

MGF 1131: MATHEMATICS IN CONTEXT



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NWFSC MGF 1131 Text

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Detailed Licensing

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1: ORGANIZING DATA

1.1: Populations and Samples

1.2: Categorizing data

1.3: Sampling methods

1.4: How to mess things up before you start

1.5: Presenting Categorical Data Graphically

1.6: Presenting Quantitative Data Graphically

1.1: Populations and Samples

Learning Objectives

After completing this section, you should be able to:

1. Distinguish between populations and samples
2. Distinguish between statistics and parameters

Before we begin gathering and analyzing data, we need to characterize the **population** we are studying. If we want to study the amount of money spent on textbooks by a typical first-year college student, our population might be all first-year students at Northwest Florida State College. Or it might be:

- All first-year community college students in the state of Florida.
- All first-year students at public colleges and universities in the state of Florida.
- All first-year students at all colleges and universities in the state of Florida.
- All first-year students at all colleges and universities in the entire United States.
- And so on.

Definition: Population

The **population** of a study is the group the collected data is intended to describe.

Sometimes, the intended population is called the **target population**, since if we design our study badly, the collected data might not actually be representative of the intended population.

Why is it important to specify the population? We might get different answers to our question as we vary the population we are studying. First-year students at the University of Florida might take slightly more diverse courses than those at NWFSC, and some of these courses may require less popular textbooks that cost more; or, on the other hand, the University Bookstore might have a larger pool of used textbooks, reducing the cost of these books to the students. Whichever the case (and it is likely that some combination of these and other factors are in play), the data we gather from NWFSC will probably not be the same as that from the University of Florida. Particularly when conveying our results to others, we want to be clear about the population we are describing with our data.

Example 1.1.1

A newspaper website contains a poll asking people their opinion on a recent news article. What is the population?

Solution

While the target (intended) population may have been all people, the real population of the survey is readers of the website.

If we were able to gather data on every member of our population, say the average (we will define "average" more carefully in a subsequent section) amount of money spent on textbooks by each first-year student at NWFSC during the 2009-2010 academic year, the resulting number would be called a **parameter**.

Definition: Parameter

A **parameter** is a value (average, percentage, etc.) calculated using all the data from a population

We seldom see parameters, however, since surveying an entire population is usually very time-consuming and expensive, unless the population is very small or we already have the data collected.

Definition: Census

A survey of an entire population is called a **census**.

You are probably familiar with two common censuses: the official government Census, which attempts to count the population of the U.S. every ten years, and voting, which asks the opinion of all eligible voters in a district. The first of these demonstrates one additional problem with a census: the difficulty in finding and getting participation from everyone in a large population, which can bias or skew the results.

There are occasionally times when a census is appropriate, usually when the population is fairly small. For example, if the manager of Starbucks wanted to know the average number of hours her employees worked last week, she should be able to pull up payroll records or ask each employee directly.

Since surveying an entire population is often impractical, we usually select a **sample** to study.

Definition: Sample

A **sample** is a smaller subset of the entire population, ideally one that is fairly representative of the whole population.

We will discuss sampling methods in greater detail in a later section. For now, let us assume that samples are chosen in an appropriate manner. If we survey a sample, say 100 first-year students at NWFSC, and find the average amount of money spent by these students on textbooks, the resulting number is called a **statistic**.

Definition: Statistic

A **statistic** is a value (average, percentage, etc.) calculated using the data from a sample.

Example 1.1.2

A researcher wanted to know how the citizens of Tallahassee felt about a voter initiative. To study this, she goes to the mall and randomly selects 500 shoppers and asks them their opinion. 60% indicate they are supportive of the initiative. What is the sample and population? Is the 60% value a parameter or a statistic?

Solution

The sample is the 500 shoppers questioned. The population is less clear. While the intended population of this survey was Tallahassee citizens, the effective population was mall shoppers. There is no reason to assume that the 500 shoppers questioned would be representative of all Tallahassee citizens.

The 60% value was based on the sample, so it is a statistic.

Your Turn 1.1.1

To determine the average length of trout in a lake, researchers catch 20 fish and measure them. What is the sample and population in this study?

Answer available at the end of this section.

Your Turn 1.1.2

A college reports that the average age of their students is 28 years old. Is this a statistic or a parameter?

Answer available at the end of this section.

Answers

Your Turn 1.1.1

The sample is the 20 fish caught. The population is all fish in the lake. The sample may be somewhat unrepresentative of the population since not all fish may be large enough to catch the bait.

Your Turn 1.1.2

This is a parameter, since the college would have access to data on all students (the population)

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1.2: Categorizing data

Learning Objectives

After completing this section, you should be able to:

1. Determine whether data is categorical or quantitative

Once we have gathered data, we might wish to classify it. Roughly speaking, data can be classified as **categorical data** or **quantitative data**.

Quantitative and categorical data

- **Categorical (qualitative) data** are pieces of information that allow us to classify the objects under investigation into various categories.
- **Quantitative data** are responses that are numerical in nature and with which we can perform meaningful arithmetic calculations.

Example 1.2.1

We might conduct a survey to determine the name of the favorite movie that each person in a math class saw in a movie theater.

When we conduct such a survey, the responses would look like: *Finding Nemo*, *The Hulk*, or *Terminator 3: Rise of the Machines*. We might count the number of people who give each answer, but the answers themselves do not have any numerical values: we cannot perform computations with an answer like "*Finding Nemo*." This would be categorical data.

Example 1.2.2

A survey could ask the number of movies you have seen in a movie theater in the past 12 months (0, 1, 2, 3, 4, ...)

This would be quantitative data.

Other examples of quantitative data would be the running time of the movie you saw most recently (104 minutes, 137 minutes, 104 minutes, ...) or the amount of money you paid for a movie ticket the last time you went to a movie theater (\$5.50, \$7.75, \$9, ...).

Sometimes, determining whether or not data is categorical or quantitative can be a bit trickier.

Example 1.2.3

Suppose we gather respondents' ZIP codes in a survey to track their geographical location.

ZIP codes are numbers, but we can't do any meaningful mathematical calculations with them (it doesn't make sense to say that 98036 is "twice" 49018 — that's like saying that Lynnwood, WA is "twice" Battle Creek, MI, which doesn't make sense at all), so ZIP codes are really categorical data.

Example 1.2.4

A survey about the movie you most recently attended includes the question "How would you rate the movie you just saw?" with these possible answers:

- 1 - It was awful
- 2 - It was just OK
- 3 - I liked it
- 4 - It was great
- 5 - Best movie ever

Again, there are numbers associated with the responses, but we can't really do any calculations with them: a movie that rates a 4 is not necessarily twice as good as a movie that rates a 2, whatever that means; if two people see the movie and one of them thinks it stinks and the other thinks it's the best ever it doesn't necessarily make sense to say that "on average they liked it."

As we study movie-going habits and preferences, we shouldn't forget to specify the population under consideration. If we survey 3-7 year-olds the runaway favorite might be *Finding Nemo*. 13-17 year-olds might prefer *Terminator 3*. And 33-37 year-olds might prefer...well, *Finding Nemo*.

Your Turn 1.2.1

Classify each measurement as categorical or quantitative

- a. Eye color of a group of people
- b. Daily high temperature of a city over several weeks
- c. Annual income

Answer available at the end of this section.

Answers

Your Turn 1.2.1

- a. Categorical
- b. Quantitative
- c. Quantitative

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1.3: Sampling methods

Learning Objectives

After completing this section, you should be able to:

1. Determine the sampling method used

As we mentioned in a previous section, the first thing we should do before conducting a survey is to identify the population that we want to study. Suppose we are hired by a politician to determine the amount of support he has among the electorate should he decide to run for another term. What population should we study? Every person in the district? Not every person is eligible to vote, and regardless of how strongly someone likes or dislikes the candidate, they don't have much to do with him being re-elected if they are not able to vote.

What about eligible voters in the district? That might be better, but if someone is eligible to vote but does not register by the deadline, they won't have any say in the election either. What about registered voters? Many people are registered but choose not to vote. What about "likely voters?"

This is the criteria used in much political polling, but it is sometimes difficult to define a "likely voter." Is it someone who voted in the last election? In the last general election? In the last presidential election? Should we consider someone who just turned 18 a "likely voter?" They weren't eligible to vote in the past, so how do we judge the likelihood that they will vote in the next election?

In November 1998, former professional wrestler Jesse "The Body" Ventura was elected governor of Minnesota. Up until right before the election, most polls showed he had little chance of winning. There were several contributing factors to the polls not reflecting the actual intent of the electorate:

- Ventura was running on a third-party ticket and most polling methods are better suited to a two-candidate race.
- Many respondents to polls may have been embarrassed to tell pollsters that they were planning to vote for a professional wrestler.
- The mere fact that the polls showed Ventura had little chance of winning might have prompted some people to vote for him in protest to send a message to the major-party candidates.

But one of the major contributing factors was that Ventura recruited a substantial amount of support from young people, particularly college students, who had never voted before and who registered specifically to vote in the gubernatorial election. The polls did not deem these young people likely voters (since in most cases young people have a lower rate of voter registration and a turnout rate for elections) and so the polling samples were subject to **sampling bias**: they omitted a portion of the electorate that was weighted in favor of the winning candidate.

Definition: Sampling bias

A sampling method is biased if every member of the population doesn't have equal likelihood of being in the sample.

So even identifying the population can be a difficult job, but once we have identified the population, how do we choose an appropriate sample? Remember, although we would prefer to survey all members of the population, this is usually impractical unless the population is very small, so we choose a sample. There are many ways to sample a population, but there is one goal we need to keep in mind: we would like the sample to be *representative of the population*.

Returning to our hypothetical job as a political pollster, we would not anticipate very accurate results if we drew all of our samples from among the customers at a Starbucks, nor would we expect that a sample drawn entirely from the membership list of the local Elks club would provide a useful picture of district-wide support for our candidate.

One way to ensure that the sample has a reasonable chance of mirroring the population is to employ *randomness*. The most basic random method is simple random sampling.

Definition: Simple random sample

A **random sample** is one in which each member of the population has an equal probability of being chosen. A **simple random sample** is one in which every member of the population and any group of members has an equal probability of being chosen.

Example 1.3.1

If we could somehow identify all likely voters in the state, put each of their names on a piece of paper, toss the slips into a (very large) hat and draw 1000 slips out of the hat, we would have a simple random sample.

In practice, computers are better suited for this sort of endeavor than millions of slips of paper and extremely large headgear.

It is always possible, however, that even a random sample might end up not being totally representative of the population. If we repeatedly take samples of 1000 people from among the population of likely voters in the state of Washington, some of these samples might tend to have a slightly higher percentage of Democrats (or Republicans) than does the general population; some samples might include more older people and some samples might include more younger people; etc. In most cases, this **sampling variability** is not significant.

Definition: Sampling variability

The natural variation of samples is called **sampling variability**.

This is unavoidable and expected in random sampling, and in most cases is not an issue.

To help account for variability, pollsters might instead use a **stratified sample**.

Definition: Stratified sampling

In **stratified sampling**, a population is divided into a number of subgroups (or strata). Random samples are then taken from each subgroup with sample sizes proportional to the size of the subgroup in the population.

Example 1.3.2

Suppose in a particular state that previous data indicated that the electorate was comprised of 39% Democrats, 37% Republicans and 24% independents. In a sample of 1000 people, they would then expect to get about 390 Democrats, 370 Republicans and 240 independents. To accomplish this, they could randomly select 390 people from among those voters known to be Democrats, 370 from those known to be Republicans, and 240 from those with no party affiliation.

Stratified sampling can also be used to select a sample with people in desired age groups, a specified mix ratio of males and females, etc. A variation on this technique is called **quota sampling**.

Definition: Quota Sampling

Quota sampling is a variation on stratified sampling, wherein samples are collected in each subgroup until the desired quota is met.

Example 1.3.3

Suppose the pollsters call people at random, but once they have met their quota of 390 Democrats, they only gather people who do not identify themselves as a Democrat.

You may have had the experience of being called by a telephone pollster who started by asking you your age, income, etc. and then thanked you for your time and hung up before asking any "real" questions. Most likely, they already had contacted enough people in your demographic group and were looking for people who were older or younger, richer or poorer, etc. Quota sampling is usually a bit easier than stratified sampling, but also does not ensure the same level of randomness.

Another sampling method is **cluster sampling**, in which the population is divided into groups, and one or more groups are randomly selected to be in the sample.

Cluster sampling

In **cluster sampling**, the population is divided into subgroups (clusters), and a set of subgroups are selected to be in the sample.

Example 1.3.4

If the college wanted to survey students, since students are already divided into classes, they could randomly select 10 classes and give the survey to all the students in those classes. This would be cluster sampling.

Other sampling methods include **systematic sampling**.

Systematic sampling

In **systematic sampling**, every n^{th} member of the population is selected to be in the sample.

Example 1.3.5

To select a sample using systematic sampling, a pollster calls every 100th name in the phone book.

Solution

Systematic sampling is not as random as a simple random sample (if your name is Albert Aardvark and your sister Alexis Aardvark is right after you in the phone book, there is no way you could both end up in the sample) but it can yield acceptable samples.

Perhaps the worst types of sampling methods are **convenience samples** and **voluntary response samples**.

Convenience sampling and voluntary response sampling

Convenience sampling is samples chosen by selecting whoever is convenient.

Voluntary response sampling is allowing the sample to volunteer.

Example 1.3.6

A pollster stands on a street corner and interviews the first 100 people who agree to speak to him. This is a convenience sample.

Example 1.3.7

A website has a survey asking readers to give their opinion on a tax proposal. This is a self-selected sample, or voluntary response sample, in which respondents volunteer to participate.

Usually voluntary response samples are skewed towards people who have a particularly strong opinion about the subject of the survey or who just have way too much time on their hands and enjoy taking surveys.

You Turn 1.3.1

In each case, indicate what sampling method was used

- Every 4th person in the class was selected
- A sample was selected to contain 25 men and 35 women
- Viewers of a new show are asked to vote on the show's website
- A website randomly selects 50 of their customers to send a satisfaction survey to
- To survey voters in a town, a polling company randomly selects 10 city blocks, and interviews everyone who lives on those blocks.

Answer available at the end of this section.

Answers

You Turn 1.3.1

- Systematic
- Stratified or Quota
- Voluntary response
- Simple random
- Cluster

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1.4: How to mess things up before you start

Learning Objectives

After completing this section, you should be able to:

1. Identify the type of bias in a study

There are number of ways that a study can be ruined before you even start collecting data. The first we have already explored – **sampling** or **selection bias**, which is when the sample is not representative of the population. One example of this is **voluntary response bias**, which is bias introduced by only collecting data from those who volunteer to participate. This is not the only potential source of bias.

Sources of bias

- **Sampling bias** – when the sample is not representative of the population
- **Voluntary response bias** – the sampling bias that often occurs when the sample is volunteers
- **Self-interest study** – bias that can occur when the researchers have an interest in the outcome
- **Response bias** – when the responder gives inaccurate responses for any reason
- **Perceived lack of anonymity** – when the responder fears giving an honest answer might negatively affect them
- **Loaded questions** – when the question wording influences the responses
- **Non-response bias** – when people refusing to participate in the study can influence the validity of the outcome

Example 1.4.1

Consider a recent study that found that chewing gum may raise math grades in teenagers[1]. This study was conducted by the Wrigley Science Institute, a branch of the Wrigley chewing gum company. This is an example of a **self-interest study**, one in which the researchers have a vested interest in the outcome of the study. While this does not necessarily ensure that the study was biased, it certainly suggests that we should subject the study to extra scrutiny

Example 1.4.2

A survey asks people, “When was the last time you visited your doctor?” This might suffer from response bias since many people might not remember exactly when they last saw a doctor and give inaccurate responses.

Sources of response bias may be innocent, such as bad memory, or as intentional as pressuring by the pollster. Consider, for example, how many voting initiative petitions people sign without even reading them.

Example 1.4.3

A survey asks participants a question about their interactions with members of other races. Here, a **perceived lack of anonymity** could influence the outcome. The respondent might not want to be perceived as racist even if they are, and give an untruthful answer.

Example 1.4.4

An employer puts out a survey asking their employees if they have a drug abuse problem and need treatment help. Here, answering truthfully might have consequences; responses might not be accurate if the employees do not feel their responses are anonymous or fear retribution from their employer.

Example 1.4.5

A survey asks, “Do you support funding research of alternative energy sources to reduce our reliance on high-polluting fossil fuels?” This is an example of a loaded or leading question—a question whose wording leads the respondent to an answer.

Loaded questions can occur intentionally by pollsters with an agenda, or accidentally through poor question wording. Also a concern is **question order**, where the order of questions changes the results. A psychology researcher provides an example[2]:

“My favorite finding is this: we did a study where we asked students, 'How satisfied are you with your life? How often do you have a date?' The two answers were not statistically related - you would conclude that there is no relationship between dating frequency and life satisfaction. But when we reversed the order and asked, 'How often do you have a date? How satisfied are you with your life?' the statistical relationship was a strong one. You would now conclude that there is nothing as important in a student's life as dating frequency.”

Example 1.4.6

A telephone poll asks the question, “Do you often have time to relax and read a book?” and 50% of the people called refused to answer the survey. It is unlikely that the results will be representative of the entire population. This is an example of **non-response bias**, introduced by people refusing to participate in a study or dropping out of an experiment. When people refuse to participate, we can no longer be so certain that our sample is representative of the population.

Your Turn 1.4.1

In each situation, identify a potential source of bias

- A survey asks how many sexual partners a person has had in the last year
- A radio station asks readers to phone in their choice in a daily poll.
- A substitute teacher wants to know how students in the class did on their last test. The teacher asks the 10 students sitting in the front row to state their latest test score.
- High school students are asked if they have consumed alcohol in the last two weeks.
- The Beef Council releases a study stating that consuming red meat poses little cardiovascular risk.
- A poll asks “Do you support a new transportation tax, or would you prefer to see our public transportation system fall apart?”

Answer available at the end of this section.

[1] Reuters. news.yahoo.com/s/nm/20090423/...k_gum_learning. Retrieved 4/27/09

[2] Swartz, Norbert. www.umich.edu/~newsinfo/MT/01...01/mt6f01.html. Retrieved 3/31/2009

Answers

Your Turn 1.4.1

Add texts here. Do not delete this text first.

- Response bias - historically, men are likely to over-report and women are more likely to under-report
- Voluntary response bias - the sample is self-selected

- c. Sampling bias - the sample may not be representative of the whole class
- d. Lack of anonymity - students may not want to admit this
- e. Self-interest study - the council wants people to consume more red meat
- f. Loaded question - the second option makes you want to choose the first option

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1.5: Presenting Categorical Data Graphically

Learning Objectives

After completing this section, you should be able to:

1. Create a frequency table to summarize categorical data
2. Create a bar chart to summarize categorical data
3. Create a pie chart to summarize categorical data

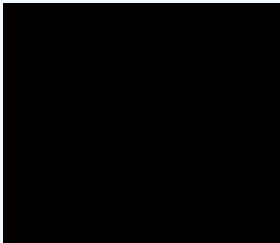
Categorical, or qualitative, data are pieces of information that allow us to classify the objects under investigation into various categories. We usually begin working with categorical data by summarizing the data into a **frequency table**.

Frequency Table

A frequency table is a table with two columns. One column lists the categories, and another for the frequencies with which the items in the categories occur (how many items fit into each category).

Example 1.5.1

An insurance company determines vehicle insurance premiums based on known risk factors. If a person is considered a higher risk, their premiums will be higher. One potential factor is the color of your car. The insurance company believes that people with some color cars are more likely to get in accidents. To research this, they examine police reports for recent total-loss collisions. The data is summarized in the frequency table below.



Sometimes we need an even more intuitive way of displaying data. This is where charts and graphs come in. There are many, many ways of displaying data graphically, but we will concentrate on one very useful type of graph, called a bar graph. In this section, we will work with bar graphs that display categorical data; the next section will be devoted to bar graphs that display quantitative data.

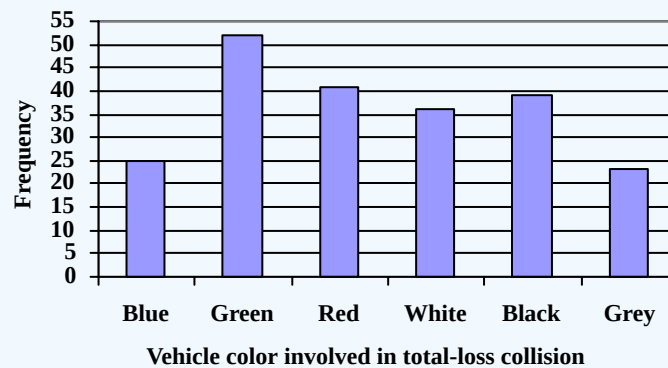
Bar graph

A **bar graph** is a graph that displays a bar for each category with the length of each bar indicating the frequency of that category.

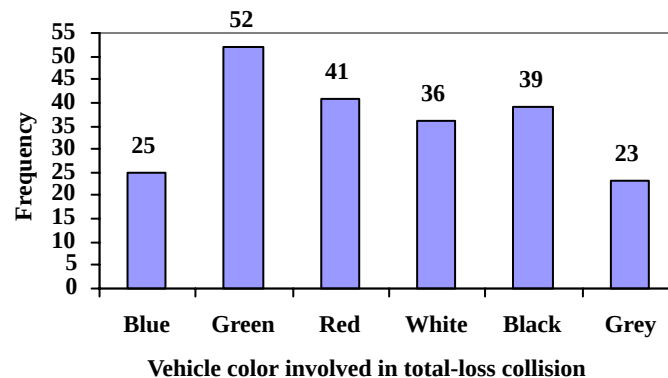
To construct a bar graph, we need to draw a vertical axis and a horizontal axis. The vertical direction will have a scale and measure the frequency of each category; the horizontal axis has no scale in this instance. The construction of a bar chart is most easily described by using an example.

Example 1.5.2

Using our car data from above, note that the highest frequency is 52. So our vertical axis needs to go from 0 to 52, but we might as well use 0 to 55, so that we can put a hash mark every 5 units.



Notice that the height of each bar is determined by the frequency of the corresponding color. The horizontal grid lines are a nice touch, but not necessary. In practice, you will find it useful to draw bar graphs using graph paper, so the grid lines will already be in place, or using technology. Instead of grid lines, we might also list the frequencies at the top of each bar, like this:



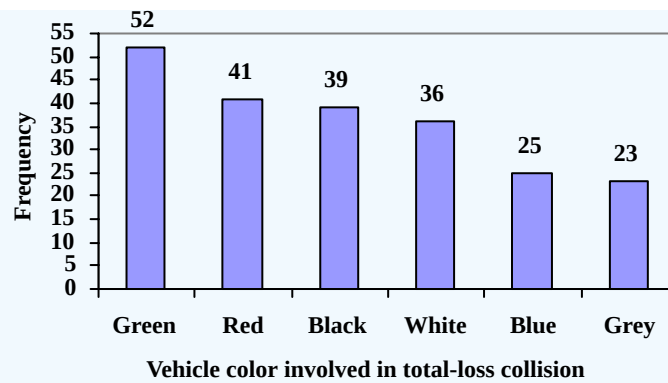
In this case, our chart might benefit from being reordered from largest to smallest frequency values. This arrangement can make it easier to compare similar values in the chart, even without grid lines. When we arrange the categories in decreasing frequency order like this, it is called a **Pareto chart**.

Pareto chart

A **Pareto chart** is a bar graph ordered from highest to lowest frequency

Example 1.5.3

Transforming our bar graph from earlier into a Pareto chart, we get:



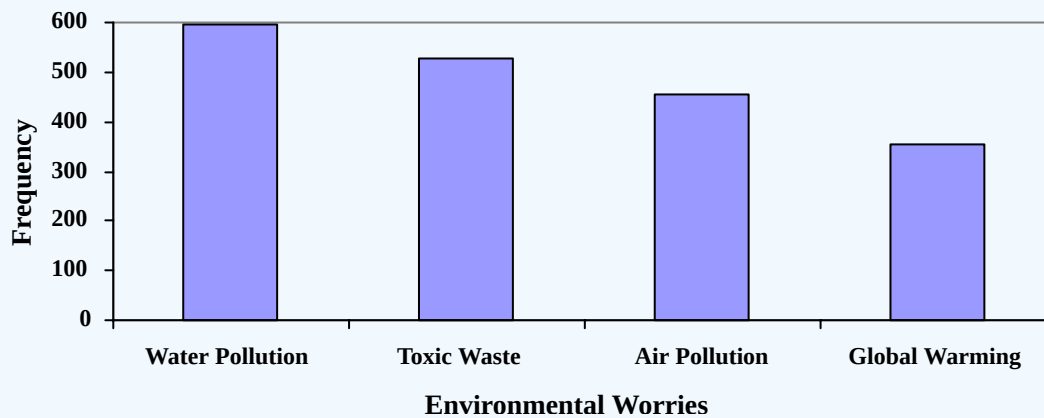
Example 1.5.4

In a survey[1], adults were asked whether they personally worried about a variety of environmental concerns. The numbers (out of 1012 surveyed) who indicated that they worried “a great deal” about some selected concerns are summarized below.



Solution

This data could be shown graphically in a bar graph:



To show relative sizes, it is common to use a pie chart.

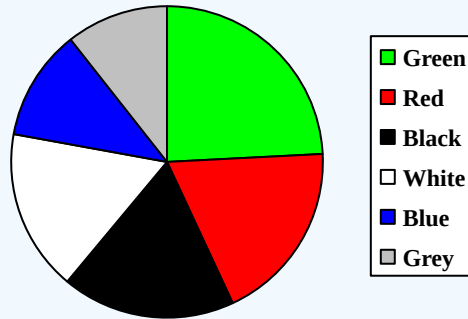
Pie Chart

A **pie chart** is a circle with wedges cut of varying sizes marked often benefit from including frequencies or relative frequencies (percents) next to the pie slices. Often,

Example 1.5.5

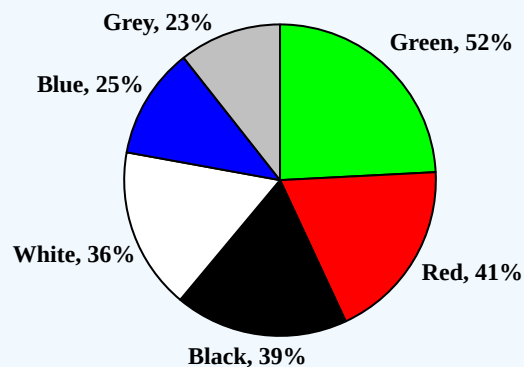
For our vehicle color data, a pie chart might look like this:

Vehicle color involved in total-loss collisions



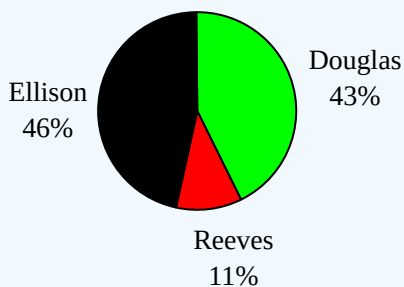
Pie charts can often benefit from including frequencies or relative frequencies (percents) in the chart next to the pie slices. Often, having the category names next to the pie slices also makes the chart clearer.

Vehicle color involved in total-loss collisions



Example 1.5.6

Voter preferences



The pie chart to the right shows the percentage of voters supporting each candidate running for a local senate seat.

If there are 20,000 voters in the district, the pie chart shows that about 11% of those, about 2,200 voters, support Reeves.

Pie charts look nice, but are harder to draw by hand than bar charts since to draw them accurately we would need to compute the angle each wedge cuts out of the circle, then measure the angle with a protractor. Computers are much better suited to drawing pie charts. Common software programs like Microsoft Word or Excel, OpenOffice.org Write or Calc, or Google Docs are able to create bar graphs, pie charts, and other graph types. There are also nummanager's salary and the average worker's saphs[2].

Your Turn 1.5.1

Create a bar graph and a pie chart to illustrate the grades on a history exam below.

A: 12 students, B: 19 students, C: 14 students, D: 4 students, F: 5 students

Answer available at the end of this section.

Don't get fancy with graphs! People sometimes add features to graphs that don't help to convey their information. Instead of plain bars, it is tempting to substitute meaningful images. This type of graph is called a **pictogram**.

Pictogram

A **pictogram** is a statistical graphic in which the size of the picture is intended to represent the frequencies or size of the values being represented.

Example 1.5.7



**Manager
Salaries**

**Worker
Salaries**

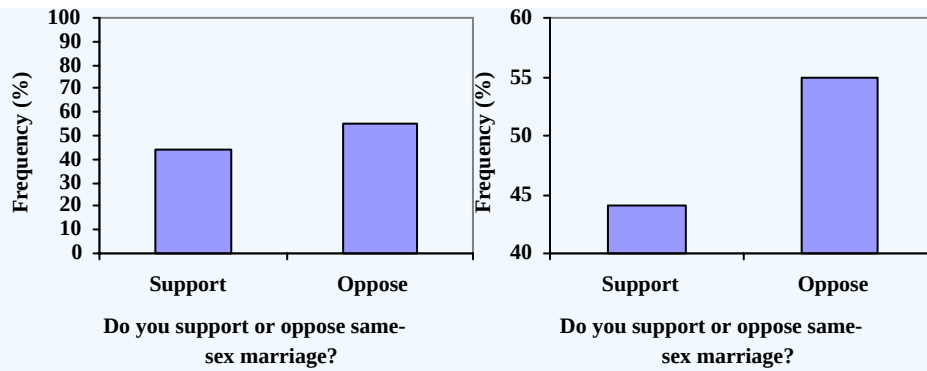
A labor union might produce the graph to the right to show the difference between the average manager's salary and the average worker's salary.

Looking at the picture, it would be reasonable to guess that the manager salaries is 4 times as large as the worker salaries – the area of the bag looks about 4 times as large. However, the manager salaries are in fact only twice as large as worker salaries, which were reflected in the picture by making the manager bag twice as tall.

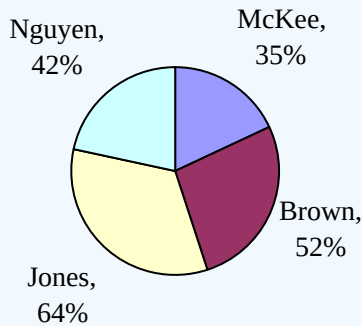
Another distortion in bar charts results from setting the baseline to a value other than zero. The baseline is the bottom of the vertical axis, representing the least number of cases that could have occurred in a category. Normally, this number should be zero.

Example 1.5.8

Compare the two graphs below showing support for same-sex marriage rights from a poll taken in December 2008[3]. The difference in the vertical scale on the first graph suggests a different story than the true differences in percentages. The second graph makes it look like twice as many people oppose marriage rights as support it.



Your Turn 1.5.2



A poll was taken asking people if they agreed with the positions of the 4 candidates for a county office. Does the pie chart present a good representation of this data? Explain.

Answer available at the end of this section.

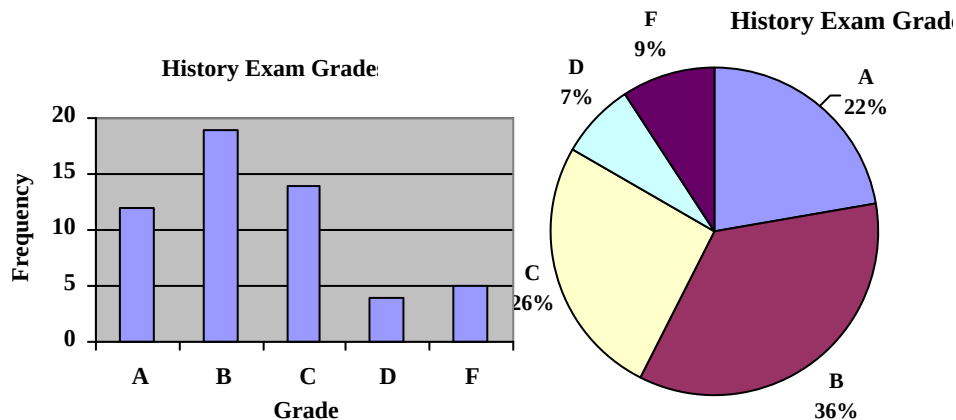
[1] Gallup Poll. March 5-8, 2009. <http://www.pollingreport.com/enviro.htm>

[2] For example: <http://nces.ed.gov/nceskids/createAgraph/> or <http://docs.google.com>

[3] CNN/Opinion Research Corporation Poll. Dec 19-21, 2008, from <http://www.pollingreport.com/civil.htm>

Answers

Your Turn 1.5.1



Your Turn 1.5.2

While the bar chart accurately depicts the relative size of the people agreeing with each candidate, the chart is confusing, since usually percents on a pie chart represent the percentage of the pie the slice represents.

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1.6: Presenting Quantitative Data Graphically

Learning Objectives

After completing this section, you should be able to:

1. Summarize quantitative data into a frequency table
2. Create a histogram for quantitative data
3. Create a graph to compare two similar sets of data

Quantitative, or numerical, data can also be summarized into frequency tables.

Example 1.6.1

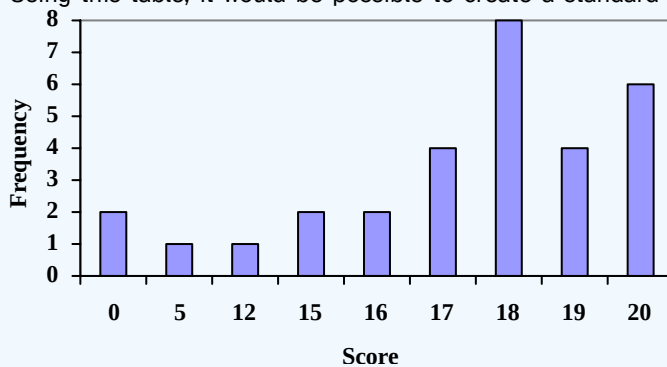
A teacher records scores on a 20-point quiz for the 30 students in his class. The scores are:

19 20 18 18 17 18 19 17 20 18 20 16 20 15 17 12 18 19 18 19 17 20 18 16 15 18 20 5 0 0

These scores could be summarized into a frequency table by grouping like values:



Using this table, it would be possible to create a standard bar chart from this summary, like we did for categorical data:



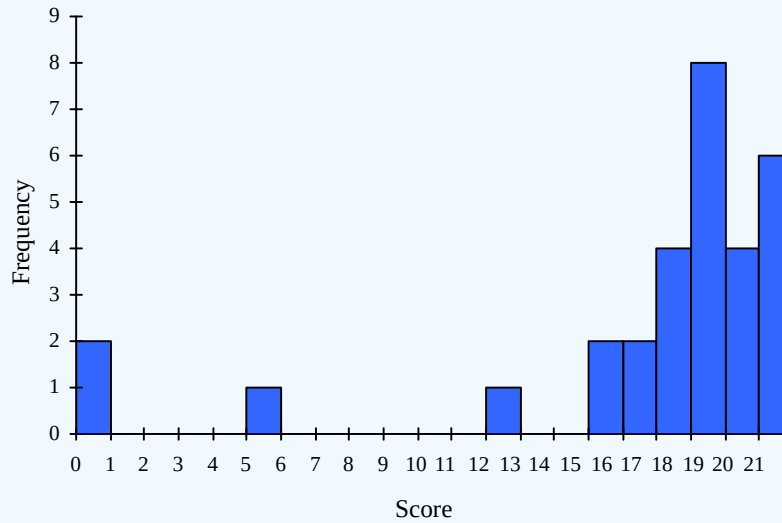
However, since the scores are numerical values, this chart doesn't really make sense; the first and second bars are five values apart, while the later bars are only one value apart. It would be more correct to treat the horizontal axis as a number line. This type of graph is called a **histogram**.

Histogram

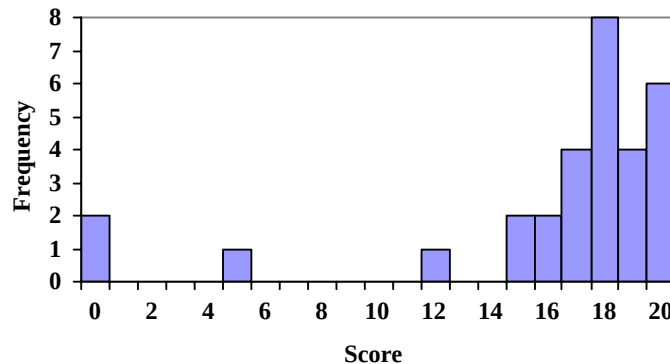
A histogram is like a bar graph, but where the horizontal axis is a number line.

Example 1.6.2

For the values above, a histogram would look like:



Notice that in the histogram, a bar represents values on the horizontal axis from that on the left hand-side of the bar up to, but not including, the value on the right hand side of the bar. Some people choose to have bars start at $\frac{1}{2}$ values to avoid this ambiguity.



Unfortunately, not a lot of common software packages can correctly graph a histogram. About the best you can do in Excel or Word is a bar graph with no gap between the bars and spacing added to simulate a numerical horizontal axis.

If we have a large number of widely varying data values, creating a frequency table that lists every possible value as a category would lead to an exceptionally long frequency table, and probably would not reveal any patterns. For this reason, it is common with quantitative data to group data into **class intervals**.

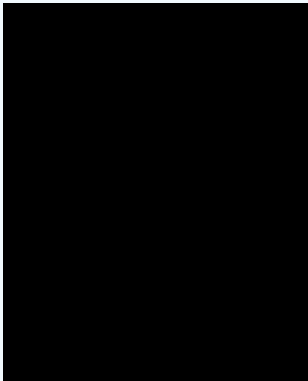
Class Intervals

Class intervals are groupings of the data. In general, we define class intervals so that:

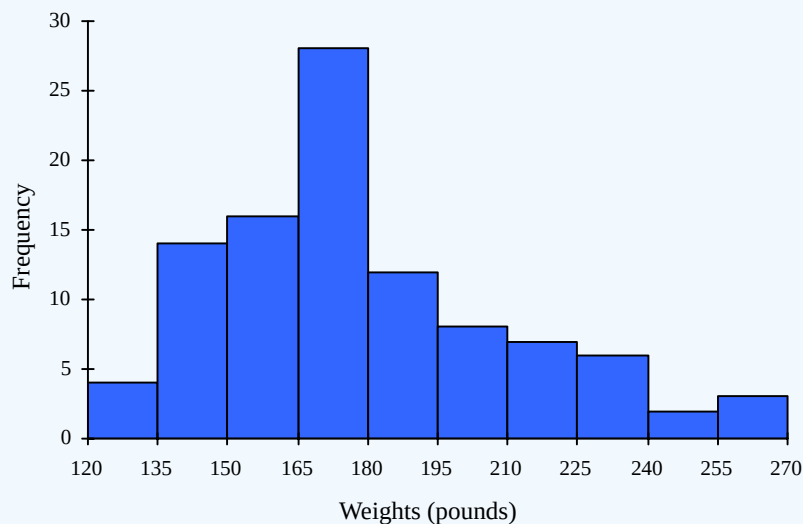
- Each interval is equal in size. For example, if the first class contains values from 120-129, the second class should include values from 130-139.
- We have somewhere between 5 and 20 classes, typically, depending upon the number of data we're working with.

Example 1.6.3

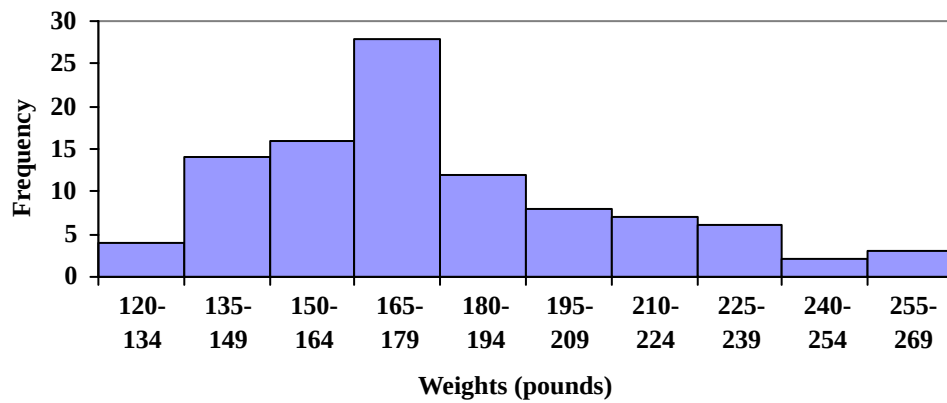
Suppose that we have collected weights from 100 male subjects as part of a nutrition study. For our weight data, we have values ranging from a low of 121 pounds to a high of 263 pounds, giving a total span of $263 - 121 = 142$. We could create 7 intervals with a width of around 20, 14 intervals with a width of around 10, or somewhere in between. Often time we have to experiment with a few possibilities to find something that represents the data well. Let us try using an interval width of 15. We could start at 121, or at 120 since it is a nice round number.



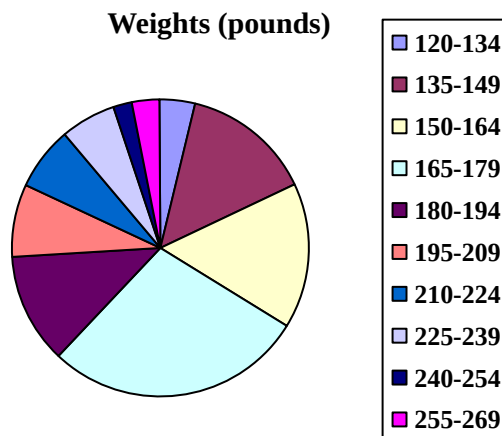
A histogram of this data would look like:



In many software packages, you can create a graph similar to a histogram by putting the class intervals as the labels on a bar chart.



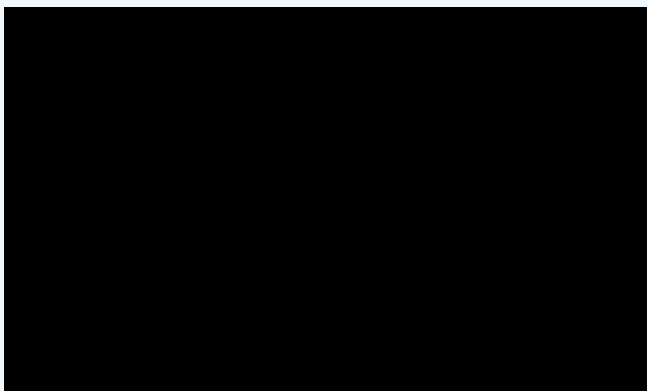
Other graph types such as pie charts are possible for quantitative data. The usefulness of different graph types will vary depending upon the number of intervals and the type of data being represented. For example, a pie chart of our weight data is difficult to read because of the quantity of intervals we used.



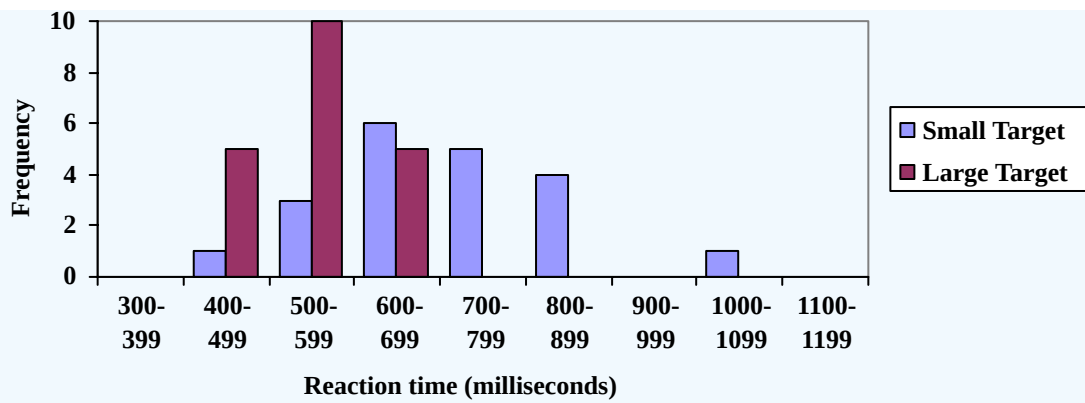
When collecting data to compare two groups, it is desirable to create a graph that compares quantities.

Example 1.6.4

The data below came from a task in which the goal is to move a computer mouse to a target on the screen as fast as possible. On 20 of the trials, the target was a small rectangle; on the other 20, the target was a large rectangle. Time to reach the target was recorded on each trial.



One option to represent this data would be a comparative histogram or bar chart, in which bars for the small target group and large target group are placed next to each other.

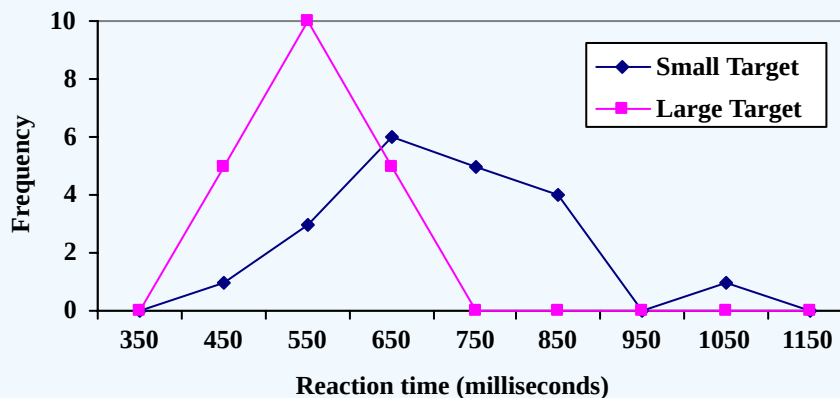


Frequency polygon

An alternative representation is a **frequency polygon**. A frequency polygon starts out like a histogram, but instead of drawing a bar, a point is placed in the midpoint of each interval at height equal to the frequency. Typically the points are connected with straight lines to emphasize the distribution of the data.

Example 1.6.5

This graph makes it easier to see that reaction times were generally shorter for the larger target, and that the reaction times for the smaller target were more spread out.



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2: ANALYZING DATA

2.1: Mean, Median, and Mode

2.2: Range and Standard Deviation

2.3: Percentiles

2.1: Mean, Median, and Mode



Figure 2.1.1: What does it mean to say someone has average height? (credit: modification of work “I’m the tallest” by Jenn Durfey/ Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

1. Calculate the mode of a dataset.
2. Calculate the median of a dataset.
3. Calculate the mean of a dataset.
4. Contrast measures of central tendency to identify the most representative average.
5. Solve application problems involving mean, median, and mode.

What exactly do we mean when we describe something as “average”? Is the height of an average person the height that more people share than any other? What if we line up every person in the world, in order from shortest to tallest, and find the person right in the middle: Is that person’s height the average? Or maybe it’s something more complicated.

Imagine a game where you and a friend are trying to guess the typical person’s height. Once the guesses are made, you bring in every person and measure their height. You and your friend figure out how far off each of your guesses were from the actual value, then square that number. The result is the number of points you earn for that person. After we check every height and award points accordingly, the person with the lower score wins (because a lower score means that person’s guess was, overall, closer to the actual values). Could we define the average height to be the number that you should guess to give you the smallest possible score?

Each of these three methods of determining the “average” is commonly used. They are all methods of measuring *centrality* (or *central tendency*). Centrality is just a word that describes the middle of a set of data. All give potentially different results, and all are useful for different reasons. In this section, we’ll explore each of these methods of finding the “average.”

The Mode

In our discussion of average heights, the first possible definition we offered was the height that more people share than any other. This is the **mode**, or the value that appears most often. If there are two modes, the data are **bimodal**. Let’s look at some examples.

If we're given a frequency distribution, the technique is straightforward: we're looking for the number with the highest frequency.

Example 2.1.1

Finding the Mode Using a Frequency Distribution

Below is a frequency distribution of the number of siblings of conflict resolution class attendees.

Number of Siblings	Frequency
0	5
1	13
2	6
3	3
4	2
5	1

What is the mode of the number of siblings?

Answer available at the end of this section.

Your Turn 2.1.1

Here is a frequency distribution for the number of people who shared a residence with people in a sample.

Number of People in the Residence	Frequency
1	12
2	13
3	8
4	6
5	1

What's the mode of the number of people in these residences?

Answer

The highest frequency is 13, so the mode is 2. The most common number of people who shared a residence was 2.

What happens if there is no number in the data that appears more than once? In that case, by our definition, every data value is a mode. But according to some other definitions, the data would have no mode. In practice, though, it doesn't really matter; if no data value appears more than once, then the mode is not helpful at all as a measure of centrality.

The Median

Let's revisit our example of trying to identify the height of the "average" person. If we lined everyone up in order by height and found the person right in the middle, that person's height is called the median, or the value that is greater than no more than half and less than no more than half of the values.

Let's look at a really simple example. Consider the following list of numbers: 11, 12, 13, 13, 14. Is the first number on the list, 11, the median? There are no values less than 11 (that's 0%), and there are four values greater than 11 (that's 80%). Since more than 50% of the data are greater than 11, the definition is violated; it's not the median. Here's a chart with the rest of the data.

Data Value	Number of Values Below	Percentage of Values Below	Number of Values Above	Percentage of Values Above
11	0	0%	4	80%
12	1	20%	3	60%
13	2	40%	1	20%
14	4	80%	0	0%

Table 2.1.1: Finding the median of a set of numbers

Only 13 has no violations, so it's the median according to the definition. In practice, we find the median just like we described in the average height example: by lining up all the data values in order from smallest to largest and picking the value in the middle. For our easy example (with data values 11, 12, 13, 13, 14), that first 13 is right in the middle; there are two values to the left and two values to the right. If there's not one value right in the middle, we pick the two closest, then choose the number exactly between them.

For example, let's say we have the data 41, 44, 46, 53. Since there are an even number of data values in our list, we can't pick the one right in the middle. The two closest to the middle are 44 and 46, so we'll choose the number halfway between those to be the median: 45. As this example shows, the median (unlike the mode) doesn't have to be a number in our original set of data.

In the examples we've looked at so far, it's been pretty easy to identify which number is right in the middle. If we had a very large dataset, though, it might be harder. Fortunately, we have some formulas to help us with that.

FORMULA

Suppose we have a set of data with n values, ordered from smallest to largest. If n is odd, then the median is the data value at position $\frac{n+1}{2}$. If n is even, then we find the values at positions $\frac{n}{2}$ and $\frac{n}{2} + 1$. If those values are named a and b , then the median is defined to be $\frac{a+b}{2}$.

Let's put those formulas to work in an example.

Example 2.1.2

Finding the Median

The data below is a list of 33 sale prices (in dollars) of a particular collectible trading card: 5, 8, 9, 10, 10, 10, 13, 14, 14, 15, 15, 15, 16, 19, 19, 20, 20, 20, 20, 25, 25, 29, 29, 30, 30, 30, 35, 35, 40, 40, 45, 60. What is the median price?

Answer available at the end of this section.

Your Turn 2.1.2

Consider the data given below. There are 17 data values. Find the median.

121, 122, 122, 125, 130, 133, 134, 134, 136, 138, 142, 145, 149, 149, 150, 153, 170

Answer available at the end of this section.

Now, let's tackle an example with an even number of values.

Example 2.1.3

Finding the Median

The data below show the number of times different crickets (of differing species, genders, etc.) chirped in a one-minute span. Find the median.

89	97	82	102	84	99
115	105	89	109	107	89
101	109	116	103	100	91
93	103	120	91	85	104
104	82	106	92	104	106

Answer available at the end of this section.

Your Turn 2.1.3

This table gives the records of the Major League Baseball teams at the end of the 2019 season. Find the median number of wins.

Team	Wins	Losses	Team	Wins	Losses
HOU	107	55	PHI	81	81
LAD	106	56	TEX	78	84
NYN	103	59	SFG	77	85
MIN	101	61	CIN	75	87
ATL	97	65	CHW	72	89
OAK	97	65	LAA	72	90
TBR	96	66	COL	71	91
CLE	93	69	SDP	70	92
WSN	93	69	PIT	69	93
STL	91	71	SEA	68	94
MIL	89	73	TOR	67	95
NYM	86	76	KCR	59	103
ARI	85	77	MIA	57	105
BOS	84	78	BAL	54	108
CHC	84	78	DET	47	114

Table 2.1.2: Wins and losses at the end of the 2019 MLB season (source: www.mlb.com)

Answer available at the end of this section.

Example 2.1.4

Finding the Median Using a Frequency Distribution

In Example 2.1.1, we looked at a frequency distribution of the number of siblings of the people who attended a conflict resolution class. Let's review that data again. What is the median number of siblings?

Number of Siblings	Frequency
0	5
1	13
2	6
3	3
4	2
5	1

Answer available at the end of this section.

Your Turn 2.1.4

In Your Turn 2.1.1, you looked at a frequency distribution for the number of people who shared a residence with people in a sample. Let's review that data again. What's the median of the number of people in these residences?

Number of People in the Residence	Frequency
1	12
2	13
3	8
4	6
5	1

Answer

If we add up all the frequencies, we find there are a total of 40 data values. Since we have an even number of data values, we need to find the values in positions $\frac{40}{2} = 20$ and $\frac{40}{2} + 1 = 21$. There are twelve 1s, so the first twelve data values are 1. Then, there are thirteen 2s. So there are 2s in positions 13 - 25. So both the twentieth and the twenty-first data values are 2.

We use the formula to compute the median: $\frac{2+2}{2} = 2$.

The Mean

Recall our example of ways we could identify the “average” height of an individual. The last method we discussed was also the most complicated. It involved a game where the player guesses a height, then figures out how far off that guess is from every single person's height. Those differences get squared and added together to get a score. Our next measure of centrality gives the lowest possible score: No other guess would beat it in the game. Given a dataset containing n total values, the mean of the

dataset is the sum of all the data values, divided by n .

This is a computation you have likely done before. In many places, including spreadsheet programs like Microsoft Excel and Google Sheets, this number is called the *average*. For statisticians, though, the word *average* has too many possible meanings, so they prefer the one we'll use: mean.

Example 2.1.5

Finding the Mean

Compute the mean of the numbers 12, 15, 17, 18, 18, and 19.

Answer available at the end of this section.

Your Turn 2.1.5

Compute the mean of the numbers 5, 8, 11, 12, 12, 12, 15, 18, and 20. Round your answer to three decimal places.

Answer available at the end of this section.

Example 2.1.6

Finding the Mean Using a Frequency Distribution

Refer again to the frequency distribution of the number of siblings people who attended a conflict resolution class reported.

Number of Siblings	Frequency
0	5
1	13
2	6
3	3
4	2
5	1

What is the mean of the number of siblings?

Answer available at the end of this section.

Your Turn 2.1.6

Refer again to the frequency distribution of the number of people who shared a residence with people in a sample. What's the mean of the number of people in these residences? Round your answer to three decimal places.

Number of People in the Residence	Frequency
1	12
2	13

Number of People in the Residence	Frequency
3	8

Answer

Step 1: We compute the mean by adding up all the data values and then dividing by the number of data values on the list.

Step 2: Adding up the frequencies, we get $12 + 13 + 8 = 33$ data values in our list.

Step 3: Let's add a column to our distribution for the products:

Number of People in Residence	Frequency	(Number of People) $\times$ (Frequency)
1	12	12
2	13	26
3	8	24

Step 4: So, the sum of all our data values is $12 + 26 + 24 = 62$. The mean is $\frac{62}{33} \approx 1.879$.

As the number of data values we are considering grows, the computation for the mean gets more and more complicated. That's why people generally trust technology to perform that computation.

Video

[Compute Measures of Centrality Using Google Sheets](#)

Note that a recent update to Google Sheets introduced a new function called "MODE.MULT," which will find every mode (not just the first one on the list).

Example 2.1.7

Using Google Sheets to Compute Measures of Centrality

The dataset "InState" contains the in-state tuitions of every college and university in the country that reported that data to the Department of Education. Find the mode, median, and mean of those in-state tuition values.

Answer available at the end of this section.

Your Turn 2.1.7

The dataset "MLB2019" gives the number of wins for every Major League Baseball team in the 2019 season. Use Google Sheets to find the mode(s), median, and mean.

Which Is Better: Mode, Median, or Mean?

If the mode, median, and mean all purport to measure the same thing (centrality), why do we need all three? The answer is complicated, as each measure has its own strengths and weaknesses. The mode is simple to compute, but there may be more than one. Further, if no data value appears more than once, the mode is entirely unhelpful. As for the mean and median, the main difference between these two measures is how each is affected by extreme values.

Consider this example: the mean and median of 1, 2, 3, 4, 5 are both 3. But what if the dataset is instead 1, 2, 3, 4, 10? The median is still 3, but the mean is now 4. What this example shows is that the mean is sensitive to extreme values, while the median isn't.

This knowledge can help us decide which of the two is more relevant for a given dataset. If it is important that the really high or really low values are reflected in the measure of centrality, then the mean is the better option. If very high or low values are not important, however, then we should stick with the median.

Example 2.1.8

Choosing Which Measure of Centrality to Use

For the following situations, decide which measure(s) of centrality would be best:

1. You found a used car that you like, and you want to know if the price is too high or too low. You find a list of sale prices for that make and model, and you see that the distribution of those prices is skewed to the right. Some of the prices at the high end are close to the original sale price of the car, so you guess that those cars might have really low mileage, or have other enhancements added on that increased the value (but which don't apply to the car you found).
2. You are asked to analyze the responses to a survey. One of the questions asked, "How strongly do you agree with the statement, 'I believe my elected representatives have my best interests in mind when they vote'?" Responses are a number between 1 and 5, with 1 representing "strongly disagree" and 5 representing "strongly agree."
3. You are asked to find the "average" household income for a zip code. Those values are skewed right.
4. Thinking back to the situation at the beginning of this section: you want to find "average" height. The data you've collected seem to be distributed symmetrically.

Answer available at the end of this section.

Answers

The mode is the value that appears the most often, which means it has the greatest frequency. Thirteen of the respondents have one sibling, more than any other number. So, the mode is 1.

Step 1: Since 33 is odd, the median is the data value at position $\frac{n+1}{2}$, where n is the number of values in the dataset. There are 33 total values, so our formula becomes $\frac{33+1}{2} = 17$. That means we want to look for the 17th number in the dataset.

Step 2: We'll want to count from the lowest value to the 17th number. We can use our list of data to do this since the prices are in order. The seventeenth number is 20, so the median is 20.

Step 1: Since 17 is odd, the median is the data value at position $\frac{n+1}{2}$, where n is the number of values in the dataset. There are 17 total values, so our formula becomes $\frac{17+1}{2} = 9$. That means we want to look for the 9th number in the dataset.

Step 2: We'll want to count from the lowest value to the 9th number. We can use our list of data to do this since the values are in order. The ninth number is 136, so the median is 136.

Step 1: In order to find the median, we first need to sort the data so that they're in order, smallest to largest:

82	82	84	85	89	89
89	91	91	92	93	97
99	100	101	102	103	103
104	104	104	105	106	106
107	109	109	115	116	120

Step 2: Next, we figure out how many data values we have. Counting them up, we see there are 30, which is even.

Step 3: Since we have an even number of data values, we need to find the values in positions $\frac{30}{2} = 15$ and $\frac{30}{2} + 1 = 16$. These are 101 and 102.

Step 4: We use the formula to compute the median: $\frac{101+102}{2} = 101.5$.

Step 1: In order to find the median, we first need to sort the data so that they're in order, smallest to largest:

47, 54, 57, 59, 67, 68, 69, 70, 71, 72, 72, 75, 77, 78, 81, 84, 84, 85, 86, 89, 91, 93, 93, 96, 97, 97, 101, 103, 106, 107

Step 2: Next, we figure out how many data values we have. Counting them up, we see there are 30, which is even.

Step 3: Since we have an even number of data values, we need to find the values in positions $\frac{30}{2} = 15$ and $\frac{30}{2} + 1 = 16$. These are 81 and 84.

Step 4: We use the formula to compute the median: $\frac{81+84}{2} = 82.5$.

If we add up all the frequencies, we find that there are 30 data values total. So the median is between the 15th and 16th values in the ordered list. There are five 0s and thirteen 1s according to the frequency distribution, so items one through five are all 0s and items six through eighteen are all 1s. Since both items fifteen and sixteen are 1s, the median is 1.

The mean is the sum of the values, divided by the number of values on the list. So, we get:

$$\frac{12 + 15 + 17 + 18 + 18 + 19}{6} = \frac{99}{6} = 16.5.$$

The mean is the sum of the values, divided by the number of values on the list. So, we get:

$$\frac{5 + 8 + 11 + 12 + 12 + 12 + 15 + 18 + 20}{9} = \frac{113}{9} \approx 12.556.$$

Step 1: We compute the mean by adding up all the data values and then dividing by the number of data values on the list.

Step 2: Adding up the frequencies, we get $5 + 13 + 6 + 3 + 2 + 1 = 30$ data values in our list.

Step 3: Now, to find the sum of all the data values, we could simply reconstruct the raw data and add up all the numbers there. But, there's an easier way: Remember that repeatedly adding a number to itself is the definition of multiplication. So, for example, since there are six 2s in our data, the sum of all those 2s must be $6 \times 2 = 12$.

Step 4: Let's add a column to our distribution for these products:

Number of Siblings	Frequency	(Number of Siblings) $\times$ (Frequency)
0	5	0
1	13	13
2	6	12
3	3	9
4	2	8
5	1	5

Step 5: So, the sum of all our data values is $0 + 13 + 12 + 9 + 8 + 5 = 47$. The mean is $\frac{47}{30} \approx 1.567$.

Step 1: To find the mode, we select an empty cell type `=MODE()`, click on the header of the column to insert a reference to the column into our formula, and then close the parentheses. When we hit the enter key, our formula is replaced with the mode: \$13,380.

Step 2: We can find the median and mean using the same process, except using the functions `"MEDIAN"` and `"AVERAGE"` in place of `"MODE"`. We find that the median is \$11,207 and the mean is \$15,476.79.

1. In this situation, the high values are not comparable to the value of the car you found and we don't want them to affect the results. Also, we're unlikely to find many repeated values, so the mode is probably not useful. Median is best.
2. Here, we want to know what a typical result is. The mean doesn't really make sense; it involves adding the numbers together, so it would treat two "strongly disagree" and two "strongly agree" responses (those add to $1 + 1 + 5 + 5 = 12$) as exactly the same as four "neutral" responses ($3 + 3 + 3 + 3 = 12$). But those are really different situations; the first shows a strong polarization in the responses, while the second represents strong indifference. The mode is probably the best choice (because the data are actually categorical), but the median would be good too.
3. The mode isn't going to be useful in this situation because it's unlikely you will find many households that have exactly the same income. The mean and median will be different because of the skew, so the choice comes down to the extreme values. Remember that the data are skewed right, so high values are prevalent. Because these high values are important for our analysis, we want them to be reflected in the results. Thus, the mean is best. That being said, the median is also useful; it allows us to say something like "50% of the households surveyed make more than" the median.
4. Because we aren't likely to find many people with exactly the same height, the mode won't be useful. Since the data are symmetric, the mean and the median will be about the same. So, it doesn't really matter which of those two we choose.

2.2: Range and Standard Deviation

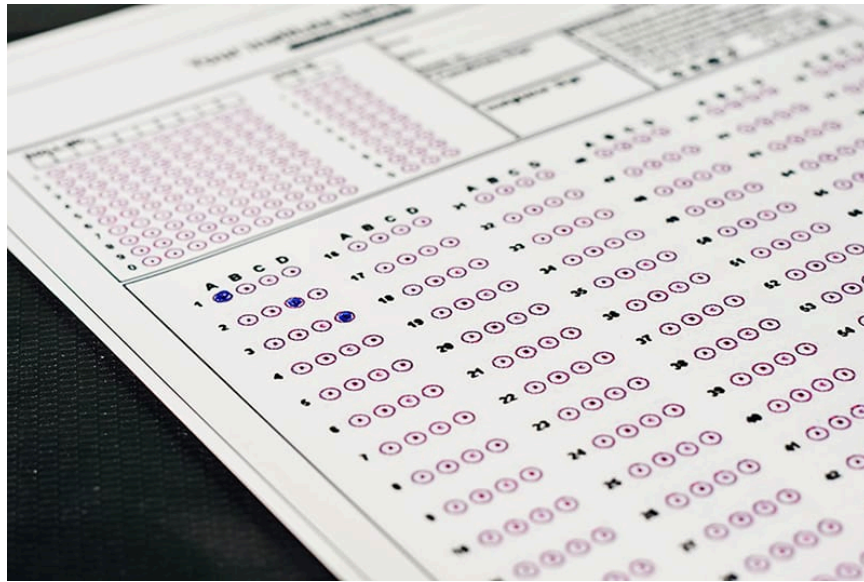


Figure 2.2.1: Measures of spread help us get a better understanding of test scores. (credit: "Standardized test exams form with answers bubbled" by Marco Verch Professional Photographer/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

1. Calculate the range of a dataset
2. Calculate the standard deviation of a dataset

Measures of centrality like the mean can give us only part of the picture that a dataset paints. For example, let's say you've just gotten the results of a standardized test back, and your score was 138. The mean score on the test is 120. So, your score is above average! But how good is it *really*? If all the scores were between 100 and 140, then you know your score must be among the best. But if the scores ranged from 0 to 200, then maybe 140 is good, but not great (though still above average). Knowing information about how the data are spread out can help us put a particular data value in better context. In this section, we'll look at two numbers that help us describe the spread in the data: the range and the standard deviation. These numbers are called measures of dispersion.

The Range

Our first measure of dispersion is the **range**, or the difference between the maximum and minimum values in the set. It's the measure we used in the standardized test example above. Let's look at a couple of examples.

Example 2.2.1

Finding the Range

You survey some of your friends to find out how many hours they work each week. Their responses are: 5, 20, 8, 10, 35, 12. What is the range?

Answer available at the end of this section.

Your Turn 2.2.1

On your morning commute, you decide to record how long you have to wait each time you get caught at a red light. Here are the times in seconds: 12, 58, 35, 79, 21. What is the range?

Answer available at the end of this section.

The range is very easy to compute, but it depends only on two of the data values in the entire set. If there happens to be just one unusually high or low data value, then the range might give a distorted measure of dispersion. Our next measure takes every single data value into account, making it more reliable.

The Standard Deviation

The standard deviation is a measure of dispersion that can be interpreted as approximately the average distance of every data value from the mean. (This distance from the mean is the “deviation” in “standard deviation.”)

FORMULA

The standard deviation is computed as follows:

$$s = \sqrt{\frac{\sum (x - \bar{x})^2}{n - 1}}$$

Here, x represents each data value, $\bar{x}$ is the mean of the data values, n is the number of data values, and the capital sigma (Σ) indicates that we take a sum.

To compute the standard deviation using the formula, we follow the steps below:

1. Compute the mean of all the data values.
2. Subtract the mean from each data value.
3. Square those differences.
4. Add up the results in step 3.
5. Divide the result in step 4 by $n - 1$
6. Take the square root of the result in step 5.

Let's see that process in action.

Example 2.2.2

Computing the Standard Deviation

You surveyed some of your friends to find out how many hours they work each week. Their responses were: 5, 20, 8, 10, 35, 12. What is the standard deviation? Round your answer to three decimal places.

Answer available at the end of this section.

Your Turn 2.2.2

On your morning commute, you decide to record how long you have to wait each time you get caught at a red light. Here are the times in seconds: 12, 58, 35, 79, 21. What is the standard deviation?

Answer available at the end of this section.

The computation for the standard deviation is complicated, even for just a small dataset. We'd never want to compute it without technology for a large dataset! Luckily, technology makes this calculation easy.

Video

[Find the Standard Deviation Using Google Sheets](#)

Example 2.2.3

Finding the Standard Deviation with Google Sheets

The data in “AvgSAT” contains the average SAT score for students attending every institution of higher learning in the US for which data is available. What is the standard deviation of these average SAT scores?

Answer available at the end of this section.

Your Turn 2.2.3

The file “InState” contains in-state tuition costs (in dollars) for every institution of higher learning in the US for which data is available. What is the standard deviation of these costs?

Answer available at the end of this section.

Answers

The maximum value in the set is 35 and the minimum is 5, so the range is $35 - 5 = 30$.

The maximum value in the set is 79 and the minimum is 12, so the range is $79 - 12 = 67$.

Let's follow the six steps mentioned previously to compute the standard deviation.

Step 1: Find the mean: $\bar{x} = \frac{5+20+8+10+35+12}{6} = 15$.

Step 2: Subtract the mean from each data value. To help keep track, let's do this in a table. In the first row, we'll list each of our data values; in the second, we'll subtract the mean of 15 from each data value.

x	5	20	8	10	35	12
$x - \bar{x}$	-10	5	-7	-5	20	-3

Step 3: Square the differences. Let's add a row to our table for those values:

x	5	20	8	10	35	12
$x - \bar{x}$	-10	5	-7	-5	20	-3
$(x - \bar{x})^2$	100	25	49	25	400	9

Step 4: Add up those squares: $100 + 25 + 49 + 25 + 400 + 9 = 608$.

Step 5: Divide the sum by $n - 1$. Since we have 6 data values, that gives us $\frac{608}{6-1} = 121.6$.

Step 6: Take the square root of the result: $\sqrt{121.6} \approx 11.027$.

Thus, the standard deviation is $s \approx 11.027$.

Let's follow the six steps mentioned previously to compute the standard deviation.

Step 1: Find the mean: $\bar{x} =$ 5=41

Step 2: Subtract the mean from each data value. To help keep track, let's do this in a table. In the first row, we'll list each of our data values; in the second, we'll subtract the mean of 41 from each data value.

x	12	58	35	79	21
-----	----	----	----	----	----

$x - \bar{x}$	-29	17	-6	38	-20
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Step 3: Square the differences. Let's add a row to our table for those values:

x	12	58	35	79	21
$x - \bar{x}$	-29	17	-6	38	-20
$(x - \bar{x})^2$	841	289	36	1444	400

Step 4: Add up those squares: $841 + 289 + 36 + 1444 + 400 = 3010$

Step 5: Divide the sum by $n - 1$. Since we have 5 data values, that gives us $\frac{3010}{5-1} = 752.5$.

Step 6: Take the square root of the result: $\sqrt{752.5} \approx 27.432$.

Thus, the standard deviation is $s \approx 27.432$.

To find the standard deviation, we click in an empty cell in our spreadsheet and then type "=STDEV(". Next, click on the letter at the top of the column containing our data; this will put a reference to that column into our formula. Then close the parentheses with and hit the enter key. The formula is replaced with the result: 125.517.

To find the standard deviation, we click in an empty cell in our spreadsheet and then type "=STDEV(". Next, click on the letter at the top of the column containing our data; this will put a reference to that column into our formula. Then close the parentheses with and hit the enter key. The formula is replaced with the result: 13,333.766.

2.3: Percentiles



Figure 2.3.1: Two students graduating with the same class rank could be in different percentiles depending on the school population. (credit: "graduation caps" by John Walker/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

1. Compute percentiles.
2. Solve application problems involving percentiles.

A college admissions officer is comparing two students. The first, Anna, finished 12th in her class of 235 people. The second, Brian, finished 10th in his class of 170 people. Which of these outcomes is better? Certainly 10 is less than 12, which favors Brian, but Anna's class was much bigger. In fact, Anna beat out 223 of her classmates, which is $\frac{223}{235} \approx 95\%$ of her classmates, while Brian bested 160 out of 170 people, or 94%. Comparing the proportions of the data values that are below a given number can help us evaluate differences between individuals in separate populations. These proportions are called percentiles. If $p\%$ of the values in a dataset are less than a number n , then we say that n is at the p th percentile.

Finding Percentiles

There are some other terms that are related to "percentile" with meanings you may infer from their roots. Remember that the word *percent* means "per hundred." This reflects that percentiles divide our data into 100 pieces. The word *quartile* has a root that means "four." So, if a data value is at the first quartile of a dataset, that means that if you break the data into four parts (because of the *quart-*), this data value comes after the first of those four parts. In other words, it's greater than 25% of the data, placing it at the 25th percentile. Quintile has a root meaning "five," so a data value at the third quintile is greater than three-fifths of the data in the set. That would put it at the 60th percentile. The general term for these is *quantiles* (the root *quant-* means "number").

In Section 2.2, we defined the *median* as a number that is greater than no more than half of the data in a dataset and is less than no more than half of the data in the dataset. With our new term, we can more easily define it: The median is the value at the 50th percentile (or second quartile).

Let's look at some examples.

Example 2.3.1

Finding Percentiles

Consider the dataset 5, 8, 12, 1, 2, 16, 2, 15, 20, 22.

1. At what percentile is the value 5?
2. What value is at the 60th percentile?

Answer available at the end of this section.

Your Turn 2.1.3

Consider the dataset 2, 5, 8, 16, 12, 1, 8, 6, 15, 4.

1. What value is at the 80th percentile?
2. At what percentile is the value 12?

Answer available at the end of this section.

In each of the examples above, the computations were made easier by the fact that the we were looking for percentiles that “came out evenly” with respect to the number of values in our dataset. Things don’t always work out so cleanly. Further, different sources will define the term *percentile* in different ways. In fact, Google Sheets has *three* built-in functions for finding percentiles, none of which uses our definition. Some of the definitions you’ll see differ in the inequality that is used. Ours uses “less than or equal to,” while others use “less than” (these correspond roughly to Google Sheets’ ‘PERCENTILE.INC’ and ‘PERCENTILE.EXC’). Some of them use different methods for *interpolating* values. (This is what we did when we first computed medians without technology; if there were an even number of data values in our dataset, we found the mean of the two values in the middle. This is an example of interpolation. Most computerized methods use this technique.) Other definitions don’t interpolate at all, but instead choose the closest actual data value to the theoretical value. Fortunately, for large datasets, the differences among the different techniques become very small.

So, with all these different possible definitions in play, what will we use? For small datasets, if you’re asked to compute something involving percentiles *without technology*, use the technique we used in the previous example. In all other cases, we’ll keep things simple by using the built-in ‘PERCENTILE’ and ‘PERCENTRANK’ functions in Google Sheets (which do the same thing as the ‘PERCENTILE.INC’ and ‘PERCENTRANK.INC’ functions; they’re “inclusive, interpolating” definitions).

Video

[Using RANK, PERCENTRANK, and PERCENTILE in Google Sheets](#)

Example 2.3.2

Using Google Sheets to Compute Percentiles: Average SAT Scores

The data in “AvgSAT” contains the average SAT score for students attending every institution of higher learning in the US for which data is available.

1. What score is at the 3rd quartile?
2. What score is at the 40th percentile?
3. At what percentile is Albion College in Michigan (average SAT: 1132)?
4. At what percentile is Oregon State University (average SAT: 1205)?

Answer available at the end of this section.

Example 2.3.3

Using Google Sheets to Compute Percentiles: In-State Tuition

The dataset "InState" contains the in-state tuitions of every college and university in the country that reported that data to the Department of Education. Use that dataset to answer these questions.

1. What tuition is at the second quintile?
2. What tuition is at the 95th percentile?
3. At what percentile is Walla Walla University in Washington (in-state tuition: \$28,035)?
4. At what percentile is the College of Saint Mary in Nebraska (in-state tuition: \$20,350)?

Answer available at the end of this section.

Answers

Before we can answer these two questions, we must put the data in increasing order: 1, 2, 2, 5, 8, 12, 15, 16, 20, 22.

1. There are three values (1, 2, and 2) in the set that are less than 5, and there are ten values in the set. Thus, 5 is at the $\frac{3}{10} \times 100 = 30$ th percentile.
2. To find the value at the 60th percentile, we note that there are ten data values, and 60% of ten is six. Thus, the number we want is greater than exactly six of the data values. Thus, the 60th percentile is 15.

Before we can answer these two questions, we must put the data in increasing order: 1, 2, 4, 5, 6, 8, 8, 12, 15, 16.

1. To find the value at the 80th percentile, we note that there are ten data values, and 80% of ten is eight. Thus, the number we want is greater than exactly eight of the data values. Thus, the 80th percentile is 15.
2. There are seven values (1, 2, 4, 5, 6, 8, and 8) in the set that are less than 12, and there are ten values in the set. Thus, 12 is at the $\frac{7}{10} \times 100 = 70$ th percentile.
1. The 3rd quartile is the 75th percentile, so we'll use the PERCENTILE function. Click on an empty cell, and type "=PERCENTILE(". Next, enter the data: click on the letter at the top of the column containing the average SAT scores. Then, tell the function which percentile we want; it needs to be entered as a decimal. So, type a comma (to separate the two pieces of information we're giving this function), then type "0.75" and close the parentheses with ")". The result should look like this (assuming the data are in column C): "=PERCENTILE(C:C, 0.75)". When you hit the enter key, the formula will be replaced with the average SAT score at the 75th percentile: 1199.
2. Using the PERCENTILE function, we find that an average SAT of 1100 is at the 40th percentile.
3. Since we want to know the percentile for a particular score, we'll use the PERCENTRANK function. Like the PERCENTILE function, we need to give PERCENTRANK two pieces of information: the data, and the value we care about. So, click on an empty cell, type "=PERCENTRANK(", and then click on the letter at the top of the column containing the data. Next, type a comma and then the value we want to find the percentile for; in this case, we'll type "1132". Finally, close the parentheses with ")" and hit the enter key. The formula will be replaced with the information we want: an average SAT of 1132 is at the 54th percentile.
4. Using the PERCENTRANK function, an average SAT of 1205 is at the 76.1th percentile.
1. The second quintile is the 40th percentile; using PERCENTILE in Google Sheets, we get \$8,400.
2. Using PERCENTILE again, we get \$44,866.
3. Using PERCENTRANK, we get the 81.6th percentile.
4. Using PERCENTRANK, we get the 74.8th percentile.

3: PROBABILITY

3.1: Proportions and Rates

3.2: Basic Probability Concepts

3.3: Working with Events

3.4: Counting

3.1: Proportions and Rates

Learning Objectives

After completing this section, you should be able to:

1. Find the rate or unit rate for a real world situation.
2. Set up and solve a proportion that represents a real world problem.

If you wanted to power the city of Seattle using wind power, how many windmills would you need to install? Questions like these can be answered using rates and proportions.

Rates

A rate is the ratio (fraction) of two quantities.

A **unit rate** is a rate with a denominator of one.

Example 3.1.1

Your car can drive 300 miles on a tank of 15 gallons. Express this as a rate.

Solution

Expressed as a rate, $\frac{300 \text{ miles}}{15 \text{ gallons}}$. We can divide to find a unit rate: $\frac{20 \text{ miles}}{1 \text{ gallon}}$, which we could also write as $20 \frac{\text{miles}}{\text{gallon}}$, or just 20 miles per gallon.

Proportion Equation

A proportion equation is an equation showing the equivalence of two rates or ratios.

Example 3.1.2

Solve the proportion $\frac{5}{3} = \frac{x}{6}$ for the unknown value x .

Solution

This proportion is asking us to find a fraction with denominator 6 that is equivalent to the fraction $\frac{5}{3}$. We can solve this by multiplying both sides of the equation by 6, giving $x = \frac{5}{3} \cdot 6 = 10$.

Example 3.1.3

A map scale indicates that $\frac{1}{2}$ inch on the map corresponds with 3 real miles. How many miles apart are two cities that are $2\frac{1}{4}$ inches apart on the map?

Solution

We can set up a proportion by setting equal two $\frac{\text{map inches}}{\text{real miles}}$ rates, and introducing a variable, x , to represent the unknown quantity – the mile distance between the cities.

$$\begin{aligned} \frac{\frac{1}{3} \text{ map inch}}{3 \text{ miles}} &= \frac{2\frac{1}{4} \text{ map inches}}{x \text{ miles}} && \text{Multiply both sides by } x \text{ and rewriting the mixed number} \\ \frac{1}{3} \cdot x &= \frac{9}{4} && \text{Multiply both sides by 3} \\ \frac{1}{2} x &= \frac{27}{4} && \text{Multiply both sides by 2 (or divide by } \frac{1}{2} \text{)} \\ x &= \frac{27}{2} = 13\frac{1}{2} \text{ miles} \end{aligned}$$

Many proportion problems can also be solved using **dimensional analysis**, the process of multiplying a quantity by rates to change the units.

Example 3.1.4

Your car can drive 300 miles on a tank of 15 gallons. How far can it drive on 40 gallons?

Solution

We could certainly answer this question using a proportion: $\frac{300 \text{ miles}}{15 \text{ gallons}} = \frac{x \text{ miles}}{40 \text{ gallons}}$.

However, we earlier found that 300 miles on 15 gallons gives a rate of 20 miles per gallon. If we multiply the given 40 gallon quantity by this rate, the *gallons* unit “cancels” and we’re left with a number of miles:

$$40 \text{ gallons} \cdot \frac{20 \text{ miles}}{1 \text{ gallon}} = \frac{40 \text{ gallons}}{1} \cdot \frac{20 \text{ miles}}{\text{gallon}} = 800 \text{ miles}$$

Notice if instead we were asked “how many gallons are needed to drive 50 miles?” we could answer this question by inverting the 20 mile per gallon rate so that the *miles* unit cancels and we’re left with gallons:

$$50 \text{ miles} \cdot \frac{1 \text{ gallon}}{20 \text{ miles}} = \frac{50 \text{ miles}}{1} \cdot \frac{1 \text{ gallon}}{20 \text{ miles}} = \frac{50 \text{ gallons}}{20} = 2.5 \text{ gallons}$$

Dimensional analysis can also be used to do unit conversions. Here are some unit conversions for reference.

Unit Conversions

Length

$$\begin{aligned} 1 \text{ foot (ft)} &= 12 \text{ inches (in)} & 1 \text{ yard (yd)} &= 3 \text{ feet (ft)} \\ 1 \text{ mile} &= 5,280 \text{ feet} \\ 1000 \text{ millimeters (mm)} &= 1 \text{ meter (m)} & 100 \text{ centimeters (cm)} &= 1 \text{ meter} \\ 1000 \text{ meters (m)} &= 1 \text{ kilometer (km)} & 2.54 \text{ centimeters (cm)} &= 1 \text{ inch} \end{aligned}$$

Weight and Mass

$$\begin{aligned} 1 \text{ pound (lb)} &= 16 \text{ ounces (oz)} & 1 \text{ ton} &= 2000 \text{ pounds} \\ 1000 \text{ milligrams (mg)} &= 1 \text{ gram (g)} & 1000 \text{ grams} &= 1 \text{ kilogram (kg)} \\ 1 \text{ kilogram} &= 2.2 \text{ pounds (on earth)} \end{aligned}$$

Capacity

$$\begin{aligned} 1 \text{ cup} &= 8 \text{ fluid ounces (fl oz)}^* & 1 \text{ pint} &= 2 \text{ cups} \\ 1 \text{ quart} &= 2 \text{ pints} = 4 \text{ cups} & 1 \text{ gallon} &= 4 \text{ quarts} = 16 \text{ cups} \\ 1000 \text{ milliliters (ml)} &= 1 \text{ liter (L)} \end{aligned}$$

* Fluid ounces are a capacity measurement for liquids. 1 fluid ounce $\approx$ 1 ounce (weight) for water only.

Example 3.1.5

A bicycle is traveling at 15 miles per hour. How many feet will it cover in 20 seconds?

Solution

To answer this question, we need to convert 20 seconds into feet. If we know the speed of the bicycle in feet per second, this question would be simpler. Since we don't, we will need to do additional unit conversions. We will need to know that 5280 ft = 1 mile. We might start by converting the 20 seconds into hours:

$$\begin{aligned}
 20 \text{ seconds} &\cdot \frac{1 \text{ minute}}{60 \text{ seconds}} \cdot \frac{1 \text{ hour}}{60 \text{ minutes}} = \frac{1}{180} \text{ hour} && \text{Now we can multiply by the 15 miles/hr} \\
 \frac{1}{180} \text{ hour} &\cdot \frac{15 \text{ miles}}{1 \text{ hour}} = \frac{1}{12} \text{ mile} && \text{Now we can convert to feet} \\
 \frac{1}{12} \text{ mile} &\cdot \frac{5280 \text{ feet}}{1 \text{ mile}} = 440 \text{ feet}
 \end{aligned}$$

We could have also done this entire calculation in one long set of products:

$$20 \text{ seconds} \cdot \frac{1 \text{ minute}}{60 \text{ seconds}} \cdot \frac{1 \text{ hour}}{60 \text{ minutes}} \cdot \frac{15 \text{ miles}}{1 \text{ hour}} \cdot \frac{5280 \text{ feet}}{1 \text{ mile}} = 440 \text{ feet}$$

Your Turn 3.1.1

A 1000 foot spool of bare 12-gauge copper wire weighs 19.8 pounds. How much will 18 inches of the wire weigh, in ounces?

Answer available at the end of this section.

Notice that with the miles per gallon example, if we double the miles driven, we double the gas used. Likewise, with the map distance example, if the map distance doubles, the real-life distance doubles. This is a key feature of proportional relationships, and one we must confirm before assuming two things are related proportionally.

Example 3.1.6

Suppose you're tiling the floor of a 10 ft by 10 ft room, and find that 100 tiles will be needed. How many tiles will be needed to tile the floor of a 20 ft by 20 ft room?

Solution

In this case, while the width the room has doubled, the area has quadrupled. Since the number of tiles needed corresponds with the area of the floor, not the width, 400 tiles will be needed. We could find this using a proportion based on the areas of the rooms:

$$\frac{100 \text{ tiles}}{100\text{ft}^2} = \frac{n \text{ tiles}}{400\text{ft}^2}$$

Other quantities just don't scale proportionally at all.

Example 3.1.7

Suppose a small company spends \$1000 on an advertising campaign, and gains 100 new customers from it. How many new customers should they expect if they spend \$10,000?

Solution

While it is tempting to say that they will gain 1000 new customers, it is likely that additional advertising will be less effective than the initial advertising. For example, if the company is a hot tub store, there are likely only a fixed number of people interested in buying a hot tub, so there might not even be 1000 people in the town who would be potential customers.

Sometimes when working with rates, proportions, and percents, the process can be made more challenging by the magnitude of the numbers involved. Sometimes, large numbers are just difficult to comprehend.

Example 3.1.8

Compare the 2010 U.S. military budget of \$683.7 billion to other quantities.

Here we have a very large number, about \$683,700,000,000 written out. Of course, imagining a billion dollars is very difficult, so it can help to compare it to other quantities.

If that amount of money was used to pay the salaries of the 1.4 million Walmart employees in the U.S., each would earn over \$488,000.

There are about 300 million people in the U.S. The military budget is about \$2,200 per person.

If you were to put \$683.7 billion in \$100 bills, and count out 1 per second, it would take 216 years to finish counting it.

Example 3.1.9

Compare the electricity consumption per capita in China to the rate in Japan.

To address this question, we will first need data. From the CIA[1] website we can find the electricity consumption in 2011 for China was 4,693,000,000,000 KWH (kilowatt-hours), or 4.693 trillion KWH, while the consumption for Japan was 859,700,000,000, or 859.7 billion KWH. To find the rate per capita (per person), we will also need the population of the two countries. From the World Bank[2], we can find the population of China is 1,344,130,000, or 1.344 billion, and the population of Japan is 127,817,277, or 127.8 million.

Solution

Computing the consumption per capita for each country:

China: $\frac{4,693,000,000,000 \text{ KWH}}{1,344,130,000 \text{ people}} \approx 3491.5 \text{ KWH per person}$

Japan: $\frac{859,700,000,000 \text{ KWH}}{127,817,277 \text{ people}} \approx 6726 \text{ KWH per person}$

While China uses more than 5 times the electricity of Japan overall, because the population of Japan is so much smaller, it turns out Japan uses almost twice the electricity per person compared to China.

[1] www.cia.gov/library/publicat.../2042rank.html

[2] <http://data.worldbank.org/indicator/SP.POP.TOT>

Answers

Your Turn 3.1.1

$$18 \text{ inches} \cdot \frac{1 \text{ foot}}{12 \text{ inches}} \cdot \frac{19.8 \text{ pounds}}{1000 \text{ feet}} \cdot \frac{16 \text{ ounces}}{1 \text{ pound}} \approx 0.475 \text{ ounces}$$

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3.2: Basic Probability Concepts

Learning Objectives

After completing this section, you should be able to:

1. Identify the sample space of an experiment or procedure.
2. Calculate the probability of a simple event.

If you roll a die, pick a card from deck of playing cards, or randomly select a person and observe their hair color, we are executing an experiment or procedure. In probability, we look at the likelihood of different outcomes. We begin with some terminology.

Events and Outcomes

The result of an experiment is called an **outcome**.

An **event** is any particular outcome or group of outcomes.

A **simple event** is an event that cannot be broken down further

The **sample space** is the set of all possible simple events.

Example 3.2.1

If we roll a standard 6-sided die, describe the sample space and some simple events.

The sample space is the set of all possible simple events: $\{1, 2, 3, 4, 5, 6\}$



Some examples of simple events:

- We roll a 1
- We roll a 5

Some compound events:

- We roll a number bigger than 4
- We roll an even number

Basic Probability

Given that all outcomes are equally likely, we can compute the probability of an event E using this formula:

$$P(E) = \frac{\text{Number of outcomes corresponding to the event } E}{\text{Total number of equally likely outcomes}}$$

Example 3.2.2

If we roll a 6-sided die, calculate

- P(rolling a 1)
- P(rolling a number bigger than 4)

Solution

Recall that the sample space is $\{1, 2, 3, 4, 5, 6\}$

- There is one outcome corresponding to “rolling a 1”, so the probability is $\frac{1}{6}$
- There are two outcomes bigger than a 4, so the probability is $\frac{2}{6} = \frac{1}{3}$

Probabilities are essentially fractions, and can be reduced to lower terms like fractions.

Example 3.2.3

Let's say you have a bag with 20 cherries: 14 sweet and 6 sour. If you pick a cherry at random, what is the probability that it will be sweet?

Solution

There are 20 possible cherries that could be picked, so the number of possible outcomes is 20. Of these 20 possible outcomes, 14 are favorable (sweet), so the probability that the cherry will be sweet is $\frac{14}{20} = \frac{7}{10}$.

There is one potential complication to this example, however. It must be assumed that the probability of picking any of the cherries is the same as the probability of picking any other. This wouldn't be true if (let us imagine) the sweet cherries are smaller than the sour ones. (The sour cherries would come to hand more readily when you sampled from the bag.) Let us keep in mind, therefore, that when we assess probabilities in terms of the ratio of favorable to all potential cases, we rely heavily on the assumption of equal probability for all outcomes.

Your Turn 3.2.1

At some random moment, you look at your clock and note the minutes reading.

- What is probability the minutes reading is 15?
- What is the probability the minutes reading is 15 or less?

Answer available at the end of this section.

Cards

A standard deck of 52 playing cards consists of four **suits** (hearts, spades, diamonds and clubs). Spades and clubs are black while hearts and diamonds are red. Each suit contains 13 cards, each of a different **rank**: an Ace (which in many games functions as both a low card and a high card), cards numbered 2 through 10, a Jack, a Queen and a King.

Example 3.2.4

Compute the probability of randomly drawing one card from a deck and getting an Ace.

Solution

There are 52 cards in the deck and 4 Aces so

$$P(\text{Ace}) = \frac{4}{52} = \frac{1}{13} \approx 0.0769$$

We can also think of probabilities as percents: There is a 7.69% chance that a randomly selected card will be an Ace.

Notice that the smallest possible probability is 0 – if there are no outcomes that correspond with the event. The largest possible probability is 1 – if all possible outcomes correspond with the event.

Certain and Impossible events

An **impossible event** has a probability of 0.

A **certain event** has a probability of 1.

The probability of any event must be $0 \leq P(E) \leq 1$.

In the course of this chapter, *if you compute a probability and get an answer that is negative or greater than 1, you have made a mistake and should check your work.*

Answers

Your Turn 3.2.1

There are 60 possible readings, from 00 to 59.

- $\frac{1}{60}$
- $\frac{16}{60}$ (counting 00 through 15)

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3.3: Working with Events

Learning Objectives

After completing this section, you should be able to:

1. Calculate the probability of an event not happening
2. Calculate the probability of two independent events occurring
3. Calculate the probability of one event or another event happening
4. Calculate conditional probabilities

Complementary Events

Now let us examine the probability that an event does **not** happen. As in the previous section, consider the situation of rolling a six-sided die and first compute the probability of rolling a six: the answer is $P(\text{six}) = \frac{1}{6}$. Now consider the probability that we do *not* roll a six: there are 5 outcomes that are not a six, so the answer is $P(\text{not a six}) = \frac{5}{6}$. Notice that

$$P(\text{six}) + P(\text{not a six}) = \frac{1}{6} + \frac{5}{6} = \frac{6}{6} = 1$$

This is not a coincidence. Consider a generic situation with n possible outcomes and an event E that corresponds to m of these outcomes. Then the remaining $n - m$ outcomes correspond to E not happening, thus

$$P(\text{not } E) = \frac{n - m}{n} = \frac{n}{n} - \frac{m}{n} = 1 - \frac{m}{n} = 1 - P(E)$$

Complement of an Event

The **complement** of an event is the event “ E doesn’t happen”

The notation $\bar{E}$ is used for the complement of event E .

We can compute the probability of the complement using $P(\bar{E}) = 1 - P(E)$

Notice also that $P(E) = 1 - P(\bar{E})$

Example 3.3.1

If you pull a random card from a deck of playing cards, what is the probability it is not a heart?

Solution

There are 13 hearts in the deck, so

$$P(\text{heart}) = \frac{13}{52} = \frac{1}{4}.$$

The probability of *not* drawing a heart is the complement:

$$P(\text{ not heart }) = 1 - P(\text{ heart }) = 1 - \frac{1}{4} = \frac{3}{4}$$

Probability of two independent events

Example 3.3.2

Suppose we flipped a coin and rolled a die, and wanted to know the probability of getting a head on the coin and a 6 on the die.

Solution

We could list all possible outcomes: $\{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}$.

Notice there are $2 \cdot 6 = 12$ total outcomes. Out of these, only 1 is the desired outcome, so the probability is $\frac{1}{12}$.

The prior example was looking at two independent events.

Independent Events

Events A and B are **independent events** if the probability of Event B occurring is the same whether or not Event A occurs.

Example 3.3.3

Are these events independent?

- A fair coin is tossed two times. The two events are (1) first toss is a head and (2) second toss is a head.
- The two events (1) "It will rain tomorrow in Houston" and (2) "It will rain tomorrow in Galveston" (a city near Houston).
- You draw a card from a deck, then draw a second card without replacing the first.

Solution

- The probability that a head comes up on the second toss is $\frac{1}{2}$ regardless of whether or not a head came up on the first toss, so these events are independent.
- These events are not independent because it is more likely that it will rain in Galveston on days it rains in Houston than on days it does not.
- The probability of the second card being red depends on whether the first card is red or not, so these events are not independent.

When two events are independent, the probability of both occurring is the product of the probabilities of the individual events.

$P(A \text{ and } B)$ for Independent Events

If events A and B are independent, then the probability of both A and B occurring is

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

where $P(A \text{ and } B)$ is the probability of events A and B both occurring, $P(A)$ is the probability of event A occurring, and $P(B)$ is the probability of event B occurring

If you look back at the coin and die example from earlier, you can see how the number of outcomes of the first event multiplied by the number of outcomes in the second event multiplied to equal the total number of possible outcomes in the combined event.

Example 3.3.4

In your drawer you have 10 pairs of socks, 6 of which are white, and 7 tee shirts, 3 of which are white. If you randomly reach in and pull out a pair of socks and a tee shirt, what is the probability both are white?

Solution

The probability of choosing a white pair of socks is $\frac{6}{10}$.

The probability of choosing a white tee shirt is $\frac{3}{7}$.

The probability of both being white is $\frac{6}{10} \cdot \frac{3}{7} = \frac{18}{70} = \frac{9}{35}$

Your Turn 3.3.1

A card is pulled from a deck of cards and noted. The card is then replaced, the deck is shuffled, and a second card is removed and noted. What is the probability that both cards are Aces?

Answer available at the end of this section.

The previous examples looked at the probability of *both* events occurring. Now we will look at the probability of *either* event occurring.

Example 3.3.5

Suppose we flipped a coin and rolled a die, and wanted to know the probability of getting a head on the coin *or* a 6 on the die.

Solution

Here, there are still 12 possible outcomes:

$$\{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}$$

By simply counting, we can see that 7 of the outcomes have a head on the coin *or* a 6 on the die *or* both – we use *or* inclusively here (these 7 outcomes are $H1, H2, H3, H4, H5, H6, T6$), so the probability is $\frac{7}{12}$. How could we have found this from the individual probabilities?

As we would expect, $\frac{1}{2}$ of these outcomes have a head, and $\frac{1}{6}$ of these outcomes have a 6 on the die. If we add these, $\frac{1}{2} + \frac{1}{6} = \frac{6}{12} + \frac{2}{12} = \frac{8}{12}$, which is not the correct probability. Looking at the outcomes we can see why: the outcome H6 would have been counted twice, since it contains both a head and a 6; the probability of both a head *and* rolling a 6 is $\frac{1}{12}$.

If we subtract out this double count, we have the correct probability: $\frac{8}{12} - \frac{1}{12} = \frac{7}{12}$.

$P(A \text{ or } B)$

The probability of either A or B occurring (or both) is

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Example 3.3.6

Suppose we draw one card from a standard deck. What is the probability that we get a Queen or a King?

Solution

There are 4 Queens and 4 Kings in the deck, hence 8 outcomes corresponding to a Queen or King out of 52 possible outcomes. Thus the probability of drawing a Queen or a King is:

$$P(\text{King or Queen}) = \frac{8}{52}$$

Note that in this case, there are no cards that are both a Queen and a King, so $P(\text{King and Queen}) = 0$. Using our probability rule, we could have said:

$$P(\text{King or Queen}) = P(\text{King}) + P(\text{Queen}) - P(\text{King and Queen}) = \frac{4}{52} + \frac{4}{52} - 0 = \frac{8}{52} = \frac{2}{13}$$

In the last example, the events were **mutually exclusive**, so $P(A \text{ or } B) = P(A) + P(B)$.

Example 3.3.7

Suppose we draw one card from a standard deck. What is the probability that we get a red card or a King?

Solution

- Half the cards are red, so $P(\text{red}) = \frac{26}{52}$
- There are four kings, so $P(\text{King}) = \frac{4}{52}$
- There are two red kings, so $P(\text{Red and King}) = \frac{2}{52}$

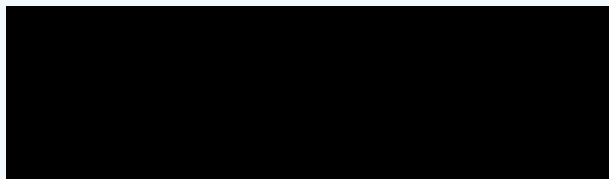
We can then calculate

$$P(\text{Red or King}) = P(\text{Red}) + P(\text{King}) - P(\text{Red and King}) = \frac{26}{52} + \frac{4}{52} - \frac{2}{52} = \frac{28}{52} = \frac{7}{13}$$

Example 3.3.8

The table below shows the number of survey subjects who have received and not received a speeding ticket in the last year, and the color of their car. Find the probability that a randomly chosen person:

- Has a red car *and* got a speeding ticket
- Has a red car *or* got a speeding ticket.



Solution

- We can see that 15 people of the 665 surveyed had both a red car and got a speeding ticket, so the probability is $\frac{15}{665} = \frac{3}{133} \approx 0.0226$.

Notice that having a red car and getting a speeding ticket are not independent events, so the probability of both of them occurring is not simply the product of probabilities of each one occurring.

b. We could answer this question by simply adding up the numbers: 15 people with red cars and speeding tickets + 135 with red cars but no ticket + 45 with a ticket but no red car = 195 people. So the probability is $\frac{195}{665} = \frac{39}{133} \approx 0.2932$.

We also could have found this probability by:

$$P(\text{had a red car}) + P(\text{got a speeding ticket}) - P(\text{had a red car and got a speeding ticket}) = \frac{150}{665} + \frac{60}{665} - \frac{15}{665} = \frac{195}{665} = \frac{39}{133}$$

Conditional Probability

Often it is required to compute the probability of an event given that another event has occurred.

Example 3.3.9

What is the probability that two cards drawn at random from a deck of playing cards will both be aces?

Solution

It might seem that you could use the formula for the probability of two independent events and simply multiply $\frac{4}{52} \cdot \frac{4}{52} = \frac{1}{169}$. This would be incorrect, however, because the two events are not independent. If the first card drawn is an ace, then the probability that the second card is also an ace would be lower because there would only be three aces left in the deck.

Once the first card chosen is an ace, the probability that the second card chosen is also an ace is called the **conditional probability** of drawing an ace. In this case the "condition" is that the first card is an ace. Symbolically, we write this as:

$P(\text{ace on second draw} \mid \text{an ace on the first draw})$.

The vertical bar "|" is read as "given," so the above expression is short for "The probability that an ace is drawn on the second draw given that an ace was drawn on the first draw." What is this probability? After an ace is drawn on the first draw, there are 3 aces out of 51 total cards left. This means that the conditional probability of drawing an ace after one ace has already been drawn is $\frac{3}{51} = \frac{1}{17}$.

Thus, the probability of both cards being aces is $\frac{4}{52} \cdot \frac{3}{51} = \frac{12}{2652} = \frac{1}{221}$.

Conditional Probability

The probability the event B occurs, given that event A has happened, is represented as

$$P(B|A)$$

This is read as "the probability of B given A "

Example 3.3.10

Find the probability that a die rolled shows a 6, given that a flipped coin shows a head.

Solution

These are two independent events, so the probability of the die rolling a 6 is $\frac{1}{6}$, regardless of the result of the coin flip.

Example 3.3.11

The table below shows the number of survey subjects who have received and not received a speeding ticket in the last year, and the color of their car. Find the probability that a randomly chosen person:

- Has a speeding ticket *given* they have a red car
- Has a red car *given* they have a speeding ticket

Solution

a) Since we know the person has a red car, we are only considering the 150 people in the first row of the table. Of those, 15 have a speeding ticket, so

$$P(\text{ticket} \mid \text{red car}) = \frac{15}{150} = \frac{1}{10} = 0.1$$

b) Since we know the person has a speeding ticket, we are only considering the 60 people in the first column of the table. Of those, 15 have a red car, so

$$P(\text{red car} \mid \text{ticket}) = \frac{15}{60} = \frac{1}{4} = 0.25.$$

Notice from the last example that $P(B|A)$ is **not** equal to $P(A|B)$.

These kinds of conditional probabilities are what insurance companies use to determine your insurance rates. They look at the conditional probability of you having an accident, given your age, your car, your car color, your driving history, etc., and price your policy based on that likelihood.

Conditional Probability Formula

If Events A and B are not independent, then

$$P(A \text{ and } B) = P(A) \cdot P(B|A)$$

Example 3.3.12

If you pull 2 cards out of a deck, what is the probability that both are spades?

Solution

The probability that the first card is a spade is $\frac{13}{52}$.

The probability that the second card is a spade, given the first was a spade, is $\frac{12}{51}$, since there is one less spade in the deck, and one less total cards.

The probability that both cards are spades is $\frac{13}{52} \cdot \frac{12}{51} = \frac{156}{2652} = \frac{1}{16} \approx 0.0588$

Example 3.3.13

If you draw two cards from a deck, what is the probability that you will get the Ace of Diamonds and a black card?

Solution

You can satisfy this condition by having Case A or Case B, as follows:

Case A) you can get the Ace of Diamonds first and then a black card or

Case B) you can get a black card first and then the Ace of Diamonds.

Let's calculate the probability of Case A. The probability that the first card is the Ace of Diamonds is $\frac{1}{52}$. The probability that the second card is black given that the first card is the Ace of Diamonds is $\frac{26}{51}$ because 26 of the remaining 51 cards are black. The probability is therefore $\frac{1}{52} \cdot \frac{26}{51} = \frac{1}{102}$.

Now for Case B: the probability that the first card is black is $\frac{26}{52} = \frac{1}{2}$. The probability that the second card is the Ace of Diamonds given that the first card is black is $\frac{1}{51}$. The probability of Case B is therefore $\frac{1}{2} \cdot \frac{1}{51} = \frac{1}{102}$, the same as the probability of Case A.

Recall that the probability of A or B is $P(A) + P(B) - P(A \text{ and } B)$. In this problem, $P(A \text{ and } B) = 0$ since the first card cannot be the Ace of Diamonds and be a black card. Therefore, the probability of Case A or Case B is $\frac{1}{102} + \frac{1}{102} = \frac{2}{102} = \frac{1}{51}$. The probability that you will get the Ace of Diamonds and a black card when drawing two cards from a deck is $\frac{1}{51}$.

Your Turn 3.3.2

In your drawer you have 10 pairs of socks, 6 of which are white. If you reach in and randomly grab two pairs of socks, what is the probability that both are white?

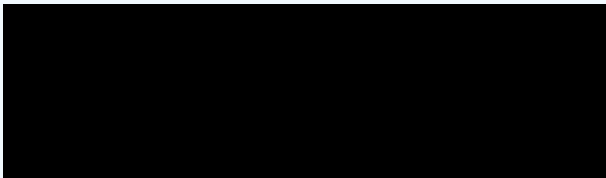
Answer available at the end of this section.

Example 3.3.14

A home pregnancy test was given to women, then pregnancy was verified through blood tests. The following table shows the home pregnancy test results. Find

a) $P(\text{not pregnant} \mid \text{positive test result})$

b) $P(\text{positive test result} \mid \text{not pregnant})$



Solution

a) Since we know the test result was positive, we're limited to the 75 women in the first column, of which 5 were not pregnant. $P(\text{not pregnant} \mid \text{positive test result}) = \frac{5}{75} = \frac{1}{15} \approx 0.067$.

b) Since we know the woman is not pregnant, we are limited to the 19 women in the second row, of which 5 had a positive test. $P(\text{positive test result} \mid \text{not pregnant}) = \frac{5}{19} \approx 0.263$.

The second result is what is usually called a false positive: A positive result when the woman is not actually pregnant.

Answers

Your Turn 3.3.1

since the second draw is made after replacing the first card, these events are independent The probability of an ace on each draw is $\frac{4}{52} = \frac{1}{13}$, so the probability of an Ace on both draws is $\frac{1}{13} \cdot \frac{1}{13} = \frac{1}{169}$

Your Turn 3.3.2

$$\frac{6}{10} \cdot \frac{5}{9} = \frac{30}{90} = \frac{1}{3}$$

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3.4: Counting

Counting? You already know how to count or you wouldn't be taking a college-level math class, right? Well yes, but what we'll really be investigating here are ways of counting *efficiently*.

Basic Counting

We will start with some more reasonable sorts of counting problems in order to develop the ideas that we will soon need.

Example 3.4.1

Suppose at a particular restaurant you have three choices for an appetizer (soup, salad or breadsticks) and five choices for a main course (hamburger, sandwich, quiche, fajita or pizza). If you are allowed to choose exactly one item from each category for your meal, how many different meal options do you have?

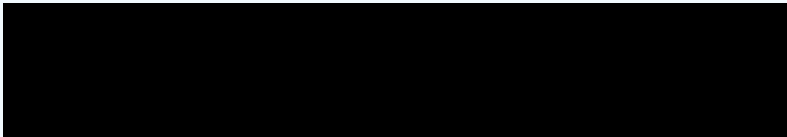
Solution

Solution 1: One way to solve this problem would be to systematically list each possible meal:

soup + hamburger	soup + sandwich	soup + quiche
soup + fajita	soup + pizza	salad + hamburger
salad + sandwich	salad + quiche	salad + fajita
salad + pizza	breadsticks + hamburger	breadsticks + sandwich
breadsticks + quiche	breadsticks + fajita	breadsticks + pizza

Assuming that we did this systematically and that we neither missed any possibilities nor listed any possibility more than once, the answer would be 15. Thus you could go to the restaurant 15 nights in a row and have a different meal each night.

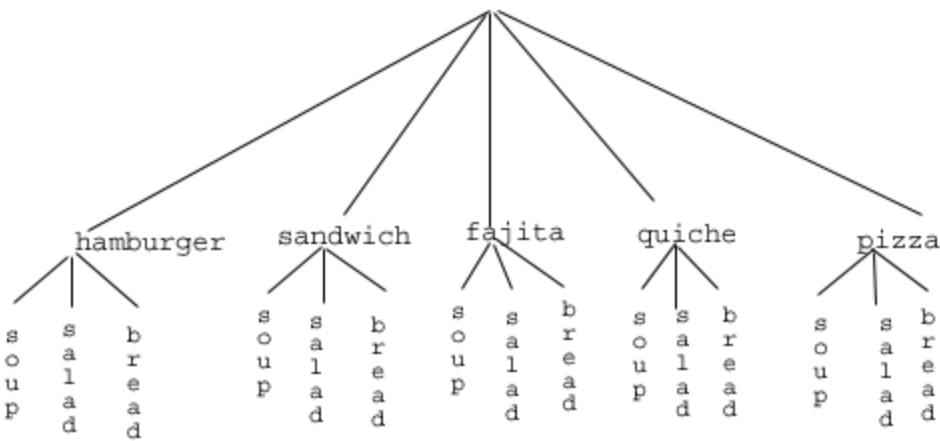
Solution 2: Another way to solve this problem would be to list all the possibilities in a table:



In each of the cells in the table we could list the corresponding meal: soup + hamburger in the upper left corner, salad + hamburger below it, etc. But if we didn't really care *what* the possible meals are, only *how many* possible meals there are, we could just count the number of cells and arrive at an answer of 15, which matches our answer from the first solution. (It's always good when you solve a problem two different ways and get the same answer!)

Solution 3: We already have two perfectly good solutions. Why do we need a third? The first method was not very systematic, and we might easily have made an omission. The second method was better, but suppose that in addition to the appetizer and the main course we further complicated the problem by adding desserts to the menu: we've used the rows of the table for the appetizers and the columns for the main courses—where will the desserts go? We would need a third dimension, and since drawing 3-D tables on a 2-D page or computer screen isn't terribly easy, we need a better way in case we have three categories to choose from instead of just two.

So, back to the problem in the example. What else can we do? Let's draw a **tree diagram**:



This is called a "tree" diagram because at each stage we branch out, like the branches on a tree. In this case, we first drew five branches (one for each main course) and then for each of those branches we drew three more branches (one for each appetizer). We count the number of branches at the final level and get (surprise, surprise!) 15.

If we wanted, we could instead draw three branches at the first stage for the three appetizers and then five branches (one for each main course) branching out of each of those three branches.

OK, so now we know how to count possibilities using tables and tree diagrams. These methods will continue to be useful in certain cases, but imagine a game where you have two decks of cards (with 52 cards in each deck) and you select one card from each deck. Would you really want to draw a table or tree diagram to determine the number of outcomes of this game?

Let's go back to the previous example that involved selecting a meal from three appetizers and five main courses, and look at the second solution that used a table. Notice that one way to count the number of possible meals is simply to number each of the appropriate cells in the table, as we have done above. But another way to count the number of cells in the table would be multiply the number of rows (3) by the number of columns (5) to get 15. Notice that we could have arrived at the same result without making a table at all by simply multiplying the number of choices for the appetizer (3) by the number of choices for the main course (5). We generalize this technique as the *basic counting rule*:

Basic Counting Rule

If we are asked to choose one item from each of two separate categories where there are m items in the first category and n items in the second category, then the total number of available choices is $m \cdot n$.

This is sometimes called the multiplication rule for probabilities.

Example 3.4.2

There are 21 novels and 18 volumes of poetry on a reading list for a college English course. How many different ways can a student select one novel and one volume of poetry to read during the quarter?

Solution

There are 21 choices from the first category and 18 for the second, so there are $21 \cdot 18 = 378$ possibilities.

The Basic Counting Rule can be extended when there are more than two categories by applying it repeatedly, as we see in the next example.

Example 3.4.3

Suppose at a particular restaurant you have three choices for an appetizer (soup, salad or breadsticks), five choices for a main course (hamburger, sandwich, quiche, fajita or pasta) and two choices for dessert (pie or ice cream). If you are allowed to choose exactly one item from each category for your meal, how many different meal options do you have?

Solution

There are 3 choices for an appetizer, 5 for the main course and 2 for dessert, so there are $3 \cdot 5 \cdot 2 = 30$ possibilities.

Example 3.4.4

A quiz consists of 3 true-or-false questions. In how many ways can a student answer the quiz?

Solution

There are 3 questions. Each question has 2 possible answers (true or false), so the quiz may be answered in $2 \cdot 2 \cdot 2 = 8$ different ways. Recall that another way to write $2 \cdot 2 \cdot 2$ is 2^3 which is much more compact.

Your Turn 3.4.1

Suppose at a particular restaurant you have eight choices for an appetizer, eleven choices for a main course and five choices for dessert. If you are allowed to choose exactly one item from each category for your meal, how many different meal options do you have?

Answer available at the end of this section.

Permutations

In this section, we will develop an even faster way to solve some of the problems we have already learned to solve by other means. Let's start with a couple examples.

Example 3.4.5

How many different ways can the letters of the word MATH be rearranged to form a four-letter code word?

Solution

This problem is a bit different. Instead of choosing one item from each of several different categories, we are repeatedly choosing items from the *same* category (the category is: the letters of the word MATH) and each time we choose an item we *do not replace* it, so there is one fewer choice at the next stage: we have 4 choices for the first letter (say we choose A), then 3 choices for the second (M, T and H; say we choose H), then 2 choices for the next letter (M and T; say we choose M) and only one choice at the last stage (T). Thus there are $4 \cdot 3 \cdot 2 \cdot 1 = 24$ ways to spell a code word with the letters MATH.

In this example, we needed to calculate $n \cdot (n - 1) \cdot (n - 2) \cdots 3 \cdot 2 \cdot 1$. This calculation shows up often in mathematics, and is called the **factorial**, and is notated $n!$

Factorial

$$n! = n \cdot (n - 1) \cdot (n - 2) \cdots 3 \cdot 2 \cdot 1$$

Example 3.4.6

How many ways can five different door prizes be distributed among five people?

Solution

There are 5 choices of prize for the first person, 4 choices for the second, and so on. The number of ways the prizes can be distributed will be $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ ways.

Now we will consider some slightly different examples.

Example 3.4.7

A charity benefit is attended by 25 people and three gift certificates are given away as door prizes: one gift certificate is in the amount of \$100, the second is worth \$25 and the third is worth \$10. Assuming that no person receives more than one prize, how many different ways can the three gift certificates be awarded?

Solution

Using the Basic Counting Rule, there are 25 choices for the person who receives the \$100 certificate, 24 remaining choices for the \$25 certificate and 23 choices for the \$10 certificate, so there are $25 \cdot 24 \cdot 23 = 13,800$ ways in which the prizes can be awarded.

Example 3.4.8

Eight sprinters have made it to the Olympic finals in the 100-meter race. In how many different ways can the gold, silver and bronze medals be awarded?

Solution

Using the Basic Counting Rule, there are 8 choices for the gold medal winner, 7 remaining choices for the silver, and 6 for the bronze, so there are $8 \cdot 7 \cdot 6 = 336$ ways the three medals can be awarded to the 8 runners.

Note that in these preceding examples, the gift certificates and the Olympic medals were awarded *without replacement*; that is, once we have chosen a winner of the first door prize or the gold medal, they are not eligible for the other prizes. Thus, at each succeeding stage of the solution there is one fewer choice (25, then 24, then 23 in the first example; 8, then 7, then 6 in the second). Contrast this with the situation of a multiple choice test, where there might be five possible answers — A, B, C, D or E — for each question on the test.

Note also that *the order of selection was important* in each example: for the three door prizes, being chosen first means that you receive substantially more money; in the Olympics example, coming in first means that you get the gold medal instead of the silver or bronze. In each case, if we had chosen the same three people in a different order there might have been a different person who received the \$100 prize, or a different gold medalist. (Contrast this with the situation where we might draw three names out of a hat to each receive a \$10 gift certificate; in this case the order of selection is *not* important since each of the three people receive the same prize. Situations where the order is *not* important will be discussed later.)

We can generalize the situation in the two examples above to any problem *without replacement* where the *order of selection is important*. If we are arranging in order r items out of n possibilities (instead of 3 out of 25 or 3 out of 8 as in the previous examples), the number of possible arrangements will be given by

$$n \cdot (n - 1) \cdot (n - 2) \cdots (n - r + 1)$$

If you don't see why $(n - r + 1)$ is the right number to use for the last factor, just think back to the first example in this section, where we calculated $25 \cdot 24 \cdot 23$ to get 13,800. In this case $n = 25$ and $r = 3$, so $n - r + 1 = 25 - 3 + 1 = 23$, which is exactly the right number for the final factor.

Now, why would we want to use this complicated formula when it's actually easier to use the Basic Counting Rule, as we did in the first two examples? Well, we won't actually use this formula all that often, we only developed it so that we could attach a special notation and a special definition to this situation where we are choosing r items out of n possibilities *without replacement* and where the *order of selection is important*. In this situation we write:

Permutations

$${}_nP_r = n \cdot (n - 1) \cdot (n - 2) \cdots (n - r + 1)$$

We say that there are ${}_nP_r$ **permutations** of size r that may be selected from among n choices *without replacement* when *order matters*.

It turns out that we can express this result more simply using factorials.

$${}_nP_r = \frac{n!}{(n-r)!}$$

In practicality, we usually use technology rather than factorials or repeated multiplication to compute permutations.

Example 3.4.9

I have nine paintings and have room to display only four of them at a time on my wall. How many different ways could I do this?

Solution

Since we are choosing 4 paintings out of 9 *without replacement* where the *order of selection is important* there are ${}_9P_4 = 9 \cdot 8 \cdot 7 \cdot 6 = 3,024$ permutations.

Example 3.4.10

How many ways can a four-person executive committee (president, vice-president, secretary, treasurer) be selected from a 16-member board of directors of a non-profit organization?

Solution

We want to choose 4 people out of 16 without replacement and where the order of selection is important. So the answer is ${}_{16}P_4 = 16 \cdot 15 \cdot 14 \cdot 13 = 43,680$.

Your Turn 3.4.2

How many 5 character passwords can be made using the letters A through Z

- if repeats are allowed
- if no repeats are allowed

Answer available at the end of this section.

Combinations

We just considered the situation where we chose r items out of n possibilities *without replacement* and where the *order of selection was important*. We now consider a similar situation in which the order of selection is *not* important.

Example 3.4.11

A charity benefit is attended by 25 people at which three \$50 gift certificates are given away as door prizes. Assuming no person receives more than one prize, how many different ways can the gift certificates be awarded?

Solution

Using the Basic Counting Rule, there are 25 choices for the first person, 24 remaining choices for the second person, and 23 for the third, so there are $25 \cdot 24 \cdot 23 = 13,800$ ways to choose three people. Suppose for a moment that Abe is chosen first, Bea second, and Cindy third; this is one of the 13,800 possible outcomes. Another way to award the prizes would be to choose Abe first, Cindy second, and Bea third; this is another of the 13,800 possible outcomes. But either way, Abe, Bea, and Cindy each get \$50, so it doesn't really matter the order in which we select them. In how many different orders can Abe, Bea and Cindy be selected? It turns out there are 6:

ABC ACB BAC BCA CAB CBA

How can we be sure that we have counted them all? We are really just choosing 3 people out of 3, so there are $3 \cdot 2 \cdot 1 = 6$ ways to do this; we didn't really need to list them all, we can just use permutations!

So, out of the 13,800 ways to select 3 people out of 25, six of them involve Abe, Bea and Cindy. The same argument works for any other group of three people (say Abe, Bea, and David or Frank, Gloria, and Hildy) so each three-person group is counted *six times*. Thus the 13,800 figure is six times too big. The number of distinct three-person groups will be $\frac{13,800}{6} = 2300$.

We can generalize the situation in this example above to any problem of choosing a collection of items *without replacement* where the *order of selection is not important*. If we are choosing r items out of n possibilities (instead of 3 out of 25 as in the previous examples), the number of possible choices will be given by ${}_nC_r = \frac{{}_nP_r}{r!}$, and we could use this formula for computation. However this situation arises so frequently that we attach a special notation and a special definition to this situation where we are choosing r items out of n possibilities *without replacement* where the *order of selection is not important*.

Combinations

$${}_nC_r = \frac{{}_nP_r}{r!}$$

We say that there are ${}_nC_r$ **combinations** of size r that may be selected from among n choices *without replacement* where *order doesn't matter*.

We can also write the combinations formula in terms of factorials:

$${}_nC_r = \frac{n!}{(n-r)!r!}$$

Example 3.4.12

A group of four students is to be chosen from a 35-member class to represent the class on the student council. How many ways can this be done?

Solution

Since we are choosing 4 people out of 35 *without replacement* where the *order of selection is not important* there are $= {}_{35}C_4$
 $= \frac{35 \cdot 34 \cdot 33 \cdot 32}{4 \cdot 3 \cdot 2 \cdot 1} = 52,360$ combinations.

Your Turn 3.4.3

The United States Senate Appropriations Committee consists of 29 members; the Defense Subcommittee of the Appropriations Committee consists of 19 members. Disregarding party affiliation or any special seats on the Subcommittee, how many different 19-member subcommittees may be chosen from among the 29 Senators on the Appropriations Committee?

Answer available at the end of this section.

In the preceding Your Turn problem we assumed that the 19 members of the Defense Subcommittee were chosen without regard to party affiliation. In reality this would never happen: if Republicans are in the majority they would never let a majority of Democrats sit on (and thus control) any subcommittee. (The same of course would be true if the Democrats were in control.) So let's consider the problem again, in a slightly more complicated form:

Example 3.4.13

The United States Senate Appropriations Committee consists of 29 members, 15 Republicans and 14 Democrats. The Defense Subcommittee consists of 19 members, 10 Republicans and 9 Democrats. How many different ways can the members of the Defense Subcommittee be chosen from among the 29 Senators on the Appropriations Committee?

Solution

In this case we need to choose 10 of the 15 Republicans and 9 of the 14 Democrats. There are ${}_{15}C_{10} = 3003$ ways to choose the 10 Republicans and ${}_{14}C_9 = 2002$ ways to choose the 9 Democrats. But now what? How do we finish the problem?

Suppose we listed all of the possible 10-member Republican groups on 3003 slips of red paper and all of the possible 9-member Democratic groups on 2002 slips of blue paper. How many ways can we choose one red slip and one blue slip? This is a job for the Basic Counting Rule! We are simply making one choice from the first category and one choice from the second category, just like in the restaurant menu problems from earlier.

There must be $3003 \cdot 2002 = 6,012,006$ possible ways of selecting the members of the Defense Subcommittee.

Probability using Permutations and Combinations

We can use permutations and combinations to help us answer more complex probability questions.

Example 3.4.14

A 4 digit PIN number is selected. What is the probability that there are no repeated digits?

Solution

There are 10 possible values for each digit of the PIN (namely: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9), so there are $10 \cdot 10 \cdot 10 \cdot 10 = 10^4 = 10000$ total possible PIN numbers.

To have no repeated digits, all four digits would have to be different, which is selecting without replacement. We could either compute $10 \cdot 9 \cdot 8 \cdot 7$, or notice that this is the same as the permutation ${}_{10}P_4 = 5040$.

The probability of no repeated digits is the number of 4 digit PIN numbers with no repeated digits divided by the total number of 4 digit PIN numbers. This probability is $\frac{{}_{10}P_4}{10^4} = \frac{5040}{10000} = 0.504$

Example 3.4.15

In a certain state's lottery, 48 balls numbered 1 through 48 are placed in a machine and six of them are drawn at random. If the six numbers drawn match the numbers that a player had chosen, the player wins \$1,000,000. In this lottery, the order the numbers are drawn in doesn't matter. Compute the probability that you win the million-dollar prize if you purchase a single lottery ticket.

Solution

In order to compute the probability, we need to count the total number of ways six numbers can be drawn, and the number of ways the six numbers on the player's ticket could match the six numbers drawn from the machine. Since there is no stipulation that the numbers be in any particular order, the number of possible outcomes of the lottery drawing is ${}_{48}C_6 = 12,271,512$. Of these possible outcomes, only one would match all six numbers on the player's ticket, so the probability of winning the grand prize is:

$$\frac{{}_6C_6}{{}_{48}C_6} = \frac{1}{12271512} \approx 0.0000000815$$

Example 3.4.16

Compute the probability of randomly drawing five cards from a deck and getting exactly one Ace.

Solution

In many card games (such as poker) the order in which the cards are drawn is not important (since the player may rearrange the cards in his hand any way he chooses); in the problems that follow, we will assume that this is the case unless otherwise stated. Thus we use combinations to compute the possible number of 5-card hands, ${}_{52}C_5$. This number will go in the denominator of our probability formula, since it is the number of possible outcomes.

For the numerator, we need the number of ways to draw one Ace and four other cards (none of them Aces) from the deck. Since there are four Aces and we want exactly one of them, there will be ${}_4C_1$ ways to select one Ace; since there are 48 non-Aces and we want 4 of them, there will be ${}_{48}C_4$ ways to select the four non-Aces. Now we use the Basic Counting Rule to calculate that there will be ${}_4C_1 \cdot {}_{48}C_4$ ways to choose one ace and four non-Aces.

Putting this all together, we have

$$P(\text{one Ace}) = \frac{({}_4C_1)({}_{48}C_4)}{{}_{52}C_5} = \frac{778320}{2598960} \approx 0.299$$

Example 3.4.17

Compute the probability of randomly drawing five cards from a deck and getting exactly two Aces.

Solution

The solution is similar to the previous example, except now we are choosing 2 Aces out of 4 and 3 non-Aces out of 48; the denominator remains the same:

$$P(\text{two Aces}) = \frac{({}_4C_2)({}_{48}C_3)}{{}_{52}C_5} = \frac{103776}{2598960} \approx 0.0399$$

It is useful to note that these card problems are remarkably similar to the lottery problems discussed earlier.

Your Turn 3.4.4

Compute the probability of randomly drawing five cards from a deck of cards and getting three Aces and two Kings.

Answers

Your Turn 3.4.1

$8 \cdot 11 \cdot 5 = 440$ menu combinations

Your Turn 3.4.2

There are 26 characters.

- a. $26^5 = 11,881,376$.
- b. ${}_{26}P_5 = 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 = 7,893,600$

Your Turn 3.4.3

Order does not matter. ${}_{29}C_{19} = 20,030,010$ possible subcommittees

Your Turn 3.4.4

$P(\text{three Aces and two Kings}) = \frac{({}_4C_3)({}_4C_2)}{{}_{52}C_5} = \frac{24}{2598960} \approx 0.0000092$

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4: FINANCIAL MATH

4.1: Introduction

4.2: Simple Interest

4.3: Compound Interest

4.4: Annuities

4.5: Payout Annuities

4.6: Loans

4.1: Introduction

We have to work with money every day. While balancing your checkbook or calculating your monthly expenditures on espresso requires only arithmetic, when we start saving, planning for retirement, or need a loan, we need more mathematics.

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4.2: Simple Interest

Learning Objectives

After completing this section, you should be able to:

1. Calculate the amount of simple interest earned
2. Calculate the APR for a loan
3. Find the interest rate being used in simple interest situations

Discussing interest starts with the **principal**, or amount your account starts with. This could be a starting investment, or the starting amount of a loan. Interest, in its most simple form, is calculated as a percent of the principal. For example, if you borrowed \$100 from a friend and agree to repay it with 5% interest, then the amount of interest you would pay would just be 5% of 100: $100(0.05) = \$5$. The total amount you would repay would be \$105, the original principal plus the interest.

Simple One-time Interest

$$I = P_0 r$$

$$A = P_0 + I = P_0 + P_0 r = P_0(1 + r)$$

where

- I is the interest
- A is the end amount: principal plus interest
- P_0 is the principal (starting amount)
- r is the interest rate (in decimal form. Example: $5\% = 0.05$)

Example 4.2.1

A friend asks to borrow \$300 and agrees to repay it in 30 days with 3% interest. How much interest will you earn?

Solution

$P_0 = \$300$ the principal
 $r = 0.03$ 3% rate
 $I = \$300(0.03) = \$9$. You will earn \$9 interest.

One-time simple interest is only common for extremely short-term loans. For longer term loans, it is common for interest to be paid on a daily, monthly, quarterly, or annual basis. In that case, interest would be earned regularly. For example, bonds are essentially a loan made to the bond issuer (a company or government) by you, the bond holder. In return for the loan, the issuer agrees to pay interest, often annually. Bonds have a maturity date, at which time the issuer pays back the original bond value.

Example 4.2.2

Suppose your city is building a new park, and issues bonds to raise the money to build it. You obtain a \$1,000 bond that pays 5% interest annually that matures in 5 years. How much interest will you earn?

Solution

Each year, you would earn 5% interest: $\$1000(0.05) = \$50$ in interest. So over the course of five years, you would earn a total of \$250 in interest. When the bond matures, you would receive back the \$1,000 you originally paid, leaving you with a total of \$1,250.

We can generalize this idea of simple interest over time.

Simple Interest over Time

$$I = P_0rt$$

$$A = P_0 + I = P_0 + P_0rt = P_0(1 + rt)$$

where

- I is the interest
- A is the end amount: principal plus interest
- P_0 is the principal (starting amount)
- r is the interest rate in decimal form
- t is time

The units of measurement (years, months, etc.) for the time should match the time period for the interest rate.

APR – Annual Percentage Rate

Interest rates are usually given as an **annual percentage rate (APR)** – the total interest that will be paid in the year. If the interest is paid in smaller time increments, the APR will be divided up.

For example, a 6% APR paid monthly would be divided into twelve 0.5% payments.

A 4% annual rate paid quarterly would be divided into four 1% payments.

Example 4.2.3: Treasury Notes

Treasury Notes (T-notes) are bonds issued by the federal government to cover its expenses. Suppose you obtain a \$1,000 T-note with a 4% annual rate, paid semi-annually, with a maturity in 4 years. How much interest will you earn?

Solution

Since interest is being paid semi-annually (twice a year), the 4% interest will be divided into two 2% payments.

$P_0 = \$1000$	the principal
$r = 0.02$	2% rate per half-year
$t = 8$	4 years = 8 half-years
$I = \$1000(0.02)(8) = \$160$. You will earn \$160 interest total over the four years.	

Your Turn 4.2.1

A loan company charges \$30 interest for a one month loan of \$500. Find the annual interest rate they are charging.

Answer available at the end of this section.

Answers

Your Turn 4.2.1

$I = \$30$ of interest

$P_0 = \$500$ principal

$r = \frac{30}{500} = 0.06$ per month

$(0.06)(12) = 0.72$ per year

They are charging an annual interest rate of 72%.

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4.3: Compound Interest

Learning Objectives

After completing this section, you should be able to:

1. Calculate the amount in an account after a specified amount of time using compound interest

With simple interest, we were assuming that we pocketed the interest when we received it. In a standard bank account, any interest we earn is automatically added to our balance, and we earn interest on that interest in future years. This reinvestment of interest is called **compounding**.

Suppose that we deposit \$1000 in a bank account offering 3% interest, compounded monthly. How will our money grow?

The 3% interest is an annual percentage rate (APR) – the total interest to be paid during the year. Since interest is being paid monthly, each month, we will earn $\frac{3\%}{12} = 0.25\%$ per month.

In the first month,

$$P_0 = \$1000$$

$$r = 0.0025(0.25\%)$$

$$I = \$1000(0.0025) = \$2.50$$

$$A = \$1000 + \$2.50 = \$1002.50$$

In the first month, we will earn \$2.50 in interest, raising our account balance to \$1002.50.

In the second month,

$$P_0 = \$1002.50$$

$$I = \$1002.50(0.0025) = \$2.51 \text{ (rounded)}$$

$$A = \$1000 + \$2.50 = \$1002.50$$

Notice that in the second month we earned more interest than we did in the first month. This is because we earned interest not only on the original \$1000 we deposited, but we also earned interest on the \$2.50 of interest we earned the first month. This is the key advantage that **compounding** of interest gives us.

Calculating out a few more months:



To find an equation to represent this, if P_m represents the amount of money after m months, then we could write the recursive equation:

$$P_0 = \$1000$$

$$P_m = (1 + 0.0025)P_{m-1}$$

You probably recognize this as the recursive form of exponential growth. If not, we could go through the steps to build an explicit equation for the growth:

$$P_0 = \$1000$$

$$P_1 = 1.0025P_0 = 1.0025(1000)$$

$$P_2 = 1.0025P_1 = 1.0025(1.0025(1000)) = 1.0025^2(1000)$$

$$P_3 = 1.0025P_2 = 1.0025(1.0025^2(1000)) = 1.0025^3(1000)$$

$$P_4 = 1.0025P_3 = 1.0025(1.0025^3(1000)) = 1.0025^4(1000)$$

Observing a pattern, we could conclude

$$P_m = (1.0025)^m(\$1000)$$

Notice that the ~~\$1000~~ in the equation was P_0 , the starting amount. We found 1.0025 by adding one to the growth rate divided by 12, since we were compounding 12 times per year.

Generalizing our result, we could write

$$P_m = P_0 \left(1 + \frac{r}{k}\right)^m$$

In this formula:

m is the number of compounding periods (months in our example)

r is the annual interest rate

k is the number of compounds per year.

While this formula works fine, it is more common to use a formula that involves the number of years, rather than the number of compounding periods. If N is the number of years, then $m = Nk$. Making this change gives us the standard formula for compound interest.

Compound Interest

$$P_N = P_0 \left(1 + \frac{r}{k}\right)^{Nk}$$

P_N is the balance in the account after N years.

P_0 is the starting balance of the account (also called initial deposit, or principal)

r is the annual interest rate in decimal form

k is the number of compounding periods in one year.

- If the compounding is done annually (once a year), $k = 1$.
- If the compounding is done quarterly, $k = 4$.
- If the compounding is done monthly, $k = 12$.
- If the compounding is done daily, $k = 365$.

The most important thing to remember about using this formula is that it assumes that we put money in the account once and let it sit there earning interest.

Example 4.3.1

A certificate of deposit (CD) is a savings instrument that many banks offer. It usually gives a higher interest rate, but you cannot access your investment for a specified length of time. Suppose you deposit \$3000 in a CD paying 6% interest, compounded monthly. How much will you have in the account after 20 years?

Solution

In this example,

$P_0 = \$3000$ the initial deposit

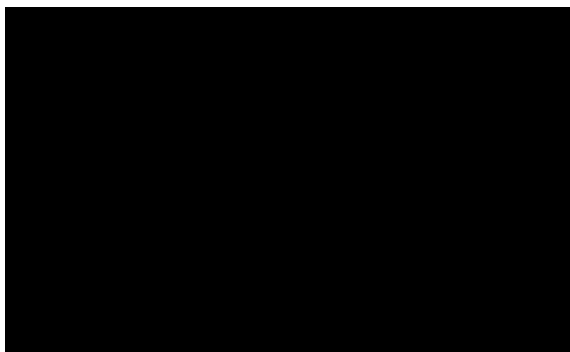
$r = 0.06$ 6% annual rate

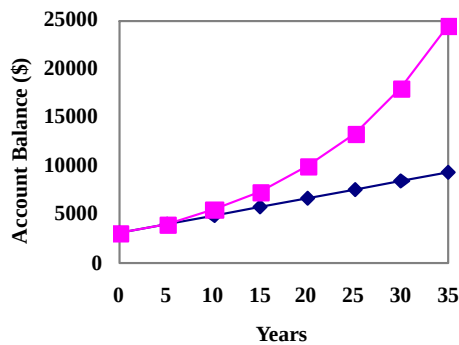
$k = 12$ 12 months in 1 year

$N = 20$ since we're looking for how much we'll have after 20 years

So $P_{20} = 3000 \left(1 + \frac{0.06}{12}\right)^{20 \times 12} = \$9930.61$ (round your answer to the nearest penny)

Let us compare the amount of money earned from compounding against the amount you would earn from simple interest





As you can see, over a long period of time, compounding makes a large difference in the account balance. You may recognize this as the difference between linear growth and exponential growth.

Evaluating exponents on the calculator

When we need to calculate something like 5^3 it is easy enough to just multiply $5 \cdot 5 \cdot 5 = 125$. But when we need to calculate something like 1.005^{240} , it would be very tedious to calculate this by multiplying 1.005 by itself 240 times! So to make things easier, we can harness the power of our scientific calculators.

Most scientific calculators have a button for exponents. It is typically either labeled like:

$[\wedge]$, $[y^x]$, or $[x^y]$

To evaluate 1.005^{240} we'd type $1.005 [\wedge] 240$, or $1.005 [y^x] 240$. Try it out - you should get something around 3.3102044758.

Example 4.3.2

You know that you will need \$40,000 for your child's education in 18 years. If your account earns 4% compounded quarterly, how much would you need to deposit now to reach your goal?

Solution

We're looking for P_0 .

$r = 0.04$ 4%
 $k = 4$ 4 quarters in 1 year
 $N = 18$ Since we know the balance in 18 years
 $P_{18} = \$40,000$ The amount we have in 18 years

In this case, we're going to have to set up the equation, and solve for P_0 .

$$40000 = P_0 \left(1 + \frac{0.04}{4} \right)^{18 \times 4}$$

$$40000 = P_0 (2.0471)$$

$$P_0 = \frac{40000}{2.0471} = \$19539.84$$

So you would need to deposit \$19,539.84 now to have \$40,000 in 18 years.

Rounding

It is important to be very careful about rounding when calculating things with exponents. In general, you want to keep as many decimals during calculations as you can. Be sure to **keep at least 3 significant digits** (numbers after any leading zeros). Rounding 0.00012345 to 0.000123 will usually give you a “close enough” answer, but keeping more digits is always better.

Example 4.3.3

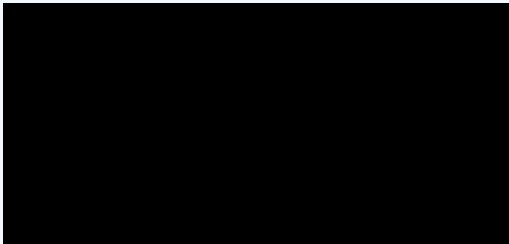
To see why not over-rounding is so important, suppose you were investing \$1000 at 5% interest compounded monthly for 30 years.

Solution

$P_0 = \$1000$ the initial deposit
 $r = 0.05$ 5%
 $k = 12$ 12 months in 1 year
 $N = 30$ since we're looking for the amount after 30 years

If we first compute $\frac{r}{k}$, we find $\frac{0.05}{12} = 0.0041666666667$

Here is the effect of rounding this to different values:



If you're working in a bank, of course you wouldn't round at all. For our purposes, the answer we got by rounding to 0.00417, three significant digits, is close enough - \$5 off of \$4500 isn't too bad. Certainly keeping that fourth decimal place wouldn't have hurt.

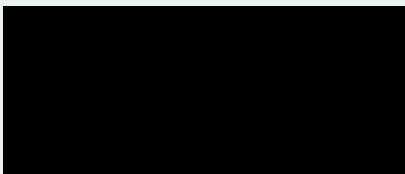
Using your calculator

In many cases, you can avoid rounding completely by how you enter things in your calculator. For example, in the example above, we needed to calculate

$$P_{30} = 1000 \left(1 + \frac{0.05}{12}\right)^{12 \times 30}$$

We can quickly calculate $12 \times 30 = 360$, giving $P_{30} = 1000 \left(1 + \frac{0.05}{12}\right)^{360}$.

Now we can use the calculator.



Using your calculator continued

The previous steps were assuming you have a “one operation at a time” calculator; a more advanced calculator will often allow you to type in the entire expression to be evaluated. If you have a calculator like this, you will probably just need to enter:

1000 [×] (1 [+] 0.05 [÷] 12) [y^x] 360 [=]

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4.4: Annuities

Learning Objectives

After completing this section, you should be able to:

1. Determine the amount of money in a savings annuity.
2. Determine the amount of money to deposit regularly into a savings annuity to earn a specified amount.

For most of us, we aren't able to put a large sum of money in the bank today. Instead, we save for the future by depositing a smaller amount of money from each paycheck into the bank. This idea is called a **savings annuity**. Most retirement plans like 401k plans or IRA plans are examples of savings annuities.

An annuity can be described recursively in a fairly simple way. Recall that basic compound interest follows from the relationship

$$P_m = \left(1 + \frac{r}{k}\right) P_{m-1}$$

For a savings annuity, we simply need to add a deposit, d , to the account with each compounding period:

$$P_m = \left(1 + \frac{r}{k}\right) P_{m-1} + d$$

Taking this equation from recursive form to explicit form is a bit trickier than with compound interest. It will be easiest to see by working with an example rather than working in general.

Suppose we will deposit \$100 each month into an account paying 6% interest. We assume that the account is compounded with the same frequency as we make deposits unless stated otherwise. In this example:

$$r = 0.06 \text{ (6\%)}$$

$$k = 12 \text{ (12 compounds/deposits per year)}$$

$$d = \$100 \text{ (our deposit per month)}$$

Writing out the recursive equation gives

$$P_m = \left(1 + \frac{0.06}{12}\right) P_{m-1} + 100 = (1.005)P_{m-1} + 100$$

Assuming we start with an empty account, we can begin using this relationship:

$$P_0 = 0$$

$$P_1 = (1.005)P_0 + 100 = 100$$

$$P_2 = (1.005)P_1 + 100 = (1.005)(100) + 100 = 100(1.005) + 100$$

$$P_3 = (1.005)P_2 + 100 = (1.005)(100(1.005) + 100) + 100 = 100(1.005)^2 + 100(1.005) + 100$$

Continuing this pattern, after m deposits, we'd have saved:

$$P_m = 100(1.005)^{m-1} + 100(1.005)^{m-2} + \dots + 100(1.005) + 100$$

In other words, after m months, the first deposit will have earned compound interest for $m - 1$ months. The second deposit will have earned interest for $m - 2$ months. Last months deposit would have earned only one month worth of interest. The most recent deposit will have earned no interest yet.

This equation leaves a lot to be desired, though – it doesn't make calculating the ending balance any easier! To simplify things,

multiply both sides of the equation by 1.005:

$$1.005P_m = 1.005(100(1.005)^{m-1} + 100(1.005)^{m-2} + \dots + 100(1.005) + 100)$$

Distributing on the right side of the equation gives

$$1.005P_m = 100(1.005)^m + 100(1.005)^{m-1} + \dots + 100(1.005)^2 + 100(1.005)$$

Now we'll line this up with like terms from our original equation, and subtract each side

$$\begin{array}{rcl} 1.005P_m & = & 100(1.005)^m + 100(1.005)^{m-1} + \dots + 100(1.005) \\ P_m & = & 100(1.005)^{m-1} + \dots + 100(1.005) + 100 \end{array}$$

Almost all the terms cancel on the right hand side when we subtract, leaving

$$1.005P_m - P_m = 100(1.005)^m - 100$$

Solving for P_m

$$0.005P_m = 100((1.005)^m - 1)$$

$$P_m = \frac{100((1.005)^m - 1)}{0.005}$$

Replacing m months with $12N$, where N is measured in years, gives

$$P_N = \frac{100((1.005)^{12N} - 1)}{0.005}$$

Recall 0.005 was $\frac{r}{k}$ and 100 was the deposit d . 12 was k , the number of deposit each year. Generalizing this result, we get the saving annuity formula.

Annuity Formula

$$P_N = \frac{d((1+\frac{r}{k})^{Nk} - 1)}{(\frac{r}{k})}$$

P_N is the balance in the account after N years.

d is the regular deposit (the amount you deposit each year, each month, etc.)

r is the annual interest rate in decimal form.

k is the number of compounding periods in one year.

If the compounding frequency is not explicitly stated, assume there are the same number of compounds in a year as there are deposits made in a year.

For example, if the compounding frequency isn't stated:

- If you make your deposits every month, use monthly compounding, $k = 12$.
- If you make your deposits every year, use yearly compounding, $k = 1$.
- If you make your deposits every quarter, use quarterly compounding, $k = 4$.

When do you use this

Annuities assume that you put money in the account on a regular schedule (every month, year, quarter, etc.) and let it sit there earning interest.

Compound interest assumes that you put money in the account once and let it sit there earning interest.

Compound interest: One deposit

Annuity: Many deposits.

Example 4.4.1

A traditional individual retirement account (IRA) is a special type of retirement account in which the money you invest is exempt from income taxes until you withdraw it. If you deposit \$100 each month into an IRA earning 6% interest, how much will you have in the account after 20 years?

Solution

In this example,

$d = \$100$ the monthly deposit
 $r = 0.06$ 6% annual rate
 $k = 12$ since we're doing monthly deposits, we'll compound monthly
 $N = 20$ we want the amount after 20 years

Putting this into the equation:

$$P_{20} = \frac{100 \left(\left(1 + \frac{0.06}{12} \right)^{20(12)} - 1 \right)}{\left(\frac{0.06}{12} \right)}$$

Remember order of operations here. Start with the parentheses. Then multiply by 100 and then divide by (.06/12).

The account will grow to \$46,204.09 after 20 years.

Notice that you deposited into the account a total of \$24,000 (\$100 a month for 240 months). The difference between what you end up with and how much you put in is the interest earned. In this case it is $\$46,204.09 - \$24,000 = \$22,204.09$.

Example 4.4.2

You want to have \$200,000 in your account when you retire in 30 years. Your retirement account earns 8% interest. How much do you need to deposit each month to meet your retirement goal?

Solution

In this example,

We're looking for d .

$r = 0.08$ 8% annual rate
 $k = 12$ since we're doing monthly deposits, we'll compound monthly
 $N = 30$ 30 years
 $P_{30} = \$200,000$ The amount we want to have in 30 years

In this case, we're going to have to set up the equation, and solve for d .

$$200,000 = \frac{d \left(\left(1 + \frac{0.08}{12} \right)^{30(12)} - 1 \right)}{\left(\frac{0.08}{12} \right)}$$

$$200,000 = d(1490.359)$$

$$d = \frac{200,000}{1490.359} = \$134.20$$

So you would need to deposit \$134.20 each month to have \$200,000 in 30 years if your account earns 8% interest

Your Turn 4.4.1

A more conservative investment account pays 3% interest. If you deposit \$5 a day into this account, how much will you have after 10 years? How much is from interest?

Answer available at the end of this section.

Answers

Your Turn 4.4.1

$d = \$5$ the daily deposit

$r = 0.03$ 3% annual rate

$k = 365$ since we're doing daily deposits, we'll compound daily

$N = 10$ we want the amount after 10 years

$$P_{10} = \frac{5 \left(\left(1 + \frac{0.03}{365} \right)^{365 \times 10} - 1 \right)}{\frac{0.03}{365}} = \$21,282.07$$

We would have deposited a total of $\$5 \cdot 365 \cdot 10 = \$18,250$, so **\$3,032.07** is from interest

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4.5: Payout Annuities

In the last section you learned about annuities. In an annuity, you start with nothing, put money into an account on a regular basis, and end up with money in your account.

In this section, we will learn about a variation called a **Payout Annuity**. With a payout annuity, you start with money in the account, and pull money out of the account on a regular basis. Any remaining money in the account earns interest. After a fixed amount of time, the account will end up empty.

Payout annuities are typically used after retirement. Perhaps you have saved \$500,000 for retirement, and want to take money out of the account each month to live on. You want the money to last you 20 years. This is a payout annuity. The formula is derived in a similar way as we did for savings annuities. The details are omitted here.

Payout Annuity Formula

$$P_0 = \frac{d(1 - (1 + \frac{r}{k})^{-Nk})}{(\frac{r}{k})}$$

P_0 is the balance in the account at the beginning (starting amount, or principal).

d is the regular withdrawal (the amount you take out each year, each month, etc.)

r is the annual interest rate (in decimal form. Example: 5% = 0.05)

k is the number of compounding periods in one year.

N is the number of years we plan to take withdrawals

Like with annuities, the compounding frequency is not always explicitly given, but is determined by how often you take the withdrawals.

When do you use this

Payout annuities assume that you take money from the account on a regular schedule (every month, year, quarter, etc.) and let the rest sit there earning interest.

Compound interest: One deposit

Annuity: Many deposits.

Payout Annuity: Many withdrawals

Example 4.5.1

After retiring, you want to be able to take \$1000 every month for a total of 20 years from your retirement account. The account earns 6% interest. How much will you need in your account when you retire?

Solution

In this example,

$d = \$1000$ the monthly withdrawal
 $r = 0.06$ 6% annual rate
 $k = 12$ since we're doing monthly withdrawals, we'll compound monthly
 $N = 20$ since we're taking withdrawals for 20 years

We're looking for P_0 ; how much money needs to be in the account at the beginning.

Putting this into the equation:

$$P_0 = \frac{1000 \left(1 - \left(1 + \frac{0.06}{12} \right)^{-20(12)} \right)}{\left(\frac{0.06}{12} \right)}$$

$$P_0 = \$139,580.77$$

You will need to have \$139,580.77 in your account when you retire.

Notice that you withdrew a total of \$240,000 (\$1000 a month for 240 months). The difference between what you pulled out and what you started with is the interest earned. In this case it is $\$240,000 - \$139,580.77 = \$100,419.23$ in interest.

Evaluating negative exponents on your calculator

With these problems, you need to raise numbers to negative powers. Most calculators have a separate button for negating a number that is different than the subtraction button. Some calculators label this $[(-)]$, some with $[+/-]$. The button is often near the $=$ key or the decimal point.

If your calculator displays operations on it (typically a calculator with multiline display), to calculate 1.005^{-240} you'd type something like: $1.005[\wedge][(-)]240$

If your calculator only shows one value at a time, then usually you hit the $(-)$ key after a number to negate it, so you'd hit: $1.005[y^x]240[(-)] =$

Give it a try - you should get $1.005^{-240} = 0.302096$

Example 4.5.2

You know you will have \$500,000 in your account when you retire. You want to be able to take monthly withdrawals from the account for a total of 30 years. Your retirement account earns 8% interest. How much will you be able to withdraw each month?

Solution

In this example,

We're looking for d .

$r = 0.08$ 8% annual rate
 $k = 12$ since we're doing monthly withdrawals
 $N = 30$ since we're taking withdrawals for 30 years
 $P_0 = \$500,000$ we are beginning with \$500,000

In this case, we're going to have to set up the equation, and solve for d .

$$500,000 = \frac{d \left(1 - \left(1 + \frac{0.08}{12} \right)^{-30(12)} \right)}{\left(\frac{0.08}{12} \right)}$$

$$500,000 = d(136.283)$$

$$d = \frac{500,000}{136.283} = \$3668.84$$

You would be able to withdraw \$3,668.84 each month for 30 years.

Your Turn 4.5.1

A donor gives \$100,000 to a university, and specifies that it is to be used to give annual scholarships for the next 20 years. If the university can earn 4% interest, how much can they give in scholarships each year?

Answer available at the end of this section.

Answers

Your Turn 4.5.1

$d =$ unknown

$r = 0.04$ 4% annual rate

$k = 1$ since we're doing annual scholarships

$N = 20$ since we're taking withdrawals for 20 years

$P_0 = \$100,000$ we are starting with \$100,000

$$100,000 = \frac{d(1 - (1 + \frac{0.04}{1})^{-20 \times 1})}{\frac{0.04}{1}}$$

$$100,000 = d(13.590)$$

Solving for d gives \$7,358.35 each year that they can give in scholarships.

It is worth noting that usually donors instead specify that only interest is to be used for scholarship, which makes the original donation last indefinitely. If this donor had specified that, $\$100,000(0.04) = \$4,000$ a year would have been available.

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4.6: Loans

Learning Objectives

After completing this section, you should be able to:

1. Calculate the payment amount on a loan.
2. Calculate the amount of loan you can afford.

In the last section, you learned about payout annuities.

In this section, you will learn about conventional loans (also called amortized loans or installment loans). Examples include auto loans and home mortgages. These techniques do not apply to payday loans, add-on loans, or other loan types where the interest is calculated up front.

One great thing about loans is that they use exactly the same formula as a payout annuity. To see why, imagine that you had \$10,000 invested at a bank, and started taking out payments while earning interest as part of a payout annuity, and after 5 years your balance was zero. Flip that around, and imagine that you are acting as the bank, and a car lender is acting as you. The car lender invests \$10,000 in you. Since you're acting as the bank, you pay interest. The car lender takes payments until the balance is zero.

Loans Formula

$$P_0 = \frac{d(1 - (1 + \frac{r}{k})^{-Nk})}{(\frac{r}{k})}$$

P_0 is the balance in the account at the beginning (the principal, or amount of the loan).

d is your loan payment (your monthly payment, annual payment, etc)

r is the annual interest rate in decimal form.

k is the number of compounding periods in one year.

N is the length of the loan, in years

Like before, the compounding frequency is not always explicitly given, but is determined by how often you make payments.

When do you use this

The loan formula assumes that you make loan payments on a regular schedule (every month, year, quarter, etc.) and are paying interest on the loan.

Compound interest: One deposit

Annuity: Many deposits.

Payout Annuity: Many withdrawals

Loans: Many payments

Example 4.6.1

You can afford \$200 per month as a car payment. If you can get an auto loan at 3% interest for 60 months (5 years), how expensive of a car can you afford? In other words, what amount loan can you pay off with \$200 per month?

Solution

In this example,

$d = \$200$ the monthly loan payment

$r = 0.03$ 3% annual rate

$k = 12$ since we're doing monthly payments, we'll compound monthly

$N = 5$ since we're making monthly payments for 5 years

We're looking for P_0 , the starting amount of the loan.

$$P_0 = \frac{200 \left(1 - \left(1 + \frac{0.03}{12} \right)^{-5(12)} \right)}{\left(\frac{0.03}{12} \right)}$$

$$P_0 = \$11,130.47$$

You can afford a \$11,130.47 loan.

You will pay a total of \$12,000 (\$200 per month for 60 months) to the loan company. The difference between the amount you pay and the amount of the loan is the interest paid. In this case, you're paying $\$12,000 - \$11,130.47 = \$869.53$ interest total.

Example 4.6.2

You want to take out a \$140,000 mortgage (home loan). The interest rate on the loan is 6%, and the loan is for 30 years. How much will your monthly payments be?

Solution

In this example,

We're looking for d .

$r = 0.06$ 6% annual rate

$k = 12$ since we're doing monthly payments, we'll compound monthly

$N = 30$ since we're making monthly payments for 30 years

$P_0 = \$140,000$ the starting loan amount

In this case, we're going to have to set up the equation, and solve for d .

$$140,000 = \frac{d \left(1 - \left(1 + \frac{0.06}{12} \right)^{-30(12)} \right)}{\left(\frac{0.06}{12} \right)}$$

$$140,000 = d(166.792)$$

$$d = \frac{140,000}{166.792} = \$839.37$$

You will make payments of \$839.37 per month for 30 years.

You're paying a total of \$302,173.20 to the loan company: \$839.37 per month for 360 months. You are paying a total of $\$302,173.20 - \$140,000 = \$162,173.20$ in interest over the life of the loan.

Your Turn 4.6.1

Janine bought \$3,000 of new furniture on credit. Because her credit score isn't very good, the store is charging her a fairly high interest rate on the loan: 16%. If she agreed to pay off the furniture over 2 years, how much will she have to pay each month?

Answer available at the end of this section.

Answers

Your Turn 4.6.1

$d =$ unknown

$r = 0.16$ 16% annual rate

$k = 12$ since we're doing monthly payments, we'll compound monthly

$N = 2$ 2 year to repay

$P_0 = 3,000$ the starting loan amount \$3,000 loan

$$3,000 = \frac{d \left(1 - \left(1 + \frac{0.16}{12} \right)^{-2 \times 12} \right)}{\frac{0.16}{12}}$$

$$3,000 = d(20.424)$$

Solving for d gives \$146.89 as monthly payments.

In total, she will pay $146.89 \cdot 24 = \$3,525.36$ to the store, meaning she will pay \$525.36 in interest over the two years.

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5: VOTING AND GRAPH THEORY

5.1: Introduction to Voting Theory

5.2: Voting Methods

5.3: Fairness in Voting Methods

5.4: Introduction to Graph Theory

5.5: Graphs

5.6: Shortest Path

5.7: Euler Circuits and the Chinese Postman Problem

5.8: Eulerization and the Chinese Postman Problem

5.9: Hamiltonian Circuits and the Traveling Salesman Problem

5.1: Introduction to Voting Theory

Suppose a friend asked you, “When did you last vote?” What would your answer be? Maybe you would tell your friend that the last time you voted was during the last presidential election, or perhaps you would tell your friend that you prefer not to vote. When thinking about voting, presidential campaigns or advertisements for reelections may come to mind, but you can cast your vote in many ways. Have you liked a post, followed a creator, friended a stranger, or clicked a heart online today? In the digital age, it's possible to vote several times a day. Voting systems are not only the machines that drive every democracy on Earth, but they are also the engines driving social media and many other aspects of life. A deeper understanding of these voting systems will enhance your ability to successfully engage with the world in which we live.

In this chapter, you will become one of the founders of the new democratic country of Imaginaria. You have a great responsibility to the people of this fledgling democracy. You have been tasked with writing the portion of the constitution that lays out voting procedures. In preparation for this important task, you will explore the various types of voting systems, from school board elections to Twitter wars. You will see how these types are alike, how they differ, and how they might be applied in Imaginaria. Most importantly, you will learn about the mathematically inherent advantages and disadvantages of various voting systems so that you can make informed choices to better the lives of the Imaginarians.

5.2: Voting Methods

Learning Objectives

After completing this section, you should be able to:

1. Apply plurality voting to determine a winner.
2. Apply runoff voting to determine a winner.
3. Apply ranked-choice voting to determine a winner.
4. Apply Borda count voting to determine a winner.
5. Apply pairwise comparison and Condorcet voting to determine a winner.
6. Apply approval voting to determine a winner.
7. Compare and contrast voting methods to identify flaws.

Today is the day that you begin your quest to collaborate on the constitution of Imaginaria! Let's begin by thinking about the selection of a leader who can serve as president. It seems straightforward; if the majority of citizens prefer a particular candidate, that candidate should win. But not all votes are decided by a simple majority. Why not? What are the options?

Majority versus Plurality Voting

When an election involves only two options, a simple majority is a reasonable way to determine a winner. A majority is a number equaling more than half, or greater than 50 percent of the total.

Let's take a look at the outcomes of U.S. presidential elections to understand more. Table 5.2.1 displays the results of the 2000 U.S. presidential election. Like most presidential elections, this election involved more than two options. If that is the case, is it possible that no single candidate will receive more than half of the votes cast?

Candidate (Party Label)	Popular Vote Total
Al Gore (Democrat)	50,999,897
George W. Bush (Republican)	50,456,002
Ralph Nader (Green)	2,882,955
Patrick J. Buchanan (Reform/Independent)	448,895
Harry Browne (Libertarian)	384,431
Howard Phillips (Constitution)	98,020
Other	134,900
Total:	105,405,100

Table 5.2.1: Results of the Popular Vote for the 2000 U.S. Presidential Election (source: <https://www.fec.gov/introduction-cam.../president2000/>)

Example 5.2.1

Majority of Popular Vote in the 2000 U.S. Presidential Election

Refer back to Table 5.2.1. Did any single candidate secure the majority of popular votes?

Answer available at the end of this section.

Your Turn 5.2.1

According to the Cook Political Report, a total of 158,394,605 votes were cast in the 2020 U.S. presidential election. Of those, 81,281,502 were cast for Joe Biden, 74,222,593 for Donald Trump, and 2,890,510 for other candidates. Although U.S. presidential elections are not determined by popular vote, did either Joe Biden or Donald Trump secure a majority of the votes?

Answer

Step 1: Calculate 50 percent of 158,394,605 by multiplying the decimal form of 50 percent, which is 0.50, by 158,394,605: $0.50 (158,394,605) = 79,197,302.5$

Step 2: Determine the minimum number of votes needed to have a majority. The minimum number of votes required is the lowest counting number that is larger than 50 percent of the votes. To have a majority, an individual candidate must have more than 79,197,302.5; so, a majority candidate must have 79,197,303 votes or more.

Step 3: Compare the number of votes each candidate received to 79,197,303. According to the data, Joe Biden had a majority of the votes.

Unlike in the 2000 U.S. presidential election, a candidate won the majority of votes in the 2020 election. It is a common occurrence for no single candidate to receive a majority of the votes in an election with more than two candidates. When this occurs, the candidate with the largest portion of the votes is said to have a **plurality**.

Example 5.2.2

Plurality of Popular Vote in the 2000 U.S. Presidential Election

Refer again to Table 5.2.1. In the 2000 U.S. presidential election, which candidate had a plurality of popular votes?

Answer available at the end of this section.

Your plans for Imaginarian elections will likely include primary elections, or preliminary elections to select candidates for a principal or general election. Table 5.2.2 displays the results of the 2018 U.S. Senate Republican primary for Maryland.

Top Four Republican Candidates	Votes	Percentage of Party Votes
Cambell, Tony	51,426	29.22%
Chaffee, Chris	42,328	24.05%
Grigorian, Christina J.	30,756	17.48%
Graziani, John R.	15,435	8.77%
Total Votes	175,981	100%

Table 5.2.2 (source: https://ballotpedia.org/United_States_Senate_Republican_primary)

Your Turn 5.2.2

Consider the results of the 2018 U.S. Senate primary for Maryland. Determine which candidate won the primary for the Republicans (R). Did the candidate win a majority or a plurality of Republican votes?

Answer

Majority:

Step 1: Calculate 50 percent of 175,981 by multiplying the decimal form of 50 percent, which is 0.50, by 175,981: $0.50 (175,981) = 87,990.5$

Step 2: Determine the minimum number of votes needed to have a majority. The minimum number of votes required is the lowest counting number that is larger than 50 percent of the votes. To have a majority, an individual candidate must have more than 87,990.5; so, a majority candidate must have 87,991 votes or more.

Step 3: Compare the number of votes each candidate received to 87,991. According to the data in Table 5.2.2, none of the candidates secured a majority.

Plurality: Tony Campbell had the most votes, so he won the plurality.

Consider how election by plurality, not majority, is the most common method of selecting candidates for public office.

Who Knew?

The U.S. Electoral College: Winning the Presidency without a Plurality

In the 2000 U.S. presidential election, Al Gore had a plurality of the popular votes, but he did not win the election. Why? This occurred because the U.S. president and vice president are elected by electors rather than a direct vote by the citizens. The electors are part of the Electoral College, a body of people representing the states. Why was the Electoral College created? The Electoral College was created as a compromise between those authors of the U.S. Constitution who believed Congress should elect the president, and those who believed the citizens should vote directly. The popular vote was not recorded until the presidential election of 1824. Since then, only five presidents have been elected without winning a plurality of the popular vote: John Quincy Adams in 1824, Rutherford B. Hayes in 1876, Grover Cleveland in 1888, George Bush in 2000, and Donald Trump in 2016.

Runoff Voting

Has your family ever debated what to have for dinner? Suppose your family is deciding on a restaurant and exactly half of you want to have pizza but the other half want hamburgers. How do you decide when the result is a tie? You need a tiebreaker!

Will the new democracy of Imaginaria need tiebreakers? When no candidate satisfies the requirements to win the election, a runoff election, or second election, is held to determine a winner.

How would runoff voting work in Imaginaria? There are many types of **runoff voting systems**, which are voting systems that utilize a runoff election when the first round does not result in a winner. The method for implementing a runoff election can vary widely, particularly in the criteria used to determine whether a candidate will be on the ballot in the second election. For example, a two-round system is a runoff voting system in which only the top candidates advance to the runoff election. In some two-round systems, only the top two candidates are on the second ballot, or it may be any candidate who secures a certain percentage of the vote will advance. The Hare Method is another runoff voting system in which only the candidate(s) with the very least votes are eliminated. This can potentially result in several rounds of runoff elections.

Example 5.2.3

Runoff Election for Condominium Association President

A condominium association elects a new president every two years by a two-round system of voting. If none of the candidates receive a majority, the association charter states that the top two candidates will be eligible to participate in a runoff election. In a particular year, five residents were nominated. The results of the first round are given in the table below.

Candidate	Votes in First Round
Abou	18
Baiocchi	10

Candidate	Votes in First Round
Campana	5
Dali	11
Eugene	4

1. Is there a winner based on the first round? Why or why not?
2. If there is a winner, who won? If there should be a runoff, who will advance to the second round?

Answer available at the end of this section.

Your Turn 5.2.3

The student government bylaws of a particular college require that a new president is elected annually by plurality voting. In the event of a tie, the bylaws require the candidate(s) with the fewest votes to be removed from the ballot and a runoff election to be held with the remaining candidates. This process is repeated until a single candidate receives a plurality and wins the election. The results of two voting rounds are given in the table below.

1. Which two candidates tied in the second round? Does this mean there will be a third election?
2. If there was a winner, who won? If there should be a runoff, who must be removed from the ballot?

Candidate	Votes in First Round	Votes in Second Round
Ferguson	158	168
Garcia	103	104
Hearn	157	180
Isaac	123	123
Jackman	58	Eliminated
Kelly	72	74
Lim	158	180

Answer

1. Hearn and Lim tied in the second round. Since they had the most votes, there needs to be another round of voting.
2. Since there needs to be another runoff, the person with the least votes will be eliminated. Kelly will be eliminated in the third round.

Steps to Determine Winner by Plurality or Majority Election with Runoff

To determine the winner by plurality or when a majority election with runoff occurs, we take these three steps:

Step 1: If a majority is required to win the election, determine the number of votes needed to achieve a majority. This is the least whole number greater than 50 percent of the total votes. If a majority is not required, move to Step 2.

Step 2: Count the number of votes for each candidate in the current round of voting. If a single candidate has enough votes to win a plurality, or a majority as appropriate, then you are done! Otherwise, eliminate a predetermined number of candidates based on the rules of the election. Elimination conditions may vary. For example, the rules may state that the candidate(s) with

the fewest votes will be eliminated (as in the Hare method), or that only the candidates meeting a certain threshold will move on (as in a two-round system). Once the appropriate candidates are eliminated, move on to Step 3.

Step 3: Hold a runoff election. If the runoff is simulated using a list of voter's preferences, renumber the preferences to reflect the remaining number of options in such a way that the original order of preference is retained. Then repeat Step 2.

Note: The second and third steps may be repeated as many times as necessary for voting procedures that allow multiple runoffs.

Example 5.2.4

Family Dinner Night

The five members of the Chionilis family—Annette, Rene, Seema, Titus, and Galen—have decided to get takeout for dinner. They are trying to decide on a restaurant. The options are Rainbow China, Dough Boys Pizza, Taco City, or Caribbean Flavor. They will use majority election with runoffs where the restaurant with the fewest votes is eliminated in each round. The preferences of each family member are listed by first initial in the table below. An entry of 1 represents the person's first choice; 2, their second; and so on. For example, Annette's second choice is Dough Boys Pizza.

Options	A	R	S	T	G
Rainbow China	1	3	3	1	3
Dough Boys Pizza	2	2	1	2	1
Taco City	3	4	2	4	2
Caribbean Flavor	4	1	4	3	4

Use the information in the table to answer the following questions.

1. Which common type of runoff voting method is this?
2. List the results of each round of voting based on this information and determine which restaurant was ultimately chosen.

Answer available at the end of this section.

Your Turn 5.2.4

There are six members on the board of a Parent Teacher Association (PTA) at a local elementary school: the president (P), the vice president (V), the recording secretary (R), the liaison to the administration (L), the treasurer (T), and the chief fundraiser (C). The board must decide which equipment to purchase for the classrooms with moneys from their annual fundraisers. The preferences of the board members are shown in the table below.

	P	V	R	L	T	C
Option A	3	4	1	3	4	1
Option B	1	3	2	1	3	2
Option C	2	2	3	2	1	4
Option D	4	1	4	4	2	3

The board uses plurality method and a runoff in the event of a tie, such that the option(s) with the least votes will be eliminated in each round. Which option will be chosen?

Answer

After the first round of voting, we have A with two votes, B with 2 votes, C with 1 vote, and D with 1 vote. Since no one has a plurality, we will eliminate both choices with one vote: C and D.

Recalculating the table with only options A and B gives us the following:

	P	V	R	L	T	C
Option A	2	2	1	2	2	1
Option B	1	1	2	1	1	2

Option A has two votes, and Option B has 4 votes. So option B is the winner!

Ranked-Choice Voting

In Example 5.2.4 and Your Turn 5.2.4, you were given a list that ordered each voter's preferences. This ordering is called a **preference ranking**. A ballot in which a voter is required to give an ordering of their preferences is a ranked ballot, and any voting system in which a voter uses a ranked ballot is referred to as ranked voting.

The vote for the Academy Awards uses a ranked ballot. The table below provides an example of a ranked ballot for the 2020 Academy Award nominees for Best Director. Voters would circle their ranking in each row.

Candidate for Best Director	Rank top choice as 1, next choice as 2, and so on.				
Martin Scorsese, <i>The Irishman</i>	1	2	3	4	5
Todd Phillips, <i>Joker</i>	1	2	3	4	5
Sam Mendes, <i>1917</i>	1	2	3	4	5
Quentin Tarantino, <i>Once Upon a Time in Hollywood</i>	1	2	3	4	5
Bong Joon-ho, <i>Parasite</i>	1	2	3	4	5

As you decide on the voting methods that will be used in your new democracy, budget must be a consideration. You might consider a particular type of ranked voting called ranked-choice voting (RCV), which simulates a series of runoff elections without the usual time and expense involved when voters must repeatedly return to the polls, like we did in Example 5.2.3.

The method of ranked-choice voting (RCV), also called instant runoff voting (IRV), is a version of the Hare Method, using *preference ranking* so that, if no single candidate receives a majority, the least popular selections can be eliminated and the results can be recounted, without the need for more elections.

Checkpoint

Ranked voting can be confused with ranked-choice voting, but ranked voting is a more general category which includes ranked-choice voting and several other voting methods.

As we explore examples of ranked voting, we will summarize the voters' preference rankings using a table in which the top row shows the number of ballots that ranked the options in the same order. Let's practice interpreting the information in this type of table.

Example 5.2.5

Interpreting the Sample Preference Summary

Refer to the table below containing voters' preference rankings to answer the following questions.

Number of Ballots	100	200	150	75
Option A	1	4	3	4
Option B	2	3	4	2
Option C	4	2	1	1
Option D	3	1	2	3

1. How many voters ranked the options in the following order: Option A in fourth place, Option B in second place, Option C in first place, and Option D in third place?
2. How many ballots in total were collected?
3. How many voters indicated that Option C was their first choice?

Answer

1. The column farthest to the right displays this ordering. The top entry in this column is 75; so, there were 75 voters who ranked the options in this way.
2. The sum of the top row gives the total number of ballots collected: $100 + 200 + 150 + 75 = 525$. So, there were 525 ballots collected.
3. In the Option C row, there are two entries of 1 which indicate a first choice for that option. These occur in the last two columns. The sum of the top entries in these columns is $150 + 75 = 225$. So, 225 voters indicated Option C as their first choice.

Your Turn 5.2.5

A kindergarten class votes on their favorite colors using a ranked ballot. Use the results in the following table to answer the questions.

1. How many students voted in total?
2. How many students voted for yellow as their favorite color?
3. How many students in the class indicated that blue was their favorite color while green was their least favorite color?

Number of Ballots	4	6	4	7
Red	2	6	2	5
Blue	1	2	1	4
Green	6	5	5	2
Yellow	5	4	6	3
Purple	4	3	4	1
Pink	3	1	3	6

Answer

1. There were a total of $4 + 6 + 4 + 7 = 21$ votes.
2. There are no columns that have a "1" for yellow. So no one chose yellow as their favorite color.
3. We are looking for a column with a "1" for blue and a "6" for green. That happens only in the first column, so there were 4 votes.

Now that we've covered how to read a summary of preference rankings, let's practice using the ranked-choice method to determine the winner of an election. Recall that ranked-choice voting is still the Hare Method where the candidate with the very least number of votes is eliminated each round until a majority is attained. The difference here is that the voters have completed

a ranked ballot, so they don't have to visit the polls multiple times. Here are the steps for ranked-choice voting.

Steps to Determine Winner by Ranked-Choice Voting

To determine the winner when ranked-choice voting occurs, we take these three steps:

Step 1: Determine the number of votes needed to achieve a majority. This is the least whole number greater than 50 percent of the total votes.

Step 2: Count the number of first place votes for each candidate. If a candidate has a majority, that candidate wins the election and we are done! Otherwise, eliminate the candidate(s) with the fewest votes and complete Step 3.

Step 3: Reallocate the votes to the remaining candidates, and repeat Step 2.

Checkpoint

Be methodical to avoid arithmetic errors. Make sure that each time you count the number of first place votes they sum to the number of ballots.

Video

[How Does Ranked-Choice Voting Work?](#)

Example 5.2.6

Most Popular Color in Kindergarten

Let's review the kindergarten class color preferences again, and this time determine which color would be selected based on these results using the ranked-choice method.

Number of Ballots	4	6	4	7
Red	2	6	2	5
Blue	1	2	1	4
Green	6	5	5	2
Yellow	5	4	6	3
Purple	4	3	4	1
Pink	3	1	3	6

Answer available at the end of this section.

Video

[Determine Winner of Election by Ranked-Choice Method \(aka Instant Runoff\)](#)

Borda Count Voting

Ranked-choice voting is one type of ranked voting that simulates multiple runoffs based on ranked ballots. Another type of ranked voting is the **Borda count method**, which uses ranked ballots that award candidates points corresponding to the number of candidates ranked lower on each ballot.

To understand how this works, let's review the favorite colors of our kindergarten class from the table below. Let's focus on the votes represented by the first column of the preference summary.

Number of Ballots	4	6	4	7
Red	2	6	2	5
Blue	1	2	1	4
Green	6	5	5	2
Yellow	5	4	6	3
Purple	4	3	4	1
Pink	3	1	3	6

Each student had six options. This first column tells us that four students ranked blue as their first choice, red as their second choice, pink as their third choice, purple as their fourth choice, yellow as their fifth choice, and green as their sixth choice. Blue was ranked higher than $6 - 1 = 5$ other colors. For each of the four students who completed their ballot in this way, blue would receive five points. Since there were four ballots with this ordering, blue would receive $5(4) = 20$ points from the first column. To determine the total points for each candidate, we have to find the sum of the points they received in each column.

To determine the winner of a contest using the Borda count method, we must compare total number of points earned by each candidate. The candidate with the most points is the winner. Each row of the preference summary corresponds to a single candidate. To find the number of points received by a particular candidate in the preference summary, or their **Borda score**, we will need to focus on the row in which that candidate appears.

Before we practice determining the winner of a Borda count election, let's examine how to find the Borda score for a single candidate.

Example 5.2.7

Most Popular Color in Kindergarten Revisited

Let's review the ballots from the kindergarten class again, as shown in the table below. This time, let's determine the Borda score received by the color purple.

Number of Ballots	4	6	4	7
Red	2	6	2	5
Blue	1	2	1	4
Green	6	5	5	2
Yellow	5	4	6	3
Purple	4	3	4	1
Pink	3	1	3	6

Answer available at the end of this section.

Checkpoint

When calculating a Borda score in one step, be careful to use the correct order of operations. Perform the subtraction inside each pair of parentheses first, then perform each multiplication, and then perform each addition.

Your Turn 5.2.6

Consider the color preferences of the kindergarten class once more. Using the Borda count method, determine the total number of points the color blue received.

Answer

$$\begin{aligned}
 &4 \times (6 - 1) + 6 \times (6 - 2) + 4 \times (6 - 1) + 7 \times (6 - 4) \\
 &= 4 \times 5 + 6 \times 4 + 4 \times 5 + 7 \times 2 \\
 &= 20 + 24 + 20 + 14 \\
 &= 78
 \end{aligned}$$

Now let's determine the winner of an election by comparing the Borda scores for each of the candidates.

Example 5.2.8

Determine the Winner by Two Ranked Voting Methods

Use the table below, which displays a sample preference summary, to answer the questions that follow.

Number of Ballots	95	90	110	115
Option A	4	4	1	1
Option B	2	2	2	2
Option C	3	1	3	4
Option D	1	3	4	3

1. Use the ranked-choice voting method to determine the winner of the election.
2. Use the Borda count method to determine the winner of the election.

Answer available at the end of this section.

The Borda count method may seem too complicated to even consider using for Imaginaria, but each voting method has its own pros and cons. The Borda count method, for example, favors compromise candidates over divisive candidates. A **compromise candidate** is not the first choice of most of the voters, but is more acceptable to the population as a whole than the other candidates. A **divisive candidate** is simultaneously the first choice of a large portion of the voters and the last choice of another large portion of the voters.

In Example 5.2.8, Candidate A was ranked first by 225 voters, but was ranked last by 185 voters. No voters ranked Candidate A as second or third. It appears that, although Candidate A had the majority of first place votes, there was a significant minority who strongly disliked them. Candidate A was a divisive candidate. Candidate B, on the other hand, was the second choice of every voter, making Candidate B a good compromise. The Borda count method chose Candidate B, a compromise candidate, that was more acceptable to the population as a whole. This scenario is cited by both opponents and proponents of the Borda count method.

Pairwise Comparison and Condorcet Voting

We have discussed two kinds of ranked voting methods so far: ranked-choice and Borda count. A third type of ranked voting is the **pairwise comparison method**, in which the candidates receive a point for each candidate they would beat in a one-on-one election and half a point for each candidate they would tie. If one candidate earns more points than the others, then that candidate wins. This method is one of several **Condorcet voting methods**, which are methods in which candidates are ranked and then compared pairwise to each other, a candidate having to beat all others in order to win. These methods vary in the way candidates are scored, and there is not always a clear winner. A candidate who wins each possible pairing is known as a **Condorcet candidate**. These terms are named after the Marquis de Condorcet, a French philosopher and mathematician who preferred the pairwise comparison method to the Hare method and made public arguments in its favor.

People in Mathematics

Marquis de Condorcet

Condorcet voting methods are named for the Marquis de Condorcet, a French philosopher and mathematician known for, among other accomplishments, writing “*Sur l'admission des femmes au droit de Cité*” (“On the Admission of Women to the Rights of Citizenship”), in 1789, the first published essay on the political rights of women.

[For more details visit this Web site.](#)

If you include a Condorcet voting method in the constitution of Imaginaria, the election supervisors may want to use a pairwise comparison matrix like the one in Figure 5.2.1. It's a tool used to list the number of wins associated with each pairing of two candidates. Each candidate will receive a point for each win and a half a point for each tie. Each pairing is listed twice, once for the number of wins of a candidate over a particular challenger and once for the number of wins of the challenger over that candidate.

Opponent	A	B	C
A	--	A over B	A over C
B	B over A	--	B over C
C	C over A	C over B	--

Figure 5.2.1: Pairwise Comparison Matrix for Three Candidates

Steps to Determine a Winner by Pairwise Comparison Method Using a Matrix

To determine the winner when the pairwise comparison method is used, we take these three steps:

Step 1: On the matrix, indicate a losing matchup by crossing out a box, $\boxtimes$, and tie match ups by drawing a slash through the box, $\boxdiv$.

Step 2: Award each candidate 1 point for a win, half a point for a tie, and 0 points for a loss.

Step 3: Identify the winner, which is the candidate with the most points.

Video

[Determine Winner of Election by Using the Pairwise Comparison Method](#)

Before you decide on the pairwise comparison method for Imaginaria, review what's involved in constructing a pairwise comparison matrix from a summary of ranked ballots. Then we can use the matrix to determine the winner of the election. Does the winner using the Borda method still win?

Example 5.2.9

Construct and Use a Pairwise Comparison Matrix

Consider the summary of ranked ballots shown in the table below. Determine the winner of an election using the pairwise comparison method.

Number of Ballots	95	90	110	115
Option A	4	4	1	1
Option B	2	2	2	2
Option C	3	1	3	4
Option D	1	3	4	3

1. Construct a pairwise comparison matrix for the sample summary of ranked ballots in the table above.
2. Use the pairwise comparison method to determine a winner.
3. Recall that in Example 5.2.8, Candidate A won by the ranked-ballot method, and Candidate B won by the Hare method. Did the same candidate win using the pairwise comparison method?
4. Is the winner a Condorcet candidate?

Answer available at the end of this section.

Checkpoint

Notice that the pairwise vote totals are not used to determine the points. Vote totals are only used to determine a win or a loss. Avoid the common error of adding the values in each row to get the points.

Three Key Questions

Before you decide if you want to use the pairwise comparison method for Imaginarian elections, let's consider three questions

that might affect your decision.

- I. Is there always a winner?
- II. If there is a winner, is the winner always a Condorcet candidate?
- II. If there is a Condorcet candidate, does that candidate always win?

Let's think about why these questions might be important to you if you chose the pairwise comparison method. First, if no candidate meets the criteria to win an election, you will need a backup plan such as a runoff election. Second, if the winner is not a Condorcet candidate, then there is at least one candidate who beat the winner in a pairwise matchup and the supporters of that candidate might question the validity of the election. Finally, if there is a Condorcet candidate who beat every other candidate in a pairwise matchup, it is reasonable to conclude that it would be unfair for anyone else to win. The rest of the examples in this section should illustrate these key concepts.

Example 5.2.10

Rock, Paper, Scissors by Pairwise Comparison

Suppose that three people are playing the game Rock, Paper, Scissors. On the count of three, each person shows a hand signal for rock, paper, or scissors. Each hand signal beats another hand signal. The group keeps having a tie because Person A always picks rock, Person B always picks paper which beats rock, and Person C always picks scissors which beats paper, and is beaten by rock! This leads to a disagreement about which choice is best. They decide to use the pairwise comparison method determine the winner. Their preference rankings are given in the following table.

Voters	A	B	C
Rock (R)	1	3	2
Paper (P)	2	1	3
Scissors (S)	3	2	1

Answer available at the end of this section.

Example 5.2.10 illustrates the answer to the first key question. The pairwise comparison method does not always result in a winner. For example, much like the game of Rock, Paper, Scissors, it is possible for a cyclic pattern to emerge in which each candidate beats the next until the last candidate who beats the first.

Your Turn 5.2.7

A pairwise comparison matrix is given. Determine the winner by the pairwise comparison method. If there is not a winner, explain why. If there is a winner, tell whether the winner is a Condorcet candidate.

Opponent \ Runner	P	Q	R	S	T	Z
P wins	--	PQ 1	PR 4	PS 6	PT 4	PZ 5
Q wins	QP 5	--	QR 6	QS 4	QT 1	QZ 2
R wins	RP 2	RQ 0	--	RS 5	RT 4	RZ 1
S wins	SP 0	SQ 2	SR 1	--	ST 6	SZ 5
T wins	TP 2	TQ 5	TR 2	TS 0	--	TZ 4
Z wins	ZP 1	ZQ 4	ZR 5	ZS 1	ZT 2	--

Figure 5.2.6: A Pairwise Comparison Matrix

Answer

Option P wins with 4 pairwise comparisons.

Now, you have the answer to the second key question. The pairwise comparison matrix in Your Turn 5.2.7 is an example of a scenario where a winner is not a Condorcet candidate.

The answer to the third question is not as clear. If there is a Condorcet candidate, does that candidate win? So far, we have not come across a contradictory example where the Condorcet candidate didn't win, but we cannot know with certainty that it is not possible by looking at examples. Instead, we will need to use some reasoning. Let's review some particular cases of elections with a certain number of candidates, and then we will try to generalize the scenario to an election with n candidates.

Example 5.2.11

Does the Condorcet Candidate Win?

1. Suppose there is an election with five candidates—A, B, C, D, and E—and that Candidate C is a Condorcet candidate. How many points did Candidate C win?
2. What is the greatest number of points that any one of the other candidates could win?
3. Is it possible for Candidate C to lose or tie?

Answer available at the end of this section.

Let's consider a general case where there are n candidates. One of the candidates is a Condorcet candidate. Since the Condorcet candidate wins all matchups, the Condorcet candidate wins $n - 1$ points. Since each of the other candidates lost to the Condorcet candidate, the most a single candidate could win is $n - 2$. Since the Condorcet candidate won $n - 1$ points and each other candidate won $n - 2$ points or fewer, the Condorcet candidate is the winner. You have your answer to the third key question! If there is a Condorcet candidate, that candidate is always the winner.

Approval Voting

The last type of voting system you will consider for your budding democracy is an approval voting system. In this system, each voter may approve any number of candidates without rank or preference for one over another (among the approved candidates), and the candidate approved by the most voters wins. This voting system has aspects in common with plurality voting and Condorcet voting methods, but it has characteristics that distinguish it from both. An approval voting ballot lists the candidates and provides the option to approve or not approve each candidate.

The term “approval voting” was not used until the 1970s (Brams, Steven J.; Fishburn, Peter C. (2007), *Approval Voting*, Springer-Verlag, p. xv, ISBN 978-0-387-49895-9), although its use has been documented as early as the 13th century (Brams, Steven J. (April 1, 2006). *The Normative Turn in Public Choice* (PDF) (Speech). Presidential Address to Public Choice Society. New Orleans, Louisiana.) Approval voting has the appeal of being simpler than ranked voting methods. It also allows an individual voter to support more than one candidate equally. This has appeal for those who do not want a split vote among a few mainstream candidates to lead to the election of a fringe candidate. It also has appeal for those who want an underdog to have a chance of success because voters will not worry about wasting their vote on a candidate who is not believed likely to win.

Example 5.2.12

Rock, Paper, Scissors, Lizard, Spock

Suppose that Person A/B/C were just about to give up on their game of Rock, Paper, Scissors when they were joined by Person D who reminded them that their updated version: Rock, Paper, Scissors, Lizard, Spock was a far superior game with the added rules that Lizard eats Paper, Paper disproves Spock, Spock vaporizes Rock, Rock crushes Lizard, Lizard poisons Spock, Spock smashes Scissors, and Scissors decapitates Lizard.

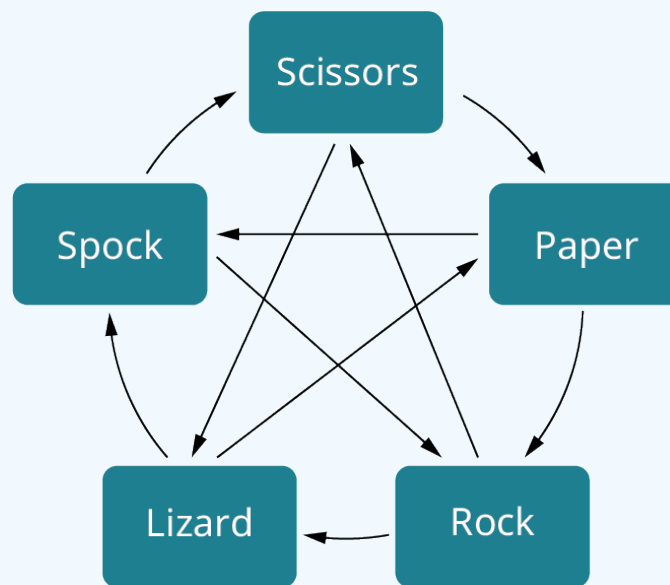


Figure 5.2.7 Rock, Paper, Scissors, Lizard, Spock Dominance

Person D encourages their friends to hold a new election. This time, for the sake of simplicity, the group decides to use approval voting to determine the best move in the game. The summary of approval ballots for Rock, Paper, Scissors, Lizard, Spock is given in the table below.

Voters	A	B	C	D
Rock	Yes	No	No	No
Paper	No	Yes	No	No
Scissors	No	No	Yes	No
Lizard	No	No	No	Yes
Spock	Yes	Yes	Yes	Yes

Answer available at the end of this section.

Your Turn 5.2.8

The Chionilis family is trying to decide on a restaurant again, but now they don't want to deal with multiple runoffs or even ranking. They will use the approval voting method shown in the following table. Each family member will approve their top two choices. Rainbow China won when multiple runoffs were used. Find the winner for tonight's dinner.

Options	A	R	S	T	G	E	M	D
Rainbow China	Yes	No	No	Yes	No	No	Yes	Yes
Dough Boys Pizza	Yes	Yes	Yes	Yes	No	Yes	No	Yes
Taco City	No	No	Yes	No	Yes	Yes	No	No
Caribbean Flavor	No	Yes	No	No	Yes	No	Yes	No

Answer

Dough Boys Pizza won with six votes.

Example 5.2.13

The Chionilis Family Is Hungry Again!

The eight members of the Chionilis family—Annette, Rene, Seema, Titus, Galen, Elena, Max and Demitri—have another decision to make. Approval voting worked out nicely the last time. They are going to use it again, but this time, Annette, Rene, Seema, and Galen are feeling a little indecisive. They can't narrow their choice down to two. They will approve their three top choices, but the other family members will only approve two. These choices are reflected in the following table. Determine the restaurant that will be chosen.

Options	A	R	S	T	G	E	M	D
Rainbow China	Yes	Yes	Yes	Yes	Yes	No	Yes	Yes
Dough Boys Pizza	Yes	Yes	Yes	Yes	No	Yes	No	Yes
Taco City	Yes	No	Yes	No	Yes	Yes	No	No
Caribbean Flavor	No	Yes	No	No	Yes	No	Yes	No

Answer available at the end of this section.

Example 5.2.14

Three Habitable Planets

In the future, humans have explored distant solar systems and found three habitable planets which could be colonized. Since it will take all available resources to colonize one planet, humans must agree on the planet. Planet A has the most comfortable climate and most plentiful resources, but it is the farthest from Earth making travel to the planet a challenge. Planet B is half the distance but will require more resources to make comfortable. Planet C is the least suitable of the three and terraforming will be required, but it is close enough to make travel between Earth and Planet C possible on a more regular basis. The table below provides the voter preferences for the colonization of each planet.

Percentage of Voters	45%	15%	40%
Planet A	1	3	3

Percentage of Voters	45%	15%	40%
Planet B	2	1	2
Planet C	3	2	1

If the entire population were able to vote, determine the winning planet using each of the methods listed below.

1. Plurality
2. Ranked-choice method
3. Borda count

Answer available at the end of this section.

The election in Example 5.2.14 involves a scenario in which there are two extreme candidates, Planet A and Planet B, and a moderate candidate, Planet C. The supporters of the extreme candidates prefer the moderate candidate to the other extremist ones. This makes Planet C a compromise candidate. In this case, both the plurality method and ranked-choice voting resulted in the election of one of the extreme candidates, but the Borda count method elected the compromise candidate in this scenario. Depending on a person's perspective, this may be perceived as a flaw in either ranked-choice and plurality systems, or the Borda count method.

In Fairness in Voting Methods, we will analyze the fairness of each voting system in greater detail using objective measures of fairness.

Answers

Step 1: Calculate 50 percent of 105,405,100 by multiplying the decimal form of 50 percent, which is 0.50, by 105,405,100: $0.50(105,405,100) = 52,702,550$

Step 2: Determine the minimum number of votes needed to have a majority. The minimum number of votes required is the lowest counting number that is larger than 50 percent of the votes. To have a majority, an individual candidate must have more than 52,702,550; so, a majority candidate must have 52,702,551 votes or more.

Step 3: Compare the number of votes each candidate received to 52,702,551. According to the data in Table 5.2.1, none of the candidates secured a majority.

Al Gore secured 50,999,897 votes, which was more than any other single candidate. Therefore, he had a plurality of the popular votes.

1. A majority of the 48 total votes is required to win. Begin by finding 50 percent of 48, which is calculated as follows: $0.50(48) = 24$. A majority is 25 or more. No candidate has a majority, so there is no winner based on the first round.
2. Abou and Dali advance to the second round.

1. the Hare Method

2. **Step 1:** Determine the number of votes necessary to have a majority. There are five family members, so 50 percent of 5 is $0.50(5) = 2.5$. A majority is three or more votes.

Step 2: Count the number of votes for each restaurant in the first election. In a list of voter preferences, the 1s represent the top choice of each voter, which corresponds to their vote in the first round.

Results of Round 1:

- Rainbow China — 2 votes
- Dough Boys — 2 votes
- Taco City — 0 votes
- Caribbean Flavor — 1 vote

No restaurant received a majority. Eliminate Taco City, which has the fewest first place votes:

Options	A	R	S	T	G
Rainbow China	1	3	3	1	3
Dough Boys Pizza	2	2	1	2	1
Caribbean Flavor	4	1	4	3	4

Step 3: Hold a runoff election. In other words, hold a second round. Since we have a list of the voters' preferences with the eliminated option removed, we will renumber the preferences as first, second, and third so that we keep the original order of preference. The result is that we will count the second-place vote of any voter whose first choice was eliminated.

Options	A	R	S	T	G
Rainbow China	1	3	2	1	2
Dough Boys Pizza	2	2	1	2	1
Caribbean Flavor	3	1	3	3	3

Step 4: Repeat the process from Step 2. Count the number votes for each restaurant in the first-round election. Since we are using a list of preferences, we need to count the number of 1s received by each restaurant.

Results of Round 2:

- Rainbow China — 2 votes
- Dough Boys — 2 votes
- Caribbean Flavor — 1 vote

No single restaurant has three votes. Eliminate Caribbean Flavor, which has the fewest first place votes.

OPTIONS	A	R	S	T	G
Rainbow China	1	3	2	1	2
Dough Boys Pizza	2	2	1	2	1

Step 5: Repeat the process from Step 3. Hold another runoff election. This will be Round 3. Renumber the voters' preferences as first and second this time.

Options	A	R	S	T	G
Rainbow China	1	2	2	1	2
Dough Boys Pizza	2	1	1	2	1

Step 6: Repeat the process from Step 2 one last time. Count the number of first place votes for each remaining restaurant.

Results of Round 3:

- Rainbow China — 2 votes
- Dough Boys — 3 votes

Determine whether any one choice has a majority. Yes! Dough Boys has three votes, so it is the winner!

Step 1: Determine whether any candidate received a majority. There were 21 ballots. Fifty percent of 21 is $0.50(21) = 10.5$. A majority is 11.

Step 2: Count the number of first place votes for each candidate. If a candidate has a majority, that candidate wins the election. Otherwise, eliminate the candidate(s) with the fewest votes.

- Red: 0
- Blue: $4 + 4 = 8$
- Green: 0
- Yellow: 0
- Purple: 7
- Pink: 6

Notice that $8 + 7 + 6 = 21$, which is the total number of ballots. Confirming this helps to catch any arithmetic or counting errors. No candidate has a majority with 11 or more votes. We must eliminate red, green, and yellow which had the fewest votes with 0 each. The remaining votes that must be counted for Round 2 are given in the table below.

Number of Ballots	4	6	4	7
Blue	1	2	1	4
Purple	4	3	4	1
Pink	3	1	3	6

Step 3: Reallocate the votes to the remaining candidates. We can do this by numbering the choices as 1, 2, and 3 in such a way that the order of preference is retained as seen in the table below:

Number of Ballots	4	6	4	7
Blue	1	2	1	2

Number of Ballots	4	6	4	7
Purple	3	3	3	1
Pink	2	1	2	3

Step 4: Repeat the process from Step 2. Count the number of first place votes for each candidate. If a candidate has a majority, that candidate wins the election. Otherwise, eliminate the candidate(s) with the fewest votes.

- Blue: $4 + 4 = 8$
- Purple: 7
- Pink: 6

Confirm that $8 + 7 + 6 = 21$. Great! No candidate has 11 or more votes. We must eliminate pink which had the fewest votes with 6. The remaining votes that must be counted for Round 2 are shown in the table below.

Number of Ballots	4	6	4	7
Blue	1	2	1	2
Purple	3	3	3	1

Step 5: Repeat the process from Step 3. Reallocate the votes to the remaining candidates. We can do this by numbering the choices as 1 and 2;

Number of Ballots	4	6	4	7
Blue	1	1	1	2
Purple	2	2	2	1

Step 6: Repeat the process from Step 2 one last time. Count the number of first place votes for each candidate. If a candidate has a majority, that candidate wins the election. Otherwise, eliminate the candidate(s) with the fewest votes.

- Blue: $4 + 6 + 4 = 14$
- Purple: 7

Blue has a majority and wins the election!

Step 1: Find the number of points received by the candidate in each column.

Column 1: $4 \times (6 - 4) = 4 \times 2 = 8$

Column 2: $6 \times (6 - 3) = 6 \times 3 = 18$

Column 3: $4 \times (6 - 4) = 4 \times 2 = 8$

Column 4: $7 \times (6 - 1) = 7 \times 5 = 35$

Step 2: Find the sum of the points received in each column. This is the total number of points received by this candidate: $8 + 18 + 8 + 35 = 69$

This process can also be combined into one step as shown here.

$$4 \times (6 - 4) + 6 \times (6 - 3) + 4 \times (6 - 4) + 7 \times (6 - 1)$$

$$= 4 \times 2 + 6 \times 3 + 4 \times 2 + 7 \times 5$$

$$= 8 + 18 + 8 + 35$$

$$= 69$$

Purple received 69 points in this election.

- Step 1:** Determine the number of votes needed to achieve a majority. The number of ballots is $95 + 90 + 110 + 115 = 410$. Fifty percent of 410 is $0.50(410) = 205$. So, 206 votes or more is a majority.

Step 2: Count the number of first place votes for each candidate.

- Option A has $110 + 115 = 225$
- Option B has 0
- Option C has 90
- Option D has 95

Since Option A has a majority, Option A is the winner by the ranked-choice method.

- The Borda scores would be:

- Option A: $95(4 - 4) + 90(4 - 4) + 110(4 - 1) + 115(4 - 1) = 675$

- Option B: $95(4 - 2) + 90(4 - 2) + 110(4 - 2) + 115(4 - 2) = 820$

- Option C: $95(4 - 3) + 90(4 - 1) + 110(4 - 3) + 115(4 - 4) = 475$

- Option D: $95(4 - 1) + 90(4 - 3) + 110(4 - 4) + 115(4 - 3) = 490$

Since Option B has a Borda score of 820 points, Option B is the winner by the Borda count method.

There are four candidates on the ballots. We will need a row and a column for each candidate in addition to the headings, so we will draw a five by five matrix.

Opponent Runner	Option A	Option B	Option C	Option D
Option A wins	--	A over B	A over C	A over D
Option B wins	B over A	--	B over C	B over D
Option C wins	C over A	C over B	--	C over D
Option D wins	D over A	D over B	D over C	--

Figure 5.2.2: Pairwise Comparison Matrix for Four Candidates

- Step 1:** Refer to Figure 5.2.2 to determine the values that belong in each cell.
 - A over B: A is preferred to B in columns 3 and 4. So, A scores $110 + 115 = 225$ points.
 - A over C: A is preferred to C in columns 3 and 4. So, A scores 225 points again.
 - A over D: Similarly, A scores 225 points.
 - B over A: B is preferred to A in columns 1 and 2. So, B scores $95 + 90 = 185$ points.

Step 2: Continuing in this way, we complete the pairwise comparison matrix, as shown in Figure 5.2.3.

Opponent Runner	Option A	Option B	Option C	Option D
Option A wins	--	AB 225	AC 225	AD 225
Option B wins	BA 185	--	BC 320	BD 315
Option C wins	CA 185	CB 90	--	CD 200
Option D wins	DA 185	DB 95	DC 210	--

Figure 5.2.3 Pairwise Comparison Matrix for Four Candidates with Vote Counts

- Step 1:** Losing pairings are crossed off with an $\boxtimes$. In the event of a tie, we will draw a slash, $\boxdiv$.
Step 2: Determine the number of points for each candidate by analyzing their row of wins. Each win is 1 point, each loss, $\boxtimes$, is 0 points, and each tie, $\boxdiv$, is half a point. Construct an additional column for each candidate's points.

Opponent \ Runner	Option A	Option B	Option C	Option D	Points
Option A wins	--	AB 225	AC 225	AD 225	3
Option B wins	BA 185	--	BC 320	BD 315	2
Option C wins	CA 185	CB 90	--	CD 200	0
Option D wins	DA 185	DB 95	DC 210	--	1

Figure 5.2.4 Pairwise Comparison Matrix for Four Candidates with Pairwise Winners and Points Column Added

Step 3: The winner by the pairwise comparison method is Option A with 3 points.

- Option A was not the winner by the Hare method.
- The winner, Option A, is a Condorcet candidate because Option A won each pairwise comparison.

Construct the comparison matrix:

Opponent \ Runner	Rock (R)	Paper (P)	Scissors (S)	Points
Rock wins	--	RP 2	RS 1	1
Paper wins	PR 1	--	PS 2	1
Scissors wins	SR 2	SP 1	--	1

Figure 5.2.5: Rock, Paper, Scissors Pairwise Comparison Matrix

There is a tie! There is no winner.

- In any pairwise election with five candidates, each candidate must compete against four other candidates. It follows that the most points a single candidate can win is four points, which would occur if the candidate won every matchup. As a Condorcet candidate, Candidate C won all the pairwise matchups against Candidates A, B, D, and E, earning four points.
- The rest of the candidates lost to Candidate C. The most points a particular candidate could win if they won matchups with each of the other three candidates is three points.
- Since Candidate C has four points and the rest of the candidates have three points or less, Candidate C is the winner. Therefore, it is not possible for Candidate C to tie or lose.

Count the number of approval votes for each candidate by counting the number of "Yes" votes in each row of the table.

- Rock: 1
- Paper: 1
- Scissors: 1
- Lizard: 1
- Spock: 4

Spock is the winning candidate, approved by four voters!

Count the number of approval votes for each restaurant by counting the number of "Yes" votes in each row.

- Rainbow China: 7
- Dough Boys Pizza: 6
- Taco City: 4
- Caribbean Flavor: 3

This time, Rainbow China won!

1. The plurality method only considers the top choice of each voter. By this system, Planet A has 45 percent of the vote, Planet B has 15 percent of the vote, and Planet C has 40 percent of the vote. Planet A wins.
2. Using either instant runoff or a two-round system, Planet B with only 15 percent of the vote will be eliminated in the first round. In Round 2, the 15 percent that voted for Planet B would vote for their second choice, Planet C. This leaves Planet A with 45 percent and Planet C with 55 percent. Planet C has a majority and wins the election.
3. To find the Borda score for each candidate, imagine there are exactly 100 voters. Then the summary of ranked ballots looks like:

Out of 100 Voters	45	15	40
Planet A	1	3	3
Planet B	2	1	2
Planet C	3	2	1

The Borda score for each candidate is as follows:

Planet A: $(3 - 1)(45) + (3 - 3)(15) + (3 - 3)(40) = 90$

Planet B: $(3 - 2)(45) + (3 - 1)(15) + (3 - 2)(40) = 115$

Planet C: $(3 - 3)(45) + (3 - 2)(15) + (3 - 1)(40) = 95$

Planet B wins.

5.3: Fairness in Voting Methods

Learning Objectives

After completing this section, you should be able to:

1. Compare and contrast fairness of voting using majority criterion.
2. Compare and contrast fairness of voting using head-to-head criterion.
3. Compare and contrast fairness of voting using monotonicity criterion.
4. Compare and contrast fairness of voting using irrelevant alternatives criterion.
5. Apply Arrow's Impossibility Theorem when evaluating voting fairness.

Now that we've covered a variety of voting methods and discussed their differences and similarities, you might be leaning toward one method over another. You will need to convince the other founders of Imaginaria that your preference will be the best for the country. Before your collaborators approve the inclusion of a voting method in the constitution, they will want to know that the voting method is a fair method. In this section, we will formally define the characteristics of a fair system. We will analyze each voting method previously discussed to determine which characteristics of fairness they have, and which they do not. In order to guarantee one ideal, we must often sacrifice others.

The Majority Criterion

One of the most fundamental concepts in voting is the idea that most voters should be in favor of a candidate for a candidate to win, and that a candidate should not win without majority support. This concept is known as the majority criterion.

With respect to the four main ranked voting methods we have discussed—plurality, ranked-choice, pairwise comparison, and the Borda count method—we will explore two important questions:

1. Which of these voting systems satisfy the majority criterion and which do not?
2. Is it always “fair” for a voting system to satisfy the majority criterion?

Keep in mind that this criterion only applies when one of the candidates has a majority. So, the examples we will analyze will be based on scenarios in which a single candidate has more than 50 percent of the vote.

Example 5.3.1

Roommates Choose Fast Food

It's final exams week and seven college students are hungry. They must get food, but from which drive thru? Their preferences are listed in the table below. The majority have listed McDonald's as their top choice. Let's calculate what the results of the election will be using various voting methods.

Voters	A	B	C	D	E	F	G
(M) McDonald's	1	1	1	1	5	5	5
(B) Burger King	2	2	2	2	2	2	2
(T) Taco Bell	4	4	3	4	3	3	1
(O) Pollo Tropical	3	5	4	3	4	1	3
(I) Pizza Hut	5	3	5	5	1	4	4

1. Which restaurant is the winner using the plurality voting method?

2. Which restaurant is the winner using the ranked-choice voting method?
3. Does the majority criterion apply? If so, for which of voting method(s), if any, did the majority criterion fail?

Answer available at the end of this section.

From Example 5.3.1, it appears that the plurality and ranked-choice voting methods satisfy the majority criterion. In general, the majority candidate always wins in a plurality election because the candidate that has more than half of the votes has more votes than any other candidate. The same is true for ranked-choice voting; and there will never be a need for a second round when there is a majority candidate. Let's examine how some of the other voting methods stand up to the majority criterion.

Example 5.3.2

Roommates Choose Fast Food

Those seven college students are hungry again! Their preferences haven't changed, as shown below. Let's calculate if the results change when we use different voting methods.

VOTERS	A	B	C	D	E	F	G
(M) McDonald's	1	1	1	1	5	5	5
(B) Burger King	2	2	2	2	2	2	2
(T) Taco Bell	4	4	3	4	3	3	1
(O) Pollo Tropical	3	5	4	3	4	1	3
(I) Pizza Hut	5	3	5	5	1	4	4

1. Which restaurant is the winner using the pairwise comparison voting method?
2. Which restaurant is the winner using the Borda count voting method?
3. Does the majority criterion apply? If so, for which voting method(s), if any, did the majority criterion fail?

Answer available at the end of this section.

Example 5.3.2 demonstrates a concept that we also saw in Borda Count Voting—the Borda method frequently favors the compromise candidate over the divisive candidate. This can happen even when the divisive candidate has a majority, as it did in this example. Although a majority of the voters were in favor of McDonald's, a significant minority was strongly opposed to McDonald's, ranking it last. Since the Borda score includes all rankings, this strong opposition has an impact on the outcome of the election.

Pairwise comparison will always satisfy the majority criterion because the candidate with the majority of first-place votes wins each pairwise matchup. While it is possible for the majority candidate to win by the Borda count method, it is not guaranteed. So, the Borda method fails the majority criterion. A summary of each voting method as it relates to the majority criterion is found in the following table.

Voting Method	Majority Criterion
Plurality	Satisfies
Ranked-choice	Satisfies
Pairwise comparison	Satisfies
Borda count	Violates

If you prefer the Borda method, you might argue that its failure to satisfy the majority criterion is actually one of its strengths. As we saw in Example 5.3.1 and Example 5.3.2, the majority have the power to vote for their own benefit at the expense of the minority. While four students were very enthusiastic about McDonald's, three students were strongly opposed to McDonald's. It is reasonable to say that the better option would be Burger King, the compromise candidate, which everyone ranked highly and no

one strongly opposed. The Borda method is designed to favor a candidate that is acceptable to the population as a whole. In this way, the Borda method avoids a downfall of strict majority rule known as the tyranny of the majority, which occurs when a minority of a population is treated unfairly because their situation is different from the situation of the majority.

People in Mathematics

Robert Dahl

In his book, *Democracy and Its Critics*, Robert Dahl wrote, “If a majority is not entitled to do so, then it is thereby deprived of its rights; but if a majority is entitled to do so, then it can deprive the minority of its rights.” Dahl was a renowned political theorist, but he is also considered to be a mathematician since his work utilizes ideas from an area of mathematics known as Game Theory (Mathematics Genealogy Project, NDSU Department of Mathematics with the American Mathematical Society).

Who Knew?

Three Branches of Government

Concerns about the consequences of majority rule are not new. In 1788, John Adams warned of the consequences of majority rule and he argued for three branches of government as a way to temper them. In the early 1800s, a young French aristocrat named Alexis de Tocqueville toured the United States and wrote *Democracy in America*, which focused on the impact of democracy on political and civil societies. He observed, even then, the dominance of the white majority over the Indigenous people and enslaved people, which was perpetuated by majority rule.

Video

[Separation of Powers and Checks and Balances](#)

Head-to-Head Criterion

Another fairness criterion you must consider as you select a voting method for Imaginaria is the Condorcet criterion, also known as the head-to-head criterion. An election method satisfies the Condorcet criterion provided that the Condorcet candidate wins the election whenever a Condorcet candidate exists. A Condorcet method is any voting method that satisfies the Condorcet criterion.

Recall from the Three Key Questions that not every election has a Condorcet candidate; the Condorcet criterion will not apply to every election. Also recall that a Condorcet candidate cannot lose an election by pairwise comparison. So, the pairwise comparison voting method is said to satisfy the Condorcet criterion.

Example 5.3.3

Spending Tax Refund

A survey asked a random sample of 100 people in the United States to rank their priorities for spending their tax refund. The options were (V) go on vacation, (S) put into savings, (D) pay off debt, or (T) other. The pairwise comparison matrix for the results is in Figure 5.3.1. Determine whether the Condorcet criterion applies.

Opponent Runner	V	S	D	T
V wins	--	VS 33	VD 37	VT 100
S wins	SV 67	--	SD 36	ST 100
D wins	DV 63	DS 64	--	DT 96
T wins	TV 0	TS 0	TD 4	--

Figure 5.3.1: Pairwise Comparison Matrix for Tax Refund Spending

Answer available at the end of this section.

Example 5.3.4

Spending Tax Refund

Let's return to the survey about tax refund spending from Example 5.3.3. We know that the Condorcet criterion applies because Option D, "Pay off debt," is a Condorcet candidate, which wins every pairwise match up.

Use the information in the ballot summary from the table below to find the winner and determine whether the Condorcet criterion is satisfied in this election when each of the following voting methods are used.

Votes	33	32	31	4
On a vacation (V)	1	3	3	2
Put into savings (S)	3	1	2	1
Pay off debt (D)	2	2	1	4
Other (T)	4	4	4	3

1. Plurality
2. Ranked-choice voting
3. Borda count

Answer available at the end of this section.

Step 6: Repeat the process from Step 2 one last time. Count the first-place votes for each candidate: V has 33, S has 67. Candidate S, "Put into savings," has a majority and wins. Since the Condorcet candidate D candidate D, "Pay off debt," didn't win, the Condorcet criterion is violated.

- Calculate the Borda score for each candidate.

$$V: 33(4 - 1) + 32(4 - 3) + 31(4 - 3) + 4(4 - 2) = 170$$

$$S: 33(4 - 3) + 32(4 - 1) + 31(4 - 2) + 4(4 - 1) = 203$$

$$D: 33(4 - 2) + 32(4 - 2) + 31(4 - 1) + 4(4 - 4) = 223$$

$$T: 33(4 - 4) + 32(4 - 4) + 31(4 - 4) + 4(4 - 3) = 4$$

Candidate D, “Pay off debt,” has the highest Borda score and wins. Since D was the Condorcet candidate, this election satisfies the Condorcet criterion.

As we have seen, the plurality method, ranked-choice voting, and the Borda count method each fail the Condorcet criterion in some circumstances. Of the four main ranked voting methods we have discussed, only the pairwise comparison method satisfies the Condorcet criterion every time. A summary of each voting method as it relates to the Condorcet criterion is found in the following table.

Voting Method	Condorcet Criterion
Plurality	Violates
Ranked-choice	Violates
Pairwise comparison	Satisfies
Borda count	Violates

Monotonicity Criterion

The citizens of Imaginaria might be surprised to learn that it is possible for a voter to cause a candidate to lose by ranking that candidate higher on their ballot. Is that fair? Most voters would say, “Absolutely not!!” This is an example of a violation of the fairness criterion called the **monotonicity criterion**, which is satisfied when no candidate is harmed by up-ranking nor helped by down-ranking, provided all other votes remain the same.

Consider a scenario in which voters are permitted a first round that is not binding, and then they may change their vote before the second round. Such a first round can be called a “straw poll.” Now, let’s suppose that a particular candidate won the straw poll. After that, several voters are convinced to increase their support, or up-rank, that winning candidate and no voters decrease that support. It is reasonable to expect that the winner of the first round will also win the second. Similarly, if some of the voters decide to decrease their support, or down-rank, a losing candidate, it is reasonable to expect that candidate will still lose in the second round.

You might be wondering why it’s called the monotonicity criterion. In mathematics, the term monotonicity refers to the quality of always increasing or always decreasing. For example, a person’s age is monotonic because it always increases, whereas a person’s weight is not monotonic because it can increase or decrease. If the only changes to the votes for a particular candidate after a straw poll are in one direction, this change is considered monotonic.

If you are going to make an informed decision about which voting method to use in Imaginaria, you need to know which of the four main ranked voting methods we have discussed—plurality, ranked-choice, pairwise comparison, and the Borda count method—satisfy the monotonicity criterion.

Example 5.3.5

Favorite Dog Breed by Plurality

The local animal shelter is having a vote-by-donation charity event. For a \$10 donation, an individual can complete a ranked ballot indicating their favorite large dog breed: standard poodle, golden retriever, Labrador retriever, or bulldog. Use the summary of ballots below to answer each question.

Votes	42	53	61	24
(S) Standard Poodle	1	3	2	1

Votes	42	53	61	24
(G) Golden Retriever	3	1	4	4
(L) Labrador Retriever	4	2	1	2
(B) Bulldog	2	4	3	3

1. Determine the winner of the election by plurality.
2. Suppose that the 53 voters in the second column increased their ranking of the winner by 1. Determine the winner by plurality with the new rankings.
3. Does this election violate the monotonicity criterion?
4. Do you think the result of part 3 is also true for plurality voting and the monotonicity criterion in general? Why or why not?

Answer available at the end of this section.

Example 5.3.6

Favorite Dog Breed by Pairwise Comparison

Earlier, we discovered that the summary of ranked ballots shown in the table below results in the pairwise comparison matrix in Figure 5.3.1. Use this information to answer the questions.

Number of Ballots	95	90	110	115
Option A	4	4	1	1
Option B	2	2	2	2
Option C	3	1	3	4
Option D	1	3	4	3

1. Determine the winner of the election by the pairwise comparison method.
2. Suppose that the 95 voters in the first column increased their ranking of the winner by 1. Determine the winner by the pairwise comparison method with the new rankings.
3. Does this election violate the monotonicity criterion?
4. Do you think the result of part 3 is true for the pairwise comparison method and the monotonicity criterion in general? Why or why not?

Answer available at the end of this section.

The last few examples illustrate that the plurality method, pairwise comparison voting, and the Borda count method each satisfy the monotonicity criterion. Of the four main ranked voting methods we have discussed, only the ranked-choice method violates the monotonicity criterion. A summary of each voting method as it relates to the Condorcet criterion is found in the table below.

Voting Method	Monotonicity Criterion
Plurality	Satisfies
Ranked-choice	Violates
Pairwise comparison	Satisfies
Borda count	Satisfies

Irrelevant Alternatives Criterion

We have covered a lot about voting fairness, but there is one more fairness criterion that you and the other Imaginarians should know. Consider this well-known anecdote that is sometimes attributed to the American philosopher Sidney Morgenbesser:

A man is told by his waiter that the dessert options this evening are blueberry pie or apple pie. The man orders the apple pie. The waiter returns and tells him that there is also a third option, cherry pie. The man says, "In that case, I would like the blueberry pie." (*Gaming the Vote: Why Elections Aren't Fair (and What We Can Do About It)*, William Pound Stone, p. 50, ISBN 0-8090-4893-0)

This story illustrates the concept of the Irrelevant Alternatives Criterion, also known as the **Independence of Irrelevant Alternatives Criterion (IIA)**, which means that the introduction or removal of a third candidate should not change or reverse the rankings of the original two candidates relative to one another. In particular, if a losing candidate is removed from the race or if a new candidate is added, the winner of the race should not change.

Example 5.3.7

Apple, Blueberry, or Cherry?

Suppose that 30 students in a class are going to vote on whether to have apple, blueberry, or cherry pie. Use the summary of ranked ballots in below to answer each question.

Number of Ballots	14	12	4
(A) Apple Pie	1	3	3
(B) Blueberry Pie	2	1	2
(C) Cherry Pie	3	2	1

1. Determine the winner of the election by plurality.
2. Which candidate would win a plurality election if cherry pie were removed from the ballot?
3. Does this election violate the IIA?

Answer available at the end of this section.

Example 5.3.8

Best Fourth Wall Breaking Stare on The Office

The NBC sitcom *The Office* ran for nine years and has been one of the most popular streamed television shows of all time. One of the trademarks of the show was that characters would often break the fourth wall to communicate with the audience just by staring directly into the camera. In fact, there is a [website dedicated to "The Office stares"](#) where you can watch over 700 of these stares! Suppose that 36 fans were asked which character had the best "The Office stare." Use the ballot summary in below to answer each question.

Number of Ballots	9	11	7	6	3
(J) Jim Halpert (John Krasinski)	1	2	4	2	4
(P) Pam Beesly-Halpert (Jenna Fischer)	4	1	2	4	3
(D) Dwight Schrute (Rainn Wilson)	2	3	3	1	2
(M) Michael Scott (Steve Carell)	3	4	1	3	1

1. Determine the winner of the election by the pairwise comparison method.
2. Determine the winner of the election by the pairwise comparison method if Michael Scott is removed from the ballot.
3. Does this election violate IIA?

Answer available at the end of this section.

We have seen that all four of the main voting systems we are working with fail the Irrelevant Alternatives Criterion (IIA). A summary of each voting method as it relates to the IIA criterion is found in the table below.

Voting Method	Irrelevant Alternatives Criterion
Plurality	Violates
Ranked-choice	Violates
Pairwise comparison	Violates
Borda count	Violates

So far, every one of the voting methods we have analyzed has failed one or more of the fairness criteria in one election or another.

Voting Method	Majority Criterion	Condorcet Criterion (Head-to-Head Criterion)	Monotonicity Criterion	(Independence of) Irrelevant Alternatives Criterion
Plurality	Satisfies	Violates	Satisfies	Violates
Ranked-choice	Satisfies	Violates	Satisfies	Violates
Pairwise comparison	Satisfies	Satisfies	Violates	Violates
Borda count	Violates	Violates	Satisfies	Violates

You might be wondering if there is a voting system you could recommend for Imaginaria that satisfies all the fairness criteria. If there is one, it remains to be discovered and it is not a voting system that is based solely on preference rankings. In 1972, Harvard Professor of Economics Kenneth J. Arrow received the Nobel Prize in Economics for proving Arrow's Impossibility Theorem, which states that any voting system, either existing or yet to be created, in which the only information available is the preference rankings of the candidates, will fail to satisfy at least one of the following fairness criteria: the majority criterion, the Condorcet criterion, the monotonicity criterion, and the independence of irrelevant alternatives criterion. This theorem only applies to a specific category of voting systems—those for which the preference ranking is the only information collected. There are other types of voting systems to which the Impossibility Theorem does not apply. For example, there is a class of voting systems called Cardinal voting systems that allow for rating the candidates in some way.

“Rating” is different from “ranking” because a voter can give different candidates the same rating. Consider the five-star rating systems used by various industries, or the thumbs up/thumbs down rating system used on YouTube. Could there be a Cardinal voting system that does not violate any of the fairness criteria we have discussed? It's possible, but more research must be done in order to prove it.

People in Mathematics

Kenneth J. Arrow

In 1972, Kenneth J. Arrow, a Harvard Professor of Economics, received the Nobel Prize in Economics jointly with Sir John Hicks, another world renowned economist, for their contributions to economic theory. In particular, Professor Arrow proved mathematically that no ranked voting system meets all four of the fairness criteria discussed in this section. The statement of this fact is known as Arrow's Impossibility Theorem. [Visit this site for more details on Professor Arrow](#)

Answers

1. For plurality voting, we only need to count the first-place votes for each candidate. In this case, McDonald's has four first place votes, which is a majority and wins the election automatically.
2. For ranked-choice voting, McDonald's also wins because it has a majority at the end of Round 1.
3. The majority criterion does apply because one candidate had a majority of the first-place votes. The majority criterion did not fail by either method, because in each case the majority candidate won.

1. For pairwise comparison, notice that McDonald's is a Condorcet candidate because it wins every pairwise comparison. So, McDonald's is the winner.
2. For the Borda count, we must calculate the Borda score for each candidate: McDonald's is 16, Burger King is 21, Taco Bell is 13, Pollo Tropical is 12, Pizza Hut is 8. The winner is Burger King!
3. Yes, the majority criterion applies because McDonald's has the majority of first place votes. The majority criterion only fails using the Borda method

The Condorcet criterion only applies when there is a Condorcet candidate. "Pay off debt" (D) is a Condorcet candidate because D wins every matchup. Yes, the Condorcet criterion applies to this election.

1. V wins 33 first place votes; S, 36; D, 31; and T, 0. So candidate S, "Put into savings," has a plurality and wins. Since the Condorcet candidate D didn't win, the Condorcet criterion is violated.
2. Use the steps outlined in Ranked-Choice Voting for determining the winner of an election by ranked-choice voting, the application of the Hare method in which instant runoffs are used.

Step 1: The number of votes needed to achieve a majority is 51.

Step 2: As illustrated in part 1, no candidate has a majority of first-place votes; so the candidate with the fewest votes, T, must be eliminated.

Step 3: Reallocate votes to the remaining candidates for the second round:

Votes	33	32	31	4
On a vacation (V)	1	3	3	2
Put into savings (S)	3	1	2	1
Pay off debt (D)	2	2	1	3

Step 4: Repeat the process from Step 2. Count the first-place votes for each candidate: V has 33 votes, S has 36 votes, D has 31. votes. Eliminate candidate D, "Pay off debt," for the third round.

Step 5: Repeat the process from Step 3. Reallocate votes to the remaining candidates for the third round:

Votes	33	32	31	4
On a vacation (V)	1	2	2	2
Put into savings (S)	2	1	1	1

1. The number of votes for each candidate are: S 66, G 53, L 61, and B 0. The winner is the standard poodle.
2. If the 53 voters in the second column rank S as 2 and L as 3, then the number of votes for each candidate are: S with 66, G with 53, L with 61, and B with 0. The winner is still the standard poodle.
3. This election does not violate the monotonicity criterion because the winner was not hurt by up-ranking.
4. In general, increasing the ranking for a winner of a plurality election will either leave them with the same or more first place votes while leaving the other candidates with the same or fewer first place votes. So a plurality election will never violate the monotonicity criterion.
1. We leave the creation of the matrix to you. By the pairwise comparison method, Option A wins with three points.
2. If the 95 voters in the first column of Figure 11.13 increased their ranking of the winner by 1, then C would fall into fourth place and A would move up to third place on those ballots. This would only affect the matchup between A and C, and the result would be that A would gain 95 votes while C would lose 95 votes. This means A would have 320 votes and C would have 90. Since A already was ahead of C, this just puts A further ahead and causes no change to the election results.
3. Since the winner A is not hurt by an up-rank, and the loser C is not helped by a down-rank, this election is fair by the monotonicity criterion.
4. Yes, the monotonicity criterion would be satisfied by the pairwise comparison method, because an up-rank of the winner can never decrease the number of pairwise wins. Similarly, a down-rank can never increase the number of pairwise wins.
1. The number of first place votes for each candidate is: A with 14, B with 12, and C with 4. Apple pie has the most first-place votes and wins the election.
2. If cherry pie is removed from the ballot, then the four voters in the third column now rank blueberry pie as their first choice. So the four votes for C now belong to B. This means that blueberry pie has 16 votes compared to the 14 votes for apple pie. Blueberry pie now wins the plurality election.
3. Yes, the election violates the IIA because the removal of a losing candidate from the ballot changed the winner of the election.
1. Construct and analyze a pairwise comparison matrix:

Opponent \ Runner	J	P	D	M	Points
J wins	--	~~JP 15~~	JD 20	JM 26	2
P wins	PJ 21	--	PD 18	~~PM 11~~	$1\frac{1}{2}$
D wins	~~DJ 16~~	DP 18	--	DM 26	$1\frac{1}{2}$
M wins	~~MJ 10~~	MP 25	~~MD 10~~	--	1

Figure 5.3.2: Pairwise Comparison Matrix for Jim, Pam, Dwight, and Michael
Jim Halpert wins with 2 points.

2. If Michael Scott is removed, the summary of ranked ballots becomes:

Number of Ballots	9	11	7	6	3
(J) Jim Halpert (John Krasinski)	1	2	3	2	3
(P) Pam Beesly-Halpert (Jenna Fischer)	3	1	1	3	2
(D) Dwight Schrute (Rainn Wilson)	2	3	2	1	1

Construct and analyze a pairwise comparison matrix:

Opponent \ Runner	J	P	D	Points
J wins	--	~~JP 15~~	JD 20	1
P wins	PJ 21	--	PD 18	$1\frac{1}{2}$
D wins	~~DJ 16~~	DP 18	--	0.5

Figure 5.3.3 Pairwise Comparison Matrix for Jim, Pam, and Dwight
Pam wins with $1\frac{1}{2}$ points.

3. Yes, this violates the IIA, because the winning candidate was hurt by the elimination of a losing candidate.

5.4: Introduction to Graph Theory

In the modern world, planning efficient routes is essential for business and industry, with applications as varied as product distribution, laying new fiber optic lines for broadband internet, and suggesting new friends within social network websites like Facebook.

This field of mathematics started nearly 300 years ago as a look into a mathematical puzzle (we'll look at it in a bit). The field has exploded in importance in the last century, both because of the growing complexity of business in a global economy and because of the computational power that computers have provided us.

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5.5: Graphs

Learning Objectives

After completing this section, you should be able to:

1. Draw a graph to represent a real life situation.
2. Answer questions about a graph using the correct vocabulary.

Drawing Graphs

Example 5.5.1

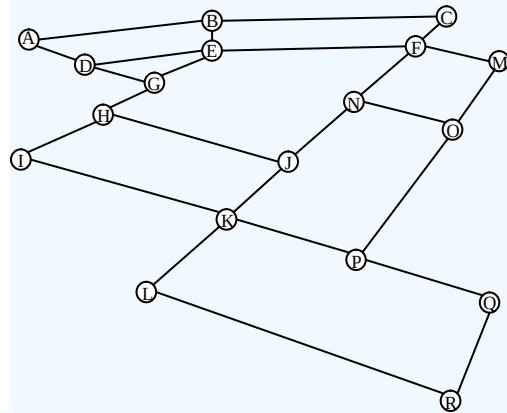
Here is a portion of a housing development from Missoula, Montana[1]. As part of her job, the development's lawn inspector has to walk down every street in the development making sure homeowners' landscaping conforms to the community requirements.



Solution

Naturally, she wants to minimize the amount of walking she has to do. Is it possible for her to walk down every street in this development without having to do any backtracking? While you might be able to answer that question just by looking at the picture for a while, it would be ideal to be able to answer the question for any picture regardless of its complexity.

To do that, we first need to simplify the picture into a form that is easier to work with. We can do that by drawing a simple line for each street. Where streets intersect, we will place a dot.

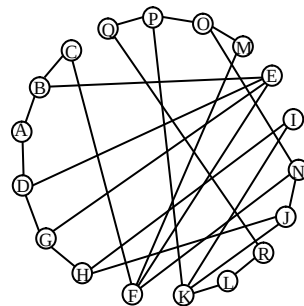
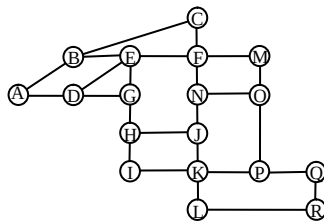


This type of simplified picture is called a **graph**.

Graphs, Vertices, and Edges

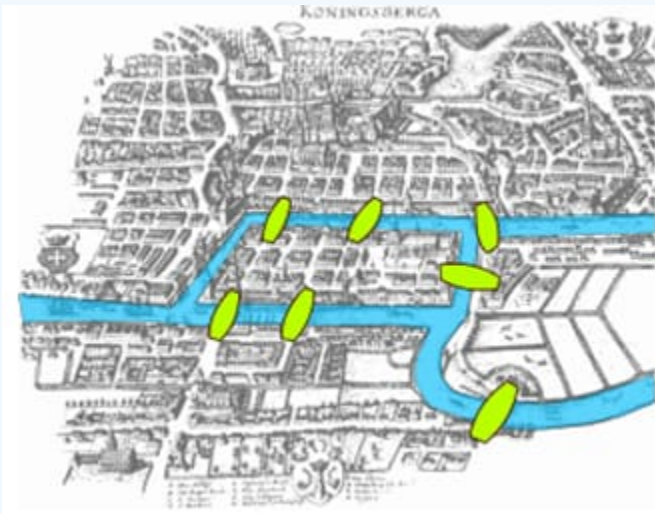
A **graph** consists of a set of dots, called **vertices**, and a set of **edges** connecting pairs of vertices.

While we drew our original graph to correspond with the picture we had, there is nothing particularly important about the layout when we analyze a graph. Both of the graphs below are equivalent to the one drawn above since they show the same edge connections between the same vertices as the original graph.



You probably already noticed that we are using the term *graph* differently than you may have used the term in the past to describe the graph of a mathematical function.

Example 5.5.2

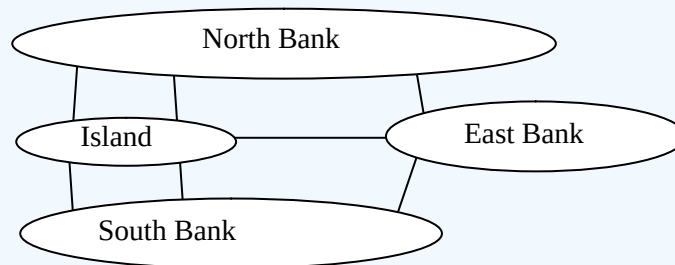


Back in the 18th century in the Prussian city of Königsberg, a river ran through the city and seven bridges crossed the forks of the river. The river and the bridges are highlighted in the picture to the right[2].

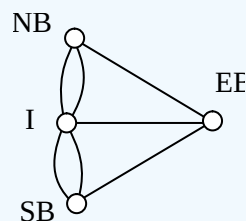
As a weekend amusement, townsfolk would see if they could find a route that would take them across every bridge once and return them to where they started.

Solution

Leonard Euler (pronounced OY-lur), one of the most prolific mathematicians ever, looked at this problem in 1735, laying the foundation for graph theory as a field in mathematics. To analyze this problem, Euler introduced edges representing the bridges:



Since the size of each land mass it is not relevant to the question of bridge crossings, each can be shrunk down to a vertex representing the location:



Notice that in this graph there are *two* edges connecting the north bank and island, corresponding to the two bridges in the original drawing. Depending upon the interpretation of edges and vertices appropriate to a scenario, it is entirely possible and reasonable to have more than one edge connecting two vertices.

While we haven't answered the actual question yet of whether or not there is a route which crosses every bridge once and returns to the starting location, the graph provides the foundation for exploring this question.

Definitions

While we loosely defined some terminology earlier, we now will try to be more specific.

Vertex

A vertex is a dot in the graph that could represent an intersection of streets, a land mass, or a general location, like “work” or “school”. Vertices are often connected by edges.

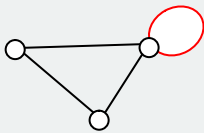
Note that vertices only occur when a dot is explicitly placed, not whenever two edges cross. Imagine a freeway overpass – the freeway and side street cross, but it is not possible to change from the side street to the freeway at that point, so there is no intersection and no vertex would be placed.

Edges

Edges connect pairs of vertices. An edge can represent a physical connection between locations, like a street, or simply that a route connecting the two locations exists, like an airline flight.

Loop

A loop is a special type of edge that connects a vertex to itself. Loops are not used much in street network graphs.



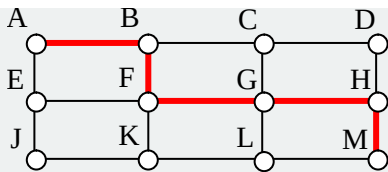
Degree of a vertex

The degree of a vertex is the number of edges meeting at that vertex. It is possible for a vertex to have a degree of zero or larger.

Degree 0	Degree 1	Degree 2	Degree 3	Degree 4

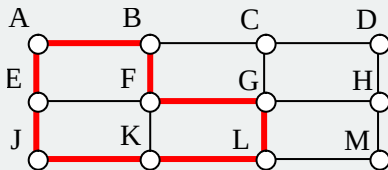
Path

A path is a sequence of vertices using the edges. Usually we are interested in a path between two vertices. For example, a path from vertex A to vertex M is shown below. It is one of many possible paths in this graph.



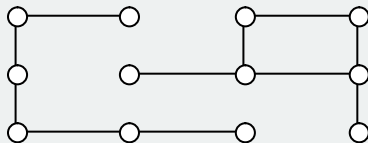
Circuit

A circuit is a path that begins and ends at the same vertex. A circuit starting and ending at vertex A is shown below.



Connected

A graph is connected if there is a path from any vertex to any other vertex. Every graph drawn so far has been connected. The graph below is **disconnected**; there is no way to get from the vertices on the left to the vertices on the right.



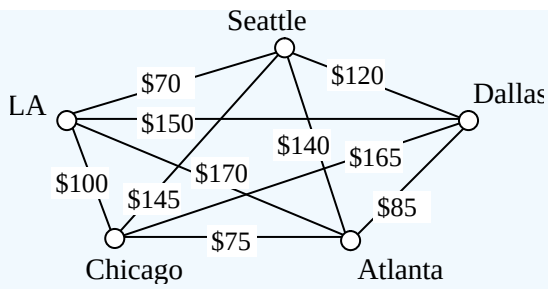
Weights

Depending upon the problem being solved, sometimes weights are assigned to the edges. The weights could represent the distance between two locations, the travel time, or the travel cost. It is important to note that the distance between vertices in a graph does not necessarily correspond to the weight of an edge.

Your Turn 5.5.1

The graph below shows 5 cities. The weights on the edges represent the airfare for a one-way flight between the cities.

- How many vertices and edges does the graph have?
- Is the graph connected?
- What is the degree of the vertex representing LA?
- If you fly from Seattle to Dallas to Atlanta, is that a path or a circuit?
- If you fly from LA to Chicago to Dallas to LA, is that a path or a circuit?



Answer available at the end of this section.

[1] Sam Beebe. <http://www.flickr.com/photos/sbeebe/2850476641/>, CC-BY

[2] Bogdan Giușcă. http://en.Wikipedia.org/wiki/File:Ko...rg_bridges.png

Answers

Your Turn 5.5.1

- 5 vertices, 10 edges
- Yes, it is connected
- The vertex is degree 4
- A path
- A circuit

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5.6: Shortest Path

Learning Objectives

After completing this section, you should be able to:

1. Find the shortest path between two vertices on a graph using Dijkstra's Algorithm

When you visit a website like Google Maps or use your Smartphone to ask for directions from home to your Aunt's house in Pasadena, you are usually looking for a shortest path between the two locations. These computer applications use representations of the street maps as graphs, with estimated driving times as edge weights.

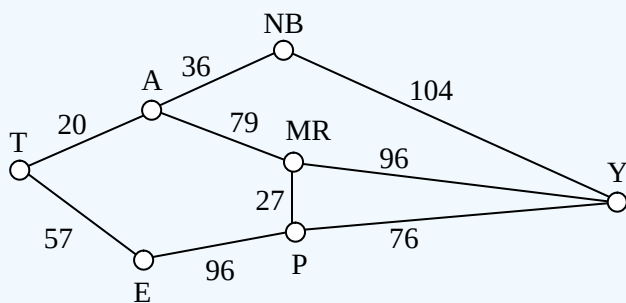
While often it is possible to find a shortest path on a small graph by guess-and-check, our goal in this chapter is to develop methods to solve complex problems in a systematic way by following **algorithms**. An algorithm is a step-by-step procedure for solving a problem. Dijkstra's (pronounced dike-stra) algorithm will find the shortest path between two vertices.

Dijkstra's Algorithm

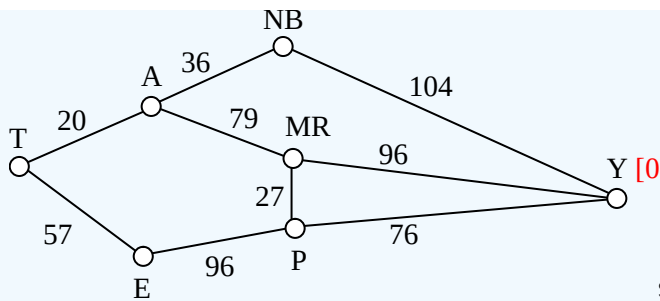
1. Mark the ending vertex with a distance of zero. Designate this vertex as current.
2. Find all vertices leading to the current vertex. Calculate their distances to the end. Since we already know the distance the current vertex is from the end, this will just require adding the most recent edge. Don't record this distance if it is longer than a previously recorded distance.
3. Mark the current vertex as visited. We will never look at this vertex again.
4. Mark the vertex with the smallest distance as current, and repeat from step 2.

Example 5.6.1

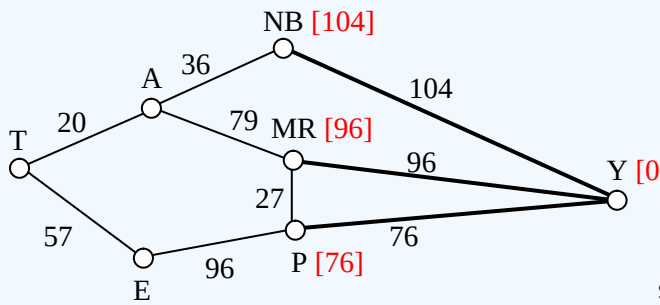
Suppose you need to travel from Tacoma, WA (vertex T) to Yakima, WA (vertex Y). Looking at a map, it looks like driving through Auburn (A) then Mount Rainier (MR) might be shortest, but it's not totally clear since that road is probably slower than taking the major highway through North Bend (NB). A graph with travel times in minutes is shown below. An alternate route through Eatonville (E) and Packwood (P) is also shown.



Solution

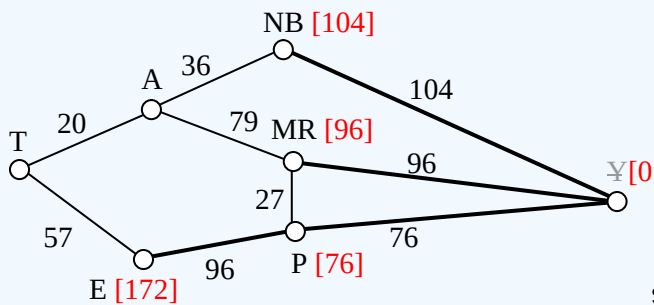


Step 1: Mark the ending vertex with a distance of zero. The distances will be recorded in [brackets] after the vertex name.



Step 2: For each vertex leading to Y, we calculate the distance to the end. For example, NB is a distance of 104 from the end, and MR is 96 from the end. Remember that distances in this case refer to the travel time in minutes.

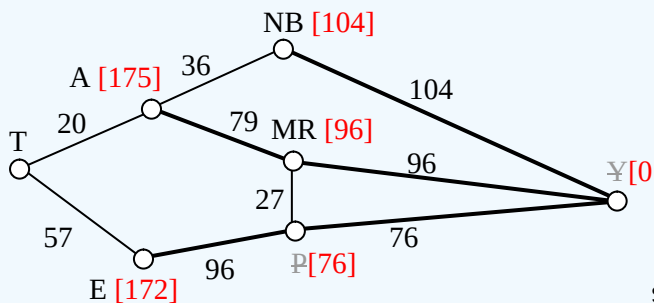
Step 3 & 4: We mark Y as visited, and mark the vertex with the smallest recorded distance as current. At this point, P will be designated current. Back to step 2.



Step 2 (#2): For each vertex leading to P (and not leading to a visited vertex) we find the distance from the end. Since E is 96 minutes from P, and we've already calculated P is 76 minutes from Y, we can compute that E is $96 + 76 = 172$ minutes from Y.

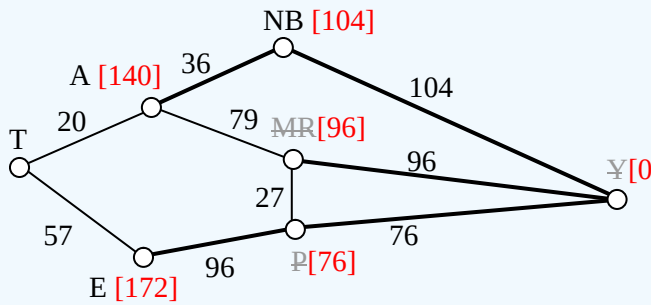
If we make the same computation for MR, we'd calculate $76 + 27 = 103$. Since this is larger than the previously recorded distance from Y to MR, we will *not* replace it.

Step 3 & 4 (#2): We mark P as visited, and designate the vertex with the smallest recorded distance as current: MR. Back to step 2.



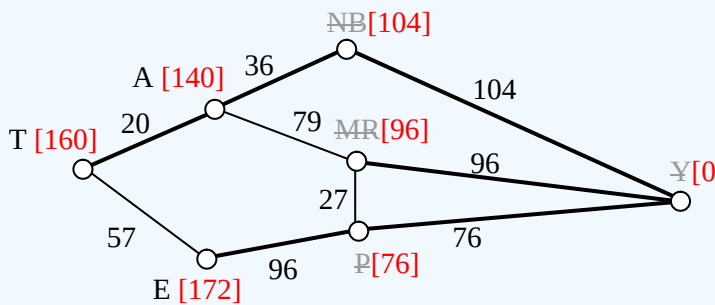
Step 2 (#3): For each vertex leading to MR (and not leading to a visited vertex) we find the distance to the end. The only vertex to be considered is A, since we've already visited Y and P. Adding MR's distance 96 to the length from A to MR gives the distance $96 + 79 = 175$ minutes from A to Y.

Step 3 & 4 (#3): We mark MR as visited, and designate the vertex with smallest recorded distance as current: NB. Back to step 2.



Step 2 (#4): For each vertex leading to NB, we find the distance to the end. We know the shortest distance from NB to Y is 104 and the distance from A to NB is 36, so the distance from A to Y through NB is $104 + 36 = 140$. Since this distance is shorter than the previously calculated distance from Y to A through MR, we replace it.

Step 3 & 4 (#4): We mark NB as visited, and designate A as current, since it now has the shortest distance.



Step 2 (#5): T is the only non-visited vertex leading to A, so we calculate the distance from T to Y through A: $20 + 140 = 160$ minutes.

Step 3 & 4 (#5): We mark A as visited, and designate E as current.

Step 2 (#6): The only non-visited vertex leading to E is T. Calculating the distance from T to Y through E, we compute $172 + 57 = 229$ minutes. Since this is longer than the existing marked time, we do not replace it.

Step 3 (#6): We mark E as visited. Since all vertices have been visited, we are done.

From this, we know that the shortest path from Tacoma to Yakima will take 160 minutes. Tracking which sequence of edges yielded 160 minutes, we see the shortest path is T-A-NB-Y.

Dijkstra's algorithm is an **optimal algorithm**, meaning that it always produces the actual shortest path, not just a path that is pretty short, provided one exists. This algorithm is also **efficient**, meaning that it can be implemented in a reasonable amount of time. Dijkstra's algorithm takes around V^2 calculations, where V is the number of vertices in a graph[1]. A graph with 100 vertices would take around 10,000 calculations. While that would be a lot to do by hand, it is not a lot for computer to handle. It is because of this efficiency that your car's GPS unit can compute driving directions in only a few seconds.

In contrast, an **inefficient** algorithm might try to list all possible paths then compute the length of each path. Trying to list all possible paths could easily take 10^{25} calculations to compute the shortest path with only 25 vertices; that's a 1 with 25 zeros after it! To put that in perspective, the fastest computer in the world would still spend over 1000 years analyzing all those paths.

Example 5.6.2

A shipping company needs to route a package from Washington, D.C. to San Diego, CA. To minimize costs, the package will first be sent to their processing center in Baltimore, MD then sent as part of mass shipments between their various processing centers, ending up in their processing center in Bakersfield, CA. From there it will be delivered in a small truck to San Diego.

The travel times, in hours, between their processing centers are shown in the table below. Three hours has been added to each travel time for processing. Find the shortest path from Baltimore to Bakersfield.

	Baltimore	Denver	Dallas	Chicago	Atlanta	Bakersfield
Baltimore	*			15	14	
Denver		*		18	24	19
Dallas			*	18	15	25
Chicago	15	18	18	*	14	
Atlanta	14	24	15	14	*	
Bakersfield		19	25			*

Solution

While we could draw a graph, we can also work directly from the table.

Step 1: The ending vertex, Bakersfield, is marked as current.

Step 2: All cities connected to Bakersfield, in this case Denver and Dallas, have their distances calculated; we'll mark those distances in the column headers.

Step 3 & 4: Mark Bakersfield as visited. Here, we are doing it by shading the corresponding row and column of the table. We mark Denver as current, shown in bold, since it is the vertex with the shortest distance.

	Baltimore	Denver [19]	Dallas [25]	Chicago	Atlanta	Bakersfield [0]
Baltimore	*			15	14	
Denver		*		18	24	19
Dallas			*	18	15	25
Chicago	15	18	18	*	14	
Atlanta	14	24	15	14	*	
Bakersfield		19	25			*

Step 2 (#2): For cities connected to Denver, calculate distance to the end. For example, Chicago is 18 hours from Denver, and Denver is 19 hours from the end, the distance for Chicago to the end is $18+19 = 37$ (Chicago to Denver to Bakersfield). Atlanta is 24 hours from Denver, so the distance to the end is $24+19 = 43$ (Atlanta to Denver to Bakersfield).

Step 3 & 4 (#2): We mark Denver as visited and mark Dallas as current.

	Baltimore	Denver [19]	Dallas [25]	Chicago [37]	Atlanta [43]	Bakersfield [0]
Baltimore	*			15	14	
Denver		*		18	24	19
Dallas			*	18	15	25
Chicago	15	18	18	*	14	
Atlanta	14	24	15	14	*	
Bakersfield		19	25			*

Step 2 (#3): For cities connected to Dallas, calculate the distance to the end. For Chicago, the distance from Chicago to Dallas is 18 and from Dallas to the end is 25, so the distance from Chicago to the end through Dallas would be $18+25 = 43$. Since this is longer than the currently marked distance for Chicago, we do not replace it. For Atlanta, we calculate $15+25 = 40$. Since this is shorter than the currently marked distance for Atlanta, we replace the existing distance.

Step 3 & 4 (#3): We mark Dallas as visited, and mark Chicago as current.

	Baltimore	Denver [19]	Dallas [25]	Chicago [37]	Atlanta [40]	Bakersfield [0]
Baltimore	*			15	14	
Denver		*		18	24	19
Dallas			*	18	15	25
Chicago	15	18	18	*	14	
Atlanta	14	24	15	14	*	
Bakersfield		19	25			*

Step 2 (#4): Baltimore and Atlanta are the only non-visited cities connected to Chicago. For Baltimore, we calculate $15+37 = 52$ and mark that distance. For Atlanta, we calculate $14+37 = 51$. Since this is longer than the existing distance of 40 for Atlanta, we do not replace that distance.

Step 3 & 4 (#4): Mark Chicago as visited and Atlanta as current.

	Baltimore [52]	Denver [19]	Dallas [25]	Chicago [37]	Atlanta [40]	Bakersfield [0]
Baltimore	*			15	14	
Denver		*		18	24	19
Dallas			*	18	15	25
Chicago	15	18	18	*	14	
Atlanta	14	24	15	14	*	
Bakersfield		19	25			*

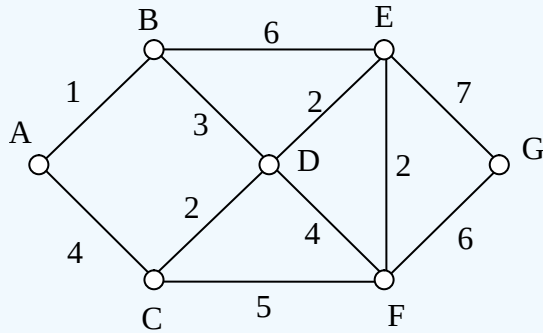
Step 2 (#5): The distance from Atlanta to Baltimore is 14. Adding that to the distance already calculated for Atlanta gives a total distance of $14+40 = 54$ hours from Baltimore to Bakersfield through Atlanta. Since this is larger than the currently calculated distance, we do not replace the distance for Baltimore.

Step 3 & 4 (#5): We mark Atlanta as visited. All cities have been visited and we are done.

The shortest route from Baltimore to Bakersfield will take 52 hours, and will route through Chicago and Denver.

Your Turn 5.6.1

Find the shortest path between vertices A and G in the graph below.



Answer available at the end of this section.

[1] It can be made to run faster through various optimizations to the implementation.

Answers

Your Turn 5.6.1

The shortest path is ABDEG, with length 13

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5.7: Euler Circuits and the Chinese Postman Problem

Learning Objectives

After completing this section, you should be able to:

1. Find an Euler path and an Euler circuit for a graph.
2. Determine if an Euler path or Euler circuit exists for a graph.
3. Apply Fleury's Algorithm to find an Euler circuit.

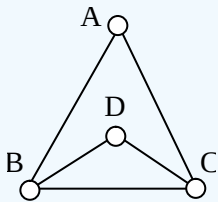
In Section 5.5, we created a graph of the Königsberg bridges and asked whether it was possible to walk across every bridge once. Because Euler first studied this question, these types of paths are named after him.

Euler Path

An **Euler path** is a path that uses every edge in a graph with no repeats. Being a path, it does not have to return to the starting vertex.

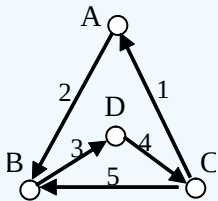
Example 5.7.1

In the graph shown below, there are several Euler paths.



Solution

One such path is CABDCB. The path is shown in arrows to the right, with the order of edges numbered.

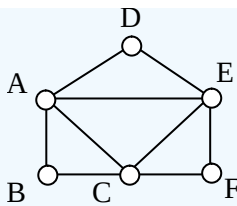


Euler Circuit

An **Euler circuit** is a circuit that uses every edge in a graph with no repeats. Being a circuit, it must start and end at the same vertex.

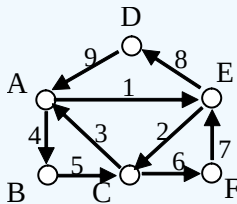
Example 5.7.2

The graph below has several possible Euler circuits.



Solution

Here's a couple, starting and ending at vertex A: ADEACEFCBA and AECABCFEDA. The second is shown in arrows.



Look back at the example used for Euler paths – does that graph have an Euler circuit? A few tries will tell you no; that graph does not have an Euler circuit. When we were working with shortest paths, we were interested in the optimal path. With Euler paths and circuits, we're primarily interested in whether an Euler path or circuit exists.

Why do we care if an Euler circuit exists? Think back to our housing development lawn inspector from Section 5.5. The lawn inspector is interested in walking as little as possible. The ideal situation would be a circuit that covers every street with no repeats. That's an Euler circuit! Luckily, Euler solved the question of whether or not an Euler path or circuit will exist.

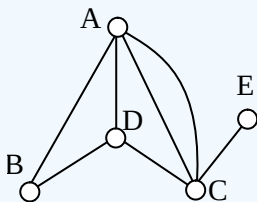
Euler's Path and Circuit Theorems

A graph will contain an Euler path if it contains at most two vertices of odd degree.

A graph will contain an Euler circuit if all vertices have even degree

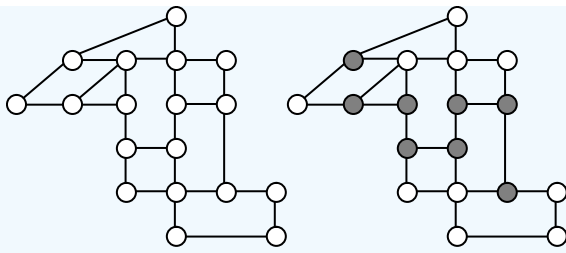
Example 5.7.3

In the graph below, vertices A and C have degree 4, since there are 4 edges leading into each vertex. B is degree 2, D is degree 3, and E is degree 1. This graph contains two vertices with odd degree (D and E) and three vertices with even degree (A, B, and C), so Euler's theorems tell us this graph has an Euler path, but not an Euler circuit.



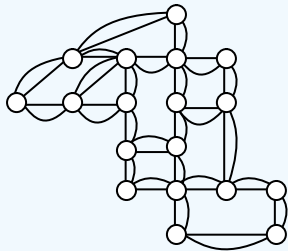
Example 5.7.4

Is there an Euler circuit on the housing development lawn inspector graph we created earlier in the chapter? All the highlighted vertices have odd degree. Since there are more than two vertices with odd degree, there are no Euler paths or Euler circuits on this graph. Unfortunately our lawn inspector will need to do some backtracking.



Example 5.7.5

When it snows in the same housing development, the snowplow has to plow both sides of every street. For simplicity, we'll assume the plow is out early enough that it can ignore traffic laws and drive down either side of the street in either direction. This can be visualized in the graph by drawing two edges for each street, representing the two sides of the street.



Notice that every vertex in this graph has even degree, so this graph does have an Euler circuit.

Now we know how to determine if a graph has an Euler circuit, but if it does, how do we find one? While it usually is possible to find an Euler circuit just by pulling out your pencil and trying to find one, the more formal method is **Fleury's algorithm**.

Fleury's Algorithm

1. Start at any vertex if finding an Euler circuit. If finding an Euler path, start at one of the two vertices with odd degree.
2. Choose any edge leaving your current vertex, provided deleting that edge will not separate the graph into two disconnected sets of edges.
3. Add that edge to your circuit, and delete it from the graph.
4. Continue until you're done.

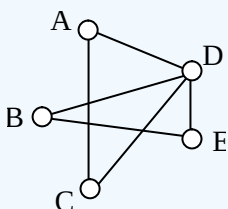
Example 5.7.6

Let's find an Euler Circuit on this graph using Fleury's algorithm, starting at vertex A.

Solution

Step 1:

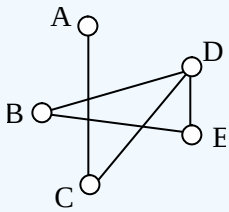
Original Graph. Choosing edge AD.



Circuit so far: AD

Step 2:

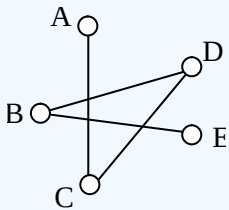
AD deleted. D is current. Can't choose DC since that would disconnect graph. Choosing DE



Circuit so far: ADE

Step 3:

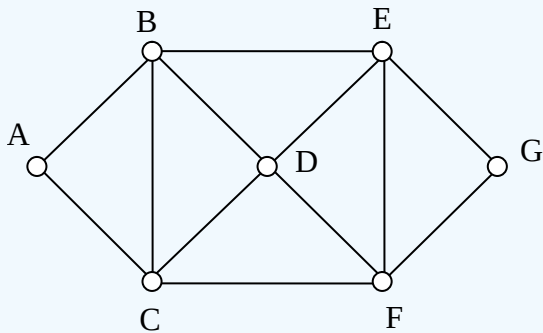
E is current. From here, there is only one option, so the rest of the circuit is determined.



Circuit: ADEBDCA

Your Turn 5.7.1

Does the graph below have an Euler Circuit? If so, find one.



Answer available at the end of this section.

Answers

Your Turn 5.7.1

Yes, all vertices have even degree so this graph has an Euler Circuit. There are several possibilities. One is: ABEGFCDFEDBCA

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5.8: Eulerization and the Chinese Postman Problem

Learning Objectives

After completing this section, you should be able to:

1. Eulerize a graph to create an Euler circuit.

Not every graph has an Euler path or circuit, yet our lawn inspector still needs to do her inspections. Her goal is to minimize the amount of walking she has to do. In order to do that, she will have to duplicate some edges in the graph until an Euler circuit exists.

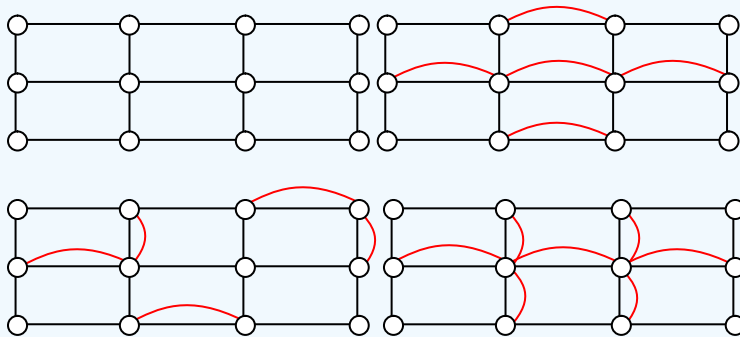
Eulerization

Eulerization is the process of adding edges to a graph to create an Euler circuit on a graph. To eulerize a graph, edges are duplicated to connect pairs of vertices with odd degree. Connecting two odd degree vertices increases the degree of each, giving them both even degree. When two odd degree vertices are not directly connected, we can duplicate all edges in a path connecting the two.

Note that we can only duplicate edges, not create edges where there wasn't one before. Duplicating edges would mean walking or driving down a road twice, while creating an edge where there wasn't one before is akin to installing a new road!

Example 5.8.1

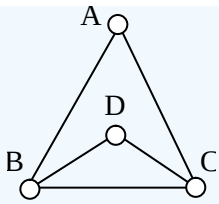
For the rectangular graph shown, three possible eulerizations are shown. Notice in each of these cases the vertices that started with odd degrees have even degrees after eulerization, allowing for an Euler circuit.



In the example above, you'll notice that the last eulerization required duplicating seven edges, while the first two only required duplicating five edges. If we were eulerizing the graph to find a walking path, we would want the eulerization with minimal duplications. If the edges had weights representing distances or costs, then we would want to select the eulerization with the minimal total added weight.

Your Turn 5.8.1

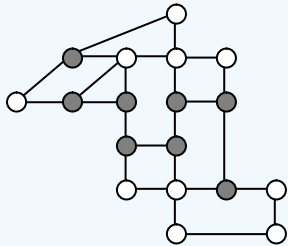
Eulerize the graph shown, then find an Euler circuit on the eulerized graph.



Answer available at the end of this section.

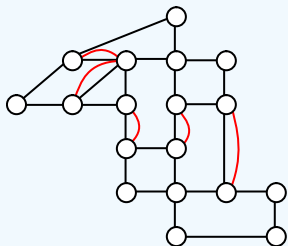
Example 5.8.2

Looking again at the graph for our lawn inspector again, the vertices with odd degree are shown highlighted. With eight vertices, we will always have to duplicate at least four edges.



Solution

In this case, we need to duplicate five edges since two odd degree vertices are not directly connected. Without weights we can't be certain this is the eulerization that minimizes walking distance, but it looks pretty good.



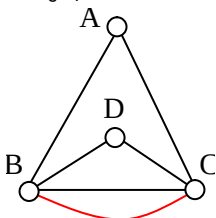
The problem of finding the optimal eulerization is called the Chinese Postman Problem, a name given by an American in honor of the Chinese mathematician Mei-Ko Kwan who first studied the problem in 1962 while trying to find optimal delivery routes for postal carriers. This problem is important in determining efficient routes for garbage trucks, school buses, parking meter checkers, street sweepers, and more.

Unfortunately, algorithms to solve this problem are fairly complex. Some simpler cases are considered in the exercises.

Answers

Your Turn 5.8.1

This graph can be eulerized by duplicating the edge BC, as shown. One possible Euler circuit on the eulerized graph is ACDBCBA



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5.9: Hamiltonian Circuits and the Traveling Salesman Problem

Learning Objectives

After completing this section, you should be able to:

1. Identify a Hamilton path and a Hamilton Circuit for a graph.
2. Apply the Brute Force Algorithm. to find a Hamilton circuit.
3. Apply the Nearest Neighbor Algorithm to find a Hamilton circuit.
4. Apply the Repeated Nearest Neighbor Algorithm to find a Hamilton circuit.
5. Apply the Sorted Edges Algorithm to find a Hamilton circuit.

In the last section, we considered optimizing a walking route for a postal carrier. How is this different than the requirements of a package delivery driver? While the postal carrier needed to walk down every street (edge) to deliver the mail, the package delivery driver instead needs to visit every one of a set of delivery locations. Instead of looking for a circuit that covers every edge once, the package deliverer is interested in a circuit that visits every vertex once.

Hamiltonian Circuits and Paths

A **Hamiltonian circuit** is a circuit that visits every vertex once with no repeats. Being a circuit, it must start and end at the same vertex. A **Hamiltonian path** also visits every vertex once with no repeats, but does not have to start and end at the same vertex.

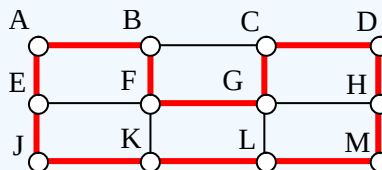
Hamiltonian circuits are named for William Rowan Hamilton who studied them in the 1800's.

Example 5.9.1

One Hamiltonian circuit is shown on the graph below. There are several other Hamiltonian circuits possible on this graph. Notice that the circuit only has to visit every vertex once; it does not need to use every edge.

Solution

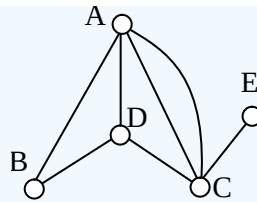
This circuit could be notated by the sequence of vertices visited, starting and ending at the same vertex: ABFGCDHMLKJEA. Notice that the same circuit could be written in reverse order, or starting and ending at a different vertex.



Unlike with Euler circuits, there is no nice theorem that allows us to instantly determine whether or not a Hamiltonian circuit exists for all graphs.[1]

Example 5.9.2

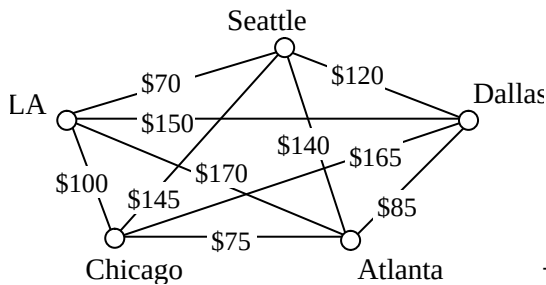
Does a Hamiltonian path or circuit exist on the graph below?



Solution

We can see that once we travel to vertex E there is no way to leave without returning to C, so there is no possibility of a Hamiltonian circuit. If we start at vertex E we can find several Hamiltonian paths, such as ECDAB and ECABD.

With Hamiltonian circuits, our focus will not be on existence, but on the question of optimization; given a graph where the edges have weights, can we find the optimal Hamiltonian circuit; the one with lowest total weight.



This problem is called the **Traveling salesman problem (TSP)** because the question can be framed like this: Suppose a salesman needs to give sales pitches in four cities. He looks up the airfares between each city, and puts the costs in a graph. In what order should he travel to visit each city once then return home with the lowest cost?

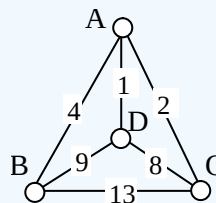
To answer this question of how to find the lowest cost Hamiltonian circuit, we will consider some possible approaches. The first option that might come to mind is to just try all different possible circuits.

Brute Force Algorithm (a.k.a. exhaustive search)

1. List all possible Hamiltonian circuits
2. Find the length of each circuit by adding the edge weights
3. Select the circuit with minimal total weight.

Example 5.9.3

Apply the Brute force algorithm to find the minimum cost Hamiltonian circuit on the graph below.



Solution

To apply the Brute force algorithm, we list all possible Hamiltonian circuits and calculate their weight:



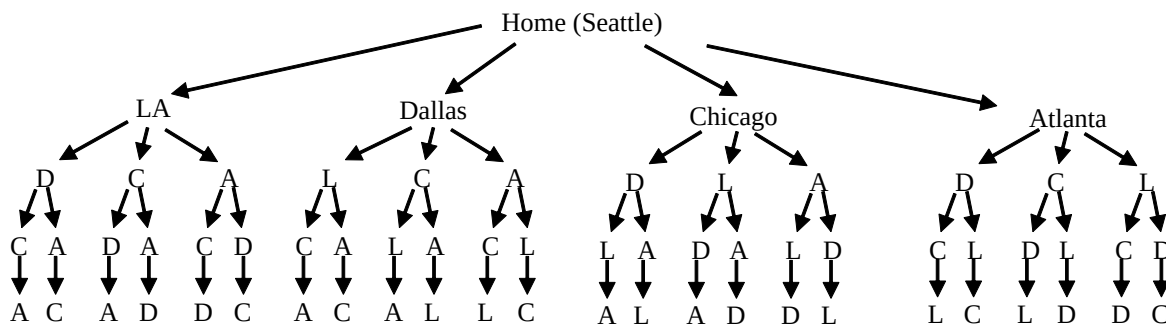
Note: These are the unique circuits on this graph. All other possible circuits are the reverse of the listed ones or start at a different vertex, but result in the same weights.

From this we can see that the second circuit, ABDCA, is the optimal circuit.

The Brute force algorithm is optimal; it will always produce the Hamiltonian circuit with minimum weight. Is it efficient? To answer that question, we need to consider how many Hamiltonian circuits a graph could have. For simplicity, let's look at the worst-case possibility, where every vertex is connected to every other vertex. This is called a **complete graph**.

Suppose we had a complete graph with five vertices like the air travel graph above. From Seattle there are four cities we can visit first. From each of those, there are three choices. From each of those cities, there are two possible cities to visit next. There is then only one choice for the last city before returning home.

This can be shown visually:



Counting the number of routes, we can see there are $4 \cdot 3 \cdot 2 \cdot 1 = 24$ routes. For six cities there would be $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ routes.

Number of Possible Circuits

For n vertices in a complete graph, there will be $(n - 1)! = (n - 1)(n - 2)(n - 3) \cdots 3 \cdot 2 \cdot 1$ routes. Half of these are duplicates in reverse order, so there are $\frac{(n-1)!}{2}$ unique circuits.

The exclamation symbol, !, is read “factorial” and is shorthand for the product shown.

Example 5.9.4

How many circuits would a complete graph with 8 vertices have?

Solution

A complete graph with 8 vertices would have $(8 - 1)! = 7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$ possible Hamiltonian circuits. Half of the circuits are duplicates of other circuits but in reverse order, leaving 2520 unique routes.

While this is a lot, it doesn't seem unreasonably huge. But consider what happens as the number of cities increase:



As you can see the number of circuits is growing extremely quickly. If a computer looked at one billion circuits a second, it would still take almost two years to examine all the possible circuits with only 20 cities! Certainly Brute Force is **not** an efficient algorithm.

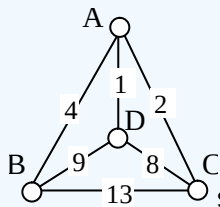
Unfortunately, no one has yet found an efficient *and* optimal algorithm to solve the TSP, and it is very unlikely anyone ever will. Since it is not practical to use brute force to solve the problem, we turn instead to **heuristic algorithms**; efficient algorithms that give approximate solutions. In other words, heuristic algorithms are fast, but may or may not produce the optimal circuit.

Nearest Neighbor Algorithm (NNA)

1. Select a starting point.
2. Move to the nearest unvisited vertex (the edge with smallest weight).
3. Repeat until the circuit is complete.

Example 5.9.5

Consider our earlier graph, shown to the right.



Starting at vertex A, the nearest neighbor is vertex D with a weight of 1.

From D, the nearest neighbor is C, with a weight of 8.

From C, our only option is to move to vertex B, the only unvisited vertex, with a cost of 13.

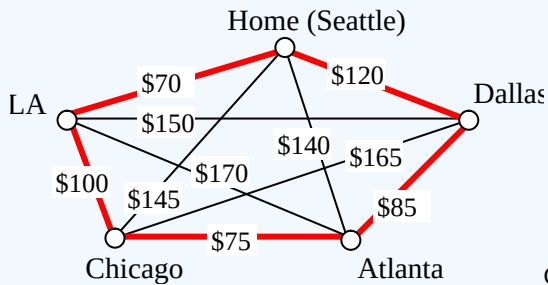
From B we return to A with a weight of 4.

Solution

The resulting circuit is ADCBA with a total weight of $1 + 8 + 13 + 4 = 26$.

We ended up finding the worst circuit in the graph! What happened? Unfortunately, while it is very easy to implement, the NNA is a **greedy** algorithm, meaning it only looks at the immediate decision without considering the consequences in the future. In this case, following the edge AD forced us to use the very expensive edge BC later.

Example 5.9.6



Consider again our salesman. Starting in Seattle, the nearest neighbor (cheapest flight) is to LA, at a cost of \$70. From there:

Solution

- LA to Chicago: \$100
- Chicago to Atlanta: \$75
- Atlanta to Dallas: \$85
- Dallas to Seattle: \$120

Total cost: \$450

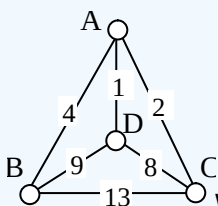
In this case, nearest neighbor did find the optimal circuit.

Going back to our first example, how could we improve the outcome? One option would be to redo the nearest neighbor algorithm with a different starting point to see if the result changed. Since nearest neighbor is so fast, doing it several times isn't a big deal.

Repeated Nearest Neighbor Algorithm (RNNA)

1. Do the Nearest Neighbor Algorithm starting at each vertex
2. Choose the circuit produced with minimal total weight

Example 5.9.7



We will revisit the graph from Example 5.9.5.

Starting at vertex A resulted in a circuit with weight 26.

Starting at vertex B, the nearest neighbor circuit is BADCB with a weight of $4+1+8+13 = 26$. This is the same circuit we found starting at vertex A. No better.

Starting at vertex C, the nearest neighbor circuit is CADBC with a weight of $2+1+9+13 = 25$. Better!

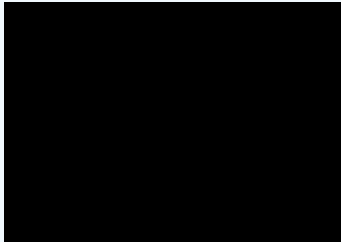
Starting at vertex D, the nearest neighbor circuit is DACBD. Notice that this is actually the same circuit we found starting at C, just written with a different starting vertex.

Solution

The RNNA was able to produce a slightly better circuit with a weight of 25, but still not the optimal circuit in this case. Notice that even though we found the circuit by starting at vertex C, we could still write the circuit starting at A: ADBCA or ACBDA.

Your Turn 5.9.1

The table below shows the time, in milliseconds, it takes to send a packet of data between computers on a network. If data needed to be sent in sequence to each computer, then notification needed to come back to the original computer, we would be solving the TSP. The computers are labeled A-F for convenience.



Find the circuit generated by the NNA starting at vertex B.

Answer available at the end of this section.

While certainly better than the basic NNA, unfortunately, the RNNA is still greedy and will produce very bad results for some graphs. As an alternative, our next approach will step back and look at the “big picture” – it will select first the edges that are shortest, and then fill in the gaps.

Sorted Edges Algorithm (a.k.a. Cheapest Link Algorithm)

1. Select the cheapest unused edge in the graph.
2. Repeat step 1, adding the cheapest unused edge to the circuit, unless:
 - a. adding the edge would create a circuit that doesn't contain all vertices, or
 - b. adding the edge would give a vertex degree 3.
3. Repeat until a circuit containing all vertices is formed.

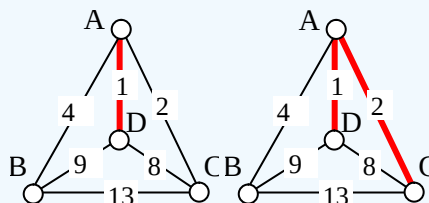
Example 5.9.8

Using the four vertex graph from earlier, we can use the Sorted Edges algorithm.

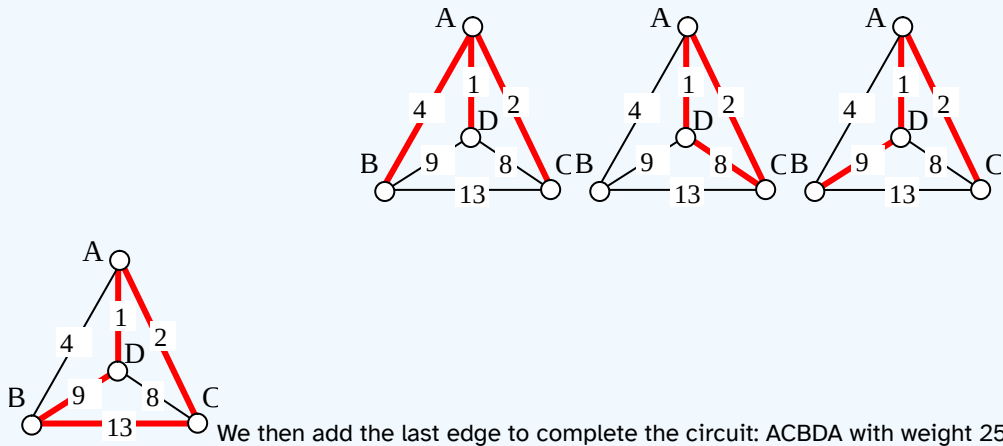
Solution

The cheapest edge is AD, with a cost of 1. We highlight that edge to mark it selected.

The next shortest edge is AC, with a weight of 2, so we highlight that edge.



For the third edge, we'd like to add AB, but that would give vertex A degree 3, which is not allowed in a Hamiltonian circuit. The next shortest edge is CD, but that edge would create a circuit ACDA that does not include vertex B, so we reject that edge. The next shortest edge is BD, so we add that edge to the graph.

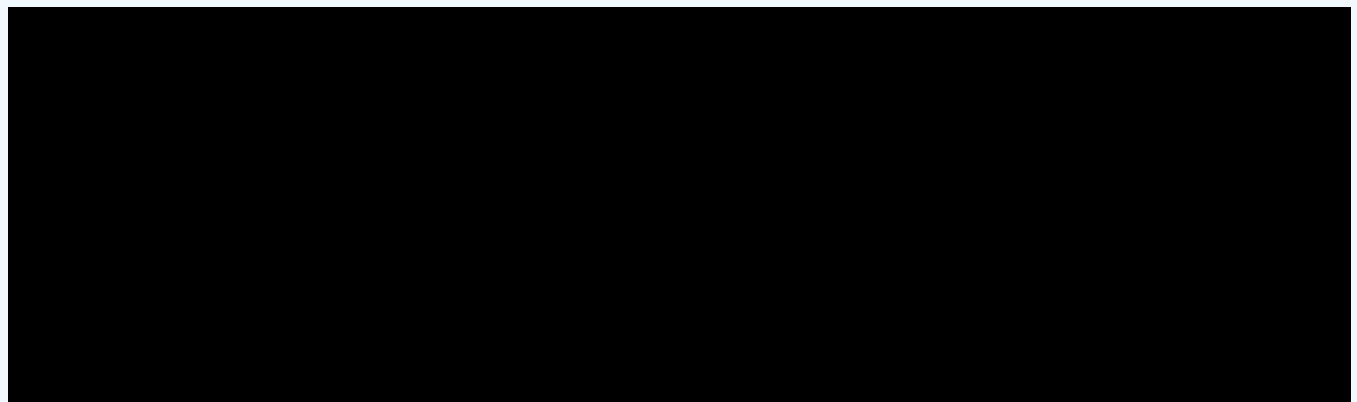


Notice that the algorithm did not produce the optimal circuit in this case; the optimal circuit is ACDBA with weight 23.

While the Sorted Edge algorithm overcomes some of the shortcomings of NNA, it is still only a heuristic algorithm, and does not guarantee the optimal circuit.

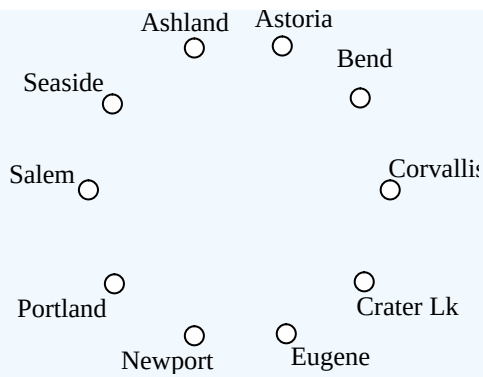
Example 5.9.9

Your teacher's band, *Derivative Work*, is doing a bar tour in Oregon. The driving distances are shown below. Plan an efficient route for your teacher to visit all the cities and return to the starting location. Use NNA starting at Portland, and then use Sorted Edges.



Solution

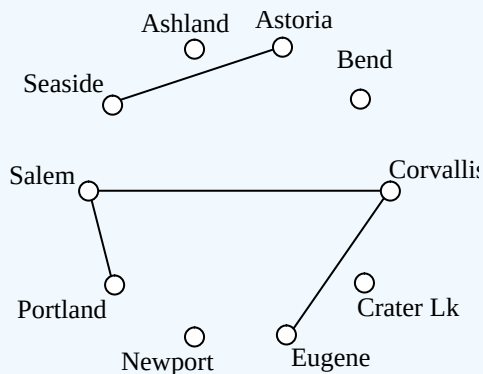
Using NNA with a large number of cities, you might find it helpful to mark off the cities as they're visited to keep from accidentally visiting them again. Looking in the row for Portland, the smallest distance is 47, to Salem. Following that idea, our circuit will be:



Portland to Salem	47
Salem to Corvallis	40
Corvallis to Eugene	47
Eugene to Newport	91
Newport to Seaside	117
Seaside to Astoria	17
Astoria to Bend	255
Bend to Ashland	200
Ashland to Crater Lake	108
Crater Lake to Portland	344
Total trip length:	1266 miles

Using Sorted Edges, you might find it helpful to draw an empty graph, perhaps by drawing vertices in a circular pattern. Adding edges to the graph as you select them will help you visualize any circuits or vertices with degree 3.

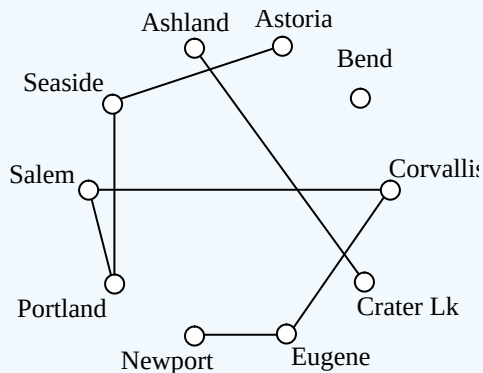
We start adding the shortest edges:



Seaside to Astoria	17 miles
Corvallis to Salem	40 miles
Portland to Salem	47 miles
Corvallis to Eugene	47 miles

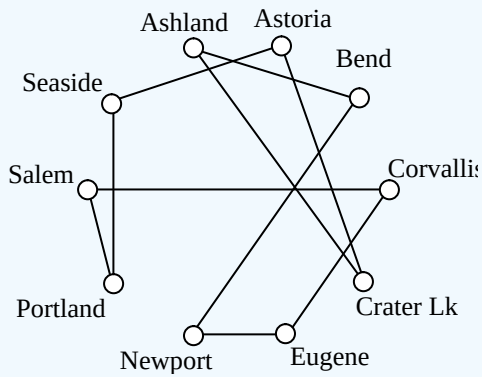
The graph after adding these edges is shown to the right. The next shortest edge is from Corvallis to Newport at 52 miles, but adding that edge would give Corvallis degree 3.

Continuing on, we can skip over any edge pair that contains Salem or Corvallis, since they both already have degree 2.



Portland to Seaside	78 miles
Eugene to Newport	91 miles
Portland to Astoria	(reject – closes circuit)
Ashland to Crater Lk	108 miles

The graph after adding these edges is shown to the right. At this point, we can skip over any edge pair that contains Salem, Seaside, Eugene, Portland, or Corvallis since they already have degree 2.



Newport to Astoria	(reject – closes circuit)
Newport to Bend	180 miles
Bend to Ashland	200 miles

At this point the only way to complete the circuit is to add:

Crater Lk to Astoria 433 miles

The final circuit, written to start at Portland, is:

Portland, Salem, Corvallis, Eugene, Newport, Bend, Ashland, Crater Lake, Astoria, Seaside, Portland.

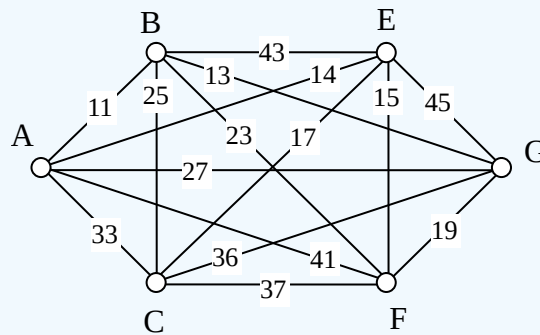
Total trip length: 1241 miles.

While better than the NNA route, neither algorithm produced the optimal route. The following route can make the tour in 1069 miles:

Portland, Astoria, Seaside, Newport, Corvallis, Eugene, Ashland, Crater Lake, Bend, Salem, Portland

Your Turn 5.9.2

Find the circuit produced by the Sorted Edges algorithm using the graph below.



Answer available at the end of this section.

[1] There are some theorems that can be used in specific circumstances, such as Dirac's theorem, which says that a Hamiltonian circuit must exist on a graph with n vertices if each vertex has degree $n/2$ or greater.

Answers

Your Turn 5.9.1

At each step, we look for the nearest location we haven't already visited.

From B the nearest computer is E with time 24.

From E, the nearest computer is D with time 11.

From D the nearest is A with time 12.

From A the nearest is C with time 34.

From C, the only computer we haven't visited is F with time 27

From F, we return back to B with time 50.

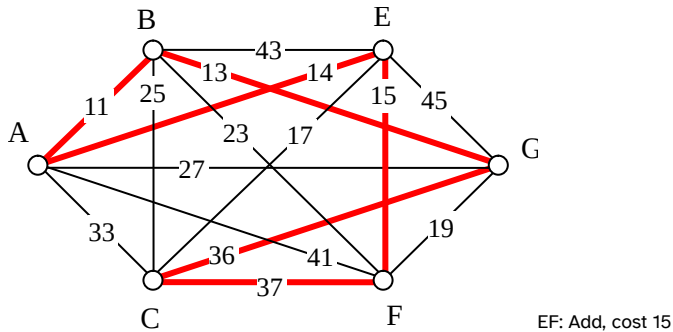
The NNA circuit from B is BEDACFB with time 158 milliseconds.

Your Turn 5.9.2

AB: Add, cost 11

BG: Add, cost 13

AE: Add, cost 14



EC: Skip (degree 3 at E)

FG: Skip (would create a circuit not including C)

BF, BC, AG, AC: Skip (would cause a vertex to have degree 3)

GC: Add, cost 36

CF: Add, cost 37, completes the circuit

Final circuit: ABGCFEA

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