

# PROFESSOR HOLZ' TOPICS IN CONTEMPORARY MATHEMATICS



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# Professor Holz' Topics in Contemporary Mathematics

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## Licensing

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## CHAPTER OVERVIEW

### 1: Logic

#### 1.1: Symbolic Language

##### 1.1.1: Introduction

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## SECTION OVERVIEW

### 1.1: Symbolic Language

#### 1.1.1: Introduction

#### 1.1.2: Statements and Quantifiers

#### 1.1.3: Compound Statements

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### 1.1.1: Introduction

What is logic? Logic is the study of reasoning, and it has applications in many fields, including philosophy, law, psychology, digital electronics, and computer science.

In law, constructing a well-reasoned, logical argument is extremely important. The main goal of arguments made by lawyers is to convince a judge and jury that their arguments are valid and well-supported by the facts of the case, so the case should be ruled in their favor. Think about Thurgood Marshall arguing for desegregation in front of the U.S. Supreme Court during the *Brown v. Board of Education of Topeka* lawsuit in 1954, or Ruth Bader Ginsburg arguing for equality in social security benefits for both men and women under the law during the mid-1970s. Both these great minds were known for the preparation and thoroughness of their logical legal arguments, which resulted in victories that advanced the causes they fought for. Thurgood Marshall and Ruth Bader Ginsburg would later become well respected justices on the U.S. Supreme Court themselves.



Figure 1.1.1.1: Logic is key to a well-reasoned argument, in both math and law. (credit: modification of work "Lady Justitia holding sword and scale bronze figurine with judge hammer on wooden table" by Jernej Furman/Flickr, CC BY 2.0)

In this chapter, we will explore how to construct well-reasoned logical arguments using varying structures. Your ability to form and comprehend logical arguments is a valuable tool in many areas of life, whether you're planning a dinner date, negotiating the purchase of a new car, or persuading your boss that you deserve a raise.

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## 1.1.2: Statements and Quantifiers



Figure 1.1.2.1: Construction of a logical argument, like that of a house, requires you to begin with the right parts. (credit: modification of work “Barn Raising” by Robert Stinnett/Flickr, CC BY 2.0)

### Learning Objectives

After completing this section, you should be able to:

1. Identify logical statements.
2. Represent statements in symbolic form.
3. Negate statements in words.
4. Negate statements symbolically.
5. Translate negations between words and symbols.
6. Express statements with quantifiers of all, some, and none.
7. Negate statements containing quantifiers of all, some, and none.

Have you ever built a club house, tree house, or fort with your friends? If so, you and your friends likely started by gathering some tools and supplies to work with, such as hammers, saws, screwdrivers, wood, nails, and screws. Hopefully, at least one member of your group had some knowledge of how to use the tools correctly and helped to direct the construction project. After all, if your house isn't built on a strong foundation, it will be weak and could possibly fall apart during the next big storm. This same foundation is important in logic.

In this section, we will begin with the parts that make up all logical arguments. The building block of any logical argument is a logical statement, or simply a statement. A logical statement has the form of a complete sentence, and it must make a claim that can be identified as being true or false.

When making arguments, sometimes people make false claims. When evaluating the strength or validity of a logical argument, you must also consider the truth values, or the identification of true or false, of all the statements used to support the argument. While a false statement is still considered a logical statement, a strong logical argument starts with true statements.

### Identifying Logical Statements



Figure 1.1.2.2: Not all roses are red. (credit: “assorted pink yellow white red roses macro” by ProFlowers/Flickr, CC BY 2.0)

An example of logical statement with a false truth value is, “All roses are red.” It is a logical statement because it has the form of a complete sentence and makes a claim that can be determined to be either true or false. It is a false statement because not all roses are red: some roses are red, but there are also roses that are pink, yellow, and white. Requests, questions, or directives may be complete sentences, but they are not logical statements because they cannot be determined to be true or false. For example, suppose someone said to you, “Please, sit down over there.” This request does not make a claim and it cannot be identified as true or false; therefore, it is not a logical statement.

### ? Exercise 1.1.2.1: Identifying Logical Statements

Determine whether each of the following sentences represents a logical statement. If it is a logical statement, determine whether it is true or false.

1. Tiger Woods won the Master’s golf championship at least five times.
2. Please, sit down over there.
3. All cats dislike dogs.

#### Answer

1. This is a logical statement because it is a complete sentence that makes a claim that can be identified as being true or false. As of 2021, this statement is true: Tiger Woods won the Master’s in 1997, 2001, 2002, 2005 and 2019.
2. This is not a logical statement because, although it is a complete sentence, this request does not make a claim that can be identified as being either true or false.
3. This is a logical statement because it is a complete sentence that makes a claim that can be identified as being true or false. This statement is false because some cats do like some dogs.

### ✓ Your Turn 1.1.2.1

Determine whether each of the following sentences represents a logical statement. If it is a logical statement, determine whether it is true or false.

1. The Buffalo Bills defeated the New York Giants in Super Bowl XXV
2. Michael Jackson’s album *Thriller* was released in 1982.
3. Would you like some coffee or tea?

## Representing Statements in Symbolic Form

When analyzing logical arguments that are made of multiple logical statements, symbolic form is used to reduce the amount of writing involved. Symbolic form also helps visualize the relationship between the statements in a more concise way in order to determine the strength or validity of an argument. Each logical statement is represented symbolically as a single lowercase letter, usually starting with the letter  $p$ .

To begin, you will practice how to write a single logical statement in symbolic form. This skill will become more useful as you work with compound statements in later sections.

### ? Exercise 1.1.2.2: Representing Statements Using Symbolic Form

Write each of the following logical statements in symbolic form.

1. Barry Bonds holds the Major League Baseball record for total career home runs.
2. Some mammals live in the ocean.
3. Ruth Bader Ginsburg served on the U.S. Supreme Court from 1993 to 2020.

#### Answer

1.  $p$  : Barry Bonds holds the Major League Baseball record for total career home runs. The statement is labeled with a  $p$ . Once the statement is labeled, use  $p$  as a replacement for the full written statement to write and analyze the argument symbolically.
2.  $q$  : Some mammals live in the ocean. The letter  $q$  is used to distinguish this statement from the statement in question 1, but any lower-case letter may be used.
3.  $r$  : Ruth Bader Ginsburg served on the U.S. Supreme Court from 1993 to 2020. When multiple statements are present in later sections, you will want to be sure to use a different letter for each separate logical statement.

### ✓ Your Turn 1.1.2.2

Write each of the following logical statements in symbolic form.

1. The movie *Gandhi* won the Academy Award for Best Picture in 1982.
2. Soccer is the most popular sport in the world.
3. All oranges are citrus fruits.

### 📌 Who Knew?

Mathematics is not the only language to use symbols to represent thoughts or ideas. The Chinese and Japanese languages use symbols known as Hanzi and Kanji, respectively, to represent words and phrases. At one point, the American musician Prince famously changed his name to a symbol representing love.

[BBC Prince Symbol Article](#)

## Negating Statements

Consider the false statement introduced earlier, “All roses are red.” If someone said to you, “All roses are red,” you might respond with, “Some roses are not red.” You could then strengthen your argument by providing additional statements, such as, “There are also white roses, yellow roses, and pink roses, to name a few.”

The negation of a logical statement has the opposite truth value of the original statement. If the original statement is false, its negation is true, and if the original statement is true, its negation is false. Most logical statements can be negated by simply adding or removing the word *not*. For example, consider the statement, “Emma Stone has green eyes.” The negation of this statement would be, “Emma Stone does not have green eyes.” The table below gives some other examples.

Table 1.1.2.1

| Logical Statement        | Negation                     |
|--------------------------|------------------------------|
| Gordon Ramsey is a chef. | Gordon Ramsey is not a chef. |

| Logical Statement                   | Negation                  |
|-------------------------------------|---------------------------|
| Tony the Tiger does not have spots. | Tony the Tiger has spots. |

The way you phrase your argument can impact its success. If someone presents you with a false statement, the ability to rebut that statement with its negation will provide you with the tools necessary to emphasize the correctness of your position.

### ? Exercise 1.1.2.3: Negating Logical Statements

Write the negation of each logical statement in words.

1. Michael Phelps was an Olympic swimmer.
2. Tom is a cat.
3. Jerry is not a mouse.

#### Answer

1. Add the word *not* to negate the affirmative statement: "Michael Phelps was not an Olympic swimmer."
2. Add the word *not* to negate the affirmative statement: "Tom is not a cat."
3. Remove the word *not* to negate the negative statement: "Jerry is a mouse."

### ✓ Your Turn 1.1.2.3

Write the negation of each logical statement in words.

1. Ted Cruz was not born in Texas
2. Adele has a beautiful singing voice.
3. Leaves convert carbon dioxide to oxygen during the process of photosynthesis.

The symbol for negation, or not, in logic is the tilde,  $\sim$ . So, not  $p$  is represented as  $\sim p$ . To negate a statement symbolically, remove or add a tilde. The negation of not (not  $p$ ) is  $p$ . Symbolically, this equation is  $\sim(\sim p) = p$ .

### ? Exercise 1.1.2.4: Negating Logical Statements Symbolically

Write the negation of each logical statement symbolically.

1.  $p$ : Michael Phelps was an Olympic swimmer.
2.  $r$ : Tom is not a cat.
3.  $\sim q$ : Jerry is not a mouse.

#### Answer

1. To negate an affirmative logical statement symbolically, add a tilde:  $\sim p$ .
2. Because the symbol for this statement is  $r$ , its negation is  $\sim r$ .
3. The symbol for this statement is  $\sim q$ , so to negate it we simply remove the  $\sim$ , because  $\sim(\sim q) = q$ . The answer is  $q$ .

### ✓ Your Turn 1.1.2.4

Write the negation of each logical statement symbolically.

1.  $\sim p$ : Ted Cruz was not born in Texas
2.  $q$ : Adele has a beautiful voice.
3.  $r$ : Leaves convert carbon dioxide to oxygen during the process of photosynthesis.

## Translating Negations Between Words and Symbols

In order to analyze logical arguments, it is important to be able to translate between the symbolic and written forms of logical statements.

### ? Exercise 1.1.2.5

Given the statements:

$r$ : Elmo is a red Muppet.

$p$ : Ketchup is not a vegetable.

1. Write the symbolic form of the statement, "Elmo is not a red Muppet."
2. Translate the statement  $\sim p$  into words.

#### Answer

1. "Elmo is not a red muppet" is the negation of "Elmo is a red muppet," which is represented as  $r$ . Thus, we would write "Elmo is not a red muppet" symbolically as  $\sim r$ .
2. Because  $p$  is the symbol representing the statement, "Ketchup is not a vegetable,"  $\sim p$  is equivalent to the statement, "Ketchup is a vegetable."

### ✓ Your Turn 1.1.2.5

Given the statements:

$r$ : Woody and Buzz Lightyear are best friends.

$\sim p$ : Wonder Woman is not stronger than Captain Marvel.

1. Write the symbolic form of the statement, "Wonder Woman is stronger than Captain Marvel."
2. Translate the statement  $\sim r$  into words

## Expressing Statements with Quantifiers of All, Some, or None

A quantifier is a term that expresses a numerical relationship between two sets or categories. For example, all squares are also rectangles, but only some rectangles are squares, and no squares are circles. In this example, *all*, *some*, and *none* are quantifiers. In a logical argument, the logical statements made to support the argument are called premises, and the judgment made based on the premises is called the conclusion. Logical arguments that begin with specific premises and attempt to draw more general conclusions are called inductive arguments.

Consider, for example, a parent walking with their three-year-old child. The child sees a cardinal fly by and points it out. As they continue on their walk, the child notices a robin land on top of a tree and a duck flying across to land on a pond. The child recognizes that cardinals, robins, and ducks are all birds, then excitedly declares, "All birds fly!" The child has just made an inductive argument. They noticed that three different specific types of birds all fly, then synthesized this information to draw the more general conclusion that all birds can fly. In this case, the child's conclusion is false.

The specific premises of the child's argument can be paraphrased by the following statements:

- Premise: Cardinals are birds that fly.
- Premise: Robins are birds that fly.
- Premise: Ducks are birds that fly.

The general conclusion is: "All birds fly!"

All inductive arguments should include at least three specific premises to establish a pattern that supports the general conclusion. To counter the conclusion of an inductive argument, it is necessary to provide a counter example. The parent can tell the child about penguins or emus to explain why that conclusion is false.

On the other hand, it is usually impossible to prove that an inductive argument is true. So, inductive arguments are considered either strong or weak. Deciding whether an inductive argument is strong or weak is highly subjective and often determined by the background knowledge of the person making the judgment. Most hypotheses put forth by scientists using what is called the "scientific method" to conduct experiments are based on inductive reasoning.

In the following example, we will use quantifiers to express the conclusion of a few inductive arguments.

### ? Exercise 1.1.2.6: Drawing General Conclusions to Inductive Arguments Using Quantifiers

For each series of premises, draw a logical conclusion to the argument that includes one of the following quantifiers: all, some, or none.

1. Squares and rectangles have four sides. A square is a parallelogram, and a rectangle is a parallelogram. What conclusion can be drawn from these premises?
2.  $1 + 2 = 3$ ,  $6 + 7 = 13$ , and  $23 + 24 = 47$ . Of these, 1 and 2, 6 and 7, and 23 and 24 are consecutive integers; 3, 13, and 47 are odd numbers. What conclusion can be drawn from these premises?
3. Sea urchins live in the ocean, and they do not breathe air. Sharks live in the ocean, and they do not breathe air. Eels live in the ocean, and they do not breathe air. What conclusion can be drawn from these premises?

#### Answer

1. The conclusion you would likely come to here is “Some four-sided figures are parallelograms.” However, it would be incorrect to say that all four-sided figures are parallelograms because there are some four-sided figures, such as trapezoids, that are not parallelograms. This is a false conclusion.
2. From these premises, you may draw the conclusion “All sums of two consecutive counting numbers result in an odd number.” Most inductive arguments cannot be proven true, but several mathematical properties can be. If we let  $n$  represent our first counting number, then  $n + 1$  would be the next counting number and  $n + (n + 1) = 2n + 1$ . Because  $2n$  is divisible by two, it is an even number, and if you add one to any even number the result is always an odd number. Thus, the conclusion is true!
3. Based on the premises provided, with no additional knowledge about whales or dolphins, you might conclude “No creatures that live in the ocean breathe air.” Even though this conclusion is false, it still follows from the known premises and thus is a logical conclusion based on the evidence presented. Alternatively, you could conclude “Some creatures that live in the ocean do not breathe air.” The quantifier you choose to write your conclusion with may vary from another person’s based on how persuasive the argument is. There may be multiple acceptable answers.

### ✓ Your Turn 1.1.2.6

For each series of premises, draw a logical conclusion to the argument that includes one of the following quantifiers: all, some, or none.

1.  $1 + 2 = 3$ ,  $5 + 6 = 11$ , and  $14 + 15 = 29$ . Of these, 1 and 2 are consecutive integers, 5 and 6 are consecutive integers, and 14 and 15 are consecutive integers. Also, their sums, 3, 11, and 29 are all prime numbers. Prime numbers are positive integers greater than one that are only divisible by one and the number itself. What conclusion can you draw from these premises?
2. A robin is a bird that lays blue eggs. A chicken is a bird that typically lays white and brown eggs. An ostrich is a bird that lays exceptionally large eggs. If a bird lays eggs, then they do not give live birth to their young. What conclusion can you draw from these premises?
3. All parallelograms have four sides. All rectangles are parallelograms. All squares are rectangles. What additional conclusion can you make about squares from these premises?

#### 📌 Checkpoint

*It is not possible to prove definitively that an inductive argument is true or false in most cases.*

### Negating Statements Containing Quantifiers

Recall that the negation of a statement will have the opposite truth value of the original statement. There are four basic forms that logical statements with quantifiers take on.

- All  $A$  are  $B$ .
- Some  $A$  are  $B$ .
- No  $A$  are  $B$ .

- Some  $A$  are not  $B$ .

The negation of logical statements that use the quantifiers *all*, *some*, or *none* is a little more complicated than just adding or removing the word *not*.

For example, consider the logical statement, "All oranges are citrus fruits." This statement expresses as a subset relationship. The set of oranges is a subset of the set of citrus fruit. This means that there are no oranges that are outside the set of citrus fruit. The negation of this statement would have to break the subset relationship. To do this, you could say, "At least one orange is not a citrus fruit." Or, more concisely, "Some oranges are not citrus fruit." It is tempting to say "No oranges are citrus fruit," but that would be incorrect. Such a statement would go beyond breaking the subset relationship, to stating that the two sets have nothing in common. The negation of " $A$  is a subset of  $B$ " would be to state that " $A$  is not a subset of  $B$ ," as depicted by the Venn diagram in Figure 1.1.2.3

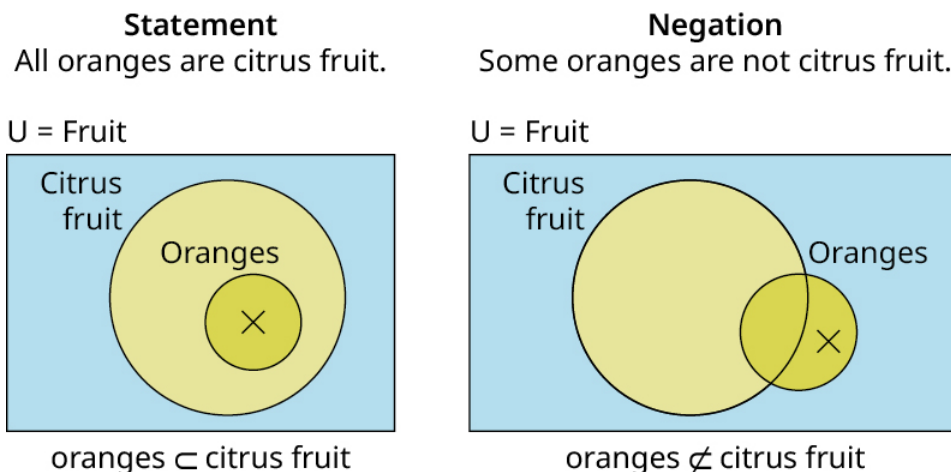


Figure 1.1.2.3: The statement, "All oranges are citrus fruit," is true, so its negation, "Some oranges are not citrus fruit," is false.

Now, consider the statement, "No apples are oranges." This statement indicates that the set of apples is disjoint from the set of oranges. The negation must state that the two are not disjoint sets, or that the two sets have a least one member in common. Their intersection is not empty. The negation of the statement, "A intersection B is the empty set," is the statement that "A intersection B is not empty," as depicted in the Venn diagram in Figure 1.1.2.4

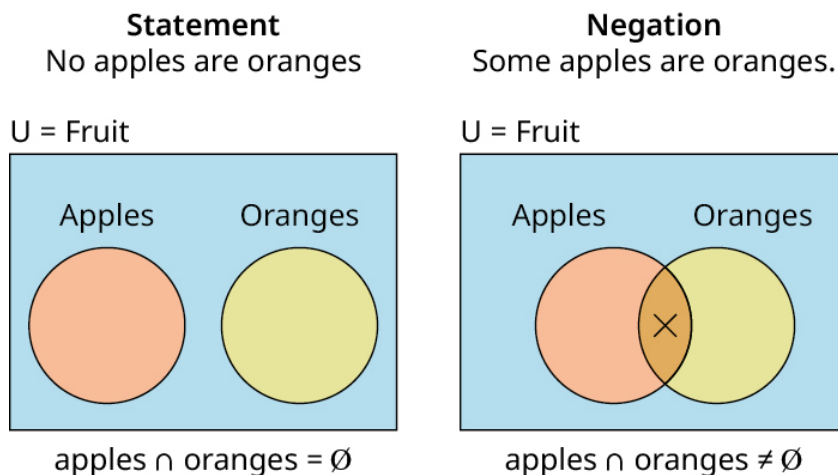


Figure 1.1.2.4: The negation of the true statement "No apples are oranges," is the false statement, "Some apples are oranges."

Table 1.1.2.2 summarizes the four different forms of logical statements involving quantifiers and the forms of their associated negations, as well as the meanings of the relationships between the two categories or sets  $A$  and  $B$ .

Table 1.1.2.2

| Logical Statements with Quantifiers | Negation of Logical Statements w/Quantifiers |
|-------------------------------------|----------------------------------------------|
|                                     |                                              |

| Logical Statements with Quantifiers                                                                                               | Negation of Logical Statements w/Quantifiers                                                                                           |
|-----------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------|
| Form: All $A$ are $B$ .<br>Means: $A$ is a subset of $B$ , $A \subset B$ .<br>All zebras have stripes. (True)                     | Form: Some $A$ are not $B$ .<br>Means: $A$ is not a subset of $B$ , $A \not\subset B$ .<br>Some zebras do not have stripes. (False)    |
| Form: Some $A$ are $B$ .<br>Means: $A$ intersection $B$ is not empty, $A \cap B \neq \emptyset$ .<br>Some fish are sharks. (True) | Form: No $A$ are $B$ .<br>Means: $A$ intersection $B$ is empty, $A \cap B = \emptyset$ .<br>No fish are sharks. (False)                |
| Form: No $A$ are $B$ .<br>Means: $A$ intersection $B$ is empty, $A \cap B = \emptyset$ .<br>No trees are evergreens. (False)      | Form: Some $A$ are $B$ .<br>Means: $A$ intersection $B$ is not empty, $A \cap B \neq \emptyset$ .<br>Some trees are evergreens. (True) |
| Form: Some $A$ are not $B$ .<br>Means: $A$ is not a subset of $B$ , $A \not\subset B$ .<br>Some horses are not mustangs. (True)   | Form: All $A$ are $B$ .<br>Means: $A$ is a subset of $B$ , $A \subset B$ .<br>All horses are mustangs. (False)                         |

We covered sets in great detail in Chapter 1. To review, " $A$  is a subset of  $B$ " means that every member of set  $A$  is also a member of set  $B$ . The intersection of two sets  $A$  and  $B$  is the set of all elements that they share in common. If  $A$  intersection  $B$  is the empty set, then sets  $A$  and  $B$  do not have any elements in common. The two sets do not overlap. They are disjoint.



### ? Exercise 1.1.2.7: Negating Statements Containing Quantifiers *All*, *Some*, or *None*

Given the statements:

$p$ : All leopards have spots.

$r$ : Some apples are red.

$s$ : No lemons are sweet.

Write each of the following symbolic statements in words.

1.  $\sim p$
2.  $\sim r$
3.  $\sim s$

#### Answer

1. The statement "All leopards have spots" is  $p$  and has the form "All  $A$  are  $B$ ." According to Table 1.1.2.2 the negation will have the form "Some  $A$  are not  $B$ ." The negation of  $p$  is the statement, "Some leopards do not have spots."
2. The statement "Some apples are red" has the form "Some  $A$  are  $B$ ." This indicates that the categories  $A$  and  $B$  overlap or intersect. According to Table 1.1.2.2 the negation will have the form, "No  $A$  are  $B$ ," indicating that  $A$  and  $B$  do not intersect. This results in the opposite truth value of the original statement, so the negation of "Some apples are red" is the statement: "No apples are red."

3. Because  $s$  is the statement: "No lemons are sweet,"  $s$  is asserting that the set of lemons does not intersect with the set of sweet things. The negation of  $s$ ,  $\sim s$ , must make the opposite claim. It must indicate that the set of lemons intersects with the set of sweet things. This means at least one lemon must be sweet. The statement, "Some lemons are sweet" is  $\sim s$ . The negation of the statement, "No  $A$  are  $B$ ," is the statement, "Some  $A$  are  $B$ ," as indicated in Table 1.1.2.2

### ✓ Your Turn 1.1.2.7

Given the statements:

$\sim p$  : Some apples are not sweet.

$r$  : No triangles are squares.

$s$  : Some vegetables are green. Write each of the following symbolic statements in words.

1.  $p$
2.  $\sim r$
3.  $\sim s$

### Check Your Understanding

1. A \_\_\_\_\_ is a complete sentence that makes a claim that may be either true or false.
2. The \_\_\_\_\_ of a logical statement has the opposite truth value of the original statement.
3. If  $p$  represents the logical statement, "Marigolds are yellow flowers," then \_\_\_\_\_ represents the statement, "Marigolds are not yellow flowers."
4. The statement  $\sim(\sim p)$  has the same truth value as the statement \_\_\_\_\_.
5. The logical statements used to support the conclusion of an argument are called \_\_\_\_\_.
6. \_\_\_\_\_ arguments attempt to draw a general conclusion from specific premises.
7. All, some, and none are examples of \_\_\_\_\_, words that assign a numerical relationship between two or more groups.
8. The negation of the statement, "All giraffes are tall," is \_\_\_\_\_.

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## 1.1.3: Compound Statements



1.1.3.1: A person seeking their driver's license must pass two tests. A compound statement can be used to explain performance on both tests at once. (credit: modification of work "Drivers License -Teen driver" by State Farm/Flickr, CC BY 2.0)

### Learning Objectives

After completing this section, you should be able to:

1. Translate compound statements into symbolic form.
2. Translate compound statements in symbolic form with parentheses into words.
3. Apply the dominance of connectives.

Suppose your friend is trying to get a license to drive. In most places, they will need to pass some form of written test proving their knowledge of the laws and rules for driving safely. After passing the written test, your friend must also pass a road test to demonstrate that they can perform the physical task of driving safely within the rules of the law.

Consider the statement: "My friend passed the written test, but they did not pass the road test." This is an example of a compound statement, a statement formed by using a connective to join two independent clauses or logical statements. The statement, "My friend passed the written test," is an independent clause because it is a complete thought or idea that can stand on its own. The second independent clause in this compound statement is, "My friend did not pass the road test." The word "but" is the connective used to join these two statements together, forming a compound statement. So, did your friend acquire their driving license?.

This section introduces common logical connectives and their symbols, and allows you to practice translating compound statements between words and symbols. It also explores the order of operations, or dominance of connectives, when a single compound statement uses multiple connectives.

### Common Logical Connectives

Understanding the following logical connectives, along with their properties, symbols, and names, will be key to applying the topics presented in this chapter. The chapter will discuss each connective introduced here in more detail.

The joining of two logical statements with the word "and" or "but" forms a compound statement called a conjunction. In logic, for a conjunction to be true, all the independent logical statements that make it up must be true. The symbol for a conjunction is  $\wedge$ . Consider the compound statement, "Derek Jeter played professional baseball for the New York Yankees, and he was a shortstop." If  $p$  represents the statement, "Derrick Jeter played professional baseball for the New York Yankees," and if  $q$  represents the statement, "Derrick Jeter was a short stop," then the conjunction will be written symbolically as  $p \wedge q$ .

The joining of two logical statements with the word "or" forms a compound statement called a disjunction. Unless otherwise specified, a disjunction is an inclusive *or* statement, which means the compound statement formed by joining two independent clauses with the word *or* will be true if a least one of the clauses is true. Consider the compound statement, "The office manager ordered cake for for an employee's birthday or they ordered ice cream." This is a disjunction because it combines the independent clause, "The office manager ordered cake for an employee's birthday," with the independent clause, "The office manager ordered ice cream," using the connective, *or*. This disjunction is true if the office manager ordered only cake, only ice cream, or they ordered both cake and ice cream. Inclusive *or* means you can have one, or the other, or both!

Joining two logical statements with the word *implies*, or using the phrase “if *first statement*, then *second statement*,” is called a conditional or *implication*. The clause associated with the “if” statement is also called the hypothesis or *antecedent*, while the clause following the “then” statement or the word *implies* is called the **conclusion** or *consequent*. The conditional statement is like a one-way contract or promise. The only time the conditional statement is false, is if the hypothesis is true and the conclusion is false. Consider the following conditional statement, “If Pedro does his homework, then he can play video games.” The hypothesis/antecedent is the statement following the word *if*, which is “Pedro does/did his homework.” The conclusion/consequent is the statement following the word *then*, which is “Pedro can play his video games.”

Joining two logical statements with the connective phrase “if and only if” is called a biconditional. The connective phrase “if and only if” is represented by the symbol,  $\leftrightarrow$ . In the biconditional statement,  $p \leftrightarrow q$ ,  $p$  is called the hypothesis or antecedent and  $q$  is called the conclusion or consequent. For a biconditional statement to be true, the truth values of  $p$  and  $q$  must match. They must both be true, or both be false.

The table below lists the most common connectives used in logic, along with their symbolic forms, and the common names used to describe each connective.

Table 1.1.3.1

| Connective              | Symbol            | Name                      |
|-------------------------|-------------------|---------------------------|
| and<br>but              | $\wedge$          | conjunction               |
| or                      | $\vee$            | disjunction, inclusive or |
| not                     | $\sim$            | negation                  |
| if . . . , then implies | $\rightarrow$     | conditional, implication  |
| if and only if          | $\leftrightarrow$ | biconditional             |

#### Checkpoint

*These connectives, along with their names, symbols, and properties, will be used throughout this chapter and should be memorized.*

#### Exercise 1.1.3.1

##### Associate Connectives with Symbols and Names

For each of the following connectives, write its name and associated symbol.

1. or
2. implies
3. but

##### Answer

A compound statement formed with the connective word *or* is called a disjunction, and it is represented by the  $\vee$  symbol.

A compound statement formed with the connective word *implies* or phrase “if . . . , then” is called a conditional statement or implication and is represented by the  $\rightarrow$  symbol.

A compound statement formed with the connective words *but* or *and* is called a conjunction, and it is represented by the  $\wedge$  symbol.

#### Your Turn 1.1.3.1

For each connective write its name and associated symbol.

1. not

2. and
3. if and only if

## Translating Compound Statements to Symbolic Form

To translate a compound statement into symbolic form, we take the following steps.

1. Identify and label all independent affirmative logical statements with a lower case letter, such as  $p$ ,  $q$ , or  $r$ .
2. Identify and label any negative logical statements with a lowercase letter preceded by the negation symbol, such as  $\sim p$ ,  $\sim q$ , or  $\sim r$ .
3. Replace the connective words with the symbols that represent them, such as  $\wedge$ ,  $\vee$ ,  $\rightarrow$ , or  $\leftrightarrow$ .

Consider the previous example of your friend trying to get their driver's license. Your friend passed the written test, but they did not pass the road test. Let  $p$  represent the statement, "My friend passed the written test." And, let  $\sim q$  represent the statement, "My friend did not pass the road test." Because the connective *but* is logically equivalent to the word *and*, the symbol for *but* is the same as the symbol for *and*; replace *but* with the symbol  $\wedge$ . The compound statement is symbolically written as:  $p \wedge \sim q$ . My friend passed the written test, but they did not pass the road test.

### Checkpoint

*When translating English statements into symbolic form, do not assume that every connective "and" means you are dealing with a compound statement. The statement, "The stripes on the U.S. flag are red and white," is a simple statement. The word "white" doesn't express a complete thought, so it is not an independent clause and does not get its own symbol. This statement should be represented by a single letter, such as  $s$ : The stripes on the U.S. flag are red and white.*

### ? Exercise 1.1.3.2: Translating Compound Statements into Symbolic Form

Let  $p$  represent the statement, "It is a warm sunny day," and let  $q$  represent the statement, "the family will go to the beach." Write the symbolic form of each of the following compound statements.

1. If it is a warm sunny day, then the family will go to the beach.
2. The family will go to the beach, and it is a warm sunny day.
3. The family will not go to the beach if and only if it is not a warm sunny day.
4. The family not go to the beach, or it is a warm sunny day.

#### Answer

1. Replace "it is a warm sunny day" with  $p$ . Replace "the family will go to the beach." with  $q$ . Next, because the connective is if ..., then place the conditional symbol,  $\rightarrow$ , between  $p$  and  $q$ . The compound statement is written symbolically as:  $p \rightarrow q$ .
2. Replace "The family will go to the beach" with  $q$ . Replace "it is a warm sunny day." with  $p$ . Next, because the connective is and, place the  $\wedge$  symbol between  $q$  and  $p$ . The compound statement is written symbolically as:  $q \wedge p$ .
3. Replace "The family will not go to the beach." with  $\sim q$ . Replace "it is not a warm sunny day" with  $\sim p$ . Next, because the connective is or, if and only if, place the biconditional symbol,  $\leftrightarrow$  between  $\sim q$  and  $\sim p$ . The compound statement is written symbolically as:  $\sim q \leftrightarrow \sim p$ .
4. Replace "The family will not go to the beach" with  $\sim q$ . Replace "it is a warm sunny day" with  $p$ . Next, because the connective is or, place the  $\vee$  symbol between  $\sim q$  and  $p$ . The compound statement is written symbolically as:  $\sim q \vee p$ .

### ✓ Your Turn 1.1.3.2

Let  $p$  represent the statement, "Last night it snowed," and let  $q$  represent the statement, "Today we will go skiing." Write the symbolic form of each of the following compound statements:

1. Today we will go skiing, but last night it did not snow.
2. Today we will go skiing if and only if it snowed last night.
3. Last night is snowed or today we will not go skiing.
4. If it snowed last night, then today we will go skiing.

## Translating Compound Statements in Symbolic Form with Parentheses into Words

When parentheses are written in a logical argument, they group a compound statement together just like when calculating numerical or algebraic expressions. Any statement in parentheses should be treated as a single component of the expression. If multiple parentheses are present, work with the inner most parentheses first.

Consider your friend's struggles to get their license to drive. Let  $p$  represent the statement, "My friend passed the written test," let  $q$  represent the statement, "My friend passed the road test," and let  $r$  represent the statement, "My friend received a driver's license." The statement  $(p \wedge q) \rightarrow r$  can be translated into words as follows: the statement  $p \wedge q$  is grouped together to form the hypothesis of the conditional statement and  $r$  is the conclusion. The conditional statement has the form "if  $p \wedge q$ , then  $r$ ." Therefore, the written form of this statement is: "If my friend passed the written test and they passed the road test, then my friend received a driver's license."

Sometimes a compound statement within parentheses may need to be negated as a group. To accomplish this, add the phrase, "it is not the case that" before the translation of the phrase in parentheses. For example, using  $p$ ,  $q$ , and  $r$  of your friend obtaining a license, let's translate the statement  $\sim (p \wedge q) \rightarrow \sim r$  into words.

In this case, the hypothesis of the conditional statement is  $\sim (p \wedge q)$  and the conclusion is  $\sim r$ . To negate the hypothesis, add the phrase "it is not the case" before translating what is in parentheses. The translation of the hypothesis is the sentence, "It is not the case that my friend passed the written test and they passed the road test," and the translation of the conclusion is, "My friend did not receive a driver's license." So, a translation of the complete conditional statement,  $\sim (p \wedge q) \rightarrow \sim r$  is: "If it is not the case that my friend passed the written test and the road test, then my friend did not receive a driver's license."

### Checkpoint

*It is acceptable to interchange proper names with pronouns and remove repeated phrases to make the written statement more readable, as long the meaning of the logical statement is not changed.*



### Compound Statements & Connectives

| Connective | Symbol   |
|------------|----------|
| and        | $\wedge$ |
| or         | $\vee$   |
| not        | $\sim$   |

Logic Part 1B: Compound Statements, Connectives and Symbols

### Exercise 1.1.3.3

Let  $p$  represent the statement, "My child finished their homework," let  $q$  represent the statement, "My child cleaned her room," let  $r$  represent the statement, "My child played video games," and let  $s$  represent the statement, "My child streamed a movie." Translate each of the following symbolic statements into words.

1.  $\sim (p \wedge q)$
2.  $(p \wedge q) \rightarrow (r \vee s)$
3.  $\sim (r \vee s) \leftrightarrow \sim (p \wedge q)$

**Answer**

1. Replace  $\neg$  with "It is not the case," and  $\wedge$  with "and." One possible translation is: "It is not the case that my child finished their homework and cleaned their room."
2. The hypothesis of the conditional statement is, "My child finished their homework and cleaned their room." The conclusion of the conditional statement is, "My child played video games or streamed a movie." One possible translation of the entire statement is: "If my child finished their homework and cleaned their room, then they played video games or streamed a movie."
3. The hypothesis of the biconditional statement is  $\sim (r \vee s)$  and is written in words as: "It is not the case that my child played video games or streamed a movie." The conclusion of the biconditional statement is  $\sim (p \wedge q)$ , which translates to: "It is not the case that my child finished their homework and cleaned their room." Because the biconditional,  $\leftrightarrow$  translates to if and only if, one possible translation of the statement is: "It is not the case that my child played video games or streamed a movie if and only if it is not the case that my child finished their homework and cleaned their room."

### ✓ Your Turn 1.1.3.3

Let  $p$  represent the statement, "My roommates ordered pizza," let  $q$  represent the statement, "I ordered wings," and let  $r$  be the statement, "Our friends came over to watch the game." Translate the following statements into words.

1.  $\sim r \rightarrow (p \vee q)$
2.  $(p \wedge q) \rightarrow r$
3.  $\sim (p \vee r)$

## The Dominance of Connectives

The order of operations for working with algebraic and arithmetic expressions provides a set of rules that allow consistent results. For example, if you were presented with the problem  $1 + 3 \times 2$ , and you were not familiar with the order of operation, you might assume that you calculate the problem from left to right. If you did so, you would add 1 and 3 to get 4, and then multiply this answer by 2 to get 8, resulting in an incorrect answer. Try inputting this expression into a scientific calculator. If you do, the calculator should return a value of 7, not 8.

Use the Scientific Calculator found under the Resources tab to the far left

The order of operations for algebraic and arithmetic operations states that all multiplication must be applied prior to any addition. Parentheses are used to indicate which operation—addition or multiplication—should be done first. Adding parentheses can change and/or clarify the order. The parentheses in the expression  $1 + (3 \times 2)$  indicate that 3 should be multiplied by 2 to get 6, and then 1 should be added to 6 to get 7:  $1 + (3 \times 2) = 7$ .

As with algebraic expressions, there is a set of rules that must be applied to compound logical statements in order to evaluate them with consistent results. This set of rules is called the dominance of connectives. When evaluating compound logical statements, connectives are evaluated from least dominant to most dominant as follows:

1. Parentheses are the least dominant connective. So, any expression inside parentheses must be evaluated first. Add as many parentheses as needed to any statement to specify the order to evaluate each connective.
2. Next, we evaluate negations.
3. Then, we evaluate conjunctions and disjunctions from left to right, because they have equal dominance.
4. After evaluating all conjunctions and disjunctions, we evaluate conditionals.
5. Lastly, we evaluate the most dominant connective, the biconditional. If a statement includes multiple connectives of equal dominance, then we will evaluate them from left to right.

See Figure 1.1.3.2 for a visual breakdown of the dominance of connectives.

| Dominance                            | Connective              | Symbol            | Evaluate                                                                                                          |
|--------------------------------------|-------------------------|-------------------|-------------------------------------------------------------------------------------------------------------------|
| Least Dominant<br>↓<br>Most Dominant | Parentheses             | ( )               | First<br>↓<br>Left to right or add parentheses to specify order because or/and have equal dominance.<br>↓<br>Last |
|                                      | Negation                | $\sim$            |                                                                                                                   |
|                                      | Disjunction/Conjunction | $\vee, \wedge$    |                                                                                                                   |
|                                      | Conditional             | $\rightarrow$     |                                                                                                                   |
|                                      | Biconditional           | $\leftrightarrow$ |                                                                                                                   |

Figure 1.1.3.2

Let's revisit your friend's struggles to get their driver's license. Let  $p$  represent the statement, "My friend passed the written test," let  $q$  represent the statement, "My friend passed the road test," and let  $r$  represent the statement, "My friend received a driver's license." Let's use the dominance of connectives to determine how the compound statement  $p \wedge \sim q \rightarrow r$  should be evaluated.

**Step 1:** There are no parentheses, which is least dominant of all connectives, so we can skip over that.

**Step 2:** Because negation is the next least dominant, we should evaluate  $\sim q$  first. We could add parentheses to the statement to make it clearer which connecting needs to be evaluated first:  $p \wedge \sim q \rightarrow r$  is equivalent to  $p \wedge (\sim q) \rightarrow r$ .

**Step 3:** Next, we evaluate the conjunction,  $\wedge$ .  $p \wedge (\sim q) \rightarrow r$  is equivalent to  $(p \wedge (\sim q)) \rightarrow r$ .

**Step 4:** Finally, we evaluate the conditional,  $\rightarrow$ , as this is the most dominant connective in this compound statement.

#### Who Knew?

When using spreadsheet applications, like Microsoft Excel or Google Sheets, have you noticed that core functions, such as sum, average, or rate, have the same name and syntax for use? This is not a coincidence. Various standards organizations have developed requirements that software developers must implement to meet the conditions set by governments and large customers worldwide. The OpenDocument Format OASIS Standard enables transferring data between different office productivity applications and was approved by the International Standards Organization (ISO) and IEC on May 6, 2006.

Prior to adopting these standards, a government entity, business, or individual could lose access to their own data simply because it was saved in a format no longer supported by a proprietary software product—even data they were required by law to preserve, or data they simply wished to maintain access to.

Just as rules for applying the dominance of connectives help maintain understanding and consistency when writing and interpreting compound logical statements and arguments, the standards adopted for database software maintain global integrity and accessibility across platforms and user bases.

#### ? Exercise 1.1.3.4

For each of the following compound logical statements, add parentheses to indicate the order to evaluate the statement. Recall that parentheses are evaluated innermost first.

- $p \wedge \sim q \vee r$
- $q \rightarrow \sim p \wedge r$
- $\sim (p \vee q) \leftrightarrow \sim p \wedge \sim q$

#### Answer

- Because negation is the least dominant connective, we evaluate it first:  $p \wedge (\sim q) \vee r$ . Because conjunction and disjunction have the same dominance, we evaluate them left to right. So, we evaluate the conjunction next, as indicated by the additional set of parentheses:  $(p \wedge (\sim q)) \vee r$ . The only remaining connective is the disjunction, so it is

evaluated last, as indicated by the third set of parentheses. The complete solution is:  $((p \wedge (\sim q)) \vee r)$  .

2. Negation has the lowest dominance, so it is evaluated first:  $q \rightarrow (\sim p) \wedge r$  . The remaining connectives are the conditional and the conjunction. Because conjunction has a lower precedence than the conditional, it is evaluated next, as indicated by the second set of parentheses:  $q \rightarrow ((\sim p) \wedge r)$  . The last step is to evaluate the conditional, as indicated by the third set of parentheses:  $(q \rightarrow ((\sim p) \wedge r))$  .
3. This statement is known as De Morgan's Law for the negation of a disjunction. It is always true. Section 2.6 of this chapter will explore De Morgan's Laws in more detail.
  - First, we evaluate the negations on the right side of the biconditional prior to the conjunction.
  - Then, we evaluate the disjunction on the left side of the biconditional, followed by the negation of the disjunction on the left side.
  - Lastly, after completely evaluating each side of the biconditional, we evaluate the biconditional. It does not matter which side you begin with.
 The final solution is:  $(\sim (p \vee q)) \leftrightarrow ((\sim p) \wedge (\sim q))$  .

#### ✓ Example 1.1.3.4

For each of the following compound logical statements, add parentheses to indicate the order in which to evaluate the statement. Recall that parentheses are evaluated innermost first.

1.  $p \vee q \wedge \sim r$
2.  $\sim p \rightarrow q \vee r$
3.  $\sim p \vee \sim q \leftrightarrow \sim (p \wedge q)$

#### 📌 WORK IT OUT: Logic Terms and Properties – Matching Game

**Materials:** For every group of four students, include 30 cards (game, trading, or playing cards), 30 individual clear plastic gaming card sleeves, and 30 card-size pieces of paper. Prepare a list of 60 questions and answers ahead of time related to definitions and problems in Statements and Quantifiers and Compound Statements. Provide each group of four students with 20 questions and their associated answers. Instruct each group to select 15 of the 20 questions, and then, for each problem selected, create one card with the question and one card with the answer. Instruct the groups to then place each problem and answer in a separate card sleeve. Once they create 15 problem cards and 15 answer cards, have students shuffle the set of cards.

To play the game, groups should exchange their card decks to ensure no team begins playing with the deck that they created. Split each four-person group into teams of two students. After shuffling the cards, one team lays cards face down on their desk in a five-by-six grid. The other team will go first.

Each player will have a turn to try matching the question with the correct answer by flipping two cards to the face up position. If a team makes a match, they get to flip another two cards; if they do not get a match, they flip the cards face down and it is the other team's turn to flip over two cards. The game continues in this manner until teams match all question cards with their corresponding answer cards. The team with the most set of matching cards wins.

In the first module of this chapter, we evaluated the truth value of negations. In the following modules, we will discuss how to evaluate conjunctions, disjunctions, conditionals, and biconditionals, and then evaluate compound logical statements using truth tables and our knowledge of the dominance of connectives.

#### Check Your Understanding

1. A \_\_\_\_\_ is a logical statement formed by combining two or more statements with connecting words, such as and, or, but, not, and if ..., then.
2. A \_\_\_\_\_ is a word or symbol used to join two or more logical statements together to form a compound statement.
3. The most dominant connective is the \_\_\_\_\_.
4. \_\_\_\_\_ are used to specify which logical connective should be evaluated first when evaluating a compound statement.
5. Both \_\_\_\_\_ and \_\_\_\_\_ have equal dominance and are evaluated from left to right when no parentheses are present in a compound logical statement.

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## SECTION OVERVIEW

### 1.2: Truth Tables

#### 1.2.1: Truth Tables- Conjunction (and), Disjunction (or), Negation (not)

#### 1.2.2: Truth Tables- Conditional, Biconditional

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### 1.2.1: Truth Tables- Conjunction (and), Disjunction (or), Negation (not)

Before we focus on truth tables, we're going to introduce some symbols that are commonly used for *and*, *or*, and *not*.

#### Symbols

The symbol  $\wedge$  is used for and:  $A$  and  $B$  is notated  $A \wedge B$

The symbol  $\vee$  is used for or:  $A$  or  $B$  is notated  $A \vee B$

The symbol  $\sim$  is used for not: not  $A$  is notated  $\sim A$

You can remember the first two symbols by relating them to the shapes for the union and intersection.  $A \wedge B$  would be the elements that exist in both sets, in  $A \cap B$ . Likewise,  $A \vee B$  would be the elements that exist in either set, in  $A \cup B$ . When we are working with sets, we use the rounded version of the symbols; when we are working with statements, we use the pointy version.

#### ✓ Example 15

Translate each statement into symbolic notation. Let  $P$  represent "I like Pepsi" and let  $C$  represent "I like Coke".

- I like Pepsi or I like Coke.
- I like Pepsi and I like Coke.
- I do not like Pepsi.
- It is not the case that I like Pepsi or Coke.
- I like Pepsi and I do not like Coke.

#### Solution

- $P \vee C$
- $P \wedge C$
- $\sim P$
- $\sim (P \vee C)$
- $P \wedge \sim C$

As you can see, we can use parentheses to organize more complicated statements.

#### ? Try it Now 2

Translate "We have carrots or we will not make soup" into symbols. Let  $C$  represent "we have carrots" and let  $S$  represent "we will make soup".

#### Answer

$$C \vee \sim S$$

Because complex Boolean statements can get tricky to think about, we can create a **truth table** to keep track of what truth values for the simple statements make the complex statement true and false.

#### Truth Table

A table showing what the resulting truth value of a complex statement is for all the possible truth values for the simple statements.

#### ✓ Example 16

Suppose you're picking out a new couch, and your significant other says "get a sectional *or* something with a chaise".

This is a complex statement made of two simpler conditions: "is a sectional", and "has a chaise". For simplicity, let's use  $S$  to designate "is a sectional", and  $C$  to designate "has a chaise".

A truth table for this situation would look like this:

| $S$ | $C$ | $S \text{ or } C$ |
|-----|-----|-------------------|
| T   | T   | T                 |
| T   | F   | T                 |
| F   | T   | T                 |
| F   | F   | F                 |

In the table, T is used for true, and F for false. In the first row, if  $S$  is true and  $C$  is also true, then the complex statement “ $S$  or  $C$ ” is true. This would be a sectional that also has a chaise, which meets our desire. (Remember that *or* in logic is not exclusive; if the couch has both features, it meets the condition.)

In the previous example about the couch, the truth table was really just summarizing what we already know about how the *or* statement work. The truth tables for the basic *and*, *or*, and *not* statements are shown below.

### Basic truth tables

#### Conjunction

| $A$ | $B$ | $A \wedge B$ |
|-----|-----|--------------|
| T   | T   | T            |
| T   | F   | F            |
| F   | T   | F            |
| F   | F   | F            |

#### Disjunction

| $A$ | $B$ | $A \vee B$ |
|-----|-----|------------|
| T   | T   | T          |
| T   | F   | T          |
| F   | T   | T          |
| F   | F   | F          |

#### Negation

| $A$ | $\sim A$ |
|-----|----------|
| T   | F        |
| F   | T        |

Truth tables really become useful when we analyze more complex Boolean statements.

### ✓ Example 17

Create a truth table for the statement  $A \vee \sim B$

#### Solution

When we create the truth table, we need to list all the possible truth value combinations for  $A$  and  $B$ . Notice how the first column contains 2 Ts followed by 2 Fs, and the second column alternates T, F, T, F. This pattern ensures that all 4 combinations are considered.

| $A$ | $B$ |
|-----|-----|
| T   | T   |
| T   | F   |
| F   | T   |
| F   | F   |

After creating columns with those initial values, we create a third column for the expression  $\sim B$ . Now we will temporarily ignore the column for  $A$  and write the truth values for  $\sim B$

| $A$ | $B$ | $\sim B$ |
|-----|-----|----------|
| T   | T   | F        |
| T   | F   | T        |
| F   | T   | F        |
| F   | F   | T        |

Next we can find the truth values of  $A \vee \sim B$ , using the first and third columns.

| $A$ | $B$ | $\sim B$ | $A \vee \sim B$ |
|-----|-----|----------|-----------------|
| T   | T   | F        | T               |
| T   | F   | T        | T               |
| F   | T   | F        | F               |
| F   | F   | T        | T               |

The truth table shows that  $A \vee \sim B$  is true in three cases and false in one case. If you're wondering what the point of this is, suppose it is the last day of the baseball season and two teams, who are not playing each other, are competing for the final playoff spot. Anaheim will make the playoffs if it wins its game or if Boston does not win its game. (Anaheim owns the tie-breaker; if both teams win, or if both teams lose, then Anaheim gets the playoff spot.) If  $A$  = Anaheim wins its game and  $B$  = Boston wins its game, then  $A \vee \sim B$  represents the situation "Anaheim wins its game or Boston does not win its game". The truth table shows us the different scenarios related to Anaheim making the playoffs. In the first row, Anaheim wins its game and Boston wins its game, so it is true that Anaheim makes the playoffs. In the second row, Anaheim wins and Boston does not win, so it is true that Anaheim makes the playoffs. In the third row, Anaheim does not win its game and Boston wins its game, so it is false that Anaheim makes the playoffs. In the fourth row, Anaheim does not win and Boston does not win, so it is true that Anaheim makes the playoffs.

### ? Try it Now 3

Create a truth table for this statement:  $\sim A \wedge B$

**Answer**

| $A$ | $B$ | $\sim A$ | $\sim A \wedge B$ |
|-----|-----|----------|-------------------|
| T   | T   | F        | F                 |
| T   | F   | F        | F                 |
| F   | T   | T        | T                 |
| F   | F   | T        | F                 |

### ✓ Example 18

Create a truth table for the statement  $A \wedge \sim (B \vee C)$

**Solution**

It helps to work from the inside out when creating a truth table, and to create columns in the table for intermediate operations. We start by listing all the possible truth value combinations for  $A$ ,  $B$ , and  $C$ . Notice how the first column contains 4 Ts followed by 4Fs, the second column contains 2Ts, 2Fs, then repeats, and the last column alternates T, F, T, F . . . This pattern ensures that all 8 combinations are considered. After creating columns with those initial values, we create a fourth column for the innermost expression,  $B \vee C$ . Now we will temporarily ignore the column for  $A$  and focus on  $B$  and  $C$ , writing the truth values for  $B \vee C$

| $A$ | $B$ | $C$ |
|-----|-----|-----|
| T   | T   | T   |
| T   | T   | F   |
| T   | F   | T   |
| T   | F   | F   |
| F   | T   | T   |
| F   | T   | F   |
| F   | F   | T   |
| F   | F   | F   |

| $A$ | $B$ | $C$ | $B \vee C$ |
|-----|-----|-----|------------|
| T   | T   | T   | T          |
| T   | T   | F   | T          |
| T   | F   | T   | T          |
| T   | F   | F   | F          |
| F   | T   | T   | T          |
| F   | T   | F   | T          |
| F   | F   | T   | T          |
| F   | F   | F   | F          |

Next we can find the negation of  $B \vee C$ , working off the  $B \vee C$  column we just created. (Ignore the first three columns and simply negate the values in the  $B \vee C$  column.)

| $A$ | $B$ | $C$ | $B \vee C$ | $\sim(B \vee C)$ |
|-----|-----|-----|------------|------------------|
| T   | T   | T   | T          | F                |
| T   | T   | F   | T          | F                |
| T   | F   | T   | T          | F                |
| T   | F   | F   | F          | T                |
| F   | T   | T   | T          | F                |
| F   | T   | F   | T          | F                |
| F   | F   | T   | T          | F                |
| F   | F   | F   | F          | T                |

Finally, we find the values of  $A$  and  $\sim(B \vee C)$ . (Ignore the second, third, and fourth columns.)

| $A$ | $B$ | $C$ | $B \vee C$ | $\sim(B \vee C)$ | $A \wedge \sim(B \vee C)$ |
|-----|-----|-----|------------|------------------|---------------------------|
| T   | T   | T   | T          | F                | F                         |
| T   | T   | F   | T          | F                | F                         |
| T   | F   | T   | T          | F                | F                         |
| T   | F   | F   | F          | T                | T                         |
| F   | T   | T   | T          | F                | F                         |
| F   | T   | F   | T          | F                | F                         |
| F   | F   | T   | T          | F                | F                         |
| F   | F   | F   | F          | T                | F                         |

It turns out that this complex expression is true in only one case: when  $A$  is true,  $B$  is false, and  $C$  is false. To illustrate this situation, suppose that Anaheim will make the playoffs if: (1) Anaheim wins, and (2) neither Boston nor Cleveland wins. IFF is the only scenario in which Anaheim will make the playoffs.

**? Try it Now 4**

Create a truth table for this statement:  $(\sim A \wedge B) \vee \sim B$

**Answer**

| $A$ | $B$ | $\sim A$ | $\sim A \wedge B$ | $\sim B$ | $(\sim A \wedge B) \vee \sim B$ |
|-----|-----|----------|-------------------|----------|---------------------------------|
| T   | T   | F        | F                 | F        | F                               |
| T   | F   | F        | F                 | T        | T                               |
| F   | T   | T        | T                 | F        | T                               |
| F   | F   | T        | F                 | T        | T                               |

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## 1.2.2: Truth Tables- Conditional, Biconditional

We discussed conditional statements earlier, in which we take an action based on the value of the condition. We are now going to look at another version of a conditional, sometimes called an implication, which states that the second part must logically follow from the first.

### Conditional

A conditional is a logical compound statement in which a statement  $p$ , called the antecedent, implies a statement  $q$ , called the consequent.

A conditional is written as  $p \rightarrow q$  and is translated as "if  $p$ , then  $q$ ".

### ✓ Example 19

The English statement "If it is raining, then there are clouds in the sky" is a conditional statement. It makes sense because if the antecedent "it is raining" is true, then the consequent "there are clouds in the sky" must also be true.

Notice that the statement tells us nothing of what to expect if it is not raining; there might be clouds in the sky, or there might not. If the antecedent is false, then the consequent becomes irrelevant.

### ✓ Example 20

Suppose you order a team jersey online on Tuesday and want to receive it by Friday so you can wear it to Saturday's game. The website says that if you pay for expedited shipping, you will receive the jersey by Friday. In what situation is the website telling a lie?

There are four possible outcomes:

- 1) You pay for expedited shipping and receive the jersey by Friday
- 2) You pay for expedited shipping and don't receive the jersey by Friday
- 3) You don't pay for expedited shipping and receive the jersey by Friday
- 4) You don't pay for expedited shipping and don't receive the jersey by Friday

Only one of these outcomes proves that the website was lying: the second outcome in which you pay for expedited shipping but don't receive the jersey by Friday. The first outcome is exactly what was promised, so there's no problem with that. The third outcome is not a lie because the website never said what would happen if you didn't pay for expedited shipping; maybe the jersey would arrive by Friday whether you paid for expedited shipping or not. The fourth outcome is not a lie because, again, the website didn't make any promises about when the jersey would arrive if you didn't pay for expedited shipping.

It may seem strange that the third outcome in the previous example, in which the first part is false but the second part is true, is not a lie. Remember, though, that if the antecedent is false, we cannot make any judgment about the consequent. The website never said that paying for expedited shipping was the *only* way to receive the jersey by Friday.

### ✓ Example 21

A friend tells you "If you upload that picture to Facebook, you'll lose your job." Under what conditions can you say that your friend was wrong?

There are four possible outcomes:

- 1) You upload the picture and lose your job
- 2) You upload the picture and don't lose your job
- 3) You don't upload the picture and lose your job
- 4) You don't upload the picture and don't lose your job

There is only one possible case in which you can say your friend was wrong: the second outcome in which you upload the picture but still keep your job. In the last two cases, your friend didn't say anything about what would happen if you didn't upload the picture, so you can't say that their statement was wrong. Even if you didn't upload the picture and lost your job anyway, your friend never said that you were guaranteed to keep your job if you didn't upload the picture; you might lose your job for missing a shift or punching your boss instead.

In traditional logic, a conditional is considered true as long as there are no cases in which the antecedent is true and the consequent is false.

#### Truth table for the conditional

| $p$ | $q$ | $p \rightarrow q$ |
|-----|-----|-------------------|
| T   | T   | T                 |
| T   | F   | F                 |
| F   | T   | T                 |
| F   | F   | T                 |

Again, if the antecedent  $p$  is false, we cannot prove that the statement is a lie, so the result of the third and fourth rows is true.

#### ✓ Example 22

Construct a truth table for the statement  $(m \wedge \sim p) \rightarrow r$

##### Solution

We start by constructing a truth table with 8 rows to cover all possible scenarios. Next, we can focus on the antecedent,  $m \wedge \sim p$ .

| $m$ | $p$ | $r$ |
|-----|-----|-----|
| T   | T   | T   |
| T   | T   | F   |
| T   | F   | T   |
| T   | F   | F   |
| F   | T   | T   |
| F   | T   | F   |
| F   | F   | T   |
| F   | F   | F   |

| $m$ | $p$ | $r$ | $\sim p$ |
|-----|-----|-----|----------|
| T   | T   | T   | F        |
| T   | T   | F   | F        |
| T   | F   | T   | T        |
| T   | F   | F   | T        |
| F   | T   | T   | F        |
| F   | T   | F   | F        |
| F   | F   | T   | T        |
| F   | F   | F   | T        |

| $m$ | $p$ | $r$ | $\sim p$ | $m \wedge \sim p$ |
|-----|-----|-----|----------|-------------------|
| T   | T   | T   | F        | F                 |
| T   | T   | F   | F        | F                 |
| T   | F   | T   | T        | T                 |
| T   | F   | F   | T        | T                 |
| F   | T   | T   | F        | F                 |
| F   | T   | F   | F        | F                 |
| F   | F   | T   | T        | F                 |
| F   | F   | F   | T        | F                 |

Now we can create a column for the conditional. Because it can be confusing to keep track of all the Ts and Fs, why don't we copy the column for  $r$  to the right of the column for  $m \wedge \sim p$ ? This makes it a lot easier to read the conditional from left to right.

| $m$ | $p$ | $r$ | $\sim p$ | $m \wedge \sim p$ | $r$ | $(m \wedge \sim p) \rightarrow r$ |
|-----|-----|-----|----------|-------------------|-----|-----------------------------------|
| T   | T   | T   | F        | F                 | T   | T                                 |
| T   | T   | F   | F        | F                 | F   | T                                 |
| T   | F   | T   | T        | T                 | T   | T                                 |
| T   | F   | F   | T        | T                 | F   | F                                 |
| F   | T   | T   | F        | F                 | T   | T                                 |
| F   | T   | F   | F        | F                 | F   | T                                 |
| F   | F   | T   | T        | F                 | T   | T                                 |
| F   | F   | F   | T        | F                 | F   | T                                 |

When  $m$  is true,  $p$  is false, and  $r$  is false- the fourth row of the table-then the antecedent  $m \wedge \sim p$  will be true but the consequent false, resulting in an invalid conditional; every other case gives a valid conditional.

If you want a real-life situation that could be modeled by  $(m \wedge \sim p) \rightarrow r$ , consider this: let  $m$  = we order meatballs,  $p$  = we order pasta, and  $r$  = Rob is happy. The statement  $(m \wedge \sim p) \rightarrow r$  is "if we order meatballs and don't order pasta, then Rob is happy". If  $m$  is true (we order meatballs),  $p$  is false (we don't order pasta), and  $r$  is false (Rob is not happy), then the statement is false, because we satisfied the antecedent but Rob did not satisfy the consequent.

For any conditional, there are three related statements, the converse, the inverse, and the contrapositive.

#### Related Statements

The original conditional is "if  $p$ , then  $q$ "  $p \rightarrow q$

The converse is "if  $q$ , then  $p$ "  $q \rightarrow p$

The inverse is "if not  $p$ , then not  $q$ "  $\sim p \rightarrow \sim q$

The contrapositive is "if not  $q$ , then not  $p$ "  $\sim q \rightarrow \sim p$

#### Example 23

Consider again the conditional "If it is raining, then there are clouds in the sky." It seems reasonable to assume that this is true.

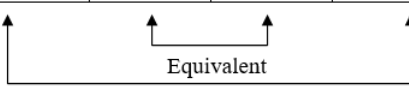
The converse would be "If there are clouds in the sky, then it is raining." This is not always true.

The inverse would be "If it is not raining, then there are not clouds in the sky." Likewise, this is not always true.

The contrapositive would be "If there are not clouds in the sky, then it is not raining." This statement is true, and is equivalent to the original conditional.

Looking at truth tables, we can see that the original conditional and the contrapositive are logically equivalent, and that the converse and inverse are logically equivalent.

|     |     | Conditional       | Converse          | Inverse                     | Contrapositive              |
|-----|-----|-------------------|-------------------|-----------------------------|-----------------------------|
| $p$ | $q$ | $p \rightarrow q$ | $q \rightarrow p$ | $\sim p \rightarrow \sim q$ | $\sim q \rightarrow \sim p$ |
| T   | T   | T                 | T                 | T                           | T                           |
| T   | F   | F                 | T                 | T                           | F                           |
| F   | T   | T                 | F                 | F                           | T                           |
| F   | F   | T                 | T                 | T                           | T                           |



### Equivalence

A conditional statement and its contrapositive are logically equivalent.

The converse and inverse of a conditional statement are logically equivalent.

In other words, the original statement and the contrapositive must agree with each other; they must both be true, or they must both be false. Similarly, the converse and the inverse must agree with each other; they must both be true, or they must both be false.

Be aware that symbolic logic cannot represent the English language perfectly. For example, we may need to change the verb tense to show that one thing occurred before another.

### ✓ Example 24

Suppose this statement is true: “If I eat this giant cookie, then I will feel sick.” Which of the following statements must also be true?

- If I feel sick, then I ate that giant cookie.
- If I don’t eat this giant cookie, then I won’t feel sick.
- If I don’t feel sick, then I didn’t eat that giant cookie.

#### Solution

- This is the converse, which is not necessarily true. I could feel sick for some other reason, such as drinking sour milk.
- This is the inverse, which is not necessarily true. Again, I could feel sick for some other reason; avoiding the cookie doesn’t guarantee that I won’t feel sick.
- This is the contrapositive, which is true, but we have to think somewhat backwards to explain it. If I ate the cookie, I would feel sick, but since I don’t feel sick, I must not have eaten the cookie.

Notice again that the original statement and the contrapositive have the same truth value (both are true), and the converse and the inverse have the same truth value (both are false).

### ? Try it Now 5

“If you microwave salmon in the staff kitchen, then I will be mad at you.” If this statement is true, which of the following statements must also be true?

- If you don’t microwave salmon in the staff kitchen, then I won’t be mad at you.
- If I am not mad at you, then you didn’t microwave salmon in the staff kitchen.
- If I am mad at you, then you microwaved salmon in the staff kitchen.

#### Answer

Choice b is correct because it is the contrapositive of the original statement.

Consider the statement “If you park here, then you will get a ticket.” What set of conditions would prove this statement false?

- You don’t park here and you get a ticket.
- You don’t park here and you don’t get a ticket.
- You park here and you don’t get a ticket.

The first two statements are irrelevant because we don't know what will happen if you park somewhere else. The third statement, however, contradicts the conditional statement "If you park here, then you will get a ticket" because you parked here but didn't get a ticket. This example demonstrates a general rule; the negation of a conditional can be written as a conjunction: "It is not the case that if you park here, then you will get a ticket" is equivalent to "You park here *and* you do not get a ticket."

### The Negation of a Conditional

The negation of a conditional statement is logically equivalent to a conjunction of the antecedent and the negation of the consequent.

$$\sim (p \rightarrow q) \text{ is equivalent to } p \wedge \sim q$$

### Example 25

Which of the following statements is equivalent to the negation of "If you don't grease the pan, then the food will stick to it" ?

- a. I didn't grease the pan and the food didn't stick to it.
- b. I didn't grease the pan and the food stuck to it.
- c. I greased the pan and the food didn't stick to it.

#### Solution

- a. This is correct; it is the conjunction of the antecedent and the negation of the consequent. To disprove that not greasing the pan will cause the food to stick, I have to not grease the pan and have the food not stick.
- b. This is essentially the original statement with no negation; the "if...then" has been replaced by "and".
- c. This essentially agrees with the original statement and cannot disprove it.

### Try it Now 6

"If you go swimming less than an hour after eating lunch, then you will get cramps." Which of the following statements is equivalent to the negation of this statement?

- a. I went swimming more than an hour after eating lunch and I got cramps.
- b. I went swimming less than an hour after eating lunch and I didn't get cramps.
- c. I went swimming more than an hour after eating lunch and I didn't get cramps.

#### Answer

Choice b is equivalent to the negation; it keeps the first part the same and negates the second part.

In everyday life, we often have a stronger meaning in mind when we use a conditional statement. Consider "If you submit your hours today, then you will be paid next Friday." What the payroll rep really means is "If you submit your hours today, then you will be paid next Friday, and if you don't submit your hours today, then you won't be paid next Friday." The conditional statement if  $t$ , then  $p$  also includes the inverse of the statement: if not  $t$ , then not  $p$ . A more compact way to express this statement is "You will be paid next Friday *if and only if* you submit your timesheet today." A statement of this form is called a **biconditional**.

### Biconditional

A biconditional is a logical conditional statement in which the antecedent and consequent are interchangeable.

A biconditional is written as  $p \leftrightarrow q$  and is translated as " $p$  if and only if  $q$ ".

Because a biconditional statement  $p \leftrightarrow q$  is equivalent to  $(p \rightarrow q) \wedge (q \rightarrow p)$ , we may think of it as a conditional statement combined with its converse: if  $p$ , then  $q$  and if  $q$ , then  $p$ . The double-headed arrow shows that the conditional statement goes from left to right and from right to left. A biconditional is considered true as long as the antecedent and the consequent have the same truth value; that is, they are either both true or both false.

### Truth table for the biconditional

| $p$ | $q$ | $p \leftrightarrow q$ |
|-----|-----|-----------------------|
| T   | T   | T                     |
| T   | F   | F                     |
| F   | T   | F                     |
| F   | F   | T                     |

Notice that the fourth row, where both components are false, is true; if you don't submit your timesheet and you don't get paid, the person from payroll told you the truth.

### ✓ Example 26

Suppose this statement is true: "The garbage truck comes down my street if and only if it is Thursday morning." Which of the following statements could be true?

- It is noon on Thursday and the garbage truck did not come down my street this morning.
- It is Monday and the garbage truck is coming down my street.
- It is Wednesday at 11:59PM and the garbage truck did not come down my street today.

#### Solution

- This cannot be true. This is like the second row of the truth table; it is true that I just experienced Thursday morning, but it is false that the garbage truck came.
- This cannot be true. This is like the third row of the truth table; it is false that it is Thursday, but it is true that the garbage truck came.
- This could be true. This is like the fourth row of the truth table; it is false that it is Thursday, but it is also false that the garbage truck came, so everything worked out like it should.

### ? Try it Now 7

Suppose this statement is true: "I wear my running shoes if and only if I am exercising." Determine whether each of the following statements must be true or false.

- I am exercising and I am not wearing my running shoes.
- I am wearing my running shoes and I am not exercising.
- I am not exercising and I am not wearing my running shoes.

#### Answer

Choices a & b are false; c is true.

### ✓ Example 27

Create a truth table for the statement  $(A \vee B) \leftrightarrow \sim C$

#### Solution

Whenever we have three component statements, we start by listing all the possible truth value combinations for  $A$ ,  $B$ , and  $C$ . After creating those three columns, we can create a fourth column for the antecedent,  $A \vee B$ . Now we will temporarily ignore the column for  $C$  and focus on  $A$  and  $B$ , writing the truth values for  $A \vee B$ .

| $A$ | $B$ | $C$ |
|-----|-----|-----|
| T   | T   | T   |
| T   | T   | F   |
| T   | F   | T   |
| T   | F   | F   |
| F   | T   | T   |
| F   | T   | F   |
| F   | F   | T   |
| F   | F   | F   |

| $A$ | $B$ | $C$ | $A \vee B$ |
|-----|-----|-----|------------|
| T   | T   | T   | T          |
| T   | T   | F   | T          |
| T   | F   | T   | T          |
| T   | F   | F   | T          |
| F   | T   | T   | T          |
| F   | T   | F   | T          |
| F   | F   | T   | F          |
| F   | F   | F   | F          |

Next we can create a column for the negation of  $C$ . (Ignore the  $A \vee B$  column and simply negate the values in the  $C$  column.)

| $A$ | $B$ | $C$ | $A \vee B$ | $\sim C$ |
|-----|-----|-----|------------|----------|
| T   | T   | T   | T          | F        |
| T   | T   | F   | T          | T        |
| T   | F   | T   | T          | F        |
| T   | F   | F   | T          | T        |
| F   | T   | T   | T          | F        |
| F   | T   | F   | T          | T        |
| F   | F   | T   | F          | F        |
| F   | F   | F   | F          | T        |

Finally, we find the truth values of  $(A \vee B) \leftrightarrow \sim C$ . Remember, a biconditional is true when the truth value of the two parts match, but it is false when the truth values do not match.

| $A$ | $B$ | $C$ | $A \vee B$ | $\sim C$ | $(A \vee B) \leftrightarrow \sim C$ |
|-----|-----|-----|------------|----------|-------------------------------------|
| T   | T   | T   | T          | F        | F                                   |
| T   | T   | F   | T          | T        | T                                   |
| T   | F   | T   | T          | F        | F                                   |
| T   | F   | F   | T          | T        | T                                   |
| F   | T   | T   | T          | F        | F                                   |
| F   | T   | F   | T          | T        | T                                   |
| F   | F   | T   | F          | F        | T                                   |
| F   | F   | F   | F          | T        | F                                   |

To illustrate this situation, suppose your boss needs you to do either project  $A$  or project  $B$  (or both, if you have the time). If you do one of the projects, you will not get a crummy review ( $C$  is for crummy). So  $(A \vee B) \leftrightarrow \sim C$  means "You will not get a crummy review if and only if you do project  $A$  or project  $B$ ." Looking at a few of the rows of the truth table, we can see how this works out. In the first row,  $A$ ,  $B$ , and  $C$  are all true: you did both projects and got a crummy review, which is not what your boss told you would happen! That is why the final result of the first row is false. In the fourth row,  $A$  is true,  $B$  is false, and  $C$  is false: you did project  $A$  and did not get a crummy review. This is what your boss said would happen, so the final result of this row is true. And in the eighth row,  $A$ ,  $B$ , and  $C$  are all false: you didn't do either project and did not get a crummy

review. This is not what your boss said would happen, so the final result of this row is false. (Even though you may be happy that your boss didn't follow through on the threat, the truth table shows that your boss lied about what would happen.)

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## 1.3: Advanced Truth Tables

Because complex Boolean statements can get tricky to think about, we can create a truth table to keep track of what truth values for the simple statements make the complex statement true and false.

### Definition: Truth Table

A table showing what the resulting truth value of a complex statement is for all the possible truth values for the simple statements.

### ✓ Example 1.3.1

Suppose you're picking out a new couch, and your significant other says "get a sectional *or* something with a chaise".

#### Solution

This is a complex statement made of two simpler conditions: "is a sectional", and "has a chaise". For simplicity, let's use  $S$  to designate "is a sectional", and  $C$  to designate "has a chaise". The condition  $S$  is true if the couch is a sectional.

A truth table for this would look like this:

| $S$ | $C$ | $S \text{ or } C$ |
|-----|-----|-------------------|
| T   | T   | T                 |
| T   | F   | T                 |
| F   | T   | T                 |
| F   | F   | F                 |

In the table, T is used for true, and F for false. In the first row, if  $S$  is true and  $C$  is also true, then the complex statement " $S$  or  $C$ " is true. This would be a sectional that also has a chaise, which meets our desire.

Remember also that *or* in logic is not exclusive; if the couch has both features, it does meet the condition.

To shorthand our notation further, we're going to introduce some symbols that are commonly used for *and*, *or*, and *not*.

### Definition: Symbols

The symbol  $\wedge$  is used for *and*:  $A$  and  $B$  is notated  $A \wedge B$ .

The symbol  $\vee$  is used for *or*:  $A$  or  $B$  is notated  $A \vee B$

The symbol  $\sim$  is used for *not*: not  $A$  is notated  $\sim A$

You can remember the first two symbols by relating them to the shapes for the union and intersection.  $A \wedge B$  would be the elements that exist in both sets, in  $A \cap B$ . Likewise,  $A \vee B$  would be the elements that exist in either set, in  $A \cup B$ .

In the previous example, the truth table was really just summarizing what we already know about how the *or* statement work. The truth tables for the basic *and*, *or*, and *not* statements are shown below.

### Basic Truth Tables

| $A$ | $B$ | $A \wedge B$ |
|-----|-----|--------------|
| T   | T   | T            |
| T   | F   | F            |
| F   | T   | F            |
| F   | F   | F            |

|          |          |                              |
|----------|----------|------------------------------|
| F        | F        | F                            |
| <b>A</b> | <b>B</b> | <b><math>A \vee B</math></b> |
| T        | T        | T                            |
| T        | F        | F                            |
| F        | T        | F                            |
| F        | F        | F                            |

|          |                            |
|----------|----------------------------|
| <b>A</b> | <b><math>\sim A</math></b> |
| T        | F                          |
| F        | T                          |

Truth tables really become useful when analyzing more complex Boolean statements.

### ✓ Example 1.3.2

Create a truth table for the statement  $A \wedge \sim(B \vee C)$ .

#### Solution

It helps to work from the inside out when creating truth tables, and create tables for intermediate operations. We start by listing all the possible truth value combinations for  $A$ ,  $B$ , and  $C$ . Notice how the first column contains 4 Ts followed by 4 Fs, the second column contains 2 Ts, 2 Fs, then repeats, and the last column alternates. This pattern ensures that all combinations are considered. Along with those initial values, we'll list the truth values for the innermost expression,  $B \vee C$ .

|          |          |          |                              |
|----------|----------|----------|------------------------------|
| <b>A</b> | <b>B</b> | <b>C</b> | <b><math>B \vee C</math></b> |
| T        | T        | T        | T                            |
| T        | T        | F        | T                            |
| T        | F        | T        | T                            |
| T        | F        | F        | F                            |
| F        | T        | T        | T                            |
| F        | T        | F        | T                            |
| F        | F        | T        | T                            |
| F        | F        | F        | F                            |

Next, we can find the negation of  $B \vee C$ , working off the  $B \vee C$  column we just created.

|          |          |          |                              |                                    |
|----------|----------|----------|------------------------------|------------------------------------|
| <b>A</b> | <b>B</b> | <b>C</b> | <b><math>B \vee C</math></b> | <b><math>\sim(B \vee C)</math></b> |
| T        | T        | T        | T                            | F                                  |
| T        | T        | F        | T                            | F                                  |
| T        | F        | T        | T                            | F                                  |
| T        | F        | F        | F                            | T                                  |
| F        | T        | T        | T                            | F                                  |

|   |   |   |   |   |
|---|---|---|---|---|
| F | T | F | T | F |
| F | F | T | T | F |
| F | F | F | F | T |

Finally, we find the values of  $A$  and  $\sim(B \vee C)$

| $A$ | $B$ | $C$ | $B \vee C$ | $\sim(B \vee C)$ | $A \wedge \sim(B \vee C)$ |
|-----|-----|-----|------------|------------------|---------------------------|
| T   | T   | T   | T          | F                | F                         |
| T   | T   | F   | T          | F                | F                         |
| T   | F   | T   | T          | F                | F                         |
| T   | F   | F   | F          | T                | T                         |
| F   | T   | T   | T          | F                | F                         |
| F   | T   | F   | T          | F                | F                         |
| F   | F   | T   | T          | F                | F                         |
| F   | F   | F   | F          | T                | F                         |

It turns out that this complex expression is only true in one case: if  $A$  is true,  $B$  is false, and  $C$  is false.

### Try It Now 1

Create a truth table for this statement:  $(\sim A \wedge B) \vee \sim B$

When we discussed conditions earlier, we discussed the type where we take an action based on the value of the condition. We are now going to talk about a more general version of a conditional, sometimes called an **implication**.

### Definition: Implications

Implications are logical conditional sentences stating that a statement  $p$ , called the antecedent, implies a consequence  $q$ .

Implications are commonly written as  $p \rightarrow q$

Implications are similar to the conditional statements we looked at earlier;  $p \rightarrow q$  is typically written as “if  $p$  then  $q$ ”, or “ $p$  therefore  $q$ ”. The difference between implications and conditionals is that conditionals we discussed earlier suggest an action – if the condition is true, then we take some action as a result. Implications are logical statements that suggest that the consequence must logically follow if the antecedent is true.

### ✓ Example 1.3.3

The English statement “If it is raining, then there are clouds in the sky” is a logical implication. It is a valid argument because if the antecedent “it is raining” is true, then the consequence “there are clouds in the sky” must also be true.

Notice that the statement tells us nothing of what to expect if it is not raining. If the antecedent is false, then the implication becomes irrelevant.

### ✓ Example 1.3.4

A friend tells you that “if you upload that picture to Facebook, you’ll lose your job”. There are four possible outcomes:

1. You upload the picture and keep your job
2. You upload the picture and lose your job

3. You don't upload the picture and keep your job
4. You don't upload the picture and lose your job

There is only one possible case where your friend was lying – the first option where you upload the picture and keep your job. In the last two cases, your friend didn't say anything about what would happen if you didn't upload the picture, so you can't conclude their statement is invalid, even if you didn't upload the picture and still lost your job.

In traditional logic, an implication is considered valid (true) as long as there are no cases in which the antecedent is true and the consequence is false. It is important to keep in mind that symbolic logic cannot capture all the intricacies of the English language.

### Truth Values For Implications

| $p$ | $q$ | $p \rightarrow q$ |
|-----|-----|-------------------|
| T   | T   | T                 |
| T   | F   | F                 |
| F   | T   | T                 |
| F   | F   | T                 |

### ✓ Example 1.3.5

Construct a truth table for the statement  $(m \wedge \sim p) \rightarrow r$

#### Solution

We start by constructing a truth table for the antecedent.

| $m$ | $p$ | $\sim p$ | $m \wedge \sim p$ |
|-----|-----|----------|-------------------|
| T   | T   | F        | F                 |
| T   | F   | T        | T                 |
| F   | T   | F        | F                 |
| F   | F   | T        | F                 |

Now we can build the truth table for the implication

| $m$ | $p$ | $\sim p$ | $m \wedge \sim p$ | $r$ | $(m \wedge \sim p) \rightarrow r$ |
|-----|-----|----------|-------------------|-----|-----------------------------------|
| T   | T   | F        | F                 | T   | T                                 |
| T   | F   | T        | T                 | T   | T                                 |
| F   | T   | F        | F                 | T   | T                                 |
| F   | F   | T        | F                 | T   | T                                 |
| T   | T   | F        | F                 | F   | T                                 |
| T   | F   | T        | T                 | F   | F                                 |
| F   | T   | F        | F                 | F   | T                                 |
| F   | F   | T        | F                 | F   | T                                 |

In this case, when  $m$  is true,  $p$  is false, and  $r$  is false, then the antecedent  $m \wedge \sim p$  will be true but the consequence false, resulting in an invalid implication; every other case gives a valid implication.

For any implication, there are three related statements, the **converse**, the **inverse**, and the **contrapositive**.

#### Definition: Related Statements

The original implication is “if p then q”  $p \rightarrow q$

The converse is: “if q then p”  $q \rightarrow p$

The inverse is “if not p then not q”  $\sim p \rightarrow \sim q$

The contrapositive is “if not q then not p”  $\sim q \rightarrow \sim p$

#### ✓ Example 1.3.6

Consider again the valid implication “If it is raining, then there are clouds in the sky”.

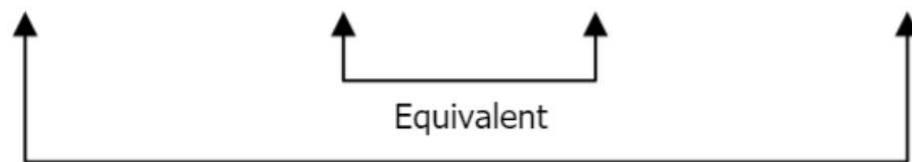
The converse would be “If there are clouds in the sky, it is raining.” This is certainly not always true.

The inverse would be “If it is not raining, then there are not clouds in the sky.” Likewise, this is not always true.

The contrapositive would be “If there are not clouds in the sky, then it is not raining.” This statement is valid, and is equivalent to the original implication.

Looking at truth tables, we can see that the original conditional and the contrapositive are logically equivalent, and that the converse and inverse are logically equivalent.

|   |   | Implication       | Converse          | Inverse                     | Contrapositive              |
|---|---|-------------------|-------------------|-----------------------------|-----------------------------|
| p | q | $p \rightarrow q$ | $q \rightarrow p$ | $\sim p \rightarrow \sim q$ | $\sim q \rightarrow \sim p$ |
| T | T | T                 | T                 | T                           | T                           |
| T | F | F                 | T                 | T                           | F                           |
| F | T | T                 | F                 | F                           | T                           |
| F | F | T                 | T                 | T                           | T                           |



#### Definition: Equivalence

A conditional statement and its contrapositive are logically equivalent.

The converse and inverse of a statement are logically equivalent.

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## SECTION OVERVIEW

### 1.4: Logical Arguments

#### 1.4.1: Evaluating Deductive Arguments with Truth Tables

#### 1.4.2: Arguments

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## 1.4.1: Evaluating Deductive Arguments with Truth Tables

Arguments can also be analyzed using truth tables, although this can be a lot of work.

### 📌 Analyzing arguments using truth tables

To analyze an argument with a truth table:

1. Represent each of the premises symbolically
2. Create a conditional statement, joining all the premises to form the antecedent, and using the conclusion as the consequent.
3. Create a truth table for the statement. If it is always true, then the argument is valid.

### ✓ Example 34

Consider the argument

Premise: If you bought bread, then you went to the store.

Premise: You bought bread.

Conclusion: You went to the store.

#### Solution

While this example is fairly obviously a valid argument, we can analyze it using a truth table by representing each of the premises symbolically. We can then form a conditional statement showing that the premises together imply the conclusion. If the truth table is a tautology (always true), then the argument is valid.

We'll let  $b$  represent "you bought bread" and  $s$  represent "you went to the store". Then the argument becomes:

Premise:  $b \rightarrow s$

Premise:  $b$

Conclusion:  $s$

To test the validity, we look at whether the combination of both premises implies the conclusion; is it true that  $[(b \rightarrow s) \wedge b] \rightarrow s$ ?

| $b$ | $s$ | $b \rightarrow s$ |
|-----|-----|-------------------|
| T   | T   | T                 |
| T   | F   | F                 |
| F   | T   | T                 |
| F   | F   | T                 |

| $b$ | $s$ | $b \rightarrow s$ | $(b \rightarrow s) \wedge b$ |
|-----|-----|-------------------|------------------------------|
| T   | T   | T                 | T                            |
| T   | F   | F                 | F                            |
| F   | T   | T                 | F                            |
| F   | F   | T                 | F                            |

| $b$ | $s$ | $b \rightarrow s$ | $(b \rightarrow s) \wedge b$ | $[(b \rightarrow s) \wedge b] \rightarrow s$ |
|-----|-----|-------------------|------------------------------|----------------------------------------------|
| T   | T   | T                 | T                            | T                                            |
| T   | F   | F                 | F                            | T                                            |
| F   | T   | T                 | F                            | T                                            |
| F   | F   | T                 | F                            | T                                            |

Since the truth table for  $[(b \rightarrow s) \wedge b] \rightarrow s$  is always true, this is a valid argument.

### ? Try it Now 13

Determine whether the argument is valid:

Premise: If I have a shovel, I can dig a hole.

Premise: I dug a hole.

Conclusion: Therefore, I had a shovel.

#### Answer

Let  $S$  = have a shovel,  $D$  = dig a hole. The first premise is equivalent to  $S \rightarrow D$ . The second premise is  $D$ . The conclusion is  $S$ . We are testing  $[(S \rightarrow D) \wedge D] \rightarrow S$

| $S$ | $D$ | $S \rightarrow D$ | $(S \rightarrow D) \wedge D$ | $[(S \rightarrow D) \wedge D] \rightarrow S$ |
|-----|-----|-------------------|------------------------------|----------------------------------------------|
| T   | T   | T                 | T                            | T                                            |
| T   | F   | F                 | F                            | T                                            |
| F   | T   | T                 | T                            | F                                            |
| F   | F   | T                 | F                            | T                                            |

This is not a tautology, so this is an invalid argument.

### ✓ Example 35

Premise: If I go to the mall, then I'll buy new jeans.

Premise: If I buy new jeans, I'll buy a shirt to go with it.

Conclusion: If I go to the mall, I'll buy a shirt.

#### Solution

Let  $m$  = I go to the mall,  $j$  = I buy jeans, and  $s$  = I buy a shirt.

The premises and conclusion can be stated as:

Premise:  $m \rightarrow j$

Premise:  $j \rightarrow s$

Conclusion:  $m \rightarrow s$

We can construct a truth table for  $[(m \rightarrow j) \wedge (j \rightarrow s)] \rightarrow (m \rightarrow s)$ . Try to recreate each step and see how the truth table was constructed.

| $m$ | $j$ | $s$ | $m \rightarrow j$ | $j \rightarrow s$ | $(m \rightarrow j) \wedge (j \rightarrow s)$ | $m \rightarrow s$ | $[(m \rightarrow j) \wedge (j \rightarrow s)] \rightarrow (m \rightarrow s)$ |
|-----|-----|-----|-------------------|-------------------|----------------------------------------------|-------------------|------------------------------------------------------------------------------|
| T   | T   | T   | T                 | T                 | T                                            | T                 | T                                                                            |
| T   | T   | F   | T                 | F                 | F                                            | F                 | T                                                                            |
| T   | F   | T   | F                 | T                 | F                                            | T                 | T                                                                            |
| T   | F   | F   | F                 | T                 | F                                            | F                 | T                                                                            |
| F   | T   | T   | T                 | T                 | T                                            | T                 | T                                                                            |
| F   | T   | F   | T                 | F                 | F                                            | T                 | T                                                                            |
| F   | F   | T   | T                 | T                 | T                                            | T                 | T                                                                            |
| F   | F   | F   | T                 | T                 | T                                            | T                 | T                                                                            |

From the final column of the truth table, we can see this is a valid argument.

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## 1.4.2: Arguments

A logical argument is a claim that a set of premises support a conclusion. There are two general types of arguments: inductive and deductive arguments.

### Argument Types

An **inductive** argument uses a collection of specific examples as its premises and uses them to propose a general conclusion.

A **deductive** argument uses a collection of general statements as its premises and uses them to propose a specific situation as the conclusion.

#### ✓ Example 1.4.2.1

The argument “when I went to the store last week I forgot my purse, and when I went today I forgot my purse. I always forget my purse when I go the store” is an inductive argument.

##### **Solution**

The premises are:

- I forgot my purse last week
- I forgot my purse today

The conclusion is:

- I always forget my purse

Notice that the premises are specific situations, while the conclusion is a general statement. In this case, this is a fairly weak argument, since it is based on only two instances.

#### ✓ Example 1.4.2.2

The argument “every day for the past year, a plane flies over my house at 2 p.m. A plane will fly over my house every day at 2 p.m.” is a stronger inductive argument, since it is based on a larger set of evidence.

### Evaluating Inductive Arguments

An inductive argument is never able to prove the conclusion true, but it can provide either weak or strong evidence to suggest it may be true.

Many scientific theories, such as the big bang theory, can never be proven. Instead, they are inductive arguments supported by a wide variety of evidence. Usually in science, an idea is considered a hypothesis until it has been well tested, at which point it graduates to being considered a theory. The commonly known scientific theories, like Newton’s theory of gravity, have all stood up to years of testing and evidence, though sometimes they need to be adjusted based on new evidence. For gravity, this happened when Einstein proposed the theory of general relativity.

A deductive argument is more clearly valid or not, which makes them easier to evaluate.

### Evaluating Deductive Arguments

A deductive argument is considered valid if all the premises are true, and the conclusion follows logically from those premises. In other words, the premises are true, and the conclusion follows necessarily from those premises.

#### ✓ Example 1.4.2.3

The argument “All cats are mammals and a tiger is a cat, so a tiger is a mammal” is a valid deductive argument.

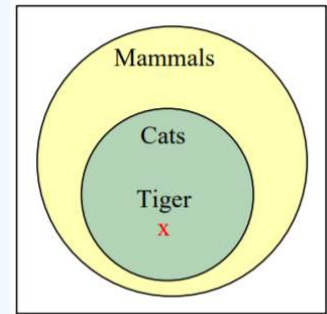
The premises are:

- All cats are mammals
- A tiger is a cat

The conclusion is:

- A tiger is a mammal

Both the premises are true. To see that the premises must logically lead to the conclusion, one approach would be to use a Venn diagram. From the first premise, we can conclude that the set of cats is a subset of the set of mammals. From the second premise, we are told that a tiger lies within the set of cats. From that, we can see in the Venn diagram that the tiger also lies inside the set of mammals, so the conclusion is valid.



### Analyzing Arguments with Venn Diagrams [1]

To analyze an argument with a Venn diagram

1. Draw a Venn diagram based on the premises of the argument
2. If the premises are insufficient to determine what determine the location of an element, indicate that.
3. The argument is valid if it is clear that the conclusion must be true

### ✓ Example 1.4.2.4

Premise: All firefighters know CPR

Premise: Jill knows CPR

Conclusion: Jill is a firefighter

#### **Solution**

From the first premise, we know that firefighters all lie inside the set of those who know CPR. From the second premise, we know that Jill is a member of that larger set, but we do not have enough information to know if she also is a member of the smaller subset that is firefighters.

Since the conclusion does not necessarily follow from the premises, this is an invalid argument, regardless of whether Jill actually is a firefighter.

It is important to note that whether or not Jill is actually a firefighter is not important in evaluating the validity of the argument; we are only concerned with whether the premises are enough to prove the conclusion.

### Try It Now 2

Determine the validity of this argument:

Premise: No cows are purple

Premise: Fido is not a cow

Conclusion: Fido is purple

In addition to these categorical style premises of the form “all \_\_\_\_”, “some \_\_\_\_”, and “no \_\_\_\_”, it is also common to see premises that are implications.

### ✓ Example 1.4.2.5

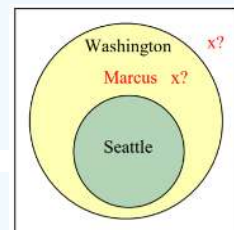
Premise: If you live in Seattle, you live in Washington.

Premise: Marcus does not live in Seattle

Conclusion: Marcus does not live in Washington

#### **Solution**

From the first premise, we know that the set of people who live in Seattle is inside the set of those who live in Washington. From the second premise, we know that Marcus does not lie in the Seattle set, but we have insufficient information to know whether or not Marcus lives in Washington or not. This is an invalid argument.



#### ✓ Example 1.4.2.6

Consider the argument “You are a married man, so you must have a wife.”

#### Solution

This is an invalid argument, since there are, at least in parts of the world, men who are married to other men, so the premise not insufficient to imply the conclusion.

Some arguments are better analyzed using truth tables.

#### ✓ Example 1.4.2.7

Consider the argument

Premise: If you bought bread, then you went to the store

Premise: You bought bread

Conclusion: You went to the store

While this example is hopefully fairly obviously a valid argument, we can analyze it using a truth table by representing each of the premises symbolically. We can then look at the implication that the premises together imply the conclusion. If the truth table is a tautology (always true), then the argument is valid.

We'll get  $B$  represent “you bought bread” and  $S$  represent “you went to the store”. Then the argument becomes:

Premise:  $B \rightarrow S$

Premise:  $B$

Conclusion:  $S$

To test the validity, we look at whether the combination of both premises implies the conclusion; is it true that  $[(B \rightarrow S) \wedge B] \rightarrow S$ ?

| $B$ | $S$ | $B \rightarrow S$ | $(B \rightarrow S) \wedge B$ | $[(B \rightarrow S) \wedge B] \rightarrow S$ |
|-----|-----|-------------------|------------------------------|----------------------------------------------|
| T   | T   | T                 | T                            | T                                            |
| T   | F   | F                 | F                            | T                                            |
| F   | T   | T                 | F                            | T                                            |
| F   | F   | T                 | F                            | T                                            |

Since the truth table for  $[(B \rightarrow S) \wedge B] \rightarrow S$  is always true, this is a valid argument.

#### ✎ Try It Now 3

Determine if the argument is valid:

Premise: If I have a shovel I can dig a hole.

Premise: I dug a hole

Conclusion: Therefore, I had a shovel

## Analyzing Arguments Using Truth Tables

To analyze an argument with a truth table:

1. Represent each of the premises symbolically
2. Create a conditional statement, joining all the premises with and to form the antecedent, and using the conclusion as the consequent.
3. Create a truth table for that statement. If it is always true, then the argument is valid.

### ✓ Example 1.4.2.8

Premise: If I go to the mall, then I'll buy new jeans

Premise: If I buy new jeans, I'll buy a shirt to go with it

Conclusion: If I got to the mall, I'll buy a shirt.

#### Solution

Let  $M$  = I go to the mall,  $J$  = I buy jeans, and  $S$  = I buy a shirt.

The premises and conclusion can be stated as:

Premise:  $M \rightarrow J$

Premise:  $J \rightarrow S$

Conclusion:  $M \rightarrow S$

We can construct a truth table for  $[(M \rightarrow J) \wedge (J \rightarrow S)] \rightarrow (M \rightarrow S)$

| $M$ | $J$ | $S$ | $M \rightarrow J$ | $J \rightarrow S$ | $(M \rightarrow J) \wedge (J \rightarrow S)$ | $M \rightarrow S$ | $[(M \rightarrow J) \wedge (J \rightarrow S)] \rightarrow (M \rightarrow S)$ |
|-----|-----|-----|-------------------|-------------------|----------------------------------------------|-------------------|------------------------------------------------------------------------------|
| T   | T   | T   | T                 | T                 | T                                            | T                 | T                                                                            |
| T   | T   | F   | T                 | F                 | F                                            | F                 | T                                                                            |
| T   | F   | T   | F                 | T                 | F                                            | T                 | T                                                                            |
| T   | F   | F   | F                 | T                 | F                                            | F                 | T                                                                            |
| F   | T   | T   | T                 | T                 | T                                            | T                 | T                                                                            |
| F   | T   | F   | T                 | F                 | F                                            | T                 | T                                                                            |
| F   | F   | T   | T                 | T                 | T                                            | T                 | T                                                                            |
| F   | F   | F   | T                 | T                 | T                                            | T                 | T                                                                            |

From the truth table, we can see this is a valid argument.

The previous problem is an example of a syllogism.

## Definition: Syllogism

A syllogism is an implication derived from two others, where the consequence of one is the antecedent to the other. The general form of a syllogism is:

Premise:  $p \rightarrow q$

Premise:  $q \rightarrow r$

Conclusion:  $p \rightarrow r$

This is sometimes called the transitive property for implication.

## ✓ Example 1.4.2.9

Premise: If I work hard, I'll get a raise.

Premise: If I get a raise, I'll buy a boat.

Conclusion: If I don't buy a boat, I must not have worked hard.

**Solution**

If we let  $W$  = working hard,  $R$  = getting a raise, and  $B$  = buying a boat, then we can represent our argument symbolically:

Premise:  $H \rightarrow R$

Premise:  $R \rightarrow B$

Conclusion:  $\sim B \rightarrow \sim H$

We could construct a truth table for this argument, but instead, we will use the notation of the contrapositive we learned earlier to note that the implication  $\sim B \rightarrow \sim H$  is equivalent to the implication  $H \rightarrow B$ . Rewritten, we can see that this conclusion is indeed a logical syllogism derived from the premises.

## ✎ Try It Now 4

Is this argument valid?

Premise: If I go to the party, I'll be really tired tomorrow.

Premise: If I go to the party, I'll get to see friends.

Conclusion: If I don't see friends, I won't be tired tomorrow.

Lewis Carroll, author of *Alice in Wonderland*, was a math and logic teacher, and wrote two books on logic. In them, he would propose premises as a puzzle, to be connected using syllogisms.

## ✓ Example 1.4.2.10

Solve the puzzle. In other words, find a logical conclusion from these premises.

- All babies are illogical.
- Nobody is despised who can manage a crocodile.
- Illogical persons are despised

**Solution**

Let  $B$  = is a baby,  $D$  = is despised,  $I$  = is illogical, and  $M$  = can manage a crocodile.

Then we can write the premises as:

$B \rightarrow I$

$M \rightarrow \sim D$

$I \rightarrow D$

From the first and third premises, we can conclude that  $B \rightarrow D$ ; that babies are despised. Using the contrapositive of the second premise,  $D \rightarrow \sim M$ , we can conclude that  $B \rightarrow \sim M$ ; that babies cannot manage crocodiles.

While silly, this is a logical conclusion from the given premises.

**Notes:**

1) Technically, these are Euler circles or Euler diagrams, not Venn diagrams, but for the sake of simplicity we'll continue to call them Venn diagrams.

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## 1.5: Arguments Forms

Rather than making a truth table for every argument, we may be able to recognize certain common forms of arguments that are valid (or invalid). If we can determine that an argument fits one of the common forms, we can immediately state whether it is valid or invalid.

### The Law of Detachment (*Modus Ponens*)

The law of detachment applies when a conditional and its antecedent are given as premises, and the consequent is the conclusion. The general form is:

Premise:  $p \rightarrow q$

Premise:  $p$

Conclusion:  $q$

The Latin name, *modus ponens*, translates to “mode that affirms”.

### Example 36

Recall this argument from an earlier example:

Premise: If you bought bread, then you went to the store.

Premise: You bought bread.

Conclusion: You went to the store.

In symbolic form:

Premise:  $b \rightarrow s$

Premise:  $b$

Conclusion:  $s$

This argument has the structure described by the law of detachment. (The second premise and the conclusion are simply the two parts of the first premise detached from each other.) Instead of making a truth table, we can say that this argument is valid by stating that it satisfies the law of detachment.

### The Law of Contraposition (*Modus Tollens*)

The law of contraposition applies when a conditional and the negation of its consequent are given as premises, and the negation of its antecedent is the conclusion. The general form is:

Premise:  $p \rightarrow q$

Premise:  $\sim q$

Conclusion:  $\sim p$

The Latin name, *modus tollens*, translates to “mode that denies”.

Notice that the second premise and the conclusion look like the contrapositive of the first premise,  $\sim q \rightarrow \sim p$ , but they have been detached. You can think of the law of contraposition as a combination of the law of detachment and the fact that the contrapositive is logically equivalent to the original statement.

### Example 37

Premise: If I drop my phone into the swimming pool, my phone will be ruined.

Premise: My phone isn't ruined.

Conclusion: I didn't drop my phone into the swimming pool.

If we let  $d$  = I drop the phone in the pool and  $r$  = the phone is ruined, then we can represent the argument this way:

Premise:  $d \rightarrow r$

Premise:  $\sim r$

Conclusion:  $\sim d$

The form of this argument matches what we need to invoke the law of contraposition, so it is a valid argument.

### ? Try it Now 14

Is this argument valid?

Premise: If you brushed your teeth before bed, then your toothbrush will be wet.

Premise: Your toothbrush is dry.

Conclusion: You didn't brush your teeth before bed.

#### Answer

Let  $b$  = brushed teeth and  $w$  = toothbrush is wet.

Premise:  $b \rightarrow w$

Premise:  $\sim w$

Conclusion:  $\sim b$

This argument is valid by the Law of Contraposition.

### 📌 The Transitive Property (Hypothetical Syllogism)

The transitive property has as its premises a series of conditionals, where the consequent of one is the antecedent of the next. The conclusion is a conditional with the same antecedent as the first premise and the same consequent as the final premise. The general form is:

Premise:  $p \rightarrow q$

Premise:  $q \rightarrow r$

Conclusion:  $p \rightarrow r$

The earlier example about buying a shirt at the mall is an example illustrating the transitive property. It describes a chain reaction: if the first thing happens, then the second thing happens, and if the second thing happens, then the third thing happens. Therefore, if we want to ignore the second thing, we can say that if the first thing happens, then we know the third thing will happen. We don't have to mention the part about buying jeans; we can simply say that the first event leads to the final event. We could even have more than two premises; as long as they form a chain reaction, the transitive property will give us a valid argument.

### ✓ Example 38

Premise: If a soccer player commits a reckless foul, she will receive a yellow card.

Premise: If Hayley receives a yellow card, she will be suspended for the next match.

Conclusion: If Hayley commits a reckless foul, she will be suspended for the next match.

If we let  $r$  = committing a reckless foul,  $y$  = receiving a yellow card, and  $s$  = being suspended, then our argument looks like this:

Premise:  $r \rightarrow y$

Premise:  $y \rightarrow s$

Conclusion:  $r \rightarrow s$

This argument has the exact structure required to use the transitive property, so it is a valid argument.

### ? Try it Now 15

Is this argument valid?

- Premise: If the old lady swallows a fly, she will swallow a spider.  
Premise: If the old lady swallows a spider, she will swallow a bird.  
Premise: If you brushed your teeth before bed, then your toothbrush will be wet.  
Premise: If you brushed your teeth before bed, then your toothbrush will be wet.  
Premise: If the old lady swallows a bird, she will swallow a cat.  
Premise: If the old lady swallows a cat, she will swallow a dog.  
Premise: If the old lady swallows a dog, she will swallow a goat.  
Premise: If the old lady swallows a goat, she will swallow a cow.  
Premise: If the old lady swallows a cow, she will swallow a horse.  
Premise: If the old lady swallows a horse, she will die, of course.  
Conclusion: If the old lady swallows a fly, she will die, of course.

#### Answer

This argument is valid by the Transitive Property, which can involve more than two premises, as long as they continue the chain reaction. The premises  $f \rightarrow s, s \rightarrow b, b \rightarrow c, c \rightarrow d, d \rightarrow g, g \rightarrow w, w \rightarrow h, h \rightarrow x$  can be reduced to  $f \rightarrow x$ . (Because we had already used  $c$  and  $d$  we decided to use  $w$  for cow and  $x$  for death. If the old lady swallows the fly, she will eventually eat a horse and die.

### 📌 Disjunctive Syllogism

In a disjunctive syllogism, the premises consist of an *or* statement and the negation of one of the options. The conclusion is the other option. The general form is:

- Premise:  $p \vee q$   
Premise:  $\sim p$   
Conclusion:  $q$

The order of the two parts of the disjunction isn't important. In other words, we could have the premises  $p \vee q$  and  $\sim q$ , and the conclusion  $p$

### ✓ Example 39

- Premise: I can either drive or take the train.  
Premise: I refuse to drive.  
Conclusion: I will take the train.

If we let  $d = I$  drive and  $t = I$  take the train, then the symbolic representation of the argument is:

- Premise:  $d \vee t$   
Premise:  $\sim d$   
Conclusion:  $t$

This argument is valid because it has the form of a disjunctive syllogism. I have two choices, and one of them is not going to happen, so the other one must happen.

### ? Try it Now 16

Is this argument valid?

Premise: Alison was required to write a 10-page paper or give a 5-minute speech.  
Premise: Alison did not give a 5-minute speech.  
Conclusion: Alison wrote a 10-page paper.

**Answer**

Let  $p$  = wrote a paper and  $s$  = gave a speech.

Premise:  $p \vee s$

Premise:  $\neg s$

Conclusion:  $p$

This argument is valid by Disjunctive Syllogism. Alison had to do one or the other; she didn't choose the speech, so she must have chosen the paper.

Keep in mind that, when you are determining the validity of an argument, you must assume that the premises are true. If you don't agree with one of the premises, you need to keep your personal opinion out of it. Your job is to pretend that the premises are true and then determine whether they force you to accept the conclusion. You may attack the premises in a court of law or a political discussion, of course, but here we are focusing on the structure of the arguments, not the truth of what they actually say.

We have just looked at four forms of valid arguments; there are two common forms that represent *invalid* arguments, which are also called *fallacies*.

 **The Fallacy of the Converse**

The fallacy of the converse arises when a conditional and its consequent are given as premises, and the antecedent is the conclusion. The general form is:

Premise:  $p \rightarrow q$

Premise:  $q$

Conclusion:  $p$

Notice that the second premise and the conclusion look like the converse of the first premise,  $q \rightarrow p$ , but they have been detached. The fallacy of the converse incorrectly tries to assert that the converse of a statement is equivalent to that statement.

 **Example 40**

Premise: If I drink coffee after noon, then I have a hard time falling asleep that night.

Premise: I had a hard time falling asleep last night.

Conclusion: I drank coffee after noon yesterday.

If we let  $c$  = I drink coffee after noon and  $h$  = I have a hard time falling asleep, then our argument looks like this:

Premise:  $c \rightarrow h$

Premise:  $h$

Conclusion:  $c$

This argument uses converse reasoning, so it is an invalid argument. There could be plenty of other reasons why I couldn't fall asleep: I could be worried about money, my neighbors might have been setting off fireworks, ...

 **Try it Now 17**

Is this argument valid?

Premise: If you pull that fire alarm, you will get in big trouble.

Premise: You got in big trouble.

Conclusion: You must have pulled the fire alarm.

**Answer**

Let  $f$  = pulled fire alarm and  $t$  = got in big trouble.

Premise:  $f \rightarrow t$

Premise:  $t$

Conclusion:  $f$

### The Fallacy of the Inverse

The fallacy of the inverse occurs when a conditional and the negation of its antecedent are given as premises, and the negation of the consequent is the conclusion. The general form is:

Premise:  $p \rightarrow q$

Premise:  $\sim p$

Conclusion:  $\sim q$

Again, notice that the second premise and the conclusion look like the inverse of the first premise,  $\sim p \rightarrow \sim q$ , but they have been detached. The fallacy of the inverse incorrectly tries to assert that the inverse of a statement is equivalent to that statement.

### ✓ Example 41

Premise: If you listen to the Grateful Dead, then you are a hippie.

Premise: Sky doesn't listen to the Grateful Dead.

Conclusion: Sky is not a hippie.

If we let  $g$  = listen to the Grateful Dead and  $h$  = is a hippie, then this is the argument:

Premise:  $g \rightarrow h$

Premise:  $\sim g$

Conclusion:  $\sim h$

This argument is invalid because it uses inverse reasoning. The first premise does not imply that all hippies listen to the Grateful Dead; there could be some hippies who listen to Phish instead.

### ? Try it Now 18

Is this argument valid?

Premise: If a hockey player trips an opponent, he will be assessed a 2-minute penalty.

Premise: Alexei did not trip an opponent.

Conclusion: Alexei will not be assessed a 2-minute penalty.

#### Answer

Let  $t$  = tripped and  $p$  = got a penalty.

Premise:  $t \rightarrow p$

Premise:  $\sim t$

Conclusion:  $\sim p$

This argument is invalid because it has the form of the Fallacy of the Inverse. Alexei may have gotten a penalty for an infraction other than tripping.

Of course, arguments are not limited to these six basic forms; some arguments have more premises, or premises that need to be rearranged before you can see what is really happening. There are plenty of other forms of arguments that are invalid. If an argument doesn't seem to fit the pattern of any of these common forms, though, you may want to use a Venn diagram or a truth table instead.

Lewis Carroll, author of *Alice's Adventures in Wonderland*, was a math and logic teacher, and wrote two books on logic. In them, he would propose premises as a puzzle, to be connected using syllogisms. The following example is one such puzzle.

#### ✓ Example 42

Solve the puzzle. In other words, find a logical conclusion from these premises.

All babies are illogical.

Nobody is despised who can manage a crocodile.

Illogical persons are despised.

Let  $b$  = is a baby,  $d$  = is despised,  $i$  = is illogical, and  $m$  = can manage a crocodile.

Then we can write the premises as:

$$b \rightarrow i$$

$$m \rightarrow \sim d$$

$$i \rightarrow d$$

Writing the second premise correctly can be a challenge; it can be rephrased as “If you can manage a crocodile, then you are not despised.”

Using the transitive property with the first and third premises, we can conclude that  $b \rightarrow d$ , that all babies are despised. Using the contrapositive of the second premise,  $d \rightarrow \sim m$ , we can then use the transitive property with  $b \rightarrow d$  to conclude that  $b \rightarrow \sim m$ , that babies cannot manage crocodiles. While it is silly, this is a logical conclusion from the given premises.

#### ✓ Example 43

Premise: If I work hard, I'll get a raise.

Premise: If I get a raise, I'll buy a boat.

Conclusion: If I don't buy a boat, I must not have worked hard.

If we let  $h$  = working hard,  $r$  = getting a raise, and  $b$  = buying a boat, then we can represent our argument symbolically:

Premise:  $h \rightarrow r$

Premise:  $r \rightarrow b$

Conclusion:  $\sim b \rightarrow \sim h$

Using the transitive property with the two premises, we can conclude that  $h \rightarrow b$ , if I work hard, then I will buy a boat. When we learned about the contrapositive, we saw that the conditional statement  $h \rightarrow b$  is equivalent to  $\sim b \rightarrow \sim h$ . Therefore, the conclusion is indeed a logical syllogism derived from the premises.

#### ? Try it Now 19

Is this argument valid?

Premise: If I go to the party, I'll be really tired tomorrow.

Premise: If I go to the party, I'll get to see friends.

Conclusion: If I don't see friends, I won't be tired tomorrow.

#### Answer

Let  $p$  = go to party,  $t$  = be tired, and  $f$  = see friends.

Premise:  $p \rightarrow t$

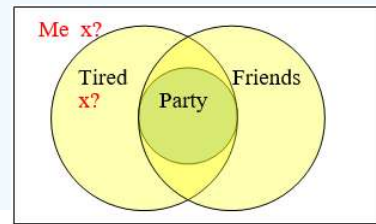
Premise:  $p \rightarrow f$

Conclusion:  $\sim f \rightarrow \sim t$

We could try to rewrite the second premise using the contrapositive to state  $\sim f \rightarrow \sim p$ , but that does not allow us to form a syllogism. If I don't see friends, then I didn't go the party, but that is not sufficient to claim I won't be tired tomorrow.

Maybe I stayed up all night watching movies.

A Venn diagram can help, if we set it up correctly. The “party” circle must be completely contained within the intersection of the other circles. We know that I am somewhere outside the “friends” circle, but we cannot determine whether I am in the “tired” circle. All we really know for sure is that I didn’t go to the party.



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## CHAPTER OVERVIEW

### 2: Sets and Counting

#### 2.1: Sets and Set Operations

##### 2.1.1: Basics of Sets

##### 2.1.2: Union, Intersection, and Complement

#### 2.2: Trees and Counting Principle

##### 2.2.1: Tree Diagrams, Tables, and Outcomes

##### 2.2.2: The Multiplication Rule for Counting

#### 2.3: Permutations and Combinations

##### 2.3.1: Permutations

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#### 2.4: Probability

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#### 2.5: Probability Trees

##### 2.5.1: Probability Using Tree Diagrams and Combinations

##### 2.5.2: Probability Using Tree Diagrams and Combinations (Exercises)

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## SECTION OVERVIEW

### 2.1: Sets and Set Operations

#### 2.1.1: Basics of Sets

#### 2.1.2: Union, Intersection, and Complement

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## 2.1.1: Basics of Sets

An art collector might own a collection of paintings, while a music lover might keep a collection of CDs. Any collection of items can form a **set**.

### 📌 Set

A **set** is a collection of distinct objects, called **elements** of the set

A set can be defined by describing the contents, or by listing the elements of the set, enclosed in curly brackets.

### ✓ Example 1

Some examples of sets defined by describing the contents:

- The set of all even numbers
- The set of all books written about travel to Chile

Some examples of sets defined by listing the elements of the set:

- $\{1, 3, 9, 12\}$
- $\{\text{red, orange, yellow, green, blue, indigo, purple}\}$

A set simply specifies the contents; order is not important. The set represented by  $\{1, 2, 3\}$  is equivalent to the set  $\{3, 1, 2\}$ .

### 📌 Notation

Commonly, we will use a variable to represent a set, to make it easier to refer to that set later.

The symbol  $\in$  means “is an element of”.

A set that contains no elements,  $\{\}$ , is called the **empty set** and is notated  $\emptyset$

### ✓ Example 2

Let  $A = \{1, 2, 3, 4\}$

To notate that 2 is element of the set, we'd write  $2 \in A$

Sometimes a collection might not contain all the elements of a set. For example, Chris owns three Madonna albums. While Chris's collection is a set, we can also say it is a **subset** of the larger set of all Madonna albums.

### 📌 Subset

A **subset** of a set  $A$  is another set that contains only elements from the set  $A$ , but may not contain all the elements of  $A$ .

If  $B$  is a subset of  $A$ , we write  $B \subseteq A$

A **proper subset** is a subset that is not identical to the original set – it contains fewer elements.

If  $B$  is a proper subset of  $A$ , we write  $B \subset A$

### ✓ Example 3

Consider these three sets

$A = \text{the set of all even numbers}$     $B = \{2, 4, 6\}$     $C = \{2, 3, 4, 6\}$

Here  $B \subset A$  since every element of  $B$  is also an even number, so is an element of  $A$ .

More formally, we could say  $B \subset A$  since if  $x \in B$ , then  $x \in A$

It is also true that  $B \subset C$ .

$C$  is not a subset of  $A$ , since  $C$  contains an element, 3, that is not contained in  $A$ .

#### ✓ Example 4

Suppose a set contains the plays “Much Ado About Nothing”, “MacBeth”, and “A Midsummer’s Night Dream”. What is a larger set this might be a subset of?

##### **Solution**

There are many possible answers here. One would be the set of plays by Shakespeare. This is also a subset of the set of all plays ever written. It is also a subset of all British literature.

#### ? Try it Now 1

The set  $A = \{1, 3, 5\}$ . What is a larger set this might be a subset of?

##### **Answer**

There are several answers: The set of all odd numbers less than 10. The set of all odd numbers. The set of all integers. The set of all real numbers.

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## 2.1.2: Union, Intersection, and Complement

Commonly sets interact. For example, you and a new roommate decide to have a house party, and you both invite your circle of friends. At this party, two sets are being combined, though it might turn out that there are some friends that were in both sets.

### Union, Intersection, and Complement

The **union** of two sets contains all the elements contained in either set (or both sets).

The union is notated  $A \cup B$

More formally,  $x \in A \cup B$  if  $x \in A$  or  $x \in B$  (or both)

The **intersection** of two sets contains only the elements that are in both sets.

The intersection is notated  $A \cap B$

More formally,  $x \in A \cap B$  if  $x \in A$  and  $x \in B$

The **complement** of a set  $A$  contains everything that is *not* in the set  $A$ .

The complement is notated  $A^c$ , or  $A^c$ , or sometimes  $\sim A$ .

### Example 5

Consider the sets:

$$A = \{ \text{red, green, blue} \} \quad B = \{ \text{red, yellow, orange} \} \quad C = \{ \text{red, orange, yellow, green, blue, purple} \}$$

- Find  $A \cup B$
- Find  $A \cap B$
- Find  $A^c \cap C$

#### Solution

a) The union contains all the elements in either set:  $A \cup B = \{ \text{red, green, blue, yellow, orange} \}$

Notice we only list red once.

b) The intersection contains all the elements in both sets:  $A \cap B = \{ \text{red} \}$

c) Here we're looking for all the elements that are not in set  $A$  and are also in  $C$ .

$$A^c \cap C = \{ \text{orange, yellow, purple} \}$$

### Try it Now 2

Using the sets from the previous example, find  $A \cup C$  and  $B^c \cap A$

#### Answer

$$A \cup C = \{ \text{red, orange, yellow, green, blue, purple} \}$$

$$B^c \cap A = \{ \text{green, blue} \}$$

Notice that in the example above, it would be hard to just ask for  $A^c$ , since everything from the color fuchsia to puppies and peanut butter are included in the complement of the set. For this reason, complements are usually only used with intersections, or when we have a universal set in place.

### Universal Set

A **universal set** is a set that contains all the elements we are interested in. This would have to be defined by the context.

A complement is relative to the universal set, so  $A^c$  contains all the elements in the universal set that are not in  $A$ .

## ✓ Example 6

- If we were discussing searching for books, the universal set might be all the books in the library.
- If we were grouping your Facebook friends, the universal set would be all your Facebook friends.
- If you were working with sets of numbers, the universal set might be all whole numbers, all integers, or all real numbers

## ✓ Example 7

Suppose the universal set is  $U =$  all whole numbers from 1 to 9. If  $A = \{1, 2, 4\}$ , then

$$A^c = \{3, 5, 6, 7, 8, 9\}$$

As we saw earlier with the expression  $A^c \cap C$ , set operations can be grouped together. Grouping symbols can be used like they are with arithmetic - to force an order of operations.

## ✓ Example 8

Suppose

$$H = \{ \text{cat, dog, rabbit, mouse} \}, F = \{ \text{dog, cow, duck, pig, rabbit} \} \quad W = \{ \text{duck, rabbit, deer, frog, mouse} \}$$

- Find  $(H \cap P) \cup W$
- Find  $H \cap (F \cup W)$
- Find  $(H \cap P) \cap W$

**Solution**

a) We start with the intersection:  $H \cap F = \{ \text{dog, rabbit} \}$

Now we union that result with  $W$  :  $(H \cap F) \cup W = \{ \text{dog, duck, rabbit, deer, frog, mouse} \}$

b) We start with the union:  $F \cup W = \{ \text{dog, cow, rabbit, duck, pig, deer, frog, mouse} \}$

Now we intersect that result with  $H$  :  $H \cap (F \cup W) = \{ \text{dog, rabbit, mouse} \}$

c) We start with the intersection:  $H \cap F = \{ \text{dog, rabbit} \}$

Now we want to find the elements of  $W$  that are not in  $H \cap F$

$$(H \cap F)^c \cap W = \{ \text{duck, deer, frog, mouse} \}$$

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## SECTION OVERVIEW

### 2.2: Trees and Counting Principle

#### 2.2.1: Tree Diagrams, Tables, and Outcomes

#### 2.2.2: The Multiplication Rule for Counting

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## 2.2.1: Tree Diagrams, Tables, and Outcomes



Figure 2.2.1.1: In genetics, the characteristics of an offspring organism depends on the characteristics of its parents. (credit: “Pea Plant” by Maria Keays/Flickr, CC BY 2.0))

### Learning Objectives

1. Determine the sample space of single stage experiment.
2. Use tables to list possible outcomes of a multistage experiment.
3. Use tree diagrams to list possible outcomes of a multistage experiment.

In the 19th century, an Augustinian friar and scientist named Gregor Mendel used his observations of pea plants to set out his theory of genetic propagation. In his work, he looked at the offspring that resulted from breeding plants with different characteristics together. For applications like this, it is often insufficient to only know in how many ways a process might end; we need to be able to list all of the possibilities. As we’ve seen, the number of possible outcomes can be very large! Thus, it’s important to have a strategy that allows us to systematically list these possibilities to make sure we don’t leave any out. In this section, we’ll look at two of these strategies.

### Single Stage Experiments

When we are talking about combinatorics or probability, the word “experiment” has a slightly different meaning than it does in the sciences. Experiments can range from very simple (“flip a coin”) to very complex (“count the number of uranium atoms that undergo nuclear fission in a sample of a given size over the course of an hour”). Experiments have unknown outcomes that generally rely on something random, so that if the experiment is repeated (or replicated) the outcome might be different. No matter what the experiment, though, analysis of the experiment typically begins with identifying its sample space.

The sample space of an experiment is the set of all of the possible outcomes of the experiment, so it’s often expressed as a set (i.e., as a list bound by braces; if the experiment is “randomly select a number between 1 and 4,” the sample space would be written  $\{1, 2, 3, 4\}$ ).

### ✓ Example 2.2.1.1: Finding the Sample Space of an Experiment

For each of the following experiments, identify the sample space.

1. Flip a coin (which has 2 faces, typically called “heads” and “tails”) and note which face is up.
2. Flip a coin 10 times and count the number of heads.
3. Roll a 6-sided die and note the number that is on top.
4. Roll two 6-sided dice and note the sum of the numbers on top.

#### Answer

1. If we use “H” to denote “heads is facing up” and “T” to denote “tails is facing up”, then the sample space is {H, T}.
2. It’s possible (though unlikely) that there will be no heads flipped; the outcome in that case would be “0.” It’s also possible (more likely, but still quite unlikely) that only one flip will result in heads. Any other whole number is possible, up to the maximum: We’re flipping the coin 10 times, so we can’t get any more than 10 heads. So, the sample space is {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}.
3. There are 6 numbers on the die: 1, 2, 3, 4, 5, and 6. So, the sample space for a single roll of the die is {1, 2, 3, 4, 5, 6}.
4. If we roll 2 dice, the smallest possible sum we could get is  $1 + 1 = 2$  and the biggest is  $6 + 6 = 12$ . Every other whole number between those two is possible. So, the sample space is {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}.

### 📌 Your Turn 2.2.1.1

Identify the sample space of each experiment.

You draw a card from a standard deck and note its suit.

You draw a card from a standard deck and note its rank.

You roll a 4-sided die and note the number on the bottom. (A 4-sided die is shaped like a pyramid, so when it comes to rest, there’s no single side facing up.)

You roll three 4-sided dice and note the sum of the numbers on the bottom.

## Multistage Experiments

Some experiments have more complicated sample spaces because they occur in stages. These stages can occur in succession (like drawing cards one at a time) or simultaneously (rolling 2 dice). Sample spaces get more complicated as the complexity of the experiment increases, so it’s important to choose a systematic method for identifying all of the possible outcomes. The first method we’ll discuss is the table.

### Using Tables to Find Sample Spaces

Tables are useful for finding the sample space for experiments that meet two criteria: (1) The experiment must have only two stages, and (2) the outcomes of each stage must have no effect on the outcomes of the other. When the stages do not affect each other, we say the stages are independent. Otherwise, the stages are dependent and so we can’t use tables; we’ll look at a method for analyzing dependent stages soon.

### ✓ Example 2.2.1.2: Determining Independence

Decide whether the two stages in these experiments are independent or dependent.

1. You flip a coin and note the result, and then flip the coin again and note the result.
2. You draw 2 cards from a standard deck (52 cards), one at a time.

#### Answer

1. No matter what happens on the first flip, the second flip has the same sample space: {H, T} (You’ll sometimes hear the phrase “The coin has no memory”). So, these stages are independent.

- Let's say that the first card you draw is  $A\spadesuit$ . The sample space for the second draw consists of all the cards *except*  $A\spadesuit$  (since that card is no longer in the deck, you can't draw it again). If instead that first card was  $2\heartsuit$ , the sample space for the second draw is different: it's every card except  $2\heartsuit$ . Since the sample space for the second card changes based on the result of the first draw, these stages are dependent.

#### 📌 Your Turn 2.2.1.2

Decide whether the two stages in these experiments are dependent or independent.

You're getting dressed to go to a party, and you plan to wear a blouse and a skirt. You choose the blouse first, then the skirt (assume that you'd be comfortable wearing any of your skirts with any of your blouses).

On further reflection, you realize that some of your skirts clash with some of your blouses. So, you choose the blouse first, and then choose a skirt that goes with your chosen blouse.

If you have a two-stage experiment with independent stages, a table is the most straightforward way to identify the sample space. To build a table, you list the outcomes of one stage of the experiment along the top of the table and the outcomes of the other stage down the side. The cells in the interior of the table are then filled using the outcomes associated with each cell's row and column. Let's look at an example.

#### ✓ Example 2.2.1.3: Using Tables to Identify Sample Spaces

Identify the sample spaces of these experiments using tables.

- You roll two dice: one 4-sided and one 6-sided.
- You're in an ice-cream shop, and you're going to get a single scoop of ice cream with a topping. The flavors of ice cream you're considering are vanilla, chocolate, and rocky road; the toppings are fudge, whipped cream, and sprinkles.
- The pea plants you're breeding have two possible pod colors: green and yellow. These colors are decided by a particular gene, which comes in two types: "G" for green, and "g" for yellow (In genetics, capital letters usually denote *dominant* genes, while lower-case letters denote *recessive* genes). Each plant has two genes. If you breed a Gg pea plant with a gg plant, the offspring plant will get one gene from each parent. What are the possible outcomes?

#### Answer

- Step 1:** Make the outline of a table, with the results of the 4-sided roll on one side and the results of the 6-sided roll on the other. In practice, it doesn't matter which you choose; for this example, we'll put the 4-sided results on top (labeling the columns of the chart) and the 6-sided results on the side (labeling the rows of the chart) as shown in the following table:

|              |   | 4-Sided Roll |   |   |   |
|--------------|---|--------------|---|---|---|
|              |   | 1            | 2 | 3 | 4 |
| 6-Sided Roll | 1 |              |   |   |   |
|              | 2 |              |   |   |   |
|              | 3 |              |   |   |   |
|              | 4 |              |   |   |   |
|              | 5 |              |   |   |   |
|              | 6 |              |   |   |   |

- Step 2:** Fill in the results in each cell of the table below. Use the notation of an ordered pair, with the row label first.

|  |  | 4-Sided Roll |   |   |   |
|--|--|--------------|---|---|---|
|  |  | 1            | 2 | 3 | 4 |
|  |  |              |   |   |   |

|              |   | 4-Sided Roll |       |       |       |
|--------------|---|--------------|-------|-------|-------|
|              |   | 1            | 2     | 3     | 4     |
| 6-Sided Roll | 1 | (1,1)        | (1,2) | (1,3) | (1,4) |
|              | 2 | (2,1)        | (2,2) | (2,3) | (2,4) |
|              | 3 | (3,1)        | (3,2) | (3,3) | (3,4) |
|              | 4 | (4,1)        | (4,2) | (4,3) | (4,4) |
|              | 5 | (5,1)        | (5,2) | (5,3) | (5,4) |
|              | 6 | (6,1)        | (6,2) | (6,3) | (6,4) |

So, for example, (3,2) represents the outcome where the 6-sided roll results in a 3, and the 4-sided roll gives us a 2. Thus, the sample space of the experiment is  $\{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,3), (4,4), (5,1), (5,2), (5,3), (5,4), (6,1), (6,2), (6,3), (6,4)\}$ .

**Step 1:** Let's put the flavors on the rows and the toppings on the columns of the following table:

|         |            | Toppings |               |           |
|---------|------------|----------|---------------|-----------|
|         |            | fudge    | whipped cream | sprinkles |
| Flavors | vanilla    |          |               |           |
|         | chocolate  |          |               |           |
|         | rocky road |          |               |           |

**Step 2:** We can fill in the cells of the table below with the resulting combinations.

|         |            | Toppings              |                               |                           |
|---------|------------|-----------------------|-------------------------------|---------------------------|
|         |            | fudge                 | whipped cream                 | sprinkles                 |
| Flavors | vanilla    | vanilla with fudge    | vanilla with whipped cream    | vanilla with sprinkles    |
|         | chocolate  | chocolate with fudge  | chocolate with whipped cream  | chocolate with sprinkles  |
|         | rocky road | rocky road with fudge | rocky road with whipped cream | rocky road with sprinkles |

So, the sample space is  $\{\text{vanilla with fudge, vanilla with whipped cream, vanilla with sprinkles, chocolate with fudge, chocolate with whipped cream, chocolate with sprinkles, rocky road with fudge, rocky road with whipped cream, rocky road with sprinkles}\}$ .

**Step 1:** We'll put the parents' genes (P1 and P2) as labels on the rows and columns of the following table:

|    |   | P2 |   |
|----|---|----|---|
|    |   | g  | g |
| P1 | G |    |   |
|    | g |    |   |

**Step 2:** We'll fill in the offspring's gene composition, listing parent 1's gene first in the table below.

|  |  |  |  |
|--|--|--|--|
|  |  |  |  |
|--|--|--|--|

|    |   | P2 |    |
|----|---|----|----|
|    |   | g  | g  |
| P1 | G | Gg | Gg |
|    | g | gg | gg |

Thus, the sample space is  $\{Gg, Gg, gg, gg\}$ . (Diagrams like this, which allow us to identify the genotypes of offspring, are called *Punnett squares* in honor of Reginald Punnett (1875–1967), who first used them in the context of genetics.)

### 📌 Your Turn 2.2.1.3

Use a table to identify the sample space of an experiment in which you flip a coin and roll a 6-sided die.

### Using Tree Diagrams to Identify Sample Spaces

In experiments where there are more than two stages, or where the stages are dependent, a tree diagram is a helpful tool for systematically identifying the sample space. Tree diagrams are built by first drawing a single point (or *node*), then from that node we draw one branch (a short line segment) for each outcome of the first stage. Each branch gets its own node at the other end (which we typically label with the corresponding outcome for that branch); from each of these, we draw another branch for each outcome of the second stage, assuming that the outcome of the first stage matches the branch we were on. If there are other stages, we can continue from there by continuing to add branches and nodes. This sounds really complicated, but it's easier to understand through an example.

#### ✓ Example 2.2.1.4: Using a Tree Diagram to Identify a Sample Space

Use a tree diagram to find the sample spaces of each of the following experiments:

1. You flip a coin 3 times, noting the outcome of each flip in order.
2. You flip a coin. If the result is heads, you roll a 4-sided die. If it's tails, you roll a 6-sided die.
3. You are planning to go on a hike with a group of friends. There are 3 trails to consider: Abel Trail, Borel Trail, and Condorcet Trail. One of your friends, Jess, requires a wheelchair; if she joins you, the group couldn't handle the rocky Condorcet Trail.

#### Answer

1. **Step 1:** Let's start by placing our first node (Figure ✓ 2.2.1.2).



- Step 2:** We'll add two branches, one for each outcome of the first coin flip, and label them (Figure ✓ 2.2.1.3).



**Step 3:** We're ready for stage two of the experiment: another coin flip. At each node, we add in branches that represent those outcomes (Figure ✓ 2.2.1.4).



Figure ✓ 2.2.1.4

Finally, we can add another set of branches for the outcomes of the third stage (Figure ✓ 2.2.1.5).

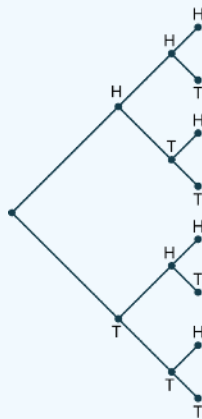


Figure ✓ 2.2.1.5

(These final nodes are called *leaves*.)

**Step 4:** We can write down the outcomes in the sample space by tracing the path out to each leaf, writing down the outcome at each node we pass through. For example, this leaf (Figure ✓ 2.2.1.6):

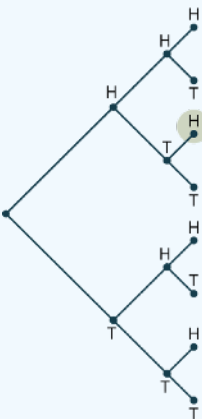


Figure ✓ 2.2.1.6

is reached via this path (Figure 7.16):

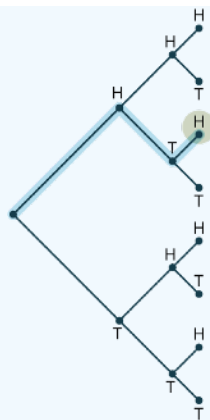


Figure ✓ 2.2.1.7

**Step 5:** We'll label that leaf as "HTH" (Figure ✓ 2.2.1.8), since the path passes through nodes labeled H, T, and H on its way out to our leaf.

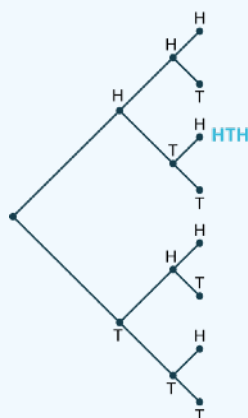


Figure ✓ 2.2.1.8

**Step 6:** We can label the remaining leaves using the same method (Figure ✓ 2.2.1.9).

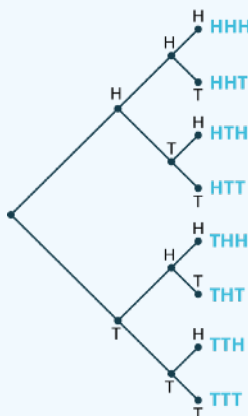


Figure ✓ 2.2.1.9

The sample space is the labels on the leaves: {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}.

**Step 1:** We'll start with our initial node (Figure ✓ 2.2.1.10).



Figure ✓ 2.2.1.10

**Step 2:** We'll add in branches for the outcomes of the first stage (Figure ✓ 2.2.1.11), which is the coin flip.



Figure ✓ 2.2.1.11

Step 3: The second stage of the experiment depends on the outcome of the first stage.

a. If the outcome of the first stage was H, then we roll a 4-sided die. So, *only* on the node for H, we'll add in the outcomes of a 4-sided die roll (Figure

✓ 2.2.1.12).

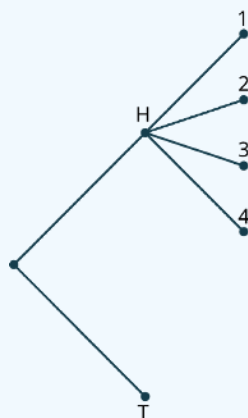


Figure ✓ 2.2.1.12

If the outcome of the first stage was T, then we roll a 6-sided die. So, we'll add those branches to the node for T (Figure ✓ 2.2.1.13).

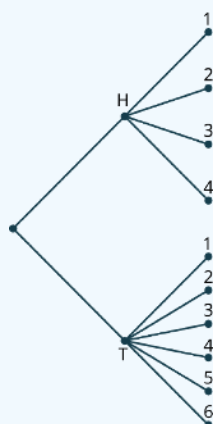


Figure ✓ 2.2.1.13

**Step 4:** We can label the leaves to get the sample space (Figure 2.2.1.14).

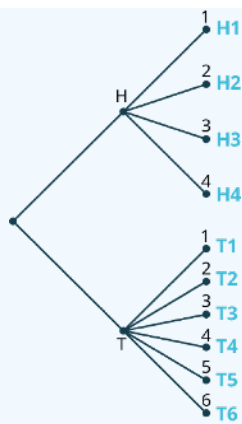


Figure ✓ 2.2.1.14

The sample space is: {H1, H2, H3, H4, T1, T2, T3, T4, T5, T6}.

**c. Step 1:** Let's label the trails A, B, and C for ease of labeling. Even though the trails are listed first in the exercise, we can't use the trail choice as our first stage: the trails available to us depend on whether Jess is able to join the trip. So, the first stage is whether Jess joins us (J) or not (N) (Figure ✓ 2.2.1.15).



Figure ✓ 2.2.1.15

**Step 2:** We list the appropriate trails on each branch (Figure ✓ 2.2.1.16).

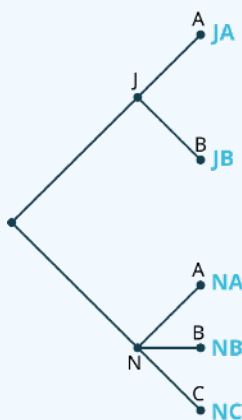


Figure ✓ 2.2.1.16

So, the sample space is {JA, JB, NA, NB, NC}

### Your Turn 2.2.1.4

You have a modified deck of cards containing only  $J♥$ ,  $Q♥$ , and  $K♥$ . You draw 2 cards without replacing them (where order matters). Use a tree diagram to identify the sample space.

### Check Your Understanding

1. You flip a coin 6 times and note the number of heads. What is the sample space of this experiment?
2. You are ordering a combo meal at a restaurant. The meal comes with either 8 or 12 chicken nuggets, and your choice of crinkle fries, curly fries, or onion rings. Create a table to help you identify the sample space containing your combo meal possibilities.
3. You need one more class to fill out your schedule for next semester. You want to take either History 101 (H), English 220 (E), or Sociology 112 (S). There are two professors teaching the history class: Anderson (A) and Burr (B); one professor teaching the English class: Carter (C); and three people teaching sociology: Johnson (J), Kirk (K), and Lambert (L). Create a tree diagram that helps you identify all your options.
4. Identify the sample space from Exercise 18.
5. Identify the sample space from Exercise 19.

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## 2.2.2: The Multiplication Rule for Counting



Figure 2.2.2.1 : The Multiplication Rule for Counting allows us to compute more complicated probabilities, like drawing two aces from a deck. (credit: "Pair of Aces – Poker" by Poker Photos/Flickr, CC BY 2.0)

### Learning Objectives

Apply the Multiplication Rule for Counting to solve problems.

One of the first bits of mathematical knowledge children learn is how to count objects by pointing to them in turn and saying: “one, two, three, ...” That’s a useful skill, but when the number of things that we need to count grows large, that method becomes onerous (or, for very large numbers, impossible for humans to accomplish in a typical human lifespan). So, mathematicians have developed short cuts to counting big numbers. These techniques fall under the mathematical discipline of combinatorics, which is devoted to counting.

### Multiplication as a Combinatorial Short Cut

One of the first combinatorial short cuts to counting students learn in school has to do with areas of rectangles. If we have a set of objects to be counted that can be physically arranged into a rectangular shape, then we can use multiplication to do the counting for us. Consider this set of objects (Figure 2.2.2.2):



Figure 2.2.2.2

Certainly we can count them by pointing and running through the numbers, but it’s more efficient to group them (Figure 2.2.2.3).

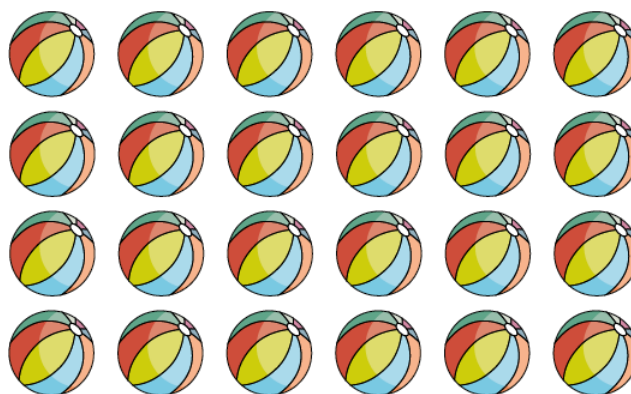


Figure 2.2.2.3

If we group the balls by 4s, we see that we have 6 groups (or, we can see this arrangement as 4 groups of 6 balls). Since multiplication is repeated addition (i.e.,  $6 \times 4 = 4 + 4 + 4 + 4 + 4 + 4$ ), we can use this grouping to quickly see that there are 24 balls.

Let's generalize this idea a little bit. Let's say that we're visiting a bakery that offers customized cupcakes. For the cake, we have three choices: vanilla, chocolate, and strawberry. Each cupcake can be topped with one of four types of frosting: vanilla, chocolate, lemon, and strawberry. How many different cupcake combinations are possible? We can think of laying out all the possibilities in a grid, with cake choices defining the rows and frosting choices defining the columns (Figure 2.2.2.4).

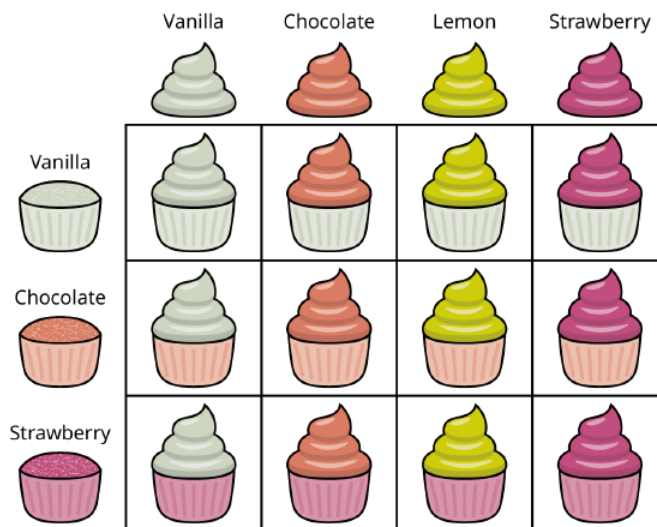


Figure 2.2.2.4

Since there are 3 rows (cakes) and 4 columns (frostings), we have  $3 \times 4 = 12$  possible combinations. This is the reasoning behind the Multiplication Rule for Counting, which is also known as the Fundamental Counting Principle. This rule says that if there are  $n$  ways to accomplish one task and  $m$  ways to accomplish a second task, then there are  $n \times m$  ways to accomplish both tasks. We can tack on additional tasks by multiplying the number of ways to accomplish those tasks to our previous product.

### ✓ Example ✓ 2.2.2.1: Using the Multiplication Rule for Counting

Every card in a standard deck of cards has two identifying characteristics: a suit (clubs, diamonds, hearts, or spades; these are indicated by these symbols, respectively: ♣, ♦, ♥, ♠) and a rank (ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, jack, queen, and king; the letters A, J, Q, and K are used to represent the words). Each possible pair of suit and rank appears exactly once in the deck. How many cards are in the standard deck?

#### Answer

Since there are 4 suits and 13 ranks, the number of cards must be  $4 \times 13 =$



Figure 2.2.2.5 Standard Deck of Cards, Sorted by Rank and Suit (credit: "Playing Cards, USS Arkansas" by Naval History & Heritage Command/Flickr, CC BY 2.0)

#### 📌 Your Turn 2.2.2.1

Joe's Pizza Shack offers pizzas with 4 different types of crust and a choice of 15 toppings. How many different one-topping pizzas can be made at Joe's?

#### ✓ Example 2.2.2.2: Using the Multiplication Rule for Counting for 4 Groups

The University Combinatorics Club has 31 members: 8 seniors, 7 juniors, 5 sophomores, and 11 first-years. How many possible 4-person committees can be formed by selecting 1 member from each class?

##### Answer

Since we have 8 choices for the senior, 7 choices for the junior, 5 for the sophomore, and 11 for the first-year, there are  $8 \times 7 \times 5 \times 11 = 3,080$  different ways to fill out the committee.

#### 📌 Your Turn 2.2.2.2

The menu for Joe's Pizza Shack offers pizzas with 4 different types of crust and a choice of 15 toppings. Suppose that Joe's also offers a choice of 3 sauces and 2 cheese blends. How many different one-topping pizzas can be made at Joe's now?

#### ✓ Example 2.2.2.3: Using the Multiplication Rule for Counting for More Groups

The standard license plates for vehicles in a certain state consist of 6 characters: 3 letters followed by 3 digits. There are 26 letters in the alphabet and 10 digits (0 through 9) to choose from. How many license plates can be made using this format?

##### Answer

Since there are 26 different letters and 10 different digits, the total number of possible license plates is  $26 \times 26 \times 26 \times 10 \times 10 \times 10 = 17,576,000$ .

#### 📌 Your Turn 2.2.2.3

At a certain college, ID cards are issued to all students, faculty, and staff. These cards have unique ID codes for each person: a letter to indicate the person's status (S for students, F for faculty, and E for staff), followed by 5 digits and finally 3 letters (these letters can be anything). How many different ID codes can be created using this scheme?

## Check Your Understanding

1. A website that lets you build custom belts has 18 different buckles and 30 different straps. How many different belts can be made using those materials?
2. A chain of chicken restaurants offers a combo that includes your choice of 3 or 5 chicken strips, along with your choice of side dish. If there are 7 side dishes, how many different ways are there to build this combo meal?
3. When you flip a coin, there are 2 possible outcomes: heads and tails. Let's say you flip a coin 10 times, and after each you write down the result of the flip (H for heads, T for tails). How many different results (strings of 10 characters, where each is either an H or a T) are possible?
4. A T-shirt company allows shoppers to customize their shirts in several ways. There are 5 sizes, 8 shirt colors, 4 designs, and 5 design colors. How many different shirts can be made?
5. Josephine is trying to build her class schedule for next semester. Because of her work schedule, she has only 4 class periods that can work for her, and she must take 4 classes. If there are 15 classes that she could take during the first period, 18 during the second, 12 during the third, and 8 during the fourth, how many different schedules could Josephine build?

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## SECTION OVERVIEW

### 2.3: Permutations and Combinations

#### 2.3.1: Permutations

#### 2.3.2: Combinations

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## 2.3.1: Permutations

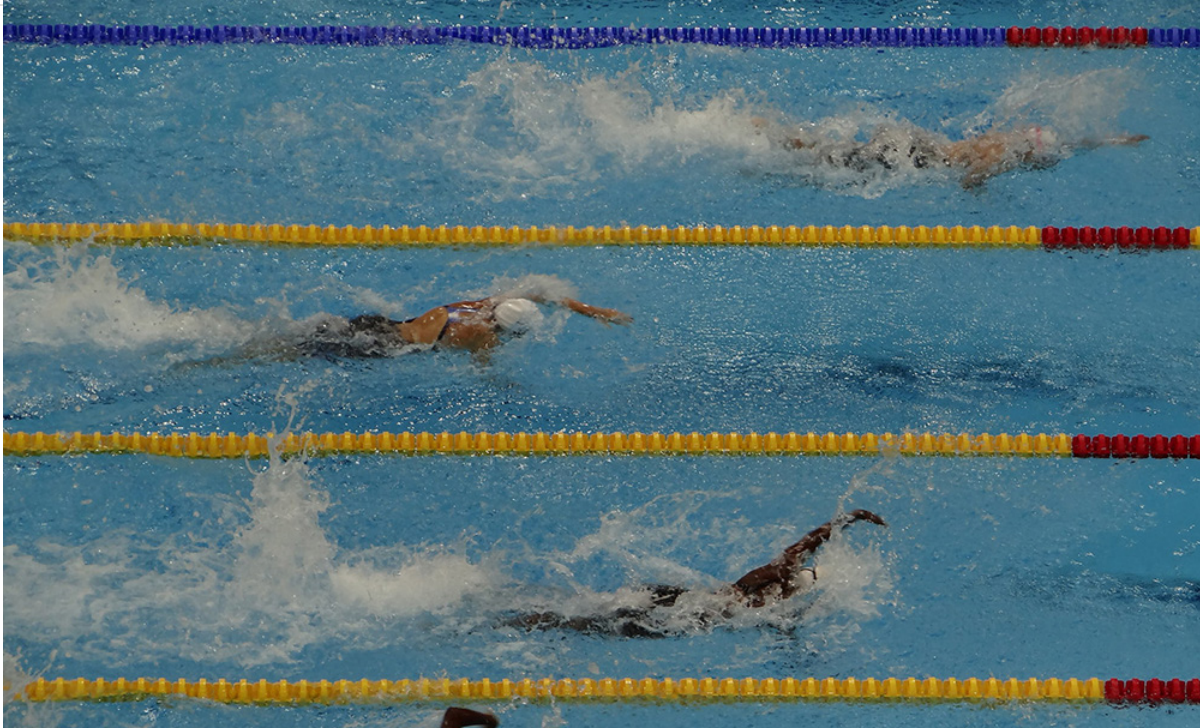


Figure 2.3.1.1: We can use permutations to calculate the number of different orders of finish in an Olympic swimming heat. (credit: "London 2012 Olympics Park Stratford London" by Gary Bembridge/Flickr, CC BY 2.0)

### Learning Objectives

1. Use the Multiplication Rule for Counting to determine the number of permutations.
2. Compute expressions containing factorials.
3. Compute permutations.
4. Apply permutations to solve problems.

Swimming events are some of the most popular events at the summer Olympic Games. In the finals of each event, 8 swimmers compete at the same time, making for some exciting finishes. How many different orders of finish are possible in these events? In this section, we'll extend the Multiplication Rule for Counting to help answer questions like this one, which relate to permutations. A permutation is an ordered list of objects taken from a given population. The length of the list is given, and the list cannot contain any repeated items.

### Applying the Multiplication Rule for Counting to Permutations

In the case of the swimming finals, one possible permutation of length 3 would be the list of medal winners (first, second, and third place finishers). A permutation of length 8 would be the full order of finish (first place through eighth place). Let's use the Multiplication Rule for Counting to figure out how many of each of these permutations there are.

### ✓ Example

#### ✓ 2.3.1.1: Using the Multiplication Rule for Counting to Find the Number of Permutations

The final heat of Olympic swimming events features 8 swimmers (or teams of swimmers).

1. How many different podium placements (first place, second place, and third place) are possible?
2. How many different complete orders of finish (first place through eighth place) are possible?

**Answer**

1. Let's start with the first place finisher. How many options are there? Since 8 swimmers are competing, there are 8 possibilities. Once that first swimmer completes the race, there are 7 swimmers left competing for second place. After the second finisher is decided, there are 6 swimmers remaining who could possibly finish in third place. Thus, there are 8 possibilities for first place, 7 for second place, and 6 for third place. The Multiplication Rule for Counting then tells us there are  $8 \times 7 \times 6 = 336$  different ways the winners' podium can be filled out.
2. To look at the complete order of finish, we can continue the pattern we can see in part 1 of this example: There are 5 possibilities for fourth place, 4 for fifth place, 3 for sixth place, 2 for seventh place, and then just 1 swimmer is left to finish in eighth place. Using the Multiplication Rule for Counting, we see that there are  $8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40,320$  possible orders of finish.

### Your Turn 2.3.1.1

You have a hand of 5 cards (that happen to create what's called a royal flush in the game of poker):  $10\spadesuit, J\spadesuit, Q\spadesuit, K\spadesuit$ , and  $A\spadesuit$ . Into how many different orders can you put those cards?

## Factorials

The pattern we see in Example 7.4 occurs commonly enough that we have a name for it: factorial.

For any positive whole number  $n$ , we define the factorial of  $n$  (denoted  $n!$  and read " $n$  factorial") to be the product of every whole number less than or equal to  $n$ . We also define  $0!$  to be equal to one. We will use factorials in a couple of different contexts, so let's get some practice doing computations with them.

### ✓ Example 2.3.1.2: Computing Factorials

Compute the following:

1.  $4!$
2.  $\frac{8!}{6!}$
3.  $\frac{9!}{3!4!}$

#### Answer

1.  $4! = 4 \times 3 \times 2 \times 1 = 24$
2. There are two ways to approach this calculation. The first way is to compute the factorials first, then divide:

$$\frac{8!}{6!} = \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{6 \times 5 \times 4 \times 3 \times 2 \times 1} = \frac{40,320}{720} = 56$$

However, there is an easier way! You may notice in the second step that there are several terms that can be canceled; that's always the case whenever we divide factorials. In this case, notice that we can rewrite the numerator like this:

$$8! = 8 \times 7 \times (6 \times 5 \times 4 \times 3 \times 2 \times 1) = 8 \times 7 \times 6!$$

With that in mind, we can proceed this way by canceling out the  $6!$ :

$$\frac{8!}{6!} = \frac{8 \times 7 \times \cancel{6!}}{\cancel{6!}} = 8 \times 7 = 56$$

That's much easier!

3. Let's approach this one using our canceling technique. When we see two factorials in either the numerator or denominator, we should focus on the larger one first. So:

$$\frac{9!}{3!4!} = \frac{9 \times 8 \times 7 \times 6 \times 5 \times \cancel{4!}}{\cancel{3!4!}} = \frac{9 \times 8 \times 7 \times \cancel{6} \times 5}{\cancel{3} \times \cancel{2} \times 1} = 9 \times 8 \times 7 \times 5 = 2,520$$

### 📌 Your Turn 2.3.1.2

Compute the following:

$$6!$$

$$\frac{12!}{10!}$$

$$\frac{8!}{4!4!}$$

## Permutations

As we've seen, factorials can pop up when we're computing permutations. In fact, there is a formula that we can use to make that connection explicit. Let's define some notation first. If we have a collection of  $n$  objects and we wish to create an ordered list of  $r$  of the objects (where  $1 \leq r \leq n$ ), we'll call the number of those permutations (read "the number of permutations of  $n$  objects taken  $r$  at a time"). We formalize the formula we'll use to compute permutations below.

### 📌 FORMULA

$${}_nP_r = \frac{n!}{(n-r)!}$$

If you wondered why we defined  $0! = 1$  earlier, it was to make formulas like this one work; if we have  $n$  objects and want to order all of them (so, we want the number of permutations of  $n$  objects taken  $n$  at a time), we get  ${}_nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1} = n!$ . Next, we'll get some practice computing these permutations.

### ✓ Example 2.3.1.3: Computing Permutations

Find the following numbers:

1. The number of permutations of 12 objects taken 3 at a time
2. The number of permutations of 8 objects taken 5 at a time
3. The number of permutations of 32 objects taken 2 at a time

**Answer**

$$1. \quad {}_{12}P_3 = \frac{12!}{(12-3)!} = \frac{12!}{9!} = \frac{12 \times 11 \times 10 \times \mathbf{9!}}{\mathbf{9!}} = 12 \times 11 \times 10 = 1,$$

$$2. \quad {}_8P_5 = \frac{8!}{(8-5)!} = \frac{8 \times 7 \times 6 \times 5 \times 4 \times \mathbf{3!}}{\mathbf{3!}} = 8 \times 7 \times 6 \times 5 \times 4 = 6,$$

$$3. \quad {}_{32}P_2 = \frac{32!}{(32-2)!} = \frac{32 \times 31 \times \mathbf{30!}}{\mathbf{30!}} = 32 \times 31 =$$

### 📌 Your Turn 2.3.1.3

Find the following numbers:

The number of permutations of 6 objects taken 2 at a time

The number of permutations of 14 objects taken 4 at a time

The number of permutations of 19 objects taken 3 at a time

### ✓ Example 2.3.1.4: Applying Permutations

1. A high school graduating class has 312 students. The top student is declared valedictorian, and the second-best is named salutatorian. How many possible outcomes are there for the valedictorian and salutatorian?
2. In the card game blackjack, the dealer's hand of 2 cards is dealt with 1 card faceup and 1 card facedown. If the game is being played with a single deck of (52) cards, how many possible hands could the dealer get?
3. The University Combinatorics Club has 3 officers: president, vice president, and treasurer. If there are 18 members of the club, how many ways are there to fill the officer positions?

#### Answer

1. This is the number of permutations of 312 students taken 2 at a time, and  $312P_2 = 97,.$
2. We want the number of permutations of 52 cards taken 2 at a time, and  $52P_2 = 2,.$
3. Here we're looking for the number of permutations of 18 members taken 3 at a time, and  $18P_3 = 4,.$

### 📌 Your Turn 2.3.1.4

One of the big draws at this year's state fair is the pig race. There are 15 entrants, and prizes are given to the top three finishers. How many different combinations of top-three finishes could there be?

### 📌 Who Knew?: Very Big Permutations

Permutations involving relatively small sets of objects can get very big, very quickly. A standard deck contains 52 cards. So, the number of different ways to shuffle the cards—in other words, the number of permutations of 52 objects taken 52 at a time—is  $52! \approx 8 \times 10^{67}$  (written out, that's an 8 followed by 67 zeroes). The estimated age of the universe is *only* about  $4 \times 10^{17}$  seconds. So, if a very bored all-powerful being started shuffling cards at the instant the universe began, it would have to have averaged at least  $\frac{8 \times 10^{67}}{4 \times 10^{17}} \approx 2 \times 10^{50}$  shuffles *per second since the beginning of time* to have covered every possible arrangement of a deck of cards. That means the next time you pick up a deck of cards and give it a good shuffle, it's almost certain that the particular arrangement you created has never been created before and likely never will be created again.

### Check Your Understanding

1. Compute  $5!$ .
2. Compute  $\frac{10!}{7!3!}$ .
3. Compute  ${}_{12}P_3$ .
4. Compute  ${}_8P_4$ .
5. The standard American edition of the board game *Monopoly* has a deck of 15 orange Chance cards. In how many different ways could the first 4 Chance cards drawn in a game appear?

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## 2.3.2: Combinations



Figure 2.3.2.1: Combinations help us count things like the number of possible card hands, when the order in which the cards were drawn doesn't matter. (credit: "IMG\_3177" by Zanaca/Flickr, CC BY 2.0)

### Learning Objectives

1. Distinguish between permutation and combination uses.
2. Compute combinations.
3. Apply combinations to solve applications.

In Permutations, we studied permutations, which we use to count the number of ways to generate an ordered list of a given length from a group of objects. An important property of permutations is that the order of the list matters: The results of a race and the selection of club officers are examples of lists where the order is important. In other situations, the order is not important. For example, in most card games where a player receives a hand of cards, the order in which the cards are received is irrelevant; in fact, players often rearrange the cards in a way that helps them keep the cards organized.

### Combinations: When Order Doesn't Matter

In situations in which the order of a list of objects doesn't matter, the lists are no longer permutations. Instead, we call them combinations.

#### ✓ Example 2.3.2.1: Distinguishing Between Permutations and Combinations

For each of the following situations, decide whether the chosen subset is a permutation or a combination.

1. A social club selects 3 members to form a committee. Each of the members has an equal share of responsibility.
2. You are prompted to reset your email password; you select a password consisting of 10 characters without repeats.
3. At a dog show, the judge must choose first-, second-, and third-place finishers from a group of 16 dogs.
4. At a restaurant, the special of the day comes with the customer's choice of 3 sides taken from a list of 6 possibilities.

#### Answer

1. Since there is no distinction among the responsibilities of the 3 committee members, the order isn't important. So, this is a combination.
2. The order of the characters in a password matter, so this is a permutation.
3. The order of finish matters in a dog show, so this is a permutation.
4. A plate with mashed potatoes, peas, and broccoli is functionally the same as a plate with peas, broccoli, and mashed potatoes, so this is a combination.

### 📌 Your Turn 2.3.2.1

Decide whether the following represent permutations or combinations:

On Halloween, you give each kid who comes to your door 3 pieces of candy, taken randomly from a candy dish.

Your class is going on a field trip, but there are too many people for one vehicle. Your instructor chooses half the class to take the first vehicle.

$$\left(\frac{12!}{10!}\right) \left(\frac{8!}{4!}\right)$$

## Counting Combinations

Permutations and combinations are certainly related, because they both involve choosing a subset of a large group. Let's explore that connection, so that we can figure out how to use what we know about permutations to help us count combinations. We'll take a basic example. How many ways can we select 3 letters from the group A, B, C, D, and E? If order matters, that number is  $5P_3 =$ . That's small enough that we can list them all out in the table below.

|     |     |     |     |     |     |
|-----|-----|-----|-----|-----|-----|
| ABC | ABD | ABE | ACB | ACD | ACE |
| ADB | ADC | ADE | AEB | AEC | AED |
| BAC | BAD | BAE | BCA | BCD | BCE |
| BDA | BDC | BDE | BEA | BEC | BED |
| CAB | CAD | CAE | CBA | CBD | CBE |
| CDA | CDB | CDE | CEA | CEB | CED |
| DCA | DAC | DAE | DBA | DBC | DBE |
| DCA | DCB | DCE | DEA | DEB | DEC |
| EAB | EAC | EAD | EBA | EBC | EBD |
| ECA | ECB | ECD | EDA | EDB | EDC |

Now, let's look back at that list and color-code it so that groupings of the same 3 letters get the same color, as shown in Figure 7.9:

|     |     |     |     |     |     |
|-----|-----|-----|-----|-----|-----|
| ABC | ABD | ABE | ACB | ACD | ACE |
| ADB | ADC | ADE | AEB | AEC | AED |
| BAC | BAD | BAE | BCA | BCD | BCE |
| BDA | BDC | BDE | BEA | BEC | BED |
| CAB | CAD | CAE | CBA | CBD | CBE |
| CDA | CDB | CDE | CEA | CEB | CED |
| DAB | DAC | DAE | DBA | DBC | DBE |
| DCA | DCB | DCE | DEA | DEB | DEC |
| EAB | EAC | EAD | EBA | EBC | EBD |
| ECA | ECB | ECD | EDA | EDB | EDC |

Figure 2.3.2.2

After color-coding, we see that the 60 cells can be seen as 10 groups (colors) of 6. That's no coincidence! We've already seen how to compute the number of permutations using the formula. To compute the number of combinations, let's count them another way using the Multiplication Rule for Counting. We'll do this in two steps:

**Step 1:** Choose 3 letters (paying no attention to order).

**Step 2:** Put those letters in order.

The number of ways to choose 3 letters from this group of 5 (A, B, C, D, E) is the number of combinations we're looking for; let's call that number  $nCr$  (read "the number of combinations of  $n$  objects taken  $r$  at a time"). We can see from our chart that this is ten (the number of colors used). We can generalize our findings this way: remember that the number of permutations of  $n$  things taken  $r$  at a time is  $nPr = \frac{n!}{(n-r)!}$ . That number is *also* equal to  $nCr \times r!$ , and so it must be the case that  $\frac{n!}{(n-r)!} = nCr \times r!$ . Dividing both sides of that equation by  $r!$  gives us the formula below.

#### FORMULA

$$nCr = \frac{n!}{r!(n-r)!}$$

$$nCr = \frac{n!}{r!(n-r)!}$$

#### ? Example 2.3.2.2: Using the Combination Formula

Compute the following:

1.  $8C_3$
2.  $12C_5$
3.  $15C_9$

**Answer**

$$1. 8C_3 = \frac{8!}{3!(8-3)!} = \frac{8 \times 7 \times 6 \times 5!}{3 \times 2 \times 1 \times 5!} = 8 \times 7 = 56$$

$$2. 12C_5 = \frac{12!}{5!(12-5)!} = \frac{12 \times 11 \times 10 \times 9 \times 8 \times 7!}{5 \times 4 \times 3 \times 2 \times 1 \times 7!} = 11 \times 9 \times 8 = 792$$

$$3. {}_{15}C_9 = \frac{15!}{9!(15-9)!} = \frac{15 \times 14 \times 13 \times 12 \times 11 \times 10 \times 9!}{9! \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} = \frac{14 \times 13 \times 11 \times 10}{4} = 5,005$$

#### 📌 Your Turn 2.3.2.2

Compute the following:

$${}_6C_4$$

$${}_{10}C_8$$

$${}_{14}C_5$$

#### ✓ Example 2.3.2.3: Applying the Combination Formula

1. In the card game Texas Hold'em (a variation of poker), players are dealt 2 cards from a standard deck to form their hands. How many different hands are possible?
2. The board game *Clue* uses a deck of 21 cards. If 3 people are playing, each person gets 6 cards for their hand. How many different 6-card *Clue* hands are possible?
3. Palmetto Cash 5 is a game offered by the South Carolina Education Lottery. Players choose 5 numbers from the whole numbers between 1 and 38 (inclusive); the player wins the jackpot of \$100,000 if the randomizer selects those numbers in any order. How many different sets of winning numbers are possible?

#### Answer

1. A standard deck has 52 cards, and a hand has 2 cards. Since the order doesn't matter, we use the formula for counting combinations:

$${}_{52}C_2 = \frac{52!}{2!(52-2)!} = \frac{52 \times 51 \times 50!}{2 \times 1 \times 50!} = \frac{52 \times 51}{2} = 1,326.$$

2. Again, the order doesn't matter, so the number of combinations is:

$${}_{21}C_6 = \frac{21!}{6!(21-6)!} = \frac{21 \times 20 \times 19 \times 18 \times 17 \times 16 \times 15!}{6 \times 5 \times 4 \times 3 \times 2 \times 1 \times 15!} = \frac{21 \times 19 \times 17 \times 16}{2} = 54,264.$$

3. There are 38 numbers to choose from, and we must pick 5. Since order doesn't matter, the number of combinations is:

$${}_{38}C_5 = \frac{38!}{5!(38-5)!} = \frac{38 \times 37 \times 36 \times 35 \times 34 \times 33!}{5 \times 4 \times 3 \times 2 \times 1 \times 33!} = 501,492.$$

#### 📌 Your Turn 2.3.2.3

At a charity event with 58 people in attendance, 3 raffle winners are chosen. All receive the same prize, so order doesn't matter. How many different groups of 3 winners can be chosen?

A sorority with 42 members needs to choose a committee with 4 members, each with equal responsibility. How many committees are possible?

#### 📌 Checkpoint

The notation and nomenclature used for the number of combinations is not standard across all sources. You'll sometimes see  $\binom{n}{r}$  instead of  ${}_nC_r$ . Sometimes you'll hear that expression read as “*n* choose *r*” as shorthand for “the number of combinations of *n* objects taken *r* at a time.”

## 📌 People in Mathematics: Early Eastern Mathematicians

Although combinations weren't really studied in Europe until around the 13th century, mathematicians of the Middle and Far East had already been working on them for hundreds of years. The Indian mathematician known as Pingala had described them by the second century BCE; Varāhamihira (fl. sixth century) and Halayudha (fl. 10th century) extended Pingala's work. In the ninth century, a Jain mathematician named Mahāvīra gave the formula for combinations that we use today.

In 10th-century Baghdad, a mathematician named Al-Karaji also knew formulas for combinations; though his work is now lost, it was known to (and repeated by) Persian mathematician Omar Khayyam, whose work survives. Khayyam is probably best remembered as a poet, with his *Rubaiyat* being his most famous work.

Meanwhile, in 11th-century China, Jia Xian also was working with combinations, as was his 13th-century successor Yang Hui.

It is not known whether the discoveries of any of these men were known in the other regions, or if the Indians, Persians, and Chinese all came to their discoveries independently. We do know that mathematical knowledge and sometimes texts did get passed along trade routes, so it can't be ruled out.

## ✓ Example 2.3.2.4: Combining Combinations with the Multiplication Rule for Counting

The student government at a university consists of 10 seniors, 8 juniors, 6 sophomores, and 4 first-years.

1. How many ways are there to choose a committee of 8 people from this group?
2. How many ways are to choose a committee of 8 people if the committee must consist of 2 people from each class?

### Answer

1. There are 28 people to choose from, and we need 8. So, the number of possible committees is  ${}_{28}C_8 = 3,108$ .
2. Break the selection of the committee members down into a 4-step process: Choose the seniors, then choose the juniors, then the sophomores, and then the first-years, as shown in the table below:

| Class      | Number of Ways to Choose Committee Representatives |
|------------|----------------------------------------------------|
| senior     | ${}_{10}C_2 =$                                     |
| junior     | ${}_{8}C_2 =$                                      |
| sophomore  | ${}_{6}C_2 =$                                      |
| first-year | ${}_{4}C_2 =$                                      |

The Multiplication Rule for Counting tells us that we can get the total number of ways to complete this task by multiplying together the number of ways to do each of the four subtasks. So, there are  $45 \times 28 \times 15 \times 6 = 113,400$  possible committees with these restrictions.

## 📌 Your Turn 2.3.2.4

How many ways are there to choose a hand of 6 cards from a standard deck with the constraint that 3 are ♠, 2 are ♥, and 1 is ♣?

## Check Your Understanding

1. Suppose you want to count the number of ways that you can arrange the apps on the home screen on your phone. Should you use permutations or combinations?
2. Your little brother is packing up for a family vacation, but there's only room for 3 of his toys. If you want to know how many possible groups of toys he can bring, should you use permutations or combinations?
3. Compute  ${}_{12}C_{10}$ .
4. Compute  ${}_{16}C_3$ .
5. You're planning a road trip with some friends. Though you have 6 friends you'd consider bringing along, you only have room for 3 other people in the car. How many different possibilities are there for your road trip squad?

6. You're packing for a trip, for which you need 3 shirts and 3 skirts. If you have 8 shirts and 5 skirts that would work for the trip, how many different ways are there to pack for the trip?

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## SECTION OVERVIEW

### 2.4: Probability

#### 2.4.1: Introduction to Probability

#### 2.4.2: Basic Concepts

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## 2.4.1: Introduction to Probability

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The probability of a specified event is the chance or likelihood that it will occur. There are several ways of viewing probability. One would be **experimental** in nature, where we repeatedly conduct an experiment. Suppose we flipped a coin over and over and over again and it came up heads about half of the time; we would expect that in the future whenever we flipped the coin it would turn up heads about half of the time. When a weather reporter says “there is a 10% chance of rain tomorrow,” she is basing that on prior evidence; that out of all days with similar weather patterns, it has rained on 1 out of 10 of those days.

Another view would be **subjective** in nature, in other words an educated guess. If someone asked you the probability that the Seattle Mariners would win their next baseball game, it would be impossible to conduct an experiment where the same two teams played each other repeatedly, each time with the same starting lineup and starting pitchers, each starting at the same time of day on the same field under the precisely the same conditions. Since there are so many variables to take into account, someone familiar with baseball and with the two teams involved might make an educated guess that there is a 75% chance they will win the game; that is, *if* the same two teams were to play each other repeatedly under identical conditions, the Mariners would win about three out of every four games. But this is just a guess, with no way to verify its accuracy, and depending upon how educated the educated guesser is, a subjective probability may not be worth very much.

We will return to the experimental and subjective probabilities from time to time, but in this course we will mostly be concerned with **theoretical** probability, which is defined as follows: Suppose there is a situation with  $n$  equally likely possible outcomes and that  $m$  of those  $n$  outcomes correspond to a particular event; then the **probability** of that event is defined as  $\frac{m}{n}$ .

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## 2.4.2: Basic Concepts

If you roll a die, pick a card from deck of playing cards, or randomly select a person and observe their hair color, we are executing an experiment or procedure. In probability, we look at the likelihood of different outcomes. We begin with some terminology.

### Events and Outcomes

The result of an experiment is called an **outcome**.

An **event** is any particular outcome or group of outcomes.

A **simple event** is an event that cannot be broken down further

The **sample space** is the set of all possible simple events.

### ✓ Example 1

If we roll a standard 6-sided die, describe the sample space and some simple events.

The sample space is the set of all possible simple events:  $\{1, 2, 3, 4, 5, 6\}$

Some examples of simple events:

We roll a 1

We roll a 5

Some compound events:

We roll a number bigger than 4

We roll an even number



### Basic Probability

Given that all outcomes are equally likely, we can compute the probability of an event  $E$  using this formula:

$$P(E) = \frac{\text{Number of outcomes corresponding to the event } E}{\text{Total number of equally - likely outcomes}}$$

### ✓ Example 2

If we roll a 6-sided die, calculate

- P(rolling a 1)
- P(rolling a number bigger than 4)

#### Solution

Recall that the sample space is  $\{1, 2, 3, 4, 5, 6\}$

- There is one outcome corresponding to “rolling a 1”, so the probability is  $\frac{1}{6}$
- There are two outcomes bigger than a 4, so the probability is  $\frac{2}{6} = \frac{1}{3}$

Probabilities are essentially fractions, and can be reduced to lower terms like fractions.

### ✓ Example 3

Let's say you have a bag with 20 cherries, 14 sweet and 6 sour. If you pick a cherry at random, what is the probability that it will be sweet?

#### Solution

There are 20 possible cherries that could be picked, so the number of possible outcomes is 20. Of these 20 possible outcomes, 14 are favorable (sweet), so the probability that the cherry will be sweet is  $\frac{14}{20} = \frac{7}{10}$ .

There is one potential complication to this example, however. It must be assumed that the probability of picking any of the cherries is the same as the probability of picking any other. This wouldn't be true if (let us imagine) the sweet cherries are smaller than the sour ones. (The sour cherries would come to hand more readily when you sampled from the bag.) Let us keep in mind, therefore, that when we assess probabilities in terms of the ratio of favorable to all potential cases, we rely heavily on the assumption of equal probability for all outcomes.

### ? Try it Now 1

At some random moment, you look at your clock and note the minutes reading.

- What is probability the minutes reading is 15?
- What is the probability the minutes reading is 15 or less?

### Answer

There are 60 possible readings, from 00 to 59.

- $\frac{1}{60}$
- $\frac{16}{60}$  (counting 00 through 15)

### 📌 Cards

A standard deck of 52 playing cards consists of four **suits** (hearts, spades, diamonds and clubs). Spades and clubs are black while hearts and diamonds are red. Each suit contains 13 cards, each of a different **rank**: an Ace (which in many games functions as both a low card and a high card), cards numbered 2 through 10, a Jack, a Queen and a King.

### ✓ Example 4

Compute the probability of randomly drawing one card from a deck and getting an Ace.

### Solution

There are 52 cards in the deck and 4 Aces so

$$P(\text{Ace}) = \frac{4}{52} = \frac{1}{13} \approx 0.0769$$

We can also think of probabilities as percents: There is a 7.69% chance that a randomly selected card will be an Ace.

Notice that the smallest possible probability is 0 – if there are no outcomes that correspond with the event. The largest possible probability is 1 – if all possible outcomes correspond with the event.

### 📌 Certain and Impossible events

An **impossible event** has a probability of 0.

A **certain event** has a probability of 1.

The probability of any event must be  $0 \leq P(E) \leq 1$ .

In the course of this chapter, *if you compute a probability and get an answer that is negative or greater than 1, you have made a mistake and should check your work.*

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## SECTION OVERVIEW

### 2.5: Probability Trees

#### 2.5.1: Probability Using Tree Diagrams and Combinations

#### 2.5.2: Probability Using Tree Diagrams and Combinations (Exercises)

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## 2.5.1: Probability Using Tree Diagrams and Combinations

### Learning Objectives

In this section, you will learn to:

1. Use probability tree diagrams to calculate probabilities
2. Use combinations to calculate probabilities

In this section, we will apply previously learnt counting techniques in calculating probabilities, and use tree diagrams to help us gain a better understanding of what is involved.

### USING TREE DIAGRAMS TO CALCULATE PROBABILITIES

We already used tree diagrams to list events in a sample space. Tree diagrams can be helpful in organizing information in probability problems; they help provide a structure for understanding probability. In this section we expand our previous use of tree diagrams to situations in which the events in the sample space are not all equally likely.

We assign the appropriate probabilities to the events shown on the branches of the tree.

By multiplying probabilities along a path through the tree, we can find probabilities for “and” events, which are intersections of events.

We begin with an example.

#### ✓ Example 2.5.1.1

Suppose a jar contains 3 red and 4 white marbles. If two marbles are drawn with replacement, what is the probability that both marbles are red?

##### **Solution**

Let  $E$  be the event that the first marble drawn is red, and let  $F$  be the event that the second marble drawn is red.

We need to find  $P(E \cap F)$ .

By the statement, “two marbles are drawn with replacement,” we mean that the first marble is replaced before the second marble is drawn.

There are 7 choices for the first draw. And since the first marble is replaced before the second is drawn, there are, again, seven choices for the second draw. Using the multiplication axiom, we conclude that the sample space  $S$  consists of 49 ordered pairs. Of the 49 ordered pairs, there are  $3 \times 3 = 9$  ordered pairs that show red on the first draw and, also, red on the second draw. Therefore,

$$P(E \cap F) = \frac{9}{49}$$

Further note that in this particular case

$$P(E \cap F) = \frac{9}{49} = \frac{3}{7} \cdot \frac{3}{7}$$

giving us the result that in this example:  $P(E \cap F) = P(E) \cdot P(F)$

#### ✓ Example 2.5.1.2

If in Example 2.5.1.1, the two marbles are drawn without replacement, then what is the probability that both marbles are red?

##### **Solution**

By the statement, “two marbles are drawn without replacement,” we mean that the first marble is not replaced before the second marble is drawn.

Again, we need to find  $P(E \cap F)$ .

There are, again, 7 choices for the first draw. And since the first marble is not replaced before the second is drawn, there are only six choices for the second draw. Using the multiplication axiom, we conclude that the sample space  $S$  consists of 42 ordered pairs. Of the 42 ordered pairs, there are  $3 \times 2 = 6$  ordered pairs that show red on the first draw and red on the second draw. Therefore,

$$P(E \cap F) = \frac{6}{42}$$

Note that we can break this calculation down as

$$P(E \cap F) = \frac{6}{42} = \frac{3}{7} \cdot \frac{2}{6}$$

Here  $3/7$  represents  $P(E)$ , and  $2/6$  represents the probability of drawing a red on the second draw, given that the first draw resulted in a red.

We write the latter as  $P(\text{red on the second} \mid \text{red on first})$  or  $P(F|E)$ . The "|" represents the word "given" or "if". This leads to the result that:

$$P(E \cap F) = P(E) \cdot P(F|E)$$

This is an important result, called the **Multiplication Rule**, which will appear again in later sections.

We now demonstrate the above results with a tree diagram.

### ✓ Example 2.5.1.3

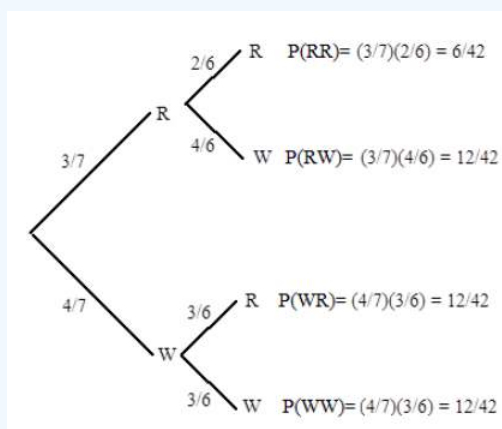
Suppose a jar contains 3 red and 4 white marbles. If two marbles are drawn without replacement, find the following probabilities using a tree diagram.

- The probability that both marbles are red.
- The probability that the first marble is red and the second white.
- The probability that one marble is red and the other white.

#### Solution

Let  $R$  be the event that the marble drawn is red, and let  $W$  be the event that the marble drawn is white.

We draw the following tree diagram.



- The probability that both marbles are red is  $P(RR) = 6/42$
- The probability that the first marble is red and the second is white is  $P(RW) = 12/42$
- For the probability that one marble is red and the other is white, we observe that this can be satisfied if the first is red and the second is white, **or** if the first is white and the second is red. The "or" tells us we'll be using the Addition Rule from Section 7.2.

Furthermore events RW and WR are mutually exclusive events, so we use the form of the Addition Rule that applies to mutually exclusive events.

Therefore

$P(\text{one marble is red and the other marble is white})$

$$\begin{aligned} &= P(RW \text{ or } WR) \\ &= P(RW) + P(WR) \\ &= 12/42 + 12/42 = 24/42 \end{aligned}$$

## USING COMBINATIONS TO FIND PROBABILITIES

Although the tree diagrams give us better insight into a problem, they are not practical for problems where more than two or three things are chosen. In such cases, we use the concept of combinations that we learned in the last chapter. This method is best suited for problems where the order in which the objects are chosen is not important, and the objects are chosen without replacement.

### ✓ Example 2.5.1.4

Suppose a jar contains 3 red, 2 white, and 3 blue marbles. If three marbles are drawn without replacement, find the following probabilities.

- $P(\text{Two red and one white})$
- $P(\text{One of each color})$
- $P(\text{None blue})$
- $P(\text{At least one blue})$

#### Solution

Let us suppose the marbles are labeled as  $R_1, R_2, R_3, W_1, W_2, B_1, B_2, B_3$ .

- $P(\text{Two red and one white})$

Since we are choosing 3 marbles from a total of 8, there are  $8C3 = 56$  possible combinations. Of these 56 combinations, there are  $3C2 \times 2C1 = 6$  combinations consisting of 2 red and one white. Therefore,

$$P(\text{Two red and one white}) = \frac{3C2 \times 2C1}{8C3} = \frac{6}{56}.$$

- $P(\text{One of each color})$

Again, there are  $8C3 = 56$  possible combinations. Of these 56 combinations, there are  $3C1 \times 2C1 \times 3C1 = 18$  combinations consisting of one red, one white, and one blue. Therefore,

$$P(\text{One of each color}) = \frac{3C1 \times 2C1 \times 3C1}{8C3} = \frac{18}{56}$$

- $P(\text{None blue})$

There are 5 non-blue marbles, therefore

$$P(\text{None blue}) = \frac{5C3}{8C3} = \frac{10}{56} = \frac{5}{28}$$

- $P(\text{At least one blue})$

By "at least one blue marble," we mean the following: one blue marble and two non-blue marbles, OR two blue marbles and one non-blue marble, OR all three blue marbles. So we have to find the sum of the probabilities of all three cases.

$$P(\text{At least one blue}) = P(1 \text{ blue, 2 non-blue}) + P(2 \text{ blue, 1 non-blue}) + P(3 \text{ blue})$$

$$P(\text{At least one blue}) = \frac{3C1 \times 5C2}{8C3} + \frac{3C2 \times 5C1}{8C3} + \frac{3C3}{8C3}$$

$$P(\text{At least one blue}) = 30/56 + 15/56 + 1/56 = 46/56 = 23/28$$

Alternately, we can use the fact that  $P(E) = 1 - P(E^c)$ . If the event  $E$  = At least one blue, then  $E^c$  = None blue. But from part c of this example, we have  $P(E^c) = 5/28$ , so  $P(E) = 1 - 5/28 = 23/28$ .

#### ✓ Example 2.5.1.5

Five cards are drawn from a deck. Find the probability of obtaining two pairs, that is, two cards of one value, two of another value, and one other card.

##### Solution

Let us first do an easier problem-the probability of obtaining a pair of kings and queens.

Since there are four kings, and four queens in the deck, the probability of obtaining two kings, two queens and one other card is

$$P(\text{A pair of kings and queens}) = \frac{4C2 \times 4C2 \times 44C1}{52C5}$$

To find the probability of obtaining two pairs, we have to consider all possible pairs.

Since there are altogether 13 values, that is, aces, deuces, and so on, there are  $13C2$  different combinations of pairs.

$$P(\text{Two pairs}) = 13C2 \cdot \frac{4C2 \times 4C2 \times 44C1}{52C5} = .04754$$

#### ✓ Example 2.5.1.6

A cell phone store receives a shipment of 15 cell phones that contains 8 iPhones and 7 Android phones. Suppose that 6 cell phones are randomly selected from this shipment. Find the probability that a randomly selected set of 6 cell phones consists of 2 iPhones and 4 Android phones.

##### Solution

There are  $8C2$  ways of selecting 2 out of the 8 iPhones.

and  $7C4$  ways of selecting 4 out of the 7 Android phones

But altogether there are  $15C6$  ways of selecting 6 out of 15 cell phones.

Therefore we have

$$P(2 \text{ iPhones and } 4 \text{ Android phones}) = \frac{8C2 \times 7C4}{15C6} = \frac{(28)(35)}{5005} = \frac{980}{5005} = 0.1958$$

#### ✓ Example 2.5.1.7

One afternoon, a bagel store still has 53 bagels remaining: 20 plain, 15 poppyseed, and 18 sesame seed bagels. Suppose that the store owner packages up a bag of 9 bagels to bring home for tomorrow's breakfast, and selects the bagels randomly. Find the probability that the bag contains 4 plain, 3 poppyseed, and 2 sesame seed.

##### Solution

There are  $20C4$  ways of selecting 4 out of the 20 plain bagels,

and  $15C3$  ways of selecting 3 out of the 15 poppyseed bagels,

and  $18C2$  ways of selecting 2 out of the 18 sesame seed bagels.

But altogether there are  $53C9$  ways of selecting 9 out of the 53 bagels.

$$\begin{aligned} P(4 \text{ plain, } 3 \text{ poppyseed, and } 2 \text{ sesame seed}) &= \frac{20C4 \times 15C3 \times 18C2}{53C9} \\ &= \frac{(4845)(455)(153)}{4431613550} & (2.5.1.1) \\ &= 0.761 \end{aligned}$$

We end the section by solving a famous problem called the **Birthday Problem**.

✓ Example 2.5.1.8: Birthday Problem

If there are 25 people in a room, what is the probability that at least two people have the same birthday?

**Solution**

Let event E represent that at least two people have the same birthday.

We first find the probability that no two people have the same birthday.

We analyze as follows.

Suppose there are 365 days to every year. According to the multiplication axiom, there are  $365^{25}$  possible birthdays for 25 people. Therefore, the sample space has  $365^{25}$  elements. We are interested in the probability that no two people have the same birthday. There are 365 possible choices for the first person and since the second person must have a different birthday, there are 364 choices for the second, 363 for the third, and so on. Therefore,

$$P(\text{No two have the same birthday}) = \frac{365 \cdot 364 \cdot 363 \cdots 341}{365^{25}} = \frac{365P_{25}}{365^{25}}$$

Since  $P(\text{at least two people have the same birthday}) = 1 - P(\text{No two have the same birthday})$ ,

$$P(\text{at least two people have the same birthday}) = 1 - \frac{365P_{25}}{365^{25}} = .5687$$

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## 2.5.2: Probability Using Tree Diagrams and Combinations (Exercises)

### SECTION 8.3 PROBLEM SET: PROBABILITIES USING TREE DIAGRAMS AND COMBINATIONS

Two apples are chosen from a basket containing five red and three yellow apples.

Draw a tree diagram below, and find the following probabilities.

|                            |                                            |
|----------------------------|--------------------------------------------|
| 1) $P(\text{both red})$    | 2) $P(\text{one red, one yellow})$         |
| 3) $P(\text{both yellow})$ | 4) $P(\text{First red and second yellow})$ |

A basket contains six red and four blue marbles. Three marbles are drawn at random.

Find the following probabilities using the method shown in Example 8.3.2. Do not use combinations.

|                                  |                                                  |
|----------------------------------|--------------------------------------------------|
| 5) $P(\text{All three red})$     | 6) $P(\text{two red, one blue})$                 |
| 7) $P(\text{one red, two blue})$ | 8) $P(\text{first red, second blue, third red})$ |

Three marbles are drawn from a jar containing five red, four white, and three blue marbles.

Find the following probabilities using combinations.

|                              |                                      |
|------------------------------|--------------------------------------|
| 9) $P(\text{all three red})$ | 10) $P(\text{two white and 1 blue})$ |
| 11) $P(\text{none white})$   | 12) $P(\text{at least one red})$     |

A committee of four is selected from a total of 4 freshmen, 5 sophomores, and 6 juniors. Find the probabilities for the following events.

|                                      |                                                         |
|--------------------------------------|---------------------------------------------------------|
| 13) At least three freshmen.         | 14) No sophomores.                                      |
| 15) All four of the same class.      | 16) Not all four from the same class.                   |
| 17) Exactly three of the same class. | 18) More juniors than freshmen and sophomores combined. |

Five cards are drawn from a deck. Find the probabilities for the following events.

|                                                                                            |                                                                                                                                                       |
|--------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------|
| 19) Two hearts, two spades, and one club.                                                  | 20) A flush of any suit ( <i>all cards of a single suit</i> ).                                                                                        |
| 21) A full house of nines and tens ( <i>3 nines and 2 tens</i> ).                          | 22) Any full house.                                                                                                                                   |
| 23) A pair of nines and a pair of tens ( <i>and the fifth card is not a nine or ten</i> ). | 24) Any two pairs ( <i>two cards of one value, two more cards of another value, and the fifth card does not have the same value as either pair</i> ). |

Jorge has 6 rock songs, 7 rap songs and 4 country songs that he likes to listen to while he exercises.

He randomly selects six (6) of these songs to create a playlist to listen to today while he exercises.

Find the following probabilities:

|                                                                |                                                                           |
|----------------------------------------------------------------|---------------------------------------------------------------------------|
| 25) $P(\text{playlist has 2 songs of each type})$              | 26) $P(\text{playlist has no country songs})$                             |
| 27) $P(\text{playlist has 3 rock, 2 rap, and 1 country song})$ | 28) $P(\text{playlist has 3 or 4 rock songs and the rest are rap songs})$ |

A project is staffed 12 people: 5 engineers, 4 salespeople, and 3 customer service representatives.

A committee of 5 people is selected to make a presentation to senior management.

Find the probabilities of the following events.

|  |
|--|
|  |
|--|

29) The committee has 2 engineers, 2 salespeople, and 1 customer service representative.

31) The committee has no engineers.

30) The committee contains 3 engineer and 2 salespeople.

32) The committee has all salespeople.

Do the following birthday problems.

33) If there are 5 people in a room, what is the probability that no two have the same birthday?

34) If there are 5 people in a room, find the probability that at least 2 have the same birthday.

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## CHAPTER OVERVIEW

### 3: Number Bases and Modular Arithmetic

#### 3.1: Binary Numbers

##### 3.1.1: Bases

##### 3.1.2: Binary (base 2)

#### 3.2: The Development and Use of Different Number Bases

#### 3.3: Addition and Subtraction with Other Bases

#### 3.4: Calculator Shortcut for Modular Arithmetic

#### 3.5: Cryptography

##### 3.5.1: Cryptography

##### 3.5.2: Substitution Ciphers

##### 3.5.3: Public Key Cryptography

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## SECTION OVERVIEW

### 3.1: Binary Numbers

#### 3.1.1: Bases

#### 3.1.2: Binary (base 2)

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### 3.1.1: Bases

As I mentioned, a **base** is simply a number that's an anchor for our place value system. It represents *how many distinct symbols we will use to represent numbers*. This implicitly sets the value of the largest quantity we can hold in one digit, before we'd need to "roll over" to two digits.

In base 10 (decimal), we use ten symbols: 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. Consequently, the number nine is the highest value we can hold in a single digit. Once we add another element to a set of nine, we have no choice but to add another digit to express it. This makes a "ten's place" because it will represent the number of sets-of-10 (which we couldn't hold in the 1's place) that the value contains.

Now why is the next place over called the "hundred's place" instead of, say, the "twenty's place"? Simply because twenty — as well as every other number less than a hundred — comfortably fits in two digits. We can have up to 9 in the one's place, and also *up to 9 in the ten's place*, giving us a total of ninety-nine before we ever have to cave in to using three digits. The number one hundred is exactly the point at which we *must* roll over to three digits; therefore, the sequence of digits 1-0-0 represents one hundred.

If the chosen base isn't obvious from context (as it often won't be in this chapter) then when we write out a sequence of digits we'll append the base as a subscript to the end of the number. So the number "four hundred and thirty-seven" will be written as  $437_{10}$ .

The way we interpret a decimal number, then, is by counting the right-most digits as a number of *individuals*, the digit to its left as the number of *groups of ten* individuals, the digit to its left as the number of groups of hundred individuals, and so on.  $5472_{10}$  is just a way of writing  $5 \times 1000 + 4 \times 100 + 7 \times 10 + 2 \times 1$ .

If we use exponential notation (remember that anything to the  $0^{\text{th}}$  power is 1), this is equivalent to:

$$5472_{10} = 5 \times 10^3 + 4 \times 10^2 + 7 \times 10^1 + 2 \times 10^0. \quad (3.1.1.1)$$

By the way, we will often use the term **least significant digit** to refer to the right-most digit (2, in the above example), and **most significant digit** to refer to the left-most (5). "Significant" simply refers to how much that digit is "worth" in the overall magnitude of the number. Obviously 239 is less than 932, so we say that the hundreds place is more significant than the other digits.

All of this probably seems pretty obvious to you. All right then. Let's use a base other than ten and see how you do. Let's write out a number *in base 7*. We have seven symbols at our disposal: 0, 1, 2, 3, 4, 5, and 6. Wait, you ask — why not 7? Because there is no digit for seven in a base 7 system, just like there is no digit for ten in a base 10 system. Ten is the point where we need *two* digits in a decimal system, and analogously, seven is the point where we'll need two digits in our base 7 system. How will we write the value seven? Just like this: **10**. Now stare at those two digits and practice saying "seven" as you look at them. All your life you've been trained to say the number "ten" when you see the digits 1 and 0 printed like that. But those two digits only represent the number ten *if you're using a base 10 system*. If you're using a base 34 system, "10" is how you write "thirty-four."

Very well, we have our seven symbols. Now how do we interpret a number like  $6153_7$ ? It's this:

$$6153_7 = 6 \times 7^3 + 1 \times 7^2 + 5 \times 7^1 + 3 \times 7^0. \quad (3.1.1.2)$$

That doesn't look so strange: it's very parallel to the decimal string we expanded, above. It looks weirder when we actually multiply out the place values:

$$6153_7 = 6 \times 343 + 1 \times 49 + 5 \times 7 + 3 \times 1. \quad (3.1.1.3)$$

So in base 7, we have a "one's place," a "seven's place," a "forty-nine's place," and a "three hundred forty-three's place." This seems unbelievably bizarre — how could a number system possibly hold together with such place values? — but I'll bet it wouldn't look funny at all if we had been born with 7 fingers. Keep in mind that in the equation above, we wrote out the place values as decimal numbers! Had we written them as base-7 numbers (as we certainly would have if base 7 was our natural numbering system), we would have written:

$$6153_7 = 6 \times 1000_7 + 1 \times 100_7 + 5 \times 10_7 + 3 \times 1_7. \quad (3.1.1.4)$$

This is exactly equivalent numerically. Because after all,  $1000_7$  is  $343_{10}$ . A quantity that looks like an oddball in one base system looks like the roundest possible number in another.

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### 3.1.2: Binary (base 2)

The other base we commonly use in computer science is base 2, or **binary**. This is because the basic unit of information in a computer is called a **bit**, which has only two values, conventionally called either “true” and “false” or “1” and “0”. Numbers (as well as everything else) are ultimately represented as colossal sequences of 1’s and 0’s, which are of course binary numbers.

The rules for interpreting place value are the same:

$$\begin{aligned} 110101_2 &= 1 \times 2^5 + 1 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 \\ &= 1 \times 32 + 1 \times 16 + 0 \times 8 + 1 \times 4 + 0 \times 2 + 1 \times 1 \\ &= 53_{10}. \end{aligned}$$

So in binary we have a one’s-place, a two’s-place, a four’s-place, an eight’s-place, and so on. We call the right-most place the **least significant bit (LSB)** and the left-most the **most significant bit (MSB)**.

Counting up from zero is really just the same as any other base, although it feels a little strange in binary because you “roll over” so often:

|          |          |
|----------|----------|
| $0_2$    | zero     |
| $1_2$    | one      |
| $10_2$   | two      |
| $11_2$   | three    |
| $100_2$  | four     |
| $101_2$  | five     |
| $110_2$  | six      |
| $111_2$  | seven    |
| $1000_2$ | eight    |
| $1001_2$ | nine     |
| $\vdots$ | $\vdots$ |

#### Converting to and from decimal

Converting from binary to decimal was demonstrated above (with  $110101_2 = 53_{10}$ .) To go the other way, we follow the algorithm from page . Let’s try it for the decimal number 49:

1. (Step 1) We first compute  $49 \bmod 2$ . Doing “mod 2” is easy: you just see whether the number is even or odd. In this case, it’s odd, so the remainder is a 1:

1

2. (Step 2) Now divide 49 by 2 and take the floor, which gives  $\lfloor 49 \div 2 \rfloor = 24$ . It’s not zero, so we perform step 2.2: make 24 our new value, move our pencil to the left of the 1, and go back to step 1.

3. (Step 1) Compute  $24 \bmod 2$ . Since 24 is even, this is zero, which we write down to the left of the 1:

01

4. (Step 2) Divide 24 by 2 and take the floor, which gives  $\lfloor 24 \div 2 \rfloor = 12$ . Make 12 our new value, move our pencil to the left of the 0, and go back to step 1.

5. (Step 1) Compute  $12 \bmod 2$ . Since 12 is even, this is zero, which we write down:

001

6. (Step 2) Divide 12 by 2 and take the floor, which gives  $\lfloor 12 \div 2 \rfloor = 6$ . Make 6 our new value, move our pencil to the left of the 0, and go back to step 1.

7. (Step 1) Compute  $6 \bmod 2$ . Since 6 is even, this is zero, which we write down:

0001

8. (Step 2) Divide 6 by 2 and take the floor, which gives  $\lfloor 6 \div 2 \rfloor = 3$ . Make 3 our new value, move our pencil to the left of the 0, and go back to step 1.

9. (Step 1) Compute  $3 \bmod 2$ . Since 3 is odd, this is one, which we write down:

10001

10. (Step 2) Divide 3 by 2 and take the floor, which gives  $\lfloor 3 \div 2 \rfloor = 1$ . This still isn't zero, so make 1 our new value, move our pencil to the left of the 0, and go back to step 1.

11. (Step 1) Compute  $1 \bmod 2$ . Since 1 is odd, this is one, which we write down:

110001

12. (Step 2) Divide 1 by 2 and take the floor, which gives  $\lfloor 1 \div 2 \rfloor = 0$ . We're done. The final answer is  $110001_2$ . Double-checking our work, we verify that indeed one 32 plus one 16 plus one 1 gives 49, which is what we started with.

### Converting to and from hex

That was pretty tedious. But converting back and forth from binary to *hex* is a snap. That's because 16 is exactly  $2^4$ , and so one hex digit is exactly equal to four binary digits. This isn't the case with base 10, where one decimal digit is equal to three binary digits... *plus* a little extra. This "not quite a whole number of digits" thing is what makes converting from decimal to binary (or decimal to hex, for that matter) so awkward.

We most commonly deal with sets of eight bits at a time, which is called a **byte**. (This is the fundamental unit of storage on pretty much every computer on earth.) Suppose I had the following byte:

$10000110_2$

Because one hex digit is exactly equal to four bits, this byte is exactly equal to:

$86_{16}$

This is because the byte can be neatly split into two parts:  $1000$ , which corresponds to the hex digit 8, and  $0110$ , which corresponds to the hex digit 6. These two halves are called **nibbles** — one byte has two nibbles, and each nibble is one hex digit. At a glance, therefore, with no multiplying or adding, we can convert from binary to hex.

Going the other direction is just as easy. If we have:

$3E_{16}$

we just convert each hex digit into the corresponding nibble:

$00111110_2$

After you do this a while, you get to the point where you can instantly recognize which hex digit goes with which nibble value. Until then, though, here's a handy table:

| nibble | hex digit |
|--------|-----------|
| 0000   | 0         |
| 0001   | 1         |
| 0010   | 2         |
| 0011   | 3         |
| 0100   | 4         |

|      |   |
|------|---|
| 0101 | 5 |
| 0110 | 6 |
| 0111 | 7 |
| 1000 | 8 |
| 1001 | 9 |
| 1010 | A |
| 1011 | B |
| 1100 | C |
| 1101 | D |
| 1110 | E |
| 1111 | F |

In case you're wondering, yes this is worth memorizing.

### Adding binary numbers

Adding two binary numbers is the same as adding in decimal, hexadecimal, or any other base: you just have to know when to “roll over the odometer,” which in this case is almost instantly, since the highest value a bit can hold is 1! Let's give it a shot:

$$\begin{array}{r}
 1\ 1\ 1\ 0\ 0\ 1_2 \\
 +\ 0\ 1\ 1\ 0\ 1\ 0_2 \\
 \hline
 \phantom{1\ 1\ 1\ 0\ 0\ 1_2} ?_2
 \end{array}
 \tag{3.1.2.1}$$

A child could follow the rules: when we add two zeroes, we get zero. Adding a one to a zero gives one. Adding two ones gives zero, and a carry to the next significant digit. And adding two ones plus a carry gives a one and a carry. See if you can follow the flow:

$$\begin{array}{r}
 1\ 1 \\
 1\ 1\ 1\ 0\ 0\ 1_2 \\
 +\ 0\ 1\ 1\ 0\ 1\ 0_2 \\
 \hline
 1\ 0\ 1\ 0\ 0\ 1\ 1_2
 \end{array}
 \tag{3.1.2.2}$$

### Capacity

How large a value can a byte store? There are 8 bits, and each one can independently have either of two values (0 or 1), so by the Fundamental Theorem of Counting, there are  $2^8$  different combinations. This works out to 256, but we can't actually store the number 256 in a byte if we're using the bit pattern  $00000000_2$  (or  $00_{16}$ ) to represent zero. The highest value would be  $11111111_2$  (or  $FF_{16}$ ), which is  $256_{10}$ .

How do we store a number larger than that? Simply use more than one byte, of course. If we used two bytes of memory, and treated them as concatenated one after the other, that would give us 16 bits, allowing us to store up to the number  $0000000000000000_2 = FFFF_{16} = 65,535_{10}$ . We'd call one of these bytes — the one representing the  $2^0$ 's place up to the  $2^7$ 's place — the least significant *byte*, and the other one — containing places  $2^8$  through  $2^{15}$  — the most significant byte. Extending to more than two bytes to accommodate even larger numbers is done in the obvious way.

### Binary representation schemes

That's mostly all there is to it. But there's one thing we haven't discussed yet, and that's *negative* numbers. We know how to represent any positive number (or zero) with an ordinary place value scheme. But how do we store a number like  $-5$ ?

There are three different schemes for treating negative numbers, each with its strengths and weaknesses.

### Unsigned

The simplest scheme is called **unsigned**, and it simply means that we don't *allow* negative numbers. For one byte, we have 256 different bit patterns at our disposal, and we might just choose to allocate them all to represent positive numbers, so as to get the widest range. This makes sense for, say, a C++ program variable called `heightInInches` which we know can never meaningfully be negative (no one has a negative height).

The advantage of this scheme is simply that we can represent the greatest possible range of positive numbers, which is sometimes the goal. Each of the alternative schemes carves off a chunk of these available bit patterns and devotes them to representing negative numbers, leaving fewer left over for positive numbers. There's no free lunch: you have to decide how you want to "spend" your available bit patterns depending on what values you need to represent.

### Sign-magnitude

The **sign-magnitude** scheme is probably the first thing you'd think of to solve the negative number representation problem. We need to store the sign of the number somehow, and a sign is inherently a two-valued thing (either positive or negative), so why not peel off one of the bits and use it to represent the sign? The remaining bits can then be used in the ordinary way to represent the magnitude of the number.

The way this is most often done is to take the left-most bit and use it as the **sign bit**. This bit now has *no other meaning*. It can't "double" as the 128's place, because then there'd be no way to distinguish between, say, 129 and  $-129$  (each would be represented with `10000001`.) No, the sign bit must be considered "spent money," and its expressive power cannot be reclaimed to also represent part of the magnitude. By convention, if the sign bit is 0 this represents a *positive* number, and a sign bit of 1 represents a *negative* number. (That might seem counterintuitive, but hey, that's the way it is.)

So this number in sign-magnitude:

`00100110`

represents the decimal number 38. That's because the sign bit (bolded, on the far left) is 0, which means the number is positive. The magnitude of the number is contained in the other 7 bits, which gives  $32 + 4 + 2 = 38$ . This number, on the other hand:

`10100110`

represents  $-38$ . The magnitude is the same, but the sign bit is 1 so this pattern now "means" a negative number.

Clearly we have reduced our range of positive numbers in exchange for the ability to also store negatives. We have 7 bits of range instead of 8, so instead of 255, our highest possible value is merely 127. On the other end, the lowest possible value is  $-127$ .

If you have sharp eyes, you may have noticed a discrepancy in the counting. With the sign-magnitude approach, we can hold numbers in the range  $-127$  to 127. But wait: that's only 255 different values, not 256! Why did we lose one value of expressive power? The answer is that the sign-magnitude scheme has *two ways* of representing *zero*. The bit pattern `00000000` is obviously zero, but so is `10000000` (which you might call "negative zero.") Using two different patterns to represent the same value is a little wasteful, but the situation is actually worse than that. Having to account for both patterns means that computer hardware using the sign-magnitude scheme is inevitably more complicated. To compare two bytes to see if they're equal, you'd think we'd just compare each bit position, and if they were all the same, the bytes would be declared equal, otherwise no. Alas, this is no longer quite that simple. The two zero patterns must be considered numerically equal, so our digital logic now has to contain a special case. "To be equal, all the bits have to be the same...oh, but actually not if the right-most seven are all zeroes in both bytes. In that case, it doesn't matter what the left-most bit contains." Maddening.

### Two's-complement

This shortcoming in the sign-magnitude scheme is remedied with the **two's-complement** scheme, which is the one actually used most often in practice. It'll seem weird at first — certainly not as intuitive as the first two — but it leads to a critically important feature that we'll look at shortly.

First, the rules. To interpret a two's-complement number, you:

1. Look at the left-most bit (just like in sign-magnitude). If it's a 0, you have a positive number. If it's a 1, you have a negative number.

2. If it's a positive number, the other 7 bits give you the magnitude (just like in sign-magnitude).
3. If, however, it's a negative number, then to discover the magnitude of that negative number you must *flip all the bits and add one*. This will give you a positive number which is the absolute value of your negative number.

Easy example: take the byte `00100110`. The left-most bit is a 0, which means it's a positive number, and as we discovered above, the remaining 7 bits give a magnitude of 38. So this is the number 38.

Harder example: take the byte `10100110`. The left-most bit is a 1, which means it's negative. Okay: negative *what*? How do we find the magnitude? Well, we "flip" all the bits (*i.e.*, invert each one from 0 to 1 or vice versa) to get:

01011001

and then add one to the result:

$$\begin{array}{r}
 \phantom{0}1 \\
 0 \ 1 \ 0 \ 1 \ 1 \ 0 \ 0 \ 1 \\
 + \phantom{0000000}1 \\
 \hline
 0 \ 1 \ 0 \ 1 \ 1 \ 0 \ 1 \ 0
 \end{array}
 \tag{3.1.2.3}$$

This black magic produces the value  $01011010_2$ , which converts to  $90_{10}$ . **This means that the original number, `10100110`, corresponds to the value  $-90$ .**

"Flipping all the bits and adding one" is the cookbook procedure for taking the complement (negative) of a number in the two's-complement scheme. It works in reverse, too. Let's start with 90 this time and crank through the process again, making sure we get  $-90$ .

Start with the binary representation of  $90_{10}$ :

$$0 \ 1 \ 0 \ 1 \ 1 \ 0 \ 1 \ 0 \tag{3.1.2.4}$$

Flip all the bits to get:

$$1 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 1 \tag{3.1.2.5}$$

and finally add one to the result:

$$\begin{array}{r}
 \phantom{0}1 \\
 1 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 1 \\
 + \phantom{0000000}1 \\
 \hline
 1 \ 0 \ 1 \ 0 \ 0 \ 1 \ 1 \ 0
 \end{array}
 \tag{3.1.2.6}$$

We get `10100110`, which was precisely the number we originally began with, and which we have already determined represents  $-90$ .

Now you may ask what we gain from all this. Surely this scheme is considerably more convoluted than the simple idea of reserving one bit as a sign bit, and treating the rest as a magnitude. But it turns out there is indeed a method to the madness. Strange as it sounds, a two's-complement representation scheme allows us to *perform addition and subtraction with a single operation*.

In first grade (or so), you learned the procedure for adding multi-digit numbers, which we've followed several times in this chapter. It involves adding the digits right-to-left and possibly "carrying." Then in second grade (or so), you learned the procedure for *subtracting* multi-digit numbers. It involves subtracting the digits right-to-left and possibly "borrowing." If you're like me, you found adding easier than subtracting. It's easy to just carry the one, but to borrow requires looking at the digit to the left, making sure that you *can* borrow from it (*i.e.*, that it's not already 0), borrowing from further left until you actually find an available non-zero value, hoping the number on the bottom is actually less than the one on the top (because otherwise you have to switch the order and then add a negative sign to the result), and keeping all of that straight as you march down the line.

Even if you didn't find subtracting more difficult than adding, though, you can't argue that it's still a completely *different* algorithm, with different rules to follow. In computer hardware, we have to implement different circuitry to perform each operation, which is more difficult, costly, error-prone, and power-draining.

The wonderful thing about two's-complement, however, is that with this scheme we actually *never need to use the subtraction algorithm*. If we want to subtract two numbers — say,  $24 - 37$  — we can instead take the complement of the second number and then add them. Instead of  $24 - 37$  we compute  $24 + (-37)$ .

Let's see it in action. Using conversion procedures, we can figure out that  $24_{10}$  is:

$$0 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0 \ 0 \quad (3.1.2.7)$$

and that *positive*  $37_{10}$  is:

$$0 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 1 \quad (3.1.2.8)$$

If we wanted to compute  $24 + 37$ , we'd just add these. But instead we're looking for  $24 - 37$ , so we'll take the complement of 37 to find  $-37$ . Flip all the bits of 37:

$$1 \ 1 \ 0 \ 1 \ 1 \ 0 \ 1 \ 0 \quad (3.1.2.9)$$

and add one:

$$\begin{array}{r} 1 \ 1 \ 0 \ 1 \ 1 \ 0 \ 1 \ 0 \\ + \phantom{00000000} 1 \\ \hline 1 \ 1 \ 0 \ 1 \ 1 \ 0 \ 1 \ 1 \end{array} \quad (3.1.2.10)$$

and so now we've determined that in the two's-complement scheme,  $-37$  is represented by  $11011011_2$ .

We're now ready to compute  $24 + (-37)$ :

$$\begin{array}{r} \phantom{00} 1 \ 1 \\ 0 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0 \ 0 \quad \leftarrow \text{this is } 24_{10} \\ + \ 1 \ 1 \ 0 \ 1 \ 1 \ 0 \ 1 \ 1 \quad \leftarrow \text{this is } -37_{10} \\ \hline 1 \ 1 \ 1 \ 1 \ 0 \ 0 \ 1 \ 1 \end{array} \quad (3.1.2.11)$$

So we have our two's-complement answer,  $11110011$ . What value does that correspond to? Well, the left-most bit is a 1, so it's a negative number. To find out what it's the negative of, flip all the bits and add one:

$$\begin{array}{r} 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0 \quad \leftarrow \text{flip the bits to get} \\ + \phantom{00000000} 1 \quad \leftarrow \text{add one} \\ \hline 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 0 \ 1 \end{array} \quad (3.1.2.12)$$

This is positive 13, which means the number we inverted to get it —  $11110011$  — must represent  $-13$ . And that is indeed the correct answer, for  $24 - 37 = -13$ .

One last word on two's-complement: what is the *range* of numbers we can represent? It turns out to be -128 to 127. The highest value is  $01111111$ , which is 127. You might think the lowest value would be represented as  $11111111$ , but if you work it out, you'll find that this is actually the number  $-1$ . The lowest number is actually the bit pattern  $10000000$ , which is  $-128$ .

## Overflow

One last sticky detail we need to cover has to do with **overflow**. When we add two numbers, there is the possibility that the result will contain one more digit than the original numbers did. You've probably seen this on a hand calculator when you press "=" and get an "E" (for "error") in the display. If there are only ten digits on your display, adding two ten-digit numbers will (sometimes) result in an eleven-digit number that your calculator can't display, and it's alerting you to that fact so you don't misinterpret the result. Here, we might add two 8-bit quantities and end up with a 9-bit quantity that can't fit in one byte. This situation is called overflow, and we need to detect when it occurs.

The rules for detecting overflow are different depending on the scheme. For *unsigned* numbers, the rule is simple: if a 1 is carried out from the MSB (far left-side), then we have overflow. So if I were to try to add  $155_{10}$  and  $108_{10}$ :

$$\begin{array}{rcccccccc}
 & 1 & 1 & 1 & 1 & & & & \\
 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & \leftarrow 155_{10} \\
 + & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & \leftarrow 108_{10} \\
 \hline
 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1
 \end{array}
 \tag{3.1.2.13}$$

then I get a carry out left into the 9th digit. Since we can only hold eight digits in our result, we would get a nonsensical answer ( $15_{10}$ ), which we can detect as bogus because the carry out indicated overflow.

Sign-magnitude works the same way, except that I have one fewer bit when I'm adding and storing results. (Instead of a byte's worth of bits representing magnitude, the left-end bit has been reserved for a special purpose: indicating the number's sign. Therefore, if I add the remaining 7-bit quantities and get a carry out left into the *eighth* digit, that would indicate overflow.)

Now with two's-complement, things are (predictably) not that easy. But it turns out they're *almost* as easy. There's still a simple rule to detect overflow, it's just a different rule. The rule is: if the carry *in* to the last (left-most) bit is *different* than the carry *out* from the last bit, then we have overflow.

Let's try adding  $103_{10}$  and  $95_{10}$  in two's-complement, two numbers which fit in our -128 to 127 range, but whose sum will not:

$$\begin{array}{rcccccccc}
 \text{carry-in} \rightarrow & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \\
 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & \leftarrow 103_{10} \\
 + & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & \leftarrow 95_{10} \\
 \hline
 \text{carry-out} \rightarrow & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0
 \end{array}
 \tag{3.1.2.14}$$

The carry-in to the last bit was 1, but the carry-out was 0, so for two's-complement this means we detected overflow. It's a good thing, too, since  $11000110$  in two's-complement represents  $-57_{10}$ , which is certainly not  $103 + 95$ .

Essentially, if the carry-in is not equal to the carry-out, that means we added two positive numbers and came up with a negative number, or that we added two negatives and got a positive. Clearly this is an erroneous result, and the simple comparison tells us that. Just be careful to realize that the rule for detecting overflow depends *totally* on the particular representation scheme we're using. A carry-out of 1 always means overflow...*in the unsigned scheme*. For two's-complement, we can easily get a carry-out of 1 with no error at all, provided the carry-in is *also* 1.

### "It's all relative"

Finally, if we come up for air out of all this mass of details, it's worth emphasizing that there is no intrinsically "right" way to interpret a binary number. If I show you a bit pattern — say,  $11000100$  — and ask you what value it represents, you can't tell me without knowing how to interpret it.

If I say, "oh, that's an unsigned number," then you'd treat each bit as a digit in a simple base 2 numbering scheme. You'd add  $2^7 + 2^6 + 2^2$  to get 196, then respond, "ah, then that's the number  $196_{10}$ ." And you'd be right.

But if I say, "oh, that's a sign-magnitude number," you'd first look at the leftmost bit, see that it's a 1, and realize you have a negative number. Then you'd take the *remaining* seven bits and treat them as digits in a simple base 2 numbering scheme. You'd add  $2^6 + 2^2$  to get 68, and then respond, "ah, then that's the number  $-68_{10}$ ." And you'd be right.

But then again, if I say, "oh, that's a two's-complement number," you'd first look at the leftmost bit, see that it's a 1, and realize you're dealing with a negative number. What is it the negative of? You'd flip all the bits and add one to find out. This would give you  $00111100$ , which you'd interpret as a base 2 number and get  $60_{10}$ . You'd then respond, "ah, then that's the number  $-60_{10}$ ." And you'd be right.

So what does  $11000100$  represent then?? Is it 196,  $-68$ , or  $-60$ ? The answer is *any of the three*, depending on what representation scheme you're using. None of the data in computers or information systems has intrinsic meaning: it all has to be interpreted according to the syntactic and semantic rules that we invent. In math and computer science, anything can be made to mean anything: after all, we invent the rules.

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## 3.2: The Development and Use of Different Number Bases

### Introduction and Basics

During the previous discussions, we have been referring to positional base systems. In this section of the chapter, we will explore exactly what a base system is and what it means if a system is “positional.” We will do so by first looking at our own familiar, base-ten system and then deepen our exploration by looking at other possible base systems. In the next part of this section, we will journey back to [Mayan civilization](#) and look at their unique base system, which is based on the number 20 rather than the number 10.

A base system is a structure within which we count. The easiest way to describe a base system is to think about our own base-ten system. The base-ten system, which we call the “decimal” system, requires a total of ten different symbols/digits to write any number. They are, of course, 0, 1, 2, ..... 9.

The decimal system is also an example of a *positional* base system, which simply means that the position of a digit gives its place value. Not all civilizations had a positional system even though they did have a base with which they worked.

In our base-ten system, a number like 5,783,216 has meaning to us because we are familiar with the system and its places. As we know, there are six ones, since there is a 6 in the ones place. Likewise, there are seven “hundred thousands,” since the 7 resides in that place. Each digit has a value that is explicitly determined by its position within the number. We make a distinction between digit, which is just a symbol such as 5, and a number, which is made up of one or more digits. We can take this number and assign each of its digits a value. One way to do this is with a table, which follows:

|           |                                                                        |                   |                        |
|-----------|------------------------------------------------------------------------|-------------------|------------------------|
| 5,000,000 | $= 5 \times 1,000,000$                                                 | $= 5 \times 10^6$ | Five million           |
| +700,000  | $= 7 \times 100,000$                                                   | $= 7 \times 10^5$ | Seven hundred thousand |
| +80,000   | $= 8 \times 10,000$                                                    | $= 8 \times 10^4$ | Eighty thousand        |
| +3,000    | $= 3 \times 1000$                                                      | $= 3 \times 10^3$ | Three thousand         |
| +200      | $= 2 \times 100$                                                       | $= 2 \times 10^2$ | Two hundred            |
| +10       | $= 1 \times 10$                                                        | $= 1 \times 10^1$ | Ten                    |
| +6        | $= 6 \times 1$                                                         | $= 6 \times 10^0$ | Six                    |
| 5,783,216 | Five million, seven hundred eighty-three thousand, two hundred sixteen |                   |                        |

From the third column in the table we can see that each place is simply a multiple of ten. Of course, this makes sense given that our base is ten. The digits that are multiplying each place simply tell us how many of that place we have. We are restricted to having at most 9 in any one place before we have to “carry” over to the next place. We cannot, for example, have 11 in the hundreds place. Instead, we would carry 1 to the thousands place and retain 1 in the hundreds place. This comes as no surprise to us since we readily see that 11 hundreds is the same as one thousand, one hundred. Carrying is a pretty typical occurrence in a base system.

However, base-ten is not the only option we have. Practically any positive integer greater than or equal to 2 can be used as a base for a number system. Such systems can work just like the decimal system except the number of symbols will be different and each position will depend on the base itself.

### Other Bases

For example, let’s suppose we adopt a base-five system. The only modern digits we would need for this system are 0,1,2,3 and 4. What are the place values in such a system? To answer that, we start with the ones place, as most base systems do. However, if we were to count in this system, we could only get to four (4) before we had to jump up to the next place. Our base is 5, after all! What is that next place that we would jump to? It would not be tens, since we are no longer in base-ten. We’re in a different numerical world. As the base-ten system progresses from  $10^0$  to  $10^1$ , so the base-five system moves from  $5^0$  to  $5^1 = 5$ . Thus, we move from the ones to the fives.

After the fives, we would move to the  $5^2$  place, or the twenty fives. Note that in base-ten, we would have gone from the tens to the hundreds, which is, of course,  $10^2$ .

Let’s take an example and build a table. Consider the number 30412 in base five. We will write this as  $30412_5$ , where the subscript 5 is not part of the number but indicates the base we’re using. First off, note that this is NOT the number “thirty thousand, four hundred twelve.” We must be careful not to impose the base-ten system on this number. Here’s what our table might look like. We will use it to convert this number to our more familiar base-ten system.

|   | Base 5         | This column converts to base-ten | In Base-Ten |
|---|----------------|----------------------------------|-------------|
|   | $3 \times 5^4$ | $= 3 \times 625$                 | $= 1875$    |
| + | $0 \times 5^3$ | $= 0 \times 125$                 | $= 0$       |
| + | $4 \times 5^2$ | $= 4 \times 25$                  | $= 100$     |
| + | $1 \times 5^1$ | $= 1 \times 5$                   | $= 5$       |
| + | $2 \times 5^0$ | $= 2 \times 1$                   | $= 2$       |
|   |                | Total                            | 1982        |

As you can see, the number  $30412_5$  is equivalent to 1,982 in base-ten. We will say  $30412_5 = 1982_{10}$ . All of this may seem strange to you, but that's only because you are so used to the only system that you've ever seen.

### ✓ Example 3

Convert  $6234_7$  to a base 10 number.

#### Solution

We first note that we are given a base-7 number that we are to convert. Thus, our places will start at the ones ( $7^0$ ), and then move up to the  $7$ 's,  $49$ 's ( $7^2$ ), etc. Here's the breakdown:

|   | Base 7           | Convert          | Base 10  |
|---|------------------|------------------|----------|
|   | $= 6 \times 7^3$ | $= 6 \times 343$ | $= 2058$ |
| + | $= 2 \times 7^2$ | $= 2 \times 49$  | $= 98$   |
| + | $= 3 \times 7$   | $= 3 \times 7$   | $= 21$   |
| + | $= 4 \times 1$   | $= 4 \times 1$   | $= 4$    |
|   |                  | Total            | 2181     |

Thus  $6234_7 = 2181_{10}$

### ? Try it Now 3

Convert  $41065_7$  to a base 10 number.

#### Answer

$$41065_7 = 9994_{10}$$

## Converting from Base 10 to Other Bases

Converting from an unfamiliar base to the familiar decimal system is not that difficult once you get the hang of it. It's only a matter of identifying each place and then multiplying each digit by the appropriate power. However, going the other direction can be a little trickier. Suppose you have a base-ten number and you want to convert to base-five. Let's start with some simple examples before we get to a more complicated one.

### ✓ Example 4

Convert twelve to a base-five number.

#### Solution

We can probably easily see that we can rewrite this number as follows:

$$12 = (2 \times 5) + (2 \times 1)$$

Hence, we have two fives and 2 ones. Hence, in base-five we would write twelve as  $22_5$ . Thus,  $12_{10} = 22_5$

### ✓ Example 5

Convert sixty-nine to a base-five number.

#### Solution

We can see now that we have more than 25, so we rewrite sixty-nine as follows:

$$69 = (2 \times 25) + (3 \times 5) + (4 \times 1)$$

Here, we have two twenty-fives, 3 fives, and 4 ones. Hence, in base five we have 234. Thus,  $69_{10} = 234_5$

#### ✓ Example 6

Convert the base-seven number  $3261_7$  to base 10

##### **Solution**

The powers of 7 are:

$$7^0 = 1$$

$$7^1 = 7$$

$$7^2 = 49$$

$$7^3 = 343$$

Etc...

$$3261_7 = (3 \times 343) + (2 \times 49) + (6 \times 7) + (1 \times 1) = 1170_{10}$$

Thus  $3261_7 = 1170_{10}$

#### ? Try it Now 4

Convert 143 to base 5

##### **Answer**

$$143_{10} = 1033_5$$

#### ? Try it Now 5

Convert the base-three number  $21021_3$  to base 10.

##### **Answer**

$$21021_3 = 196_{10}$$

*In general, when converting from base-ten to some other base, it is often helpful to determine the highest power of the base that will divide into the given number at least once. In the last example,  $5^2 = 25$  is the largest power of five that is present in 69, so that was our starting point. If we had moved to  $5^3 = 125$ , then 125 would not divide into 69 at least once.*

Converting from Base 10 to Base  $b$

1. Find the highest power of the base  $b$  that will divide into the given number at least once and then divide.
2. Write down the whole number part, then use the remainder from division in the next step.
3. Repeat step two, dividing by the next highest power of the base  $b$ , writing down the whole number part (including 0), and using the remainder in the next step.
4. Continue until the remainder is smaller than the base. This last remainder will be in the “ones” place.
5. Collect all your whole number parts to get your number in base  $b$  notation.

#### ✓ Example 7

Convert the base-ten number 348 to base-five.

##### **Solution**

The powers of five are:

$$5^0 = 1$$

$$5^1 = 5$$

$$5^2 = 25$$

$$5^3 = 125$$

$$5^4 = 625$$

Etc..

since 348 is smaller than 625, but bigger than 125, we see that  $5^3 = 125$  is the highest power of five present in 348. So we divide 125 into 348 to see how many of them there are:

$$348 \div 125 = 2 \text{ with remainder } 98$$

We write down the whole part, 2, and continue with the remainder. There are 98 left over, so we see how many 25's (the next smallest power of five) there are in the remainder:

$$98 \div 25 = 3 \text{ with remainder } 23$$

We write down the whole part, 3, and continue with the remainder. There are 23 left over, so we look at the next place, the 5's:

$$23 \div 5 = 4 \text{ with remainder } 3$$

This leaves us with 3, which is less than our base, so this number will be in the "ones" place. We are ready to assemble our base-five number:

$$348 = (2 \times 5^3) + (3 \times 5^2) + (4 \times 5^1) + (3 \times 1)$$

Hence, our base-five number is 2343. We'll say that  $348_{10} = 2343_5$

### ✓ Example 8

Convert the base-ten number 4509 to base-seven.

#### Solution

The powers of 7 are:

$$7^0 = 1$$

$$7^1 = 7$$

$$7^2 = 49$$

$$7^3 = 343$$

$$7^4 = 2401$$

$$7^5 = 16807$$

Etc...

The highest power of 7 that will divide into 4509 is  $7^4 = 2401$

With division, we see that it will go in 1 time with a remainder of 2108. So we have 1 in the  $7^4$  place.

The next power down is  $7^3 = 343$ , which goes into 2108 six times with a new remainder of 50. So we have 6 in the  $7^3$  place.

The next power down is  $7^2 = 49$ , which goes into 50 once with a new remainder of 1. So there is a 1 in the  $7^2$  place.

The next power down is  $7^1$  but there was only a remainder of 1, so that means there is a 0 in the 7's place and we still have 1 as a remainder.

That, of course, means that we have 1 in the ones place.

$$4509 \div 7^4 = 1 \text{ R } 2108$$

$$2108 \div 7^3 = 6 \text{ R } 50$$

$$50 \div 7^2 = 1 \text{ R } 1$$

$$1 \div 7^1 = 0 \text{ R } 1$$

$$1 \div 7^0 = 1$$

$$4509_{10} = 16101_7$$

Putting all of this together means that  $4509_{10} = 16101_7$

**? Try it Now 6**

Convert  $657_{10}$  to a base 4 number.

**Answer**

$$657_{10} = 22101_4$$

**? Try it Now 7**

Convert  $8377_{10}$  to a base 8 number.

**Answer**

$$8377_{10} = 20271_8$$

### Another Method For Converting From Base 10 to Other Bases

As you read the solution to this last example and attempted the “Try it Now” problems, you may have had to repeatedly stop and think about what was going on. The fact that you are probably struggling to follow the explanation and reproduce the process yourself is mostly due to the fact that the non-decimal systems are so unfamiliar to you. In fact, the only system that you are probably comfortable with is the decimal system.

As budding mathematicians, you should always be asking questions like “How could I simplify this process?” In general, that is one of the main things that mathematicians do...they look for ways to take complicated situations and make them easier or more familiar. In this section we will attempt to do that.

To do so, we will start by looking at our own decimal system. What we do may seem obvious and maybe even intuitive but that’s the point. We want to find a process that we readily recognize works and makes sense to us in a familiar system and then use it to extend our results to a different, unfamiliar system.

Let’s start with the decimal number,  $4863_{10}$ . We will convert this number to base 10. Yeah, I know it’s already in base 10, but if you carefully follow what we’re doing, you’ll see it makes things work out very nicely with other bases later on. We first note that the highest power of 10 that will divide into 4863 at least once is  $10^3 = 1000$ . *In general, this is the first step in our new process; we find the highest power that a given base that will divide at least once into our given number.*

We now divide 1000 into 4863:

$$4863 \div 1000 = 4.863$$

This says that there are four thousands in 4863 (obviously). However, it also says that there are 0.863 thousands in 4863. This fractional part is our remainder and will be converted to lower powers of our base (10). If we take that decimal and multiply by 10 (since that’s the base we’re in) we get the following:

$$0.863 \times 10 = 8.63$$

Why multiply by 10 at this point? We need to recognize here that 0.863 thousands is the same as 8.63 hundreds. Think about that until it sinks in.

$$(0.863)(1000) = 863$$

$$(8.63)(100) = 863$$

These two statements are equivalent. So, what we are really doing here by multiplying by 10 is rephrasing or converting from one place (thousands) to the next place down (hundreds).

$$0.863 \times 10 \Rightarrow 8.63$$

$$(\text{Parts of Thousands}) \times 10 \Rightarrow \text{Hundreds}$$

What we have now is 8 hundreds and a remainder of 0.63 hundreds, which is the same as 6.3 tens. We can do this again with the 0.63 that remains after this first step.

$$0.63 \times 10 \Rightarrow 6.3$$

$$\text{Hundreds} \times 10 \Rightarrow \text{Tens}$$

So we have six tens and 0.3 tens, which is the same as 3 ones, our last place value.

Now here's the punch line. Let's put all of the together in one place:

$$\begin{array}{rcl}
 4863 \div 1000 & = & \textcircled{4}.863 \\
 \downarrow & & \\
 0.863 \times 10 & = & \textcircled{8}.63 \\
 \downarrow & & \\
 0.63 \times 10 & = & \textcircled{6}.3 \\
 \downarrow & & \\
 0.3 \times 10 & = & \textcircled{3}.0
 \end{array}$$

Note that in each step, the remainder is carried down to the next step and multiplied by 10, the base. Also, at each step, the whole number part, which is circled, gives the digit that belongs in that particular place. What is amazing is that this works for any base! So, to convert from a base 10 number to some other base,  $b$ , we have the following steps we can follow:

#### 📌 Converting from Base 10 to Base $b$ : Another method

1. Find the highest power of the base  $b$  that will divide into the given number at least once and then divide.
2. Keep the whole number part, and multiply the fractional part by the base  $b$ .
3. Repeat step two, keeping the whole number part (including 0), carrying the fractional part to the next step until only a whole number result is obtained.
4. Collect all your whole number parts to get your number in base  $b$  notation.

We will illustrate this procedure with some examples.

#### ✓ Example 9

Convert the base 10 number,  $348_{10}$ , to base 5.

##### Solution

This is actually a conversion that we have done in a previous example. The powers of five are:

$$5^0 = 1$$

$$5^1 = 5$$

$$5^2 = 25$$

$$5^3 = 125$$

$$5^4 = 625$$

Etc...

The highest power of five that will go into 348 at least once is  $5^3$

We divide by 125 and then proceed.

$$\begin{array}{rcl}
 348 \div 5^3 & = & \textcircled{2}.784 \\
 \downarrow & & \\
 0.784 \times 5 & = & \textcircled{3}.92 \\
 \downarrow & & \\
 0.92 \times 5 & = & \textcircled{4}0.6 \\
 \downarrow & & \\
 0.6 \times 5 & = & \textcircled{3}.0
 \end{array}$$

By keeping all the whole number parts, from top bottom, gives 2343 as our base 5 number. Thus,  $2343_5 = 348_{10}$

We can compare our result with what we saw earlier, or simply check with our calculator, and find that these two numbers really are equivalent to each other.

#### ✓ Example 10

Convert the base 10 number,  $3007_{10}$ , to base 5.

##### Solution

The highest power of 5 that divides at least once into 3007 is  $5^4 = 625$ . Thus, we have:

$$3007 \div 625 = (4).8112$$

$$0.8112 \times 5 = (4).056$$

$$0.056 \times 5 = (0).28$$

$$0.28 \times 5 = (1)0.4$$

$$0.4 \times 5 = (2)0.0$$

This gives us that  $3007_{10} = 44012_5$ . Notice that in the third line that multiplying by 5 gave us 0 for our whole number part. We don't discard that! The zero tells us that a zero in that place. That is, there are no  $5^2$ 's in this number

This last example shows the importance of using a calculator in certain situations and taking care to avoid clearing the calculator's memory or display until you get to the very end of the process.

### ✓ Example 11

Convert the base 10 number,  $63201_{10}$ , to base 7.

#### Solution

The powers of 7 are:

$$7^0 = 1$$

$$7^1 = 7$$

$$7^2 = 49$$

$$7^3 = 343$$

$$7^4 = 2401$$

$$7^5 = 16807$$

etc...

The highest power of 7 that will divide at least once into 63201 is  $7^5$ . When we do the initial division on a calculator, we get the following:

$$63201 \div 7^5 = 3.760397453$$

The decimal part actually fills up the calculators display and we don't know if it terminates at some point or perhaps even repeats down the road. So if we clear our calculator at this point, we will introduce error that is likely to keep this process from ever ending. To avoid this problem, we leave the result in the calculator and simply subtract 3 from this to get the fractional part all by itself. DO NOT ROUND OFF! Subtraction and then multiplication by seven gives:

$$63201 \div 7^5 = (3).760397453$$

$$0.760397453 \times 7 = (5).322782174$$

$$0.322782174 \times 7 = (2).259475219$$

$$0.259475219 \times 7 = (1).816326531$$

$$0.816326531 \times 7 = (5).714285714$$

$$0.714285714 \times 7 = (5).000000000$$

Yes, believe it or not, that last product is exactly 5, *as long as you don't clear anything out on your calculator*. This gives us our final result:  $63201_{10} = 352155_7$ .

If we round, even to two decimal places in each step, clearing our calculator out at each step along the way, we will get a series of numbers that do not terminate, but begin repeating themselves endlessly. (Try it!) We end up with something that doesn't make any sense, at least not in this context. So be careful to use your calculator cautiously on these conversion problems.

Also, remember that if your first division is by  $7^5$ , then you expect to have 6 digits in the final answer, corresponding to the places for  $7^5$ ,  $7^4$  and so on down to  $7^0$ . If you find yourself with more than 6 digits due to rounding errors, you know something went wrong

### ? Try it Now 8

Convert the base 10 number,  $9352_{10}$ , to base 5.

#### Answer

$$9352_{10} = 244402_5$$

**? Try it Now 9**

Convert the base 10 number,  $1500_{10}$ , to base 3.

Be careful not to clear your calculator on this one. Also, if you're not careful in each step, you may not get all of the digits you're looking for, so move slowly and with caution.

**Answer**

$$1500_{10} = 2001120_3$$

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### 3.3: Addition and Subtraction with Other Bases

#### Addition in Other Bases

As we saw in the previous section with the Mayan numeration system, we can add or subtract in other bases. Below are a series of steps, but, overall, we add as usual while finding its equivalent number from base 10 to the new base.

##### Adding in Base $b$

1. Rewrite the addition vertically, if not already.
2. Start in the ones place (as usual), but find the number the sum represents in base  $b$ .
3. If the sum is larger than base  $b$ , then carry over to the  $b^1$  place value.
4. Repeat steps 2 and 3 for the  $b^2$ ,  $b^3$ , ... place values.

##### ✓ Example 3.3.1

Add in base two:  $111_{\text{two}} + 11_{\text{two}}$

##### **Solution**

Writing this vertically, we get

$$\begin{array}{r} 111_{\text{two}} \\ + 11_{\text{two}} \\ \hline \end{array}$$

Let's add the ones place as usual. If we were in base 10,  $1+1=2$ ; 2 in base 10 is equivalent to 0 in base 2 and we carry 1 over to the 2's place. Recall, base two is  $\{0,1\}$ . Hence, the ones place is 0:

$$\begin{array}{r} 11^{+1}1_{\text{two}} \\ + 1 \quad 1_{\text{two}} \\ \hline 1 \quad 0_{\text{two}} \end{array}$$

Adding in the 2's place,  $1+1+1=3$  in base 10, but 3 is 1 in base 2 and we carry 1 to the  $2^2$ 's place:

$$\begin{array}{r} 1^{+1}11_{\text{two}} \\ + \quad 11_{\text{two}} \\ \hline 0 \quad 10_{\text{two}} \end{array}$$

Adding in the  $2^2$ 's place, we get  $1+1=2$  in base 10, but 2 in base 10 is 0 in base 2 and we carry a 1 over to the  $2^3$ 's place:

$$\begin{array}{r} +1 \quad 111_{\text{two}} \\ + \quad 11_{\text{two}} \\ \hline 1 \quad 010_{\text{two}} \end{array}$$

Hence,  $111_{\text{two}} + 11_{\text{two}} = 1010_{\text{two}}$ . Note, another way to do this is convert each number to base 10 and add as usual, then convert the result back into base 2.

##### ✓ Example 3.3.2

Add in base five:  $44_{\text{five}} + 42_{\text{five}}$

##### **Solution**

Let's try this by converting each number to base 10, adding them, then converting the sum back into base 5:

$$44_{\text{five}} = 4(5) + 4(1) = 24_{\text{ten}}$$

$$42_{\text{five}} = 4(5) + 2(1) = 22_{\text{ten}}$$

Next, we add the base 10 numbers:

$$24 + 22 = 46_{\text{ten}}$$

Converting 46 in base ten into a number in base 5, we use the previous sections' technique and obtain  $141_{\text{five}}$ . Thus, 46 in base 10 is equivalent to 141 in base 5.

## Subtraction in Other Bases

### Subtracting in Base $b$

Rewrite the subtraction vertically, if not already.

Start in the ones place (as usual), but find the number the difference represents in base  $b$ .

If the ones place of the minuend is smaller than the ones place of the subtrahend, then borrow from the place value to the left in that base  $b$ . Then subtract as usual.

Repeat steps 2 and 3 for the  $b^2$ ,  $b^3$  place values.

### ✓ Example 3.3.3

Subtract in base 5:  $240_{\text{five}} - 40_{\text{five}}$

#### Solution

$$\begin{array}{r} 240_{\text{five}} \\ -40_{\text{five}} \\ \hline \end{array}$$

If we take the ones place,  $0-0=0$ , which is 0 in base 5, then the ones place stays 0.

$$\begin{array}{r} 240_{\text{five}} \\ -40_{\text{five}} \\ \hline 0 \end{array}$$

Now let's take the 5's place:  $4-4=0$ . Since 0 is in base 5, then the 5's place is 0.

$$\begin{array}{r} 240_{\text{five}} \\ -40_{\text{five}} \\ \hline 00 \end{array}$$

Let's look at the  $5^2$ 's place. Notice, we can just drop the two down and get

$$\begin{array}{r} 240_{\text{five}} \\ -40_{\text{five}} \\ \hline 200_5 \end{array}$$

Thus,  $240_{\text{five}} - 40_{\text{five}} = 200_{\text{five}}$ .

What if we have to borrow? We can subtract with borrowing easily in base 10, but what if we wanted to subtract two numbers that included borrowing? Let's see.

### ✓ Example 3.3.4

Subtract in base 5:  $404_{\text{five}} - 323_{\text{five}}$

#### Solution

Rewriting this vertically, we get

$$\begin{array}{r} 404_{\text{five}} \\ -323_{\text{five}} \\ \hline \end{array}$$

Subtracting in the ones place, we get  $4-3=1$ . Since 1 is in base 5, then the ones place is 1.

$$\begin{array}{r} 404_{five} \\ -323_{five} \\ \hline 1 \end{array}$$

Looking at the 5's place, notice that 0 is less than 2 and we have to borrow from the  $5^2$ 's place. Recall, we are in base 5, so when we borrow, we still reduce 4 to 3, but we carry 5 since we are in base 5:

$$\begin{array}{r} 4^3 0^5 4_{five} \\ -3^2 2^3 3_{five} \\ \hline 0^3 3^1 1_{five} \end{array}$$

Now, we subtract as usual:  $5-2=3$ . Since 3 is in base 5, then the 5's place is 3. Subtracting in the  $5^2$ 's place, we get  $3-3=0$ ; hence, the  $5^2$ 's place is 0.

Thus,  $404_{five} - 323_{five} = 31_{five}$ .

## Conclusion

In this chapter, we have briefly sketched the development of numbers and our counting system, with the emphasis on the "brief" part. There are numerous sources of information and research that fill many volumes of books on this topic. Unfortunately, we cannot begin to come close to covering all of the information that is out there.

We have only scratched the surface of the wealth of research and information that exists on the development of numbers and counting throughout human history. It is important to note that the system that we use every day is a product of thousands of years of progress and development. It represents contributions by many civilizations and cultures. It does not come down to us from the sky, a gift from the gods. It is not the creation of a textbook publisher. It is indeed as human as we are, as is the rest of mathematics. Behind every symbol, formula, and rule there is a human face to be found, or at least sought.

Furthermore, we hope that you now have a basic appreciation for just how interesting and diverse number systems can get. Also, we're pretty sure that you have also begun to recognize that we take our own number system for granted so much that when we try to adapt to other systems or bases, we find ourselves truly having to concentrate and think about what is going on.

## Try It Now Answers

1.  $1+6\times 3+3\times 6+2\times 12=61$  cats.

2. From left to right:

Cord 1 = 2,162

Cord 2 = 301

Cord 3 = 0

Cord 4 = 2,070

3.  $41065_7 = 9994_{10}$

4.  $143_{10} = 1033_5$

5.  $21021_3 = 196_{10}$

6.  $657_{10} = 22101_4$

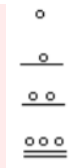
7.  $8377_{10} = 20271_8$

8.  $9352_{10} = 244402_5$

9.  $1500_{10} = 2001120_3$

10. 1562

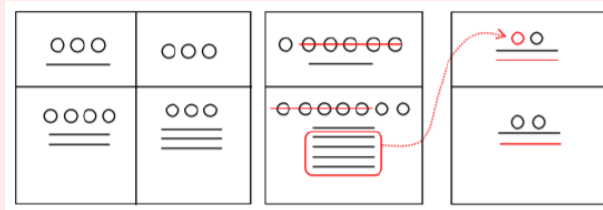
11.  $10553_{10} = 1,6,7,13_{20}$



12.  $5617_{10} = 14,0,17_{20}$ . Note that there is a zero in the 20's place, so you'll need to use the appropriate zero symbol in between the ones and 400's places.



13. A sample solution is shown.



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## 3.4: Calculator Shortcut for Modular Arithmetic

### Modular arithmetic

If you think back to doing division with whole numbers, you may remember finding the whole number result and the remainder after division.

#### Modulus

The **modulus**<sup>[1]</sup> is another name for the remainder after division.

For example,  $17 \bmod 5 = 2$ , since if we divide 17 by 5, we get 3 with remainder 2.

Modular arithmetic is sometimes called clock arithmetic, since analog clocks wrap around times past 12, meaning they work on a modulus of 12. If the hour hand of a clock currently points to 8, then in 5 hours it will point to 1. While  $8 + 5 = 13$ , the clock wraps around after 12, so all times can be thought of as modulus 12. Mathematically,  $13 \bmod 12 = 1$ .

#### Example 1

Compute the following:

1.  $10 \bmod 3$
2.  $15 \bmod 5$
3.  $27 \bmod 5$

#### Answers

1. Since 10 divided by 3 is 3 with remainder 1,  $10 \bmod 3 = 1$
2. Since 15 divided by 5 is 3 with no remainder,  $15 \bmod 5 = 0$
3.  $2^7 = 128$ . 128 divide by 5 is 25 with remainder 3, so  $2^7 \bmod 5 = 3$

#### Try it Now

Compute the following:

1.  $23 \bmod 7$
2.  $15 \bmod 7$
3.  $2034 \bmod 7$

Recall that when we divide 17 by 5, we could represent the result as 3 remainder 2, as the mixed number  $3\frac{2}{5}$ , or as the decimal 3.4. Notice that the modulus, 2, is the same as the numerator of the fractional part of the mixed number, and that the decimal part 0.4 is equivalent to the fraction  $\frac{2}{5}$ . We can use these conversions to calculate the modulus of not-too-huge numbers on a standard calculator.

#### Modulus on a Standard Calculator

To calculate  $a \bmod n$  on a standard calculator

1. Divide  $a$  by  $n$
2. Subtract the whole part of the resulting quantity
3. Multiply by  $n$  to obtain the modulus

#### Example 2

Calculate  $31345 \bmod 419$ .

#### Answer

|                                                                                                                   |                                                             |
|-------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------|
|  $31345 \div 419 = 74.8090692$ | Now subtract 74 to get just the decimal remainder           |
|  $74.8090692 - 74 = 0.8090692$ | Multiply this by 419 to get the modulus                     |
|  $0.8090692 \times 419 = 339$  | This tells us 0.8090692 was equivalent to $\frac{339}{419}$ |

In the text above, only a portion of the decimal value was written down. In practice, you should try to avoid writing down the intermediary steps, and instead allow your calculator to retain as many decimal values as it can.

---

1. Sometimes, instead of seeing  $17 \bmod 5 = 2$ , you'll see  $17 \equiv 2 \pmod{5}$ . The  $\equiv$  symbol means “congruent to” and means that 17 and 2 are equivalent, after you consider the modulus 5. [↩](#)
- 

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## SECTION OVERVIEW

### 3.5: Cryptography

#### 3.5.1: Cryptography

#### 3.5.2: Substitution Ciphers

#### 3.5.3: Public Key Cryptography

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### 3.5.1: Cryptography

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When people need to secretly store or communicate messages, they turn to cryptography. Cryptography involves using techniques to obscure a message so outsiders cannot read the message. It is typically split into two steps: encryption, in which the message is obscured, and decryption, in which the original message is recovered from the obscured form.

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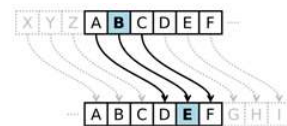
## 3.5.2: Substitution Ciphers

One simple encryption method is called a **substitution cipher**.

### Substitution Cipher

A substitution cipher replaces each letter in the message with a different letter, following some established mapping.

A simple example of a substitution cipher is called the **Caesar cipher**, sometimes called a shift cipher. In this approach, each letter is replaced with a letter some fixed number of positions later in the alphabet. For example, if we use a shift of 3, then the letter A would be replaced with D, the letter 3 positions later in the alphabet. The entire mapping would look like: [1]



Original: ABCDEFGHIJKLMNOPQRSTUVWXYZ

Maps to: DEF GHIJ KLMNOP QRSTUV WXYZ ABC

### ✓ Example 1

Use the Caesar cipher with shift of 3 to encrypt the message: “We ride at noon”

#### Solution

We use the mapping above to replace each letter. W gets replaced with Z, and so forth, giving the encrypted message: ZH ULGH DW QRRQ.

Notice that the length of the words could give an important clue to the cipher shift used. If we saw a single letter in the encrypted message, we would assume it must be an encrypted A or I, since those are the only single letters than form valid English words.

To obscure the message, the letters are often rearranged into equal sized blocks. The message ZH ULGH DW QRRQ could be written in blocks of three characters as

ZHU LGH DWQ RRRQ.

### ✓ Example 2

Decrypt the message GZD KNK YDX MFW JXA if it was encrypted using a shift cipher with shift of 5.

#### Solution

We start by writing out the character mapping by shifting the alphabet, with A mapping to F, five characters later in the alphabet.

Original: ABCDEFGHIJKLMNOPQRSTUVWXYZ

Maps to: F GHIJ KLMNOP QRSTUV WXYZABCDE

We now work backwards to decrypt the message. The first letter G is mapped to by B, so B is the first character of the original message. Continuing, our decrypted message is

BUY FIF TYS HAR ESA.

Removing spaces we get BUYFIFTYSHARESA. In this case, it appears an extra character was added to the end to make the groups of three come out even, and that the original message was “Buy fifty shares.”

### ? Try it Now 1

Decrypt the message BNW MVX WNH if it was encrypted using a shift cipher with shift 9 (mapping A to J).

#### Answer

SEND MONEY

Notice that in both the ciphers above, the extra part of the alphabet wraps around to the beginning. Because of this, a handy version of the shift cipher is a cipher disc, such as the Alberti cipher disk shown here[2] from the 1400s. In a cipher disc, the inner wheel could be turned to change the cipher shift. This same approach is used for “secret decoder rings.”



The security of a cryptographic method is very important to the person relying on their message being kept secret. The security depends on two factors:

1. The security of the method being used
2. The security of the encryption key used

In the case of a shift cipher, the method is “a shift cipher is used.” The encryption key is the specific amount of shift used.

Suppose an army is using a shift cipher to send their messages, and one of their officers is captured by their enemy. It is likely the method and encryption key could become compromised. It is relatively hard to change encryption methods, but relatively easy to change encryption keys.

During World War II, the Germans’ Enigma encryption machines were captured, but having details on the encryption method only slightly helped the Allies, since the encryption keys were still unknown and hard to discover. Ultimately, the security of a message cannot rely on the method being kept secret; it needs to rely on the key being kept secret.

### Encryption Security

The security of any encryption method should depend only on the encryption key being difficult to discover. It is not safe to rely on the encryption method (algorithm) being kept secret.

With that in mind, let’s analyze the security of the Caesar cipher.

### Example 3

Suppose you intercept a message, and you know the sender is using a Caesar cipher, but do not know the shift being used. The message begins EQZP. How hard would it be to decrypt this message?

#### Solution

Since there are only 25 possible shifts, we would only have to try 25 different possibilities to see which one produces results that make sense. While that would be tedious, one person could easily do this by hand in a few minutes. A modern computer could try all possibilities in under a second.

| Shift | Message | Shift | Message     | Shift | Message | Shift | Message |
|-------|---------|-------|-------------|-------|---------|-------|---------|
| 1     | DPYO    | 7     | XJSI        | 13    | RDMC    | 19    | LXGW    |
| 2     | COXN    | 8     | WIRH        | 14    | QCLB    | 20    | KWVU    |
| 3     | BNWM    | 9     | VHQG        | 15    | PBKA    | 21    | JVEU    |
| 4     | AMVL    | 10    | UGPF        | 16    | OAJZ    | 22    | IUDT    |
| 5     | ZLUK    | 11    | TFOE        | 17    | NZIY    | 23    | HTCS    |
| 6     | YKTJ    | 12    | <b>SEND</b> | 18    | MYHX    | 24    | GSBR    |
|       |         |       |             |       |         | 25    | FRAQ    |

In this case, a shift of 12 (A mapping to M) decrypts EQZP to SEND. Because of this ease of trying all possible encryption keys, the Caesar cipher is not a very secure encryption method.

### Brute Force Attack

A brute force attack is a method for breaking encryption by trying all possible encryption keys.

To make a brute force attack harder, we could make a more complex substitution cipher by using something other than a shift of the alphabet. By choosing a random mapping, we could get a more secure cipher, with the tradeoff that the encryption key is harder to

describe; the key would now be the entire mapping, rather than just the shift amount.

#### ✓ Example 4

Use the substitution mapping below to encrypt the message “March 12 0300”

Original:  $\mathrm{ABCDEFGHIJKLMNOPQRSTUVWXYZ0123456789}$

Maps to: 2BQF5WR TD8IJ6HLCOSUVK3A0X9YZN1G4ME7P

#### Solution

Using the mapping, the message would encrypt to 62SQT ZN Y1YY

#### ? Try it Now 2

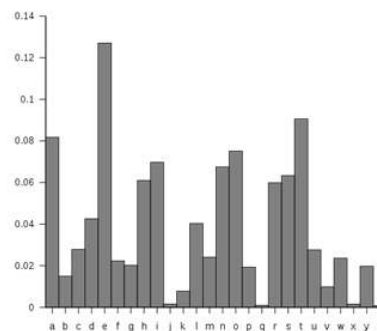
Use the substitution mapping from Example 4 to decrypt the message C2SVX2VP

#### Answer

PARTY AT 9

While there were only 25 possible shift cipher keys (35 if we had included numbers), there are about  $10^{40}$  possible substitution ciphers[3]. That’s much more than a trillion trillions. It would be essentially impossible, even with supercomputers, to try every possible combination. Having a huge number of possible encryption keys is one important part of key security.

Unfortunately, this cipher is still not secure, because of a technique called frequency analysis, discovered by Arab mathematician Al-Kindi in the 9<sup>th</sup> century. English and other languages have certain letters than show up more often in writing than others.[4] For example, the letter E shows up the most frequently in English. The chart to the right shows the typical distribution of characters.

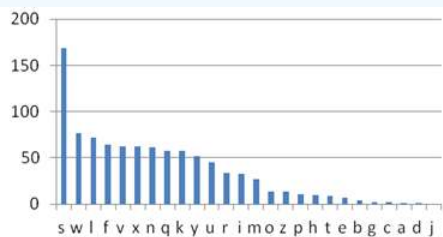


#### ✓ Example 5

The chart to the right shows the frequency of different characters in some encrypted text. What can you deduce about the mapping?

#### Solution

Because of the high frequency of the letter S in the encrypted text, it is very likely that the substitution maps E to S. Since W is the second most frequent character, it likely that T or A maps to W. Because C, A, D, and J show up rarely in the encrypted text, it is likely they are mapped to from J, Q, X, and Z.



In addition to looking at individual letters, certain pairs of letters show up more frequently, such as the pair “th.” By analyzing how often different letters and letter pairs show up an encrypted message, the substitution mapping used can be deduced[5].

[1] [en.Wikipedia.org/w/index.php?title=File:Caesar3.svg&page=1](https://en.wikipedia.org/w/index.php?title=File:Caesar3.svg&page=1). PD

[2] [en.Wikipedia.org/wiki/File:Alberti\\_cipher\\_disk.JPG](https://en.wikipedia.org/wiki/File:Alberti_cipher_disk.JPG)

[3] There are 35 choices for what A maps to, then 34 choices for what B maps to, and so on, so the total number of possibilities is  $35 \times 34 \times 33 \times \dots \times 2 \times 1 = 35! = \text{about } 10^{40}$

[4] [en.Wikipedia.org/w/index.php?title=File:English\\_letter\\_frequency\\_\(alphabetic\).svg&page=1](https://en.wikipedia.org/w/index.php?title=File:English_letter_frequency_(alphabetic).svg&page=1) PD

[5] For an example of how this is done, see [en.Wikipedia.org/wiki/Frequency\\_analysis](https://en.wikipedia.org/wiki/Frequency_analysis)

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### 3.5.3: Public Key Cryptography

Suppose that you are connecting to your bank's website. It is possible that someone could intercept any communication between you and your bank, so you'll want to encrypt the communication. The problem is that all the encryption methods we've discussed require that both parties have already agreed on a shared secret encryption key. How can you and your bank agree on a key if you haven't already?

This becomes the goal of public key cryptography – to provide a way for two parties to agree on a key without a snooping third party being able to determine the key. The method relies on a one-way function; something that is easy to do one way, but hard to reverse. We will explore the Diffie-Hellman-Merkle key exchange method.

As an example, let's consider mixing paint. It's easy to mix paint to make a new color, but much harder to separate a mixed paint into the two original colors used.[1][2]

Using this analogy, Alice and Bob publicly agree on a common starter color. Each then mixes in some of their own secret color. They then exchange their mixed colors.

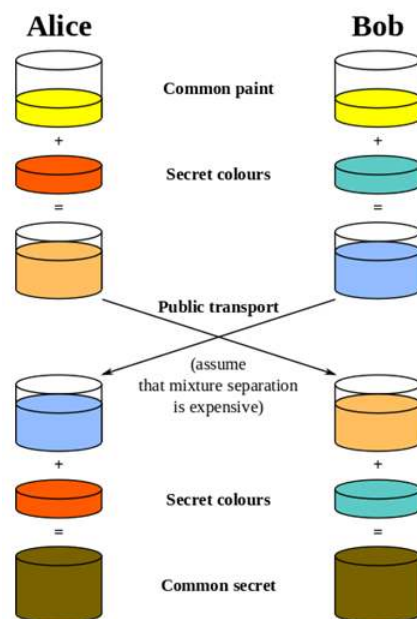
Since separating colors is hard, even if a snooper were to obtain these mixed colors, it would be hard to obtain the original secret colors.

Once they have exchanged their mixed colors, Alice and Bob both add their secret color to the mix they obtained from the other person. In doing so, both Alice and Bob now have the same common secret color, since it contains a mix of the original common color, Alice's secret color, and Bob's secret color.

They now have a common secret color they can use as their encryption key, even though neither Alice nor Bob knows the other's secret color.

Likewise, there is no way for a snooper to obtain the common secret color without separating one of the mixed colors.

To get this process to work for computer communication, we need to have the process result in a shared common number to act as the common secret encryption key. For this, we need a numerical one-way function.



#### Modular arithmetic

If you think back to doing division with whole numbers, you may remember finding the whole number result and the remainder after division.

##### Modulus [3]

The **modulus** is another name for the remainder after division.

For example,  $17 \bmod 5 = 2$ , since if we divide 17 by 5, we get 3 with remainder 2.

Modular arithmetic is sometimes called clock arithmetic, since analog clocks wrap around times past 12, meaning they work on a modulus of 12. If the hour hand of a clock currently points to 8, then in 5 hours it will point to 1. While  $8 + 5 = 13$ , the clock wraps around after 12, so all times can be thought of as modulus 12. Mathematically,  $13 \bmod 12 = 1$ .

##### ✓ Example 11

Compute: a)  $10 \bmod 3$  b)  $15 \bmod 5$  c)  $2^7 \bmod 5$

##### Solution

a) Since 10 divided by 3 is 3 with remainder 1,  $10 \bmod 3 = 1$

b) Since 15 divided by 5 is 3 with no remainder,  $15 \bmod 5 = 0$

c)  $2^7 = 128$ . 128 divided by 5 is 25 with remainder 3, so  $2^7 \bmod 5 = 3$

### ? Try it Now 5

Compute: a)  $23 \bmod 7$  b)  $15 \bmod 7$  c)  $2034 \bmod 7$

#### Answer

a) 2 b) 1 c) 4

Recall that when we divide 17 by 5, we could represent the result as 3 remainder 2, as the mixed number  $3\frac{2}{5}$ , or as the decimal 3.4. Notice that the modulus, 2, is the same as the numerator of the fractional part of the mixed number, and that the decimal part 0.4 is equivalent to the fraction  $\frac{2}{5}$ . We can use these conversions to calculate the modulus of not-too-huge numbers on a standard calculator.

### 📌 Modulus on a Standard Calculator

To calculate  $a \bmod n$  on a standard calculator

1. Divide  $a$  by  $n$
2. Subtract the whole part of the resulting quantity
3. Multiply by  $n$  to obtain the modulus

### ✓ Example 12

Calculate  $31345 \bmod 419$

#### Solution

$31345/419 = 74.8090692$       Now subtract 74 to get just the decimal remainder

$74.8090692 - 74 = 0.8090692$       Multiply this by 419 to get the modulus

$0.8090692 \times 419 = 339$       This tells us 0.8090692 was equivalent to  $\frac{339}{419}$

In the text above, only a portion of the decimal value was written down. In practice, you should try to avoid writing down the intermediary steps, and instead allow your calculator to retain as many decimal values as it can.

## The one-way function

When you use a prime number  $p$  as a modulus, you can find a special number called a generator,  $g$ , so that  $g^n \bmod p$  will result in all the values from 1 to  $p - 1$

| $n$ | $3^n$ | $3^n \bmod 7$ |
|-----|-------|---------------|
| 1   | 3     | 3             |
| 2   | 9     | 2             |
| 3   | 27    | 6             |
| 4   | 81    | 4             |
| 5   | 243   | 5             |
| 6   | 729   | 1             |

In the table to the top, notice that when we give values of  $n$  from 1 to 6, we get out all values from 1 to 6. This means 3 is a generator when 7 is the modulus.

This gives us our one-way function. While it is easy to compute the value of  $g^n \bmod p$  when we know  $n$ , it is difficult to find the exponent  $n$  to obtain a specific value.

For example, suppose we use  $p = 23$  and  $g = 5$ . If I pick  $n$  to be 6, I can fairly easily calculate  $5^6 \bmod 23 = 15625 \bmod 23 = 8$

If someone else were to tell you  $5^n \bmod 23 = 7$ , it is much harder to find  $n$ . In this particular case, we'd have to try 22 different values for  $n$  until we found one that worked – there is no known easier way to find  $n$  other than brute-force guessing.

While trying 22 values would not take too long, when used in practice much larger values for  $p$  are used, typically with well over 500 digits. Trying all possibilities would be essentially impossible.

## The key exchange

Before we can begin the key exchange process, we need a couple more important facts about modular arithmetic.

### Modular Exponentiation Rule

$$(a^b \bmod n) = (a \bmod n)^b \bmod n$$

#### ✓ Example 13

Compute  $12^5 \bmod 7$  using the exponentiation rule.

##### Solution

Evaluated directly:  $12^5 = 248,832$ , so  $12^5 \bmod 7 = 248,832 \bmod 7 = 3$

Using the rule above,  $12^5 \bmod 7 = (12 \bmod 7)^5 \bmod 7 = 5^5 \bmod 7 = 3125 \bmod 7 = 3$

You may remember a basic exponent rule from algebra:

$$(a^b)^c = a^{bc} = a^{cb} = (a^c)^b$$

For example:

$$64^2 = (4^3)^2 = 4^6 = (4^2)^3 = 16^3$$

We can combine the modular exponentiation rule with the algebra exponent rule to define the modular exponent power rule.

### Modular Exponent Power Rule

$$(a^b \bmod n)^c \bmod n = (a^{bc} \bmod n) = (a^c \bmod n)^b \bmod n$$

#### ✓ Example 14

Verify the rule above if  $a = 3$ ,  $b = 4$ ,  $c = 5$ , and  $n = 7$

##### Solution

$$(3^4 \bmod 7)^5 \bmod 7 = (81 \bmod 7)^5 \bmod 7 = 4^5 \bmod 7 = 1024 \bmod 7 = 2$$

$$(3^5 \bmod 7)^4 \bmod 7 = (243 \bmod 7)^4 \bmod 7 = 5^4 \bmod 7 = 625 \bmod 7 = 2, \text{ the same result.}$$

#### ? Try it Now 6

Use the modular exponent rule to calculate  $10000 \bmod 7$ , by noting  $10000 = 10^4$

##### Answer

$$10000 \bmod 7 = 10^4 \bmod 7 = (10 \bmod 7)^4 \bmod 7 = 3^4 \bmod 7 = 81 \bmod 7 = 4$$

This provides us the basis for our key exchange. While it will be easier to understand in the following example, here's the process:

1. Alice and Bob agree publically on values a prime  $p$  and generator  $g$ .
2. Alice picks some secret number  $a$ , while Bob picks some secret number  $b$ .
3. Alice computes  $A = g^a \bmod p$  and sends it to Bob.
4. Bob computes  $B = g^b \bmod p$  and sends it to Alice.
5. Alice computes  $B^a \bmod p$ , which is  $(g^b \bmod p)^a \bmod p$ .

6. Bob computes  $A^b \bmod p$ , which is  $(g^a \bmod p)^b \bmod p$ .

The modular exponent power rule tells us  $(g^a \bmod p)^b \bmod p = (g^b \bmod p)^a \bmod p$ , so Alice and Bob will arrive at the same shared value to use as a key, even though neither knows the other's secret number, and no eavesdropper can determine this value knowing only  $g, p, A$ , and  $B$

### ✓ Example 15

Alice and Bob publically share a generator and prime modulus. In this case, we'll use 3 as the generator and 17 as the prime.

#### Solution

|                                                                                   | Alice                                  | Common info          | Bob                                    |
|-----------------------------------------------------------------------------------|----------------------------------------|----------------------|----------------------------------------|
| Alice and Bob publically share a generator and prime modulus.                     | $g = 3, p = 17$                        |                      | $g = 3, p = 17$                        |
| Each then secretly picks a number $n$ of their own.                               | $n = 8$                                | secret number        | $n = 6$                                |
| Each calculates $g^n \bmod p$                                                     | $3^8 \bmod 17 = 16$                    |                      | $3^6 \bmod 17 = 15$                    |
| They then exchange these resulting values.                                        | $A = 16$<br>$B = 15$                   |                      | $B = 15$<br>$A = 16$                   |
| Each then raises the value they received to the power of their secret $n \bmod p$ | $B^n \bmod p =$<br>$15^8 \bmod 17 = 1$ | mix in secret number | $A^n \bmod p =$<br>$16^6 \bmod 17 = 1$ |
| The result is the shared secret key.                                              | 1                                      | shared secret key    | 1                                      |

The shared secrets come out the same because of the modular exponent power rule

$(a^b \bmod n)^c \bmod n = (a^c \bmod n)^b \bmod n$ . Alice computed  $(3^6 \bmod 17)^8 \bmod 17$  while Bob computed  $(3^8 \bmod 17)^6 \bmod 17$ , which the rule says will give the same results.

Notice that even if a snooper were to obtain both exchanged values  $A = 16$  and  $B = 15$ , there is no way they could obtain the shared secret key from these without having at least one of Alice or Bob's secret numbers. There is no easy way to obtain the secret numbers from the shared values, since the function was a one-way function.

Using this approach, Alice and Bob can now use the shared secret key obtained as the key for a standard encryption algorithm like DES or AES.

### ? Try it Now 7

Suppose you are doing a key exchange with Kylie using generator 5 and prime 23. Your secret number is 2. What number do you send to Kylie? If Kylie sends you the value 8, determine the shared secret key.

#### Answer

To compute the number we'd send to Kylie, we raise the generator to the power of our secret number modulus the prime:  
 $5^2 \bmod 23 = 25 \bmod 23 = 2$ .

If Kylie sends us the value 8, we determine the shared secret by raising her number to the power of our secret number modulus the prime:  $8^2 \bmod 23 = 64 \bmod 23 = 18$ . 18 would be the shared secret.

## RSA

There are several other public-key methods used, including RSA, which is very commonly used. RSA involves distributing a public encryption key, which anyone can use to encrypt messages to you, but which can only be decrypted using a separate private

key. You can think of this as sending an open padlock to someone – they can lock up information, but no one can unlock it without the key you kept secret.

RSA's security relies on the difficulty of factoring large numbers. For example, it's easy to calculate that 53 times 59 is 3127, but given the number 12,317 that is a product of two primes, it's much harder to find the numbers that multiply to give that number. It's exponentially harder when the primes each have 100 or more digits. Suppose we find two primes  $p$  and  $q$  and multiply them to get  $n = pq$ . This number will be very hard to factor. If we also know  $p$  and  $q$ , there are shortcuts to find two numbers  $e$  and  $d$  so that  $m^{ed} \bmod n = m \bmod n$  for all numbers  $m$ . Without knowing the factorization of  $n$ , finding these values is very hard.

To use RSA, we generate two primes  $p$  and  $q$  and multiply them to get  $n = pq$ . since we know the factorization, we can easily find  $e$  and  $d$  so  $m^{ed} = m \bmod n$ . Now, we lock away  $p$  and  $q$ . We then send the values  $e$  and  $n$  out publically. To encrypt a message  $m$ , the sender computes  $S = m^e \bmod n$ . As we saw earlier, the modulus is a one-way function which makes the original message very hard to recover from  $S$ . However, we have our private key  $d$  we can use to decrypt the message. When we receive the secret message  $S$ , we compute  $S^d \bmod n = (m^e)^d \bmod n = m^{ed} \bmod n = m \bmod n$ , recovering the original message[4].

#### ✓ Example 16

Suppose that Alice has computed  $n = 3127$ ,  $e = 3$ , and  $d = 2011$ . Show how Bob would encrypt the message 50 and how Alice would then decrypt it.

#### Solution

Bob would only know his message,  $m = 50$  and Alice's public key:  $n = 3127$  and  $e = 3$ . He would encrypt the message by computing  $m^e \bmod n : 50^3 \bmod 3127 = 3047$

Alice can then decrypt this message using her private key  $d$  by computing  $S^d \bmod n$ :

$$3047^{2011} \bmod 3127 = 50$$

This method differs from Diffie-Hellman-Merkle because no exchange process is needed; Bob could send Alice an encrypted message using Bob's public key without having to communicate with Alice beforehand to determine a shared secret key. This is especially handy for applications like encrypting email, where both parties might not be online at the same time to perform a Diffie-Hellman-Merkle style key exchange.

[1] [en.Wikipedia.org/w/index.php?...nge.svg&page=1](http://en.Wikipedia.org/w/index.php?...nge.svg&page=1)

[2] For a video overview of this process, see [http://www.youtube.com/watch?v=YEBfamv-\\_do](http://www.youtube.com/watch?v=YEBfamv-_do)

[3] Sometime, instead of seeing  $17 \bmod 5 = 2$ , you'll see  $17 \equiv 2 \pmod{5}$ . The  $\equiv$  symbol means "congruent to" and means that 17 and 2 are equivalent, after you consider the modulus 5.

[4] Many details have been left out, including how  $e$  and  $d$  are determined, and why this all works. For a bit more detail, see [http://www.youtube.com/watch?v=wXB-V\\_Keiu8](http://www.youtube.com/watch?v=wXB-V_Keiu8), or <http://doctrina.org/How-RSA-Works-With-Examples.html>

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## CHAPTER OVERVIEW

### 4: Population Growth Models

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[4.3: Exponential \(Geometric\) Growth](#)

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## 4.1: Linear (Algebraic) Growth

Marco is a collector of antique soda bottles. His collection currently contains 437 bottles. Every year, he budgets enough money to buy 32 new bottles. Can we determine how many bottles he will have in 5 years, and how long it will take for his collection to reach 1000 bottles?

While both of these questions you could probably solve without an equation or formal mathematics, we are going to formalize our approach to this problem to provide a means to answer more complicated questions.

Suppose that  $P_n$  represents the number, or population, of bottles Marco has after  $n$  years. So  $P_0$  would represent the number of bottles now,  $P_1$  would represent the number of bottles after 1 year,  $P_2$  would represent the number of bottles after 2 years, and so on. We could describe how Marco's bottle collection is changing using:

$$P_0 = 437$$

$$P_n = P_{n-1} + 32$$

This is called a **recursive relationship**. A recursive relationship is a formula which relates the next value in a sequence to the previous values. Here, the number of bottles in year  $n$  can be found by adding 32 to the number of bottles in the previous year,  $P_{n-1}$ . Using this relationship, we could calculate:

$$P_1 = P_0 + 32 = 437 + 32 = 469$$

$$P_2 = P_1 + 32 = 469 + 32 = 501$$

$$P_3 = P_2 + 32 = 501 + 32 = 533$$

$$P_4 = P_3 + 32 = 533 + 32 = 565$$

$$P_5 = P_4 + 32 = 565 + 32 = 597$$

We have answered the question of how many bottles Marco will have in 5 years. However, solving how long it will take for his collection to reach 1000 bottles would require a lot more calculations.

While recursive relationships are excellent for describing simply and cleanly *how* a quantity is changing, they are not convenient for making predictions or solving problems that stretch far into the future. For that, a closed or explicit form for the relationship is preferred. An **explicit equation** allows us to calculate  $P_n$  directly, without needing to know  $P_{n-1}$ . While you may already be able to guess the explicit equation, let us derive it from the recursive formula. We can do so by selectively not simplifying as we go:

$$P_1 = 437 + 32 = 437 + 1(32)$$

$$P_2 = P_1 + 32 = 437 + 32 + 32 = 437 + 2(32)$$

$$P_3 = P_2 + 32 = (437 + 2(32)) + 32 = 437 + 3(32)$$

$$P_4 = P_3 + 32 = (437 + 3(32)) + 32 = 437 + 4(32)$$

You can probably see the pattern now, and generalize that

$$P_n = 437 + n(32) = 437 + 32n$$

Using this equation, we can calculate how many bottles he'll have after 5 years:

$$P_5 = 437 + 32(5) = 437 + 160 = 597$$

We can now also solve for when the collection will reach 1000 bottles by substituting in 1000 for  $P_n$  and solving for  $n$ .

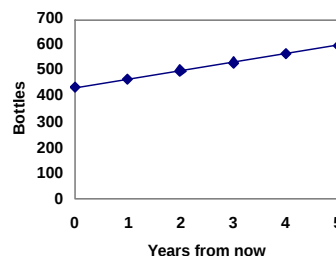
$$1000 = 437 + 32n$$

$$563 = 32n$$

$$n = 563/32 = 17.59$$

So Marco will reach 1000 bottles in 18 years.

In the previous example, Marco's collection grew by the *same number* of bottles every year. This constant change is the defining characteristic of linear growth. Plotting the values we calculated for Marco's collection, we can see the values form a straight line, the shape of linear growth.



## Linear Growth

If a quantity starts at size  $P_0$  and grows by  $d$  every time period, then the quantity after  $n$  time periods can be determined using either of these relations:

Recursive form:

$$P_n = P_{n-1} + d$$

Explicit form:

$$P_n = P_0 + dn$$

In this equation,  $d$  represents the **common difference** – the amount that the population changes each time  $n$  increases by 1

### ✓ Connection to prior learning – slope and intercept

You may recognize the common difference,  $d$ , in our linear equation as slope. In fact, the entire explicit equation should look familiar – it is the same linear equation you learned in algebra, probably stated as  $y = mx + b$ .

In the standard algebraic equation  $y = mx + b$ ,  $b$  was the  $y$ -intercept, or the  $y$  value when  $x$  was zero. In the form of the equation we're using, we are using  $P_0$  to represent that initial amount.

In the  $y = mx + b$  equation, recall that  $m$  was the slope. You might remember this as “rise over run”, or the change in  $y$  divided by the change in  $x$ . Either way, it represents the same thing as the common difference,  $d$ , we are using – the amount the output  $P_n$  changes when the input  $n$  increases by 1.

The equations  $y = mx + b$  and  $P_n = P_0 + dn$  mean the same thing and can be used the same ways, we're just writing it somewhat differently.

### ✓ Example 1

The population of elk in a national forest was measured to be 12,000 in 2003, and was measured again to be 15,000 in 2007. If the population continues to grow linearly at this rate, what will the elk population be in 2014?

#### Solution

To begin, we need to define how we're going to measure  $n$ . Remember that  $P_0$  is the population when  $n = 0$ , so we probably don't want to literally use the year 0. Since we already know the population in 2003, let us define  $n = 0$  to be the year 2003. Then

$$P_0 = 12,000.$$

Next we need to find  $d$ . Remember  $d$  is the growth per time period, in this case growth per year. Between the two measurements, the population grew by  $15,000 - 12,000 = 3,000$  but it took  $2007 - 2003 = 4$  years to grow that much. To find the growth per year, we can divide:  $3000 \text{ elk} / 4 \text{ years} = 750 \text{ elk in 1 year}$ .

Alternatively, you can use the slope formula from algebra to determine the common difference, noting that the population is the output of the formula, and time is the input.

$$d = \text{slope} = \frac{\text{change in output}}{\text{change in input}} = \frac{15,000 - 12,000}{2007 - 2003} = \frac{3000}{4} = 750$$

We can now write our equation in whichever form is preferred.

Recursive form:

$$P_0 = 12,000$$

$$P_n = P_{n-1} + 750$$

Explicit form:

$$P_n = 12,000 + 750n$$

To answer the question, we need to first note that the year 2014 will be  $n = 11$ , since 2014 is 11 years after 2003. The explicit form will be easier to use for this calculation:

$$P_{11} = 12,000 + 750(11) = 20,250 \text{ elk}$$

### ✓ Example 2

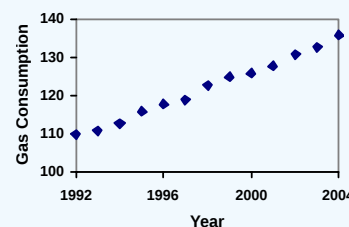
Gasoline consumption in the US has been increasing steadily. Consumption data from 1992 to 2004 is shown below[1]. Find a model for this data, and use it to predict consumption in 2016. If the trend continues, when will consumption reach 200 billion gallons?

| Year                                   | .92 | .93 | .94 | .95 | .96 | .97 | .98 | .99 | .00 | .01 | .02 | .03 | .04 |
|----------------------------------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| Consumption<br>(billion of<br>gallons) | 110 | 111 | 113 | 116 | 118 | 119 | 123 | 125 | 126 | 128 | 131 | 133 | 136 |

### Solution

Plotting this data, it appears to have an approximately linear relationship:

While there are more advanced statistical techniques that can be used to find an equation to model the data, to get an idea of what is happening, we can find an equation by using two pieces of the data – perhaps the data from 1993 and 2003.



Letting  $n = 0$  correspond with 1993 would give  $P_0 = 111$  billion gallons.

To find  $d$ , we need to know how much the gas consumption increased each year, on average. From 1993 to 2003 the gas consumption increased from 111 billion gallons to 133 billion gallons, a total change of  $133 - 111 = 22$  billion gallons, over 10 years. This gives us an average change of  $22 \text{ billion gallons} / 10 \text{ year} = 2.2 \text{ billion gallons per year}$ .

Equivalently,

$$d = \text{slope} = \frac{\text{change in output}}{\text{change in input}} = \frac{133 - 111}{10 - 0} = \frac{22}{10} = 2.2 \text{ billion gallons per year}$$

We can now write our equation in whichever form is preferred.

Recursive form:

$$P_0 = 111$$

$$P_n = P_{n-1} + 2.2$$

Explicit form:

$$P_n = 111 + 2.2n$$

Calculating values using the explicit form and plotting them with the original data shows how well our model fits the data.

We can now use our model to make predictions about the future, assuming that the previous trend continues unchanged. To predict the gasoline consumption in 2016:

$$n = 23(2016 - 1993 = 23 \text{ years later})$$

$$P_{23} = 111 + 2.2(23) = 161.6$$

Our model predicts that the US will consume 161.6 billion gallons of gasoline in 2016 if the current trend continues.

To find when the consumption will reach 200 billion gallons, we would set  $P_n = 200$ , and solve for  $n$ :

$$\begin{aligned} P_n &= 200 && \text{Replace } P_n \text{ with our model} \\ 111 + 2.2n &= 200 && \text{Subtract 111 from both sides} \\ 2.2n &= 89 && \text{Divide both sides by 2.2} \\ n &= 40.4545 \end{aligned}$$

This tells us that consumption will reach 200 billion about 40 years after 1993, which would be in the year 2033.

### ✓ Example 3

The cost, in dollars, of a gym membership for  $n$  months can be described by the explicit equation  $P_n = 70 + 30n$ . What does this equation tell us?

#### Solution

The value for  $P_0$  in this equation is 70, so the initial starting cost is \$70. This tells us that there must be an initiation or start-up fee of \$70 to join the gym.

The value for  $d$  in the equation is 30, so the cost increases by \$30 each month. This tells us that the monthly membership fee for the gym is \$30 a month.

### ? Try it Now 1

The number of stay-at-home fathers in Canada has been growing steadily[2]. While the trend is not perfectly linear, it is fairly linear. Use the data from 1976 and 2010 to find an explicit formula for the number of stay-at-home fathers, then use it to predict the number if 2020.

| Year                           | 1976   | 1984   | 1991   | 2000   | 2010   |
|--------------------------------|--------|--------|--------|--------|--------|
| Number of stay-at-home fathers | 20,610 | 28,725 | 43,530 | 47,665 | 53,555 |

#### Answer

Letting  $n = 0$  correspond with 1976, then  $P_0 = 20,610$ .

From 1976 to 2010 the number of stay-at-home fathers increased by

$$53,555 - 20,610 = 32,945$$

This happened over 34 years, giving a common different  $d$  of  $32,945/34 = 969$

$$P_n = 20,610 + 969n$$

Predicting for 2020, we use  $n = 44$

$$P_{44} = 20,610 + 969(44) = 63,246 \text{ stay-at-home fathers in 2020.}$$

[1] [www.bts.gov/publications/nati...ble\\_04\\_10.html](http://www.bts.gov/publications/national/ble_04_10.html)

[2] [www.fira.ca/article.php?id=140](http://www.fira.ca/article.php?id=140)

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## 4.2: Linear (Algebraic) Growth

Marco is a collector of antique soda bottles. His collection currently contains 437 bottles. Every year, he budgets enough money to buy 32 new bottles. Can we determine how many bottles he will have in 5 years, and how long it will take for his collection to reach 1000 bottles?

While both of these questions you could probably solve without an equation or formal mathematics, we are going to formalize our approach to this problem to provide a means to answer more complicated questions.

Suppose that  $P_n$  represents the number, or population, of bottles Marco has after  $n$  years. So  $P_0$  would represent the number of bottles now,  $P_1$  would represent the number of bottles after 1 year,  $P_2$  would represent the number of bottles after 2 years, and so on. We could describe how Marco's bottle collection is changing using:

$$P_0 = 437$$

$$P_n = P_{n-1} + 32$$

This is called a **recursive relationship**. A recursive relationship is a formula which relates the next value in a sequence to the previous values. Here, the number of bottles in year  $n$  can be found by adding 32 to the number of bottles in the previous year,  $P_{n-1}$ . Using this relationship, we could calculate:

$$P_1 = P_0 + 32 = 437 + 32 = 469$$

$$P_2 = P_1 + 32 = 469 + 32 = 501$$

$$P_3 = P_2 + 32 = 501 + 32 = 533$$

$$P_4 = P_3 + 32 = 533 + 32 = 565$$

$$P_5 = P_4 + 32 = 565 + 32 = 597$$

We have answered the question of how many bottles Marco will have in 5 years. However, solving how long it will take for his collection to reach 1000 bottles would require a lot more calculations.

While recursive relationships are excellent for describing simply and cleanly *how* a quantity is changing, they are not convenient for making predictions or solving problems that stretch far into the future. For that, a closed or explicit form for the relationship is preferred. An **explicit equation** allows us to calculate  $P_n$  directly, without needing to know  $P_{n-1}$ . While you may already be able to guess the explicit equation, let us derive it from the recursive formula. We can do so by selectively not simplifying as we go:

$$P_1 = 437 + 32 = 437 + 1(32)$$

$$P_2 = P_1 + 32 = 437 + 32 + 32 = 437 + 2(32)$$

$$P_3 = P_2 + 32 = (437 + 2(32)) + 32 = 437 + 3(32)$$

$$P_4 = P_3 + 32 = (437 + 3(32)) + 32 = 437 + 4(32)$$

You can probably see the pattern now, and generalize that

$$P_n = 437 + n(32) = 437 + 32n$$

Using this equation, we can calculate how many bottles he'll have after 5 years:

$$P_5 = 437 + 32(5) = 437 + 160 = 597$$

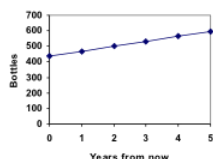
We can now also solve for when the collection will reach 1000 bottles by substituting in 1000 for  $P_n$  and solving for  $n$

$$1000 = 437 + 32n$$

$$563 = 32n$$

$$n = 563/32 = 17.59$$

So Marco will reach 1000 bottles in 18 years.



In the previous example, Marco's collection grew by the *same number* of bottles every year. This constant change is the defining characteristic of linear growth. Plotting the values we calculated for Marco's collection, we can see the values form a straight line, the shape of linear growth.

### Linear Growth

If a quantity starts at size  $P_0$  and grows by  $d$  every time period, then the quantity after  $n$  time periods can be determined using either of these relations:

Recursive form:

$$P_n = P_{n-1} + d$$

Explicit form:

$$P_n = P_0 + d n$$

In this equation,  $d$  represents the **common difference** – the amount that the population changes each time  $n$  increases by 1

### Connection to prior learning – slope and intercept

You may recognize the common difference,  $d$ , in our linear equation as *slope*. In fact, the entire explicit equation should look familiar – it is the same linear equation you learned in algebra, probably stated as  $y = mx + b$ .

In the standard algebraic equation  $y = mx + b$ ,  $b$  was the  $y$ -intercept, or the  $y$  value when  $x$  was zero. In the form of the equation we're using, we are using  $P_0$  to represent that initial amount.

In the  $y = mx + b$  equation, recall that  $m$  was the slope. You might remember this as “rise over run”, or the change in  $y$  divided by the change in  $x$ . Either way, it represents the same thing as the common difference,  $d$ , we are using – the amount the output  $P_n$  changes when the input  $n$  increases by 1.

The equations  $y = mx + b$  and  $P_n = P_0 + d n$  mean the same thing and can be used the same ways, we're just writing it somewhat differently.

### Example 1

The population of elk in a national forest was measured to be 12,000 in 2003, and was measured again to be 15,000 in 2007. If the population continues to grow linearly at this rate, what will the elk population be in 2014?

To begin, we need to define how we're going to measure  $n$ . Remember that  $P_0$  is the population when  $n = 0$ , so we probably don't want to literally use the year 0. Since we already know the population in 2003, let us define  $n = 0$  to be the year 2003. Then

$$P_0 = 12,000.$$

Next we need to find  $d$ . Remember  $d$  is the growth per time period, in this case growth per year. Between the two measurements, the population grew by  $15,000 - 12,000 = 3,000$ , but it took  $2007 - 2003 = 4$  years to grow that much. To find the growth per year, we can divide:  $3000 \text{ elk} / 4 \text{ years} = 750 \text{ elk in 1 year}$ .

Alternatively, you can use the slope formula from algebra to determine the common difference, noting that the population is the output of the formula, and time is the input.

$$d = \text{slope} = \frac{\text{change in output}}{\text{change in input}} = \frac{15,000 - 12,000}{2007 - 2003} = \frac{3000}{4} = 750$$

We can now write our equation in whichever form is preferred.

Recursive form:

$$P_0 = 12,000$$

$$P_n = P_{n-1} + 750$$

Explicit form:

$$P_n = 12,000 + 750n$$

To answer the question, we need to first note that the year 2014 will be  $n = 11$ , since 2014 is 11 years after 2003. The explicit form will be easier to use for this calculation:

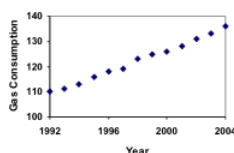
$$P_{11} = 12,000 + 750(11) = 20,250 \text{ elk}$$

## Example 2

Gasoline consumption in the US has been increasing steadily. Consumption data from 1992 to 2004 is shown below<sup>[1]</sup>. Find a model for this data, and use it to predict consumption in 2016. If the trend continues, when will consumption reach 200 billion gallons?

| Year                             | '92 | '93 | '94 | '95 | '96 | '97 | '98 | '99 | '00 | '01 | '02 | '03 | '04 |
|----------------------------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| Consumption (billion of gallons) | 110 | 111 | 113 | 116 | 118 | 119 | 123 | 125 | 126 | 128 | 131 | 133 | 136 |

Plotting this data, it appears to have an approximately linear relationship:



While there are more advanced statistical techniques that can be used to find an equation to model the data, to get an idea of what is happening, we can find an equation by using two pieces of the data – perhaps the data from 1993 and 2003.

Letting  $n = 0$  correspond with 1993 would give  $P_0 = 111$  billion gallons.

To find  $d$ , we need to know how much the gas consumption increased each year, on average. From 1993 to 2003 the gas consumption increased from 111 billion gallons to 133 billion gallons, a total change of  $133 - 111 = 22$  billion gallons, over 10 years. This gives us an average change of  $22 \text{ billion gallons} / 10 \text{ year} = 2.2 \text{ billion gallons per year}$ .

Equivalently,

$$d = \text{slope} = \frac{\text{change in output}}{\text{change in input}} = \frac{133 - 111}{10 - 0} = \frac{22}{10} = 2.2 \text{ billion gallons per year}$$

We can now write our equation in whichever form is preferred.

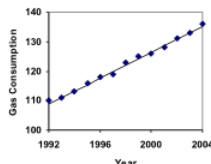
Recursive form:

$$P_0 = 111$$

$$P_n = P_{n-1} + 2.2$$

Explicit form:

$$P_n = 111 + 2.2n$$



Calculating values using the explicit form and plotting them with the original data shows how well our model fits the data.

We can now use our model to make predictions about the future, assuming that the previous trend continues unchanged. To predict the gasoline consumption in 2016:

$$n = 23 \text{ (2016} - 1993 = 23 \text{ years later)}$$

$$P_{23} = 111 + 2.2(23) = 161.6$$

Our model predicts that the US will consume 161.6 billion gallons of gasoline in 2016 if the current trend continues.

To find when the consumption will reach 200 billion gallons, we would set  $Pn = 200$ , and solve for  $n$ :

$$Pn = 200 \quad \text{Replace } Pn \text{ with our model}$$

$$111 + 2.2n = 200 \quad \text{Subtract 111 from both sides}$$

$$2.2n = 89 \quad \text{Divide both sides by 2.2}$$

$$n = 40.4545$$

This tells us that consumption will reach 200 billion about 40 years after 1993, which would be in the year 2033.

### Example 3

The cost, in dollars, of a gym membership for  $n$  months can be described by the explicit equation  $Pn = 70 + 30n$ . What does this equation tell us?

The value for  $P_0$  in this equation is 70, so the initial starting cost is \$70. This tells us that there must be an initiation or start-up fee of \$70 to join the gym.

The value for  $d$  in the equation is 30, so the cost increases by \$30 each month. This tells us that the monthly membership fee for the gym is \$30 a month.

### Try it Now 1

The number of stay-at-home fathers in Canada has been growing steadily<sup>[2]</sup>. While the trend is not perfectly linear, it is fairly linear. Use the data from 1976 and 2010 to find an explicit formula for the number of stay-at-home fathers, then use it to predict the number if 2020.

### When good models go bad

When using mathematical models to predict future behavior, it is important to keep in mind that very few trends will continue indefinitely.

### Example 4

Suppose a four year old boy is currently 39 inches tall, and you are told to expect him to grow 2.5 inches a year.

We can set up a growth model, with  $n = 0$  corresponding to 4 years old.

Recursive form:

$$P_0 = 39$$

$$P_n = P_{n-1} + 2.5$$

Explicit form:

$$Pn = 39 + 2.5n$$

So at 6 years old, we would expect him to be

$$P_2 = 39 + 2.5(2) = 44 \text{ inches tall}$$

Any mathematical model will break down eventually. Certainly, we shouldn't expect this boy to continue to grow at the same rate all his life. If he did, at age 50 he would be

$$P_{46} = 39 + 2.5(46) = 154 \text{ inches tall} = 12.8 \text{ feet tall!}$$

When using any mathematical model, we have to consider which inputs are reasonable to use. Whenever we **extrapolate**, or make predictions into the future, we are assuming the model will continue to be valid.

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1. [http://www.bts.gov/publications/national\\_transportation\\_statistics/2005/html/table\\_04\\_10.html](http://www.bts.gov/publications/national_transportation_statistics/2005/html/table_04_10.html) ↵

2. <http://www.fira.ca/article.php?id=140> ↵

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### 4.3: Exponential (Geometric) Growth

Suppose that every year, only 10% of the fish in a lake have surviving offspring. If there were 100 fish in the lake last year, there would now be 110 fish. If there were 1000 fish in the lake last year, there would now be 1100 fish. Absent any inhibiting factors, populations of people and animals tend to grow by a percent of the existing population each year.

Suppose our lake began with 1000 fish, and 10% of the fish have surviving offspring each year. Since we start with 1000 fish,  $P_0 = 1000$ . How do we calculate  $P_1$ ? The new population will be the old population, plus an additional 10%. Symbolically:

$$P_1 = P_0 + 0.10P_0$$

Notice this could be condensed to a shorter form by factoring:

$$P_1 = P_0 + 0.10P_0 = 1P_0 + 0.10P_0 = (1 + 0.10)P_0 = 1.10P_0$$

While 10% is the **growth rate**, 1.10 is the **growth multiplier**. Notice that 1.10 can be thought of as “the original 100% plus an additional 10%”

For our fish population,

$$P_1 = 1.10(1000) = 1100$$

We could then calculate the population in later years:

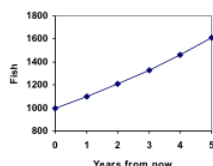
$$P_2 = 1.10P_1 = 1.10(1100) = 1210$$

$$P_3 = 1.10P_2 = 1.10(1210) = 1331$$

Notice that in the first year, the population grew by 100 fish, in the second year, the population grew by 110 fish, and in the third year the population grew by 121 fish.

While there is a constant *percentage* growth, the actual increase in number of fish is increasing each year.

Graphing these values we see that this growth doesn't quite appear linear.



To get a better picture of how this percentage-based growth affects things, we need an explicit form, so we can quickly calculate values further out in the future.

Like we did for the linear model, we will start building from the recursive equation:

$$P_1 = 1.10P_0 = 1.10(1000)$$

$$P_2 = 1.10P_1 = 1.10(1.10(1000)) = 1.10^2(1000)$$

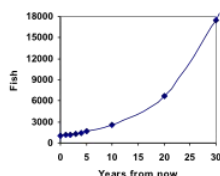
$$P_3 = 1.10P_2 = 1.10(1.10^2(1000)) = 1.10^3(1000)$$

$$P_4 = 1.10P_3 = 1.10(1.10^3(1000)) = 1.10^4(1000)$$

Observing a pattern, we can generalize the explicit form to be:

$$P_n = 1.10^n(1000), \text{ or equivalently, } P_n = 1000(1.10^n)$$

From this, we can quickly calculate the number of fish in 10, 20, or 30 years:



$$P_{10} = 1.10^{10}(1000) = 2594$$

$$P_{20} = 1.1020(1000) = 6727$$

$$P_{30} = 1.1030(1000) = 17449$$

Adding these values to our graph reveals a shape that is definitely not linear. If our fish population had been growing linearly, by 100 fish each year, the population would have only reached 4000 in 30 years compared to almost 18000 with this percent-based growth, called **exponential growth**.

In exponential growth, the population grows proportional to the size of the population, so as the population gets larger, the same percent growth will yield a larger numeric growth.

#### Exponential Growth

If a quantity starts at size  $P_0$  and grows by  $R\%$  (written as a decimal,  $r$ ) every time period, then the quantity after  $n$  time periods can be determined using either of these relations:

Recursive form:

$$P_n = (1+r) P_{n-1}$$

Explicit form:

$$P_n = (1+r)^n P_0 \quad \text{or equivalently, } P_n = P_0 (1+r)^n$$

We call  $r$  the **growth rate**.

The term  $(1+r)$  is called the **growth multiplier**, or common ratio.

#### Example 5

Between 2007 and 2008, Olympia, WA grew almost 3% to a population of 245 thousand people. If this growth rate was to continue, what would the population of Olympia be in 2014?

As we did before, we first need to define what year will correspond to  $n = 0$ . Since we know the population in 2008, it would make sense to have 2008 correspond to  $n = 0$ , so  $P_0 = 245,000$ . The year 2014 would then be  $n = 6$ .

We know the growth rate is 3%, giving  $r = 0.03$ .

Using the explicit form:

$$P_6 = (1+0.03)^6 (245,000) = 1.19405(245,000) = 292,542.25$$

The model predicts that in 2014, Olympia would have a population of about 293 thousand people.

#### Evaluating exponents on the calculator

To evaluate expressions like  $(1.03)^6$ , it will be easier to use a calculator than multiply 1.03 by itself six times. Most scientific calculators have a button for exponents. It is typically either labeled like:

$\wedge$ ,  $y^x$ , or  $x^y$ .

To evaluate  $1.03^6$  we'd type  $1.03 \wedge 6$ , or  $1.03 yx 6$ . Try it out – you should get an answer around 1.1940523.

#### Try it Now 2

India is the second most populous country in the world, with a population in 2008 of about 1.14 billion people. The population is growing by about 1.34% each year. If this trend continues, what will India's population grow to by 2020?

#### Example 6

A friend is using the equation  $P_n = 4600(1.072)^n$  to predict the annual tuition at a local college. She says the formula is based on years after 2010. What does this equation tell us?

In the equation,  $P_0 = 4600$ , which is the starting value of the tuition when  $n = 0$ . This tells us that the tuition in 2010 was \$4,600.

The growth multiplier is 1.072, so the growth rate is 0.072, or 7.2%. This tells us that the tuition is expected to grow by 7.2% each year.

Putting this together, we could say that the tuition in 2010 was \$4,600, and is expected to grow by 7.2% each year.

### Example 7

In 1990, the residential energy use in the US was responsible for 962 million metric tons of carbon dioxide emissions. By the year 2000, that number had risen to 1182 million metric tons<sup>[1]</sup>. If the emissions grow exponentially and continue at the same rate, what will the emissions grow to by 2050?

Similar to before, we will correspond  $n = 0$  with 1990, as that is the year for the first piece of data we have. That will make  $P_0 = 962$  (million metric tons of CO<sub>2</sub>). In this problem, we are not given the growth rate, but instead are given that  $P_{10} = 1182$ .

When  $n = 10$ , the explicit equation looks like:

$$P_{10} = (1+r)^{10} P_0$$

We know the value for  $P_0$ , so we can put that into the equation:

$$P_{10} = (1+r)^{10} 962$$

We also know that  $P_{10} = 1182$ , so substituting that in, we get

$$1182 = (1+r)^{10} 962$$

We can now solve this equation for the growth rate,  $r$ . Start by dividing by 962.

$$\frac{1182}{962} = \{(1+r)^{10}\}$$

Take the 10th root of both sides

$$\sqrt[10]{\frac{1182}{962}} = 1+r$$

Subtract 1 from both sides

$$r = \sqrt[10]{\frac{1182}{962}} - 1 = 0.0208 = 2.08\%$$

So if the emissions are growing exponentially, they are growing by about 2.08% per year. We can now predict the emissions in 2050 by finding  $P_{60}$

$$P_{60} = (1+0.0208)^{60} 962 = 3308.4 \text{ million metric tons of CO}_2 \text{ in 2050}$$

### Rounding

As a note on rounding, notice that if we had rounded the growth rate to 2.1%, our calculation for the emissions in 2050 would have been 3347. Rounding to 2% would have changed our result to 3156. A very small difference in the growth rates gets magnified greatly in exponential growth. For this reason, it is recommended to round the growth rate as little as possible.

If you need to round, **keep at least three significant digits** – numbers after any leading zeros. So 0.4162 could be reasonably rounded to 0.416. A growth rate of 0.001027 could be reasonably rounded to 0.00103.

### Evaluating roots on the calculator

In the previous example, we had to calculate the 10th root of a number. This is different than taking the basic square root,  $\sqrt{\phantom{x}}$ . Many scientific calculators have a button for general roots. It is typically labeled like:

$$\sqrt[n]{\phantom{x}}, \sqrt[x]{\phantom{y}}, \text{ or } \sqrt[y]{\phantom{x}}$$

To evaluate the 3rd root of 8, for example, we'd either type  $3 \sqrt[x]{\phantom{y}}$  8, or  $8 \sqrt[y]{\phantom{x}}$  3, depending on the calculator. Try it on yours to see which to use – you should get an answer of 2.

If your calculator does not have a general root button, all is not lost. You can instead use the property of exponents which states that  $\sqrt[n]{x} = x^{1/n}$ . So, to compute the 3rd root of 8, you could use your calculator's exponent key to evaluate  $8^{1/3}$ . To do this, type:

$$8 y^x ( 1 \div 3 )$$

The parentheses tell the calculator to divide 1/3 before doing the exponent.

### Try it Now 3

The number of users on a social networking site was 45 thousand in February when they officially went public, and grew to 60 thousand by October. If the site is growing exponentially, and growth continues at the same rate, how many users should then expect two years after they went public?

### Example 8

Looking back at the last example, for the sake of comparison, what would the carbon emissions be in 2050 if emissions grow linearly at the same rate?

Again we will get  $n = 0$  correspond with 1990, giving  $P0 = 962$ . To find  $d$ , we could take the same approach as earlier, noting that the emissions increased by 220 million metric tons in 10 years, giving a common difference of 22 million metric tons each year.

Alternatively, we could use an approach similar to that which we used to find the exponential equation. When  $n = 10$ , the explicit linear equation looks like:

$$P10 = P0 + 10d$$

We know the value for  $P0$ , so we can put that into the equation:

$$P10 = 962 + 10d$$

Since we know that  $P10 = 1182$ , substituting that in we get

$$1182 = 962 + 10d$$

We can now solve this equation for the common difference,  $d$ .

$$1182 - 962 = 10d$$

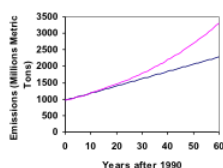
$$220 = 10d$$

$$d = 22$$

This tells us that if the emissions are changing linearly, they are growing by 22 million metric tons each year. Predicting the emissions in 2050,

$$P60 = 962 + 22(60) = 2282 \text{ million metric tons.}$$

You will notice that this number is substantially smaller than the prediction from the exponential growth model. Calculating and plotting more values helps illustrate the differences.



So how do we know which growth model to use when working with data? There are two approaches which should be used together whenever possible:

- 1) Find more than two pieces of data. Plot the values, and look for a trend. Does the data appear to be changing like a line, or do the values appear to be curving upwards?
- 2) Consider the factors contributing to the data. Are they things you would expect to change linearly or exponentially? For example, in the case of carbon emissions, we could expect that, absent other factors, they would be tied closely to population values, which tend to change exponentially.

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1. <http://www.eia.doe.gov/oiaf/1605/ggrpt/carbon.html> ↩

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- 

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## 4.4: Logarithmic Functions

### Learning Objectives

- Convert from logarithmic to exponential form.
- Convert from exponential to logarithmic form.
- Evaluate logarithms.
- Use common logarithms.
- Use natural logarithms.

In 2010, a major earthquake struck Haiti, destroying or damaging over 285,000 homes. One year later, another, stronger earthquake devastated Honshu, Japan, destroying or damaging over 332,000 buildings, like those shown in Figure 4.4.1. Even though both caused substantial damage, the earthquake in 2011 was 100 times stronger than the earthquake in Haiti. How do we know? The magnitudes of earthquakes are measured on a scale known as the Richter Scale. The Haitian earthquake registered a 7.0 on the Richter Scale whereas the Japanese earthquake registered a 9.0.



Figure 4.4.1: Devastation of March 11, 2011 earthquake in Honshu, Japan. (credit: Daniel Pierce).

The **Richter Scale** is a base-ten logarithmic scale. In other words, an earthquake of magnitude 8 is not twice as great as an earthquake of magnitude 4. It is

$$10^{8-4} = 10^4 = 10,000$$

times as great! In this lesson, we will investigate the nature of the Richter Scale and the base-ten function upon which it depends.

### Converting from Logarithmic to Exponential Form

In order to analyze the magnitude of earthquakes or compare the magnitudes of two different earthquakes, we need to be able to convert between logarithmic and exponential form. For example, suppose the amount of energy released from one earthquake were 500 times greater than the amount of energy released from another. We want to calculate the difference in magnitude. The equation that represents this problem is  $10^x = 500$ , where  $x$  represents the difference in magnitudes on the Richter Scale. How would we solve for  $x$ ?

We have not yet learned a method for solving exponential equations. None of the algebraic tools discussed so far is sufficient to solve  $10^x = 500$ . We know that  $10^2 = 100$  and  $10^3 = 1000$ , so it is clear that  $x$  must be some value between 2 and 3, since  $y = 10^x$  is increasing. We can examine a graph, as in Figure 4.4.1, to better estimate the solution.

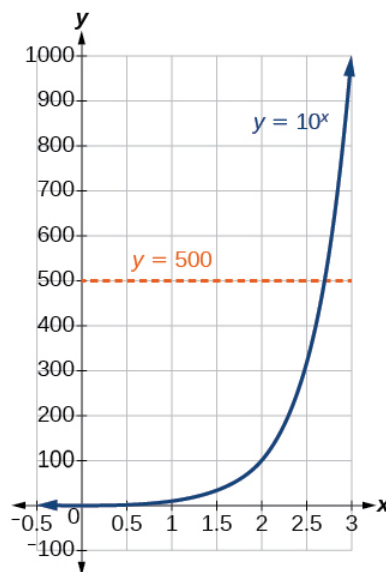


Figure 4.4.2

Estimating from a graph, however, is imprecise. To find an algebraic solution, we must introduce a new function. Observe that the graph in Figure 4.4.2 passes the horizontal line test. The exponential function  $y = b^x$  is one-to-one, so its inverse,  $x = b^y$  is also a function. As is the case with all inverse functions, we simply interchange  $x$  and  $y$  and solve for  $y$  to find the inverse function. To represent  $y$  as a function of  $x$ , we use a logarithmic function of the form  $y = \log_b(x)$ . The base  $b$  **logarithm** of a number is the exponent by which we must raise  $b$  to get that number.

We read a logarithmic expression as, “The logarithm with base  $b$  of  $x$  is equal to  $y$ ,” or, simplified, “log base  $b$  of  $x$  is  $y$ .” We can also say, “ $b$  raised to the power of  $y$  is  $x$ ,” because logs are exponents. For example, the base 2 logarithm of 32 is 5, because 5 is the exponent we must apply to 2 to get 32. Since  $2^5 = 32$ , we can write  $\log_2 32 = 5$ . We read this as “log base 2 of 32 is 5.”

We can express the relationship between logarithmic form and its corresponding exponential form as follows:

$$\log_b(x) = y \Leftrightarrow b^y = x, b > 0, b \neq 1 \quad (4.4.1)$$

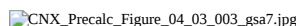
Note that the base  $b$  is always positive.

$$\log_b(x) = y$$

Think  
 $b$  to the  $y = x$

Because logarithm is a function, it is most correctly written as  $\log_b(x)$ , using parentheses to denote function evaluation, just as we would with  $f(x)$ . However, when the input is a single variable or number, it is common to see the parentheses dropped and the expression written without parentheses, as  $\log_b x$ . Note that many calculators require parentheses around the  $x$ .

We can illustrate the notation of logarithms as follows:



Notice that, comparing the logarithm function and the exponential function, the input and the output are switched. This means  $y = \log_b(x)$  and  $y = b^x$  are inverse functions.

#### DEFINITION OF THE LOGARITHMIC FUNCTION

A logarithm base  $b$  of a positive number  $x$  satisfies the following definition.

For  $x > 0$ ,  $b > 0$ ,  $b \neq 1$ ,

$$y = \log_b(x) \text{ is equivalent to } b^y = x \quad (4.4.2)$$

where,

- we read  $\log_b(x)$  as, “the logarithm with base  $b$  of  $x$ ” or the “log base  $b$  of  $x$ .”

- the logarithm  $y$  is the exponent to which  $b$  must be raised to get  $x$ .

Also, since the logarithmic and exponential functions switch the  $x$  and  $y$  values, the domain and range of the exponential function are interchanged for the logarithmic function. Therefore,

- the domain of the logarithm function with base  $b$  is  $(0, \infty)$ .
- the range of the logarithm function with base  $b$  is  $(-\infty, \infty)$ .

#### 📌 Q&A: Can we take the logarithm of a negative number?

*No. Because the base of an exponential function is always positive, no power of that base can ever be negative. We can never take the logarithm of a negative number. Also, we cannot take the logarithm of zero. Calculators may output a log of a negative number when in complex mode, but the log of a negative number is not a real number.*

#### 📌 How to: Given an equation in logarithmic form $\log_b(x) = y$ , convert it to exponential form

- Examine the equation  $y = \log_b x$  and identify  $b$ ,  $y$ , and  $x$ .
- Rewrite  $\log_b x = y$  as  $b^y = x$ .

#### ✓ Example 4.4.1: Converting from Logarithmic Form to Exponential Form

Write the following logarithmic equations in exponential form.

- $\log_6(\sqrt{6}) = \frac{1}{2}$
- $\log_3(9) = 2$

##### **Solution**

First, identify the values of  $b$ ,  $y$ , and  $x$ . Then, write the equation in the form  $b^y = x$ .

a.  $\log_6(\sqrt{6}) = \frac{1}{2}$

Here,  $b = 6$ ,  $y = \frac{1}{2}$ , and  $x = \sqrt{6}$ . Therefore, the equation  $\log_6(\sqrt{6}) = \frac{1}{2}$  is equivalent to

$$6^{\frac{1}{2}} = \sqrt{6}$$

b.  $\log_3(9) = 2$

Here,  $b = 3$ ,  $y = 2$ , and  $x = 9$ . Therefore, the equation  $\log_3(9) = 2$  is equivalent to

$$3^2 = 9$$

#### ? Exercise 4.4.1

Write the following logarithmic equations in exponential form.

- $\log_{10}(1,000,000) = 6$
- $\log_5(25) = 2$

##### **Answer a**

$\log_{10}(1,000,000) = 6$  is equivalent to  $10^6 = 1,000,000$

##### **Answer b**

$\log_5(25) = 2$  is equivalent to  $5^2 = 25$

## Converting from Exponential to Logarithmic Form

To convert from exponents to logarithms, we follow the same steps in reverse. We identify the base  $b$ , exponent  $x$ , and output  $y$ . Then we write  $x = \log_b(y)$ .

### ✓ Example 4.4.2: Converting from Exponential Form to Logarithmic Form

Write the following exponential equations in logarithmic form.

a.  $2^3 = 8$

b.  $5^2 = 25$

c.  $10^{-4} = \frac{1}{10,000}$

#### Solution

First, identify the values of  $b$ ,  $y$ , and  $x$ . Then, write the equation in the form  $x = \log_b(y)$ .

a.  $2^3 = 8$

Here,  $b = 2$ ,  $x = 3$ , and  $y = 8$ . Therefore, the equation  $2^3 = 8$  is equivalent to  $\log_2(8) = 3$ .

b.  $5^2 = 25$

Here,  $b = 5$ ,  $x = 2$ , and  $y = 25$ . Therefore, the equation  $5^2 = 25$  is equivalent to  $\log_5(25) = 2$ .

c.  $10^{-4} = \frac{1}{10,000}$

Here,  $b = 10$ ,  $x = -4$ , and  $y = \frac{1}{10,000}$ . Therefore, the equation  $10^{-4} = \frac{1}{10,000}$  is equivalent to

$$\log_{10}\left(\frac{1}{10,000}\right) = -4.$$

### ? Exercise 4.4.2

Write the following exponential equations in logarithmic form.

a.  $3^2 = 9$

b.  $5^3 = 125$

c.  $2^{-1} = \frac{1}{2}$

#### Answer a

$3^2 = 9$  is equivalent to  $\log_3(9) = 2$

#### Answer b

$5^3 = 125$  is equivalent to  $\log_5(125) = 3$

#### Answer c

$2^{-1} = \frac{1}{2}$  is equivalent to  $\log_2\left(\frac{1}{2}\right) = -1$

## Evaluating Logarithms

Knowing the squares, cubes, and roots of numbers allows us to evaluate many logarithms mentally. For example, consider  $\log_2 8$ . We ask, "To what exponent must 2 be raised in order to get 8?" Because we already know  $2^3 = 8$ , it follows that  $\log_2 8 = 3$ .

Now consider solving  $\log_7 49$  and  $\log_3 27$  mentally.

- We ask, "To what exponent must 7 be raised in order to get 49?" We know  $7^2 = 49$ . Therefore,  $\log_7 49 = 2$
- We ask, "To what exponent must 3 be raised in order to get 27?" We know  $3^3 = 27$ . Therefore,  $\log_3 27 = 3$

Even some seemingly more complicated logarithms can be evaluated without a calculator. For example, let's evaluate  $\log_{\frac{2}{3}} \frac{4}{9}$  mentally.

- We ask, "To what exponent must  $\frac{2}{3}$  be raised in order to get  $\frac{4}{9}$ ?" We know  $2^2 = 4$  and  $3^2 = 9$ , so

$$\left(\frac{2}{3}\right)^2 = \frac{4}{9}.$$

Therefore,

$$\log_{\frac{2}{3}} \left(\frac{4}{9}\right) = 2.$$

**How to:** Given a logarithm of the form  $y = \log_b(x)$ , evaluate it mentally

1. Rewrite the argument  $x$  as a power of  $b$ :  $b^y = x$ .
2. Use previous knowledge of powers of  $b$  identify  $y$  by asking, "To what exponent should  $b$  be raised in order to get  $x$ ?"

#### ✓ Example 4.4.3: Solving Logarithms Mentally

Solve  $y = \log_4(64)$  without using a calculator.

##### **Solution**

First we rewrite the logarithm in exponential form:  $4^y = 64$ . Next, we ask, "To what exponent must 4 be raised in order to get 64?"

We know

$$4^3 = 64$$

Therefore,

$$\log_4(64) = 3$$

#### ? Exercise 4.4.3

Solve  $y = \log_{121}(11)$  without using a calculator.

##### **Answer**

$$\log_{121}(11) = \frac{1}{2} \text{ (recalling that } \sqrt{121} = (121)^{\frac{1}{2}} = 11)$$

#### ✓ Example 4.4.4: Evaluating the Logarithm of a Reciprocal

Evaluate  $y = \log_3 \left(\frac{1}{27}\right)$  without using a calculator.

##### **Solution**

First we rewrite the logarithm in exponential form:  $3^y = \frac{1}{27}$ . Next, we ask, "To what exponent must 3 be raised in order to get  $\frac{1}{27}$ ?"

We know  $3^3 = 27$ , but what must we do to get the reciprocal,  $\frac{1}{27}$ ? Recall from working with exponents that  $b^{-a} = \frac{1}{b^a}$ . We use this information to write

$$3^{-3} = \frac{1}{3^3} = \frac{1}{27}$$

Therefore,  $\log_3 \left( \frac{1}{27} \right) = -3$ .

#### ? Exercise 4.4.4

Evaluate  $y = \log_2 \left( \frac{1}{32} \right)$  without using a calculator.

**Answer**

$$\log_2 \left( \frac{1}{32} \right) = -5$$

### Using Common Logarithms

Sometimes we may see a logarithm written without a base. In this case, we assume that the base is 10. In other words, the expression  $\log(x)$  means  $\log_{10}(x)$ . We call a base 10 logarithm a *common logarithm*. Common logarithms are used to measure the Richter Scale mentioned at the beginning of the section. Scales for measuring the brightness of stars and the pH of acids and bases also use common logarithms.

#### 📌 DEFINITION OF THE COMMON LOGARITHM

A common logarithm is a logarithm with base 10. We write  $\log_{10}(x)$  simply as  $\log(x)$ . The common logarithm of a positive number  $x$  satisfies the following definition.

For  $x > 0$ ,

$$y = \log(x) \text{ is equivalent to } 10^y = x \quad (4.4.3)$$

We read  $\log(x)$  as, “the logarithm with base 10 of  $x$ ” or “log base 10 of  $x$ .”

The logarithm  $y$  is the exponent to which 10 must be raised to get  $x$ .

#### 📌 How to: Given a common logarithm of the form $y = \log(x)$ , evaluate it mentally

1. Rewrite the argument  $x$  as a power of 10:  $10^y = x$ .
2. Use previous knowledge of powers of 10 to identify  $y$  by asking, “To what exponent must 10 be raised in order to get  $x$ ?”

#### ✓ Example 4.4.5: Finding the Value of a Common Logarithm Mentally

Evaluate  $y = \log(1000)$  without using a calculator.

**Solution**

First we rewrite the logarithm in exponential form:  $10^y = 1000$ . Next, we ask, “To what exponent must 10 be raised in order to get 1000?” We know

$$10^3 = 1000$$

Therefore,  $\log(1000) = 3$ .

### ? Exercise 4.4.5

Evaluate  $y = \log(1,000,000)$

**Answer**

$$\log(1,000,000) = 6$$

**How to:** Given a common logarithm with the form  $y = \log(x)$ , evaluate it using a calculator

1. Press **[LOG]**.
2. Enter the value given for  $x$ , followed by **[ ) ]**.
3. Press **[ENTER]**.

### ✓ Example 4.4.6: Finding the Value of a Common Logarithm Using a Calculator

Evaluate  $y = \log(321)$  to four decimal places using a calculator.

**Solution**

- Press **[LOG]**.
- Enter 321, followed by **[ ) ]**.
- Press **[ENTER]**.

Rounding to four decimal places,  $\log(321) \approx 2.5065$

**Analysis**

Note that  $10^2 = 100$  and that  $10^3 = 1000$ . Since 321 is between 100 and 1000, we know that  $\log(321)$  must be between  $\log(100)$  and  $\log(1000)$ . This gives us the following:

$$\begin{aligned} 100 &< 321 < 1000 \\ 2 &< 2.5065 < 3 \end{aligned}$$

### ? Exercise 4.4.6

Evaluate  $y = \log(123)$  to four decimal places using a calculator.

**Answer**

$$\log(123) \approx 2.0899$$

### ✓ Example 4.4.7: Rewriting and Solving a Real-World Exponential Model

The amount of energy released from one earthquake was 500 times greater than the amount of energy released from another. The equation  $10^x = 500$  represents this situation, where  $x$  is the difference in magnitudes on the Richter Scale. To the nearest thousandth, what was the difference in magnitudes?

**Solution**

We begin by rewriting the exponential equation in logarithmic form.

$$10^x = 500$$

$$\log(500) = x \text{ Use the definition of the common log.}$$

Next we evaluate the logarithm using a calculator:

- Press **[LOG]**.
- Enter 500, followed by **[ ) ]**.
- Press **[ENTER]**.
- To the nearest thousandth,  $\log(500) \approx 2.699$

The difference in magnitudes was about 2.699.

### ? Exercise 4.4.7

The amount of energy released from one earthquake was 8,500 times greater than the amount of energy released from another. The equation  $10^x = 8500$  represents this situation, where  $x$  is the difference in magnitudes on the Richter Scale. To the nearest thousandth, what was the difference in magnitudes?

#### Answer

The difference in magnitudes was about 3.929

## Using Natural Logarithms

The most frequently used base for logarithms is  $e$ . Base  $e$  logarithms are important in calculus and some scientific applications; they are called **natural logarithms**. The base  $e$  logarithm,  $\log_e(x)$ , has its own notation,  $\ln(x)$ . Most values of  $\ln(x)$  can be found only using a calculator. The major exception is that, because the logarithm of 1 is always 0 in any base,  $\ln 1 = 0$ . For other natural logarithms, we can use the  $\ln$  key that can be found on most scientific calculators. We can also find the natural logarithm of any power of  $e$  using the inverse property of logarithms.

### 📌 DEFINITION OF THE NATURAL LOGARITHM

A natural logarithm is a logarithm with base  $e$ . We write  $\log_e(x)$  simply as  $\ln(x)$ . The natural logarithm of a positive number  $x$  satisfies the following definition.

For  $x > 0$ ,

$$y = \ln(x) \text{ is equivalent to } e^y = x$$

We read  $\ln(x)$  as, “the logarithm with base  $e$  of  $x$ ” or “the natural logarithm of  $x$ .”

The logarithm  $y$  is the exponent to which  $e$  must be raised to get  $x$ .

Since the functions  $y = e^x$  and  $y = \ln(x)$  are inverse functions,  $\ln(e^x) = x$  for all  $x$  and  $e^{\ln(x)} = x$  for  $x > 0$ .

### 📌 How to: Given a natural logarithm with the form $y = \ln(x)$ , evaluate it using a calculator

1. Press **[LN]**.
2. Enter the value given for  $x$ , followed by **[ ) ]**.
3. Press **[ENTER]**.

### ✓ Example 4.4.8: Evaluating a Natural Logarithm Using a Calculator

Evaluate  $y = \ln(500)$  to four decimal places using a calculator.

#### Solution

- Press **[LN]**.
- Enter 500, followed by **[ ) ]**.
- Press **[ENTER]**.

Rounding to four decimal places,  $\ln(500) \approx 6.2146$

### ? Exercise 4.4.8

Evaluate  $\ln(-500)$ .

#### Answer

It is not possible to take the logarithm of a negative number in the set of real numbers.

## Media

Access this online resource for additional instruction and practice with logarithms.

- [Introduction to Logarithms](#)

## Key Equations

|                                        |                                                                                 |
|----------------------------------------|---------------------------------------------------------------------------------|
| Definition of the logarithmic function | For $x > 0$ , $b > 0$ , $b \neq 1$ , $y = \log_b(x)$ if and only if $b^y = x$ . |
| Definition of the common logarithm     | For $x > 0$ , $y = \log(x)$ if and only if $10^y = x$ .                         |
| Definition of the natural logarithm    | For $x > 0$ , $y = \ln(x)$ if and only if $e^y = x$ .                           |

## Key Concepts

- The inverse of an exponential function is a logarithmic function, and the inverse of a logarithmic function is an exponential function.
- Logarithmic equations can be written in an equivalent exponential form, using the definition of a logarithm. See Example 4.4.1.
- Exponential equations can be written in their equivalent logarithmic form using the definition of a logarithm See Example 4.4.2.
- Logarithmic functions with base  $b$  can be evaluated mentally using previous knowledge of powers of  $b$ . See Example 4.4.3 and Example 4.4.4.
- Common logarithms can be evaluated mentally using previous knowledge of powers of 10. See Example 4.4.5.
- When common logarithms cannot be evaluated mentally, a calculator can be used. See Example 4.4.6.
- Real-world exponential problems with base 10 can be rewritten as a common logarithm and then evaluated using a calculator. See Example 4.4.7.
- Natural logarithms can be evaluated using a calculator Example 4.4.8.

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## 4.5: Solve Exponentials for Time- Logarithms

Earlier, we found that since Olympia, WA had a population of 245 thousand in 2008 and had been growing at 3% per year, the population could be modeled by the equation

$$P_n = (1+0.03)^n (245,000), \text{ or equivalently, } P_n = 245,000(1.03)^n.$$

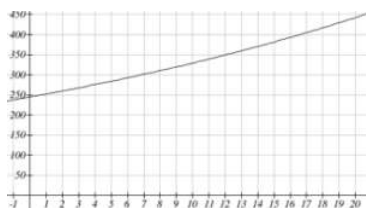
Using this equation, we were able to predict the population in the future.

Suppose we wanted to know when the population of Olympia would reach 400 thousand. Since we are looking for the year  $n$  when the population will be 400 thousand, we would need to solve the equation

$$400,000 = 245,000(1.03)^n \quad \text{dividing both sides by 245,000 gives}$$

$$1.6327 = 1.03^n$$

One approach to this problem would be to create a table of values, or to use technology to draw a graph to estimate the solution.



From the graph, we can estimate that the solution will be around 16 to 17 years after 2008 (2024 to 2025). This is pretty good, but we'd really like to have an algebraic tool to answer this question. To do that, we need to introduce a new function that will undo exponentials, similar to how a square root undoes a square. For exponentials, the function we need is called a logarithm. It is the inverse of the exponential, meaning it undoes the exponential. While there is a whole family of logarithms with different bases, we will focus on the common log, which is based on the exponential  $10^x$ .

### Common Logarithm

The common logarithm, written  $\log(x)$ , undoes the exponential  $10^x$

This means that  $\log(10^x) = x$ , and likewise  $10^{\log(x)} = x$

This also means the statement  $10^a = b$  is equivalent to the statement  $\log(b) = a$

$\log(x)$  is read as "log of  $x$ ", and means "the logarithm of the value  $x$ ". It is important to note that this is *not* multiplication – the log doesn't mean anything by itself, just like  $\sqrt{\phantom{x}}$  doesn't mean anything by itself; it has to be applied to a number.

### Example 9

Evaluate each of the following

a)  $\log(100)$       b)  $\log(1000)$       c)  $\log(10000)$       d)  $\log(1/100)$       e)  $\log(1)$

a)  $\log(100)$  can be written as  $\log(10^2)$ . Since the log undoes the exponential,  $\log(10^2) = 2$

b)  $\log(1000) = \log(10^3) = 3$

c)  $\log(10000) = \log(10^4) = 4$

d) Recall that

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•   $\log\left(\frac{1}{100}\right) = \log\left(\left(\frac{1}{10}\right)^{-2}\right) = -2$

e) Recall that  $x^0 = 1$ .  $\log(1) = \log(10^0) = 0$

It is helpful to note that from the first three parts of the previous example that the number we're taking the log of has to get *10 times bigger* for the log to increase in value by 1.

Of course, most numbers cannot be written as a nice simple power of 10. For those numbers, we can evaluate the log using a scientific calculator with a log button.

### Example 10

Evaluate  $\log(300)$

Using a calculator,  $\log(300)$  is approximately 2.477121

With an equation, just like we can add a number to both sides, multiply both sides by a number, or square both sides, we can also take the logarithm of both sides of the equation and end up with an equivalent equation. This will allow us to solve some simple equations.

### Example 11

a) Solve  $10x = 1000$

b) Solve  $10x = 3$

c) Solve  $2(10x) = 8$

a) Taking the log of both sides gives  $\log(10x) = \log(1000)$

Since the log undoes the exponential,  $\log(10x) = x$ . Similarly  $\log(1000) = \log(10^3) = 3$ .

The equation simplifies then to  $x = 3$ .

b) Taking the log of both sides gives  $\log(10x) = \log(3)$ .

On the left side,  $\log(10x) = x$ , so  $x = \log(3)$ . We can approximate this value with a calculator.  $x \approx 0.477$

c) Here we would first want to isolate the exponential by dividing both sides of the equation by 2, giving  $10x = 4$ .

Now we can take the log of both sides, giving  $\log(10x) = \log(4)$ , which simplifies to  $x = \log(4) \approx 0.602$

This approach allows us to solve exponential equations with powers of 10, but what about problems like  $2 = 1.03^n$  from earlier, which have a base of 1.03? For that, we need the exponent property for logs.

### Properties of Logs: Exponent Property

$$\log(A^r) = r \log(A)$$

To show why this is true, we offer a proof.

Since the logarithm and exponential undo each other,  $10^{\log A} = A$ .

So

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Utilizing the exponential rule that states

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,

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So then  $\log(A^r) = \log(10^{r \log A})$

Again utilizing the property that the log undoes the exponential on the right side yields the result

$$\log(A^r) = r \log A$$

### Example 12

Rewrite  $\log(25)$  using the exponent property for logs

$$\log(25) = \log(5^2) = 2\log(5)$$

This property will finally allow us to answer our original question.

### Solving exponential equations with logarithms

1. Isolate the exponential. In other words, get it by itself on one side of the equation. This usually involves dividing by a number multiplying it.
2. Take the log of both sides of the equation.
3. Use the exponent property of logs to rewrite the exponential with the variable exponent multiplying the logarithm.

4. Divide as needed to solve for the variable.

### Example 13

If Olympia is growing according to the equation,  $Pn = 245(1.03)^n$ , where  $n$  is years after 2008, and the population is measured in thousands. Find when the population will be 400 thousand.

We need to solve the equation

$$400 = 245(1.03)^n \quad \text{Begin by dividing both sides by 245 to isolate the exponential}$$

$$1.633 = 1.03^n \quad \text{Now take the log of both sides}$$

$$\log(1.633) = \log(1.03^n) \quad \text{Use the exponent property of logs on the right side}$$

$$\log(1.633) = n \log(1.03) \quad \text{Now we can divide by } \log(1.03)$$

$$\frac{\log(1.633)}{\log(1.03)} = n$$

We can approximate this value on a calculator

$$n \approx 16.591$$

Alternatively, after applying the exponent property of logs on the right side, we could have evaluated the logarithms to decimal approximations and completed our calculations using those approximations, as you'll see in the next example. While the final answer may come out slightly differently, as long as we keep enough significant values during calculation, our answer will be close enough for most purposes.

### Example 14

Polluted water is passed through a series of filters. Each filter removes 90% of the remaining impurities from the water. If you have 10 million particles of pollutant per gallon originally, how many filters would the water need to be passed through to reduce the pollutant to 500 particles per gallon?

In this problem, our “population” is the number of particles of pollutant per gallon. The initial pollutant is 10 million particles per gallon, so  $P_0 = 10,000,000$ . Instead of changing with time, the pollutant changes with the number of filters, so  $n$  will represent the number of filters the water passes through.

Also, since the amount of pollutant is *decreasing* with each filter instead of increasing, our “growth” rate will be negative, indicating that the population is decreasing instead of increasing, so  $r = -0.90$ .

We can then write the explicit equation for the pollutant:

$$Pn = 10,000,000(1 - 0.90)^n = 10,000,000(0.10)^n$$

To solve the question of how many filters are needed to lower the pollutant to 500 particles per gallon, we can set  $Pn$  equal to 500, and solve for  $n$ .

$$500 = 10,000,000(0.10)^n$$

Divide both sides by 10,000,000

$$0.00005 = 0.10^n$$

Take the log of both sides

$$\log(0.00005) = \log(0.10^n)$$

Use the exponent property of logs on the right side

$$\log(0.00005) = n \log(0.10)$$

Evaluate the logarithms to a decimal approximation

$$-4.301 = n(-1)$$

Divide by -1, the value multiplying  $n$

$$4.301 = n$$

It would take about 4.301 filters. Of course, since we probably can't install 0.3 filters, we would need to use 5 filters to bring the pollutant below the desired level.

#### Try it Now 4

India had a population in 2008 of about 1.14 billion people. The population is growing by about 1.34% each year. If this trend continues, when will India's population reach 1.2 billion?

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## CHAPTER OVERVIEW

### 5: Financial Mathematics

- 5.1: Simple Interest
- 5.2: Compound Interest
- 5.3: Annuities
- 5.4: Loans
- 5.5: Solving For Time

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## 5.1: Simple Interest

Discussing interest starts with the principal, or amount your account starts with. This could be a starting investment, or the starting amount of a loan. Interest, in its most simple form, is calculated as a percent of the principal. For example, if you borrowed \$100 from a friend and agree to repay it with 5% interest, then the amount of interest you would pay would just be 5% of 100:  $\$100(0.05) = \$5$ . The total amount you would repay would be \$105, the original principal plus the interest.

### Simple One-Time Interest

$$I = P_0 r$$

$$A = P_0 + I = P_0 + P_0 r = P_0(1 + r)$$

$I$  is the interest

$A$  is the end amount: principal plus interest

$P_0$  is the principal (starting amount)

$r$  is the interest rate (in decimal form. Example: 5% = 0.05)

### ✓ Example 5.1.1

A friend asks to borrow \$300 and agrees to repay it in 30 days with 3% interest. How much interest will you earn?

#### **Solution**

The principal:

$$P_0 = \$300$$

3% rate:

$$r = 0.03$$

Therefore:

$$I = \$300(0.03) = \$9$$

You will earn \$9 interest.

One-time simple interest is only common for extremely short-term loans. For longer-term loans, it is common for interest to be paid on a daily, monthly, quarterly, or annual basis. In that case, interest would be earned regularly. For example, bonds are essentially a loan made to the bond issuer (a company or government) by you, the bond holder. In return for the loan, the issuer agrees to pay interest, often annually. Bonds have a maturity date, at which time the issuer pays back the original bond value.

### ✓ Example 5.1.2

Suppose your city is building a new park, and issues bonds to raise the money to build it. You obtain a \$1,000 bond that pays 5% interest annually that matures in 5 years. How much interest will you earn?

#### **Solution**

Each year, you would earn 5% interest:  $\$1000(0.05) = \$50$  in interest. So, over the course of five years, you would earn a total of \$250 in interest. When the bond matures, you would receive back the \$1,000 you originally paid, leaving you with a total of \$1,250.

We can generalize this idea of simple interest over time.

 Simple Interest Over Time

$$I = P_0rt$$

$$A = P_0 + I = P_0 + P_0rt = P_0(1 + rt)$$

$I$  is the interest

$A$  is the end amount: principal plus interest

$P_0$  is the principal (starting amount)

$r$  is the interest rate in decimal form.

$t$  is time

 Definition: APR (Annual Percentage Rate)

Interest rates are usually given as an **annual percentage rate (APR)** – the total interest that will be paid in the year. If the interest is paid in smaller time increments, the APR will be divided up.

For example, a 6% APR paid monthly would be divided into twelve 0.5% payments. A 4% annual rate paid quarterly would be divided into four 1% payments.

 Example 5.1.3

Treasury Notes (T-notes) are bonds issued by the federal government to cover its expenses. Suppose you obtain a \$1,000 T-note with a 4% annual rate, paid semi-annually, with a maturity in 4 years. How much interest will you earn?

**Solution**

Since interest is being paid semi-annually (twice a year), the 4% interest will be divided into two 2% payments.

The principal:

$$P_0 = \$1000$$

3% rate:

$$r = 0.02$$

4 years = 8 half-years

$$t = 8$$

Therefore:

$$I = \$1000(0.02)(8) = \$160$$

You will earn \$160 interest total over the four years.

 Try It Now 1

A loan company charges \$30 interest for a one month loan of \$500. Find the annual interest rate they are charging.

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## 5.2: Compound Interest

With simple interest, we were assuming that we pocketed the interest when we received it. In a standard bank account, any interest we earn is automatically added to our balance, and we earn interest on that interest in future years. This reinvestment of interest is called **compounding**.

Suppose that we deposit \$1,000 in a bank account offering 3% interest, compounded monthly. How will our money grow?

The 3% interest is an annual percentage rate (APR) – the total interest to be paid during the year. Since interest is being paid monthly, each month, we will earn  $\frac{3\%}{12} = 0.25\%$  per month.

In the first month,

$$\begin{aligned}P_0 &= \$1000 \\r &= 0.0025(0.25\%) \\I &= \$1000(0.0025) = \$2.50 \\A &= \$1000 + \$2.50 = \$1,002.5\end{aligned}$$

In the first month, we will earn \$2.50 in interest, raising our account balance to \$1,002.50. In the second month,

$$\begin{aligned}P_0 &= \$1,002.50 \\I &= \$1002.50(0.0025) = \$2.51(\text{rounded}) \\A &= \$1002.50 + \$2.51 = \$1005.01\end{aligned}$$

Notice that in the second month we earned more interest than we did in the first month. This is because we earned interest not only on the original \$1,000 we deposited, but we also earned interest on the \$2.50 of interest we earned the first month. This is the key advantage that **compounding** of interest gives us.

Calculating out a few more months:

| Month | Starting Balance | Interest earned | Ending Balance |
|-------|------------------|-----------------|----------------|
| 1     | 1000.00          | 2.50            | 1002.50        |
| 2     | 1002.50          | 2.50            | 1005.01        |
| 3     | 1005.01          | 2.51            | 1007.52        |
| 4     | 1007.52          | 2.52            | 1010.04        |
| 5     | 1010.04          | 2.53            | 1012.57        |
| 6     | 1012.57          | 2.53            | 1015.10        |
| 7     | 1015.10          | 2.54            | 1017.64        |
| 8     | 1017.64          | 2.54            | 1020.18        |
| 9     | 1020.18          | 2.55            | 1022.73        |
| 10    | 1022.73          | 2.56            | 1025.29        |
| 11    | 1025.29          | 2.56            | 1027.85        |
| 12    | 1027.85          | 2.57            | 1030.42        |

To find an equation to represent this, if  $P_m$  represents the amount of money after  $m$  months, then we could write the recursive equation:

$$\begin{aligned}P_0 &= \$1000 \\P_m &= (1 + 0.0025)P_{m-1}\end{aligned}$$

You probably recognize this as the recursive form of exponential growth. If not, we could go through the steps to build an explicit equation for the growth:

$$P_0 = \$1000$$

$$P_1 = 1.0025P_0 = 1.0025(1000)$$

$$P_2 = 1.0025P_1 = 1.0025(1.0025(1000)) = 1.0025^2(1000)$$

$$P_3 = 1.0025P_2 = 1.0025(1.0025^2(1000)) = 1.0025^3(1000)$$

$$P_4 = 1.0025P_3 = 1.0025(1.0025^3(1000)) = 1.0025^4(1000)$$

Observing a pattern, we could conclude

$$P_m = (1.0025)^m(\$1000)$$

Notice that the \$1000 in the equation was  $P_0$ , the starting amount. We found 1.0025 by adding one to the growth rate divided by 12 since we were compounding 12 times per year. Generalizing our result, we could write

$$P_m = P_0 \left(1 + \frac{r}{k}\right)^m$$

In this formula:

- $m$  is the number of compounding periods (months in our example)
- $r$  is the annual interest rate
- $k$  is the number of compounds per year.

While this formula works fine, it is more common to use a formula that involves the number of years, rather than the number of compounding periods. If  $N$  is the number of years, then  $m = Nk$ . Making this change gives us the standard formula for compound interest.

#### Compound Interest

$$P_N = P_0 \left(1 + \frac{r}{k}\right)^{Nk}$$

- $P_N$  is the balance in the account after  $N$  years.
- $P_0$  is the starting balance of the account (also called initial deposit, or principal)
- $r$  is the annual interest rate in decimal form
- $k$  is the number of compounding periods in one year.

If the compounding is done annually (once a year),  $k = 1$ .

If the compounding is done quarterly,  $k = 4$ .

If the compounding is done monthly,  $k = 12$ .

If the compounding is done daily,  $k = 365$ .

The most important thing to remember about using this formula is that it assumes that we put money in the account once and let it sit there earning interest.

#### ✓ Example 5.2.1

A certificate of deposit (CD) is a savings instrument that many banks offer. It usually gives a higher interest rate, but you cannot access your investment for a specified length of time. Suppose you deposit \$3000 in a CD paying 6% interest, compounded monthly. How much will you have in the account after 20 years?

##### **Solution**

In this example,

The initial deposit:

$$P_0 = \$3000$$

6% annual rate:

$$r = 0.06$$

12 months in 1 year:

$$k = 12$$

Since we're looking for how much we'll have after 20 years

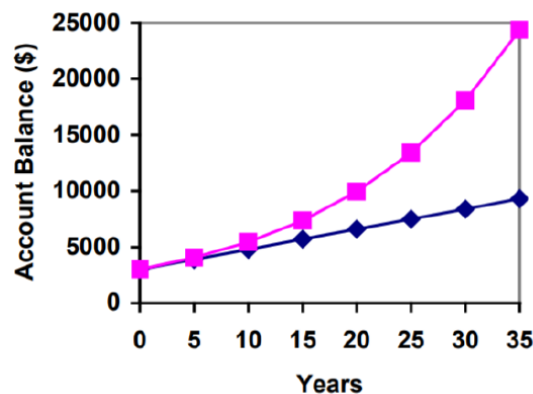
$$N = 20$$

So,

$$P_{20} = 3000 \left( 1 + \frac{0.06}{12} \right)^{20 \times 12} = \$9930.61 \text{ (round your answer to the nearest penny)}$$

Let us compare the amount of money earned from compounding against the amount you would earn from simple interest.

| Years | Simple Interest (\$15 per month) | 6% compounded monthly = 0.5% each month. |
|-------|----------------------------------|------------------------------------------|
| 5     | \$3900                           | \$4046.55                                |
| 10    | \$4800                           | \$5458.19                                |
| 15    | \$5700                           | \$7362.28                                |
| 20    | \$6600                           | \$9930.61                                |
| 25    | \$7500                           | \$13394.91                               |
| 30    | \$8400                           | \$18067.73                               |
| 35    | \$9300                           | \$24370.65                               |



As you can see, over a long period of time, compounding makes a large difference in the account balance. You may recognize this as the difference between linear growth and exponential growth.

#### Evaluating Exponents on the Calculator

When we need to calculate something like  $5^3$  it is easy enough to just multiply  $5 \cdot 5 \cdot 5 = 125$ . But when we need to calculate something like  $1.005^{240}$ , it would be very tedious to calculate this by multiplying 1.005 by itself 240 times! So to make things easier, we can harness the power of our scientific calculators.

Most scientific calculators have a button for exponents. It is typically either labeled like:

$\square$ ,  $y^x$ , or  $x^y$

To evaluate  $1.0052^{40}$  we'd type  $1.0052\square 40$ , or  $1.0052y^x 40$ . Try it out - you should get something around 3.3102044758

### ✓ Example 5.2.2

You know that you will need \$40,000 for your child's education in 18 years. If your account earns 4% compounded quarterly, how much would you need to deposit now to reach your goal?

#### Solution

In this example, we're looking for  $P_0$ .

4%:

$$r = 0.04$$

4 quarters in 1 year:

$$k = 4$$

Since we know the balance in 18 years:

$$N = 18$$

The amount we have in 18 years:

$$P_{18} = \$40000$$

In this case, we're going to have to set up the equation and solve for  $P_0$ .

$$40000 = P_0 \left( 1 + \frac{0.04}{4} \right)^{18 \times 4}$$

$$40000 = P_0 (2.0472)$$

$$P_0 = \frac{40000}{2.0472} = \$19539.84$$

So, you would need to deposit \$19,539.84 now to have \$40,000 in 18 years.

### Definition: Rounding

It is important to be very careful about rounding when calculating things with exponents. In general, you want to keep as many decimals during calculations as you can. Be sure to **keep at least 3 significant digits** (numbers after any leading zeros). Rounding 0.00012345 to 0.000123 will usually give you a "close enough" answer, but keeping more digits is always better.

### ✓ Example 5.2.3

The reason we shouldn't "over-round" is displayed by this example. Suppose you were investing \$1,000 at 5% interest compounded monthly for 30 years.

#### Solution

The initial deposit:

$$P_0 = \$1000$$

5%:

$$r = 0.05$$

12 months in 1 year:

$$k = 12$$

Since we're looking for the amount after 30 years:

$$N = 30$$

If we first compute  $\frac{r}{k}$ , we find  $\frac{0.05}{12} = 0.00416666666667$

Here is the effect of rounding this to different values:

| $\frac{r}{k}$ rounded to: | Gives $P_{30}$ to be: | Error    |
|---------------------------|-----------------------|----------|
| 0.004                     | \$4208.59             | \$259.15 |
| 0.0042                    | \$4521.45             | \$53.71  |
| 0.00417                   | \$4473.09             | \$5.35   |
| 0.004167                  | \$4468.28             | \$0.54   |
| 0.0041667                 | \$4467.80             | \$0.06   |
| No Rounding               | \$4467.74             |          |

If you're working in a bank, of course you wouldn't round at all. For our purposes, the answer we got by rounding to 0.00417, three significant digits, is close enough - \$5 off of \$4,500 isn't too bad. Certainly, keeping that fourth decimal place wouldn't have hurt.

### Using Your Calculator

Using your calculator In many cases, you can avoid rounding completely by how you enter things in your calculator. For example, in the example above, we needed to calculate

$$P_{30} = 1000 \left( 1 + \frac{0.05}{12} \right)^{12 \times 30}$$

We can quickly calculate  $12 \times 30 = 360$ , giving  $P_{30} = 1000 \left( 1 + \frac{0.05}{12} \right)^{360}$

Now we can use the calculator.

| Type This        | Calculator shows |
|------------------|------------------|
| $0.05 \div 12 =$ | 0.00416666666667 |
| $+ 1 =$          | 1.00416666666667 |
| $y^x 360 =$      | 4.46774431400613 |
| $\times 1000 =$  | 4467.74431400613 |

The previous steps were assuming you have a "one operation at a time" calculator; a more advanced calculator will often allow you to type in the entire expression to be evaluated. If you have a calculator like this, you will probably just need to enter:

$$1000 \times (1 + 0.05 \div 12) y^x 360 =$$

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## 5.3: Annuities

For most of us, we aren't able to put a large sum of money in the bank today. Instead, we save for the future by depositing a smaller amount of money from each paycheck into the bank. This idea is called a savings annuity. Most retirement plans like 401k plans or IRA plans are examples of savings annuities.

An annuity can be described recursively in a fairly simple way. Recall that basic compound interest follows from the relationship

$$P_m = \left(1 + \frac{r}{k}\right) P_{m-1}$$

For a savings annuity, we simply need to add a deposit,  $d$ , to the account with each compounding period:

$$P_m = \left(1 + \frac{r}{k}\right) P_{m-1} + d$$

Taking this equation from recursive form to explicit form is a bit trickier than with compound interest. It will be easiest to see by working with an example rather than working in general.

Suppose we will deposit \$100 each month into an account paying 6% interest. We assume that the account is compounded with the same frequency as we make deposits unless stated otherwise. In this example:

$$r = 0.06(6\%)$$

$$k = 12 \text{ (12 compounds/deposits per year)}$$

$$d = \$100 \text{ (our deposit per month)}$$

Writing out the recursive equation gives

$$P_m = \left(1 + \frac{0.06}{12}\right) P_{m-1} + 100 = (1.005)P_{m-1} + 100$$

Assuming we start with an empty account, we can begin using this relationship:

$$P_0 = 0$$

$$P_1 = (1.005)P_0 + 100 = 100$$

$$P_2 = (1.005)P_1 + 100 = (1.005)(100) + 100 = 100(1.005) + 100$$

$$P_3 = (1.005)P_2 + 100 = (1.005)(100(1.005) + 100) + 100 = 100(1.005)^2 + 100(1.005) + 100$$

Continuing this pattern, after  $m$  deposits, we'd have saved

$$P_m = 100(1.005)^{m-1} + 100(1.005)^{m-2} + \dots + 100(1.005) + 100$$

In other words, after  $m$  months, the first deposit will have earned compound interest for  $m - 1$  months. The second deposit will have earned interest for  $m - 2$  months. Last months deposit would have earned only one month worth of interest. The most recent deposit will have earned no interest yet.

This equation leaves a lot to be desired, though – it doesn't make calculating the ending balance any easier! To simplify things, multiply both sides of the equation by 1.005:

$$1.005P_m = 1.005(100(1.005)^{m-1} + 100(1.005)^{m-2} + \dots + 100(1.005) + 100)$$

Distributing on the right side of the equation gives

$$1.005P_m = 100(1.005)^m + 100(1.005)^{m-1} + \dots + 100(1.005)^2 + 100(1.005)$$

Now we'll line this up with like terms from our original equation, and subtract each side

$$1.005P_m = 100(1.005)^m + 100(1.005)^{m-1} + \dots + 100(1.005)$$

$$P_m = 100(1.005)^{m-1} + \dots + 100(1.005) + 100$$

Almost all the terms cancel on the right-hand side when we subtract, leaving

$$1.005P_m - P_m = 100(1.005)^m - 100$$

Solving for  $P_m$

$$0.005P_m = 100((1.005)^m - 1)$$

$$P_m = \frac{100((1.005)^m - 1)}{0.005}$$

Replacing  $m$  months with  $12N$ , where  $N$  is measured in years, gives

$$P_N = \frac{100((1.005)^{12N} - 1)}{0.005}$$

Recall  $0.005$  was  $\frac{r}{k}$  and  $100$  was the deposit  $d$ .  $12$  was  $k$ , the number of deposit each year. Generalizing this result, we get the saving annuity formula.

#### Annuity Formula

$$P_N = \frac{d \left( \left( 1 + \frac{r}{k} \right)^{Nk} - 1 \right)}{\left( \frac{r}{k} \right)}$$

where

- $P_N$  is the balance in the account after  $N$  years.
- $d$  is the regular deposit (the amount you deposit each year, each month, etc.)
- $r$  is the annual interest rate in decimal form.
- $k$  is the number of compounding periods in one year.
- If the compounding frequency is not explicitly stated, assume there are the same number of compounds in a year as there are deposits made in a year.

If the compounding frequency isn't stated then use these as rule of thumb:

- If you make your deposits every month, use monthly compounding,  $k = 12$ .
- If you make your deposits every year, use yearly compounding,  $k = 1$ .
- If you make your deposits every quarter, use quarterly compounding,  $k = 4$ .

and so on.

#### When do you use this

Annuities assume that you put money in the account *on a regular schedule (every month, year, quarter, etc.)* and let it sit there earning interest.

Compound interest assumes that you put money in the account *once* and let it sit there earning interest.

- Compound interest: *One* deposit
- Annuity: *Many* deposits.

#### ✓ Example 5.3.1

A traditional individual retirement account (IRA) is a special type of retirement account in which the money you invest is exempt from income taxes until you withdraw it. If you deposit \$100 each month into an IRA earning 6% interest, how much will you have in the account after 20 years?

##### Solution

In this example,

The monthly deposit:

$$d = \$100$$

6% annual rate:

$$r = 0.06$$

Since we're doing monthly deposits, we'll compound monthly:

$$k = 12$$

We want the amount after 20 years:

$$N = 20$$

Putting this into the equation, we get

$$P_{20} = \frac{100 \left( \left( 1 + \frac{0.06}{12} \right)^{20(12)} - 1 \right)}{\left( \frac{0.06}{12} \right)}$$

$$P_{20} = \frac{100 \left( (1.005)^{240} - 1 \right)}{(0.005)}$$

$$P_{20} = \frac{100 (3.310 - 1)}{(0.005)}$$

$$P_{20} = \frac{100 (2.310)}{(0.005)} = \$46,200$$

The account will grow to \$46,200 after 20 years.

Notice that you deposited into the account a total of \$24,000 (\$100 a month for 240 months). The difference between what you end up with and how much you put in is the interest earned. In this case, it is  $\$46,200 - \$24,000 = \$22,200$

### ✓ Example 5.3.2

You want to have \$200,000 in your account when you retire in 30 years. Your retirement account earns 8% interest. How much do you need to deposit each month to meet your retirement goal?

#### Solution

In this example, we're looking for  $d$ .

8% annual rate:

$$r = 0.08$$

Since we're depositing monthly:

$$k = 12$$

30 years:

$$N = 30$$

The amount we want to have in 30 years:

$$P_{30} = \$200,000$$

In this case, we're going to have to set up the equation, and solve for  $d$ .

$$200000 = \frac{d \left( \left( 1 + \frac{0.08}{12} \right)^{30(12)} - 1 \right)}{\left( \frac{0.08}{12} \right)}$$

$$200000 = \frac{d \left( (1.00667)^{360} - 1 \right)}{(0.00667)}$$

$$200000 = d(1491.57)$$

$$d = \frac{200000}{1491.57} = \$134.09$$

So, you would need to deposit \$134.09 each month to have \$200,000 in 30 years if your account earns 8% interest

In general, if we need to obtain the amount of the deposits, we can just rewrite the annuity formula as

$$d = \frac{P_N \cdot \frac{r}{k}}{\left(1 + \frac{r}{k}\right)^{Nk} - 1}$$

### Try It Now 2

A more conservative investment account pays 3% interest. If you deposit \$5 a day into this account, how much will you have after 10 years? How much is from interest?

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## 5.4: Loans

In the last section, you learned about payout annuities.

In this section, you will learn about conventional loans (also called amortized loans or installment loans). Examples include auto loans and home mortgages. These techniques do not apply to payday loans, add-on loans, or other loan types where the interest is calculated up front.

One great thing about loans is that they use exactly the same formula as a payout annuity. To see why, imagine that you had \$10,000 invested at a bank, and started taking out payments while earning interest as part of a payout annuity, and after 5 years your balance was zero. Flip that around, and imagine that you are acting as the bank, and a car lender is acting as you. The car lender invests \$10,000 in you. Since you're acting as the bank, you pay interest. The car lender takes payments until the balance is zero.

### Loans Formula

$$P_0 = \frac{d \left( 1 - \left( 1 + \frac{r}{k} \right)^{-Nk} \right)}{\left( \frac{r}{k} \right)}$$

$P_0$  is the balance in the account at the beginning (the principal, or amount of the loan).

$d$  is your loan payment (your monthly payment, annual payment, etc)

$r$  is the annual interest rate in decimal form.

$k$  is the number of compounding periods in one year.

$N$  is the length of the loan, in years

Like before, the compounding frequency is not always explicitly given, but is determined by how often you make payments.

### When do you use this

The loan formula assumes that you make loan payments *on a regular schedule (every month, year, quarter, etc.)* and are paying interest on the loan.

Compound interest: *One deposit*

Annuity: *Many deposits*

Payout Annuity: *Many withdrawals*

Loans: *Many payments*

### ✓ Example 5.4.1

You can afford \$200 per month as a car payment. If you can get an auto loan at 3% interest for 60 months (5 years), how expensive of a car can you afford? In other words, what amount loan can you pay off with \$200 per month?

#### **Solution**

In this example,

The monthly loan payment:

$$d = \$200$$

3% annual rate:

$$r = 0.03$$

Since we're doing monthly withdrawals, we'll compound monthly:

$$k = 12$$

Since we're making monthly payments for 5 years:

$$N = 5$$

We're looking for  $P_0$ , the starting amount of the loan.

$$P_0 = \frac{200 \left( 1 - \left( 1 + \frac{0.03}{12} \right)^{-5(12)} \right)}{\left( \frac{0.03}{12} \right)}$$

$$P_0 = \frac{200 \times \left( 1 - (1.0025)^{-60} \right)}{(0.0025)}$$

$$P_0 = \frac{200 (1 - 0.861)}{(0.0025)} = \$11,120$$

You will need to have \$11,120 in your account when you retire.

Notice that you withdrew a total of \$12,000 (\$200 a month for 60 months). The difference between what you pulled out and what you started with is the **interest paid**. In this case, it is  $\$12,000 - \$11,120 = \$880$  in interest.

#### ✓ Example 5.4.2

You want to take out a \$140,000 mortgage (home loan). The interest rate on the loan is 6%, and the loan is for 30 years. How much will your monthly payments be?

##### Solution

In this example, we're looking for  $d$ .

6% annual rate:

$$r = 0.06$$

Since we're paying monthly:

$$k = 12$$

30 years:

$$N = 30$$

The starting loan amount:

$$P_{30} = \$140,000$$

In this case, we're going to have to set up the equation, and solve for  $d$ .

$$140000 = \frac{d \left( 1 - \left( 1 + \frac{0.06}{12} \right)^{-30(12)} \right)}{\left( \frac{0.06}{12} \right)}$$

$$140000 = \frac{d \left( 1 - (1.005)^{-360} \right)}{(0.005)}$$

$$140000 = d(166.792)$$

$$d = \frac{140000}{166.792} = \$839.37$$

You will make payments of \$839.37 per month for 30 years.

In general, if we wanted to obtain the amount of each payment, we can rewrite the loan formula as

$$d = \frac{P_0 \cdot \frac{r}{k}}{\left(1 - \left(1 + \frac{r}{k}\right)^{-Nk}\right)}$$

#### Try It Now 4

Janine bought \$3,000 of new furniture on credit. Because her credit score isn't very good, the store is charging her a fairly high interest rate on the loan: 16%. If she agreed to pay off the furniture over 2 years, how much will she have to pay each month?

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## 5.5: Solving For Time

Often, we are interested in how long it will take to accumulate money or how long we'd need to extend a loan to bring payments down to a reasonable level.

### Note

This section assumes you've covered solving exponential equations using logarithms, either in prior classes or in the growth models chapter.

### ✓ Example 5.5.1

If you invest \$2,000 at 6% compounded monthly, how long will it take the account to double in value?

#### Solution

This is a compound interest problem since we are depositing money once and allowing it to grow. In this problem,

The initial deposit:

$$P_0 = \$2000$$

6% annual rate:

$$r = 0.06$$

12 months in 1 year:

$$k = 12$$

So, our general equation is  $P_N = 2000 \left(1 + \frac{0.06}{12}\right)^{N \times 12}$ . We also know that we want our ending amount to be double of \$2,000, which is \$4,000, so we're looking for  $N$  so that  $P_N = 4,000$ . To solve this, we set our equation for  $P_N$  equal to 4,000.

Divide both sides by 2,000.

$$4000 = 2000 \left(1 + \frac{0.06}{12}\right)^{N \times 12}$$

To solve for the exponent, take the log of both sides.

$$2 = (1.005)^{12N}$$

Use the exponent property of logs on the right side.

$$\log(2) = \log((1.005)^{12N})$$

Now we can divide both sides by  $12 \log(1.005)$

$$\log(2) = 12N \log(1.005)$$

$$\frac{\log(2)}{12 \log(1.005)} = N$$

Approximating this to three decimal places, we obtain  $N = 11.581$ . Thus, it will take about 11.581 years for the account to double in value.

Note that your answer may come out slightly differently if you had evaluated the logs to decimals and rounded during your calculations, but your answer should be close. For example, if you rounded  $\log(2)$  to 0.301 and  $\log(1.005)$  to 0.00217, then your final answer would have been about 11.577 years.

### ✓ Example 5.5.2

If you invest \$100 each month into an account earning 3% compounded monthly, how long will it take the account to grow to \$10,000?

#### Solution

This is a savings annuity problem since we are making regular deposits into the account.

The monthly deposit:

$$d = \$100$$

3% annual rate:

$$r = 0.03$$

Since we're doing monthly deposits, we'll compound monthly:

$$k = 12$$

We don't know  $N$ , but we want  $P_N$  to be \$10,000

Putting this into the equation, we get:

$$10000 = \frac{100 \left( \left( 1 + \frac{0.03}{12} \right)^{N(12)} - 1 \right)}{\left( \frac{0.03}{12} \right)}$$

Simplifying the fractions a bit:

$$10000 = \frac{100 \left( (1.0025)^{12N} - 1 \right)}{(0.0025)}$$

We want to isolate the exponential term,  $1.0025^{12N}$ , so multiply both sides by 0.0025

Divide both sides by 100:

$$25 = 100 \left( (1.0025)^{12N} - 1 \right)$$

Add 1 to both sides:

$$0.25 = (1.0025)^{12N} - 1$$

Now take the log of both sides:

$$1.25 = (1.0025)^{12N}$$

Use the exponent property of logs:

$$\log(1.25) = \log((1.0025)^{12N})$$

Divide by  $12 \log(1.0025)$

$$\begin{aligned} \log(1.25) &= 12N \log(1.0025) \\ \frac{\log(1.25)}{12 \log(1.0025)} &= N \end{aligned}$$

Approximating to three decimal places, we get  $N = 7.447$  years. Thus, it will take about 7.447 years to grow the account to \$10,000

### 👉 Try It Now 5

Joel is considering putting a \$1,000 laptop purchase on his credit card, which has an interest rate of 12% compounded monthly. How long will it take him to pay off the purchase if he makes payments of \$30 a month?

## Try It Now Answers

1.  $I = \$30$  of interest

$P_0 = \$500$  principal

$r = \text{unknown}$

$t = 1$  month

Using  $I = P_0 r t$ , we get  $30 = 500 \cdot r \cdot 1$ . Solving, we get  $r = 0.06$ , or 6%. Since the time was monthly, this is the monthly interest. The annual rate would be 12 times this: 72% interest.

2. The daily deposit:  $d = \$5$

3% annual rate:  $r = 0.03$

Since we're doing daily deposits, we'll compound daily:  $k = 365$

We want the amount after 10 years:  $N = 10$

$$P_{10} = \frac{5 \left( \left( 1 + \frac{0.03}{365} \right)^{365 \times 10} - 1 \right)}{\frac{0.03}{365}} = \$21,282.07$$

We would have deposited a total of  $\$5 \cdot 365 \cdot 10 = \$18250$ , so  $\$3,032.07$  is from interest

3.  $d = \text{unknown}$

4% annual rate:  $r = 0.04$

Since we're doing annual scholarships:  $k = 1$

20 years:  $N = 20$

We're starting with \$100,000:  $P_0 = 100,000$

$$100000 = \frac{d \left( 1 - \left( 1 + \frac{0.04}{1} \right)^{-20 \times 1} \right)}{\frac{0.04}{1}}$$

Solving for  $d$  gives \$7,358.18 each year that they can give in scholarships.

It is worth noting that usually donors instead specify that only interest is to be used for scholarship, which makes the original donation last indefinitely. If this donor had specified that,  $\$100,000(0.04) = \$4,000$  a year would have been available.

4.  $d = \text{unknown}$

16% annual rate:  $r = 0.16$

Since we're making monthly payments:  $k = 12$

2 years to repay:  $N = 2$

We're starting with a \$3,000 loan:  $P_0 = 3,000$

$$3000 = \frac{d \left( 1 - \left( 1 + \frac{0.16}{12} \right)^{-2 \times 12} \right)}{\frac{0.16}{12}}$$

Solving for  $d$  gives \$146.89 as monthly payments.

In total, she will pay \$3,525.36 to the store, meaning she will pay \$525.36 in interest over the two years.

5. The monthly payments:  $d = \$30$

12% annual rate:  $r = 0.12$

Since we're making monthly payments:  $k = 12$

2 years to repay:  $N = 2$

We're starting with a \$1,000 loan:  $P_0 = 1,000$

We are solving for  $N$ , the time to pay off the loan

$$1000 = \frac{30 \left( 1 - \left( 1 + \frac{0.12}{12} \right)^{-N(12)} \right)}{\frac{0.12}{12}}$$

Solving for  $N$  gives 3.396. It will take about 3.4 years to pay off the purchase.

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