching and lifelong learning (Finan, 2006).

Before diving into specific strategies, take a moment to reflect on what it truly means to think mathematically and to *do* mathematics. You're not alone in this journey—we will work together to develop your mathematical mindset. Let's dive into some problems and explore the different ways you naturally approach problem solving!

Wonder, Play, Grow

Draw curves connecting A to A, B to B, and C to C. Your curves cannot cross or even touch each other, they cannot cross through any of the lettered boxes, and they cannot go outside the large box or even touch it's sides.



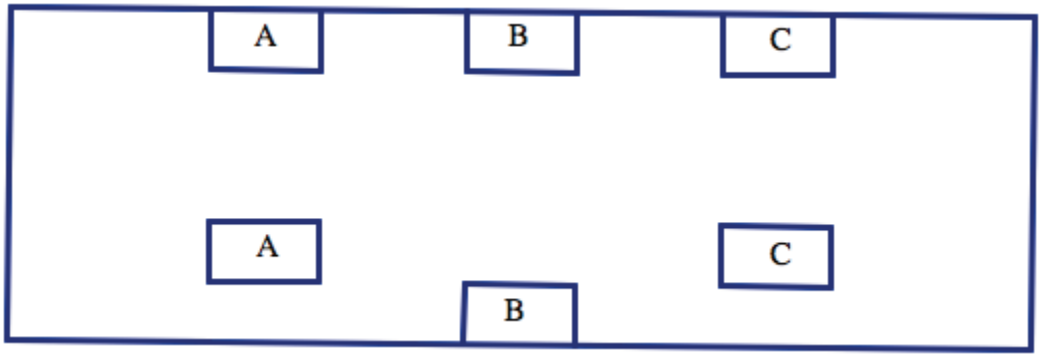
Think About and Discuss

After you have worked on the problem on your own for a while, talk through your ideas with a partner (even if you have not solved it).

- What did you try?
- What makes this problem difficult?
- Can you change the problem slightly so that it would be easier to solve?

Do you wish something in the problem was different? Would it then be easier to solve the problem?

For example, what if ABC problem had a picture like this:



Can you solve this case and use it to help you solve the original case? Think about moving the boxes around once the lines are already drawn.

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Audio

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Here is one possible solution.

YouTube Video

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View this YouTube Video online

Productive problem-solving is more than just finding the right answer—it is a dynamic process that involves exploration, creativity, and critical thinking. When we engage with problems, we are not merely applying formulas but developing a mathematical mindset that fosters persistence and adaptability. Effective problem solvers analyze problems deeply, recognize patterns, and refine their approaches through iteration and reflection. As we move forward, we will explore specific strategies that enhance our problem-solving abilities, including visualization, logical reasoning, and heuristic approaches. These strategies will not only help us tackle mathematical challenges but also cultivate the habits of mind that make mathematical thinking a powerful tool in everyday life.

Problem or Exercise?

The main activity of mathematics is solving problems. However, what most people experience in most mathematics classrooms is practice exercises. An exercise is different from a problem.

In a **problem**, you probably don't know at first how to approach solving it. You don't know what mathematical ideas might be used in the solution. Part of solving a problem is understanding what is being asked, and knowing what a solution should look like. Problems often involve false starts, making mistakes, and lots of scratch paper!

In an **exercise**, you are often practicing a skill. You may have seen a teacher demonstrate a technique, or you may have read a worked example in the book. You then practice on very similar assignments, with the goal of mastering that skill.

Note

What is a problem for some people may be an exercise for other people who have more background knowledge! For a young student just learning addition, this might be a problem:

$$\textit{Fill in the blank to make a true statement } \_\_ + 4 = 7.$$

But for you, that is an exercise!

Both problems and exercises are important in mathematics learning. But we should never forget that the ultimate goal is to develop more and better skills (through exercises) so that we can solve harder and more interesting problems.

Learning math is a bit like learning to play a sport. You can practice a lot of skills:

- hitting hundreds of forehands in tennis so that you can place them in a particular spot in the court,
- breaking down strokes into the component pieces in swimming so that each part of the stroke is more efficient,
- keeping control of the ball while making quick turns in soccer,
- shooting free throws in basketball,
- catching high fly balls in baseball,

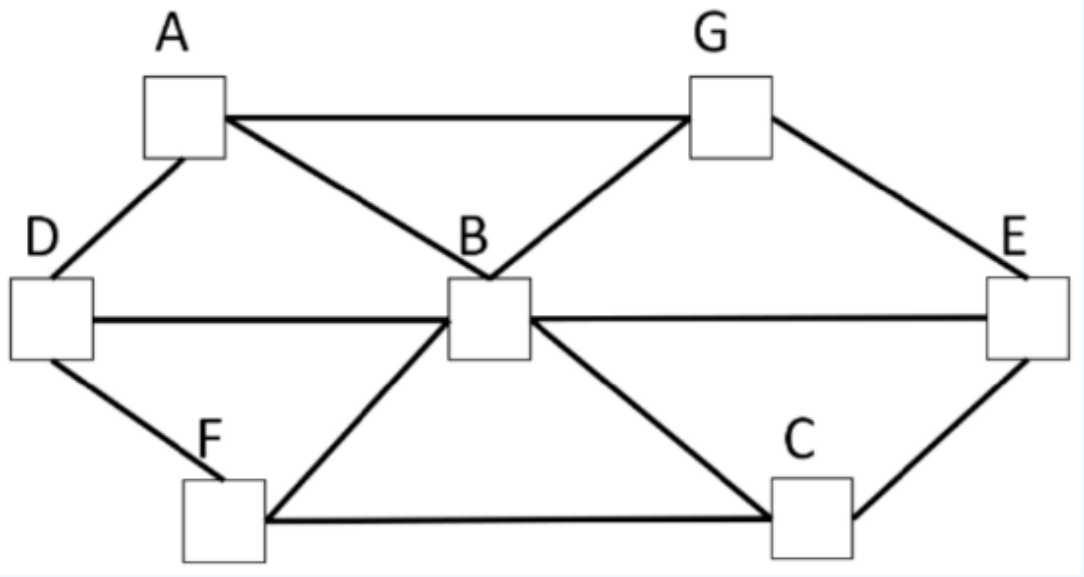
and so on.

But the point of the sport is to play the game. You practice the skills so that you are better at playing the game. In mathematics, solving problems is playing the game!

Example 1.3.1

Using the digits 1 to 7, at most one time each, place a digit in each circle so that the sum of the numbers in 3 squares (the middle horizontal line or 2 diagonals) are the same.

e.g A+B+C or D+B+E or F+B+G



Is this a problem or an exercise?

Answer available at the end of this section.

Answers

Example 1.3.1

This is a **problem**, not just an exercise, because it requires thinking, strategy, and logical reasoning rather than simply applying a known formula or procedure.

Posisble Solution: (Did you find another solution? Is there another one?)

B=4

A+B+C=1 + 4 + 7 = 12

D+B+E = 3 + 4 + 5 = 12

F+B+G = 6 + 4 + 2 = 12

**Problem Source: https://www.openmiddle.com/number-pattern/*

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1.3.1: Written Retrieval and Problem Solving Practice

Student Learning Objectives

SLO 1.1 Students will apply and evaluate multiple problem-solving strategies, including Polya's 4-step process, mathematical thinking, and algebraic reasoning to mathematical problems and real-world scenarios.

SLO 1.2 Students will analyze, translate, solve, and interpret mathematical problems and real-world scenarios.

Wonder, Play, Grow

Solve these problems using any method you can think of at this point. Do not worry if you get stuck, just try something. In the next section we will formalize problem solving strategies for you to consider.

1. *Grade 1:* Some bicycles and tricycles are on the playground. There are seven seats and nineteen wheels. How many bicycles are there? How many tricycles are there? *(Solve without using Algebra.)*
2. *Grade 2:* Pam has three quarters. She wants to buy candy, which costs 54¢. How much change will she have after she buys the candy?
3. *Grade 3:* It takes a rocket 8 seconds to travel one mile. How long will it take the rocket to travel 7 miles?
4. *Grade 4:* Irwin decided to create designs using his cereal. In total, he created nine designs and used 63 pieces of cereal. How many cereal pieces were used in each design? Was there an equal number of pieces in each design?
5. *Grade 5:* Leo has 10 animals at his ranch, which are turtles, bunnies, and chickens. If there were 5 turtles, and 3 bunnies, what fraction of the animals are chickens?
6. *Grade 6:* Ellen wants to build a rectangular fenced area in her yard that is 10 meters by 8 meters. The fence will be made of glass panes between posts. The hardware store owner suggests installing a post every 2 meters. The posts cost $3.42 each. Ellen will also need to buy a glass pane to place between each pair of posts. The glass panes cost $50.84 per meter long. She also needs to buy one gate for $31.99. How much do all of the materials cost before tax?

Solutions

1) 4 bicycles and 3 tricycles.

2) Pam will have 21¢ left.

3) 56 seconds.

4) Each design has 7 pieces of cereal, and yes, they are equal.

5) 1/5 of the animals are chickens.

6) $1,022.35 before tax.

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1.4: Problem Solving Strategies Part 1

Think back to the ABC Problem. What did you do to solve it? Even if you did not figure it out completely by yourself, you probably worked towards a solution and figured out some things that *did not* work.

Unlike exercises, there is never a simple recipe for solving a problem. You can get better and better at solving problems, both by building up your background knowledge and by simply practicing. As you solve more problems (and learn how other people solved them), you learn strategies and techniques that can be useful. But no single strategy works every time.

How to Solve It

George Pólya was a great champion in the field of *teaching* effective problem solving skills. He was born in Hungary in 1887, received his Ph.D. at the University of Budapest, and was a professor at Stanford University (among other universities). He wrote many mathematical papers along with three books, most famously, “How to Solve it.” Pólya died at the age 98 in 1985.[1]



George Pólya, circa 1973

1. Image of Pólya by Thane Plambeck from Palo Alto, California (Flickr) [CC BY 2.0 (http://creativecommons.org/licenses/by/2.0)], via Wikimedia Commons ↵

In 1945, Pólya published the short book *How to Solve It*, which gave a four-step method for solving mathematical problems:

1. First, you have to understand the problem.
2. After understanding, then devise a plan.
3. Carry out the plan.
4. Look back on your work. How could it be better?

This is all well and good, but how do you actually do these steps?!?! Steps 1. and 2. are particularly mysterious! How do you “make a plan?” That is where you need some tools in your toolbox, and some experience to draw upon.

Much has been written since 1945 to explain these steps in more detail, but the truth is that they are more art than science. This is where math becomes a creative endeavor (and where it becomes so much fun). We will articulate some useful problem solving strategies, but no such list will ever be complete. This is really just a start to help you on your way. The best way to become a skilled problem solver is to learn the background material well, and then to solve a lot of problems!

In our course, you will encounter various strategies to support the "devise a plan" step in problem solving. These include:

- Guess and Test

- Draw a Picture or Diagram
- Use a Variable
- Look for a Pattern
- Make a List or Table
- Solve a Simpler Problem

By practicing these techniques, you will gain insight into how children approach mathematics and learn strategies to guide them effectively. In this class, we aim to develop productive forms of mathematical thinking that will serve you and your future students in many facets of life. Through this process, you will not only deepen your own understanding of mathematics but also learn how to cultivate a problem-solving mindset in your students, empowering them to see mathematics as a tool for understanding and shaping the world.

Polya's Full List of Problem Solving Strategies

The following list outlines various strategies, or heuristics, that can be used to approach and solve problems effectively. These strategies help build logical thinking, creativity, and confidence in tackling complex challenges.

Guess and Test: Make an initial guess, test it, and refine the guess until a solution is found. Useful when there are limited possible answers.

Draw a Picture or Diagram: Creating a visual representation of the problem can clarify relationships and make the problem easier to solve.

Use a Variable: Representing unknown quantities with variables can help in setting up and solving equations.

Look for a Pattern: Identifying patterns in numbers or data can lead to predictions and solutions.

Make a List or Table: Organizing information systematically can reveal relationships and simplify problem-solving.

Solve a Simpler Problem: Breaking down a complex problem into a smaller, more manageable version can provide insights into the larger problem.

Work Backwards: Start with the desired outcome and reverse the steps to find the starting point.

Use Direct Reasoning: Apply logical steps in a straightforward manner to reach a conclusion.

Use Indirect Reasoning: Consider what cannot be true to eliminate possibilities and find the correct answer.

Use Properties of Numbers: Apply knowledge of prime numbers, factors, odd/even properties, etc., to simplify calculations.

Solve an Equivalent Problem: Reframe the problem in a different but equivalent way that may be easier to solve.

Use Cases: Break the problem into distinct scenarios and solve each separately.

Solve an Equation: Use algebraic methods to represent and solve the problem mathematically.

Look for a Formula: Utilize established formulas for area, volume, or other measurable attributes.

Do a Simulation: Model the problem using a controlled test or experiment.

Use a Model: Employ physical objects or manipulatives to represent the problem, particularly in geometry.

Use Dimensional Analysis: Analyze and convert units to ensure correct calculations.

Identify Subgoals: Break a complex problem into smaller steps to solve sequentially.

Use Coordinates: Represent problems graphically using a coordinate system.

Use Symmetry: Recognize and apply symmetry to simplify problem-solving.

Problem-Solving Strategy 1: Guess and Test

The "guess and test" strategy is about starting with a guess and seeing how close you are to the right answer. If your guess isn't correct, you use what you learned to make a better guess and try again. It's like trial and error, but each try gives you clues to help you get closer to the solution.

This strategy works well when:

- There aren't too many possible answers to try.
- You want to understand the problem better by testing different possibilities.
- You can test possible answers in an organized way.

People often use guess and test when they don't know a faster way to solve the problem or don't have the tools to solve it more efficiently. It's a simple method, but it can help you get to the answer! (Finan, 2006).

Example 1.4.1

Last week, Alex borrowed money from several of his friends. He finally got paid at work, so he brought cash to school to pay back his debts. First he saw Brianna, and he gave her 1/4 of the money he had brought to school. Then Alex saw Chris and gave him 1/3 of what he had left after paying Brianna. Finally, Alex saw David and gave him 1/2 of what he had remaining. Who got the most money from Alex?

Answer available at the end of this section.

Problem Solving Strategy 2 (Draw a Picture or Diagram).

The "draw a picture" strategy helps you solve problems by creating a visual representation. It's especially useful for understanding and solving certain types of problems. Here's when to use it:

This method works best when:

- Physical Situations: If the problem involves objects, locations, or movement, drawing can help you see how they relate to each other.
- Geometric Problems: For questions about shapes, area, perimeter, or volume, a drawing can clarify the problem.
- Visual Patterns: If the problem can be represented visually, drawing might reveal patterns or relationships.
- Need for Understanding: If the problem is confusing, drawing a picture can organize your thoughts and make the problem clearer

Wonder, Play Grow

Let's try to solve the previous example using the "Draw a Picture or Diagram" method. Draw a square to represent all of Alex's money. Then shade 1/4 of the square — that's what he gave away to Brianna. How can the picture help you finish the problem? After you have worked on the problem yourself using this strategy (or if you are completely stuck), you can watch someone else's solution.

YouTube Video

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Example 1.4.2

There are 5 people in a room. Each person exchanges exactly one fist bump with every other person. How many total fist bumps take place?

Answer available at the end of this section.

Problem Solving Strategy 3 (Use a Variable)

A variable is a powerful tool to represent unknown values or relationships in math problems. Sometimes, a problem asks you to figure out a number. You can use a variable (like x or n) to stand for the unknown number. Then, use the information in the problem to create an equation that you can solve to find the number. (FIRST)

This method works best when:

- The problem can be written as an equation.
- There's an unknown number connected to numbers you already know.
- You need to create a formula or general rule.

Example 1.4.3

A local community center held a charity gaming event where two types of tickets were sold: a **child ticket** for $2 and an **adult ticket** for $5. In total, **600 tickets** were sold, raising **$1,500**. How many child tickets and how many adult tickets were sold?

Answer available at the end of this section.

Note: Keep in mind that many problems can be solved using different strategies. Just because 1 strategy seems to work for you, another strategy may work just as well or come to mind more quickly for someone else. This means you should be open to hearing what others think, how they approach a problem, and understand that there may be more than one away to find a solution. As a future teacher you want to encourage your students to solve problems in multiple ways and not give up if the first method does not result in a solution, but that they will have learned something in their first attempt.

Think About and Discuss

Analyze each problem scenario and decide which problem solving strategy you would first attempt to use to solve each problem. Do this individually first before discussing this with your peers.

Scenario 1:	Scenario 2:	Scenario 3:
Lulu the Clown drops a super ball from the top of a building. She releases the ball exactly 64 feet from the ground. When the ball bounces on the ground it rebounds to a height that is half of its starting height. If the ball is allowed to bounce eight times, how high will it rebound after the eighth bounce?	*There was a box of cookies on the table. Maura was hungry because she hadn't had breakfast, so she ate half the cookies. Then Mario came along and noticed the cookies. He thought they looked good, and had not packed lunch, so he took two-thirds of what was left in the box. Suzie came by and decided to take three-fourths of the remaining cookies with her to her next class. Then Ariel came dashing up and took one cookie to munch on. When Kristen looked at the cookie box, she saw that there was just one cookie left. "How many cookies were there in the box to begin with?" she asked Maura.*	A number leaves a remainder of 2 when divided by 3 and a remainder of 1 when divided by 4. What is the smallest such number?

Answers

Example 1.4.1

Let's use a **guess and test** approach.

Step 1: Make a Guess

Suppose Alex initially brought **$120** to school.

He gives **1/4** to Brianna: 120×1/4=30

Remaining money: **$120 - 30 = 90**

He gives **1/3** of the remaining money to Chris: 90×1/3=30

Remaining money: **$90 - 30 = 60**

He gives **1/2** of the remaining money to David: 60×1/2=30

Each person received $30, meaning they all got the same amount.

Step 2: Adjust the Guess

Let's **try a different initial amount**, say **$240**.

Brianna gets: 240×1/4=60

Remaining: **$240 - 60 = 180**

Chris gets: 180×1/3=60

Remaining: **$180 - 60 = 120**

David gets: 120×1/2=60

E**ach person received $60**, meaning they all got the same amount.

Step 3: Generalizing

From our tests, it appears that the fractions ensure **each person receives the same amount**, regardless of the initial amount.

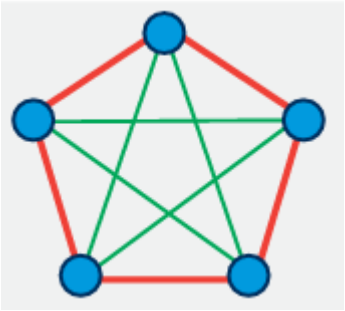
Final Answer: No one got the most money; Brianna, Chris, and David all received the same amount.

Example 1.4.2

To calculate this, you can THINK of it as the first person fist bumps 4 others, then the second person fist bumps 3 others (as they already fist bumped the first person), and so on. This adds up to 4 + 3 + 2 + 1 = 10 fist bumps.

OR (we can draw it out)

Each circle represents a person and each line segment represents one fist bump. This forms a complete network with ten total connections, or 10 total fist bumps.



Example 1.4.3

Let's use the following variables to represent the number of each ticket type:

x: Number of **child tickets** sold (cost $2 each) , y: Number of **adult tickets** sold (cost $5 each)

Step 1: Set up the equations.

Total tickets equation:

$x+y=600$

Total revenue equation:

$2x+5y=1500$

Step 2: Solve the system of equations.

From the first equation, $x+y=600$, express x in terms of y, $x=600-y$.

Substitute $x=600-y$ into the second equation $2x+5y=1500$

$2(600-y)+5y=1500$

Simplify and solve for y:

$1200-2y+5y=1500$

$3y=300$

$y=100$.

Substitute $y=100$ back into $x=600-y$ to find x:

$x=600-100=500$

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1.4.1: Written Retrieval and Problem Solving Practice

Applying Polya's Four-Step Problem-Solving Method

Purpose

This assignment is designed to enhance your problem-solving skills using Polya's four-step method. By engaging with a mathematical scenario, you will practice understanding a problem, devising a solution plan, executing the plan, and reflecting on your process. These skills are critical for success in mathematics and problem-solving across disciplines.

Learning Objectives:

SLO 1.1 Students will apply and evaluate multiple problem-solving strategies, including Polya's 4-step process, mathematical thinking, and algebraic reasoning to mathematical problems and real-world scenarios.

SLO 1.2 Students will analyze, translate, solve, and interpret mathematical problems and real-world scenarios.

Task

Select one of the three problem scenarios below and solve it using Polya's four-step method. Your response should be structured in the following manner:

1. **Understand the Problem:** Restate the problem in your own words, identifying key information and variables. What are you trying to find? What is known? What is unknown?
2. **Devise a Plan:** Outline the steps you will take to solve the problem. Identify the mathematical concepts, formulas, or strategies you will use.
3. **Carry Out the Plan:** Solve the problem step-by-step, showing all necessary calculations. Explain each step concisely.
4. **Look Back & Reflect:** Analyze your solution. Could you have solved it differently? Were there any difficulties? How does this problem relate to others you have encountered?

Scenario 1:	**Scenario 2:**	**Scenario 3:**
Lulu the Clown drops a super ball from the top of a building. She releases the ball exactly 64 feet from the ground. When the ball bounces on the ground it rebounds to a height that is half of its starting height. If the ball is allowed to bounce eight times, how high will it rebound after the eighth bounce?	*There was a box of cookies on the table. Maura was hungry because she hadn't had breakfast, so she ate half the cookies. Then Mario came along and noticed the cookies. He thought they looked good, and had not packed lunch, so he took two-thirds of what was left in the box. Suzie came by and decided to take three-fourths of the remaining cookies with her to her next class. Then Ariel came dashing up and took one cookie to munch on. When Kristen looked at the cookie box, she saw that there was just one cookie left. "How many cookies were there in the box to begin with?" she asked Maura.*	A number leaves a remainder of 2 when divided by 3 and a remainder of 1 when divided by 4. What is the smallest such number?

Criteria for Success

Your work will be assessed based on the following:

- **Completeness:** Did you follow all four steps of Polya's method?

- **Clarity:** Are your explanations and calculations easy to follow?
- **Accuracy:** Did you use correct mathematical procedures?
- **Reflection:** Did you thoughtfully evaluate your solution and process?

Problem Solving Reflection

After solving the problem, reflect on your experience using the following prompts, be prepared to discuss your reflectrions in class.:

- Identify one key takeaway from this problem-solving process.
- Describe an "aha" moment you had while working through the problem.
- What questions do you still have about the problem or Polya's method?
- How did working individually or with others influence your understanding?
- How can you apply what you learned to future problem-solving situations?

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1.5: Problem Solving Strategies Part 2

Problem Solving Strategy 4 (Look for a Pattern)

The "look for a pattern" strategy is a helpful way to solve problems, especially in math. It involves finding repeating sequences, relationships, or changes that can lead to a solution. Here's how and when to use this strategy (Musser et al., 2013):

The "Look for a Pattern" Strategy Works Best When:

- **Working with lists of data:** Trends or recurring relationships can be identified.
- **Dealing with number sequences:** Adding, subtracting, or multiplying patterns reveal the next numbers or explain the sequence.
- **Solving complex problems:** Examining examples or special cases helps uncover a general pattern.
- **Making predictions or generalizations:** Patterns lead to creating rules or formulas.
- **Organizing information in tables:** Rows or columns often reveal clear patterns.
- **Exploring shapes or growth:** Numerical or geometric patterns, such as growing figures, can be identified.
- **Using recursive patterns:** Rules that build on previous steps (e.g., "add 2 each time") emerge.
- **Simplifying computations:** Patterns can make complex calculations easier.
- **Finding patterns within patterns:** Multi-layered patterns can provide additional insight

Wonder, Play, Grow

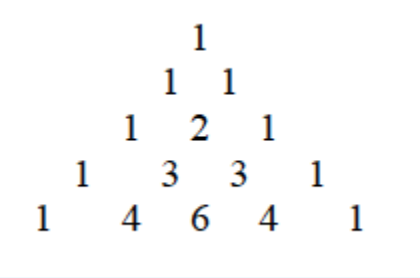
An example of a numerical pattern-finding activity is the following: ____4, 7, ____, 13, ____, ____

What are the missing numbers and how can we describe the pattern?

Answer available at the end of this section.

Example 1.5.1

Complete row 6 and row 7 for the following pattern:



Answer available at the end of this section.

Problem Solving Strategy 5 (Make a Organized List or Table)

The "make a list" strategy is a problem-solving technique where you systematically list possible solutions, outcomes, or important information to better understand and solve a problem. It helps ensure that all possibilities are considered and none are overlooked. (Feikes et al., 2018; Finan, 2006)

When to Use the "Make a List" Strategy:

- **Counting Possibilities:** Helps track all possible arrangements or outcomes.
- **Finding Patterns:** Reveals relationships in numbers generated by a rule or formula.
- **Limited Answers:** Useful when the number of possible solutions is small.
- **Organizing Information:** Simplifies complex data and ensures nothing is missed.
- **Breaking Down Problems:** Helps manage complex problems by listing special cases.
- **Visualizing Arrangements:** Useful for arranging objects or seating plans.
- **Working with Probabilities:** Identifies sample spaces and relationships between sets.
- **Handling Multiple Variables:** Organizes different factors to find all possible combinations.

Think About and Discuss

Before reviewing the next example, consider how you would implement the "Make a List or Table" strategy for the following. Discuss with your peers how you may go about answering the following:

You're at your favorite restaurant, and you're putting together the perfect meal. You can choose **3 types of sandwiches**, **2 kinds of sides**, and **2 different drinks**.

How many different meal combinations can you create if each meal consists of one sandwich, one side, and one drink?

Example 1.5.2

You're at your favorite restaurant, and you're putting together the perfect meal. You can choose **3 types of sandwiches**, **2 kinds of sides**, and **2 different drinks**.

How many different meal combinations can you create if each meal consists of one sandwich, one side, and one drink?

Answer available at the end of this section.

Problem Solving Strategy 6 (Solve a Simpler Problem).

The **"solve a simpler problem"** strategy helps break down complex problems into smaller, more manageable parts. By working with an easier version of the problem, you can identify patterns, understand relationships, and develop a strategy to solve the original problem.(Finan, 2006)

When to Use This Strategy:

- **Complicated computations** → Using smaller numbers makes it easier to see patterns.
- **Very large or very small numbers** → Simplifying helps uncover relationships that apply to the original problem.
- **Direct solution is too complex** → Breaking the problem into smaller steps makes it more approachable.
- **Gaining a better understanding** → A simpler version can clarify the problem's structure.
- **Multiple variables** → Reducing variables simplifies problem-solving.
- **Working backward** → If the final result is clear but the start is not, solving a simpler case first can help.

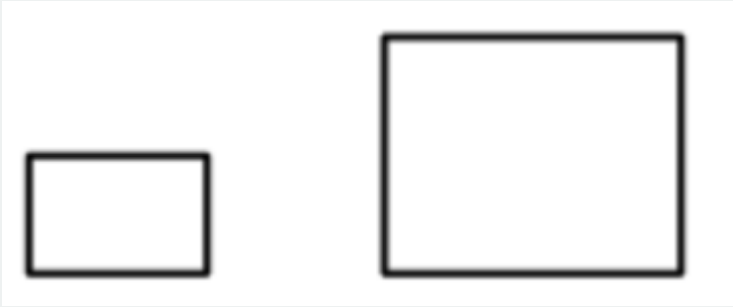
Example 1.5.3

A band wants to decide the order of 5 songs in their concert set list. How many ways can they arrange them? Before looking at the answer, consider your plan of action.

Answer available at the end of this section.

Think About and Discuss

Analyze each problem scenario and decide which problem solving strategy you would first attempt to use to solve each problem. Do this individually first before discussing this with your peers.

Scenario 1	Scenario 2	Scenario 3
Without a calculator, find each sum. **1+ 3 =** **1+ 3 + 5 =** **1 + 3 + 5 + 7 =** **1 + 3 + 5 + 7 + 9 =** . . . **1 + 3 + 5 + 7 + 9 + ...+ 19=?**	**If you double the sides of a rectangle, what happens to the area?** 	**Find the sum** $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + ... + \frac{1}{2^{10}}$.

Understanding Number Sequences and Inductive Reasoning

Definition: Number Sequence

A **number sequence** is a list of numbers arranged in a specific order, where each number in the sequence is called a **term**. For example, the counting numbers **1, 2, 3, 4, ...** form a sequence. The **three dots (...), called an ellipsis**, mean the pattern continues forever.

Key Idea:

Sequences play a crucial role in early mathematics education, as they help young learners develop number sense, recognize patterns, and lay the foundation for algebraic thinking.

Definition: Inductive Reasoning

Inductive reasoning is a way of making predictions based on patterns. Instead of checking every number in a sequence, we look at a small part of it and use that to guess what happens later.

For example, look at this sequence:
5, 10, 15, 20, 25, ...

If we continue for a few more terms:
5, 10, 15, 20, 25, 30, 35, 40, 45, 50, ...

We notice that every **even** term ends in **0 or 5**. Using this pattern, we can predict that the **100th** term will also end in **0**, without listing all 100 numbers.

Key Idea:

Inductive reasoning helps us **find patterns and make predictions** without checking every single number! It is a fundamental aspect of early mathematics learning. It enables children to explore, predict, and discover mathematical relationships, setting the stage for more formal mathematical reasoning in later grades. Encouraging students to explain their reasoning, compare solutions, and refine their conjectures fosters deeper mathematical understanding.

Connections to Mathematical Thinking

Inductive and deductive reasoning are two key approaches to logical thinking in mathematics. Inductive reasoning moves from specific observations to general conclusions by identifying patterns, making predictions, and forming conjectures. However, its conclusions are not always certain. For example, noticing a repeating pattern in a number sequence and assuming it continues is an example of inductive reasoning. In contrast, deductive reasoning starts with general principles and applies them to specific cases to reach definite conclusions. It is used to construct logical arguments, prove theorems, and validate mathematical statements. While elementary students primarily use inductive reasoning to explore patterns, deductive reasoning becomes more important as they advance in their mathematical thinking. Both types of reasoning work together—induction helps discover patterns, while deduction helps prove them. (Burton, 1984; Feikes et al., 2018)

Answers

Wonder, Play, Grow

First blank: 1 **Second blank:** 10 **Third blank:** 16

This is an **arithmetic sequence** with a **common difference** of +3+3+3. In other words, each term is found by **adding 3** to the previous term.

Example 1.5.1

This is is **Pascal's Triangle**, a famous triangular array of numbers. Each number is found by adding the two numbers directly above it in the previous row.

Row 6: 1 5 10 10 5 1, Row 7: 1 6 15 20 15 6 1

NOTE: Make sure to explore the link to all the wonderful patterns that can be found in Pascal's Triangle

www.mathsisfun.com/pascals-triangle.html

Example 1.5.2

3 sandwiches: (S1, S2, S3) **2 sides**: (D1, D2) **2 drinks**: (Dr1, Dr2)

We list all possible sandwiches in one column, each paired with all possible sides and drinks.

Sandwich	Side	Drink
S1	D1	Dr1
S1	D1	Dr2
S1	D2	Dr1
S1	D2	Dr2
S2	D1	Dr1
S2	D1	Dr2
S2	D2	Dr1
S2	D2	Dr2
S3	D1	Dr1
S3	D1	Dr2
S3	D2	Dr1
S3	D2	Dr2

By listing all possible combinations, we see that there are **12 different meal combinations**.

Example 1.5.3

Imagine the band only had 1 **songs** instead of 5. How many ways could they arrange them? 1 Way

Imagine the band only had 2 **songs** instead of 5. How many ways could they arrange them? 2 Ways

Imagine the band only had 3 **songs** instead of 5. How many ways could they arrange them? 6 Ways

Imagine the band only had 4 **songs** instead of 5. How many ways could they arrange them? 24 Ways

Final Answer:

Solving a simpler problem allows you to find a pattern. 1x1=1, 1x2=2, 1x2x3=6, 1x2x3x4=24, ... what is the pattern for each smaller problem?

There are **120 different ways** to arrange the 5 songs in the concert set list. (1x2x3x4x5=120)

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1.5.1: Written Retrieval and Problem Solving Practice

Applying Polya's Four-Step Problem-Solving Method

Purpose

This assignment is designed to enhance your problem-solving skills using Polya's four-step method. By engaging with a mathematical scenario, you will practice understanding a problem, devising a solution plan, executing the plan, and reflecting on your process. These skills are critical for success in mathematics and problem-solving across disciplines.

Learning Objectives:

SLO 1.1 Students will apply and evaluate multiple problem-solving strategies, including Polya's 4-step process, mathematical thinking, and algebraic reasoning to mathematical problems and real-world scenarios.

SLO 1.2 Students will analyze, translate, solve, and interpret mathematical problems and real-world scenarios.

Task

Select one of the three problem scenarios below and solve it using Polya's four-step method. Your response should be structured in the following manner:

1. **Understand the Problem:** Restate the problem in your own words, identifying key information and variables. What are you trying to find? What is known? What is unknown?
2. **Devise a Plan:** Outline the steps you will take to solve the problem. Identify the mathematical concepts, formulas, or strategies you will use.
3. **Carry Out the Plan:** Solve the problem step-by-step, showing all necessary calculations. Explain each step concisely.
4. **Look Back & Reflect:** Analyze your solution. Could you have solved it differently? Were there any difficulties? How does this problem relate to others you have encountered?

Scenario 1	**Scenario 2**	**Scenario 3**
Without a calculator, find each sum. **1+ 3 =** **1+ 3 + 5 =** **1 + 3 + 5 + 7 =** **1 + 3 + 5 + 7 + 9 =** . . . **1 + 3 + 5 + 7 + 9 + ...+ 19=?**	**If you double the sides of a rectangle, what happens to the area?** 	**Find the sum** $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + ... + \frac{1}{2^{10}}$.

Criteria for Success

Your work will be assessed based on the following:

- **Completeness:** Did you follow all four steps of Polya's method?
- **Clarity:** Are your explanations and calculations easy to follow?
- **Accuracy:** Did you use correct mathematical procedures?
- **Reflection:** Did you thoughtfully evaluate your solution and process?

Problem Solving Reflection

After solving the problem, reflect on your experience using the following prompts, be prepared to discuss your reflectrions in class.:

- Identify one key takeaway from this problem-solving process.
- Describe an "aha" moment you had while working through the problem.
- What questions do you still have about the problem or Polya's method?
- How did working individually or with others influence your understanding?
- How can you apply what you learned to future problem-solving situations?

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1.6: Problem Solving Practice

This section provides you with more problem solving problems to consider, review, and even try. As we discussed before, problem-solving is at the heart of learning mathematics, offering students far more than just the ability to find answers—it helps them develop critical thinking, reasoning, and analytical skills that extend beyond the classroom. I hope you have some time to engage with these example problems before you get started with your own work. It is important to remember that problem-solving encourages perseverance, creativity, and a growth mindset— you learn that struggle is a natural part of learning and that there are multiple ways to approach a challenge. Considering your future students, when they see math as a powerful tool for understanding and navigating the world, they become more engaged, motivated, and confident in their ability to tackle complex problems. You have the power to help them with that, but it first starts with you developing your own productive math mindset. Play the song to see what that sounds like. 😀

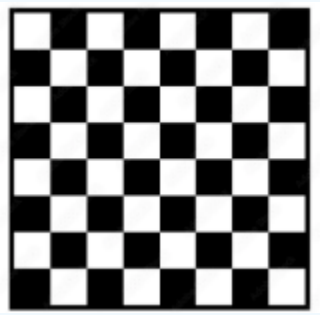
Math Mindsets SONG

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this Interactive Content online

Example 1.6.1

How many squares, of any possible size, are on a 8 × 8 chess board? (The answer is not 64... It's a lot bigger!) Remember Pólya's first step is to understand the problem. If you are not sure what is being asked, or why the answer is not just 64, be sure to ask someone!



Answer available at the end of this section.

size of board	# of 1 × 1 squares	# of 2 × 2 squares	# of 3 × 3 squares	# of 4 × 4 squares	...
1 by 1	1	0	0	0	
2 by 2	4	1	0	0	
3 by 3	9	4	1	0	
...					

SEE IF THIS INTERACTIVE APP HELPS YOU VISUALIZE THE SOLUTION

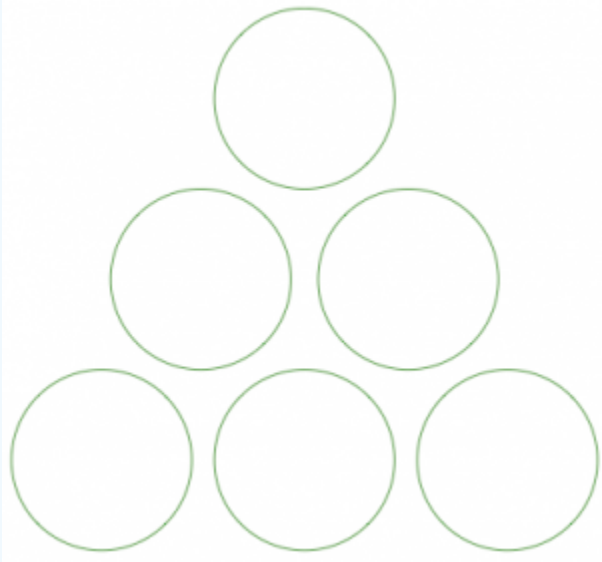
Example 1.6.2

Certain pipes are sold in lengths of 6 inches, 8 inches, and 10 inches. How many different lengths can you form by attaching three sections of pipe together?

Answer available at the end of this section.

Example 1.6.3

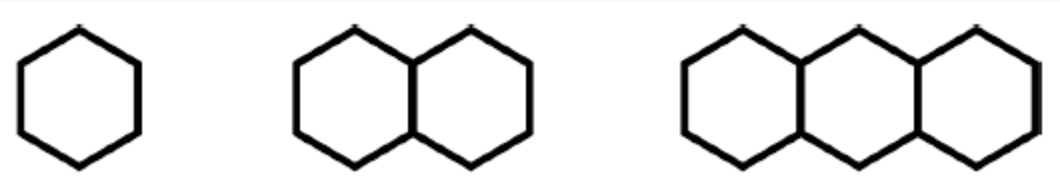
Place the digits 1, 2, 3, 4, 5, 6 in the circles so that the sum on each side of the triangle is 12. Each circle gets one digit, and each digit is used exactly once.



Answer available at the end of this section.

Example 1.6.4

As pictured in the table below, the hexagon train made with 1 hexagon has a perimeter of 6; the hexagon train made with 2 hexagons has a perimeter of 10; and the hexagon train made with 3 hexagons has a perimeter of 14. What is the perimeter of the hexagon train made with 4 hexagons? 5 hexagons? 100 hexagons?



Answer available at the end of this section.

Answers

Example 1.6.1

It is clear that you want to draw a picture for this problem, but even with the picture it can be hard to know if you have found the correct answer. The numbers get big, and it can be hard to keep track of your work. Maybe we can try a simpler problem.

Pólya suggested this strategy: "If you can't solve a problem, then there is an easier problem you can solve: find it." He also said: "If you cannot solve the proposed

problem, try to solve first some related problem. Could you imagine a more accessible related problem?" In this case, an 8 × 8 chess board is pretty big. Can you solve the problem for smaller boards? Like 1 × 1? 2 × 2? 3 × 3? Working with smaller boards might give you some insight and help you devise your plan (that is Pólya's step (2))

For example, in this problem you might keep track of how many 1 × 1 squares are on each board, how many 2 × 2 squares on are each board, how many 3 × 3 squares are on each board, and so on. You could keep track of the information in a table: See if you can do the calculations or see the pattern for the 4 by 4 size. You will get 16, 9, 4, 1 squares. If you continue the same pattern for the 5 by 5 and so on until you check the 8 by 8, you will end up with the total number of squares in an 8×8 chessboard, 204.

Video Explanation

https://youtu.be/wYGZA3DHAKk?si=Tfe8t6lZUgop3n8B

Example 1.6.2

Step 1: Consider all possible selections of three pipes

Each section of pipe can be either **6, 8, or 10 inches** long. Since we are selecting **three** sections, we consider all possible sums of three choices.

Step 2: List all possible sums

We compute the sum for all possible combinations (allowing repetition):

All three sections the same:

- 6+6+6=18
- 8+8+8=24
- 10+10+10=30

Two of one type, one of another:

- 6+6+8=20
- 6+6+10=22
- 8+8+6=22
- 8+8+10=26
- 10+10+6=26
- 10+10+8=28

All three different:

- 6+8+10=24

Step 3: List the unique sums

The different possible sums are:

18,20,22,24,26,28,30

There are **7** different possible pipe lengths that can be formed.

Example 1.6.3

To solve this puzzle, we need to place the digits **1, 2, 3, 4, 5, and 6** in the circles so that the sum along each side of the triangle equals **12**. Here is a possible arrangement. Top = 5, Middle Left = 1, Bottom Left = 6, Middle Right = 3, Bottom Right = 4, Bottom Middle = 2.

Example 1.6.4

Each additional hexagon increases the perimeter by **4**, as two sides overlap with the previous hexagon. The pattern follows the rule: Perimeter = 4 (figure number) + 2 .

4 Hexagons: P = 4(4) + 2 = 18

5 Hexagons: P = 5(4) + 2 = 22

100 Hexagons: P = 100(4) + 2 = 402

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1.6.1: Written Retrieval and Problem Solving Practice

Student Learning Objectives

SLO 1.1 Students will apply and evaluate multiple problem-solving strategies, including Polya's 4-step process, mathematical thinking, and algebraic reasoning to mathematical problems and real-world scenarios.

SLO 1.2 Students will analyze, translate, solve, and interpret mathematical problems and real-world scenarios.

The mild assignment 🌶 is a shorter and less challenging assignment. The medium assignment🌶🌶 is right in the middle in terms of length and level of difficulty. Finally, the spicy assignment 🌶🌶🌶 is a longer assignment that is the most challenging.

(Spicy 🌶🌶🌶 Problems are OPTIONAL, but go for it when you have the time to challenge yourself)

Exercise 1.6.1.1 🌶🌶

Sylvia and Morgan were asked to add two whole numbers Instead, Sylvia subtracted the two numbers and got 10, and Morgan multiplied them and got 651. What was the correct sum?

Answer available at the end of this section.

Exercise 1.6.1.2 🌶🌶

Show that 13 can be expressed as the sum of two consecutive numbers. Then decide whether every odd number can be expressed as the sum of two consecutive counting numbers.

Answer available at the end of this section.

Exercise 1.6.1.3 🌶🌶

Using the numbers 9, 8, 7, 6, 5, and 4 once each, find the following:

a) The largest possible sum for TWO 3-digit numbers.

b) the smallest (positive) difference for TWO 3-digit numbers.

Answer available at the end of this section.

Exercise 1.6.1.4 🌶🌶

A textbook is opened and the product on the page numbers of the two facing pages is 3,080. What are the numbers of the pages?

Answer available at the end of this section.

Exercise 1.6.1.5 🌶🌶

Using the symbols +, - , x , ÷ once each, fill in the following three blanks to make a true equation (Parentheses allowed).

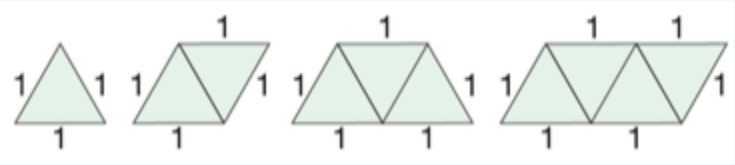
a) 6 __ 6 __ 6 __ 6 __ = 13

b) 5 __ 5 __ 5 __ 5 __ = 55

Answer available at the end of this section.

Exercise 1.6.1.6

Find the perimeters and complete the table.



# of triangles	1	2	3	4	5	6	10		n
perimeter								40	

Answer available at the end of this section.

Exercise 1.6.1.7

Find the missing term in each pattern.

a) 10, 17, ____, 37, 50, 65

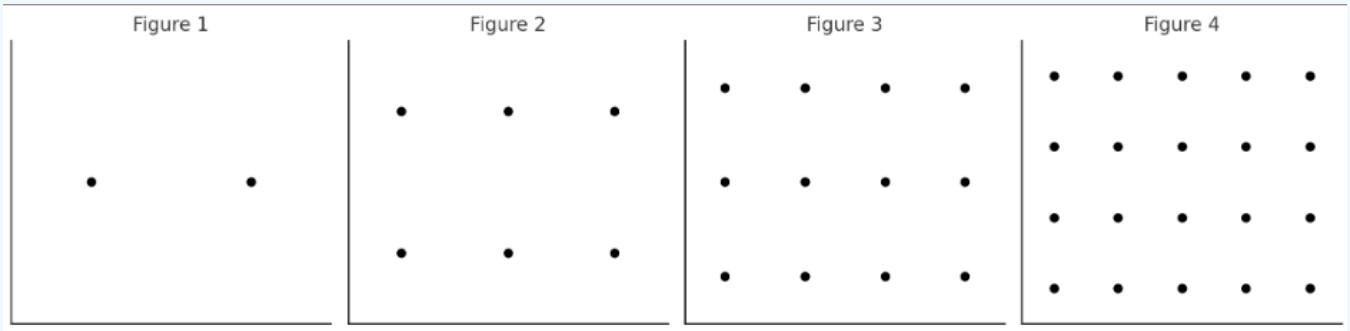
b) 1, $\frac{3}{2}$, ___, $\frac{7}{8}$, $\frac{9}{16}$

c) 243, 324, 405, ___, 567

Answer available at the end of this section.

Exercise 1.6.1.8

Rectangular numbers are whole numbers that are represented by certain rectangular arrays of dots. The first four rectangular numbers are shown.



a) Make a sketch of the 6th rectangular number.

b) How many dots will be in the 10th rectangular number?

c) Come up with a description or formula for the number of dots in the **nth** rectangular number.

Answer available at the end of this section.

Exercise 1.6.1.9

The **triangular numbers** are the whole numbers that are represented by certain triangular arrays of dots. The first four triangular numbers are shown.



a) Make a sketch to represent the 6th triangular number.

b) How many dots will be in the 10th triangular number?

c) Is there a triangular number that has 91 dots in its shape? If so, which one?

d) Come up with a description or formula for the number of dots in the **nth** rectangular number

Answer available at the end of this section.

Exercise 1.6.1.10

Gabby extended the pattern 2, 4, 8 to be 2, 4, 8, 16, 32, ... Caitlin extended the same pattern to be 2, 4, 8, 14, 22, ... They asked you who was correct. How should you respond?

Answer available at the end of this section.

Exercise 1.6.1.11

The Fibonacci sequence was defined to be the sequence 1,1,2,3,5,8,13,21, ..., where each successive number is the sum of the preceding two. Observe the following pattern:

1 + 1 = 3 – 1

1 + 1 + 2 = 5 - 1

1 + 1 + 2 + 3 = 8 - 1

1 + 1 + 2 + 3 + 5 = 13 - 1

Write out six more terms of the Fibonacci sequence, and use the sequence to predict the answer to:

1 + 1 + 2 + 3 + 5 + ... + 144

without actually computing the sum. Then use your calculator to check your result.

Answer available at the end of this section.

Exercise 1.6.1.12 🌶🌶🌶

How many palindromes are there between 10 and 1000? (A palindrome is a number like 525 that reads the same backward or forward.)

Answer available at the end of this section.

Answers

🌶🌶 **Exercise** 1.6.1.1

31 and 21

🌶🌶**Exercise** 1.6.1.2

6+7=13,

Let's generalize:

Let n and n+1 be any two consecutive counting numbers.
Their sum is:

n+(n+1)=2n+1

This is always **an odd number**, since it's one more than an even number.

🌶🌶**Exercise** 1.6.1.3

a) Strategy: Assign the **three largest digits** to the **hundreds** place. Try to form the **two largest possible 3-digit numbers**. 1641 is the largest.

b) To **minimize the difference**, the two numbers should be as **close as possible**, but **not equal**, and digits must be **distinct**. 87 is the smallest.

🌶🌶**Exercise** 1.6.1.4

55 and 56

Exercise 1.6.1.5🌶🌶

a) 6 + (6/6) + 6 is an option. b) 5 x (5 + 5) + 5 is an option.

Exercise 1.6.1.6🌶🌶

Perimeter: 3, 4, 5, 6, 7, 12, n+2

38 triangles has perimeter is 40

Exercise 1.6.1.7🌶🌶

a) 26 b) 5/4 c) 486

Exercise 1.6.1.8🌶🌶

a) 6 rows × 7 columns of dots (total **42**)

b) 10th rectangular number = **110 dots**

c) n x (n+1)

Exercise 1.6.1.9🌶🌶

It is worth your time to learn about how triangular numbers hold a special beauty in mathematics and nature, revealing deep connections between number patterns, geometry, and real-world structures.

a) We build the triangle by stacking rows:

Row 1: 1 dot

Row 2: 2 dots

Row 3: 3 dots

Row 4: 4 dots

Row 5: 5 dots

Row 6: 6 dots

b) 55 Dots - (10 x 11) / 2 c) YES, 13th Triangular Number, (13 x 14)/2 = 91 d) n(n+1)/2

Exercise 1.6.1.10🌶🌶

Gabby extended the pattern as **2, 4, 8, 16, 32, ...**

This follows a **sequence** where each term is multiplied by 2.

Caitlin extended the pattern as **2, 4, 8, 14, 22, ...**

This follows a **sequence** where each term increases by 2 more each time. Up by 2, then up by 4, then up by 6, ...

Since both sequences follow a logical rule (inductive reasoning), neither Gabby nor Caitlin is *incorrect*—they just interpreted the pattern differently.

Exercise 1.6.1.11🌶🌶🌶

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...

(377 - 1) = 376 = 1+1+2+3+5+8+13+21+34+55+89+144

Exercise 1.6.1.12🌶🌶🌶

99, 9 two digit and 90 three digit.

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1.6.2: Online Retrieval and Problem Solving Practice

Student Learning Objectives

SLO 1.1 Students will apply and evaluate multiple problem-solving strategies, including Polya's 4-step process, mathematical thinking, and algebraic reasoning to mathematical problems and real-world scenarios.

SLO 1.2 Students will analyze, translate, solve, and interpret mathematical problems and real-world scenarios.

Checking for Understanding (CFU): Use the following questions to check your understanding. The results are not part of your grade. So feel free to try multiple problems or to try problems multuple times to ensure you are understanding.

CFU 1.6.2.1

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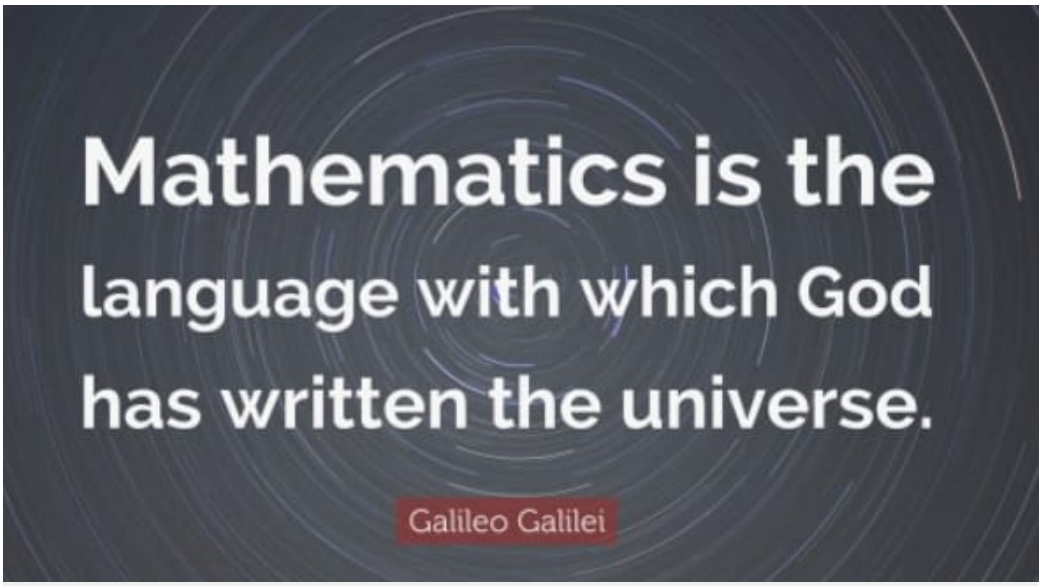
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1.7: Mathematics as a Language

This section might seem like a bit of a sidetrack from the idea of problem solving, but in fact it is not. Mathematics is a social endeavor. We do not just solve problems and then put them aside. We need to know how to communicate our mathematics to others. With that in mind, many prospective elementary teachers may not initially think of mathematics as a language, yet it has its own symbols, structures, and ways of communicating ideas. Like spoken and written languages, math relies on a system of notation and grammar that allows for precise expression and logical reasoning (Wager and Stinson, 2012). Just as letters form words and sentences, mathematical symbols create expressions and equations that convey meaning. When teachers understand math as a language, they can present it more flexibly, helping students see connections rather than isolated rules.This perspective also highlights math as a tool for communication—students should be encouraged to verbalize their reasoning, bridging the gap between informal descriptions and formal mathematical vocabulary (Seda and Brown, 2021)

Viewing math as a language fosters a sense-making approach rather than rote memorization, which is crucial for reducing math anxiety in both teachers and students (Boaler, 2022). It also acknowledges the cultural context of mathematics, as different societies have developed unique ways of representing and thinking about numbers. By recognizing these diverse perspectives, teachers can make math more accessible and meaningful for all students (Seda and Brown, 2021). Ultimately, when educators embrace the linguistic nature of mathematics, they empower students to become fluent, confident mathematical thinkers who see math as a creative and relevant part of their lives (Wager and Stinson, 2012). Mathematics is more than just a subject to be learned in school; it is a language for understanding and interacting with the world (NCTM, 2020)



Mathematical Statements

Definition

A mathematical statement is a complete sentence that is either true or false, but not both at once.

NOTE: In mathematics, we **reserve the term *mathematical statement*** for statements that are:

- About **mathematical objects** (numbers, sets, functions, structures, logical relationships)
- Or used in **mathematical reasoning** or **formal logic systems**

"Modern, pure mathematics is primarily concerned with statements about mathematical objects. Mathematical objects are things such as: integers, real numbers, sets, functions, etc." (Devlin, 2012 Math Thinking). Examples of mathematical statements are:

- (1) There are infinitely many prime numbers.
- (2) For every real number a, the equation $x^2 + a = 0$ has a real root.
- (3) $\sqrt{2}$ is irrational.

Not only are mathematicians interested in statements of the above kind, they are, above all, interested in knowing which statements are true and which are false. For instance, in the above examples, (1), and (3) are true whereas (2) is false. The truth or falsity in each case is demonstrated not by observation, measurement, or experiment, as in the sciences, but by a proof. (Devlin, 2012 Math Thinking)

So a “statement” in mathematics cannot be a question, a command, or a matter of opinion. It is a complete, grammatically correct sentence (with a subject, verb, and usually an object). It is important that the statement is either true or false, though you may not know which! (Part of the work of a mathematician is figuring out which sentences are true and which are false.)

Think About and Discuss

For each English sentence below, decide if it is a mathematical statement or not. If it is, is the statement true or false (or are you unsure)? If it is not a mathematical statement, in what way does it fail?

1. Blue is the prettiest color.
2. 60 is an even number.
3. Honolulu is the capital of Hawaii.
4. This sentence is false.
5. 1/2 = 2/4.
6. There are a total of 204 squares on an 8 × 8 chess board.

Now write some mathematical statements and some English sentences that fail to be mathematical statements.

Notice that “1/2 = 2/4” is a perfectly good mathematical statement. It does not look like an English sentence, but read it out loud. The subject is “1/2.” The verb is “equals.” And the object is “2/4.” This is a very good test when you write mathematics: try to read it out loud. Even the equations should read naturally, like English sentences. "Honolulu is the capital of Hawaii" is a **factual statement** from geography, but it does not concern mathematical objects. Statement (4) is different from the others. It is called a **paradox**: a statement that is self-contradictory. If it is true, then we conclude that it is false. (Why?) If it is false, then we conclude that it is true. (Why?) Paradoxes are no good as mathematical statements, because it cannot be true and it cannot be false.

Mathematical vs. Everyday Language: Mathematics requires precision, and the use of language in mathematics is often more precise than in everyday settings. When using language, people often rely on context to determine what their words convey. However, in mathematics, the context must be specific (Devlin, 2012).

And / or

Consider this sentence:

After work, I will go to the beach, or I will do my grocery shopping.

In everyday English, that probably means that if I go to the beach, I will not go shopping. I will do one or the other, but not both activities. This is called an “exclusive or.”

We can usually tell from context whether a speaker means “either one or the other or both,” or whether he means “either one or the other but not both.” (Some people use the awkward phrase “and/or” to describe the first option.)

Remember that in mathematical communication, though, we have to be very precise. We cannot rely on context or assumptions about what is implied or understood.

Definition

In mathematics, the word “or” always means “one or the other or both.”

The word “and” always means “both are true.”

Think About and Discuss

For each sentence below:

- Decide if the choice $x = 3$ makes the statement true or false.
- Choose a different value of x that makes the statement true (or say why that is not possible).
- Choose a different value of x that makes the statement false (or say why that is not possible).

1. x is odd or x is even.
2. x is odd and x is even.
3. $x > 5$ or $x < 5$.
4. $x > 5$ and $x < 5$
5. $x \cdot 1 = x$ or $x \cdot 0 = x$.
6. $x \cdot 1 = x$ and $x \cdot 0 = x$.

Quantifiers

Think About and Discuss

You are handed an envelope filled with money, and you are told “Every bill in this envelope is a $100 bill.”

- What would convince you beyond any doubt that the sentence is true? How could you convince someone else that the sentence is true?
- What would convince you beyond any doubt that the sentence is false? How could you convince someone else that the sentence is false?

Suppose you were given a different sentence: “There is a $100 bill in this envelope.”

- What would convince you beyond any doubt that the sentence is true? How could you convince someone else that the sentence is true?
- What would convince you beyond any doubt that the sentence is false? How could you convince someone else that the sentence is false?

What is the difference between the two sentences? How does that difference affect your method to decide if the statement is true or false?

Some mathematical statements have this form:

- “Every time...”
- “For all numbers. . . ”
- “For every choice. . . ”
- “It’s always true that. . . ”

These are *universal* statements. Such statements claim that something is always true, no matter what.

- To prove a universal statement is false, you must find an example where it fails. This is called a **counterexample** to the statement.
- To prove a universal statement is true, you must either check every single case, or you must find a logical reason why it would

be true. (Sometimes the first option is impossible, because there might be infinitely many cases to check. You would never finish!)

Some mathematical statements have this form:

- "Sometimes..."
- "There is some number. . . "
- "For some choice. . . "
- "At least once..."

These are *existential* statements. Such statements claim there is some example where the statement is true, but it may not always be true.

- To prove an existential statement is true, you may just find the example where it works.
- To prove an existential statement is false, you must either show it fails in every single case, or you must find a logical reason why it cannot be true. (Sometimes the first option is impossible!)

Think About and Discuss

For each statement below, do the following:

- Decide if it is a universal statement or an existential statement. (This can be tricky because in some statements the quantifier is "hidden" in the meaning of the words.)
- Decide if the statement is true or false, and do your best to justify your decision.

1. Every odd number is prime.
2. For all positive numbers $x, x^2 > x$.
3. There is some number x such that $x^3 < x$.
4. Addition (of real numbers) is commutative (a + b = b + a).

Look back over your work. You will probably find that some of your arguments are sound and convincing while others are less so. In some cases you may "know" the answer but be unable to justify it. That is okay for now! Divide your answers into four categories:

1. I am confident that the justification I gave is good.
2. I am not confident in the justification I gave.
3. I am confident that the justification I gave is not good, or I could not give a justification.
4. I could not decide if the statement was true or false.

Conditional Statements

Wonder, Play, Grow

You have a deck of cards where each card has a letter on one side and a number on the other side. Your friend claims: "If a card has a vowel on one side, then it has an even number on the other side."

These cards are on a table.



Which cards must you flip over to be certain that your friend is telling the truth?

After you have thought about the problem on your own for a while, discuss your ideas with a partner. Do you agree on which cards you must check? Try to come to an agreement on an answer you both believe.

Here is another very similar problem, yet people seem to have an easier time solving this one:

Wonder, Play, Grow

You are in charge of a party where there are young people. Some are drinking alcohol, others soft drinks. Some are old enough to drink alcohol legally, others are under age. You are responsible for ensuring that the drinking laws are not broken, so you have asked each person to put his or her photo ID on the table. At one table, there are four young people:

- One person has a can of beer, another has a bottle of Coke, but their IDs happen to be face down so you cannot see their ages.
- You can, however, see the IDs of the other two people. One is under the drinking age, the other is above it. They both have fizzy clear drinks in glasses, and you are not sure if they are drinking soda water or gin and tonic.

Which IDs and/or drinks do you need to check to make sure that no one is breaking the law?

After you have thought about the problem on your own for a while, discuss your ideas with a partner. Do you agree on which IDs/Drinks you must check? Compare these two problems. Which question is easier and why?

Definition

A **conditional statement** can be written in the form

If some statement then some statement.

Where the first statement is the **hypothesis** and the second statement is the **conclusion**.

These are each conditional statements, though they are not all stated in “if/then” form. Identify the hypothesis of each statement. (You may want to rewrite the sentence as an equivalent “if/then” statement.)

1. If the tomatoes are red, then they are ready to eat.
   The tomatoes are red. / The tomatoes are ready to eat.
2. An integer n is even if it is a multiple of 2.
   n is even. / n is a multiple of 2.
3. If n is odd, then n is prime.
   n is odd. / n is prime.
4. The team wins when JJ plays.
   The team wins. / JJ plays.

Remember that a mathematical statement must have a definite truth value. It is either true or false, with no gray area (even though

we may not be sure which is the case). How can you tell if a conditional statement is true or false? Surely, it depends on whether the hypothesis and the conclusion are true or false. But how, exactly, can you decide?

The key is to think of a conditional statement like a promise, and ask yourself: under what condition(s) will I have broken my promise?

Example 1.7.1:

Here is a conditional statement:

> If I win the lottery, then I'll give each of my students $1,000.

There are four things that can happen:

- **True hypothesis, true conclusion:** I do win the lottery, and I do give everyone in class $1,000. I kept my promise, so the conditional statement is TRUE.
- **True hypothesis, false conclusion:** I do win the lottery, but I decide not to give everyone in class $1,000. I broke my promise, so the conditional statement is FALSE.
- **False hypothesis, true conclusion:** I do not win the lottery, but I am exceedingly generous, so I go ahead and give everyone in class $1,000. I did not break my promise! (Do you see why?) So the conditional statement is TRUE.
- **False hypothesis, false conclusion:** I do not win the lottery, so I do not give everyone in class $1,000. I did not break my promise! (Do you see why?) So the conditional statement is TRUE.

What can we conclude from this? **A conditional statement is false only when the hypothesis is true and the conclusion is false.** In every other instance, the promise (as it were) has not been broken. If a mathematical statement is not false, it must be true.

Example 1.7.2:

Here is another conditional statement:

> If you live in Honolulu, then you live in Hawaii.

Is this statement true or false? It seems like it should depend on who the pronoun "you" refers to, and whether that person lives in Honolulu or not. Let us think it through:

- Sookim lives in Honolulu, so the hypothesis is true. Since Honolulu is in Hawaii, she does live in Hawaii. The statement is true about Sookim, since both the hypothesis and conclusion are true.
- DeeDee lives in Los Angeles. The statement is true about DeeDee since the hypothesis is false.

So in fact it does not matter! The statement is true either way. The right way to understand such a statement is as a universal statement: "Everyone who lives in Honolulu lives in Hawaii."

This statement is true, and here is how you might justify it: "Pick a random person who lives in Honolulu. That person lives in Hawaii (since Honolulu is in Hawaii), so the statement is true for that person. I do not need to consider people who do not live in Honolulu. The statement is automatically true for those people, because the hypothesis is false!"

Example 1.7.3:

How do we show a (universal) conditional statement is false?

You need to give a specific instance where the hypothesis is true and the conclusion is false. For example:

> If you are a good swimmer, then you are a good surfer.

Do you know someone for whom the hypothesis is true (that person is a good swimmer) but the conclusion is false (the person is not a good surfer)? Then the statement is false!

Wonder, Play, Grow

For each conditional statement, decide if it is true or false. Justify your answer.

1. If $2 \times 2 = 4$ then $1 + 1 = 3$.
2. If a number is even, then the number has a 4 in the one's place.
3. If the product of two numbers is 0, then one of the numbers is 0.
4. If $2 \times 2 = 5$ then $1 + 1 = 3$.

On your own and with others, come up with two conditional statements that are true and one that is false. Share your three statements with peers, but do not say which are true and which is false. See if they can figure it out!

REAL WORLD ANALOGIES

Weather Promise

> "If it rains tomorrow, I will bring my umbrella."

- If it **rains** and I **don't bring the umbrella**, I **broke my promise**. (False)
- If it **rains** and I **bring it**, I **kept my promise**. (True)
- If it **doesn't rain**, you can't say I broke my promise, because the promise only applied **if it rained**. So it counts as **true** (kept).

Nonsense Promise

> "If I am the Queen of England, then I will give you a million dollars."

Since I'm **not** the Queen, this never applies. So you can't say I lied—I just said something that doesn't trigger.

A conditional only promises something about the world if the hypothesis is true. When the hypothesis is false, it "says nothing," so by definition it counts as satisfied (true).

Mathematics serves as a universal language, allowing for precise communication of ideas, logical reasoning, and problem-solving (CDE, 2013; NCTM, 2020). Just as spoken and written languages have grammar and structure, math employs symbols and notation to convey complex relationships and patterns. For future elementary school teachers, understanding math as a language is crucial because it transforms the subject from a rigid set of rules into a dynamic tool for communication and reasoning. When educators embrace this perspective, they can help students see mathematical concepts as interconnected rather than isolated, fostering deeper comprehension and reducing math anxiety. By encouraging students to verbalize their thought processes and relate mathematical ideas to real-world contexts, teachers create an inclusive and engaging learning environment where all students can become confident, fluent mathematical thinkers. (Seda and Brown, 2021; Wager and Stinson, 2012).

Retrieval Practice

Interactive Content

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1.7.1: Written Retrieval and Problem Solving Practice

Student Learning Objectives

SLO 1.4 Students will be able to analyze and use mathematical language, symbols, and structures to communicate mathematical ideas effectively, demonstrating an understanding of precision and logic in problem-solving.

Mathematics as a Language

Objective:

This assignment is designed to encourage you to think about the nature of mathematics as a language, how mathematical statements function, and the role of proof in mathematical reasoning.

Instructions:

Respond to the following questions thoughtfully. Some questions require written explanations, while others involve problem-solving.

The mild assignment 🌶 is a shorter and less challenging assignment. The medium assignment🌶🌶 is right in the middle in terms of length and level of difficulty. Finally, the spicy assignment 🌶🌶🌶 is a longer assignment that is the most challenging.

(Spicy 🌶🌶🌶 Problems are OPTIONAL, but go for it when you have the time to challenge yourself)

Exercise 1.7.1.1 🌶🌶

Mathematics as a Language:

In your own words, explain why mathematics can be considered a language. Provide one example that highlights how mathematical symbols function similarly to words in a spoken language.

Answer available at the end of this section.

Exercise 1.7.1.2 🌶🌶

Truth and Falsity in Mathematics:
Consider the statement: "For all integers n, if n^2 is even, then n is even."
a) Is this statement true or false?
b) If true, justify it using logical reasoning. If false, provide a counterexample.

Answer available at the end of this section.

Exercise 1.7.1.3 🌶🌶

Symbolic Ambiguity:
Consider the following statement: "A number squared is always greater than or equal to the original number."
a) Test this statement with three different values (one positive, one negative, and zero).
b) Is the statement always true? If not, refine it so that it is always correct.

Answer available at the end of this section.

Exercise 1.7.1.4 🌶️🌶️

Mathematical Definitions Matter

- The word "even" in everyday language means "flat" or "smooth," but in mathematics, it has a precise meaning. Why is it important for mathematical terms to have strict definitions? Give an example of a mathematical term that has a different meaning in everyday life.

Answer available at the end of this section.

Exercise 1.7.1.5 🌶️🌶️🔷

Mathematics and Logical Structure

Rewrite the following statement using an "if-then" format, which is common in mathematical reasoning:
"*A number is divisible by 10 whenever it ends in 0.*"

Answer available at the end of this section.

Mini Project: "Teach It Forward"

💡 **Assignment:** Mathematics is a language, and the best way to learn a language is to **speak it!** Your task is to **explain** one key idea from our lesson to someone in your life—a friend, family member, roommate, or even a pet (though you'll have to imagine their responses!🐶🐱).

🔹 **Step 1: Choose Your Audience**
Pick someone to share your learning with—this could be a younger sibling, a parent, a grandparent, a friend who isn't taking this class, or even your pet (but write as if they responded!).

🔹 **Step 2: Explain Clearly & Creatively**
Choose one of the following ways to communicate your explanation:

- **Write a short dialogue** as if you're having a conversation with them.
- **Create a comic strip** showing your explanation.
- **Post** a social media response sharing your explanation.
- **Record a short video** of you teaching them and summarize their reactions.
- **Write an email** to your audience explaining the idea in a way that makes sense to them.

🔹 **Step 3: Reflect on the Experience**
After your explanation, write a **short reflection (3-5 sentences)** on:

- How did they respond?
- What was the most challenging part of explaining it?
- Did they ask any questions that made you think differently?

🎯 **Key Topics to Choose From:**

1. What does it mean to say that math is a language?
2. Why do mathematicians use symbols instead of words?
3. What makes a mathematical statement different from a regular sentence?
4. How do mathematical proofs help us know something is true?
5. Why is precision in mathematical language important?

📌 **Goal:** The goal is **not** to be overly technical, but to make the concept understandable to someone who doesn't think about math every day. Keep it clear, fun, and engaging!

Answers

🌶️🌶️**Exercise** 1.7.1.1

Share your responses with others to see what they think. But the math operators (+, -, ...) are good examples of the symbol telling you to do something with two numbers.

🌶️🌶️**Exercise** 1.7.1.2

a) True

b) To test its truth, we can use **proof by contrapositive**. The contrapositive of the statement is: if n is ODD, then n^2 is ODD. We can also directly prove using algebra. Let n be even, which means it is of the form 2k, where k is any whole number. Then n^2 is $(2k)^2 = 4k^2$. Anything times 4 is always an even number since it is multiple of 4. Multiples of 4 are divisible by 2.

(NOTE: A statement and its contrapositive are **logically equivalent**. That means if one is true, the other is also true - This is NOT ON THE TEST)

🌶️🌶️**Exercise** 1.7.1.3

a) 2^2=4 (true), $(-3)^2$=9 (true), 0^2=0 true, 0.5^2 = 0.25 (false)

b) The original statement **is not always true** for numbers between 0 and 1. "A number squared is greater than or equal to the original number for all real numbers except those between 0 and 1."

🌶️🌶️**Exercise** 1.7.1.4

It's important for mathematical terms to have **strict definitions** because mathematics relies on clarity, precision, and logical consistency. In everyday language, words can have multiple meanings or be interpreted loosely, but in mathematics, even small misunderstandings can lead to incorrect reasoning or wrong answers. Strict definitions ensure that everyone using the term is thinking about the exact same concept.

In everyday life:
A "line" might mean something short—like "Stand in line" or "Draw a line under your name." People often think of a line as a short segment.

In mathematics:
A **line** is **infinitely long** in both directions.
It has **no endpoints**, and it's perfectly straight.

🌶️🌶️🔶 **Exercise** 1.7.1.5

If a number ends in 0, then it is divisible by 10.

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1.8: Project - Mathematizing the World

Student Learning Objectives

(SLO 1.3) Students will **create** a mathematical problem-solving scenario inspired by real-world observations and **analyze** it using strategies such as counting, measuring, identifying patterns, or quantifying spatial relationships.

Mathematizing the World

> As a future elementary school teacher, you have the incredible opportunity to help children see the world through a mathematical lens. Mathematizing the world means recognizing and interpreting the math present in everyday experiences—uncovering the patterns, relationships, and structures that shape the way we live (Burton, 1984; Feikes, Schwingendorf, and Gregg, 2018). It's not just about teaching formulas or procedures; it's about fostering a sense of curiosity and understanding. When you approach math with this mindset, you make the subject more meaningful and relevant, showing your students that math isn't confined to textbooks—it's all around them, woven into their daily lives (Boaler, 2022; Feikes, Schwingendorf, and Gregg, 2018; Wager and Stinson, 2012).
>
> Embracing mathematization in your teaching will help you create a classroom where students feel confident, capable, and excited about math. By connecting math to real-world experiences, you'll help your students develop essential problem-solving, reasoning, and critical-thinking skills. You'll also promote equity by incorporating diverse perspectives and experiences, ensuring that all children see themselves as mathematical thinkers. Shifting the focus from memorization to understanding will reduce math anxiety, build confidence, and foster a lifelong appreciation for math. Ultimately, by helping your students see the beauty and power of mathematics, you'll empower them to engage with the world in a deeper, more meaningful way (Boaler, 2022; CDE, 2013; Wager and Stinson, 2012).

Purpose

The purpose of this assignment is to help you explore the connection between mathematics and the real world by identifying mathematical concepts in everyday objects, images, or videos. By engaging in mathematical thinking and problem-solving inspired by real-world observations, you will practice applying math in practical, meaningful ways and learn how to communicate your thought process clearly. This assignment will help you see how mathematics is relevant in your everyday environment and encourage curiosity and inquiry-based learning.

Skills

This assignment will help you practice the following skills essential for success in this course and in your professional life:

- **Problem-Solving**: Identifying and solving mathematical problems based on real-world scenarios.
- **Mathematical Thinking**: Applying concepts such as counting, measurement, estimation, patterns, and spatial relationships to real-life contexts.
- **Inquiry and Curiosity**: Asking meaningful mathematical questions based on observations of the world around you.
- **Communication**: Clearly presenting your mathematical thinking and problem-solving process using visual or digital tools (Google Slides, video, poster).

(SLO 1.3) Students will **create** a mathematical problem-solving scenario inspired by real-world observations and **analyze** it using strategies such as counting, measuring, identifying patterns, or quantifying spatial relationships.

Tasks

Step 1: Select a Real-World Image, Video, or Object

1. **Find a Mathematical Source**: Choose an image, video, or physical object that includes both mathematical features (e.g., invites counting, equal grouping, estimation, measurement, patterns) and contextual features that connect to real-world ideas or evoke curiosity.
2. **Observe and Wonder**:
    - Identify (3) things you notice and (3) things you wonder about regarding the image, video, or object. You may ask peers, co-workers, and loved ones for their ideas too. ***NOTE: If you cannot provide enough noticings and wonderings, then consider a new image, video, or object.***

Step 2: Pose Mathematical Questions

1. **Create Questions**: Pose at least 3 questions based on your image, video, or object that can be answered using K-12 level mathematics. These questions should involve counting, measuring, identifying patterns, or other mathematical processes.

Step 3: Solve One Question

1. **Pursue a Solution**: Choose one of your posed questions and solve it using appropriate problem-solving strategies and mathematical thinking. Clearly explain your reasoning and the steps you took to solve the problem.
    - Use mathematical strategies such as measurement, pattern identification, spatial quantification, etc.
    - Show your work and reasoning in a clear, organized way.

Step 4: Present Your Work

1. 1. 1. **Create a Presentation**: Present your mathematical problem and solution in one of the following formats:
        - **Google Slides**: Create a slide presentation that includes your image/video/object, the questions you posed, and your solution with explanations.
        - **Video**: Record a video of you explaining the mathematical problem, your thought process, and the solution.
        - **Poster**: Design a poster that visually represents the problem, your solution, and your mathematical reasoning.

Mathematizing the World ExamplesLinks to an external site.

Submission Guidelines

1. **Submit Your Presentation**:
    - If using Google Slides, submit the link to your completed presentation.
    - If creating a video, upload the video file or share the video link.
    - If designing a poster, upload a digital image of the poster or submit the physical poster in class (depending on your instructor's guidelines).
2. **Ensure Clarity**: Make sure your presentation clearly communicates your observations, questions, problem-solving process, and solution.
3. **Deadline**: Submit your work by XX/XX.

Rubric

Mathematizing Your World V.1

Criteria	Ratings			Pts
This criterion is linked to a Learning OutcomeReal World Selection Step 1: Select a Real-World Image, Video, or Object Find a Mathematical Source: Choose an image, video, or physical object that includes both mathematical features (e.g., invites counting, equal grouping, estimation, measurement, patterns) and contextual features that connect to real-world ideas or evoke curiosity.	10 pts Excellent Choice The chosen image/video/object clearly has strong mathematical features (e.g., patterns, measurements) and deeply connects to real-world ideas. It evokes significant curiosity, allowing for high engagement and creativity.	5 pts Good Starting Point The selected image/video/object has clear mathematical features and some connection to real-world ideas. Potential some engagement and some level of curiosity is possible.	0 pts No Submission	10 pts
This criterion is linked to a Learning OutcomeObserve and Wonder: Identify (3) things you notice and (3) things you wonder about regarding the image, video, or object. You may ask peers, co-workers, and loved ones for their ideas too.	30 pts Excellent Identified 3+ insightful observations and 3+ insightful wonderings that demonstrate deep curiosity. Demonstrates the ability to think critically and engage with the mathematical aspects of the object in detail.	20 pts Solid Understanding Made 3 observations and 3 thoughtful wonderings that show curiosity and connection to the mathematical aspects of the object.	10 pts Progressing Made fewer than 3 observations and wonderings. AND/OR Observations lack depth or relevance to mathematics.	0 pts No Submission	30 pts

Criteria	Ratings				Pts
This criterion is linked to a Learning OutcomeMathematical Questions Create Questions: Pose at least 3 questions based on your image, video, or object that can be answered using K-12 level mathematics. These questions should involve counting, measuring, identifying patterns, or other mathematical processes.	30 pts Excellent Posed 3+ creative and well-thought-out questions that are deeply connected to K-12 mathematics. Questions are insightful, reflecting advanced inquiry into measurement, patterns, counting, etc.	20 pts Solid Understanding Posed 3 solid questions connected to K-12 mathematics. The questions demonstrate curiosity and are appropriate for solving real-world problems using mathematical concepts. The connection to mathematical concepts is present but weak. OR the questions can be solved with simple computations and no problem solving.	10 pts Progressing Posed fewer than 3 questions, or the questions are vague, unclear, or only loosely connected to K-12 mathematics. Lack of meaningful inquiry into mathematical processes.	0 pts No Submission	30 pts
This criterion is linked to a Learning OutcomeProblem Solving Pursue a Solution: Choose one of your posed questions and solve it using appropriate problem-solving strategies and mathematical thinking. Clearly explain your reasoning and the steps you took to solve the problem. Use mathematical strategies such as measurement, pattern identification, spatial quantification, etc. Show your work and reasoning in a clear, organized way.	30 pts Excellent Solved the chosen question with excellent clarity and precision. Demonstrates strong problem-solving skills using advanced mathematical strategies (e.g., measurement, pattern recognition). Thorough explanation with well-organized work and reasoning.	20 pts Solid Understanding Solved the chosen question correctly, showing good problem-solving skills and the use of appropriate strategies. The work and reasoning are mostly clear, with minor areas for improvement. OR Solved the problem incorrectly but demonstrates strong problem-solving skills using advanced mathematical strategies (e.g., measurement, pattern recognition). Thorough explanation with well-organized work and reasoning.	10 pts Progressing Attempted to solve the question with some success, though there are gaps in the reasoning or errors in the math. The explanation is present but lacks depth or clarity.	0 pts No Submission	30 pts

Criteria	Ratings				Pts
This criterion is linked to a Learning OutcomePresent Your Work Step 4: Present Your Work Create a Presentation: Present your mathematical problem and solution in one of the following formats: Google Slides: Create a slide presentation that includes your image/video/object, the questions you posed, and your solution with explanations. Video: Record a video of you explaining the mathematical problem, your thought process, and the solution. Poster: Design a poster that visually represents the problem, your solution, and your mathematical reasoning.	20 pts Excellent The presentation is well-organized, and clearly communicates the mathematical problem, the process, and the solution. The format (slides, video, poster) is used to enhance understanding and visual appeal.	14 pts Adequate The presentation communicates the mathematical problem and solution, but the organization or clarity may be lacking. The chosen format is functional but could be enhanced for better engagement or understanding.	7 pts Needs More The presentation is unclear, disorganized, or incomplete. The format (slides, video, poster) is used poorly, resulting in confusion or lack of engagement with the mathematical ideas.	0 pts No Submission	20 pts

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1.9: Chapter 1 Resources

Chapter Example Videos

Chapter 1.1

YouTube Video

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Chapter 1.2

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Chapter 1.3

YouTube Video

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Chapter 1.4

YouTube Video

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Chapter 1.5

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Chapter 1.6

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Chapter 1.7

Chapter Extras

Chapter Bibliography

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2: SETS, COUNTING, AND NUMERATION- CONNECTING EARLY MATH TO ADVANCED IDEAS

Mathematics is all about finding patterns, organizing information, and understanding relationships. One of the most powerful tools for doing this is Set Theory (NCTM, 2020) But why should you, as a future elementary school teacher, care about set theory? The answer is simple: set theory is everywhere! It forms the foundation for many mathematical ideas and helps young learners develop essential thinking skills that will serve them throughout their education (Feike, Schwingendorf, and Gregg, 2018).

Set theory provides a structured way to classify, group, and analyze information. It gives us the tools to define and manipulate collections of objects, which is a crucial skill in problem-solving and logical reasoning. Whether we are talking about numbers, shapes, or real-world items, sets help us make sense of the world around us (Finnan, 2006). In fact, concepts from set theory are used in fields as diverse as computer science, probability, and logic (CDE, 2013).

Elementary teachers need to understand the big ideas and language of set theory to better grasp how young learners develop mathematical thinking (Burtob, 1984; NCTM 2020). Even before children formally learn about sets, they naturally engage in set-based reasoning by sorting objects by color, shape, or size—fundamental set operations. When they count apples in a basket or compare toy piles, they apply set concepts like cardinality (how many) and ordinality (order), building number sense and reasoning. In the classroom, sets help students learn sorting, classifying, and comparing numbers, concepts that also appear in science (grouping species), language arts (identifying vowels and consonants), and daily activities like organizing supplies. A strong foundation in set theory allows teachers to recognize these early learning experiences and support students in making deeper mathematical connections (Feike, Schwingendorf, and Gregg, 2018).

Set theory is not just theoretical—it has practical applications in daily life. More than a math topic, set theory is a way of thinking that helps us understand and navigate the world, shaping how you teach and how students build a strong mathematical foundation. By the end of the chapter we will explore each of the student learning objectives below.

Chapter Learning Outcomes

- **Essential Questions**:
  - How do different number systems shape the way we understand mathematics today?
  - In what ways can sets and whole numbers represent ideas in the world around us?
- **Learning that Transfers**:
  - **Beyond memorization:** Instead of just memorizing number facts, students will gain flexible reasoning skills that allow them to understand and explain mathematical structures.
  - **Real-world connections:** Understanding set theory and numeration systems will enable future teachers to guide students in recognizing mathematical patterns in everyday contexts (e.g., grouping in counting, understanding different number representations like Roman numerals on clocks).
  - **Problem-solving and adaptability:** By studying different number systems, students will develop the ability to approach unfamiliar problems with logical reasoning—an essential skill for both teaching and lifelong learning.

Student Learning Objectives

- SLO 2.1 Students will **demonstrate** an understanding of foundational set theory by defining and manipulating sets and using set operations (e.g., union, intersection, complement) to construct and analyze new sets.
- SLO 2.2 Students will **analyze** and **apply** the properties and functions of whole numbers, distinguishing between their various uses (cardinal, ordinal, and identification) and explaining relationships such as greater than and less than through set-based representations.
- SLO 2.3 Students will **compare** Egyptian, Roman, and Hindu-Arabic numeration systems, identifying differences in place value, positionality, and arithmetic operations, and evaluate each system's advantages and limitations.
- SLO 2.4 Students will **demonstrate** proficiency with the Hindu-Arabic numeration system by converting between bases, expanding numbers, and articulating the system's structure, base, and numeral naming conventions.

SET THEORY RAP

Audio

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Morgan and Grandpa's Treasure Hunt

Morgan woke up with a smile so wide,
A day at Grandpa's—she beamed with pride!
But Grandpa had a plan, oh what a treat,
A treasure hunt, with math to meet!

"To find the treasure, we must be wise,
We'll look for sets—so use your eyes!"
Grandpa said with a twinkle bright,
Morgan nodded—this felt just right!

"A set's a group, they share a trait,
Like round things here—oh won't this be great!"
Morgan searched and soon she found,
A soccer ball, an apple round.

"Now find me things that all are red,
Look around, use your head!"
Morgan grabbed a flower bright,
A ruby leaf, her backpack light.

"Now tell me dear, what things appear,
In both those sets? Look very near!"
Morgan thought, then gave a clap,
"The apple's round and red—oh snap!"

"That's an intersection, you see?"
Grandpa chuckled, filled with glee.
"Some sets don't meet, they stand apart,
Like noisy things and things with heart!"

They sorted, grouped, and laughed a lot,
Morgan learned with every spot.
She turned to Grandpa, her heart so light,
"This treasure hunt has been just right!"

Grandpa grinned, his arms stretched wide,
"The treasure's here, deep inside!
Math's not just numbers, dull and cold,
It helps us sort, it makes us bold!"

Morgan giggled, her mind so bright,
"Best treasure hunt, pure delight!"

Story Written by ChatGPT with thoughtful prompt engineering

2.1: The Language of Sets

2.1.1: Written Retrieval and Problem Solving Practice

2.1.2: Online Retrieval and Problem Solving Practice

2.1.3: Problem Solving with Sets

2.2: Whole Numbers and Numeration Systems

2.2.1: Written Retrieval and Problem Solving Practice

2.2.2: Online Retrieval and Problem Solving Practice

2.3: Hindu-Arabic Numeration System

2.3.1: Written Retreival and Problem Solving Practice

2.3.2: Online Retrieval and Problem Solving Practice

2.4: Chapter 2 Resources

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2.1: The Language of Sets

As a future elementary school teacher, understanding set theory is essential to developing mathematical thinking and reasoning. Set theory provides a structured way to categorize and organize objects, making it a powerful tool for defining numbers and mathematical operations. The fundamental ideas of grouping, comparing, and relating collections of objects naturally align with how young children develop number sense (Finnan, 2006; Feike, Schwingendorf, and Gregg, 2018). By studying set theory, you gain insights into the logical foundations of arithmetic, helping you guide students through their early stages of mathematical understanding. Additionally, set notation offers a universal mathematical language that enhances problem-solving and communication across disciplines, equipping you with the tools to teach mathematics in a more structured and meaningful way (Feike et al., 2018).

One of the most accessible and visual applications of set theory is the use of Venn diagrams, which help illustrate relationships between different sets and make abstract concepts more tangible for learners. Understanding how to represent unions, intersections, and complements of sets graphically will enhance your ability to teach students how to classify, compare, and organize information—an essential skill not only in math but across subjects. Mastering these concepts will strengthen your ability to teach foundational math skills, foster deep mathematical thinking, and build problem-solving confidence in your students. Whether you're explaining counting, arithmetic operations, or logical reasoning, set theory will become an invaluable part of your instructional toolkit, helping you create engaging and effective learning experiences (Feike, Schwingendorf, and Gregg, 2018).

Before we jump into the formal language of Sets, let's take a glimpse of what some of these ideas look like for children. Use this video of UCLA Math Educator Dr. Megan Franke **(Supporting the Mathematical Brilliance of Young People)** sharing how early learners bring brilliant ideas into the classroom and how this looks when they start doing mathematics. View the video from 2:39 to 7:15, then answer Dr. Franke's question, "what do you notice that is brilliant?"

YouTube Video

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Make sure to hear what others found brilliant before moving on. We will begin to see how what we saw in the video emerges in this chapter.

Sets

The Language of Sets

A *collection of objects* is called a **set** and the objects are called **elements** or **members** of the set. Sets can be defined in three common ways: **(1)** a verbal description, **(2)** a listing of the elements separated by commas, with braces ("{" and "}") used to enclose the list of elements, and (**3**) set-builder notation. For example, consider the set of all primary colors. This set can be defined in multiple ways: (1) "the set of primary colors", (2) Listing: {Red, Blue, Yellow}, (2) Set-builder notation: C= {x | x is a primary color}.

Sets are usually denoted by capital letters such as A, B, C and so on. The symbols **"∈"** and **"∉"** are used to indicate that an object is or is not an element of a set, respectively. For example, if C represents the set of all primary colors, then Red ∈ C and Purple ∉ C. The set without elements is called the empty set (or null set) and is denoted by **{ }** or the symbol **∅.**

🧠**Let's Listen to Learn**❤️

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Two sets A and B are **equal,** written A = B, if and only if they have precisely the same elements.

Definition: One to One (1-1) Correspondence

A 1-1 correspondence between two sets A and B is a pairing of the elements of A with the elements of B so that each element of A corresponds to exactly one element of B, and vice versa. If there is a 1-1 correspondence between sets A and B, we write A ~ B and say that A and B are **equivalent** or **matching** sets.

The Birthday Party (Analogy)

Imagine you are organizing a birthday party with ten (10) children attending. You prepare ten party bags. If every child gets exactly one bag and no bags are left over, you have a **one-to-one correspondence** between children and hats. This means the number of children and the number of hats are the same.

Question: What scenario would cause us to have the sets (party bags and children) NOT be equivalent?

Think About and Discuss

- Think about the concept of **one-to-one correspondence**—when each element in one set is paired with exactly one element in another set, with no extras or missing pieces.
- Consider examples from your daily life where this idea appears. (Hint: Think about matching, pairing, or distributing things evenly.)
- Jot down at least **two examples** from real life that illustrate one-to-one correspondence.

In groups of **3-4 students**, share your examples with each other.

- How do your examples fit the definition of one-to-one correspondence?
- How does understanding this concept help in everyday life or in teaching math?

Definition: Subset

A set ***A*** is a ***subset*** of a set ***B*** if and only if every element of set ***A*** is an element of set ***B***. We write $A \subseteq B$**.**

*To show that one set is **not** a subset of another set, **there would have to be an element in the first set that is not in the second set.***

NOTE: If $A \subseteq B$ and B has an element that is not in A, we write $A \subset B$ and say that A is a **proper subset** of B (Figure 2.1.1). For example, {a, b}⊂{a, b, c}, since {a, b}⊆{a, b, c} and c is in the second set but not in the first.

🧠**Let's Listen to Learn**❤️

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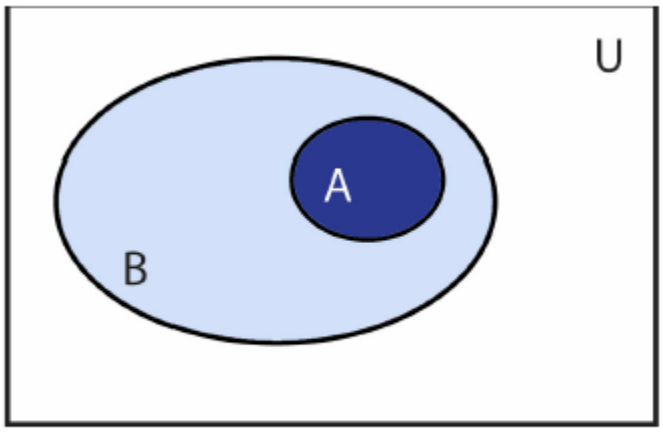


Figure 2.1.1: A is a **proper subset** of B.

There is one thing about sets that confuses a lot of people. By definition, we say that a set is a collection of objects. But in reality, a set can have no elements in it. A set containing no elements is called the ***empty set*** or ***null set***. Look at the definition of a subset very carefully. Is the null set is a subset of every set? I hope this was your conclusion: ***The null set is a subset of every set.*** [ANALOGY] Imagine a strict school rule posted on an empty classroom door: "Every student in this room wearing a neon-green cape must hop on one foot." Now, suppose you open the door and look inside—the room is completely empty. There are zero students. Can you find even one student wearing a neon-green cape who is refusing to hop? Of course not, because there are no students at all! Because you cannot find a single counterexample, the rule has not been broken. In logic, this is called being **vacuously true**—a condition is automatically satisfied simply because there are no items available to violate it.

Venn Diagrams

Relationships between sets can be visualized using Venn diagrams. Sets are represented by circles included in a rectangle that represents the universal set, i.e., the set of all elements being considered in a particular discussion. Figure 2.1.1 displays sets A and B inside a universal set U. Set A is comprised of everything inside circle A, and set B is comprised of everything inside circle B, including set A. We will come back in future sections to discuss and use Venn diagrams to provide a clear, intuitive way for students to classify and compare groups of objects or numbers. By comparing different sets, students can grasp the idea of greater than, less than, and equal to.

Example 2.1.1

Let A be the set of all counting numbers less than 8. This set is described by words (verbal description). The list of its elements provides another description A = {1, 2, 3, 4, 5, 6, 7}. Convert this set into Set-builder notation.

Answer available at the end of this section.

Example 2.1.2

A = {a, b, c, d, e} and B={a, c, f, e, b, g, d}. Is it true that $A \subseteq B$ **?** Is it true that $A \subset B$ **?**

Answer available at the end of this section.

Example 2.1.3

Are these statements TRUE for ANY set A?

a) $A \subset A$ **b)** $\emptyset \subset A$

Answer available at the end of this section.

Sets of Numbers

We work with different sets of numbers that play an important role in mathematics.

- Counting numbers are {1, 2, 3, 4, 5, 6, ...}
- Whole numbers include zero and counting numbers {0, 1, 2, 3, 4, 5, 6, ...}
- Even counting numbers include counting numbers that are multiples of two {2, 4, 6, 8, 10, ...}
- Odd counting numbers include counting numbers that are not multiples of two {1, 3, 5, 7, 9, ...}

Finite / Infinite Sets

Sets can be classified into two main types: **finite and infinite**. A set is considered finite if it contains no elements (an empty set) or if its elements can be counted in a way that eventually reaches an end. In contrast, an infinite set continues indefinitely without ever stopping. More precisely, a set is finite if (1) it is empty or (2) its elements can be placed in a one-to-one correspondence with the set {1, 2, 3, ..., n}, where *n* is some whole number. If a set does not meet these conditions, meaning it cannot be matched with a finite list of counting numbers, then it is classified as infinite.

Example 2.1.4

Determine whether the following sets are finite or infinite.

a) {100, 200, 300} b) {a, b, c, d, ... y, z} c) {5, 10, 15, 20, 25, ...}

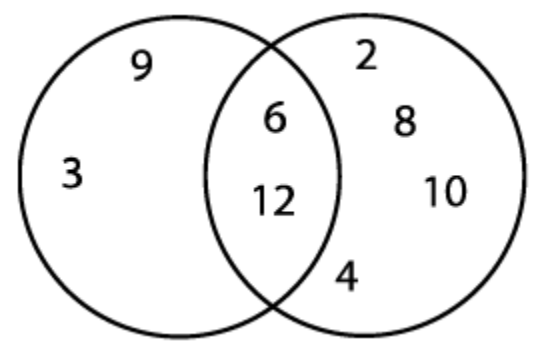
Answer available at the end of this section.

Cardinality

Something else to consider when working with sets is how many elements are in a given set. If we count the number of elements in a set, we call that number the **cardinality** or **cardinal number** of that set. To denote the cardinality of set **C={a, b, c}**, we write n(**C**) = 3. Think of the *n* as the ***number*** *of objects in the set*. When you are asked about the cardinality of a set, the answer is a number. We say that two sets are **equivalent** if they each have the *same* ***number*** *of elements* which is to say they have the *same cardinality*. We use the symbol, ~, to denote equivalence. For instance, if set **P** and **H** each contain four elements, we can write **P** ~ **H**. REMEMBER: Two sets are **equal** if they have the ***exact same elements***.

Wonder, Play, Grow

Where would you place the following numbers in the Venn diagram? 20, 21, 23, 24



Discuss with your peers where you would place the numbers, why. Be prepared to share your reasoning.

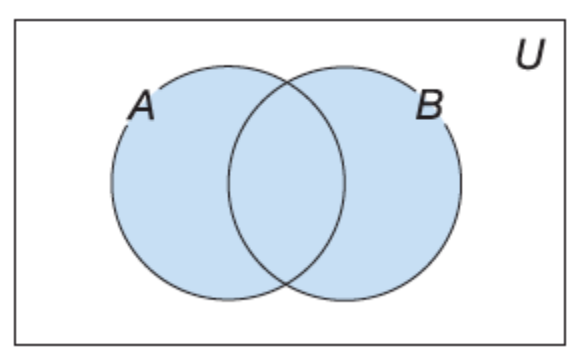
Answer available at the end of this section.

Operations of Sets

One of the most important ways we organize mathematical ideas is through **sets**. By studying set operations, we can see how math provides a precise way to describe relationships between groups of objects. This is one way that mathematics serves as a universal language—it allows us to clearly communicate ideas about numbers, patterns, and logical relationships. Let's explore some simple set operations and how they help us understand and organize information.

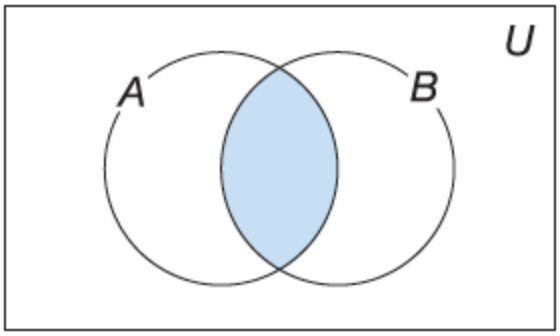
Definition: Union of Sets $A \cup B$

The union of two sets A and B, written $A \cup B$, is the set that consists of all elements belonging either to A or to B (or to both). Informally, $A \cup B$ is formed by putting all the elements of A and B together.



Definition: Intersection of Sets $A \cap B$

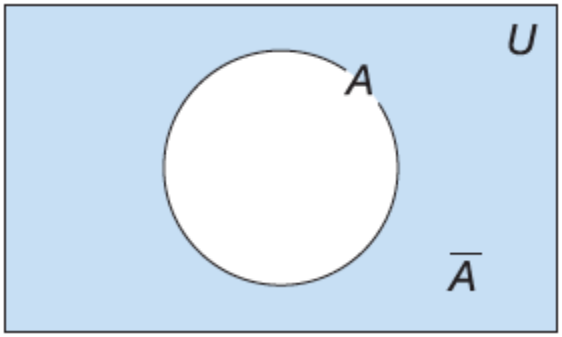
The intersection of sets A and B, written $A \cap B$, is the set of all elements common to sets A and B. Thus the intersection of A and B is the set of elements shared by A and B.



NOTE: Two sets are ***disjoint*** if their intersection is empty

Definition: Complement of Set A $\overline{A}$

The complement of a set A, written $\overline{A}$, is the set of all elements in the universe, U, that are not in A.

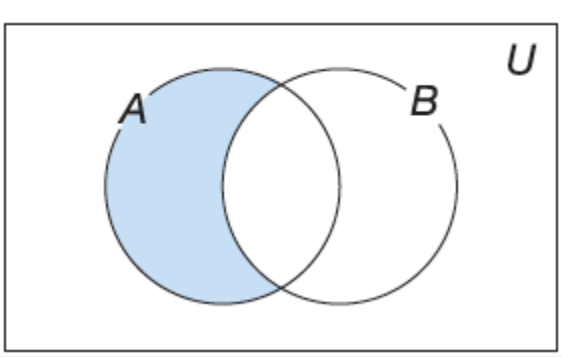


NOTE: The ***complement*** of a set **A**, can be written $\mathbf{A^c}$ or $\bar{A}$ or **A'**

For example, if **U**={1, 2, 3, 4, 5} and **H** = {3, 4}, then $\bar{H}$ = {1, 2, 5).

Definition: Difference of Sets A - B

The set difference (or relative complement) of set B from set A, written A – B, is the set of all elements in A that are not in B.



So if **F**= {1, 2, 3 ,4} and **G** = {1, 3, 5, 7}, then to find **F – G**, list the elements of **F** and cross off any that are in G: {1,2,3,4}. The answer is: {2, 4)

Audio

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Example 2.1.5

Try the following problems in which the universe is the first 9 counting numbers. If U represents the universe, we have U = {1, 2, 3, 4, *5,* 6, 7, 8, 9} Three subsets of U are defined as follows: **A** = {1, 2, 3, 4, 5} **B** = {2, 4, 6, 8} **C** = {3, 5, 7}

	a. **A** ∩ **B** = _____
b. **B** ∩ **C** = _____	c. **A** ∪ **B** = ____
d. **A** ∪ **C** = _____	e. **B** ∪ **C** = ____
f. $\mathbf{A}^C$ = _____	g. $\mathbf{B}^C$ = ____
h. $\mathbf{C}^C$ = _____	i. **A** – **B** = _____
j. **B** – **A** = ____	k. **A** – **C** = _____
l. **C** – **A** = ____	m. **B** – **C** = _____
n. **C** – **B** = _____	

*Did you notice that the order matters **for** difference, but not for union or intersection?*

Answer available at the end of this section.

Retrieval Practice

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Wonder, Play, Grow

Work with at least one peer to try these “more involved” problems. These will take more than one step to do. You might need scratch paper. **U** = {1, 2, 3, 4, 5, 6, 7, 8, 9}. The subsets are: **A**= {1, 2, 3, 4, 5} **B** = {2, 4, 6, 8} **C** = {3, 5, 7}

a. (**A** ∪ **C**) – **B** = ____	b. $(\mathbf{A} - \mathbf{C})^C$ = ____
c. (**A** ∩ **B**) ∪ **C** = ____	d. (**A** ∪ **C**) ∩ **B** = ____

e. $(\mathbf{A} \cup \mathbf{C})^C$ = ___	f. $\mathbf{A}^C \cap \mathbf{B}^C$ = ___

Answer available at the end of this section.

There is one more operation we will define on sets. So far, you've worked with union, intersection, difference and complement. Now we will learn what it means to take the Cartesian product of two sets, A and B. If you are finding the Cartesian product, the answer is a set that contains ordered pairs. The order matters! The first element must come from the set written to the left of the $\times$ and the second element must come from the set written to the right of the $\times$.

Cartesian product

The **Cartesian product** of set **A** with set **B**, which is written **A** $\times$ **B** and is read as '**A** cross **B**" is the set of all possible ordered pairs (a,b), where a $\in$ **A** and b $\in$ **B**.

NOTE: Using set builder notation, we can write ◇×◇={(a,b) | a∈◇ and b∈◇}.

More Info: Cartesian Products IRL

Example 2.1.6

If A = {x, y, z} and B = {a, b}, find A $\times$ B and B $\times$ A.

Answer available at the end of this section.

Wonder, Play, Grow

If a set **B** contains 5 elements (in other words, n(**B**) = 5) and a set **C** contains 7 elements (or n(**C**) = 7), then how many elements (where each element is an ordered pair) are in the Cartesian product for each of the following?

a. n(B $\times$ C) = ___	b. n(B $\times$ B) = ___	c. n(C $\times$ C) = ___

Answer available at the end of this section.

Answers

Example 2.1.1

A = {a is a counting number | a < 8}

Here, "a is a counting number" specifies that the set consists of numbers of a certain type and a < 8 specifies that we want to consider the numbers possessing this particular property.

Example 2.1.2

Both are true because all elements of A are in B (making A⊆B true), and B has extra elements not in A (making A⊂B true as a proper subset). Make sure to understand the distinction between the two. Remember, math is a language. In this section we will see the language of sets come to life.

Example 2.1.3

a) is False. a) Every element of A belongs to A, A cannot be a proper subset of itself and b) is True. The argument behind this statement is a little bit twisted. The empty set contains no elements. The definition of a subset requires that *every element* of the first set belongs to the second set.

Because the empty set has **no elements**, there is never an element you can point to that violates this rule. You can never find an element inside the empty set that isn't inside set A. Therefore, by default and logical necessity, the empty set is a subset of every set in existence, $\emptyset \subseteq \mathbf{A}$.

Example 2.1.4

a) finite since it can be matched with the set {1, 2, 3} b) finite since it can be matched with the set {1, 2, 3, ..., 26} c) infinite set

The **left circle** contains numbers **divisible by 3** (3, 9, 6, 12). The **right circle** contains numbers **even numbers** (2, 4, 6, 8, 10, 12).

20: Even number → Goes in the right circle. **21**: Divisible by 3 → Goes in the left circle. **23**: Neither even nor divisible by 3 → Stays outside the circles.

Example 2.1.5

a. {2, 4}
b. Ø
c. {1, 2, 3, 4, 5, 6, 8}
d. {1., 2, 3, 4, 5, 7}
e. {2, 3, 4, 5, 6, 7, 8}
f {6, 7, 8, 9}
g. {1, 3, 5, 7, 9}
h. {1, 2, 4, 6, 8, 9}
i. {1, 3, 5}
j. {6, 8}
k. {1, 2, 4}
l. {7}
m. {2, 4, 6, 8}
n. {3, 5, 7}

YouTube Video

a. {1, 3, 5, 7}
b. {3, 5, 6, 7, 8, 9}
c. {2, 3, 4, 5}
d. {2, 4, 6, 8}
e. {6, 8, 9}
f. { 7, 9}

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Example 2.1.6

$A \times B = \{(x, a), (x, b), (y, a), (y, b), (z, a), (z, b)\}$
$B \times A = \{(a, x), (b, x), (a, y), (b, y), (a, z), (b, z)\}$
Notice there were 3 elements in one set and two in the other. The Cartesian product has six elements in it, six ordered pairs.

a) 35 b) 25 c) 49

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2.1.1: Written Retrieval and Problem Solving Practice

Student Learning Objectives

SLO 2.1 Students will demonstrate an understanding of foundational set theory by defining and manipulating sets and using set operations (e.g., union, intersection, complement) to construct and analyze new sets.

The mild assignment 🌶 is a shorter and less challenging assignment. The medium assignment🌶🌶 is right in the middle in terms of length and level of difficulty. Finally, the spicy assignment 🌶🌶🌶 is a longer assignment that is the most challenging.

(Spicy 🌶🌶🌶 Problems are OPTIONAL, but go for it when you have the time to challenge yourself)

Exercise 2.1.1.1 🌶

Which of the following sets are equal to {4, 5, 6}?

a) {5, 4, 6}

b) {7, 8}

c) Whole numbers less than 7.

d) Whole numbers greater than 3 and less than 7.

e) {e, f, g}

Answer available at the end of this section.

Exercise 2.1.1.2 🌶

True or False

a) $5 \in \{5, 6, 7, 8\}$

b) $5 \notin \{2, 3, 4, 6\}$

c) $\{1, 2, 3\} \subseteq \{1, 2, 3\}$

d) $\{x, y, z\} \subset \{x, y, z\}$

Answer available at the end of this section.

Exercise 2.1.1.3 🌶

Determine which of the following sets are equivalent to the set {x, y, z}.

a) {w, x, y, z}

b) {1 , 5, 17}

c) {w, w, w}

d) {•, ♦, ►}

Answer available at the end of this section.

Exercise 2.1.1.4

a) List all of the subsets of {a, b, c}.

b) List all the proper subsets of {@, ©}

Answer available at the end of this section.

Exercise 2.1.1.5

True or False?

a) The set {10̸5, 11̸0, 115, 12̸0, ...} is an infinite set.

b) The set of whole numbers less than 6 is an infinite set.

c) The set of all books in your house is an finite set.

d) If A is infinite, and B is a subset of A, then B is always an infinite set.

Answer available at the end of this section.

Exercise 2.1.1.6

Show two different one-to-one correspondence between the sets {0̸, 1, 2, 3} and {w, x, y, z}.

Answer available at the end of this section.

Exercise 2.1.1.7

Given F = { 2, 4, 6, 8, 10̸, ...} and G = {4, 8, 12, 16, ...} answer the following questions.

a) Find $A \cap B$

b) Find $A \cup B$

c) Is G a subset of F? Explain your thinking.

d) Is F equivalent to G? Explain your thinking.

Answer available at the end of this section.

Exercise 2.1.1.8

Find the Cartesian Product for the following.

a) {a} x {b, c}

b) {5} x {x, y, z}

c) {2, 3} x {3, 2}

d) {x, y, z} x {5}

Answer available at the end of this section.

Exercise 2.1.1.9

A = {1, 2, 3, 4} , B = {4, 5, 6, 7}, and C = {1, 2, 7, 8}. List elements of the sets...

a) $A \cap B$

b) $A \cup B$

c) $(A \cup B) \cap C$

d) $(A \cup C) - B$

Answer available at the end of this section.

Exercise 2.1.1.10

Given A = {0, 1, 2, 3, 4, 5}, B={0, 2, 4, 6, 8, 10}, and C = {0, 4, 8}, find each of the following.

a) $B \cap C$

b) B - C

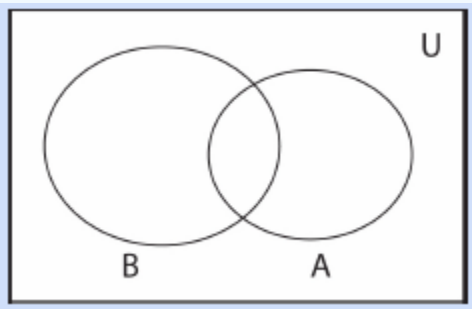
c) $(A \cup B) \cap C$

d) $A \cap \emptyset$

Answer available at the end of this section.

Exercise 2.1.1.11

Draw Venn diagrams of the sets A and B. Shade the area that represents the result of operations.



a) $\overline{A} \cap B$ b) $\overline{A} \cup \overline{B}$ c) $B - (A \cap B)$

Answer available at the end of this section.

Exercise 2.1.1.12

Kayla says that if two sets are equivalent, they must be equal. Christina says that if two sets are equal, then they must be equivalent. Who is correct? Explain.

Answer available at the end of this section.

Answers

Exercise 2.1.1.1

a) equal b) NOT equal c) Not equal d) equal e) Not equal

Exercise 2.1.1.2

a) True b) True c) True d) False

Exercise 2.1.1.3

a) No b) Yes c) No d) yes

Exercise 2.1.1.4

a) {a}, {a, b}, {a, b, c}, {a, c}, {b, c}, {b}, {c}, {}

b) {@} , {©}, {}

Exercise 2.1.1.5

a) True b) False c) True d) false

Exercise 2.1.1.6

Here are two different one-to-one correspondences between the sets {0, 1, 2, 3} and {w, x, y, z}:

Correspondence 1:

0 ↔ w 1 ↔ x 2 ↔ y 3 ↔ z

Correspondence 2:

0 ↔ x 1 ↔ w 2 ↔ z 3 ↔ y

Exercise 2.1.1.7

a) {4, 8, 12, 16, ...} = G

b) {2, 4, 6, 8, 10, ...} = F

c) Yes, all of the elements of G are an element of F.

d) Yes - because we can match each number in the first set to exactly one number in the second set, without missing any. This means they have the **same size**, even though both are infinite! In elementary set theory, **two sets are considered *equivalent*** if there is a **one-to-one correspondence. T**hat is, the elements can be matched up **exactly**, with nothing left over in either set.

Exercise 2.1.1.8

a) {(a,b),(a,c)} b) {(5,x),(5,y),(5,z)} c) Ordered pairs: {(2,3),(2,2),(3,3),(3,2)} d) {(x,5),(y,5),(z,5)}

NOTE: Sets resulting from b) and d) are NOT equal. Each set contains three ordered pairs, but **the order of elements in the pairs is different**.

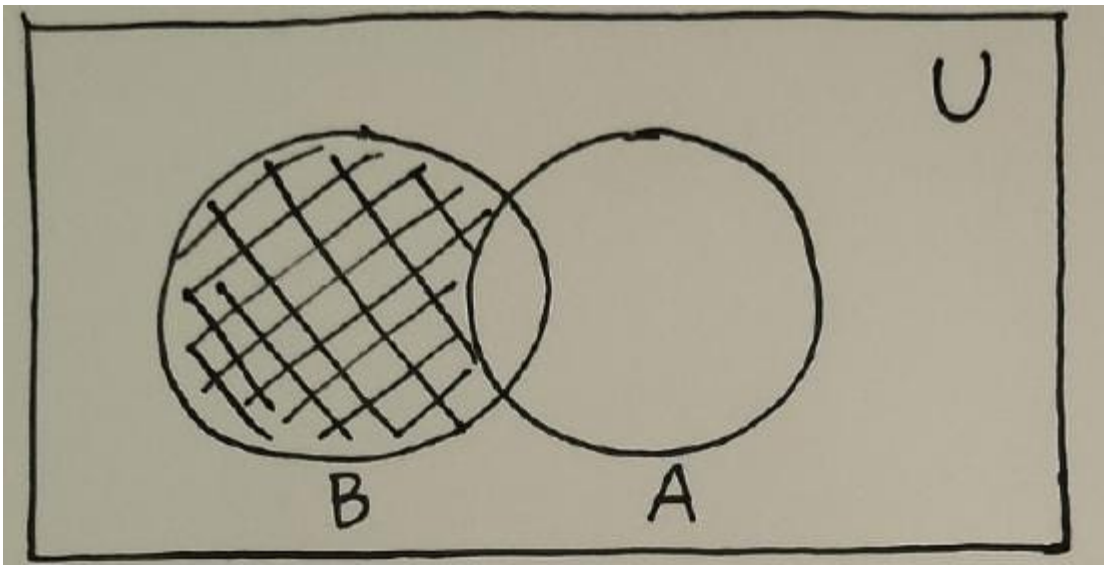
Exercise 2.1.1.9

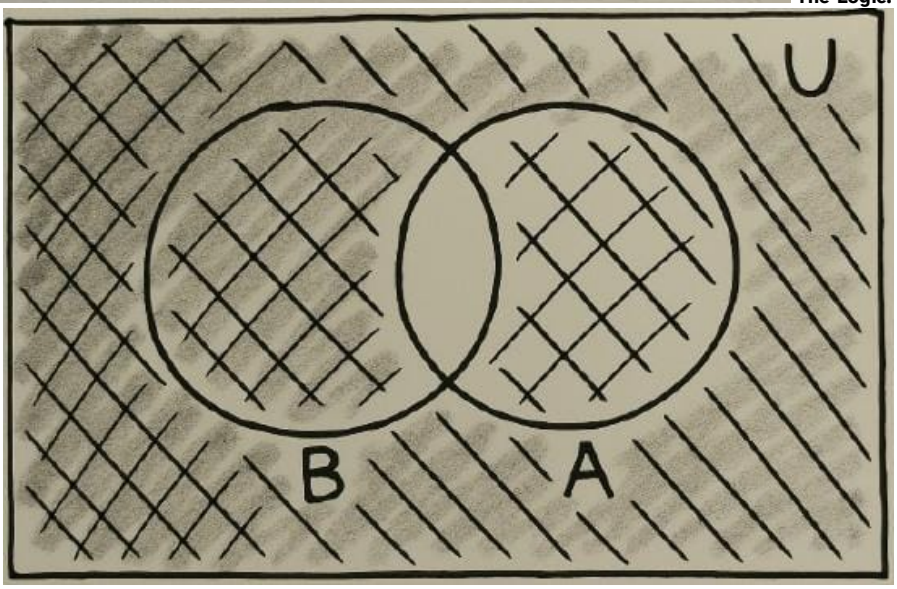
a) {4} b) {1, 2, 3, 4, 5, 6, 7} c) {1, 2, 7} d) {1, 2, 3, 8}

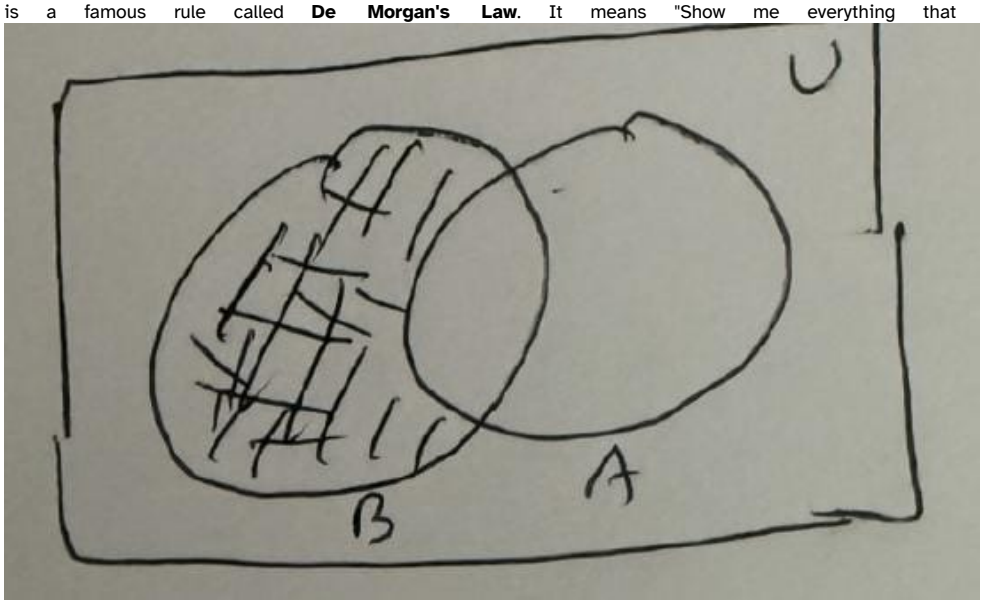
Exercise 2.1.1.10

a) {0, 4, 8} b) {2, 6, 10} c) {0, 4, 8} d) ∅

Exercise 2.1.1.11

a) 

The Logic: "Show me things that are NOT in A **AND** are in B." b) 

The Logic: This is a famous rule called **De Morgan's Law**. It means "Show me everything that is not in the shared middle." c) 

The Logic: "Take all of Circle B, then subtract the shared middle part."

Exercise 2.1.1.12🌶🌶

Kayla is incorrect, but **Christina is correct**.

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2.1.2: Online Retrieval and Problem Solving Practice

ADAPT 2.1.2.1

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2.1.3: Problem Solving with Sets

Problem solving with Venn diagrams combines the principles of productive problem solving, Pólya's four-step process, and the visual strengths of set theory. Productive problem solving emphasizes sense-making, reasoning, perseverance, and using multiple strategies. When applied to set problems, these ideas often involve using Venn diagrams to represent and explore relationships like union, intersection, and complement. Venn diagrams help students visualize how sets overlap or differ, organize information, and identify unknowns. This visual and strategic approach not only supports accurate solutions but also fosters deeper conceptual understanding.

Student Learning Objectives

SLO 2.1 Students will demonstrate an understanding of foundational set theory by defining and manipulating sets and using set operations (e.g., union, intersection, complement) to construct and analyze new sets.

Problem Solving Problems

Here are some problems for you to try. Let's start with you playing with representations and notation.

Problem 1

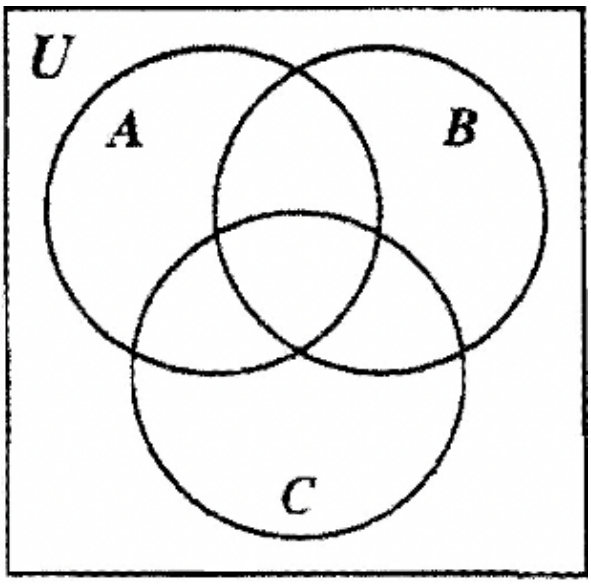
Click on the numbers in the Venn diagram to shade those particular area in order to match the prompted set notation.

_____

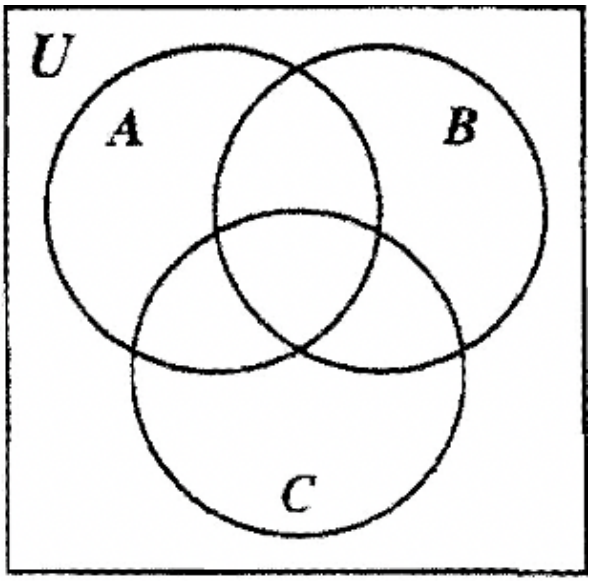
Problem 2

Shade the appropriate areas.

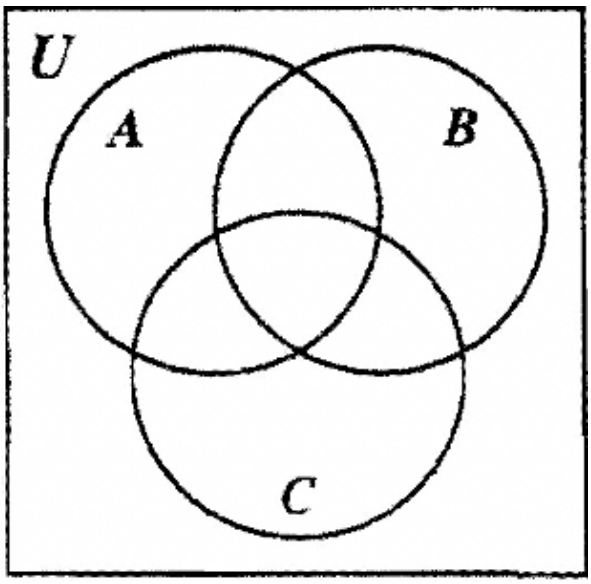
a) (**A** ∩ **B**) ∩ **C**



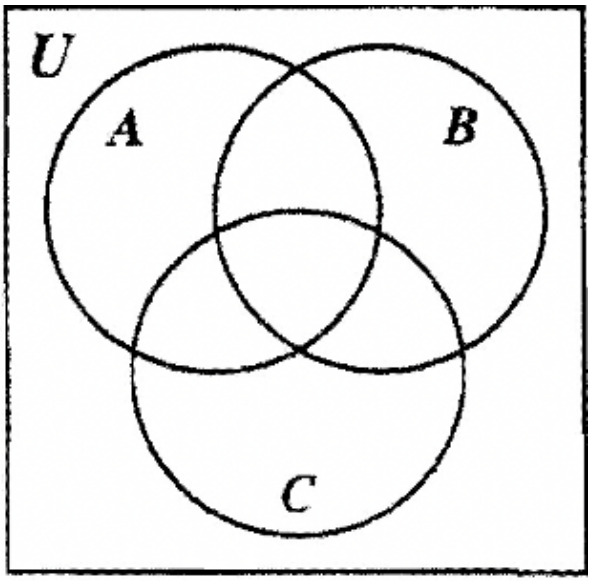
b) (**A** ∪ (**B** ∪ **C**))



c) $(\mathbf{C} \cap \mathbf{A})^c$



d) $(\mathbf{A} \cap \mathbf{B} \cap \mathbf{C})^c$



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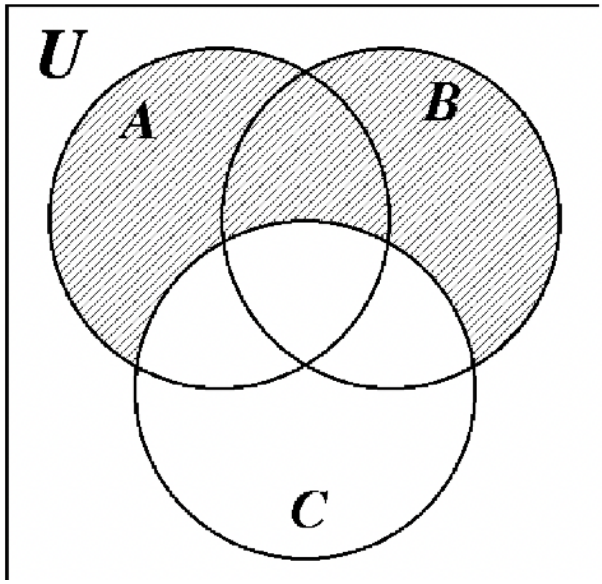
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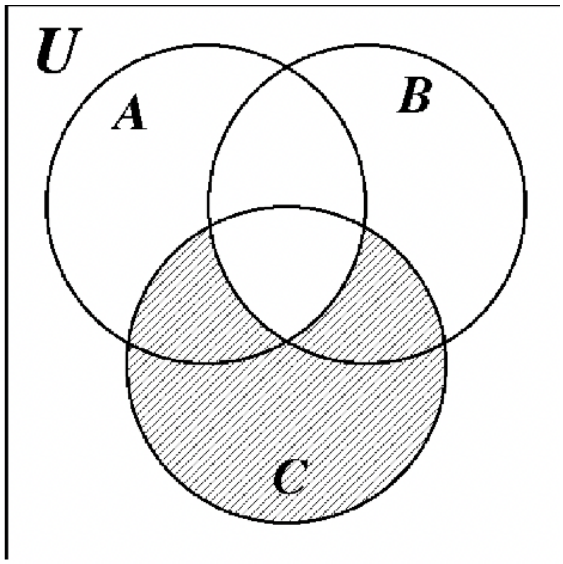
Problem 3

Identify the proper set notation for each Venn Diagram.

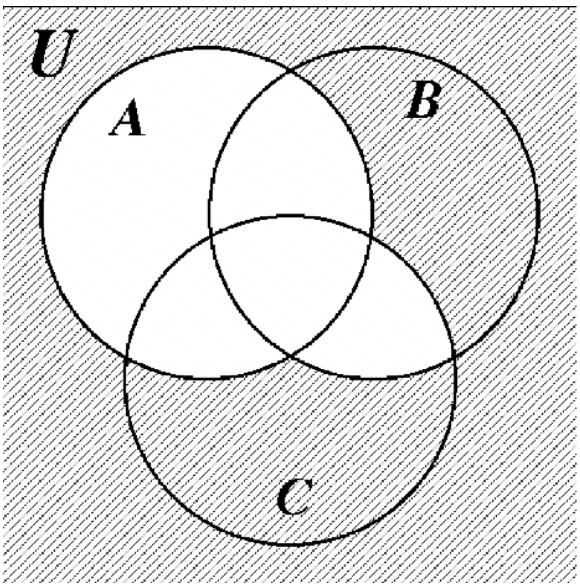
a)



b)



c)



Answer

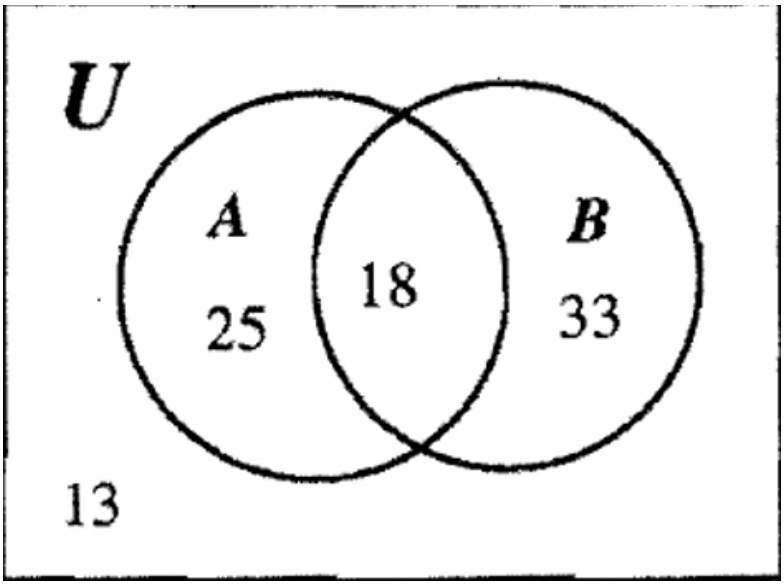
a) (**A** ∪ **B**) – **C**

b) C – (**A** ∩ **B**)

c) A^c – (**B** ∩ **C**)

Problem 4

Look at the Venn diagram below and answer the questions about the cardinality of the regions indicated.



a. $n(\mathbf{A})$ = ____	b. $n(\mathbf{B})$ = ____
c. $n(\mathbf{A}^c)$ = ____	d. $n(\mathbf{B}^c)$ = ____
e. $n(\mathbf{A} \cap \mathbf{B})$ = ____	f. $n(\mathbf{A - B})$ = ____
g. $n((\mathbf{A} \cup \mathbf{B})^c)$ = ____	h. $n(\mathbf{U})$ = ____

Imagine that the above problem was really a model for what kind of fruit a group of people packed in their lunches. **A** stood for those who packed apples and **B** stood for those who packed bananas. Assume apples and bananas are the only kind of fruit available. Then the questions you answered were these:

a. How many packed apples?
b. How many packed bananas?
c. How many didn't pack apples?
d. How many didn't pack bananas?
e. How many packed both kinds of fruit?
f. How many packed apples only?
g. How many didn't pack either fruit?
h. How many people were surveyed?

Look to see if you would have looked in the same regions if the questions were asked in this way.

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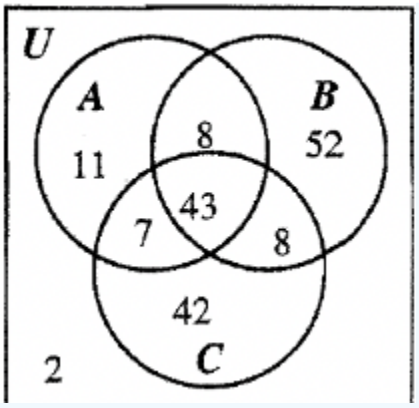
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Problem 5

Let's see how well you do working with a three set Venn diagram. The Venn diagram below shows the cardinality of each region. A survey of college students showed the grades they received on their last report card. **A** stands for getting an A, **B** stands for getting a B and **C** stands for getting a C. Some people took only one course whereas others took more. Look at the Venn diagram

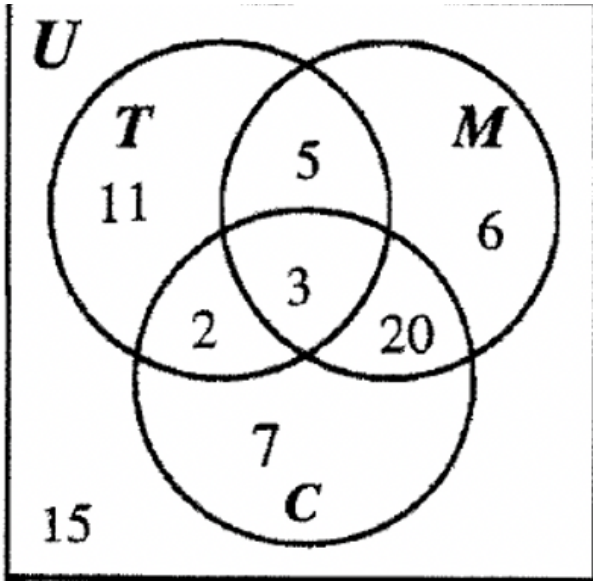
and answer the following questions. *Extra Credit: Instead of just writing the number, write down what the question is asking* ***symbolically***. *In other words, if the question is "How many received both A's and B's?" you would write* $n(\mathbf{A} \cap \mathbf{B}) = 51$.



a. How many got all A's? ___
b. How many got A's? ___
c. How many got B's and C's? ___
d. How many got only A's or B's? ____
e. How many didn't get a C? ____
f. How many got B's or C's? ____
g. How many got B's, but not C's? ___
h. How many got at least two different grades of A, B or C? ___
i. How many got exactly two different grades of A, B or C? ___

Problem 6

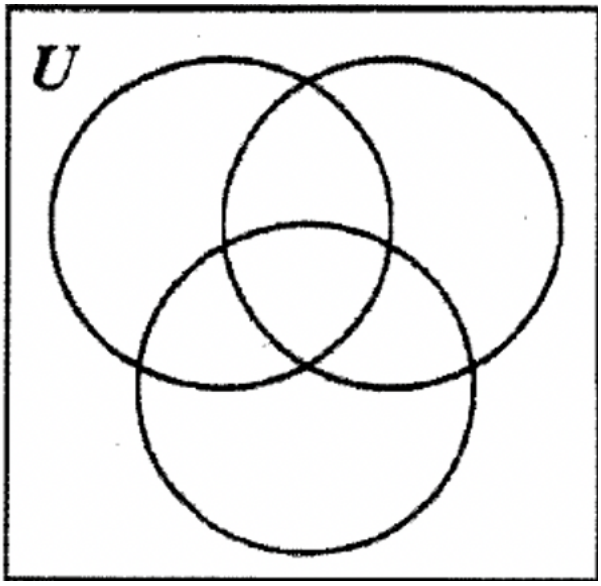
In another survey, questions were asked to see how many people were in their teens, how many were married and how many had children. **T** represents teens, **M** represents those who were married and **C** represents those having children. Look at the Venn diagram which shows the cardinality of each region to answer the following questions:



a. How many people were married? ___
b. How many teens had children? ___
c. How many people were in the survey? ___
d. How many people did not have children? ___
e. How many teens weren't married? ___
f. How many teens were married with children? ___
g. How many teens were in the survey? ___
h. How many people were unmarried with children? ___
i. How many people were married with children? ___

Problem 7

A survey of 60 college students was taken asking about which meals they regularly ate at school. Thirty students usually ate breakfast at school, 26 usually ate lunch there and 23 usually ate dinner at school. Six students said they ate all three meals at school. Eleven ate both breakfast and lunch there whereas 15 only ate breakfast at school and 12 only ate lunch there. Label the sets and fill in the Venn diagram below representing this information and answer the following questions:



a. How many students ate both lunch and dinner at school?
b. How many didn't eat any meals at school?
c. How many only ate dinner at school?
d. How many ate breakfast and/or dinner at school?
e. How many ate exactly one meal at school?
f. How many ate at least two meals at school?
g. How many ate both lunch and dinner at school, but not breakfast?

Problem 8

At a computer repair shop, 50 laptops were inspected. Suppose that 23 laptops needed a new screen and 34 laptops needed a new keyboard.

a) What is the least number of laptops that could have needed both?

b) What is the greatest number of laptops that could have needed both?

c) What is the greatest number of laptops that could have needed neither?

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Answer

a) Least that needed both: 7
b) Greatest that needed both: 23
c) Greatest that needed neither: 16

Go For It! 🌶🌶🌶

Let **U**={1, 2, 3, 4, 5, 6, 7, 8, 9} with these subsets: **A**={6, 7, 8}, **B**={2, 3}, **C**={3, 4}, **D**={5}, **E**={1, 2, 3, 4, 5, 6}. Arrange these sets in a Venn diagram that clearly shows their relationship to each other. It's a bit tricky because there are five subsets, but you can do it!

Interactive APP for Problems 7 and 8

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2.2: Whole Numbers and Numeration Systems

From Sets to Whole Numbers

Mathematics begins with understanding how we group and organize objects—this is where set theory comes in. As we recall, a set is simply a collection of objects, and this basic idea helps us define numbers and operations in a logical way. When young children count, they naturally group objects into sets (Feike, Schwingendorf, and Gregg, 2018). For example, when they see two apples and three bananas, they are subconsciously working with sets. Mathematicians formalize this idea by defining numbers as the sizes of sets—what we call cardinal numbers.

A key concept in counting is one-to-one correspondence, where each object in one set is paired with exactly one object in another set. This forms the basis for understanding numbers and operations like addition and subtraction. When we combine two sets, we are performing addition; when we take objects away, we are performing subtraction. Our numeration systems, such as the Hindu-Arabic system, build on these foundational ideas to represent numbers efficiently (NCTM, 2020). Understanding the connection between sets and whole numbers gives us a strong framework for learning mathematical operations and problem-solving (Feike, Schwingendorf, and Gregg, 2018).

Think About and Discuss

What do you remember about learning about numbers or how to count? Have you ever helped to teach a young child how to count? If so, what was it like? How long do you think it takes for someone to learn how to count for the first time?

Task 1:

- Now take a look at the video below from 25:52-27:32 (youtube: courtesy of UCLA Professor Dr. Megan Franke) to see how this 4-year old counts the bears.
- **Pause at 27:32.**

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

(https://youtu.be/5Bl7SX7l6Qs?si=IUN7...h-jKI_f&t=1552)

Task 2

- What does she know? Discuss with your peers before playing the video to hear what the audience thinks.
- Play the video 27:32-29:31
- How does what the audience and Dr. Franke discuss compare to what you noticed?
- What are you wondering about now?

With all of this brilliant thinking in mind, lets learn about numbers.

One difficulty with the concept of numbers as we use them today is that *they are concepts* and not necessarily used only for counting objects. The number four, for instance, is used in many ways –four years old, room 4, four feet high, 4 children, four hours, four minutes after eight, mail station 4, 4 fingers, a grade of 4 on a quiz, count to 4, four times bigger than something, 4th place, four wishes, fourth in line, call 444-4444, code 44, etc.

So, what is a number?

Well, a **number** is an abstract idea that represents a quantity. You can't see or touch a number. The symbols that we see and touch *that are used to represent numbers* are actually *numerals*. A **numeral** is made up of one or more symbols. In the Hindu-Arabic system we use, the numerals are made up of one or more of these ten symbols –0,1,2,3,4,5,6,7,8 and 9. For instance, in the Hindu-Arabic system, the number three is represented by writing the numeral three which we write like this: 3 and the number two million is represented by writing the numeral two million which we write like this: 2,000,000. Notice, it took one symbol to write the number for three and seven symbols to write the number for two million (we don't count the commas as symbols because they aren't essential to writing the numeral). Although it took seven symbols to write the numeral two million, note that there are *only two different symbols* used to write that numeral –a two (2) and a zero (0).

There are three common uses of numbers. Describing how many elements (quantity) are in a finite set is the most common use of the counting numbers. When used this way, the number is referred to as a **cardinal number.** If a 1-1 correspondence between two sets exists, what does that imply about the cardinality of the two sets? Another way numbers are used concerns order. For instance, someone might be the fifth place winner or you might be eighth in line. When used in this way, numbers are called **ordinal numbers**. There are also numbers used for identification such as your phone number, zip code, social security number or credit card number. These are called **identification numbers** and it is the actual order of the symbols that is important –not the value of the number.

🧠**Let's Listen to Learn**❤️

Audio

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More Information: How we use Numbers

Ordering Whole Numbers

The relationship between sets, one-to-one correspondence, and ordering whole numbers is essential for understanding fundamental mathematics. Sets serve as collections of objects, and their size, referred to as a cardinal number, helps define numerical values. The concept of one-to-one correspondence establishes equivalence between sets by pairing each element of one set with exactly one element of another. This principle is crucial in understanding whole numbers, as it allows for meaningful comparisons between different quantities. Additionally, ordering whole numbers emerges from counting and comparing sets—if one set corresponds to a proper subset of another, it represents a smaller number. For instance, recognizing that a set of two elements is a subset of a set of five elements illustrates why 2 is less than 5. Ultimately, set theory provides the foundation for defining numbers, one-to-one correspondence enables comparisons, and these comparisons lead to the logical ordering of whole numbers.

Definition: Ordering Whole Numbers

Let a = n(A) and b = n(B). Then a < b (read "a is less than b") or b > a (read "b is greater than a") if A is equivalent to a proper subset of B.

Recall: n(A) is the cardinality of set A, the number of elements in set A

There are several ways to compare whole numbers. Three common methods include:

- **Counting Chant:** Whole numbers can be ordered by reciting the counting sequence ("one, two, three..."). For example, **2 is less than 5** because "two" comes before "five" when counting.
- **One-to-One Correspondence:** Comparing numbers by matching objects in sets. For example, **2 is less than 5** because a set with two elements corresponds to a proper subset of a set with five elements.

- **Whole-Number Line:** Numbers are arranged in order on a number line, starting from **0** on the left and increasing by one as you move to the right. (Musser, 2011)

Example 2.2.1

Determine the greater of the two numbers 3 and 4 in three different ways.

Answer available at the end of this section.

Numeration Systems

Today, most but not all civilizations around the world use the Hindu-Arabic system of counting and numerals 1,2,3,4,5,6,7,8,9,10,11,12,... There are many advantages to this number system. While you might think this common, widely used system of counting the obvious choice, you might consider both the advantages and disadvantages of Hindu-Arabic as we learn about other numeration systems.

Consider the **STROKE** system where there was only one symbol, a simple vertical stroke, to express a number. A vertical stroke like this, |, expressed one. To express the number seven, you would write seven strokes: ||||||| .

Think About and Discuss

a. Show how you would write the numeral eleven in STROKE:

b. Describe in words how to write the numerals five hundred twelve ; and two million in STROKE. Don't attempt to write the actual strokes!! We only have a semester!

c. Name some advantages and disadvantages of the STROKE system, compared with the Hindu-Arabic system.

Wonder, Play, Grow

The tally system improved the STROKE system by introducing the concept of **grouping**. You are probably familiar with this system where the fifth stroke was placed across the previous four so it would be easier to read since you can count by fives. STROKE is a single symbol system, but Tally can be thought of as a two symbol system, although the second symbol is really just made up of five strokes. Note the difference in writing the numeral twenty-eight in the two systems.

STROKE system:	*Tally system:*
||||||||||||||||||||||||||||	

a. Write the numeral for 17 in STROKE.

b. Write the numeral for 17 in Tally.

c. If you think of a single stroke (|) as one symbol and the 5-stroke tally () as a separate symbol, then it takes eight symbols to express the numeral for 28. In STROKE, it took 28 symbols to write the same number. How many symbols does it take to express the numeral for 172 in TALLY?

Egyptian Numeral System

The Old **Egyptian Numeral System** which was developed around 3400 B.C., employs an **additive system** of counting, where

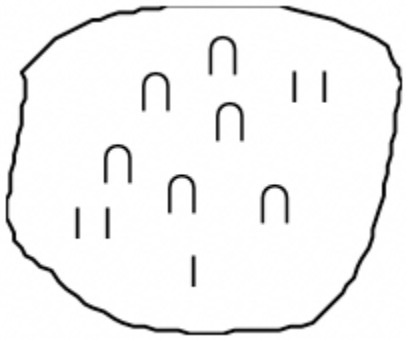
the symbols of the numeral do not have to be in any special order or even on a line going from left to right. The seven symbols in this ancient system are listed below, along with their Hindu-Arabic equivalents.

					
Staff	Heel bone	Scroll	Lotus flower	Pointing finger	Polliwog
One	Ten	Hundred	Thousand	Ten thousand	Hundred thousand
1	10	100	1,000	10,000	100,000

Here are three different ways to write the numeral *sixty five* in the Old Egyptian system:

	or	

or as shown below with all the symbols enclosed in a visual set



You simply add up the values of each of the symbols to get the answer.

Definition: Additive Number System

In an **additive number system,** the value of a written number is the sum of the face values of the symbols that make up the number. The only symbol necessary for an additive number system is a symbol for 1, however many additive number systems contain other symbols.

Remember: The order in which the symbols are written is irrelevant in an additive system.

In addition to being able to write the symbols of the numeral in any order, you may also use a *different combination* of symbols used to represent the same number. Here are two different ways to write the number for *one hundred twenty three*:



Example 2.2.2

Express each Egyptian numeral below as a Hindu-Arabic numeral. Show work.

a) 

b) 

c) 

Answer available at the end of this section.

Roman Numerals

Even though **Roman Numerals** are rare in today's society, they are still used and expected to be understood. They are taught in grades three through five, depending on the district. They can be seen in clocks, the Super Bowl, Film Credits for the copyright date like MCMLXII, preface of textbooks and others like Star Wars Episode VI and WWII.

2.2.1: Basic Table of Roman Numerals

Value	Symbol
1	I
5	V
10	X
50	L
100	C
500	D
1000	M

2.2.2: The Complete Table of Roman Numerals

	1	2	3	4	5	6	7	8	9
ONES	I	II	III	IV	V	VI	VII	VIII	IX
TENS	X	XX	XXX	XL	L	LX	LXX	LXXX	XC
HUNDREDS	C	CC	CCC	CD	D	DC	DCC	DCCC	CM
THOUSANDS	M	MM	MMM	$\overline{\text{IV}}$	$\overline{\text{V}}$	$\overline{\text{VI}}$	$\overline{\text{VII}}$	$\overline{\text{VIII}}$	$\overline{\text{IX}}$
TEN THOUSANDS	$\bar{X}$	$\overline{XX}$	$\overline{XXX}$	$\overline{XL}$	$\bar{L}$	$\overline{LX}$	$\overline{LXX}$	$\overline{LXXX}$	$\overline{XC}$
HUNDRED THOUSANDS	$\bar{C}$	$\overline{CC}$	$\overline{CCC}$	$\overline{CD}$	$\bar{D}$	$\overline{DC}$	$\overline{DCC}$	$\overline{DCCC}$	$\overline{CM}$

The **Roman numeral system** is an ancient numeration system that represents numbers using combinations of letters from the Latin alphabet (I, V, X, L, C, D, M). This system is primarily **additive**, meaning values are determined by summing the symbols (e.g., **VII = 5 + 2 = 7**). However, it also incorporates a **subtractive** principle, where a smaller numeral placed before a larger one indicates subtraction (e.g., **IV = 5 - 1 = 4** and **IX = 10 - 1 = 9**). Additionally, Roman numerals rely on **grouping**, as repeated symbols represent multiples of a value (e.g., **XXX = 10 + 10 + 10 = 30**). Unlike place-value systems, Roman numerals do not use zero, and their structure makes arithmetic operations more complex compared to modern numeration systems.

Example 2.2.3

For all numbers except 4 and 9, we **ADD** the Roman Numerals together, in order from left to right, greatest value to lowest value.

$$
\begin{aligned}
11 &= 10 + 1 = XI \\
8 &= 5 + 1 + 1 + 1 = VIII \\
123 &= 100 + 10 + 10 + 1 + 1 + 1 = CXXIII \\
3816 &= 1000 + 1000 + 1000 + 500 + 100 + 100 + 100 + 10 + 5 + 1 = MMMDCCCXVI \\
7002 &= 5000 + 1000 + 1000 + 1 + 1 = \overline{V}IIII
\end{aligned}
$$

Example 2.2.4

For any number that includes a 4 or a 9, we subtract. When we are looking at a Roman Number expression and we see a Roman character **OUT OF ORDER**, which is the clue to **SUBTRACT**!

$$
\begin{aligned}
9 &= 10 - 1 = IX \\
4 &= 5 - 1 = IV \\
1400 &= 1000 + (500 - 100) = MCD \\
29452 &= 10000 + 10000 + 10000 - 1000 + (500 - 100) + 50 + 1 + 1 = \overline{IXXX}CDLII
\end{aligned}
$$

Retreival Practice

Practice converting Roman Numerals into Hindu-Arabic numeral until you feel comfortable with the process.

Interactive Author:chris cambré - https://www.geogebra.org/m/ttsxcbzn#material/aJ5q37Bs

Wonder, Play Grow

Write each Hindu-Arabic numeral as a Roman numeral using the least amount of symbols:

a. 32:	b. 561:
c. 708:	d. 2053:

Write each Roman numeral as a Hindu-Arabic numeral and show work:

a. MMDCLXXXVII: ___	b. MCCXXXII:____

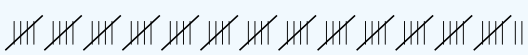
The Roman numeration system developed and incorporated changes over time. One limitation of a simple additive system is that you eventually run out of symbols for very large numbers. The Romans devised a clever way to take care of this problem. If a bar was placed over a symbol or set of symbols, it indicated that number was to be multiplied by a thousand. More than one bar could be used on any given symbol or set of numerals. For example, two bars indicated the number was to be multiplied by 1000 1000 (or one million). Simply put, symbols with one bar over them are in the thousands, those with two bars over them are in the millions, those with three bars over them are in the billions and so on. Thus, the Roman numeration system is ***multiplicative*** as well as ***additive.***

Example 2.2.5

For each of the numeration systems we've learned about so far, state how many different symbols a person has to memorize to understand the system. For Roman, do not consider symbols with bars over them as different symbols.

a. Hindu-Arabic: _____	b. STROKE: ____	c. Tally: ____	d. Egyptian: ____	e. Roman: ____

Let's write the numeral 67 in the different systems and note how many total symbols (not necessarily different symbols) it takes to express the number 67 in each system.

Hindu-Arabic:	We write: 67	2 symbols
STROKE:	We write 67 strokes, shown below ||||||||||||||||||||||||||||||||||||||||||||||||	67 symbols
Tally:	WE write 13 tally groups and 2 strokes, shown below 	15 symbols
Egyptian:	We write:	13 symbols
Roman:	We write: LXVII	5 symbols

Answer available at the end of this section.

Mayan Numerals

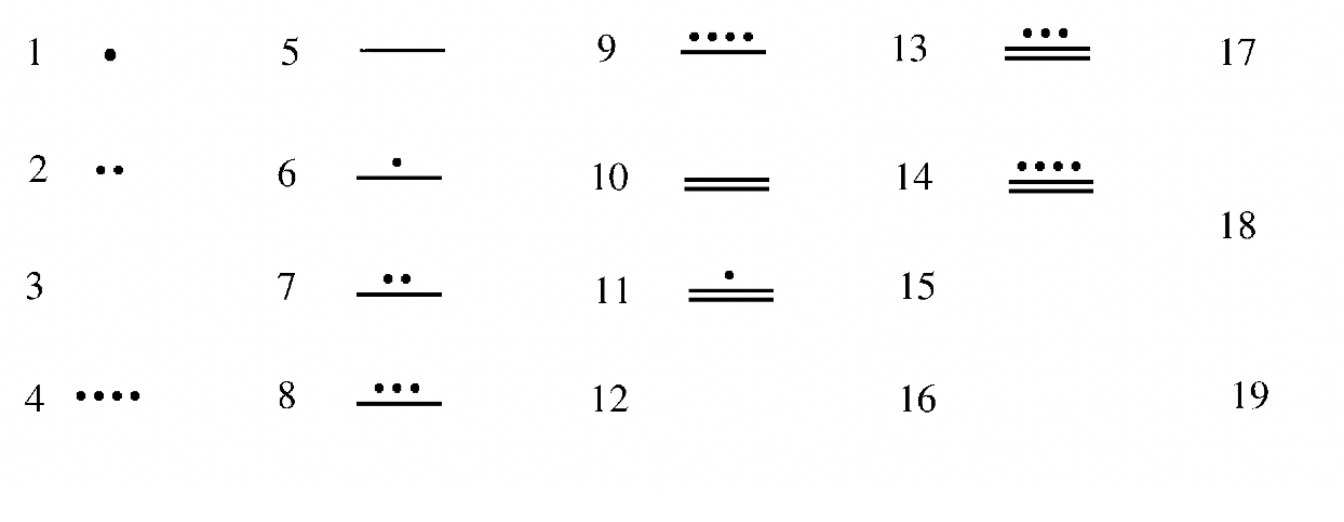
Mayan Numerals were developed by the Mayan priests of southern Mexico and Central America around 300B.C. It is believed to be the earliest **positional numeration system** incorporating a zero and using it for a placeholder. The Mayan numbers are based on three symbols:a dot, a bar, and a symbol for zero, or completion, usually a shell. It has a vertical place-value system.

Positional Number System

A **positional number system** is a method of representing numbers in which the position of each numeral (symbol) affects its value.

Example 2.2.6

Some Mayan Numerals are shown below. Try to figure out the pattern and then fill in the missing numerals.



Answer available at the end of this section.

From what you have seen of this system so far, it might look like a simple additive system. One might guess that the numeral for 20 would be four line segments and that the numeral for 103 would be twenty line segments and three dots. At first glance, it is very similar to the tally system. However, the Mayans used a vertical ***positional* system.** The bottom level represented how many units (or ones), the second level up represented how many 20's, the third level up represented how many 360's (20 18), the fourth level up represented how many 7200's (20 18 20), the fifth level up represented how many 144,000's (20 18 20 20), etc. Except between the second and third level, each place value increased by a multiple of 20. It is almost a Base Twenty system except for that strange third level. Why the third level is 18 times the second level is explained later. Looking at the chart to the left, which shows the first four place values, may help you understand the system.

7200's
360's
20's
1's

To try and make sense of all of this, we will look at some Mayan numerals that have more than one position now. The numerals from one to nineteen only utilize the bottom level so it isn't apparent that the Mayan system is positional until you count past nineteen.

To the right is a two-level Mayan numeral. There is a 16 in the bottom level, representing 16 ones, or 16 (16×1), plus there is a 7 in the second level up, representing 7 groups of twenty, or 140 (7×20). We add the values of each level, 16 + 140, so the numeral you see represents 156.



You have seen two basic symbols in this system so far –a dot ( ), which represents the number one and a line segment ( ), which represents the number five. As previously mentioned, there needs to be a symbol for zero to incorporate the use of place value. For that, the Mayans used a shell that looked something like this:

To the right is a two level Mayan numeral with a zero in the bottom level, representing zero ones, or 0, plus a 13 in the second level up representing 13 groups of twenty, or 260 (13×20). So after adding the values together (0 + 260), the numeral you see represents the number 260.



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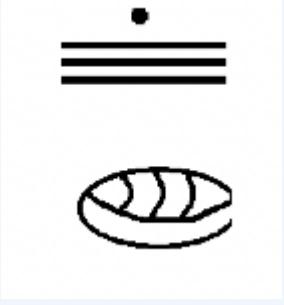
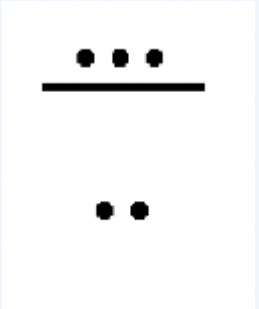
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Example 2.2.7

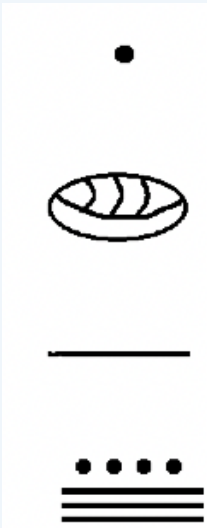
State the Hindu-Arabic equivalent of each Mayan numeral. Show how you obtained your answers. Note that proper space should be left between each place value. Otherwise, someone might incorrectly conclude the number shown for a. represents 14, which would be the answer if there was no space.

a. ____	b. ____	c. ____
		

Answer available at the end of this section.

Now, we'll go on to some three and four level Mayan numeral. Remember that the place value for the third level up is 360 and the place value for the fourth level up is 7200. See if you can figure these out on your own first.

Example 2.2.8

a. ___	b. ___	c. ___	d. ___	e. ___
				

Answer available at the end of this section.

Wonder, Play, Grow

It's a little trickier to start with a Hindu-Arabic numeral and convert it to Mayan, but with a little patience and practice, you'll be doing it quickly and accurately! Before showing a method for doing this, try the following exercise. Think about the place values of the various levels in the Mayan system.

Work with your peers to try and figure out (Problem Solve) how to write these Mayan Numerals for each of the following numbers.

a. 36	b. 387	c. 1234	d. 7654

To convert a number to a Mayan numeral, the first thing you'll have to determine is how many levels the numeral will have. Remember the levels: 1, 20, 360, 7200, 144000, ...

Check Your Answer: Online Mayan Numeral Converter

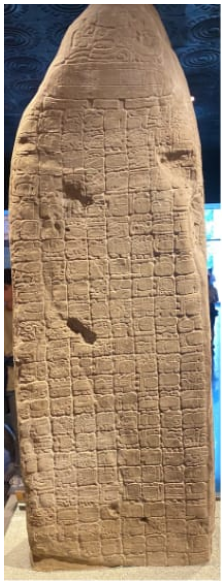
Mayan History

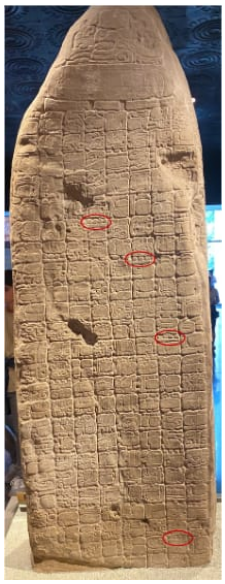
Another New World culture that utilized a place value numeration system was the ancient Maya. This civilization occupied a vast region encompassing southern Mexico and parts of present-day Guatemala, El Salvador, and Honduras. The Mayan civilization thrived for over 2,000 years, reaching its peak between 300 and 900 A.D. One of its most remarkable achievements was the development of an intricate hieroglyphic writing system consisting of approximately 1,000 glyphs. These glyphs could be either

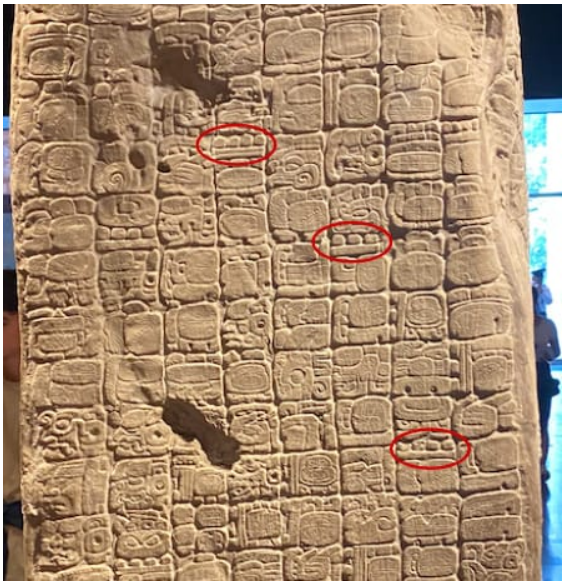
sound-based or meaning-based, though the majority of those that have survived remain undeciphered.

Following 900 A.D., the Mayan civilization experienced a sudden and dramatic decline—often referred to as The Great Collapse—during which its once-thriving cities were abandoned. The reasons behind this mass exodus remain a mystery, with theories ranging from natural disasters and epidemic diseases to warfare. Despite the Spanish Conquest, which began in the early 1500s, elements of Mayan culture persisted. The struggle against Spanish rule was marked by relentless brutality and lasted nearly a century before the last independent Mayan kingdom was finally subdued in 1597.

The Mayan calendar year consisted of 365 days, divided into 18 months of 20 days each, with an additional 5-day period. This structure led to the adoption of a base-20, or vigesimal, counting system. Numbers were represented in two symbolic forms. The priestly class used elaborate glyphs depicting grotesque deity faces to signify the numbers 1 through 19, which were commonly used in carved stone inscriptions marking significant events. Meanwhile, the common people recorded numbers using a combination of bars and dots, where a short horizontal bar signified 5 and a dot represented 1. A distinctive feature of this system was the inclusion of a stylized shell symbol to denote zero—one of the earliest known uses of a symbol for that number (Boyer, 2011; Burton, 1984). Below are real photos of Mayan Glyhphs found at the **Museo Nacional de Antropología in Mexico City.** See if you can find more Mayan Numerals on the pictures - you may need to zoom in.











Photos Courtesy of Marty Romero

Number System Attributes

System	Additive	Subtractive	Multiplicative	Positional	Place Value	Zero
Tally	Yes	No	No	No	No	No
Egyptian	Yes	No	No	No	No	No
Roman	Yes	Yes	Yes	No	No	No
Mayan	Yes	No	Yes	Yes	Yes	Yes

Mayan Number Web App Exploration and Retrieval Practice

Answers

Example 2.2.1

1) "one", "two", "three", "four" - Chant Counting - "four" comes after "three" or "three" comes before "four", 3 < 4.

2) **Create two sets**:

Set A has **3 objects** (e.g., 🟦🟦🟦)

Set B has **4 objects** (e.g., 🟥🟥🟥🟥)

Using one-to-one correspondence, we determine that **4 is greater than 3**, because after matching each object from Set A with Set B, one object is left unmatched in Set B.

3) One a number line, three is the left of four, so 3 < 4.



Example 2.2.2

a) 10 + 10 + 1000 + 100000 + 100000 + 100000 = 301020

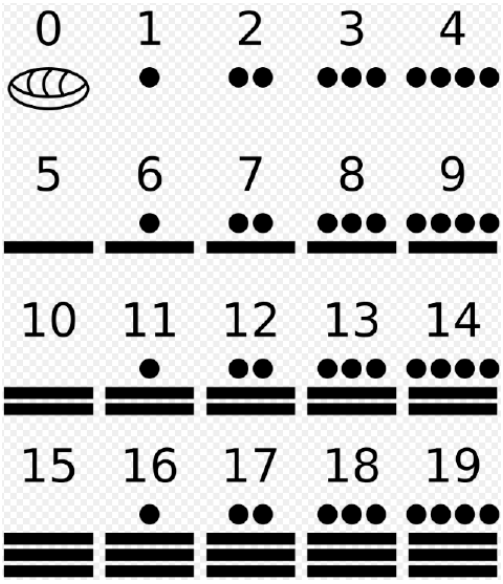
b) 10000 + 10 + 10 + 100000 = 110020

c) 4(1000000) + 2(100) + 7 + 10000 + 3(100) = 4010507

Example 2.2.5

a) 10 different symbols b) 1 symbol c) 2 symbols d) 7 different symbols e) 7 different symbols

Example 2.2.6



Example 2.2.7

a) 10(1) + 4(20) = 90 b) 0 + 16(20) = 320 c) 2(1) + 8(20) = 162

Example 2.2.8

a) 14(1) + 0 + 11(360) = 3974 b) 6(1) + 7(20) + 5(360) = 1946 c) 0 + 3(20) + 9(360) = 3300
d) 14(1) + 2(20) + 10(360) + 4(7200) = 32454 e) 19(1) + 5(20) + 0 + 1(7200) = 7319

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2.2.1: Written Retrieval and Problem Solving Practice

Student Learning Objectives

SLO 2.2 Students will analyze and apply the properties and functions of whole numbers, distinguishing between their various uses (cardinal, ordinal, and identification) and explaining relationships such as greater than and less than through set-based representations.

SLO 2.3 Students will compare Egyptian, Roman, and Hindu-Arabic numeration systems, identifying differences in place value, positionality, and arithmetic operations, and evaluate each system's advantages and limitations.

The mild assignment 🌶 is a shorter and less challenging assignment. The medium assignment🌶🌶 is right in the middle in terms of length and level of difficulty. Finally, the spicy assignment 🌶🌶🌶 is a longer assignment that is the most challenging.

(Spicy 🌶🌶🌶 Problems are OPTIONAL, but go for it when you have the time to challenge yourself)

Exercise 2.2.1.1 🌶

Identify how the number is used in the sentence: cardinal, ordinal, or identification.

a) Sammy came in first place.

b) A cat is an animal with four legs.

c) I work in suite 201.

Answer available at the end of this section.

Exercise 2.2.1.2 🌶🌶

Explain how we know that 6 > 3.

Answer available at the end of this section.

Exercise 2.2.1.3 🌶🌶

Change to Roman Numerals.

a) 14

b) 52

c) 299

Answer available at the end of this section.

Exercise 2.2.1.4 🌶🌶

Change to Mayan Numerals.

a) 24

b) 152

c) 760

Answer available at the end of this section.

Exercise 2.2.1.5

Change to Hindu-Arabic Numerals.

a) MCMXCI

b) CMLXXVI

c) MMMCCXLV

Answer available at the end of this section.

Exercise 2.2.1.6

Change to Hindu-Arabic Numerals.

a)  b) 

Answer available at the end of this section.

Exercise 2.2.1.7

Change to Hindu-Arabic Numerals.

a)  b)  c) 

Answer available at the end of this section.

Exercise 2.2.1.8

Write sentences that show the number 33 used in each of the following ways: As a cardinal number, as an ordinal number, and as an identification number.

Answer available at the end of this section.

Exercise 2.2.1.9 🌶🌶🌶

Some children go through reversal stage; that is they confuse 13 and 31, 27 and 72, 59 and 95. What numerals would give Roman children similar difficulties?

Answer available at the end of this section.

Exercise 2.2.1.10 🌶🌶🌶

Which way is a child most likely to first learn the ordering of whole numbers less than 10: i) using the counting chant, ii) using the set method, iii) using the number line method? Explain.

Answer available at the end of this section.

Problem Solving Practice

Design a **new numeration system** using symbols of your choice. Your system must:

- Represent the numbers 1–50 clearly
- Use at least 5 different symbols
- Explain how your system is similar to or different from one of the systems we have studied so far.

You must also **justify**:

- Is your system positional or non-positional?
- What are the pros/cons of using your system compared to others?

Answers

Exercise 2.2.1.1 🌶

a) ordinal b) cardinal c) identification

🌶🌶 **Exercise** 2.2.1.2

1) Via chant counting, 6 comes after 3. On a number line, 6 is to the right of 3. One-to-One correspondence, there are 3 more objects in a set of 6 coparted to a set of 3 objects.

🌶🌶 **Exercise** 2.2.1.3

a) XIV b) LII c) CCXCIX

🌶🌶 **Exercise** 2.2.1.4



a) b) c)

🌶🌶 **Exercise** 2.2.1.5

a) 1991 b) 976 c) 3245

🌶🌶 **Exercise** 2.2.1.6

a) 2,342 b) 13,026

🌶🌶 **Exercise** 2.2.1.7

a) 0 + 18(20) + 6(360) = 2,520 b) 7 + 14(20) + 0 + 6(7200) = 43487 c) 4 + 6(20) + 0 + 14(7200) = 100,942

🌶🌶 **Exercise** 2.2.1.8

Cardinal number (used for counting):
🔷 *There are **33 students** in Mrs. Rivera's class this semester.*

Ordinal number (used for order or position):
🔷 *Lina finished in **33rd place** in the citywide spelling bee.*

Identification number (used as a label or code):
🔷 *The player wearing jersey number **33** scored the winning goal.*

🌶️🌶️🌶️**Exercise** 2.2.1.9

VI (6) and **IV (4) -** One is 5 + 1, the other is 5 - 1.

XI (11) and **IX (9)** - One is 10 + 1, the other is 10 - 1.

These confusions happen because Roman numerals rely heavily on **symbol order** to determine whether values are added or subtracted—just like place value in our Hindu-Arabic system.

🌶️🌶️🌶️**Exercise** 2.2.1.10

A child is **most likely to first learn the ordering of whole numbers less than 10 using the counting chant** (**i**). Children often know how to "count to 10" before they can accurately count objects or compare sets. **ii) Set method (likely later)** This involves understanding that **larger sets have more objects** than smaller ones (e.g., 6 apples is more than 4 apples). It requires **conceptual understanding of quantity and one-to-one correspondence**, which comes **after** memorizing the counting sequence.

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2.2.2: Online Retrieval and Problem Solving Practice

ADAPT 2.2.2.1

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2.3: Hindu-Arabic Numeration System

The Hindu-Arabic Number System

The Hindu-Arabic numeral system, also known as the decimal or base-ten system, is the most widely used numeration system in the world today. It was developed by Indian scholars as early as 800 BCE and later transmitted to the Western world by Arabic mathematicians. The system's efficiency in arithmetic operations and its adaptability to complex mathematical computations contributed to its widespread adoption over other ancient systems like Roman numerals and Egyptian hieroglyphs.

Key Features:

1. **Digits:** The system uses ten symbols (0, 1, 2, 3, 4, 5, 6, 7, 8, 9) that can be combined to represent all possible numbers.
2. **Grouping by Ten:** Numbers are grouped in sets of ten, making it a base-ten system, likely influenced by the human ten fingers.
3. **Place Value (Positional Number System):** The value of a digit depends on its position in the numeral. For example, in the number 5984, the 5 represents thousands, the 9 represents hundreds, the 8 represents tens, and the 4 represents ones.
4. **Use of Zero:** A significant innovation of this system is the concept of zero as a placeholder, which allows for efficient representation of large numbers.
5. **Expanded Form (Additive and Multiplicative Number System):** Numbers can be written as the sum of each digit multiplied by its place value (e.g., 5984 = 5×1000 + 9×100 + 8×10 + 4).

Think About and Discuss

Before we proceed to learn about our base-10 number system (Hindu-Arabic) lets review what we know about the early learning of math for those you will be teaching.

Task 1: Considering what we learned so far, what are some of the first concepts of number and counting that children learn? Discuss with others and be ready to share with the class.

Task 2: Now view this video from Math Educator Graham Fletchy about his thoughts on early learning of number and counting. This video explores the progression of early number concepts and counting.

Task 3: Based on the progression of early number sense, what surprised you the most about how children develop mathematical understanding? What questions do you have now? Discuss with others and be ready to share with the class.

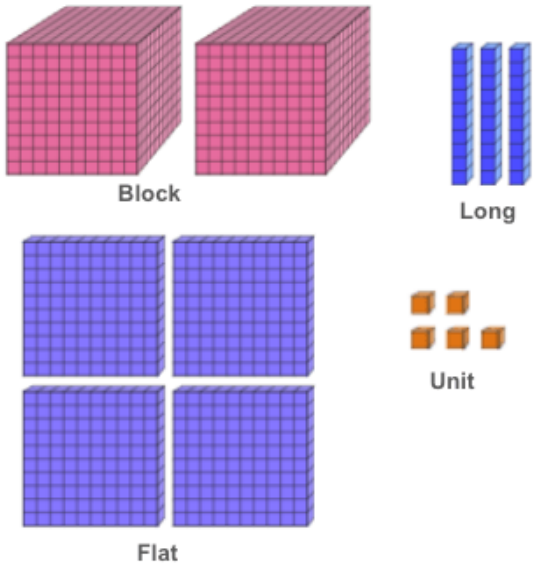
The progression of early number sense—subitizing, more/less comparisons, counting, cardinality, and conservation—lays the foundation for understanding **place value and the base-ten system.**

Models for Our Number System

The Hindu-Arabic numeration system operates on a base-ten structure, grouping numbers into sets of ten due to our natural inclination to count on our fingers. This fundamental principle replaces ten ones with one ten, ten tens with one hundred, and so forth. This concept is crucial for children developing number sense, and manipulatives such as Unifix cubes and Base Ten Blocks serve as effective teaching tools.



Unifix cubes, also known as linker cubes or multi-links, are particularly beneficial for young learners, allowing them to physically stack ten individual cubes together to form one ten, reinforcing the concept of regrouping in addition and subtraction. For instance, when solving 27 + 38, children can visually and physically group ten ones into a new ten, illustrating the process of carrying or trading. Similarly, in subtraction, breaking a ten into ten ones supports the understanding of place value. These hands-on experiences enable children to see ten both as a collection of individual units and as a single unit, fostering a flexible grasp of number relationships. (CML)



Base Ten Blocks provide another visual and tactile way for students to grasp place value, with different-sized blocks representing units (1), longs (10), flats (100), and blocks (1,000). Base Ten Blocks are common in elementary classrooms. They remain valuable for small-group instruction and remediation, particularly in helping students understand multi-digit operations and the borrowing and carrying processes in addition and subtraction. (CML)

Example 2.3.1

Express the number 5,302 in expanded form.

Answer available at the end of this section.

Numbers up to the Thousands Place

Use the interactive below to create numbers expressed in expanded form.

Interactive

Word Names for Hindu-Arabic numerals

Associated with each Hindu–Arabic numeral is a word name. Some of the English names are below.

0 zero 10 ten

1 one 11 eleven

2 two 12 twelve

3 three 13 thirteen (three plus ten)

4 four 14 fourteen (four plus ten)

5 five 24 twenty-four (2 tens plus 4 ones)

6 six 67 sixty-seven (6 tens plus seven ones)

7 seven 305 two hundred five (3 hundreds plus five ones)

8 eight 1,274 one thousand two hundred seventy-four

9 nine 25,100 twenty-five thousand one hundred

In numerals containing more than three digits, groups of three digits are sets off by commas. For example, the number 864,456,221,653,127,851 is read: "eight hundred sixty four quadrillion four hundred fifty six trillion two hundred twenty one billion six hundred fifty three million one hundred twenty seven thousand eight hundred fifty one". Notice that the word and does not appear in any of these names: it is reserved to separate the decimal portion of a numeral from the whole-number portion.

Numbers as Language

Many children initially perceive multi-digit numbers as a sequence of individual digits rather than understanding their place value (Fuson, 2004). For instance, they may see 435 as "four, three, five" rather than recognizing it as four hundreds, three tens, and five ones. The position of the number means something in the language of mathematics. To address this, elementary curricula introduce expanded notation, such as 576 = (5 × 100) + (7 × 10) + (6 × 1), aiming to help students grasp the meaning behind each digit. True place value comprehension extends beyond knowing that the seven in 576 represents seven tens; it includes understanding that there are 57 tens within 576. Despite significant instructional time devoted to naming place values, many students struggle with the broader concept because they focus on labels rather than the relationships between numbers. Connected to students learning the names of numbers, they put together the three distinct ideas, number (concept) - numeral (symbol) - name (word) in order to understand the Hindu–Arabic numeration system. Figure 2.3.1

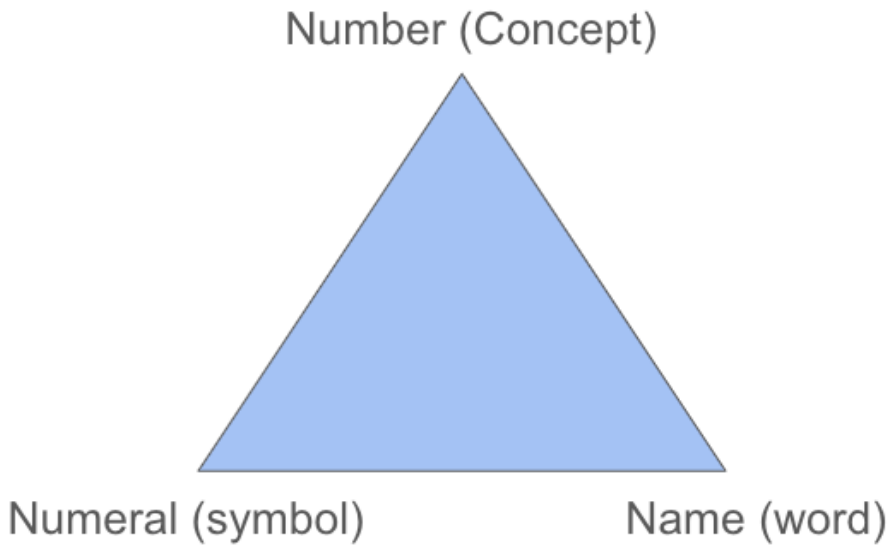


Figure 2.3.1

Wonder, Play, Grow

Task 1: Discuss with your peers- "Why do you think our number system is called 'base 10'?" Be prepared to share your answer with the class.

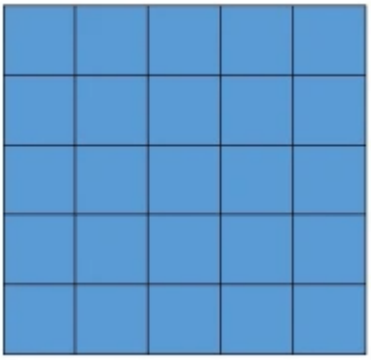
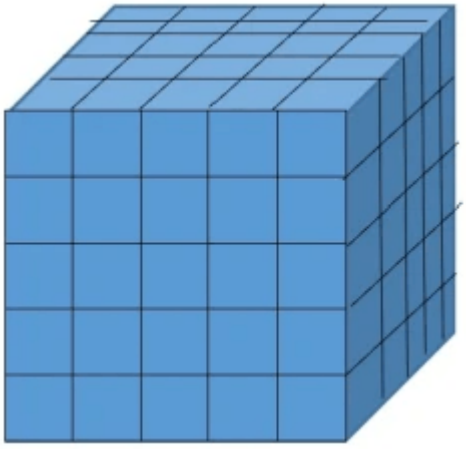
Task 2: Represent 342 in expanded form and with base-10 blocks.

Task 3: Calculate 10^3, 10^2, 10^1, 10^0. How do these values help explain why we call our number system base-10? Be prepared to share your answer with the class.

5,302 in base-10 expanded form. available at the end of this section.

Nondecimal Numeration Systems

Our Hindu-Arabic numeration system is structured around grouping by tens. To gain a deeper understanding of this system and the challenges children may encounter when learning it, it can be helpful to examine alternative systems that use different bases. For instance, imagine a numeration system based on a single hand (five fingers). In this case, numbers would be grouped in sets of five. If objects such as sticks were used, they would be bundled in groups of five. For example, the number 17 in base ten would be represented as three bundles of five with two left over in base five, written as 32_5 (or 32_{five}) . It is crucial to read this correctly as “three two base five,” rather than “thirty-two,” since the place values differ from those in base ten. When considering the language aspect of base 5, we see that the digit 3 and digit 2 represent something different because the subscript symbol 5 is included in the numeral. When we see the subscript symbol, it tells us how to assign a place value for each digit.

Base 5 Blocks	**Units are $5^0=1$**	**Longs are $5^1=5$**	**Flats are $5^2=25$**	**Blocks are $5^3=125$**
				

🧠Let's Listen to Learn❤️

Audio

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Think About and Discuss

Considering that base-10 has nine digits (0, 1, 2, 3, 4, 5, 6, 7, 8, 9) to represent all of our numbers, how many and which digits would we use for a base-5 number system? Once you and your peers agree on that then discuss how to count in base-5.

Keep in mind that counting using base five names differs from counting in base ten. What are the first 10 base-5 numerals?



Base-5 Counting available at the end of this section.

Example 2.3.2

a) Make 213_5 using base five blocks, then convert into base ten.

b) Make 2142_5 using base five blocks, then convert into base ten.

Answer available at the end of this section.

Play, Wonder, Grow

Convert the number 81 into base 5, using base 5 blocks. It may also be a good problem solving strategy to consider a place value table to organize your thinking.

25	5	1
?	?	?

Answer available at the end of this section.

Example 2.3.3

Convert the number 341 into base 5

Answer available at the end of this section.

🧠Let's Listen to Learn❤️

Audio

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Other Bases

Base-10 and Base-5 are not the only systems that can represent numbers. The **base-2 system**, also known as the **binary system**, is a number system that uses only two digits: **0 and 1**. Unlike our familiar **base-10 (decimal) system,** where each place value represents a power of 10, the binary system is based on **powers of 2**. The binary system is **fundamental to computing and digital electronics** because modern computers and electronic devices operate using electrical circuits that have **two states**:

- **ON (1)**
- **OFF (0)**

Since binary numbers correspond directly to these two states, they are the most efficient way for computers to **store, process, and transmit data**.

When you see a Base Two numeral like 11011_{two}, you need to understand what each place value stands for. *In all bases*, the rightmost place value is the units or 1's place. Next, the number name "two" to the right of the numeral tells you this is a Base Two numeral, which means each place value as you move left increases by a multiple of 2. Below is a chart showing the first eight place values in Base Two. They are written in Base Ten.

128	64	32	16	8	4	2	1

To check that the Base Two numeral 11011_{two} really is 27_{ten}, put it into a chart with just five place values showing (since it's a five-digit numeral) and check the total value of the numeral. Look at the numeral written to the right. The place value for each digit is written below each digit of the numeral. The numeral is written in bold so it does not get confused with the place values. You check this exactly the way you did for the Mayan numerals:

$$\begin{aligned}(1 \text{ group of } 16) + (1 \text{ group of } 8) + (0 \text{ groups of } 4) + (1 \text{ group of } 2) + (1 \text{ group of } 1) &= (\mathbf{1} \times 16) + (\mathbf{1} \times 8) + (\mathbf{0} \times 4) + (\mathbf{1} \times 2) + (\mathbf{1} \times 1) \\ &= 16 + 8 + 2 + 1 \\ &= \mathbf{27}.\end{aligned}$$

All right! It really works! In fact, it's really easy to check the arithmetic for a Base Two numeral because for each place value containing a 1, that place value gets added to the total, whereas those place values containing a zero do not get added to the total. There's no multiplication you have to do –you can skip directly to the addition: 16 + 8 + 2 + 1 = **27**.

$$\boxed{\begin{array}{ccccc}\mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \hline 16 & 8 & 4 & 2 & 1\end{array}\ \textbf{two}}$$

Let's convert $\mathbf{10\ 011\ 010}_{\text{two}}$ to Base Ten. Again, we'll write it with the place values shown under each digit and add up the place values containing a 1. This numeral converts to 128 + 16 + 8 + 2 = **154**.

$$\boxed{\begin{array}{cccccccc}\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \hline 128 & 64 & 32 & 16 & 8 & 4 & 2 & 1\end{array}\ \textbf{two}}$$

Example 2.3.4

Convert each Base Two numeral to Base Ten.

a) $10\ 011_{\text{two}}$	b) $1\ 000\ 001_{\text{two}}$	c) $111\ 111_{\text{two}}$

Answer available at the end of this section.

Wonder, Play, Grow

Convert the following numerals to Base Ten. Work with your peers to figure it out.

a. $10\text{Ø}2_3$ = ____
b. $10\text{Ø}23_4$ = ____
c. You make up a number in base-6 and have a peer convert it to base-10.

Did You Know

DID YOU KNOW: In Base Eleven, you need eleven different symbols (or digits) for possible placeholders. The problem we encounter is that we only have these ten recognizable digits –0,1,2,3,4,5,6,7,8 and 9. *Remember that 10 is made up of two separate digits –it isn't a single symbol!!* So we need to introduce a new symbol to represent ten in Base Eleven and higher. The convention is to use the letter "T". Similarly, we use the letter "E" to represent the number eleven in Base Twelve and higher and the letter "W" to represent the number twelve in Base Thirteen and higher. Since the second place value from the right represents the base you are working in, you never need a separate symbol to represent the value of the base or any number higher than the base. That's why there are only 0's and 1's and *no 2*'s in Base Two and why there are only 0's, 1's, 2's, 3's and 4's and *no 5*'s in Base Five, etc.

Let's convert the numeral $TE5_{\text{twelve}}$ to Base Ten. To remember the place values, we might want to draw a Base Twelve place value chart for three digits as shown to the right.

Then, $TE5_{\text{twelve}} = 10 \times 144 + 11 \times 12 + 5 \times 1 = 1440 + 132 + 5 = \mathbf{1577}$.

Pay close attention to the difference between $2T9_{\text{eleven}}$ and 2109_{eleven}.

Example 2.3.5

Convert each of the following to a Base Ten numeral. Show work

a. 110_{two} =	e. 110_{six} =
b. 110_{three} =	f. 110_{seven} =
c. 110_{four} =	g. 110_{eight} =
d. 110_{five} =	h. 110_{nine} =

Answer available at the end of this section.

Base 5 and Base 2 Conversion Retrieval Practice

Let's see what Dr. Howie Hua has to say about this.

YouTube Video

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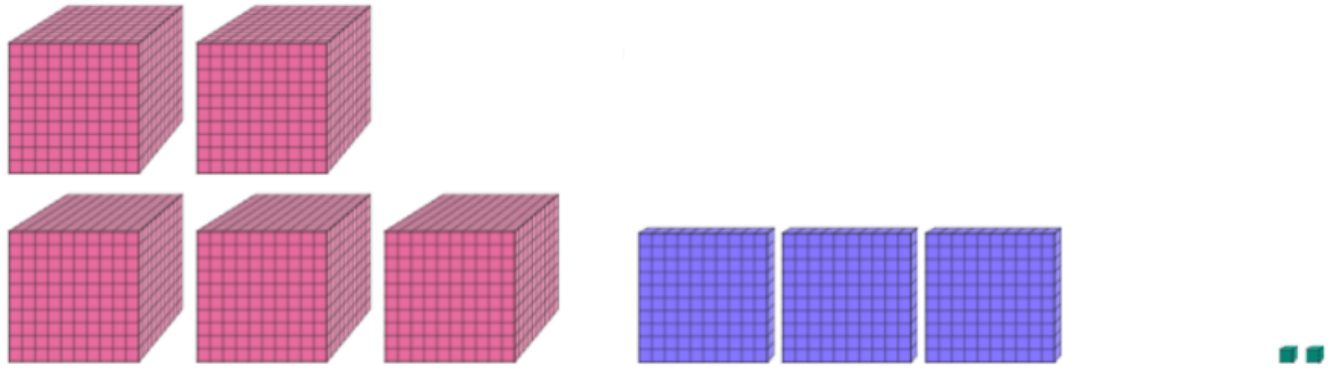
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Answers

Example 2.3.1

5(1000) + 3(100)+ 0(10) + 2(1) = 5,302

This is like saying we have 5 groups of 1000 (block), 3 groups of 100 (flat), 0 groups of 10 (long) , and 2 group 1 (unit).



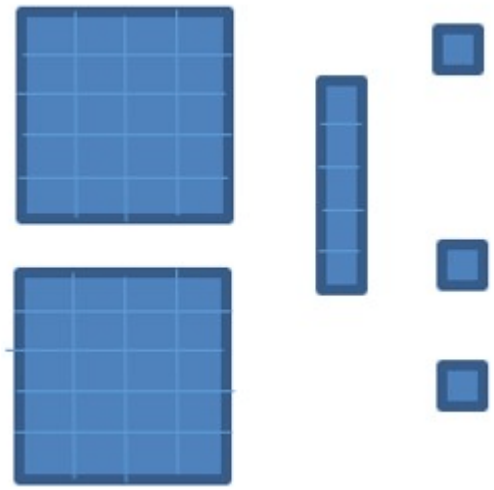
Notice that we didn't represent 5,302 using "unifix" cubes. :)

$5(10^3) + 3(10^2) + 0(10^1) + 1(10^0) = 5(1000) + 5(100) + 0(10) + 2(1) = 5,302$

Think About and Discuss

$1_5, 2_5, 3_5, 4_5, 10_5, 11_5, 12_5, 13_5, 14_5, 20_5$

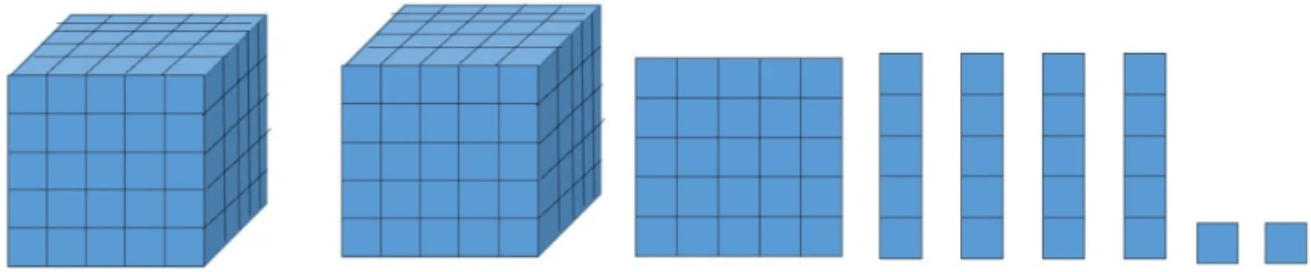
Example 2.3.2



a)

$2(5^2) + 2(5^1) + 3(5^0) = 2(25) + 1(5) + 3(1) = 58$

b)



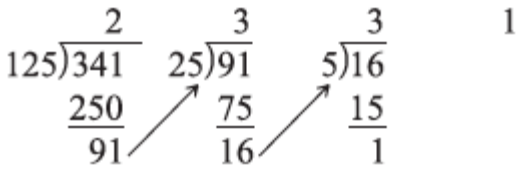
$2(5^3) + 1(5^2) + + 4(5^1) + 2(5^0) = 2(125) + 1(25) + 4(5) + 2(1) = 422$

311_5

Example 2.3.3

2331_5

You can use base blocks to model this situation, but you may also want to consider a more systematic method can be used to convert 341 to its base five numeral. First, find the highest power of 5 that will divide into 341 (the "how many" approach); that is, which is the greatest among 1, 5, 25, 125, 625, 3125, and so on, that will divide into 341? The answer in this case is 125. The rest of that procedure uses long division, each time dividing the remainder by the next smaller place value.



Remainder Approach:

🔁 **Step 1: Divide 341 by 5**

341 ÷ 5 = 68 with a remainder of **1**

Write down the **remainder**.

🔁 **Step 2: Divide 68 by 5**

68 ÷ 5 = 13 with a remainder of **3**

Write down the **remainder**.

🔁 **Step 3: Divide 13 by 5**

13 ÷ 5 = 2 with a remainder of **3**

Write down the **remainder**.

🔁 **Step 4: Divide 2 by 5**

2 ÷ 5 = 0 with a remainder of **2**

Write down the **remainder**.

📌 **Step 5: Read the remainders from bottom to top**

From last to first: **2 3 3 1**

So, **341 in base 5 is: 2331**

Example 2.3.4

a) = $(1\times2^4 + 0\times2^3 + 0\times2^2 + 1\times2^1 + 1\times2^0) = (16 + 0 + 0 + 2 + 1) = 19$

b) = $(1\times2^6 + 0\times2^5 + 0\times2^4 + 0\times2^3 + 0\times2^2 + 0\times2^1 + 1\times2^0) = (64 + 0 + 0 + 0 + 0 + 0 + 1) = 65$

c) = $(1\times2^5 + 1\times2^4 + 1\times2^3 + 1\times2^2 + 1\times2^1 + 1\times2^0) = (32 + 16 + 8 + 4 + 2 + 1) = 63$

Example 2.3.5

TO convert each of the numbers 110 from different bases into **Base Ten (Decimal)**. We'll use this formula:

$110_{base}=1\times base^2+1\times base^1+0\times base^0$

a) 6 b) 12 c) 20 d) 30 e) 42 f) 56 g) 72 h) 90

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2.3.1: Written Retreival and Problem Solving Practice

Student Learning Objectives

2.4 Students will demonstrate proficiency with the Hindu-Arabic numeration system by converting between bases, expanding numbers, and articulating the system's structure, base, and numeral naming conventions.

The mild assignment 🌶 is a shorter and less challenging assignment. The medium assignment🌶🌶 is right in the middle in terms of length and level of difficulty. Finally, the spicy assignment 🌶🌶🌶 is a longer assignment that is the most challenging.

(Spicy 🌶🌶🌶 Problems are OPTIONAL, but go for it when you have the time to challenge yourself)

Exercise 2.3.1.1 🌶

Write each of the following numbers in expanded notation.

a) 75 b) 302 c) 6814 d) 50,004,030

Answer available at the end of this section.

Exercise 2.3.1.2 🌶

Write each of the following expressions in standard place value form.

a) $2(1000) + 3(100) + 4(10) + 5(1)$

b) $6(100{,}000) + 5(100) + 7(1)$

c) $9(10^6) + 1(10^4) + 2(10^2) + 5(10^0)$

d) $2(10^9) + 4(10^6) + 6(10^5) + 8(10^1)$

Answer available at the end of this section.

Exercise 2.3.1.3 🌶

State the place value of the digit 2 in each numeral.

a) 7234 b) 4142 c) 2168

Answer available at the end of this section.

Exercise 2.3.1.4 🌶

The following numbers are written in words. Rewrite each one using Hindu–Arabic numerals.

a) Six million five hundred three thousand fifty-nine

b) Three hundred six billion four hundred fifty-three thousand

Answer available at the end of this section.

Exercise 2.3.1.5

Write each of the following numerals in expanded notation.

a) 14_5 b) 26_7 c) 321_4 d) 10101_2

Answer available at the end of this section.

Exercise 2.3.1.6

Convert the following base 2 numerals into base ten numerals.

a) 101_2 b) 1111_2 c) 10101_2

Answer available at the end of this section.

Exercise 2.3.1.7

Convert the following base five numerals into base ten numerals.

a) 12_5 b) 204_5 c) 1322_5

Answer available at the end of this section.

Extra Help 2.3.8 - "Does it Fit" Method

YouTube Video

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YouTube Video

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Exercise 2.3.1.8

Convert each base ten numeral into a numeral in the base requested.

a) 126 in base 5

b) 400 in base 5

b) 17 in base 2

d) 71 in base 2

Answer available at the end of this section.

Exercise 2.3.1.9

a. Write out the base five numerals in order from 1 to 100_5.

b. Write out the base two numerals in order from 1 to 10000_2.

Answer available at the end of this section.

Exercise 2.3.1.10

What is wrong with the numeral 24_3?

Answer available at the end of this section.

Exercise 2.3.1.11

Write out the first 10 base four numerals.

Answer available at the end of this section.

Exercise 2.3.1.12

The base twelve numeration system has the following twelve symbols:0,1,2,3,4,5,6,7,8,9,T,E.

Change each of the following numerals to base ten numerals.

a) 142_{12} b) $T9_{12}$ c) $ETET_{12}$

Answer available at the end of this section.

Problem Solving Practice

1) Directions: Using the digits 0 through 9 at most one time each, place a digit in each box to make a two-digit number that has a value between given numbers.

$$\square\square, 35, \square\square, 41, \square\square, \square\square, 88, \square\square, 107$$

Courtesy of Open Middle

Answer

One Potential Answer : 20,35,37,41,45,76,88,89,107

2) Find the number.

I am greater than 52 tens.

I am less than half of 1200.

If you round me to the nearest 100, I become 500.

I am even.

Who am I?

Answer

The number can be **any even number between 520 and 548**, such as **524, 526, 528, 530, ..., 548**.

Converting Base Into Base 5 and 2

Interactive Content

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Answers

Exercise 2.3.1.1

a) 7(10) + 5(1)

b) 3(100) + 0(10) + 2(1)

c) 6(1000) + 8(100) + 1(10) + 4(1)

d) $5(10^7) + 4(10^3) + 3(10^1)$

Exercise 2.3.1.2🌶

a) 2,345 b) 600,507 c) 9,010,205 d) 2,004,500,080

Exercise 2.3.1.3🌶

a) hundreds b) ones c) thousands

Exercise 2.3.1.4🌶

a) 6,503,059 b) 306,000,453,000

Exercise 2.3.1.5🌶

a) 1(5) + 4(1) b) 2(7) + 6(1) c) 3(16) + 2(4) + 1(1) d) 1(16) + 1(4) + 1(1)

Exercise 2.3.1.6🌶🌶

a) 4 + 1 = 5 b) 8 + 4 + 2 + 1 = 15 c) 16 + 4 + 1 = 21

Exercise 2.3.1.7🌶🌶

a) 1(5) + 2(1) = 7 b) 2(25) + 4(1) = 54 c) 1(125) + 3(25) + 2(5) + 2(1) = 212

Exercise 2.3.1.8🌶🌶🌶

a) 1001_5 b) 3100_5 c) 10001_2 d) 1000111_2

Exercise 2.3.1.9🌶🌶

a) 1, 2, 3, 4, 10, 11, 12, 13, 14, 20, 21, 22, 23, 24, 30, 31, 32, 33, 34, 40, 41, 42, 43, 44, 100

Exercise 2.3.1.10🌶🌶

In base 3, we only use the symbols 0, 1, 2.

Exercise 2.3.1.11🌶🌶🌶

1, 2, 3, 10, 11, 12, 13, 20, 21, 22

Exercise 2.3.1.12🌶🌶🌶

a) $1(12^2) + 4(12^1) + 2(12^0) = 194$

b) $10(12^1) + 9(12^0) = 129$

c) $11(12^3) + 10(12^2) + 11(12^1) + 10(12^0) = 20590$

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2.3.2: Online Retrieval and Problem Solving Practice

Base 5 and Base 2 Conversion Retrieval Practice

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2.4: Chapter 2 Resources

Chapter Example Videos

Chapter 2.1

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Chapter 2.2

YouTube Video

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Chapter 2.3

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Chapter Extras

Set Operation Practice

YouTube Video

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3: BUILDING NUMBER SENSE- UNDERSTANDING WHOLE NUMBER OPERATIONS AND THEIR PROPERTIES

Numbers are everywhere! From counting seats on a bus to sharing churros at a snack stand (Morgan and Grandpa, see story below :)), whole number operations—addition, subtraction, multiplication, and division—help us make sense of the world around us. In this chapter, we will explore these essential operations using models and strategies that highlight their fundamental properties. We will see how numbers relate through commutativity, associativity, and distributive properties, and how inequalities and exponents fit into the bigger picture of mathematical thinking. By understanding these concepts deeply, we can develop strong mental computation skills that will serve us in various real-life situations.

As future elementary teachers, mastering whole number operations is more than just learning procedures—it's about fostering number sense in young learners. Through this chapter, we will uncover how whole number properties help solve everyday problems and why they are crucial for effective math instruction.

Chapter Learning Objectives

Student Learning Objectives

- SLO 3.1 Students will demonstrate a comprehensive understanding of whole number operations, including addition, subtraction, multiplication, and division, by modeling each operation with set and measurement models and analyzing their properties.
- SLO 3.2 Students will apply strategies for mental computation in addition, subtraction, and multiplication across different bases and demonstrate mastery of basic facts using commutativity, associativity, and distributive properties.
- SLO 3.3 Students will analyze and apply the properties of whole numbers in relation to inequality, including transitivity, addition, and multiplication properties of "less than," as well as use addition to establish comparative relationships.
- SLO 3.4 Students will demonstrate an understanding of whole number exponents by defining exponents through repeated multiplication and proving fundamental exponent rules.
- SLO 3.5 Students will evaluate numerical expressions involving whole numbers using the conventional order of operations (including parentheses, exponents, multiplication, division, addition, and subtraction), to arrive at a unique and correct solution.

Essential Questions

- **How do the properties and operations of whole numbers help us make sense of and solve real-world problems?**
- **How can understanding whole number concepts and strategies support effective math instruction for young learners?**

Learning That Transfers

Students will analyze and apply whole number operations—addition, subtraction, multiplication, and division—to real-world contexts by developing mathematical models that demonstrate problem-solving strategies for everyday situations and elementary-level teaching. They will evaluate and adapt these models to support young learners' conceptual understanding and problem-solving skills.

Number Sense Groove

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View this Interactive Content online

Morgan and Grandpa's Mathematical Adventure

One bright morning, as Morgan arose, Grandpa said, "Let's go where the number wind blows! Numbers are grand, they're everywhere seen, Let's find their magic in places between!"

They hopped on a bus—oh what a delight! Numbers on seats, left and right. "See here, my dear," Grandpa said with a grin, "Addition is easy, let's begin!"

Morgan looked, eyes open wide, "Two kids sit here, and three more beside. If we add them together, it's easy to do, Two plus three, why look, it's just five—woohoo!"

Down at the park, the swings went high, Morgan saw seagulls swoop through the sky. "Grandpa, if eight birds fly off in a dash, And three fly back, what's left in a flash?"

Grandpa chuckled, "That's subtraction, my dear! Take eight, remove three, and five remain here!" Morgan beamed, "Oh, it's quite clear! Numbers are fun, let's find more near!"

They stopped for a snack at a sweet little stand, Grandpa held three churros in his hand. "If each one holds two bites just right, How many bites will we have in sight?"

Morgan thought, "Three times two, Means six sweet bites—one for me, one for you!" Grandpa clapped, "That's multiplication, you see, Numbers at work, so brilliantly!"

Then at the pond, a puzzle was found, Ten ducks lined up, splashing around. "They split into groups, two per a row, How many rows do they make in a show?"

Morgan counted, "One, two, three, four, five, Dividing by two keeps math alive! Ten ducks split makes five neat lines, Division, Grandpa, is full of designs!"

The sun dipped low, the sky turned gold, Morgan felt proud, so brave and bold. "Numbers are everywhere, big and small, And now I see, I can use them all!"

3.1: Addition and Subtraction

3.1.1: Written Retrieval and Problem Solving Practice

3.1.2: Online Retrieval and Problem Solving Practice

3.2: Developing Multiplicative and Divisional Thinking

3.2.1: Written Retrieval and Problem Solving Practice

3.2.2: Online Retrieval and Problem Solving Practice

3.3: Ordering Whole Numbers and Exponents

3.3.1: Written Retrieval and Problem Solving Practice

3.3.2: Online Retrievel and Problem Solving Practice

3.4: Counting Collections- Uncovering Number Sense and Mathematical Thinking Project
3.5: Chapter 3 Resources

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3.1: Addition and Subtraction

Addition and Subtraction

Addition and subtraction are fundamental building blocks of mathematical understanding, forming the basis for more advanced numerical reasoning. Young children first encounter these operations through real-world experiences, often using fingers, objects, or verbal counting strategies to find sums and differences. At this stage, addition is seen as putting together, while subtraction is commonly understood as taking away. As their understanding deepens, children begin to recognize the relationship between the two operations, using known addition facts to solve subtraction problems and vice versa. Encouraging students to explore various ways to compose and decompose numbers fosters a flexible understanding of number relationships, laying the groundwork for number sense and mental computation (Feike et al., 2018).

As future teachers—or even as mentors to younger family members—you have the opportunity to shape how children experience mathematics from the very beginning. By providing meaningful experiences with addition and subtraction, you can help young learners move beyond memorization to develop a deep understanding of numbers and operations. Encouraging strategies such as choral counting, subitizing, and using real-world contexts will empower children to see math as a sense-making activity rather than a set of disconnected rules. (Franke et al., 2018; NCTM 2020) Helping students develop confidence and fluency in these early concepts will not only set them up for success in later mathematics but will also nurture a positive relationship with math that lasts a lifetime.

Think About and Discuss

Here are some more moments of brilliance from our younger mathematicians. Before you take a look at the videos by famed math educator Marylin Burns, think about how an elementary school student would solve 90-75.

Task 1: Discuss with peers how you think they may solve the problem and try to explain it using math reasoning or math expressions.

Task 2: View the videos.

Meeyah 90-__=75	**Rocco 90-__=75**
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https://youtu.be/FVfF7n2FeiQ?si=oMIOGwS6D2cWYVD1	https://youtu.be/zHeqBZrT-lo?si=IU9pPx-gG6-vaSRC

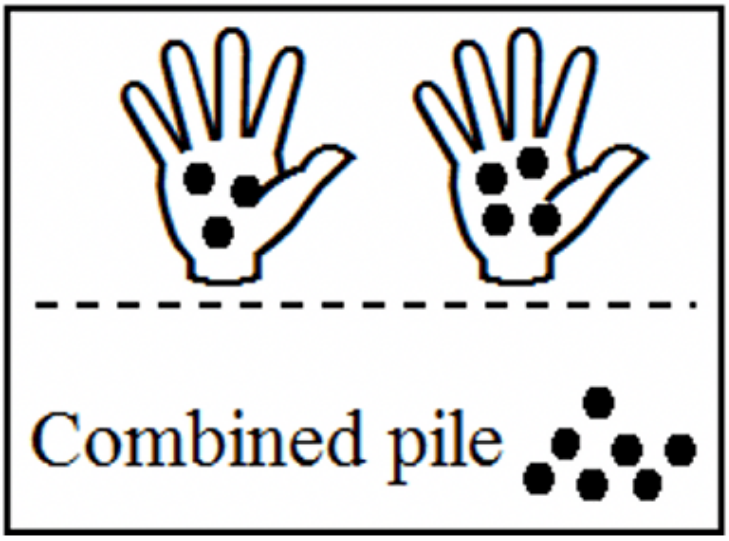
Task 3: Imagine you were Rocco's teacher, what feedback would you give him about his mathematical thinking and reasoning.

Let's take this mathematical brilliance we witnessed to explore this section's content on addition and subtraction.

Addition

Whole numbers are made up of zero and the counting numbers 0, 1, 2, 3, 4, 5, . . . We begin our work by exploring how to add

two *whole numbers* together. To add 3 + 4, we find a set with three elements and a different set with four elements (having no elements in common with the first set). We combine the two sets and then count how many elements are in the union.



So to add two numbers, 3 and 4, we must find a set A such that n(A) = 3 and a set B such that n(B) = 4. Then, 3 + 4 is how many elements are in $A \cup B$.

Definition: Set Theory Addition of Whole Numbers

Let a and b be any two whole numbers. If A and B are disjoint sets with a = n(A) and b = n(B), then a + b = n(A ∪ B).

The number a + b, read "a plus b," is called the sum of a and b, and a and b are called addends or summands of a + b.

Addition is called a binary operation because two ("bi") numbers are combined to produce a unique (one and only one) number. Multiplication is another example of a binary operation with numbers. Intersection, union, and set difference are binary operations using sets.

Example 3.1.1

a) Use the set theory definition of addition to show that 3 + 2 = 5.

b) Use the set theory definition of addition to show that ∅ + 4 = 4.

Answer available at the end of this section.

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Audio

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Measurement Model

Addition can also be represented on the whole-number line. In the measurement model, addition of whole numbers is represented by directed arrows of whole number lengths along the whole number line. Example 2 + 5. Figure: 3.1.1

Place an arrow length 2 starting at 0. Then place an arrow length 5 starting at the end of the first arrow. The sum is the total distance from 0.

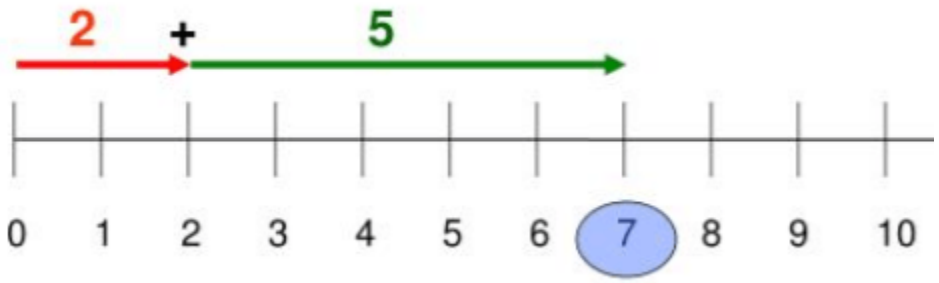


Figure 3.1.1

Learning the addition properties of whole numbers helps children develop number sense, construct relationships among facts, and build fluency in mental math and problem-solving. These properties also lay the foundation for algebraic reasoning by allowing students to generalize operations and make sense of number relationships (NCTM, 2020).

Properties of Whole-Number Addition

Commutative Property for Whole-Number Addition

Let a and b be any whole numbers. Then a + b = b + a.

Associative Property for Whole-Number Addition

Let a, b, and c be any whole numbers. Then (a + b) + c = a + (b + c).

Identity Property for Whole-Number Addition

There is a unique whole number, namely 0, such that for all whole numbers a, a + 0 = a = 0 + a. Because of this property, zero is called the additive identity or the identity for addition.

Most of us have been adding numbers together for years and years and so this property may seem obvious — it is second nature that the order in which we add two numbers is irrelevant. But if you take the opportunity to ask a child who is just learning to add the following two questions, even one right after the other, you may notice the child can do the first one quickly and easily but then struggles a little longer at the second one. That is usually the case if the child hasn't yet discovered the c**ommutative property of addition.** As adults, we take this property for granted.

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First question: What is 7 + 2? Second question: What is 2 + 7?

Think About and Discuss

How do you think a child might figure out the answer to the first question and how might he or she figure out how to do the second question? Be specific and assume the answers haven't been memorized yet! Why might the second question be perceived as a harder problem?

Answer available at the end of this section.

The **Associative Property of Addition** states that if *a* , *b* and *c* are any three numbers, then (a + b) + c = a + (b + c)

The root in the word "associative" is *associate*. Think about whether the middle number will associate with the first number or the last number for the first computation. As for adding numbers in your head, it may be a lot easier to do it one way instead of the other. For instance, to add 58 + 39 + 41, you could think (58 + 39) + 41 = 97 + 41 = 138 or you could think 58 + (39 + 41) = 58 + 80 = 138. 97 + 41 and 58 + 80 both equal 138, but that isn't obvious until after you add. I prefer to add the 39 and 41 together first. The associative property allows that. But again, the associative property is something that most adults take for granted and use but they don't really think about it.

Think About and Discuss

Here is another example. Although the sum of the three numbers is the same for each problem, the way you do the computations are different.

(84 + 56) + 73	84 + (56 + 73)
140 + 73	84 + 129
213	213

Task 1: Use "Thinking Strategies" to compute the sums.

a) 9 + 7 b) 32 + 51 c) 125 + 46 d) 37 + (42 + 13)

Task 2: Make up an example of your own showing an application of the associative property, then share it with others to see how they solve your problem.

Example 3.1.2

State which property (Commutative or Associative Property of Addition) is being used in each equation. Ask yourself: Is the difference between the left and right side due to order (commutative property) or parentheses (associative property)?

a. ____	(99 + 76) + 38 = (76 + 99) + 38
b. ____	(65 + 22) + 56 = 56 + (65 + 22)
c. ____	(57 + 88) + 43 = 57 + (88 + 43)
d. ____	(a + b) + (c + d) = ((a + b) + c) + d

Answer available at the end of this section.

Closure Property for Whole Numbers

The sum of any two whole numbers is a whole number.

The ***closure property of addition*** for whole numbers. We say the set of whole numbers is closed under addition because when you add any two whole numbers together, you get another unique whole number.

A set is closed under addition if the sum of any two elements in the set (these could be the same elements or two different elements) produces a unique element in the same set. I*n general, a set is closed under an operation if when you perform the operation on any two elements in the set, the answer is an element of the same set.*

To show that a set is **not closed** under addition, you must give a **counterexample** showing that when you add two numbers in the set, you get a sum that is not in the set. The two numbers you choose can be the same number or different numbers.

Example 3.1.3

a) Is the set {1, 2, 3, 4, 5} closed under addition?

b) Is {4} closed under addition?

c) Is {5, 10, 15, 20, . . .} closed under addition?

Answer available at the end of this section.

The Magic of Numbers: A Journey of Discovery (Inspired by Where Mathematics Comes From, Lakoff and Nunez, 2000

Imagine a world where numbers are like puzzle pieces, and every time we try a new math operation—like adding, subtracting, or dividing—we want the pieces to fit perfectly. But sometimes, they don't!

Take natural numbers, for example: 1, 2, 3, 4, and so on. If we add any two natural numbers, we always get another natural number. That's what we call **closure**—when we perform an operation, we stay within the same number set.

But what happens when we subtract? Suddenly, we hit a roadblock: **5 - 5 = 0**—oops! Zero wasn't in our original set. So, we had to **expand our number system** to include it. Then, we tried **3 - 5** and found we needed negative numbers! Later, we realized that division was another challenge: **3 ÷ 5** isn't a natural number, so we created fractions.

This pattern of discovery—finding gaps and expanding the number system—is the heart of mathematics. **Closure isn't just a math rule; it's a powerful force that drives new ideas and deeper understanding.** It's like opening new doors in a never-ending mathematical adventure.

By exploring closure, we help students see math as an exciting journey rather than just a set of rules. **When they ask, "Why do we need negative numbers?" or "Where did fractions come from?"—they're stepping into the same process mathematicians have followed for centuries.**

Let's encourage that curiosity and show students how math grows to fit the needs of our thinking!

Subtraction

Think About and Discuss

What Was Rebecca Thinking?

Marylin Burns is back, think about how a young elementary school student would solve 7 - 3.

Task 1: View the video of Rebecca solving the problem.

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https://youtu.be/TC7I_SdKWtA?si=yjHkTt1zW1d_Wb7H

Task 2: Discuss with your peers Rebecca's strategy.

Task 3: Use Rebecca's strategy to solve two other subtraction problems: 31 – 15 and 52 – 25. Are there other ways ("thinking strategies") to solve those problems? Discuss.

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There are two distinct approaches to subtraction. The one most of us are familiar with is the ***Take-Away Method.*** A typical way someone might introduce the idea of subtraction is by saying "If I put five apples in the bowl and then *take away* two of the apples, how many are left in the bowl?" The illustration below shows this as a subtraction problem where after two apples are removed from the bowl, there are three apples remaining Figure 3.1.1. Also, the problem "If you walk 7 miles from home and turn back to walk 4 miles toward home, how many miles are you from home?" can be solved with a measurement model using the takeaway approach Figure 3.1.2.

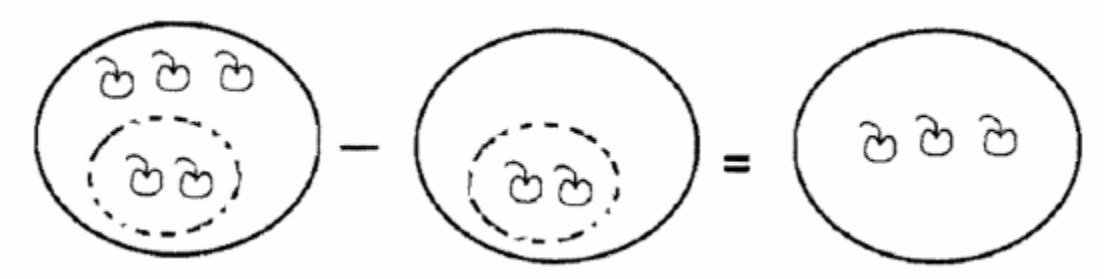


Figure 3.1.1: Take-Away Approach Set Model

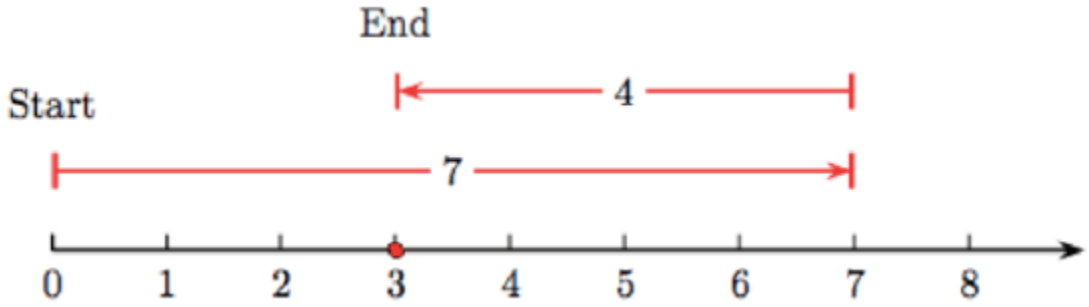


Figure 3.1.2: Take-Away Approach Measurement Model

Definition: Subtraction of Whole Numbers using Set Theory

If B is a subset of A, then n(A) – n(B) = n(A – B)

Subtraction Vocabulary: For **x – y = z**, **x** is called the *minuend*, y is called the *subtrahend*, and z (the answer) is called the *difference*.

Example 3.1.4

Use the set theory definition of subtraction to show that 5 – 2 = 3.

Let A = {v, w, x, y, z} and B = {w, z}. Since n(A) = 5, n(B) = 2 and B $\subseteq$ A,

$$\begin{aligned} 5–2 &= n(A)–n(B) && \text{by substituting } n(A) \text{ for 5 and } n(B) \text{ for 2} \\ &= n(A–B) && \text{by the set theory definition of subtraction} \\ &= n(\{v, x, y\}) && \text{by computing } A–B \\ &= 3 && \text{by counting the elements in } A–B \end{aligned}$$

Therefore, 5 – 2 = 3.

Definition: Subtraction Missing-Addend Approach

Let a and b be any two whole numbers. **a – b** is the whole number **c** such that **a = b + c**. In other words, if **c** is added to the subtrahend, **b**, the sum is the minuend, **a**. The answer, **c**, is called the *missing addend*.

The missing-addend approach can be modeled using both set and measurement models. The number line diagram above illustrates the measurement model, showing how subtraction can be visualized by identifying the missing distance (or addend)

between two numbers. For example, if a person starts at 3 and needs to reach 7, the missing addend is 4, reinforcing the idea that 7–3 can be thought of as "What do I add to 3 to get 7?". Additionally, children sometimes struggle with missing-addend problems when they are presented in isolation rather than in word problem contexts. Despite these challenges, the missing-addend approach remains a powerful way to deepen students' understanding of subtraction, particularly through real-world applications and the use of visual models like the number line.

Wonder, Play, Grow

Solve each problem and identify which approaches (SET, MEASUREMENT, TAKE-AWAY, MISSING ADDEND) you used.

a) Charles has 23 rocks that he found at the park. On his way home he lost 8 of them. How many does he have now?

b) Nancy helped Mario on his paper route. Nancy delivered 8 papers and Mario delivered 23 papers. How many more papers did Mario deliver than Nancy?

c) Emma wants to run 12 miles this week. So far, she has run 3 miles. How many more miles does she need to run to reach her goal?

d) Make up two word problems that would require the subtraction problem 8 – 3 to be computed. The first should use the take-away approach and the second should use the missing addend approach. Explain and show how you would solve each word problem using the given approach.

Answer available at the end of this section.

If you think of subtraction in terms of the missing addend approach, then we say that the statements a – b = c and a = b + c are equivalent to each other. Consider the statement, 8 – 2 = 6. It is equivalent to the statement 8 = 2 + 6. We also know that 2 + 6 = 6 + 2 because of the commutative property of addition. Therefore, 8 = 6 + 2, which in turn is equivalent to the statement 8 – 6 = 2. This gives us four facts about how to relate the numbers 2, 6 and 8 using addition and subtraction

8 – 2 = 6
8 = 2 + 6
8 – 6 = 2
8 = 6 + 2

Some teachers relate subtraction and addition by using the idea of "fact families" such as the four facts above, which is one fact family.

It is important to note that each addition statement gives us two subtraction statements, which is how many people learn their subtraction facts. Because of this relationship between addition and subtraction, once a child learns basic addition facts, the subtraction facts naturally follow.

Example 3.1.5

Write down two subtraction statements that are equivalent to an addition statement.

Answer available at the end of this section.

Think About and Discuss

Is there a whole-number answer for every whole-number subtraction problem? In other words, is subtraction of whole numbers closed? Explain your answer and provide a counterexample if it is not closed.

Example 3.1.6

Determine which sets, if any, are closed under subtraction. Provide a counterexample if a set is not closed.

a. {0}	b. {0, 2, 4, 6, ...}

Answer available at the end of this section.

Interactive Element

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Answers

Example 3.1.1

a) Let A = {x, y, z} and B = {m, r} Since n(A) = 3, n(B) =2, and $A \cap B$ = Ø then

3+2 = n(A)+n(B)	by substituting n(A) for 3 and n(B) for 2
= n($A \cup B$)	by the set theory definition of addition
= n({x, y, z, m, r})	by computing A
= 5	by counting the elements in A

Therefore, 3 +2 = 5.

b) Let A = { } and B = {a, b, c, d}. Since n(A) = 0, n(B) = 4 and $A \cup B$ = Ø, then

0 + 4 = n(A) + n(B)	by substituting n(A) for 0 and n(B) for 4
= n($A \cup B$)	by the set theory definition of addition
= n({a, b, c, d})	by computing $A \cup B$
= 4	by counting the elements in $A \cup B$

Therefore, 0 + 4 = 4

Think About and Discuss

When adding two numbers together, some people start with the first number and *count on* the second number. So if you think about 7 + 2, start with 7 and count two more in your head — eight, nine. For 2 + 7, start with 2 and count seven more in your head — three, four, five, six, seven, eight, nine. Although the answer is the same, 7 + 2 was quicker and easier to keep track of. Just think about the difference between 1000 + 1 and 1 + 1000 using this counting on method! Thank goodness for the commutative property of addition!

Example 3.1.2

a) commutative b) commutative c) associative d) associative

Example 3.1.3

a) Imagine you had two hats, each containing these five numbers in the set written on pieces of paper. So each hat contains the set {1, 2, 3, 4, 5}. If you pull a number out of each hat, you might get the same number twice or you might get different numbers. Some of the sums are 1 + 1 = 2, 2 + 3 = 5, 1 + 3 = 4, 3 + 3 = 6, etc. Note that the first three sums 2, 5 and 4 are in the set {1, 2, 3, 4, 5}. **But for a set to be closed, the sum of *every possible* two numbers in the set must yield a number in the original set.** Since 3 + 3 = 6, (where the two addends, 3 and 3, are in the original set, but the sum 6 is NOT in the original set), then the set {1, 2, 3, 4, 5} is not closed. There are any number of counterexamples you can use to show it is not closed. 3 + 3 = 6 will do. Or someone else might write 4 + 5 = 9. What are two other counterexamples you could use to show {1, 2, 3, 4, 5} is not closed under addition?

b) The only possibility to try is 4 + 4, which is 8, and 8 is not in the set. So the set is not closed. The counterexample to show it is not closed is 4 + 4 = 8.

c) Start by trying a few examples: 5 + 5 = 10 which is in the set; 10 + 15 = 25 which is in the set (note the three dots which means the next few numbers are 25, 30 and 35. It looks like it is closed. But how can you convince someone that the sum of any two numbers in the set will always be in the original set. Note that the set is composed of multiples of 5. Note any two multiples of 5 can be written as 5 times something, like 5x and 5y. Let 5x and 5y denote two arbitrary elements in the original set. Adding them together, we get 5x + 5y = 5(x+y) which is another multiple of 5 (since it is 5 times something). So, you should be convinced that the sum of any two elements in the set is a multiple of 5, which is what the original set consisted of. Therefore, {5, 10, 15, 20, . . .} is closed under addition.

a) SET Model with Take-Away Approach b) SET Model, and you can use either the Take-Away or Missing Addend Approach

c) Measurement Model, and you can use either the Take-Away or Missing Addend Approach

Example 3.1.5

4 - 1 = 3, 4 - 3 = 1, 3 + 1 = 4

Example 3.1.6

a) Closed - since 0 - 0 = 0. b) Not Closed since 6-8 = -2 or 12-20 = -8, you get negative numbers so those values are not in the original set.

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3.1.1: Written Retrieval and Problem Solving Practice

Student Learning Objectives

SLO 3.1 Students will demonstrate a comprehensive understanding of whole number operations, including addition, subtraction, multiplication, and division, by modeling each operation with set and measurement models and analyzing their properties.

SLO 3.2 Students will apply strategies for mental computation in addition, subtraction, and multiplication across different bases and demonstrate mastery of basic facts using commutativity, associativity, and distributive properties.

The mild assignment 🌶 is a shorter and less challenging assignment. The medium assignment🌶🌶 is right in the middle in terms of length and level of difficulty. Finally, the spicy assignment 🌶🌶🌶 is a longer assignment that is the most challenging.

(Spicy 🌶🌶🌶 Problems are OPTIONAL, but go for it when you have the time to challenge yourself)

Exercise 3.1.1.1 🌶🌶

Find 1 + 2 using both the Set Theory Addition of Whole Numbers and the Measurement Model.

Answer available at the end of this section.

Exercise 3.1.1.2 🌶🌶

Identify the property or properties being illustrated.

a. 7 + 5 = 5 + 7
b. 53 + 47 = 50 + 50
c. 1 + 0 = 1

Answer available at the end of this section.

Exercise 3.1.1.3 🌶🌶

Which of the following sets are closed under addition? Why or why not?

a) {10, 20, 30, 40, ...}

b) {1, 2, 3}

c) {0}

d) Whole numbers greater than 10.

Answer available at the end of this section.

Exercise 3.1.1.4 🌶🌶

Addition can be simplified using the associative property of addition. For example,

26 + 57 = 26 + (4 + 53) = (26 + 4) + 53

= 30 + 53 = 83

Complete the following statements.

a) 29 + 68 = 30 + _____ = _____

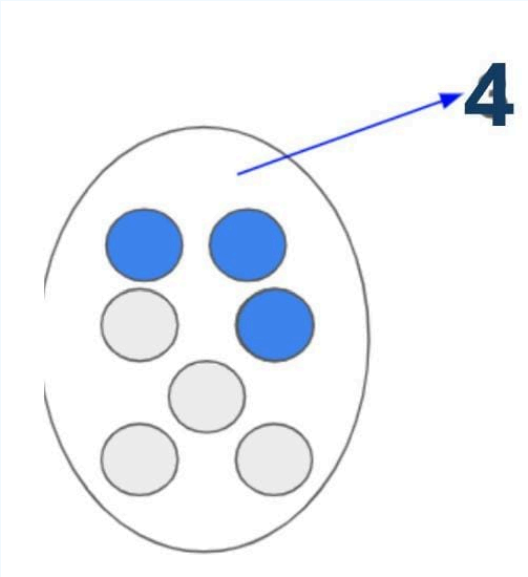
b) 37 + 23 = 40 + _____ = _____

Answer available at the end of this section.

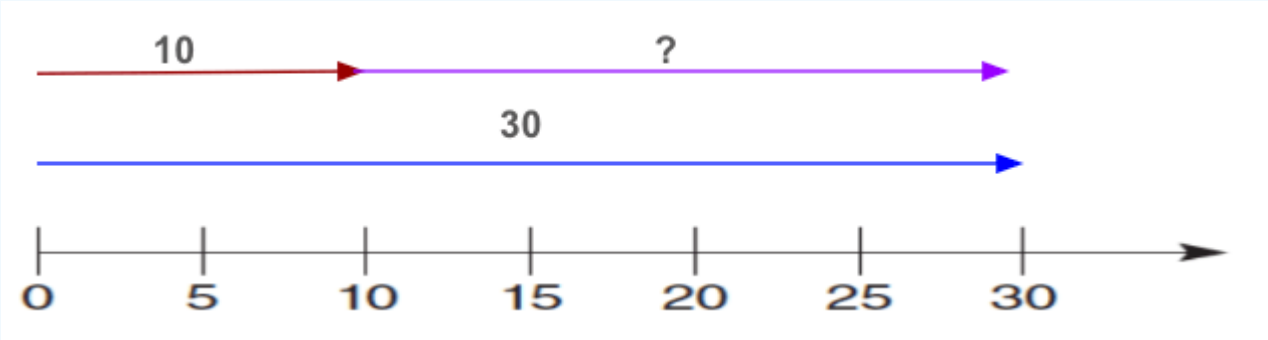
Exercise 3.1.1.5

For the following figures, identify the problem being illustrated,the model, and the conceptual approach being used.

a)



b)



Answer available at the end of this section.

Exercise 3.1.1.6

a) State a subtraction word problem involving 7 – 3, the missing-addend approach.

b) State a subtraction word problem involving 7– 3, the take-away approach.

c) Sketch the models for each of the problems in part a) and b)

Answer available at the end of this section.

Exercise 3.1.1.7

Each situation described next involves a subtraction problem. Name the approach in the problem and write an equation to fit the problem.

a) An high school teacher started the year with a budget of $200 to be spent on school supplies. By the end of December, $120 had been spent. How much money remained in the budget?

b) Litzy is saving money for a trip to Mexico over spring break. The package tour she is interested in costs $1795. From her part-time job she has saved $1240 so far. How much more money must she save?

Answer available at the end of this section.

Four Fact Family

A **four fact family** is a group of **four related math facts** that use the **same three numbers** and show the relationship between **addition and subtraction** or **multiplication and division**.

🔢 **For addition and subtraction, a four fact family looks like this:**

If the numbers are **3, 5, and 8**:

- **3 + 5 = 8**
- **5 + 3 = 8**
- **8 – 3 = 5**
- **8 – 5 = 3**

Exercise 3.1.1.8

a) Complete the following addition table in base five. Remember to use the thinking strategies and the idea of four fact families to help.

+	0	1	2	3	4
0					
1					
2					
3					
4					

(base five)

b) Find each difference, the table from part a) may be helpful

i) $13_5 - 4_5$ ii) $11_5 - 3_5$ iii) $10_5 - 2_5$

c) Complete the four fact family in base 5: $3_5 + 4_5 = 12_5$

Answer available at the end of this section.

Problem Solving Problems

1) Suppose that S is a set of whole numbers closed under addition. S contains 3, 27, and 72.

a. List six other elements in S.

b. Why must 24 be in S?

2) A palindrome is any number that reads the same backward and forward. For example, 131 and 47674 are palindromes. In the accompanying example, the process of reversing the digits and adding the two numbers has been repeated until a palindrome is obtained.

$$\begin{array}{r} 19 \\ +\ 91 \\ \hline 110 \\ +\ 11 \\ \hline 121 \end{array}$$

a. Try this method with the following numbers.

i. 39 ii. 87 iii. 32

b. Find a number for which the procedure takes more than three steps to obtain a palindrome.

Answers

🌶🌶**Exercise** 3.1.1.1

Set Theory Model of Addition

This model views addition as **combining two disjoint sets** (sets with no overlapping elements).

Step-by-Step:

Let Set A = {★} → represents 1 object

Let Set B = {▲, ●} → represents 2 objects

These sets are disjoint (no shared elements)

✅ **So:**

1 + 2 = 3 by combining the sets. Count the total elements: **3**

We now combine Set A and Set B: Set A ∪ Set B = {★, ▲, ●}

Measurement Model of Addition

This model views addition as a process of **moving along a number line** or measuring length by placing units end-to-end.

Step-by-Step:

Start at **0** on the number line. Move **1 unit** to the right → now at **1.** From there, move **2 more units** → end at **3**

✅ **So:**

1 + 2 = 3 by measuring a total length of 3 units.

🌶🌶**Exercise** 3.1.1.2

a) Commutative b) Associative: 53 + 47 = 50 + (3 + 47) c) Identity Property

🌶🌶**Exercise** 3.1.1.3

a) Closed - Multiples of 10 always results in a multiple of 10 b) Not Closed, 2+3=5, 5 is not in the set. c) Closed since 0 + 0 is 0. d) Closed since when you add a whole number greater than 10 with another whole number greater than 10, the sum is always greater than 10.

🌶🌶**Exercise** 3.1.1.4

a) 30 + 67 = 97 b) 40 + 20 = 60

🌶🌶**Exercise** 3.1.1.5

a) Set Model, Take-Away Approach, 7 - 3 = 4

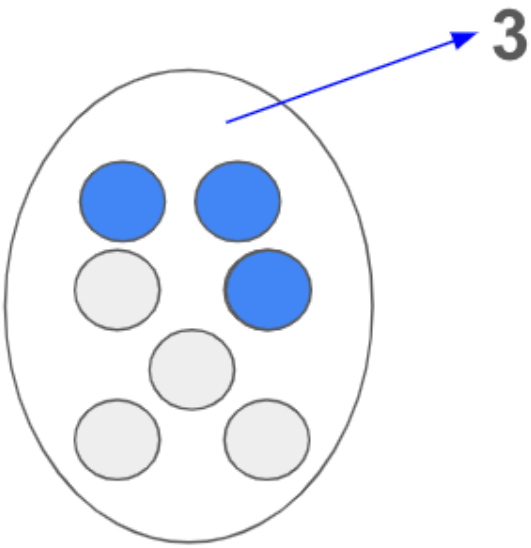
b) Measurement Model, Missing Addend Approach, 30 - 10

Exercise 3.1.1.6

a) Missing Addend Approach: Sofia had 3 apples. She wants to have 7 apples. How many more apples does she need to get?



b) Take-Away Approach: Sofia had 7 apples. She gave 3 apples to her friend. How many apples does she have left?



Exercise 3.1.1.7

a) Missing - Addend Approach: 120 + ? = 200. OR take-away approach: 200 - 120 = ?

b) Missing - Addend Approach: 1240 + ? = 1795 OR take-away approach: 1795 - 1240 = ?

Exercise 3.1.1.8

a)

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	10
2	2	3	4	10	11
3	3	4	10	11	12
4	4	10	11	12	13

b) i) $13_5 - 4_5 = 4_5$ ii) $11_5 - 3_5 = 3_5$ iii) $10_5 - 2_5 = 3_5$

c) $3_5 + 4_5 = 12_5$, $4_5 + 3_5 = 12_5$, $12_5 - 4_5 = 3_5$, $12_5 - 3_5 = 4_5$

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3.1.2: Online Retrieval and Problem Solving Practice

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3.2: Developing Multiplicative and Divisional Thinking

Children develop multiplication and division understanding through informal strategies before mastering formal procedures. Initially, they rely on repeated addition, tally marks, or halving strategies to make sense of multiplication, while division often begins with fair-sharing or repeated subtraction (Feike, Schwingendorf, and Gregg, 2018). As they build number sense, place value understanding, and fluency in addition, they transition from additive to multiplicative thinking—a crucial milestone in mathematical reasoning (NCTM, 2020). Encouraging children to explore and construct their own problem-solving strategies fosters deeper comprehension, rather than rote memorization of algorithms. Through hands-on experiences, mental math strategies, and recognizing mathematical properties like commutativity, associativity, and distributivity, children develop a rich, conceptual understanding of multiplication and division that supports their long-term mathematical success (Feike, Schwingendorf, and Gregg, 2018).

Multiplication

Think About and Discuss

More Math Moments of Brilliance - What is Nathan Thinking?

Task 1: Think about and then discuss with peers how you think a young child could solve the problem 6 x 5.

Task 2: Watch the short video of Nathan calculating 6 x 5.

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View this YouTube Video online

Task 3: WOW! What did he do? Why does it work? Think and About and discuss with peers.

Task 4: Does his process always work? Come up with examples and/or counter examples.

🧠**Let's Listen to Learn**❤️

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We use three models to discuss multiplication: the **repeated addition model**, the **rectangular array model**, and the **Cartesian-product model**. Also, we investigate the properties of multiplication.

Definition: Multiplication of Whole Numbers using the repeated addition approach

If a and b are whole numbers, then a $\times$ b = b + b+ b+ b + ... + b, where there are a addends of b in this sum.

I like to think of using the definition like this:

3×4 means 3 fours added together or $3 \times 4 = 4 + 4 + 4 = 12$

4×3 means 4 threes added together, or $4 \times 3 = 3 + 3 + 3 + 3 = 12$

5×2 means 5 twos added together, or $5 \times 2 = 2 + 2 + 2 + 2 + 2 = 10$

2×5 means 2 fives added together, or $2 \times 5 = 5 + 5 = 10$

Note that although 3×4 gives the same answer as 4×3, it does not *mean* the same thing! One means $4 + 4 + 4$ and the other means $3 + 3 + 3 + 3$.

These problems can be modeled using the **set model** and the **measurement model.** These models look similar to the ones that we used for addition, since we are merely adding repeatedly. Figure 3.2.1

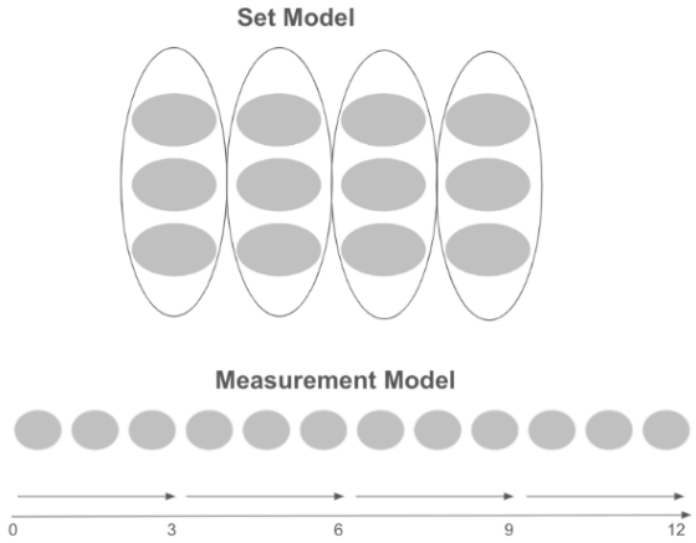


Figure 3.2.1

Before we delve any further into this discussion, I want you to use the repeated addition definition to compute the following:

Example 3.2.1

Use the repeated addition definition of multiplication to compute the following. First, write out the meaning of the multiplication, and then compute the answer.

a. $6 \times 3 =$

b. $2 \times 9 =$

c. $9 \times 2 =$

Answer available at the end of this section.

Now, it may seem obvious to you that 3×4 is the same answer as 4×3. That's because you've no doubt been using the commutative property of addition for years. Whether or not you really thought about why it was true is another thing. But, although 3×4 gives the same answer as 4×3, it does not mean the same thing. One means $4 + 4 + 4$ and the other means $3 + 3 + 3 + 3$. To a child just learning about addition and multiplication, there is no reason for them to conclude that one sum $(4 + 4 + 4)$ happens to give the same answer as the other sum $(3 + 3 + 3 + 3)$. For instance, if I was asked to compute the following two sums, it wouldn't be evident to me at all that the answer to each would be end up being the same:

17 + 17 + 17 + 17 and 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4

Actually, I find it quite amazing that they are equal. But, indeed they are, because the first sum comes from the multiplication problem 4 $\times$ 17 (since there are 4 seventeens added together), and the second sum comes from the multiplication problem 17 $\times$ 4 (since there are 17 fours added together). Later, when we discuss a more geometrical definition for multiplication, it becomes obvious why the commutative property of multiplication is true.

Commutative Property for Multiplication

The commutative property of multiplication for whole numbers states that if a and b are *any* two numbers, then

$$a \times b = b \times a.$$

Remember that the commutative property means there has been an order change. To verify the commutative property of multiplication, you need to use the repeated addition definition of multiplication and add from left to right, one number at a time. The next example shows how to verify the commutative property of multiplication, given a set of two numbers.

Associative Property for Multiplication

The *associative property of multiplication* states that if a, b and c are *any* three numbers, then $(a \times b) \times c = a \times (b \times c)$**.**

Before going on to any more properties, we'll practice verifying that the associative property of multiplication is true for some numbers in base ten later on, we'll do it in other bases, too. The associative property requires three numbers (they don't have to be three different numbers) and there are different expressions on each side of the equation. The difference between the left and right sides is that the parentheses are around a different pair of numbers. Remember that the associative property means there is a parentheses change, whereas the commutative property means there has been an order change. To verify, you need to use the order of operations when simplifying each side. If each expression simplifies to same thing, the equation is true. In the order of operations, you always simplify what is in parentheses first. The following are some examples of how to verify the associative property of multiplication, given a set of three numbers.

Example 3.2.2

Verify the associative property of multiplication is true using the three numbers {4, 5, 7} in your example.

Answer available at the end of this section.

Distributive Property for Multiplication

The distributive property of multiplication over addition for whole numbers states that if a, b and c are *any* three numbers, then $a \times (b + c) = (a \times b) + (a \times c)$.

Note: $a \times (b - c) = (a \times b) - (a \times c)$, when $b \geq c$.

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The distributive property makes more complex multiplications more manageable by leveraging simpler, known multiplication facts. That visual breakdown **bridges symbolic understanding** and **physical representation**, helping the math "stick." You can see why the property makes sense.

Now, it's time to practice verifying the distributive property of multiplication over addition is true for some numbers in base ten — later on, we'll do it in other bases, too. The distributive property requires three numbers (they don't have to be three different numbers) and there are two sides of an equation to verify. Note the difference between the left and right sides. To verify, you need to use the order of operations when simplifying each side. In the order of operations, you always simplify what is in parentheses first. If each expression simplifies to same thing, the equation is true. The following are some examples of how to verify the distributive property of multiplication, given a set of three numbers.

Example 3.2.3

Verify the distributive property of multiplication is true using the three numbers {4, 5, 7} in your example.

Answer available at the end of this section.

Wonder, Play, Grow

Compute the following products using thinking strategies.

a) 2 x 34 b) 5(37 x 2) c) 7(25)

Identity Property for Multiplication

The number 1 is the unique whole number such that for every whole number a,

$a \times 1 = (1 \times a) = a$.

Closure Property for Multiplication for Whole Numbers

The product of two whole numbers is a whole number.

When two odd whole numbers are multiplied together, the product is odd; thus the set of odd numbers is closed under multiplication. Closure is a useful idea, since if we are multiplying two (or more) odd numbers and the product we calculate is even,we can conclude that our product is incorrect. The set {2, 5, 8,11,14, . . . } is not closed under multiplication, since 2 ¥ 5 = 10 and 10 is not in the set.

Multiplication Property for Zero

For every whole number, a,

a x 0 = 0 x a = 0.

There is another way to define multiplication using Set Theory. In order to do this, we have to remember how to take the **Cartesian product** of two sets, A and B.

Defintiion: Multiplication of Whole Numbers - Cartesian Product

The **Cartesian product** of set A with set B, which is written $A \times B$ and is read as "A cross B" is the set of *all possible* ordered pairs (a,b), where a ∈ A and b ∈ B.

If you are finding the Cartesian product, the answer is a set that contains ordered pairs. Using set builder notation, we can write ⍰×⍰={(a,b) | a∈⍰ and b∈⍰}.

Example 3.2.4

a) If A = {x, y, z} and B = {a, b}, find $A \times B$.

b) If E = {1, 2} and F = {2, 3}, find $E \times F$.

Answer available at the end of this section.

Notice that when finding a Cartesian product, all possible pairs are formed where the first element comes from the first and the second element comes from the second set. Also observe that in Example 3.2.4 (a), there are three elements in the first set, two in the second, and six in their Cartesian product, and that 3 x 2 = 6. Similarly, in (b), these sets can be used to find the whole-number product 2 x 2 = 4.

Note

For completion, it is good to know the Set Theory Definition for Multiplying. **For any two sets A and B,** $\mathrm{n(A) \cdot n(B) = n(A \times B)}$**.** *Note: There are no restrictions on what elements you choose for sets A and B. They can be disjoint or they can have elements in common. It's up to you.*

Here's a more geometrical approach to defining multiplication of two whole numbers.

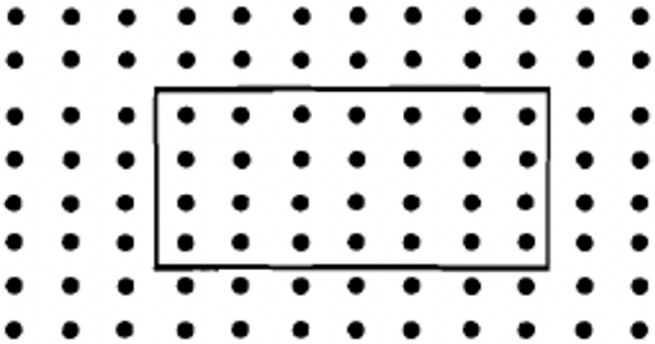
Definition: Rectangular Array Model

To find the product of any two whole numbers, m and n, make a rectangular array of objects having m rows and n columns. The product of m and n, written $m \times n$, equals the number of objects in the array.

The cool thing about using this geometrical approach is that *any* objects can be used, and one only needs to be able to count to get the answer. You don't need to use addition in order to multiply! I personally advocate having lots of graph paper around for anyone just learning to multiply. To compute 4×7, you or the learner can mark off a rectangle on the graph paper that has 4 rows and 7 columns of boxes. The number of boxes in the rectangle is the answer to the problem. This is a useful tool for first learning a particular multiplication fact, or for anyone (adults, too) who forgot a fact and needs to figure it out again. The "objects" used on a graph paper are the individual boxes on the graph paper.

Here is how six different students might geometrically show how to multiply 4×7.

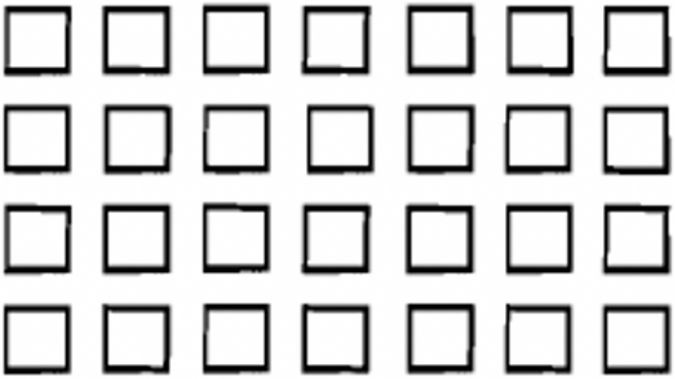
This student took some dotted paper and marked off a rectangular array containing 4 rows and 7 columns of dots.



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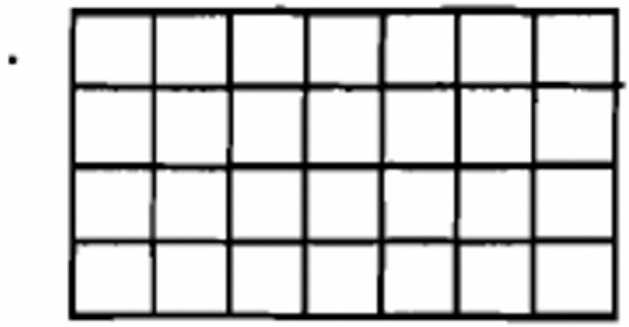


This student arranged four rows and seven columns of squares into a rectangular array.



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sign

This student drew a rectangular array having four rows and seven columns.



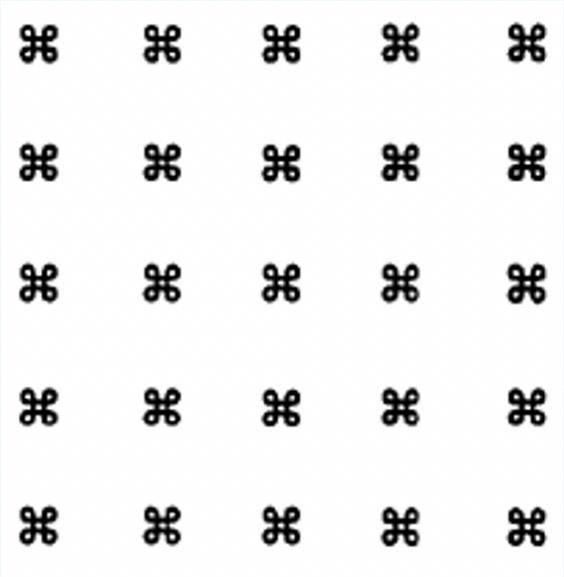
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There is no point to making anyone stress out if they forgot, didn't memorize or couldn't recall the answer to 4×7. The point is to *KNOW WHAT MULTIPLICATION MEANS* so you **can** figure a problem out again if you have to. Of course, there is merit to using flashcards and eventually memorizing basic multiplication facts like your multiplication tables. If not, it takes longer to do problems, you aren't able to focus on more advanced issues, and you might not trust your own mind. You shouldn't have to think about multiplication every time you have to multiply two numbers together, but you should *be able* to think about it. Knowing ***how*** to get the answer is better than having memorized a bunch of facts if you don't know where they came from, what multiplication really means, or how to figure it out again in case you forget the answer to some fact. And, as far as I'm concerned, nothing's wrong with using your fingers, either! Let people do what is comfortable for them, if they understand and can get the right answer, that's GREAT!

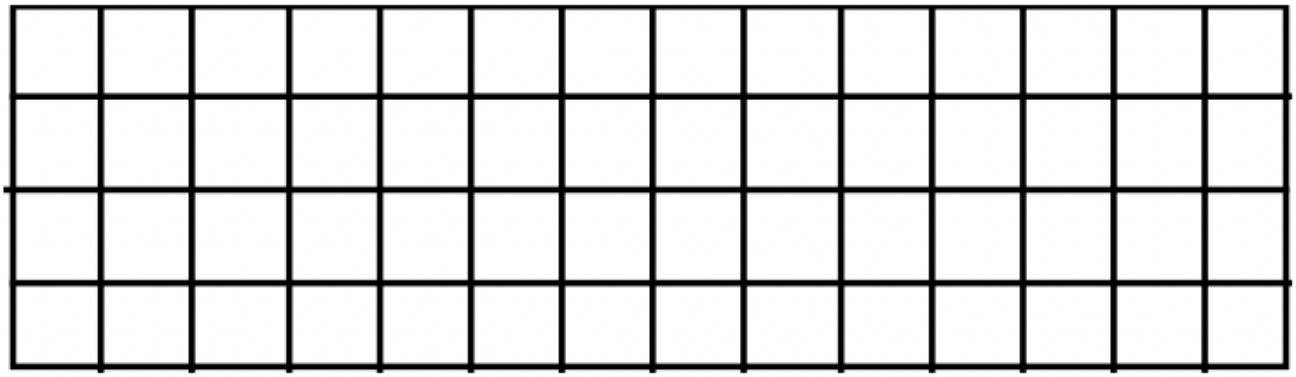
Example 3.2.5

Write the multiplication problem represented by each geometrical representation:

a. ___



c. ___



Answer available at the end of this section.

Wonder, Play, Grow

Task 1: Without using paper and the traditional way to multiply, find 18 x 5. (the answer is 90).

Task 2: View the video (maybe twice).

YouTube Video

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View this YouTube Video online

Task 3: Write down at least two ways you heard students solve the problem.Task 4: Make up a multiplication problem and solve it using thinking strategies. Share with a peer.

Division

Young children, especially those who do not yet have fluency with division facts or an understanding of multiplication as the inverse of division, solve division problems using two distinct strategies: **Partitioning Model (Fair Sharing) and Measurement Model (Repeated Subtraction).** These two approaches correspond to the two major conceptual models of division. Consider $18 \div 3 = 6$.

Partitioning (Fair Sharing Model)

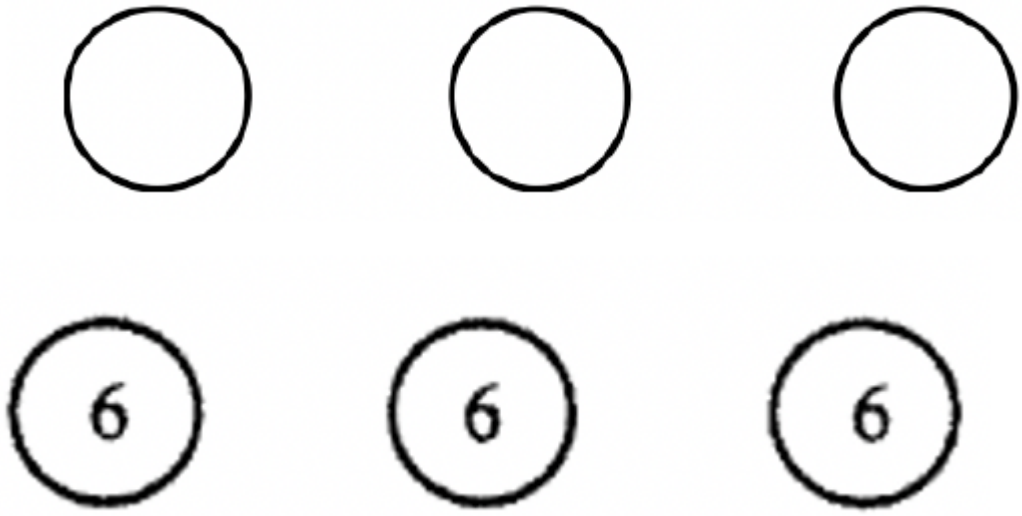
Partitioning division involves dividing a total quantity into a specified number of equal groups and determining how many objects are in each group. This model is often more intuitive for young children since it aligns with real-life experiences of sharing equally.

Fair Sharing Example

Problem: "If you have 18 cookies and want to divide the cookies up equally amongst your three children, how many cookies should you give to each child?"

How a Child Might Solve It:

1. Draw 3 circles to represent the 3 children.
2. Distribute one tally mark at a time into each circle until all 18 tally marks have been used.
3. Count the number of tally marks in each group.
4. Answer: **Each child has 6 cookies.**



Key Characteristics of Partitioning Division:

- The number of groups is given.
- The number of elements in each group is unknown (this is what needs to be found).

Measuremen

Repeated subtracti
This model is some
repeated units.

Measurement Divi

Problem: "If you ha
baggies will you be

How a Child Might

1. Make groups of
2. Continue until a
3. Count the numb
4. Answer: **There**

Alternatively, a child

- 18 - 3 = 15, 18
- → **Count the r**



Key Characteristic

- The size of eac
- The number of

Partitioning into (Equal) Subsets (Fair Share Model)

In this interpretation, *the divisor is the number of subsets in which to disburse something*. The dividend, 18, must be divided amongst 3 equal subsets (3 is the divisor). The quotient (answer to the division problem) will be the number of elements (6) placed in each subset (the kids, in this case.) The task is to **partition 18 cookies into 3 equal subsets**.

Repeated Subtraction (Measurement Model)

In this interpretation, *the divisor is the number of elements that will go into each subset*. Each time the divisor (3) is subtracted from the dividend (18), a subset is formed. After all elements have been disbursed, the quotient (answer) is found by counting how many subsets were formed.

Word Problems requiring division generally fall into the category of Repeated Subtraction or Partitioning into Subsets. After reading a word problem, you should be able to analyze and interpret it as a problem that requires **Partitioning into Subsets** or

Measurement / Repeated Subtraction to solve the problem, and then be able to draw a picture using that interpretation. Also, for any division problem given, you should be able to write two distinct word problems, one for each interpretation, and draw a picture for each interpretation, and be able to explain and show all the steps necessary to solve the problem.

🧠**Let's Listen to Learn**❤️

Audio

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View this Audio online

Think About and Discuss

Understanding these two fundamental models of division, future teachers can help young learners develop a solid mathematical foundation and flexible problem-solving skills.

1) Work with your peers to identify whether the following problems are using measurement (MD) or partitive division (PD) (if in doubt, try drawing a picture to represent the situation.

a) Jim tied 30 sticks into 3 equal bundles. How many sticks were in each bundle?

b) 24 balls are packed into boxes of 6. How many boxes are there?

c) Mr. Lin tied 195 books into bundles of 5 each. How many bundles were there?

d) 6 children shared 84 balloons equally. How many balloons did each child get?

2) Work with peers to make up a word problem for the following using the procedure: (Be prepared to share with the class)

a) measurement division for 84 ÷ 21.

b) partitive division for 90 ÷ 5.

Division is the operation that undoes multiplication. Students can rely on this fact to learn basic division facts. For instance, since 6 · 7 = 42, then $42 \div 6 = 7$ and $42 \div 7 = 6$. ***Below is a formal definition of division.***

Definition: Missing-Factor Approach to Division

If a and b are any whole numbers, **a ÷ b is defined to be the unique number, c, under the following conditions:** $a \div b = c$ **if and only if a = b · c.** If there is no number, c, that exists, or if there is more than one unique number that can be put in for c, then $a \div b$ is said to be ***undefined***.

The symbol **a ÷ b** is read “a divided by b.” Also, a is called the dividend, b is called the divisor, and c is called the quotient or missing factor.

Example 3.2.6

Use the missing-factor approach of division to find the answer to $\mathbf{21 \div 3}$**.**

First, note that $21 \div 3$ = ___ means 21 = 3 · ____ Second, try to fill in the blank in the multiplication problem with a unique number to make the equation, 21 = 3 · ____, true: 21 = 3 ·$\underline{7}$. If the number filled in is the only possible solution (unique), then that is the answer to the division problem. Therefore, $21 \div 3 = \underline{7}$. *You do not need to write this paragraph for your solution. Below is how to write the solution.*

Answer available at the end of this section.

Example 3.2.7

Use the missing-factor approach of division to find the following quotients:

a. $0 \div 13$
b. $12 \div 1$
c. $X \div 1$ (Assume X is any number.)
d. $0 \div Y$ (Assume Y is any nonzero number.)

Answer available at the end of this section.

Division Properties

Example 7c illustrates that any number divided by 1 is itself. In other words, $\mathrm{a} \div 1 = \mathrm{a}$ **for all values of a**.

Example 7d illustrates that zero divided by any *nonzero* whole number is 0. In other words, $0 \div \mathrm{a} = 0$**, for all nonzero values of a**.

Let's explore what happens when we divided by zero.

Think About and Discuss

Discuss with your peers how to use the missing-factor approach to figure out the following quotients $8 \div 0$ and $0 \div 0$.

Once you have a chance to give and take ideas, take a look at an explanation.

Explanation available at the end of this section.

Division by Zero

Division by 0 is undefined

To prove that division by zero is undefined, two cases must be shown. Below is a complete proof to show that division by zero is undefined.

Problem: Prove that division by zero is undefined

Solution: The goal is to show that $a \div 0$ is undefined for all values of a. There are two cases to consider Case 1: $a \neq 0$; Case 2: a = 0.

Case 1: Let $a \neq 0$. There is no unique number that could be put in the blank to make the equation, a = 0 · ___, true because the left-hand side of the equation is not zero (since $a \neq 0$), and any number put in the blank will make the right hand side of the equation zero. Therefore, when $a \neq 0$, $a \div 0$ is not defined.

Case 2: Let a = 0. There is no unique number that can be put in the blank to make the equation 0 = 0 · ___ true, since any number will make it true. Therefore, $0 \div 0$ is indeterminate.

It takes practice writing this proof. You'll need to practice writing it several times to master the problem. One option is to memorize the proof shown above by writing it out a few times until you really get the idea. This would make a good exam question.

Definition: Division Algorithm

If a and b are any whole numbers with b ≠ 0, then there exist unique whole numbers q and r such that a = bq + r, where 0 ≤ r < b.

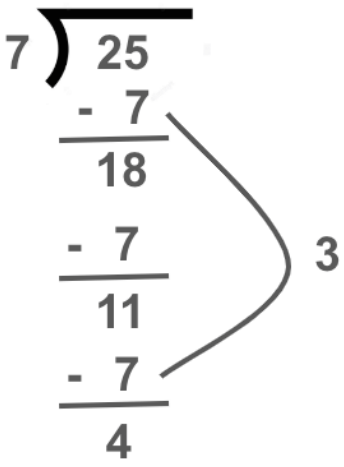
Here b is called the divisor, q is called the quotient, and r is the remainder. Notice that the remainder is always less than the divisor. Also, when the remainder is 0, this result coincides with the usual dei nition of whole-number division.

Example 3.2.8

Find the quotient and remainder for 25 ÷ 7.

Answer available at the end of this section.

The example above can also be represented using repeated subtraction. In general, to find a ÷ b using the repeated-subtraction approach, subtract b successively from a and from the resulting differences until a remainder r is reached, where r < b. The number of times b is subtracted is the quotient q. Figure 3.2.2.



Symbolic Representation Repeated Subtraction

Figure 3.2.2

Figure 3.2.3 visually illustrates the key relationships among the four basic whole-number operations. It shows how multiplication builds on repeated addition, subtraction is understood through the missing-addend approach, and other operations connect similarly. A crucial takeaway from this diagram is that a strong grasp of addition forms the foundation for success in subtraction, multiplication, and division.

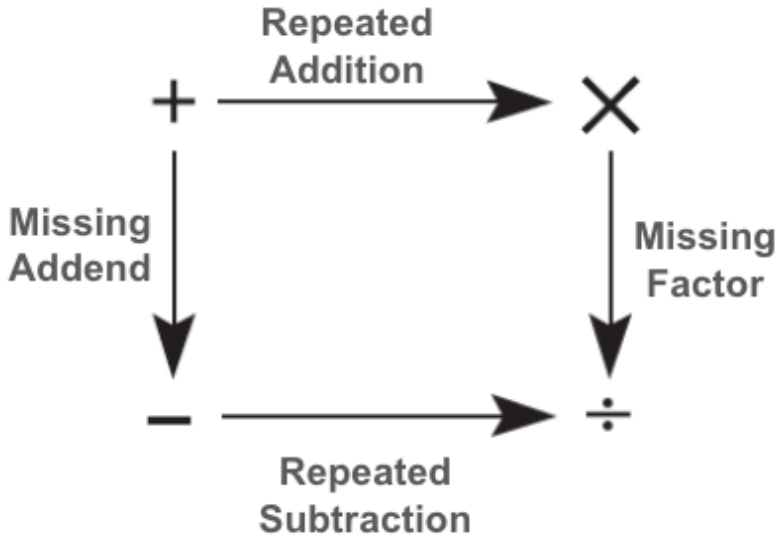


Figure 3.2.3

Future Teacher Thinking: Understanding the Progression of Multiplication and Division

Mathematics instruction in multiplication and division is most effective when it unfolds over time, allowing students to move through a developmental progression. The YouTube videos by Graham Fletcher on the "Progression of Multiplication" and "Progression of Division" demonstrate how students benefit from beginning with concrete representations, then moving to pictorial models, and eventually to abstract notation. This is known as the Concrete-Pictorial-Abstract (CPA) approach.

Take a look at the videos and discuss with your peers about what is still circling in your head, what really resonated with you, and what is one thing you do not want to forget.

YouTube Video *This interactive element is not available in this format. Visit the link or scan the QR code to access it online.* View this YouTube Video online	YouTube Video *This interactive element is not available in this format. Visit the link or scan the QR code to access it online.* View this YouTube Video online
Progression of Multiplication	**Progression of Division**

Key Concepts:

- Concrete to Abstract Learning (CPA): Students begin with hands-on experiences (like manipulatives and drawings), which ground their understanding before moving into symbolic representations and algorithms.
- Models in Division: Two key conceptual models are used—measurement (repeated subtraction) and fair share. These models help students understand the meaning of division in different contexts.

- Multiplication and Division Representations: Arrays, area models, and base-10 blocks help bridge the gap between understanding operations and executing them efficiently.
- From Inefficient to Efficient: While drawing objects or subtracting repeatedly may seem slow, these methods build the foundation for flexible strategies like partial products and partial quotients.
- Inverse Relationship: The two operations are closely connected. Arrays and area models visually demonstrate how multiplication and division are inverse operations.
- Distributive Property: This property becomes clearer to students when represented with area models or base-10 blocks, making sense of abstract equations.
- Standard Algorithms: Rather than being introduced first, standard algorithms are shown later—once students understand why the steps work.
- Teacher's Role: Educators are encouraged to let the context drive the mathematics, enabling students to make sense of problems and explore solutions before being taught procedures

Answers

Example 3.2.1

a) 3 + 3 + 3 + 3 + 3 + 3 = 18

b) 9 + 9 = 18

c) 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 = 18

Example 3.2.2

Note: Since the associative property states $(a \times b) \times c = a \times (b \times c)$, one of the numbers given is put in for a, the other for b and the other for c. The choice is yours. For example, we can show any of the following six equations are true:

a. $(4 \times 5) \times 7 = 4 \times (5 \times 7)$ or d. $(4 \times 7) \times 5 = 4 \times (7 \times 5)$
b. $(5 \times 4) \times 7 = 5 \times (4 \times 7)$ or e. $(7 \times 4) \times 5 = 7 \times (4 \times 5)$
c. $(7 \times 5) \times 4 = 7 \times (5 \times 4)$ or f. $(5 \times 7) \times 4 = 5 \times (7 \times 4)$.

Example 3.2.3

Note: Since the distributive property states $a \times (b + c) = (a \times b) + (a \times c)$, one of the numbers given is put in for a, the other for b and the other for c. The choice is yours. For example, we can show any of the following six equations are true:

a. $4 \times (5 + 7) = (4 \times 5) + (4 \times 7)$ or d. $5 \times (7 + 4) = (5 \times 7) + (5 \times 4)$
b. $4 \times (7 + 5) = (4 \times 7) + (4 \times 5)$ or e. $7 \times (4 + 5) = (7 \times 4) + (7 \times 5)$
c. $5 \times (4 + 7) = (5 \times 4) + (5 \times 7)$ or f. $7 \times (5 + 4) = (7 \times 5) + (7 \times 4)$

Example 3.2.4

a) $A \times B$ = {(x, a), (x, b), (y, a), (y, b), (z, a), (z, b)}

b) $E \times F$ = {(1, 2), (1, 3), (2, 2), (2, 3)}

Example 3.2.5

a) 5 x 5 (number of objects) b) 3 x 10 (number of objects) c) 4 x 14 (number of squares) d) 3 x 7 (number of squares)

Example 3.2.6

Since $21 = 3 \cdot \underline{7}$, then $21 \div 3 = \underline{7}$.

Example 3.2.7

a) $0 \div 13$ = 0 since 0 = 13 x 0

b) $12 \div 1$ = 12 since 12 = 12 x 1

c) $X \div 1$ = X since X = X x 1

d) $0 \div Y$ = 0 since 0 = Y x 0

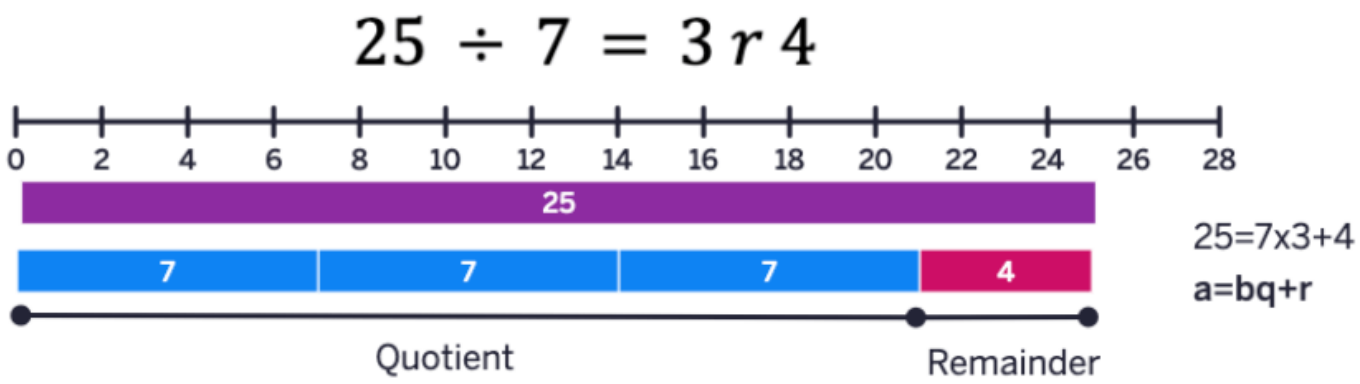
Think About and Discuss

There is no number that will make the equation, 8 = 0 · ___ true, since any number put in the blank will make the right hand side of the equation zero, which can never equal the left-hand side of the equation, which is 8. Therefore, $8 \div 0$ is not defined.

There is no unique number that can be put in the blank to make the equation 0 = 0 · ___ true, since any number will make it true. Therefore, $0 \div 0$ is called indeterminate.

Example 3.2.8

7 x 3 = 21, so 25 = 7 x 3 + 4, The quotient is 3 and the remainder is 4.



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3.2.1: Written Retrieval and Problem Solving Practice

Student Learning Objectives

SLO 3.1 Students will demonstrate a comprehensive understanding of whole number operations, including addition, subtraction, multiplication, and division, by modeling each operation with set and measurement models and analyzing their properties.

SLO 3.2 Students will apply strategies for mental computation in addition, subtraction, and multiplication across different bases and demonstrate mastery of basic facts using commutativity, associativity, and distributive properties.

The mild assignment 🌶 is a shorter and less challenging assignment. The medium assignment🌶🌶 is right in the middle in terms of length and level of difficulty. Finally, the spicy assignment 🌶🌶🌶 is a longer assignment that is the most challenging.

(Spicy 🌶🌶🌶 Problems are OPTIONAL, but go for it when you have the time to challenge yourself)

Exercise 3.2.1.1 🌶

What multiplication problems and which models are suggested by the following diagrams? Then determine which approach is being modeled.

i) Cartesian product approach

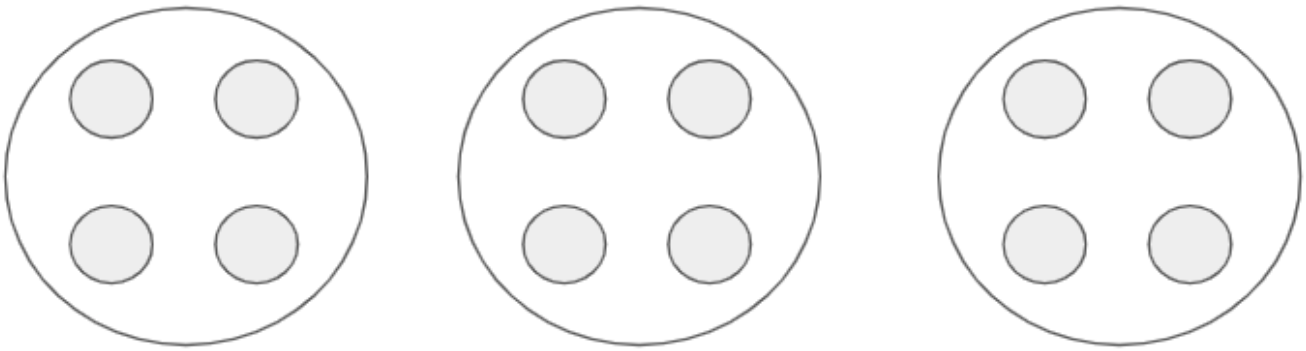
ii) Set model; rectangular array approach

iii) Set model; repeated-addition approach

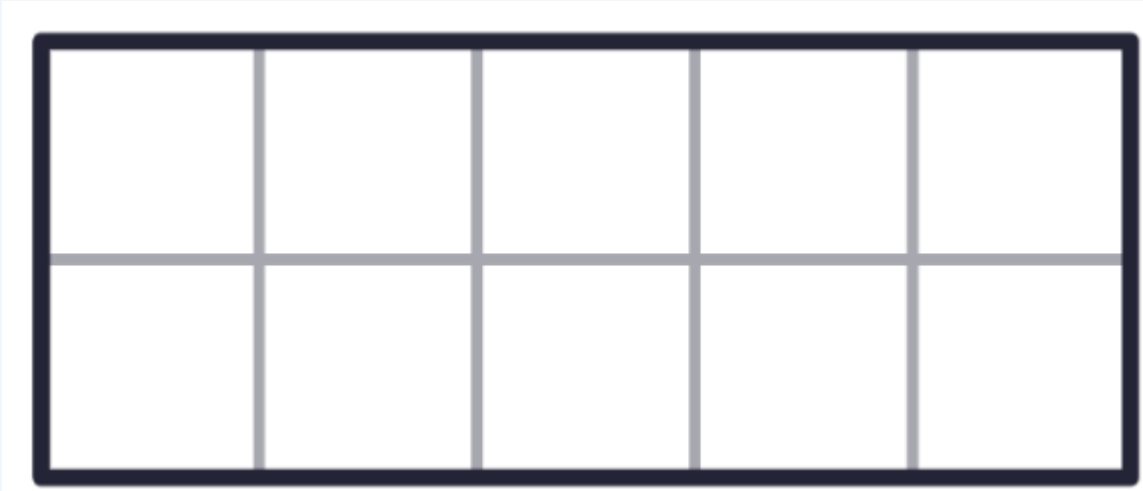
iv) Measurement model; rectangular array approach

v) Measurement model; repeated-addition approach

a)

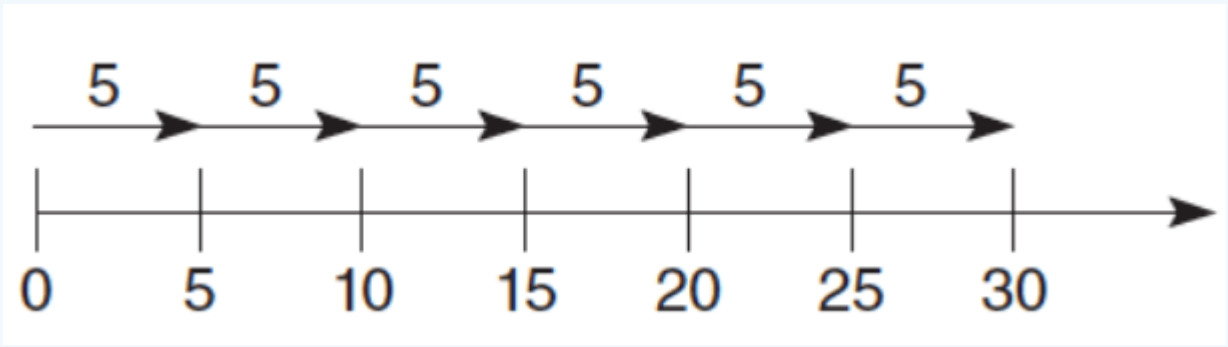


c)



d

e)



Answer available at the end of this section.

Exercise 3.2.1.2

Illustrate 4 x 5 using each of the following models.

(a) Measurement model; rectangular array approach

(b) Set model; repeated-addition approach

Answer available at the end of this section.

Exercise 3.2.1.3

Which of the following sets are closed under multiplication? Why or why not?

a) (2, 4}

b) {0, 2, 4, 6, 8, ...}

c) { 3, 5, 7, 9, 11, ...}

d) $\{2^1, 2^2, 2^3, 2^4, ...\}$

Answer available at the end of this section.

Exercise 3.2.1.4

What properties of whole number multiplication justify these equations?

(a) 4 x 9 = 9 x 4

(b) 4 x (6 + 2) = 4 x 6 + 4 x 2

(c) 0 x 12 = 0

(d) 5 x (9 x 11) = (5 x 9) x 11

(e) 7 x 3 + 7 x 8 = 7 x (3 + 8).

Answer available at the end of this section.

Exercise 3.2.1.5

Rewrite each of the following expressions using the distributive property for multiplication over addition or subtraction. Your answer should contain no parentheses.

(a) 4 (60 + 37)

(b) 3 x (29 + 30 + 6)

(c) a(7 – b + z).

Answer available at the end of this section.

Exercise 3.2.1.6

The distributive property of multiplication over addition can be used to perform some calculations mentally. For example, to find 13 x 12, you can think

13 x 12 = 13 x (10 + 2) = 13 x 10 + 13 x 2 = 130 + 26 = 156

How could each of the following products be rewritten using the distributive property so that it is more easily computed mentally?

a) 49(11) b) 35(101) c) 23(21) d) 35(99)

Answer available at the end of this section.

Exercise 3.2.1.7

a) Write a sharing division problem that would have the equation 18 ÷ 6 as part of its solution.

b) Write a measurement division problem that would have the equation 28 ÷ 4 as part of its solution.

Answer available at the end of this section.

Exercise 3.2.1.8

Rewrite each of the following division problems as a multiply problem.

a) 50 ÷ 5 = 10

b) 100 ÷ x = 25

c) x ÷ 6 = 20

Answer available at the end of this section.

Exercise 3.2.1.9

Find the quotient and reminder for the following. [Try repeated subtraction]

a) 43 ÷ 8

b) 62 ÷ 15

c) 8 ÷ 12

Answer available at the end of this section.

Exercise 3.2.1.10

Which of the following properties hold for division of whole numbers?

a. Closure

b. Commutativity

c. Associativity

d. Identity

Answer available at the end of this section.

Exercise 3.2.1.11

Complete the following multiplication table in base five.

×	0	1	2	3	4
0					
1					
2					
3					
4					

Answer available at the end of this section.

Problem Solving Practice

Problem Solving 3.2.1.1

How many possible remainders (including zero) are there when dividing by the following numbers? How many possible quotients are there?

a. 2 b. 12 c. 23

Answer available at the end of this section.

Wonder, SHARE, Grow

Task 1: View each of the videos from Math Educator Howie Hua.

A great multiplication strategy to have: doubling and halving

YouTube Video

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View this YouTube Video online

https://youtu.be/QkrVbTlMbNU?si=oqiozv-3svgSlZ98

A quick way to multiply by 11

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View this YouTube Video online

https://youtu.be/W7YLbYGGTv4?si=IIzSQO9eioiy0WEN

A neat way of multiplying by 5

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View this YouTube Video online

https://youtu.be/JGiCFRI_PZE?si=E7leFSNg1SDxpqw9

How can rectangles help us with multiplication?

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View this YouTube Video online

https://youtu.be/IHxIIna0PmQ?si=pO9XEibcqDgbP0ga

Task 2: Pick one of the methods from the videos and teach someone in your life how to do it.

Task 3: Be prepared to share in class what happened and how the person reacted.

Exploring Cartesian Products

Interactive Content

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View this Interactive Content online

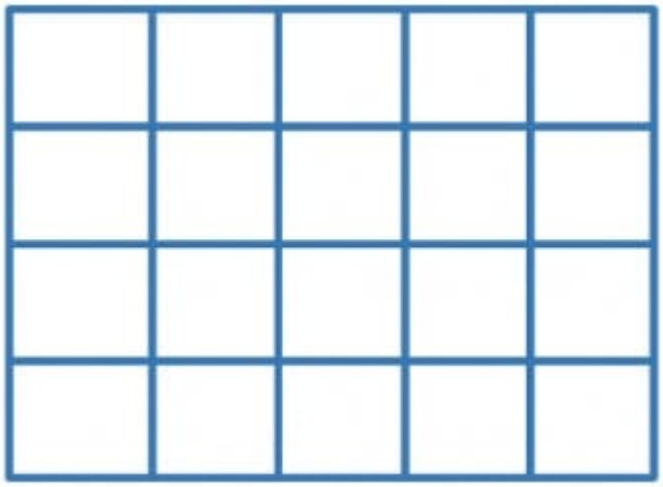
Answers

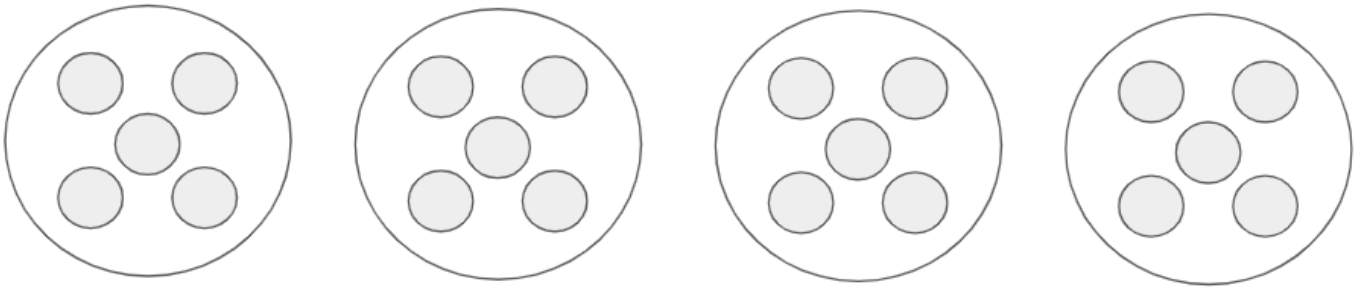
Exercise 3.2.1.1

a) 3 x 4, Set model; repeated-addition approach

b) {a, b} x {1, 2, 3} OR 2 x 3 = 6, Cartesian product approach
c) 2 x 5, Measurement model; rectangular array approach
d) 3 x 2, Set model; rectangular array approach
e) 6 x 5, Measurement model; repeated-addition approach

Exercise 3.2.1.2

a) 

b) 

Exercise 3.2.1.3

a) NO, a counter example, 2x4 = 8
b) YES, the product of any two even numbers is always even.
c) YES, the product of any two odd numbers is always odd.
d) YES, the product of any powers of 2 is always a power of 2

Exercise 3.2.1.4

a) commutative b) distributive c) multiplication property of zero d) associative e) distributive

Exercise 3.2.1.5

a) 4(60) + 4(37)
b) 3(29) + 3(30) + 3(6)
c) 7a - ab + az

Exercise 3.2.1.6

a) 49 (10 + 1) = 490 + 49 = 539
b) 35 (100 + 1) = 3500 + 35 = 3535
c) 23 (20 + 1) = 460 + 23 = 483
d) 35 (100 - 1) = 3500 - 35 = 3465

Exercise 3.2.1.7

a) "A teacher has 18 markers and wants to divide them equally among 6 students. How many markers does each student get?"
b) "A baker has 28 cookies and wants to pack them into boxes, with 4 cookies in each box. How many boxes can the baker fill?"

Exercise 3.2.1.8

a) 50 = (5)(10) b) 100 = (x)(25) c) x = (6)(20)

Exercise 3.2.1.9

a) q=5 r = 3 b) q=4 r = 2 c) q=0 r = 8

Exercise 3.2.1.10

a) Division is NOT Closed
b) Division is NOT commutative.

c) Division is NOT associative.

d) **Division has an identity element, which is 1.**

🌶🌶🌶**Exercise** 3.2.1.11

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	11	13
3	0	3	11	14	22
4	0	4	13	22	31

🌶🌶🌶**Problem Solving** 3.2.1.1

a) Dividing by 2, Possible remainders: 0,1 → 2 possible remainders Possible quotients: Any nonnegative integer → Infinite possible quotients

b) Dividing by 12 Possible remainders: 0,1,2,...,11 → 12 possible remainders Possible quotients: Any nonnegative integer → Infinite possible quotients

c) Dividing by 23 Possible remainders: 0,1,2,...,22→ 23 possible remainders Possible quotients: Any nonnegative integer → Infinite possible quotients

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3.2.2: Online Retrieval and Problem Solving Practice

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3.3: Ordering Whole Numbers and Exponents

Ordering and Inequality

Before introducing the formal definition of "less than" for whole numbers, students should develop an intuitive understanding of ordering through familiar experiences with counting and comparing sets. The counting chant provides a natural foundation—children recognize that "two" comes before "five," establishing an inherent sense of numerical order. Kindergartners engage with ordering by comparing groups of objects, using one-to-one matching to determine which set has more or fewer items. Visual tools such as number lines further reinforce the relative size of numbers. Additionally, set comparisons help illustrate that if one set is a proper subset of another, its corresponding number is smaller. These early experiences with counting, matching, and set relationships create a strong conceptual base before introducing formal symbolic definitions of comparison (CDE, 2013; Finnan, 2006).

🧠**Let's Listen to Learn**❤️

Audio

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Definition: "Greater Than" for Whole Numbers

The whole number m *is greater than* the whole number n if there is a whole number k such that m = n + k.

In mathematics, the symbol used to denote "is greater than" is: " > ". Therefore, " 7 > 5 " is read "Seven is greater than five".

Definition: "Less Than" for Whole Numbers

The whole number r *is less than* the whole number s if there is a whole number k such that r + k = s.

In mathematics, the symbol used to denote "is less than" is: " < ". Therefore, " 4 < 9 " is read "Four is less than nine".

Example 3.3.1

Verify each of the following (I've done part a for you):

a. 9 > 3 because $9 = 3 + 6$	b. 30 > 22 because ___
c. 156 > 96 because ___	d. 19 < 30 because ___

Answer available at the end of this section.

Think About and Discuss

Task 1: Think about how the definitions for > and < can be restated in the most simplest ways so that you are able to explain what they mean to someone else.

Task 2: Now consider the following: "Less than" and "Greater than" can be used to order numbers on a number line. Fill in each blank with *right* or *left*.

a. If m < n, then m is to the _____ of n on the number line
b. If m > n, then m is to the _____ of n on the number line.

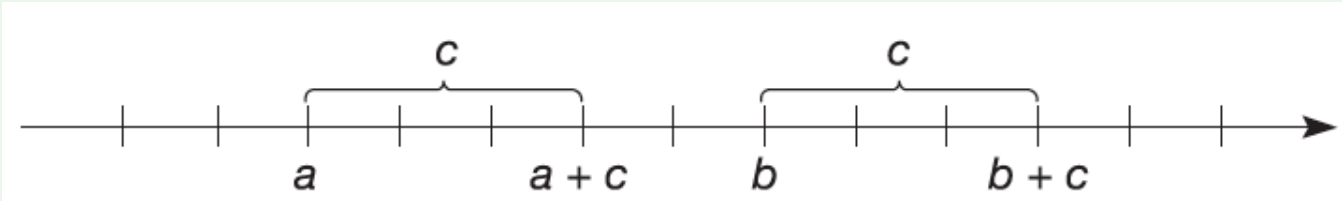
Task 3: How is "Chant Counting" related to what you discussed and learned in Task 2?

Task 4: Discuss with your peers if this statement is true or false. "*If a and b are whole numbers, then exactly one of the following is true: a = b or a < b or a > b.*"

Properties

Transitive Property of "Less Than" for Whole Numbers: For all whole numbers a, b, and c, if a < b and b < c, then a < c.

Less Than and Addition Property for Whole Numbers: If a < b, then a + c < b + c.



Less Than and Multiplication Property for Whole Numbers: If a < b and c ≠ 0, then ac < bc.

Note: This property will have to be modified when we include negative numbers.

Exponential Notation

Just as multiplication can be defined as repeated addition and division may be viewed as repeated subtraction, the concept of exponent can be used to simplify situations involving repeated multiplication.

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Factors

In multiplication, the numbers being multiplied together are called **factors**. In repeated multiplication, all the factors are the same. In nonrepeated multiplication, none of the factors are the same.

Example 3.3.2

$18 \cdot 18 \cdot 18 \cdot 18$ Repeated Multiplication of 18. All four factors, 18 are the same.

$x \cdot x \cdot x \cdot x \cdot x$ Repeated multiplication of x. All five factors, x, are the same.

$3 \cdot 7 \cdot a$ Nonrepeated multiplication. None of the factors are the same.

Exponential Notation

Exponential Notation If a is any whole number and n is a whole number n ≠ 0, then

$$a^n = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{n \text{ factors of } a}$$

An exponent records the number of identical factors in multiplication.

Note that the definition for exponential notation only has meaning for whole number exponents. We will extend this notation to include other numbers as exponents later.

The number "n" is called the exponent or power of "a", and "a" is called the base. The number a^n is read "a to the nth power" or "a to the power of n".

The symbol x^2 is often read as "x squared," and x^3 is often read as "x cubed." A natural question is "Why are geometric terms appearing in the exponent expression?" The answer for x^3 is this: x^3 means $x \cdot x \cdot x$. In geometry, the volume of a rectangular box is found by multiplying the length by the width by the depth. A cube has the same length on each side. If we represent this length by the letter x then the volume of the cube is $x \cdot x \cdot x$, which, of course, is described by x^3. (Can you think of why x^2 is read as x squared?)

Example 3.3.3

Take a look at these completed problems - ***what questions do you have?***

a) $7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 = 7^6$

The repeated factor is 7. The exponent 6 records the fact that 7 appears 6 times in the multiplication.

b) $(2y)(2y)(2y) = (2y)^3$

The repeated factor is $2y$. The exponent 3 records the fact that the factor $2y$ appears 3 times in the multiplication.

c) $2yyy = 2y^3$

The repeated factor is y. The exponent 3 records the fact that the factor y appears 3 times in the multiplication.

d)$(a+b)(a+b)(a-b)(a-b)(a-b) = (a+b)^2(a-b)^3$

The repeated factors are $(a+b)$ and $(a-b)$, $(a+b)$ appearing 2 times and $(a-b)$ appearing 3 times.Definition of Exponents

Below are a set of rules that students often just memorize without understanding. The rules of exponents for whole numbers are not just a set of procedures to be memorized, but rather a logical extension of the definition of multiplication that allows for efficient representation and manipulation of repeated multiplication. Understanding these rules is part of making sense of

mathematical patterns and connections, and the notation and rules themselves contribute to the understanding of mathematics as a precise and powerful language for expressing mathematical ideas and relationships.

The Product Rule for Exponents

The first rule we wish to develop is the rule for multiplying two exponential quantities having the **same base** and whole number exponents.

Product Rule for Exponents

If a is a whole number and n and m are whole numbers not equal to 0, then

$a^n a^m = a^{n+m}$

To multiply two exponential quantities having the same base, add the exponents. Keep in mind that the exponential quantities being multiplied **must** have the **same base** for this rule to apply.

Proof

$$a^m \cdot a^n = \underbrace{(a \cdot a \cdots a)}_{m \text{ factors}} \cdot \underbrace{(a \cdot a \cdots a)}_{n \text{ factors}} = \underbrace{a \cdot a \cdots a}_{m+n \text{ factors}} = a^{m+n}$$

The Quotient Rule for Exponents

The second rule we wish to develop is the rule for dividing two exponential quantities having the same base and whole number exponents.

Quotient Rule for Exponents

If a is a whole number and n and m are whole numbers not equal to 0,

$\frac{a^n}{a^m} = a^{n-m}, x \neq 0.$

Proof

$\frac{a^n}{a^m} = c$ ff and only if $a^m = a^n \cdot c$. Since $a^n \cdot a^{m-n} = a^{n+(m-n)} = a^m$, we have $c = a^{m-n}$. Therefore $a^m \div a^n = a^{m-n}$.

These next rules are all related to powers.

The Power Rules

Power Rule for Powers

If a is a whole number and n and m are whole numbers not equal to 0,
$(a^n)^m = a^{n \cdot m}$

To raise a power to a power, multiply the exponents.

Power Rule for Products

If a and b are whole numbers are n is a whole number not equal to 0,
$(ab)^n = a^n b^n$

To raise a product to a power, apply the exponent rule to each and every factor.

Power Rule for Quotients

If a and b are whole numbers and n is a whole number not equal to 0,

$(\frac{a}{b})^n = \frac{a^n}{b^n}, y \neq 0$

To raise a quotient to a power, distribute the exponent to both the numerator and denominator.

Example 3.3.4

Rewrite each of the following expressions using a single exponent.

a) $2^3 \cdot 2^5$	b) $3^7 \cdot 3^9$	c) Tricky One: $2^3 \cdot 5^3$
d) $(5^3)^4$	e) $(7^8)^3$	f) $5^7 \div 5^2$
g) $\frac{7^7}{7^4}$	h) $\frac{5^7}{4^7}$	i) $3^7 \div 3^7$

Answer available at the end of this section.

Zero as an Exponent

For the Quotient Rule let's study the case when the exponents are the same. When the exponents are the same, say n, the subtraction $n - n$ produces 0. Thus, by the quotient rule of exponents, $\frac{x^n}{x^n} = x^{n-n} = x^0$

But what number, if any, does x^0 represent? Let's think for a moment about our experience with division in arithmetic. We know that any nonzero number divided by itself is one.

$\frac{8}{8} = 1, \frac{43}{43} = 1, \frac{258}{258} = 1$

Since the letter x can represent some nonzero whole number divided by itself. Then $\frac{x^n}{x^n} = 1$.

But we have also established that if $x \neq 0$, $\frac{x^n}{x^n} = x^0$. We now have that $\frac{x^n}{x^n} = x^0$ and $\frac{x^n}{x^n} = 1$. This implies that $x^0 = 1, x \neq 0$.

Exponents can now be zero.

Zero as an Exponent

$a^0 = 1$ for all whole number $a \neq 0$

Video Explanation Why $a^0=1$

YouTube Video

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https://youtu.be/tOqJ7TT2Stk?si=apz4LIPzT8oFRUUu

0^0?

Any number, other than 0, raised to the power of 0, is 1. 0^0 has no meaning (it does not represent a number). Notice that 0^0 is not defined. To see why, consider the following two patterns.

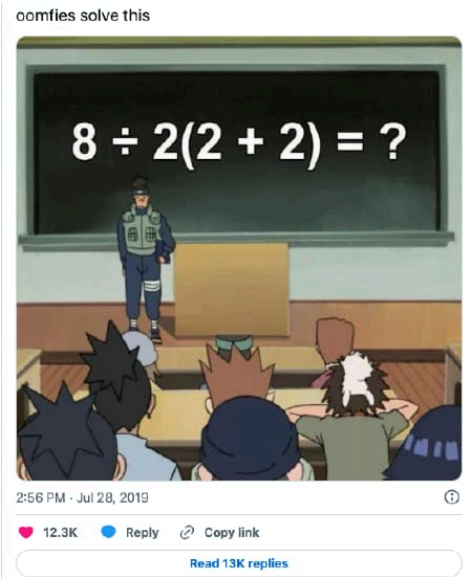
Pattern 1	Pattern 2
$3^0=1$	$0^3=0$
$2^0=1$	$0^2=0$
$1^0=1$	$0^1=0$
$0^0=?$	$0^0=?$

Pattern 1 suggests that 0^0 should be 1 and pattern 2 suggests that 0^0 should be 0. Thus to avoid such an inconsistency, 0^0 is **undefined.**

Orders of Operation

The order of operations fits into the broader essential questions of elementary mathematics by supporting students in developing foundational number sense, problem-solving skills, and mathematical reasoning. While young learners do not typically engage with formalized order of operations rules (like PEMDAS) early on, they build an understanding of operations through activities involving grouping, patterns, and number relationships.

The Simple Reason a Viral Math Equation Broke the Internet



The order in which we evaluate expressions can be ambiguous. Take, for example, the expression $4 - 2 \cdot 8$. If we perform the subtraction first, then we get 16 as a result (the question mark over the equal sign indicates that the result is questionable).

$$\begin{aligned} 4 - 2 \cdot 8 &\stackrel{?}{=} 2 \cdot 8 \\ &\stackrel{?}{=} 16 \end{aligned}$$

On the other hand, if we perform the multiplication first, then we get -12 as a result.

$$\begin{aligned} 4 - 2 \cdot 8 &\stackrel{?}{=} 4 - 16 \\ &\stackrel{?}{=} -12 \end{aligned}$$

So, what are we to do?

Orders of Operation

RULES GUIDING ORDER OF OPERATIONS

When evaluating expressions, proceed in the following order.

1. Evaluate all exponents that appear in the expression.
2. Perform all additions and subtractions in the order that they appear in the expression, moving left to right.

PEMDAS - mathematicians have agreed that the proper order of operations shall be Parentheses, Exponents, Multiplication and Division, Addition and Subtraction (PEMDAS). Although multiplication is listed before division, these operations are done left to right in order of appearance. Similarly, addition and subtraction are done left to right in order of appearance. The pneumonic device Please Excuse My Dear Aunt Sally is often used to remember this order.

Example 3.3.5

Use the proper order of operations to simplify the following expressions.

a) 5^3 - $4(1 + 2)^2$

b) 11 - 4 ÷ 2(5) + 3

Answer available at the end of this section.

Wonder, Play, Grow

Work with your peers to go through each of the parts of the table. By the end you should be able to complete the last problem.

Find the Error

$$\begin{aligned} & 7+9-5+2 \\ &= 16-5+2 \\ &= 16-7 \\ &= 9 \end{aligned}$$

Error Identified

$$\begin{aligned} & 7+9-5+2 \\ &= 16-5+2 \\ &= 16-7 \\ &= 9 \end{aligned}$$

Fix the Error

$$\begin{aligned} & \underbrace{7+9}_{\text{Add}}-5+2 \\ =\ & \underbrace{16-5}_{\text{Subtract}}+2 \\ =\ & \underbrace{11+2}_{\text{Add}} \\ =\ & 13 \end{aligned}$$

Find the Error

$$\begin{aligned} & 5(8)^2-3 \\ &= (40)^2-3 \\ &= 1600-3 \\ &= 1597 \end{aligned}$$

Error Identified

$$\begin{aligned} & 7+9-5+2 \\ &= 16-5+2 \\ &= 16-7 \\ &= 9 \end{aligned}$$

Fix the Error

Your Turn - Use the proper order of operations to simplify the following expression after you Find the Error.

$$\begin{aligned} & 7+6(8+5)^2 \\ &= 7+6(13)^2 \\ &= 7+78^2 \\ &= 6091 \end{aligned}$$

Orders of Operation Retrieval Practice

Correcting Errors in an Order of Operations

Answers

Example 3.3.1

b) 30 = 22 + 8

c) 156 = 96 + 60

d) 19 + 11 = 30

Example 3.3.4

a) 2^8	b) 3^{16}	c) 10^3
d) 5^{12}	e) 7^{24}	f) 5^5
g) 7^3	h) $(5/4)^7$	i) 3^0 or 1 (hmm?)

Example 3.3.5

a) = 5^3 - $4(3)^2$

= 125 - 4(9)

= 125 - 36

= 89

b) = 11 - 2(5) + 3

= 11 - 10 + 3

=4

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3.3.1: Written Retrieval and Problem Solving Practice

Student Learning Objectives

SLO 3.3 Students will analyze and apply the properties of whole numbers in relation to inequality, including transitivity, addition, and multiplication properties of "less than," as well as use addition to establish comparative relationships.

SLO 3.4 Students will demonstrate an understanding of whole number exponents by defining exponents through repeated multiplication and proving fundamental exponent rules.

SLO 3.5 Students will evaluate numerical expressions involving whole numbers using the conventional order of operations (including parentheses, exponents, multiplication, division, addition, and subtraction), to arrive at a unique and correct solution.

The mild assignment 🌶 is a shorter and less challenging assignment. The medium assignment🌶🌶 is right in the middle in terms of length and level of difficulty. Finally, the spicy assignment 🌶🌶🌶 is a longer assignment that is the most challenging.

(Spicy 🌶🌶🌶 Problems are OPTIONAL, but go for it when you have the time to challenge yourself)

Exercise 3.3.1.1

Find the nonzero whole number K in the definition of “less than” that verifies the following statements.

a. 11 < 19 b. 43 > 27

Answer available at the end of this section.

Exercise 3.3.1.2

Using the definitions of < and > given in this section, write four inequality statements based on the fact that 3 + 8 = 11.

Answer available at the end of this section.

Exercise 3.3.1.3

Using exponents, rewrite the following expressions in a simpler form.

a) 3x3x3x3x3 b) (2)(3)(2)(2)(3)

c) a•b•b•a d) (2)(3)(4)(2)(3)(4)

Answer available at the end of this section.

Exercise 3.3.1.4

Evaluate each of the following without a calculator and order them from largest to smallest.

5^2 , 4^3 , 3^4 , 2^5

Answer available at the end of this section.

Exercise 3.3.1.5

Rewrite each with a single exponent.

a) $5^3 \bullet 5^5$ b) $2^7 \bullet 6^7$ c) $8 \bullet 2^3$ d) $3^{12} \div 3^4$ e) $25^2 \div 5^2$ f) $9 \bullet 12^3 \bullet 2$

Answer available at the end of this section.

View FULL Explanation 3.3.1.5

🌟 Masterclass: Rewriting Expressions with Single Exponents

Rewriting expressions is about more than just finding an answer; it's about using **Mathematical Agency** to simplify the world around us. Let's look at how we solve these six different challenges by identifying their unique structures.

Strategy 1: Managing the Same Base

When the bases match, we simply add or subtract the exponents to count the total factors in our "chain."

a) $5^3 \bullet 5^5$
Action: Add the exponents (3 + 5).
Result: 5^8

d) $3^{12} \div 3^4$
Action: Subtract the exponents (12 - 4) because division "cancels out" factors.
Result: 3^8

Strategy 2: Different Bases, Same Exponents

When the powers match, we can perform the operation on the bases first!

b) $2^7 \bullet 6^7$
Action: Multiply the bases (2 • 6) and keep the power.

Result: 12^7

e) $25^2 \div 5^2$
Action: Divide the bases (25 ÷ 5) and keep the power.
Result: 5^2

Strategy 3: Structural Transformation

Sometimes we have to rewrite a number to "reveal" the hidden base.

c) $8 \cdot 2^3$
Action: Recognize that 8 is actually $\mathbf{2^3}$. Now we have $2^3 \cdot 2^3$.
Result: 2^6

f) $9 \cdot 12^3 \cdot 2$
Action: This is our "Challenge Problem." Since the bases don't match, we break them into primes: $(3^2) \cdot (2^2 \cdot 3)^3 \cdot (2^1) = 3^2 \cdot 2^6 \cdot 3^3 \cdot 2^1$
Simplified Result: $3^5 \cdot 2^7$

💡 **Humanizing Tip:** Notice how problem (f) doesn't "collapse" into one single base. In math, it is perfectly okay for numbers to keep their individual identity while being part of a larger relationship! Validation of this complexity is what makes us stronger mathematicians.

Exercise 3.3.1.6

Find x.

a) $4^7 \cdot 4^x = 4^{10}$ b) $(2^x)^4 = 2^{20}$ c) $3^x \cdot 2^x = 6^x$

Answer available at the end of this section.

Exercise 3.3.1.7

Simplify each of the following expressions.

a) $15 - 3(7 - 2)$ b) $3^2 \cdot 4 - 2(5-4)^4$

Answer available at the end of this section.

Exercise 3.3.1.8

Insert parentheses in the expression so it will be equal to the numerical value.

22 - 2 • 7 - 3 = 137

Answer available at the end of this section.

Exercise 3.3.1.9

Find all the whole numbers x such that 3 + x < 8.

Answer available at the end of this section.

Exercise 3.3.1.10

The statement a < x < b is equivalent to writing a < x and x < b and is called a compound inequality. We often read a < x < b as "x is between a and b." For the questions that follow, assume that a, x, and b are whole numbers.

If a < x < b and c is a nonzero whole number, is it always true that a + c < x + c < b + c?

Try several examples to test your conjecture.

Answer available at the end of this section.

Problem Solving Practice

1) Pizzas come in four different sizes, each with or without a choice of four ingredients. How many ways are there to order a pizza? 🌶🌶

2) Order these numbers from smallest to largest using properties of exponents and mental methods. 🌶🌶 🌶

3^{22} , 4^{14} , 9^{10} , 8^{1}

Rules of Exponents APP

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Answers

Exercise 3.3.1.1

a) K=8 b) K=16

Exercise 3.3.1.2

3 > 11, 8 < 11, 11> 8, 11 > 3

Exercise 3.3.1.3

a) 3^5 b) $2^3 \cdot 3^2$ c) a^2b^2 or $(ab)^2$ d) $(2 \cdot 3 \cdot 4)^2$

Exercise 3.3.1.4

5^2 , 2^5 , 4^3 , 3^4

Exercise 3.3.1.5

a) 5^8 b) 12^7 c) 2^6 d) 3^8 e) 25 f) $2^7 \cdot 3^5$

Exercise 3.3.1.6

a) x = 3 b) x = 5 c) x = any whole number

Exercise 3.3.1.7

a) 0 b) 34

Exercise 3.3.1.8

(22 - 2) • 7 - 3

Exercise 3.3.1.9

x = {0, 1, 2, 3, 4}

Exercise 3.3.1.10

Let a=3, x=5, and b=7. So we have: 3<5<7

Let c=2, then adding c to each term:

3+2<5+2<7+2

5 < 7 < 9

This is still a true compound inequality - TRY OTHERS to show addition preserves the order of inequalities.

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3.4: Counting Collections- Uncovering Number Sense and Mathematical Thinking Project

Purpose

This assignment helps you develop essential **mathematical teaching skills** by engaging in hands-on exploration of **counting strategies** with individuals of different ages. By documenting, analyzing, and reflecting on how different people count, you will:

- **Enhance your understanding of how counting skills develop** from early childhood to adulthood.
- **Practice analyzing mathematical thinking** in real-world scenarios.
- **Learn to identify and address misconceptions** related to counting and early operations (addition, subtraction, multiplication, and division).
- **Develop skills in video analysis and peer feedback**, critical for professional reflection and growth.

Connections Beyond This Course:

- This project **simulates real-world classroom experiences**, helping you prepare for **student observations** as a future teacher.
- Understanding how people count differently enhances your ability to **differentiate instruction** for diverse learners.
- Peer review skills will be valuable when collaborating with **colleagues in professional learning communities**.

Tasks (What You Need to Do)

Step 1: Prepare Your Counting Collection

- Gather a **set of 50–100 small objects** (e.g., buttons, beads, Legos, paper clips).
- Choose objects that allow for **different grouping strategies** (by 1s, 2s, 5s, 10s).
- Prepare a **recording sheet or notebook** to document observations.

Step 2: Engage Three Participants

- Select **three individuals of different ages**:
  - **A child (ages 4–8)**
  - **A peer (teenager or young adult)**
  - **An adult (30+)**
- Present the same collection to each participant separately.
- Ask them to count the objects **in any way they prefer** while you observe and video record the interaction. ***(Make sure to get their approval to be recorded)***

Step 3: Record & Observe the Counting Strategies

- **Video record** each session (**1-3 minutes per participant**). The participant's face does not need to be visible—hands and objects are sufficient.
- Take notes on:
  - **How do they count?** (One by one? In groups? Estimating?)
  - **What tools or gestures do they use?** (Fingers, trays, verbal counting)
  - **How efficient is their method?** (Do they backtrack? Use grouping?)
  - **Do they self-correct or adjust strategies?**
  - **Are there any errors or misconceptions?**

Step 4: Mathematical Analysis & Discussion

After the counting session, ask at least two follow-up questions appropriate for their age:

- *"What happens if we add/subtract 10 objects?"*
- *"How could we divide these equally among 4 people?"*
- *"If we doubled this collection, how many would we have?"*
- *"Could you count in a different way to check your answer?"*

A Full List of Potential Questions

Counting Strategies & Nunmber Sense

- Can you think of another way to count the collection?
- If I rearrange the objects into a different shape or pattern, does that change how you count?
- Why did you decide to count the way you did?
- Would you count differently if you had more objects? Fewer objects?

Grouping, Addition & Subtraction

- Can you group these objects in a different way? What do you notice?
- If I add 5 more objects, how many do we have now? How did you figure that out?
- If I take away 10 objects, what is left? How do you know?
- Is there a way to count this that makes adding easier?

Multiplication & Division Connections

- If we made equal groups of 5, how many groups would we have?
- If I double the collection, how many objects would there be?
- Can we split these into 4 equal groups? How many would be in each group?
- What happens if we divide the objects among 3 people? Would any be left over?

Patterns & Mathematical Properties

- Did you notice any patterns while counting?
- What would happen if we skip-counted instead of counting one by one?
- If we arrange the objects in rows and columns, how does that help us count?
- If we count backward from the total, what does that look like?

Reflection & Problem-Solving

- What was the easiest part about counting this collection? What was the hardest?
- How would you teach someone younger than you how to count this collection?
- What mistakes do you think someone might make when counting this collection?
- If you were counting a really large number of objects (like 500), how would you do it efficiently?

Step 5: Video Reflection & Analysis Submission

- **Select one** of your three recorded sessions for deeper analysis.
- Submit either:
  - **A 5–7 minute video reflection** where you analyze the counting strategies used, discuss misconceptions, and connect findings to teaching, OR
  - **A 2–3 page written analysis** covering the same points.

Your analysis should include:

✅ **Description of the interaction** (who they counted with, setting, what happened).

✅ **Mathematical analysis** (counting strategies, efficiency, operations explored).

✅ **Key takeaways** (misconceptions, learning styles, connections to teaching).

✅ **Implications for teaching** (How would you help a student who counted this way?

Step 6: Peer Review & Discussion Board Participation

- Upload your **video reflection OR written analysis** to the class discussion board.
- Review **two peers' submissions**, using the structured **Peer Review Protocol** below.

Peer Review Protocol

Each student must review **two classmates' submissions** by following this structured feedback framework:

✅ **Describe** (What do you notice?)

- Summarize the counting strategies used in the video.
- Identify **one strength** in how the participant engaged the counter.

🤔 **Analyze** (What patterns or teaching implications emerge?)

- What insights did the student make about counting strategies?
- Did they analyze efficiency, errors, or operations deeply?
- Were there any gaps in the explanation?

📌 **Suggest** (What feedback would strengthen the analysis?)

- What additional questions could they explore?
- Could they connect this to broader teaching strategies?

💡 **Reflect** (What did you learn from this?)

- How does this change your understanding of how children count?
- How might this influence your teaching approach?

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3.5: Chapter 3 Resources

Chapter Example Videos

Chapter 3.1

YouTube Video

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Chapter 3.2

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Chapter Extras

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4: STRENGTHENING NUMBER SENSE - STRATEGIES, ALGORITHMS, AND ESTIMATION

Understanding and teaching whole-number computation goes beyond memorizing procedures—it involves developing number sense, strategic thinking, and problem-solving skills (NCTM, 2020). As future elementary educators, you will guide students through the journey of making sense of numbers, fostering their ability to reason mathematically, and helping them see computation as a flexible, sense-making process (Boaler, 2022). This chapter explores mental math techniques, estimation strategies, and standard and nontraditional algorithms to equip you with the knowledge to support diverse learners. By integrating hands-on models, place value understanding, and historical perspectives on computation, you will learn how to build a foundation that encourages deep mathematical thinking rather than rote memorization (Devlin, 2012). As you engage with this content, consider how you can cultivate a classroom environment that values multiple solution paths, mathematical reasoning, and a growth mindset—helping students see themselves as capable and competent problem solvers.

Chapter Learning Objectives

Student Learning Objectives

SLO 4.1 Students will apply mental math and computational estimation techniques, such as using compatible numbers, properties, and rounding strategies, to solve mathematical problems efficiently and accurately.

SLO 4.2 Students will compare how basic and scientific calculators interpret and solve expressions, distinguishing between arithmetic logic (step-by-step entry) and algebraic logic (order of operations) while using calculator functions effectively (e.g., memory, exponent, and parentheses keys) to computer accurate results.

SLO 4.3 Students will model and justify standard written algorithms for basic operations (addition, subtraction, multiplication, division) using concrete models, place value, and mathematical properties.

SLO 4.4 Students will compute sums, differences, products, and quotients in various bases (2–12) using standard algorithms and nontraditional methods, such as lattice and scaffold techniques.

Essential Questions

- **What strategies can be used to develop a deep understanding of number relationships rather than relying on rote memorization?**
- **How do different estimation techniques (e.g., rounding, compatible numbers) help in real-world problem-solving?**

Learning That Transfers

Students will analyze and apply flexible computational strategies in varied mathematical and real-world scenarios so they can begin to think about how they will design instructional methods that help elementary learners develop number sense and mathematical reasoning across disciplines.

Computation Fun

Interactive Content

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Morgan's Math Adventure: Counting the World Around Us

One bright morning, with the sun shining high, Morgan saw numbers up in the sky! Up ran to Grandpa, a question to share—"Why do numbers dance everywhere?"

Grandpa chuckled, as wise grandpas do, “Numbers help us, through and through! From planning a trip to seeing new sights, Math is what helps us get things just right!”

Off on a road trip, they went with delight, Counting the miles and watching the sights. “We’ve gone 39 miles, the trip is so grand, The whole drive is ninety—where do we stand?”

Grandpa grinned, “Let’s round to make sense, Forty is close, and ninety is dense! About fifty to go, not quite exact, But close enough to keep us on track!”

They stopped at a fruit stand along the way, Morgan saw melons stacked up in display. “If there are seven rows, each with five, How many melons are stacked up high?”

Grandpa smiled, “Multiply and see, Seven times five? That’s thirty-five with me!” Then Morgan asked, “But what if some sell? If ten are bought, how many to tell?”

Grandpa thought, “Thirty-five take away ten, That leaves twenty-five melons again!”

Further ahead, they saw birds in the sky, Some perched on a fence, some soaring high. “If sixty birds sit, but half fly away, How many are left to rest and stay?”

Morgan paused, then started to do, “Half of sixty? That’s thirty left for you!”

The ocean appeared, waves crashing with cheer, Morgan felt proud as they finally drew near. “Numbers are everywhere, big and small, And now I see, I can use them all!”

Grandpa winked, “That’s exactly right! Now keep on counting, all day and night! With estimation, algorithms, and a flexible view, You’ll solve math problems like mathematicians do!”

4.1: Fluent and Flexible- Mental Math and Estimation for Problem Solvers

4.1.1: Written Retrieval and Problem Solving Practice

4.1.2: Mental Math Video Discussion Board

4.1.3: Online Retrieval and Problem Solving Practice

4.2: Embracing Diversity of Algorithms for Addition and Subtraction

4.2.1: Written Retrieval and Problem Solving Retrieval Practice

4.2.2: Addition and Subtraction Progressions Discusison Board
4.2.3: Online Retrieval and Problem Solving Practice

4.3: Embracing Diversity of Algorithms for Multiplication and Division

4.3.1: Written Retrieval and Problem Solving Practice
4.3.2: Online Retrieval and Problem Solving Practice

4.4: Chapter 4 Resources

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4.1: Fluent and Flexible- Mental Math and Estimation for Problem Solvers

As future elementary school teachers, it is essential to understand why mental math and estimation are foundational skills for students' mathematical development. Mental math fosters numerical fluency, allowing students to move beyond rote memorization and engage with numbers conceptually (NCTM, 2020). When students develop strong mental math strategies, they become more confident problem solvers who can think flexibly, adapt strategies to different contexts, and deepen their overall number sense (Boaler, 2022). Estimation, in particular, is a critical skill that helps students make sense of numerical values, gauge reasonableness, and approach problem-solving with efficiency. By incorporating estimation into everyday mathematical activities, teachers can help students develop a sense of magnitude and the ability to make quick, reasonable judgments about numbers (Feike et al., 2018).

Additionally, mental math encourages students to explore multiple strategies and develop a flexible mindset toward computation, which is essential for long-term mathematical success. Teachers play a key role in creating environments where students feel encouraged to experiment with numbers, share their thinking, and understand that mathematics is about reasoning and relationships, not just procedures. By prioritizing mental math and estimation, educators empower students to become confident, resourceful, and capable mathematicians (NCTM, 2020).

Mental Math

Interactive Element

Before viewing the Tik Tok video below from Math Educator Howie Hua, think about all of the ways a child would mentally (without writing something down) solve 81 - 28.

@howie_hua Different ways to subtract besides the standard algorithm #math #mathematics #maths #mathteacher #mathtok ♬ original sound - Howie Hua

Think About and Discuss

Now that you had a chance to play with some mental math strategies, let's see what Howie Hua thinks about this.

Want to be better at mental math? Try these strategies!

Task 1: View the video.

Task 2: Discuss with others what Howie did for the following problems.

38 + 97

16 x 15

Task 3: Write down what he did so can try his mental math strategies on the following examples.

Example 4.1.1

Use thinking strategies to compute.

a) 47 + 39

b) 16 x 4

c) 12 x 45

Answer available at the end of this section.

Example 4.1.2

Use thinking strategies to compute.

a) 15 + (27 + 25)

b) (8 x 7) x 25

c) 177 ÷ 3

Answer available at the end of this section.

Compatible Numbers and Compensation

Compatible numbers are numbers whose sums, differences, products, or quotients are easy to calculate mentally.

Compensation is the process of reformulating a sum, difference, product, or quotient to one that is more readily obtained mentally.

Before you move on to the list of mental math strategies that are introduced, make sure you know what the terms mean and when you are using compatible numbers and the process of compensation.

Mental Math Strategies

Mental Addition

- Left-to-Right Approach: In this model, to add 347 and 129, we first add the hundreds (300 + 100) then the tens (40 + 20), and then the ones (7+9), to obtain 476.
- Compensation: To find the sum 67 + 29, we add 67 + 30 (since 30 is the next multiple of 10 greater than 29) to obtain 97 and then we subtract 1 from 97 to compensate for the extra 1 that was added to obtain 97–1 = 96.
- Using Compatible Numbers: For example, if we are trying to find the sum 130+50+70+20+50 we will add first the numbers 130 and 70 to obtain 200, the numbers 50 and 50 to obtain 100, then the numbers 200 and 100 to obtain 300 and finally we add 20 to 300 to obtain 320.
- Breaking Up and Bridging: To find the sum 67 + 36 we find first the sum 67 + 30 = 97 and then the sum 97 + 6 = 103.

Mental Subtraction

- Left-to-Right: To find the difference 93 – 38 we find first 90 – 30 = 60, then 8 – 3 = 5 and finally 60 – 5 = 55. To find the difference, 47 – 32, we find first 40 – 30 = 10, then 7 – 2 = 5 and finally 10 + 5 = 15.
- Breaking Up and Bridging: To find the difference 67–36 we first find 67 – 30 = 37 and then 37 – 6 = 31.
- Compensation To find 47–29 is the same as finding 48–30 = 18. That is we add the same number to both addends. This is known as the equal addition method for subtraction.
- Drop the Zeros: To find 8700 – 500 we find 87 – 5 = 82 and then add the two zeros to the right of 82 to obtain 8200.
- Compatible Numbers: To find the difference 170–50–30–50 we can find first 170 – 30 = 140, then 50 + 50 = 100 and finally 140 – 100 = 40.

Mental Multiplication

- Compatible Numbers: To find the product 2 x 9 x 5 x 20 x 5 we can rearrange the product in the form 9 x (2 x 5) x (20 x 5) = 9 x 10 x 100 = 9000.
- Left-to-Right: To find the product 3 x 123 we can look at it as the expression 3 x 100 + 3 x 20 + 3 x 3 = 369.
- Multiplying Powers of 10: To find the product 12,000 x 110,000 we multiply the numbers 12 x 11 = 132 and then add 7 zeros to the right to obtain 1,320,000,000.

Mental Division

- Compatible Numbers To find 105 ÷ 3 we look for two numbers that are divisible by 3 and whose sum is 105, namely, 90 and 15. We then divide both numbers by 3 to obtain 90 ÷ 3 = 30 and 15 ÷ 3 = 5. Finally add the quotients to obtain 105 ÷ 3 = 30 + 5 = 35.

Wonder, Play, Grow

First try to solve your assigned problem, then share with others how you did it. Be prepared to present your ideas to the class. Try to name a method or process you used to compute your value.

a.197 + 248

b.125 x 44

c. 273 – 139

d. 321 + 437

e. 3 x 432

f. 320 x 25

🧠**Let's Listen to Learn**❤️

Audio

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Interactive App: Explore Mental Math Fluency ▼

🧠 **Check Your Thinking (Before You Click Play):**
Fluency isn't about speed; it's about flexibility. How can you decompose numbers mentally to solve these efficiently? Look for patterns rather than just rushing the timer.

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Estimation

Wonder, Play, Grow

Fermi Problems-Fermi problems are estimation problems that require making reasonable assumptions to arrive at an approximate solution. Named after physicist Enrico Fermi, these problems involve breaking down complex questions into simpler components and using rough calculations to reach an answer.

Techniques from Fermi questions very similar to how we solve problems in real life:

- Open-ended problem where we have to figure out what information we need and how to use it
- Can make assumptions about information that isn't given in the problem
- Use the tools we have to get the best answer we can when an "exact" answer may not be possible.

Task 1- Below is a classic example of a "Fermi Problem". We will work as a class to talk through how to answer the question.

How far could you walk in one year?

Problem Solving Fermi available at the end of this section.

Task 2 - Below are two more Fermi Problems. Decide as a group which one to attempt to answer. And then go for it. Be prepared to show your thinking.

Option 1: How many students will an elementary teacher teach in their career?

Option 2: How many hours does someone born in the United States sleep in their lifetime?

Task 3 - Create your own Fermi Problem. Think about something in your daily life as a student or future teacher that you could estimate using logical reasoning. Start with a **big question** that begins with "How many...?" or "How much...?" or “How long...?” —something that doesn’t have an obvious answer but can be broken down into smaller, reasonable estimates.

Estimation and rounding are fundamental skills in developing young children's mathematical thinking. These skills allow children to make sense of numbers, assess reasonableness, and build confidence in problem-solving. Estimation helps children approximate numerical solutions, fostering number sense and mental math abilities. By using strategies such as computational estimation and rounding, children can quickly determine whether an answer is reasonable without relying solely on exact calculations. This ability is particularly important in real-world situations, such as estimating costs while shopping or predicting time durations.

Computational estimation involves finding an approximate answer to a numerical problem. It helps determine whether a solution is reasonable by using various estimation strategies. An estimate is a value that is close to the exact answer. Here, we explore two key estimation strategies (range and front-end strategy estimation).

Range Estimation

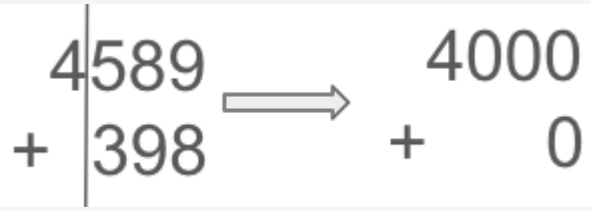
Consider the operation 294 x 53. The product 300 x 60 = 18000 is an overestimate of 294 x 53 whereas the product 200 x 50 = 10000 is an underestimate. We say that the product 294 x 53 is in the range from 10000 to 18000.

Example 4.1.3

Find a range estimate for the sum 3741 + 1252.

Answer available at the end of this section.

Front-End Strategy

One-Column Front End Estimation	Two-Column Front End Estimation	Front-End with Adjustments
Draw a line separating the leading digit(s). Add the numbers to the left of the column and ignore the numbers to the right. Thus, 4589 + 398 ≈ 4000. The symbol ≈ means ”is approximated by”.  Similarly, 372 + 53 + 417 ≈ 300 + 0 + 400 = 700.	This one improves the previous estimate. To see how this strategy works, consider the sum 372 + 53 + 417. Draw a line separating the first two leading digits. Add the numbers to the left of the line and ignore the ones to the right. Thus, 372 + 53 + 417 ≈ 370 + 50 + 410 = 830.	This method enhances the one-column front-end methods. After adding the numbers to the left of the line, adjust the answer by considering the digits in the next column to the right. For example, to estimate 498 + 251 we first do 400 + 200 = 600 and then 98 + 51 ≈ 100 + 50 = 150. Thus, 498 + 251 ≈ 600 + 150 = 750.

Example 4.1.4

Estimate using the method indicated.

a) 503 x 813 using one-column front-end method

b) 1200 x 35 using range estimation

c) 4376 – 1889 using two-column front-end

d) 3257 + 874 using front-end adjustment.

Answer available at the end of this section.

Rounding

Rounding may be the most well-known methods for computational estimation. Its goal is to simplify complicated numbers by replacing them with more manageable (compatible) ones. Since the purpose is to find an estimate, various rounding techniques can be applied. Unlike other estimation techniques, rounding is commonly used both before performing a calculation and to refine the final answer.

When rounding a number to the nearest ten, hundred, thousand, or another place value, follow these steps:

1. Identify and underline the digit in the place value you are rounding to.
2. Look at the digit immediately to the right of the underlined number:
   - If it is 5 or higher, increase the underlined digit by one and change all digits to the right to zero.
   - If it is 4 or lower, keep the underlined digit the same and replace all digits to the right with zero

Future Teacher Note

🧠Let's Listen to Learn❤️

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When teaching rounding, it's essential to help students not just memorize steps but also develop a strong conceptual understanding. Here are **some important reminders** you might want to emphasize, along with tips to support deep learning and engagement:

Understand Place Value First

- Before rounding, students must clearly understand what each digit in a number represents (ones, tens, hundreds, thousands, etc.).
- Use place value charts or base-ten blocks to build this understanding.

Think About What It Means

- Rounding makes numbers easier to work with—it's about **approximating** to make mental math or estimation quicker.

Math Mindsets and Thinking Strategies

- Encourage students to explain **why** a number rounds up or stays the same.
- Let them estimate and **justify** whether their answer is "reasonable."
- Mistakes are part of the process—help students see them as learning opportunities.

Promote Deeper Learning

- **Use real-world contexts** (e.g., rounding prices or distances).
- **Visuals help**: number lines are great for showing the "rounding jump."
- **Group discussion**: Have students compare answers and reasoning.
- **Games and Estimation Challenges**: Add fun with rounding bingo, estimation jars, or quick "Which is closer?" questions.

Example 4.1.5

Round 467 + 221 to the nearest...

a) hundred

b) ten

Answer available at the end of this section.

Example 4.1.6

Round each number to the position indicated.

a) 5280 to the nearest thousand

b) 115,234 to the nearest ten thousand

c) 115,234 to the nearest hundred thousand

d) 2,325 to the nearest ten.

Answer available at the end of this section.

Rounding Retrieval Practice

Rounding Whole Numbers

Future Teacher Note

While technology tools such as calculators, computers, and AI can efficiently perform calculations, they should not replace the foundational development of number sense, estimation, and computational flexibility in young learners. These mental skills are critical for helping students make sense of mathematical ideas, detect errors, and develop confidence in their reasoning. When students understand the *why* behind the math, they are better equipped to use technology as a tool for efficiency—not a crutch. By grounding students in flexible thinking and estimation strategies early on, we prepare them to engage with digital tools thoughtfully and critically as they grow into problem-solvers in a tech-driven world.

Using Technology

Given the ease of access to technology, it is wise that we have an awareness of how to use it to do calculations. Keep in mind from the Common Core State Standards emphasizes that proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations.

Use the Desmos Scientific Calculator to compute the following. When possible, first mentally calculate before checking the results.

a) $(4 + 2 \times 5) \div 7 + 3$

b) $8 \div 2^2 + 3 \times 2^2$

c) $17 - 4(5 - 2)$

d) $40 \div 5 \times 2^3 - 2 \times 3$

e) 120000×50000

Answers

a) 5 b) 14 c) 5 d) 58 e) 6000000000 or 6×10^9 (We will learn about scientific notation later)

Answers

Example 4.1.1

Keep in mind, there may be other ways to solve each problem.

a) $(40 + 30) + (7 + 9) = 70 + 16 = 86$

b) $(8 \times 2) \times 4 = 8 \times 8 = 64$

c) $6 \times 2 \times 45 = 6 \times 90 = 540$

Example 4.1.2

Keep in mind, there may be other ways to solve each problem.

a) (15 + 25) + 27 = 40 + 27 = 67

b) 7 x 2 x 4 x 25 = 14 x 100 = 1400

c) (180 - 3) ÷ 3 = 180/3 - 3/3 = 60 - 1 = 59

Understand the Problem

- Restate the question in simple terms.
- Identify what needs to be estimated.
- List any assumptions and things we need to know.

Devise a Plan

- Break the problem into smaller, manageable parts.
- Identify key factors and use logical approximations.
- Choose a strategy (multiplication, division, proportions, etc.).

Carry Out the Plan

- Perform rough calculations using estimates.
- Keep track of units and check for consistency.
- Adjust if the result seems unreasonable.

Look Back & Reflect

- Does the answer make sense?
- Verify using a different approach if possible.
- Consider how changes in assumptions affect the result.

Example 4.1.3

Lower Bound (underestimate): 3000 + 1000 = 4000 , Upper Bound (overestimate): 4000 + 1500 = 5500

We say that the sum 3741 + 1252 is in the range from 4000 to 5500. NOTE: Other compatible numbers could have been used to generate and estimate. (eg. 3800 + 1300 = 5100 or 3500 + 1500 = 5000). It all depends on which values make sense to you. These different adjustments are related to the next strategy.

Example 4.1.4

a) 503 x 813 ≈ 500 x 800 = 400000.

b) 1200 x 35 is in the range from 1200 x 30 = 36000 to 1200 x 40 = 48000.

c) 4376 – 1889 ≈ 4300 – 1800 = 2500.

d) We first do 3000 + 800 = 3800 and then 250 + 80 = 330 so that 3257 + 874 ≈ 3800 + 330 = 4100.

Example 4.1.5

(a) Since 467 ≈ 500 and 221 ≈ 200 then 467 + 221 ≈ 700.

(b) In this case, 467 ≈ 470 and 221 ≈ 220 so that 467 + 221 ≈ 690.

NOTE: If we add first before rounding, then we would have the exact answer - there would be no need to round the answer after the exact computation.

Example 4.1.6

a) 5000 b) 120,000 c) 100,000 d) 2,330

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4.1.1: Written Retrieval and Problem Solving Practice

Student Learning Objectives

4.1 Students will apply mental math and computational estimation techniques, such as using compatible numbers, properties, and rounding strategies, to solve mathematical problems efficiently and accurately.

4.2 Students will compare how basic and scientific calculators interpret and solve expressions, distinguishing between arithmetic logic (step-by-step entry) and algebraic logic (order of operations) while using calculator functions effectively (e.g., memory, exponent, and parentheses keys) to computer accurate results.

The mild assignment 🌶 is a shorter and less challenging assignment. The medium assignment🌶🌶 is right in the middle in terms of length and level of difficulty. Finally, the spicy assignment 🌶🌶🌶 is a longer assignment that is the most challenging.

(Spicy 🌶🌶🌶 Problems are OPTIONAL, but go for it when you have the time to challenge yourself)

Exercise 4.1.1.1 🌶

Perform each of the following computations mentally and explain what technique you used to find the answer.

a) 40 + 160 + 29 + 31

b) 3670 – 474

c) 75 + 28

Answer available at the end of this section.

Exercise 4.1.1.2 🌶

Use compatible numbers to compute each of the following mentally.

a) 2 · 9 · 5 · 6

b) 82 + 37 + 18 + 13

Answer available at the end of this section.

Exercise 4.1.1.3 🌶🌶

Calculate mentally using properties of operations, i.e. commutative, associative, distributive.

a) (37 + 25) + 43

b) 47 · 15 + 47 · 85

c) (4 x 13) x 25

d) 26 · 24 – 21 · 24.

Answer available at the end of this section.

Exercise 4.1.1.4

Find each of the following differences using compensation method.

a) 43 – 17

b) 132 – 96

c) 250 – 167.

Answer available at the end of this section.

Exercise 4.1.1.5

Estimate using compatible number estimation.

a) 51 x 212

b) 3112 ÷ 62

c) 103 x 87.

Answer available at the end of this section.

Exercise 4.1.1.6

a) Find a range estimate for the sum 3741 + 1252.

b) Find a range estimate for the sum 289 x 12.

c) Find a range estimate for the sum 4787 ÷ 17.

Answer available at the end of this section.

Exercise 4.1.1.7

Round each of these to the position indicated.

a) 947 to the nearest hundred.

b) 27,462,312 to the nearest million.

c) 2561 to the nearest thousand.

Answer available at the end of this section.

Exercise 4.1.1.8

Rounding to the left-most digit, calculate approximate values for each of the following:

a) 681 + 241

b) 678 – 431

c) 257 x 364

d) 28,329 ÷ 43.

Answer available at the end of this section.

Exercise 4.1.1.9

Use estimation to tell whether the following calculator answers are reasonable. Explain why or why not.

(a) 657 + 542 + 707 = 193,346

(b) 26 x 47 = 1,222.

Answer available at the end of this section.

Exercise 4.1.1.10

Estimate each of the following using the methods: (i) range, (ii) one-column front end, (iii) two-column front end

$$\begin{array}{r} 741 \\ 98 \\ +\ 34 \\ \hline \end{array} \qquad \begin{array}{r} 1745 \\ 1398 \\ 567 \\ +\ 99 \\ \hline \end{array}$$

Answer available at the end of this section.

Exercise 4.1.1.11

Simplify the following expressions using a calculator.

a) 135 – 7(48 – 33)

b) $32 \cdot (50 - 7)^2$

c) $\frac{1-2(3^2-4^3)-1}{5+3 \bullet 2}$

Answer available at the end of this section.

Exercise 4.1.1.12

Vanessa estimated 31 · 179 in the three ways shown below.

(i) 30 · 200 = 6000

(ii) 30 · 180 = 5400

(iii) 31 · 200 = 6200

Without finding the actual product, which estimate do you think is closer to the actual product? Explain.

Answer available at the end of this section.

Exercise 4.1.1.13 🌶️🌶️🌶️

There is a shortcut for multiplying a whole number by 99. For example, consider 15 x 99.

a) Why does 15 x 99 = (15 x 100) – (15 x 1)?

b) Compute 15 x 99 mentally, using the formula in part a)

c) Compute 95 x 99 mentally, using the same method. Then make up your own problems to try.

Answer available at the end of this section.

Problem Solving Practice

1) Develop a shortcut for multiplying by 25 mentally in a computation such as 24 x 25. Then Compute 44 x 25 using the same shortcut. Make sure to explain your method to someone and let them try a few problems.

2) Amari multiplied 3472 and 259 on their calculator and wrote down the answer of 89,248. How can you see immediately using estimation that they made an error? Can you see how ther error was made?

3) Conner tells you that multiplying by 5 is a lot like dividing by 2. For example, 48 x 5 = 240, but it is easier just to go 48 ÷ 2 = 24 and then affix a zero at the end. Will his method always work? Explain.

4) **Closest Quotient to 250**

Directions: Using the digits 2 to 9 at most one time each, place a digit in each box to find the closest quotient to 250.



5) **Multi-Digit Multiplication**

Directions: Using the digits 0 to 9 at most one time each, place a digit in each box to create a true equation with the greatest possible product.



Problem 4 and 5 Source: Open Middle

Answers

🌶 **Exercise** 4.1.1.1

a) **Step 1: Group numbers for easier addition** Combine numbers that add up to round numbers: **40 + 160 = 200, 29 + 31 = 60, Step 2: Add results 200 + 60**

b) First subtract 400 from 3679: **3670 – 400 = 3270**

Then subtract the remaining 74: First do **3270 – 70 = 3200** Then **3200 – 4 = 3196**

c) Step 1: Use compensation to simplify Round **28** up to **30** (easier to add): **75 + 30 = 105, Step 2: Subtract the extra 2 you added, 105 – 2 = 103**

🌶 **Exercise** 4.1.1.2

a) (2 x 5) x (9 x 6) = 540

b) (82 + 18) + (37 + 13) = 150

🌶🌶 ⚠ **Exercise** 4.1.1.3

a) (37 + 43) + 25 = 105

b) 47(15 + 85) = 4700

c) (4 x 25 x 13) = 1300

d) 24 (26 - 21) = 120

🌶🌶 **Exercise** 4.1.1.4

a) 46 - 20 = 26

b) 136 - 100 = 36

c) Count up: 33 + 50 = 83 OR (250-170+3) = 83

🌶🌶 ⚠ **Exercise** 4.1.1.5

a) 50 x 200 = 10000 b) 3000/60 = 50 c) 100 x 90 = 9000

🌶 **Exercise** 4.1.1.6

a) The lower estimate or underestimate is 4000 and the upper estimate or overestimate is 6000. So the sum is in the range from 4000 to 6000.

b) The lower estimate or underestimate is (280 x 10 = 2800) and the upper estimate or overestimate is (300 x 15 = 4500).

c) The lower estimate or underestimate is (4600 / 20 = 230) and the upper estimate or overestimate is (4800/16=300)

🌶 **Exercise** 4.1.1.7

a) 900 b) 27,000,000 c) 3,000

🌶🌶 ⚠ **Exercise** 4.1.1.8

a) 700 + 200 = 900 b) 700 - 400 = 300 c) 300 x 400 = 1200 d) 28,00 / 40 = 70

Exercise 4.1.1.9🌶 🌶

a) Not Reasonable: 700 + 500 + 700 = 1900 b) 25 x 50 = 1250, Reasonable

🌶 🌶 **Exercise** 4.1.1.10

i) Range: 700 + 0 + 0 = 700, 800 + 100 + 50 = 950, between 700 and 950 ii) 700 + 0 + 0 = 700 iii) 700 + 100 + 0 = 800

i) Range: 1700 + 1300 + 500 + 0 = 3500, 1800 + 1400 + 600 + 100 = 3900, between 3500 and 3900 ii) 2000 + 1400 + 500 + 0 = 3900 iii) 1700 + 1400 + 600 + 0 = 3700

🌶 **Exercise** 4.1.1.11

a) 30 b) 59168 c) 10

🌶🌶 **Exercise** 4.1.1.12

ii) since it uses numbers closer to the original numbers

🌶 🌶 🌶 **Exercise** 4.1.1.13

a) distributive property b) 1500 - 15 = 1485 c) 9500 - 95 = 9405

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4.1.2: Mental Math Video Discussion Board

Video Discussion Board

ADAPT Discussion Board 4.1.2.1

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Alternative Discussion Board

Discussion Board Assignment: Making Number Sense Make Sense

Read this article - **The building blocks of math that students need to excel — but aren't always getting**

https://hechingerreport.org/the-buil...need-to-excel/

In your post, respond thoughtfully to the following prompts. You may write in paragraph form or use bullet points to organize your ideas clearly.

Part 1 – Understanding the Article

1. **What is number sense, and why is it described as a building block of math?**
2. **Reflect on Melissa Williams' classroom example.** What stood out to you about the activity? What does it tell us about how young children develop mathematical thinking?

Part 2 – Personal Reflection

4. **One idea from the article that surprised you.** Why did it stand out?
5. **A connection to your own math learning experience as a child.** When (if ever) do you remember building number sense?

Part 3 – Try This!

7. End your post with a **"Try This!" idea** — a short description of an activity, game, or conversation you could use to help students, your own children, or some young child from your life develop number sense. (Be creative and specific!)

Replies to Peers (Due by [insert date])

Reply to at least **two classmates**. In your replies, you can:

- Share feedback or encouragement
- Ask a clarifying or probing question
- Offer a way to **extend or modify** their "Try This!" idea to reach diverse learners

Keep your responses kind, constructive, and engaging — we're building a community of future educators!

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4.1.3: Online Retrieval and Problem Solving Practice

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4.2: Embracing Diversity of Algorithms for Addition and Subtraction

As we transition from exploring mental math strategies and the use of calculators, we now turn our attention to written methods for addition and subtraction—commonly known as algorithms. It's important to recognize that there is no single, universally adopted algorithm for these operations. In fact, classrooms and cultures around the world use a variety of methods, each with its own strengths and historical roots. As a future teacher, your role is not only to understand the commonly taught algorithms but also to appreciate the diversity of strategies students may bring or develop on their own. A student's invented method, when accurate and consistent, can be a powerful demonstration of mathematical reasoning and ownership. This chapter invites you to explore multiple algorithms for addition and subtraction, including some with historical significance and others designed to deepen conceptual understanding. You'll learn to apply these algorithms in both Base Ten and alternative number systems. More importantly, you'll reflect on how to create a learning environment that celebrates multiple solution paths, fosters mathematical thinking, and empowers students to see themselves as capable, creative problem solvers.

The DOT METHOD for adding columns of single digits

Here is a method for adding several digits together without having to keep a large running total in your head. The idea is that you only add two single digits together at a time. Any time you add and get a number over 9, you put down a dot and only keep the unit's digit in your memory. The dot stands for the ten. For instance, since 6 + 7 = 13 (which is 10 + 3), you write a dot and think 3. *Think 6 + 7 = Dot 3*. In the example below, to add eight numbers together, a common approach would be to add the first two digits together, then keep adding on the next digit until all eight digits have been added. In other words, someone might think: 6 + 7 is 13, 13 + 8 is 21, 21 + 3 is 24, 24 + 5 is 29, 29 + 9 is 38, 38 + 4 is 42 and 42 + 7 is 49. Although this yields the correct answer, it's easy to lose track along the way. With the DOT METHOD shown, a dot is placed by a number if after it is added on, the sum is ten or more. At each step of the way, I've written in parentheses what number you think of after adding each digit. To start, see the eight digits to be added. Step 1: 6 + 7 = 13 so write a dot and think 3; Step 2: 3 + 8 = 11 so write a dot and think 1; Step 3: 1 + 3 = 4 so think 4; Step 4: 4 + 5 =9 so think 9; Step 5: 9 + 9 = 18 so write a dot and think 8; Step 6: 8 + 4 = 12 so write a dot and think 2; Step 7: 2 + 7 = 9 so write a 9 under the bar since all digits have been added. Count the dots (4), which stands for how many tens (40) and write in front of the 9. The answer is 49. Try it on your own now!

Start

$$\begin{array}{r} 6 \\ 7 \\ 8 \\ 3 \\ 5 \\ 9 \\ 4 \\ \underline{+7} \end{array}$$

$$\begin{array}{r} 7\cdot \\ 8\cdot \\ 3 \\ 5 \\ 9\cdot \\ 4\cdot \\ \underline{+7} \\ 49 \end{array}$$

In the example just shown on the previous page, every single step, including what you think is shown. In reality, this is a quick algorithm and if you looked at the problem after someone did it, the work on the left is all you would see. To the right are are two

more examples. Actually, each is the same sum of numbers written in a different order, which is another way to check addition! *A quick check when adding only two numbers is to reverse the order and add again.*

$$\begin{array}{r} 3 \\ 9\cdot \\ 6 \\ 5\cdot \\ 4 \\ 7\cdot \\ 7\cdot \\ +8 \\ \hline 49 \end{array}$$

Wonder, Play, Grow

Use the dot method to add the following columns of digits shown below.

$$\begin{array}{r} 3 \\ 9 \\ 4 \\ 5 \\ 2 \\ 6 \\ 8 \\ 6 \\ +7 \\ \hline \end{array}$$

By the way, you could put the dots to the left or right of the digit, do whatever is most comfortable for you. If you are adding without paper and pencil, use your fingers to keep track of the dots! Instead of writing a dot, put up a finger.

What is the beauty of this method?

The dot method could also be used to add several digits written in a horizontal format. The dots could be placed *over* the numerals. Try it on the following problems.

a. 7 + 6 + 4 + 5 + 9 + 6 + 8 + 3 + 4 + 9 + 8 = ___
b. 4 + 9 + 6 + 2 + 9 + 7 + 7 + 6 + 8 + 4 + 8 = ___
c. 7 + 9 + 8 + 5 + 9 + 4 + 2 + 6 + 9 + 8 + 7 = ___

Standard Addition Algorithm

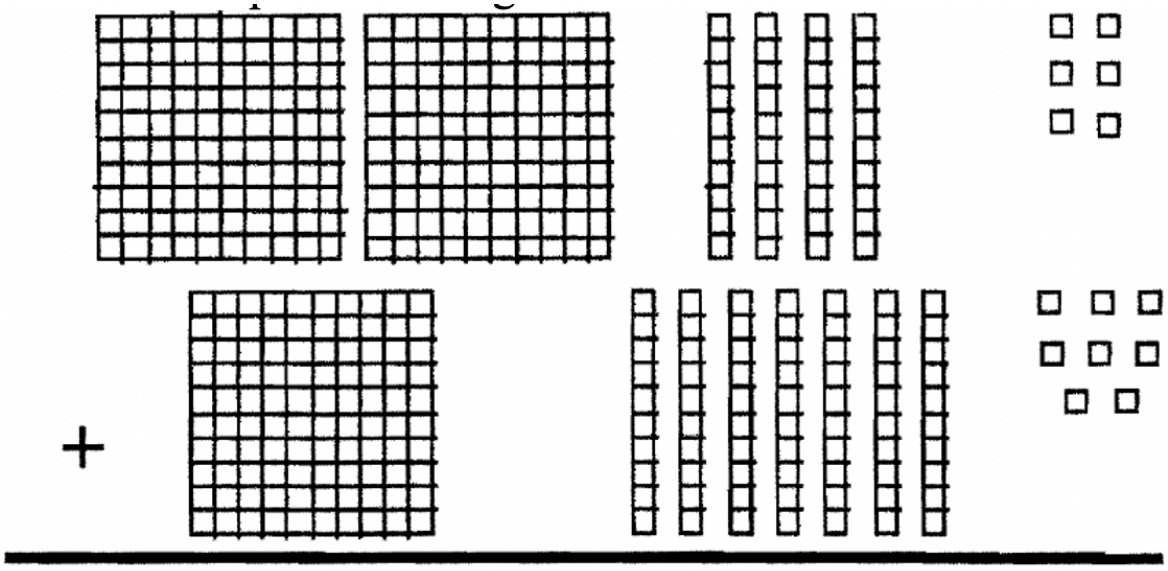
The most common addition algorithm is the *Right To Left Standard Addition Algorithm*, often referred to as the *Standard Addition Algorithm*. This is the one almost everyone grew up learning. You start on the right and add the digits. The unit's digit is put down under the line and the ten's digit is carried to the top of the next column to the left. Get out your Base Ten Blocks now to understand what is really going on.

Consider the addition problem 246 + 178. From our knowledge of place value, we know that 246 is 2(100) + 4(10) + 6(1) (or 200 + 40 + 6) and 178 is 1(100) + 7(10) + 8(1) (or 100 + 70 + 8). Using Base Ten blocks, 246 would be represented with 2 flats, 4 longs and 6 units whereas 178 would be represented with 1 flat, 7 longs and 8 units. So, the addition problem can be thought of in the following way:

Using Base Blocks

$$\begin{array}{r} 2 \text{ flat(s)} \ + 4 \text{ long(s)} \ + 6 \text{ unit(s)} \\ +1 \text{ flat(s)} \ + 7 \text{ long(s)} \ + 8 \text{ unit(s)} \\ \hline \end{array}$$

This addition problem using blocks is shown below:



Online Resource Base 10 Blocks

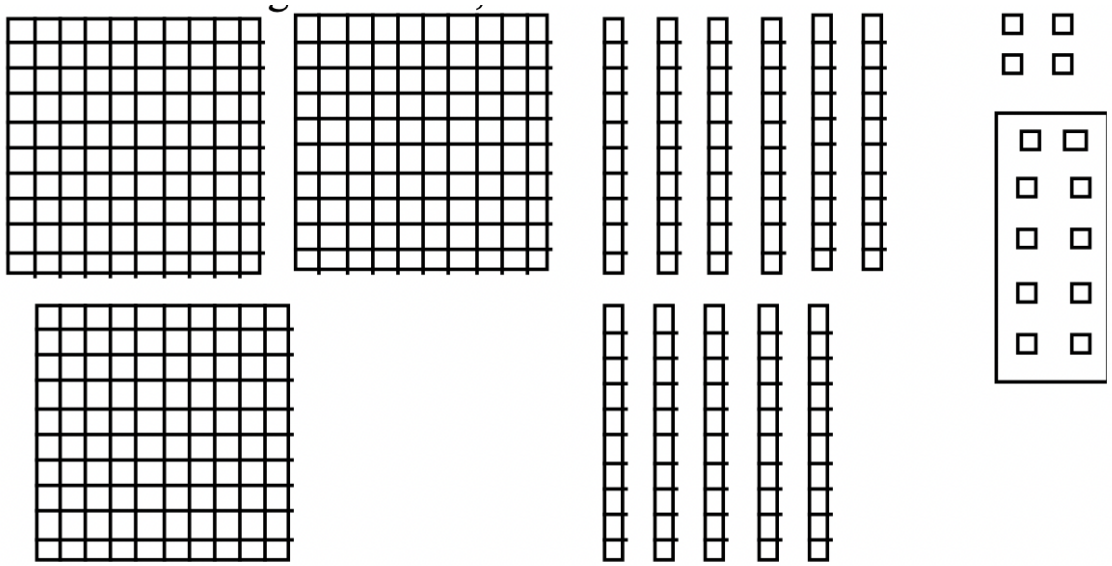
If we add, we get:

$$\begin{array}{r} 2 \text{ flat(s)} \ + 4 \text{ long(s)} \ + 6 \text{ unit(s)} \\ +1 \text{ flat(s)} \ + 7 \text{ long(s)} \ + 8 \text{ unit(s)} \\ \hline 3 \text{ flat(s)} \ + 11 \text{ long(s)} \ + 14 \text{ unit(s)} \end{array}$$

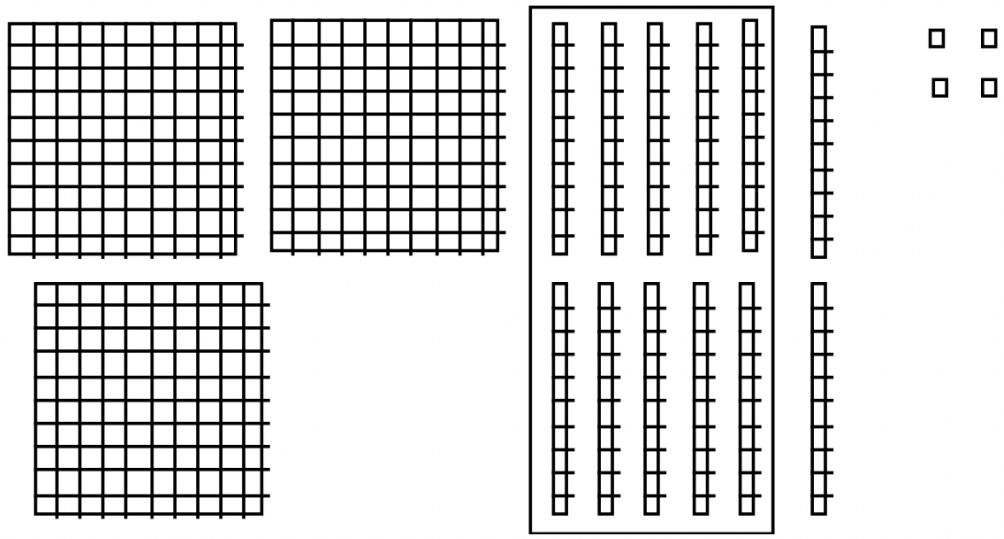
Trades can now be made. Ten of the 14 units can be exchanged for a long. That leaves us with 3 flats, 12 longs and 4 units. The sum and trades are illustrated with blocks on the next page.

After trading, sum = 3 flat(s) + 12 long(s) + 4 unit(s)

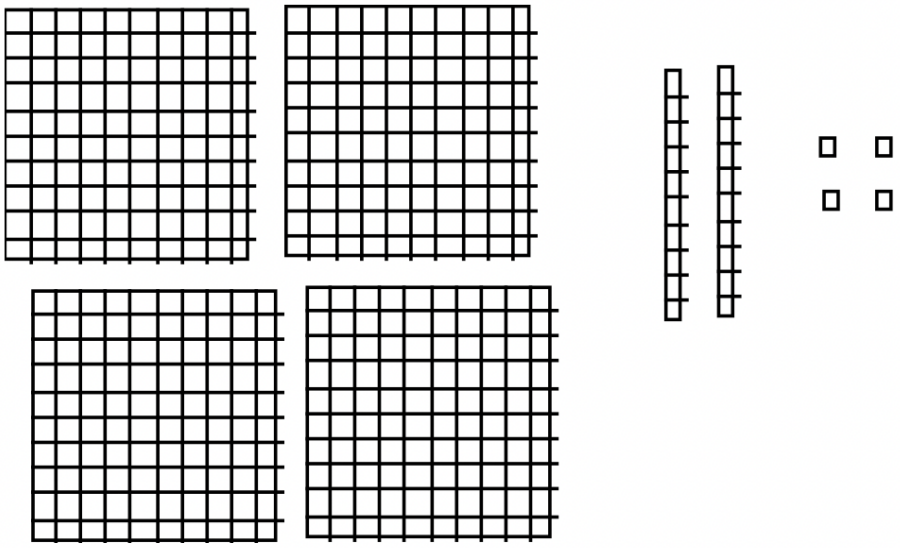
After combining the blocks, the sum is shown below.



The next step is to trade in ten units for a long. I've put a box around the units to be traded above. This yields 3 flats, 12 longs and 4 units as shown below.



Now ten of the 12 longs can be exchanged for a flat. I've put a box around ten of the longs to be traded above. After the exchange, the sum is 4 flats, 2 longs and 4 units, which is 424. This is illustrated below.



After making all traded, the final sum = 4 flat(s) + 2 long(s) + 4 unit(s)

We're in Base Ten, so the answer is 424. Therefore, 246 + 178 = 424.

	B	F	L	U
		2	4	6
+		1	7	8
		3	11	14
		3	12	4
		4	2	4

In the previous example of adding 246 + 178 using the Base Blocks, ten longs could have been exchanged for a flat at the same time that ten units were exchanged for a long. I only did one trade at a time because in the Standard Addition Algorithm, we don't combine all the blocks (or place values) at once. First, the units are added together to get 14. The 4 is written down in the unit's column and the other 10 units are written as 1 long in the next column to signify that there is another long. Now, there are 1 + 4 + 7 longs to add up, which is 12. The 2 is written down in the longs column and the other 10 units are written as 1 flat in the next column to signify that there is another flat. Finally, there are 1 + 2 + 1 flats to add up, which is 4. So the sum is 4 flats, 2 longs and 4 units, which is 424. The exchanges being made is what carrying is all about. To the right is another way of keeping track of blocks, flats, longs and units. Notice the trades being made at each step. If the largest addend has x digits, I make x + 1 columns to allow for carrying. In this problem, there are three digits for each addend, so I made four columns to allow for a possible block being made.

Example 4.2.1

Add the following using the Standard Algorithm.

a) 28 + 45

b) 324 + 78

Answer available at the end of this section.

Base 10-Blocks

Why does such a procedure work? Why "carry" the "1"? Why add by columns? To many people, these procedures remain a great mystery-we add this way because we were told to-it's simply done by rote with no understanding. Children in general learn abstract notions by first experiencing them concretely with devices they can actually see, touch, and manipulate. The use of concrete teaching aids-such as base-ten blocks-helps provide insight into the creation of the standard algorithm for addition. **Online Resource Base 10 Blocks**

Another way to add is by using expanded notation. Our first example, 246 + 178, can be written as (200 + 40 + 6) + (100 + 70 + 8). Using the commutative and associative properties, this sum can be written as (200 + 100) + (40 + 70) + (6 + 8) or 300 + 110 + 14 = 300 + (100 + 10) + (10 + 4) = (300 + 100) + (10 + 10) + 4 = 400 + 20 + 4 = 424. When written out like this, it is perhaps more clear what is really being added as opposed to doing it by rote without thinking about the place value of each digit.

Example 4.2.2

Add by using expanded notation. Show all steps.

a) 43 + 47

b) 288 + 524

Answer available at the end of this section.

Wonder, Play, Grow (Intermediate Algorithm - Partial Sums)

Suppose two different students add two numbers together as shown in the two problems to the right. Both have very similar methods. Explain what each student is doing and why it makes sense. Then, make up two more problems and add using one of these methods.

859	859
+467	+467
16	1200
110	110
+1200	+16
1326	1326

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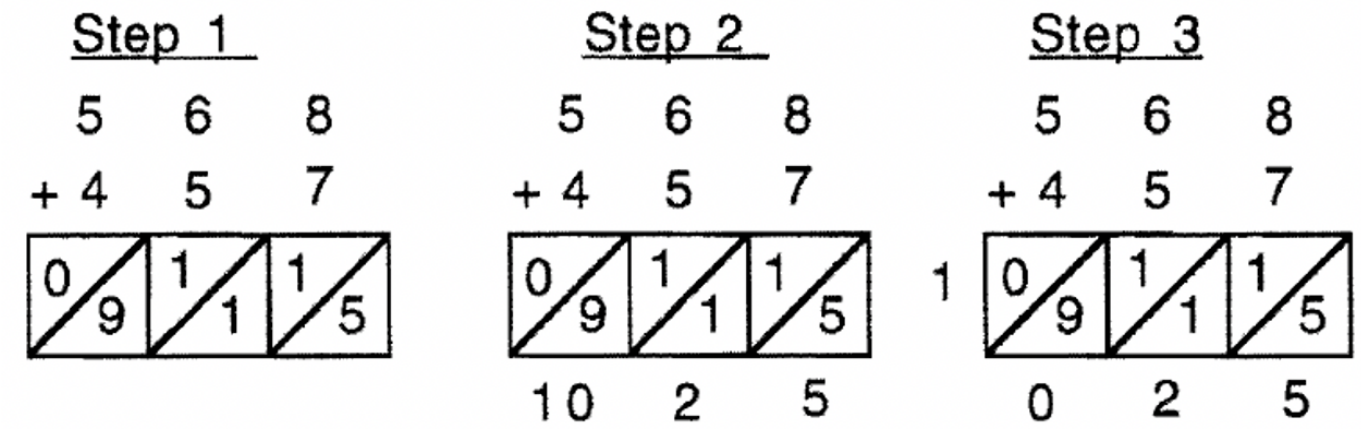
The methods outlined in the **Wonder, Play, Grow** are known as the **Intermediate Algorithm.** The **intermediate algorithm** for addition is a method that bridges **conceptual understanding** (like expanded form or base-ten models) and the **standard algorithm**. It emphasizes **place value** and **partial sums**, making it easier for students to understand *why* addition works, not just how to do it. See the video below to see more examples of the intermediae algorithm.

YouTube Video

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Below is another addition algorithm, called the **Lattice Method for Addition,** used for adding two numbers together. First, add down the columns, then down the diagonals. The addition problem is 568 + 457 and the answer is 1,025. See if you can understand how to do it and understand why it works. We'll use the lattice method again when we do multiplication.



Example 4.2.3

Use the lattice method to compute 628 + 456. Show your work.

Answer available at the end of this section.

Example 4.2.4

Use the method of your choice to compute the following:

a) $412_5 + 23_5$	b) $1011_2 + 101_2$
c) $4104_5 + 432_5$	d) $11001_2 + 1101_2$

Answer available at the end of this section.

Standard Algorithm for Substraction

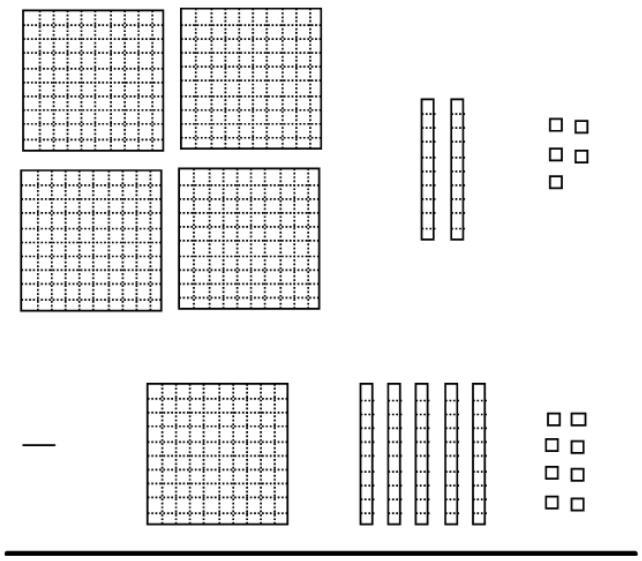
The most common subtraction algorithm is the ***Right to Left Standard Subtraction Algorithm***, which is where you start in the ones column and subtract, then move to the left and subtract at each column. The problem, of course, is when the top digit is less than the bottom digit and you have to regroup. Get out your Base Ten blocks now to see what is really going on.

Consider the subtraction problem 425 –158. From our knowledge of place value, we know that 425 is is 4 $\times$ 100 + 2 $\times$ 10 + 5 (or 400 + 20 + 5) and 158 is 1 $\times$ 100 + 5 $\times$ 10 + 8 (or 100 + 50 + 8). Using Base Ten blocks, 425 would be represented with 4 flats, 2 longs and 5 units whereas 158 would be represented with 1 flat, 5 longs and 8 units. So, the subtraction problem can be thought of in the following way:

Using Base Blocks

$$\begin{array}{r} (4 \text{ flat(s)} + 2 \text{ long(s)} + 5 \text{ unit(s)}) \\ -(1 \text{ flat(s)} + 5 \text{ long(s)} + 8 \text{ unit(s)}) \\ \hline \end{array}$$

Using blocks, the subtraction problem is shown below:

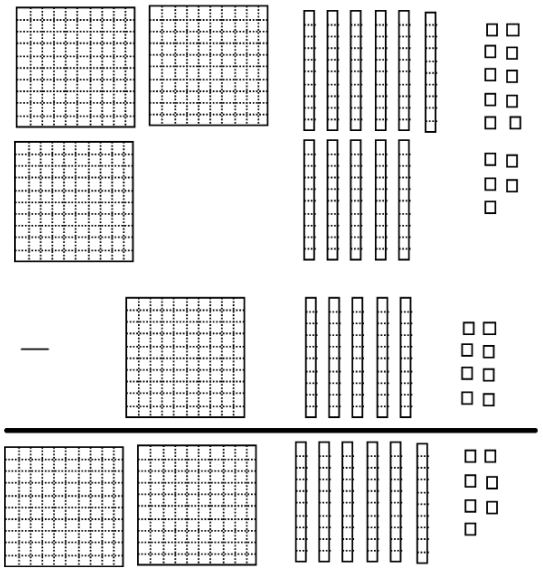


We can't subtract 8 units from 5 units, so a long is exchanged (sometimes know as "borrowing") for a ten units, which gives 4 flats, 1 long and 15 units in the minuend. Now, we also can't subtract 5 longs from 1 long, so 1 flat is exchanged ("borrowed") for ten longs, which gives 3 flats, 11 longs and 15 units in the minuend. Now we can subtract at each column, which is shown on the next page. You can use either the take away approach or the missing addends approach to subtract. **Use your base blocks to work through these problems.**

Using Base Blocks

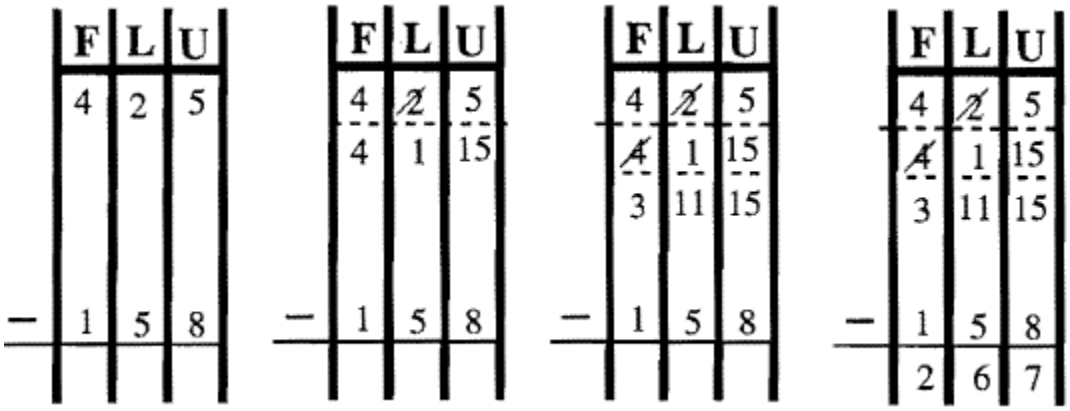
$$\begin{array}{r} (3 \text{ flat(s)} + 11 \text{ long(s)} + 15 \text{ unit(s)}) \\ -(1 \text{ flat(s)} + 5 \text{ long(s)} + 8 \text{ unit(s)}) \\ 2 \text{ flat(s)} + 6 \text{ long(s)} + 7 \text{ unit(s)}) = \underline{267} \end{array}$$

The subtraction problem using blocks is shown below:



Since we are working in Base Ten, the answer (difference) is 267. So, 425 –158 = 267.

The problem can be worked using a chart where enough space is left between the minuend and subtrahend so exchanges can be made in the minuend. Notice the steps shown if you work this problem using a Base chart as shown below.



The traditional, standard subtraction algorithm is not necessarily the best one and there are many other ones you'll be learning in the rest of this Exercise Set.

The next algorithm was taught to many people before the "new math" and so I call it the **Oldtimer's Algorithm**. It's similar to the Standard Algorithm in that you start on the right and move to the left. But you don't regroup (formally known as borrowing). First look again at an example of using the Standard Algorithm to subtract 95 – 28. Since, you can't subtract 8 from 5, you regroup from the 9 by crossing out the 9 and writing 8 so you can put a 1 in front of the 2. Now, in the unit's column, 15 – 8 = 7 and in the ten's column, 8 – 2 = 6. The answer is 67. This new method starts out the same way. You can't subtract 8 from 5, but instead of regrouping from the 9, you *add a 1 to the number under the 9,* and put a 1 in front of the 5. Notice that in the unit's column, 15 – 8 = 7 and in the ten's column, instead of 8 –2, you have 9 –3 which is the same answer of 6. The advantage of this method is if you put a 1 at the top of one column, you compensate by adding a 1 to the bottom of the next column. You don't have to worry about crossing things off and making a mess and you don't have to think about the digit in the column to the left until you go to that column to subtract – no trying to regroup and getting hung up there! Look at the two methods side by side. If you use the new method, you have to leave space between the subtrahend and subtraction bar in case you need to put a 1 there.

$$\begin{array}{r} 8 \\ \not{9}^{1}5 \\ -\ 2\ 8 \\ \hline 6\ 7 \end{array} \qquad \begin{array}{r} 9^{1}5 \\ 2\ 8 \\ -\ 1 \\ \hline 6\ 7 \end{array}$$

Figure 4.2.1: **Oldtimer's Algorithm Comparison**

Example 4.2.5

Subtract the following using the Standard Algorithm and Oldtimer's Algorithm. Check your answers by adding up.

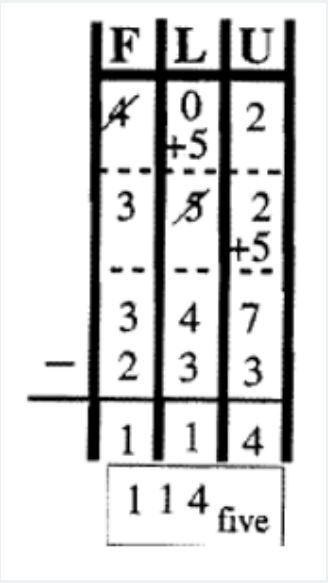
a) $\begin{array}{r} 314 \\ -221 \\ \hline \end{array}$

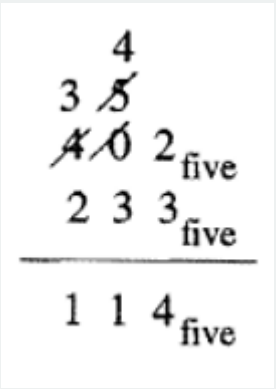
b) $\begin{array}{r} 5036 \\ -3418 \\ \hline \end{array}$

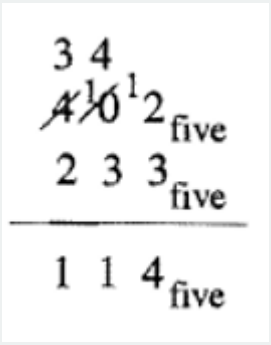
Answer available at the end of this section.

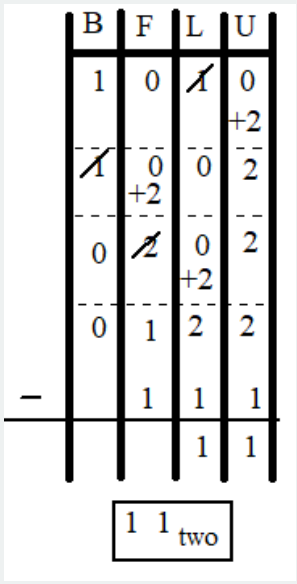
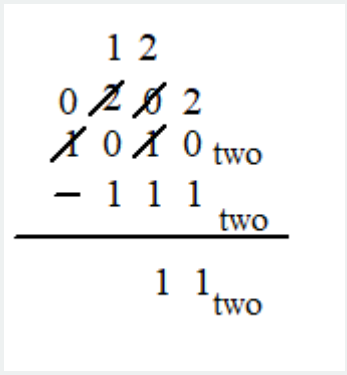
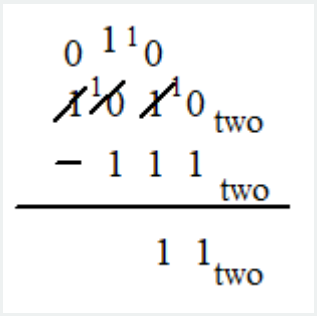
Subtraction in Other Bases

$\begin{array}{r} 402_{\text{five}} \\ -233_{\text{five}} \\ \hline \end{array}$







1010_{two} -111_{two}			

$$\begin{array}{r} 114_5 \\ +233_5 \\ \hline 402_5 \end{array}$$ Check:

$$\begin{array}{r} 11_2 \\ +111_2 \\ \hline 1010_2 \end{array}$$ Check:

In the sample problems above, when I needed to make an exchange and there was a zero in the place value to the left, the exchange had to be made at a higher place value first. Then, exchanges are made down the line until you can subtract. This case happens in Base Ten all the time when you subtract a number from 100, 1000, 10000, etc. Most people just start crossing off each zero and putting 9's above them out of habit, without having any idea why they are doing it. To subtract 235 from 1000, first you should cross off the 1, put a zero above it and put a 1 in front of the zero in the hundred's place (which is trading in 1000 for 10 hundreds), then you should cross off that 10 in the hundred's place, put a 9 above it and put a 1 in front of the zero in the ten's place (which is exchanging 10 hundreds for 9 hundreds and 10 tens), then you should cross off that 10, put a 9 above it and a 1 in front of the zero in the one's place (which is exchanging 10 tens for 9 tens and 10 ones). Then, you can subtract as shown in the example to the right. It's important to really understand what's going on here because children usually have a very hard time when there is a zero to cross off.

$$\begin{array}{r} 0\ 9\ 9 \\ \cancel{1}\cancel{0}\ \cancel{1}\cancel{0}\ {}^{1}0 \\ -\ 2\ 3\ 5 \\ \hline 7\ 6\ 5 \end{array}$$

Example 4.2.6

Use the different methods to compute the following:

a) $412_5 - 23_5$	b) $1011_2 - 101_2$
c) $4104_5 - 432_5$	d) $11001_2 - 1101_2$

Answer available at the end of this section.

Ways To Subract Without "Borrowing"

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https://youtu.be/Sq3VrltjdkI?si=1vMFtjr-SMcMt0DN

https://youtu.be/Fj20yMuvOjE?si=3s5BdTA_iwwesuFL

After viewing the videos from Math Educator Dr. Howie Hua, use one or more of his techniques to solve the following:

i) 123 - 45

ii) 1000 - 235

Then share with someone what you did. :)

Answers

Example 4.2.1

a) **Ones place**: 8 + 5 = 13 → write 3, carry over 1 **Tens place**: 2 + 4 = 6, plus 1 carried over = 7

1 ← carry-over
28
+ 45
____
73

b) **Ones place**: 4 + 8 = 12 → write 2, carry over 1 **Tens place**: 2 + 7 = 9, plus 1 carried over = 10 → write 0, carry over 1 **Hundreds place**: 3 + 1 = 4

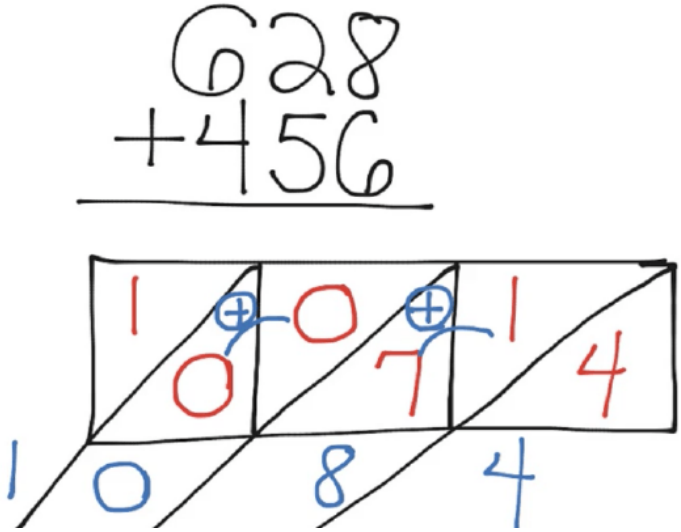
1 1 ← carry-overs
324
+ 78
____
402

Example 4.2.2

a) **Add tens and ones separately** Tens: 40 + 40 = 80, Ones: 3 + 7 = 10, **Add the partial sums** 80 + 10 = 90

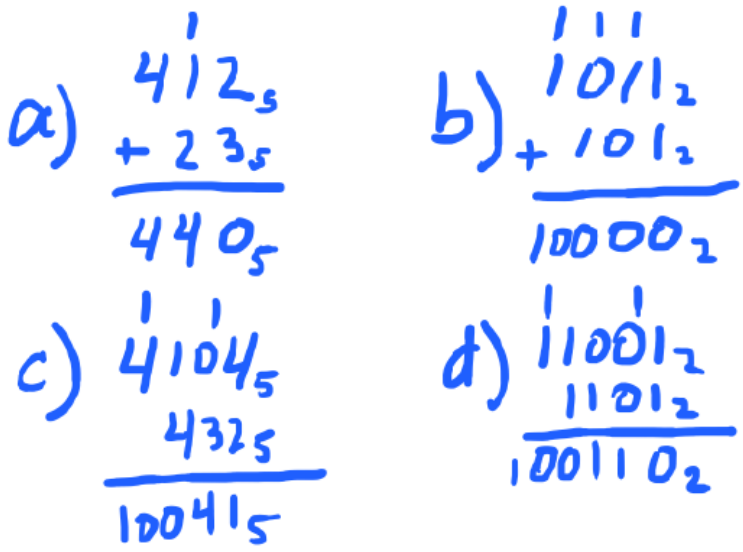
b) **Add hundred, tens and ones separately** Hundreds: 200 + 500 = 700, Tens: 80 + 50 = 130, Ones: 8 + 4 = 12, **Add the partial sums** 700 + 130 + 12 = 842

Example 4.2.3



Example 4.2.4

a) 440_5 b) 10000_2 c) 10041_5 d) 100110_2



Standard Algorithm:

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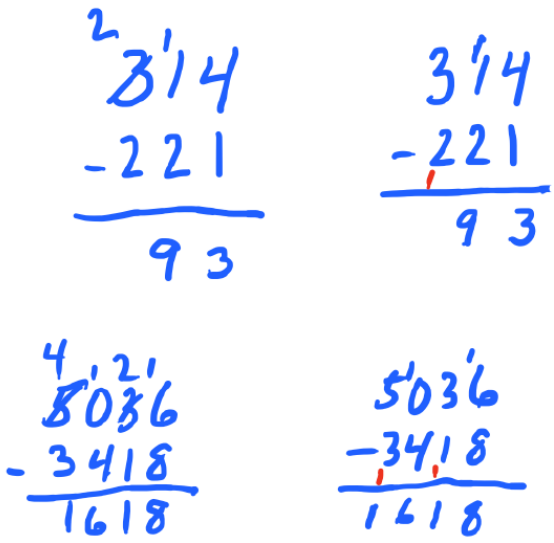
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Example 4.2.5

a) 93

b) 1618



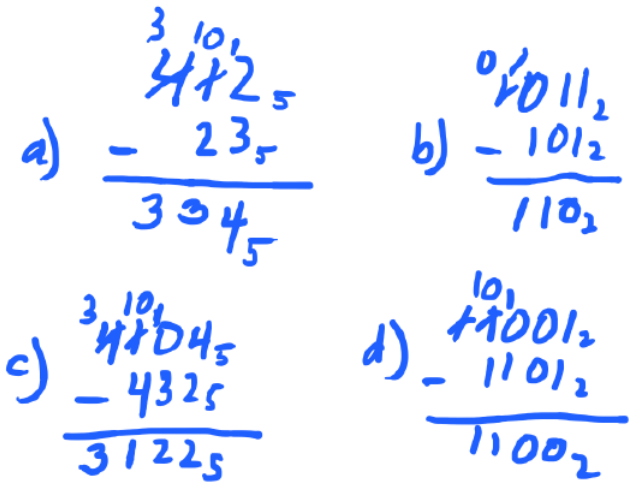
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Example 4.2.6

a) 334_5 b) 1110_2 c) 3122_5 d) 1100_2



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4.2.1: Written Retrieval and Problem Solving Retrieval Practice

Student Learning Objectives

SLO 4.3 Students will model and justify standard written algorithms for basic operations (addition, subtraction, multiplication, division) using concrete models, place value, and mathematical properties.

SLO 4.4 Students will compute sums, differences, products, and quotients in various bases (2–12) using standard algorithms and nontraditional methods, such as lattice and scaffold techniques.

The mild assignment 🌶 is a shorter and less challenging assignment. The medium assignment🌶🌶 is right in the middle in terms of length and level of difficulty. Finally, the spicy assignment 🌶🌶🌶 is a longer assignment that is the most challenging.

(Spicy 🌶🌶🌶 Problems are OPTIONAL, but go for it when you have the time to challenge yourself)

Exercise 4.2.1.1 🌶🌶

Sketch how the base ten blocks would be used.

a) 267 + 32 b) 247 + 86

Answer available at the end of this section.

Base 10 Blocks Online Resource

Exercise 4.2.1.2 🌶🌶

Use this expanded form of the addition algorithm to compute the following sums.

a) 351 + 75 b) 564 + 437

Answer available at the end of this section.

Exercise 4.2.1.3 🌶🌶

Use the Intermediate Algorithm to compute the following sums.

a) 598 + 486 b) 1667 + 865

Answer available at the end of this section.

Exercise 4.2.1.4 🌶🌶

Compute the following sums using the lattice method.

a) 582 + 369 b) 7654 + 987

Answer available at the end of this section.

Exercise 4.2.1.5

Give an advantage and/or a disadvantage of each of the following methods for addition.

a) Expanded Form b) Standard Algorithm

Answer available at the end of this section.

Exercise 4.2.1.6

Use the Dot Method to add 1 + 3 + 5 + 7 + 9.

Answer available at the end of this section.

Exercise 4.2.1.7

Use the method of your choice to compute the following:

a) $12_5 + 23_5$	b) $101_2 + 11_2$
c) $412_5 + 43_5$	d) $1101_2 + 1101_2$

Answer available at the end of this section.

Exercise 4.2.1.8

Sketch how the base ten blocks would be used to find each difference. (Check you answer using the Standard Algorithm)

a) 85 - 62 b) 343 - 224

Answer available at the end of this section.

Exercise 4.2.1.9

Use the Oldtimer's Algorithm to find each difference.

a) 1234 - 765 b) 7531 - 642

Answer available at the end of this section.

Exercise 4.2.1.10

Order these computations from easiest to hardest. a) 809 - 306 b) 8 - 3 c) 82 - 67

Answer available at the end of this section.

Exercise 4.2.1.11

Use one of Howie Hua's strategies to compute the following.

a) 1000 - 789 b) 101 - 17

https://youtu.be/Sq3VrltjdkI?si=1vMFtjr-SMcMt0DN

https://youtu.be/Fj20yMuvOjE?si=3s5BdTA_iwwesuFL

Answer available at the end of this section.

Exercise 4.2.1.12

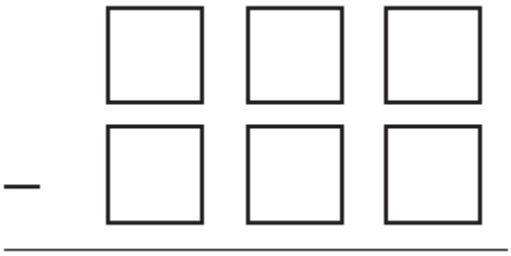
Use the method of your choice to compute the following:

a) $142_5 - 12_5$	b) $1011_2 - 100_2$
c) $432_5 - 43_5$	d) $11101_2 - 1100_2$

Answer available at the end of this section.

Problem Solving Practice

1) Place the digits 3, 5, 6, 2, 4, 8 in the boxes to obtain the following differences.



a) The Greatest Difference

b) The Least Difference

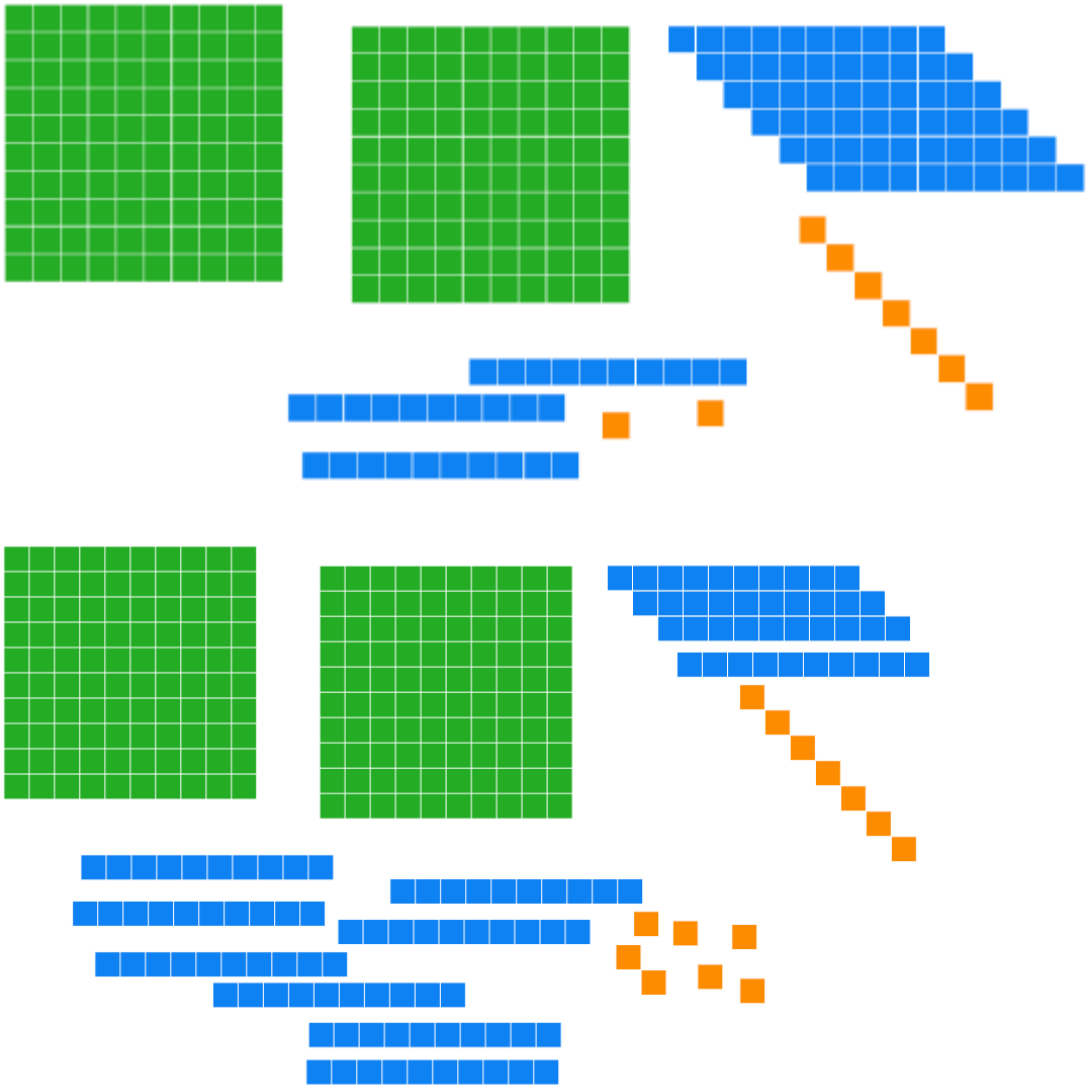
2) To stimulate his child in the pursuit of mathematics, a math professor offered to pay them $8 for every equation correctly solved and to fine them $5 for every incorrect solution. At the end of 26 problems, neither owed any money to the other. How many did the child solve correctly?

Answers

Exercise 4.2.1.1

Use this online resource to access base-10 blocks.

a) 299 b) 333

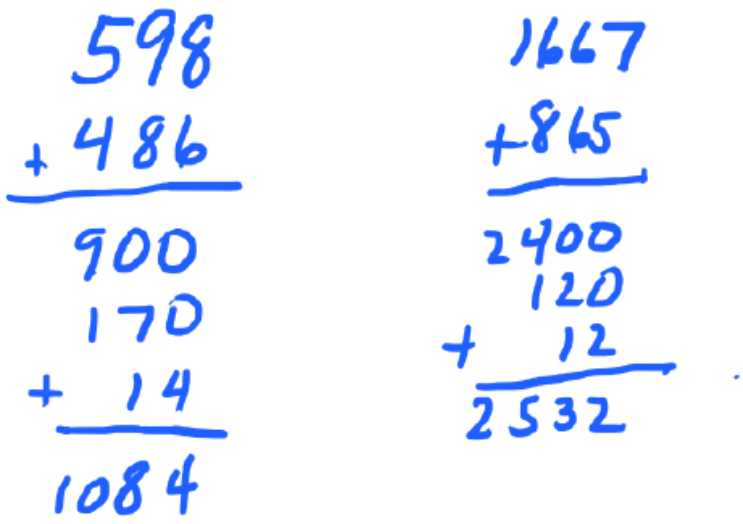


Exercise 4.2.1.2🌶🌶

a) 300 + (50 + 70) + (1 + 5) = 300 + 120 + 6 = 426

b) (500 + 400) + (60 + 30) + (4 + 7) = 900 + 90 + 11 = 1001

Exercise 4.2.1.3🌶🌶



Exercise 4.2.1.4🌶🌶

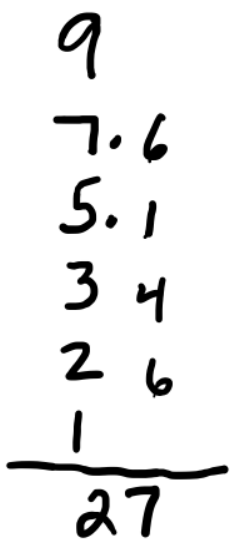


Exercise 4.2.1.5🌶🌶

a) **Expanded Form Advantage:** Builds strong place value understanding by showing the value of each digit. **Disadvantage:** Time-consuming and can confuse students with too many steps.

b) **Standard Algorithm Advantage:** Quick and efficient, ideal for solving problems accurately. **Disadvantage:** Often used without understanding, which can limit conceptual learning.

Exercise 4.2.1.6🌶🌶



Exercise 4.2.1.7🌶🌶

a) 40_5 b) 1000_2 c) 10105 d) 11000_2

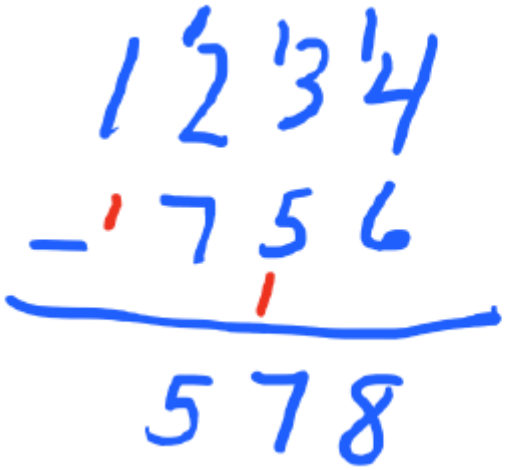
Exercise 4.2.1.8🌶🌶

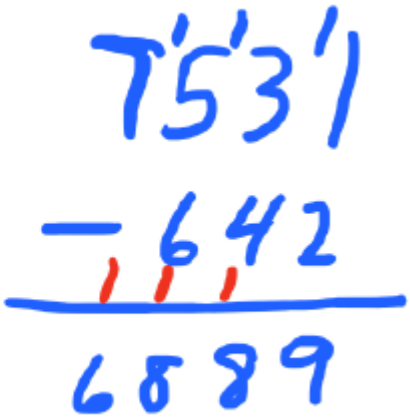
Use this online resource to access base-10 blocks.

a) Video Solution

b) Video Solution

Exercise 4.2.1.9🌶🌶





🌶🌶**Exercise** 4.2.1.10

The difficulty increases with the number of digits and the need for **regrouping.** According to sources like *Mathematical Mindsets* by Jo Boaler and *Adding It Up*, conceptual understanding and number sense are crucial, especially for multi-digit operations. Problems with more digits and potential regrouping steps typically challenge learners' understanding of place value and operation algorithms more than single-digit facts. c) or a) may be the hardest with b) being the easiest

🌶🌶🌶**Exercise** 4.2.1.11

You got this! a) 999-788=211 b) 99-15=84 OR 3+80+1=84 (count up)

🌶🌶**Exercise** 4.2.1.12

a) 130_5 b) 111_2 c) 334_5 d) 10001_2

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4.2.2: Addition and Subtraction Progressions Discusison Board

Discussion Board Assignment: Addition and Subtraction Progressions

YouTube Video

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In your post, respond thoughtfully to the following prompts. You may write in paragraph form or use bullet points to organize your ideas clearly.

Part 1 – Understanding Big Ideas

1. Why is it important for students to understand addition and subtraction conceptually before being introduced to the standard algorithm?
2. The speaker mentions that models shouldn't "expire." What does that mean?

Part 2 – Understanding Details

3. **According to the transcript, the standard algorithm for addition and subtraction becomes an expectation at what grade level?**

- A. End of 2nd grade
- B. Middle of 3rd grade
- C. End of 4th grade
- D. Start of 5th grade

4. **In kindergarten, students begin to learn addition and subtraction by:**

- A. Memorizing math facts
- B. Using calculators
- C. Counting and joining sets
- D. Writing numbers in expanded form

5. **What mathematical idea is introduced when students begin seeing numbers like 13 as 10 and 3 more?**

- A. Estimation
- B. Subitizing
- C. Unitizing
- D. Decomposition

6. **Which of the following best describes the progression from concrete to abstract understanding in addition and subtraction?**

- A. Memorize → Practice → Test
- B. Count → Guess → Calculate

- C. Model → Represent → Abstract
- D. Listen → Copy → Recall

Part 3 – Ask A Question

7. After watching the video, use one of the question starters below to ask a thoughtful question that encourages further reflection, discussion, or exploration. Your question should be designed to spark thinking—not just recall information.

   Possible Question Starters
   - I'm wondering how this approach would work when…" "
   - What would happen if a student struggles with…?"
   - "How might we support students who don't yet understand…?"
   - "Why do you think it's important that we…?"
   - "Could someone explain more about the idea of…?"
   - "What connections can we make between _____ and _____"

Replies to Peers (Due by [insert date])

Reply to at least **two classmates**. In your replies, you can:

- Share feedback or encouragement
- Ask a clarifying or probing question
- Offer a possible answer to a question.

Keep your responses kind, constructive, and engaging — we're building a community of future educators!

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4.2.3: Online Retrieval and Problem Solving Practice

Base 5 Arithmetic Retrieval Practice

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4.3: Embracing Diversity of Algorithms for Multiplication and Division

Multiplication

The standard multiplication algorithm is efficient but relies heavily on place value, multiplication facts, and the distributive property. While it streamlines complex calculations, it often hides the underlying math concepts, making it hard for students to make sense of what they're doing. This lack of conceptual understanding can lead to errors and "buggy" algorithms when students forget a step or misalign place values. Without meaning behind the steps, students struggle to check if their answers are reasonable. In contrast, students can create their own multiplication strategies or use alternative thinking strategies, which tend to be more intuitive and flexible. These alternative methods, like repeated addition or doubling, often evolve into more efficient algorithms as their understanding deepens.

Many people use repeated addition all the time to multiply. Consider the multiplication problem 26 times 17. Basically, this means add 26 seventeens added together. This could be done by adding them together one at a time. That would take a long time. Another approach would be to add 10 seventeens together first to get 170, then add 10 more seventeens together to get 170 again, then add 5 more seventeens together which is 85, and then add one more seventeen. Adding 10 seventeens (170), 10 seventeens (170), 5 seventeens (85) and 1 seventeen (17) gives 170 + 170 + 85 + 17 = 442, which is exactly the answer to 26 times 17. That should be no surprise since we added 26 (10+10+5+1) seventeens together. Someone else might add 10 seventeens (170), 10 seventeens (170), 3 seventeens (51) and 3 more seventeens (51) to get 170 + 170 + 51 + 51 = 340 + 102 =442. Figure 4.3.1

🧠Let's Listen to Learn❤️

Audio

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$$(10 \times 17) + (10 \times 17) + (5 \times 17) + (1 \times 17) = 170 + 170 + 85 + 17 = 442$$

OR

$$(10 \times 17) + (10 \times 17) + (3 \times 17) + (3 \times 17) = 170 + 170 + 51 + 51 = 442$$

Figure 4.3.1

One reason it's advantageous to do multiplication using repeated addition is that many problems can be done in your head this way. But, of course, every new method takes practice and getting used to, and it's common for people to revert back to using their old, familiar methods even if it takes longer and requires paper. That's okay, too. It is, however, very important that a teacher knows several ways so that he or she can provide more opportunities for students. What seems obvious to one student is foreign to another, so you give alternatives and present different ways of looking at things.

Example 4.3.1

Use some form of repeated addition to compute the following multiplication problems. Show how you did each one.

a. 23×13

b. 14×14

Answer available at the end of this section.

One new algorithm (for most of you reading this) is called **Duplation (Egyptian Multiplication)**. This is actually a very old algorithm used in ancient times by the Egyptians. It also uses the principle that multiplication is simply repeated addition. It is called duplation because it uses the idea of doubling numbers to get the answer. Let's look again at the first multiplication we did in this section. It was 26×17. Since 26 seventeens must be added together, the number 26 is broken down into powers of two. It's easy to get powers of two: simply start with 1, then double it to get 2, then double it to get 4, then double it to get 8, then double it to get 16, etc. To figure out how far you have to go in the doubling process, stop if doubling gives a number bigger than the number you are trying to break down (26, in this case). Now, all this sounds more complicated than it really is. Before explaining too much more, notice how you'll write it. Write 26 at the top and underline it. Underneath, start with 1, then, double continuously, stopping at 16 since if you doubled another time, you'd get a number bigger than 26. To the right of 26, write the number 17 and underline it.

Underneath it, first write 17, then double 17 to get 34, then double 34 to get 68, double 68 to get 136, etc., until you have the same amount of numbers under the 17 as you do under the 26.

$\underline{26}$	$\times$	$\underline{17}$
1	17	
2	34	
4	68	
8	136	
16	272	
……	……	

Do you see the corresponding numbers? 1 corresponds with 17, because 17 is 1×17, 2 corresponds with 34, because 34 is 2×17, 4 corresponds with 68, because 68 is 4×17, 8 corresponds with 136, because 136 is 8×17, and 16 corresponds with 272, because 272 is 8×17. Now, you may need to think about this for a few minutes. The left side keeps track of how many of some number you are adding together. So, the 1×17 should be clear. So, 2×17 is simply 17 doubled. Now if 2×17 is 34, then 4×17 is twice as many as 2×17, so double 34 to get 68. This continues on and on. So, if 8×17 is 136, then if we double 136, we should have 16×17. Isn't it neat how we know that $16 \times 17 = 272$ and we just double a few numbers to get there?

Okay, now we only need to add 26 seventeens together. Simply start at the bottom of the first column, and check off numbers that add up to 26 (this is like doing it in base two). After you check off the numbers in the left column, circle or point to their corresponding numbers in the right column. It's those numbers in the right column that you add together to get the answer. Watch how the rest of the problem is done:

26	×	17
1		17
✓2	----->34	
4		68
✓8	-----136	
✓16	-----272	

272 + 136 + 34 = **442** (yep, it's the same answer again!)

If that all sounds complicated, maybe math educator Dr. Howie Hua can better explain in this video.

YouTube Video

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https://youtu.be/whnAk43GGM0?si=4wYzcnRozBcGGih

Here are two more examples for you to study. I'll do 15×35 and 35×15.

15	×	35
✓1	-----35	
✓2	-----70	
✓4	-----140	
✓8	-----280	

280+140+70+35=**525**

35	×	15
✓1	-----15	
✓2	-----30	
4		60
8		120
16		240
✓32	-----480	

480+30+15=**525**

Example 4.3.2

Multiply 13×29 using duplation (Egyptian Multiplication). Show all of your work. Make it clear what is added together for each problem.

Answer available at the end of this section.

The method you are now going to learn is called the **LATTICE METHOD** and it could be used for multiplying 2 digit or 3 digit or even bigger numbers together. The example we'll do is 47×32.

STEP 1: To begin, put one of the numbers at the top (47) of a rectangle (that has a space for each digit in the number) and the other number along the side (32) of the same rectangle (that has a space for each digit). In this case, we need a rectangle that has two spaces across the top and 2 spaces along the side. Then, you make diagonals. Look to the right.

STEP 2: Multiply each of the digits on the top by each of the digits along the side, and put the answer where they would meet. For instance, look where the 12 goes when you multiply 4 times 3.

STEP 3: There are three other multiplications to do: 7 times 3, 4 times 2 and 7 times 2. Do you see how all the answers are filled in? It doesn't matter which ones you multiply and put in first.

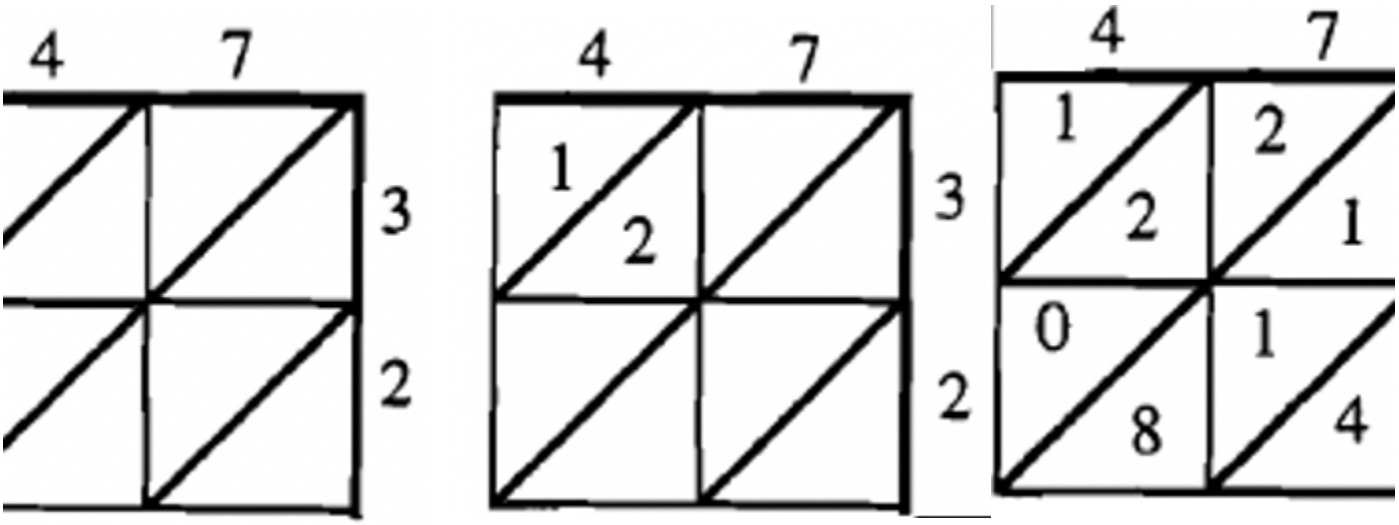
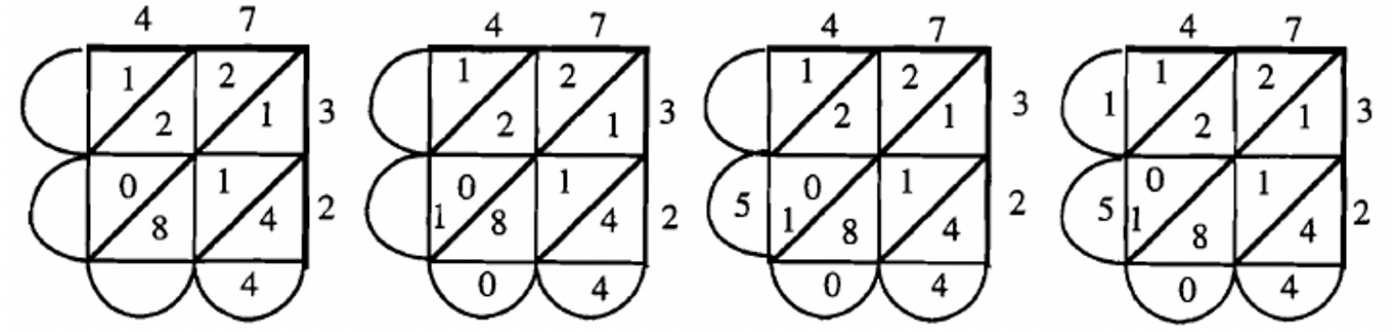


Figure 4.3.2:

STEP 4: There are four diagonals (made up of little triangles) slanting to the right inside the rectangle. Starting with the one in the right bottom corner, add the digits inside each diagonal and fill put the sum in the "cup" under its diagonal. In the first diagonal is 4, so 4 is put in the "cup" under its diagonal (bottom right of rectangle). In the next diagonal are three digits a 1, 1 and 8 which adds up to 10, so put a 0 in the cup under that diagonal (at the bottom to the left) and a 1 (for the carry) in one of the triangles to the left. The next diagonal has a 2, 2, 0 and 1, which adds up to *5,* and goes in the "cup" under its diagonal (left, bottom). In the last diagonal, there is just a 1, so a 1 is put in the "cup"under its diagonal (left, top). Look at how I filled in the numbers one at a time.



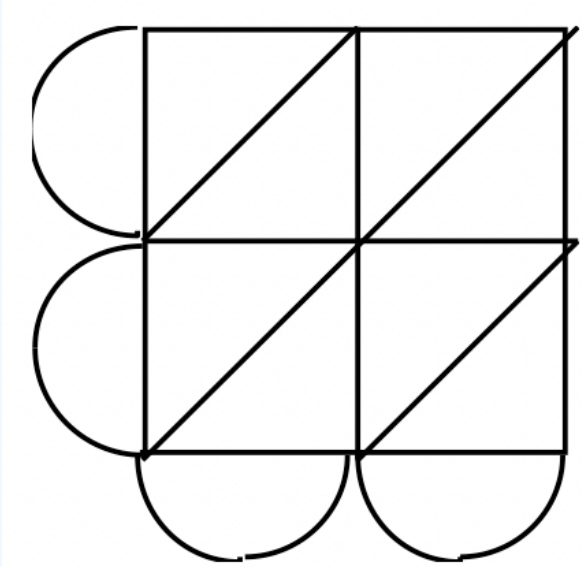
STEP 5: The numbers you filled in under each diagonal in the "cups" (outside the rectangle) form the answer. It is read by starting at the left top and reading down and then across the bottom. In this case, the answer is 1504. Isn't that cool? That's all there is to it!

Most of my students love this very old Lattice method of multiplying. It works as easily for small numbers as it does for large numbers. First, you simply do a series of simple single digit multiplications, and then you add, with carrying. Actually you could use the left to right (with underlining for carrying) method of addition for the last step. The main disadvantage is that you have to draw a lattice, which is a small price to pay to make multiplication easier and less complicated. Most likely, it takes less time than using the standard algorithm with carrying, because there's less chance of error and confusion with all that multiplying and carrying, etc. I propose all students learn multiplication this way first, and let them stay with it if they like it. Who needs to shorten it up and make it more complicated than it really is? Ah, well, I take it you get my point.

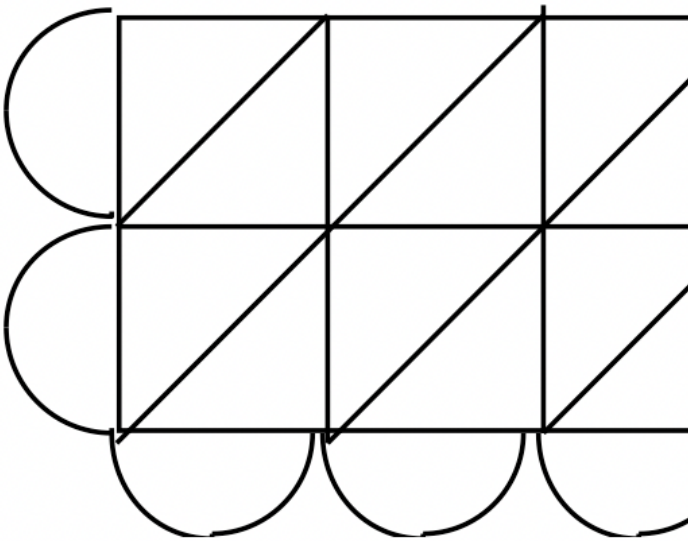
Example 4.3.3

Use the lattices provided in a) and b) to multiply the given numbers. Then write the answer in the space provided. For c, draw your own lattice to compute the multiplication. By the way, you don't have to draw the little cups.

a. 63×57 = ___



b. 342×85 = ___



c. 58×67 = ___ Draw your own lattice.

Answer available at the end of this section.

Interactive Element

Standard Algorithm for Multiplication

Before we take a look at the standard algorithm for multiplication, Figure 4.3.3 shows different conceptual ways to see multiplication. The concrete model is a way of seeing repeated addition using base-10 blocks, while the Place Value Representation uses expanded forms and distributive property to calculate the product.

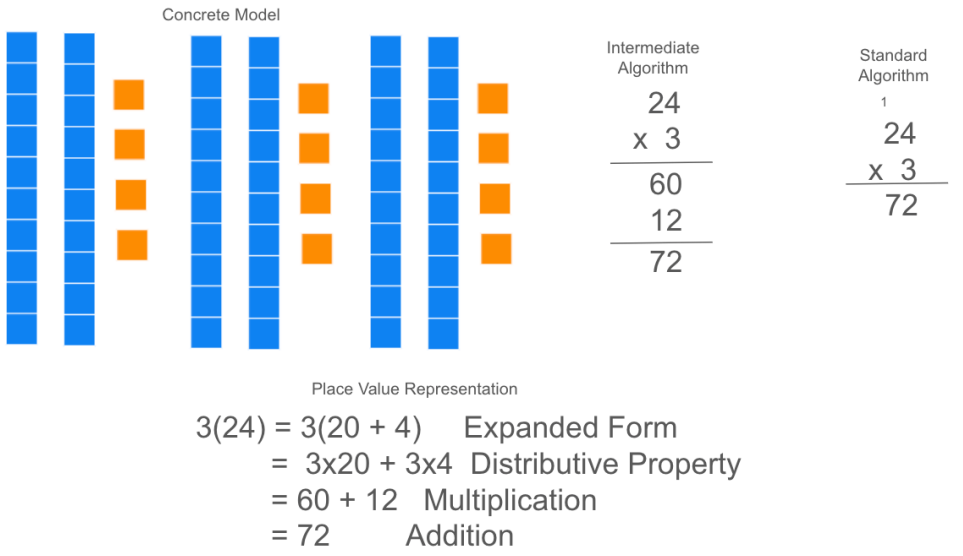


Figure 4.3.3

Finally, you could use the standard multiplication algorithm to compute products. This is the shortest algorithm in terms of space on the paper, but you have to be fairly adept at mental calculations. However, before we do what we learned in elementary school , we can use the Intermediate Algorithm (Partial Products Algorithm) to compute multiplication problems - See Figure 4.3.3.

Consider 346 x 70. Doing the **intermediate algorithm** or **partial products algorithm** is similar to how we used the distributive property to multiply. For instance, to multiply, rewrite the problem like this:

$$\begin{aligned}
&346 \times 72 = (300 + 40 + 6) \times (70 + 2) \\
&= (300 + 40 + 6) \times 70 + (300 + 40 + 6) \times 2 \\
&= 300 \times 70 + 40 \times 70 + 6 \times 70 + 300 \times 2 + 40 \times 2 + 6 \times 2 \\
&= 21000 + 2800 + 420 + 600 + 80 + 12 \\
&= 24912
\end{aligned}$$

The partial products are shown in the third line in the example above. They are:

$$300 \times 70, 40 \times 70, 6 \times 70, 300 \times 2, 40 \times 2 \text{ and } 6 \times 2$$

It's easier to write this in a vertical format. I'll show this problem two ways. It doesn't matter which partial products you multiply first. The second way it is shown is more similar to our standard algorithm, where we start with the unit's value of the bottom number and eventually work up to the larger place values.

346		346	
$\times \underline{72}$		$\times \underline{72}$	
21000	(300×70)	12	(2×6)
2800	(40×70)	80	(2×40)
420	(6×70)	600	(2×300)
600	(300×2)	420	(70×6)
80	(40×2)	2800	(70×40)
$\underline{12}$	(6×2)	$\underline{21000}$	(70×300)
24912		24912	

On the left the first three partial products are 346×70 and the second three partial products are 346×2. On the right, the first three partial products are 2×346 and the second three partial products are 70×346.

Our standard algorithm is simply a shortening up of the partial products algorithm. We don't write all the zeroes and we doing the carrying involved with adding more than one partial product at a time in our head. Sure, it's shorter, but it is easier to make

mistakes.

$$\begin{array}{rl} 346 & \\ \times\underline{72} & \\ 692 & (2 \times 346) \\ \underline{24220} & (70 \times 346) \\ 24912 & \end{array}$$

Usually we just move over one place to the left to account for the zero in the one's place and don't write the zero in 24220. We "think" 7 times 346.

Example 4.3.4

For each problem, write out the partial products in a vertical format to multiply the numbers together. Show your work, identifying the multiplication for each partial product. Check your answer using the standard algorithm.

a) $\begin{array}{r} 27 \\ \times\underline{91} \end{array}$

b) $\begin{array}{r} 538 \\ \times\underline{34} \end{array}$

Answer available at the end of this section.

Wonder, Play, Grow

Task 1: Considering the diverse sets of algorithms we learned to multiply, how would you do the following base-5 and base-2 problems? Try the first two independently, make sure to make an effortful attempt even if you are not sure. Try something.

i) $34_{\text{five}} \times 2_{\text{five}}$

ii) $101_{\text{two}} \times 10_{\text{two}}$

Task 2: Now share with others what you did. Decide together what is the best way to proceed to complete these problems.

Task 3: Compute the following.

iii) $34_{\text{five}} \times 12_{\text{five}}$

iv) $134_{\text{five}} \times 32_{\text{five}}$

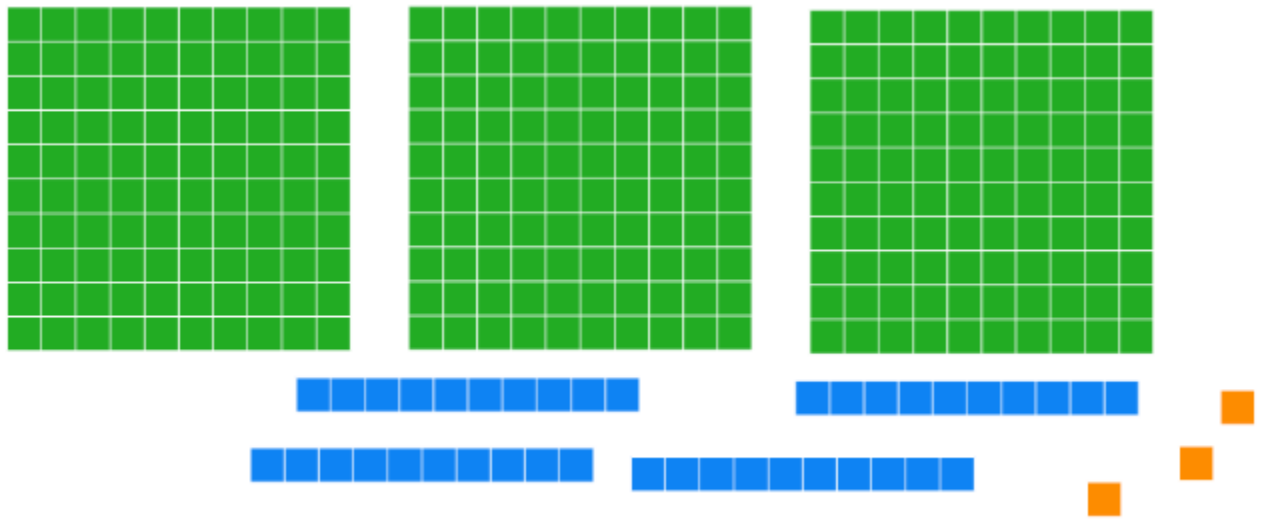
iV) $1011_{\text{two}} \times 110_{\text{two}}$

Division

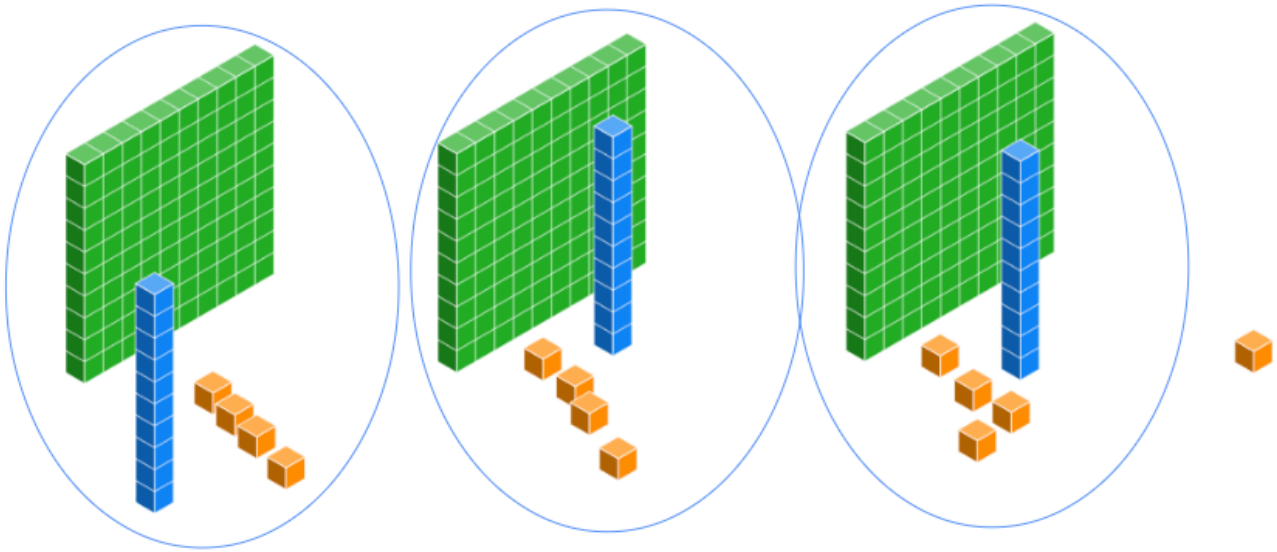
While calculators can now do much of this work for us, understanding long division—especially with one- or two-digit divisors—is still useful and helps us think mathematically. To make this clearer, we'll use base ten blocks and repeated subtraction to build a strong visual and conceptual foundation before moving into the standard algorithm. Remember, division isn't just one way of doing things—there are many strategies, and we'll look at several so you can choose the one that makes the most sense to you.

Let's recall the **division algorithm** (it states that if a and b are any whole numbers with $b \neq 0$, there exist unique whole numbers q and r such that $a = bq + r$, where $0 \leq r < b$). Remember its just a way of expressing division using a simple idea: if you have two whole numbers, like 17 and 5, you can figure out how many times 5 fits into 17 and what's left over. In math language, we say that $17 = 5 \times 3 + 2$. That means 5 fits into 17 three times with 2 left over. This idea is behind the long division standard algorithm, a method you may have seen before. The following Figure 4.3.1 illustrates the sharing division using base ten blocks and the

standard algorithm. Find the quotient and remainder of 343 divided by 3, 343 = 3 x 114 + 1



343



Three equal groups with one remaining - Sharing Division

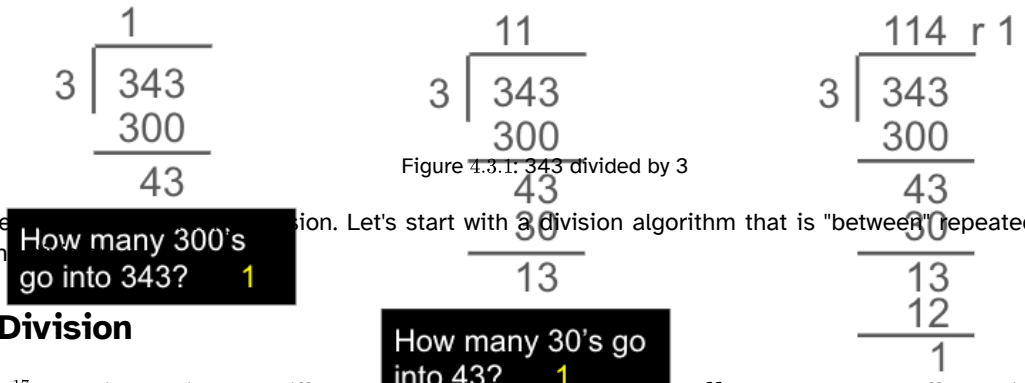


Figure 4.3.1: 343 divided by 3

There are othe ision. Let's start with a division algorithm that is "between" repeated subtraction of the divisor and lon

Scaffold Division

Let's consider $\frac{17}{3}$. We will do this three different w **scaffold**. To make a scaffold, write the dividend, then draw a line starting to the left of the dividend and going up and around to the right g as far downward as needed (it will not be clear how far this is until we are finished, so start short a d as you continue through the problem). Write the divisor to the left of the part of the scaffold that goes around the dividend. See below.

Standard Algorithm of Division

In the first example below, we subtract 3 repeatedly (no short cuts!). We keep track of how many threes we subtract at each step to the right of the scaffold. Since 3 is subtracted one at a time, and a total of five 3's are subtracted before we get to 2 (any more and we would get a negative value as a result, so we stop), we have 5 ones to the right of the scaffold.

In the second one, first 2 threes are subtracted at once, then 2 more, and then 1 more. Notice that the numbers 2, 2, and 1 are written to the right of the scaffold. We stop because the last difference is 2, which means we cannot take away another 3. That means, again, we subtracted a total of 5 threes.

In the third one, first 4 threes are subtracted at once (and 4 is written to the right of the scaffold), and then only 1 more (and 1 is written to the right of the scaffold). Again, we stop because the difference is 2. And again this means we subtracted a total of 5 threes.

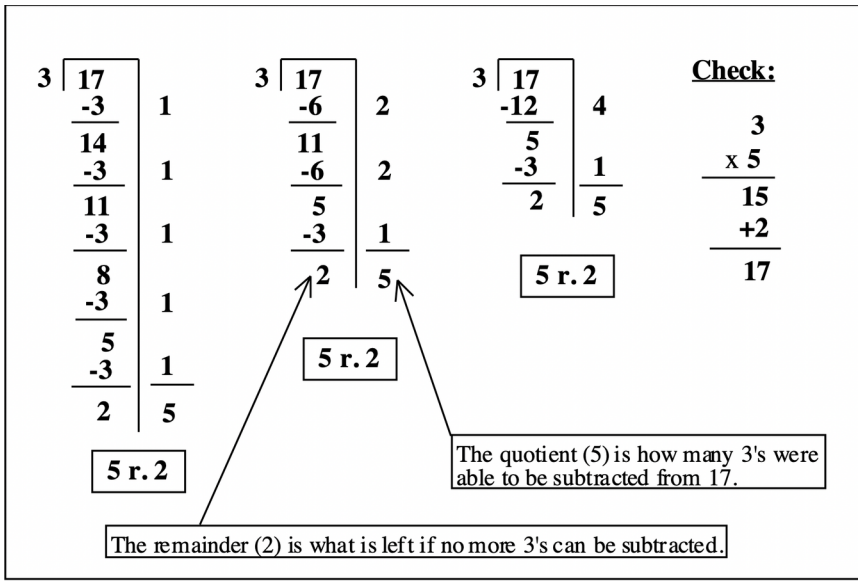
In each case, we got that $17 \div 3 = 5$ with a remainder of 2.

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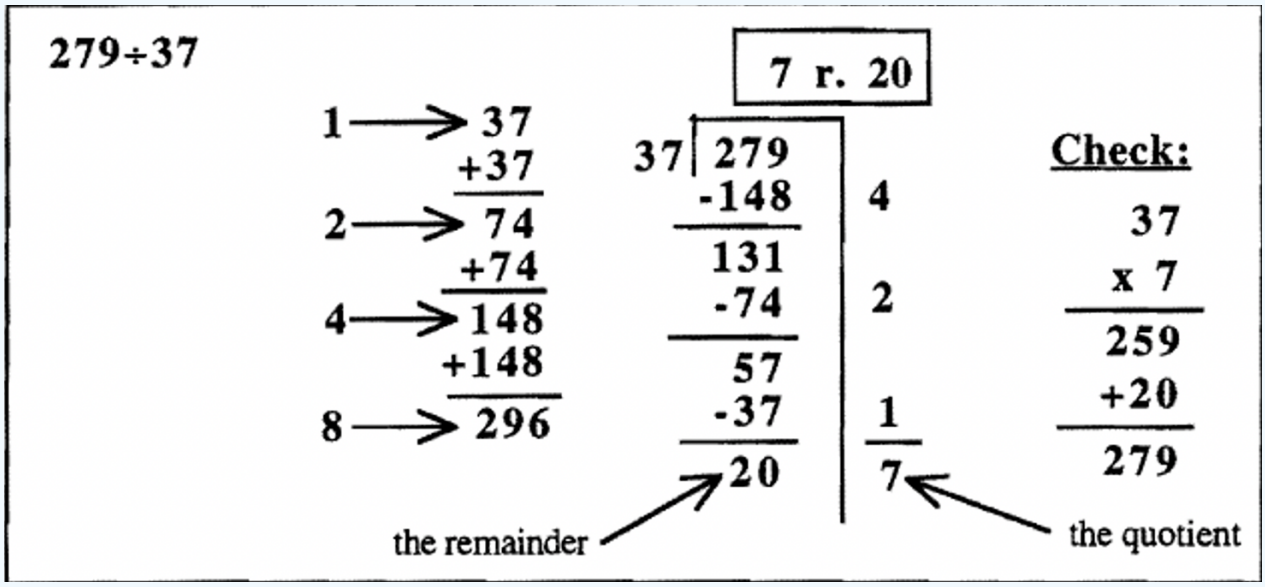
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Long division with large numbers is often difficult because you have to multiply in your head or use a lot of space for scratch work to find the largest possible factor for each multiplication step. It's easy to make mistakes! But, by using scaffold division, you can keep the scratch work organized, and you do not have to find the *largest* number times one number "goes into" another number.

Example 4.3.5

Divide $\frac{279}{37}$ using a scaffold.

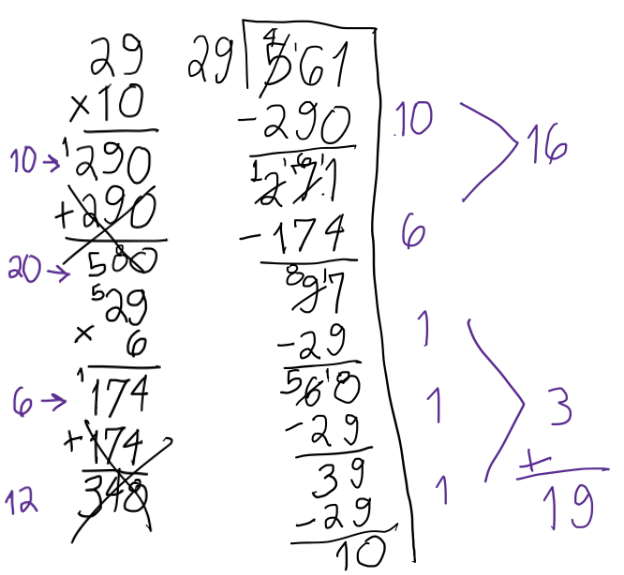


Wonder, Play, Grow

Task 1: Try the following problem on your own using scaffold division: $361 \div 53$.

Now that you have tried it, let's do it together. The way we will do this might look different than what you did, but that does not mean your work is incorrect! Scaffold division lets the user choose what works best for them.

Task 2: Analyze this work done by a young student. Identify which division problem they are completing and check if their work is correct. Discuss with your peers what you see.



Answer available at the end of this section.

Example 4.3.6

Divide $\frac{8871}{231}$ using the scaffold and the standard long division algorithm.

Answer available at the end of this section.

Answers

Example 4.3.1

a) (10 x 23) + (3 x 23) = 230 + 69 = 299

b) (10 x 14) + (2 x 14) + (2 x 14) = 140 + 28 + 28 = 196

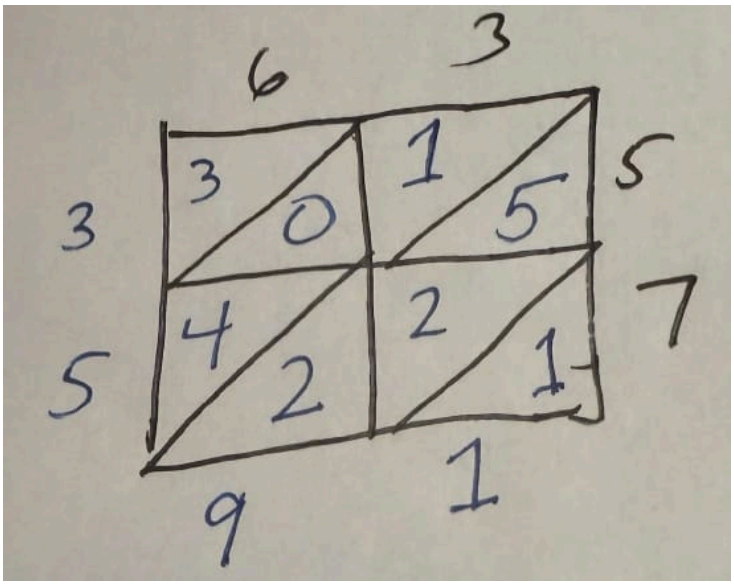
Example 4.3.2

13	×	29
∗1		29
2		58
∗4		116
∗8		232
……		……

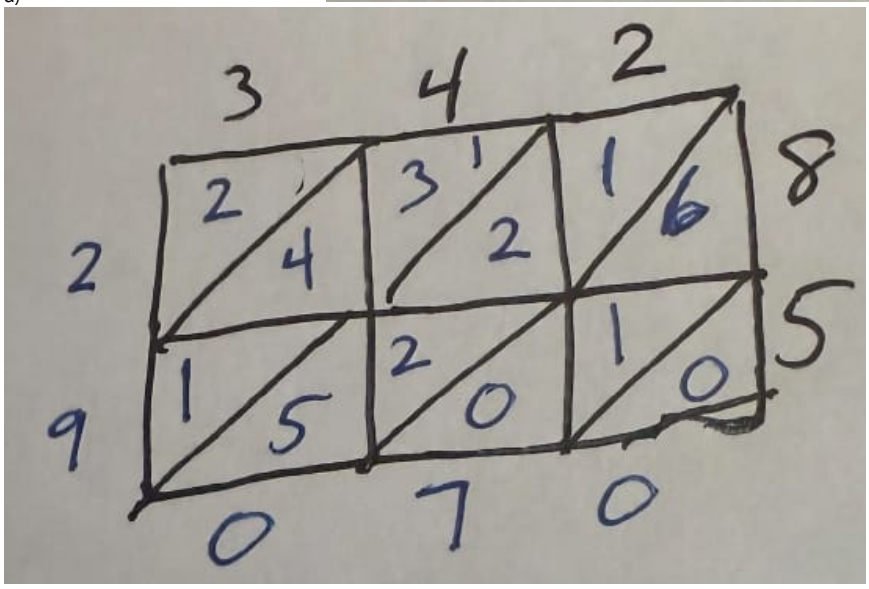
13 x 29 = 232 + 116 + 29 = 377

Example 4.3.3

a)



b)



c) 3886

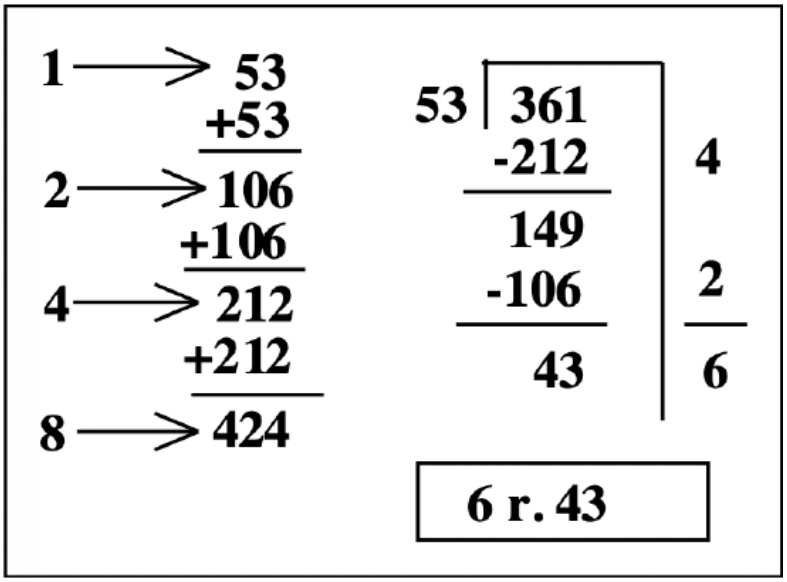
Example 4.3.4

a)

$$\begin{array}{rr} & 27 \\ \times & \underline{91} \\ & 7 \quad (1 \times 7) \\ & 20 \quad (1 \times 20) \\ & 630 \quad (90 \times 7) \\ & \underline{1800} \quad (90 \times 20) \\ & 2457 \end{array}$$

b)

$$\begin{array}{rr} & 538 \\ \times & \underline{34} \\ & 2000 \quad (4 \times 500) \\ & 120 \quad (4 \times 30) \\ & 32 \quad (4 \times 8) \\ & 240 \quad (30 \times 8) \\ & 900 \quad (30 \times 30) \\ & \underline{15000} \quad (30 \times 500) \\ & 2457 \end{array}$$

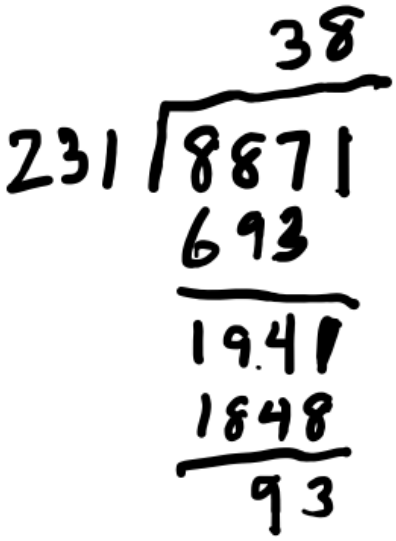


To check, multiply 53×6 and add 43. You should have gotten 361.

Example 4.3.6



Scaffold



Long Division

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4.3.1: Written Retrieval and Problem Solving Practice

Student Learning Objectives

SLO 4.3 Students will model and justify standard written algorithms for basic operations (addition, subtraction, multiplication, division) using concrete models, place value, and mathematical properties.

SLO 4.4 Students will compute sums, differences, products, and quotients in various bases (2–12) using standard algorithms and nontraditional methods, such as lattice and scaffold techniques.

The mild assignment 🌶 is a shorter and less challenging assignment. The medium assignment🌶🌶 is right in the middle in terms of length and level of difficulty. Finally, the spicy assignment 🌶🌶🌶 is a longer assignment that is the most challenging.

(Spicy 🌶🌶🌶 Problems are OPTIONAL, but go for it when you have the time to challenge yourself)

Exercise 4.3.1.1 🌶

Suppose you want to introduce a fourth grader to the standard algorithm for computing 24 x 4. Explain how to find the product with base-ten blocks.

Answer available at the end of this section.

Exercise 4.3.1.2 🌶🌶

a) Compute 83 x 42 with the intermediate algorithm.

b) Compute 83 x 42 with the standard algorithm.

c) Compute 83 x 42 with a repeated addition process.

Answer available at the end of this section.

Exercise 4.3.1.3 🌶🌶

Compute 234 x 36 with lattice multiplication and Egyptian multiplication.

Answer available at the end of this section.

Exercise 4.3.1.4 🌶🌶

A student says 32 x 15 is the same as 16 x 30, which is 480, you say what to them?

Answer available at the end of this section.

Exercise 4.3.1.5 🌶🌶

Solve the following problems using a different method for each.

a) 27 x 17 b) 768 x 891 c) 1258 x 45

Answer available at the end of this section.

Exercise 4.3.1.6

Find each product.

a) $123_5 \times 41_5$ b) $4120_5 \times 121_5$

c) $111_2 \times 11_2$ d) $1010_2 \times 101_2$

Answer available at the end of this section.

Any Base Calculator

Exercise 4.3.1.7

Use the area model (pictorial representation) to multiply the following. Howie Hua explains how to do it.

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https://youtu.be/VWLON0nEoAQ

a) 25 x 32

b) 156 x 42

Answer available at the end of this section.

Exercise 4.3.1.8

Use the scaffold method for division and the standard algorithm.

a) 899 ÷ 13 b) 5697 ÷ 23

Answer available at the end of this section.

Exercise 4.3.1.9

Show how to find the quotient and remainder using repeated subtraction.

a) 89 ÷ 22

Answer available at the end of this section.

Exercise 4.3.1.10

Divide $\frac{200}{24}$ using both the scaffold and standard long division algorithm.

Answer available at the end of this section.

Exercise 4.3.1.11

Making sense of the long division using the sharing model. Divide 432 by 3 using the standard algorithm for division. Then review how Howie Hua explains what this means using a sharing model.

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https://youtu.be/Hvp69HcY_5k

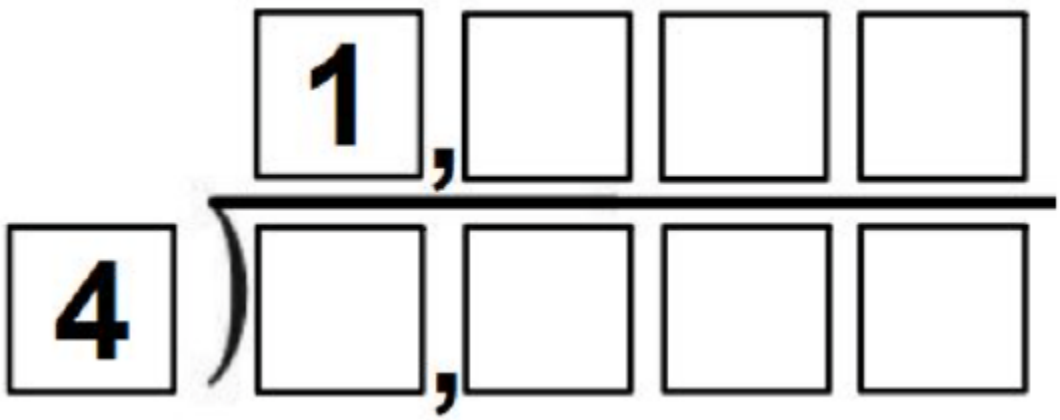
Make up your own division problem and explain to someone how to connect the standard algorithm to the sharing idea Howie Hua discussed.

Answer available at the end of this section.

Problem Solving Problems

1) Division Fill in the Blanks (no remainder)

Directions: Fill in the blanks using all different non-zero digits (except the numbers 1 and 4, which have already been used) to make the greatest possible quotient.



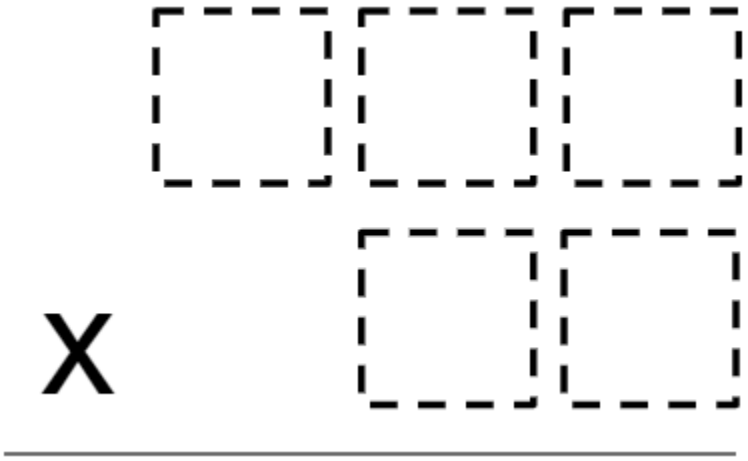
Source: Open Middle

Hint

What number should ideally be placed in the hundreds place of the quotient? Why? Can the units' digit of the dividend be odd if the quotient has no remainder?

2) Multiplying Products to Get as Close to 10000

Directions: Using the digits 1-9 only once, create two factors that will result in a product as close to 10,000, without going over.



Source: Open Middle

Hint

Which place value is most helpful to think about? Why?

Answers

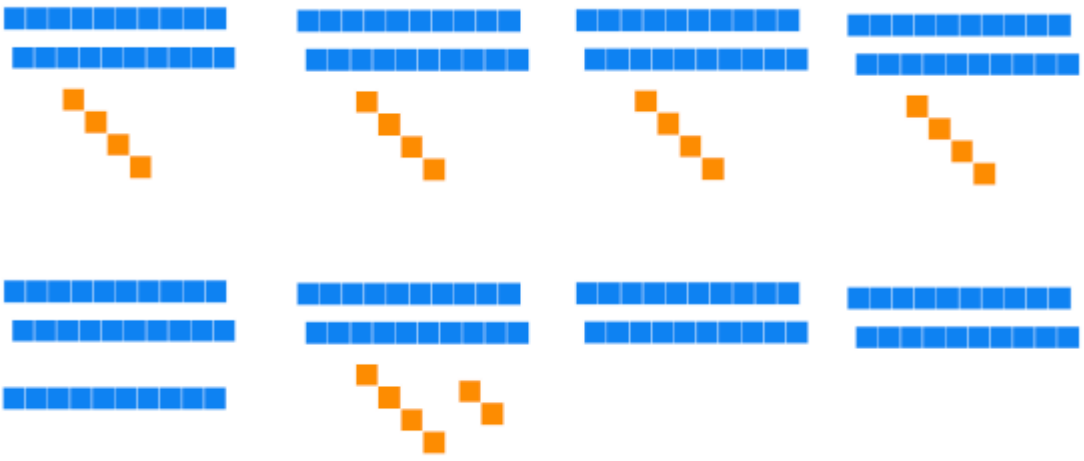
Exercise 4.3.1.1

Make 4 groups of 24. This means: 4 groups of 2 tens → 4×2=8 tens, 4 groups of 4 ones → 4×4=16 ones

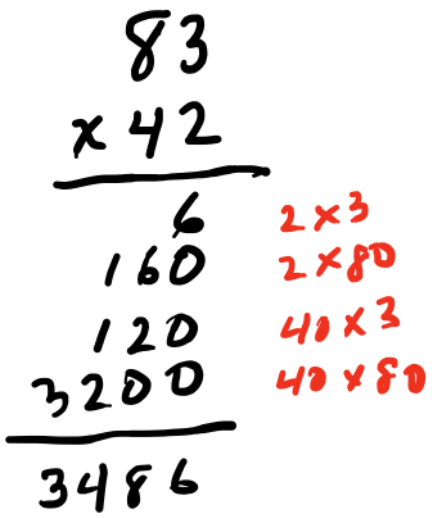
16 unit cubes = 1 rod (10) and 6 unit cubes

Add the 1 rod to the 8 rods = 9 rods = 90, Remaining 6 unit cubes

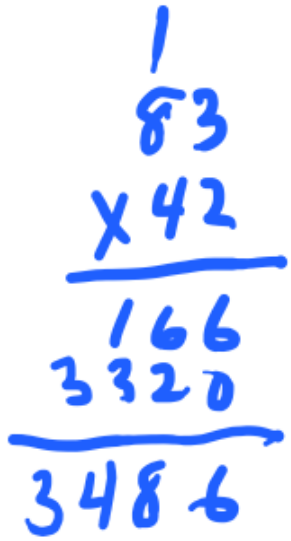
90 (from rods) + 6 (from unit cubes) = **96**



Exercise 4.3.1.2

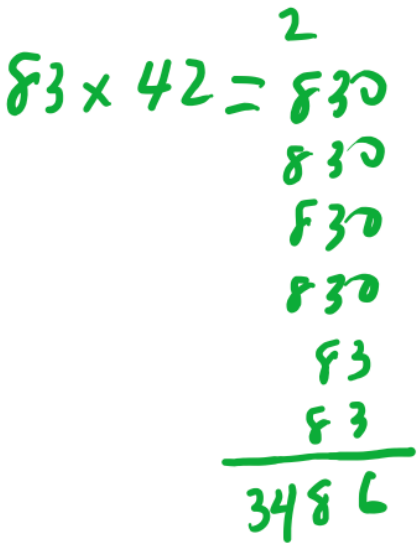


a)

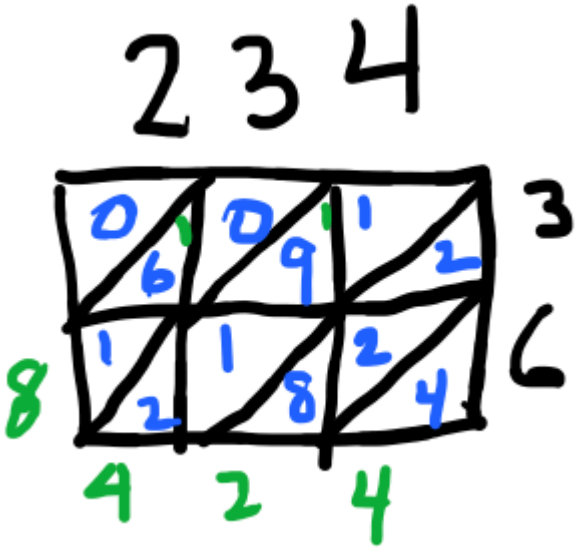


b)

c)

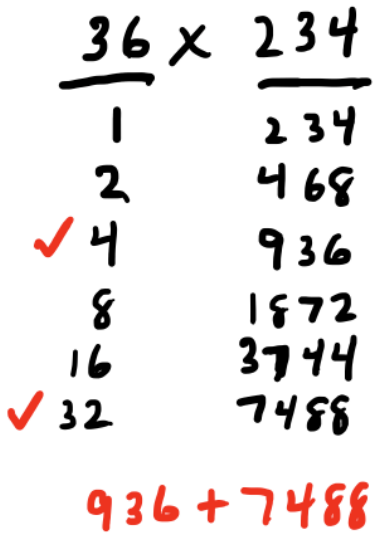


Exercise 4.3.1.3



Lattice:

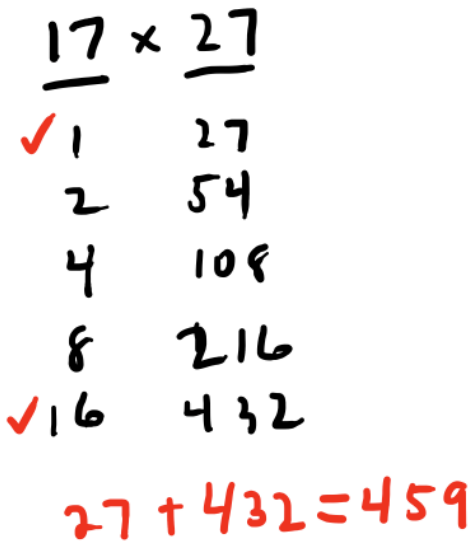
Eqgyptian:



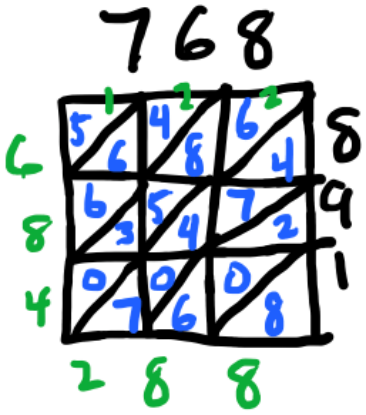
Exercise 4.3.1.4

"You're absolutely right that both expressions give the same product, 480. That's a great observation, and it shows you're thinking flexibly about numbers. Can you explain how you knew they would give the same result?" 16 x 2 x 15 = 16 x 30, *This response celebrates the mathematical thinking, reinforces concepts of number sense and equivalence, and encourages the student to articulate their reasoning, a key aspect of developing mathematical understanding.*

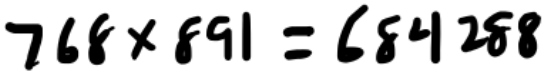
Exercise 4.3.1.5



a)



b)



c)

```
  1258
x   45
------
    40
   250
  1000
 5000
   320
  2000
  8000
 40000
------
 56610
```

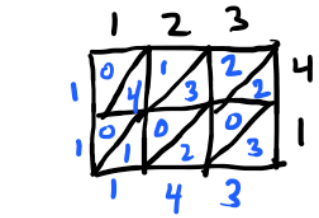
Exercise 4.3.1.6

a) 11143_5

b) 1104020_5

c) 10101_2

d) 1100010_2



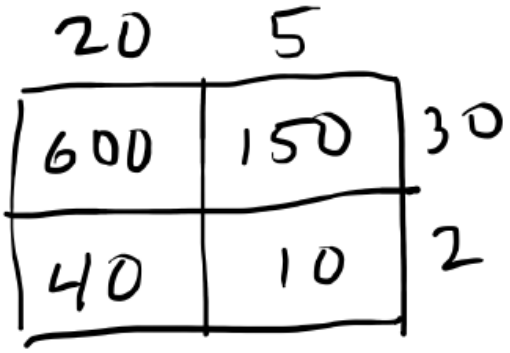
$123_5 \times 41_5 = 11143$

```
  4120
   121
------
  4120
132400
412000
------
1104020
```

```
  111
x  11
-----
  111
 1110
-----
10101
```

```
  1010
x  101
------
  1010
 0000
10100
------
110010
```

Exercise 4.3.1.7

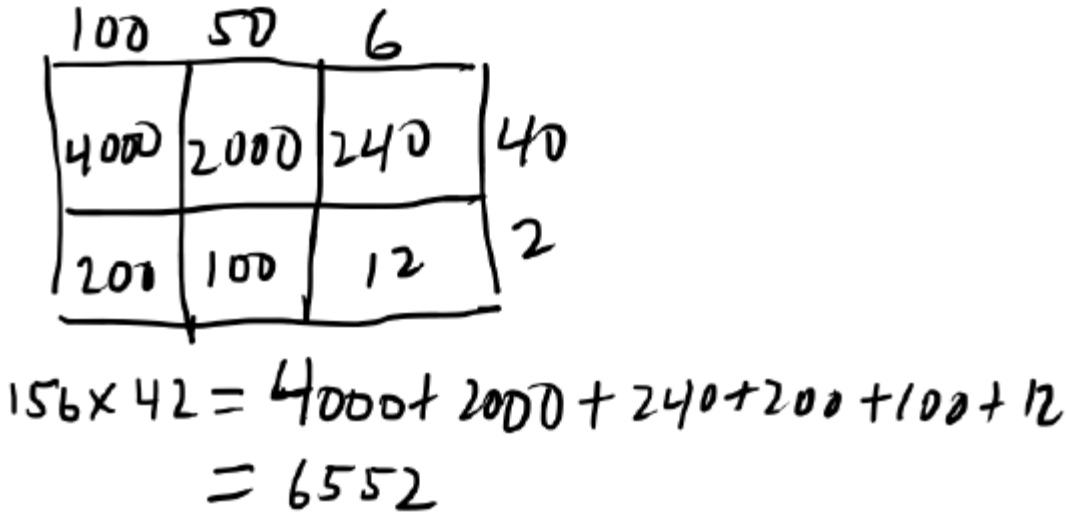


$25 \times 32 = 600 + 150 + 40 + 10$

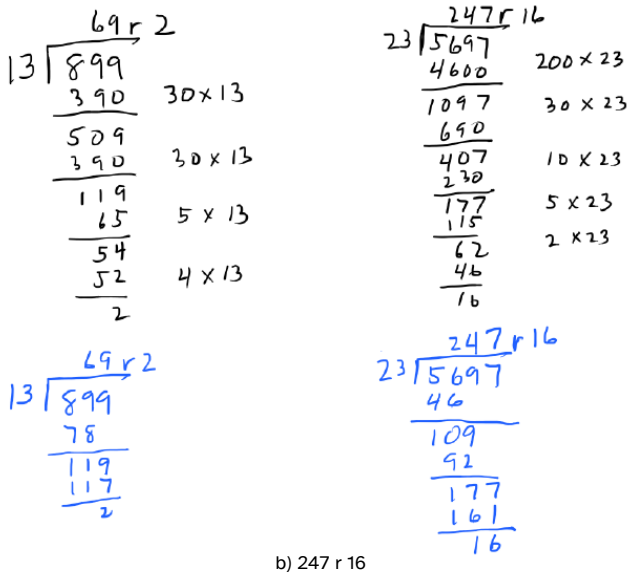
$= 800$

a)

b)



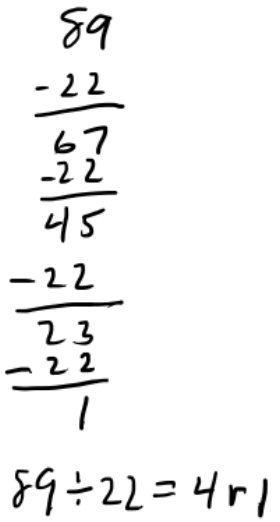
Exercise 4.3.1.8



a) 69 r 2

b) 247 r 16

Exercise 4.3.1.9



89 ÷ 22 = 4 r 1

Exercise 4.3.1.10

200÷24

1 → 24
+24
2 → 48
+48
4 → 96
+96
8 → 192

8 r. 8

24 | 200 | 8
-192 | 8
8

Check:

24
x 8
192
+8
200

Exercise 4.3.1.11

144
3 | 432
3
13
12
12
12
0

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4.4: Chapter 4 Resources

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Chapter 4.3

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Chapter Extras

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5: THE FOUNDATIONS OF NUMBER THEORY

At first glance, number theory may seem distant from the work of an elementary classroom. But beneath the surface of early math instruction lies the deep structure of our number system—and that's where number theory lives. It is the study of the relationships, patterns, and properties of counting numbers, and it provides powerful tools for understanding how numbers work (Feike et al., 2018).

For future elementary teachers, studying number theory is about more than solving abstract problems—it's about building a deep, flexible number sense that allows you to recognize patterns, make connections, and respond meaningfully to students' thinking (Boaler, 2022). Concepts like divisibility, factors, and multiples are the foundation for simplifying fractions, understanding operations, and developing algebraic thinking. While your students may not yet learn formal number theory, your ability to recognize its presence in their reasoning will help you guide them toward mathematical confidence and curiosity. In this chapter, we explore how number theory empowers teachers to see numbers not just as answers, but as meaningful, interconnected ideas (Boaler, 2022).

Chapter Learning Objectives

Student Learning Objectives

- SLO 5.1 Students will analyze and apply properties of numbers, including divisibility rules and the Fundamental Theorem of Arithmetic, to determine the prime or composite nature of numbers and represent them using prime factorization
- SLO 5.2 Students will demonstrate proficiency in identifying prime numbers using methods such as the Sieve of Eratosthenes and analyzing the relationship between prime factors and number properties
- SLO 5.3 Students will compute the greatest common factor (GCF) and least common multiple (LCM) of pairs of numbers using various methods, including prime factorization, set intersection, and algorithms
- SLO 5.4 Students will evaluate the relationships between the GCF, LCM, and the product of two numbers, demonstrating a deeper understanding of number relationships and their practical applications.

Essential Questions

- **What patterns in numbers help us understand their structure and relationships?**
- **How can recognizing a number's factors and multiples help us solve real-world problems?**

Learning That Transfers

Students will recognize and apply number structures and relationships to solve problems, make generalizations, and support students' mathematical reasoning in elementary classrooms. Students will use number relationships (such as factors, multiples, and prime decomposition) to make sense of and solve real-world problems involving patterns, quantities, and fair sharing, both in daily life and in the elementary classroom.

Computation Fun

Interactive Content

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NEED AUDIO FOR STORY

Morgan's Prime Time at the Number Museum

One bright Saturday, with a curious mind, Morgan was ready to learn and unwind. Grandpa said, "Let's go, I've got a surprise— A magical place with numbers that rise!"

They packed up their snacks, took the bus into town, To visit a place of much number renown. "The Number Museum!" the banner read high, With a spiral of digits that danced in the sky.

Morgan asked, "Grandpa, what will we see?" He winked and replied, "Come discover with me!"

The first hall they entered had doors numbered bold— "Some are quite special," the tour guide told. "See, here are the primes, standing alone, Only one and themselves, they call their own!"

"Two, three, and five," Morgan read with a squint, "Seven and eleven—do they give us a hint?" Grandpa laughed, "They sure do, my dear, They can't be divided, that's why they're here!"

In the next hall they saw a sieve made of gold, With numbers falling through in a beautiful fold. "The Sieve of Eratosthenes," Grandpa did say, "It helps us find primes in a magical way!"

Morgan reached up and gave it a spin, Out came the primes with a jubilant grin. "But what of the rest?" she asked with a stare, "Are they just leftover? Do we not care?"

"Oh no!" said Grandpa, "They're just as grand, They're called composites, and they too have a stand! They break into pieces, like six becomes two times three, That's called prime factorization, you see!"

They stopped by a puzzle, two numbers in place, "Find their GCF!" lit up the space. Morgan thought hard, then gave a shout, "Twelve and sixteen? Four is what jumps out!"

"Now," said Grandpa, "Let's flip the show, Can you find the LCM? Give it a go!" Morgan grinned, "That's twenty-four, The smallest they share, and nothing more!"

With gears and with wheels, a machine did display, A fact Grandpa whispered as they walked away: "If you multiply numbers and divide by GCF, You'll get the LCM—it's like number math chef!"

Morgan laughed loud as they reached the last room, Where stars filled the ceiling and digits did zoom. "Numbers are magic!" she said with a cheer, "And primes are the ones who make patterns appear!"

As they left the museum, the sky turning gold, Morgan felt curious, clever, and bold. Numbers weren't just for school or a grade— They were puzzles and stories just waiting to be made!

5.1: Primes, Composites, and Divisibility
5.1.1: Written Retrieval and Problem Solving Practice
5.1.2: Online Retrieval and Problem Solving Practice
5.2: Greatest Common Factors and Least Common Multiples
5.2.1: Written Retrieval and Problem Solving Practice
5.2.2: Online Retreival and Problem Solving Practice
5.3: Chapter 5 Resources

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5.1: Primes, Composites, and Divisibility

While the formal study of prime and composite numbers typically occurs in later elementary grades, it plays a crucial role in deepening children's understanding of multiplication, division, number sense, fractions, and lays a foundation for more advanced mathematical concepts like algebra. Recognizing the patterns and unique properties of these numbers can also contribute to fostering mathematical curiosity and an appreciation for the structure of our number system.

Wonder, Play, Grow  Squares, Rectangles, or Sticks? A Number Shape Investigation

Let's explore what numbers can *become* when we build with unit squares.

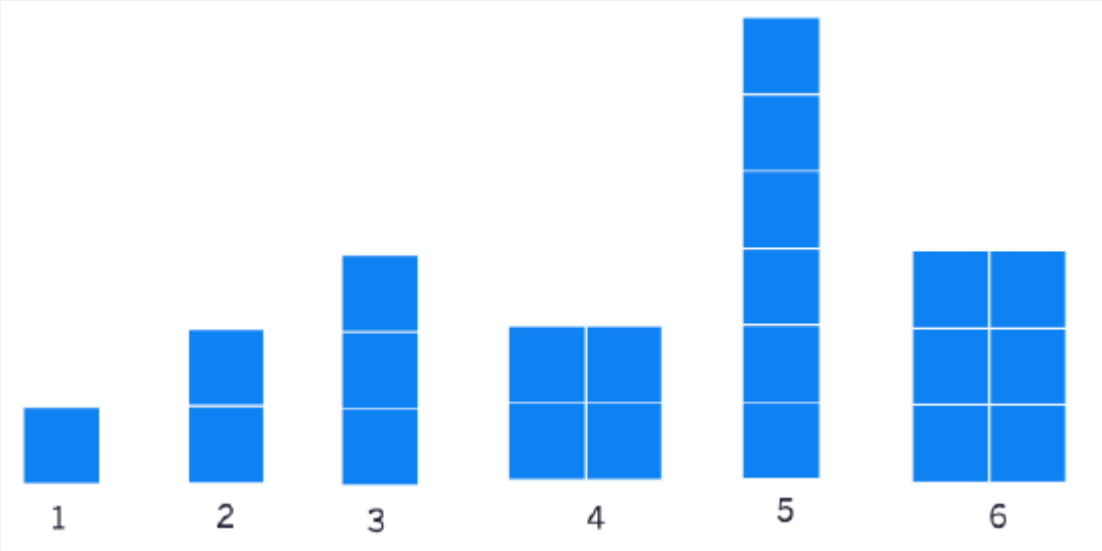
We'll imagine each number as **blue squares**  , and ask:

- Can we arrange them into a **perfect square**?
- Or can they only be laid out in a **stick** (a single row or column)?
- Can we arrange them into a **rectangle** that's not a square nor a stick? *(FYI: A square and a stick are technically not rectangles)*

Examples:

- **1** square: This is a square! Just one small square. ✅
- **2** squares: We can only make a stick (1×2 or 2×1). ❌ Not a square.
- **3** squares: Still just a stick. ❌
- **4** squares: Yes! A 2×2 square! ✅, a stick too.
- **5** squares: Only a stick again. ❌
- **6** squares: Not a square, but we can make a 2×3 rectangle and a stick. ✅

Math Note available at the end of this section.



TASK 1: For each number from **1 to 17**, ask: (Interactive Shapes you can use to help with the exploration)

- Can I make a **square** (equal rows and columns)?
- Can I make a **rectangle** with more than 1 row and more than 1 column, but not equal?
- Or can I only make a **stick** (just one row or column)?

Task 2:

- Which numbers can only be **sticks**?
- Which can form **rectangles**?
- Which can form **squares**?
- After 17, what number is the next stick, rectangle, and square?
- What do you notice and wonder about now?

Adapted from nrich Numbers as Shapes

🧠**Let's Listen to Learn**❤️

Audio

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Prime Numbers and Composites

There is something deeply poetic about prime numbers. These indivisible building blocks of the number world sit like hidden treasures in the vast sea of integers—mysterious, unpredictable, and infinitely many. From ancient Greek mathematicians marveling at their simplicity, to modern cryptographers relying on them to secure digital life, primes have always held a magnetic pull on human curiosity. If this piques your interest in anyway, and before we jump into the ideas prime and composite numbers, take a look at the article **Why prime numbers still fascinate mathematicians, 2,300 years later.**

Definition: Prime and Composite Numbers

A counting number with exactly two different factors (itself and 1) is called a prime number, or a prime. A counting number with more than two factors is called a composite number, or a composite.

Factor available at the end of this section.

For example, 2, 3, 5, 7, 11 are primes, since they have only themselves and 1 as factors; 4, 6, 8, 9, 10 are composites, since they each have more than two factors; 1 is neither prime nor composite, since 1 is its only factor. An algorithm used to find primes is called the Sieve of Eratosthenes. Here is a worksheet that takes you through the process or we can see our favorite math educator Dr. Howie Hua explain **How to quickly find all prime numbers from 1 to 100 #math.**

YouTube Video

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1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Composite numbers have more than two factors and can be expressed as the product of two smaller numbers. Figure 5.1.1 shows how a composite (48) can be expressed as the product of smaller numbers using factor trees.

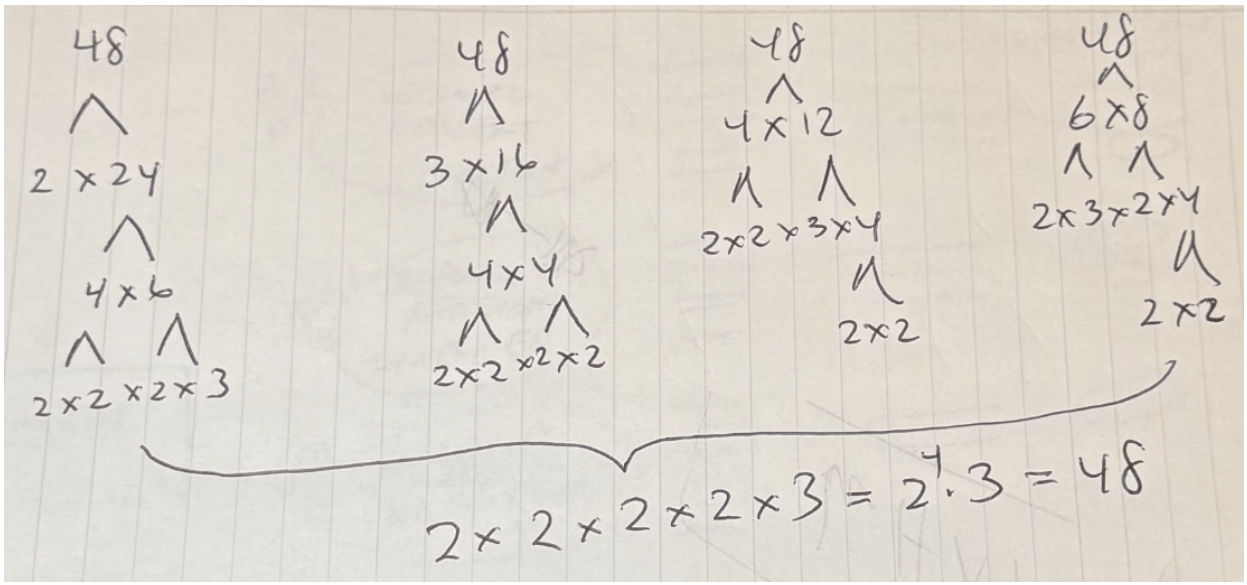


Figure 5.1.1: Factor Trees for 48

Notice that 48 was expressed as the product of two factors in several different ways. However, when we kept factoring until we reached primes, each method led us to the same prime factorization, namely $2^4 \times 3 = 48$.

Theorem 5.1.1 Fundamental Theorem of Arithmetic

Each composite number can be expressed as the product of primes in exactly one way (except for the order of the factors).

Example 5.1.1

Express each number as a product of primes.

a) 84 b) 180 c) 324

Answer available at the end of this section.

Prime Factors - Step by Step

Enter a whole number and explore it's decomposition into prime factors.

We will now look at some shortcuts (in addition to using a calculator) to help us find prime factors.

Definition: Divides

"a divides b" if there exists a whole number, n, such that an = b. This is written **"a|b"** if there exists a whole number, **n**, such that **an = b**, In words, **a|b** if and only if **a** is a *factor* of **b**. When **a** divides **b**, we can also say that **a** is a *divisor* of **b**, **a** is a *factor* of **b**, **b** is a *multiple* of **a**, and **b** is d*ivisible* by **a**.

When you see 12/3, this means 12 "**divided by**" 3. The slash that slants to the right is another way to write the division sign, ÷. 12/3 (or $12 \div 3$) is a division problem, and the answer is 4.

Here is something altogether different. If I say "3 **divides** 12", I am making a *statement*. "3 divides 12" is not a division problem that needs to be done. It is a statement that happens to be true. The symbol used to represent the word "divides" is a vertical line. So, "3 **divides** 12" can be written "3|12". Again, this is a statement, a fact, not a division problem. The way to express "does not divide" is to put a slash through the symbol: $\not|$

So, how do I know "3 ***divides*** 12" is a true statement? The definition of "divides" follows:

Okay, so what exactly does "there exists a whole number, n, such that an = b" mean? Well, it means that the first number times some whole number equals the second number. Or you can think "a divides b" is true if $b \div a$ is a whole number.

Let's get back to why "3 **divides** 12" is a true statement. Hmmm...you can be formal and ask yourself: Is there a whole number n such that 3n = 12? Or, you can simply ask yourself: Is 3 a factor (or divisor) of 12? In either case, the answer is yes, so the statement is true.

Let's work a little more on the difference between a statement using "divides", and an actual division problem.

Example 5.1.2

Determine if each of the following is a statement or if it is a division problem. If it is a statement, state if it is true or false and back up your answer. If it is a division problem, state the answer to the division problem.

a) 4|20 b) 20/6 c) 15|3 d) 6 divides 20 e) 3 divides 21

Answer available at the end of this section.

Think About and Discuss

Think about whether the following are True or False. Then explain to your peers your answers.

i) 8 is a divisor of 96

ii) 216 is a multiple of 6

iii) 51 is divisible by 17

iv) $(2^2 \bullet 3) \mid (2^3 \bullet 3^2 \bullet 5)$

Test for Divisibility

Divisibility tests may seem old-fashioned in a world with calculators, but they are *deeply important* for young math learners. Divisibility tests help young learners develop number sense, logical reasoning, and mental math skills, fostering a deeper understanding of mathematical patterns and relationships. Unlike calculators that provide answers, divisibility rules teach students how to think mathematically. Keep in mind that list below is not intended for you to memorize, but to understand where they come from and how it is all related to the ideas of factors, multiples, and divisors.

Listed below are the divisibility tests that you should know. It's important to realize that each of the tests only work for the number specified. In other words, you can't use the divisibility test for 3 to determine is 7 divides a number.

Divisibility Tests

All counting numbers can be considered factors of zero. In other words, for all counting numbers, m, m|0 is always true since there is always some number times m that equals zero, namely zero itself. Assume n is a positive whole number. The following are divisibility tests to determine what numbers divide into n, or what numbers are factors of n.

- Divisibility Test for 2: 2|n if the last digit of n is even (0, 2, 4, 6, or 8)
- Divisibility Test for 3: 3|n if and only if the sum of its digits is divisible by 3.
- Divisibility Test for 4: 4|n if 4|(the number represented by the last two digits of n)
- Divisibility Test for 5: 5|n if the last digit of n is 0 or 5
- Divisibility Test for 6: 6|n if 2|n AND 3|n. (Think of 6 as $2 \cdot 3$)
- Divisibility Test for 7: The steps of this test are described below.
- Step 1: Cross off the one's digit of the number to get a number with one less place value.
- Step 2: Double the one's digit you crossed off and subtract from the new number without the one's digit.
- Step 3: If 7 divides the number you get after subtracting, it divides the original number. Otherwise, it doesn't. If you aren't sure, repeat the procedure on the new number by going back to step 1.
- Divisibility Test for 8: 8|n if 8|(the number represented by the last three digits of n)
- Divisibility Test for 9: 9|n if the sum of its digits is divisible by 9.
- Divisibility Test for 10: 10|n if the last digit of n is 0
- Divisibility Test for 11: 11|n if the difference between the sum of the digits in the places that are even powers of 10 and the sum of the digits in the places that are odd powers of 10 is divisible by 11
- Divisibility Test for 15: 15|n if 5|n AND 3|n

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Example 5.1.3

Use the divisibility test for 2 to determine if the following is true or false 1. 2|97 This is false because the last digit (7) is not even 2. 2|356 This is true because the last digit (6) is even	**Use the divisibility test for 4 to determine if the following is true or false.** 1. 4|527 This is false because 4 does not divide 27, (since 4 is not a factor of 27). 2. 4|25,356 This is true because 4|56.	**Use the divisibility test for 5 to determine if the following is true or false.** 1. 5|527 This is false because 7 is the last digit of 527 2. 5|25,335 This is true because 5 is the last digit of 25,335
Use the divisibility test for 3 to determine if the following is true or false. 1. 3|97 This is false because 3 does not divide 16 (9+ 7) 2. 3|738 This is true because 3 divides 18 (7+3+8)	**Use the divisibility test for 6 to determine if the following is true or false.** 1. 6|627 This is false because 2 does not divide 7 (the last digit, which is not even). 2. 6|7,620 This is true because 2|7,620 (since the last digit, 0, is even) AND 3 divides 15 (7+6+2+0)	**Use the divisibility test for 11 to determine if the following is true or false.** a. 11|4,365: Add 4 + 6 = 10 Add 3 + 5 = 8 Subtract 10 – 8 = 2 Since 11 doesn't divide 2, then 11 doesn't divide 4365. Therefore, 11|4365 is false. b. 11|542,879,216: Add 5 + 2 + 7 + 2 + 6 = 22 Add 4 + 8 + 9 + 1 = 22 Subtract 22 – 22 = 0 Since 11|0, then 11|542,879,216 is true.

Example 5.1.4

What divides evenly into 3495?

Number	Check it!	Yes or No?
2	Is 3495 even?	no
3	Add the digits: 3 + 4 + 9 + 5 = 21 and 21 divides 3 evenly	yes
4	Can 95 divide 4 evenly?	no

Number	Check it!	Yes or No?
5	Ends in 0 or 5?	yes
6	Divisible by both 2 AND 3?	
8	Can 495 divide 8 evenly?	
9	Add the digits: 3 + 4 + 9 + 5 = 21 and 21 does not divide 9 evenly	
10	Ends in 0?	
11		
15		

Answer available at the end of this section.

Wonder, Play, Grow

Work with your peers to play around with the ideas below.

Let a, m, n, and k be whole numbers where $a \neq 0$.

Task 1: Is the following statement true? If a|b and a|c, then a|(b+c).

Task 2: Is the following statement true? If a|b and a|c, then a|(b-c), b≥c.

Now that you played around with some number theory like a mathematician would do, you probably recognize that 3 | 27 and 3 | 9. It is also true that 3 | (27 + 9) and that 3 | (27 – 9). This is an instance of the following theorem.

Theorem 5.1.1

Let a, b, c, and k be whole numbers where $a \neq 0$.

a. If a | b and a | c, then a | (a + b).

b. If a | b and a | c, then a | (a – b) for m ≥ n.

c. If a | b, then a | kb.

Note on theorems in mathematics: While the direct content of most theorems may not be taught in elementary school, the underlying principles of logical reasoning, justification, and the interconnectedness of mathematical ideas, which are central to theorems, are crucial for effective elementary mathematics teaching. Learning about theorems empowers future teachers with a deeper understanding of the structure and nature of mathematics, enabling them to teach with greater clarity, confidence, and a focus on sense-making for their students. Learning theorems prepares you to teach not just procedures, but the *thinking* and

truth-seeking that makes math meaningful.

Think About and Discuss

Is 137 a prime number? How can we quickly check that? Discuss with your peers how they figured it out.

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Prime Factor Test

To test for prime factors of a number n, one need only search for prime factors p of n, where $\diamondsuit^2 \leq \diamondsuit$ (or $p \leq \sqrt{n}$)

Example 5.1.5

Determine whether the following numbers are prime or composite.

a) 299

b) 401

Answer available at the end of this section.

Answers

Audio

From geometry we know that a square is actually a rectangle too, but a rectangle is not a square. And the "stick" in this investigation is technically a rectangle too.

Definition: Prime and Composite Numbers

A **factor** of a whole number is another whole number that divides it **exactly**, with no remainder.

Example 5.1.1

a) 84 = 4 x 21 = 2 x 2 x 3 x 7 = 2^2 x 3 x 7

b) 180 = 10 x 18 = 2 x 5 x 2 x 3 x 3 = 2^2 x 3^2 x 5

c) 324 = 4 x 81 = 2 x 2 x 3 x 3 x 3 x 3 = 2^2 x 3^4

Example 5.1.2

a) This is a true statement, because 4 is a factor of 20 (since $4 \cdot 5 = 20$)

b) This is a division problem. The answer is 3 r. 2

c) This is a false statement because 15 is not a factor of 3

d) This is a false statement because 6 is not a factor of 20

e) This is a true statement because 3 is a factor of 21 (since $3 \cdot 7 = 21$)

Example 5.1.4

2 - no, 3 - yes, 4 - no, 5 - yes, 6 - no, 8 - no, 9 - no, 10 - no , 11 - no, 15 - yes

Example 5.1.5

a) Only the prime factors 2 through 17 need to be checked, since $17^2 < 299 < 19^2$ (check this on your calculator) or because $\sqrt{299} \approx 17.29$. None of the numbers 2, 3, 5, 7, or 11 is a factor, but since 299 = 13 x 23, the number 299 is composite.

b) Only primes 2 through 19 need to be checked, since $\sqrt{401} \approx 20$. Since none of the primes 2 through 19 are factors of 401, we know that 401 is a prime. (The tests for divisibility show that 2, 3, 5, and 11 are not factors of 401. A calculator, tests for divisibility, or long division can be used to check 7, 13, 17, and 19.)

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5.1.1: Written Retrieval and Problem Solving Practice

Student Learning Objectives

5.1 Students will analyze and apply properties of numbers, including divisibility rules and the Fundamental Theorem of Arithmetic, to determine the prime or composite nature of numbers and represent them using prime factorization.

5.2 Students will demonstrate proficiency in identifying prime numbers using methods such as the Sieve of Eratosthenes and analyzing the relationship between prime factors and number properties.

The mild assignment 🌶 is a shorter and less challenging assignment. The medium assignment🌶🌶 is right in the middle in terms of length and level of difficulty. Finally, the spicy assignment 🌶🌶🌶 is a longer assignment that is the most challenging.

(Spicy 🌶🌶🌶 Problems are OPTIONAL, but go for it when you have the time to challenge yourself)

Exercise 5.1.1.1 🌶

Find the prime factorization for each of the following Numbers.

a) 36 b) 54 c) 204 d) 1000

Answer available at the end of this section.

Exercise 5.1.1.2 🌶

Determine which of the following are true. If true, state how know. If false, explain.

a) 3 | 9 b) 12 | 6

c) 3 is a divisor of 21. d) 6 is a factor of 3.

e) 4 is a factor of 16. f) 0 | 5

g) 11|11 h) 48 is a multiple of 16.

Answer available at the end of this section.

Exercise 5.1.1.3 🌶🌶

Use the divisibility tests for 2 and 4 to determine if the following is true or false. Support your answer with a reason using the appropriate divisibility test.

a) 2|9,712 ____
b) 2|5,643 ____
c) 4|5,690 ____
d) 4|63,868 ____

Answer available at the end of this section.

Exercise 5.1.1.4

Which of the following are multiples of 3? of 4? of 9?

a) 12,345 b) 1,114,500

Answer available at the end of this section.

Exercise 5.1.1.5

Use the divisibility tests for 5 and 10 to determine if the following is true or false. Support your answer with a reason using the appropriate divisibility test.

a) 5|9,750 _____
b) 10|50,005 _____
c) 5|5,556 _____

Answer available at the end of this section.

Exercise 5.1.1.6

Use the test for divisibility by 11 to determine which of the following numbers are divisible by 11.

a) 28,282 b) 1,992

Answer available at the end of this section.

Exercise 5.1.1.7

If 27 divides m, what else must divide m?

Answer available at the end of this section.

Exercise 5.1.1.8

To determine if 467 is prime, we must check to see if it is divisible by any numbers other than 1 and itself. List all of the numbers that must be checked as possible factors to see if 467 is prime.

Answer available at the end of this section.

Exercise 5.1.1.9

Use the divisibility test for 15 to determine if the following is true or false. Support your answer with a reason using the divisibility test for 15.

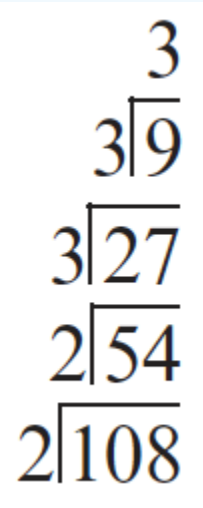
a) 15|9,753
b) 15|6,645

Answer available at the end of this section.

Exercise 5.1.1.10

A factor tree is not the only way to find the prime factorization of a composite number. Another method is to divide the number first by 2 if possible, then divide the quotient by 2 if possible, then by 3, then by 5, and so on, until all possible divisions by prime numbers have been performed.

For example, to find the prime factorization of 108, you might organize your work as follows to conclude that $108 = 2^2 \times 3^3$



Use this method to find the prime factorization of the following numbers.

a) 160 b) 4200

Answer available at the end of this section.

Problem Solving Practice

1) Find the smallest counting number that is divisible by the numbers 2 through 7.

2) Thelma and Louise are serving cookies at a school party. If they arrange the cookies in groups of 2, 3, 4, 5, or 6, they always have exactly one cupcake left over. What is the smallest number of cupcakes they could have?

3) Two primes that differ by 2 are called twin primes. For example, 5 and 7, 11 and 13, 29 and 31 are twin primes. Find all the twin primes less than 120. (The existence of infinitely many twin primes has not been proved) **[Search online for the largest known twin primes 😊]**

Answers

Exercise 5.1.1.1

a) $2^2 \times 3^2$

b) 2×3^3

c) $2^2 \times 3 \times 17$

d) $2^3 \times 5^3$

Exercise 5.1.1.2

a) True; "3 | 9" means **3 divides 9**. Since 9 ÷ 3 = 3 (an integer), this is true.

b) False; "12 | 6" means **12 divides 6**. But 6 ÷ 12 = 0.5 (not an integer), so this is false.

c) True: 21 ÷ 3 = 7, so 3 is a divisor of 21.
d) False; 3 ÷ 6 = 0.5, which is not an integer, so 6 is **not** a factor of 3.
e) True; 16 ÷ 4 = 4, so 4 is a factor of 16.
f) False; "0 | 5" means 0 divides 5. But **division by 0 is undefined**, so this is false.
g) True; 11 ÷ 11 = 1, which is an integer. So 11 divides 11.
h) True; 48 ÷ 16 = 3 , so 48 **is** a multiple of 16.

🌶️🌶️**Exercise** 5.1.1.3

a) True; Last digit = **2**, which is **even**, so divisible by 2.
b) False; Last digit = **3**, which is **odd**, so not divisible by 2.
c) False; Last two digits = **90,** 90 ÷ 4 = 22.5 → Not an integer.
d) True; Last two digits = **68,** 68 ÷ 4 = 17 → An integer.

🌶️🌶️**Exercise** 5.1.1.4

a)
Sum of digits: 1 + 2 + 3 + 4 + 5 = **15**
- 15 is divisible by **3**, but **not by 9**.

Last two digits: 45
- 45 ÷ 4 = 11.25 → not divisible by **4**

b)
Sum of digits: 1 + 1 + 1 + 4 + 5 + 0 + 0 = **12**
- 12 is divisible by **3**, but **not by 9**

Last two digits: 00
- 00 is divisible by **4** (and by everything),

🌶️**Exercise** 5.1.1.5

a) True, ends in 0
b) False, does not end in 0
c) False, does not end in 0 or 5

🌶️🌶️**Exercise** 5.1.1.6

a)
Digits: 2 - 8 + 2 - 8 + 2 =
→ 2 - 8 = -6
→ -6 + 2 = -4
→ -4 - 8 = -12
→ -12 + 2 = **-10**
-10 is not divisible by 11
❌ **Not divisible by 11**

b)
→ 1 - 9 = -8
→ -8 + 9 = 1
→ 1 - 2 = **-1**
-1 is not divisible by 11
❌ **Not divisible by 11**

Exercise 5.1.1.7🌶️🌶️

1, 3, 9, 27

Exercise 5.1.1.8🌶️🌶️

To determine if 467 is prime, you must check divisibility by:
2, 3, 5, 7, 11, 13, 17, and 19
If 467 is not divisible by any of these, then **467 is prime**.

🌶️🌶️**Exercise** 5.1.1.9

a)
✅ **Divisible by 3?**
Sum of digits: 9 + 7 + 5 + 3 = **24** → 24 ÷ 3 = 8 → ✔️ Yes
❌ **Divisible by 5?**
Ends in **3**, which is **not** 0 or 5 → ✖️ No
Since it's not divisible by **5**,
❌ **9,753 is not divisible by 15**

b)
✅ **Divisible by 3?**
Sum of digits: 6 + 6 + 4 + 5 = **21** → 21 ÷ 3 = 7 → ✔️ Yes
✅ **Divisible by 5?**
Ends in **5** → ✔️ Yes
Since it's divisible by **both 3 and 5**,
✅ **6,645 is divisible by 15**

Exercise 5.1.1.10🌶🌶🌶

a) 160 = 2^5 x 5

Let's divide by primes:

160 ÷ 2 = 80

80 ÷ 2 = 40

40 ÷ 2 = 20

20 ÷ 2 = 10

10 ÷ 2 = 5

5 is a prime number

b) 4200 = 2^3 x 3 x 5^2 x 7

Let's divide step-by-step:

4200 ÷ 2 = 2100

2100 ÷ 2 = 1050

1050 ÷ 2 = 525

525 ÷ 3 = 175

175 ÷ 5 = 35

35 ÷ 5 = 7

7 is a prime number

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5.1.2: Online Retrieval and Problem Solving Practice

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5.2: Greatest Common Factors and Least Common Multiples

Consider the following scenarios for the Wonder, Play, Grow. These concrete situations help us think about the content for this section.

Wonder, Play, Grow

Pick One! Problem Solve with some of your peers.

Scenario 1: Two buses stop at the same bus station—one every 12 minutes, another every 15 minutes. If they both arrive at the station at noon, when will they next arrive together?
Scenario 2: You're tiling a rectangular floor that's 8 feet by 12 feet with square tiles. What's the largest size square tile you can use without cutting any tiles?

🧠Let's Listen to Learn❤️

Audio

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Greatest Common Factor (GCF)

Knowing how to find the GCF is not just a procedural skill; it fosters a deeper understanding of number relationships, provides essential tools for working with fractions, and lays crucial groundwork for future algebraic concepts. As a future elementary school teacher, understanding the significance of the GCF will enable you to teach it more effectively, emphasizing the underlying concepts and its practical applications (Feike, Schwingendorf, and Gregg, 2018).

Definition: Greatest Common Factor

The Greatest Common Factor is the largest whole number that divides evenly two or more nonzero whole numbers. The greatest common factor of a and b will be denoted by GCF(a,b).

Note: By definition, GCF(a, a) = a.

Three ways for finding GCF(a,b) are presented. The first one uses sets, the second one uses prime factorizations, and the third one uses the division algorithm.

Set Intersection Method

In this method we list all of the factors of each number, then list the common factors and choose the largest one.

Example 5.2.1

Find the GCF of 54 and 60, GCF(54, 60).

Answer available at the end of this section.

Prime Factorization Method

In this method we list the prime factors, then multiply the common prime factors.

Example 5.2.2

Find the GCF of 54 and 60, GCF(54, 60).

Answer available at the end of this section.

Wonder, Play, Grow

Task 1: Work with your peers to find the Greatest Common Factor (GCF). Be sure to try both approaches.

1. 40 and 90
2. 75 and 25
3. 168 and 85
4. 90, 120, and 150

Task 2: Let's start to think about another way to find the GCF. Can you think of any? 🤔

We will let math educator Dr. Howie Hua explain another method in this video called the Euclidean Algorithm.

YouTube Video

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View this YouTube Video online

https://youtu.be/rOnw1CJZBaI?si=yxa_nnk_FUsa0tQl

What do you think? We will try it in the example below.

The Euclidean Algorithm

In order to apply the Euclidean Algorithm we need the results of the following theorem. Thus, to find the GCF of any two numbers, this theorem can be applied repeatedly until a remainder of zero is obtained. The final divisor that leads to the zero remainder is the GCF of the two numbers. This is Dr. Howie Hua does in his video, without writing it out in this format.

Theorem 5.2.1

If a and b are whole numbers with a ≥ b and a = bq + r, where r < b, then GCF(a, b) = GCF(r, b).

How to use the Euclidean Algorithm to find the greatest common factor of two numbers:

Write the GCF of the two numbers in parentheses (remember the order of the numbers is irrelevant). The smaller of the two numbers will be one of the numbers in the next parentheses. To get the other number, divide the larger number by the smaller number, and put the **remainder** in parentheses. If the the smaller number is a factor of the larger number, that means it will divide evenly, so there will be no remainder. That means the remainder is 0. Remember to put the remainder in the parentheses, not the quotient! If one of the new numbers in the parentheses is zero, the other number is the GCF; if one the new numbers is 1, 1 is the GCF. Otherwise, repeat the process until one of the two numbers is 0 or 1. An example is shown below.

Example 5.2.3

GCF(546, 390)	= GCF(390, 156)	(*390* is smaller than 546, $546 \div 390$ = 1 r *156*)
	= GCF(156, 78)	(*156* is smaller than 390, $390 \div 156$ = 2 r *78*)
	= GCF(78, 0)	(78 is smaller than 156, $156 \div 78$ = 2 r 0)
	= ***78***	***Put the remainder (0), NOT 2, in the parentheses!***
		78 is the GCF(78, 78); The answer is a number!

Check: First, make sure 78 is a factor of 546 and 390: $546 = 78 \cdot 7$ and $390 = 78 \cdot 5$. Second, check to make sure the other factors of each (7 and 5) are relatively prime. If so, then 78 is not only a factor of 546 and 390, but is in fact the greatest common factor of each since they have no other common factors (because 7 and 5 have no common factors

CAUTION: THE MOST COMMON MISTAKE PEOPLE MAKE WHEN USING THE EUCLIDEAN ALGORITHM IS ON THE LAST STEP, WHEN THE SMALLER NUMBER DIVIDES EVENLY INTO THE LARGER NUMBER, WHICH *MEANS THE REMAINDER IS ZERO*. IT DOESN'T MATTER WHAT THE QUOTIENT IS – IT'S THE *REMAINDER* THAT MATTERS – 0 GOES IN THE PARENTHESES!!! Also, zero is not a factor of any number except zero, so the GCF can't be zero. On the other hand, every number is a factor of zero. So, when zero is one of the numbers in parentheses, the other number is the GCF. **REMEMBER TO ALWAYS CHECK YOUR Answer!!!**

One More Method - Just For Fun 😄😊🥳✨

Since every common factor of a and b is also a common factor of b and a – b, the pairs (a, b) and (a – b, b) have the same common factors. So GCF(a, b) and GCF(a – b, b) must also be the same.

Theorem 5.2.2

If a and b are whole numbers, with a ≥ b, then GCF(a, b) = GCF(a – b, b).

Let's see what you think about this method.

Example 5.2.4

Find the GCF(546, 390).

Answer available at the end of this section.

Think About and Discuss

Consider all of the ways in which we can find the GCF. When would you use one method over the others? Discuss with your peers before trying the following two problems.

Decide on a method for each before finding the GCF.

i) GCF(24, 42)

ii) GCF (504, 600)

iii) GCF(30, 48, 144)

Math Fact

If two numbers have no factors in common, they are called **relatively prime**. In other words, **if GCF(a, b) = 1, then a and b are relatively prime**. The numbers, a and b, might both be prime, both be composite, or one might be prime and the other composite.

4 and 15 are relatively prime, although neither is prime. Can you think of others?

Least Common Multiples

Just as with the procedures for finding the GCF, the concept of the Least Common Multiple (LCM) is about understanding fundamental number relationships, gaining essential skills for working with fractions, and building a crucial foundation for future mathematical studies, including algebra (Feike et al., 2018). As a future elementary teacher, being able to explain the "why" behind the LCM and its applications will empower your students to see mathematics as a connected and meaningful subject. In Scenario 1 of the Wonder, Play, Grow activity we saw a concrete example where finding the LCM can help you solve a real world problem. Now we will take a look at the procedures for finding the LCM.

Definition: Multiples

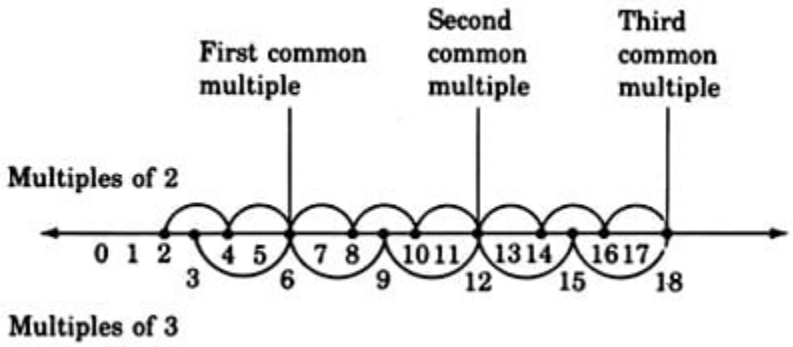
When a whole number is multiplied by other whole numbers, with the exception of Multiples zero, the resulting products are called *multiples* of the given whole number.

Multiples of 2	**Multiples of 3**	**Multiples of 8**	**Multiples of 10**
2·1=2	3·1=3	8·1=8	10·1=10
2·2=4	3·2=6	8·2=16	10·2=20
2·3=6	3·3=9	8·3=24	10·3=30
2·4=8	3·4=12	8·4=32	10·4=40
2·5=10	3·5=15	8·5=40	10·5=50
...	...	...	...

Common Multiples

There will be times when we are given two or more whole numbers and we will need to know if there are any multiples that are common to each of them. If there are, we will need to know what they are. For example, some of the multiples that are common to 2 and 3 are 6, 12, and 18. We can visualize common multiples using the number line.

Notice that the common multiples can be divided by both whole numbers.



The Least Common Multiple (LCM)

Notice that in our number line visualization of common multiples (above) the first common multiple is also the smallest, or *least common multiple,* abbreviated by *LCM*.

As with the GCFs, there are multiple ways for finding LCM(a,b): we will start with the set intersection method and the prime factorization method.

Definition: Least Common Multiple (LCM)

The least common multiple of a and b, denoted by LCM(a,b), is the smallest nonzero whole number that is a multiple of both a and b.

Set Intersection Method - In this method list the nonzero multiples of each number until a first common multiple appears. This number is the LCM(a,b).

Example 5.2.5

Find the LCM of 54 and 30

Solution

Step 1: List the Multiples

54 → 54, 108, 162, 216, 270, 324,...

30 → 30, 60, 90, 120, 150, 180, 210, 240, 270, 300,...

Step 2: Circle the Common multiple(s)

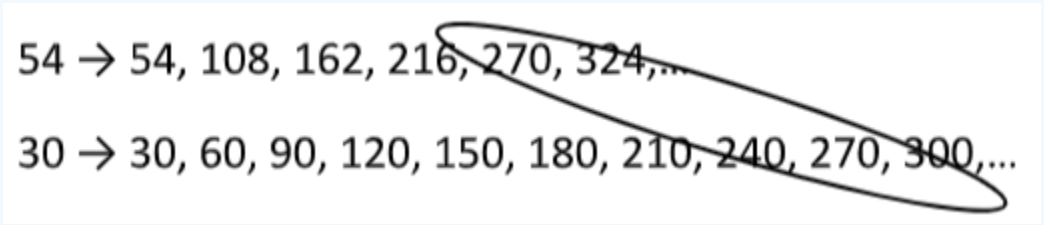


Figure 5.5.2

Step 3: Choose the Least common multiple = 270

Prime Factorization Method - To find the LCM using this method we first find the prime factorization of each number. Then take each of the primes that are factors of either of the given numbers. The LCM is the product of these primes, each raised to the greatest power of the prime that occurs in either of the prime factorizations

STEPS:

1. Write the prime factorization of each number, using exponents on repeated factors.
2. Write each base that appears in each of the prime factorizations.
3. To each base, attach the largest exponent that appears on it in the prime factorizations.
4. The LCM is the product of the numbers found in step 3.

Example 5.2.6

Find the LCM(9,12).

Step 1: $9 = 3 \cdot 3 = 3^2$
$12 = 2 \cdot 6 = 2 \cdot 2 \cdot 3 = 2^2 \cdot 3$

Step 2: The bases that appear in the prime factorizations are 2 and 3.

Step 3: The largest exponents appearing on 2 and 3 in the prime factorizations are, respectively, 2 and 2 (or 2^2 from 12, and 3^2 from 9).

Step 4: The LCM is the product of these numbers. LCM $=2^2 \cdot 3^2 = 4 \cdot 9 = 36$

Thus, 36 is the smallest number that both 9 and 12 divide into without remainders.

Example 5.2.7

Find the LCM(90,630).

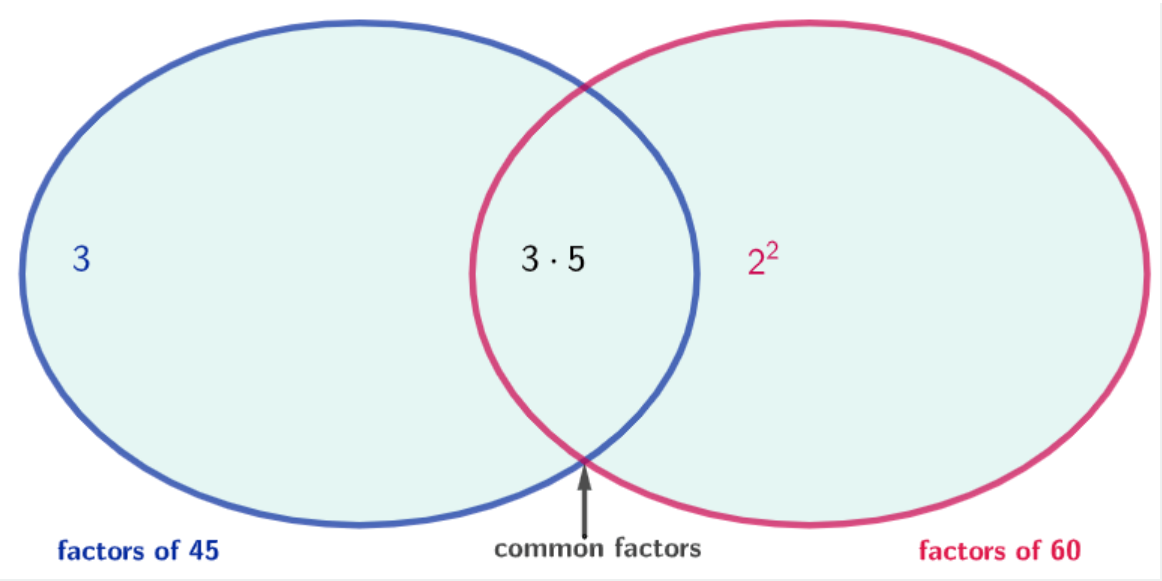
Answer available at the end of this section.

Wonder, Play, Grow

A visual way to see the LCM and GCF of two numbers is with a Venn-diagram. Below is a Venn-diagram containing the factors of 45 and 60. Take a look at the visual as see how we can use it to determine the LCM(45, 60) and GCF(45, 60).

$45 = 3 \times 3 \times 5 = 3^2 \times 5$

$60 = 2 \times 2 \times 3 \times 5 = 2^2 \times 3 \times 5$



Work with peers to find the LCM(45, 60) and GCF(45, 60) using the Venn-diagram. Once you see how it works, make your own diagram to find the following: LCM(24, 54) and GCF(24, 54).

🧠**Let's Listen to Learn**❤️

Audio

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Some Cool Number Theory

The "coolness" of number theory stems from its blend of accessible yet challenging problems, its rich historical and even mystical past, the surprising patterns it unveils, and its unexpected importance in the modern world (Devlin, 2012). It demonstrates that even seemingly simple questions about numbers can lead to profound and fascinating mathematical inquiries. Here are some things related to what we are learning.

After inspiring your students, some of them ask you the following questions, what do you say?

How are GCF and LCM related?

Theorem 5.2.3

Let a and b be any two whole numbers. Then GCF(a, b) x LCM(a, b) = ab.

How many primes numbers are there?

Theorem 5.2.4

There is an infinite number of primes.

Is there is a way to predict how many factors a number has?

Theorem 5.2.5

Suppose that a counting number n is expressed as a product of distinct primes with their respective exponents, say n = $(p1^{n1})(p2^{n2})...(pm^{nm})$. Then the number of factors of n is the product (n1+1)(n2+1)....(nm+1).

eg. The number 12 = 2 x 2 x 3 = $(2^2)(3^1)$, so the number of factors 12 has is (2+1)(1+1) = 6 factors. We know that 1, 2, 3, 4, 6, and 12 are factors of 12, so it confirms the theorem.

Example 5.2.8

Find the number of factors for 144.

Answer available at the end of this section.

Are there any other cool ways to find the GCF and LCM? Dr. Howie Hua may have an answer.

YouTube Video

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https://youtu.be/Js5ATEh-498?si=QKQTt1CzofObitS1

GCF and LCM Explorer

Interactive Content

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Answers

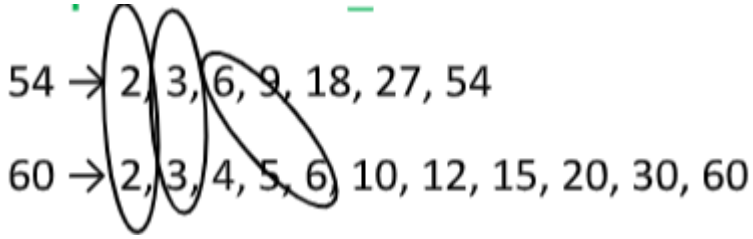
Example 5.2.1

Step 1: List all the Factors, starting with 2

54 → 2, 3, 6, 9, 18, 27, 54

60 → 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60

Step 2: Circle the Common factors



Step 3: Choose the Greatest circled pair

Answer: GCF (54, 60) = 6

Example 5.2.2

Step 1: Find the Prime Factorization

60 → 2 x 2 x 3 x 5 = 2^2 x 3 x 5

54 → 2 x 3 x 3 x 3 = 2 x 3^3

Step 2: Find the common prime factors

Notice that the prime factorizations of 36 and 54 both have one 2 and two 3s in common. So, we simply multiply these common prime factors to find the greatest common factor.

Step 3: Multiply the common prime factors

That is, GCF(54, 60) = 2 x 3 = 6

Example 5.2.4

GCF(546, 390) = GCF(390, 156) = GCF(234, 156) = GCF(156, 78) = GCF(78, 78) = 78

Example 5.2.7

Step 1: $90 = 2 \cdot 45 = 2 \cdot 3 \cdot 15 = 2 \cdot 3 \cdot 3 \cdot 5 = 2 \cdot 3^2 \cdot 5$
$630 = 2 \cdot 315 = 2 \cdot 3 \cdot 205 = 2 \cdot 3 \cdot 3 \cdot 35 = 2 \cdot 3 \cdot 3 \cdot 5 \cdot 7 = 2 \cdot 3^2 \cdot 5 \cdot 7$

Step 2: The bases that appear in the prime factorizations are 2, 3, 5, and 7.

Step 3: The largest exponents that appear on 2, 3, 5, and 7 are, respectively, 1, 2, 1, and 1.

2^1 from either 90 or 630
3^2 from either 90 or 630
5^1 from either 90 or 630
7^1 from 630

Step 4: The LCM is the product of these numbers. LCM $= 2 \cdot 3^2 \cdot 5 \cdot 7 = 2 \cdot 9 \cdot 5 \cdot 7 = 630$

Thus, 630 is the smallest number that both 90 and 630 divide into with no remainders.

Example 5.2.8

144 = 2^4 x 3^2. So, the number of factors of 144 is (4 + 1)(2 + 1) = 15.

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5.2.1: Written Retrieval and Problem Solving Practice

Student Learning Objectives

SLO 5.3 Students will compute the greatest common factor (GCF) and least common multiple (LCM) of pairs of numbers using various methods, including prime factorization, set intersection, and algorithms.

SLO 5.4 Students will evaluate the relationships between the GCF, LCM, and the product of two numbers, demonstrating a deeper understanding of number relationships and their practical applications.

The mild assignment 🌶 is a shorter and less challenging assignment. The medium assignment🌶🌶 is right in the middle in terms of length and level of difficulty. Finally, the spicy assignment 🌶🌶🌶 is a longer assignment that is the most challenging.

(Spicy 🌶🌶🌶 Problems are OPTIONAL, but go for it when you have the time to challenge yourself)

Exercise 5.2.1.1 🌶🌶

Use the set intersection method to find the GCFs.

a) GCF(42, 28) b) GCF(60, 84)

Answer available at the end of this section.

Exercise 5.2.1.2 🌶🌶

Use the prime factorization method to find the GCFs.

a) GCF(36, 42) b) GCF(24, 66)

Answer available at the end of this section.

Exercise 5.2.1.3 🌶🌶

Using the Euclidean algorithm, find the following GCFs.

a) GCF(39, 91) b) GCF(72, 160)

Answer available at the end of this section.

Exercise 5.2.1.4 🌶🌶

Use the set intersection method to find the following LCMs.

a) LCM(24, 30) b) LCM(42, 28)

Answer available at the end of this section.

Exercise 5.2.1.5

Find the following LCMs using the prime factorization method.

a) LCM(60, 72) b) LCM(35, 110)

Answer available at the end of this section.

Exercise 5.2.1.6

Try some alternative method or one you want to practice to find the LCM and GCF.

GCF(125, 210) and LCM(125, 210)

Answer available at the end of this section.

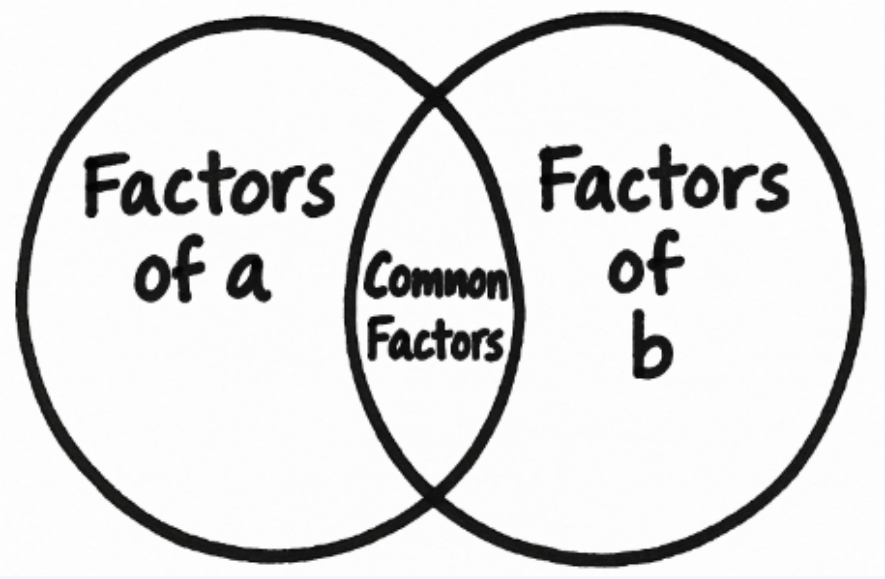
Exercise 5.2.1.7

Two counting numbers are relatively prime if the greatest common factor of the two numbers is 1. Which of the following pairs of numbers are relatively prime?

a) 4 and 9 b) 24 and 123

Answer available at the end of this section.

Exercise 5.2.1.8



For each of the pairs of numbers in parts a – c below

i. sketch a Venn diagram with the prime factors of a and b in the appropriate locations.

ii. find GCF(a, b) and LCM(a, b).

a) a = 63, b = 90

b) a = 16, b = 49

Answer available at the end of this section.

Exercise 5.2.1.9

How many factors do each of the following numbers have?

a) 360

b) 102

c) $3^3 \times 5^2 \times 7$

Answer available at the end of this section.

Exercise \({5{.}2{.}1{.}\eatSpaces{10\)

Find the LCM(33, 110, 484).

Answer available at the end of this section.

Problem Solving Problems

1) A recipe for peanut butter cookies will make 15 cookies. A recipe for chocolate cookies will make two dozen cookies. If you want to have the same number of each type of cookie, what is the least number of each that you will need to make using complete recipes?



2) What is the least number of cards that could satisfy the following three conditions?

- If all the coins are put in two equal piles, there is one coin left over.
- If all the coins are put in three equal piles, there is one coin left over.
- If all the coins are put in five equal piles, there is one coin left over.

3) Two numbers are said to be betrothed if the sum of all proper factors greater than 1 of one number equals the other, and vice versa. For example, 48 and 75 are betrothed, since 48 = 3 + 5 + 15 + 25 proper factors of 75 except for 1, and 75 = 2 + 3 + 4 + 6 + 8 + 12 + 16 + 24, proper factors of 48 except for 1.

Determine which of the following pairs are betrothed.

a) (140, 195) b) (1575, 1648)

Proper Factor (or proper divisor)

A **positive factor** of the number that is **less than the number itself**. **Proper factors** of 12: 1,2,3,4,6 (We **do not** include 12 itself.)

4) Hilda's car gets 34 miles per gallon and Naomi's gets 8 miles per gallon. When traveling from Washington, D.C., to Philadelphia, they both used a whole number of gallons of gasoline. How far is it from Philadelphia to Washington, D.C.?

5) **Problem**:
Ms. Rivera is organizing school supply kits for students.

- She has **84 pencils** and **60 erasers**.
- She wants to divide them into kits so that **each kit has the same number of pencils and the same number of erasers**, with **no supplies left over**.

What is the **greatest number of kits** she can make? How many pencils and how many erasers will each kit contain?

6) **Problem**:
Two friends, Mia and Jaden, go jogging around a park that forms a loop.

- Mia finishes one loop every **12 minutes**.
- Jaden finishes one loop every **18 minutes**.

If they both start jogging at the same time and keep running at their own paces, **after how many minutes will they be at the starting point together again?** If the park opens at 6:00 a.m., what is the **next time** they will meet at the starting point?

Answers

Exercise Extra close brace or missing open brace 🌶🌶

a)

Factors of 42:

1,2,3,6,7,14,21,42

Factors of 28:

1,2,4,7,14,28

Common factors:

1,2,7,14

✅ **GCF = 14**

b)

Factors of 60:

1,2,3,4,5,6,10,12,15,20,30,60

Factors of 84:

1,2,3,4,6,7,12,14,21,28,42,84

Common factors:

1,2,3,4,6,121, 2, 3, 4, 6, 121,2,3,4,6,12

✅ **GCF = 12**

🌶🌶**Exercise** 5.2.1.2

a)

Prime Factorization:

$36=2^2 \times 3^2$

$42=2 \times 3 \times 7$

✅ **GCF = 2 x 3 = 6**

b)

Prime Factorization:

$24=2^3 \times 3$

$66=2 \times 3 \times 11$

✅ **GCF = 2 x 3 = 6**

🌶🌶**Exercise** 5.2.1.3

a) We'll start with the larger number first.

$91 \div 39 = 2$ remainder **13**

→ $91 = 2 \cdot 39 + 13$

39÷13=3 remainder **0**
→ 39=3⋅13+0
✅ **GCF = 13**
b)
160÷72=2 remainder **16**
→ 160=2⋅72+16
72÷16=4 remainder **8**
→ 72=4⋅16+8
16÷8=2 remainder **0**
→ 16=2⋅8+0

🌶🌶**Exercise** 5.2.1.4
a)
Multiples of 24:
24,48,72,96,120,144,...24, 48, 72, 96, 120, 144, ...24,48,72,96,120,144,...
Multiples of 30:
30,60,90,120,150,...30, 60, 90, 120, 150, ...30,60,90,120,150,...
Common multiples:
→ 120,240,...120, 240
✅ **LCM = 120**
b)
Multiples of 42:
42,84,126,168,210,252,294,...
Multiples of 28:
28,56,84,112,140,168,...
Common multiples:
→ 84,168,...84, 168,
✅ **LCM = 84**

Exercise 5.2.1.5 🌶🌶
a)
Prime Factorizations:
$60=2^2 \cdot 3 \cdot 5$
$72=2^3 \cdot 3^2$
LCM:
$LCM=2^3 \cdot 3^2 \cdot 5=8 \cdot 9 \cdot 5=72 \cdot 5=360$
b)
Prime Factorizations:
$35=5 \cdot 7$
$110=2 \cdot 5 \cdot 11$
LCM:
$LCM=2 \cdot 5 \cdot 7 \cdot 11=770$

Exercise 5.2.1.6 🌶🌶
GCF(125, 210) = 5
LCM(125, 210) = 5250

Exercise 5.2.1.7 🌶🌶
a) 4 and 9 → Relatively prime
b) 24 and 123 → Not relatively prime

🌶🌶🌶**Exercise** 5.2.1.8
a) GCF(63, 90) = 9, LCM(63, 90) = 630
b) GCF(16, 49) = 1, LCM(16, 49) = 784

Exercise 5.2.1.9 🌶🌶
a) 24 Factors b) 8 Factors c) 24 factors

Exercise \(\PageIndex{10}\) 🌶🌶
1. 33 = 3 ⋅ 11
   110 = 2 ⋅ 55 = 2 ⋅ 5 ⋅ 11
   484 = 2 ⋅ 242 = 2 ⋅ 2 ⋅ 121 = 2 ⋅ 2 ⋅ 11 ⋅ 11 = $2^2 \cdot 11^2$
2. The bases that appear in the prime factorizations are 2, 3, 5, and 11.
3. The largest exponents that appear on 2, 3, 5, and 11 are, respectively, 2, 1, 1, and 2.

   2^2 from 484
   3^1 from 33
   5^1 from 110
   11^2 from 484
4. The LCM is the product of these numbers.

LCM = $2^2 \cdot 3 \cdot 5 \cdot 11^2$
= $4 \cdot 3 \cdot 5 \cdot 121$
= 7260 ; Thus, 7260 is the smallest number that 33, 110, and 484 divide into without remainders.

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- 8.2: Primes and GCF by Julie Harland is licensed CC BY-NC 4.0. Original source: https://sites.google.com/site/harlandclub/my-books/math-64.
- 1.4: The Least Common Multiple by Denny Burzynski and Wade Ellis, Jr. is licensed CC BY 4.0.

5.2.2: Online Retreival and Problem Solving Practice

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5.3: Chapter 5 Resources

Chapter Example Videos

Chapter 5.1

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Chapter 5.2

YouTube Video

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Chapter Extras

Interactive Learning Apps

Interactive Content

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6: THE STORY OF FRACTIONS

Long ago, people started noticing that whole numbers just weren't enough. Imagine trying to divide a pizza between three friends or pour half a can of paint—you can't describe those situations using just whole numbers! So, people *invented* fractions to solve real-world problems, "fractions were invented because it was not convenient to describe many problem situations using only whole numbers" (Musser et al., 2014). Around 1600 B.C.E., the ancient Egyptians were among the first to use fractions in a big way. They used them to track land, food, and time. Their system was based on unit fractions—fractions like ½ or ⅓, always with a top number of 1. Even when they needed something like 2/3, they would break it down into a sum of unit fractions. Their methods were different from ours, but their need was the same: to describe parts of a whole. (Burton, 1984)

Just like kids today naturally split snacks or toys into fair shares, ancient people had to figure out how to share and divide things too. Young children often show early understanding of halves, thirds, and fourths even before they learn formal math. This makes fractions one of the oldest—and most human—mathematical ideas. (Feike et al., 2018). As math developed, so did the way we write and use fractions. Our modern notation (like ¾) is a tool created to make life easier. But understanding what a fraction *means*—a number that represents part of a whole—is way more important than just writing it correctly. (Burton, 1984)

Why does this matter for your future students? Because fractions are a bridge. Once students understand them, they can move on to decimals, percentages, algebra, and even higher math. The *California Math Framework* (2013) emphasizes that building a strong number sense, including a deep understanding of fractions, sets the stage for success in middle and high school math. Yes, fractions can be tricky. In fact, they're considered one of the most challenging topics in math. But here's the twist: that difficulty means it's even more important to build meaning first, not just teach rules. Understanding the story behind fractions—the real-life needs that inspired them—can help both teachers and students feel more confident.

In the end, fractions show us something powerful: Math was created to solve real problems, and it keeps evolving to help us understand the world better. And that's something your students deserve to see from the start.

Chapter Learning Objectives

Student Learning Objectives

- SLO 6.1 Students will model and compare fractions using visual, verbal, and symbolic representations, and explain how these models support conceptual understanding and simplification strategies.
- SLO 6.2 Students will analyze and compute sums and differences of fractions, applying addition and subtraction properties and using estimation techniques to verify the reasonableness of results of real-world problems.
- SLO 6.3 Students will use visual models and real-life contexts to represent and compute multiplication and division of fractions, justifying the methods they choose and interpreting the results.
- SLO 6.4 Students will evaluate the application of mental math and estimation techniques for operations with fractions, including rounding, range estimation, and using compatible numbers to simplify calculations.

Essential Questions

- **How do conceptual and visual representations of fractions deepen understanding and support meaningful real-world application—especially in teaching and parenting contexts?**
- **What misconceptions do students commonly have about fractions, and how can we design instruction to address them?**

Learning That Transfers

Students will connect conceptual understanding of fraction operations to real-world contexts, using multiple representations and strategies to explain and model these ideas for others—especially young learners.

One Part Love One Part Math

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Morgan's Fraction Fiesta

It was cold in December, the sky turning gray, When Morgan arrived for Tamale Day.
The house smelled of chili, of masa and steam— It was time to cook up Abuelito's dream!

"Morgan," said Grandpa, "today's just for you, We'll make tamales—red, green, and blue!
(Well not blue... but sweet ones with raisins and spice!) You'll measure and mix and cut things just right."

He laid out the masa, a warm golden mound, In a bowl so big it could spin 'round and 'round.
"We'll need to divide this in parts of a whole—Each type of tamale has its own role."

"One-half of the masa is spicy and bold, With green chile filling, just like I told.
One-fourth will be sweet, with cinnamon flair, The rest will have beans and some cheese we can share."

Morgan looked close, her mind in a spin— "How do you know that these parts all fit in?" Grandpa just grinned and pulled out a tray, And showed how the pieces could fit the whole way.

She spread the masa with careful precision, Laughed when she made her very first division.
"Half for the green, one-fourth for the sweet...This fraction thing? It's actually neat!"

Next came the toppings—"Now don't go too fast! We've got 24 tamales to wrap up and cast.
If there are six people, all hungry and loud, How many tamales for each of the crowd?"

Morgan thought hard, then gave him a grin—"That's four each!" she beamed. "Let the feast begin!"

"But wait!" said Grandpa, "Let's count them again, What if five more cousins arrive at ten?"
Morgan blinked twice, "Oh my... that's eleven! Now 24 tamales must stretch like heaven!"

She estimated quickly, rounding with care—"Two for each, maybe three if we share!"
They made a new batch, dividing once more, Using thirds, halves, and even three-fourths galore.

As they worked, she added and mixed with pride, Explaining her math with Grandpa by her side.
"When I take 1/2 and add 1/4 in the pot, I get 3/4, and that's quite a lot!"

They multiplied fillings—"Three kinds by eight!" And divided them fairly upon every plate.
They laughed and they learned, just as families do, Mixing culture and numbers into something brand new.

And when the last tamal was done—The family gathered, everyone!
They ate and they smiled, with stories to say, And Morgan knew math in a whole different way.

So now when she sees a pie or a stew, She thinks of the math in the cooking they do.
Fractions aren't just for numbers or tests—They live in our homes, in our hearts, in our quests.

6.1: Concept of Fraction, Equivalence, and Order
6.1.1: Written Retrieval and Problem Solving Practice
6.1.2: Online Retreival and Problem Solving Practice
6.2: Fraction Addition and Subtraction
6.2.1: Written Retrieval and Problem Solving Practice
6.2.2: Online Retrieval and Problem Solving Practice
6.3: Multiplication and Division
6.3.1: Written Retrieval and Problem Solving Practice
6.3.2: Fraction Discussion Board
6.3.3: Online Retrieval and Problem Solving Practice
6.3.4: Problem Solving Task
6.4: Chapter 6 Resources

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6.1: Concept of Fraction, Equivalence, and Order

The early chapters in our book focused on building a solid understanding of the whole number system, which is essential for long-term success in mathematics. Now, we shift our attention to fractions. Fractions were developed because many real-world problems cannot be accurately described using only whole numbers. As you work through this chapter, you'll see how a strong grasp of whole numbers provides a helpful foundation for making sense of fraction concepts.

Importance of Fractions

Students often struggle with fractions because they view them as two separate whole numbers rather than a single number representing a part of a whole. This misunderstanding prevents them from seeing the relationship between the numerator and denominator, and from using helpful benchmarks like 0, ½, and 1. Unlike whole numbers, fractions rarely appear in children's everyday experiences beyond simple examples like ½ or ¼, making them less intuitive. Additionally, students tend to incorrectly apply whole number reasoning to fractions, by believing, for instance, that 1/20 is greater than 1/10 because 20 is larger than 10. (Musser et al., 2014; NCTM, 2020).

Another major challenge is the abstract nature of fractions, which can represent multiple ideas—part-whole, measurement, division, ratio, or operator—depending on the context. Many students only encounter the part-whole model, limiting their understanding. Cognitive overload also plays a role, as tasks with fractions often require juggling several steps, especially when working with unlike denominators (Feike et al., 2018). Moreover, instruction that emphasizes procedural rules—like "invert and multiply"—before students fully grasp the meaning of fractions leads to superficial learning. When students are taught procedures without context, they may memorize steps but lack the conceptual understanding needed for success in more advanced math (NCTM, 2020).

Fractions Children Learn in Elementary School

As we learn the concepts and procedures associated with fractions, it is important that you have context of what students learn about fractions in elementary school. Below is a video explaining how children progress with their understanding of fractions. Also included is a list of some of the standards from the Common Core State Standards (CDE, 2013).

YouTube Video

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Graham Fletchy - https://youtu.be/9IwVAXmrWK8?si=0DCRvo6Lmqb_1zEq

Common Core Standards (CDE, 2013)

Grade 3: Introduction to Fractions

3.NF.1: Understand a fraction 1/b as one part of a whole divided into *b* equal parts; a/b as *a* parts of size 1/b.

3.NF.2: Represent fractions on a number line.
2a: Mark 1/b on a number line between 0 and 1.
2b: Mark a/b by partitioning and counting lengths of 1/b.

3.NF.3: Explain equivalence and compare fractions:
3a: Understand equivalence by same size or location.
3b: Generate simple equivalent fractions.
3c: Express whole numbers as fractions.
3d: Compare fractions with same numerator or denominator using visual models.

Grade 4: Operations and Equivalence

4a–4c: Visualize, apply repeated addition, and solve word problems.

4.NF.1: Explain why a fraction a/b is equivalent to (n × a)/(n × b).

4.NF.2: Compare fractions with different numerators and denominators.

4.NF.3: Build fractions from unit fractions:
3a: Understand addition/subtraction as joining/separating parts.
3b: Decompose fractions into sums.
3c: Add/subtract mixed numbers.
3d: Solve word problems with like denominators.

4.NF.4: Multiply a fraction by a whole number

Grade 5: All Operations with Fractions

5.NF.1-2: Use equivalent fractions to add/subtract fractions with unlike denominators.

5.NF.3: Interpret a fraction as division (e.g., 2 ÷ 3 = 2/3).

5.NF.4: Multiply a fraction or whole number by a fraction.

4a: Visualize multiplication.
4b: Find area of rectangles with fractional side lengths.

5.NF.5: Interpret multiplication outcomes relative to 1.

5.NF.6: Solve real-world problems involving multiplication of fractions and mixed numbers.

5.NF.7: Divide unit fractions by whole numbers and vice versa:

Wonder, Play, Grow

Fraction Talks: Before discussing anything with others, review each picture and determine what fraction of space is occupied by each color. Take some time to think about it and about how you can defend your answer with the class.

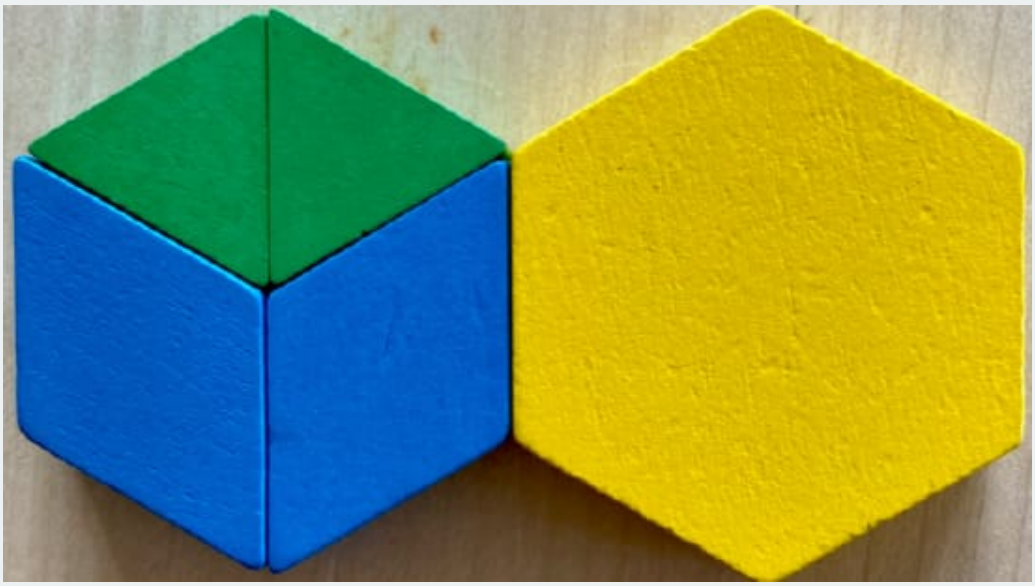
Picture 1:



Picture 2:



Picture 3:



Concept of Fraction

In elementary math, fractions are used in **two distinct ways**:

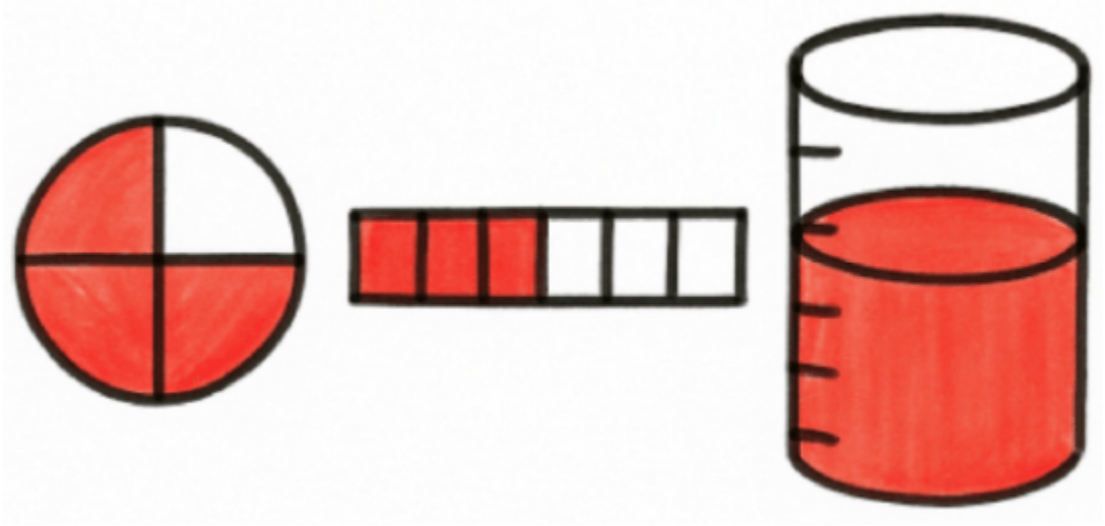
🧠**Let's Listen to Learn**❤️

Audio

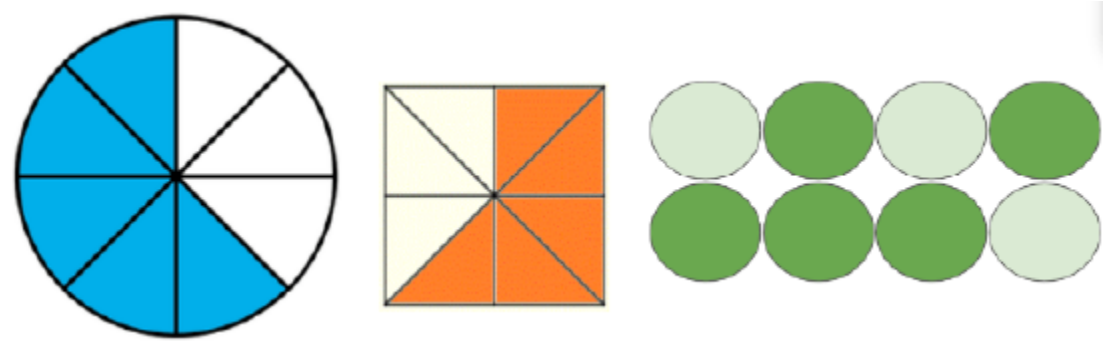
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Part-to-Whole Model:
This is the most familiar introduction to fractions. A fraction like **3/4** represents 3 parts out of 4 *equal parts* of a whole. The **numerator** (3) tells how many parts are being considered, and the **denominator** (4) tells what kind of parts (fourths, fifths, etc.). This model emphasizes fractions as a way to **name parts of a whole**, whether it's pieces of a pie (¾), segments of a stick (3/6), or cups of liquid (4/6).



Fractions as Numbers:
Fractions are also abstract numbers that represent a **relative amount**, independent of any particular object. For instance, **5/8** means the same proportion regardless of the size or shape of the whole—it expresses a point on the number line, just like whole numbers do. This view shifts the meaning of fractions from a specific object-based context to a general **numerical** concept. Each of the figures below shows the relative amount of 5/8.



Definition: Fractions

A fraction is a number that can be represented by an ordered pair of whole numbers $(\frac{a}{b})$ (or (a/b)), where $b \neq 0$. In set notation, the set of fractions is

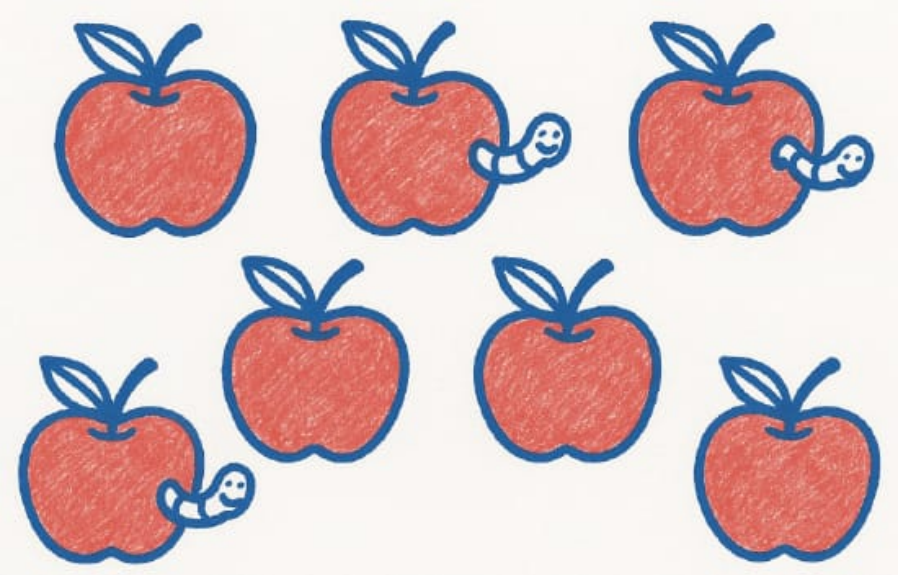
$$F = \left\{ \frac{a}{b} \,\middle|\, a \text{ and } b \text{ are whole numbers, } b \neq 0 \right\}. \tag{6.1.1}$$

Language Note: The word fraction comes from the Latin word fractius which means "to break". When an object is divided into an equal number of parts then each part is called a fraction. The number a is called the numerator, derived from the Latin word numeros, meaning "number", and represents the number of parts in consideration. The number b is called the denominator, derived from the Latin word denominare, meaning "namer", and represents how many equal parts in the unit. Keep in mind that this number can never be zero since division by zero is undefined.

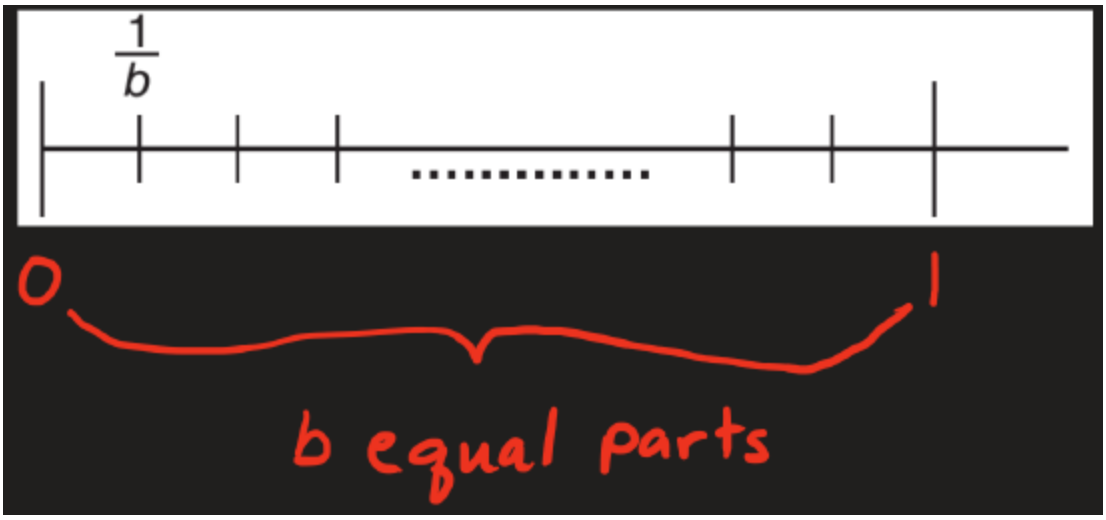
Visual Representation of a Fraction

Several physical and pictorial representations are useful in the elementary school classroom to illustrate fraction concepts.

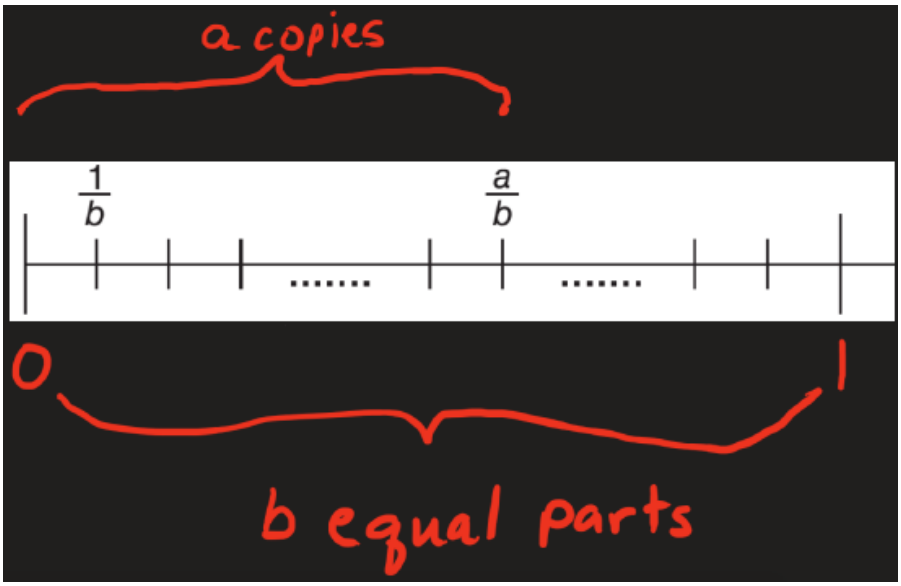
SET MODEL - The figure below shows a set of 7 apples that contains a subset of 3 that are wormy. Therefore, we would say that $\frac{3}{7}$ of the apples are wormy.



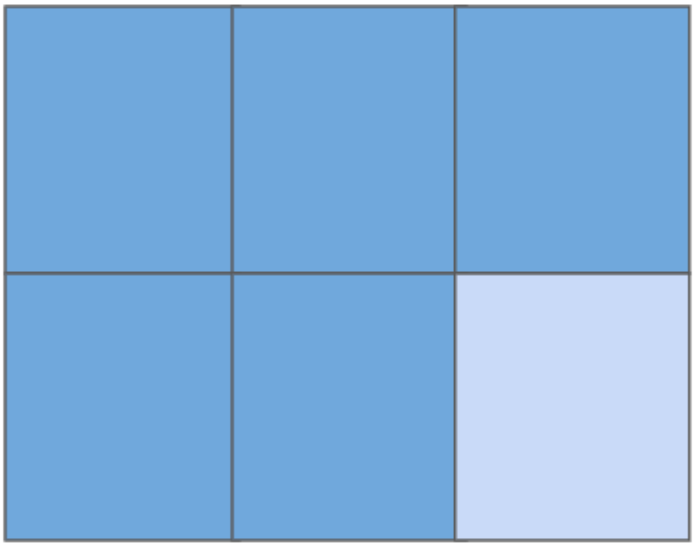
NUMBER LINE (Measurement or Length Model) - Fractions can be shown on a number line by dividing the segment from 0 to 1 into equal parts. The fraction $\frac{1}{b}$ is the point at the end of the first segment when that interval is divided into **b** equal parts.



To locate $\frac{a}{b}$, start at 0 and count **a** segments of length $\frac{1}{b}$. The end of the **a**th segment marks the location of **a/b**.

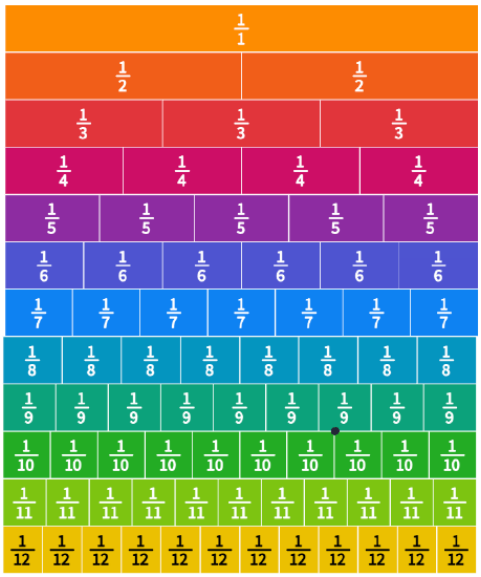


REGION/AREA MODEL - A shape is chosen to represent the unit and is then subdivided into equal parts. A fraction is visualized by coloring some of the parts as shown in the figure below.



FRACTION STRIPS (Bars) -

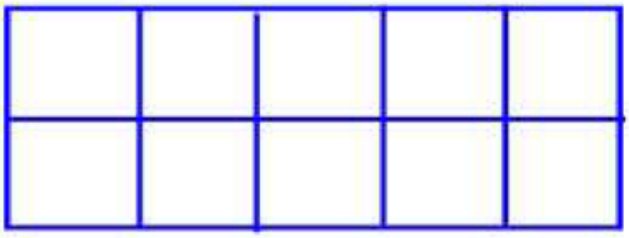
Here the unit is defined by a rectangular strip. A fraction $\frac{a}{b}$ is modeled by shading **a** parts of the **b** equally sized subrectangles. Sample fraction strips are shown in figure below.



Example 6.1.1

Explain how to complete each diagram below to represent $\frac{3}{10}$.





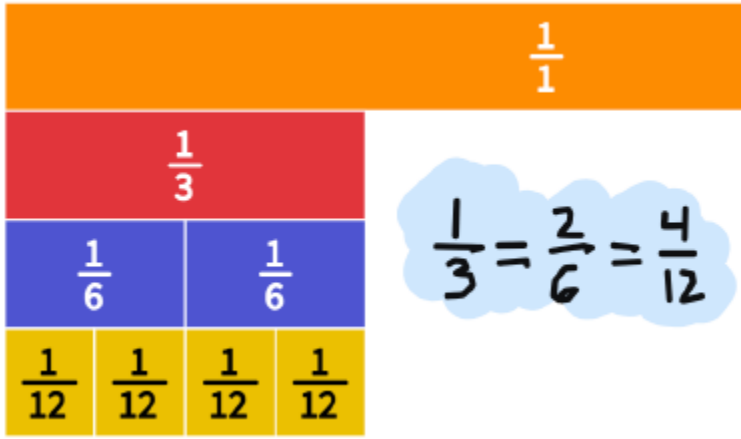
Answer available at the end of this section.

There are 3 different types of fractions:

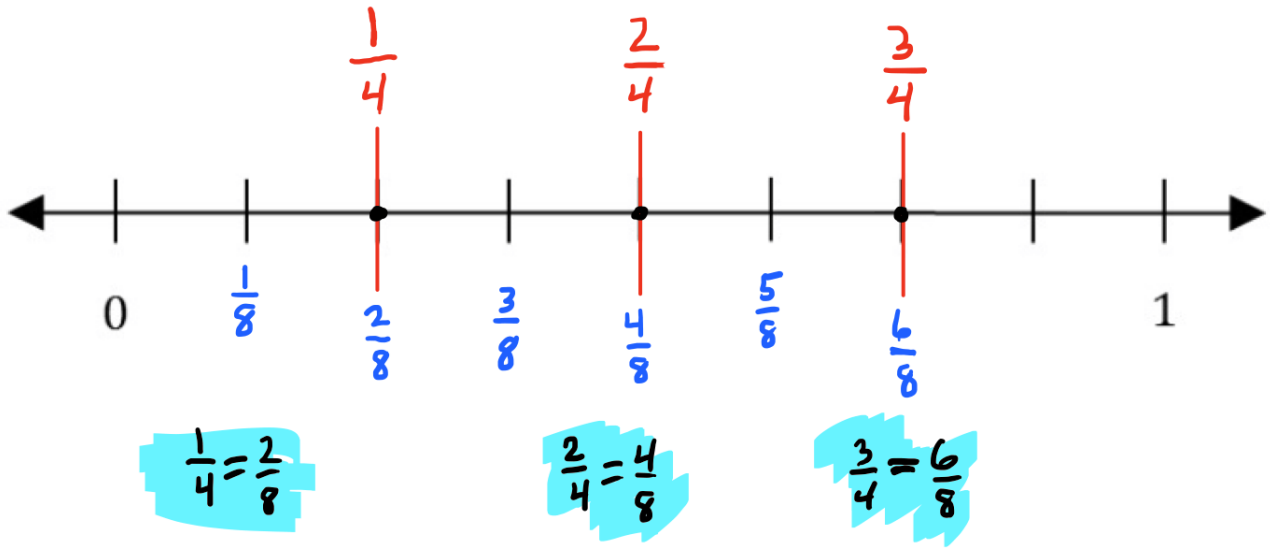
- Proper Fractions - Proper fractions have the numerator part smaller than the denominator part, for example, $\frac{2}{5}$.
- Improper Fractions - Improper fractions have the numerator part greater or equal to the denominator for example, $\frac{7}{3}$.
- Mixed Fractions - Mixed fractions have a whole number plus a fraction, for example, $3\frac{1}{2}$.

Equivalent Fractions

Two fractions that represent the same relative amount are said to be **equivalent fractions.** As suggested by the fraction strips below, the fractions all represent the same relative amount as $\frac{1}{3}$. And make a note that whole number 1 can be represented by a fraction $\frac{1}{1}$.



Equivalent fractions are also located at the same point on a number line. The equivalence can be seen by their common location on the number line as seen below.



It is useful to have a simple test to determine whether fractions such as $\frac{3}{4}$ and $\frac{6}{8}$ represent the same relative amount without having to draw a picture of each representation.

Definition: Fraction Equality

Let $\frac{a}{b}$ and $\frac{c}{d}$ be any fractions. Then $\frac{a}{b}=\frac{c}{d}$ if and only if **ad= bc.**

NOTE: Two fractions are equal fractions if and only if their cross-products, that is, products **ad** and **bc** obtained by cross-multiplication, are equal.

Theorem 6.1.1

Let $\frac{a}{b}$ be any fraction and **n** a nonzero whole number. Then $\frac{a}{b}=\frac{an}{bn}=\frac{na}{nb}$.

Example 6.1.2

Verify the following equations using the definition of fraction equality or the preceding theorem.

a) $\frac{6}{14}=\frac{9}{21}$

b) $\frac{6}{21}=\frac{60}{210}$

Answer available at the end of this section.

Occasionally, the term reducing is used to describe the process in 6.2 example b). However, the term reducing can be misleading, since fractions are not reduced in size (the relative amount they represent) but only in complexity (the numerators and denominators are smaller). We use the term **“simplify”** or say 60/210 is in “simplified form” when it is written as 2/7.

Example 6.1.3

Find the simplest form for each fraction.

a) $\frac{72}{240}$

b) $\frac{9}{32}$

c) $\frac{399}{483}$

Answer available at the end of this section.

Ordering Fractions

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Audio

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Think About and Discuss

Before discussing these ideas with your peers, make sure to independently respond to the prompts.

Task 1: Consider each fraction strip (1/2, 1/3, 1/4, 1/5, 1/6, 1/8, 1/9, 1/10, and 1/12). One fraction is bigger than another if it has more length than the other fraction. Compare them and put them in order from smallest to largest, and write them in order below, using the less than symbol (<). Where would 1/7 go?

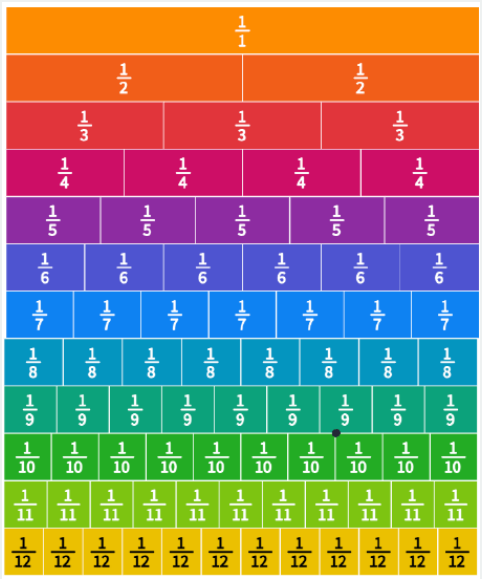
Task 2: Use fraction strips to compare 5/6, 5/8, 5/9, 5/10, and 5/12. Put them in order from smallest to largest using the less than symbol (<). Where would 5/7 go?

Task 3: Use fraction strips to compare these fractions: 1/2, 2/3, 3/4, 4/5, 5/6, 7/8, 8/9, 9/10, and 11/12. Put them in order from smallest to largest using the less than symbol (<).

Task 4: Discuss with your peers and explain any patterns you learned.

Online Fraction Strips

https://polypad.amplify.com/p#fractions



The concepts of less than and greater than in fractions are extensions of the respective whole-number concepts. As it was in the case of whole numbers and what you learned in the Think About and Discuss activity, the smaller of two fractions is to the left of the larger fraction on the fraction number line. Also, what we learned suggest the following definition, where fractions having common denominators can be compared simply by comparing their numerators (which are whole numbers).

Definition: Less Than for Fractions

Let $\frac{a}{c}$ and $\frac{b}{c}$ be any fractions. Then $\frac{a}{c} < \frac{b}{c}$ if and only if **a < b.**

NOTE: Although the definition is stated for “less than,” a corresponding statement holds for “greater than.” Similar statements hold for “less than or equal to” and “greater than or equal to.”

Theorem 6.1.2

Cross-Multiplication of Fraction Inequality

Let $\frac{a}{b}$ and $\frac{c}{d}$ be any fractions. Then $\frac{a}{b}<\frac{c}{d}$ if and only if **ad**

Example 6.1.4

Arrange in order.

a) $\frac{7}{8}$ **and** $\frac{9}{11}$

b) $\frac{2}{9}$ **and** $\frac{7}{36}$

b) $\frac{17}{32}$ **and** $\frac{19}{40}$

Answer available at the end of this section.

Another COOL way to compare fractions - Dr. Howie Hua does it again.

YouTube Video

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

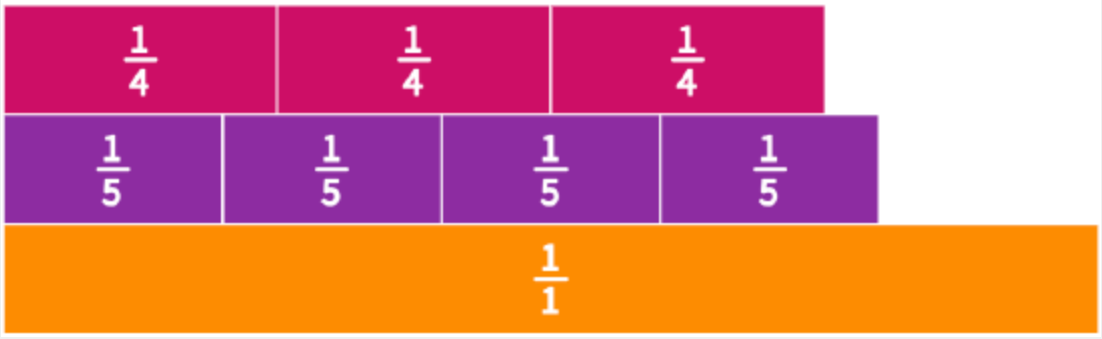
View this YouTube Video online

https://youtu.be/7pbiKIYsWF0?si=NFS_08iO8gCRYqRG

Wonder, Play, Grow

Let's see what else Dr. Hua is up to? Work with your peers to find a fraction that is between $\frac{3}{4}$ and $\frac{4}{5}$. For an extra challenge, can you find more than one?





Using common denominators is great to find a fraction between two others. For example, 3/4 = 30/40 and 4/5 = 32/40. Therefore we know that 3/4 < 31/40 < 4/5. However there is simple technique of adding the numerators and denominators to determine a fraction between two other fractions.

Theorem 6.1.3

Let $\frac{a}{b}$ and $\frac{c}{d}$ be any fractions where $\frac{a}{b} < \frac{c}{d}$. Then

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}.$$

Example 6.1.5

Find a fraction between these pairs of fractions.

a) $\frac{7}{11}$ **and** $\frac{8}{11}$

b) $\frac{9}{13}$ **and** $\frac{12}{17}$

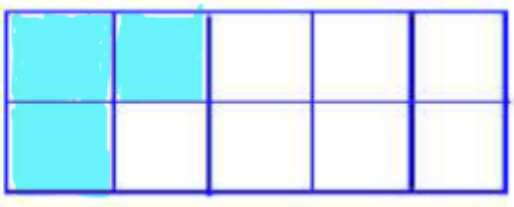
Answer available at the end of this section.

Fratction Finder

Answers

Example 6.1.1





Example 6.1.2

a) (6 x 21) = (9 x 14) = 126, equivalent fractions.

b) $\frac{60}{210} = \frac{30x2}{30x7} = \frac{2}{7}$ **and** $\frac{6}{21} = \frac{3x2}{3x7} = \frac{2}{7}$**, equivalent fractions.**

Example 6.1.3

a) $\frac{72}{240} = \frac{3x24}{10x24}$ **= 3/10 b) 9/32 c)** $\frac{399}{483} = \frac{19x21}{23x21}$**=19/23**

Example 6.1.4

a) 7/8 > 9/11 b) 7/36 < 2/9 c) 19/40 < 17/32

Example 6.1.5

a) $\frac{7}{11} = \frac{14}{22}$ and $\frac{8}{11} = \frac{16}{22}$. Hence $\frac{15}{22}$ is between $\frac{7}{11}$ and $\frac{8}{11}$.

b) $\frac{9}{13} = \frac{9 \cdot 17}{13 \cdot 17} = \frac{153}{13 \cdot 17}$ and $\frac{12}{17} = \frac{12 \cdot 13}{17 \cdot 13} = \frac{156}{17 \cdot 13}$. Hence both $\frac{154}{13 \cdot 17}$ and $\frac{155}{13 \cdot 17}$ are between $\frac{9}{13}$ and $\frac{12}{17}$.

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6.1.1: Written Retrieval and Problem Solving Practice

Student Learning Objectives

6.1 Students will model and compare fractions using visual, verbal, and symbolic representations, and explain how these models support conceptual understanding and simplification strategies.

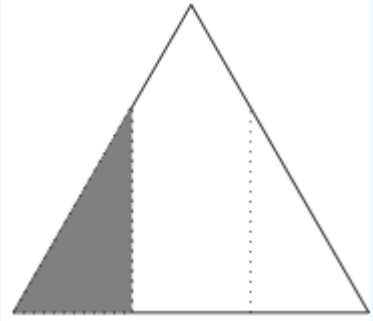
The mild assignment 🌶 is a shorter and less challenging assignment. The medium assignment🌶🌶 is right in the middle in terms of length and level of difficulty. Finally, the spicy assignment 🌶🌶🌶 is a longer assignment that is the most challenging.

(Spicy 🌶🌶🌶 Problems are OPTIONAL, but go for it when you have the time to challenge yourself)

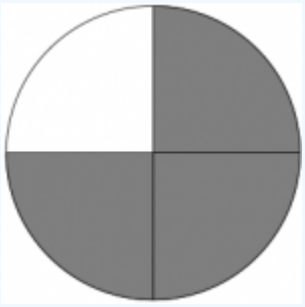
Exercise 6.1.1.1 🌶🌶

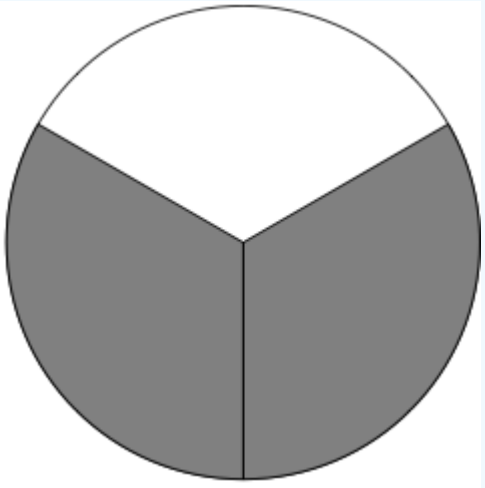
For each of the following visual representations of fractions, there is a corresponding **incorrect** symbolic expression.

- Why is the symbolic representation incorrect?
- What might elementary students find confusing in these visual representations?



a) $\frac{1}{3}$





b) $\frac{2}{3} > \frac{3}{4}$

Answer available at the end of this section.

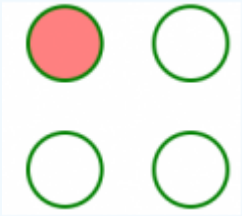
Exercise 6.1.1.2 🌶

What fraction is represented by the shaded portion?

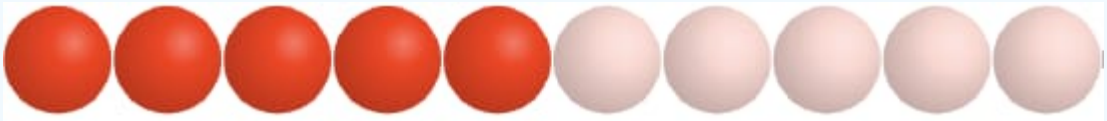
a)



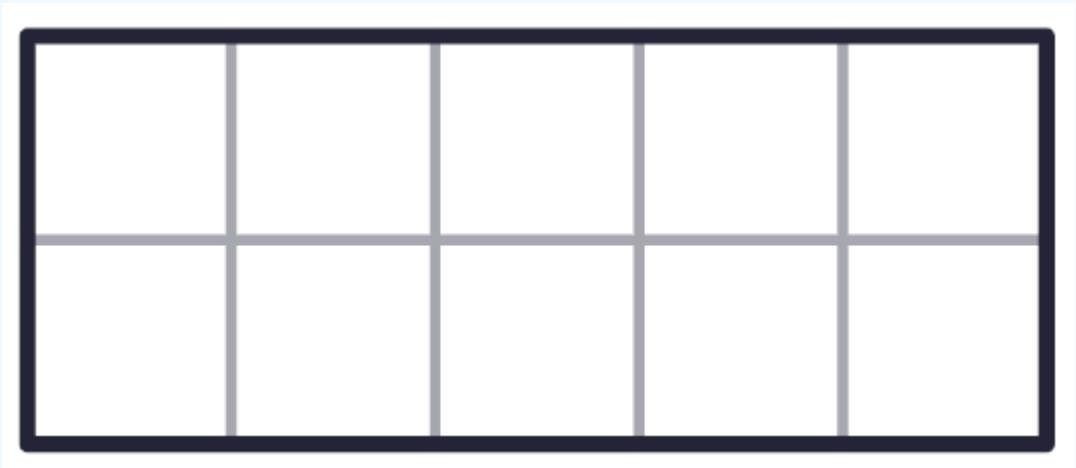
b)



c)



d)



Answer available at the end of this section.

Exercise 6.1.1.3

Represent $\frac{4}{5}$ using the given models.

a) Set Model

b) Fraction Strips

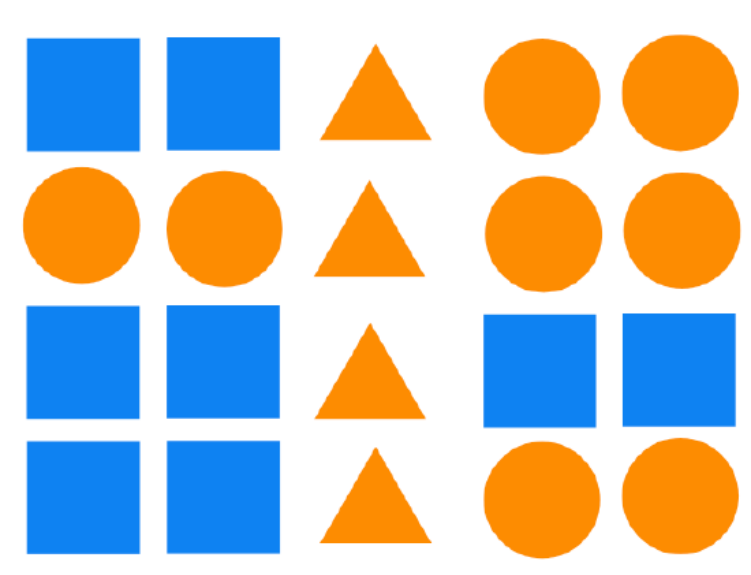
c) Number Line

d) Region Model

Answer available at the end of this section.

Exercise 6.1.1.4

Using the diagram below, represent each of the following as a fraction of all shapes shown.



a) What fraction is made of circular shapes?

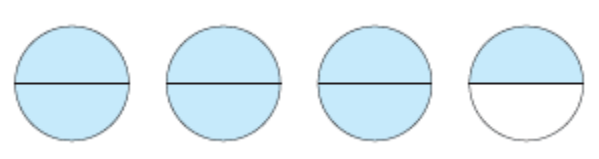
b) What fraction is made of square shapes?

c) What fraction is not made of triangular shapes?

Answer available at the end of this section.

Exercise 6.1.1.5

a) Write the fraction below as both a mixed number and improper fraction. Assume one circle is one whole.



b) A student sees the fraction above and states the value is 7/8? What are they seeing?

Answer available at the end of this section.

Exercise 6.1.1.6

Determine which of the following pairs are equal.

a) $\frac{5}{8}, \frac{625}{1000}$

b) $\frac{24}{36}, \frac{50}{72}$

c) $\frac{240}{72}$, $\frac{10}{4}$

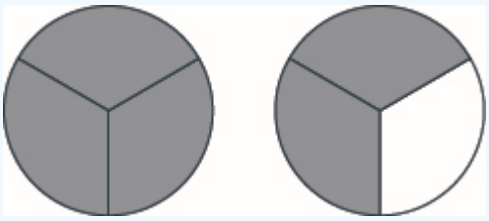
Answer available at the end of this section.

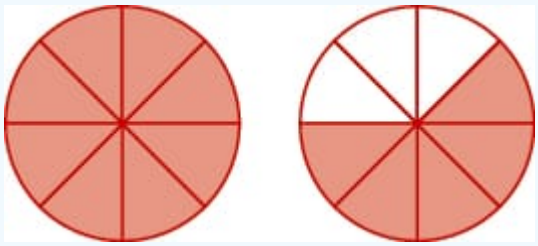
Try the Desmos Fraction Button to help check your work.

$\frac{12}{15}$ $= \frac{4}{5}$

Exercise 6.1.1.7

Name the improper fraction and mixed number modeled.

a) 

b) 

Answer available at the end of this section.

Exercise 6.1.1.8

Rewrite in simplest form (a-d).

a) $\frac{25}{175}$

b) $\frac{21}{30}$

c) $\frac{108}{126}$

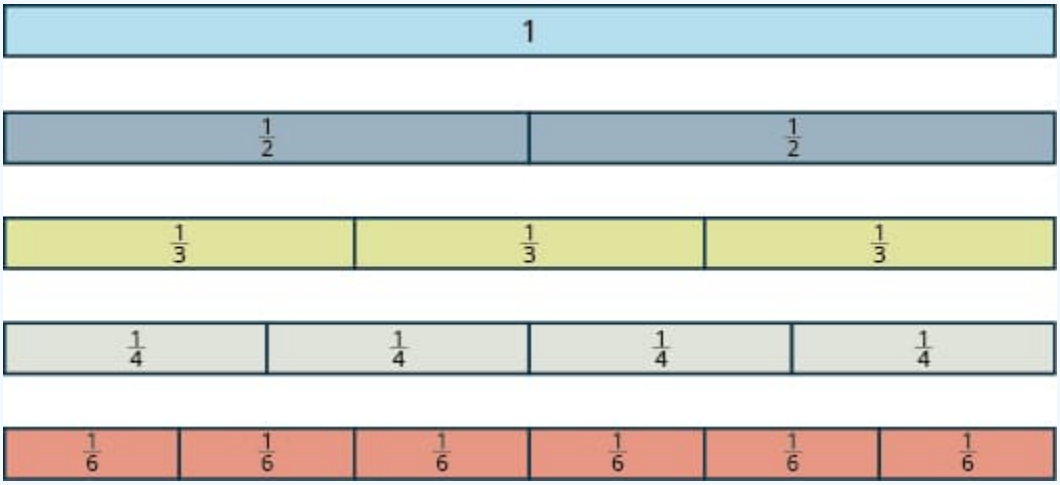
d) $\frac{220}{100}$

e) Find three fractions equivalent to $\frac{2}{5}$.

Answer available at the end of this section.

Exercise 6.1.1.9

Use the diagram of fraction strips to determine which fraction is larger.



a) $\frac{1}{2}$, $\frac{2}{3}$

b) $\frac{3}{4}$, $\frac{5}{6}$

c) $\frac{2}{3}$, $\frac{4}{6}$

Answer available at the end of this section.

Exercise 6.1.1.10

Arrange each of the following from smallest to largest.

a) $\frac{1}{15}$, $\frac{1}{16}$, $\frac{1}{17}$

b) $\frac{11}{15}$, $\frac{13}{15}$, $\frac{12}{15}$

Answer available at the end of this section.

Exercise 6.1.1.11

Of each group of fractions, which is the smallest?

a) $\frac{14}{27}$, $\frac{3}{7}$, $\frac{9}{20}$

b) $\frac{6}{13}$, $\frac{17}{39}$, $\frac{25}{51}$

Answer available at the end of this section.

Exercise 6.1.1.12

Find a fraction between each pair.

a) $\frac{1}{11}$, $\frac{3}{20}$

b) $\frac{3}{7}$, $\frac{4}{9}$

Answer available at the end of this section.

Problem Solving Practice

1) The Green Party received one-eleventh of the 6,186,279 votes cast. How many votes did the party receive?

2) The school library contains about 5280 books. If five-twelfths of the books are for the teens, how many such books are there in the library?

3) Think About and Discuss: What quick method can you use to determine if a fraction is greater than 1? What quick method can you use to determine if a fraction is greater than $\frac{1}{2}$?

Organize these fractions from smallest to largest using *benchmarks*: 0 to $\frac{1}{2}$, $\frac{1}{2}$ to 1, and greater than 1. Justify your choices.

$$\frac{25}{23}, \quad \frac{4}{7}, \quad \frac{17}{35}, \quad \frac{2}{9}, \quad \frac{14}{15}.$$

Arrange each group of fractions in *ascending order*. Keep track of your thinking and your methods.

- $\frac{7}{17}$, $\frac{4}{17}$, $\frac{12}{17}$
- $\frac{3}{7}$, $\frac{3}{4}$, $\frac{3}{8}$
- $\frac{5}{6}$, $\frac{7}{8}$, $\frac{3}{4}$
- $\frac{8}{13}$, $\frac{12}{17}$, $\frac{1}{6}$
- $\frac{5}{6}$, $\frac{10}{11}$, $\frac{2}{3}$

4) Mr. Li shows this picture to his class and asks what number is shown by the shaded region.

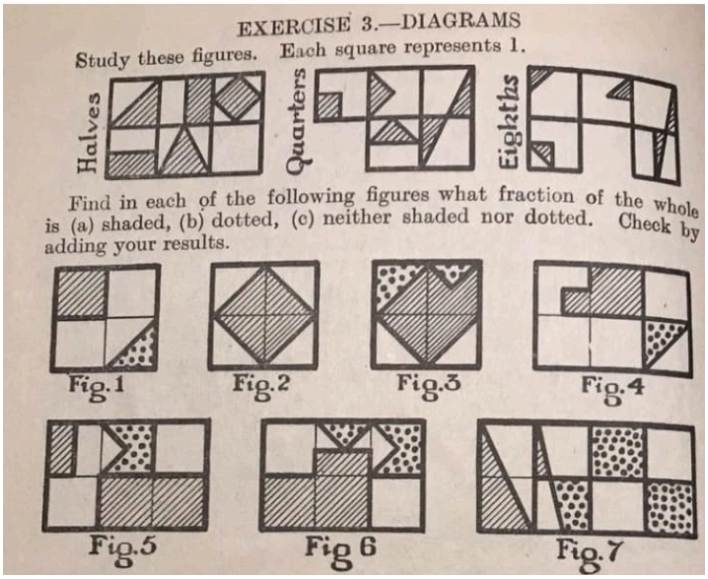


- Kendra says the shaded region represents the number 5.
- Dylan says it represents $2\frac{1}{2}$.
- Kiana says it represents $\frac{5}{8}$.
- Nate says it is $1\frac{1}{4}$.

Mr. Li exclaims, “Everyone is right!”

Do you believe Mr. Li in the example above? His statement is accurate. See if you can justify each answer by explaining what each student thought was the *unit* in Mr. Li’s picture. Try the following and share your answers with another person. Did you and the other person get the same answers?

5) Here is a series of activities from an old textbook. (The book seems to be part of a large series from the late 1920s & early 1930s: *Fundamental Arithmetic*, by P. B. Ballard). Have fun trying to find which fraction of the whole is shaded, dotted, and neither nor dotted. Each square represents 1 whole.



Answers

Exercise 6.1.1.1

a) The triangle is **not divided into three equal parts**, yet it's labeled **1/3**. In order for a shape to represent 1/3, the **whole must be divided into three equal-sized sections**, and **one** of them must be shaded. This could lead to misconceptions about the importance of **equal parts** when defining fractions.

b) The image suggests that **2/3 is greater than 3/4**, but **mathematically, this is incorrect**: Students might **compare the shaded areas directly** without considering that the **circles are different sizes**, leading to a false comparison. They may wrongly conclude that more shaded area = bigger fraction, without realizing that fractions are always relative to the **whole**.

Exercise 6.1.1.2

a) 2/7 b) 1/4 c) 5/10 or 1/2 d) 0/10 or 0

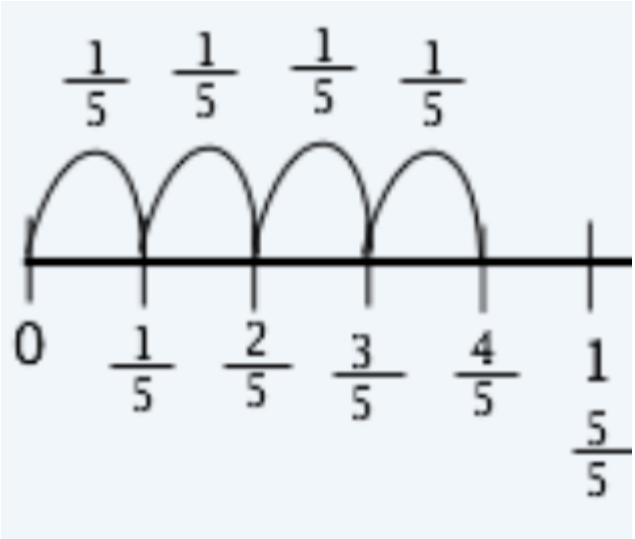
Exercise 6.1.1.3

a)



b)





c)

d)



Exercise 6.1.1.4

a) 8/20 or 2/5 b) 8/20 or 2/5 c) 4/20 or 1/5

Exercise 6.1.1.5

a) 7/2 or $3\frac{1}{2}$

b) There are **4 circles**, each divided into **2 halves** → total of **8 halves**. Out of those 8 halves, **7 are shaded**. So, if **each half-circle is considered one of 8 equal parts**, then it can be 7/8. To interpret it as **7/8**, students must understand that sometimes a **fractional unit is smaller than one whole object**, and in this case, **each half-circle is treated as 1/8 of a larger whole**. This highlights the importance of **defining the unit or whole** in fraction representations.

Exercise 6.1.1.6

a) equal b) not equal c) not equal

Exercise 6.1.1.7

a) 5/3 or $1\frac{2}{3}$

b) 13/8 or $1\frac{5}{8}$

Exercise 6.1.1.8

a) $\frac{25}{175} = \frac{1}{7}$
b) $\frac{21}{30} = \frac{7}{10}$
c) $\frac{108}{126} = \frac{6}{7}$
d) $\frac{220}{100} = \frac{11}{5}$

e) To find a fraction equivalent to $\frac{2}{5}$, we multiply the numerator and denominator by the same number (but not zero). Let us multiply them by 2, 3, and 5.

$$\frac{2 \cdot 2}{5 \cdot 2} = \frac{4}{10} \qquad \frac{2 \cdot 3}{5 \cdot 3} = \frac{6}{15} \qquad \frac{2 \cdot 5}{5 \cdot 5} = \frac{10}{25}$$

So, $\frac{4}{10}$, $\frac{6}{15}$, and $\frac{10}{25}$ are equivalent to $\frac{2}{5}$.

Exercise 6.1.1.9

a) 2/3 b) 5/6 c) equivalent fractions

Exercise 6.1.1.10

a) 1/17, 1/16, 1/15

b) 11/15, 12/15, 13/15

Exercise 6.1.1.11

a) 3/7

b) 17/39

Exercise 6.1.1.12

a) 4/34 or 2/17 is a possible answer

b) 7/16 is a possible answer

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- 4.5: What is a Fraction? Revisited by Michelle Manes is licensed CC BY-SA 4.0. Original source: pressbooks.oer.hawaii.edu/mathforelementaryteachers.

6.1.2: Online Retreival and Problem Solving Practice

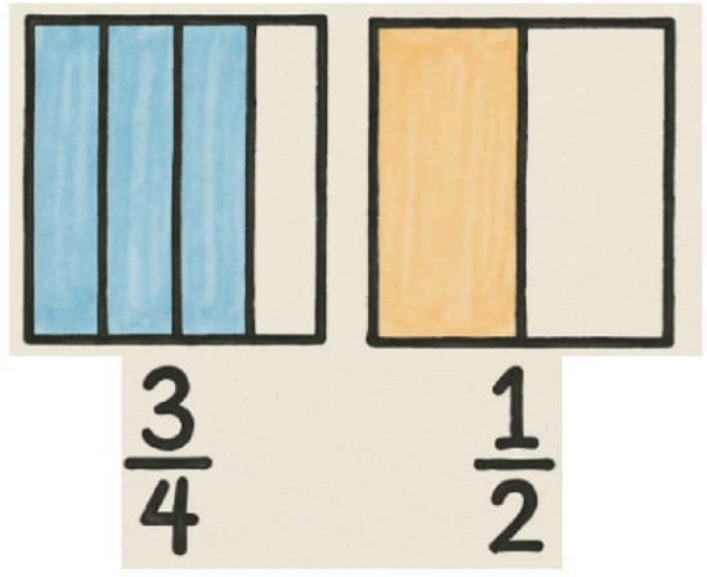
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6.2: Fraction Addition and Subtraction

Adding and subtracting fractions is a foundational yet often challenging concept for elementary students. Many struggle to see fractions as single quantities, instead interpreting them as two separate whole numbers—leading to common errors like adding numerators and denominators directly (e.g., thinking 1/3 + 1/2 = 2/5). These misconceptions are compounded by confusion around the need for common denominators, which stems from a lack of understanding that fractions must represent parts of the same whole. Procedural instruction without conceptual grounding further exacerbates the issue, leaving students overwhelmed and prone to mistakes. To address this, future teachers must emphasize conceptual understanding through visual models, real-world contexts, and a clear focus on the meaning behind operations, fostering a more accurate and intuitive grasp of fraction addition and subtraction (NCTM, 2020).

Think About and Discuss

When first learning fraction operations, students view the addition of fractions as adding the numerators and adding the denominators: $\frac{3}{4} + \frac{1}{2} = \frac{4}{6}$. Using this example, think about why such a method for addition is unreasonable. You may want to use fractions strips to help with your understanding. Be prepared to discuss your ideas with your peers.



Model Fraction Addition

How many quarters are pictured, if you even know what these are (😆)? One quarter plus 2 quarters equals 3 quarters.



Figure $6.2.1$

Remember, quarters are really fractions of a dollar. Quarters are another way to say fourths. So the picture of the coins shows that

$$\frac{1}{4} \qquad \frac{2}{4} \qquad \frac{3}{4}$$
$$one\ quarter + two\ quarters = three\ quarters$$

So again, we see that

$$\frac{1}{4}+\frac{2}{4}=\frac{3}{4}$$

Addition of fractions is an extension of whole-number addition and can be motivated using models. To find the sum of $\frac{3}{5}$ + $\frac{1}{5}$, consider the following models Figure 6.2.1: region, fraction strip, number-line model.

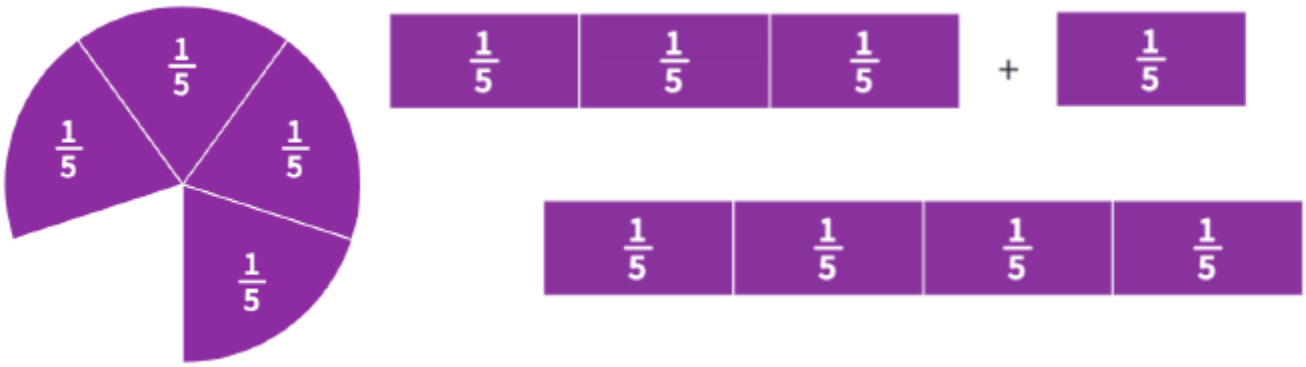


Figure 6.2.2: Region, Fraction Strip, Number Line

We see from those models that

$$\frac{3}{5}+\frac{1}{5}=\frac{4}{5}$$

Definition: Addition of Fractions Like Denominators

If a, b, and c are numbers where $c \neq 0$, then

$$\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c} \tag{6.2.1}$$

To add fractions with a common denominator, add the numerators and place the sum over the common denominator.

Example 6.2.1

Use a model to find the sum. Show a diagram to illustrate your model.

$$\frac{1}{8}+\frac{4}{8}$$

Answer available at the end of this section.

Think About and Discuss

How about ⅔ + ¼? Use fractions strips to build a model for this sum. Be prepared to discuss your ideas with your peers.

MODEL available at the end of this section.

Theorem 6.2.1

Addition of Fractions with Unlike Denominators

Let $\frac{a}{b}$ and $\frac{c}{d}$ be any fractions. Then $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$

PROOF available at the end of this section.

In words, to add fractions with unlike denominators, find equivalent fractions with common denominators. Then the sum will be represented by the sum of the numerators over the common denominator

Example 6.2.2

Find the following sums and simplify.

a) $\frac{5}{9}+\frac{3}{4}$

b) $\frac{17}{15}+\frac{5}{12}$

c) $\frac{3}{5} + (\frac{4}{7} + \frac{2}{5})$

Answer available at the end of this section.

Use the Desmos Scientific Calclator to check your work.

Improper Fractions and Mixed Numbers

As we saw in the previous example, the number 1 + 4/7 can be expressed as the mixed number and as an improper fraction. For example, $2\frac{1}{2}$, can be seen as 2 + 1/2, which can be represented as 4/2 + 1/2 = 5/2. Any mixed number can be changed to an improper fraction, and vice versa.

Example 6.2.3

a) Express $3\frac{2}{5}$ as an improper fraction.

b) Express 36/7 as a mixed number.

Answer available at the end of this section.

Think About and Discuss

When converting a mixed number to an improper fraction, why do we multiply the whole number times the denominator of the fractional part, add this product to the numerator of the fractional part, and write the result over the denominator of the fractional part?

1.**Multiply the whole number by the denominator.**

2.**Add the numerator to this product.**

3. **Write this new sum over the original denominator.**

IMPORTANT: Children may know the rule for converting a mixed number to an improper fraction, but they often do not understand why they do this. Make sure you do before moving on.

Fraction Subtraction

Subtraction of fractions can be viewed in two ways as we did with whole-number subtraction—either as (1) take-away or (2) using the missing-addend approach. The same rules follow as fraction addition.

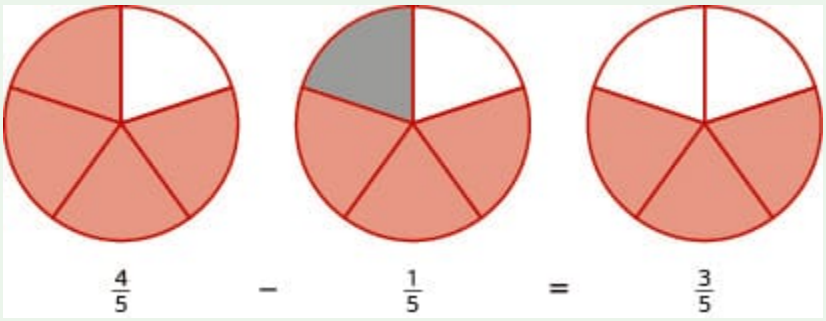
Definition: Fraction Subtraction

If a, b, and c are numbers where $c \neq 0$, then

$$\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c} \tag{6.2.2}$$

To subtract fractions with a common denominator, we subtract the numerators and place the difference over the common denominator.

$\frac{4}{5} - \frac{1}{5} = \frac{4-1}{5} = \frac{3}{5}$.



Theorem 6.2.2

Subtraction of Fractions with Unlike Denominators

Let $\frac{a}{b}$ and $\frac{c}{d}$ be any fractions. Then $\frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}$

NOTE: In this case we have $\frac{a}{b} \geq \frac{c}{d}$.

Example 6.2.4: difference

Find the following differences.

a) $\frac{4}{7} - \frac{3}{8}$.

b) $\frac{25}{12} - \frac{7}{18}$.

Answer available at the end of this section.

Interactive Content

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Mental Math and Estimation

Mental Math and Estimation for Addition and Subtraction Mental math and estimation techniques similar to those used with whole numbers can be used with fractions.

Example 6.2.5

Estimate using the indicated techniques or your own techniques that come to mind.

a) $3\frac{3}{7} + 6\frac{2}{5}$ using range estimation.

b) $15\frac{1}{4} - 7\frac{3}{5}$ using front-end with adjustment.

c) $3\frac{5}{6} + 5\frac{1}{8} + 8\frac{3}{8}$ using rounding to the nearest 1/2 or whole.

Answer available at the end of this section.

Think About and Discuss

Which is larger? $\frac{3}{8}$ or $\frac{2}{3}$?

Think of a way to figure that out without a calculator and then be prepared to discuss this with your peers.

Using Benchmark Fractions

Expected of fifth grade students is the ability to use benchmark fractions and number sense of fractions to estimate mentally and assess the reasonableness of answers (CDE, 2013). One way to estimate the size of a fraction is to compare it to common benchmarks like 0, $\frac{1}{2}$, and 1. For example, to compare $\frac{5}{6}$ and $\frac{5}{9}$, you might reason that $\frac{5}{6}$ is very close to 1, while $\frac{5}{9}$ is a little more than $\frac{1}{2}$ (since $\frac{4.5}{9}$)= $\frac{1}{2}$. Therefore, you can estimate that $\frac{5}{6}$ is larger than $\frac{5}{9}$. You can also focus on the size of the unit fraction (the fraction with a numerator of 1). For example, when comparing $\frac{1}{8}$ and $\frac{1}{3}$, you can reason that thirds are larger parts than eighths (if the whole is the same), so $\frac{1}{3}$ is larger.

Wonder, Play, Grow

Work collaboratively to share your thinking about the following questions. Keep in mind everything you have learned and that there will multiple ways to respond.

i) Which is larger, 5/6 or 5/9?

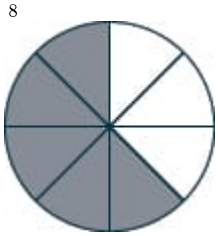
ii) Which is smaller, 2/11 or 1/9?

It's important to note that estimation skills develop slowly and require a strong conceptual understanding of fractions. Encouraging children to think about the meaning of fractions and use visual models can support the development of these estimation abilities (Feike et al., 2018).

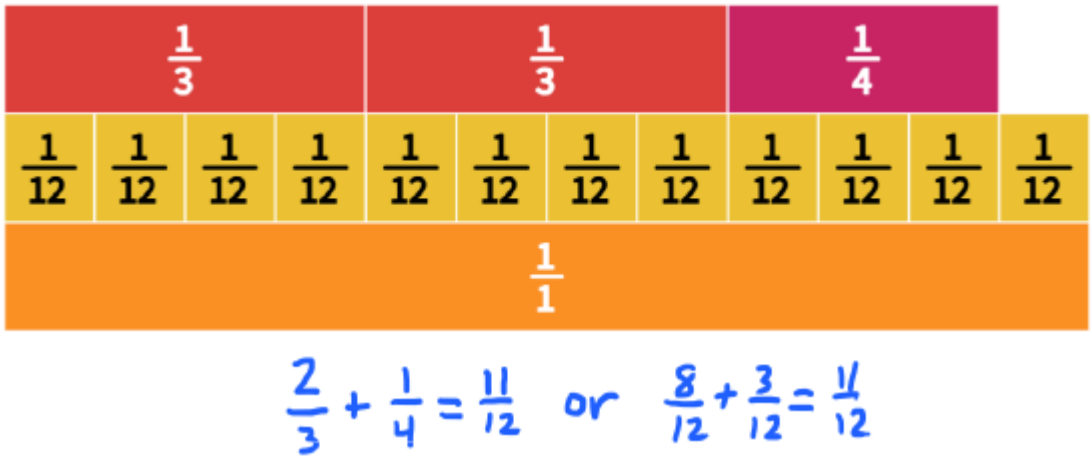
Answers

Example 6.2.1

$\frac{5}{8}$



Think About and Discuss



$\frac{2}{3} + \frac{1}{4} = \frac{11}{12}$ or $\frac{8}{12} + \frac{3}{12} = \frac{11}{12}$

🧠**Let's Listen to Learn**❤️

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Theorem 6.2.1

$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad+bc}{bd}$

Example 6.2.2

a) 20/36 + 27/36 = 47/36

b) Do not cross multiply, it may be easier to first find a common denominator. 68/60 + 25/60 = 93/60. 93/60 simplifies to 31/20.

c) (3/5 + 2/5) + 4/7 = 1 + 4/7 = $1\frac{4}{7}$ or 1+4/7 = 7/7 + 4/7 = 11/7

Example 6.2.3

a) 3 + 2/5 = 15/5 + 2/5 = 17/5. Do you know a shortcut?

b) 36/7 = 35/7 + 1/7 = 5 + 1/7 = $5\frac{1}{7}$. Do you know another way to do this?

Divide the numerator by the denominator.

- The **quotient** is the **whole number** part of the mixed number.
- The **remainder** becomes the **new numerator**.

Keep the same denominator from the original fraction.

Example 6.2.4**: difference**

a) $\frac{4}{7}-\frac{3}{8}=\frac{4\bullet 8}{7\bullet 8}-\frac{3\bullet 7}{8\bullet 7}=\frac{32-21}{56}$ = 11/56

b) NOTE: LCM(12, 18)=36

$\frac{25}{12}-\frac{7}{18}=\frac{25\bullet 3}{12\bullet 3}-\frac{7\bullet 2}{18\bullet 2}=\frac{75-14}{36}$ = 61/36

Example 6.2.5

a) $3\frac{3}{7}+6\frac{2}{5}$ is between 3 + 6 = 9 and 4 + 7 = 11.

b) $15\frac{1}{4}-7\frac{3}{5}\approx 7\frac{1}{2}$ since 15-7 = 8 and $\frac{3}{5}\approx\frac{1}{2}$.

c) $3\frac{5}{6}+5\frac{1}{8}+8\frac{3}{8}\approx 4+5+8\frac{1}{2}=17\frac{1}{2}$

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6.2.1: Written Retrieval and Problem Solving Practice

Student Learning Objectives

6.2 Students will analyze and compute sums and differences of fractions, applying addition and subtraction properties and using estimation techniques to verify the reasonableness of results of real-world problems.

6.4 Students will evaluate the application of mental math and estimation techniques for operations with fractions, including rounding, range estimation, and using compatible numbers to simplify calculations. The mild assignment 🌶 is a shorter and less challenging assignment. The medium assignment🌶🌶 is right in the middle in terms of length and level of difficulty. Finally, the spicy assignment 🌶🌶🌶 is a longer assignment that is the most challenging.

(Spicy 🌶🌶🌶 Problems are OPTIONAL, but go for it when you have the time to challenge yourself)

Exercise 6.2.1.1

Find $\frac{2}{5}+\frac{1}{3}$ using a common denominator approach and a number line model.

Answer available at the end of this section.

Exercise 6.2.1.2

Find $\frac{5}{9}+\frac{7}{12}$ using three different denominators.

Answer available at the end of this section.

Exercise 6.2.1.3

Find the following sums and express your answer in simplest form.

a) $\frac{3}{7}+\frac{5}{7}$

b) $\frac{8}{13}+\frac{4}{51}$

c) $\frac{6}{100}+\frac{7}{1000}$

d) $\frac{5}{9}+\frac{7}{12}$

e) $2\frac{1}{5}+3\frac{3}{10}$

f) $1\frac{2}{3}+2\frac{3}{4}$

Answer available at the end of this section.

Exercise 6.2.1.4

Change the following mixed numbers to improper fractions.

a) $3\frac{5}{6}$

b) $5\frac{1}{5}$

Answer available at the end of this section.

Exercise 6.2.1.5

Perform the following subtractions.

a) $\frac{9}{13} - \frac{5}{13}$

b) $\frac{3}{7} - \frac{2}{9}$

c) $\frac{11}{100} - \frac{99}{1000}$

d) $2\frac{2}{3}$ - $1\frac{1}{4}$

Answer available at the end of this section.

Exercise 6.2.1.6

Use the properties of fraction addition and subtraction to calculate each of the following sums mentally.

a) $(\frac{9}{7} + \frac{1}{9})$ - $\frac{2}{7}$

b) $1\frac{9}{13} + \frac{5}{6}$ + $\frac{4}{13}$

Answer available at the end of this section.

Exercise 6.2.1.7

Estimate using your methods of choice.

a) $9\frac{4}{7}$ - $6\frac{1}{8}$

b) $8\frac{7}{9}$ + $3\frac{6}{13}$

c) $5\frac{1}{3}$ + $4\frac{4}{5}$ + $5\frac{6}{7}$

Answer available at the end of this section.

Exercise 6.2.1.8

Use estimation techniques to order the fractions from least to greatest.

$\frac{5}{9}, \frac{88}{97}, \frac{9}{20}$

Answer available at the end of this section.

PROBLEM SOLVING PRACTICE

1) After a party, Lupe has $\frac{5}{12}$ of a cheese pizza, $\frac{1}{3}$ of a pepperoni pizza, and $\frac{1}{6}$ of a veggie pizza left. Will all the slices fit into 1 pizza box? Explain your reasoning.

2) Rafael ate one-fourth of a pie and Rocco ate one-third of it. What fraction of the pie did they eat?

3) John spent a quarter of his life as a boy growing up, one-sixth of his life in college, and one-half of his life as a teacher. He spent his last six years in retirement. How old was he when he died?

4) Alex, Bryan, and Cynthia are about to eat lunch, and they have two sandwiches to share.

- Draw a picture to show how they could equally share the sandwiches. How much of a sandwich does each person get?
- Write an equation involving addition to show how together these parts make up the 2 sandwiches. Explain how the equation you wrote represents this situation.
- Write an equation involving multiplication to show how all the parts make up the 2 sandwiches. Explain how the equation you wrote represents this situation.
- Write an equation using multiplication to show the fraction of a sandwich each student gets. Explain how the equation you wrote represents this situation.
- Write an equation using division to show the fraction of a sandwich each student gets. Explain how the equation you wrote represents this situation.

5) For each of the following word problems, determine whether or not 2/5 + 3/10 represents the problem. Explain your decision.

a) Wesley ran 2/5 of a mile on Monday and 3/10 of a mile on Tuesday. How far did he run those two days?

b) Andy made 2 free throws out of 5 free throw attempts. Jose made 3 free throws out of 10 free throw attempts. What is the fraction of free throw attempts that the two boys made together?

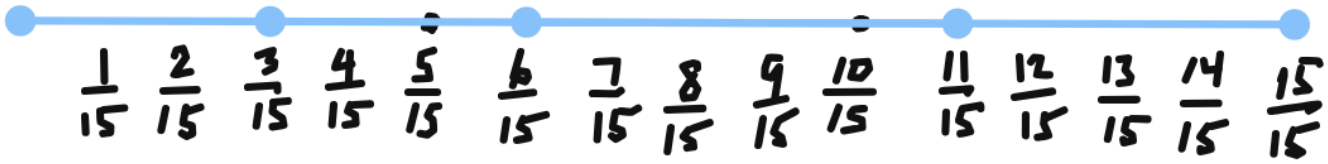
c) Andy made 2 free throws out of 5 free throw attempts. Jose made 3 free throws out of 10 free throw attempts. What is the fraction of free throw attempts that the two boys made together?

Answers

Exercise 6.2.1.1

$\frac{2}{5} + \frac{1}{3} = \frac{6}{15} + \frac{5}{15} = \frac{11}{15}$





Exercise 6.2.1.2

$\frac{5}{9}+\frac{7}{12}=\frac{20}{36}+\frac{21}{36}$

$\frac{5}{9}+\frac{7}{12}=\frac{40}{72}+\frac{42}{72}$

$\frac{5}{9}+\frac{7}{12}=\frac{60}{108}+\frac{63}{108}$

Exercise 6.2.1.3

a) $\frac{3+5}{7}=\frac{8}{7}$

b) $\frac{408+52}{663}=\frac{460}{663}$

c) $\frac{60}{1000}+\frac{7}{1000}\ \frac{607}{1000}$

d) $\frac{20+21}{36}=\frac{41}{36}$

e) $\frac{11}{5}+\frac{33}{10}=\frac{22+33}{10}=\frac{55}{10}$

f) $\frac{5}{3}+\frac{11}{4}=\frac{20+33}{12}=\frac{53}{12}$

Exercise 6.2.1.4

a) $\frac{23}{6}$ b) $\frac{26}{5}$

Exercise 6.2.1.5

a) $\frac{9-5}{13}=\frac{4}{13}$

b) $\frac{27-14}{63}=\frac{13}{63}$

c) $\frac{110}{1000}-\frac{99}{1000}=\frac{11}{1000}$

d) $\frac{8}{3}-\frac{5}{4}=\frac{32-15}{12}=\frac{17}{12}$

Exercise 6.2.1.6

a) $1\frac{1}{9}$ b) $2\frac{5}{6}$

Exercise 6.2.1.7

a) ≈ $3\frac{1}{2}$ b) ≈ $12\frac{1}{2}$ c) ≈ 16

Exercise 6.2.1.8

5/9 which is **a little more than 1/2**, but **less than 1**.

88/97 so this is **very close to 1**.

9/20 is **just less than 1/2**.

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6.2.2: Online Retrieval and Problem Solving Practice

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6.3: Multiplication and Division

Fraction multiplication and division are among the most conceptually challenging areas of elementary mathematics—and yet they are often taught through memorized rules that students don't truly understand (Feike et al., 2018). Research and practice show that **conceptual understanding must come first**. Children need to grasp *what* they're doing and *why* it makes sense before they are expected to perform symbolic procedures fluently. This means centering instruction around **models, representations, and real-world contexts**. Area models, number lines, diagrams, and hands-on tools help children visualize fraction operations and construct their own meaning (NCTM, 2020).

Ultimately, procedural fluency is important—but only when built on a strong foundation of understanding. When students can *see* and *explain* what it means to multiply or divide fractions, they become confident and competent mathematicians, not just rule-followers.

Wonder, Play, Grow

Your group will be assigned one of the problems below. Before collaborating, take a few minutes to work independently on your problem. In order to solve the problem, think of producing a solution by drawing a picture or using a number line. Try NOT to use procedures at this point. You may want to use the virtual fraction bars to help you see the solutions. https://polypad.amplify.com/p#fraction-bars

Task 1 - Alisa had a ½ liter of juice in a bottle. She drank ¾ of the juice that was in the bottle. How many liters of juice did she drink?

Task 2 - The distance between Rosa's house and her school is ¾ mile. She ran ⅓ of the way to school. How many miles did she run?

The tasks can be found at Illustrative Math. https://tasks.illustrativemathematic...t-standards/NF

Multiplication

Trying to extend the repeated-addition approach of whole-number multiplication to fraction multiplication is an interesting challenge. Consider the following cases (Musser, 2014).

Case 1: A Whole Number times a Fraction

$$3 \times \frac{1}{4} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$$

$$8 \times \frac{1}{2} = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 4$$

The repeated-addition approach works well since the first factor is a whole number.

Case 2: A Fraction Times a Whole Number

$$\frac{1}{2} \times 8$$

We cannot apply the repeated-addition approach, since that would literally say to add 8 one-half times. But if multiplication is to be commutative, then 1/2 x 8 would have to be equal to 8 x 1/2=4, as in case 1. We can then interpret 1/2 x 8 as taking "one-half" of 8, which is 4.

Case 3: A Fraction time a Fraction

Next, it is impossible to use the repeated-addition approach, so we apply the technique of case 2.

$$\frac{1}{3} \times \frac{2}{5} \text{ means } \frac{1}{3} \text{ of } \frac{2}{5}$$

First picture 2/5, subdividing a rectangular region vertically into 5 equal parts. Then horizontally subdividing a region into three equal parts (1/3). The intersecting shaded region (small rectangles = 1/21) consists of 2 of the 15 smallest rectangle. Therefore, $\frac{1}{3} \times \frac{2}{5} = \frac{2}{15}$, see Figure 6.3.1.

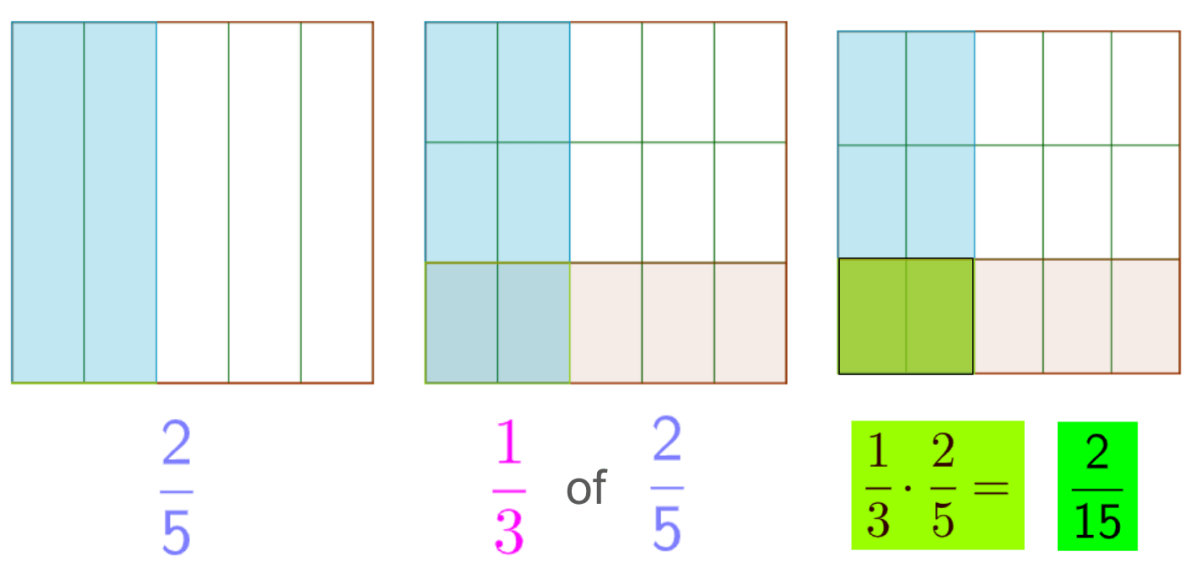


Figure 6.3.1: Fraction Times a Fraction

Example 6.3.1

Use a diagram (case 3) to model $\frac{1}{2} \cdot \frac{3}{4}$.

Answer available at the end of this section.

YouTube Video

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Before you view the video, see if you can model $\frac{1}{4} \cdot \frac{3}{5}$ and $\frac{2}{3} \cdot \frac{4}{7}$.

This leads to the definition of fraction multiplication. To multiply fractions, we multiply the numerators and multiply the denominators. Then we write the fraction in simplified form.

Definition: Fraction Multiplication

If $a, b, c,$ and d are numbers where $b \neq 0$ and $d \neq 0$, then

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \tag{6.3.1}$$

Example 6.3.2

Multiply, and write the answer in simplified form: $\frac{3}{4} \cdot \frac{1}{5}$.

Answer available at the end of this section.

Example 6.3.3

Multiply $2 \cdot \frac{3}{7}$.

Answer available at the end of this section.

Wonder, Play, Grow

Work together to find each product and then check your work with the Desmos Scientific Calculator.

a) $\frac{3}{4} \cdot \frac{28}{15}$

b) $\frac{2}{3} \cdot \frac{5}{13}$

c) $2\frac{1}{3} \cdot \frac{2}{5}$

d) $\frac{18}{13} \cdot \frac{39}{72}$

e) $\frac{50}{15} \cdot \frac{39}{55}$

f) $3\frac{4}{7} \cdot 2\frac{1}{2}$

GIVE ONE TAKE ONE: Make up a multiplication problem, find the product, then share your problem with a peer. They will then share the problem they created. You can then check each other's work.

Properties of Fraction Multiplication

Let $\frac{a}{b}$ be any fraction.

Multiplicative Identity Property for Fraction Multiplication: $\frac{a}{b} \cdot 1 = \frac{a}{b} = 1 \cdot \frac{a}{b}$

Multiplicative Inverse Property for Fraction Multiplication: For every nonzero fraction $\frac{a}{b}$, there is a unique fraction $\frac{b}{a}$ such that $\frac{a}{b} \cdot \frac{b}{a} = 1$.

IMPORTANT NOTE: $\frac{b}{a}$ is called the **multiplicative inverse or reciprocal** of $\frac{a}{b}$.

Understanding reciprocals is essential because they are the key to dividing with fractions. Dividing by a fraction is conceptually and procedurally more complex than multiplying, but the reciprocal simplifies the process: dividing by a fraction is the same as multiplying by its reciprocal. This "invert and multiply" rule transforms a tricky division into a familiar multiplication (Finan, 2006). When students get to algebra, the reciprocal is also critical in solving equations, where it allows us to isolate variables by "undoing" a fraction multiplication (Musser et al., 2014). Simply put, a reciprocal reverses the numerator and denominator, and when multiplied by the original fraction, the result is always 1—making it a powerful tool for both conceptual understanding and procedural fluency.

Example 6.3.4

Find the reciprocal of each number. Then check that the product of each number and its reciprocal is 1.

a. $\frac{4}{9}$
b. 7

Answer available at the end of this section.

Division

Division of fractions is a difficult concept for many children (and adults). This difficulty is partly due to the lack of simple concrete models (Musser et al. , 2014). While the foundation for multiplication and division begins as early as second grade by developing an understanding of place value and a part/whole conception of numbers the formal study of division of fractions typically occurs later (Feike et al., 2018). At this stage, a strong emphasis on conceptual understanding using models, real-world contexts, and the connection to the meaning of division is crucial, rather than solely relying on rote procedures.

Why is $12 \div 3 = 4$? We previously modeled this with counters. How many groups of 3 counters can be made from a group of 12 counters?



Figure $6.3.2$

There are 4 groups of 3 counters. In other words, there are four 3s in 12. So, $12 \div 3 = 4$.

What about dividing fractions? Suppose we want to find the quotient: $\frac{1}{2} \div \frac{1}{6}$. We need to figure out how many $\frac{1}{6}$s there are in $\frac{1}{2}$. We can use fraction tiles to model this division. We start by lining up the half and sixth fraction tiles as shown in Figure $6.3.3$. Notice, there are three $\frac{1}{6}$ tiles in $\frac{1}{2}$, so $\frac{1}{2} \div \frac{1}{6} = 3$.

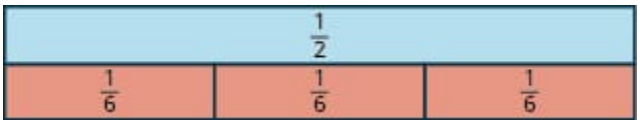


Figure $6.3.3$

Example 6.3.5

Model: $\frac{1}{4} \div \frac{1}{8}$.

Answer available at the end of this section.

Example 6.3.6

Model: $2 \div \frac{1}{4}$.

Answer available at the end of this section.

Using fraction tiles, we showed that $\frac{1}{2} \div \frac{1}{6} = 3$. Notice that $\frac{1}{2} \cdot \frac{6}{1} = 3$ also. How are $\frac{1}{6}$ and $\frac{6}{1}$ related? They are reciprocals. This leads us to the procedure for fraction division.

Definition: Fraction Division

If $a, b, c,$ and d are numbers where $b \neq 0$, $c \neq 0$, and $d \neq 0$, then

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} \tag{6.3.2}$$

To divide fractions, multiply the first fraction by the reciprocal of the second.

We need to say $b \neq 0$, $c \neq 0$ and $d \neq 0$ to be sure we don't divide by zero.

Example 6.3.7

Divide, and write the answer in simplified form: $\frac{7}{18} \div \frac{14}{27}$.

Answer available at the end of this section.

Now that we just learned the "invert and multiply" rule. Here are some other ways you may not have seen to divide fractions.

Common Denominator Method

Think About and Discuss

Solve each of the following fraction division problems until you see a pattern.

$\frac{6}{4} \div \frac{3}{4}$ $\quad \frac{6}{10} \div \frac{3}{10}$ $\quad \frac{8}{9} \div \frac{4}{9}$ $\quad \frac{6}{33} \div \frac{2}{33}$ $\quad \frac{5}{4} \div \frac{2}{4}$ $\quad \frac{5}{2} \div \frac{2}{2}$ $\quad \frac{5}{10} \div \frac{2}{10}$

What do you notice?

This leads to our first alternative fraction division method:

Common denominator method

If two fractions have the same denominator, then when you divide them, you can just divide the numerators. In symbols,

$$\frac{a}{d} \div \frac{b}{d} = \frac{a}{b}.$$

Missing Factor Method

We know that we can always turn a division problem into a "missing factor" multiplication problem. Can that help us compute fraction division? Sometimes!

Example 6.3.8

For each division problem, rewrite it as a missing factor multiplication question. Then find the quotient using what you know about multiplying fractions.

a. $\frac{9}{10} \div \frac{3}{5}$
b. $\frac{25}{12} \div \frac{5}{2}$

Answer available at the end of this section.

Unfortunately, the missing factor method doesn't always work out so nicely. For example,

$$\frac{3}{4} \div \frac{1}{3} = \_$$

can be rewritten as

$$\frac{1}{3} \cdot \_ = \frac{3}{4}.$$

There isn't a nice ratio of whole numbers that obviously fills in the blank, but we'll come back to this idea and resolve it soon.

Dividing Across

Let's let Dr. Howie Hua explain this process before we try it for ourselves.

YouTube Video

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Example 6.3.9

Divide.

a. $\frac{8}{15} \div \frac{2}{3}$
b. $\frac{4}{7} \div \frac{3}{2}$

Answer available at the end of this section.

Wonder, Play, Grow

Find the following quotients using the most convenient division method. Then share with your peers how you completed the problems.

a) $\frac{17}{4} \div \frac{4}{11}$

b) $\frac{3}{4} \div \frac{5}{7}$

c) $\frac{6}{25} \div \frac{2}{5}$

d) $\frac{5}{19} \div \frac{13}{11}$

Fraction Division Retrieval Practice

Complex Fractions

Consider the following situation. You have 3/4 of a liter of juice left. Each small cup holds 1/3 of a liter. **Question:** How many cups can you fill?

To solve this, you would compute $\frac{\frac{3}{4}}{\frac{1}{3}}$. This is an example of a complex fraction. Although fractions are made of whole numbers in elementary school, numbers other than whole numbers are used in numerators and denominators. Complex fractions are used to divide fractions.

$$\frac{\frac{3}{4} \cdot \frac{3}{1}}{\frac{1}{3} \cdot \frac{3}{1}} = \frac{\frac{3}{4} \cdot \frac{3}{1}}{1} = \frac{9}{4}$$

Notice that the multiplicative inverse of the denominator 1/3 was used to form a complex fraction form of one.

Example $6.3.10$

Find $\frac{1}{2} \div \frac{3}{5}$

Answer available at the end of this section.

Multiplication and Division Exploration

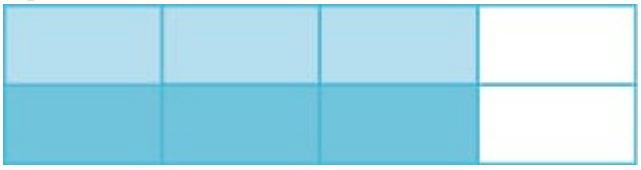
Answers

Example $6.3.1$

First shade in $\frac{3}{4}$ of the rectangle.



We will take $\frac{1}{2}$ of this $\frac{3}{4}$, so we heavily shade $\frac{1}{2}$ of the shaded region.



Notice that 3 out of the 8 pieces are heavily shaded. This means that $\frac{3}{8}$ of the rectangle is heavily shaded. Therefore, $\frac{1}{2}$ of $\frac{3}{4}$ is $\frac{3}{4}$, or $\frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8}$.

Example 6.3.2

	$\frac{3}{4} \cdot \frac{1}{5}$
Multiply the numerators, and multiply the denominators.	$\frac{3 \cdot 1}{4 \cdot 5}$
Simplify.	$\frac{3}{20}$

There are no common factors, so the fraction is simplified.

Example 6.3.3

to multiply $2 \cdot \frac{3}{7}$, we think of "2" as $\frac{2}{1}$, and compute this way

$$2 \cdot \frac{3}{7} = \frac{2}{1} \cdot \frac{3}{7} = \frac{2 \cdot 3}{1 \cdot 7} = \frac{6}{7}.$$

Example 6.3.4

a)

The reciprocal of $\frac{4}{9}$ is $\frac{9}{4}$.

$\frac{4}{9} \cdot \frac{9}{4}$

$\frac{36}{36}$

1 ✓

b)

$\frac{7}{1}$

$\frac{1}{7}$

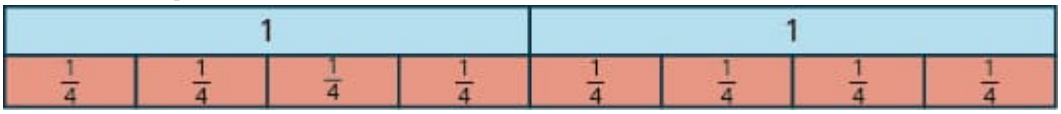
$7 \cdot \left(\frac{1}{7}\right) = 1$ ✓

Example 6.3.5

We want to determine how many $\frac{1}{8}$s are in $\frac{1}{4}$. Start with one $\frac{1}{4}$ tile. Line up $\frac{1}{8}$ tiles underneath the $\frac{1}{4}$ tile.

Example 6.3.6

We are trying to determine how many $\frac{1}{4}$s there are in 2. We can model this as shown.



Because there are eight $\frac{1}{4}$s in 2, $2 \div \frac{1}{4} = 8$.

Example 6.3.7

Add texts here. Do not delete this text first.

1. Multiply the first fraction by the reciprocal of the second. $\frac{7}{18} \cdot \frac{27}{14}$
2. Multiply. $\frac{7 \cdot 27}{18 \cdot 14}$
3. Rewrite showing common factors. $\frac{\cancel{7} \cdot \cancel{9} \cdot 3}{\cancel{9} \cdot \cancel{7} \cdot 2}$
4. Remove common factors. $\frac{3}{2 \cdot 2}$

5. Simplify. $\frac{3}{4}$

Example 6.3.8

a) $\frac{3}{5} \cdot \frac{?}{?} = \frac{9}{10}$

$\frac{?}{?} = \frac{3}{2}$, so $\frac{9}{10} \div \frac{3}{5} = \frac{3}{2}$

b) $\frac{5}{2} \cdot \frac{?}{?} = \frac{25}{12}$

$\frac{?}{?} = \frac{5}{6}$, so $\frac{25}{12} \div \frac{5}{2} = \frac{5}{6}$

Example 6.3.9

a. $\frac{8}{15} \div \frac{2}{3} = \frac{8 \div 2}{15 \div 3} = \frac{4}{5}$ - WOW Right?

b. $\frac{4}{7} \div \frac{3}{2} = \frac{8}{14} \div \frac{21}{14} = \frac{8 \div 21}{14 \div 14} = \frac{8/21}{1} = \frac{8}{21}$

Example 6.3.10

$$\frac{\frac{1}{2} \cdot \frac{5}{3}}{\frac{3}{5} \cdot \frac{5}{3}} = \frac{\frac{1}{2} \cdot \frac{5}{3}}{1} = \frac{5}{6}$$

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6.3.1: Written Retrieval and Problem Solving Practice

Student Learning Objectives

6.3 Students will use visual models and real-life contexts to represent and compute multiplication and division of fractions, justifying the methods they choose and interpreting the results.

6.4 Students will evaluate the application of mental math and estimation techniques for operations with fractions, including rounding, range estimation, and using compatible numbers to simplify calculations.

The mild assignment 🌶 is a shorter and less challenging assignment. The medium assignment🌶🌶 is right in the middle in terms of length and level of difficulty. Finally, the spicy assignment 🌶🌶🌶 is a longer assignment that is the most challenging.

(Spicy 🌶🌶🌶 Problems are OPTIONAL, but go for it when you have the time to challenge yourself)

Exercise 6.3.1.1 🌶🌶

Use a number line to show how $\frac{3}{5} \cdot 2$ is conceptually different than $2 \cdot \frac{3}{5}$.

Answer available at the end of this section.

Exercise 6.3.1.2 🌶

Multiply, and write the answer in simplified form:

a) $\frac{1}{3} \cdot \frac{2}{7}$.

b) $\frac{3}{5} \cdot \frac{7}{8}$.

Answer available at the end of this section.

Exercise 6.3.1.3 🌶🌶

Multiply using a rectangular region model:

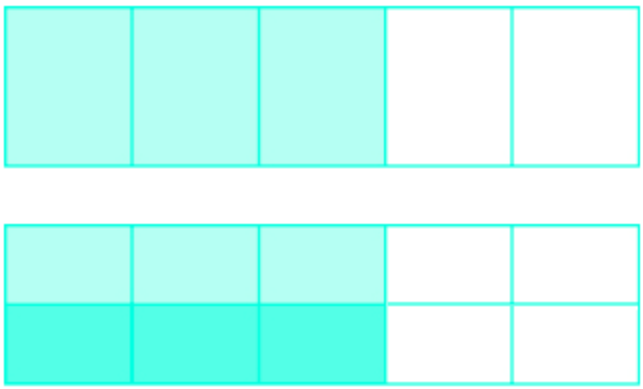
a) $\frac{3}{8} \cdot \frac{5}{6}$.

b) $\frac{1}{3} \cdot \frac{3}{4}$.

Answer available at the end of this section.

Exercise 6.3.1.4 🌶🌶

Which fraction multiplication is being modeled?



Answer available at the end of this section.

Exercise 6.3.1.5

How are these two problems different? Consider drawing a picture to help with your thinking.

1. Pam had $\frac{2}{3}$ of a cake in her refrigerator, and she ate $\frac{1}{2}$ of it. How much total cake did she eat?
2. On Monday, Pam ate $\frac{2}{3}$ of a cake. On Tuesday, Pam ate $\frac{1}{2}$ of a cake. Both cakes were the same size. How much total cake did she eat?

Answer available at the end of this section.

Exercise 6.3.1.6

Find the reciprocal:

a. $\frac{5}{7}$
b. $\frac{1}{8}$
c. $\frac{11}{4}$
d. 14

Answer available at the end of this section.

Exercise 6.3.1.7

Perform the following operations and express your answer in simplest form.

a) $\frac{5}{7} \cdot \frac{5}{6} \cdot \frac{7}{5}$

b) $\frac{6}{25} \cdot \frac{5}{9} \cdot \frac{3}{2}$

c) $\frac{2}{9} \cdot \frac{4}{3} \cdot \frac{5}{7}$

d) $3\frac{1}{2} \cdot \frac{7}{15} \cdot 2\frac{3}{5}$

Answer available at the end of this section.

Exercise 6.3.1.8

Use fraction strips to model and solve each problem

a) $\frac{1}{3} \div \frac{1}{6}$.

b) $2 \div \frac{1}{3}$

Answer available at the end of this section.

Exercise 6.3.1.9

Use the common-denominator method to divide the following fractions.

a) $\frac{15}{17} \div \frac{3}{17}$.

b) $\frac{4}{7} \div \frac{3}{7}$.

Answer available at the end of this section.

Exercise 6.3.1.10

Use the fact that the numerators and denominators divide evenly to simplify the following quotients.

a) $\frac{15}{16} \div \frac{3}{4}$.

b) $\frac{39}{56} \div \frac{3}{8}$.

c) $\frac{17}{24} \div \frac{17}{12}$.

Answer available at the end of this section.

Exercise 6.3.1.11

Find the following quotients using the most convenient of the three methods for division. Express your answer in simplest form.

a) $\frac{5}{7} \div \frac{4}{9}$.

b) $\frac{3}{11} \div \frac{8}{22}$.

c) $6\frac{1}{4} \div 1\frac{2}{3}$.

Answer available at the end of this section.

Exercise 6.3.1.12

Change each of the following complex fractions into ordinary fractions.

a) $\frac{\frac{7}{9}}{\frac{2}{3}}$

b) $\frac{\frac{5}{4}}{\frac{4}{5}}$

Answer available at the end of this section.

Exercise 6.3.1.13

Estimate using compatible numbers.

a) $29\frac{6}{7} \cdot 5\frac{1}{8}$.

b) $56\frac{1}{4} \div 7\frac{2}{3}$.

c) $5\frac{1}{10} \cdot 4\frac{17}{18} \cdot \frac{99}{100}$.

Answer available at the end of this section.

Problem Solving Practice

1) Rafael wanted to order half a medium pizza at a restaurant. The waiter told him that a medium pizza could be cut into 6 or 8 slices. Would he prefer 3 out of 6 slices or 4 out of 8 slices? Rafael replied that since he wasn't very hungry, he would prefer 3 out of 6 slices. Explain what is wrong with Rafael's reasoning.

2) Don purchased a bulk package of candy that weighs 5 pounds. He wants to sell the candy in little bags that hold $\frac{1}{4}$ pound. How many little bags of candy can he fill from the bulk package?

3) All but 1/16 of the students enrolled at a particular elementary school participated in "Family Fun Night" activities. If a total of 405 students were involved in the evening's activities, how many students attend the school?

4) Each morning Sammy walks to school. At one-third of the way she passes a grocery store, and halfway to school she passes a bicycle shop. At the grocery store, her watch says 7:40 and at the bicycle shop it says 7:45. When does Sammy reach her school?

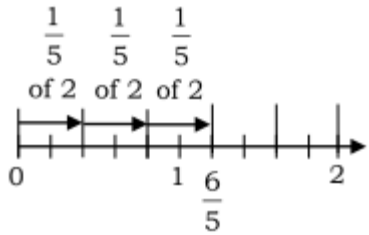
5) Can you figure out the shortcut for multiplying by 25.

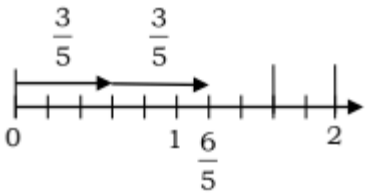
$25 \cdot 36 = \frac{?}{?} \cdot 36 = 100 \cdot \frac{36}{4} = 900$

TRY 25 x 44, 25 x 120.

Answers

Exercise 6.3.1.1





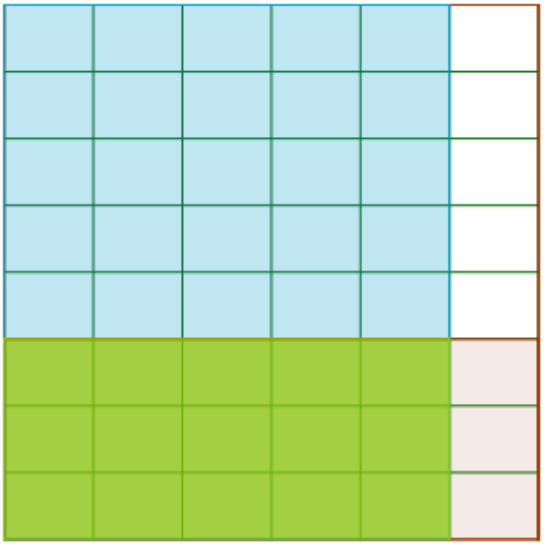
Exercise 6.3.1.2

a) $\frac{2}{21}$

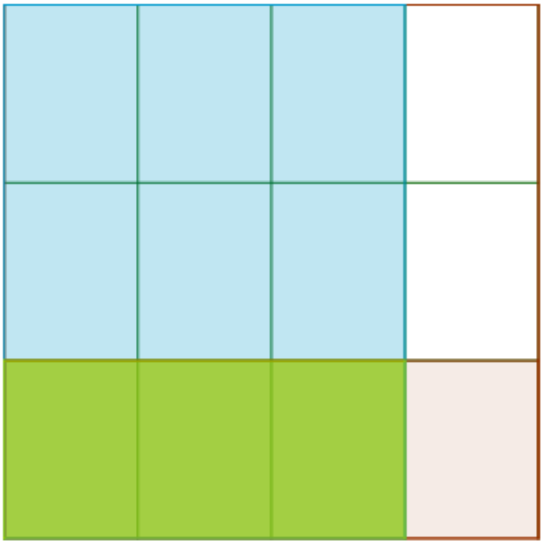
b) $\frac{21}{40}$

Exercise 6.3.1.3

a) $\frac{15}{48}=\frac{5}{16}$



b) $\frac{3}{12}=\frac{1}{4}$



Exercise 6.3.1.4

a) $\frac{1}{2} \cdot \frac{3}{5} = \frac{3}{10}$

Exercise 6.3.1.5

1. This is a multiplication problem. 1/2 x 2/3 = 1/3 of the cake.
2. She ate 2/3 + 1/2 of a cake, this 1 and 1/6 of a cake.

When a problem includes a phrase like "\(\frac{2}{3}\) of ...," students are taught to treat "of" as multiplication, and to use that to solve the problem. As the above problems show, in some cases this makes sense, and in some cases it does not. It is important to read carefully and understand what a problem is asking, not memorize rules about "translating" word problems.

Exercise 6.3.1.6

a) $\frac{7}{5}$

b) 8

c) $\frac{4}{11}$

d) $\frac{1}{14}$

Exercise 6.3.1.7

a) 5/6

b) 1/5

c) 40/189

d) 637/150 = 4 $\frac{37}{150}$

Exercise 6.3.1.8



Exercise 6.3.1.9

a) 5

b) 4/3

Exercise 6.3.1.10

a) 5/4

b) 13/7

c) 1/2

Exercise 6.3.1.11

a) 45/28 = 1 $\frac{17}{28}$

b) 3/4

c) 15/4 = 3 $\frac{3}{4}$

Exercise $6.3.1.12$🌶🌶

a) 7/6

b) 25/16

Exercise $6.3.1.13$🌶

a) 30 · 5 = 150

b) 56 ÷ 8 = 7

c) 5 · 5 · 1 = 25

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6.3.2: Fraction Discussion Board

📚 Discussion Board Prompt: Making Fractions Stick – A SAY-MEAN-MATTER Exploration

Instructions:
Read the article *[What Neuroscience Can Tell Us About Making Fractions Stick]* and reflect on it using the SAY-MEAN-MATTER framework. Then, reply meaningfully to at least two classmates, engaging with their ideas, asking questions, or offering connections to your own insights or classroom experiences.

SAY:
Choose a quote or section from the article that stood out to you. Write it out word-for-word (with citation). What is the author saying?

MEAN:
Interpret the quote in your own words. What does this mean in the context of teaching and learning math?

MATTER:
Why does this matter for you as a future teacher of elementary mathematics? How might this influence how you teach fractions, or how you create a classroom environment that supports deep, conceptual math learning?

🤝 Peer Response Guidelines:

- Choose two peers to respond to.
- Extend their thinking by:
  - Asking a thoughtful question,
  - Sharing a related classroom experience or strategy, or
  - Connecting to a concept from another course reading

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6.3.3: Online Retrieval and Problem Solving Practice

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6.3.4: Problem Solving Task

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6.4: Chapter 6 Resources

Chapter Example Videos

Chapter 6.1

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Chapter 6.2

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Chapter 6.3

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Chapter Extras

Multiplication and Division Exploration

Chapter Bibliography

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7: FROM PARTS TO PERCENTS- DECIMALS, RATIOS, PROPORTIONS, PERCENTAGES

A strong foundation in fractions built from early grades is critical for students' later success with decimals,percentages, ratios, and proportions (CDE, 2013). In upper elementary (grades 4–6), students extend their understanding by connecting fractions to decimals and percents, recognizing all as representations of rational numbers (Feike et al., 2018). Teachers should prioritize deep conceptual understanding over memorization of procedures, using multiple models like number lines and visual diagrams to make connections clear. Building on students' intuitive ideas, teachers help bridge whole-number operations to fractional reasoning, preparing students for proportional reasoning in middle school mathematics (NCTM, 2020).

By Grade 7, students are expected to have developed a unified understanding of number, recognizing fractions, decimals (including finite and repeating representations), and percents as different representations of rational numbers.

Chapter Learning Objectives

Student Learning Objectives

- SLO 7.1 Students will demonstrate understanding of decimal numbers by representing them visually (using number lines, hundreds squares, and grids), translating between word names and expanded forms, and ordering them based on place value and fraction equivalence.
- SLO 7.2 Students will compute with decimals and justify operations, including multiplying or dividing by powers of 10, using estimation techniques to evaluate sums, differences, products, and quotients effectively.
- SLO 7.3 Students will analyze and convert between fractions, decimals, and percentages, distinguishing between terminating, repeating, and nonterminating decimal representations and classifying their corresponding fractions.
- SLO 7.4 Students will solve problems involving ratios, rates, and proportions, comparing part-to-part and part-to-whole quantities, interpreting units, and applying strategies such as cross multiplication and scaling.
- SLO 7.5 Students will apply percentages to solve real-world problems by representing n% on grids, converting between percentages and fractions or decimals, and solving percent problems using equations, proportions, and calculators.
- SLO 7.6 Students will use scientific notation to represent decimal numbers efficiently and interpret scientific notation results in mathematical and scientific contexts.

Essential Questions

- **How can understanding the connections between fractions, decimals, and percentages help us teach children to make sense of numbers in everyday life?**
- **What strategies help students reason about ratios, rates, and proportions in meaningful, real-world contexts?**

Learning That Transfers

Students will connect decimal, fraction, percentage, ratio, and scientific notation concepts to real-world applications, modeling and explaining these relationships using visual, verbal, and symbolic methods to support elementary learners' mathematical reasoning and confidence.

One Part Love One Part Math

Interactive Content

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Morgan and Grandpa Go to the Game

Today was the day of the big baseball game,
Morgan and Grandpa were thrilled that they came!
With caps on their heads and gloves on their hands,
They dreamed of wild pitches and grand slamming stands.

They hurried to find their bright numbered seat,
The smell of hot dogs drifted up from the street.
"Let's grab a quick bite," said Grandpa with cheer,
"And be ready to shout when the players appear!"

At the food stand, a big banner flew,
"Buy two hot dogs and save quite a few!"
Morgan grinned wide and clapped her small hands,
"One-fourth off the price—that's a pretty good plan!"

They grabbed their hot dogs and ketchup to squeeze,
And found their seats in the front with the breeze.
But Morgan kept looking up at the big board—
Tracking the players, how many had scored!

One batter had 8 hits out of 20 at-bats,
Morgan did quick math beneath her ball cap:
"That's 8 out of 20...or 2 out of 5!
That's 40 percent to keep hopes alive!"

Another young player had 3 out of 9—
"A third!" Morgan shouted, "That's pretty fine!
One out of three is about 33—
That's the percent that player should see!"

The crowd roared loud when the home team was ahead,
Morgan and Grandpa both nodded and said,
"If we win 6 games out of a set of 10,
That's 60 percent—we're tough to defend!"

Ratios and percents flew all through the air,
Morgan kept thinking of fractions with care.

Later that inning, the sun beat down hot,
So Grandpa said, “Ice cream’s what we’ve gotta get bought!”
The sign said, “One scoop free when you buy three!”
Morgan laughed, "That's a deal for you and me!"

"One out of four is free," she explained with delight,
"That's 25 percent savings, and that feels just right!"

They licked their cold treats and watched the next play,
Another home run made their bright, happy day!

Morgan looked up at the field and the sky,
"Math is all over, just give it a try!
Ratios, percents—they're part of the game,
From hot dogs to scoreboards, it’s all just the same."

Grandpa just smiled, gave a tip of his hat,
"You’re seeing the world through math—imagine that!"

7.1: Decimals

7.1.1: Written Retrieval and Problem Solving Practice

7.1.2: Online Retrieval and Problem Solving Practice

7.1.3: AI Learning Prompts 7.1

7.2: Understanding Decimal Operations

7.2.1: Written Retrieval and Problem Solving Practice

7.2.2: Online Retrieval and Problem Solving Practice

7.2.3: AI Learning Prompts 7.2

7.3: Ratios and Proportions

7.3.1: Written Retrieval and Problem Solving Practice

7.3.2: Online Retrieval and Problem Solving Practice

7.3.3: AI Learning Prompts 7.3

7.4: Percentages

7.4.1: Written Retrieval and Problem Solving Practice

7.4.2: Online Retrieval and Problem Solving Practice

7.4.3: AI Learning Prompts 7.4

7.5: Chapter 7 Resources

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7.1: Decimals

The study of decimals is introduced after students have begun developing an understanding of fractions. Specifically, children begin studying decimals in the fourth grade. In this grade, instructional time focuses on developing an understanding of decimals, including the connections between fractions and decimals. One thing to keep in mind is that representing a number using a decimal system is just another way to represent a fraction. As we will see, the number $1\frac{3}{10}$ is the same value as 1.3, or that 0.75 is just another way to represent ¾. You may be thinking, why do we need to have different ways to represent the same values? That is an excellent question. We will explore these ideas in this section.

Name and Write Decimals

The word decimal comes from the Latin "decem", meaning ”ten”. Decimals fit naturally into our base-10 place value system, making them easier for most people to understand and manipulate, especially since this is the same system we use for whole numbers. As mentioned, decimals are another way of writing fractions whose denominators are powers of 10.

$$\begin{aligned} 0.1 &= \tfrac{1}{10} && 0.1 \text{ is "one tenth"} \\ 0.01 &= \tfrac{1}{100} && 0.01 \text{ is "one hundredth} \\ 0.001 &= \tfrac{1}{1{,}000} && 0.001 \text{ is "one thousandth} \\ 0.0001 &= \tfrac{1}{10{,}000} && 0.0001 \text{ is "one ten-thousandth"} \end{aligned} \tag{7.1.1}$$

Notice that “ten thousand” is a number larger than one, but “one ten-thousandth” is a number smaller than one. The “th” at the end of the name tells you that the number is smaller than one.

When we name a whole number, the name corresponds to the place value based on the powers of ten. We read 10,000 as “ten thousand” and 10,000,000 as “ten million.” Likewise, the names of the decimal places correspond to their fraction values. Figure 7.1.1 shows the names of the place values to the left and right of the decimal point.

Ten Thousands	Thousands	Hundreds	Tens	Ones	Decimal Point	Tenths	Hundreths	Thousandths	Ten Thousandths
10000	1000	100	10	1	.	0.1	0.01	0.001	0.0001
10000	1000	100	10	1	.	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$	$\frac{1}{10000}$

Figure 7.1.1: Place value of decimal numbers are shown to the left and right of the decimal point.

Referring to **Figure 7.1.1**, the number 543.21 shows that the decimal point is placed between the ones column and the tenths column to show where the whole-number portion ends and where the decimal (or fractional) portion begins. Decimals are read as if they were written as fractions and the decimal point is read “and.” The number 543.21 is written in its expanded form as 5(100) + 4(10) + 3(1) + 2(1/10) + 1(1/100).

We see that 543.21 = 543 $\frac{21}{100}$ is read “five hundred forty three and twenty one hundredths “. Note that the word "and" should only be used to indicate where the decimal point is located.

Example 7.1.1

a) Write as a decimal: thirteen and sixty-eight thousandths.

b) Write as a decimal: five and ninety-four ten-thousandths.

c) Write 0.234 as a fraction.

d) Write 0.24 as a fraction.

Answer available at the end of this section.

Figure 7.1.2 shows how a hundreds square can be used to represent tenths and hundredths. Notice that the large square represents 1, one vertical strip represents 0.1, and each one of the smallest squares represents 0.01. A number line can also be used to picture decimals. The number line in **Figure 7.1.3** shows the location of various decimals between 0 and 1.

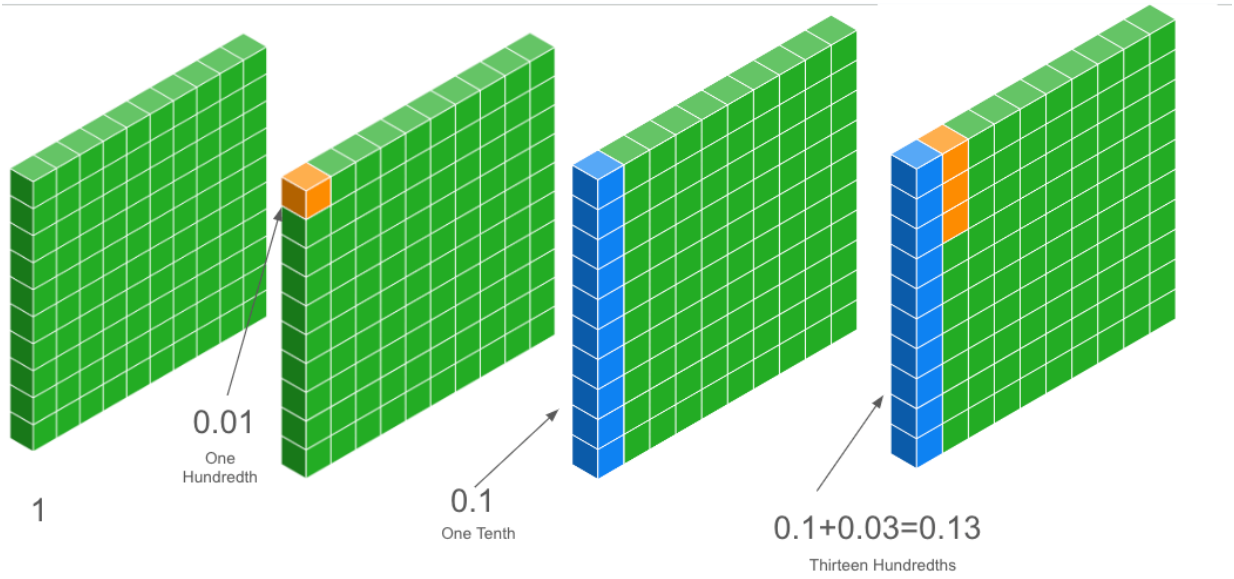


Figure 7.1.2: Hundred Squares/Grids

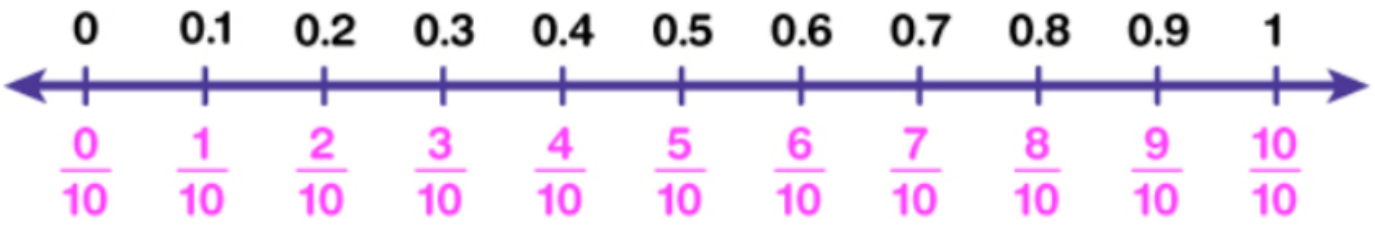


Figure 7.1.3: A horizontal number line ranging from 0 to 1 is shown, divided into ten equal parts.

Example 7.1.2

Rewrite each of these numbers in decimal form, and state the decimal name.

a) $\frac{7}{100}$

b) $\frac{123}{10000}$

c) $1\frac{7}{8}$

Answer available at the end of this section.

Wonder, Play, Grow

Use whatever method you would like to find the decimal representations of the following numbers.

$$\frac{1}{3}, \frac{3}{8}, \frac{2}{7}, \frac{4}{5}, \frac{9}{36}, \frac{25}{30}, \frac{7}{15}, \frac{27}{40}$$

Work with your peers to identify a rule that allows you to tell whether a fraction's decimal form will terminate just by examining the fraction.

🧠**Let's Listen to Learn**❤️

Audio

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Theorem 7.1.1

Fractions with Terminating Decimal Representations

Let a/b be a fraction in simplest form. Then a/b has a terminating decimal representation if and only if b contains only 2s and/or 5s in its prime factorization (since b can be expanded to a power of 10).

Well, what do you think? Can $\frac{63}{72}$ be written as a terminating decimal? _____ Explain why or why not.

Answer

The reason $\frac{63}{72}$ can be written as a terminating decimal is because in its ***simplified*** form, it only has 2's and/or 5's as its prime factors. Here is how to finish this problem by simplifying first, and then multiplying by any needed factors of 2 or 5 to get the same number of each:

$$\frac{63}{72} = \frac{3\cdot3\cdot7}{2\cdot2\cdot2\cdot3\cdot3} = \frac{7}{2\cdot2\cdot2} \cdot \frac{5\cdot5\cdot5}{5\cdot5\cdot5} = \frac{875}{1000} = 0.875$$

Ordering Decimals

Zeroes, which are behind a decimal's last non-zero digit, can be added or removed without changing the value of the decimal. If you look at some equivalent fraction, you'll see why this should be true. For example,

$$\frac{14}{100} = \frac{140}{1000} = \frac{1400}{10000} = \frac{14000}{100000}.$$

These are all equivalent to $\frac{14}{100}$ because the numerator and denominator was multiplied by some power of 10 (10, 100 or 1000) to get one of the other equivalent fractions. If we replace each of the four fractions above with their decimal equivalents, we get 0.14 = 0.140 = 0.1400 = 0.14000. Let's call any zeroes at the end of a decimal's last non-zero digit "**trailing zeroes**". Then, we could conclude that any number starting with .14 that has trailing zeroes is also equivalent, like 0.1400000.

Two decimals are equal only if one can be made to look identical to the other by adding or removing trailing zeroes. You can also determine if they are equivalent by removing any excess trailing zeroes from each to see if they are identical.

Case 1: To compare two or more decimals that are not equal, but have the same number of digits after the decimal point, you can write each as a fraction with the same denominator and then compare the numerator. For instance 0.14 is less than 0.21 since fourteen hundredths is less than twenty-one hundredths. Basically, in this case, it's just like comparing whole numbers. You'll be able to determine which is larger by comparing each number as if there was no decimal point. But, keep in mind this only makes sense if the numbers you are looking at have the same number of digits after the decimal point.

Case 2: To compare two or more decimals that are not equal that do not all have the same number of digits after the decimal point, first write each decimal with the same number of digits after the decimal point (by adding trailing zeroes to one or more if necessary). By doing that, you are comparing tenths with tenths, or hundredths with hundredth, etc., as you did in exercise 5.

Example 7.1.3

Compare each of the following decimals using <, = or >.

a) 3.5 and 0.9

b) 35.06 and 35.0600

c) 0.8900 and 0.098

d) 0.08933 and 0.0894

Answer available at the end of this section.

NOTE: In addition to looking at each digit from left to right - just like whole numbers - you can also compare terminating decimals by using a hundredths chart, a number line, and turning them into fractions.

Mental Math and Estimation, and Rounding

Before learning how to add, subtract, multiply, or divide decimals, you'll first build your skills with mental math and estimation—just like you did with whole numbers and fractions—to help you feel confident working with decimal problems.

Example 7.1.4

Use compatible (decimal) numbers, properties, and/or compensation to calculate the following mentally.

a) 1.7 + (3.2 + 4.3)

b) (0.5 x 6.7) x 4

c) 6 x 8.5

d) 3.76 + 1.98

e) 7.32 – 4.94

Answer available at the end of this section.

Decimal and Fraction Equivalents

DECIMAL	FRACTION
0.05	1/20
0.1	1/10
0.125	1/8
0.2	1/5
0.25	1/4
0.375	3/8
0.4	2/5
0.5	1/2
0.6	3/5
0.625	5/8
0.75	3/4
0.8	4/5
0.875	7/8

Figure 7.1.4:Fraction Equivalents

Example 7.1.5

Find these products using fraction equivalents.

a) 68 x 0.5

b) 0.25 x 48

c) 0.2 x 375

d) 0.05 x 280

Answer available at the end of this section.

You can multiply or divide decimals by powers of 10 using mental math, just like you would with whole numbers.

Example 7.1.6

Find the following products and quotients by converting to fractions.

a) 3.14 x 1000

b) 27.018 x 100000

c) 123.4 ÷ 100

d) 0.89 ÷ 10000

Answer available at the end of this section.

This mental mathematics leads to the Theorem 7.1.2.

Theorem 7.1.2

Multiplying/Dividing Decimals by Powers of 10

Let n be any decimal number and m represent any nonzero whole number. Multiplying a number n by 10^m is equivalent to forming a new number by moving the decimal point of n to the right m places. Dividing a number n by 10^m is equivalent to forming a new number by moving the decimal point of n to the left m places.

Using the theorem, we can rewrite Example 7.1.6 in the following way:

a) 3.14×10^3

b) 27.018×10^5

c) $123.4 \div 10^2$

d) $0.89 \div 10^4$

Round Decimals

Rounding decimals is very much like rounding whole numbers. We will round decimals with a method based on the one we used to round whole numbers. Overall, rounding decimals helps students understand the structure of the decimal system through place value and provides a valuable tool for estimating, checking computations, and simplifying numbers in both mathematical exercises and real-world contexts

We summarize the steps for rounding a **decimal** here.

ROUND DECIMALS.

1. Locate the given place value and mark it with an arrow.
2. Underline the digit to the right of the place value.
3. Is this digit greater than or equal to 5?
   - Yes—add 1 to the digit in the given place value.
   - No—do not change the digit in the given place value.
4. Rewrite the number, deleting all digits to the right of the rounding digit.

Example 7.1.7

a) Round 18.379 to the nearest hundredth.

b) Round 1.047 to the nearest hundredth.

c) Round 29.512 to the nearest whole number.

Answer available at the end of this section.

Rounding Decimals Retieval Practice

One of the most effective techniques for estimating with decimals is rounding them to values that produce compatible whole numbers or fractions.

Example 7.1.8

Estimate.

a) 203.4 × 47.8

b) 31 ÷ 1.93

c) 75 × 0.24

d) 124 ÷ 0.74

Answer available at the end of this section.

Answers

Example 7.1.1

a) 13.068 b) 5.00094 c) $\frac{234}{1000} = \frac{117}{500}$ d) $\frac{24}{100} = \frac{6}{25}$

Example 7.1.2

a) 0.07, "seven hundredths"

b) $\frac{123}{10{,}000} = \frac{100}{10{,}000} + \frac{20}{10{,}000} + \frac{3}{10{,}000} = \frac{1}{100} + \frac{2}{1{,}000} + \frac{3}{10{,}000} = 0.0123,$

c) $1\frac{7}{8} = 1 + \frac{7}{8} = 1 + \frac{7 \cdot 5 \cdot 5 \cdot 5}{2 \cdot 2 \cdot 2 \cdot 5 \cdot 5 \cdot 5} = 1 + \frac{875}{1000} = 1 + \frac{800}{1000} + \frac{70}{1000} + \frac{5}{1000} = 1 + \frac{8}{10} + \frac{7}{100} + \frac{5}{1000}$ = 1.875

NOTE: All of the fractions in this example have denominators whose only prime factors are 2 or 5. Such fractions can always be expressed in decimal form, since they have equivalent fractional forms whose denominators are powers of 10.

Example 7.1.3

a) 3.5 > 0.9

b) 35.06 = 35.0600

c) 0.8900 > 0.098

d) 0.08933 < 0.0894

Example 7.1.4

a) 1.7 + (3.2 + 4.3) = (1.7 + 4.3) + 3.2 = 6 + 3.2 = 9.2. Here 1.7 and 4.3 are compatible numbers with respect to addition, since their sum is 6.

b) (0.5 x 6.7) x 4 = 6.7 x (0.5 x 4) = 6.7 x 2 = 13.4.

c) Using distributivity, 6 x 8.5 = 6(8 + 0.5) = 6 x 8 + 6 x 0.5 = 48 + 3 = 51.

d) 3.76 + 1.98 = 3.74 + 2 = 5.74 using additive compensation.

e) 7.32 − 4.94 = 7.38 − 5 = 2.38 by equal additions.

Example 7.1.5

a) 68 x 1/2 = 34

b) 1/4 x 48 = 12

c) 1/5 x 375 = 75

d) 1/20 x 280 = 14

Example 7.1.6

a) $\frac{314}{100} \times \frac{1000}{1}$= 3140

b) $\frac{27018}{1000} \times \frac{100000}{1}$ = 2701800

c) $\frac{1234}{10} \div \frac{100}{1} = \frac{1234}{10} \times \frac{1}{100}$ = 1.234

d) $\frac{89}{100} \div \frac{10000}{1} = \frac{89}{100} \times \frac{1}{10000}$ = 0.000089

Example 7.1.7

a)

Step 1. Locate the given place value and mark it with an arrow.		hundredths place ↓ 18.379
Step 2. Underline the digit to the right of the given place value.		hundredths place ↓ 18.379

Step 3. Is this digit greater than or equal to 5? **Yes:** Add 1 to the digit in the given place value. **No:** Do not change the digit in the given place value.	Because 9 is greater than or equal to 5, add 1 to the 7.	18.37 9 add 1 delete
Step 4. Rewrite the number, removing all digits to the right of the rounding digit.		18.38 18.38 is 18.379 rounded to the nearest hundredth.

b) 1.05

c) 30

Example 7.1.8

a) 203.4 × 47.8 ≈ 200 × 50 = **10,000**

b) 31 ÷ 1.93 ≈ 30 ÷ 2 = **15**

c) 75 × 0.24 = 75 × ¼ × 0.96 ≈ 76 × ¼ = **19**

d) 124 ÷ 0.74 ≈ 124 ÷ ¾ = 124 × 4/3 ≈ 123 × 4/3 = **164**

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7.1.1: Written Retrieval and Problem Solving Practice

Student Learning Objectives

7.1 Students will demonstrate understanding of decimal numbers by representing them visually (using number lines, hundreds squares, and grids), translating between word names and expanded forms, and ordering them based on place value and fraction equivalence.

7.2 Students will compute with decimals and justify operations, including multiplying or dividing by powers of 10, using estimation techniques to evaluate sums, differences, products, and quotients effectively.

7.3 Students will analyze and convert between fractions, decimals, and percentages, distinguishing between terminating, repeating, and nonterminating decimal representations and classifying their corresponding fractions.

The mild assignment 🌶 is a shorter and less challenging assignment. The medium assignment🌶🌶 is right in the middle in terms of length and level of difficulty. Finally, the spicy assignment 🌶🌶🌶 is a longer assignment that is the most challenging.

(Spicy 🌶🌶🌶 Problems are OPTIONAL, but go for it when you have the time to challenge yourself)

Exercise 7.1.1.1 🌶

Write each of the following sums in decimal form.

a) $4(10) + 3 + 2(1/10) + 4(1/1000)$

b) $2(1/10)^2 + 2(1/10)^4$

c) $3(10)^2 + 6 + 4(1/10)^2 + 2(1/10)^3$

Answer available at the end of this section.

Exercise 7.1.1.2 🌶🌶

Write each of the following decimals (i) in its expanded form and (ii) as a fraction.

a) 0.35

b) 2.183

c) 34.2005

Answer available at the end of this section.

Exercise 7.1.1.3 🌶🌶

Write the following expressions as decimal numbers.

a) Six hundred forty-six thousand

b) Six hundred forty-six thousandths

c) Six hundred and forty-six thousandths

Answer available at the end of this section.

Exercise 7.1.1.4

Determine, without converting to decimals, which of the following fractions has a terminating decimal representation.

a) 21/45

b) 62/125

c) 63/90

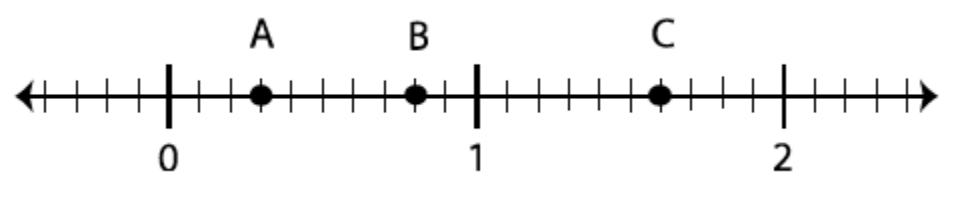
d) 39/60

e) 54/60

Answer available at the end of this section.

Exercise 7.1.1.5

What numbers are represented by the points labeled A, B and C?



Answer available at the end of this section.

Exercise 7.1.1.7

Arrange the following numbers in order from smallest to largest.

a) 0.48, 0.084, 0.84

b) 681.345, 681.354, 680.9999

c) 1.9, 1.09, 1.99, 1.099

Answer available at the end of this section.

Exercise 7.1.1.8

Multiply 2.58

a) by 10

b) by 1000

c) by 10^5

Answer available at the end of this section.

Exercise 7.1.1.9

Round 6.582 to the nearest

a) hundredth

b) tenth

c) whole number.

Answer available at the end of this section.

Exercise 7.1.1.10

Round 15.2175 to the nearest

a) thousandth

b) hundredth

c) tenth.

Answer available at the end of this section.

Exercise 7.1.1.11

Calculate mentally by using fraction equivalents.

a) 0.25 × 44

b) 0.75 × 80

c) 35 × 0.4

d) 0.2 × 65

Answer available at the end of this section.

Exercise 7.1.1.12

A student reads the number 2146 as “two thousand one hundred and forty-six.” What is wrong with this reading?

Answer available at the end of this section.

Problem Solving Problems

1) Discuss with peers how your thought process to estimate by rounding to compatible numbers and fraction equivalents.

a) 47.1 ÷ 2.9

b) 0.23 × 88

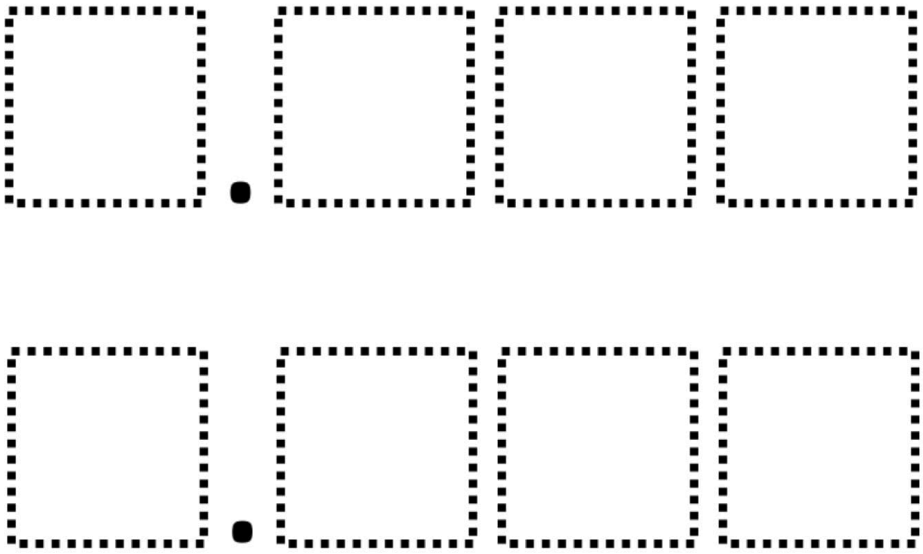
c) 126 × 0.21

d) 56,324 × 0.25

e) 14,897 ÷ 750

f) 0.59 × 474

2) **Comparing Decimals*** Directions: Using the digits 0 to 9 at most one time each, place a digit in each box to create two decimals that are close to 5 as possible but also equally far away from 5.

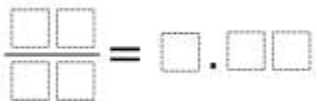


Answer

The closest decimals you can get are 4.987 and 5.013, which are each 0.013 away from 5.

*This problem was graciously taken from the Open Middle Website

3) **Converting Between Fractions and Decimals**** Directions: Using the digits 0 to 9 at most one time each, fill in the boxes so that the fraction equals the decimal.

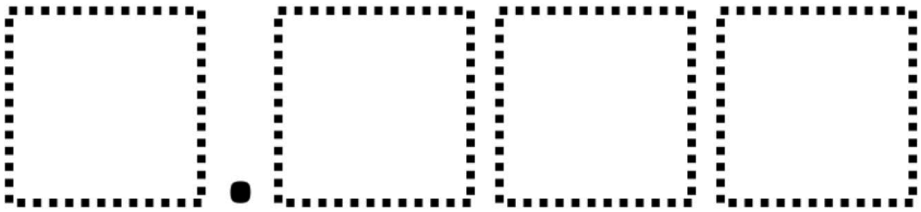


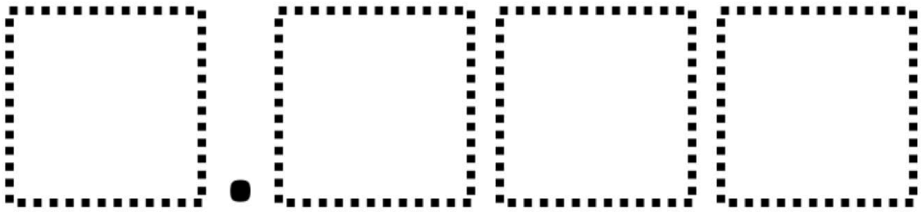
Answer

Number of Unique Solutions: 56 - Below are two of them.
1: 17 / 25 = 0.68
2: 17 / 68 = 0.25

**This problem was graciously taken from the Open Middle Website

4) ***Rounding Decimals** Directions: Using the digits 0 to 9 at most one time each, place a digit in each box to create two different decimals that are equivalent when rounded to the nearest tenth.





Answer

There are many possible answers including 5.971 and 6.043 which both round to 6.0.

***This problem was graciously taken from the Open Middle Website

Answers

Exercise 7.1.1.1🌶️

a) 43.204

b) 0.0202

c) 306.042

Exercise 7.1.1.2🌶️🔶🔶

a) 3(1/10) + 5(1/100) = 35/100 = 7/50

b) 2 + 1(1/10) + 8(1/100) + 3(1/1000) = $2\frac{183}{1000}$

c) 3(10) + 4 + 2(1/10) + 5(10000) = $34\frac{2005}{10000}$ = $34\frac{401}{2000}$

Exercise 7.1.1.3🌶️🌶️

a) 646,000 b) 0.646 c) 600.046

🌶️🌶️**Exercise** 7.1.1.4

a) 21 ÷ 3 = 7, 45 ÷ 3 = 15 → 7/15
15 = 3 × 5 → has a 3, so non-terminating

b) Already in lowest terms.
125 = 5 × 5 × 5 → only 5s → ✅ terminating

c) 63 ÷ 9 = 7, 90 ÷ 9 = 10 → 7/10
10 = 2 × 5 → only 2s and 5s → ✅ terminating

d) GCF of 39 and 60 is 3 → 13/20
20 = 2 × 2 × 5 → only 2s and 5s → ✅ terminating

e) GCF is 6 → 9/10
10 = 2 × 5 → only 2s and 5s → ✅ terminating

Exercise 7.1.1.5

Point **A** is located at 0.4.

Point **B** is located at 0.8.

Point **C** is located at 1.6.

Exercise 7.1.1.7🌶️

a) 0.084 < 0.48 < 0.84

b) 680.9999 < 681.345 < 681.354

c) 1.09 < 1.099 < 1.9 < 1.99

Exercise 7.1.1.8🌶️

a) 25.8

b) 2580

c) 258,000

Exercise 7.1.1.9🌶

a) 6.58 b) 6.6 c) 7

Exercise 7.1.1.10🌶

a) 15.218 b) 15.22 c) 15.2

Exercise 7.1.1.11🌶🌶

a) 1/2 x 44 = 22 b) 3/4 of 80, 80/4 then times 3 is 60 c) 35 x 2/5 = (35/5 x 2) = 14 d) 1/5 of 65 = 65/5 13

Exercise 7.1.1.12🌶🌶

Correct reading: **“Two thousand one hundred forty-six”**

The "and" is reserved for a decimal point.

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7.1.2: Online Retrieval and Problem Solving Practice

Online Retrieval Practice

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7.1.3: AI Learning Prompts 7.1

AI Prompt Library

Decimals — Representation, Ordering & Fraction-Decimal Connections

MATH 204 · SANTA ANA COLLEGE

Choose the tab that matches how you want to learn right now. Each section has ready-to-paste prompts — copy one into any AI chat to start a learning conversation. Use the Follow-Up Question Card at the bottom of each section to keep the conversation going.

📖 Explain It | 🔍 Explore It | 🔀 Practice It

Section 1 — Direct Explanation Prompts

Learning Objective: Understand what decimal numbers are, how they relate to fractions with powers-of-10 denominators, and how to represent, translate between forms, and order decimals — especially as a future elementary teacher who will need to explain these ideas to children.

Type 1 — Just Explain It

PROMPT 1

"Can you explain to me what decimals really are and how they connect to fractions? I want to understand why 0.75 is the same thing as 3/4. Cover the ideas of place value, expanded form, and powers of 10, and use plain language — I'm a future elementary teacher and I need to be able to explain this to kids someday. After you explain it, ask me if anything is unclear."

PROMPT 2

"I need a clear explanation of how to order and compare decimals. My big confusion is when decimals have different numbers of digits — like comparing 0.8 to 0.12. Explain the role of place value and trailing zeroes, give me a concrete example, and then check in with me to see if I have questions."

Type 2 — Explain It and Connect It

PROMPT 3

"I already understand fractions pretty well — equivalent fractions, simplifying, all that. Can you explain decimals in a way that builds directly on what I already know about fractions? I want to see how a decimal is really just a fraction whose denominator is a power of 10, and how that connects to the place value system I already use for whole numbers. Help me see the bridge before you go into details."

PROMPT 4

"I know how to add and multiply fractions, and I know how to do decimal arithmetic with the standard algorithms. But I've never really connected the two — I just do them separately. Can you explain why converting a decimal to a fraction and doing fraction arithmetic gives me the same answer as the decimal algorithm? Use a specific example like 0.25 × 0.4 to show me. Then ask me if it clicks or if I need another angle. I'm asking because I'm going to be a teacher and I want my students to see the connection, not just follow two separate sets of rules."

📋 Follow-Up Question Card — Decimals & Fraction-Decimal Connections

Got the explanation — now keep going. Pick one of these moves:

Question Card

MOVE	WHAT TO TYPE
Say it differently	*"I'm still fuzzy on why place value makes decimals work like fractions. Can you explain it a different way — maybe using a hundreds square or a number line visual instead of just the notation?"*
Give me an example	*"Can you walk me through converting 7/8 into a decimal step by step, and show me why the denominator's prime factors matter for whether the decimal terminates?"*
Check me	*"Can you give me a quick check — ask me to explain in my own words why 0.5 and 0.50 are the same value, and tell me if my reasoning is solid?"*
Go deeper	*"Why do some fractions like 1/3 turn into repeating decimals while others like 1/4 terminate? What is it about the denominator that determines this?"*

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7.2: Understanding Decimal Operations

Operations with Decimals

The algorithms for adding, subtracting, multiplying, and dividing decimals are straightforward extensions of the methods used with whole numbers. In essence, while decimal algorithms provide efficient methods for computation, the arithmetic of fractions serves as a conceptual foundation and a tool for explaining and justifying *why* those decimal algorithms work (Musser et al., 2014).

Think About and Discuss

Perform the following computations:

- 3.8 + 4.37
- 3.8 × 4.37

For each computation, discuss with your peers:

a) *How the decimal point is positioned when setting up the problem and how it appears in the final answer.*

b) *Why this method for placing the decimal point makes sense and leads to a correct solution.*

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Audio

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Addition

Addition of decimal numbers is quite similar to addition of whole numbers. For example, suppose that we are asked to add 2.34 and 5.25. We could change these decimal numbers to **mixed fractions** and add.

$$\begin{aligned} 2.34 + 5.25 &= 2\frac{34}{100} + 5\frac{25}{100} \\ &= 7\frac{59}{100} \end{aligned}$$

However, we can also line the decimal numbers on their decimal points and add vertically, as follows.

$$\begin{array}{r} 2.34 \\ +5.25 \\ \hline 7.59 \end{array}$$

Note that this alignment procedure produces the same result, “seven and fifty nine hundredths.” The lining up of the decimals is like we are finding a common denominator. This motivates the following procedure for adding decimal numbers.

Adding Decimals

To add decimal numbers, proceed as follows:

1. Place the numbers to be added in vertical format, aligning the decimal points.
2. Add the numbers as if they were whole numbers.
3. Place the decimal point in the answer in the same column as the decimal points above it.

Example 7.2.1

Add using fractions and the decimal addition algorithm.

a) 2.56 + 8.95

b) 0.0084 + 70.183

Answer available at the end of this section.

Before looking at another example, let's recall an important observation.

Important Observation - "Trailing Zeros"

Adding zeros to the end of the fractional part of a decimal number does not change its value. Similarly, deleting trailing zeros from the end of a decimal number does not change its value.

For example, we could add two zeros to the end of the fractional part of 7.25 to obtain 7.2500. The numbers 7.25 and 7.2500 are identical as the following argument shows:

$$\begin{aligned} 7.2500 &= 7\frac{2500}{10000} \\ &= 7\frac{25}{100} \\ &= 7.25 \end{aligned}$$

Subtraction

Subtraction of decimal numbers proceeds in much the same way as addition of decimal numbers.

Subtracting Decimals

To subtract decimal numbers, proceed as follows:

1. Place the numbers to be subtracted in vertical format, aligning the decimal points.
2. Subtract the numbers as if they were whole numbers.
3. Place the decimal point in the answer in the same column as the decimal points above it.

Example 7.2.2

a) Subtract 12.23 from 33.57.

b) Find the difference: 13.3 − 8.572.

Answer available at the end of this section.

Multiplication

Let's look at multiplying decimals. Keep in mind the common student misconception that "multiplication always makes things bigger." While this idea holds when multiplying by whole numbers greater than one, it does not hold true for fractions or decimal numbers less than one (Feike et al., 2018). This misunderstanding highlights the importance of building conceptual understanding of multiplication beyond whole-number contexts.

Multiplying decimal numbers involves two steps: (1) multiplying the numbers as whole numbers, ignoring the decimal point, and (2) placing the decimal point in the correct position in the product or answer.

For example, consider (0.7)(0.08), which asks us to find the product of “seven tenths” and “eight hundredths.” We multiply 7 with 8, and move the decimal three places to the left. The product is 0.056, or “fifty six thousandths,” which as a fraction is 56/1000.

We could change these decimal numbers to fractions, then multiply.

$$\begin{aligned}(0.7)(0.08) &= \frac{7}{10} \cdot \frac{8}{100} \\ &= \frac{56}{1000} \\ &= 0.056\end{aligned}$$

Example 7.2.3

Multiply by changing to fractions first, (2.34)(1.2).

Answer available at the end of this section.

Multiplying Decimal Numbers

To multiply two decimal numbers, perform the following steps:

1. Ignore the decimal points in the factors and multiply the two factors as if they were whole numbers.
2. Count the number of digits to the right of the decimal point in each factor. Sum these two numbers.
3. Place the decimal point in the product so that the number of digits to the right of the decimal point equals the sum found in step 2.

Example 7.2.4

Use the steps outlined in Multiplying Decimal Numbers to find the product in Example 7.2.3.

Answer available at the end of this section.

Division

In this and following sections we make use of the terms *divisor*, *dividend*, *quotient*, and *remainder*.

Divisor, Dividend, Quotient, and Remainder

This schematic reminds readers of the position of these terms in the division process.

$$\begin{array}{r} \text{quotient} \\ \text{divisor})\ \overline{\text{dividend}} \\ \vdots \\ \overline{\text{remainder}} \end{array}$$

Now that these terms are defined, we begin the discussion of division of decimal numbers. Suppose that we wish to divide 637 by 100. We could do this in fraction form, change the result to a mixed fraction, then the mixed fraction to decimal form.

$$\frac{637}{100} = 6\frac{37}{100} = 6.37$$

We can also arrange the division much as we would the division of two whole numbers.

$$\begin{array}{r} 6.37 \\ 100\overline{)637.00} \\ \underline{600} \\ 37\,0 \\ \underline{30\,0} \\ 7\,00 \\ \underline{7\,00} \\ 0 \end{array}$$

Note that adding two zeros after the decimal point in the dividend does not change the value of 637. Further, note that we proceed as if we are dividing two whole numbers, placing the decimal point in the quotient directly above the decimal point in the dividend.

These observations lead to the following algorithm.

Dividing a Decimal by a Whole Number

To divide a decimal number by a whole number, proceed as follows:

1. Set up the long division as you would the division of two whole numbers.
2. Perform the division as if the numbers were both whole numbers, adding zeros to the right of the decimal point in the dividend as necessary to complete the division.
3. Place the decimal point in the quotient immediately above the decimal point in the dividend.

Although the algorithm above is what we will use to divide decimals, it is important to know the relationship between the decimal placement and the fraction process of dividing decimals, see the example below to reinforce this idea.

$$154.63 \div 4.7 = \left(\frac{15463}{100}\right) \div \left(\frac{47}{10}\right) = \frac{15463 \times 10}{100 \times 47} = \frac{154630}{470} = 32.9 \tag{7.2.1}$$

Adding Zeroes to the Right of the Decimal Point

Usually, one does not immediately see how many zeros to the right of the decimal point in the dividend are needed. These zeros are usually added at each step of the division, until the division is completed or the user is willing to terminate the process and accept only an estimate of the quotient.

Example 7.2.5

Divide: 155.2 ÷ 25.

Answer available at the end of this section.

Decimal Divisors When the divisor contains a decimal point, we have a little work to do before we begin the division process. Suppose that we wish to divide 1.25 by 2.5. In fraction form, we could start with

$$\frac{1.25}{2.5},$$

then clear the decimals from the denominator by multiplying both numerator and denominator by 10. *Note: Recall that multiplying by 10 moves the decimal point one place to the right.*

$$\begin{aligned}\frac{1.25}{2.5} &= \frac{1.25 \cdot 10}{2.5 \cdot 10} \\ &= \frac{12.5}{25}\end{aligned}$$

Thus, dividing 1.25 by 2.5 is equivalent to dividing 12.5 by 25. This we know how to do.

$$\begin{array}{r} 0.5 \\ 25\overline{)12.5} \\ 12\ 5 \\ \hline 0 \end{array}$$

Thus, 1.25 divided by 2.5 is 0.5.

Instead of working in fraction form, we can take care of positioning the decimal point in the long division framework. Start with:

$$2.5\overline{)1.25}$$

Move the decimal point in the divisor to the end of the divisor, then move the decimal point in the dividend an equal number of places.

$$2.5\overline{)1.2\ 5}$$

Thus, the division becomes

$$25\overline{)12.5}$$

and we proceed as above to find the quotient. This discussion motivates the following algorithm.

Dividing by a Decimal Divisor

If the divisor contains a decimal, proceed as follows:

1. Move the decimal to the end of the divisor.
2. Move the decimal in the dividend an equal number of places.

We can then complete the division using the rules for dividing a decimal by a whole number.

Classifying Repeating Decimals

If a fraction is written with a power of ten in the denominator, it's basic to write the same number in decimal form. You did that in Exercise 2. Any fraction that is written with a power of 10 in the denominator can be written as a terminal decimal. This means that it's possible to write the number with trailing zeroes. But what if the fraction doesn't have a power of 10 (like 10, 100, 1000, etc.) in the denominator? Sometimes those can be tricky!

If a fraction WITHOUT a power of ten in the denominator CAN be written as an equivalent fraction WITH a power of ten in the denominator, then it can be written as a terminating decimal.

For instance, $\frac{1}{2}$ can be written as $\frac{5}{10}$ (by multiplying the numerator and denominator by 5)

Therefore, $\frac{1}{2} = \frac{5}{10} = 0.5$ (remember to read this as five-tenths)

Well, that one wasn't too hard, but what about $\frac{7}{80}$? The question is whether you can multiply the denominator, 80, by something to get 10, 100, 1000, 10000, etc. There is no whole number you can multiply 80 by to get 10, 100 or 1000. But, if you multiply 80 times 125, it equals 100000. So, by multiplying both 7 and 80 by 125, we get the equivalent fraction $\frac{875}{10000}$ for $\frac{7}{80}$ which can now be written as the decimal 0.0875.

How about writing $\frac{5}{6}$ as a terminating decimal? Well, there is nothing you can multiply 6 by to get 10 or 100 or 1000 or 10000. Is there maybe some number we could multiply 6 by to get some higher power of ten? Well, that's a good question! Actually, there isn't, but how could you be sure? You certainly can't try every power of ten since there are an infinite number of them to try.

It would be nice if there was an easy way to determine if any given fraction could be written as a terminating decimal. The key is to consider the factors of the denominator of a fraction that can be written as a terminating decimal. If a fraction can be written as a terminating decimal, then there is some equivalent fraction where the denominator must be a power of ten: 10 or 100 or 1000, etc. **Powers of ten only have 2s and 5s as its prime factors and nothing else.**

Think About and Discuss

1) What is the decimal representation for $\frac{5}{6}$? Discuss with others how to find it. Be prepared to share with the class.

2) How about $\frac{63}{72}$? If we prime factor the denominator, we get $2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$. Can $\frac{63}{72}$ be written as a terminating decimal? _____ Explain why or why not.

Theorem 7.2.1

Fractions with Repeating, Nonterminating Decimal Representations

Let $\frac{a}{b}$ be a fraction written in **simplest form**. Then $\frac{a}{b}$ has a repeating decimal representation that does not terminate if and only if b has a prime factor other than 2 or 5.

Example 7.2.6

Which of the following fractions have repeating, nonterminating decimal representations?

a) $\frac{3}{4}$

b) $\frac{9}{20}$

c) $\frac{7}{40}$

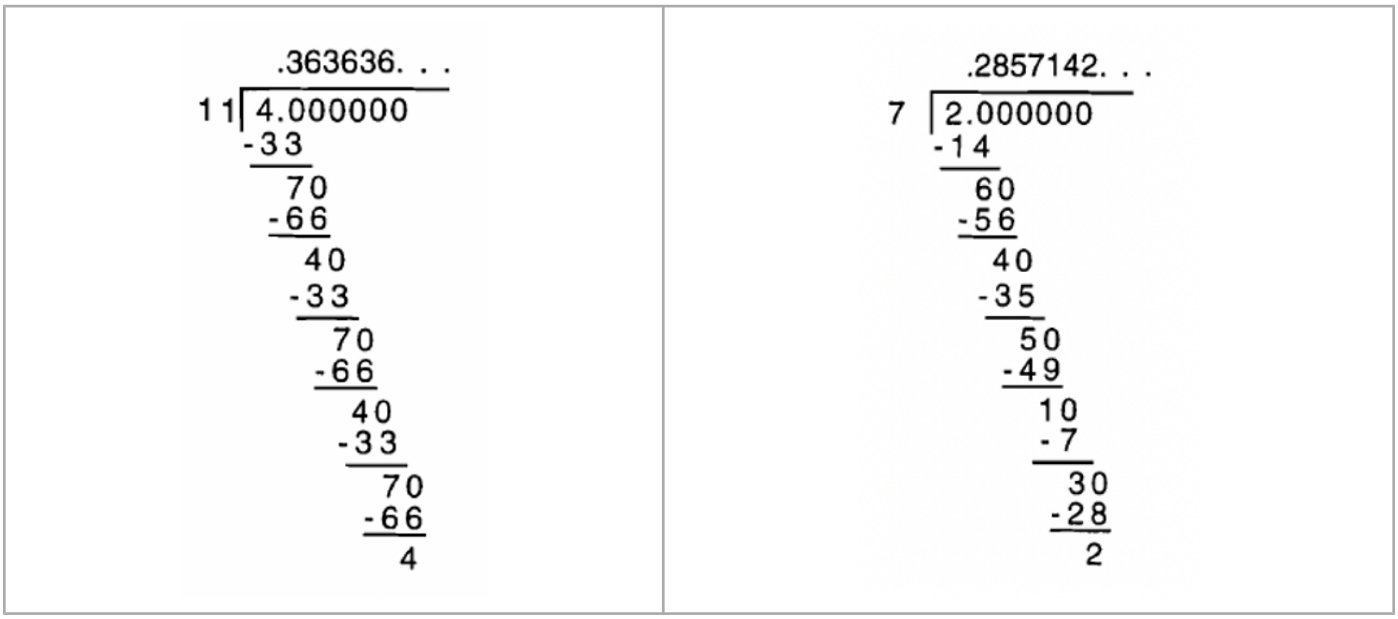
d) $\frac{18}{25}$

e) $\frac{6}{28}$

Then, **check your answer with a calculator** by taking the original fraction and dividing the numerator by the denominator.

Answer available at the end of this section.

Let's take a look at what happens when you do long division to compute $\frac{4}{11}$ and $\frac{2}{7}$. In both of these cases, eventually, you get a remainder that you got previously, so the computation repeats itself.



Rational Numbers are defined to be numbers which can be expressed as the ratio of two integers. Fractions as we usually refer to them (without decimal points or square roots, etc. in the numerator or denominator) are rational numbers. Since all fractions can be written as terminating or nonterminating (infinite, repeating) decimals, then all terminating and repeating decimals are also rational numbers. In our work, we wrote some terminating decimals as fractions. Since all nonterminating decimals came from a fraction, now we want to be able to go the other way around and write a repeating decimal as a fraction. We've got a nifty trick for doing just that!

Let's say we wanted to write $0.\overline{72}$ or 0.727272727272... as a fraction.

First, notice this is a very different number from 0.72, since 0.72 is seventy-two hundredths, or $\frac{72}{100}$. This simplifies to $\frac{18}{25}$. So,0 .72 and $\frac{18}{25}$ are the same number. You can check to make sure $\frac{18}{25}$ is the correct fraction by using a calculator and dividing 18 by 25. The calculator should read 0.72, which is where we started.

Okay, so how do we write $0.\overline{72}$ as a fraction? Since this is a repeating decimal that has infinitely many digits, there is no single power of ten we can put in the denominator. The trick is to use algebra to write a number in a way that eliminates the infinitely repeating part of the decimal. First, let's write out $0.\overline{72}$ the long way:

$0.\overline{72}$ = .727272727272... and call it the number x. So x = 0.727272727272...

Remember that if you multiply a decimal by 10, the decimal point moves one place to the right, and when you multiply a decimal by 100, the decimal point moves two places to the right, etc.

Example 7.2.7

Express $0.\overline{72}$ in fractional form.

Answer available at the end of this section.

Here's the trick for changing repeating decimals to simplified fractions:

Call the number you are trying to write as a decimal a variable, like n or x.

If there is one repeating digit, compute 10x; if there are two repeating digits, compute 100x; if there are three repeating digits, compute 1000x, etc. This aligns the repeating decimals up with each other so that the tail of x and the other number (10x, 100x, 1000x, etc.) is the same.

Then subtract x from the other number (10x, 100x, 1000x, etc.) The tails of both numbers will come off and you should have an algebraic problem to solve at this point. Make sure you write your answer as a reduced fraction with only integers in the numerator and denominator.

Here are two examples. *If you write the number in decimal form without the bar over the number, remember to put the ellipses (three dots) at the end!*

Write $0.\bar{2}$ as a simplified fraction.

Let x = $0.\bar{2}$ = .222222... Since only one digit is repeating, multiply x by 10.

Then 10x = 2.222222...

$$\begin{aligned} 10x &= 2.22222.... \\ -x &= .2222..... \\ 9x &= 2 \end{aligned}$$

$$x = \frac{2}{9}$$

Use a calculator to check that $\frac{2}{9} = 0.\bar{2}$

Convert the Repeating Decimal

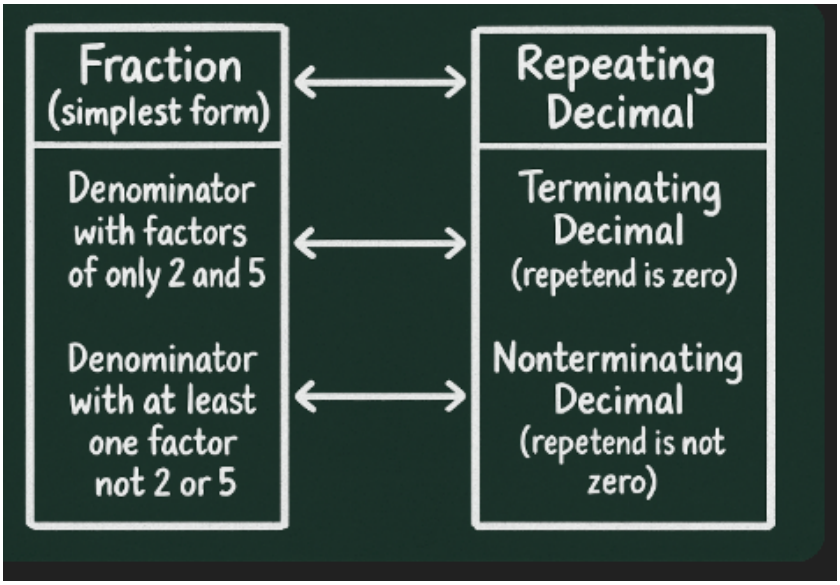


Figure 7.2.1: Relationship between fractions in simplest form and repeating decimals.

A **repetend** is the **digit or group of digits that repeats indefinitely** in the decimal representation of a rational number

Important Note: (Figure 7.2.1) Every fraction has a repeating decimal representation, and every repeating decimal has a fractional representation.

The rationals make up a very small part of the real numbers. To complete the real number system, we have to talk about the **irrational numbers**. Every real number is either rational or irrational. Those numbers which cannot be written as the ratio of two relatively prime integers is irrational. Those decimal numbers which neither terminate nor repeat are irrational. So, what do they look like?

One of the most commonly known irrational numbers is π. It is the number that is the ratio of the circumference of a circle to its diameter. Sounds like it's a ratio of two integers, but it's not! π can only be approximated. The most common approximation is 3.14 or 22/7. Neither of these is equal to π because both of these are rational numbers, and π is not!

Other irrational numbers are square roots of numbers that are not perfect squares, or cube roots of numbers that are not perfect cubes, etc.

For instance, these numbers are irrational: $\sqrt{5}, \sqrt{3}, \sqrt{12}, \sqrt{20}, \sqrt[3]{35}, \sqrt[4]{72}$

Scientific Notation

Multiplying and dividing large numbers can sometimes be assisted by first expressing them in scientific notation. Numbers are said to be in **scientific notation** when written in the form $a \times 10^n$, where $1 \leq a < 10$.

Scientific Notation	Standard Notation
1.43×10^8 meters	143,800,000 meters
1.2×10^{24} kilograms	1,200,000,000,000,000,000,000,000 kilograms
5.88×10^8 meters	588,000,000 meters

Once large numbers are expressed in scientific notation, they can be multiplied and divided more easily, as shown below.

Example 7.2.8

Compute the following using scientific notation:

a) 54,500,000,000 × 346,000,000

b) 1,200,000,000,000 ÷ 62,500,000

Answer available at the end of this section.

Answers

Example 7.2.1

a) $2.56 + 8.95 = \frac{256}{100} + \frac{895}{100} = \frac{256+895}{100} = \frac{1151}{100} = 11.51$ b) $0.0084 + 70.183 = \frac{84}{10000} + \frac{70183}{1000} = \frac{84}{10000} + \frac{701830}{10000} = \frac{701914}{10000} = 70.1914$

a) 2.56 + 8.95

```
   2.56
+  8.95
-------
  11.51
```

- Line up the decimal points:

b) 0.0084 + 70.183

```
   0.0084
+ 70.1830
---------
  70.1914
```

- Pad with zeros to match decimal places:

Example 7.2.2

a) $\frac{3357}{100} - \frac{1223}{100} = \frac{3357-1223}{100} = \frac{2134}{100} = 21.34$

Arrange the numbers in vertical format, aligning their decimal points in a column, then subtract. Note that we subtract 12.23 **from** 33.57.

$$\begin{array}{r} 33.57 \\ -12.23 \\ \hline 21.34 \end{array}$$

Hence, 33.57 – 12.23 = 21.34.

b) $\frac{133}{10} - \frac{8572}{1000} = \frac{13300}{1000} - \frac{8572}{1000} = \frac{13300-8572}{1000} = \frac{4728}{1000} = \frac{1182}{250}$

Arrange the numbers in vertical format, aligning their decimal points in a column. Note that we add trailing zeros to the fractional part of 13.3 to improve columnar alignment.

$$\begin{array}{r} 13.300 \\ -8.572 \\ \hline 4.728 \end{array}$$

Hence, 13.3 – 8.572 = 4.728.

Example 7.2.3

Change the decimal numbers “two and thirty four hundredths” and “one and two tenths” to fractions, then multiply.

$$\begin{aligned} (2.34)(1.2) &= 2\frac{34}{100} \cdot 1\frac{2}{10} && \text{Change decimals to mixed fractions.} \\ &= \frac{234}{100} \cdot \frac{12}{10} && \text{Change mixed to improper fractions.} \\ &= \frac{234 \cdot 12}{100 \cdot 10} = \frac{2808}{1000} && \text{Multiply numerators and denominators.} \\ &= 2\frac{808}{1000} && \text{Change to mixed fraction.} \\ &= 2.808 && \text{Change back to decimal form.} \end{aligned} \tag{7.2.2}$$

Example 7.2.4

1. The first step is to multiply the factors 2.34 and 1.2 as whole numbers, ignoring the decimal points.

$$\begin{array}{r} 234 \\ \times 12 \\ \hline 468 \\ 234 \\ \hline 2808 \end{array}$$

2. The second step is to find the sum of the number of digits to the right of the decimal points in each factor. Note that 2.34 has two digits to the right of the decimal point, while 1.2 has one digit to the right of the decimal point. Thus, we have a total of three digits to the right of the decimal points in the factors.

3. The third and final step is to place the decimal point in the product or answer so that there are a total of three digits to the right of the decimal point. Thus,

$$(2.34)(1.2) = 2.808.$$

Note that this is precisely the same solution found in Example 1.

What follows is a convenient way to arrange your work in vertical format.

$$\begin{array}{r} 2.34 \\ \times 1.2 \\ \hline 468 \\ 2\ 34 \\ \hline 2.808 \end{array}$$

Example 7.2.5

Arrange as if using long division to divide whole numbers, and begin.

$$\begin{array}{r} 6.2 \\ \hline 25\overline{)155.2} \\ 150 \\ \hline 5\ 2 \\ 5\ 0 \\ \hline 2 \end{array}$$

However, if we add one more additional zero, the division completes with a zero remainder.

$$\begin{array}{r} 6.208 \\ \hline 25\overline{)155.200} \\ 150 \\ \hline 5\ 2 \\ 5\ 0 \\ \hline 200 \\ 200 \\ \hline 0 \end{array}$$

Thus, 155.2 divided by 25 is 6.208.

Example 7.2.6

a) Terminates

b) Terminates

c) Terminates

d) Terminates

e) Non-Terminating, Repeating

Example 7.2.7

Okay, let's work with x = 0.727272727272... and 100x = 72.727272727272...

a. In algebra, what is 100x - x? _____ (Hint: Subtract coefficients.) **99x**

b. Compute 100x - x another way: Line up the decimal points and subtract 0.727272727272... from 72.727272727272... Notice that if the decimals are lined up, the

"tails" at the end of both repeating decimals are exactly the same so it should be easy to subtract. Show the work below.

72.727272727272... - 0.727272727272... = 72

c. Write an equation so that the answer to part a equals the answer to part b. Then use algebra to solve for x. Show work.

99x = 72

d. Simplify the fraction you got for x in part c. Show work.

x = 72/99

e. Use a calculator to rewrite the fraction you got in part c as a decimal:

Is it equal to $0.\overline{72}$? **YES - 72/99 =** $0.\overline{72}$**.**

If so, you must have written $0.\overline{72}$ as the correct fraction!

Example $7.2.8$

a) $54{,}500{,}000{,}000 \times 346{,}000{,}000 = (5.45 \times 10^{10}) \times (3.46 \times 10^{8})$
$= (5.45 \times 3.46) \times 10^{10+8}$
$= 18.857 \times 10^{18}$
$= 1.8857 \times 10^{19}$

b) $1{,}200{,}000{,}000{,}000 \div 62{,}500{,}000 = (1.2 \times 10^{12}) \div (6.25 \times 10^{7})$
$= (1.2/6.25) \times 10^{12-7}$
$= 0.192 \times 10^{5}$
$= 1.92 \times 10^{4}$

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7.2.1: Written Retrieval and Problem Solving Practice

Student Learning Objectives

7.2 Students will compute with decimals and justify operations, including multiplying or dividing by powers of 10, using estimation techniques to evaluate sums, differences, products, and quotients effectively.

7.3 Students will analyze and convert between fractions, decimals, and percentages, distinguishing between terminating, repeating, and nonterminating decimal representations and classifying their corresponding fractions.

7.6 Students will use scientific notation to represent decimal numbers efficiently and interpret scientific notation results in mathematical and scientific contexts.

The mild assignment 🌶 is a shorter and less challenging assignment. The medium assignment🌶🌶 is right in the middle in terms of length and level of difficulty. Finally, the spicy assignment 🌶🌶🌶 is a longer assignment that is the most challenging.

(Spicy 🌶🌶🌶 Problems are OPTIONAL, but go for it when you have the time to challenge yourself)

Exercise 7.2.1.1 🌶🌶

Perform the following operations using the decimal algorithms and by also changing the decimal to fractions.

i. 38.52 + 9.251 ii. 534.51 – 48.67

Answer available at the end of this section.

Exercise 7.2.1.2 🌶🌶

Perform the following operations using the decimal algorithms and by also changing the decimal to fractions.

i. 5.23 x 0.034 ii. 8.272 ÷ 1.76

Answer available at the end of this section.

Exercise 7.2.1.3 🌶🌶

By thinking about the process of the division algorithm for decimals, determine which of the following division problems have the same quotient.

a) $52\overline{)2080}$
b) $0.052\overline{)0.208}$
c) $0.52\overline{)20.8}$

Answer available at the end of this section.

Exercise 7.2.1.4 🌶

Write each of the following in scientific notation.

a) 97,000

b) 1,000,000

c) 64,020,000

d) The longest human life on record was more than 122 years, or about 3,850,000,000 seconds. Express this number of seconds in scientific notation.

Answer available at the end of this section.

Exercise 7.2.1.6

Perform the following operations, and express answers in scientific notation.

a) $(2.3 \times 10^2) \times (3.5 \times 10^4)$

b) $(7.3 \times 10^3) \times (8.6 \times 10^6)$

Answer available at the end of this section.

Exercise 7.2.1.7

Find the following quotients, and express the answers in scientific notation.

a) $\frac{2.6 \times 10^{27}}{1.3 \times 10^3}$

b) $\frac{1.2 \times 10^{23}}{2.4 \times 10^{18}}$

Answer available at the end of this section.

Exercise 7.2.1.8

Write each of the following using a bar over the repetend.

a) 0.7777 . . . b) 0.181818 . . . c) ◈ 0.47121212 . . .

Answer available at the end of this section.

Exercise 7.2.1.9

Divide: $0.36\overline{)4.392}$

Answer available at the end of this section.

Exercise 7.2.1.10

Mulitply: (8.235)(2.3).

Answer available at the end of this section.

Exercise 7.2.1.11

Determine which of the following is equivalent to 0.383432432432...

a) 0.383432432

b) 0.383432432...

c) 0.3834323432...

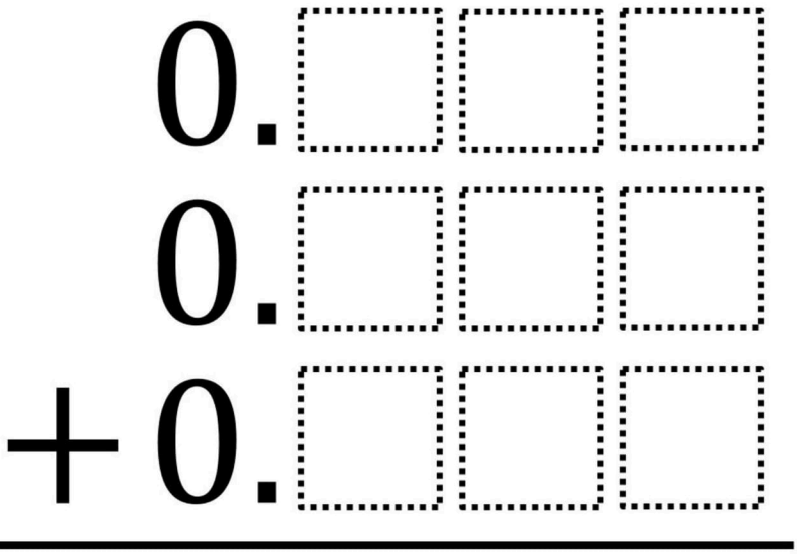
d) $0.38343\overline{3243}$

Answer available at the end of this section.

Problem Solving Problems

1) We know that 1/3 = 0.333333....., 2/3 = 0.666666..., so based on that pattern, what does 3/3 equal? Given the response based on patterns, some people argue that 0.999999 = 1. What do you think?

2) ***Adding Decimals to Make Them As Close to One as Possible** Directions: Using the digits 1 to 9 at most one time each, fill in the boxes to make three decimals whose sum is as close to 1 as possible.



Answer

There are many possible answers - including 0.125 + 0.376 + 0.498 = 0.999, 0.198 + 0.376 + 0.425 = 0.999

* This problem was graciously taken from the Open Middle Website.

3) ****Multiplying Decimals to Make a Whole Number Product** Directions: Using the digits 1 to 9 at most one time each, fill in the boxes to make a whole number product.



Answer

Number of Unique Solutions: 14 , Here are two: 1.6 x 3.75 = 6, 9.6 x 8.75 = 84

**This problem was graciously taken from the Open Middle Website.

Answers

Exercise 7.2.1.1

i) 38.520 $\frac{3852}{100}+\frac{9251}{1000}=\frac{38520}{1000}+\frac{9251}{1000}=\frac{47771}{1000}$
+ 9.251

47.771

ii) 534.51 $\frac{53451}{100}-\frac{4867}{100}=\frac{48584}{100}$
- 48.67

485.84

Exercise 7.2.1.2

i) 523×34=17782, 5.23×0.034=0.17782

i. $5.23\times 0.034=\frac{523}{100}\cdot\frac{34}{1000}=\frac{17782}{100000}=0.17782$

ii. $8.272\div 1.76=\frac{8272}{1000}\div\frac{176}{100}=\frac{8272}{1000}\cdot\frac{100}{176}=\frac{827200}{176000}=4.7$

Exercise 7.2.1.3

a) and c) have the same quotient.

Exercise 7.2.1.4

a) 9.7 x 10^4 b) 1.0 x 10^6 c) 6.402 x 10^7 d) 3.85 x 10^{10}

Exercise 7.2.1.6

a) $(2.3\times 10^2)\times(3.5\times 10^4)=8.05\times 10^6$

b) $(7.3\times 10^3)\times(8.6\times 10^6)=62.78\times 10^9=6.278\times 10^{10}$

Exercise 7.2.1.7

a) $\frac{2.6\times 10^{27}}{1.3\times 10^3}=2\times 10^{24}$

b) $\frac{1.2\times 10^{23}}{2.4\times 10^{18}}=0.5\times 10^5=5\times 10^4$

Exercise 7.2.1.8

a) 7/9 b) 18/99 c) 4665/9900 = 933/1980 (Hint: need to let x = 0.471212..., and use 10000x and 10x to find the fraction)

Exercise 7.2.1.9

Move the decimal in the divisor to the end of the divisor. Move the decimal in the dividend an equal number of places (two places) to the right.

$$0.36\overline{)4.39\,2}$$

Now we can follow the algorithm for dividing a decimal number by a whole number.

$$
\begin{array}{r}
12.2 \\
36\overline{)439.2} \\
36 \\
\hline
79 \\
72 \\
\hline
7\,2 \\
7\,2 \\
\hline
0
\end{array}
$$

Thus, 4.392 divided by 0.36 is 12.2.

Exercise 7.2.1.10🌶️

$$
\begin{array}{r}
8.235 \\
\times 2.3 \\
2\ 4705 \\
16\ 470 \\
\hline
18.9405
\end{array}
$$

The factor 8.235 has three digits to the right of the decimal point; the factor 2.3 has one digit to the right of the decimal point. Therefore, there must be a total of four digits to the right of the decimal point in the product or answer.

Exercise 7.2.1.11 🌶️

Only b) is equivalent.

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7.2.2: Online Retrieval and Problem Solving Practice

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7.2.3: AI Learning Prompts 7.2

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Decimal Operations & Fraction-Decimal Connections — SLOs 7.2 & 7.3

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💡 Direct Explanation | 🔍 Conceptual Learning | 🛠️ Problem-Solving

SECTION 1

Direct Explanation Prompts

Learning Objective: Understand what it means to compute with decimals using place value and scaling, how multiplying or dividing by powers of 10 shifts the decimal point and why, and how fractions and decimals are two representations of the same quantity — including why some fractions produce terminating decimals while others repeat.

1 **Type 1 — Just Explain It**

PROMPT 1 Copy

"Can you explain to me how multiplying and dividing by powers of 10 works with decimals? I don't just want the rule about moving the decimal — I want to understand why the decimal shifts and what's actually happening to the number's place value. Use plain language and a concrete example, and after you explain it, ask me if anything is unclear."

PROMPT 2 Copy

"Can you explain the difference between terminating and repeating decimals? I want to understand why some

fractions like 1/4 give a clean decimal but others like 1/3 go on forever. Cover what 'terminating' and 'repeating' actually mean, and what it has to do with the fraction's denominator. Use a couple of examples to make it clear, and then check in with me."

PROMPT 3 Copy

"Can you explain how to convert a repeating decimal back into a fraction? I want to understand the method — like how would I turn 0.121212… or 0.342342… into a fraction? Walk me through the algebra behind it in plain language, explain why we multiply by a power of 10 to line up the repeating parts, and then check in with me to make sure I followed."

2 Type 2 — Explain It and Connect It

PROMPT 4 Copy

"I understand place value with whole numbers — like how moving a digit one place to the left makes it 10 times bigger. Can you explain decimal operations in a way that connects to that? I want to see how the same place value logic I already know extends to the right side of the decimal point, especially when multiplying or dividing by 10, 100, or 1000."

PROMPT 5 Copy

"I already know how to do basic fraction-to-decimal conversion by dividing the numerator by the denominator. But I don't really understand why some of those divisions stop and some keep going. Can you explain it by connecting it to what I know about prime factorization? Help me see how the factors in the denominator determine whether the decimal terminates or repeats."

📋 Follow-Up Question Card — Decimal Operations & Fraction-Decimal Connections

Got the explanation — now keep going. Pick one of these moves:

Move	What to type
Say it differently	*"I'm still confused about why the decimal point moves when you multiply by a power of 10. Can you explain it a different way — maybe using a place value chart or a visual instead of words?"*

Move	What to type
Give me an example	*"Can you show me a specific fraction and walk me through how I'd figure out whether its decimal terminates or repeats — step by step with actual numbers?"*
Check me	*"Can you ask me a question to see if I really understand the difference between a terminating and a repeating decimal, and not just the definitions?"*
Go deeper	*"Why do only the prime factors 2 and 5 in the denominator lead to terminating decimals? What's special about 2 and 5 compared to other primes?"*

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7.3: Ratios and Proportions

The formal introduction and detailed study of ratios and proportional reasoning primarily occur later than the early elementary grades, specifically in middle school. However, early mathematics learning plays a crucial foundational role in preparing students for later understanding of ratios (CDE, 2013).

Ratios and proportions are essential concepts in the middle school mathematics curriculum, offering a powerful way to understand relationships between quantities. In everyday life, ratios appear all around us. For instance, a community might have 2 parks for every 5000 residents. In a classroom, the student-to-laptop ratio might be 3 to 2, meaning there are three students for every two laptops available. Further, Learning about ratios and proportions develops relational reasoning and thinking, the ability to compare two quantities and recognize invariant relationships. This form of mathematical thinking is foundational for advanced topics like rates of change, which are central to calculus.

Ratios

A ratio is a comparison of two quantities using division. Some ratios show a **part-to-part relationship**. For example, the ratio of left-handed students to right-handed students in a class could be 1 to 4. Other ratios represent **part-to-whole or whole-to-part relationships**. If there are 10 students who have dogs in the class of 25, then the ratio of students who have a dog (part) to all students (whole) is 10 to 25. Conversely, the ratio of all students (whole) to those who have a dog (part) would be 25 to 10. We will also see that this ratio can be stated as 2 to 5 or 5 to 2.

Ratio

A *ratio* is the quotient of two numerical quantities or two quantities with the same physical units written as an ordered pair of numbers.

There are a number of equivalent ways of expressing ratios, three of which we will use in this text: fraction notation, “to” notation, and “colon” notation.

- 3/4 is a ratio, read as “the ratio of 3 to 4.”
- 3 to 4 is a ratio, read as “the ratio of 3 to 4.”
- 3:4 is a ratio, read as “the ratio of 3 to 4.”
- In general, a/b, a to b, or a:b.

NOTE: Unlike fractions, there are instances of ratios in which b could be zero.

Example 7.3.1

Express each of the following ratios as a fraction reduced to lowest terms: a) 36 to 24, and b) 0.12 : 0.18.

Answer available at the end of this section.

Example 7.3.2

For the rectangle that follows, express the ratio of length to width as a fraction reduced to lowest terms.



Answer available at the end of this section.

Rates

We now introduce the concept of *rate*, a special type of ratio.

Rate

A *rate* is a quotient of two measurements with different units.

Example 7.3.3

An automobile travels 224 miles on 12 gallons of gasoline. Express the ratio distance traveled to gas consumption as a fraction reduced to lowest terms. Write a short sentence explaining the physical significance of your solution. Include units in your description.

Answer available at the end of this section.

Unit Rates

When making comparisons, it is helpful to have a rate in a form where the denominator is 1. Such rates are given a special name.

Unit Rate

A *unit rate* is a rate whose denominator is 1.

Example 7.3.4

a) Herman drives 120 miles in 4 hours. Find his average rate of speed.

b) Aditya works 8.5 hours and receives $95 for his efforts. What is his hourly salary rate?

Answer available at the end of this section.

Definition: Equality of Ratios

For any proportion of the form $\frac{a}{b} = \frac{c}{d}$, where $b \neq 0, d \neq 0$, its cross products are equal.

$a \cdot d = b \cdot c$

$\frac{a}{b} \times \frac{c}{d}$

Example 7.3.5

One automobile travels 422 miles on 15 gallons of gasoline. A second automobile travels 354 miles on 13 gallons of gasoline. Which automobile gets the better gas mileage?

Answer available at the end of this section.

Use the Definition of Proportion

When two ratios or rates are equal, the equation relating them is called a proportion.

Definition: Proportion

A proportion is an equation of the form $\frac{a}{b} = \frac{c}{d}$, where $b \neq 0, d \neq 0$.

The proportion states two ratios or rates are equal. The proportion is read "a is to b, as c is to d".

The equation $\frac{1}{2} = \frac{4}{8}$ is a proportion because the two fractions are equal. The proportion $\frac{1}{2} = \frac{4}{8}$ is read "1 is to 2 as 4 is to 8". If we compare quantities with units, we have to be sure we are comparing them in the right order.

To determine if a proportion is true, we find the **cross products** of each proportion. To find the cross products, we multiply each denominator with the opposite numerator (diagonally across the equal sign). The results are called a cross product because of the cross formed. If, and only if, the given proportion is true, that is, the two sides are equal, then the cross products of a proportion will be equal.

$8 \cdot 1 = 8 \quad 2 \cdot 4 = 8$

$\frac{1}{2} \times \frac{4}{8}$

$9 \cdot 2 = 18 \quad 3 \cdot 6 = 18$

$\frac{2}{3} \times \frac{6}{9}$

Example 7.3.6

Determine whether each relationship is a proportion:

a) 4:9 and 12:28

b) \$1.50 for 6 ounces is equivalent to \$2.25 for 9 ounces.

Answer available at the end of this section.

The strategy for solving applications that we have used earlier in this chapter, also works for proportions, since proportions are equations. When we set up the proportion, we must make sure the units are correct—the units in the numerators match and the units in the denominators match.

Example 7.3.7

Josiah went to Mexico for spring break and changed $325 dollars into Mexican pesos. At that time, the exchange rate had $1 U.S. is equal to 12.54 Mexican pesos. How many Mexican pesos did he get for his trip?

Answer

Identify what you are asked to find.	How many Mexican pesos did Josiah get?
Choose a variable to represent it.	Let $p =$ number of pesos.
Write a sentence that gives the information to find it.	If $1 U.S. is equal to 12.54 Mexican pesos, then $325 is how many pesos?
Translate into a proportion.	$\frac{\$}{\text{pesos}} = \frac{\$}{\text{pesos}}$
Substitute given values.	$\frac{1}{12.54} = \frac{325}{p}$
The variable is in the denominator, so find the cross products and set them equal.	$p \cdot 1 = 12.54(325)$
Simplify.	$c = 4{,}075.5$
Check if the answer is reasonable.	
Yes, $100 would be $1,254 pesos. $325 is a little more than 3 times this amount.	
Write a complete sentence.	Josiah has 4075.5 pesos for his spring break trip.

Example 7.3.8

In a scale drawing, 0.5 centimeter represents 35 miles.
a. How many miles will 4 centimeters represent?
b. How many centimeters will represent 420 miles?

Answer available at the end of this section.

Scaling is when one quantity changes in a consistent ratio with another. For fun, they some problems below.

Scaling

Wonder, Play, Grow

Solve the following problem in as many different ways as possible. Compare and contrast your methods with those of your peers.

Lena has a new smoothie recipe that uses a 117-ounce container of almond milk every 6.5 weeks. If she keeps making smoothies at the same rate, how many ounces of almond milk will she need for a full year?

Advanced Connections

Proportional reasoning prepares you to understand rates of change, such as velocity (change in distance over time) or slope of a function. In calculus, these ideas are formalized with derivatives. For instance, the concept of unit rate — "miles per hour" — becomes the instantaneous rate of change in calculus. Even related rates problems in calculus (e.g., how fast a shadow lengthens as someone walks) are grounded in the ability to set up ratios and proportions between changing quantities. Understanding ratios and proportions isn't just about solving everyday problems — it's about developing the habits of mind that support mathematical modeling and reasoning about change. These skills are foundational as you move toward algebra, functions, and eventually calculus, where the relationships between changing quantities are central.

Answers

Example 7.3.1

a) To express the ratio “36 to 24” as a fraction, place 36 over 24 and reduce.

$$\begin{aligned}\frac{36}{24} &= \frac{3\cdot 12}{2\cdot 12} && \text{Factor.}\\ &= \frac{3\cdot \cancel{12}}{2\cdot \cancel{12}} && \text{Cancel common factor.}\\ &= \frac{3}{2}\end{aligned}$$

Thus, the ratio 36 to 24 equals 3/2.(b) To express the ratio “0.12:0.18” as a fraction, place 0.12 over 0.18 and reduce.

$$\begin{aligned}\frac{0.12}{0.18} &= \frac{(0.18)(100)}{(0.18)(100)} && \text{Multiply numerator and denominator by 100.}\\ &= \frac{12}{18} && \text{Move each decimal 2 places right.}\\ &= \frac{2\cdot 6}{3\cdot 6} && \text{Factor.}\\ &= \frac{2\cdot \cancel{6}}{3\cdot \cancel{6}} && \text{Cancel.}\\ &= \frac{2}{3}\end{aligned}$$

Thus, the ratio 0.12:0.18 equals 2/3.

Example 7.3.2

The ratio length to width can be expressed as a fraction and reduced as follows.

$$\begin{aligned}\frac{\text{length}}{\text{width}} &= \frac{3\frac{1}{4}\text{ ft}}{2\frac{1}{2}\text{ ft}} && \text{Length to width as a fraction.}\\ &= \frac{3\frac{1}{4}\,\cancel{\text{ft}}}{2\frac{1}{2}\,\cancel{\text{ft}}} && \text{Cancel common units.}\\ &= \frac{\frac{13}{4}}{\frac{5}{2}} && \text{Mixed to improper fractions.}\end{aligned}$$

Invert and multiply, factor, and cancel common factors.

$$\begin{aligned}&= \frac{13}{4}\cdot\frac{2}{5} && \text{Invert and multiply.}\\ &= \frac{26}{20} && \text{Multiply numerators and denominators.}\\ &= \frac{13\cdot 2}{10\cdot 2} && \text{Factor numerator and denominator.}\\ &= \frac{13\cdot \cancel{2}}{10\cdot \cancel{2}} && \text{Cancel common factors.}\\ &= \frac{13}{10}\end{aligned}$$

Hence, the ratio length to width is 13/10.

Example 7.3.3

Place miles traveled over gallons of gasoline consumed and reduce.

$$\begin{aligned}\frac{224\text{ mi}}{12\text{ gal}} &= \frac{56\cdot 4\text{ mi}}{3\cdot 4\text{ gal}} && \text{Factor.}\\ &= \frac{56\cdot \cancel{4}\text{ mi}}{3\cdot \cancel{4}\text{ gal}} && \text{Cancel common factor.}\\ &= \frac{56\text{ mi}}{3\text{ gal}}\end{aligned}$$

Thus, the rate is 56 miles to 3 gallons of gasoline. In plain-speak, this means that the automobile travels 56 miles on 3 gallons of gasoline.

Example 7.3.4

a)

Place the distance traveled over the time it takes to drive that distance.

$$\frac{120 \text{ miles}}{4 \text{ hours}} = \frac{30 \text{ miles}}{1 \text{ hour}} \quad \text{Divide: } 120/4 = 30.$$
$$= 30 \text{ miles/hour}$$

Hence, Herman's average rate of speed is 30 miles per hour.

b)

Let's place money earned over hours worked to get the following rate:

$$\frac{95 \text{ dollars}}{8.5 \text{ hours}}$$

We will get a much better idea of Aditya's salary rate if we express the rate with a denominator of 1. To do so, divide.

$$\frac{95 \text{ dollars}}{8.5 \text{ hours}} = \frac{11.18 \text{ dollars}}{1 \text{ hour}} \quad \text{Divide: } 95/8.5 \approx 11.18.$$
$$= 11.18 \text{ dollars/hour.}$$

That is, his salary rate is 11.18 dollars per hour.

Example 7.3.5

Decimal division (rounded to the nearest tenth) reveals the better gas mileage.

In the case of the first automobile, we get the following rate:

$$\frac{422 \text{ mi}}{15\text{gal}}$$

In the case of the second autombile, we get the following rate:

$$\frac{354 \text{ mi}}{13 \text{ gal}}$$

In the case of the first automobile, the mileage rate is 28.1 mi/1 gal, which can be read "28.1 miles per gallon." In the case of the second automobile, the mileage rate is 27.2 mi/1 gal, which can be read "27.2 miles per gallon." Therefore, the first automobile gets the better gas mileage.

Example 7.3.6

a) To determine if the equation is a proportion, we find the cross products. If they are equal, the equation is a proportion.

$$\frac{4}{9} = \frac{12}{28} \qquad 28 \cdot 4 = 112,\ 9 \cdot 12 = 108 \qquad \frac{4}{9} \neq \frac{12}{28}$$

Since the cross products are not equal, $28 \cdot 4 \neq 9 \cdot 12$, $28 \cdot 4 \neq 9 \cdot 12$, the relatiobship is not a proportion.

b) $\frac{\$}{ounces} = \frac{\$}{ounces}$, $\frac{1.50}{6} = \frac{2.25}{9}$

$9 \cdot 1.50 = 6 \cdot 2.25$, \$1.50 for 6 ounces is equivalent to \$2.25 for 9 ounces.

Example 7.3.8

a) $\frac{0.5}{35} = \frac{4}{x}$. Solving, we obtain $x = \frac{35 \cdot 4}{0.5}$, or $x = 280$.

b) $\frac{0.5}{35} = \frac{y}{420}$, or $\frac{0.5 \times 420}{35} = y$. Therefore, $y = \frac{210}{35} = 6$ centimeters.

NOTE: We could have been solved mentally by using a technique called scaling up/scaling down, that is, by multiplying/dividing each number in a ratio by the same number.

$0.5 \text{ centimeter} : 35 \text{ miles} = 1 \text{ centimeter} : 70 \text{ miles}$
$= 2 \text{ centimeters} : 140 \text{ miles}$
$= 4 \text{ centimeters} : 280 \text{ miles.}$

See if you can do part b) using this technique.

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7.3.1: Written Retrieval and Problem Solving Practice

Student Learning Objectives

SLO 7.4 Students will solve problems involving ratios, rates, and proportions, comparing part-to-part and part-to-whole quantities, interpreting units, and applying strategies such as cross multiplication and scaling.

The mild assignment 🌶 is a shorter and less challenging assignment. The medium assignment🌶🌶 is right in the middle in terms of length and level of difficulty. Finally, the spicy assignment 🌶🌶🌶 is a longer assignment that is the most challenging.

(Spicy 🌶🌶🌶 Problems are OPTIONAL, but go for it when you have the time to challenge yourself)

Exercise 7.3.1.1 🌶

Write each ratio as a fraction:

a) 15 : 27

b) 45 : 18

Answer available at the end of this section.

Exercise 7.3.1.2 🌶🌶

The ratio of students to teachers at a particular high school is 100 : 6.

a) What is the ratio of teachers to students?
b) What fraction of the total number of students and teachers are teachers?
c) How many teachers are in the school?
d) How many students are in the school if there are 24 teachers?

Answer available at the end of this section.

Exercise 7.3.1.3 🌶

Convert each of the following into the corresponding unit rate.

a) 250 miles/11.6 gallons

b) 25 dollars/3.5 hours

c) 75 feet per 10 seconds

Answer available at the end of this section.

Exercise 7.3.1.4 🌶

a) Are the ratios 3 to 4 and 8:6 equal?

b) Are the ratios 7:1 and 4:81 equal?

Answer available at the end of this section.

Exercise 7.3.1.4

Alicia works 8 hours and makes $100. Connie works 10 hours and makes $122. Which woman works at the larger hourly rate?

Answer available at the end of this section.

Exercise 7.3.1.5

Solve each proportion for n.

a) $\frac{n}{70} = \frac{6}{21}$ b) $\frac{n}{84} = \frac{3}{14}$ c) $\frac{7}{n} = \frac{42}{48}$ d) $\frac{12}{n} = \frac{18}{45}$

Answer available at the end of this section.

Exercise 7.3.1.6

The average American produces about 1,600 pounds of paper trash per year (365 days). How many pounds of paper trash does the average American produce each day? (Round to the nearest tenth of a pound.)

Answer available at the end of this section.

Exercise 7.3.1.7

Kevin wants to keep his heart rate at 160 beats per minute while training. During his workout he counts 27 beats in 10 seconds. How many beats per minute is this? Has Kevin met his target heart rate?

Answer available at the end of this section.

Exercise 7.3.1.8

A recipe making 2 dozen cookies requires $1\frac{3}{4}$ cups of flour, among other ingredients. If the baker wishes to make twice that number of cookies, how much flour is required?

Answer available at the end of this section.

Exercise 7.3.1.9

Paul is shopping for laundry detergent. At the grocery store, the liquid detergent is priced at $\$14.99$ for 64 loads of laundry and the same brand of powder detergent is priced at $\$15.99$ for 80 loads.

Which detergent has the lowest cost per load?

Answer available at the end of this section.

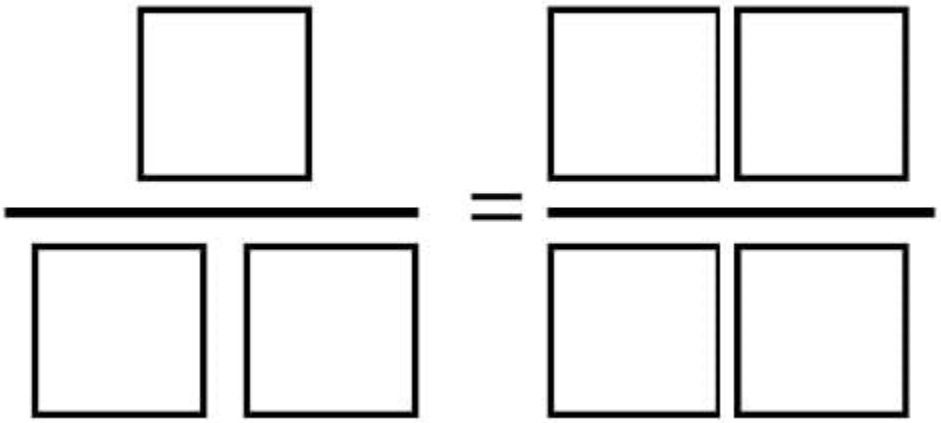
Problem Solving Problems

1) Dan runs a small eco-friendly cleaning business and uses a 117-ounce bottle of liquid detergent every 6.5 weeks. He wants to

switch to a subscription service that ships him a year's supply all at once to save on packaging. If the subscription comes in 64-ounce bottles, how many bottles should Dan order to last him the entire year?

2) Monica and Bianca went to a flower shop to buy some roses. Bianca bought a bouquet with 5 pink roses. Monica bought a bouquet with two dozen roses, some red and some yellow. She has 3 red roses in her bouquet for every 5 yellow roses. How many red roses are in Monica's bouquet?

3) **Equivalent Ratios 2** Using each of the digits 0-6 only once, make two equivalent ratios (also known as a proportion).



Answer

Number of Unique Solutions: 6

1: 4/13 = 20/65
2: 4/20 = 13/65
3: 5/10 = 23/46
4: 5/10 = 32/64
5: 5/23 = 10/46
6: 5/32 = 10/64

Problem Source: Open Middle

4) **A Golden Crown ?**

The King asks Archimedes if his crown is made from pure gold.

- He knows that the crown is either pure gold or it may have some silver in it.
- Archimedes figures out that the volume of the crown is 125 cm^3 and that its mass is 1.8 kilograms.
- He also knows that 1 kilogram of gold has a volume of about 50 cm^3 and 1 kilogram of silver has a volume of about 100 cm^3.

i) Is the crown pure gold? Explain how you know.

ii) If the crown is not pure gold, then how much silver is in it? Show all your work.

Problem Source: MAP Math Assessment Project

Answers

Exercise 7.3.1.1🌶️
a) 15 to 27, 15/27, 5/9
b) 45 to 18, 45/18. 5/2

Exercise 7.3.1.2🌶️🌶️
a) 6:100 or 3:50
b) 6/106
c) Cannot determine.
d) Use cross multiplication or scaling up process: 400 students

Exercise 7.3.1.3

a) 21.6 miles per gallon b) 7.1 dollars per hour c) 7.5 feet per second

Exercise 7.3.1.4

a) Not equal.

$\frac{3}{4} \neq \frac{8}{6}$

$\frac{8}{6} = \frac{4}{3}$

b) Not Equal

$\frac{7}{1} = 7 \quad \text{and} \quad \frac{4}{81} \approx 0.0494$

$\therefore 7 : 1 \neq 4 : 81$

Exercise 7.3.1.4

Alicia's rate: $\frac{100}{8} = 12.50$ dollars/hour

Connie's rate: $\frac{122}{10} = 12.20$ dollars/hour

$\therefore$ Alicia has the higher hourly rate.

Exercise 7.3.1.5

a) n=20 b) n=18 c) n=8 d) n=30

Exercise 7.3.1.6

$\frac{1600 \text{ pounds}}{365 \text{ days}} \approx 4.3836$ pounds/day

Rounded to the nearest tenth: $\boxed{4.4 \text{ pounds per day}}$

Exercise 7.3.1.7

Kevin counts 27 beats in 10 seconds

To find beats per minute: $27 \times 6 = 162$ bpm

Since $162 > 160$, Kevin has met his target heart rate.

Exercise 7.3.1.8

Twice 2 dozen is 4 dozen cookies. Let *x* represent the amount of flour needed for 4 dozen cookies. Assuming an equal rate for 2 dozen cookies (2 dozen requires 1 3 4 cups of flour), we set up the following proportion. Again, using units helps us craft the correct proportion.

$$\frac{2 \text{ dozen}}{1\frac{3}{4} \text{ cups}} = \frac{4 \text{ dozen}}{x \text{ cups}}$$

Note how including the units aids in the setup of the proportion. Now, let's drop the units and solve for *x*.

$$\frac{2}{1\frac{3}{4}} = \frac{4}{x} \quad \text{Original proportion.}$$

$$2x = 1\frac{3}{4} \cdot 4 \quad \text{Cross multiply.}$$

$$2x = \frac{7}{4} \cdot 4 \quad \text{Change to improper fraction.}$$

$$2x = 7 \quad \text{Multiply.}$$

Divide both sides of the equation by 2 and finish.

$$\frac{2x}{2} = \frac{7}{2} \quad \text{Divide both sides by 2.}$$

$$x = \frac{7}{2}$$

Change the improper fraction to a mixed fraction. Thus, it will take $3\frac{1}{2}$ cups of flour to make 4 dozen cookies.

Exercise 7.3.1.9

To compare the prices, we first find the unit price for each type of detergent.

	Liquid	Powder
Write as a rate.	$\frac{\$14.99}{64 \text{ loads}}$	$\frac{\$15.99}{80 \text{ loads}}$
Find the unit price.	$\frac{\$0.234...}{1 \text{ load}}$	$\frac{\$0.199...}{1 \text{ load}}$
Round to the nearest cent.	$0.23/load (23 cents per load.)	$0.20/load (20 cents per load)

Now we compare the unit prices. The unit price of the liquid detergent is about $0.23 per load and the unit price of the powder detergent is about $0.20 per load. The powder is the better buy.

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7.3.3: AI Learning Prompts 7.3

MATH 204

AI Prompt Library — Section 7.3

Ratios, Rates, and Proportions

Choose a tab below based on how you want to learn. Copy any prompt and paste it into an AI chatbot to start a learning conversation. Each section ends with a Follow-Up Question Card to help you keep the conversation going.

Section 1: Explain It | Section 2: Explore It | Section 3: Solve It

Learning Objective: Understand what ratios, rates, and proportions are — including the difference between part-to-part and part-to-whole comparisons, how units work in rates, and why cross multiplication is a valid strategy (not just a shortcut trick) for solving proportions.

TYPE 1 Just Explain It

For when you want a clear, direct explanation before you try anything.

PROMPT 1 Copy

"Can you explain the difference between a ratio, a rate, and a proportion in plain language? Cover what each term means, how they're related, and give me a simple real-world example of each. After you explain it, ask me if anything is unclear so we can talk it through."

PROMPT 2 Copy

"I'm learning about ratios and I keep getting confused about part-to-part versus part-to-whole. Can you explain the difference clearly — what each one means, when you'd use each one, and why it matters that they're not the same thing? Use a classroom example, like comparing boys to girls in a class, and after you explain, check if I have questions."

TYPE 2 Explain It and Connect It

For when you want the new idea connected to something you already know.

PROMPT 3 Copy

"I already understand fractions pretty well. Can you explain ratios and proportions in a way that connects to what I already know about fractions? I want to see how ratios are like fractions but also how they're different — especially since ratios can compare part-to-part, not just part-to-whole. Help me see the connection before going into the full explanation."

PROMPT 4

Copy

"I know how to multiply both sides of an equation by the same number. Can you use that to explain why cross multiplication works in proportions? I don't want to just memorize the trick — I want to understand why it's mathematically valid. Connect it back to equivalent ratios and what it means for two fractions to be equal."

📋 Follow-Up Question Card — Ratios and Proportions

Got the explanation — now keep going. Pick one of these moves:

Move	What to type
Say it differently	"I'm still fuzzy on part-to-part vs. part-to-whole. Can you explain the difference using a completely different example — maybe a recipe or a sports team instead of a classroom?"
Give me an example	"Can you show me a real-world situation where I'd set up a proportion and solve it step by step, using a unit pricing scenario like comparing two brands at the grocery store?"
Check me	"Can you ask me a question to see if I understand the difference between a ratio and a rate — and whether I can identify which is which in a word problem?"
Go deeper	"Why do teachers say you shouldn't just teach cross multiplication as a trick? What goes wrong when students memorize the procedure without understanding equivalent ratios?"

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7.4: Percentages

Motivating young learners to engage with percentages is essential because percentages are all around them—on sales tags, test scores, nutrition labels, and sports statistics. When students see how percentages connect to their daily lives and future aspirations, the math becomes relevant and meaningful. Effective teachers build on this relevance by designing lessons that are engaging and connected to students' interests, helping them experience success and joy in learning (NCTM, 2020).

Percents as Decimals and Fractions

Percent (from the Latin word per centum meaning "per hundred") is denoted by n% = $\frac{n}{100}$ where the symbol % indicates percent. The "cent" part of the word percent means "hundred". It is important to know that percents are an alternative representation for fractions and decimals; and be able to convert between all forms, Figure 7.4.1.

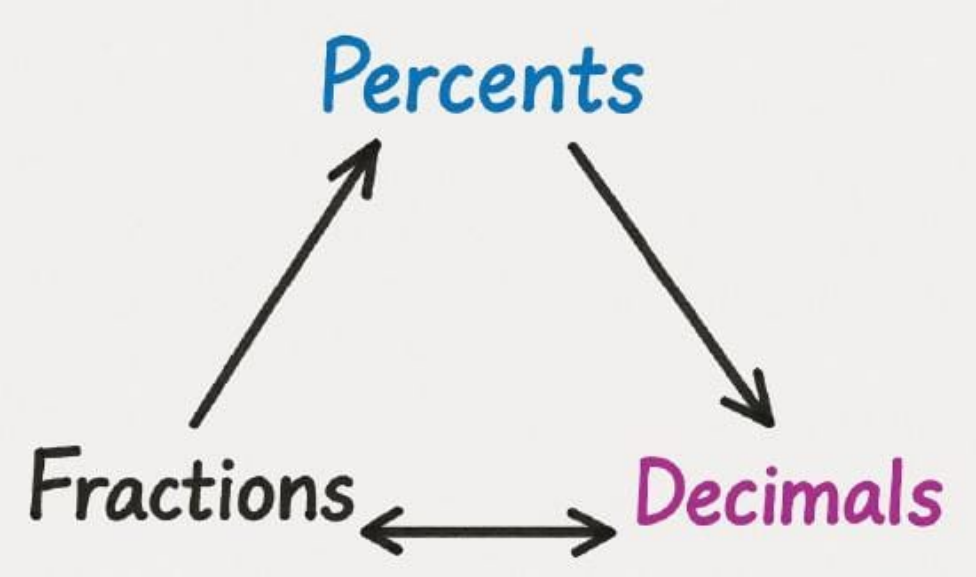


Figure 7.4.1:Percents are alternative representations of fractions and decimals

Case 1: Percents to Fractions

Use the definition of percent. For example, 23% = $\frac{23}{100}$ by the meaning of percent.

Case 2: Percents to Decimals

To convert a percent directly to a decimal, you drop the % symbol and move the number's decimal point two places to the left. This shortcut works because dividing a number by 100 (as indicated by the "per hundred" definition) is equivalent to moving the decimal point two places to the left.

23% = 0.23, 135% = 1.35, 0.56% = 0.0056

Case 3: Decimals to Percents

To convert a decimal to a percent, you move the decimal point two places to the right and then add the % sign to the resulting number. This process essentially reverses the steps used to convert a percent to a decimal (case 2), where the decimal point is moved two places to the left. Also, fractions can be converted to decimals (using a calculator or long division), and then case 3

can be applied.

Case 4: Fractions to Percents

Some fractions that have terminating decimals can be converted to percents by expressing the fraction with a denominator of 100.

Example 7.4.1

a) Write $\frac{3}{5}$ as a decimal and percent.

b) Write 0.52 as a fraction and percent.

c) Write 216% as a decimal and fraction.

Answer available at the end of this section.

Solving Percent Problems

Solving percent problems involves understanding the "per hundred" concept and its connection to fractions and decimals, recognizing the common problem types, and employing various methods including grids, proportions, equations, and mental math strategies. Effective teaching supports students in developing conceptual understanding, connects problems to relevant contexts, addresses language and context barriers, and encourages exploration of multiple solution strategies.

Percent Equations As Proportions

Some people prefer to solve percent equations by using the proportion method. The proportion method for solving percent problems involves a percent proportion. A **percent proportion** is an equation where a percent is equal to an equivalent ratio.

For example, $60\% = \frac{60}{100}$ and we can simplify $\frac{60}{100} = \frac{3}{5}$. Since the equation $\frac{60}{100} = \frac{3}{5}$ shows a percent equal to an equivalent ratio, we call it a percent proportion.

$$\frac{amount}{base} = \frac{percent}{100}$$

$$\frac{3}{5} = \frac{60}{100}$$

Percent Proportion

Translate to a proportion. What number is 75% of 90 ?

$$\underbrace{\text{What number}}_{\text{amount}} \text{ is } \underbrace{75\%}_{\text{percent}} \text{ of } \underbrace{90?}_{\text{base}}$$

$$\frac{n}{90} = \frac{75}{100}$$

Example 7.4.2

Translate to a proportion.

a) What number is 40% of 85?

b) 19 is 25% of what number?

c) What percent of 27 is 9 ?

Answer available at the end of this section.

We can use a visual representation to help us solve percent problems using bar drawings and proportions, start by representing the whole amount as a bar divided into sections representing percentages. Then, use the given information to determine the value of each section Figure 7.4.2. Technically, we can use the drawings to set up equivalent proportions to solve for n. For example, we can write $\frac{n}{85}=\frac{40}{100}$, or $\frac{85}{n}=\frac{100}{40}$, or $\frac{n}{40}=\frac{85}{100}$, or $\frac{40}{n}=\frac{100}{85}$. This flexibility the drawing provides may make it easier to set up and solve a problem.

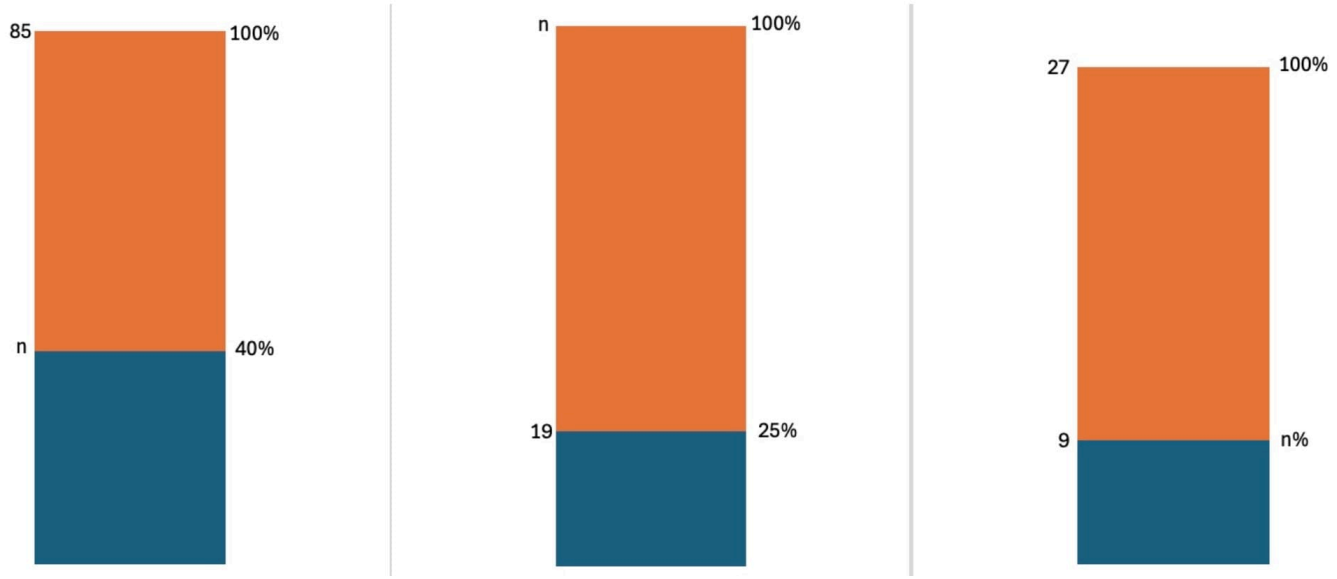


Figure 7.4.2: Visual Represenations for Percents as Proportions, $\frac{n}{85}=\frac{40}{100}$, $\frac{19}{n}=\frac{25}{100}$, $\frac{9}{27}=\frac{n}{100}$

Now that we have written percent equations as proportions and represented in a visual, we are ready to solve the equations.

Example 7.4.3

a) 125% of 25 is what number?

b) 6.5% of what number is $1.56 ?

c) What percent of 72 is 9 ?

Answer available at the end of this section.

Algebra Approach to Percent Equations

The following equations illustrate these three forms, where x represents an unknown and takes the place of percent, base, or amount depending on what is given and what is unknown in the problem.

Equation Approach

$$\begin{array}{lll} \text{(a)} & p\% \text{ of } n \text{ is } x & \left(\frac{p}{100}\right) n = x \\ \text{(b)} & p\% \text{ of } x \text{ is } a & \left(\frac{p}{100}\right) x = a \\ \text{(c)} & x\% \text{ of } n \text{ is } a & \left(\frac{x}{100}\right) n = a \end{array}$$

Once we have obtained one of these three equations, the solution, x, can be found. In equation (a), we multiply p/100 and n. In equations (b) and (c) we solve for the missing factor x.

Example 7.4.4

Use the equation approach to solve.

a) 125% of 25 is what number?

b) 6.5% of what number is $1.56 ?

c) What percent of 72 is 9 ?

Answer available at the end of this section.

Now that we have the strategies to refer to, and have practiced solving basic percent equations, we are ready to solve percent applications. Be sure to ask yourself if your final answer makes sense—since many of the applications we'll solve involve everyday situations, you can rely on your own experience.

Example 7.4.5

Dejohn and his girlfriend enjoyed a dinner at a restaurant, and the bill was $68.50. They want to leave an 18% tip. If the tip will be 18% of the total bill, how much should the tip be?

Answer available at the end of this section.

Example 7.4.6

The label on Masao's breakfast cereal said that one serving of cereal provides 85 milligrams (mg) of potassium, which is 2% of the recommended daily amount. What is the total recommended daily amount of potassium?

Nutrition Facts

Serving Size: 1 cup (47g)
Servings Per Container: About 7

Amount Per Serving	Cereal	With Milk
Calories	180	230
Calories from Fat	10	20
	% Daily Value*	
Total Fat 1g	**2%**	**2%**
Saturated Fat 0g	**0%**	**0%**
Trans Fat 0g		
Polyunsaturated Fat 0.5g		
Monounsaturated Fat 0.5g		
Cholesterol 0mg	**0%**	**0%**
Sodium 190mg	**8%**	**11%**
Potassium 85mg	**2%**	**8%**
Total Carbohydrate 40g	**13%**	**15%**
Dietary Fiber 1g	**4%**	**4%**
Sugars 8g		
Protein 3g		

Answer available at the end of this section.

Example 7.4.7

Mitzi received some gourmet brownies as a gift. The wrapper said each brownie was 480 calories, and had 240 calories of fat. What percent of the total calories in each brownie comes from fat?

Answer available at the end of this section.

Wonder, Play, Grow

Work with your peers to complete your assigned percentage task. It may involve some creative thinking and multiple steps to complete, but you got this. Just start playing with the numbers and see what works and doesn't work.

Option 1: Concert tickets originally cost $100. Resellers list them for $250. Additionally, resale sites charge a 15% service fee on the sale price.

- What is the total price a buyer pays after the service fee?
- What is the percentage increase from the original $100 price to the final price?
- How much profit (as a percent) does the reseller make before fees?

Option 2: A refrigerator was on sale at the appliance store for 20% off. As a loyalty member, Marcus received a coupon from the store for an additional 30% off any current price in the store. If he uses the coupon to buy the refrigerator, the price would be $487.20 before taxes. What was the original price?

Option 3:

A workers' union negotiates a 3-year contract with these annual raises:

- 3% raise in Year 1
- 4% raise in Year 2
- 5% raise in Year 3

The base salary at the start of Year 1 is **$50,000**.

- What will the worker's salary be at the end of Year 3?
- What is the total percent increase from the original $50,000 salary?
- Why is the total increase not equal to 12%?

Do not look at the answers until you have some agreed upon solutions that seem reasonable.

Answer available at the end of this section.

Mental Math and Estimation Using Fraction Equivalents

Since many common percentages are easily expressed as simple fractions, it's often faster to calculate percentages mentally by using their fraction equivalents (see Figure 7.4.4). Just like with proportions, you can also estimate percentages by choosing fractions that are close and easy to work with.

5%	$\frac{1}{20}$
10%	$\frac{1}{10}$
20%	$\frac{1}{5}$
25%	$\frac{1}{4}$
$33\frac{1}{3}\%$	$\frac{1}{3}$
50%	$\frac{1}{2}$
$66\frac{2}{3}\%$	$\frac{2}{3}$
75%	$\frac{3}{4}$

Figure 7.4.4: Fraction Percent Equivalents

Example 7.4.8

Find the following percents mentally, using fraction equivalents.

a) 25% of 44

b) 75% of 24

c) 38% of 50

d) 35% of 20

Answer available at the end of this section.

Example 7.4.9

Estimate the following percents mentally, using fraction equivalents.

a) 48% of 73

b) 32% of 95

c) 123% of 59

d) 81% of 24

Answer available at the end of this section.

Before doing from retrieval practice below, let's learn more from Dr. Howie Hua.

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Mental Math Retrieval Practice - Percentages

Use what you know about percentages and fractions to help you calculate each percentage.

Answers

Example 7.4.1

a) 3/5=60/100, 0.60, 60% b) 52/100=13/25. 52% c) 216/100=54/25, 2.16

Example 7.4.2

a) $\frac{n}{85}=\frac{40}{100}$

100n = 3400

n=34

b) $\frac{19}{n}=\frac{25}{100}$

1900 = 25n

n=76

c) $\frac{9}{27}=\frac{n}{100}$

900=27n

n≈33.33

Example 7.4.3

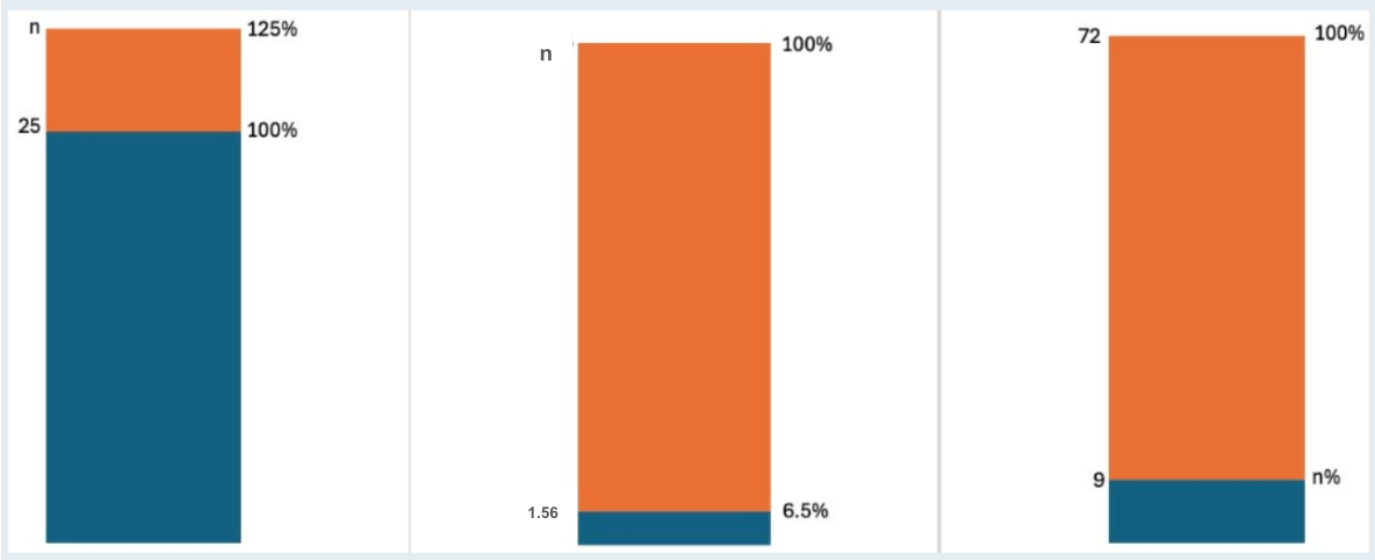


Figure 7.4.3: Visual for each problem.

125% is 25 of what number ?

percent base amount

a)

$\frac{n}{25}=\frac{125}{100}$

100n=3125

n=31.25

125% of 25 is 31.25.

6.5% of what number is $1.56?

percent base amount

b)

$\frac{n}{1.56}=\frac{100}{6.5}$

n=(1.56x100)/6.5

n=24

6.5% of $24 is $1.56.

What percent of 72 is 9?

percent base amount

c)

$\frac{9}{72}=\frac{n}{100}$

n=900/72=12.5

12.5% of 72 is 9.

Example 7.4.4

a) 1.25x25=31.25

b) 0.065x = 1.56, x = 1.56/0.065=24

c) $\frac{x}{100}$ ·72=9, x = (900/72) = 12.5

Example 7.4.5

The tip is 18% of $68.50.

$t = 0.18 . \$68.50$

t=12.33

If we approximate the bill to $70 and the percent to 20%, we would have a tip of $14. So a tip of $12.33 seems reasonable.

NOTE: you can also use other methods to calculate the tip. 18/100=n/68.50

Example 7.4.6

$$\underbrace{85\text{ mg}}_{85} \quad \underbrace{\text{is}}_{=} \quad \underbrace{2\%}_{0.02} \quad \underbrace{\text{of}}_{\cdot} \quad \underbrace{a}_{a}\,?$$

$$\frac{85}{0.02} = \frac{0.02a}{0.02}$$

a=4250,

NOTE: you can also use other methods to calculate the amount. 2/100=85/n, OR 1% = 42.5mg, so 100%=4250 mg

Example 7.4.7

$$\underbrace{\text{What percent}}_{p} \quad \underbrace{\text{of}}_{\cdot} \quad \underbrace{480}_{480} \quad \underbrace{\text{is}}_{=} \quad \underbrace{240?}_{240}$$

$$\frac{p \cdot 480}{480} = \frac{240}{480}$$

p=0.5, or 50%

NOTE: you can also use other methods to calculate the percentage. n/100 = 240/480

Option 1: $287.50, 187.5%, 150% Option 2: $870 Option 3: $56238, 12.476%, Year 2's 4% applies to the new, higher salary after Year 1's raise. Year 3's 5% applies to an even higher salary, percent increases compound, they build on a new amount.

Example 7.4.8

a) 1/4 x 44 = 11 b) 3/4 x 24 = 18 c) 38% of 50 is the same as 50% of 38, 1/2 x 38 = 19 d) 35% of 20 is the same as 20% of 35, 1.5 x 35 = 7

Example 7.4.9

a) 50% of 72 is 36 b) 33 1/3% of 96 is 32 c) 100% of 60 and 25% of 60 is 60 + 15, 75 d) 25% of 80 is 20

*You may have other and better ways to estimate. :)

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7.4.1: Written Retrieval and Problem Solving Practice

Student Learning Objectives

7.5 **Students will apply percentages to solve real-world problems by representing n% on grids, converting between percentages and fractions or decimals, and solving percent problems using equations, proportions, and calculators.**

The mild assignment 🌶 is a shorter and less challenging assignment. The medium assignment🌶🌶 is right in the middle in terms of length and level of difficulty. Finally, the spicy assignment 🌶🌶🌶 is a longer assignment that is the most challenging.

(Spicy 🌶🌶🌶 Problems are OPTIONAL, but go for it when you have the time to challenge yourself)

Exercise 7.4.1.1 🌶🌶

Complete the chart.

FRACTION	DECIMAL	PERCENT
_____	_____	50%
_____	0.35	_____
$\frac{1}{4}$	_____	_____
$\frac{1}{8}$	_____	_____
_____	0.0125	_____
_____	_____	125%
_____	0.75	_____

Answer available at the end of this section.

Exercise 7.4.1.2 🌶

Write each of the following in all three forms: decimal, percent, and fraction (in simplest form).

a) 66%

b) 0.28

c) $\frac{1}{8}$

Answer available at the end of this section.

Exercise 7.4.1.3

Mentally calculate each of the following.

a) 10% of 46.8

b) 5% of 32.8

c) 15% of 32.8

Answer available at the end of this section.

Exercise 7.4.1.4

Mentally calculate each of the following.

a) 92 is 50% of _____.

b) 20 is 25% of _____.

c) 22% of 50 is _____.

d) 12% of 75 is _____.

Answer available at the end of this section.

Exercise 7.4.1.5

Mentally find the following percents using fraction equivalents.

a) 25% of 64

b) 75% of 44

c) 20% of 520

d) 60% of 450

Answer available at the end of this section.

Exercise 7.4.1.6

Estimate.

a) 39% of 81

b) 58.9% of 29

c) 9.4% of 315

d) 5.1% of 239

Answer available at the end of this section.

Exercise 7.4.1.7

a) Translate and solve using proportions: 36 is 75% of what number?

b) Translate and solve using proportions: 9.5% of what number is $1.17 ?

Answer available at the end of this section.

Exercise 7.4.1.8

Answer the following and round to one decimal place.

a) Find 1/2% of 140.

b) Find 32.7% of 160.

c) Find 3.5% of 40.

d) What percent of 24 is 11?

e) 120 is 20% of what number?

f) 125% of a number is 1.1. Find the number.

Answer available at the end of this section.

Exercise 7.4.1.9

A car was purchased for $17,120 including a 7% sales tax. What was the price of the car before tax?

Answer available at the end of this section.

Exercise 7.4.1.10

At the campus coffee cart, a medium coffee costs $1.65 . MaryAnne brings $2.00 with her when she buys a cup of coffee and leaves the change as a tip. What percent tip does she leave?

Answer available at the end of this section.

Exercise 7.4.1.11

Complete the table.

Fraction	Decimal	Percent
$\frac{1}{2}$		
	0.45	
		18%
$\frac{1}{3}$		
	0.008	
2		

Answer available at the end of this section.

Exercise 7.4.1.12

A refrigerator and range were purchased and a 5% sales tax was added to the purchase price. If the total bill was $834.75, how much did the refrigerator and range cost?

Answer available at the end of this section.

Problem Solving Problems

1) The average price of a gallon of gas in one city in June 2024 was $3.71 . $3.71 . The average price in that city in July was $3.64. Find the percent decrease.

HINT: Finding the percent decrease is very similar to finding the percent increase, but now the amount of decrease is the difference between the original amount and the final amount. Then we find what percent the amount of decrease is of the original amount.

2) In 2025, the governor proposed raising community college fees from $44 per unit to $50 per unit. Find the percent increase. (Round to the nearest tenth of a percent.)

HINT: To find the percent increase, first we find the amount of increase, which is the difference between the new amount and the original amount. Then we find what percent the amount of increase is of the original amount.

3) A pair of slacks was made of material that was expected to shrink 10%. If the manufacturer makes the 34-inch inseam of the slacks 10% longer, what will the inseam measure after shrinkage?

4) Which results in a higher price: a 10% markup followed by a 10% discount, or a 10% discount followed by a 10% markup?

Answers

Exercise 7.4.1.1

FRACTION	DECIMAL	PERCENT
1/2	0.5	50%
7/20	0.35	35%
1/4	0.25	25%
1/8	0.125	12.5%
1/80	0.0125	1.25%
5/4	1.25	125%

FRACTION	DECIMAL	PERCENT
3/4	0.75	75%

Exercise 7.4.1.2
a) 33/50, 0.66
b) 7/25, 28%
c) 0.125, 12.5%

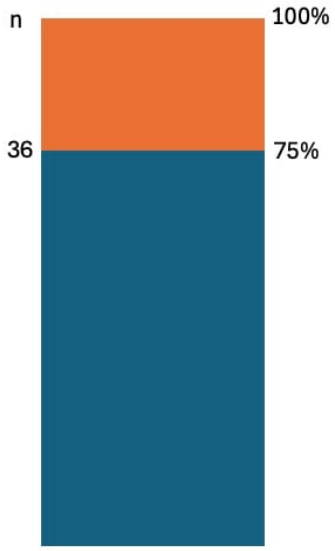
Exercise 7.4.1.3
a) 4.68
b) 2.34
c) 7.02 (4.68 + 2.34)

Exercise 7.4.1.4
a) 184 b) 80 c) 11 d) 9

Exercise 7.4.1.5
a) 16 b) 33 c) 104 d) 270

Exercise 7.4.1.6
a) 40% of 80 is 32
b) 60% of 30 is 18
c) 10% of 315 is 31.5
d) 5% of 240 is 12

Exercise 7.4.1.7
a) 36 is 75% of **48**.



b) $1.17 is 9.5% of **$12.32**.

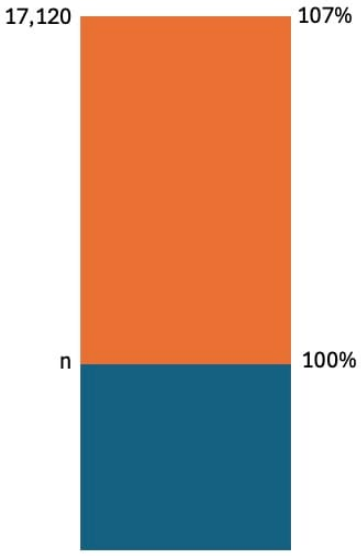


Exercise 7.4.1.8🌶🌶

a) 1400 x 0.005 = 0.7 b) 160 x 0.327 = 52.32
c) 40 x 0.035 = 1.4 d) 11/24 = 0.458, 45.8%
e) 120/0.20 = 600 f) 1.1/1.25 = 0.88

Exercise 7.4.1.9🌶🌶

The price of the car before tax was **$16,000**.



Exercise 7.4.1.10🌶🌶

Exercise 7.4.1.11🌶🌶

Fraction	Decimal	Percent
1/2	0.5	50%
9/20	0.45	45%
9/50	0.18	18%
1/3	0.333...	33 1/3%
1/125	0.008	0.8%
2/1	2	200%

Exercise 7.4.1.12🌶🌶

$\frac{x}{834.75}=\frac{100}{105}$ or x+0.05x=834.75

The refrigerator and range **cost $795 before tax**.

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7.5: Chapter 7 Resources

Chapter Example Videos

Chapter 7.1

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Chapter 7.4

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Chapter Extras

Chapter Bibliography

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8: INTEGERS

As future elementary school teachers, it's important to understand that many students struggle with integer operations because these concepts challenge their existing ideas about how numbers work. For children who have learned that larger numbers always mean more, the introduction of negative numbers—values less than zero—can feel abstract and even nonsensical. Adding to the challenge, students often confuse the rules for adding and subtracting integers, especially when subtracting negatives, which may seem illogical without a solid conceptual foundation. Multiplying and dividing with negative numbers brings further difficulty, as the idea that two negatives make a positive can seem arbitrary. Unlike whole number operations, which often come with clear, concrete models, integer operations can be harder to visualize and internalize. That's why it's essential to use a variety of teaching strategies—like manipulatives, real-world analogies, games, and digital tools—to help students build a deeper and more intuitive understanding. These approaches allow each child to engage with the material in a way that makes sense to them, transforming what can be a confusing topic into one that is meaningful, memorable, and even fun.

Chapter Learning Objectives

Student Learning Objectives

SLO 8.1 Students will represent integers using models such as the integer number line and colored chips, and demonstrate addition, subtraction, and ordering of integers using these tools.

SLO 8.2 Students will analyze and apply the properties of integer operations, including addition, subtraction, multiplication, and division, to solve mathematical problems and verify relationships such as closure, commutativity, and distributivity.

SLO 8.3 Students will explain integer multiplication and division through repeated addition, patterns, and missing-factor approaches, and compute using negative exponents and scientific notation.

SLO 8.4 Students will evaluate relationships between integers, applying the properties of inequalities (e.g., transitivity, multiplication by a positive or negative number) and theorems such as additive cancellation and opposite of the opposite

Essential Questions

- How do negative numbers extend the number system to make subtraction and other operations more complete and meaningful in real-world contexts?
- In what ways do the properties of integer operations lay the groundwork for understanding algebraic reasoning and solving equations?"

Learning That Transfers

Students will be able to explain how extending the number system to include integers supports flexible problem-solving in both real-world and mathematical contexts, and apply the properties of integer operations to model and solve algebraic equations.

From Plus to Minus, the Golf Ball Reminds Us

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NEED AUDIO FOR STORY

From Plus to Minus, the Golf Ball Reminds Us

One sunny day, with the breeze nice and cool,
Morgan and Grandpa set off with a tool—
A putter, some balls, and a pencil in hand,
To play mini golf at "The Green Wacky Land."

“Today,” Grandpa said, “we’ll keep score the right way,
With numbers that rise, and numbers that stray.
You’ll see how the numbers can go up and down,
Not just above zero, but also below ground!”

The first hole was tricky, with curves left and right,
Morgan got through with two strokes—what a sight!
Grandpa took four, with a laugh and a grin,
“Remember in golf, the low score will win!”

“So your score is two, and mine is four high,
But to show who is winning, let’s give this a try:
We’ll subtract my four strokes from yours—so neat!
That’s two minus four… hmm, that’s negative two, sweet!”

“Negative two? What’s that mean?” Morgan frowned,
“Are we digging below the score into ground?”
Grandpa just chuckled, “It’s numbers that dive,
They’re just as important to keep math alive!”

“Positive numbers go up—high and tall,
But negative numbers go down, just like fall.
They help us when scores need to measure below,
When you’re ‘under par’—it’s good math to know!”

At Hole Number Six, things started to swing,
Morgan made a birdie—*one under—zing!*
“That’s minus one, since you beat par by a bit,
A negative score shows you’re winning, that’s it!”

Grandpa had a bogey, one over, poor guy,
“Plus one for me,” he said with a sigh.
“Let’s show it on the line—see this chart here?
It helps us keep track, makes it all very clear.”

They counted up scores with each hole they played,
Positive, negative, swaps that they made.
Adding negatives pulled scores back to small,
Subtracting a negative? That's adding, after all!

Morgan grinned wide, "So subtracting your loss,
Is like giving me points? Well, I'll be the boss!"
"You've got it!" said Grandpa, "It's patterns we find,
Opposites, adding, they all intertwine."

At the last hole, the scorecard was done,
Morgan had *negative five*—she had won!
Grandpa had *positive six* on his side,
"But I had more fun!" Grandpa smiled wide.

On the way home, Morgan sat up with a grin,
"Negative numbers help show when we win.
Adding, subtracting, they tell quite a tale,
With numbers that climb and with numbers that sail."

Grandpa just smiled, gave a wink and a clap,
"You're thinking in math—what a marvelous map!
From scores to equations, from golf to the sky,
Integers help us all multiply!"

8.1: Addition and Subtraction

8.1.1: Written Retrieval and Problem Solving Practice

8.1.2: Online Retrieval and Problem Solving Practice

8.1.3: AI Learning Prompts 8.1

8.2: Multiplying and Dividing

8.2.1: Written Retrieval and Problem Solving Practice

8.2.2: Online Retrieval and Problem Solving Practice

8.2.3: AI Learning Prompts 8.2

8.3: Wonder Play Grow - Digit Disguise

8.4: Integers - The Story

8.5: Chapter 8 Resources

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8.1: Addition and Subtraction

🧠Let's Listen to Learn❤️

Audio

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Whole numbers and fractions are valuable tools for solving a wide range of problems in everyday life. However, many real-world situations call for the use of negative numbers. For instance, negative numbers are essential for representing temperatures below zero, elevations below sea level, financial losses in the stock market, and overdrawn bank accounts. In this chapter, we explore the set of integers—numbers that include all whole numbers along with their negative counterparts, which are the opposites of the nonzero whole numbers.

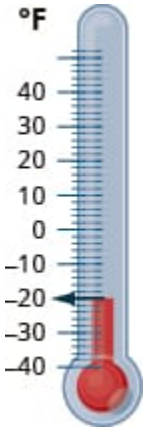


Figure 8.1.1: Temperatures below zero are described by negative numbers.

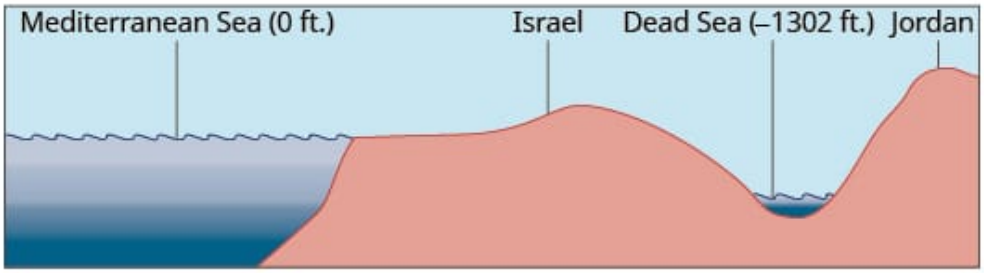


Figure 8.1.2: The surface of the Mediterranean Sea has an elevation of $0\,\text{ft}$. The diagram shows that nearby mountains have higher (positive) elevations whereas the Dead Sea has a lower (negative) elevation.

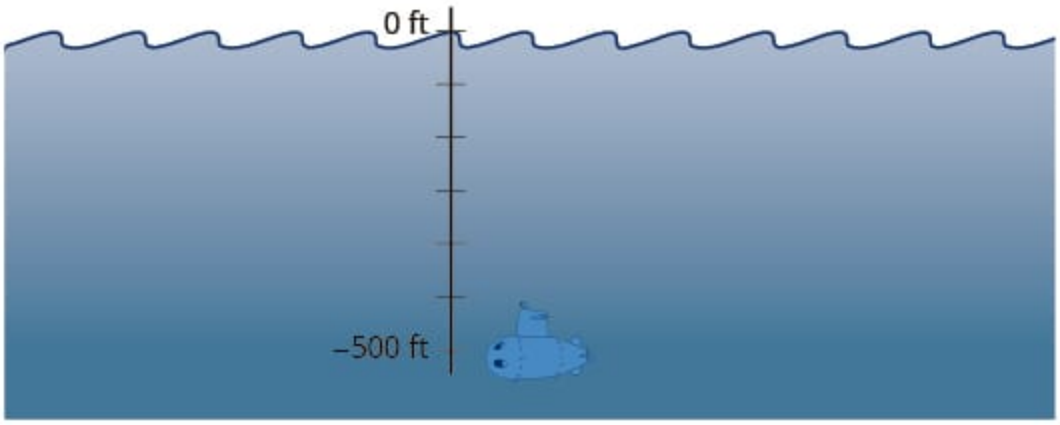


Figure 8.1.3: Depths below sea level are described by negative numbers. A submarine 500 ft below sea level is at -500 ft.

Wonder, Play, Grow

Before we start learning the specifics of integers, let's consider the different situations and scenarios where negative numbers arise. Write a word problem for each of the following expressions.

1. -20 + 8
2. -20 - 8
3. -20 - (-8)

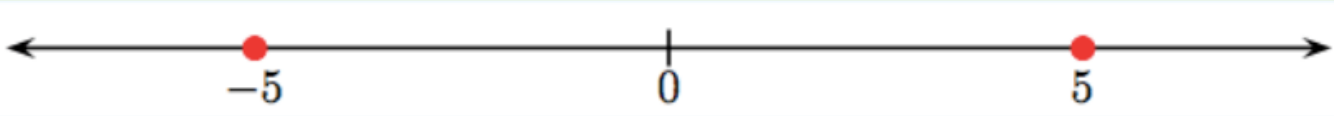
Be prepared to share with the class.

The opposite or negative of any whole number is easily located on the number line.

Number Line Locations

To locate the opposite (or negative) of any whole number, first locate the whole number on the number line. The opposite is the reflection of the whole number through the origin (zero).

To find its opposite, reflect the number 5 through the origin. This will be the location of the opposite (negative) of the whole number 5, which we indicate by the symbol –5.



The symbol –5 is pronounced in one of two ways: (1) “negative five,” or (2) “the opposite of five.”

In similar fashion, we can locate the opposite or negative of any whole number by reflecting the whole number through the origin (zero), which leads to the image shown in Figure 8.1.4.

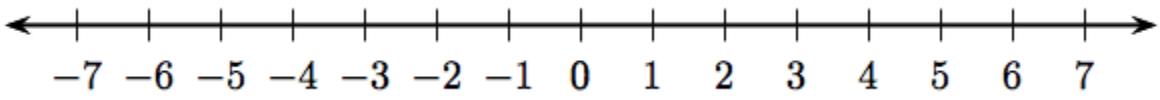


Figure 8.1.4: The opposite (negative) of any whole number is a reflection of that number through the origin (zero).

The Integers

The collection of numbers arranged on the number line in Figure 8.4.1 extend indefinitely to the right, and because the numbers on the left are reflections through the origin, the numbers also extend indefinitely to the left. This collection of numbers is called the set of *integers*.

The Integers

The infinite collection of numbers

$$\{\ldots, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, \ldots\}$$

is called the **set of integers.**

The ellipsis . . . at each end of this infinite collection means “etcetera,” as the integers continue indefinitely to the right and left. Thus, for example, both 23,456 and –117, 191 are elements of this set and are therefore integers.

Ordering the Integers

As we saw with the whole numbers, as you move to the right on the number line, the numbers get larger; as you move to the left, the numbers get smaller. Just as we did with positive numbers, we can use inequality symbols to show the ordering of positive and negative numbers, Figure 8.1.5.

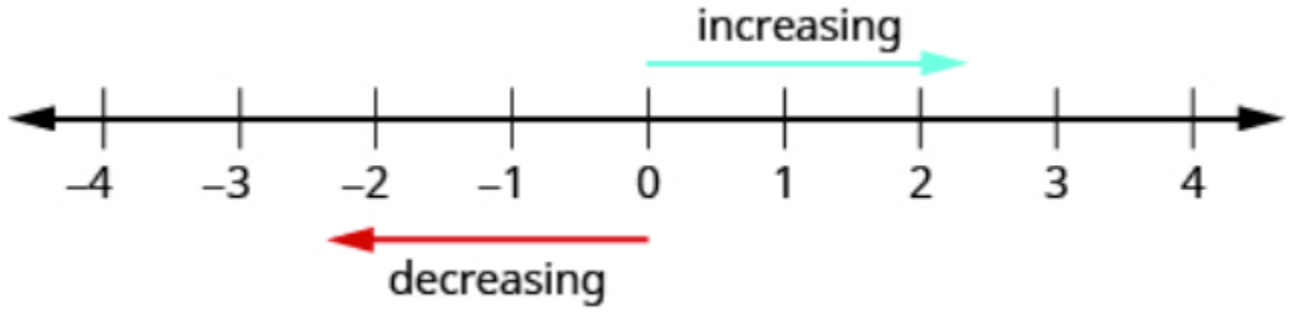
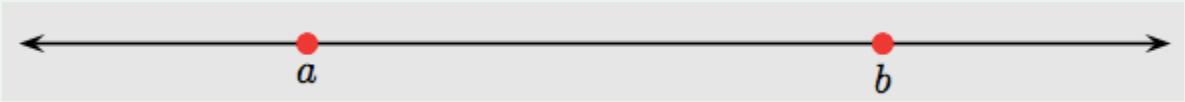


Figure 8.1.5

Order on the Number Line

Let *a* and *b* be integers located on the number line so that the point representing the integer *a* lies to the left of the point representing the integer *b*.



Then the integer a is “less than” the integer b and we write

$$a < b$$

Alternatively, we can also say that the integer *b* is “larger than” the integer *a* and write

$$b > a.$$

Example 8.1.1

Replace each shaded box with < (less than) or > (greater than) so the resulting inequality is a true statement.

$-3 \square 5 \qquad -2 \square -4$

Answer available at the end of this section.

We stated earlier that every integer has a unique number called its "opposite" or "negative." Thus, the integer –5 is the opposite (negative) of the integer 5. Thus, we can say that the pair –5 and 5 are opposites. Each is the opposite of the other. Logically, this leads us to the conclusion that the opposite of –5 is 5. In symbols, we would write

$$-(-5) = 5.$$

Opposites of Opposites

Let *a* be an integer. Then the "opposite of the opposite of *a* is *a*." In symbols, we write

$$-(-a) = a.$$

We can also state that the "negative of a negative *a* is *a*.

For example: –(–13) = 13

Model Addition of Integers

Now that we have located positive and negative numbers on the number line, it is time to discuss arithmetic operations with integers.

Most students are comfortable with the addition and subtraction facts for positive numbers. But doing addition or subtraction with both positive and negative numbers may be more difficult. This difficulty relates to the way the brain learns.

The brain learns best by working with objects in the real world and then generalizing to abstract concepts. Toddlers learn quickly that if they have two cookies and their older brother steals one, they have only one left. This is a concrete example of $2-1$. Children learn their basic addition and subtraction facts from experiences in their everyday lives. Eventually, they know the number facts without relying on cookies.

Addition and subtraction of negative numbers have fewer real world examples that are meaningful to us. Math teachers have several different approaches, such as number lines, banking, temperatures, and so on, to make these concepts real.

Set Model

Addition means to put together or form the union of two disjoint sets. We will model addition and subtraction of negatives with two color counters. We let a blue counter represent a positive and a red counter will represent a negative.



If we have one positive and one negative counter, the value of the pair is zero. The counters form a neutral pair. The value of this neutral pair is zero as summarized in Figure 8.1.6.

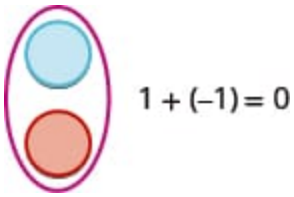


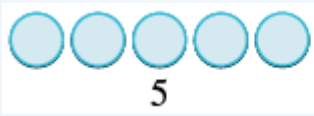
Figure 8.1.6 A blue counter represents $+1$. A red counter represents -1. Together they add to zero.

We will model using

$$5+3 \quad -5+(-3) \quad -5+3$$

Set Model Addition

Model: 5 + 3 .

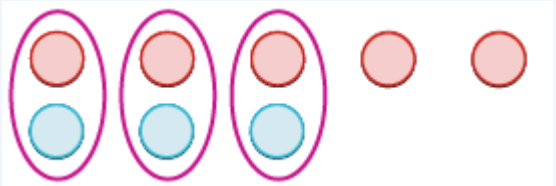
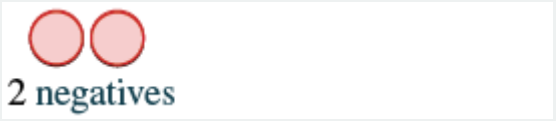
Interpret the expression.	$5+3$ means the sum of 5 and 3.
Model the first number. Start with 5 positives.	
Model the second number. Add 3 positives.	
Count the total number of counters.	
The sum of 5 and 3 is 8.	$5+3=8$

Model: –5 + (–3) .

Interpret the expression.	$-5+(-3)$ means the sum of -5 and -3.
Model the first number. Start with 5 negatives.	
Model the second number. Add 3 negatives.	
Count the total number of counters.	
The sum of –5 and –3 is –8.	$-5+-3=-8$

Model: –5 + 3 .

Interpret the expression.	$-5+3$ means the sum of -5 and 3.

Model the first number. Start with 5 negatives.	
Model the second number. Add 3 positives.	
Remove any neutral pairs.	
Count the result.	
The sum of –5 and 3 is –2.	$-5+3=-2$

Notice that there were more negatives than positives, so the result is negative.

Measurement Model

Addition means to put directed arrows end to end starting at zero. Note that positive integers are represented by arrows pointing to the right and negative integers by arrows pointing to the left (Figure 8.1.7-9).

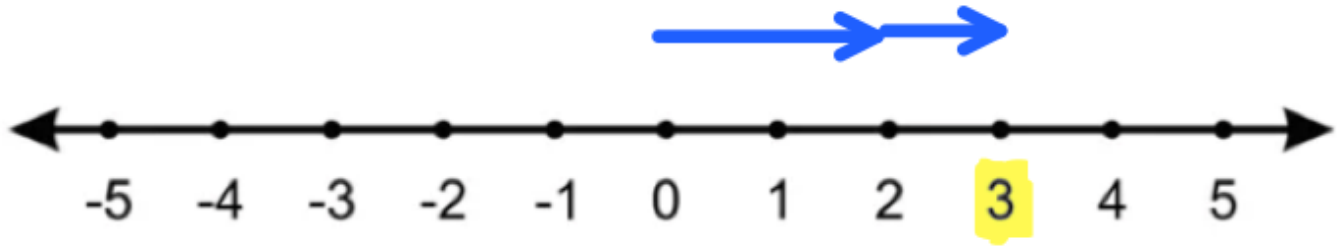


Figure 8.1.7: 2 + 1

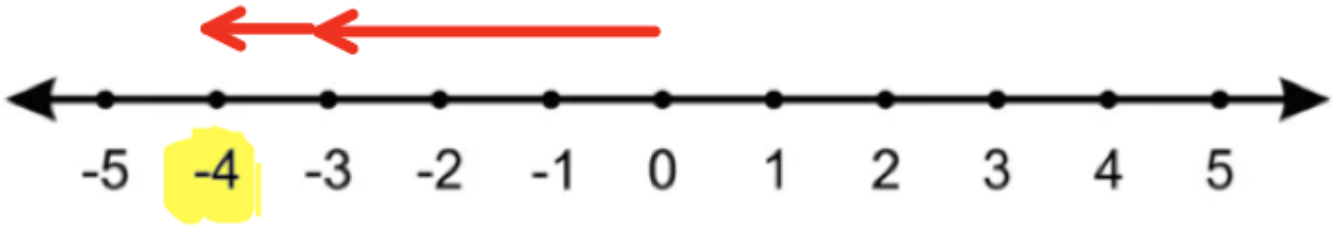


Figure 8.1.8: -3 + (-1)

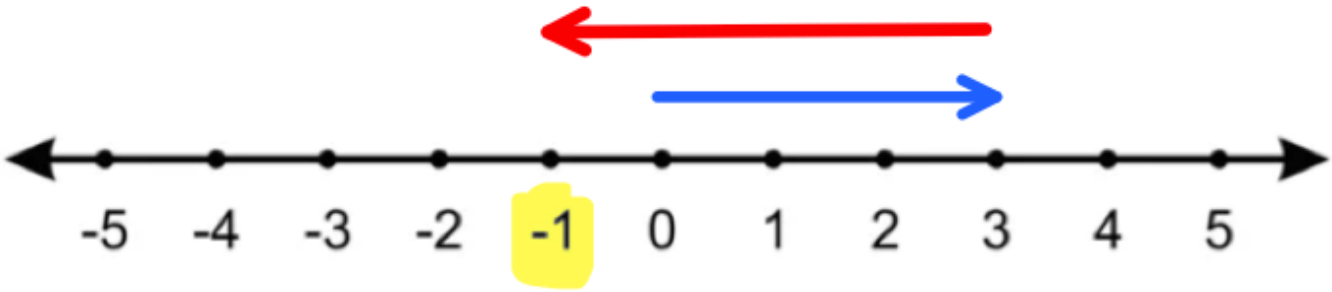


Figure 8.1.9: 3 + (-4)

Example 8.1.2

Model each addition.

a) −3 + 6

b) 4 + (−5)

c) -2 + (−3)

Answer available at the end of this section.

This conceptual work leads us to the following definition of integer addition.

Definition: Addition of Integers

Let a and b be integers.

1. Adding zero: a + 0 = 0 + a = a.

2. Adding two positives: If a and b are positive, they are added as whole numbers.

3. Adding two negatives: If a and b are positive (hence −a and −b are negative), then (−a) + (−b) = −(a + b), where a + b is the whole-number sum of a and b.

4. Adding a positive and a negative:

a. If a and b are positive and a ≥ b, then a + (−b) = a − b, where a − b is thewhole-number difference of a and b.

b. If a and b are positive and a < b, then a + (−b) = −(b − a), where b − a is the whole-number difference of a and b.

Example 8.1.3

Calculate the following using the definition of integer addition.

a) 3 + 0 b) 3 + 4 c) (−3) + (−4)

d) 7 + (−3) e) 3 + (−7) f) 5 + (−5)

Answer available at the end of this section.

Additive Inverse Property for Integer Addition

For each integer a there is a unique integer, written –a, such that a + (–a) = 0. The integer –a is called the additive inverse of a. In words, this property states that any number plus its additive inverse is zero.

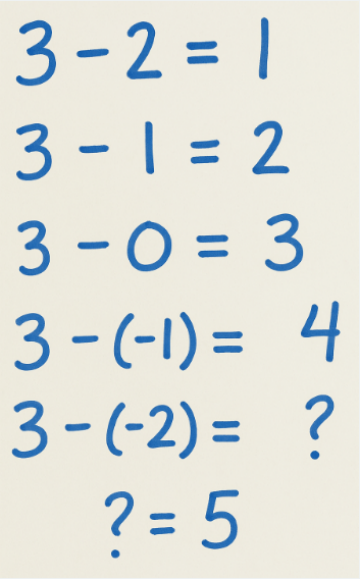
Algebraic Reasoning: When solving an equation such as x + 4 = 3, the additive inverse of 4, namely –4, is added to both sides of the equation as follows: x + 4 + (–4) = 3 + (–4). This shows that x = –1.

Model Subtraction of Integers

Children learn how to subtract numbers through their everyday experiences. Real-life experiences serve as models for subtracting positive numbers, and in some cases, such as temperature, for adding negative as well as positive numbers. But it is difficult to relate subtracting negative numbers to common life experiences. Most people do not have an intuitive understanding of subtraction when negative numbers are involved. Math teachers use several different models to explain subtracting negative numbers.

Subtraction of integers can be viewed in several ways.

Pattern Approach: What do you notice? Come up with your own examples after you see what is going on.



Subtracting a negative number is the same as **adding the positive version** of that number.

Adding the Opposite: Find 2 - 5.

Let a and b be any integers. Then a – b = a + (–b).

2 - 5 = 2 + (-5) = -3

Take Away Approach:

Remember, the blue counters represent positive numbers and the red counters represent negative numbers. Perhaps when you were younger, you read $5 - 3$ as *five take away three*. When we use counters, we can think of subtraction the same way.

Model: 5 – 3 .



5 - 3 = 2.

Model: –5 –(-3) .



–5 – (–3) = –2.

Model: - 5 – 3 .



But there are no positives to take away.

Add neutral pairs until you have 3 positives.





–5 – 3 = –8

TRY 5 - (-3) available at the end of this section.

Let's see what Dr. Howie Hua has to say about the Take-Away Approach.

YouTube Video

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Example 8.1.4

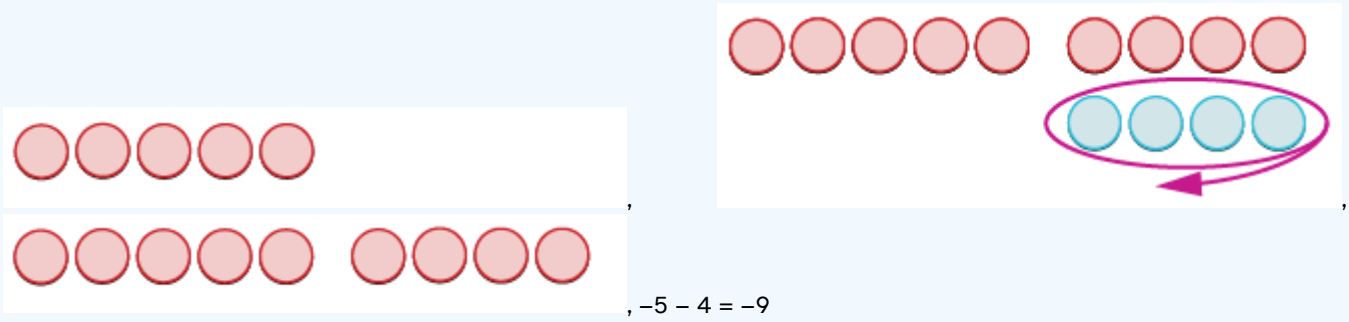
Model each subtraction using a different approach.

a) –5 – 4

b) 6 – (–6)

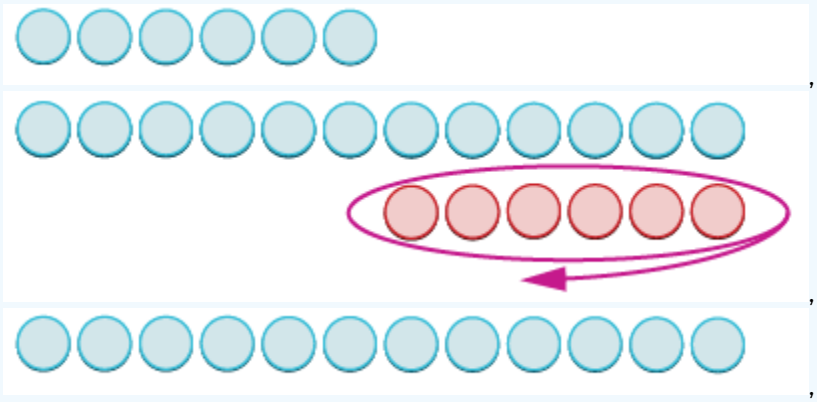
c) −8 − (−3)

Answer



, −5 − 4 = −9

b) 6 + (6) = 12



, 6 − (−6) = 12

c) -8 + (3) = -5

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Think About and Discuss

Recall that another approach to subtraction, the missing-addend approach, was used in whole-number subtraction. For example, 6 − 2 = n if and only if 6 = 2 + n. In this way, subtraction can be done by referring to addition. This method can also be extended to integer subtraction.

Missing-Addend Approach: Let a, b, and c be any integers. Then a − b = c if and only if a = b + c.

Think about and discuss with your peers how the Missing-Addend Approach can be used to find 4 - (-2). Then come up with another example that solidifies your understanding.

Answer available at the end of this section.

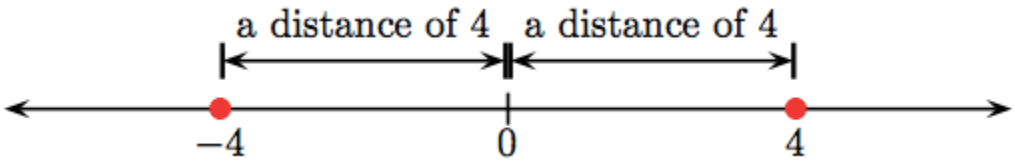
Absolute Value (Optional Section for this Course - But Still Important for Future Teachers)

We define the absolute value of an integer.

Absolute Value

The absolute value of an integer is defined as its distance from the origin (zero).

It is important to note that distance is always a nonnegative quantity (not negative); i.e., distance is either positive or zero. As an example, we've shaded the integers –4 and 4 on a number line.



The number line above shows two cases:

- The integer –4 is 4 units from zero. Because absolute value measures the distance from zero, | – 4| = 4.
- The integer 4 is also 4 units from zero. Again, absolute value measures the distance from zero, so |4| = 4.

Let's look at another example.

Example 8.1.5

Determine the value of each expression: a) | – 7|, b) |3|, and c) |0|.

Answer available at the end of this section.

Example 8.1.6

Determine the value of each expression: a) –(–8) and b) –| – 8|.

Answer available at the end of this section.

An alternate definition of absolute value is:

$$|a| = \begin{cases} a & \text{if } a \text{ is positive or zero.} \\ -a & \text{if } a \text{ is negative.} \end{cases} \tag{8.1.1}$$

Example 8.1.7

Using this definition, calculate the following values.

a) |– 3|

b) |7|

c) |0|

Answer available at the end of this section.

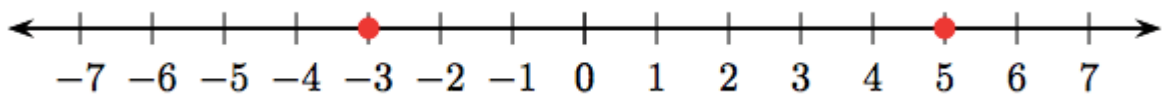
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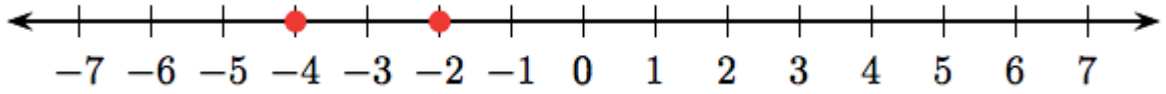
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Answers

Example 8.1.1



Note that –3 lies to the left of 5, so: -3 < 5.



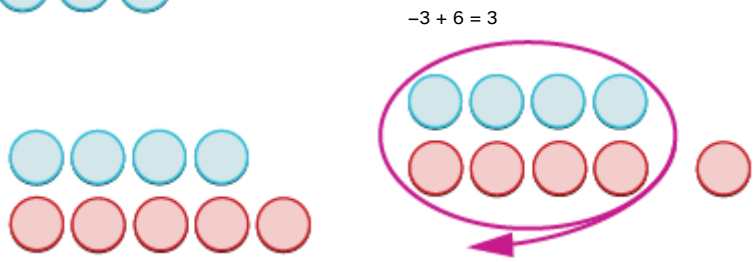
Note that –2 lies to the right of –4, so: -2 > -4

Example 8.1.2

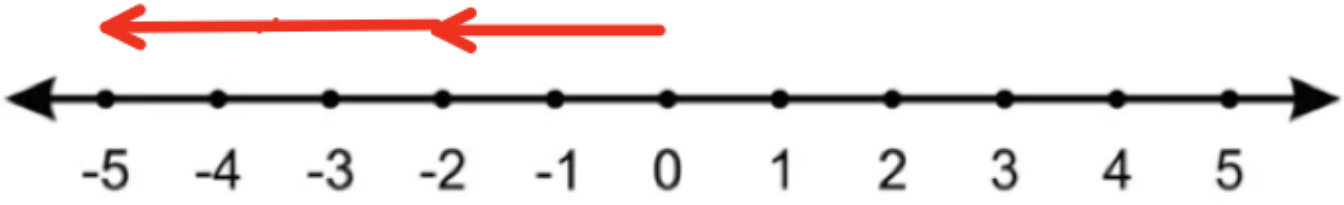
a)



b)



c)



4 + (−5) = −1

−2 + (−3) = −5

Example 8.1.3

a. Adding zero: 3 + 0 = 3

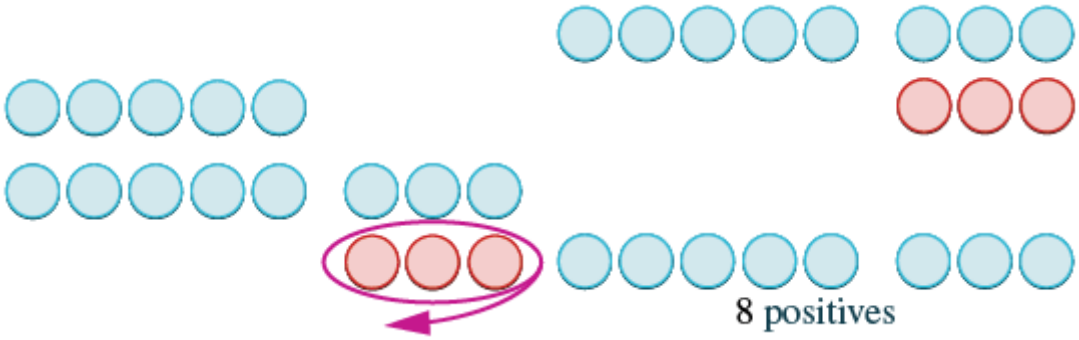
b. Adding two positives: 3 + 4 = 7

c. Adding two negatives: (−3) + (−4) = −(3 + 4) = −7

d. Adding a positive and a negative: 7 + (−3) = 7 − 3 = 4

e. Adding a positive and a negative: 3 + (−7) = −(7 − 3) = −4

f. Adding a number and its opposite: 5 + (−5) = 0



5 − (−3) = 8

Think About and Discuss

4 − (−2) = c if and only if 4 = (−2) + c. But 4 = −2 + 6. Therefore, c=6.

Example 8.1.5

The absolute value of any integer is equal to the distance that number is from the origin (zero) on the number line. Thus:

a) The integer −7 is 7 units from the origin; hence, | − 7| = 7.

b) The integer 3 is 3 units from the origin; hence, |3| = 3.

c) The integer 0 is 0 units from the origin; hence, |0| =0.

Example 8.1.6

These are distinctly different problems.

a) The opposite of −8 is 8. That is, −(−8) = 8.

b) However, in this case, we take the absolute value of −8 first, which is 8, then the opposite of that result to get −8. That is,

$$\begin{aligned} -|-8| &= -(8) \quad \text{First: } |-8| = 8. \\ &= -8 \quad \text{Second: The opposite of 8 is } -8. \end{aligned}$$

Example 8.1.7

a) 3 b) 7 c) 0

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8.1.1: Written Retrieval and Problem Solving Practice

Student Learning Objectives

SLO 8.1 Students will represent integers using models such as the integer number line and colored chips, and demonstrate addition, subtraction, and ordering of integers using these tools.

SLO 8.2 Students will analyze and apply the properties of integer operations, including addition, subtraction, multiplication, and division, to solve mathematical problems and verify relationships such as closure, commutativity, and distributivity.

The mild assignment 🌶 is a shorter and less challenging assignment. The medium assignment🌶🌶 is right in the middle in terms of length and level of difficulty. Finally, the spicy assignment 🌶🌶🌶 is a longer assignment that is the most challenging.

(Spicy 🌶🌶🌶 Problems are OPTIONAL, but go for it when you have the time to challenge yourself)

Exercise 8.1.1.1 🌶

Which of the following are integers?

a) $-\frac{1}{10}$

b) 123

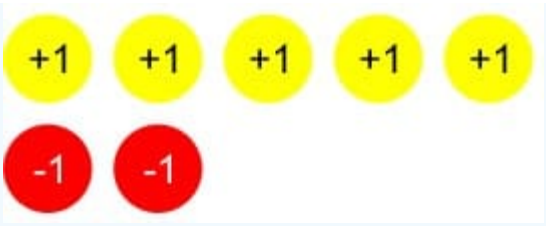
c) -2.1

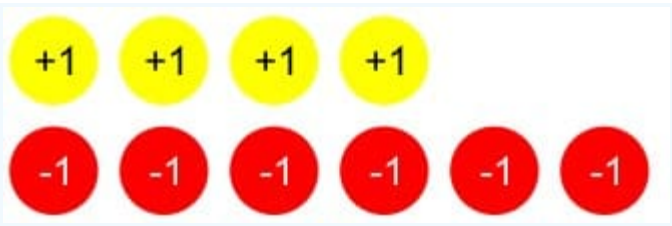
d) 3.0

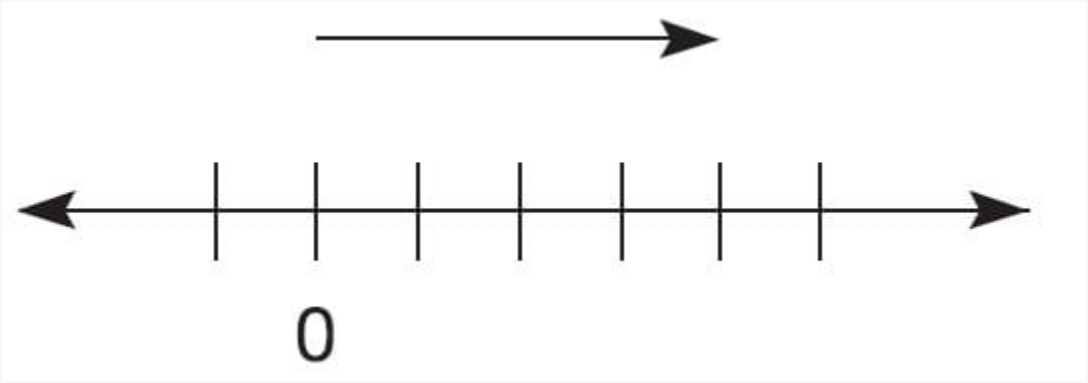
Answer available at the end of this section.

Exercise 8.1.1.2 🌶

Identify each of the integers represented by the following models.

a) 

b) 

c) 

Answer available at the end of this section.

Exercise 8.1.1.3

Use the set model and number-line model to represent each of the following integers.

a) 2 b) -3 c) 0

Answer available at the end of this section.

Exercise 8.1.1.4

a) Arrange the integers 5, –6, 0, and 2 on a number line.

b) Arrange the integers –2, –4, 3, and –6 on a number line.

Answer available at the end of this section.

Exercise 8.1.1.5

Write the opposite of each integer.

a) 3 b) -2 c) -(-4) d) x

Answer available at the end of this section.

Exercise 8.1.1.6

Show how you could find the following sums (i) using a number-line model and (ii) using positive and negative chips.

a) -3 + 5 b) 2 + (-1)

Answer available at the end of this section.

Exercise 8.1.1.7

Show how you could find the following differences by adding the opposite.

a) -3 - 5 b) 2 - (-1)

Answer available at the end of this section.

Exercise 8.1.1.8

Model each subtraction expression:

a) $2 - 8$

b) -3 - (-8)

Answer available at the end of this section.

Properties of Integer Addition

Let a, b, and c be any integers.

- Closure Property for Integer Addition: a + b is an integer.
- Commutative Property for Integer Addition: a + b = b + a
- Associative Property for Integer Addition: (a + b) + c = a + (b + c)

Exercise 8.1.1.9

Use thinking strategies to compute the following sums. Identify your strategy.

a) -14 + 6 b) 12 + (-3) c) -81 + (111 + 81) d) 84 + (-51) + (-30)

Answer available at the end of this section.

Exercise 8.1.1.10

a) The temperature in Buffalo, NY, one morning started at 7 degrees below zero Fahrenheit. By noon, it had warmed up 32 degrees. What was the temperature at noon?

b) A scuba diver was swimming 16 feet below the surface and then dove down another 17 feet. What is her new depth?

c) Dinesh hiked from Mt. Whitney, the highest point in California, to Death Valley, the lowest point. The elevation of Mt. Whitney is 14,497 above sea level and the elevation of Death Valley is 282 feet below sea level. What is the difference in elevation between Mt. Whitney and Death Valley?

Answer available at the end of this section.

Exercise 8.1.1.11

Simplify: 7 + (–4 – 3) – 9 .

Answer available at the end of this section.

Exercise 8.1.1.12

If r is an arbitrary negative integer and s is an arbitrary integer, which of the following is TRUE?

a) -r is negative b) -s is negative c) -r is positive d) r - s is positive

Answer available at the end of this section.

Exercise 8.1.1.13

Order each of the following pairs of numbers using $<$ or $>$:

a) 14 ___ 6 b) -1 ___ 9 c) -1 ___ -4 d) 2 ___ -20

Answer available at the end of this section.

Addition Approach: The integer a is less than the integer b, written a < b, if and only if there is a positive integer p such that a + p = b. Thus –5 < –3, since –5 + 2 = –3, and –7 < 2, since –7 + 9 = 2. Equivalently, a < b if and only if b – a is positive (since b – a = p). For example, –27 < –13, since –13 – (–27) = 14, which is positive. The integer a is greater than the integer b, written a > b, if and only if b < a. Thus, the discussion of greater than is analogous to that of less than. Similar definitions can be made for ≤ and ≥.

Exercise 8.1.1.14

Determine the smallest integer in the set {3, 0, – 5, 9, – 8} using the addition approach.

Answer available at the end of this section.

Exercise 8.1.1.15

Evaluate the following absolute values.

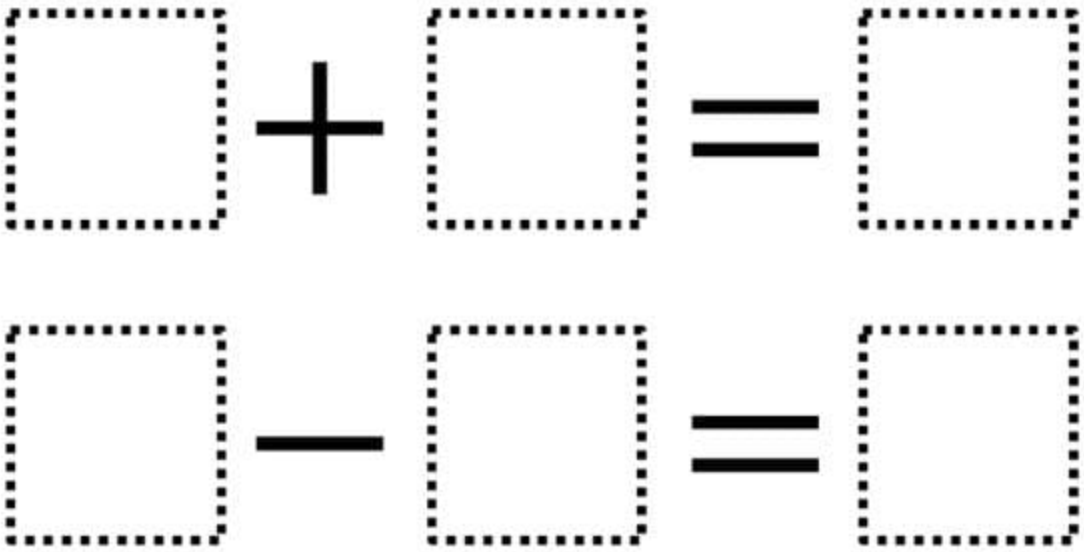
a) |5| b) |4 - 2| c) |2 - 4| d) |2| - |4|

Answer available at the end of this section.

Problem Solving Problems

1) Darlene had a balance of $115 in her checking account at the beginning of the month. She deposited $384 in the account and then made purchases for $153, $86, $196, $34, and $79. Then she made a deposit of $123. If at any time during the month the account is overdrawn, a $10 service charge is deducted. At the end of the month, what was Dixie's balance?

2) Directions: Using the integers -3 to 3, at most one time each, fill in the blanks to make each equation true.



Problem Source: Open Middle

3) Here is our favorite Math Educator Dr. Howie Hua. Take a look at how he uses negative values to complete subtraction problems without borrowing. After you get the hang of the process, try 72 - 38. Then see if you can make your own problems to show others how to use this method.

YouTube Video

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View this YouTube Video online

Answers

Exercise 8.1.1.1

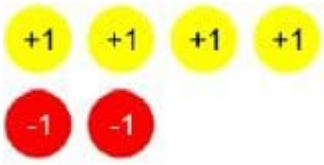
a) NO b) YES c) NO d) YES

Exercise 8.1.1.2

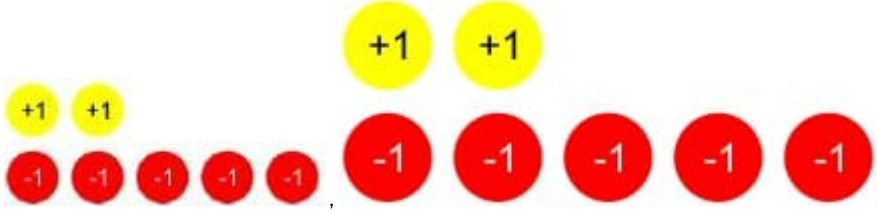
a) 3 b) -2 c) 4

Exercise 8.1.1.3

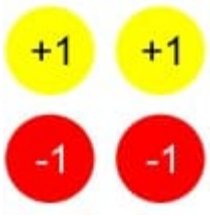
a) Multiple ways to represent as a set model,



b) Multiple ways to represent as a set model,

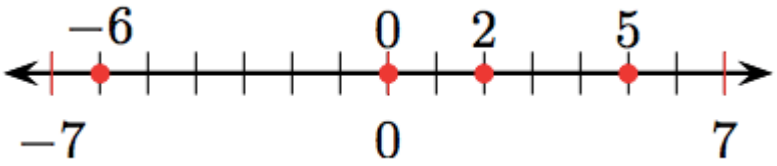


,

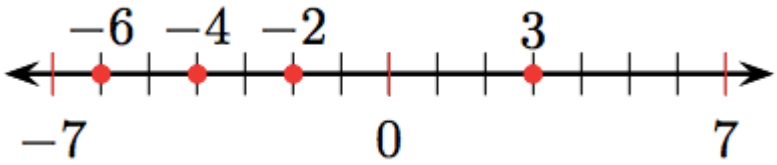


c) Multiple ways to represent as a set model, , Point at 0 on a number line.

Exercise 8.1.1.4



a)

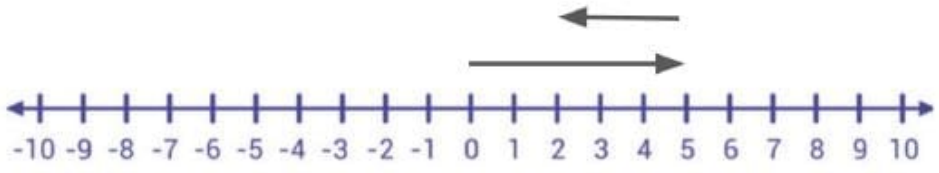


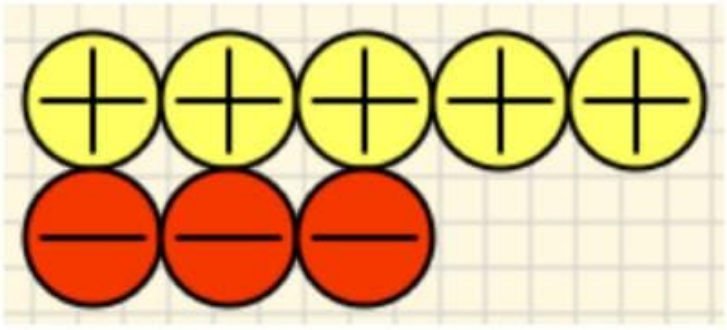
b)

Exercise 8.1.1.5

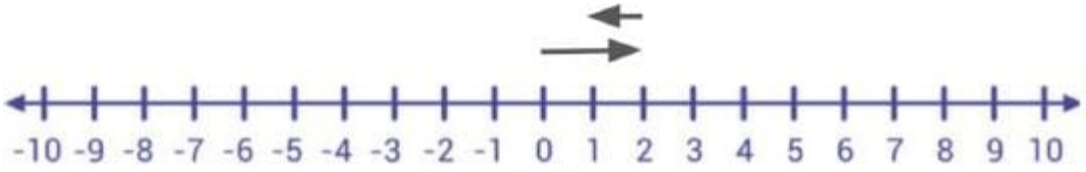
a) -3 b) 2 c) -4 d) -x

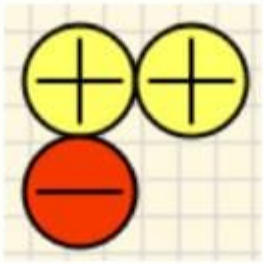
Exercise 8.1.1.6





a)



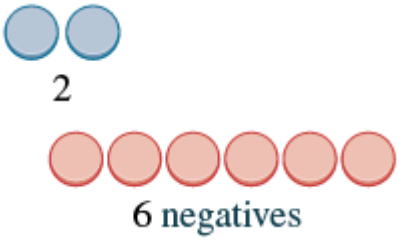


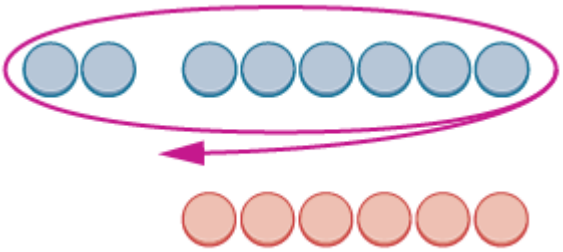
b)

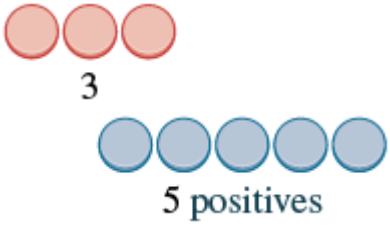
Exercise 8.1.1.7

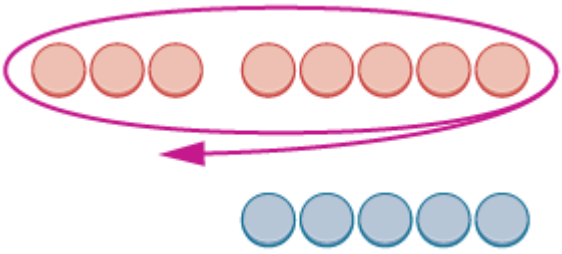
a) -3 + (-5) = -8 b) 2 + 1 = 3

Exercise 8.1.1.8

a) 

, 
, 2 - 8 = -6

b) 

, 
, -3 - (-8) = 5

Exercise 8.1.1.9

a) -8 + (-6) + 6 = -8 b) 9 + 3 + (-3) = 9 c) -81 + 81 + 111 = 111 d) 80 + 4 + (-50) + (-30) + (-1) = 3

Exercise 8.1.1.10

a) –7 + 32 = 25 degrees

b) -16 + (-17) = -33, 33 feet below

c) 14,497 – (–282) = 14779, The difference in elevation is 14,779 feet.

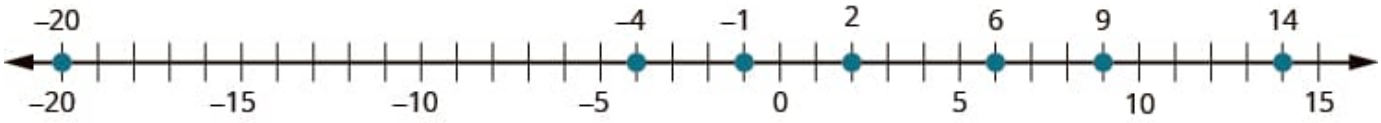
Exercise 8.1.1.11

-9

Exercise 8.1.1.12

Add texts here. Do not delete this text first.

Exercise 8.1.1.13



a) 14 > 6 b) -1 < 9 c) -1 > -4 d) 2 > -20

Exercise 8.1.1.14🌶🌶

–8 < –5, since (–8) + 3 = –5. Also, since any negative integer is less than 0 or any positive integer, –8 must be the smallest.

Exercise 8.1.1.15🌶🌶

a) 5 b) 2 c) 2 d) -2

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8.1.3: AI Learning Prompts 8.1

📚 AI Prompt Library

Integers — Addition & Subtraction (Chapter 8.1)

MATH 204 · MATHEMATICS FOR ELEMENTARY TEACHERS

How to use this page: Choose the tab that matches how you want to learn right now. **Explain It** gives you a clear explanation you can read and absorb. **Explore It** starts a back-and-forth conversation to help you build understanding. **Practice It** coaches you through problems step by step. Copy any prompt and paste it into an AI chatbot (like ChatGPT or Claude) to start your session.

💡 Explain It | 🔍 Explore It | 🛠️ Practice It

Section 1 — Direct Explanation Prompts

Understand what integers are, how the number line and colored chip models work for adding and subtracting integers, why zero pairs matter, what absolute value actually means, and how properties like closure and commutativity apply to integer operations.

TYPE 1 — JUST EXPLAIN IT

PROMPT 1 COPY

Can you explain how to add and subtract integers using the colored chip model? I want to understand what zero pairs are, why we sometimes have to add zero pairs before we can subtract, and what "adding zero" really means in that context. Use plain language and a concrete example, and after you explain it, ask me if anything is unclear.

PROMPT 2 COPY

Can you explain what absolute value actually means? I keep hearing it described as "the opposite" or "just make it positive," but I feel like that's not the full picture. Explain the real definition in plain language with a couple of examples, and then check if I understood it.

TYPE 2 — EXPLAIN IT AND CONNECT IT

PROMPT 3 COPY

I already understand how to add and subtract whole numbers. Can you explain integer addition and subtraction in a way that builds from what I already know? I'm confused about why sometimes adding two numbers gives a smaller result and subtracting gives a bigger result. Help me see how this connects to whole number arithmetic before going into the full explanation.

PROMPT 4 COPY

I know that whole number addition is commutative and closed. Can you explain what happens when we extend those same properties to integers — does closure still hold? Does commutativity? What about subtraction — does it stay commutative with integers? Connect it to what I already know about whole number properties and help me see what changes and what stays the same.

📋 Follow-Up Question Card — Integer Addition & Subtraction

Got the explanation — now keep going. Pick one of these moves:

Follow-Up Question Card

Move	What to type
Say it differently	I'm still confused about zero pairs. Can you explain why adding a positive and a negative chip equals zero using a different analogy — maybe something like money or temperature?
Give me an example	Can you show me step by step how to use the chip model to solve 3 – (–5), including the part where you have to add zero pairs?
Check me	Can you ask me a question to see if I really understand the difference between absolute value and opposite?
Go deeper	Why is integer subtraction not commutative, but integer addition is? Can you show me a specific example that proves subtraction fails the commutative property?

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8.2: Multiplying and Dividing

Multiply Integers

Since multiplication is mathematical shorthand for repeated addition, our counter model can easily be applied to show multiplication of integers. Let's look at this concrete model to see what patterns we notice. We will use the same examples that we used for addition and subtraction.

We remember that 5 x 3 $= 3 + 3 + 3 + 3 + 3$. So what is 5 x (-3)? -3 + -3 + -3 + -3 + -3 = -15, Figure 8.2.1

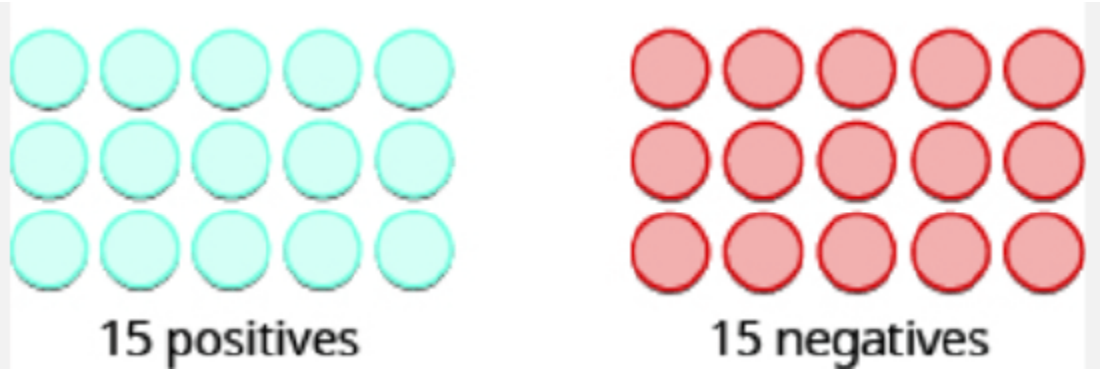


Figure 8.2.1: Add 3, 5 times, Add (-3), 5 Times

What about (-5) x 3? In this case the first number –5 is negative, which indicates that we should "take away 5 groups of 3 blue" rather than combine. When the first number is positive, the groups are combined into a new set, Figure 8.2.1. When the first number is negative, the groups are taken away from a set that has a value of 0. In order to take something away from a set with a value of 0, we must add some chips with a value of 0 to the set Figure 8.2.2. This is done by adding an equal number of red and blue chips to the set. After taking away 5 groups of 3 blue chips Figure 8.2.3, the resulting set has 15 red chips or a value of –15, Figure 8.2.4.

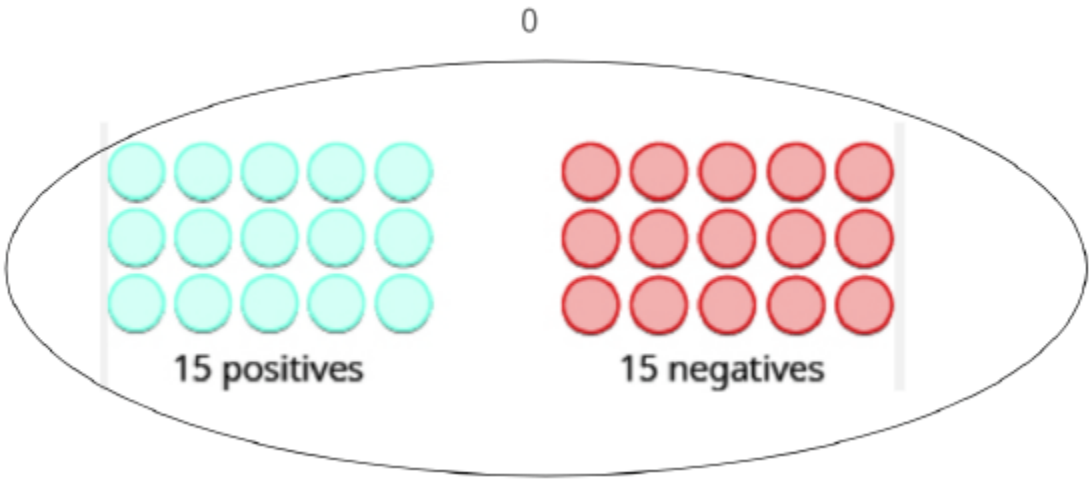


Figure 8.2.2: Value is 0

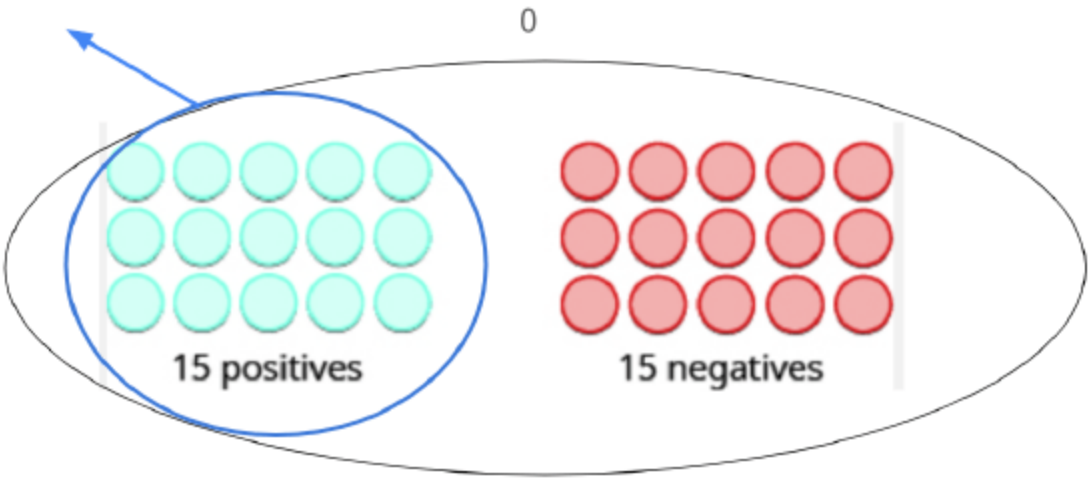


Figure 8.2.3: Take Away 5 groups of 3

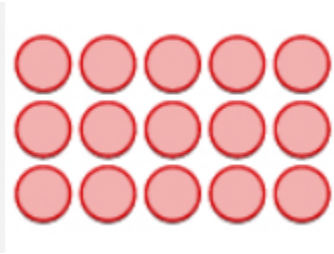


Figure 8.2.4: -15 is remaining

What about (-5) x (-3)? We can do the same thing with the chips, see Figure 8.2.2, but this time take away 5 groups of 3 red chips, the resulting set has a value of 15. Another way to view this, and the bigger question often asked, ***how come a negative times a negative is a positive,*** we can also use a pattern (inductive reasoning) to respond.

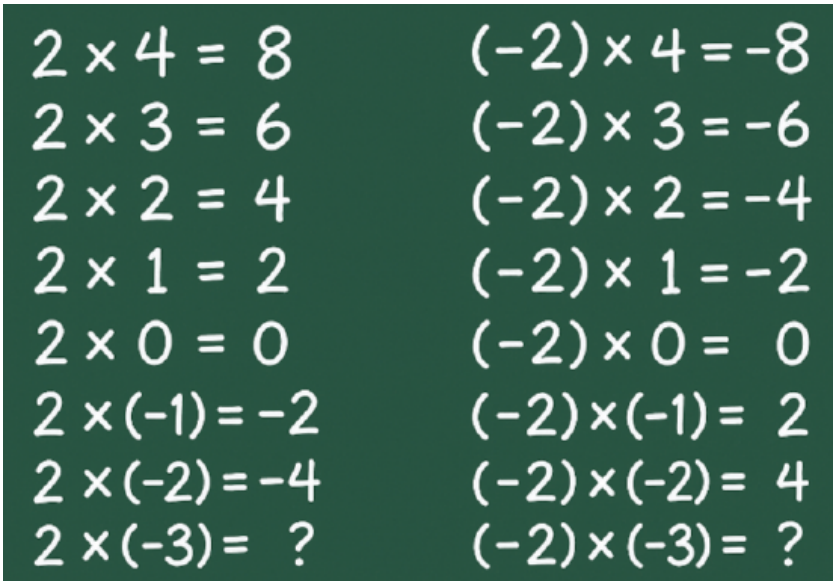


Figure 8.2.5:Rules for Integer Multiplication Motivated by Patterns

This pattern (values decreasing by 2, left column Figure 8.2.5) extended suggests that 2 x (–1) = –2, 2 x (–2) = –4, 2 x (–3) = –6, and so on. A similar pattern can be used to suggest what the product of two negative integers should be, as follows. This pattern (values increase by 2, right column Figure 8.2.5) suggests that (–2)(–1) = 2, (–2)(–2) = 4, (–2)(–3) = 6, and so on.

Here is another way to think about why it makes sense that a negative times a negative is a positive - Thank you Dr. Howie Hua.

YouTube Video

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Notice that for multiplication of two signed numbers, when the signs are the same, the product is positive, and when the signs are different, the product is negative.

Multiplication of Signed Numbers

The sign of the product of two numbers depends on their signs.

Same signs	Product
•Two positives •Two negatives	Positive Positive

Different signs	Product
•Positive • negative •Negative • positive	Negative Negative

The number-line model, the patterns, and the chips model all lead to the following definition.

Definition: Multiplication of Integers

Let a and b be integers.

1. Multiplying by zero: a x 0 = 0 = 0 x a.

2. Multiplying two positives: If a and b are positive, they are multiplied as whole numbers.

3. Multiplying a positive and a negative: If a is positive and b is positive (thus –b is negative), then a(–b) = – (ab), where ab is the whole-number product of a and b. That is, the product of a positive and a negative is negative.

4. Multiplying two negatives: If a and b are positive, then(–a)(–b) = ab, where ab is the whole-number product of a and b. That is, the product of two negatives is positive.

Example 8.2.1

Multiply each of the following, but keep in mind the definition to deepen your understanding.

a) (4)(0) b) 5(-5) c) (-4)(-4) d) (-4)(5)(-6) e) (-a)(-b)(-c)

Answer available at the end of this section.

When we multiply a number by 1, the result is the same number. What happens when we multiply a number by -1? Let's multiply a

positive number and then a negative number by -1 to see what we get.

$-1 \cdot 4$	$-1(-3)$
-4	3

Each time we multiply a number by -1, we get its opposite.

Theorem 8.2.1

Multiplying a number by -1 gives its opposite.

$$-1a = -a$$

Example 8.2.2

Multiply each of the following.

a) -1(9) b) -1(-6) c) -1(-1)(3)

Answer available at the end of this section.

Although students can memorize the rules of integer multiplication, it is important for future teachers to understand that these rules are foundational for algebra and algebraic reasoning. More than just procedural fluency, a deep conceptual understanding of why multiplying a negative by a negative yields a positive, or why the sign rules behave as they do, supports students' ability to generalize patterns, construct arguments, and make sense of symbolic expressions. This conceptual grounding becomes essential as students transition from arithmetic to algebra, where they manipulate variables and solve equations involving signed numbers.

Definition: Properties of Integer Multiplication

Let a, b, and c be any integers.

- Closure Property for Integer Multiplication: ab is an integer.
- Commutative Property for Integer Multiplication: ab = ba
- Associative Property for Integer Multiplication: (ab)c = a(bc)
- Identity Property for Integer Multiplication: 1 is the unique integer such that a x 1 = a = 1 x a for all a.
- Distributivity of Multiplication over Addition of Integers: a(b + c) = ab + ac.
- Multiplicative Cancellation Property: Let a, b, c be any integers with c ≠ 0. If ac = bc, then a = b.
- Zero Divisors Property: Then ab = 0 if and only if a = 0 or b = 0 or a and b both equal zero.

Algebraic Reasoning: This property makes it possible to solve equations such as (x – 2)(x + 3) = 0. Because of this property, it is known that either x – 2 = 0 or x + 3 = 0.

Divide Integers

Division of Signed Numbers

The sign of the quotient of two numbers depends on their signs.

Same signs	Quotient
•Two positives •Two negatives	Positive Positive

Different signs	Quotient
•Positive & negative •Negative & positive	Negative Negative

Remember, you can always check the answer to a division problem by multiplying. This is demonstrated with the following definition.

Definition: Division of Integers

Let a and b be any integers, where b ≠ 0. Then a ÷ b = c if and only if a = b · c for a unique integer c.

Example 8.2.3

Divide each of the following, if possible:

a) 12 ÷ (–3) b) (–15) ÷ (–5) c) (–8) ÷ 2 d) 7 ÷ (–2) e) 0 ÷ (-2)

Answer available at the end of this section.

Just as we saw with multiplication, when we divide a number by 1, the result is the same number. What happens when we divide a number by -1? Let's divide a positive number and then a negative number by -1 to see what we get.

$8 \div (-1)$	$-9 \div (-1)$
-8	9
-8 is the opposite of 8	9 is the opposite of -9

When we divide a number by, -1 we get its opposite.

Division by -1

Dividing a number by -1 gives its opposite.

$$a \div (-1) = -a$$

Now that we have finalized our work with the operations of integers, we can take a look at the properties of Ordering Integers

Properties of Ordering of Integers

Let a, b, and c be any integers, p a positive integer, and n a negative integer.

- Transitive Property for Less Than: If a < b and b < c, then a < c.
- Property of Less Than and Addition: If a < b, then a + c < b + c.
- Property of Less Than and Multiplication by a Positive: If a < b, then ap < bp.
- Property of Less Than and Multiplication by a Negative: If a < b, then an > bn.

Negative Exponents and Scientific Notation

Think About and Discuss

Recall that for whole-number exponents, the following properties hold:

$$5^4 = 5 \cdot 5 \cdot 5 \cdot 5 \qquad 5^0 = 1 \qquad \frac{5^5}{5^3} = 5^{5-3} = 5^2 \qquad 5^5 \cdot 5^3 = 5^{5+3} = 5^8 \tag{8.2.1}$$

It is important that the properties of negative exponents are consistent with the properties of exponents above.

- If the properties were consistent, what would 5^{-2} be equal to?
- Justify your conclusion with your peers.
- After you have your discussion, use the interactive below to check your thinking. Then explore other bases to solidify your thinking.

Explore

In our earlier study of whole numbers, we encountered exponents as a shorthand for repeated multiplication. Building on this understanding, we now explore how the concept of exponents extends to include integer values.

$$\begin{aligned} a^3 &= a \cdot a \cdot a \\ a^2 &= a \cdot a \\ a^1 &= a \\ a^0 &= 1 \\ a^{-1} &= \frac{1}{a} \\ a^{-2} &= \frac{1}{a^2} \\ a^{-3} &= \frac{1}{a^3} \\ &\vdots \\ &\text{etc.} \end{aligned} \tag{8.2.2}$$

Definition: Negative Integer Exponent

Let a be any nonzero number and n be a positive integer. Then $a^{-n} = \frac{1}{n}$.

We care about **negative exponents** because they are essential for understanding and working with mathematical concepts that go beyond whole number operations, especially in areas like algebra, science, and real-world applications. Further, negative exponents are a necessary development to make the rules of exponents consistent across all integers, provide a compact and convenient notation, and are crucial for representing numbers in scientific notation and building the system of rational and real exponents.

Algebraic Reasoning: Negative integer exponents give students a way to see that $\frac{1}{7^3} \cdot 7^3 = 1$. Because $\frac{1}{7^3} = 7^{-3}$, $\frac{1}{7^3} \cdot 7^3 = 7^{-3} \cdot 7^3 = 7^{-3+3} = 7^0 = 1$. Knowing this is a way for solving algebraic equations.

Scientific Notation

We previously used scientific notation that was introduced in terms of very large numbers and positive exponents. With the introduction of negative exponents, we can now use scientific notation to represent very small numbers. Some examples are below.

Scientific Notation

What?	How much?	Scientific Notation
Distance from Earth to Sun	150,000,000,000 meters	1.5×10^{11}
Mass of average human cell	0.000000000001 kilograms	1×10^{-12}
Diameter of the moon	3,474,000 meters	3.474×10^{6}
Time for light to travel 1 mm	0.00000000333 seconds	3.33×10^{-9}
Estimated # of cells in human body	100,000,000,000,000 cells	1×10^{14}

When a number is written in scientific notation, the **exponent** tells you if the term is a large or a small number. A positive exponent indicates a large number and a negative exponent indicates a small number that is between 0 and 1. Since we already studied big numbers, we will only focus on small numbers in this section.

To write a small number (between 0 and 1) in scientific notation, you move the decimal to the *right* and the exponent will have to be *negative,* as in the following example.

$$\begin{aligned} 0.\underset{\longrightarrow}{00004} &= 00.0004 \times 10^{-1} \\ &= 000.004 \times 10^{-2} \\ &= 0000.04 \times 10^{-3} \\ &= 00000.4 \times 10^{-4} \\ &= 000004. \times 10^{-5} \\ 0.00004 &= 4 \times 10^{-5} \end{aligned}$$

You may notice that the decimal point was moved five places to the *right* until you got to the number 4, which is between 1 and 10. The exponent is -5.

Example 8.2.4

Write the following numbers in scientific notation.

a) 0.0000000000035

b) 0.0000000102

c) 0.00000000000000793

Answer available at the end of this section.

Example 8.2.5

Write the following in decimal notation.

a) 4.8×10^{-4}

b) 3.08×10^{-6}

Answer available at the end of this section.

Multiplying and Dividing Numbers Expressed in Scientific Notation

Numbers that are written in scientific notation can be multiplied and divided rather simply by taking advantage of the properties of numbers and the rules of exponents that you may recall. To multiply numbers in scientific notation, first multiply the numbers that aren't powers of 10 (the *a* in $a \times 10^n$). Then multiply the powers of ten by adding the exponents.

This will produce a new number times a different power of 10. All you have to do is check to make sure this new value is in scientific notation. If it isn't, you convert it.

Let's look at some examples.

Example

$$\left(3 \times 10^8\right)\left(6.8 \times 10^{-13}\right)$$

Regroup using the commutative and associative properties.

$$\left(3 \times 6.8\right)\left(10^8 \times 10^{-13}\right)$$

Multiply the coefficients.

$$\left(20.4\right)\left(10^8 \times 10^{-13}\right)$$

Multiply the powers of 10 using the Product Rule. Add the exponents.

$$20.4 \times 10^{-5}$$

Convert 20.4 into scientific notation by moving the decimal point one place to the left and multiplying by 10^1.

$$\left(2.04 \times 10^1\right) \times 10^{-5}$$

Group the powers of 10 using the associative property of multiplication.

$$2.04 \times \left(10^1 \times 10^{-5}\right)$$

Multiply using the Product Rule—add the exponents.

$$2.04 \times 10^{1+(-5)}$$

$\left(3 \times 10^8\right)\left(6.8 \times 10^{-13}\right) = 2.04 \times 10^{-4}$

Think About and Discuss

Using what you learned for the problem above, discuss with your peers how we would solve $\frac{2.4 \times 10^7}{4.8 \times 10^5}$. Use the video below to check your answer.

YouTube Video

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Example 8.2.6

Express each of the following numbers in scientific notation.

a) (42000)(0.000002)

b) $\dfrac{240000}{0.0006}$

Answer available at the end of this section.

Answers

Example 8.2.1

a) 0 b) -25 c) 16 d) 120 e) -abc

NOTES: a) Multiply by 0 rule b) -(5x5)=-25 c) (-4)(-4) = (4x4)=16 d) -(4x5)(-6) = (-20)(-6) = 120 e) ab(-c)=-(abc)=-abc

Example 8.2.2

a) opposite of 9, -9 b) opposite of -6, 6 c) opposite of -1, 1. Then 1(3)=3

Example 8.2.3

a) 12 ÷ (−3) = c if and only if 12 = (−3) c. From multiplication, 12 = (−3)(−4). Since (−3) · c = (−3)(−4), by multiplicative cancellation, c = −4.
b) (−15) ÷ (−5) = c if and only if −15 = (−5) · c. From multiplication, −15 = (−5) · 3. Since (−5) · c = (−5) · 3, by multiplicative cancellation, c = 3.
c) (−8) ÷ 2 = c if and only if (−8) = 2 · c. Thus c = −4, since 2(−4) = −8.
d) 7 ÷ (−2) = c if and only if 7 = (−2) · c. There is no such integer c. Therefore, 7 ÷ (−2) is undefined in the integers.
e) Division into zero, 0 ÷ (-2) = 0

Example 8.2.4

a) $0.00000000000035 = 3.5 \times 10^{-12}$, we moved the decimal 12 times to get to a number between 1 and 10

b) $0.00000000102 = 1.02 \times 10^{-8}$

c) $0.0000000000000793 = 7.93 \times 10^{-15}$

Example 8.2.5

a) $4.8 \times 10^{-4} = .00048$

b) $3.08 \times 10^{-6} = 0.00000308$

Example 8.2.6

a) $(4.2\text{x}10^{4})(2.0\text{x}10^{-6})=8.4\text{x}10^{-2}$

b) $(2.4\text{x}10^{5})(6.0\text{x}10^{-4})=0.4\text{x}10^{1}=4.0\text{x}10^{0}$

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8.2.1: Written Retrieval and Problem Solving Practice

Student Learning Objectives

SLO 8.2 **Students will analyze and apply the properties of integer operations, including addition, subtraction, multiplication, and division, to solve mathematical problems and verify relationships such as closure, commutativity, and distributivity.**

SLO 8.3 **Students will explain integer multiplication and division through repeated addition, patterns, and missing-factor approaches, and compute using negative exponents and scientific notation.**

SLO 8.4 **Students will evaluate relationships between integers, applying the properties of inequalities (e.g., transitivity, multiplication by a positive or negative number) and theorems such as additive cancellation and opposite of the opposite.**

The mild assignment 🌶 is a shorter and less challenging assignment. The medium assignment🌶🌶 is right in the middle in terms of length and level of difficulty. Finally, the spicy assignment 🌶🌶🌶 is a longer assignment that is the most challenging.

(Spicy 🌶🌶🌶 Problems are OPTIONAL, but go for it when you have the time to challenge yourself)

Exercise 8.2.1.1

Illustrate the following products on an integer number line.

a) $2 \times (-5)$

b) $3 \times (-4)$

Answer available at the end of this section.

Exercise 8.2.1.2

Find the following products.

a) $(-2)(-5)(-3)$

b) $(-10)(7)(-6)$

c) $5[(-2)(13) + 5(-4)]$

Answer available at the end of this section.

Exercise 8.2.1.3

Represent the following products using chips and give the results.

a) $(2) \times (-4)$

b) $(-2) \times (-1)$

Answer available at the end of this section.

Exercise 8.2.1.4

Using the distributive property, expand each of the following products.

a) −6(x + 2)

b) −5(a − b)

Answer available at the end of this section.

Exercise 8.2.1.5

Find each quotient.
a) (−5 + 5) ÷ (−2)
b) [144 ÷ (−12)] ÷ (−3)
c) 144 ÷ [−12 ÷ (−3)]

Answer available at the end of this section.

Exercise 8.2.1.6

Without doing the indicated calculations, determine whether the answers are positive, negative, or zero. Explain your reasoning.
a) (−3)(−7)(−5) ÷ (−15)

b) (−27) ÷ 3 × (−4) ÷ (−3)

c) 35(−4) ÷ 5 × 0 × (−2)

Answer available at the end of this section.

Exercise 8.2.1.7

Write each of the following as a fraction without exponents.

a) 10^{-3} b) 2^{-5} c) 4^{-4}

Answer available at the end of this section.

Exercise 8.2.1.8

a) $(-3)^3$ b) $(-5)^2$ c) $-(5)^2$

Answer available at the end of this section.

Exercise 8.2.1.9

Convert as indicated.
a) 0.000079 to scientific notation

b) 3×10^{-4} to standard notation

c) 0.00000012003 to scientific notation

Answer available at the end of this section.

Exercise 8.2.1.10

Explain how to determine the smaller of –11 and –9 using the following techniques.

a) The number-line approach

b) The addition approach

Answer available at the end of this section.

Exercise 8.2.1.11

Express the following in scientific notation.
a) $(9.3 \times 10^{-3})(3 \times 10^{5})$

b) $(5.5 \times 10^{5}) \div (1.1 \times 10^{-3})$

Answer available at the end of this section.

Exercise 8.2.1.12

Compute each of the following problems without using a calculator.

a) 37 + (–43) b) (–7)(–6)

c) 45 – (–3) d) 16 ÷ (–2)

e) (–13)(4) f) [–24 – (–27)] × (–3)

Answer available at the end of this section.

Exercise 8.2.1.13

A student does not believe that –100 < –500. He argues that a debt of $100 is greater than a debt of $500. How would you convince him that the inequality is true?

Answer available at the end of this section.

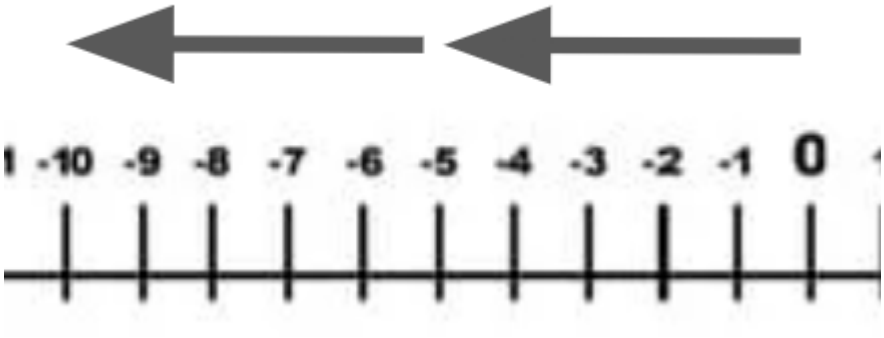
Problem Solving Problems

1) A shopper asked for 50 cents worth of apples. The shopper was surprised when she received five more than the previous week. Then she noticed that the price had dropped 10 cents per dozen. What was the new price per dozen?

2) Which is larger: 5.1×10^{4} or 7.8×10^{5}? Explain your reasoning without a calculator.

Answers

Exercise 8.2.1.1

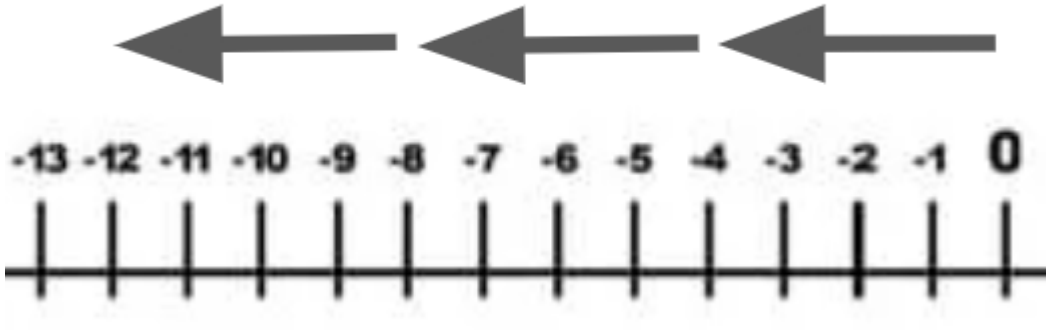


a)

2×(−5)=−10

The arrows show two equal jumps of −5 starting from 0.

This models multiplication as repeated addition: −5+(−5)=−10



b)

3×(−4)=−12

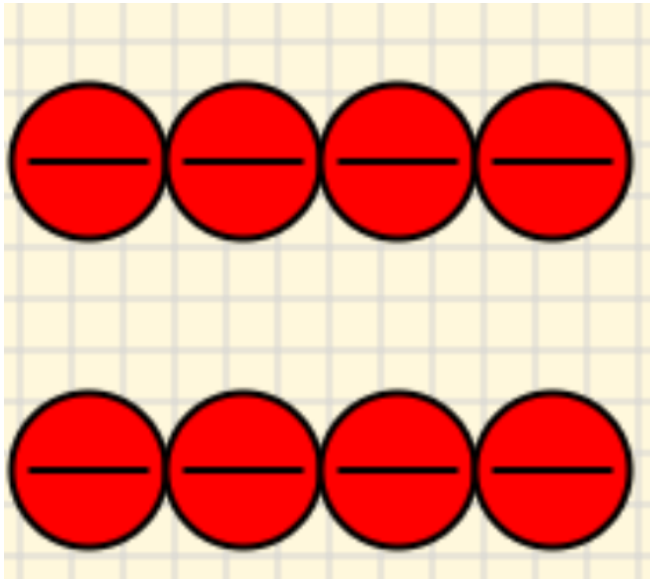
The arrows show three equal jumps of −4-4−4 starting from 0.

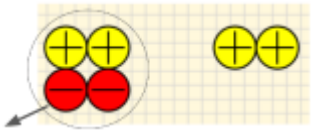
This represents repeated addition:−4+(−4)+(−4)=−12.

Exercise 8.2.1.2

a) -30 b) 420 c) 5(-35)=-175

Exercise 8.2.1.3

a) 

b)  Take away 2 group of 1 negative chip, leaves two positive chips.

Exercise 8.2.1.4

a) -6x + (-12) b) -5a + 5b

Exercise 8.2.1.5

a) 0 b) 4 c) 36

Exercise 8.2.1.6

a) 7 b) -12 c) 0

Exercise 8.2.1.7

a) 1/1000 b) 1/32 c) 1/256

Exercise 8.2.1.8

a) -27 b) 25 c) -25

Exercise 8.2.1.9

a) 7.9×10^{-5}

b) 0.0003

c) 1.2003×10^{-6}

Exercise 8.2.1.10

a) -11 is to the left of -9 on the number line, so -11 is smaller.

b) -11 + ? = -9, ?=2, since I have to add 2 to -11 to get -9, we know -9 is a larger value.

Exercise 8.2.1.11

a) $(9.3 \times 3) \times (10^{-3} \times 10^{5}) = 27.9 \times 10^{2} = 2.79 \times 10^{3}$

b) $(5.5 / 1.1) \times (10^{5} / 10^{-3}) = 5 \times 10^{2}$

Exercise 8.2.1.12

a) -6 b) 42 c) 48 d) -8 e) -52 f) -9

Exercise 8.2.1.13

On the number line, values increase as you move to the right. So: –10 is to the **left** of –5. Therefore, –10<–5.

Explain that in math, “greater than” and “less than” refer to position on the number line, **not** how bad or serious something feels. This situation is a perfect example of what Keith Devlin calls developing *mathematical thinking*: understanding that symbols (like "<") have meanings that may not always align with everyday language or intuition. This kind of conceptual understanding is central to building number sense, as emphasized in *Adding It Up* by the National Research Council.

Using Absolute Value: “Yes, a $10 debt has an absolute value of 10, and a $5 debt has an absolute value of 5. That tells you how big the debts are. But on a number line, the $10 debt puts you further into the negatives, so it’s actually a smaller number.”

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8.2.3: AI Learning Prompts 8.2

AI Prompt Library

Section 8.2 — Multiplying and Dividing Integers

MATH 204 · MATHEMATICS FOR ELEMENTARY TEACHERS

This page gives you **three different ways** to use AI as a learning partner for this topic. Pick the approach that matches how you learn best right now — you can always come back and try a different one. **Copy any prompt below**, paste it into an AI chat (like ChatGPT or Claude), and start a conversation. Each section ends with a **Follow-Up Question Card** to help you keep the conversation going.

📖 Explain It to Me | 🔍 Help Me Explore | 🛠️ Coach Me Through It

LEARNING OBJECTIVE

Understand what integer multiplication and division mean — through repeated addition, patterns, and missing-factor approaches — and why the rules for signs work the way they do. Understand what negative exponents and scientific notation represent, why properties like closure, commutativity, and distributivity hold for integers, and how inequality properties apply.

Type 1 — Just Explain It

For when you want a clear, straight explanation before anything else.

PROMPT 1 Copy

Can you explain to me why a negative number times a negative number equals a positive number? I don't want just the rule — I want to actually understand why it has to be that way. Use plain language, walk me through a pattern or a

concrete example, and after you explain it, ask me if anything is unclear.

PROMPT 2

Copy

I'm confused about negative exponents. When I see something like 3^{-2}, I want to make the answer negative, but I know that's wrong. Can you explain what a negative exponent actually means in plain language? Cover why it makes the number smaller instead of negative, connect it to how we define exponents for whole numbers, and then give me a concrete example with scientific notation. After you explain, check in with me.

Type 2 — Explain It and Connect It

For when you already know something related and want the new idea connected to it.

PROMPT 3

Copy

I understand multiplication of whole numbers as repeated addition — like 3 × 4 means 4 + 4 + 4. But when one or both of the numbers are negative, I lose the thread. Can you explain how the definition of multiplication as repeated addition extends to integers? Help me see how the whole-number idea still connects, and where the pattern approach fills in the gaps. Then ask me if I see the connection.

PROMPT 4

Copy

I learned about properties like commutativity and the distributive property for whole numbers, and they made sense. But now with integers — especially with negative numbers — I'm not sure those properties still hold or why they would. Can you explain how closure, commutativity, and distributivity work for integer multiplication, in a way that connects back to what I already knew about whole numbers? After you explain it, ask me something to see if I'm following.

📋 Follow-Up Question Card — Integer Multiplication & Division

Got the explanation — now keep going. Pick one of these moves:

Question Card

MOVE	WHAT TO TYPE
Say it differently	I'm still not sure I get why negative × negative = positive. Can you explain it using a completely different approach — maybe with a number line model or a real-world scenario instead of patterns?
Give me an example	Can you show me a specific problem where someone converts a number into scientific notation using a negative exponent, and walk me through what each part means?
Check me	Can you give me a quick question to see if I understand the difference between a negative exponent making a number smaller versus making it negative?
Go deeper	Why does multiplying both sides of an inequality by a negative number flip the direction of the inequality sign? What would go wrong if we didn't flip it?

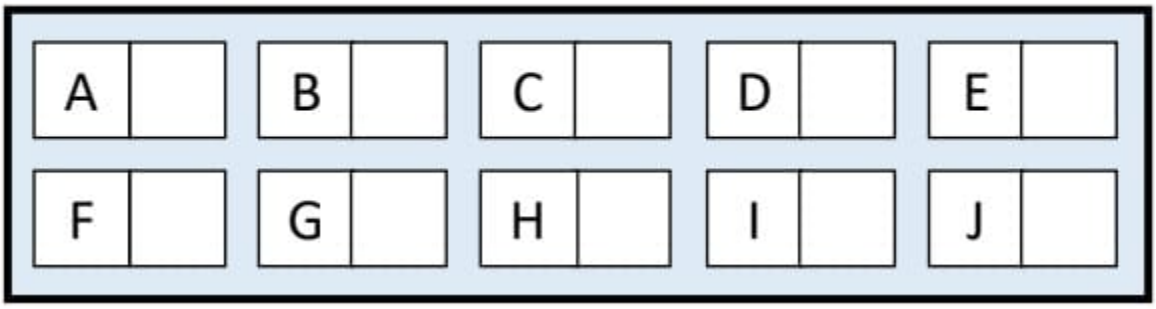
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8.3: Wonder Play Grow - Digit Disguise

Digit Disguise

DIGIT DISGUISES: A game of algebraic deduction by David K Butler 2019

MINE



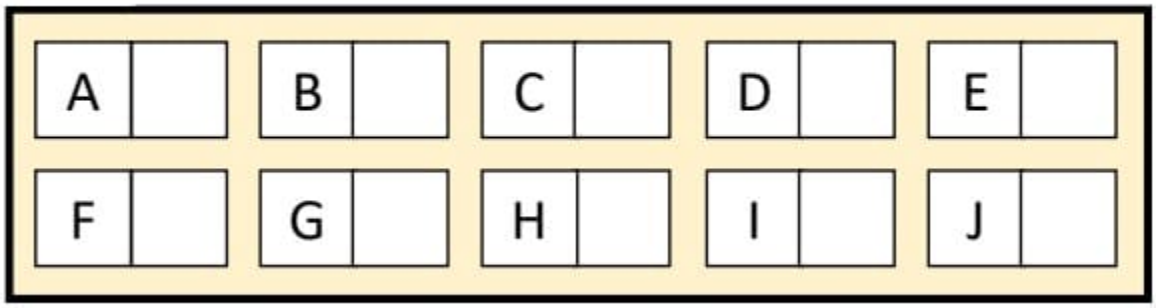
Set Up

- Each player/team take a sheet, and without letting the other player see, write the ten digits 0 to 9, each in a separate box in the "MINE" grid.
- Set up so you can write in the “THEIRS” grid without the other player/team seeing the “MINE” grid. (See reverse side for more instructions.)

How to Play

- Each number has been disguised as a letter. You need to find out which number each of the other player/team’s letters is.
- On your turn, ask for the result of a calculation involving exactly two different letters and one of the operations +, -, ×, ÷. Some examples are A – B or C÷J or H+D or E×G.
- The other player/team answers your question truthfully, either telling you which of their letters is the result, or telling you the result is not a letter. Nobody ever says a number.
- You can write notes in response anywhere on the paper if you like.
- Now it is the other player/team’s turn.
- Once during the game, instead of asking for a calculation, you can say what number you think goes with every letter. If you are right, you win! If you are wrong, you lose! Either way the game is over.

THEIRS



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8.4: Integers - The Story

A Brief History of Negative Numbers for Future Elementary Teachers

Understanding negative numbers is essential not only for students but also for teachers who must present them with clarity and context. The journey toward accepting and using negative numbers spans many cultures and centuries, marked by gradual conceptual and symbolic developments.

When Did Negative Numbers Gain Real-World Context?

The real-world contextualization of negative numbers began with accounting and debt in ancient China and India, but it only became common in Europe around the 17th–18th centuries with the rise of commercial bookkeeping.

Before that, negative numbers were more of a mathematical oddity or "fictitious" tool for solving equations. Only when money, trade, and data tracking demanded precision did negatives find a home in daily life.

Current Practical, Real-World Applications:

1. Banking and Finance

- **Overdrafts**: If you withdraw more money than you have in your bank account, your balance shows a negative number.
- **Debts**: Owing money is commonly expressed with negative numbers.
- **Stock Market**: Losses in investments are indicated with negative values.

2. Temperature and Weather

- **Thermometers**: Temperatures below zero (especially in Celsius or Fahrenheit) are negative.
- **Climate Analysis**: Negative changes in temperature are used in climate models and predictions.

3. Business and Accounting

- **Profit/Loss Calculations**: A business running at a loss represents that loss with negative values.
- **Inventory Management**: Negative numbers may flag errors or shortages (e.g., when inventory is oversold).

4. Elevation and Geography

- **Sea Level**: Locations below sea level, like the Dead Sea region, are expressed in negative meters or feet.
- **Diving and Mining**: Depths below the surface are measured with negative numbers.

5. Science and Engineering

- **Electric Charges**: Electrons carry a negative charge.
- **Forces and Direction**: In physics, direction matters. Movement or forces in opposite directions are assigned negative signs.

6. Gaming and Technology

- **Scores**: In some games, penalties result in negative scores.
- **Programming**: Negative numbers are often used in algorithms, such as representing movement in reverse or array indexing in Python.

7. Time and History

- **Historical Dates**: Years before the common era (BCE) are often shown as negative numbers (e.g., -500 for 500 BCE).

- **Time Zones**: Time differences behind GMT (Greenwich Mean Time) are negative (e.g., GMT-8).

Early Traces and Cultural Contributions

Babylonian Mathematics (c. 2000 BCE)

- Equations occasionally implied negative values, but these were not recognized as legitimate numbers.
- No concept of negative numbers as stand-alone entities.

Chinese Mathematics (c. 100 BCE and onward)

- The *Nine Chapters on the Mathematical Art* introduced a positional rod system:
  - **Black rods** for positive numbers, **red rods** for negative.
  - Blank spaces indicated zero.
- Despite their use, Chinese mathematicians often limited solutions to positive roots due to practical applications.

Indian Mathematics (c. 600 CE)

- **Brahmagupta** systematized operations with negatives and zero:
  - Clearly stated rules for arithmetic with negatives.
  - Recognized negative solutions to quadratic equations.
- Conceptually, Indian mathematicians treated zero and negatives as valid numerical entities—unlike the Greeks.

Greek Mathematics (Classical Era)

- Emphasized geometry over arithmetic.
- Rejected negative and irrational numbers as "nonsensical" because they lacked physical-geometric interpretation.

Arabic Mathematics (8th–14th centuries)

- Aware of irrational roots and zero, but typically avoided negative numbers.
- Solutions to equations emphasized positive values only.

European Acceptance (1200s–1800s)

Early Resistance

- **Fibonacci (13th century):** Rejected negative roots, though in some financial problems interpreted them as losses.
- **Michael Stifel (16th century):** Used negatives in computations but called them "absurd numbers" with no true status.

Gradual Recognition

- **Girolamo Cardan:** Acknowledged negative roots but called them "fictitious."
- **Albert Girard:** Used negative and imaginary numbers, visually representing them as line segments in the opposite direction—early number line ideas.
- **René Descartes:** Systematized signs in polynomials, distinguished "real" and "imaginary" roots.

Institutional Acceptance

- **John Wallis (17th century):** Advocated for negative numbers, likening them to land lost to the sea.
- **William Rowan Hamilton (19th century):** Attempted to rebuild the logic of negative numbers, interpreting them as steps backward in time.
- **Richard Dedekind (1872):** Grounded irrational and real numbers in a formal logical structure.
- **Hermann Hankel:** Argued numbers should be viewed as abstract constructs rather than purely geometrical entities.

Key Educational Takeaways

1. **Cultural Context Matters:** Different societies interpreted numbers through the lens of their practical and philosophical systems.
2. **Symbolism Preceded Conceptual Acceptance:** Symbols for negative numbers appeared before full conceptual recognition.
3. **Historical Struggle Mirrors Student Struggles:** Just as scholars once struggled to accept negative numbers, students often need time and context to internalize them.

For the Elementary Classroom

- Use number lines and real-world contexts (e.g., debts, temperatures).
- Normalize the idea that "less than zero" is meaningful and logical.
- Emphasize patterns in operations to build intuitive understanding.

Page Sources

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8.5: Chapter 8 Resources

Chapter Example Videos

8.1 Part 1

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